Ample Dividing

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Abstract

We construct a stable one-based, trivial theory with a reduct which is not trivial. This answers a question of John B. Goode. Using this, we construct a stable theory which is n-ample for all natural numbers n, and does not interpret an infinite group.
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Introduction

The constructions of Hrushovski which produce new strongly minimal sets [8], strictly stable $\aleph_0$-categorical structures [7], and supersimple $\aleph_0$-categorical structures [9] are now very familiar. In those which do not involve an infinite field, the independence relation of non-forking satisfies a property called *CM-triviality* ([8], Proposition 10; 'CM-trivial' is equivalent to 'not 2-ample' defined below), which restricts its complexity. It is a major open problem to decide whether there are strongly minimal sets which are not CM-trivial and which do not interpret an infinite field. The work of Zil'ber which interprets Hrushovski's constructions in the context of complex analytic functions gives this problem additional significance.

At present, this problem looks beyond reach, so we should perhaps settle for less: we look for stable structures which are not CM-trivial and do not involve an infinite field. The first such example was given by Baudisch and Pillay in [1]. They construct an ω-stable structure (of infinite rank) which is non-CM-trivial. Their example is constructed as an incidence structure of points, lines and planes satisfying axioms which bear the same relation to properties of points, lines and planes in euclidean space as Lachlan's pseudoplane axioms bear to the properties of points and lines. Baudisch and Pillay therefore refer to their example as a (free) pseudospace.

However, outside the context of finite rank structures another notion is relevant. Recall ([5]) that a stable theory is *trivial* if, for every three tuples a, b, c of elements and any set A of parameters from some model, if a, b, c are pairwise independent over A, then a, b, c are independent over A. A superstable trivial theory with all types having finite U-rank is one-based ([5], Proposition 9), and this is stronger than CM-triviality. Baudisch and Pillay show that their example is trivial: therefore it lacks much of the flavour which would have to be present in a finite rank example. Of course we can obtain an ω-stable, non-trivial, non-CM-trivial structure by taking the disjoint union of the Baudisch-Pillay example with, say, a vector space, but this is really avoiding the issue.

In [12], Pillay extended the notion of CM-triviality into a hierarchy of geometric complexity for stable theories.

Definition 0.1 Suppose $n \geq 1$ is a natural number. A complete stable theory T is *n-ample* if (in some model of T, possibly after naming some parameters) there exist tuples $a_0, \ldots, a_n$ such that:
(i) $a_n \not\perp a_0$;
(ii) $a_n \perp_{a_i} a_0 \ldots a_{i-1}$ for $1 \leq i < n$;
(iii) $\mathrm{acl}(a_0) \cap \mathrm{acl}(a_1) = \mathrm{acl}(\emptyset)$;
(iv) $\mathrm{acl}(a_0 \ldots a_{i-1} a_i) \cap \mathrm{acl}(a_0 \ldots a_{i-1} a_{i+1}) = \mathrm{acl}(a_0 \ldots a_{i-1})$ for $1 \leq i < n$.
Here acl is algebraic closure in the T^{eq} sense.

Clearly $(n+1)$-ample implies n-ample, and Pillay observes that T is not 1-ample iff it is one-based, and it is 2-ample iff it is not CM-trivial. Moreover, a stable structure which interprets an infinite field is n-ample for all n. We remark in passing that for $n > 2$ it seems to us to be more natural to replace (ii) in Pillay's definition by:
(ii)$'$ $a_n \ldots a_{i+1} \perp_{a_i} a_0 \ldots a_{i-1}$ for $1 \leq i < n$
(or equivalently by the requirement that $x_{i+1} \perp_{a_i} a_0 \ldots a_{i-1}$). For example, Pillay's definition of 3-ampleness appears to allow that possibility that $a_0 \in \mathrm{acl}(a_2)$.

It is plausible that the construction of [1] could be extended to give an (infinite rank) ω-stable trivial structure which is n-ample for $n > 2$, although the technical difficulties are already quite severe in [1]. In this paper we give a different type of construction in which there is really no additional work involved in going from 2-ampleness to n-ample for all n. Moreover, unlike in [1], the structures we produce are not trivial, and the n-ampleness is witnessed by elements having the same strong type. However, our structures are stable, but not superstable, and it is an interesting problem to find a superstable structure with these properties. Another problem is to construct a regular type p (in a stable theory) whose geometry is 2-ample (by which we mean conditions as above given by tuples of realizations of p and where algebraic closure is replaced by p-closure). Both of these problems retain more of the geometric character of the problem of constructing a 2-ample strongly minimal set than we have achieved here.

We construct our structures as reducts of one-based, trivial stable structures. It is well-known that a reduct (where one discards some of the existing structure) of a one-based theory need not be one-based (although this cannot happen in a finite rank structure [3]). The easiest example (from [3] and due to Hodges) is as follows. One considers directed graphs with no directed cycles in which each vertex has infinitely many predescessors but only one successor (- we shall say 'descendant' in the sequel). This gives a complete, stable, one-based trivial theory. If we consider the graph reduct where one forgets the orientation of the edges, the result is no longer one-based: its models are disjoint unions of trees with all vertices of infinite valency, and the complete type of an edge gives a type-definable pseudoplane (the *free pseudoplane*). Chapter 4 of [11] is a convenient reference for this material.

In Hodges' example the reduct is still trivial. In [5], the question is posed as to whether a reduct of a stable trivial theory can be non-trivial. In Section 1 we show that it can be. Essentially we change the condition 'every vertex has one descendant' in the previous example to 'every vertex has at most 2 descendants.' From this class of directed graphs together with embeddings which add no more descendants, one axiomatises a generic structure which is stable, one-based and trivial. The (undirected) graph reduct is stable, but no longer one-based nor trivial (Theorem 1.9).

The difference between our example and Hodges' example may be explained as follows (- these remarks are essentially due to the Referee). In both cases the graphs and directed graphs have a notion of closure, which turns out to be algebraic closure in the model-theoretic sense. Two closed sets A and B are independent over their intersection provided that: (i) they are in free amalgamation over $A \cap B$; (ii) their union $A \cup B$ is closed. The first condition is of a trivial nature, but not necessarily the second. In the case of the directed graphs, it follows automatically as closure is closure under descendants and so the union of two closed sets is closed. For the undirected graphs, in Hodges' example closure is closure under shortest paths between pairs of points. Connected components are trees, so if $A \cup B$, $B \cup C$ and $A \cup C$ are closed it follows that $A \cup B \cup C$ is closed. This is precisely what does not happen in our example.

Goode's question remains open for superstable theories (having a type of infinite rank). It would be good to know if the example in [1] can be seen as a reduct of a one-based structure (although, of course, as this is trivial, it would not resolve Goode's question). More interestingly, one could ask whether the ω-stable structures of infinite rank given by Hrushovski's constructions are reducts of trivial (one-based) structures.[1]

The n-ample structure M of Theorem 2.11 is also constructed as a reduct of a trivial one-based structure N. In particular, no infinite group is interpretable in M, as no infinite group is interpretable in N (because it is trivial). On N one has binary relations $V_1, V_2, \ldots$, each of which gives a directed graph with all vertices having at most 2 descendants, as in Section 1. In the reduct we will again forget the direction of the edges to give relations $W_1, W_2, \ldots$. The theory of N is constructed so that the existence of various undirected paths is preseved under descendant-closed embeddings, and in the reduct we also include binary predicates $P^{i,r}$ for the existence of these types of paths. Roughly speaking, the intuition is as in the example of Baudisch and Pillay. One should think of W_1 as giving a point-line incidence relation (on M); W_2 a line-plane incidence relation and so on. Then, for example, the predicate $P^{1,2}(x, y)$ indicates the existence of a path $W_1(x, z), W_2(z, y)$: that is, a line z incident with both x and y. Thus, one thinks of $P^{1,2}$ as giving point-plane incidence. (In Proposition 2.13 we show that this intuition is actually fairly precise: the main correction we need to make is to add parameters to ensure that the relations W_i give pseudoplanes.)

We have worked throughout with directed graphs with every vertex having at most 2 descendants. Of course, we could replace 2 here by any larger integer, and this can be done independently for each of V_i. Thus one obtains (very cheaply) continuum many examples from Section 2. It might be interesting to investigate whether these constructions can be generalised to relations of higher arity (- so not just based on graphs and digraphs).[2]

Acknowledgements. The Author is very much indebted to the Referee of the original version of this paper. In that version, we worked with unary algebras rather than directed graphs, and missed the strong form of the

[1]The Author has recently shown that this is the case [4].

[2]Again, see [4].

amalgamation lemma (2.3). Consequently, we were unable to axiomatise the generic corresponding to N and regarded it as a stable Robinson theory. Thus in the original version the n-ample reduct was not known to be fully first-order stable. It was the suggestion of the Referee to work with directed graphs and to amplify the original description of the example which now forms Section 1. The observation that this example provides an answer to Goode's question is due to the Referee. The Referee is also to be thanked for pointing out a number of inaccuracies in the original version, and for demanding more explanation and less notation.

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1 Goode's Question

1.1 Directed Graphs

We work in a first-order language with a single binary relation symbol $V(x, y)$, pronounced 'y is a *descendant* of x.' Let T' be the theory whose models are the directed graphs with no directed cycles, and in which all vertices have at most two descendants. If $B \models T'$ and $X \subseteq B$ we write $\mathrm{cl}'_B(X)$ for the closure of X in B under the operation of taking descendants. As any vertex has at most two descendants, $\mathrm{cl}'_B(X)$ is contained in the algebraic closure of X. We write $A \leq' B$ if A contains all of its descendants in B. This closure is disintegrated: $\mathrm{cl}'_B(X) = \bigcup_{x \in X} \mathrm{cl}'_B(x)$.

We have the following amalgamation property for models of T'. Suppose $B, C \models T'$ and $A \subseteq B$, $A \leq' C$. Then the disjoint union F of B and C over A (with directed edges those of B and C) is again a model of T' and $B \leq' F$. We refer to F as the free amalgam of B and C over A.

We now describe the theory mentioned in the abstract. Form T'_1 by adjoining to T' sentences of the form:

$$\forall \bar{x} \exists \bar{y} (\Delta_X(\bar{x}) \to \Delta_{X,A}(\bar{x}, \bar{y}) \wedge \text{`}\mathrm{cl}'(\bar{x}\bar{y}) = \mathrm{cl}'(\bar{x}) \cup \bar{y}\text{'})$$

where A is a finite model of T', $X \leq' A$, $\Delta_X(\bar{x})$ denotes the basic diagram of X and $\Delta_{X,A}(\bar{x}, \bar{y})$ denotes the basic diagram of A, where the variables $\bar{y}$ represent the elements of $A \setminus X$. The condition '$\mathrm{cl}'(\bar{x}\bar{y}) = \mathrm{cl}'(\bar{x}) \cup \bar{y}$' is expressed in a first-order way by saying that any descendant of a variable in $\bar{y}$ is one of the variables in $\overline{xy}$. Thus a model M of T' is a model of T'_1 iff for all finite subsets X of M and $X \leq' A \models T'$ with A finite, there is an embedding over X of A into M whose image A_1 has closure $\mathrm{cl}'_M(X) \cup A_1$. Note that by compactness we have also have the following. Suppose M is an ω-saturated model of T'_1, $X \leq' M$ is the closure of a finite set, and $X \leq' A \models T'$ where A is the closure of a finite set. Then there exists an embedding over X of A into M with closed image.

Lemma 1.1 *The theory T'_1 is consistent and complete. Moreover, n-tuples $\bar{a}, \bar{b}$ in models M, N of T'_1 have the same types iff the map $\bar{a} \mapsto \bar{b}$ extends to an isomorphism between $\mathrm{cl}'_M(\bar{a})$ and $\mathrm{cl}'_N(\bar{b})$.*

Proof: Consistency is by a Fraïssé construction using the amalgamation property. If the types of $\bar{a}$ and $\bar{b}$ are the same, then clearly we have an isomorphism between their closures. For the rest, it is enough to show that if M, N are ω-saturated models of T'_1, then the set of isomorphisms between closures of finite subsets of M and N is a back-and-forth system (cf. [13], Chapitre 5.b or [14], Section 5.2). But this follows at once from the remarks immediately preceding the lemma. □

In the terminology of [2], the theory T'_1 describes the semigenerics for the class of models of T' with embeddings given by $\leq'$. (Of course, general results from [2] also give the above lemma: see in particular 1.26–1.32.) Note that T'_1 is near model complete, but not model complete: elementary embedding between models of T'_1 is the same as closed embedding.

Suppose $M \models T'_1$. If $B \leq' M$ and $\bar{a}$ is a tuple in M, then $\mathrm{tp}_M(\bar{a}/B)$ is determined by the quantifier-free type of $\mathrm{cl}'_M(\bar{a}B)$, and this is the free amalgam over $B \cap \mathrm{cl}'_M(\bar{a})$ of B and $\mathrm{cl}'_M(\bar{a})$. In particular, as the closure of a finite set is countable, the number of 1-types over B is at most $max(2^{\aleph_0}, |B|^{\aleph_0})$. So T'_1 is stable.

With the above notation, we next show that $\mathrm{tp}(\bar{a}/B)$ does not divide over $C = B \cap \mathrm{cl}'_M(\bar{a})$. Without loss we may assume that M is a large saturated model of T'_1. Suppose $(B_i : i < \omega)$ is any sequence of translates of B over C. Let X be the union of these and let Y be the free amalgam of X and $\mathrm{cl}'(\bar{a})$ over C. Then $X \leq' Y$ so we may assume (by the saturation), that $Y \leq' M$. Let $\bar{a}_1$ be the copy of $\bar{a}$ inside Y. Then $\mathrm{cl}'(\bar{a}_1) \cap B_i = C$ and $\mathrm{cl}'(\bar{a}_1)$ and B_i are freely amalgamated over C. So $\mathrm{tp}(\bar{a}_1 B_i) = \mathrm{tp}(\bar{a}B)$, as required.

In summary, we have:

Lemma 1.2 *The theory T_1' is stable and if A, B, C are subsets of a model of T_1', then $A \mathop{\smile\!\!\!\!\!\!\mid}_C B \Leftrightarrow \mathrm{cl}'(AC) \cap \mathrm{cl}'(BC) = \mathrm{cl}'(C)$. Moreover, T_1' is 1-based and trivial.*

Proof: We have stability already, and as dividing is the same as forking in a stable theory, we also have one direction of the double implication. The other direction follows from the observation that algebraic closure is given by cl' in a model of T_1'. The description of independence gives 1-basedness, and triviality follows from the fact that the closure cl' is disintegrated. $\square$

1.2 Undirected reducts

We now consider reducts where we forget the orientation of the directed edges. So we take the reducts in the language consisting of the definable relation $W(x, y) \leftrightarrow V(x, y) \vee V(y, x)$. As T' is a universal theory in a relational language, the class $\mathcal{G}$ of reducts of models of T' is first-order axiomatizable (see, for example [6], Theorem 6.6.7) by universal sentences T. In particular, a graph is in $\mathcal{G}$ iff all of its finite subgraphs are in $\mathcal{G}$.

We refer to an expansion of $B \in \mathcal{G}$ to a model to T' as an *orientation* of B.

Lemma 1.3 *A graph B is in $\mathcal{G}$ iff every finite subgraph of B has a vertex of valency ≤ 2 (in the subgraph).*

Proof. First, suppose $B \in \mathcal{G}$ and $A \subseteq B$ is finite. Take some orientation of B. As A is finite and has no oriented cycles, there is a vetex in A which is not a descendant of any other vertex in A. Thus, in the subgraph on A this vertex has valency ≤ 2.

For the converse, we may assume that B is finite. We construct an orientation of B as follows. Take a vertex $b_0 \in B$ of valency ≤ 2 and orient its edges outwards (- so it is not a descendant). Do the same on the subgraph on $B \setminus \{b_0\}$. Repeating this gives the required orientation. $\square$

If $A \subseteq B \in \mathcal{G}$, write $A \leq B$ to mean that there is an orientation of B in which A is a closed subset (that is, it contains all of its descendants). Note that if we have an orientation of B in which A is closed, the induced orientation of A can be replaced by any other orientation, and we still have an orientation of B (in which A is closed).

Lemma 1.4 *(i) If $A \leq B \in \mathcal{G}$ and $X \subseteq B$, then $A \cap X \leq X$.*
(ii) If $A \leq B \leq C \in \mathcal{G}$, then $A \leq C$.

Proof. (i) Take an orientation on B in which A contains all of its descendants. Then any descendant of a vertex in $A \cap X$ which lies in X must also lie in $A \cap X$.

(ii) Take an orientation of C in which B is closed. Replace the induced orientation on B by one in which A is closed. The result is still an orientation of C, and in it, A is closed. $\square$

Lemma 1.5 *Suppose $B, C \in \mathcal{G}$, $A \leq B$ and $A \subseteq C$. Then the disjoint union, F, of B and C over A is in $\mathcal{G}$ and $C \leq F$.*

Proof. Take an orientation on C. As $A \leq B$, the orientation on A induced by this can be extended to an orientation of B in which A is closed. Taking the disjoint union over A of these gives an orientation of F in which C is closed. $\square$

Again, we refer to F in the above as the free amalgam of B and C over A.

We now associate a closure with $\leq$. Suppose $X \subseteq B \in \mathcal{G}$. We define $\mathrm{cl}_B(X) = \bigcap\{C : X \subseteq C \leq B\}$, that is, the intersection of the closures of X in all possible orientations of B. If B is finite, then it follows from Lemma 1.4 that $\mathrm{cl}_B(X) \leq B$. The following characterization of cl_B gives this in general.

Lemma 1.6 *Suppose $X \subseteq B \in \mathcal{G}$. Then:*
(i) $\mathrm{cl}_B(X)$ is the union of all finite $Y \subseteq B$ such that the only vertices of valency ≤ 2 in the subgraph on Y lie in $X \cap Y$.
(ii) $\mathrm{cl}_B(X) \leq B$ and $\mathrm{cl}_B(X) = \bigcup\{\mathrm{cl}_B(X_0) : X_0 \subseteq X$ finite $\}$.

Proof. First, suppose that $X \subseteq A \leq B$ and Y is as in (i). Then $X \cap Y \subseteq A \cap Y \leq Y$. Take an orientation of Y in which $A \cap Y$ contains all of its descendants. If $Y \setminus A \cap Y$ is non-empty, it contain a vertex which is not a descendant of any vertex in Y in this orientation: but this is a contradiction as its valency is at least 3 in Y. Thus $Y \subseteq A$.

Let Z denote the union of such sets Y. From the previous paragraph, we have $Z \subseteq \mathrm{cl}_B(X)$. To show that $Z = \mathrm{cl}_B(X)$ and $\mathrm{cl}_B(X) \leq B$ it will suffice to prove that $Z \leq B$. Once we have this, the finitary character of cl_B follows from the description of cl_B in (i).

We do this first in the case where $B \setminus Z$ is finite (and non-empty). We have to produce an orientation of B in which all descendants of vertices in Z are in Z. Note that we can choose some orientation on Z and then there are only finitely many possibilities for the orientation on B: any edge between a vertex in $B \setminus Z$ and Z must be directed towards Z, so all that has to be determined is the orientation on the edges in $B \setminus Z$.

To show that there is some orientation (extending the given one on Z) we follow the proof of Lemma 1.3: it is enough to show that there is a vertex in $B \setminus Z$ of valency ≤ 2 in B, and proceed inductively. Suppose there is no such vertex. Let $S \subseteq Z$ be such that every vertex of $B \setminus Z$ is adjacent to at least 3 vertices of $S \cup (B \setminus Z)$. Each vertex in S is contained in some finite set Y as in (i). Taking the union of these with $B \setminus Z$, we obtain a finite subgraph in which the only vertices of valency ≤ 2 are in X. In particular, $B \setminus Z \subseteq Z$, a contradiction.

We have shown that if $Z \subseteq B_1 \subseteq B$ and $B_1 \setminus Z$ is finite, then a given orientation on Z can be extended to one on B_1 (with Z closed) in at least one of only finitely many ways. Thus, the general case follows by a compactness argument. □

We now consider the reduct T_1 of the theory T_1' (to the language consisting of $W(x, y)$). This is complete and stable (because T_1' is), and we shall show that T_1 is not trivial, thereby providing an answer to the question of Goode. Before doing this, we give an axiomatization of T_1 and characterize non-forking in its models. This is not strictly necessary in order to demonstrate that T_1 is not trivial, but it seems worthwhile.

If X is a finite subset of $B \in \mathcal{G}$ and $m \in \mathbb{N}$ let $\mathrm{cl}_B^m(X)$ be the union of sets $Y \subseteq B$ of size $\leq m$ in which the only vertices of valency ≤ 2 lie in $X \cap Y$. This is X-definable (uniformly in $|X|$), and the union of these sets (as m ranges over $\mathbb{N}$) is $\mathrm{cl}_B(X)$. Also note that $\mathrm{cl}_B^m(X)$ is finite. Otherwise, there exist infinitely many such Y (of size $\leq m$). By a Ramsey argument, we may assume that some infinite subcollection $\{Y_i : i < \omega\}$ of these have common pairwise intersection X_1 contained in X, and that they are all isomorphic over X. Consider $\bigcup_{i \leq 2} Y_i$ and discard from this any vertex in X_1 which is not adjacent to a vertex in one (equivalently, all) of the $Y_i \setminus X$. The result is a finite graph in which every vertex has valency at least 3: a contradiction.

For $m \in \mathbb{N}$ and finite $X \leq A \in \mathcal{G}$ consider the sentence $\sigma_{X,A}^m$ given by:

$$\forall \bar{x} \exists \bar{y} (\Delta_X(\bar{x}) \to \Delta_{X,A}(\bar{x}, \bar{y}) \wedge \text{`}\mathrm{cl}^m(\bar{x}\bar{y}) = \mathrm{cl}^m(\bar{x}) \cup \bar{y}\text{'})$$

where as before Δ_X and $\Delta_{X,A}$ denote the basic diagrams of X and A with the appropriate subdivision of the variables.

Lemma 1.7 *Together with T, the sentences $\sigma^m_{X,A}$ axiomatize T_1. Moreover, n-tuples $\bar{a}, \bar{b}$ in models M_1, M_2 of T_1 have the same types iff the map $\bar{a} \mapsto \bar{b}$ extends to an isomorphism between $\mathrm{cl}_{M_1}(\bar{a})$ and $\mathrm{cl}_{M_2}(\bar{b})$.*

Proof. First, we show that these sentences are in T_1. Suppose M' is a model of T_1' and X' is a finite subset of M' whose reduct is isomorphic to X. Extend the induced orientation on X to an orientation A' of A in which X is a closed subset. By the axiomatization of T_1', there is an embedding of A' into M' over X' whose image A_1 has closure $A_1 \cup \mathrm{cl}'_{M'}(X')$. In the reduct, this image witnesses the condition required for $\sigma^m_{X,A}$ (for any m).

Note that in any model M of T, $\mathrm{cl}_M(X)$ is contained in the algebraic closure of X, so the isomorphism type of $\mathrm{cl}_M(X)$ is implied by the type of X in M. Also, if M is an ω-saturated model of T and the sentences $\sigma^m_{X,A}$, then by compactness it has the following genericity property: if $B \leq M$, $B \leq C \in \mathcal{G}$, and B, C are closures of finite sets, then there is an embedding over B of C into M with closed image. So by the back-and-forth method (as in Lemma 1.1) T and these sentences axiomatize a complete theory, which must be T_1, and we also have the description of types as in the statement of the lemma. □

We remark that for each of T, T' there is a unique countable model in which the closures of finite sets are finite: these are the *generic* models (for the respective amalgamation classes of finite structures).

Lemma 1.8 *Suppose M is a (large, saturated) model of T_1 and $A, B, C \leq M$ are small subsets such that $A \cap B = C$, $A \cup B \leq M$ and $A \cup B$ is the free amalgam over C of A and B. Then $A \mathop{\smile\!\!\!\!\!\mid}_C B$.*

Conversely if a, b, c are small tuples in M and $a \mathop{\smile\!\!\!\!\!\mid}_c b$, then $A = \mathrm{cl}_M(ac)$, $B = \mathrm{cl}_M(bc)$, $C = \mathrm{cl}_M(c)$ satisfy the above conditions.

Proof. The proof that $\mathrm{tp}(A/B)$ does not divide over C is essentially as in the previous case (see the argument preceding Lemma 1.2). Temporarily refer to the independence given by sets in this configuration as 'strong independence.'

For the converse, it is enough to show that types of tuples of elements of M over closed sets are stationary (by homogeneity, the first part already gives us one type of non-forking extension). As we are in a stable theory it is enough, by the finite equivalence relation theorem, to show that any imaginary in the algebraic closure of $C \leq M$ is in its definable closure.

So suppose a is a finite tuple of elements of M and $\theta(x, y)$ is a C-definable f.e.r. on $\mathrm{tp}(a/C)$. Let $A = \mathrm{cl}_M(Ca)$. We can use the genericity property to find translates A_i (for $i < \omega$) of A over C such that $\bigcup_{i<\omega} A_i$ is the free amalgam over C of the A_i and for every n we have $\bigcup_{i<n} A_i \leq M$. Let a_i be the copy of a inside A_i. The a_i are strongly independent and indiscernible over C. As θ has finitely many classes, the a_i must all be in the same θ-class. So realisations of $\mathrm{tp}(a/C)$ which are strongly independent over C are in the same θ-class.

Now suppose a' is any realisation of $\mathrm{tp}(a/C)$. There is a realisation a'' of $\mathrm{tp}(a/C)$ which is strongly independent from a, a' over C (just consider the free amalgam of A and $\mathrm{cl}_M(Caa')$ over C). Then it is easy to see that a, a'' are strongly independent over C, as are a', a''. Thus $\theta(a, a')$, as required. $\square$

We remark that it follows from this description of independence that T_1 is CM-trivial.

Theorem 1.9 *The theory T_1 is not trivial.*

Proof. Let M be a saturated model of T_1. Consider the graph B with four vertices a_1, a_2, a_3, b and edges $\{a_i, b\}$. This can be oriented by giving b exactly two descendants, so one may regard B as a closed subset of M, and any singleton and any pair from $\{a_1, a_2, a_3\}$ is closed in M. Thus, by Lemma 1.8, the a_i are pairwise independent over the empty set. On the other hand, $b \in \mathrm{cl}_M(a_1, a_2, a_3)$, so $a_1, a_2, a_3 \not\leq M$, whence a_1 is not independent from a_2, a_3 (over the empty set). $\square$

A more elaborate construction shows that T_1 is not k-trivial for any $k \in \mathbb{N}$: there exists a set of $(k+2)$ non-independent points in which any $k+1$-subset is independent. Indeed, define graphs S_k recursively as follows. S_1 consist of points $a_{1,1}, a_{1,2}, a_{1,3}, b$ with $a_{1,j}$ adjacent to b, for each j. From S_i we construct S_{i+1} by adding new vertices $a_{i+1,j}$ for $1 \leq j \leq i+3$ and new edges $\{a_{i,j}, a_{i+1,j}\}, \{a_{i,j}, a_{i+1,j+1}\}$ for $1 \leq j \leq i+2$. It is easy to see that $S_k \in \mathcal{G}$ (for any k). Moreover, the whole graph is in the closure of $a_{k,1}, \ldots, a_{k,k+2}$ and any proper subset of this set of vertices is closed. The first of these statements follows easily from Lemma 1.6. The second is more difficult, but can be done by producing an orientation of the graph with $a_{k,i}$ deleted in which there are no descendants of vertices $a_{k,j}$.

Once we have this, if we regard S_k as a closed subset of a model of T_1, then $\{a_{k,1}, \ldots, a_{k,k+2}\}$ is not independent, but any $(k+1)$-subset is (by Lemma 1.8).

1.3 Further remarks

We conclude this section with two observations. The first is that T_1 is not superstable; the second is a curious connection between our example and Hrushovski's constructions.

We start with a construction which encodes finitely branching trees as subgraphs of M which are closures of single points.

Definition 1.10 Suppose we are given the following data $\mathcal{T}$:
• a rooted, finitely branching tree Θ of height ω;
• a collection $(B_t : t \in \Theta)$ of connected finite graphs in which all vertices have valency 3;
• for each $t \in \Theta$ an edge $e_t = \{a_t, b_t\}$ of B_t;
• for each $t \in \Theta$ and each immediate successor r of t, a vertex $v_r \in B_t$.

We write t^+ for the set of (immediate) successors of t in the tree Θ, and t^- for the (immediate) predecessor of t. We let $R_t = \{v_r : r \in t^+\}$, and we assume the v_r are distinct, and $a_t, b_t \notin R_t$. Furthermore, we assume that:
• R_t is a coclique in B_t;
• the subgraph on $B_t \setminus R_t$ with e_t removed is connected.

For example, we can take Θ as the binary tree and each B_t the graph given by the vertices and edges of a cube in which the v_r are a pair of diagonally opposite vertices.

We form a graph $B = B_{\mathcal{T}}$ by joining the graphs B_t together along the tree Θ, as follows. The vertex set of B consists of a new vertex x_0 and the disjoint union of the vertices of the B_t. The edges of B are as in the B_t, with the following exceptions:
• the edges e_t are removed;
• for each non-root vertex r in Θ we form new edges $\{v_r, a_r\}, \{v_r, b_r\}$;
• if t is the root of Θ, we form new edges $\{x_0, a_t\}, \{x_0, b_t\}$.

Lemma 1.11 *With $\mathcal{T}$ and B as above we have:*
(i) $B \models T$*;*
(ii) $\mathrm{cl}_B(x_0) = B$*;*
(iii) $\mathrm{cl}_B(v_r) = \{v_r\} \cup \bigcup_{r' \geq r} B_{r'}$.

Proof. (i) Suppose for a contradiction that X is a finite subset of B on which the induced subgraph has no vertex of valency at most 2. We show that $x_0 \in X$, which is a contradiction.

Suppose $X \cap B_t \neq \emptyset$. Then $X \cap B_t \not\subseteq R_t$ as R_t is a coclique whose vertices are adjacent to only two vertices outside B_t. If $x \in X \cap (B_t \setminus R_t)$ then all neighbours of x lie in X as there are only 3 of them. So as $B_t \setminus R_t$ (without the edge e_t) is connected, we have that all vertices of $B_t \setminus R_t$ are in X, in particular, $a_t \in X$. But then it follows that $v_t \in X \cap B_{t^-}$, and we can proceed down the tree to obtain $x_0 \in X$.

(ii) If T' is a finite initial segment of T then x_0 is the only vertex of valency ≤ 2 in $\{x_0\} \cup \bigcup_{t \in T'} B_t$. So the statement follows from Lemma 1.6.

(iii) This is similar to (ii). □

Using this it is easy to construct $2^{\aleph_0}$ non-isomorphic graphs $B_{\mathcal{T}}$. All of these can, of course, be realised as closed subsets of some model of T_1, thus T_1 is not small: there are continuum many 1-types over the empty set. Furthermore, we can now see that T_1 is not superstable. Essentially, the point is that a closed subset of a finitely generated closed set need not be finitely generated. More formally, in the above construction, take the tree Θ to be the binary tree $2^{<\omega}$ and let R be an infinite antichain in Θ. Let a be a point in some saturated model M of T_1 whose closure is isomorphic to $B_{\mathcal{T}}$. Let $c_t \in \mathrm{cl}_M(a)$ be the point corresponding to the vertex v_t in $B_{\mathcal{T}}$, and $C = \{c_r : r \in R\}$. Let $b \in M$ be of the same type over C as a and independent from a over C. So in particular, $\mathrm{cl}_M(a) \cap \mathrm{cl}_M(b) = \mathrm{cl}_M(C)$. On the other hand, a, b are not independent over any finite subset of C, as the algebraic closure of any set over which they are independent has to contain $\mathrm{cl}_M(C)$. The argument also shows that $\mathrm{tp}(a/\emptyset)$ is of infinite weight (and so T_1 cannot be superstable): the set $\{c_r : r \in R\}$ is independent over the empty set, but $a \not\downarrow c_r$ for each $r \in R$.

We now turn to what we see as an interesting connection between our example and Hrushovski's constructions.[3] We recall briefly some of the definitions for these.

Definition 1.12 If $k \in \mathbb{R}^{\geq 0}$ and B is a finite graph let $\delta_k(B) = k|B| - e(B)$, where $e(B)$ is the number of edges in B. For $A \subseteq B$ write $A \leq_k B$ if whenever $A \subseteq C \subseteq B$, then $\delta_k(C) \geq \delta_k(A)$.

[3] These connections are made clearer in [4].

This notion of embedding can be extended to infinite graphs and in general $A \leq_k B$ iff for all finite $X \subseteq B$ we have $A \cap X \leq_k X$.

Of particular interest for Hrushovski's constructions is the class of all finite graphs A with $\emptyset \leq_k A$. This class (with embeddings given by $\leq_k$) is an amalgamation class and if k is a natural number, then the theory T_k^H of the corresponding generic structure is ω-stable of rank $\omega \cdot k$. Moreover independence in its models is described as in Lemma 1.8, but with $\leq_k$ in place of $\leq$. We shall be concerned with the case $k = 2$.

Lemma 1.13 *If $B \in \mathcal{G}$ and $A \leq B$, then $A \leq_2 B$. In particular, $\emptyset \leq_2 B$.*

Proof. It is enough to do this when B is finite. We show by induction on $|C \setminus A|$ that if $A \subseteq C \subseteq B$, then $\delta_2(A) \leq \delta_2(C)$. Indeed, as $A \leq C$ there is $c \in C \setminus A$ which is of valency at most 2 in C. Let $C_1 = C \setminus \{c\}$. By inductive assumption $\delta_2(A) \leq \delta_2(C_1)$, and by definition of c, $\delta_2(C) \geq \delta_2(C_1)$. □

From this it follows that any graph in $\mathcal{G}$ can be $\leq_2$-embedded as a subgraph of some model of T_2^H. Note that closure with respect to $\leq_2$ (on a graph in $\mathcal{G}$) is contained in, but can be smaller than, the closure with respect to $\leq$ (in fact the closure of a finite set with respect to $\leq_2$ is finite). In particular, suppose $M \leq_2 M_2$ where M is a saturated model of T_1 and M_2 is a saturated model of T_2^H, and we have $A, B, C \leq M$ with A, B independent (in the sense of T_1) over C. Then A, B are independent over C in M_2, in the sense of T_2^H. (With a little extra effort the condition that $A, B \leq M$ can be removed.) On the other hand, if $A, B, C \leq M$ and A, B are independent over C in the sense of T_2^H, we can have $A \cup B \not\leq M$, so A, B are not independent over C in the sense of T_1.

There is nevertheless a sort of converse to all of this.

Lemma 1.14 *Suppose B is a finite graph and $\emptyset \leq_2 A \leq_2 B$. Then the edges of B can be directed so that: B has no directed cycles; any vertex has at most 4 descendants; and A contains all of its descendants.*

Proof. As in the proof of Lemma 1.3, it will suffice to show that if $A \subset C \subseteq B$, then there is a vertex in $C \setminus A$ with valency ≤ 4 in the subgraph on C.

The sum of the valencies in C of vertices in $C \setminus A$ is $2e(C) - 2e(A)$. So as $\delta_2(C) = 2|C| - e(C) \geq 2|A| - e(A)$, this is at most $4|C \setminus A|$. So the average valency in C of vertices in $C \setminus A$ is at most 4. Thus there is a vertex in $C \setminus A$ with valency ≤ 4, as required. $\Box$

So a model of T_2^H can be embedded as a closed substructure of a variant of our example (where one allows at most 4 descendants in the orientations). Of course, one can then repeat, and obtain a chain of Hrushovski's examples (with $k = 2, 4, 8, \ldots$) alternated with variants on our examples (allowing $2, 4, 8, \ldots$ descendants).

2 An ample structure

2.1 Directed structures

In this section we work with a first-order language in a signature consisting of denumerably many 2-ary relation symbols $V_1(x, y), V_2(x, y), \ldots$. The class $\mathcal{C}_0'$ of structures is given by the following (first-order) axioms. The relations V_i are disjoint; each V_i gives a directed graph in which all vertices have at most 2 descendants; the directed graph given by the union of the V_i has no directed cycles. Note that in such a structure we have a notion of closure as in the previous section: one closes under descendants for all the V_i. We shall again write $\mathrm{cl}'_B(X)$ for the closure of X in B and $A \leq' B$ to indicate that A contains all of its descendants in B. This closure is disintegrated and contained in algebraic closure. To express various things in a first-order way we will also use the notation $\mathrm{cl}'_{m,B}(X)$ for the closure of X in B under the operation of taking V_i-descendants for $i \leq m$.

We will again consider undirected reducts, but we will also retain information about the existence of certain paths between pairs of vertices when we pass to the reduct. Now, in the directed graphs, the existence of a particular type of path between two vertices is not in general preserved between closed substructures. So we shall impose extra axioms on our structures to guarantee this.

Definition 2.1 Write $W_i(x, y)$ iff $V_i(x, y) \vee V_i(y, x)$.

(i) If $i, r \geq 1$ and $A \in \mathcal{C}_0'$ an *(i, r)-path from a_0 to a_r in A* is a sequence $a_0, \ldots, a_r$ of elements of A with $W_i(a_0, a_1), W_{i+1}(a_1, a_2), \ldots, W_{i+r-1}(a_{r-1}, a_r)$. It is a *nice* (i, r)-path if there is $l \leq r$ with $V_{i+k}(a_k, a_{k+1})$ for $k < l$ and $V_{i+k}(a_{k+1}, a_k)$ for $l \leq k$. We refer to a_l here as the *node* of the path. So a directed (i, r)-path is nice iff it consists of two descending paths (one possibly empty) with a common terminal vertex (the node). Write $A \models P^{i,r}(a,b)$ if there is an (i,r)-path in A from a to b.

(ii) The class $\mathcal{C}' \subseteq \mathcal{C}_0'$ consists of structures A which satisfy the following additional axioms $\theta_{i,r}$ (for $r \geq 2$). Suppose $a_0, a_1, \ldots, a_r$ is an (i,r)-path in A and $V_i(a_1,a_0), V_{i+1}(a_1,a_2), V_{i+2}(a_2,a_3), \ldots, V_{i+r-1}(a_{r-1},a_r)$. Then there is a nice (i,r)-path from a_0 to a_r in A.

(iii) We denote by $\hat{T}'$ the axioms for $\mathcal{C}'$.

Of course, $P^{i,1}$ is superfluous as it is the same thing as W_i, but it is convenient to have a uniform notation in the following arguments.

Lemma 2.2 *(i) Let $A \in \mathcal{C}'$ and suppose $a_0, \ldots, a_r$ is an (i,r)-path in A. Then there is a nice (i,r)-path in A starting at a_0 and ending at a_r.*
(ii) If $A \leq' B \in \mathcal{C}'$ and $a, b \in A$ then $A \models P^{i,r}(a,b) \Leftrightarrow B \models P^{i,r}(a,b)$.

Proof. (i) This is by induction on r. The base case $r = 2$ follows quickly from the axioms $\theta_{i,2}$. For the inductive step, note first that we may assume $a_1, \ldots, a_r$ is a nice $(i+1, r-1)$-path, with node a_k. If $V_i(a_0,a_1)$, then $a_0, \ldots, a_r$ is a nice (i,r)-path. So suppose $V_i(a_1,a_0)$. If $k = 1$, there is no problem (we have a nice (i,r)-path with node a_0). If $k = r$ we can appeal directly to $\theta_{i,r}$ to get a nice (i,r)-path from a_0 to a_r. Finally, if $1 < k < r$ we can apply $\theta_{i,k}$ to get a nice (i,k)-path from a_0 to a_k. Adjoining $a_{k+1}, \ldots, a_r$ to this we get a nice (i,r)-path, as required.

(ii) One direction is clear. For the other, if $B \models P^{i,r}(a,b)$, then by (i) there is a nice (i,r)-path from a to b in B, and as $a, b \in A \leq' B$, this lies entirely within A. $\square$

The amalgamation property for $\hat{T}'$ is as before. Once again we refer to the disjoint union of two structures in $\mathcal{C}'$ over a common substructure as their *free amalgam* over the substructure.

Lemma 2.3 *Suppose $B, C \in \mathcal{C}'$ and $A \subseteq B$, $A \leq' C$. Then the free amalgam F of B and C over A is in $\mathcal{C}'$ and $B \leq' F$.*

Proof. It is clear that $F \in \mathcal{C}_0'$ and $B \leq' F$. So it remains to show that $F \models \theta_{i,r}$. Let $a_0, a_1, \ldots, a_r \in F$ be as in the definition of $\theta_{i,r}$. We must show that there is a nice (i,r)-path from a_0 to a_r in F.

Note that each a_i is in $\mathrm{cl}'_F(a_1)$, so if $a_1 \in B$, then there is no problem (as $B \leq' F$ and $B \models \theta_{i,r}$). Thus we may assume $a_1 \in C \setminus A$. If all the a_i are in C then again there is no problem as $C \models \theta_{i,r}$. If not, let $j > 1$ be as small as possible with $a_j \notin C$. Note that a_0 in C and $j > 2$ as a_1 is not adjacent to any vertex outside C, and similarly $a_{j-1} \in A$. Also $a_j, \ldots, a_r \in B$ as $B \leq F$. As $C \models \theta_{i,j-1}$, there is a nice $(i, j-1)$-path in C from a_0 to a_{j-1}. Denote this by $c_0, c_1, \ldots, c_{j-1}$ and let c_s be the node. Then $c_s, c_{s+1}, \ldots, c_{j-1} \in \mathrm{cl}'_C(a_{j-1})$ and so are in A. Thus $c_s, \ldots, c_{j-1}, a_j, \ldots, a_r$ is an $(i+s, r-s)$-path in B. Thus (by Lemma 2.2) there is a nice $(i+s, r-s)$-path $b_s, \ldots, b_r$ from c_s to a_r in B, and then $c_0, \ldots, c_{s-1}, c_s, b_{s+1}, \ldots, b_r$ is a nice (i, r)-path in F from a_0 to a_r. $\square$

Now let $\hat{T}'_1$ consist of $\hat{T}'$ and all sentences of the form:

$$\forall \bar{x} \exists \bar{y} (\Delta_X(\bar{x}) \rightarrow \Delta_{X,A}(\bar{x}, \bar{y}) \wedge \text{`}\mathrm{cl}'_m(\bar{x}\bar{y}) = \mathrm{cl}'_m(\bar{x}) \cup \bar{y}\text{'})$$

where $A \in \mathcal{C}'$ is finite, $X \leq' A$, $\Delta_X(\bar{x})$ denotes the basic diagram of X and $\Delta_{X,A}(\bar{x}, \bar{y})$ denotes the basic diagram of A, where the variables $\bar{y}$ represent the elements of $A \setminus X$. The condition '$\mathrm{cl}'_m(\bar{x}\bar{y}) = \mathrm{cl}'_m(\bar{x}) \cup \bar{y}$' is expressed in a first-order way by saying that any V_i-descendent of a variable in $\bar{y}$ is one of the variables in $\overline{xy}$, for $i \leq m$.

Note that if X is the closure of a finite set inside some ω-saturated model M of $\hat{T}'_1$, and $X \leq' A \models \hat{T}'$, where A is also the closure of a finite set, then, by compactness, there exists an embedding over X of A into M with closed image. One can then argue exactly as for T'_1 (as in Lemmas 1.1 and 1.2) to obtain:

Lemma 2.4 *(i) The theory $\hat{T}'_1$ is consistent and complete. Moreover, n-tuples $\bar{a}, \bar{b}$ in models M, N of $\hat{T}'_1$ have the same type iff the map $\bar{a} \mapsto \bar{b}$ extends to an isomorphism between $\mathrm{cl}'_M(\bar{a})$ and $\mathrm{cl}'_N(\bar{b})$.*
(ii) The theory $\hat{T}'_1$ is stable and if A, B, C are subsets of a model N of $\hat{T}'_1$, then $A \mathop{\smile\!\!\!\!\!\!|}_C B \Leftrightarrow \mathrm{cl}'_N(AC) \cap \mathrm{cl}'_N(BC) = \mathrm{cl}'_N(C)$. Moreover, $\hat{T}'_1$ is 1-based and trivial. $\square$

2.2 Reducts

We now consider the class $\mathcal{C}$ of reducts of structures in $\mathcal{C}'$ to the signature consisting of the (definable) predicates $W_i(x, y)$ and $P^{i,r}(x, y)$. (Of course, these predicates are definable in the original language rather than being a subset of it, so the usage of the word 'reduct' is somewhat incorrect, particularly as the $P^{i,r}$ are not even quantifier-free definable.) This is not closed under substructures. For example, take $A = \{a, b, c\} \in \mathcal{C}'$ with $V_1(a, b), V_2(c, b)$ in

A. Then in the reduct we have $P^{1,2}(a,c)$, so clearly $\{a,c\}$ is not the reduct of a structure in $\mathcal{C}'$.

We again refer to an expansion of a structure in $\mathcal{C}$ to a structure in $\mathcal{C}'$ (with the correct meaning of W_i and $P^{i,r}$) as an *orientation* of the structure. We say that structures in $\mathcal{C}'$ with the same domain are *equivalent* if their reducts are equal (i.e. they are both orientations of the same structure in $\mathcal{C}$). If $A \subseteq B \in \mathcal{C}$, we write $A \leq B$ to mean that there is an orientation of B in which A is a closed subset. As $\hat{T}_1'$ is complete the reducts of its models all have the same theory $\hat{T}_1$. This is of course also complete and stable. We shall show that it is n-ample for all $n \in \mathbb{N}$.

Before proceeding, we introduce a convenient piece of notation.

Notation 2.5 For any structure in $\mathcal{C}'$, the union of the relations V_i has no cycles and so its transitive closure is a partial order, and this can be extended to a total order. Thus we can describe an orientation on $A \in \mathcal{C}$ by specifying an ordering on its points: if $A \models W_i(a,b)$ and b is less than a in the ordering then b is a V_i-descendant of a in the orientation (of course, not all orderings give orientations). If A is denumerable, we will usually describe an ordering on its points by enumerating them $a_0, a_1, a_2, \ldots$: the understanding being that a_i is less than a_j in the ordering for $i < j$.

One difference from the previous case is that there is no closure operation associated with $\leq$: it can happen that $A_1, A_2 \leq B \in \mathcal{C}$ and $A_1 \cap A_2 \not\leq B$. For example, suppose B has points a, b_1, b_2, c and relations $W_1(a,b_i)$, $W_2(c,b_i)$, $P^{1,2}(a,c)$ (for $i = 1,2$). This has orientations b_1, a, c, b_2 and b_2, a, c, b_1 so $B \in \mathcal{C}$ and $\{a,b_1,c\}, \{a,b_2,c\} \leq B$. On the other hand $\{a,c\} \not\leq B$, as $P^{1,2}(a,c)$.

Despite this, the class $(\mathcal{C}, \leq)$ does have some of the good properties of the earlier example $(\mathcal{G}, \leq)$. We first describe the appropriate notion of free amalgamation. Note that if $a_0, a_1, a_2 \in A \in \mathcal{C}$ and $A \models W_1(a_0,a_1) \wedge W_2(a_1,a_2)$, then $A \models P^{1,2}(a_0,a_2)$. So we cannot expect to amalgamate structures over a common substructure by taking the union of the relations on the structures: we may have to add some new instances of the relations $P^{i,r}$. Free amalgamation does this in the minimal way possible.

Definition 2.6 Suppose $A, B_1, B_2 \in \mathcal{C}$ and $A \subseteq B_1, B_2$. By the *free amalgam* of B_1 and B_2 over A we mean the structure F whose domain is the disjoint union of B_1 and B_2 over A and whose relations consist of the unions of the relations on B_1 and B_2 together with new instances of relations $P^{i,r}$ as follows. If $b_1 \in B_1 \setminus A$ and $b_2 \in B_2 \setminus A$ then $F \models P^{i,r}(b_1, b_2)$ iff there is $k < r$ and $a \in A$ with $B_1 \models P^{i,k}(b_1, a)$ and $B_2 \models P^{i+k,r-k}(a, b_2)$. Similarly with the roles of B_1 and B_2 interchanged.

Lemma 2.7 *(i) Suppose $A \leq' C \in \mathcal{C}'$ and C_1 is obtained from C by replacing the substructure on A by an equivalent structure A_1. Then $C_1 \in \mathcal{C}'$.*
(ii) If $A \leq B \leq C \in \mathcal{C}$ then $A \leq C$.
(iii) If $A \leq B, C \in \mathcal{C}$, then the free amalgam F of B and C over A is in $\mathcal{C}$ and $B, C \leq F$.

Proof. (i) Easily $C_1 \in \mathcal{C}'_0$, so it is enough to show that if $a_0, \ldots, a_r$ is a nice (i, r)-path in C, then there is a nice (i, r)-path in C_1 from a_0 to a_r. We may assume that some $a_j \in A$. Let s be the smallest j with $a_j \in A$ and t the largest. As $A \leq C$ we have $a_s, \ldots, a_t \in A$ and the node of the path is amongst these. If $s = t$ there is no problem, so assume $s < t$. Then $A \models P^{i+s,t-s}(a_s, a_t)$, so the same is true in A_1 (as the reducts of A and A_1 are the same). Thus there is a nice $(i + s, t - s)$-path $b_s, \ldots, b_t$ in C_1 from $a_s = b_s$ to $a_t = b_t$. Then $a_0, a_s, b_{s+1}, \ldots, b_{t-1}, a_t, \ldots, a_r$ is a nice (i, r)-path in C_1, as required.

(ii) There is an orientation of C in which B is the domain of a closed substructure. Replace the orientation on B by one in which A is the domain of a closed substructure. By (i), the result is an orientation of C in which A is the domain of a closed substructure.

(iii) Take an orientation of B in which A is the domain of a closed substructure. By (i) the induced orientation of A can be extended to an orientation of C. The free amalgamation of these over A gives an orientation of F in which B is the domain of a closed substructure. $\square$

We do not have a convenient axiomatization of $\hat{T}_1$ as we do for T_1: the difficulty is in expressing $\leq$. Nevertheless, as $\hat{T}'_1$ is complete and recursively axiomatized, the same is true of $\hat{T}_1$ and it is therefore decidable.

Henceforth, we work with a large, saturated model N of $\hat{T}'_1$ (necessarily uncountable) and take its reduct M, which will be a saturated model of $\hat{T}_1$. We will first show that M is homogeneous and universal for small structures in $(\mathcal{C}, \leq)$ (where 'small' means of cardinality less than $|M|$). The main point is the following.

Proposition 2.8 *Suppose $A \leq' N$ is small and $A_1 \in \mathcal{C}'$ is equivalent to A. Let N_1 be the structure obtained by replacing A by A_1 in N. Then N_1 is a saturated model of $\hat{T}_1'$.*

Proof. By Lemma 2.7 (i) we have $N_1 \models \hat{T}'$. So it will be be enough to show that N_1 satisfies the following 'genericity' condition. Suppose $B \leq' N_1$ and $B \leq' D \in \mathcal{C}'$ is small. Then there is an embedding $\delta : D \to N_1$ which is the identity on B, and which satisfies $\delta(D) \leq' N_1$. Indeed, if this condition holds, then N and N_1 are back-and-forth equivalent (as in Lemma 2.4), so $N_1 \models \hat{T}_1'$, and saturation is then clear from the description of types in Lemma 2.4.

As A_1 and B are closed in N_1 we have $B_1 = A_1 \cup B \leq' N_1$. Let D_1 be the free amalgam over B of B_1 and D. So $A_1 \leq' B_1 \leq' D_1$ and $D \leq' D_1 \in \mathcal{C}'$. If we replace A_1 by the equivalent structure A in B_1 we obtain $A \leq' B_2 \leq' N$. Doing the same thing in D_1 we obtain $D_2 \in \mathcal{C}'$ (by Lemma 2.7) with $A \leq' B_2 \leq' D_2$. By saturation of N (i.e. the above genericity property), there is an embedding $\alpha : D_2 \to N$ which is the identity on B_2 and which has closed image in N. Now, D is not necessarily the domain of a closed substructure of D_2, but if we replace the structure on A by A_1 in both D_2 and N, the map α gives us an embedding $D_1 \to N_1$ (- same map, different structures!) with closed image and which is the identity on B_1. If we restrict this to $D \leq' D_1$, we get the required embedding δ. $\Box$

Corollary 2.9 *(i) If $A \subseteq M$ is small, then $A \leq M$ iff there is an orientation of M which is a saturated model of $\hat{T}_1'$ in which A is closed.*
(ii) If $A \leq M$ is small and $\beta : A \to B$ is an embedding of A into some small $B \in \mathcal{C}$ with $\beta(A) \leq B$, then there exists an embedding $\gamma : B \to M$ with $\gamma \circ \beta$ the identity on A and $\gamma(B) \leq M$.
(iii) If $A_1, A_2 \leq M$ are small and $\alpha : A_1 \to A_2$ is an isomorphism, then α can be extended to an automorphism of M.

Proof. (i) Suppose $A \leq M$ is small. Let P be an orientation of M in which A is closed. There is a small subset B containing A which is closed in both P and N. Let B_1 denote the structure on B in P. So $A \leq' B_1$. Replace the structure on B in N by the equivalent structure B_1. By Proposition 2.8 the result is still a saturated model N_1 of $\hat{T}_1'$. So we have $A \leq' B_1 \leq' N_1$ and N_1 is an orientation of M which is saturated and in which A is closed.

(ii) This follows from (iii) and the fact that any small $B \in \mathcal{C}$ can be $\leq$-embedded in M.

(iii) By Proposition 2.8 and (i), there exist orientations N_1, N_2 of M which are saturated models of $\hat{T}_1'$ with A_1, A_2 (respectively) closed subsets and in which α gives an isomorphism of the oriented structures on A_1, A_2. By Lemma 2.4 (i) this is a partial elementary map, so by uniqueness of saturated models, it extends to an isomorphism between N_1 and N_2. Passing back to the reduct, we obtain an automorphism of M which extends α. $\square$

We do not have a full characterization of forking in M. However, the following is useful.

Lemma 2.10 *Suppose A, B, C are small subsets of M with $A \cap B = C \leq M$; $A, B \leq A \cup B \leq M$ and $A \cup B$ the free amalgam over C of A and B. Then $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$.*

Proof. This is similar to the proof of Lemma 1.2: we show that $\mathrm{tp}_M(A/B)$ does not divide over C. Let $(B_i : i < \omega)$ be a sequence of translates over C of $B = B_0$. So in particular $B_i \leq M$. First, we show that there is a small $D \leq M$ with $B_i \leq D$ for all $i < \omega$. To see this, note that for each i there is an orientation N_i of M in which C and B_i are closed. As the closure of a small set is small in any orientation, there is a small subset D which contains all the B_i and which is closed in N and all the N_i. It follows that $B_i \leq D \leq M$ for all i.

Let F be the free amalgam over C of D with a copy over C of A (call it A_1). By Corollary 2.9(ii), we may assume that $F \leq M$. As $A_1 \leq F \leq M$ we have that A and A_1 have the same type over C. Now we claim that $A_1, B_i \leq A_1 \cup B_i \leq F$ for each i. Indeed, there is an orientation D' of D in which C, B_i are closed. Extend the orientation C' on C to an orientation A_1' of A_1 (using Lemma 2.7). The free amalgam (in $\mathcal{C}'$) of A_1' and D' over C' is an orientation of F in which A_1, B_i and $A_1 \cup B_i$ are closed. The establishes the claim and also shows that $A_1 \cup B_i$ is the free amalgam over C of A_1 and B_i. Thus $\mathrm{tp}_M(B_i A_1) = \mathrm{tp}_M(BA)$ for all i, by Corollary 2.9(iii). $\square$

Theorem 2.11 *The structure M is non-trivial and n-ample for all $n \in \mathbb{N}$. Take $A = \{a_0, \ldots, a_n, \ldots\} \leq M$ such that $W_i(a_{i-1}, a_i)$ and $P^{i+1,j-i}(a_i, a_j)$ (for $j \geq i+1$), and no other atomic relations hold on A. Then $a_i \leq M$ for each i and these have the same strong type over $\emptyset$. Moreover, for all n:*
(i) $a_n \ldots a_{i+1} \mathop{\smile\!\!\!\!\!\!\mid}_{a_i} a_0 \ldots a_{i-1}$ for $i < n$;
(ii) $a_n \not\mathop{\smile\!\!\!\!\!\!\mid} a_0$, and in fact $P^{1,n}(a_0, y)$ divides over $\emptyset$;
(iii) $\mathrm{acl}(a_0) \cap \mathrm{acl}(a_1) = \mathrm{acl}(\emptyset)$;
(iv) $\mathrm{acl}(a_0 \ldots a_{i-1} a_i) \cap \mathrm{acl}(a_0 \ldots a_{i-1} a_{i+1}) = \mathrm{acl}(a_0 \ldots a_{i-1})$ for all i.

Proof. Non-triviality is exactly as in Theorem 1.9 (just using W_1), and we will not repeat the argument. For the rest, we use the notational convention of (2.5) to specify various orientations.

First note that $a_0, a_1, \ldots$ is an orientation of A, so we can indeed find such points in M. Moreover, for any i, the enumeration $a_i, a_{i-1}, \ldots, a_0$ is also an orientation of the initial segment $\{a_0, \ldots, a_i\} \leq A$, so in particular $\{a_i\} \leq M$. In fact, one can now see that for any i, A is the free amalgam over $\{a_i\}$ of $\{a_i, a_{i-1}, \ldots, a_0\}$ and $\{a_i, a_{i+1}, \ldots\} \leq A$. By Lemma 2.10, this gives (i).

As $a_i \leq M$, the a_i have the same type over $\emptyset$ (by Corollary 2.9(iii)). We can argue as in the proof of Lemma 1.8 to show that $\mathrm{tp}(a_1/\emptyset)$ is stationary and it then follows that the a_i have the same strong type over $\emptyset$.

(ii) Note that $M \models P^{1,n}(a_0, a_n)$, so it is enough to prove the second assertion. Let $C = \{c_i : i < \omega\} \leq M$ have all atomic relations empty. Note that $c_i \leq C \leq M$, so $(c_i : i < \omega)$ is an indiscernible sequence over $\emptyset$, and we may assume $c_0 = a_0$. We show that no subset of $\{P^{1,n}(c_i, y) : i < \omega\}$ of size greater than 2^n is realised in M. Indeed, take an orientation of M in which C is closed. Let $d \in M$ and suppose $M \models P^{1,n}(c_i, d)$. This is witnessed by a nice $(1, n)$-path in any orientation of M and (as C is closed in our particular orientation and there are no realisations of W_j in C) it follows that this nice $(1, n)$-path is directed from d to c_i in our orientation. But the number of such directed paths (for fixed d, and fixed orientation) is at most 2^n, so the number of possible c_i reachable by such a path is at most 2^n.

(iii) Suppose $e \in \mathrm{acl}(a_0) \cap \mathrm{acl}(a_1)$. There exists a sequence $(c_j : j < \omega)$ with $c_0 = a_1$, $W_1(a_0, c_j)$ for all j, no other atomic relations holding on $C = \{a_0, c_0, c_1, \ldots\}$, and $C \leq M$. Then $a_0 c_j \leq M$ and the c_j are all of the same type over a_0. The same is true of any pair of the c_j, thus, as e is algebraic over a_0, we have that c_0, c_1 have the same type over a_0, e.

It follows that $e \in \mathrm{acl}(c_0) \cap \mathrm{acl}(c_1)$. But any enumeration of C which starts with c_0, c_1, a_0 gives an orientation of C, so $\{c_0, c_1\} \leq M$. Thus $c_0 \mathop{\smile\!\!\!\!\!\mid} c_1$ by Lemma 2.10, and therefore $e \in \mathrm{acl}(\emptyset)$.

(iv) This is similar to (iii). Fix i. Let $\bar{a} = (a_0, \ldots, a_{i-1})$ and $\hat{a} = (a_{i-1}, \ldots, a_0)$. Suppose $e \in \mathrm{acl}(\bar{a}a_i) \cap \mathrm{acl}(\bar{a}a_{i+1})$. There exist distinct $(c_j : j < \omega)$ with $D = \{\bar{a}, a_{i+1}, c_j : j < \omega\} \leq M$, $c_0 = a_i$, $W_i(a_{i-1}, c_j)$, $W_{i+1}(c_j, a_{i+1})$ and the only other instances of atomic relations holding on D being those $P^{l,r}$ forced by the (l, r)-paths. For each j, any enumeration of D starting off with $c_j, \hat{a}, a_{i+1}$ gives an orientation of D, so $c_j \hat{a} a_{i+1} \leq M$ and therefore the c_j are of the same type over $\bar{a}a_{i+1}$. Thus (as $e \in \mathrm{acl}(\bar{a}a_{i+1})$) we may assume

c_0, c_1 are of the same type over $\bar{a}a_{i+1}e$. So $e \in \mathrm{acl}(\bar{a}c_0) \cap \mathrm{acl}(\bar{a}c_1)$. But any enumeration of D starting with $\bar{a}, c_0, c_1, a_{i+1}$ gives an orientation of D, so $c_0 \perp_{\bar{a}} c_1$ by Lemma 2.10. Thus $e \in \mathrm{acl}(\bar{a})$, as required. □

Remarks 2.12 Note that we could have worked throughout with W_i, V_i for $i \leq n$, with n fixed. The argument shows that the resulting structure is n-ample. We conjecture that it is not $(n+1)$-ample, but have not attempted to verify this.

2.3 Pseudospaces in M

It is not completely clear what the precise definition of 'pseudospace' should be (the term is also not defined in [1]). Ideally, one would like to define the combinatorial notion of an 'n-pseudospace' so that a stable structure is n-ample iff it type-interprets an n-pseudospace. Of course, we have this for $n = 1$: this is Lachlan's notion of a pseudoplane. In vague terms, however, an n-pseudospace should consist of points, lines, planes, ... which satisfy various 'geometric' incidence properties.

We show how to build such a structure in M. In the example below, one could think of the loci of $a_0, \ldots, a_n$ over B as (canonical parameters for) points, lines,..., n-flats, ... with the various 2-types $(a_i a_j / B)$ giving incidence relations between these.

Proposition 2.13 *Let M be the structure constructed in the previous section.*
(i) There are points $A = \{a_i, b_{i+1}, c_i, d_{i+1} : i < \omega\}$ with $A \leq M$ and only the following atomic relations (and the instances of the $P^{i,r}$ they imply) on A (see Figure 1): for $i \geq 1$
$W_i(a_{i-1}, a_i)$, $W_i(b_i, b_{i+1})$, $W_{i+1}(c_{i-1}, c_i)$, $W_i(a_{i-1}, c_{i-1})$, $W_i(b_i, a_i)$, $W_i(d_i, b_i)$, $W_{i+1}(d_i, c_i)$.

If $B = \{b_{i+1}, c_i, d_{i+1} : i < \omega\}$, then $B \leq A$.

With this notation, we have, for all $i < \omega$:
(ii) $a_i \notin \mathrm{acl}(Ba_0, \ldots, a_{i-1}, a_{i+1}, \ldots)$;
(iii) the locus of (a_i, a_{i+1}) over B is a pseudoplane.

Proof. (i) Using Figure 1 to identify the instances of the relations $P^{i,r}$, one checks that

$$d_1, d_2, \ldots, b_1, b_2, \ldots, c_0, c_1, c_2, \ldots, a_0, a_1, a_2, \ldots$$

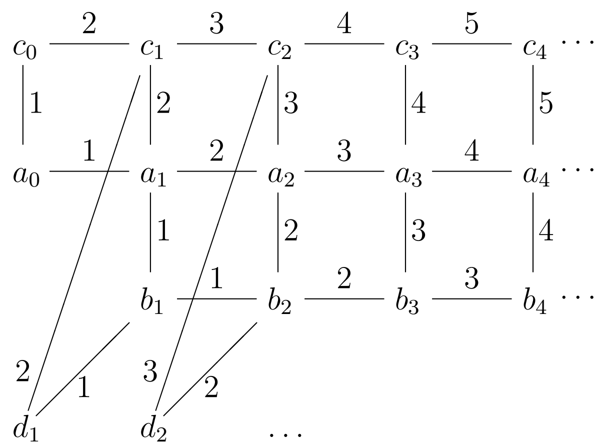

Figure 1: The pseudospace

gives an orientation of A.

(ii) Let $i < \omega$. Consider the structure $E = A \cup \{e_j : j < \omega\}$ where the quantifier free type of e_j over $A \setminus \{a_i\}$ is the same as that of a_i, and there are no other basic relations on E other than what is implied by this. Then $Ba_0, a_1, \ldots, e_0, e_1, e_2, \ldots$ is an orientation of E with A as a closed substructure, so we may assume that the e_j are in M. We may interchange a_i with any of the e_j and still have an orientation of E. Thus a_i, e_j have the same type over $Ba_0, \ldots, a_{i-1}, a_{i+1}, \ldots$.

(iii) Suppose a'_i is a translate of a_i over Ba_{i+1}. We need to check that $a_{i+1} \in \mathrm{acl}(a_i a'_i)$. Indeed, suppose $a_{i+1} = a^1_{i+1}, \ldots, a^r_{i+1}$ are translates over $Ba_i a'_i$. If $r \geq 3$, then the graph with edge set W_{i+1} on the points b_{i+1}, a_i, a'_i, $a^1_{i+1}, \ldots, a^r_{i+1}$ has all vertices being of valency at least 3, which contradicts the existence of an orientation on M. Thus $r \leq 2$.

Similarly, suppose a'_{i+1} is a translate of a_{i+1} over Ba_i, and $a_i = a^1_i, \ldots, a^r_i$ are translates over $Ba_{i+1}a'_{i+1}$. Again, if $r \geq 3$ then the graph with edge set W_{i+1} on the points $c_i, a_{i+1}, a'_{i+1}, a^1_i, \ldots, a^r_i$ has all vertices of valency ≥ 3, which is again a contradiction. □

Remarks 2.14 Conditions (ii) and (iii) are probably weaker than n-ampleness. In the example we also have that:
(iv) $a_0, \ldots, a_{i-1} \mathop{\smile\!\!\!\!\!\!|}_{Ba_i} a_{i+1} \ldots$.

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