On $\mathcal{P}_\kappa\lambda$-combinatorics using a third cardinal

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Dedicated to the memory of Donna Carr

Abstract

Let κ and λ be infinite cardinals. Then $\mathcal{P}_\kappa\lambda$ denotes the set $\{A \subseteq \lambda : |A| < \kappa\}$. We consider some aspects of $\mathcal{P}_\kappa\lambda$-combinatorics, where a third cardinal is introduced into the discussion. Hence the context becomes that of two regular cardinals $\aleph_0 \le \mu < \kappa$, where $\kappa \le \lambda$. Classical results of $\mathcal{P}_\kappa\lambda$ combinatorics are discussed in the described three cardinal setting. It is shown that a diamond-like principle is consistent with $2^\lambda < 2^{\lambda^{<\mu}}$. This implies that the club filter on $\mathcal{P}_\kappa\lambda$ can have its saturation number equal to $2^{\lambda^{<\mu}}$ and greater than 2^λ, strengthening a previous result of H.D. Donder and P. Matet.

Further, we consider a three-cardinal strengthening of the notion of normality for a filter, here called μ-normality (this notion is a weakening of the known notion of strong normality). It is shown that if for all $\sigma < \kappa$ we have $\sigma^{<\mu} < \kappa$, then there is a minimal such filter on $\mathcal{P}_\kappa\lambda$. [1]

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0 Introduction

Let κ and λ be infinite cardinals. Then $\mathcal{P}_\kappa\lambda$ denotes the set $\{A \subseteq \lambda : |A| < \kappa\}$. Combinatorics of $\mathcal{P}_\kappa\lambda$ has been well studied (see the references mentioned here for a sample of the results), and it exhibits many properties similar to the combinatorics of ordinals. Apart from the combinatorial interest, the study of $\mathcal{P}_\kappa\lambda$ is interesting from the point of view of its connections with a central notion of modern set theory, that of embeddings between universes of set theory.

We study the combinatorics of $\mathcal{P}_\kappa\lambda$ by introducing a third regular cardinal, $\mu \in [\aleph_0, \kappa)$. Although this idea is very natural, and in fact, T. Menas [Me 2] studied a context of three cardinals regarding some questions about measures, $\mathcal{P}_\kappa\lambda$-combinatorics using a third cardinal does not seem to have been studied before. A point to note about the three-cardinal approach is that it does not use stationary coding (see e.g [Zw]) or similar techniques for which $2^{\lambda^{<\kappa}} = 2^\lambda$ has to be assumed. As a consequence, we obtain results which are meaningful when this equality fails. An example of such a result is that we show that it is consistent to have $2^\lambda < 2^{\lambda^{<\mu}}$ and that there are $2^{\lambda^{<\mu}}$ almost disjoint stationary sets in $\mathcal{P}_\kappa\lambda$. H.-D. Donder and P.Matet showed in [DoMa] that it is consistent to have 2^λ such sets. The results we obtain are examples of how a result about $\mathcal{P}_\kappa\lambda$ can be obtained in a three cardinal setting, and there are many other results one could discuss. Particularly amenable to this method are various reflection and guessing principles.

An interesting case of cardinal arithmetic for us is $\sigma < \kappa \Longrightarrow \sigma^{<\mu} < \kappa$. It turns out that under these assumptions μ behaves with respect to $\mathcal{P}_\kappa\lambda$ very much like $\mu = 2$ would (i.e. in the classical $\mathcal{P}_\kappa\lambda$ context). In particular, there is a natural strengthening of normality, μ-normality, and there is a minimal (fine) μ-normal filter on $\mathcal{P}_\kappa\lambda$, which we call $\mathcal{C}^\mu_{\kappa,\lambda}$. We obtain some basic properties of this filter. We note that the notion of κ-normality in the $\mathcal{P}_\kappa\lambda$ context was introduced by D. Carr in [Ca], and a stronger version of normality than the one studied here was studied by D. Carr, J.-P. Levinski and D. Pelletier in [CaLePe].

The paper is organized as follows. After describing the notation and some basic facts, we go on to isolate the notion of μ-normality in §2. We introduce $\mathcal{C}^{\mu}_{\kappa,\lambda}$ and consider some basic properties. In §3 we give a $\diamondsuit$-like principle and study some of its properties, and then derive the above described result about the non-saturation of the club filter on $\mathcal{P}_\kappa\lambda$.

1 Notation and basics

In this paper, we use A. Kanamori's book [Ka], Chapter 25, as a standard reference. In particular, we use terms like "κ-closed" and "κ-directed", in place of the alternative terminology as in "$(<\kappa)$-closed".

Notation 1.1. (1) If $\theta \leq \lambda$, then

$$S^{\lambda}_{<\theta} \stackrel{\text{def}}{=} \{\alpha < \lambda :\, \aleph_0 \leq \mathrm{cf}(\alpha) < \theta\}$$

and

$$S^{\lambda}_{\theta} \stackrel{\text{def}}{=} \{\alpha < \lambda :\, \mathrm{cf}(\alpha) = \theta\}.$$

(2) If $\aleph_0 \leq \mathrm{cf}(\mu) = \mu < \mathrm{cf}(\kappa) = \kappa$, we let

$$S^{[\lambda]^{<\kappa}}_{\mu} \stackrel{\text{def}}{=} \{x \in \mathcal{P}_\kappa\lambda :\, \sup(x) \in S^{\lambda}_{\mu} \setminus x\}.$$

Convention 1.2. Throughout the paper, unless said otherwise, we assume

$$\aleph_0 \leq \mathrm{cf}(\mu) = \mu < \mathrm{cf}(\kappa) = \kappa \leq \lambda.$$

We state a well known fact, whose proof we include for lack of a specific reference.

Fact 1.3. (1) Suppose that C is a club of λ. Then

$$C^* \stackrel{\text{def}}{=} \{x \in \mathcal{P}_\kappa\lambda :\, \sup(x) \in C \setminus x\}$$

is a club of $\mathcal{P}_\kappa\lambda$.

(In fact, note that the set $\{x \in \mathcal{P}_\kappa\lambda :\, \sup(x) = \lambda\}$ is cobounded when $\mathrm{cf}(\lambda) < \kappa$).

(2) Suppose that C^* is a club of $\mathcal{P}_\kappa\lambda$. Then

$$C \stackrel{\text{def}}{=} \{\gamma < \lambda : (\exists x \in C^*)[\gamma = \sup(x) \notin x]\}$$

is unbounded in λ.

(3) If $\aleph_0 \le \text{cf}(\mu) = \mu < \text{cf}(\kappa) = \kappa$, then $S_\mu^{[\lambda]^{<\kappa}}$ is stationary in $\mathcal{P}_\kappa\lambda$.

Proof of the Fact.

(1) Suppose $x = \bigcup_{\alpha<\alpha^*} x_\alpha$ for some $\alpha^* < \kappa$ and a $\subseteq$-increasing sequence

$$\langle x_\alpha : \alpha < \alpha^* \rangle$$

from C^* is given. Let $\gamma_\alpha \stackrel{\text{def}}{=} \sup(x_\alpha)$. Hence $\sup(x) = \sup\{\gamma_\alpha : \alpha < \alpha^*\}$, which is an element of C.

Suppose we are given $x \in \mathcal{P}_\kappa\lambda$. Let $\gamma \stackrel{\text{def}}{=} \sup(x)$ and $\delta > \gamma$ be a limit ordinal $< \lambda$ with $\text{cf}(\delta) < \kappa$. Let $\bar{a}$ be any cofinal sequence in δ, whose range has cardinality $< \kappa$, and let $y \stackrel{\text{def}}{=} x \cup \text{Rang}(\bar{a})$.

(2) Suppose we are given $\alpha < \lambda$. Let $x \in C^*$ with $\alpha \in x$. By induction on $n < \omega$ we build a $\subseteq$-increasing sequence $\langle x_n : n < \omega \rangle$ in $\mathcal{P}_\kappa\lambda$ of elements of C^*, with $x_0 = x$ and $\sup(x_{n+1}) > \sup(x_n)$. Let $y \stackrel{\text{def}}{=} \cup_{n<\omega} x_n$ and $\gamma \stackrel{\text{def}}{=} \sup(y)$. Hence $\gamma > \alpha$ and $\gamma \in C$.

(3) Let C be a club of $\mathcal{P}_\kappa\lambda$. We shall define $x \in C \cap S_\mu^{[\lambda]^{<\kappa}}$. By induction on $i < \mu$ we define a continuously $\subseteq$-increasing sequence $\langle x_i : i < \mu \rangle$ of elements of C such that $\sup(x_i) < \sup(x_{i+1})$. Let $x \stackrel{\text{def}}{=} \cup_{i<\mu} x_i$.

$\bigstar_{1.3}$

2 The minimal μ-normal filter on $\mathcal{P}_\kappa\lambda$

Definition 2.1. A proper filter $\mathcal{D}$ on $\mathcal{P}_\kappa\lambda$ is said to be *μ-normal* iff for every family $\{C_a : a \in [\lambda]^{<\mu}\}$ of elements of $\mathcal{D}$, we have

$$\Delta_{a\in[\lambda]^{<\mu}} C_a \stackrel{\text{def}}{=} \{b \in \mathcal{P}_\kappa\lambda : (\forall a \in [b]^{<\mu})\,[b \in C_a]\} \in \mathcal{D}.$$

Remark 2.2. Hence normality is the same as 2-normality. In fact, it is the same as n-normality for any $n \leq \aleph_0$. D. Carr [Ca] introduced the notion of κ-normality in the context of $\kappa = \mu$. Our notion of normality is a weakening of strong normality in the sense of [CaLePe] and coincides with it when $\kappa = \mu^+$.

Observation 2.3. If $\mu_1 \leq \mu_2$, then μ_2-normality implies μ_1-normality.

Theorem 2.4. The following are equivalent.

(i) $\mathcal{P}_\kappa\lambda$ carries a μ-normal filter,

(ii) $\{C(f) : f : [\lambda]^{<\mu} \to \mathcal{P}_\kappa\lambda\}$ generates a (proper) filter, where for $f : [\lambda]^{<\mu} \to \mathcal{P}_\kappa\lambda$

$$C(f) \stackrel{\text{def}}{=} \{x \in \mathcal{P}_\kappa\lambda : f``([x]^{<\mu}) \subseteq \mathcal{P}(x)\},$$

(iii) for every $f : [\lambda]^{<\mu} \to \mathcal{P}_\kappa\lambda$ the set $C(f)$ is stationary,

(iv) for all $\sigma < \kappa$ we have $\sigma^{<\mu} < \kappa$.

Before proving the theorem, we introduce some notation and prove some Claims.

Definition 2.5. (1) $C \subseteq \mathcal{P}_\kappa\lambda$ is μ-*almost closed* iff it is closed under increasing unions of chains whose length i^* satisfies

$$\mu \leq \mathrm{cf}(i^*) \leq i^* < \kappa.$$

(2) $C \subseteq \mathcal{P}_\kappa\lambda$ is a μ-*aclub* iff it is μ-almost closed and unbounded.

(3) A family B of subsets of $\mathcal{P}_\kappa\lambda$ is said to be μ-*directed* iff for every $B' \subseteq B$ with $|B'| < \mu$, there is $b \in B$ such that $\bigcup B \subseteq b$.

(4)

$$\mathrm{a}\mathcal{C}^\mu_{\kappa,\lambda} \stackrel{\text{def}}{=} \{D \subseteq \mathcal{P}_\kappa\lambda : D \text{ contains a } \mu\text{-aclub of } \mathcal{P}_\kappa\lambda\}.$$

(5) $\mathcal{C}^{\mu}_{\kappa,\lambda}$ is the filter on $\mathcal{P}_{\kappa}\lambda$ generated by the set of all $C \subseteq \mathcal{P}_{\kappa}\lambda$ which are unbounded and closed under unions of μ-directed subsets of size $< \kappa$ (such sets are called *μ-dclubs* of $\mathcal{P}_{\kappa}\lambda$).

(6) $\mathrm{s}\mathcal{C}^{\mu}_{\kappa,\lambda}$ is the filter on $\mathcal{P}_{\kappa}\lambda$ generated by the set of all $C \subseteq \mathcal{P}_{\kappa}\lambda$ which are unbounded, closed under unions of μ-directed subsets of size $< \kappa$, *and* for every $\emptyset \neq B \subseteq C$ we have $\bigcap B \in C$. Such sets C are called *μ-sclubs* of $\mathcal{P}_{\kappa}\lambda$.

Remark 2.6. Note that the above filters are well defined, that is, in each case the generators indeed generate a filter.

Observation 2.7. $\mathrm{a}\mathcal{C}^{\mu}_{\kappa,\lambda} \supseteq \mathcal{C}^{\mu}_{\kappa,\lambda} \supseteq \mathrm{s}\mathcal{C}^{\mu}_{\kappa,\lambda}$.

Claim 2.8. Suppose that $\sigma < \kappa \Longrightarrow \sigma^{<\mu} < \kappa$.

Then C is a μ-sclub of $\mathcal{P}_{\kappa}\lambda$ iff there is a function $f : [\lambda]^{<\mu} \to \mathcal{P}_{\kappa}\lambda$ such that $C = C(f)$ (see Theorem 2.4 for a definition of $C(f)$).

Proof of the Claim. From right to left: given $B \subseteq C$, let $x \stackrel{\mathrm{def}}{=} \bigcap B$. Hence if $a \in [x]^{<\mu}$, we have that $(\forall b \in B)(a \in [b]^{<\mu})$. Hence $f(a) \subseteq x$ and $x \in C$.

Suppose $B \subseteq C$ is μ-directed and $|B| < \kappa$, and let $x = \bigcup B$. Given $a \in [x]^{<\mu}$, there is $B_0 \subseteq B$ with $|B_0| < \mu$ and $a \subseteq \bigcup B_0$. Let $b \in B$ be such that $\bigcup B_0 \subseteq b$. Hence $a \in [b]^{<\mu}$ and $f(a) \subseteq b \subseteq x$.

To show that $C(f)$ is unbounded, take $x_0 \in \mathcal{P}_{\kappa}\lambda$ and by induction on $\alpha \leq \mu$ build an increasing sequence $\langle x_\alpha : \alpha \leq \mu \rangle$ such that $f``([x_\alpha]^{<\mu})$ is contained in $[x_{\alpha+1}]^{<\kappa}$, and $x_\delta = \cup_{\alpha<\delta} x_\alpha$ for δ a limit. The choice is made possible by the cardinal arithmetic assumptions. Now notice that $[x_\mu]^{<\mu} = \cup_{\alpha<\mu}[x_\alpha]^{<\mu}$.

From left to right: given $a \in [\lambda]^{<\mu}$ we define $f(a) \stackrel{\mathrm{def}}{=} \bigcap\{x \in C : x \supseteq a\}$, which is well defined as C is unbounded. As C is a μ-sclub, it is closed under arbitrary nonempty intersections, so for every $a \in [\lambda]^{<\mu}$ we have that $f(a) \in C$. It is easily seen that all $x \in C$ are closed under f. Suppose that $x \in \mathcal{P}_{\kappa}\lambda$ is closed under f. We shall show that $x \in C$.

We have that

$$x = \cup_{a\in[x]^{<\mu}} a = \cup_{a\in[x]^{<\mu}} f(a).$$

Then $F \stackrel{\text{def}}{=} \{f(a) : a \in [x]^{<\mu}\}$ is a subset of C of size $< \kappa$, and $x = \bigcup F$. If $B \subseteq F$ is of size $< \mu$, say $B = \{f(a_i) : i < i(*) < \mu\}$, let $a = \cup_{i<i(*)} a_i$. Hence $a \in [x]^{<\mu}$, and now notice that $f(a) \in F$ and $i < i(*) \implies f(a_i) \subseteq f(a)$. Hence F is μ-directed, so $x \in C$. $\bigstar_{2.8}$

Claim 2.9. Suppose that $\sigma < \kappa \implies \sigma^{<\mu} < \kappa$. Then $\mathrm{s}\mathcal{C}^{\mu}_{\kappa\lambda} = \mathcal{C}^{\mu}_{\kappa\lambda}$.

Proof of the Claim. By Claim 2.8 and Observation 2.7, it suffices to show that for every μ-dclub $C \subseteq \mathcal{P}_\kappa\lambda$, there is $f : [\lambda]^{<\mu} \to \mathcal{P}_\kappa\lambda$ such that $C \supseteq C(f)$. Let such C be given.

We first define g whose domain is the set of all increasing sequences $\langle \alpha_i : i < i^* \rangle$ of elements of λ for which $i^* < \mu$. The value of $g(\langle \alpha_i : i < i^* \rangle)$ is defined by induction on i^*.

If $\underline{i^* = 0}$, let $g(\langle\rangle)$ be any element of C of cardinality $\geq \mu$. If $\underline{i^* = 1}$, let $g(\langle \alpha \rangle)$ be any element of C with $g(\langle\rangle) \cup \{\alpha\} \subseteq g(\langle \alpha \rangle)$.

For $\underline{i^* > 1}$, consider

$$F \stackrel{\text{def}}{=} \{g(\langle \beta_j : j < j^* \rangle) : j^* < i^* \ \& \ \{\beta_j : j < j^*\} \subseteq \{\alpha_i : i < i^*\}\}.$$

By the cardinal arithmetic assumed, F is a set of cardinality $< \kappa$. So, there is an element $g(\langle \alpha_i : i < i^* \rangle)$ of C such that $g(\langle \alpha_i : i < i^* \rangle) \supseteq \bigcup F$.

Note that $g(\langle \alpha_i : i < i^* \rangle) \supseteq \{\alpha_i : i < i^*\}$, if $\{\alpha_i : i < i^*\} \in \mathrm{Dom}(g)$.

Having defined g, we let $f(A) \stackrel{\text{def}}{=} g(\langle \alpha_i : i < i^* \rangle)$. Here $\{\alpha_i : i < i^*\}$ is the increasing enumeration of A, for $A \in [\lambda]^{<\mu}$. Suppose now that $x \in \mathcal{P}_\kappa\lambda$ is closed under f. We shall show that $x \in C$.

First observe that $|x| \geq \mu$, as $f(\emptyset) \subseteq x$. Let

$$B \stackrel{\text{def}}{=} \{g(\langle \alpha_i : i < i^* \rangle) : \{\alpha_i : i < i^*\} \subseteq x \text{ is increasing} \ \& \ i^* < \mu\}.$$

Hence we have that $x = \bigcup B$ and $B \subseteq C$ is of size $< \kappa$. It suffices to show that B is μ-directed.

Let $j^* < \mu$ and $\{g(\langle \alpha^j_i : i < i^*_j \rangle) : j < j^*\}$ be given. As $|x| \geq \mu$, it is possible to find an increasing sequence $\bar{\beta} = \langle \beta_\zeta : \zeta < \zeta^* \rangle$ of elements of x such that $\zeta^* > \sup\{i^*_j : j < j^*\}$, and $\cup_{j<j^*}\{\alpha^j_i : i < i^*_j\} \subseteq \{\beta_\zeta : \zeta < \zeta^*\}$.

Then $g(\bar{\beta})$ is an element of A which contains all $g(\langle \alpha_i^j : i < i_j^* \rangle)$ as subsets. $\bigstar_{2.9}$

Claim 2.10. Every μ-normal fine filter on $\mathcal{P}_\kappa\lambda$ contains all sets $C(f)$ for $f : [\lambda]^{<\mu} \to \mathcal{P}_\kappa\lambda$.

Proof of the Claim. Let $\mathcal{D}$ be a fine μ-normal filter on $\mathcal{P}_\kappa\lambda$. Suppose we are given $f : [\lambda]^{<\mu} \to \mathcal{P}_\kappa\lambda$. For $a \in [\lambda]^{<\mu}$, we let $D_a = \{x \in \mathcal{P}_\kappa\lambda : f(a) \subseteq x\}$. Hence $D_a \in \mathcal{D}$. We claim that $C(f) \supseteq \Delta_{a\in[\lambda]^{<\mu}} D_a$, so $C(f) \in \mathcal{D}$.

If $x \in \Delta_{a\in[\lambda]^{<\mu}} D_a$, then $(\forall a \in [x]^{<\mu})[x \in D_a]$, so x is closed under f. $\bigstar_{2.10}$

Claim 2.11. Suppose that $\sigma < \kappa \Longrightarrow \sigma^{<\mu} < \kappa$. Then $\mathcal{C}^\mu_{\kappa,\lambda}$ is a fine μ-normal (κ-closed) filter on $\mathcal{P}_\kappa\lambda$. (Hence, by Claims 2.9 and 2.10, it is a minimal such filter).

Proof of the Claim. We use Claims 2.8 and 2.9. Given $a \in \mathcal{P}_\kappa\lambda$, defining f on $[\lambda]^{<\mu}$ to be identically a shows that $\{x \in \mathcal{P}_\kappa\lambda : a \subseteq x\}$ is in $\mathcal{C}^\mu_{\kappa,\lambda}$.

To show μ-normality, consider an arbitrary family $\{f_a : a \in [\lambda]^{<\mu}\}$ of functions $f_a : [\lambda]^{<\mu} \to \mathcal{P}_\kappa\lambda$. We show that $\Delta_{a\in[\lambda]^{<\mu}} C(f_a) \in \mathcal{C}^\mu_{\kappa,\lambda}$. This is done by observing that the set in question contains (in fact is equal to) $C(f)$ for $f : [\lambda]^{<\mu} \to \mathcal{P}_\kappa\lambda$ defined by letting $f(c) = \bigcup_{d,e\subseteq c} f_d(e)$. One has first to note that f is well defined. To this end, if $c \in [\lambda]^{<\mu}$, we use our cardinal arithmetic assumptions to observe that $|\mathcal{P}(c)| < \kappa$. Hence $f(c)$ is the union of $< \kappa$ sets each of size $< \kappa$, and so has size $< \kappa$, by the regularity of κ. Now suppose that $x \in C(f)$. Hence for all $a \in [x]^{<\mu}$ we have $f(a) \subseteq x$. In particular, for all $a, b \in [x]^{<\mu}$ we have that $f(a \cup b) \subseteq x$. By the definition of $f(a \cup b)$, for all $a, b \in [x]^{<\mu}$ we have $f_a(b) \subseteq x$, which gives us that for any $a \in [x]^{<\mu}$ we have $x \in C(f_a)$. So, $x \in \Delta_{a\in[\lambda]^{<\mu}} C(f_a)$. $\bigstar_{2.11}$

Proof of the Theorem. (i)$\Longrightarrow$(ii). Let $\mathcal{C}$ be a (proper) μ-normal filter on $\mathcal{P}_\kappa\lambda$. By Claim 2.10, $\mathcal{C}$ contains all sets $C(f)$ for $f : [\lambda]^{<\mu} \to \mathcal{P}_\kappa\lambda$. Hence the filter generated by these sets is proper.

(ii)$\Longrightarrow$(iii). Let $\mathcal{D}$ be the filter generated by all $C(f)$, and assume that $\mathcal{D}$ is proper. Suppose that $C(f)$ is non-stationary, and let C be a club of $\mathcal{P}_\kappa\lambda$ which misses $C(f)$. By a theorem of Menas [Me 1], there is $g : [\lambda]^2 \to \mathcal{P}_\kappa\lambda$ with $C(g) \subseteq C$. Extending g to $[\lambda]^{<\mu}$ by letting $g(x) = \emptyset$ unless $|x| = 2$, we obtain a function from $[\lambda]^{<\mu}$ into $\mathcal{P}_\kappa\lambda$ such that $C(g) \cap C(f) = \emptyset$, in contradiction with $\mathcal{D}$ being proper.
(iii)$\Longrightarrow$(iv). Suppose that (iii) is true but (iv) fails, and let $\sigma < \kappa$ be the minimal such that $\sigma^{<\mu} \geq \kappa$.

Let $\{x_\zeta : \zeta < \kappa\}$ be distinct elements of $[\sigma]^{<\mu}$. Define f on $[\lambda]^{<\mu}$ by letting

$$f(x) \stackrel{\text{def}}{=} \{\zeta : x_\zeta \subseteq x\} \text{ if } |x| < \sigma,$$

and let $f(x) \stackrel{\text{def}}{=} \emptyset$ otherwise. We have that f is into $\mathcal{P}_\kappa\lambda$, as if for some $x \in [\lambda]^{<\mu}$ we were to have $|f(x)| = \kappa$, this would contradict the minimality of σ. Suppose now that $C(f)$ is stationary. We have that $\{x \in \mathcal{P}_\kappa\lambda : \sigma \subseteq x\}$ is a club. Hence $C \cap C(f)$ is stationary. But if x were to be in $C \cap C(f)$, then, as $\sigma \subseteq x$ and $x \in C(f)$, we would have that for all $\zeta < \kappa$, also $\zeta \in x$, contradicting that $|x| < \kappa$.
(iv)$\Longrightarrow$(i). Claim 2.11 shows that $\mathcal{C}^\mu_{\kappa,\lambda}$ is such a filter.

Theorem 2.12. Suppose that $\mu^{<\mu} = \mu$ and $\kappa = \mu^{+n}$ for some natural number $n \geq 1$. Then every μ-aclub of $\mathcal{P}_\kappa\lambda$ is μ-directed closed.

Proof. Suppose we are given C a μ-aclub of $\mathcal{P}_\kappa\lambda$ and $B \subseteq C$ which is μ-directed and $|B| < \kappa$. We show that $\bigcup B \in C$. Let $B = \{x_i : i < i^*\}$. The proof is by induction on $\sigma \stackrel{\text{def}}{=} |i^*|$.

Case 1. $\sigma < \mu$.

As B is μ-directed, there is $x \in B$ such that $\bigcup B \subseteq x$. Hence $\bigcup B = x \in C$.

Case 2. $\sigma = \mu$.

By reenumerating, without loss of generality, $i^* = \mu$. By induction on $i < \mu$ we define $B_i \subseteq C$ such that

(i) $x_i \in B_i$,

(ii) B_i is μ-directed, moreover, B_i has a unique largest element,

(iii) $|B_i| < \mu$,

(iv) $i < j < \mu \Longrightarrow B_i \subseteq B_j$.

Let $B_0 \stackrel{\text{def}}{=} \{x_0\}$. If B_j for $j < i$ is defined, let $B'_i \stackrel{\text{def}}{=} \bigcup_{j<i} B_j \cup \{x_i\}$. Hence $|B'_i| < \mu$. Hence there is $b_i \in B$ such that $\bigcup B'_i \subseteq b_i$. Let $B_i \stackrel{\text{def}}{=} B'_i \cup \{b_i\}$.

By the induction hypothesis, $\bigcup B_i \in C$ for all $i < \mu$, and as we have $\bigcup B = \bigcup_{i<\mu} \bigcup B_i$, we finish by the fact that C is μ-almost closed.

Case 3. $\sigma = \mu^{+m}$ for some $1 \leq m < n$.

Again, without loss of generality, $\sigma = i^*$. By induction on $i < \sigma$ we define $B_i \subseteq C$ such that

(i) $x_i \in B_i$,

(ii) B_i is μ-directed,

(iii) $|B_i| \leq \mu^{+m-1}$,

(iv) $i < j < \mu^{+m} \Longrightarrow B_i \subseteq B_j$.

The conclusion follows by the induction hypothesis as in the previous case. To do the induction, let $B_0 \stackrel{\text{def}}{=} \{x_0\}$, and suppose B_j for $j < i$ has been given. Let $B'_i \stackrel{\text{def}}{=} \bigcup_{j<i} B_j \cup \{x_i\}$. By induction on $\zeta < \mu$ we define D_ζ as follows:

(a) $D_0 = B'_i$,

(b) $D_{\zeta+1} \supseteq D_\zeta$, and for every sequence $\bar{b}$ of length $< \mu$ from D_ζ, there is an element $x_{\bar{b}}$ of $D_{\zeta+1}$ such that $\cup\mathrm{Rang}(\bar{b}) \subseteq x_{\bar{b}}$,

(c) $D_\zeta = \bigcup_{\xi<\zeta} D_\xi$ for ζ a limit,

(d) $|D_\zeta| \leq \mu^{+m-1}$,

(e) $D_\zeta \subseteq C$.

There are no problems in the induction, thanks to the cardinal arithmetic assumed. Namely, to define $D_{\zeta+1}$, choose for each $\bar{b} \in {}^{<\mu}D_\zeta$ an element $x_{\bar{b}}$ of C as required, so choosing $\leq (\mu^{+m-1})^{<\mu} = \mu^{+m-1}$ elements of C. Then let $D_{\zeta+1} = D_\zeta \cup \{x_{\bar{b}} : \bar{b} \in {}^{<\mu}D_\zeta\}$.

At the end of the induction used to choose D_ζ for $\zeta < \mu$, we let B_i be $\bigcup_{\zeta<\mu} D_\zeta$. This set is μ-directed because every sequence of $< \mu$ elements of B_i is in fact a sequence of $< \mu$ elements of some D_ζ, say $\bar{b}$. Hence $\cup\mathrm{Rang}(\bar{b}) \subseteq x_{\bar{b}}$, where $x_{\bar{b}}$ was chosen when $D_{\zeta+1}$ was being defined. $\bigstar_{2.12}$

Corollary 2.13. Suppose that $\mu^{<\mu} = \mu$ and $\kappa = \mu^{+n}$ for some $n \geq 1$. Then $\mathrm{s}\mathcal{C}^\mu_{\kappa,\lambda} = \mathcal{C}^\mu_{\kappa,\lambda} = \mathrm{a}\mathcal{C}^\mu_{\kappa,\lambda}$.

Remark 2.14. It is an open question if Theorem 2.12 or Corollary 2.13 is true if just $\sigma < \kappa \Longrightarrow \sigma^{<\mu} < \kappa$ is assumed.

We do however remark that in general, the filter $\mathcal{C}^\mu_{\kappa,\lambda}$ may be a proper superset of the club filter on $\mathcal{P}_\kappa\lambda$. This can be seen by considering

$$C = \{x \in \mathcal{P}_\kappa\kappa : x \text{ is a limit cardinal of cofinality } \geq \mu\}$$

for κ a weakly inaccessible cardinal and $\aleph_0 < \mu < \kappa$. To see that C does not contain a club, suppose for a contradiction that D were such, and take any increasing ω sequence from it.

Lacking any knowledge on the equivalence between $\mathcal{C}^\mu_{\kappa,\lambda}$ and $\mathrm{a}\mathcal{C}^\mu_{\kappa,\lambda}$ except that given by Corollary 2.13, it is worth noting the following claims.

Claim 2.15. $\mathrm{a}\mathcal{C}^\mu_{\kappa,\lambda}$ is a (proper) κ-complete filter on $\mathcal{P}_\kappa\lambda$. In fact, the intersection of $< \kappa$ μ-aclubs of $\mathcal{P}_\kappa\lambda$ is a μ-aclub.

Proof of the Claim. Given $\langle C_i : i < i^* < \kappa\rangle$ we shall prove that $C \stackrel{\mathrm{def}}{=} \bigcap_{i<i^*} C_i$ is a μ-aclub of $\mathcal{P}_\kappa\lambda$, provided each C_i is. It is obvious that C is μ-almost closed. We prove it is also unbounded. The proof is by induction on i^*. Suppose we are given $x \in \mathcal{P}_\kappa\lambda$.

If $\underline{i^* < \mu}$, we fix a bijection $f : i^* \times \mu \to \mu$ such that for all $i < i^*$ and $\beta < \gamma$ we have $f(i,\beta) < f(i,\gamma)$. We build a strictly increasing chain $\langle x_\alpha : \alpha < \mu\rangle$ in $\mathcal{P}_\kappa\lambda$ by induction on α, and so that

(a) $x_0 \supseteq x$,

(b) If $f(i,\beta) = \alpha$, then $x_\alpha \in C_i$.

Let $x_\mu \stackrel{\text{def}}{=} \cup_{\alpha<\mu} x_\alpha$. Hence $x_\mu \supseteq x$ and $x_\mu \in \mathcal{P}_\kappa\lambda$, and for every $i < i^*$ we have that $x_\mu = \cup_{\beta<\mu} x_{f(i,\beta)}$, so $x_\mu \in C_i$.

Now we can assume that $i < j < i^* \Longrightarrow C_i \supseteq C_j$.

If $\underline{i^* \geq \mu \text{ is not regular}}$, the conclusion follows by the induction hypothesis. If $\underline{i^* \geq \mu \text{ is regular}}$, we build by induction on $i < i^*$ an increasing sequence $\langle x_i : i < i^* \rangle$ such that $x_0 \supseteq x$ and $x_i \in C_i$, and take $y \stackrel{\text{def}}{=} \cup_{i<i^*} x_i$. $\bigstar_{2.15}$

Claim 2.16. If $\rho \leq \mu$ and $\sigma < \kappa \Longrightarrow \sigma^{<\rho} < \kappa$, <u>then</u> a$\mathcal{C}^\mu_{\kappa,\lambda}$ is ρ-normal.

Proof of the Claim. Suppose we are given $\{C_a : a \in [\lambda]^{<\rho}\}$, a family of elements of $\mathcal{C}^\mu_{\kappa,\lambda}$. Let $C \stackrel{\text{def}}{=} \Delta_{a\in[\lambda]^{<\rho}} C_a$.

Suppose that $\langle x_i : i < i^* < \kappa \rangle$ is an increasing chain from C and $\mathrm{cf}(i^*) \geq \mu$, and let $x \stackrel{\text{def}}{=} \bigcup_{i<i^*} x_i$. If $a \in [x]^{<\rho}$, then there is $i < i^*$ such that $a \in [x_j]^{<\rho}$ for all $j \geq i$. Hence, for all $j \geq i$ we have $x_j \in C_a$, so $x \stackrel{\text{def}}{=} \bigcup_{j\geq i} x_j \in C_a$. Hence $x \in C$.

Suppose that $x_0 \in \mathcal{P}_\kappa\lambda$ is given. By induction on $\alpha < \mu$ we build an increasing sequence $\langle x_\alpha : \alpha < \mu \rangle$ of elements of $\mathcal{P}_\kappa\lambda$, such that

$$x_{\alpha+1} \in \bigcap\{C_a : a \in [x_\alpha]^{<\rho}\}.$$

The induction goes through by the completeness of the filter and the cardinal arithmetic assumption. Let $x \stackrel{\text{def}}{=} \cup_{\alpha<\mu} x_\alpha$. It is easy to check that $x \in C$. $\bigstar_{2.16}$

3 A three-cardinal version of diamond

H.D. Donder and P. Matet proved in [DoMa], Proposition 9.6. that it is consistent that the club filter on $\mathcal{P}_\kappa\lambda$ is not 2^λ-saturated. In this section we formulate a principle which can be used to see that even a larger non-saturation number is possible, and we prove some consistency results about this principle.

Notation 3.1. For a stationary set $S \subseteq \mathcal{P}_\kappa\lambda$, we let $\diamondsuit(S)$ mean that there is a sequence $\langle A_x : x \in S\rangle$ such that $A_x \subseteq x$ and

$$(\forall A \subseteq \lambda)\,[\{x \in S : A \cap x = A_x\} \text{ is stationary}].$$

(Note: This principle was introduced by T. Jech in [Jec], where it was called $\spadesuit(\kappa, \lambda, S)$).

(d) $\diamondsuit^\mu(S)$ means that there is a sequence $\langle A_x : x \in S\rangle$ such that $A_x \subseteq [x]^{<\mu}$ and

$$(\forall A \subseteq [\lambda]^{<\mu})\,[\{x \in S : A \cap [x]^{<\mu} = A_x\} \text{ is stationary}].$$

Claim 3.2. Assume $\diamondsuit^\mu(\mathcal{P}_\kappa\lambda)$. <u>Then</u> the club filter on $\mathcal{P}_\kappa\lambda$ is not $2^{\lambda^{<\mu}}$-saturated.

Proof of the Claim. Let $\langle A_x : x \in \mathcal{P}_\kappa\lambda\rangle$ be a $\diamondsuit^\mu(\mathcal{P}_\kappa\lambda)$-sequence. For $B \subseteq [\lambda]^{<\mu}$, let $\mathcal{G}(B) \stackrel{\text{def}}{=} \{x \in \mathcal{P}_\kappa\lambda : B \cap [x]^{<\mu} = A_x\}$. If $B_1 \neq B_2$ are subsets of $[\lambda]^{<\mu}$, let $X \in B_1 \Delta B_2$, and let C be a club of $\mathcal{P}_\kappa\lambda$ such that $x \in C \implies X \subseteq x$. We now claim that $\mathcal{G}(B_1) \cap \mathcal{G}(B_2) \cap C = \emptyset$, so $\mathcal{G}(B_1) \cap \mathcal{G}(B_2)$ is non-stationary. To see this, suppose that $X \in B_1 \setminus B_2$, the remaining case being handled analogously, and let $x \in \mathcal{G}(B_1) \cap \mathcal{G}(B_2) \cap C$. Then $A_x = B_1 \cap [x]^{<\mu}$, so $X \in A_x$ (note that $|X| < \mu$ as $X \in B_1$ and $B_1 \subseteq [\lambda]^{<\mu}$). Hence, as $A_x = B_2 \cap [x]^{<\mu}$, we obtain that $X \in B_2$, a contradiction. $\bigstar_{3.2}$

In addition to the conventions from earlier in the paper, for most of this section we assume that λ is regular.

The following easily follows by a technique used in T. Jech's [Jec], where the same is proved but with $S^* = \mathcal{P}_\kappa\lambda$.

Lemma 3.3. For κ a successor cardinal $\leq \lambda$ and S^* a stationary subset of $\mathcal{P}_\kappa\lambda$, the principle $\diamondsuit(S^*)$ holds in L .

Proof of the Lemma. In [Jec], use the proof of 3.8, defining S_P for $P \in S^*$ only. In (3.24) change the requirement (5) to apply only to $P \in C \cap S^*$. Note that Lemma 3.9 proves that there is a club K^* of $P \in \mathcal{P}_\kappa\lambda$ such that $M_P \cap \lambda = P \in C$. After the proof of Lemma 3.9, choose $P \in S^* \cap K^*$, and continue the proof from [Jec]. $\bigstar_{3.3}$

Lemma 3.4. Assume that λ is regular, while $\diamondsuit(S^\lambda_\kappa)$ holds and

$$(\forall \sigma < \lambda)(\sigma^{<\kappa} < \lambda).$$

Then there is a sequence $\langle B_\alpha : \alpha < \lambda \rangle$ such that $B_\alpha \subseteq [\alpha]^{<\kappa}$, and for every $D \subseteq [\lambda]^{<\kappa}$, the set of all α such that $D \cap [\alpha]^{<\kappa} = B_\alpha$ is stationary.

Proof of the Lemma. We use the coding technique of K. Kunen [Ku]. Let $\langle A_\alpha : \alpha \in S^\lambda_\kappa \rangle$ exemplify $\diamondsuit(S^\lambda_\kappa)$. As $\lambda^{<\kappa} = \lambda$, there is a bijection F from λ onto $[\lambda]^{<\kappa}$.

Let

$$C \stackrel{\text{def}}{=} \{\alpha < \lambda : \kappa < \alpha \;\&\; [a \in [\alpha]^{<\kappa} \Longrightarrow F^{-1}(a) \in \alpha] \;\&\; F``(\alpha) \subseteq [\alpha]^{<\kappa}\}.$$

We shall show that C is an unbounded subset of λ which contains all its closure points of cofinality $\geq \kappa$.

Given $\alpha_0 < \lambda$, let $\alpha_1 \stackrel{\text{def}}{=} \mathrm{Max}\{\alpha_0, \kappa\}$ and now by induction on $i < \kappa$ build $\alpha_i < \lambda$ strictly increasing with i, such that $a \in [\alpha_i]^{<\kappa} \Longrightarrow F^{-1}(a) < \alpha_{i+1}$, and $F``(\alpha_i) \subseteq [\alpha_{i+1}]^{<\kappa}$. This is possible, as $(\forall \sigma < \lambda)(\sigma^{<\kappa} < \lambda)$ and λ is regular. For $i < \kappa$ a limit ordinal, we let $\alpha_i \stackrel{\text{def}}{=} \sup\{\alpha_j : j < i\}$. Now let $\beta \stackrel{\text{def}}{=} \sup\{\alpha_i : i < \kappa\}$. Hence $\beta \in S^\lambda_\kappa$. It is easily checked that $\beta \in C$, so C is unbounded. It is similarly seen that C contains its limit points of cofinality $\geq \kappa$.

Now we define $\langle B_\alpha : \alpha < \lambda \rangle$ by letting $B_\alpha \stackrel{\text{def}}{=} F``(A_\alpha)$ if $\alpha \in C$, and $\emptyset$ otherwise. Certainly $B_\alpha \subseteq [\alpha]^{<\kappa}$ for all α.

Given $B \subseteq [\lambda]^{<\kappa}$, let $A \stackrel{\text{def}}{=} \{F^{-1}(x) : x \in B\}$. Hence $A \subseteq \lambda$. Let $S \stackrel{\text{def}}{=} \{\alpha \in S^\lambda_\kappa : A \cap \alpha = A_\alpha\}$, so S is stationary. Hence $S \cap C$ is stationary.

[Why? Let $D \stackrel{\text{def}}{=} \mathrm{cl}(C)$, and notice that $D \cap S = C \cap S$.]

Given $\alpha \in S \cap C$ and $a \in B \cap [\alpha]^{<\kappa}$, we have that $F^{-1}(a) < \alpha$. Hence $F^{-1}(a) \in A \cap \alpha = A_\alpha$. Hence $a \in F``(A_\alpha) = B_\alpha$. On the other hand, if $a \in B_\alpha$, then $\gamma \stackrel{\text{def}}{=} F^{-1}(a) \in A_\alpha = A \cap \alpha$. Hence $a \in [\alpha]^{<\kappa} \cap B$. $\bigstar_{3.4}$

Theorem 3.5. Suppose $\aleph_0 < \mathrm{cf}(\mu) = \mu < \mathrm{cf}(\kappa) = \kappa < \mathrm{cf}(\lambda) = \lambda$, and

(a) $\diamondsuit(S_\mu^{[\lambda]^{<\kappa}})$,

(b) $\diamondsuit(S_\mu^\lambda)$,

(c) $(\forall \sigma < \lambda)(\sigma^{<\mu} < \lambda)$.

Then $\diamondsuit^\mu(S_\mu^{[\lambda]^{<\kappa}})$ holds.

Proof. By (b) and (c) and Lemma 3.4, there is $\langle B_\alpha : \alpha \in S_\mu^\lambda \rangle$ such that $B_\alpha \subseteq [\alpha]^{<\mu}$ for all $\alpha \in S_\mu^\lambda$, and for every $B \subseteq [\lambda]^{<\mu}$, the set

$$\{\alpha \in S_\mu^\lambda : B \cap [\alpha]^{<\mu} = B_\alpha\}$$

is stationary. Let $\{A_x : x \in S_\mu^{[\lambda]^{<\kappa}}\}$ exemplify $\diamondsuit(S_\mu^{[\lambda]^{<\kappa}})$. For $x \in S_\mu^{[\lambda]^{<\kappa}}$, we define $D_x \stackrel{\text{def}}{=} \bigcup_{\alpha \in A_x \bigcap S_\mu^\lambda} (B_\alpha \cap \mathcal{P}(x))$.

We claim that $\{D_x : x \in S_\mu^{[\lambda]^{<\kappa}}\}$ is as required. Certainly we have $D_x \subseteq [x]^{<\mu}$ for each x.

Suppose we are given $B \subseteq [\lambda]^{<\mu}$. Let $T \stackrel{\text{def}}{=} \{\alpha \in S_\mu^\lambda : B \cap [\alpha]^{<\mu} = B_\alpha\}$. Hence T is stationary. Let $C \stackrel{\text{def}}{=} \mathrm{cl}(T)$. By Fact 1.3(1), the set

$$C^* \stackrel{\text{def}}{=} \{x \in \mathcal{P}_\kappa\lambda : \sup(x) \in \mathrm{cl}(T) \setminus x\}$$

is a club of $\mathcal{P}_\kappa\lambda$.

Hence, there is a stationary set of $x \in S_\mu^{[\lambda]^{<\kappa}}$ such that $x \in C^*$ and $A_x = T \cap x$. Let x be an element of such a set and let $\gamma \stackrel{\text{def}}{=} \sup(x)$. Hence $\gamma = \sup(A_x)$. As $\mathrm{cf}(\gamma) = \mu$, we have $[\gamma]^{<\mu} = \cup_{\alpha \in A_x} [\alpha]^{<\mu}$. If $\alpha \in A_x$, we have that $\mathrm{cf}(\alpha) = \mu$ and $B \cap [\alpha]^{<\mu} = B_\alpha$, as $\alpha \in T$.

If $a \in B \cap \mathcal{P}(x)$, then for some $\alpha \in A_x$ we have $a \in B \cap [\alpha]^{<\mu} = B_\alpha$, so $a \in D_x$. If $a \in D_x$, we can similarly see that $a \in B \cap \mathcal{P}(x)$. Hence $B \cap \mathcal{P}(x) = D_x$. $\bigstar_{3.5}$

Corollary 3.6. Assume $V = L$. Let $\kappa > \mu > \aleph_0$ be regulars, and $\lambda > \kappa$ be regular such that λ is not the successor of a singular cardinal of cofinality $< \mu$. In addition assume that κ is a successor cardinal. Then $\diamondsuit(S_\mu^{[\lambda]^{<\kappa}})$.

The following Claim 3.7 is proved similarly to the above results, and it does not require λ to be regular.

Claim 3.7. Assume that $\lambda^{<\mu} = \lambda$ and for all $\sigma < \kappa$ we have $\sigma^{<\mu} < \kappa$, and let F be a bijection $F : [\lambda]^{<\mu} \to \lambda$.

Then

(1) $S \stackrel{\text{def}}{=} \{B \in \mathcal{P}_\kappa\lambda : F``([B]^{<\mu}) = B\}$ is stationary.

(2) $\diamondsuit(S) \Longrightarrow \diamondsuit^\mu(S)$.

Proof of the Claim. (1) Given C a club of $\mathcal{P}_\kappa\lambda$, we build an increasing continuous sequence $\langle B_\alpha : \alpha < \mu\rangle$ of elements of C, starting with any element of C and such that $F``([B_\alpha]^{<\mu}) \subseteq B_{\alpha+1}$ for α even, and $F^{-1}(B_\alpha) \subseteq B_{\alpha+1}$ for α odd. The union of the elements of the sequence is in $S \bigcap C$.
(2) Let $\langle A_x : x \in S\rangle$ exemplify $\diamondsuit(S)$. For $x \in S$ let $A_x^* \stackrel{\text{def}}{=} F^{-1}(A_x)$, and now verify that $\langle A_x^* : x \in S\rangle$ is a $\diamondsuit^\mu(S)$-sequence. $\bigstar_{3.7}$

The following is analogous to Theorem 3.7. of [Jec]:

Theorem 3.8. Suppose that V satisfies

$$\sigma < \kappa \Longrightarrow 2^{\sigma^{<\mu}} \leq \kappa.$$

Then:

(1) There is a κ-closed (i.e. closed under sequences of length $< \kappa$), κ^+-cc forcing notion Q which adds a $\diamondsuit^\mu(\mathcal{P}_\kappa\lambda)$-sequence.

(2) If in addition $2^\lambda = \lambda^{<\mu} = \lambda^+$, then Q preserves that $2^{\lambda^{<\mu}} > 2^\lambda$.

Proof. (1) Let Q be given by $Q \stackrel{\text{def}}{=}$

$$\{p:\ \mathrm{Dom}(p) \subseteq \mathcal{P}_\kappa\lambda \ \&\ |\mathrm{Dom}(p)| < \kappa \ \&\ (\forall P \in \mathrm{Dom}(p))[p(P) \subseteq [P]^{<\mu}]\},$$

ordered by inclusion. Q is easily seen to be κ-closed, by taking unions. To see that Q has κ^+-cc, take conditions $\{p_\alpha :\ \alpha < \kappa^+\}$ from Q, whose domains without loss of generality form a Δ-system with root D^*. Now, the number of functions from D^* into $[\bigcup D^*]^{<\mu}$ is at most κ, so without loss of generality all p_α agree on D^*.

To see that Q is as required, we take $p \in Q$ such that

$$p \Vdash \text{“}\underset{\sim}{T} \subseteq [\lambda]^{<\mu} \ \&\ \underset{\sim}{\sigma} \text{ is a club of } \mathcal{P}_\kappa\lambda\text{”},$$

and find $q \geq p$ (meaning q is stronger than p) and $P \in \mathcal{P}_\kappa\lambda$ such that

$$q \Vdash \text{“}P \in \underset{\sim}{\sigma} \ \&\ \underset{\sim}{T} \cap [P]^{<\mu} = q(P)\text{”}.$$

By induction on $i < \mu$ we define a continuous increasing sequence $\langle p_i :\ i < \mu\rangle$ of conditions in Q and a continuously $\subseteq$-increasing sequence $\langle P_i :\ i < \mu\rangle$ of elements of $\mathcal{P}_\kappa\lambda$, such that

(a) $p_0 \geq p$ and $p_i \Vdash$ “$P_i \in \underset{\sim}{\sigma}$”,

(b) p_{i+1} decides the value of $\underset{\sim}{T} \cap [P_i]^{<\mu}$ (possible as $\sigma < \kappa \Longrightarrow \sigma^{<\mu} < \kappa$).

At the end let $P \stackrel{\text{def}}{=} \bigcup_{i<\mu} P_i$ and $q' \stackrel{\text{def}}{=} \cup_{i<\mu} p_i$. As $[P]^{<\mu} = \bigcup_{i<\mu}[P_i]^{<\mu}$, we have that $q' \Vdash$ “$\underset{\sim}{T} \cap [P]^{<\mu} = B$” for some B. Let $q \stackrel{\text{def}}{=} q' \cup \{(P, B)\}$.

(2) Considering the closure of the forcing and the number of antichains, we see that $2^\lambda < 2^{\lambda^+}$ is preserved. $\bigstar_{3.8}$

Corollary 3.9. It is consistent that $2^{\lambda^{<\mu}} > 2^\lambda$ and the club filter on $\mathcal{P}_\kappa\lambda$ is not $2^{\lambda^{<\mu}}$-saturated.

We finally remark that the three-cardinal diamond can be seen to behave similarly to other diamond principles, in that it has consequences on cardinal arithmetic and implies some other combinatorial principles. Also, it has a formulation in terms of homogeneity principles à la Matet (see [Ma]). If $1 \leq \theta < \kappa$ and $\sigma < \kappa \Longrightarrow \sigma^{<\mu} < \kappa$, it can be shown that $\diamondsuit^\mu(\mathcal{P}_\kappa\lambda)$ is equivalent to a similar principle in which θ guesses for A_x are allowed at each $x \in \mathcal{P}_\kappa\lambda$.

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Sadržaj

Za beskonačne kardinalne brojeve κ i λ, označimo sa $\mathcal{P}_\kappa\lambda$ skup svih podskupova $A \subseteq \lambda$ koji imaju veličinu manju od κ. Kombinatorika skupova ove vrste je tema mnogog istraživanja, jer ne samo da ima osobine slične osobinama kombinatorike ordinalnih brojeva, nego je i povezana sa jednim od centralnih tema savremene teorije skupova, pojmom elementarnog uglavljenja jedne teorijsko-skupovne univerze, u drugu.

U ovom članku istražujemo kombinatoriku $\mathcal{P}_\kappa\lambda$ sa tačke gledišta nekog trećeg kardinalnog broja μ, koji zadovoljava $\aleph_0 \leq \mu < \kappa$. Pokazaćemo da paralele klasičnih rezultata $\mathcal{P}_\kappa\lambda$-kombinatorike ostaju tačne i u ovom generaliziranom kontekstu, i izmedju ostalog, dokazaćemo da je princip sličan $\diamondsuit$ nekontradiktoran sa predpostavkom $2^\lambda < 2^{\lambda^{<\mu}}$. Ovo povlači da filter zatvorenih neograničenih skupova u $\mathcal{P}_\kappa\lambda$ može imati karakteristiku zasićenosti (dakle najveću moguću veličinu familije podskupova $\mathcal{P}_\kappa\lambda$ od kojih se svaka dva medjusobno presjecaju u skupu čiji je komplement u datom filteru), veću nego 2^λ. Ovaj teorem je jači nego i povlači prethodno poznati rezultat Donder-a i Matet-a, da ova karakteristika može biti 2^λ.

Dalji rezultati ovoga članka uključuju generalizaciju pojma normalnosti za filtere na pojam ovdje nazvan μ-normalnost, o kojemu smo dokazali da pod predpostavkom $\sigma^{<\mu} < \kappa$ for all $\sigma < \kappa$, postoji minimalni μ-normalni filter na $\mathcal{P}_\kappa\lambda$.