Weak reflection at the successor of a singular cardinal

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Abstract

The notion of stationary reflection is one of the most important notions of combinatorial set theory. We investigate weak reflection, which is, as its name suggests, a weak version of stationary reflection. Our main result is that modulo a large cardinal assumption close to 2-hugeness, there can be a regular cardinal κ such that the first cardinal weakly reflecting at κ is the successor of a singular cardinal. This answers a question of Cummings, Džamonja and Shelah. [1]

0 Introduction and the statement of the results.

Stationary reflection is a compactness phenomenon in the context of stationary sets. To motivate its investigation, let us consider first the situation of a regular uncountable cardinal κ and a closed unbounded subset C of κ. For every limit point α of C we have that $C \cap \alpha$ is closed unbounded in α. Now let us ask the same question, but starting with a set S which is stationary, not necessarily club, in κ. Is there necessarily $\alpha < \kappa$ such that $S \cap \alpha$ is stationary in α-or as this situation is known in set theory, S reflects at α? The answer to this question turns out to be very intricate, and in fact the notion of stationary reflection is one of the most studied notions of combinatorial set theory. This is the case not only because of the historical significance stationary reflection achieved through by now classical work of R. Jensen [Je] and later work of J.E. Baumgartner [Ba], L. Harrington and S. Shelah [HaSh 99], M. Magidor [Ma] and many later papers, but also because of the large number of applications it has in set theory and allied areas. In set theory, stationary reflection is known to have deep connections with various guessing and coherence principles, the simplest one of which is Jensen's $\square$ ([Je]), and the notions from pcf theory, such as good scales (for a long list of results in this area, as well as an excellent list of references, we refer the reader to [CuFoMa]), and some connections with saturation of normal filters ([DjSh 545]). In set-theoretic topology, various kinds of spaces have been constructed from the assumption of the existence of a non-reflecting stationary set (for references see [KuVa]), and in model theory versions of

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stationary reflection have been shown to have a connection with decidability of monadic second-order logic ([Sh 80]).

We investigate the notion of weak reflection, which, as the name suggests, is a weakening of the stationary reflection. For a regular cardinal κ, we say that $\lambda > \kappa$ weakly reflects at κ iff for every function $f : \lambda \to \kappa$, there is $\delta < \lambda$ of cofinality κ (we say $\delta \in S^\lambda_\kappa$) such that $f \restriction e$ is not strictly increasing for any e a club of δ. The negation of this principle is a strong form of non-reflection, called strong non-reflection. The notions were introduced by Džamonja and Shelah in [DjSh 545] in connection with saturation of normal filters, as well as the guessing principle $\clubsuit^*_{-\lambda}(\lambda^+)$, which is a relative of another popular guessing principle, $\clubsuit$. It is proved in [DjSh 545] that, in the case when $\lambda = \mu^+$ and $\aleph_0 < \kappa = \mathrm{cf}(\mu) < \mu$, <u>if</u> weak reflection of λ at κ holds relativized to every stationary subset of S^λ_κ, <u>then</u> $\clubsuit^*_{-\mu}(S^\lambda_\kappa)$ holds. The exact statement of the principle is of no consequence to us here, so we omit the definition. We simply note that this statement is stronger than just $\clubsuit^*_{-\mu}(\lambda)$, which holds just from the given cardinal assumptions.

Weak reflection was further investigated by Cummings, Džamonja and Shelah in [CuDjSh 571], more about which will be mentioned in a moment. A very interesting application of weak reflection was given by Cummings and Shelah in [CuSh 596], where they used it as a tool to build models where stationary reflection holds for some cofinalities but badly fails for others.

It was proved in [DjSh 545] that if there is λ which weakly reflects at κ, the first such λ is a regular cardinal. It is also not difficult to see that the first such λ cannot be weakly compact. On the other hand, in [CuDjSh 571] Cummings, Džamonja and Shelah proved that, modulo the existence of certain large cardinals, it is consistent to have a cardinal λ which weakly reflects at unboundedly many regular κ below it, and strongly non-reflects at unboundedly many others. The forcing notion used in this can be used to get models where there is κ such that the first cardinal weakly reflecting at κ is the successor of a regular cardinal.

A question we attempted in these investigations but did not succeed in resolving was if it is consistent to have κ for which the first λ which weakly reflects at κ is the successor of a singular cardinal. In this paper we answer this question positively, modulo the existence of a certain large cardinal whose strength is in the neighbourhood of (and seemingly less than) being 2-huge. In our model both κ and λ are successors of singulars. Cummings has meanwhile obtained in [Cu] an interesting result indicating that it would be very difficult to obtain a result similar to ours with e.g. $\kappa = \aleph_2$ and $\lambda = \aleph_{\omega+1}$, as there is an interplay with a closely related compactness phenomenon. To state our results more precisely, let us now give the exact definition of weak reflection and the statement of our main theorem.

Definition 0.1. Given $\aleph_0 < \kappa = \mathrm{cf}(\kappa)$ and $\lambda > \kappa$. We say that λ *weakly reflects at* κ iff for every function $f : \lambda \to \kappa$, there is $\delta \in S^\lambda_\kappa$ such that $f \restriction e$ is not strictly increasing for any e a club of δ.

Theorem 0.2. (1) Let V be a universe in which, for simplicity, GCH holds and let μ_0 be a cardinal such that there is an elementary embedding $\mathbf{j} : V \to M$ with the following properties:

(i) $\mathrm{crit}(\mathbf{j}) = \mu_0$,

(ii) For some κ^* which is the successor of a singular cardinal θ^* and for some χ, we have

$$\mu_0 < \kappa^* < \mu_1 \stackrel{\mathrm{def}}{=} \mathbf{j}(\mu_0) < \lambda^* \stackrel{\mathrm{def}}{=} \mathbf{j}(\kappa^*) < \mathrm{cf}(\chi) = \chi < \mu_2 \stackrel{\mathrm{def}}{=} \mathbf{j}(\mu_1),$$

(iii) ${}^{\chi}M \subseteq M$.

<u>Then</u> there is a generic extension of V in which cardinals and cofinalities $\geq \mu_0$ are preserved, GCH holds above μ_0, and the first λ weakly reflecting at κ^* is λ^* (hence, the successor of a singular).

(2) In (1), we can replace the requirement that κ^* is the successor of a singular by "$\varphi(\kappa^*)$ holds" for any of the following meanings of $\varphi(x)$:

(a) x is inaccessible,

(b) x is strongly inaccessible,

(c) x is Mahlo,

(d) x is strongly Mahlo,

(e) x is α-(strongly) inaccessible for $\alpha < x$,

(f) x is α-(strongly) Mahlo for $\alpha < x$,

and have the same conclusion (hence in place of "λ^* is the successor of a singular", V^P will satisfy $\varphi(\lambda^*)$).
(3) With the same assumptions as in (1), there is a generic extension of V in which $\kappa^* = \aleph_{53}$ and λ^*, the successor of a singular, is the first cardinal weakly reflecting at κ^*.

Remark 0.3. (1) Our assumptions follow if μ_0 is 2-huge and GCH holds. The integer 53 in the statement of part (3) above is to a large extent arbitrary.
(2) Notice that κ^*-cc forcing notions preserve that κ^* is a regular uncountable cardinal and that λ^* is the first cardinal weakly reflecting at κ^*, as well as the fact that λ^* is the successor of a singular cardinal, but not necessarily the fact that κ^* is the successor of a singular cardinal. It is natural to consider the possibility of $\kappa^* = \aleph_{\omega+1}$ and $\lambda^* = \aleph_{\omega+\omega+1}$ but we have not considered this for the moment.

The proof of (1) of the Theorem uses as a building block a forcing notion used by Cummings, Džamonja and Shelah in [CuDjSh 571], which introduces a function witnessing strong non-reflection of a given cardinal λ to a cardinal κ. An important feature of this forcing is that it has a reasonable degree of (strategic) closure, provided that strong non-reflection of θ at κ already holds for $\theta \in (\kappa, \lambda)$, and hence it can be iterated. This forcing is a rather homogeneous forcing, so the term forcing associated with it has strong decision properties. The forcing that we actually use is a term forcing associated with a certain product of the strong non-reflection forcing and a Laver-like preparation. Using this, we force the strong non-reflection of θ at κ^* for all $\theta < \lambda^*$ (say $\theta > \kappa^*$, as the alternative situation is trivial), and the point is to prove that in the extension λ^* weakly reflects on κ^*. If we are given a condition and a name forced to be a strongly non-reflecting function, we can use the large cardinal assumptions to pick a certain model N, for which we are able to build a generic condition, whose existence contradicts the choice of the name. To build the generic condition we use the preparation and the fact that we are dealing with a term forcing. Proofs of (2) and (3) are easy modifications of the proof of (1).

We recall some facts and definitions.

Notation 0.4. (1) Reg stands for the class of regular cardinals.
(2) If p, q are elements of a forcing notion, then $p \leq q$ means that q is an extension of p. For a forcing notion Q we assume that $\emptyset_Q$ is the minimal member of Q.
(3) For p a condition in the limit of an iteration $\langle P_\alpha, \underset{\sim}{Q}_\beta : \alpha \leq \alpha^*, \beta < \alpha^* \rangle$, we let

$$\mathrm{Dom}(p) \stackrel{\text{def}}{=} \{\beta < \alpha^* : \neg(p \restriction \beta \Vdash \text{“}p(\beta) = \emptyset_{\underset{\sim}{Q}_\beta}\text{”})\}.$$

(4) The statement that λ weakly reflects at κ is denoted by $WR(\lambda, \kappa)$. Its negation (including the situation $\lambda \leq \kappa$) is denoted by $SNR(\lambda, \kappa)$.

Remark 0.5. It is easily seen that λ weakly reflects at κ iff $|\lambda|$ does, so we can without loss of generality, when discussing weak reflection of λ to κ assume that λ is a cardinal.

Definition 0.6. (1) For a forcing notion and a limit ordinal ε, we define the game $\partial(P, \varepsilon)$ as follows. The game is played between players I and II, and it lasts ε steps, unless a player is forced to stop before that time. For $\zeta < \varepsilon$, we denote the ζ-th move of I by p_ζ, and that of II by q_ζ. The requirements are that I commences by $\emptyset_P$ and that for all ζ we have $p_\zeta \leq q_\zeta$, while for $\xi < \zeta$ we have $q_\xi \leq p_\zeta$.

I wins a play Γ of $\partial(P, \varepsilon)$ iff Γ lasts ε steps.
(2) For P and ε as above, we say that P is *ε-strategically closed* iff I has a winning strategy in $\partial(P, \varepsilon)$. We say that P is *$(< \varepsilon)$-strategically closed* iff it is ζ-strategically closed for all $\zeta < \varepsilon$.

1 Proofs.

We give the proof of Theorem 0.2. The main point is the proof of part (1) of the Theorem. With minimal changes, this proof can be adapted to prove the other parts of the theorem. The necessary changes are described at the end of the section. Let V, $\mathbf{j}$ and the cardinals mentioned in the assumptions of the Theorem be fixed and satisfy the assumptions. Note that the elementarity of $\mathbf{j}$ guarantees that λ^* is the successor of a singular cardinal. We shall build a generic extension in which λ^* remains the successor of a singular cardinal and is made to be the first cardinal which weakly reflects at κ^*. Let us commence by describing the forcing for obtaining strong non-reflection at a given (favourably prepared) pair of cardinals.

Definition 1.1. Suppose that we are given cardinals κ and σ satisfying $\aleph_0 < \kappa = \mathrm{cf}(\kappa) < \sigma$.

$\mathbb{P}(\kappa, \sigma)$ is the forcing notion whose elements are functions p with $\mathrm{dom}(p)$ an ordinal $< \sigma$, the range $\mathrm{rge}(p) \subseteq \kappa$, and the property

$$[\beta \in S^{\sigma}_{\kappa} \ \&\ \beta \subseteq \mathrm{dom}(p)] \Longrightarrow (\exists c \text{ a club of } \beta)\, [p \restriction c \text{ is strictly increasing}],$$

ordered by extension.

Fact 1.2 (Cummings, Džamonja and Shelah, [CuDjSh 571]). Let κ and σ be such that $\mathbb{P}(\kappa, \sigma)$ is defined, then

(1) $|\mathbb{P}(\kappa, \sigma)| \leq |{}^{\sigma>}\kappa| = \kappa^{<\sigma}$.

(2) Suppose that for all $\theta \in (\kappa, \sigma)$ we have $SNR(\theta, \kappa)$. Then $\mathbb{P}(\kappa, \sigma)$ is $(<\sigma)$-strategically closed.

(3) If GCH holds then $\mathbb{P}(\kappa, \sigma)$ preserves cardinals and cofinalities.

We recall the following

Definition 1.3. A set A of ordinals is an *Easton set* iff

$$\sigma \in \mathrm{Reg} \cap (\sup(A) + 1) \Longrightarrow \sup(A \cap \sigma) < \sigma.$$

The following forcing notion enforces strong non-reflection through an iteration of forcing of the form $\mathbb{P}(\kappa, \sigma)$.

Definition 1.4. Given $\aleph_0 < \mathrm{cf}(\kappa) = \kappa < \lambda$.

$Q_{(\kappa,\lambda)}$ is the result of a reverse Easton support iteration of $\mathbb{P}(\kappa, \sigma)$ for $\sigma = \mathrm{cf}(\sigma) \in (\kappa, \lambda)$. More precisely, let

$$\bar{Q} = \langle Q_\alpha, \underset{\sim}{R}_\beta : \alpha \leq \lambda, \beta < \lambda \rangle,$$

where

(1) $\Vdash_{Q_\alpha}$ "$\underset{\sim}{R}_\alpha = \{\emptyset\}$" unless $\alpha \in \mathrm{Reg} \cap (\kappa, \lambda)$, in which case

$$\Vdash_{Q_\alpha} \text{"}\underset{\sim}{R}_\alpha = \underset{\sim}{\mathbb{P}}(\kappa, \alpha)\text{"}.$$

(2) For $\alpha \leq \lambda$ we define by induction on α that

$p \in Q_\alpha$ iff p is a function with domain α such that for all $\gamma < \alpha$ we have $p \restriction \gamma \in Q_\gamma$ and $\Vdash_{Q_\gamma}$ "$p(\gamma) \in \underset{\sim}{R}_\gamma$" and letting

$$\mathrm{Dom}(p) = \{\gamma < \alpha : \neg(\Vdash_{Q_\gamma} \text{"}p(\gamma) = \emptyset_{\underset{\sim}{R}_\gamma}\text{"})\}$$

we have that $\mathrm{Dom}(p)$ is an Easton subset of α.

(3) $p \leq q$ iff for all $\beta < \lambda$ we have $q \restriction \beta \Vdash_{Q_\beta}$ "$q(\beta) \geq p(\beta)$".

Fact 1.5 (Cummings, Džamonja, Shelah, [CuDjSh 571]). Let $\bar{Q}$, κ and λ be as in Definition 1.4. For all $\alpha \leq \lambda$:

(1) Whenever α is regular, $|Q_\alpha| \leq \alpha^{<\alpha}$,

(2) $\Vdash_{Q_\alpha}$ "$|\underset{\sim}{R}_\alpha| \leq \kappa^{<|\alpha|}$",

(3) If $\alpha \geq \aleph_0$, then $Q_{\alpha+1}$ has $(|\alpha|^{<|\alpha|})^+$-cc. In addition, if α is strongly Mahlo, then Q_α has α-cc.

(4) $\Vdash_{Q_\alpha}$ "$\underset{\sim}{R}_\alpha$ is $(< \alpha)$-strategically closed".

(5) For all $\beta < \alpha$, we have that $\underset{\sim}{Q}_\alpha / Q_\beta$ is $(< \beta)$-strategically closed.

(6) Q_α preserves all cardinals and cofinalities $\geq (|\alpha|^{<|\alpha|})^+$, and all strongly inaccessible cardinals and cofinalities $\leq |\alpha|$ [2], as well as all cardinals and cofinalities $\leq (\kappa)^+$. If α is strongly Mahlo, then Q_α preserves cardinalities and cofinalities $\geq \alpha$.

(7) $\Vdash_{Q_\alpha}$ "$SNR(\kappa, \beta)$" for all $\beta < \alpha$.

The forcing notion we shall use will be a term forcing associated with iterations of the above type. We first need some general notation for such forcing.

Notation 1.6. (1) For a forcing notion Q of the form $Q = P_1 * \underset{\sim}{P}_2$, we denote by $Q^\otimes$ the term forcing associated with Q, defined by

$$Q^\otimes \stackrel{\text{def}}{=} \{(\emptyset_{P_1}, \underset{\sim}{q}) : \underset{\sim}{q} \text{ is a canonical } P_1\text{-name for a condition in } P_2\},$$

(in particular $Q^\otimes \subseteq Q$), with the order inherited from Q. Following the usual practice, we may write $(\emptyset, q)$ in place of $(\emptyset_{P_1}, \underset{\sim}{q})$ when the meaning is clear from the context.

(2) For a triple (R, κ, λ) with $\mathrm{cf}(\kappa) = \kappa < \lambda$, and R a forcing notion preserving the fact that $\lambda > \kappa$ is a cardinal and $\kappa = \mathrm{cf}(\kappa) > \aleph_0$, we define $Q^\otimes_{(R,\kappa,\lambda)}$ to be $[R * \underset{\sim}{Q}_{(\kappa,\lambda)}]^\otimes$.

(3) In the situation when the notation $Q^\otimes_{(R,\kappa,\kappa^*)}$ makes sense, we abbreviate it as $Q^\otimes_{(R,\kappa)}$.

Observation 1.7. $Q^\otimes_{(R,\kappa,\lambda)}$, when defined, is $(< \kappa^+)$-strategically closed.

The forcing notion we shall use will have a preparatory component, $\mathbb{P}^-$ described below, which will be followed by a component made up of term forcings described above. We give a precise definition in the following.

[2]Note that this implies that if GCH holds, all cardinalities and cofinalities are preserved.

Definition 1.8. We define $\mathbb{P}^-$ to be the forcing whose elements are functions h, with $\mathrm{dom}(h)$ an Easton subset of μ_0 consisting of cardinals, with the property that

$$\alpha < \beta \in \mathrm{dom}(h) \cup \{\mu_0\} \Longrightarrow h(\alpha) \in \mathcal{H}(\beta)^3.$$

The order on $\mathbb{P}^-$ is the extension.

In order to be able to define the next component of the forcing we need to ascertain a preservation property of $\mathbb{P}^-$. Recall that we are assuming that $V \models GCH$.

Claim 1.9. Forcing with $\mathbb{P}^-$ preserves cardinals and cofinalities $\geq \mu_0$, and GCH above and at μ_0. If $p \in \mathbb{P}^-$ and $\theta \in \mathrm{dom}(p)$ is (strongly) inaccessible, then p forces that the cofinality of any $\sigma \leq \mu_0$ whose V-cofinality is $> \theta$, remains $> \theta$, while 2^θ remains θ^+.

Proof of the Claim. First notice that $|\mathbb{P}^-| = \mu_0$, so $\mathbb{P}^-$ has μ_0^+-cc and preserves cardinals and cofinalities $\geq \mu_0^+$, as well as GCH above and at μ_0.

Now suppose that $p \in \mathbb{P}^-$ and p forces that for some $\sigma \leq \mu_0$ and $\theta < \mathrm{cf}(\sigma)$ with $\theta \in \mathrm{dom}(p)$ inaccessible, the cofinality of σ in $V[G]$ is $\leq \theta$. Let

$$P_{<\theta} \stackrel{\mathrm{def}}{=} \{q \restriction \theta : q \in \mathbb{P}^- \ \& \ q \geq p\}$$

and

$$P_{\geq\theta} \stackrel{\mathrm{def}}{=} \{q \restriction [\theta, \mu_0) : q \in \mathbb{P}^- \ \& \ q \geq p\},$$

both ordered by the extension. The mapping $q \mapsto (q \restriction [\theta, \mu_0), q \restriction \theta)$ shows that $\mathbb{P}^-/p \stackrel{\mathrm{def}}{=} \{q \in \mathbb{P}^- : q \geq p\}$ is isomorphic to $P_{\geq\theta} \times P_{<\theta}$ (we are using the fact that $\theta \in \mathrm{dom}(p)$). We have that $P_{\geq\theta}$ is $(< \theta^+)$-closed, so $P_{<\theta}$ adds a cofinal function from θ to σ. However, $|P_{<\theta}| \leq \theta$ (as θ is strongly inaccessible), and so it preserves cardinals and cofinalities $\geq \theta^+$, a contradiction.

We can similarly observe that

$$p \Vdash_{\mathbb{P}^-} \text{“}2^\theta = \theta^+\text{”}.$$

$\bigstar_{1.9}$

[3]Note that this implies that each $h(\alpha)$ is bounded in μ_0.

Definition 1.10. (1) For $\mu < \mu_0$ strongly inaccessible let $\underset{\sim}{\mathbb{R}}_\mu$ and $\underset{\sim}{\kappa}_\mu$ be the following $\mathbb{P}^-$-names:

for a condition $p \in \mathbb{P}^-$, <u>if</u> $\mu \in \mathrm{dom}(p)$ and

$$p(\mu) = (\kappa, R) \text{ with } \mu < \mathrm{cf}(\kappa) = \kappa < \mu_0,$$

and $R \in \mathcal{H}(\mu_0)$ a forcing notion which preserves the fact that κ is a regular uncountable cardinal (and it by necessity preserves that κ^* is a cardinal larger than κ), <u>then</u> p forces $\underset{\sim}{\kappa}_\mu$ to be κ and $\underset{\sim}{\mathbb{R}}_\mu$ to be $\underset{\sim}{Q}^{\otimes}_{(R,\underset{\sim}{\kappa}_\mu)}$. <u>If</u> $\mu \in \mathrm{Dom}(p)$ but μ or $p(\mu)$ do not satisfy the conditions above, or p has no extension q with $\mu \in \mathrm{Dom}(q)$, <u>then</u> p forces $\underset{\sim}{\kappa}_\mu = 0$ and $\underset{\sim}{\mathbb{R}}_\mu$ to be the trivial forcing, which will for notational purposes be thought of as $\{(\emptyset, \emptyset)\}$. In these circumstances we think of $R_\mu = \{\emptyset\}$.

It follows from Claim 1.9 that the above definition is correct and that over a dense subset of $\mathbb{P}^-$ each $\underset{\sim}{\mathbb{R}}_\mu$ is a $\mathbb{P}^-$-name of a forcing notion from V, $\underset{\sim}{\kappa}_\mu$ is a $\mathbb{P}^-$-name of an ordinal $< \mu_0$. In the following item (2), clearly $\prod_{\mu<\mu_0} \underset{\sim}{\mathbb{R}}_\mu$ is a $\mathbb{P}^-$-name of a product of forcing notions from V, but $\underset{\sim}{\mathbb{R}}$ below is forced not to be from V.

(2) For a $\mathbb{P}^-$-name $\underset{\sim}{f} \in \prod_{\mu<\mu_0} \underset{\sim}{\mathbb{R}}_\mu$ and $\alpha \leq \mu_0$, let

$$\underset{\sim}{A}_{\underset{\sim}{f},\alpha} \stackrel{\text{def}}{=} \{\mu < \mu_0 : \underset{\sim}{f}(\mu) = (\emptyset, \underset{\sim}{q}) \text{ with } \neg(\Vdash_{\underset{\sim}{\mathbb{R}}_\mu} \text{“}\alpha \notin \mathrm{Dom}(\underset{\sim}{q})\text{”})\}.$$

(3) Let $\underset{\sim}{\mathbb{R}}$ be a $\mathbb{P}^-$-name for:

$$\left\{ \underset{\sim}{f} \in \prod_{\mu<\mu_0} \underset{\sim}{\mathbb{R}}_\mu : (\forall \alpha \leq \mu_0) \, [\underset{\sim}{A}_{\underset{\sim}{f},\alpha} \text{ is an Easton set }] \right\},$$

ordered by the order inherited from $\prod_{\mu<\mu_0} \underset{\sim}{\mathbb{R}}_\mu$.

We proceed to discuss the preservation properties of the forcing notions we defined.

Notation 1.11. If we write $(p, \underset{\sim}{\bar{r}}) \in \mathbb{P}^- * \underset{\sim}{\mathbb{R}}$, we mean that $p \in \mathbb{P}^-$ and

$$\Vdash_{\mathbb{P}^-} \underset{\sim}{\bar{r}} = \langle (\emptyset_{R_\mu}, \underset{\sim}{r}(\mu)) : \mu < \mu_0 \rangle.$$

Definition 1.12. (1) Given $(p,\bar{\underset{\sim}{r}}) \in \mathbb{P}^- * \underset{\sim}{\mathbb{R}}$ and $\sigma = \mathrm{cf}(\sigma) < \mu_0$. For $(q,\bar{\underset{\sim}{s}}) \in \mathbb{P}^- * \underset{\sim}{\mathbb{R}}$, we define

(i) $(q,\bar{\underset{\sim}{s}}) \geq_{\mathrm{pr},\sigma} (p,\bar{\underset{\sim}{r}})$ iff

(α) $(q,\bar{\underset{\sim}{s}}) \geq (p,\bar{\underset{\sim}{r}})$,

(β) $q \restriction (\sigma+1) = p \restriction (\sigma+1)$,

(γ) for $\mu < \mu_0$ with $\neg(q \Vdash$ "$\underset{\sim}{\mathbb{R}}_\mu$ is trivial"), we have

$$(q,\emptyset_{\underset{\sim}{R}_\mu}) \Vdash \text{"if } \underset{\sim}{\kappa}_\mu < \sigma, \text{ then } \underset{\sim}{s}(\mu) \restriction (\underset{\sim}{\kappa}_\mu,\sigma] = \underset{\sim}{r}(\mu) \restriction (\underset{\sim}{\kappa}_\mu,\sigma]\text{"}.$$

(ii) $(q,\bar{\underset{\sim}{s}}) \geq_{\mathrm{apr},\sigma} (p,\bar{\underset{\sim}{r}})$ iff

(α) $(q,\bar{\underset{\sim}{s}}) \geq (p,\bar{\underset{\sim}{r}})$,

(β) $q \restriction (\sigma+1,\mu_0) = p \restriction (\sigma+1,\mu_0)$,

(γ) for $\mu < \mu_0$ with $\neg(q \Vdash$ "$\underset{\sim}{\mathbb{R}}_\mu$ is trivial"), we have

$$(q,\emptyset_{\underset{\sim}{R}_\mu}) \Vdash \text{"}\underset{\sim}{s}(\mu) \restriction (\sigma,\kappa^*) = \underset{\sim}{r}(\mu) \restriction (\sigma,\kappa^*)\text{"}.$$

(2) For $(p,\bar{\underset{\sim}{r}}) \in \mathbb{P}^- * \underset{\sim}{\mathbb{R}}$ and $\sigma = \mathrm{cf}(\sigma) \leq \mu_0$, we let

$$Q^-_{(p,\bar{\underset{\sim}{r}}),\sigma} \stackrel{\mathrm{def}}{=} \{(q,\bar{\underset{\sim}{s}}) :\ (q,\bar{\underset{\sim}{s}}) \geq_{\mathrm{apr},\sigma} (p',\bar{\underset{\sim}{r}}') \text{ for some } (p',\bar{\underset{\sim}{r}}') \leq_{\mathrm{pr},\sigma} (p,\bar{\underset{\sim}{r}})\},$$

ordered as a suborder of $\mathbb{P}^- * \underset{\sim}{\mathbb{R}}$.

Claim 1.13. Given $(p,\bar{\underset{\sim}{r}}) \leq (q,\bar{\underset{\sim}{s}})$ in $\mathbb{P}^- * \underset{\sim}{\mathbb{R}}$, and a regular $\sigma < \mu_0$.

<u>Then</u> there is a unique $(t,\bar{\underset{\sim}{z}})$ such that

$$(p,\bar{\underset{\sim}{r}}) \leq_{\mathrm{pr},\sigma} (t,\bar{\underset{\sim}{z}}) \leq_{\mathrm{apr},\sigma} (q,\bar{\underset{\sim}{s}}).$$

Proof of the Claim. Let $t \stackrel{\mathrm{def}}{=} p \restriction (\sigma+1) \cup q \restriction (\sigma+1,\mu_0)$. Hence $t \in \mathbb{P}^-$ and $p \leq t \leq q$ (note that $\sigma+1 \notin \mathrm{dom}(p)$). We define a $\mathbb{P}^- * \underset{\sim}{\mathbb{R}}$-name $\bar{\underset{\sim}{z}}$ by letting for $\mu < \mu_0$

$$\underset{\sim}{z}(\mu) \stackrel{\mathrm{def}}{=} \begin{cases} \underset{\sim}{r}(\mu) \restriction (\underset{\sim}{\kappa}_\mu,\sigma] \frown \underset{\sim}{q}(\mu) \restriction (\sigma,\kappa^*] & \text{if defined} \\ \underset{\sim}{r}(\mu) & \text{otherwise.} \end{cases}$$

$\bigstar_{1.13}$

Notation 1.14. $(t, \bar{\underset{\sim}{z}})$ as in Claim 1.13 is denoted by $\operatorname{intr}((p, \bar{\underset{\sim}{r}}), (q, \bar{\underset{\sim}{s}}))$.

Claim 1.15. For $\sigma = \operatorname{cf}(\sigma) < \mu_0$, the forcing notion $(\mathbb{P}^- * \underset{\sim}{\mathbb{R}}, \leq_{\mathrm{pr},\sigma})$ is $(< \sigma + 2)$-strategically closed.

Proof of the Claim. For every $\mu < \mu_0$ we have by Fact 1.5(5) applied to σ^+ (or $\sigma+2$, recalling that for coordinates $\zeta \in (\sigma, \sigma^+)$ in the iteration $\underset{\sim}{Q}_{\underset{\sim}{\kappa}_\mu, \kappa^*)}$ the factor $\underset{\sim}{R}_\zeta$ is trivial)

$$\Vdash_{\mathbb{P}^-} \text{“}\underset{\sim}{\mathbb{R}}_\mu \text{ non-trivial} \Longrightarrow \underset{\sim}{Q}_{(\underset{\sim}{\kappa}_\mu, \kappa^*)} / \underset{\sim}{Q}_{(\underset{\sim}{\kappa}_\mu, \sigma]} \text{ is } (< \sigma + 2)\text{-strategically closed.”}$$

Hence we can find names $\underset{\sim}{\mathrm{St}}^\sigma_\mu$ of the winning strategies exemplifying the corresponding instances of the above statement.

Suppose that $\zeta \leq \sigma + 1$ and $\langle p_\xi = \langle p^0_\xi, \bar{\underset{\sim}{p}}^1_\xi \rangle : \xi < \zeta \rangle$, $\langle q_\xi = \langle q^0_\xi, \bar{\underset{\sim}{q}}^1_\xi \rangle : \xi < \zeta \rangle$ are sequences of elements of $\mathbb{P}^- * \underset{\sim}{\mathbb{R}}$ such that

(1) For all $\xi < \zeta$ we have $p_\xi \leq_{\mathrm{pr},\sigma} q_\xi$,

(2) For all $\xi < \zeta$ and $\varepsilon < \xi$ we have $q_\varepsilon \leq_{\mathrm{pr},\sigma} p_\xi$ and for $\mu < \mu_0$ with $\neg(p^0_\xi \Vdash \text{“}\underset{\sim}{\mathbb{R}}_\mu \text{ is trivial”})$, we have $(p^0_\xi, (\emptyset_{\underset{\sim}{R}_\mu}, \underset{\sim}{p}^1_\xi(\mu) \restriction (\underset{\sim}{\kappa}_\mu, \sigma])) \Vdash_{\mathbb{P}^- * \underset{\sim}{\mathbb{R}}_\mu * \underset{\sim}{Q}_{(\underset{\sim}{\kappa}_\mu, \sigma]}}$

$$\text{“}\underset{\sim}{p}^1_\xi(\mu) \restriction (\sigma, \kappa^*) = \underset{\sim}{\mathrm{St}}^\sigma_\mu(\langle \underset{\sim}{p}^1_\varepsilon(\mu) \restriction (\sigma, \kappa^*) : \varepsilon < \xi \rangle, \langle \underset{\sim}{q}^1_\varepsilon(\mu) \restriction (\sigma, \kappa^*) : \varepsilon < \xi \rangle)\text{”}$$

We define p_ζ as follows. First let $p^0_\zeta \stackrel{\text{def}}{=} \bigcup \{ q^0_\xi : \xi < \zeta \}$. Notice that $p^0_\zeta \in \mathbb{P}^-$ and $p^0_\zeta \restriction (\sigma + 1) = p^0_0 \restriction (\sigma + 1)$.

For $\mu < \mu_0$ with $\neg(p_\xi \Vdash \text{“}\underset{\sim}{\mathbb{R}}_\mu \text{ is trivial”})$, we let $\underset{\sim}{p}^1_\zeta(\mu)$ be the name given by

$$\underset{\sim}{p}^1_\zeta(\mu) \restriction (\underset{\sim}{\kappa}_\mu, \sigma] \stackrel{\text{def}}{=} \underset{\sim}{p}^1_0(\mu) \restriction (\underset{\sim}{\kappa}_\mu, \sigma]$$

and

$$\underset{\sim}{p}^1_\zeta(\mu) \restriction (\sigma, \kappa^*) \stackrel{\text{def}}{=} \underset{\sim}{\mathrm{St}}^\sigma_\mu(\langle \underset{\sim}{p}^1_\xi(\mu) \restriction (\sigma, \kappa^*) : \xi < \zeta \rangle, \langle \underset{\sim}{q}^1_\xi(\mu) \restriction (\sigma, \kappa^*) : \xi < \zeta \rangle).$$

The conclusion follows because we have just described a winning strategy for I in $\partial((\mathbb{P}^- * \underset{\sim}{\mathbb{R}}, \leq_{\mathrm{pr},\sigma}), \xi)$. $\bigstar_{1.15}$

Claim 1.16. Suppose $(p, \underset{\sim}{\bar{r}}) \Vdash$ "$\underset{\sim}{\tau} : \sigma \to \mathrm{Ord}$", where σ is regular $< \mu_0$.

<u>Then</u> there is $(q, \underset{\sim}{\bar{s}}) \geq_{\mathrm{pr},\sigma} (p, \underset{\sim}{\bar{r}})$ and a $Q^-_{(q,\underset{\sim}{\bar{s}}),\sigma}$-name $\underset{\sim}{\tau}'$ such that

$$(q, \underset{\sim}{\bar{s}}) \Vdash \text{“}\underset{\sim}{\tau} = \underset{\sim}{\tau}'\text{”}.$$

Proof of the Claim. We define a play of $\partial((\mathbb{P}^- * \underset{\sim}{\mathbb{R}}, \leq_{\mathrm{pr},\sigma}), \sigma + 1)$ as follows.

Player I starts by playing $(p, \underset{\sim}{\bar{r}}) \stackrel{\mathrm{def}}{=} p_0$. At the stage $\zeta \leq \sigma$, player II chooses $q^*_\zeta \geq p_\zeta$ such that q^*_ζ forces a value to $\underset{\sim}{\tau}(\zeta)$, and we let $q_\zeta \stackrel{\mathrm{def}}{=} \mathrm{intr}(p_\zeta, q^*_\zeta)$. At the stage $0 < \zeta < \sigma$, we let I play according to the winning strategy for $\partial((\mathbb{P}^- * \underset{\sim}{\mathbb{R}}, \leq_{\mathrm{pr},\sigma}), \sigma + 1)$ applied to $(\langle p_\xi : \xi < \zeta\rangle, \langle q_\xi : \xi < \zeta\rangle)$. At the end, we let $(q, \underset{\sim}{\bar{s}}) = p_\sigma$. This process defines $\underset{\sim}{\tau}'$ by letting $\underset{\sim}{\tau}'(\zeta)$ be the $Q^-_{(q,\underset{\sim}{\bar{s}}),\sigma}$-name such that for $(q^*, \underset{\sim}{\bar{t}}) \in Q^-_{(q,\underset{\sim}{\bar{s}}),\sigma}$ we have

$$(q^*, \underset{\sim}{\bar{t}}) \Vdash \text{“}\underset{\sim}{\tau}'(\zeta) = \underset{\sim}{\tau}(\zeta)\text{”}.$$

Note that $q^*_\zeta \in Q^-_{(q,\underset{\sim}{\bar{s}}),\sigma}$ and $\underset{\sim}{\tau}'$ is a $Q^-_{(q,\underset{\sim}{\bar{s}}),\sigma}$-name. $\bigstar_{1.16}$

Claim 1.17. If $(p, \underset{\sim}{\bar{r}}) \in \mathbb{P}^- * \underset{\sim}{\mathbb{R}}$, and $\sigma < \mu_0$ is inaccessible with $\sigma \in \mathrm{dom}(p)$, <u>then</u> $Q^-_{(p,\underset{\sim}{\bar{r}}),\sigma}$ has μ_0-cc.

Proof of the Claim. Given $\bar{q} = \langle q_i = \langle q^0_i, \underset{\sim}{q}^1_i\rangle : i < \mu_0\rangle$, with $q_i \in Q^-_{(p,\underset{\sim}{\bar{r}}),\sigma}$. Suppose for contradiction that the range of this sequence is an antichain.

We have that for all $i < \mu_0$

$$\mathrm{dom}(q^0_i) \cap (\sigma + 1, \mu_0) \subseteq \mathrm{dom}(p) \cap (\sigma + 1, \mu_0).$$

As $\mathrm{dom}(p)$ is an Easton set, without loss of generality we have that all $\mathrm{dom}(q^0_i) \cap (\sigma + 1, \mu_0)$ are the same. If this set has the largest element, let us denote its successor by ρ. Otherwise, let $\rho \stackrel{\mathrm{def}}{=} |\sup[\mathrm{dom}(p) \cap (\sigma + 1, \mu_0)]|^+$. In either case, we have that for all i and $\alpha \in \mathrm{dom}(q^0_i \restriction (\sigma + 1, \mu_0))$ the relation $q^0_i(\alpha) \in \mathcal{H}(\rho)$ holds. Hence for all $i < \mu_0$

$$\langle q^0_i(\alpha) : \alpha \in \mathrm{dom}(q^0_i) \cap (\sigma + 1, \mu_0)\rangle \in \mathcal{H}(\rho),$$

as $\mathrm{dom}(q^0_i)$ is an Easton set, so without loss of generality all q^0_i are the same. As $\sigma \in \mathrm{dom}(p)$, for each i we have that $\sigma \in \mathrm{dom}(q^0_i)$ and hence $q^0_i \restriction \sigma \subseteq \mathcal{H}(\sigma)$

and we can assume that all $q^0_i \restriction \sigma$ are the same, and hence all q^0_i are the same condition in $\mathbb{P}^-$, which we shall call q^*. Let G^- be $\mathbb{P}^-$-generic over V with $q^* \in G^-$. Hence in $V[G^-]$ the sequence $\langle q^1_i \stackrel{\text{def}}{=} (\emptyset, \underset{\sim}{\bar{q}}^i) : i < \mu_0 \rangle$ is an antichain in $\prod_{\mu<\mu_0} [R_\mu * \underset{\sim}{Q}_{(\kappa_\mu,\kappa^*)}]^\otimes$, and by the choice of the initial sequence, we have that $\langle (\emptyset, \underset{\sim}{q}^i(\mu) \restriction (\sigma+1)) : \mu < \mu_0 \rangle : i < \mu_0 \rangle$ is an antichain in $\prod_{\mu<\mu_0} [R_\mu * \underset{\sim}{Q}_{(\kappa_\mu,\sigma)}]^\otimes$. For every $i < \mu_0$,

$$A_i \stackrel{\text{def}}{=} \{\mu < \mu_0 : \underset{\sim}{q}^i_\mu \restriction (\sigma+1) \neq \emptyset\}$$

is the union of $\leq \sigma$ Easton sets. Hence by Claim 1.9 (i.e. as in $V[G^-]$ the cardinal μ_0 is still strongly Mahlo), without loss of generality, A_i's form a Δ-system with root A^*. Note that A^* is a bounded subset of μ_0. We can conclude that

$$\langle \langle \emptyset, \underset{\sim}{q}^i_\mu \restriction (\sigma+1) \rangle : \mu \leq \sup(A^*) \rangle : i < \mu_0 \rangle$$

forms an antichain in $\prod_{\mu \leq \sup(A^*)} [R_\mu * \underset{\sim}{Q}_{(\kappa_\mu,\sigma)}]^\otimes$, in contradiction with Fact 1.5 (in fact the number of possible values is $< \mu_0$. $\bigstar_{1.17}$

Claim 1.18. Forcing with $\mathbb{P}^- * \underset{\sim}{\mathbb{R}}$ preserves cardinals and cofinalities $\geq \mu_0$.

Proof of the Claim. Suppose cofinalities $\geq \mu_0$ are not preserved and let θ be the first cofinality $\geq \mu_0$ destroyed. Hence θ is regular, and for some $\underset{\sim}{\tau}$, condition $(p, \underset{\sim}{\bar{r}})$ and some regular $\sigma < \theta$, we have $(p, \underset{\sim}{\bar{r}}) \Vdash$ "$\underset{\sim}{\tau} : \sigma \to \theta$ is cofinal".

<u>Case 1</u>. $\sigma < \mu_0$. Without loss of generality, σ is inaccessible, and by increasing σ and p if necessary, we can assume that $\sigma \in \mathrm{dom}(p)$. By Claim 1.16, there is $(q, \underset{\sim}{\bar{s}}) \geq (p, \underset{\sim}{\bar{r}})$ and a $Q^-_{(q,\underset{\sim}{\bar{s}}),\sigma}$-name $\underset{\sim}{\tau}'$ such that $(q, \underset{\sim}{\bar{s}}) \Vdash$ "$\underset{\sim}{\tau} = \underset{\sim}{\tau}'$". Hence $(q, \underset{\sim}{\bar{s}}) \Vdash$ "$\underset{\sim}{\tau}' : \sigma \to \theta$ is cofinal", contradicting the fact that $Q^-_{(p,\underset{\sim}{\bar{r}}),\sigma}$ satisfies μ_0-cc, which follows from Claim 1.17.

<u>Case 2</u>. $\sigma \geq \mu_0$.

As for every $\mu < \mu_0$ with $\underset{\sim}{\mathbb{R}}_\mu$ non-trivial we have

$$\Vdash_{\mathbb{P}^- * \underset{\sim}{R}_\mu} \text{"}\underset{\sim}{Q}_{(\underset{\sim}{\kappa}_\mu,\kappa^*)} / \underset{\sim}{Q}_{(\underset{\sim}{\kappa}_\mu,\sigma]} \text{ is } (< (\sigma+1))\text{-strategically closed"},$$

there is $(q, \underset{\sim}{\bar{s}}) \geq (p, \underset{\sim}{\bar{r}})$ and a $\mathbb{P}^- * \prod_{\mu<\mu_0}[\underset{\sim}{R}_\mu * \underset{\sim}{Q}_{(\underset{\sim}{\kappa}_\mu,\sigma)}]^{\otimes}$-name $\underset{\sim}{\tau}'$ such that $(q, \underset{\sim}{\bar{s}}) \Vdash$ "$\underset{\sim}{\tau}' : \sigma \to \theta$ is cofinal". But this forcing has σ^+-cc, a contradiction. $\bigstar_{1.18}$

Combining Claim 1.18 with Fact 1.5, we obtain

Corollary 1.19. Forcing with $\mathbb{P}^- * \underset{\sim}{\mathbb{R}} * \underset{\sim}{Q}_{(\kappa^*,\lambda^*)}$ preserves cardinalities and cofinalities $\geq \mu_0$, it preserves GCH from μ_0 on, and forces $SNR(\theta, \kappa^*)$ for $\theta \in (\kappa^*, \lambda^*)$.

By our assumptions it follows that the predecessor of λ^* is singular, so λ^* remains the successor of a singular after we have forced by $\mathbb{P}^- * \underset{\sim}{\mathbb{R}} * \underset{\sim}{Q}_{(\kappa^*,\lambda^*)}$, and clearly κ^* remains regular.

Now we have arranged the situation so that we are left with the main point of the argument, which is that after forcing with $\mathbb{P}^- * \underset{\sim}{\mathbb{R}} * \underset{\sim}{Q}_{(\kappa^*,\lambda^*)}$ we shall have weak reflection of λ^* at κ^*. The basic forcing for enforcing strong non-reflection and the actual forcing we have used posses a convenient homogeneity property that will become relevant in the main argument, so we formulate these in the following. As the proof below suggests, the homogeneity these forcing notions have is actually stronger than the mild homogeneity defined in the following definition, but as all we need is mild homogeneity, we formulate our claims using that notion.

Definition 1.20. We shall call a forcing notion P *mildly homogeneous* iff for every formula $\varphi(x_0, \ldots, x_{n-1})$ of the forcing language of P and $a_0, \ldots, a_{n-1}$ (canonical names of) objects in V, we have $\emptyset_P \parallel$ "$\varphi(a_0, \ldots, a_{n-1})$".

Claim 1.21. (1) $\mathbb{P}(\sigma, \lambda)$ is mildly homogeneous, for $\aleph_0 < \mathrm{cf}(\sigma) = \sigma < \lambda$,

(2) $Q_{(\kappa,\lambda)}$ is mildly homogeneous, for $\aleph_0 < \mathrm{cf}(\kappa) = \kappa < \lambda$.

(3) If R is mildly homogeneous and $Q^{\otimes}_{(R,\kappa,\lambda)}$ is well defined, <u>then</u> $Q^{\otimes}_{(R,\kappa,\lambda)}$ is mildly homogeneous.

(4) It is forced by $\mathbb{P}^-$ that $\underset{\sim}{\mathbb{R}}$ is mildly homogeneous.

(5) It is forced by $\mathbb{P}^-$ that $\underset{\sim}{\mathbb{R}} * \underset{\sim}{Q}_{(\kappa^*,\lambda^*)}$ is mildly homogeneous.

Proof of the Claim. (1) Suppose not, and let $p, q \in P \stackrel{\text{def}}{=} \mathbb{P}(\sigma,\lambda)$ force contradictory statements about $\varphi(a_0,\ldots,a_{n-1})$ for some $a_0,\ldots,a_{n-1} \in V$. Let $\alpha = \mathrm{dom}(q)$ and consider the function $F = F_{(\sigma,\lambda)} : P \to P$ such that $F(f) = g$ iff $q \subseteq g$ and for $i \in \mathrm{dom}(f)$ we have $g(\alpha + i) = f(i)$.

This function is an isomorphism between P and $P/q \stackrel{\text{def}}{=} \{g \in P : g \supseteq q\}$, so it induces an isomorphism between the canonical P-names for objects in V and the canonical P/q-names for the same objects. In particular, $F(p)$ forces in P/q the same statements about $a_0,\ldots,a_{n-1}$ that p does in P. If G is P-generic over V such that $F(p) \in G$ (then also $q \in G$). As $q \in G$ and $F(p)$ force contradictory statements about $a_0,\ldots,a_{n-1}$, we obtain a contradiction.
(2) Suppose that $p, q \in Q = Q_{(\kappa,\lambda)}$ force contradictory statements about $\varphi(a_0,\ldots,a_{n-1})$ for some $a_0,\ldots,a_{n-1} \in V$. We define a function $F^+ : Q \to Q$ so that such that $F(f) = g$ iff $\mathrm{Dom}(g) = \mathrm{Dom}(q) \cup \mathrm{Dom}(f)$ and for $\zeta \in \mathrm{Dom}(g)$ we have that $q \restriction \zeta \Vdash$ "$g(\zeta) = F_{(\kappa,\zeta)}(p(\zeta))$, where $F_{(\kappa,\zeta)}$ is defined as in (1) above. One can now check that F^+ is an isomorphism between Q and Q/q, and then the conclusion follows as in (1).
(3)-(5) Similar proofs. $\bigstar_{1.21}$

Remark 1.22. The homogeneity properties discussed in Claim 1.21 are not enjoyed by the forcing notion $\mathbb{P}^-$.

Main Claim 1.23. After forcing with $\mathbb{P} \stackrel{\text{def}}{=} \mathbb{P}^- * \underset{\sim}{\mathbb{R}} * \underset{\sim}{Q}_{(\kappa^*,\lambda^*)}$, the weak reflection of λ^* holds at κ^*.

Proof of the Main Claim. Suppose otherwise, and let $p^* = (p, \underset{\sim}{q}, \underset{\sim}{r})$ force $\underset{\sim}{\tau}$ to be a function exemplifying the strong non-reflection of λ^* at κ^*. As $\underset{\sim}{\mathbb{R}} * \underset{\sim}{Q}_{(\kappa^*,\lambda^*)}$ is forced to be mildly homogeneous by Claim 1.21, without loss of generality $p^* = (p, \emptyset, \emptyset)$. We proceed through a series of lemmas that taken together suffice to prove the Claim.

Lemma 1.24. There are cardinals μ, κ and χ' and a model $N \prec \mathcal{H}(\chi)$ such that

(i) $N \cap \mu_0$ is an inaccessible cardinal $\mu < \mu_0$,

(ii) $\mathrm{otp}(N \cap \lambda^*) = \kappa^*$,

(iii) $\mathrm{otp}(N \cap \mu_1) = \mu_0$,

(iv) ${}^{\mu >}N \subseteq N$,

(v) $(N, \in)$ is isomorphic to $\mathcal{H}(\chi')$ for some regular $\chi' < \chi$,

(vi) $|N \cap \kappa^*|$ is a regular cardinal $\kappa < \mu_0$, in fact $\mathrm{otp}(N \cap \kappa^*) = \kappa$,

(vii) $\kappa^*, \mu_0, \mu_1, \lambda^*, \mathbb{P}, p^*, \underset{\sim}{\tau} \in N$.

Proof of the Lemma. We use the notation of the main Theorem 0.2, in particular M described in the assumptions of the Theorem. Since we assume that ${}^{\chi}M \subseteq M$ and $\chi^{<\chi} = \chi$ we conclude that $\mathbf{j}``(\mathcal{H}(\chi)) \in M$. Now we can use elementarity to obtain $N \in V$ whose relationship to $\mathcal{H}(\chi)$ mirrors that of the relationship between $\mathbf{j}``(\mathcal{H}(\chi))$ and $\mathbf{j}(\mathcal{H}(\chi))$ in M. In M we have that $\mathbf{j}``(\mathcal{H}(\chi)) \cap \mathbf{j}(\mu_0) = \mu_0$ since the critical point of $\mathbf{j}$ is μ_0, so in M this is an inaccessible cardinal $< \mathbf{j}(\mu_0)$, and so we can obtain that N satisfies (i) above. Requirement (ii) follows because in M we have

$$\mathbf{j}(\lambda^*) \cap \mathbf{j}``(\mathcal{H}(\chi)) = \mathbf{j}``(\lambda^*)$$

since $\lambda^* < \chi$, and hence this set has order type $\lambda^* = \mathbf{j}(\kappa^*)$. We argue similarly for requirement (iii), using that $\mu_1 < \chi$ and $\mathbf{j}(\mu_0) = \mu_1$. For (iv) we choose the definition of μ as the "mirror" image of μ_0 and the fact that $\mathbf{j}``(\mathcal{H}(\chi))$ is closed under sequences of length $< \mu_0$, as $\mu_0$ is the critical point of the embedding. For (v) we use the fact that $\mathbf{j}``(\mathcal{H}(\chi))$ is isomorphic to $\mathcal{H}(\chi)$, which can be said in M because M is closed under χ sequences, and (vi) and (vii) are handled similarly to the above. $\bigstar_{1.24}$

We fix N as guaranteed by Lemma 1.24 and let $\delta \overset{\mathrm{def}}{=} \sup(N \cap \lambda^*)$. Some simple consequences of the choice of N are given in the following

Lemma 1.25. The following hold

(i) $\delta \in S^{\lambda^*}_{\kappa^*}$,

(ii) $N \cap \delta$ is a stationary subset of δ, and it remains such after forcing with $\mathbb{P}$.

Proof of the Lemma. The first claim holds because $\mathrm{otp}(N \cap \lambda^*)$ is κ^*. As for (ii), notice that the set E defined as the closure of $N \cap \delta$ is a club of δ. Letting $S \stackrel{\mathrm{def}}{=} S^{\delta}_{\aleph_0} \cap N$ we have that $[\alpha \in E \ \&\ \mathrm{cf}(\alpha) = \aleph_0] \Longrightarrow \alpha \in S$ (the analogue of this is true even with"$\mathrm{cf}(\alpha) < \mu$" in place of "$\mathrm{cf}(\alpha) = \aleph_0$"). As $\mathbb{P}$ is an $(< \omega_1)$-closed forcing notion, S remains stationary after forcing with $\mathbb{P}$ and hence so does $N \cap \delta$. $\bigstar_{1.25}$

The next step of our argument is to extend p to a $\mathbb{P}^-$-generic condition over N, which is done as in the following. We use the notation Most_N to denote the function of the Mostowski collapse of the structure $(N, \in)$.

Note that one of the properties of N is that it is isomorphic to $\mathcal{H}(\chi')$ for some regular cardinal $\chi' < \chi$, in particular the Mostowski collapse of N is $\mathcal{H}(\chi')$, and it follows from the other properties of N that $\kappa^* \in \mathcal{H}(\chi')$ and $\kappa^* = \mathrm{Most}_N(\lambda^*)$.

As $p^* \in N$ and $N \cap \mu_0 = \mu$, we have that $\mathrm{dom}(p) \subseteq \mu$ and since $\mathbb{P}^- \in N$ also for all $\sigma \in \mathrm{dom}(p)$ we have $p(\sigma) \in \mathcal{H}(\mu)$. Hence we can extend p to the condition

$$p^+ \stackrel{\mathrm{def}}{=} p \cup \{\langle \mu, (\kappa, \mathrm{Most}_N((\mathbb{P}^- * \underset{\sim}{\mathbb{R}})^N))\rangle\}.$$

Note that this is a well defined element of $\mathbb{P}^-$ because $\kappa < \mu_0$ (see clause (vi)) and $\mathrm{Most}_N((\mathbb{P}^- * \underset{\sim}{\mathbb{R}})^N)$ is a well defined element of $\mathcal{H}(\mu_0)$ and a forcing notion that preserves the fact that κ is a regular uncountable cardinal. In fact we have

Claim 1.26. The condition p^+ a well defined extension of p which is $\mathbb{P}^-$-generic over N.

Proof of the Claim. We have already discussed the fact that p^+ is a well defined extension of p. By the choice of its last coordinate it is actually $\mathbb{P}^-$-generic over N, since any condition $q \geq p^+$ will satisfy that for all $\alpha \in N \cap \mathrm{dom}(q)$ we have $q(\alpha) \in \mathcal{H}(\mu)$. $\bigstar_{1.26}$

Claim 1.27. The condition p^+ forces that

$$\underset{\sim}{\kappa}_\mu = \kappa \text{ and } \underset{\sim}{R}_\mu = \mathrm{Most}_N((\mathbb{P}^- * \underset{\sim}{\mathbb{R}})^N),$$

hence that $\underset{\sim}{\mathbb{R}}_\mu$ is $[\mathrm{Most}_N((\mathbb{P}^- * \underset{\sim}{\mathbb{R}})^N) * \underset{\sim}{Q}_{(\kappa,\kappa^*)}]^{\otimes}$.

Proof of the Claim. This follows by the definition of $\underset{\sim}{\kappa}_\mu$ and $\underset{\sim}{\mathbb{R}}_\mu$. $\bigstar_{1.27}$

Claim 1.28. The condition $(p^+, \emptyset)$ is $\mathbb{P}^- * \underset{\sim}{\mathbb{R}}$-generic over N.

Proof of the Claim. This follows by the previous claims and the definition of term forcing. $\bigstar_{1.28}$

Using the definition of the Mostowski collapse, one can establish a connection between $Q^N_{(\kappa^*,\lambda^*)}$ and $Q_{(\kappa,\kappa^*)}$. Namely, let F be the inverse of the Mostowski collapse of $N \cap \lambda^*$, so an order preserving function from κ^* onto $N \cap \lambda^*$. We also let $F(\kappa^*) = \lambda^*$. If $N \vDash$ "$\sigma = \mathrm{cf}(\sigma) \in (\kappa^*, \lambda^*)$", then $F^{-1}(\sigma)$ is an ordinal in (κ, κ^*) and if for such σ we have $N \vDash$ "$\beta < \sigma$ is an ordinal", then $F^{-1}(\beta)$ is an ordinal $< F^{-1}(\sigma)$. For such β, if $N \vDash$ "r is a function from β to κ^*", then

$$\{(\alpha, F^{-1}(\gamma)) : \alpha < F^{-1}(\beta) \ \& \ N \vDash \text{“}r(F(\alpha)) = \gamma\}$$

is a function from $F^{-1}(\beta)$ into $F^{-1}(\kappa^*)$. Continuing in a similar fashion, we can see that F^{-1} induces a mapping from $\mathbb{P}(\kappa^*, \sigma)^N$ into $\mathbb{P}(\kappa, F^{-1}(\sigma))$, which we again denote by F^{-1}. Suppose that $N \vDash$ "$\alpha \le \lambda^*$ and $A \subseteq \alpha$ is an Easton set", then $F^{-1}(\alpha) \le \kappa^*$ and $F^{-1}(A) \subseteq F^{-1}(\alpha)$ is an Easton set, for if $\sigma \in \mathrm{Reg} \cap \sup(F^{-1}(A)) + 1$ then $N \vDash$ "$F(\sigma) \in \mathrm{Reg} \cap \sup(A) + 1$" and so $\sup(F^{-1}(A) \cap \sigma) < \sigma$. This shows that F^{-1} induces a mapping from $Q^N_{(\kappa^*,\lambda^*)}$ into $Q_{(\kappa,\kappa^*)}$, which we again denote by F^{-1}.

This process can be pushed one step further. As p^+ is $\mathbb{P}^-$-generic over N, it forces that $N[\underset{\sim}{G}] \cap V = N$, so using that $\mathbb{P}^- \in N$ and an analysis similar to the one carried so far, one shows that over p^+ it is forced that $[\mathrm{Most}_N((\mathbb{P}^- * \underset{\sim}{\mathbb{R}})^N) * \underset{\sim}{Q}_{(\kappa,\kappa^*)}]^{\otimes}$ is $\mathrm{Most}_N(([(\mathbb{P}^- * \underset{\sim}{\mathbb{R}}) * \underset{\sim}{Q}_{(\kappa^*,\lambda^*)}]^{\otimes})^N)$ and hence that there is a function F^{-1} that induces a mapping between the $\mathbb{P}^-$-names for conditions in the latter forcing and those in $[(\mathbb{P}^- * \underset{\sim}{\mathbb{R}}) * \underset{\sim}{Q}_{(\kappa^*,\lambda^*)}]^{\otimes}$. This now shows that p^+ forces that $\underset{\sim}{\mathbb{R}}_\mu$ is $\mathrm{Most}_N(([(\mathbb{P}^- * \underset{\sim}{\mathbb{R}}) * \underset{\sim}{Q}_{(\kappa^*,\lambda^*)}]^{\otimes})^N)$.

Let $\underset{\sim}{H}$ be the canonical $\mathbb{P}^-$-name for a subset of $\mathrm{Most}_N([(\mathbb{P}^- * \underset{\sim}{\mathbb{R}}) * \underset{\sim}{Q}_{(\kappa^*,\lambda^*)}]^{\otimes})^N)$ such that p^+ forces $\underset{\sim}{H}$ to be $\mathrm{Most}_N([(\mathbb{P}^- * \underset{\sim}{\mathbb{R}}) * \underset{\sim}{Q}_{(\kappa^*,\lambda^*)}]^{\otimes})^N)$-generic. The Mostowski collapse induces a mapping that maps $\underset{\sim}{H}$ into a $\mathbb{P}^-$-name for a subset $\underset{\sim}{H}^*$ of $([(\mathbb{P}^- * \underset{\sim}{\mathbb{R}}) * \underset{\sim}{Q}_{(\kappa^*,\lambda^*)}]^{\otimes})^N$.

We proceed to define q as follows: $q \stackrel{\text{def}}{=} (p^+, \emptyset_{\mathbb{R}}, \underset{\sim}{r})$, where $\underset{\sim}{r}$ is a $\mathbb{P}^- * \underset{\sim}{\mathbb{R}}$-name over $(p^+, \emptyset_{\underset{\sim}{\mathbb{R}}})$ of a condition in $\underset{\sim}{Q}_{(\kappa^*,\lambda^*)}$ defined by letting

$$\mathrm{Dom}(\underset{\sim}{r}) \stackrel{\text{def}}{=} \bigcup \{\mathrm{Dom}(\underset{\sim}{h}) : p^+ \Vdash_{\mathbb{P}^-} \text{“}(\emptyset_{\underset{\sim}{\mathbb{R}}}, \underset{\sim}{h}) \in \underset{\sim}{H}^*\text{”}\},$$

and for θ with $(p^+, \emptyset_{\underset{\sim}{\mathbb{R}}}) \Vdash$ “$\theta \in \mathrm{Dom}(\underset{\sim}{r})$”, we let

$$\underset{\sim}{r}(\theta) \stackrel{\text{def}}{=} \bigcup \{\underset{\sim}{h}(\theta) : (p^+, \emptyset_{\underset{\sim}{\mathbb{R}}}) \Vdash \text{“}\theta \in \mathrm{Dom}(\underset{\sim}{h}) \ \& \ \underset{\sim}{h} \in \underset{\sim}{H}^*\text{”}\}.$$

We have to verify that q is a condition in $\mathbb{P}$ and we also claim that $q \geq p^*$. Let us check the relevant items by proving a series of short Lemmas:

Lemma 1.29. If $(p^+, \emptyset_{\underset{\sim}{\mathbb{R}}}) \Vdash$ “θ is strongly inaccessible $\in (\kappa^*, \lambda^*)$”, then

$$(p^+, \emptyset_{\underset{\sim}{\mathbb{R}}}) \Vdash \text{“}|\mathrm{Dom}(\underset{\sim}{r}) \cap \theta| < \theta\text{”}.$$

Proof of the Lemma. We have that $(p^+, \emptyset_{\underset{\sim}{\mathbb{R}}})$ forces that

$$\mathrm{Dom}(\underset{\sim}{r}) \cap \theta \subseteq \bigcup \{\mathrm{Dom}(\underset{\sim}{f}) \cap \theta : \underset{\sim}{f} \in \underset{\sim}{H}^*\}$$

and $|\underset{\sim}{H}^*| \leq \kappa^* < \theta$ (as $|N \cap \lambda^*| = \kappa^*$), so $\mathrm{Dom}(\underset{\sim}{r}) \cap \theta$ is forced to be bounded in θ. $\bigstar_{1.29}$

Lemma 1.30. If $(p^+, \emptyset_{\underset{\sim}{\mathbb{R}}}) \Vdash$ “$\theta \in \mathrm{Dom}(\underset{\sim}{r})$”, then $(p^+, \emptyset_{\underset{\sim}{\mathbb{R}}}) \Vdash$ “$\underset{\sim}{r}(\theta)$ is a function whose domain is an ordinal $< \theta$ and range a subset of κ^*”.

Proof of the Lemma. As $(p^+, \emptyset_{\underset{\sim}{\mathbb{R}}}) \Vdash$ “$\underset{\sim}{H}^*$ is directed”, we have that $(p^+, \emptyset_{\underset{\sim}{\mathbb{R}}}) \Vdash$ “$\underset{\sim}{r}(\theta)$ is a function”. If

$$(p^+, \emptyset_{\underset{\sim}{\mathbb{R}}}) \Vdash \text{“}\theta \in \mathrm{Dom}(\underset{\sim}{h}) \ \& \ F(\underset{\sim}{h}) \in \underset{\sim}{H}\text{”},$$

then $(p^+, \emptyset_{\underset{\sim}{\mathbb{R}}})$ forces

$$\text{“}(\forall\sigma \in \mathrm{Dom}(F(\underset{\sim}{h})))\,[F(\underset{\sim}{h})(\sigma) \text{ is a function with domain } \in \sigma]\text{”},$$

so by the definition of F and the fact that $|\underset{\sim}{H}^*| \leq \kappa^* < \theta = \mathrm{cf}(\theta)$ we have

$$(p^+, \emptyset_{\underset{\sim}{\mathbb{R}}}) \Vdash \text{“dom}(\underset{\sim}{h}(\theta)) \text{ is an element of } \theta\text{”}$$

and $(p^+, \emptyset_{\underset{\sim}{\mathbb{R}}}) \Vdash$ “$\mathrm{dom}(\underset{\sim}{r}(\theta))$ is an element of θ”. $\bigstar_{1.30}$

Lemma 1.31. $(p^+, \emptyset_{\underset{\sim}{\mathbb{R}}}) \Vdash$ “$\underset{\sim}{r} \in \underset{\sim}{Q}_{(\kappa^*, \lambda^*)}$”.

Proof of the Lemma. We know by Lemma 1.30 that $(p^+, \emptyset_{\underset{\sim}{\mathbb{R}}}) \Vdash$ “$\mathrm{Dom}(\underset{\sim}{r}) \subseteq \lambda^*$ is an Easton set”. It is forced by $(p^+, \emptyset_{\underset{\sim}{\mathbb{R}}})$ that for relevant θ the set $\mathrm{dom}(\underset{\sim}{r}(\theta))$ is the union of a subset of $\mathcal{H}(\theta^{++}) \cap N$ which has cardinality $\leq |\theta^{++} \cap N| < \kappa^*$, and clearly $\{h(\theta) : h \in \underset{\sim}{H}^*\}$ has no last element, by density and genericity. Hence the sup of this union has cofinality $< \kappa^*$ (as having cofinality $\geq \kappa^*$ is preserved by the forcing $\mathbb{P}^- * \underset{\sim}{\mathbb{R}}$), so $\underset{\sim}{r}(\theta)$ is forced to be in $\underset{\sim}{\mathbb{P}}(\theta, \kappa^*)$, and hence $\underset{\sim}{r}$ is forced to be an element of $\underset{\sim}{Q}_{(\kappa^*, \lambda^*)}$. $\bigstar_{1.31}$

Lemma 1.32. $(p^+, \emptyset_{\underset{\sim}{\mathbb{R}}})$ forces that if $\theta \in \mathrm{Dom}(r)$ then for all $\varepsilon \in S^{\theta}_{\kappa^*}$, there is a club e of ε on which $\underset{\sim}{r}(\theta)$ is strictly increasing, for $\theta \in \mathrm{dom}(\underset{\sim}{r})$.

Proof of the Lemma. Again modulo what is forced by $(p^+, \emptyset_{\underset{\sim}{\mathbb{R}}})$, the set $\{h(\theta) : h \in \underset{\sim}{H}^*, \theta \in \mathrm{Dom}(h)\}$ is linearly ordered, as the domains are ordinals, so the conclusion follows because the analogue holds for each $h \in \underset{\sim}{H}^*$. $\bigstar_{1.32}$

Conclusion 1.33. The condition $q = (p^+, \emptyset, \underset{\sim}{r})$ is an element of $\mathbb{P}$ and is above any condition of the form $(p^+, \emptyset, h)$ such that $(p^+, \emptyset)$ forces that $h \in \underset{\sim}{H}^*$. Consequently, q is $\mathbb{P}$-generic over N.

We shall now see that q forces $\underset{\sim}{\tau}$ to be constant on a stationary subset of δ, a contradiction, as $\delta \in S^{\lambda^*}_{\kappa^*}$, and remains there after forcing with $\mathbb{P}$. We need to consider what q forces about $\underset{\sim}{\tau}(\alpha)$ for $\alpha \in N \cap \lambda^*$. Such $\underset{\sim}{\tau}(\alpha)$ is a

$\mathbb{P}$-name of an ordinal $< \kappa^*$, by the choice of $\underset{\sim}{\tau}$, see the beginning of the proof of the Main Claim. Let

$$\mathcal{I}_\alpha \stackrel{\text{def}}{=} \{(\emptyset, \emptyset, \underset{\sim}{t}) \in \mathbb{P} : (\emptyset, \emptyset, \underset{\sim}{t}) \text{ forces } \underset{\sim}{\tau}(\alpha) \text{ to be equal to a } \mathbb{P}^- * \underset{\sim}{\mathbb{R}}\text{-name}\}.$$

Hence $\mathcal{I}_\alpha \in N$. As $\mathbb{P}^- * \underset{\sim}{\mathbb{R}}$ forces that $\underset{\sim}{Q}_{(\kappa^*,\lambda^*)}$ is (κ^*+1)-strategically closed (this is by Fact 1.5(5) applied to any $\beta \in (\kappa^*+2, (\kappa^*)^+)$, and as the cardinality of $\mathbb{P}^- * \underset{\sim}{\mathbb{R}}$ is $\leq \kappa^*$, we have that $\mathcal{I}_\alpha$ is dense in $[(\mathbb{P}^- * \underset{\sim}{\mathbb{R}}) * \underset{\sim}{Q}_{(\kappa^*,\lambda^*)}]^{\otimes}$. By the definition of $\underset{\sim}{H}^*$, there is $(\emptyset, \emptyset, \underset{\sim}{h}) \in \mathcal{I}_\alpha \cap N$ such that $(\emptyset, \emptyset, \underset{\sim}{h}) \leq q$. Let $\underset{\sim}{\tau}'$ exemplify this, so $\underset{\sim}{\tau}' \in N$.

Hence q forces $\underset{\sim}{\tau}(\alpha)$ to be in the set of all $\underset{\sim}{\tau}'_{\underset{\sim}{G}} \in N[\underset{\sim}{G}]$, where $\underset{\sim}{\tau}'$ is a $\mathbb{P}^- * \underset{\sim}{\mathbb{R}}$-name of an ordinal $< \kappa^*$. The cardinality of the set

$$T = \{\underset{\sim}{\tau}' \in N : \underset{\sim}{\tau}' \text{ is a } \mathbb{P} * \underset{\sim}{\mathbb{R}}\text{-name for an ordinal } < \kappa^*\}$$

is forced to be $\leq |\mathcal{P}(\kappa^*) \cap N|$, which is $< \mu_0$. Since $\alpha \in N$ was arbitrary, q forces the range of $\underset{\sim}{\tau} \restriction (N \cap \delta)$ to be a set of size $< \mu_0 < \kappa^*$, hence $\underset{\sim}{\tau}$ will be constant on a stationary subset of $N \cap \delta$ (as $N \cap \delta$ is stationary). More elaborately, one checks the stationarity in $V^{\mathbb{P}^- * \underset{\sim}{\mathbb{R}} * \underset{\sim}{Q}_{(\kappa^*,\lambda^*)}}$. Forcing with $\underset{\sim}{Q}_{(\kappa^*,\lambda^*)}$ adds no subsets to κ^*, hence is irrelevant. Forcing with $\mathbb{P}^- * \underset{\sim}{\mathbb{R}}$ preserves the uncountability of cofinalities, so as $N \cap \delta$ contains $\{\gamma \in e : \mathrm{cf}(\gamma) = \aleph_0\}$ for some club e of δ, clearly $N \cap \delta$ is stationary in $V^{\mathbb{P}^- * \underset{\sim}{\mathbb{R}} * \underset{\sim}{Q}_{(\kappa^*,\lambda^*)}}$, and κ^* is still a cardinal, hence we are done. $\bigstar_{1.23}$

Part (2) of Theorem 0.2 Same proof.

Part (3) of Theorem 0.2 In short, follow the forcing from (1) by a Levy collapse. We are making use of the following

Claim 1.34. Suppose λ weakly reflects at κ and P is a κ-cc forcing.
<u>Then</u> λ weakly reflects at κ in V^P.

Proof of the Claim. Suppose that

$$p \Vdash_P \text{“}\underset{\sim}{f} : \lambda \to \kappa \ \&\ \underset{\sim}{E} \subseteq \lambda \text{ is a club ”}.$$

We define $f' : \lambda \to \kappa$ by letting $f(\alpha) \stackrel{\text{def}}{=} \sup\{\gamma < \kappa : \neg(p \Vdash \text{“}\underset{\sim}{f}(\alpha) \neq \gamma\text{”})\}$. As P is κ-cc, the range of f' is indeed contained in κ. Let $\delta \in S^\lambda_\kappa$ be such that $f' \restriction S$ is constant on a stationary set $S \subseteq \delta$ (the existence of such δ follows as $WR(\lambda, \kappa)$ holds). Hence $p \Vdash$ "$\underset{\sim}{f} \restriction S$ is bounded", so $\underset{\sim}{f}$ does not witness $SNR(\lambda, \kappa)$ in V^P, as S remains stationary. $\bigstar_{1.34}$

So, for example to get $\kappa^* = \aleph_n$, we could in $V^{\mathbb{P}}$ from (1) first make GCH hold below μ_0 by collapsing various cardinals below μ_0, and then collapse κ^* to $\aleph_n$.

$\bigstar_{0.2}$

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