On versions of ♣ on cardinals larger than $\aleph_1$

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[1]

Abstract

We give two results on guessing unbounded subsets of λ^+. The first is a positive result and applies to the situation of λ regular and at least equal to $\aleph_3$, while the second is a negative consistency result which applies to the situation of λ a singular strong limit with $2^\lambda > \lambda^+$. The first result shows that in ZFC there is a guessing of unbounded subsets of $S^{\lambda^+}_\lambda$. The second result is a consistency result (assuming a supercompact cardinal exists) showing that a natural guessing fails.

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A result of Shelah in [?] shows that if $2^\lambda = \lambda^+$ and λ is a strong limit singular, then the corresponding guessing holds.

Both results are also connected to an earlier result of Džamonja-Shelah in which they showed that a certain version of $\clubsuit$ holds at a successor of singular just in ZFC. The first result here shows that the result of Fact **??** can to a certain extent be extended to the successor of a regular. The negative result here gives limitations to the extent to which one can hope to extend the mentioned Džamonja-Shelah result. [2]

0 Introduction and background

The combinatorial principle $\clubsuit$ is a weakening of $\diamondsuit$ which, at $\aleph_1$, means that there is a sequence $\langle A_\delta : \delta \text{ limit } < \omega_1\rangle$ such that every A_δ is an unbounded subset of δ, and for every unbounded subset A of ω_1, there are stationarily many δ such that $A_\delta \subseteq A$. One can weaken this statement in various ways, for example requiring $|A_\delta \setminus A| < \aleph_0$ in place of $A_\delta \subseteq A$ above, and in general the weakened statements are not equivalent to $\clubsuit$ (as opposed to the situation with $\diamondsuit$, see Džamonja-Shelah [?] and Kunen's [?] respectively), and are not provable in ZFC. The question we consider here is if the corresponding situation holds at cardinals larger than $\aleph_1$. As an example of earlier results in this direction and connected to the statement of our main theorem, we mention a result of Erdös, Dushnik and Miller, and a much more recent one of Shelah. Erdös, Dushnik and Miller prove in [?], that if $\alpha < \lambda^+$, then α can be written as $\bigcup_{n<\omega} A_n$, where for each n we have $\mathrm{otp}(A_n) < \lambda^n$ (compare this with Note **??** below). Shelah proves in [?] that if $\lambda = \mathrm{cf}(\lambda) > \kappa > \aleph_0$, then there is a sequence

$$\langle (C_\delta = \{\alpha_{\delta,\varepsilon} : \varepsilon < \lambda\}, h_\delta) : \delta \in S^{\lambda^+}_\lambda \rangle$$

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with each $\{\alpha_{\delta,\varepsilon} : \varepsilon < \lambda\}$ a continuous increasing sequence with $\sup_{\varepsilon<\lambda} \alpha_{\delta,\varepsilon} = \delta$, and $h_\delta : C_\delta \to \kappa$ an onto function, such that for every club E of λ^+, for stationarily many $\delta < \lambda^+$ we have, for every $i < \kappa$

$$\{\varepsilon < \lambda : \alpha_{\delta,\varepsilon}, \alpha_{\delta,\varepsilon+1} \in E \ \&\ h(\alpha_{\delta,\varepsilon}) = i\}$$

is stationary in λ (see Notation **??** below for $S^{\lambda^+}_\lambda$). Note that it is not known if the analogous result holds when "$\alpha_{\delta,\varepsilon}, \alpha_{\delta,\varepsilon+1} \in E$" is replaced by "$\alpha_{\delta,\varepsilon}, \alpha_{\delta,\varepsilon+1}, \alpha_{\delta,\varepsilon+2} \in E$". If it does, it would have interesting consequences regarding the generalized Suslin hypothesis, see [**?**].

We prove that if λ is regular and at least equal to $\aleph_3$, then just in ZFC a version of $\clubsuit$ by which unbounded subsets of $S^{\lambda^+}_\lambda$ (i.e. the ordinals $< \lambda^+$ of cofinality λ) are guessed, holds. If $\lambda \geq \aleph_2, 2^{\aleph_0}$ we obtain a similar version of guessing. See Theorem **??**.

Another result along these lines is one of Džamonja and Shelah from [**?**]:

Definition 0.1. Suppose that λ is a cardinal. $\clubsuit^*_{-\lambda}(\lambda^+)$ is the statement saying that there is a sequence $\langle \mathcal{P}_\delta : \delta \text{ limit } < \lambda^+\rangle$ such that

(i) $\mathcal{P}_\delta$ is a family of $\leq |\delta|$ unbounded subsets of δ,

(ii) For $a \in \mathcal{P}_\delta$ we have $\mathrm{otp}(a) < \lambda$,

(iii) For all $X \in [\lambda^+]^{\lambda^+}$, there is a club C of λ^+ such that for all $\delta \in C$ limit, there is $a \in \mathcal{P}_\delta$ such that

$$\sup(a \cap X) = \delta.$$

Fact 0.2 (Džamonja-Shelah). [**?**] If $\aleph_0 < \kappa = \mathrm{cf}(\lambda) < \lambda$, <u>then</u> $\clubsuit^*_{-\lambda}(\lambda^+)$.

See the discussion below for a related result of Shelah from [**?**].

Our Theorem **??** can be understood as an extension of the theorem from [**?**] to the successor of a regular κ for some $\kappa \geq \aleph_3$, with the exception that our guessing has less guesses at each δ (just one), but the guessing is obtained stationarily often, as opposed to club often. Also note that the result in Theorem **??** is in some sense complementary to the "club guessing" results of Shelah ([?] for example) because here we are guessing unbounded subsets of λ^+ which are not necessarily clubs, but on the other hand, there are limitations on the cofinalities.

In §**??**, we investigate successors of singulars. On the one hand, we can hope to improve or at least modify in a non-trivial way the above result from [?]. If λ is a strong limit singular, <u>and</u> $2^\lambda = \lambda^+$, it has already been done by Shelah in [?]:

Fact 0.3 (Shelah). [?] Suppose that λ is a strong limit singular with $2^\lambda = \lambda^+$, and S is a stationary subset of $S^{\lambda^+}_{\mathrm{cf}(\lambda)}$.

<u>Then</u> there is a sequence

$$\left\langle \langle \bar{\alpha}^\delta = \alpha_{\delta,i} : i < \mathrm{cf}(\lambda)\rangle : \delta \in S^{\lambda^+}_{\mathrm{cf}(\lambda)} \right\rangle$$

such that $\bar{\alpha}^\delta$ increases to δ, and for every $\theta < \lambda$ and $f \in {}^{\lambda^+}\theta$, there are stationarily many δ such that

$$(\forall^* i)\,[f(\alpha_{\delta,2i}) = f(\alpha_{\delta,2i+1})].$$

Here, the quantifier $\forall^* i$ means "for all but $< \mathrm{cf}(\lambda)$ many". For more on guessing of unbounded sets, see [?].

In §**??**, we show that to a large extent the assumption that $2^\lambda = \lambda^+$, is necessary above. See Theorem **??**.

We finish this introduction by recalling some notation and facts which will be used in the following sections.

Notation 0.4. (1) Suppose that $\kappa = \mathrm{cf}(\kappa) < \delta$. We let

$$S^\delta_\kappa \stackrel{\mathrm{def}}{=} \{\alpha < \delta : \mathrm{cf}(\alpha) = \kappa\}.$$

(2) Suppose that $C \subseteq \alpha$. We let

$$\mathrm{acc}(C) \stackrel{\mathrm{def}}{=} \{\beta \in C : \beta = \sup(C \cap \beta)\},$$

and $\mathrm{nacc}(C) \stackrel{\mathrm{def}}{=} C \setminus \mathrm{acc}(C)$, while $\lim(C) \stackrel{\mathrm{def}}{=} \{\delta < \alpha : \delta = \sup(C \cap \delta)\}$.

Definition 0.5. Suppose that $\lambda \geq \aleph_1$ and γ is an ordinal, while $A \subseteq \lambda^+$. For $S \subseteq \lambda^+$, we say that S *has a square of type* $\leq \gamma$ *nonaccumulating in* A iff there is a sequence $\langle e_\alpha : \alpha \in S \rangle$ such that

(i) $\beta \in e_\alpha \Longrightarrow \beta \in S \ \&\ e_\beta = e_\alpha \cap \beta$,

(ii) e_α is a closed subset of α,

(iii) If $\alpha \in S \setminus A$, then $\alpha = \sup(e_\alpha)$ (so $\mathrm{nacc}(e_\alpha) \subseteq A$ for all $\alpha \in S$),

(iv) $\mathrm{otp}(e_\alpha) \leq \gamma$.

Fact 0.6 (Shelah). [[**?**]§4, [**?**]III§2] Suppose that

$$\lambda = \mathrm{cf}(\lambda) = \theta^+ > \theta = \mathrm{cf}(\theta) > \kappa = \mathrm{cf}(\kappa).$$

Further suppose that $S \subseteq S^\lambda_\kappa$ is stationary. Then there is $S_1 \subseteq \lambda$ on which there is a square of type $\leq \kappa$, nonaccumulating on A=the successor ordinals, and $S_1 \cap S$ is stationary.

Remark 0.7. In the proof of Fact **??** we can replace A=the successor ordinals with $A = S^\lambda_\sigma$ for any $\sigma = \mathrm{cf}(\sigma) < \kappa$.

Definition 0.8 (Shelah). [**?**] Suppose that $\delta < \lambda$ and $e \subseteq \delta$, while $E \subseteq \lambda$. We define

$$\mathrm{gl}(e, E) \stackrel{\mathrm{def}}{=} \{\sup(\alpha \cap E) : \alpha \in e \ \&\ \alpha > \min(E)\}.$$

Observation 0.9. Suppose that e and E are as in Definition **??**, and both e and $E \cap \delta$ are clubs of δ. Then, we have that $\mathrm{gl}(e, E)$ is a club of δ with $\mathrm{otp}(\mathrm{gl}(e, E)) \leq \mathrm{otp}(e)$.

If e is just closed in δ, and $E \cap \delta$ is a club of δ then $\mathrm{gl}(e, E)$ is closed and $\mathrm{otp}(\mathrm{gl}(e, E)) \leq \mathrm{otp}(e)$.

Fact 0.10 (Shelah). [**?**] Suppose that $\mathrm{cf}(\kappa) = \kappa < \kappa^+ < \mathrm{cf}(\lambda) = \lambda$. Further suppose that $S \subseteq S^\lambda_\kappa$ is stationary and $\langle e_\delta : \delta \in S \rangle$ is a sequence such that each e_δ is a club of δ. Then there is a club E^* of λ such that the sequence

$$\bar{c} = \langle c_\delta \stackrel{\mathrm{def}}{=} \mathrm{gl}(e_\delta, E^*) : \delta \in S \cap E^* \rangle$$

has the property that for every club E of λ, there are stationarily many δ such that $c_\delta \subseteq E$.

Observation 0.11. Suppose that $\mathrm{cf}(\kappa) = \kappa < \kappa^+ < \lambda$ and λ is a successor cardinal. Further assume that $S_1 \subseteq S^\lambda_\kappa$ is stationary, while $A = S^\lambda_\sigma$ for some $\sigma = \mathrm{cf}(\sigma) < \kappa$, possibly $\sigma = 1$. Then there is stationary $S_2 \subseteq S_1$ and a square $\langle e_\delta : \delta \in S_2 \rangle$ of type $\leq \kappa$ nonaccumulating in A, such that each e_δ is a set of limit ordinals and $S_1 \cap S_2$ is stationary, while

$$E \text{ a club of } \lambda \Longrightarrow \{\delta \in S_1 \cap S_2 : e_\delta \subseteq E\} \text{ is stationary.}$$

[Why? The proof of this can be found in [[**?**], III, §2], but as it easily follows from the Facts we already quoted, we shall give a proof. By Fact **??** and Remark **??**, there is $S_3 \subseteq \lambda$ with a square $\langle e_\alpha : \alpha \in S_3 \rangle$ of type $\leq \kappa$ nonaccumulating in A, and such that $S_1 \cap S_3$ is stationary. By Fact **??**, there is club E^* of λ as in the conclusion of Fact **??**, with $S_3 \cap S_1$ in place of S. Now, letting

$$S_2 \stackrel{\text{def}}{=} \{\sup(\alpha \cap E^*) : \alpha \in \bigcup_{\delta \in S_3} e_\delta \cup \{\delta\} \ \&\ \alpha > \min(E^*)\} \cap S_3,$$

and for $\delta \in S_2$, letting $c_\delta \stackrel{\text{def}}{=} \mathrm{gl}(e_\delta, E^*)$, we observe that $S_2 \cap S_1$ is stationary (as $S_2 \cap S_1 \supseteq S_1 \cap S_3 \cap \mathrm{acc}(E^*)$), and $\langle c_\delta : \delta \in S_2 \rangle$ is a square of type $\leq \kappa$ nonaccumulating in A, while

$$E \text{ a club of } \lambda \Longrightarrow \{\delta \in S_1 \cap S_2 : c_\delta \subseteq E\} \text{ is stationary.}]$$

Notation 0.12. Reg stands for the class of regular cardinals.

1 A ZFC version of $\clubsuit$

Theorem 1.1. (1) Suppose that

(a) $\lambda = \mathrm{cf}(\lambda) > \kappa = \mathrm{cf}(\kappa) > \theta = \mathrm{cf}(\theta) \geq \aleph_1$.

(b) $S^* \subseteq S^{\lambda^+}_\theta$ is stationary, moreover

$$S_1 \stackrel{\mathrm{def}}{=} \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \kappa \ \&\ S^* \cap \delta \text{ is stationary}\}$$

is stationary. (e.g. $S^* = S^{\lambda^+}_\theta$.)

Then there is a stationary $S' \subseteq S^*$ and $\langle E_\delta : \delta \in S' \rangle$ such that

(i) E_δ is a club of δ with $\mathrm{otp}(E_\delta) < \lambda^\omega \cdot \kappa$,

(ii) for every unbounded $A \subseteq S^{\lambda^+}_\lambda$, for stationarily many $\delta \in S'$, we have

$$\delta = \sup(A \cap \mathrm{nacc}(E_\delta)).$$

(2) If above we allow $\theta = \aleph_0$, but request $\lambda \geq 2^{\aleph_0}$, the conclusion of (1) remains true.

Remark 1.2. We explain why the assumption that S_1 is stationary in item (b) above implies that S^* is stationary. Notice that $\lambda^+ > \kappa^+$, hence club guessing holds between λ^+ and κ. In fact, by Fact **??**, we can assume that this is exemplified by a sequence $\langle c_\delta : \delta \in S \subseteq S_1 \rangle$. Now suppose that C is a club of λ^+, and let $E \stackrel{\mathrm{def}}{=} \mathrm{acc}(C)$. Let $\delta \in S$ be such that $c_\delta \subseteq E$. Hence, c_δ is a club of δ, so $c_\delta \cap S^* \neq \emptyset$, implying that $C \cap S^* \neq \emptyset$.

Obviously, S_1 being stationary is a necessary condition for our conclusion, as if S_1 were to be non-stationary, we could assume $S' \subseteq S^* \setminus S_1$, and $E_\delta \cap S^* = \emptyset$ for $\delta \in S'$. This would be a contradiction with (ii) above when $A = S^*$.

Proof. Let $S_0 \stackrel{\mathrm{def}}{=} S^{\lambda^+}_\theta$ and let $A^* \stackrel{\mathrm{def}}{=} S^{\lambda^+}_{\aleph_1}$.

By Observation **??** with λ^+ in place of λ and A^* in place of A, there is a $S_2 \subseteq S^{\lambda^+}_{\leq\kappa}$ such that there is a square $\bar{e} = \langle e_\delta : \delta \in S_2 \rangle$ of type $\leq \kappa$ nonaccumulating in A^*, the set $S_1 \cap S_2$ is stationary, and, moreover, for every E a club of λ^+, the set $\{\delta \in S_1 \cap S_2 : e_\delta \subseteq E\}$ is stationary. [Why can we assume that $S_2 \subseteq S^{\lambda^+}_{\leq\kappa}$? As if $\alpha \in S_2$ and $\alpha = \sup(e_\alpha)$, we have that $\mathrm{cf}(\alpha) \leq |e_\alpha| \leq \kappa$ and if $\alpha \in S_2$ and $\alpha > \sup(e_\alpha)$ we have that $\alpha \in A^*$, hence $\mathrm{cf}(\alpha) = \aleph_1 \leq \kappa$.

Let $S' \stackrel{\text{def}}{=} S^* \cap S_2$, so be stationary. [Why? As otherwise there is a club C of λ^+ with $S' \cap C = \emptyset$. Let $E \stackrel{\text{def}}{=} \text{acc}(C)$ and let $\delta \in S_1 \cap S_2 \cap E$ be such that $e_\delta \subseteq E$. As $\text{cf}(\delta) = \kappa \neq \aleph_1$, we have that $\delta \notin A^*$, and hence e_δ is a club of δ. On the other hand, $S^* \cap \delta$ is stationary in δ, hence $e_\delta \cap S^* \neq \emptyset$, a contradiction with $e_\delta \subseteq C$, as $e_\delta \subseteq S_2$ by the definition of a square (see Definition **??**). So any point in $e_\delta \cap S^*$ is in $C \cap S'$, contrary to the choice of C.]

Claim 1.3. There is a function $g : S' \to \omega$ such that for every club E of λ^+, there are stationarily many $\delta \in S_1 \cap S_2$ such that $e_\delta \subseteq E$ and

$$(\forall n < \omega)[E \cap \delta \cap g^{-1}(\{n\}) \text{ is stationary in } \delta].$$

Proof of the Claim. For $\delta \in S'$, we choose a sequence $\bar{\xi}^\delta = \langle \xi_{\delta,i} : i < \theta \rangle$ increasing with limit δ, and such that $\xi_{\delta,i} \in e_\delta$ and $\text{otp}(e_{\xi_{\delta,i}})$ depends only on i and $\text{otp}(e_\delta)$, but not on δ. [The point is of course that $\text{otp}(e_\delta)$ is in general larger than θ.] For each $i < \theta$, we define a function $h_i : S' \to \kappa$ by letting

$$h_i(\delta) \stackrel{\text{def}}{=} \text{otp}(e_{\xi_{\delta,i}}).$$

Subclaim 1.4. For each $\delta \in S_1 \cap S_2$ we can find $i(\delta) < \theta$ such that with $i = i(\delta)$,

$$A_i^\delta \stackrel{\text{def}}{=} \{\beta \in e_\delta : \{\gamma \in e_\delta \cap S' : \xi_{\gamma,i} = \beta\} \text{ is stationary } \}$$

is unbounded in δ.

Proof of the Subclaim. If this fails for some $\delta \in S_1 \cap S_2$, then each A_i^δ for $i < \theta$ is bounded in δ. As $\theta < \kappa = \text{cf}(\delta)$, we have $\beta^* \stackrel{\text{def}}{=} \sup_{i<\theta} \sup(A_i^\delta) < \delta$. As $\delta \in S_1$, we have that $e_\delta \cap S^*$ is stationary in δ. For every $\gamma \in e_\delta \cap S^* \setminus \beta^*$, we in particular have that $\gamma \in S'$, so $\bar{\xi}^\gamma$ is defined. Hence, for such γ there is $i_\gamma < \theta$ such that $\gamma > \xi_{\gamma,i_\gamma} > \beta^*$. By Fodor's Lemma, there is ξ^* such that for stationarily many γ we have $\xi_{\gamma,i_\gamma} = \xi^*$, and applying the same lemma again, we can without loss of generality assume that for some $i^* < \theta$ we have $i_\gamma = i^*$ for stationarily many γ for which $\xi_{\gamma,i_\gamma} = \xi^*$. But then $\xi^* > \beta^*$ and yet $\xi^* \in A_{i^*}^\delta$, a contradiction. $\bigstar$**??**

Subclaim 1.5. For some $i(*) < \theta$, the set

$$S^{**} \stackrel{\text{def}}{=} \{\delta \in S_1 \cap S_2 : i(\delta) = i(*)\}$$

is stationary, and $\bar{e} \restriction S^{**}$ still guesses clubs of λ^+.

Proof of the Subclaim. Otherwise, for each $i < \theta$ such that

$$T_i \stackrel{\text{def}}{=} \{\delta \in S_1 \cap S_2 : i(\delta) = i\}$$

is stationary, there is a club C_i of λ^+ such that for no $\delta \in T_i$ do we have $e_\delta \subseteq C_i$. Let $C \stackrel{\text{def}}{=} \bigcap\{C_i : T_i \text{ stationary }\}$, hence a club of λ^+, and let E be a club of λ^+ such that

$$[i < \theta \;\&\; T_i \text{ not stationary}] \Longrightarrow T_i \cap E = \emptyset.$$

Let $\delta \in S_1 \cap S_2$ be such that $e_\delta \subseteq \mathrm{acc}(E \cap C)$. Hence $\delta \in E \cap T_{i(\delta)}$, so $T_{i(\delta)}$ is stationary. On the other hand, we have $e_\delta \subseteq C_{i(\delta)}$, a contradiction. $\bigstar$**??**

Now notice that $\kappa > \aleph_1$, so club guessing holds between κ and $\aleph_0$, i.e. there is a sequence $\langle w_\zeta : \zeta \in S^\kappa_{\aleph_0}\rangle$ such that $w_\zeta \subseteq \zeta$ and $\mathrm{otp}(w_\zeta) = \omega$ for each ζ, while for every club C of κ, there are stationarily many ζ with $w_\zeta \subseteq C$. Let $W \stackrel{\text{def}}{=} \{w_\zeta : \zeta \in S^\kappa_{\aleph_0}\}$. For $\beta \in S'$, let

$$w^\zeta_\beta \stackrel{\text{def}}{=} \{\gamma \in e_\beta : \mathrm{otp}(e_\beta \cap \gamma) \in w_\zeta\}.$$

For each $\zeta \in S^\kappa_{\aleph_0}$, we define $g_\zeta : S' \to \omega$ by letting

$$g_\zeta(\gamma) \stackrel{\text{def}}{=} \mathrm{otp}(h_{i(*)}(\gamma) \cap w_\zeta).$$

If some g_ζ is as required in Claim **??**, then we are done. Otherwise, for each ζ there is a club E_ζ of λ^+ with

$$[\delta \in S^{**} \cap E_\zeta \;\&\; e_\delta \subseteq E_\zeta] \Longrightarrow (\exists n \stackrel{\text{def}}{=} n_{\delta,\zeta})\,[E_\zeta \cap \delta \cap g_\zeta^{-1}(\{n\}) \text{ is non-stationary in } \delta].$$

Let $E \stackrel{\text{def}}{=} \bigcap_{\zeta<\kappa} E_\zeta$. Let $\delta^* \in S^{**} \cap E$ be such that $e_{\delta^*} \subseteq E$. As $\delta^* \in S^{**}$, we have that $A^{\delta^*}_{i(*)}$ is unbounded in δ^*. Let $C_{\delta^*} \stackrel{\text{def}}{=} \lim(A^{\delta^*}_{i(*)}) \cap e_{\delta^*}$, hence a club of δ^*. As $\mathrm{cf}(\delta^*) = \kappa$, and so $\mathrm{otp}(e_{\delta^*}) = \kappa$, we have that

$$C^-_{\delta^*} \stackrel{\text{def}}{=} \{\mathrm{otp}(\gamma \cap e_{\delta^*}) : \gamma \in C_{\delta^*}\}$$

is a club of κ. Hence there is $\zeta^* \in S^\kappa_{\aleph_0}$ such that $w_{\zeta^*} \subseteq C^-_{\delta^*}$. Let $\{\xi_0, \xi_1, \xi_2, \ldots\}$ be the increasing enumeration of w_{ζ^*}, and stipulate $\xi_{-1} = 0$. For $l < \omega$, let γ_l be the unique $\gamma \in C_{\delta^*}$ such that $\mathrm{otp}(\gamma_l \cap e_{\delta^*}) = \xi_l$, and let $\gamma_{-1} \stackrel{\mathrm{def}}{=} \min(C_{\delta^*})$. Hence for every $l < \omega$ we have

$$A^{\delta^*}_{i(*)} \cap [\gamma_{l-1}, \gamma_l) \neq \emptyset.$$

Hence for some $\beta_l \in [\gamma_{l-1}, \gamma_l)$, we have that

$$\{\gamma \in e_{\delta^*} \cap S' :\ \xi_{\gamma, i(*)} = \beta_l\}$$

is stationary in δ^*. This means that for each $l < \omega$, the set $g^{-1}_{\zeta^*}(\{l\}) \cap e_{\delta^*} \cap S'$ is stationary in δ^*, a contradiction. $\bigstar$**??**

Continuation of the Proof of Theorem **??**. We now choose $\bar{c} = \langle c_\alpha :\ \alpha < \lambda^+ \rangle$, so that

(α) For every α, we have that c_α is a club of α with $\mathrm{otp}(c_\alpha) \leq \lambda$, and for α a limit ordinal $\beta \in \mathrm{acc}(c_\alpha) \Longrightarrow \mathrm{cf}(\beta) < \lambda$, while if $\alpha = \beta + 1$, then $c_\alpha = \{\beta\}$.

(β) If $\delta \in S_2$, then $c_\delta \supseteq e_\delta$,

(γ) If $\delta \in S_2$ and $\sup(e_\delta) = \delta$, then $c_\delta = e_\delta$,

Now for any limit $\delta < \lambda^+$ we choose by induction on $n < \omega$ a club C^n_δ of δ of order type $\leq \lambda^{n+1}$, using the following algorithm:
Let $C^0_\delta \stackrel{\mathrm{def}}{=} c_\delta$. Let

$$C^{n+1}_\delta \stackrel{\mathrm{def}}{=} C^n_\delta \cup \{\alpha :\ (\exists \beta \in \mathrm{nacc}(C^n_\delta))\, [\sup(\beta \cap C^n_\delta) < \alpha < \beta \ \&\ \alpha \in c_\beta]\}.$$

Note 1.6. (1) The above algorithm really gives C^n_δ which is a club of δ with

$$\mathrm{otp}(C^n_\delta) \leq \lambda^{n+1}.$$

If $\delta \in S_2$ and $\sup(e_\delta) = \delta$, then $\mathrm{otp}(C^\delta_n) \leq \lambda^n \cdot \kappa$.

[Why? We prove this by induction on n. It is clearly true for $n = 0$. Assume its truth for n. Clearly C^{n+1}_δ is unbounded in δ, let us show that it is closed. Suppose $\alpha = \sup(C^{n+1}_\delta \cap \alpha) < \delta$. If $\alpha = \sup(C^n_\delta \cap \alpha)$, then $\alpha \in C^n_\delta \subseteq C^{n+1}_\delta$ by the induction hypothesis. So, assume

$$\alpha^* \overset{\text{def}}{=} \sup(C^n_\delta \cap \alpha) < \alpha$$

and $\alpha \notin C^n_\delta$. Let $\langle \alpha_i : i < \mathrm{cf}(\alpha)\rangle$ be an increasing to α sequence in $(\alpha^*, \alpha) \cap C^{n+1}_\delta$. Hence for every i there is $\beta_i \in \mathrm{nacc}(C^n_\delta)$ such that $\alpha_i \in c_{\beta_i}$ and $\sup(C^n_\delta \cap \beta_i) < \alpha_i$. As $\sup(C^n_\delta \cap \alpha) = \alpha^* < \alpha_i$ and $\alpha \notin C^n_\delta$, we have $\beta_i > \alpha$, for every i, as $\beta_i \in \mathrm{nacc}(C^n_\delta)$. Suppose that $i \neq j$ and $\beta_i < \beta_j$. Hence $\sup(C^n_\delta \cap \beta_j) \geq \beta_i > \alpha_j$, a contradiction. So, there is β such that $\beta_i = \beta$ for all i, hence $\{\alpha_i : i < \mathrm{cf}(\alpha)\} \subseteq c_\beta$. As c_β is closed, and $\alpha < \beta$, we have $\alpha \in c_\beta$, and by the definition of C^{n+1}_δ we have $\alpha \in C^{n+1}_\delta$.

As for every β we have $\mathrm{otp}(c_\beta) \leq \lambda$, and by the induction hypothesis $\mathrm{otp}(C^n_\delta) \leq \lambda^{n+1}$, we have $\mathrm{otp}(C^{n+1}_\delta) \leq \lambda^{n+2}$.

Similarly, if $\delta \in S_2$ and $\sup(e_\delta) = \delta$, clearly $\mathrm{otp}(C^n_\delta) \leq \lambda^n \cdot \kappa$.]

(2) For every n, we have $\mathrm{acc}(C^n_\delta) \setminus \bigcup_{m<n} C^m_\delta \subseteq S^{\lambda^+}_{<\lambda}$.

[Why? Again by induction on n. For $n = 0$ it follows as $\mathrm{otp}(c_\delta) \leq \lambda$. Suppose this is true for C^n_δ. The analysis from the proof of (1) shows that for $\alpha \in \mathrm{acc}(C^{n+1}_\delta) \setminus C^n_\delta$, there is β such that $\alpha \in \mathrm{acc}(c_\beta)$, hence $\mathrm{cf}(\alpha) < \lambda$.]

(3) For every limit $\delta < \lambda^+$, we have $S^\delta_\lambda = \bigcup_{n<\omega} \mathrm{nacc}(C^n_\delta) \cap S^\delta_\lambda$.

[Why? Fix such δ and let $\alpha \in S^\delta_\lambda$. By item (2), it suffices to show that $\alpha \in C^n_\delta$ for some n. Suppose not, so let $\gamma_n \overset{\text{def}}{=} \min(C^n_\delta \setminus \alpha)$ for $n < \omega$. Hence $\langle \gamma_n : n < \omega\rangle$ is a non-increasing sequence of ordinals $> \alpha$, and so there is n^* such that $n \geq n^* \Longrightarrow \gamma_n = \gamma_{n^*}$. In particular we have that $\gamma_{n^*} \in \mathrm{nacc}(C^{n^*}_\delta)$. Let $\beta \in c_{\gamma_{n^*}} \setminus \alpha$. Hence $\sup(\gamma_{n^*} \cap C^{n^*}_\delta) < \alpha \leq \beta < \gamma_{n^*}$. By the definition of $C^{n^*+1}_\delta$, we have $\beta \in C^{n^*+1}_\delta$, a contradiction.]

Now for each $\delta \in S'$ we define

$$E_\delta \overset{\text{def}}{=} e_\delta \cup \bigcup \{C^{g(\delta)}_\alpha \setminus \sup(e_\delta \cap \alpha) : \alpha \in \mathrm{nacc}(e_\delta)\}.$$

Note first that E_δ is a club of δ, for $\delta \in S'$.

[Why? Clearly, E_δ is unbounded. Suppose $\gamma = \sup(E_\delta \cap \gamma) < \delta$. Without loss of generality we can assume $\gamma \notin e_\delta$. Let $\gamma^* \stackrel{\text{def}}{=} \sup(e_\delta \cap \gamma) < \gamma$. For every $\beta \in E_\delta \cap (\gamma^*, \gamma)$, there is $\alpha_\beta \in \operatorname{nacc}(e_\delta) \cap S'$ such that $\beta \in C^{g(\delta)}_{\alpha_\beta} \setminus \sup(e_\delta \cap \alpha_\beta)$. By the choice of γ^*, every such $\alpha_\beta > \gamma$. Suppose that $\beta_1 \neq \beta_2 \in E_\delta \cap (\gamma^*, \gamma)$ and $\alpha_{\beta_1} < \alpha_{\beta_2}$. Hence $\sup(e_\delta \cap \alpha_{\beta_2}) \geq \alpha_{\beta_1}$, a contradiction. So all α_β are a fixed α. Hence $\gamma < \alpha$ is a limit point of $C^{g(\delta)}_\alpha$, and we are done, as $C^{g(\delta)}_\alpha$ is closed.]

Also note that $\operatorname{otp}(E_\delta) < \lambda^\omega \cdot \kappa$.

Suppose that $A \subseteq S^{\lambda^+}_\lambda$ is unbounded and it exemplifies that $\langle E_\delta : \delta \in S' \rangle$ fails to satisfy the requirements of Theorem **??**. Hence there is a club E of λ^+ such that

$$\delta \in E \cap S' \Longrightarrow \sup(A \cap \operatorname{nacc}(E_\delta)) < \delta.$$

Let $E^* \stackrel{\text{def}}{=} \operatorname{acc}(E) \cap \{\delta : \delta = \sup(A \cap \delta)\}$, hence a club of λ^+. Let $\delta^* \in S_1 \cap S_2 \cap E^*$ be such that $e_{\delta^*} \subseteq E^*$ and for all $n < \omega$, the set $\delta^* \cap g^{-1}(\{n\})$ is stationary in δ^*.

For $\alpha \in \operatorname{nacc}(e_{\delta^*})$ we have that $A \cap \alpha$ is unbounded in α. Now we use Note **??**(3). As $A \subseteq S^{\lambda^+}_\lambda$ we have $A \cap \alpha \subseteq S^\alpha_\lambda$. So $A \cap \alpha = \bigcup_{n<\omega} \operatorname{nacc}(C^n_\alpha) \cap A \cap \alpha$, by the above mentioned Note. As $\alpha \in \operatorname{nacc}(e_{\delta^*})$ and $\operatorname{nacc}(e_{\delta^*}) \subseteq S^{\lambda^+}_{\aleph_1}$, there is $n < \omega$ such that $A \cap \operatorname{nacc}(C^n_\alpha)$ is unbounded in α. Let $n^*(\alpha)$ be the smallest such n. There is n^* such that

$$\sup\{\alpha \in \operatorname{nacc}(e_{\delta^*}) : n^*(\alpha) = n^*\} = \delta^*,$$

as $\operatorname{cf}(\delta^*) > \aleph_0$. Let

$$e \stackrel{\text{def}}{=} \{\beta \in \operatorname{acc}(e_{\delta^*}) : \beta = \sup\{\alpha \in \beta \cap \operatorname{nacc}(e_{\delta^*}) : n^*(\alpha) = n^*\}\},$$

hence e is a club of δ^*. By the choice of δ^*, the set $g^{-1}(\{n^*\}) \cap \delta^*$ is stationary in δ^*. So, there is $\beta \in e$ such that $g(\beta) = n^*$. In particular, $\beta \in S'$. For every $\alpha \in \operatorname{nacc}(e_\beta)$ such that $n^*(\alpha) = n^*$, we have that $A \cap \operatorname{nacc}(C^{n^*}_\alpha)$ is unbounded in α. However,

$$C^{n^*}_\alpha \setminus \sup(\alpha \cap e_\beta) = E_\beta \cap [\sup(\alpha \cap e_\beta), \beta),$$

hence $\mathrm{nacc}(C^{n^*}_\alpha)\setminus\sup(\alpha\cap e_\beta)\subseteq\mathrm{nacc}(E_\beta)$. Now, on the one hand $\beta\in E\cap S'$, so $\alpha^*\stackrel{\mathrm{def}}{=}\sup(A\cap\mathrm{nacc}(E_\beta))<\beta$, but on the other hand, the set of $\alpha\in\mathrm{nacc}(e_\beta)$ with $n^*(\alpha)=n^*$ is unbounded in β, hence there is $\gamma\in A$ with $\gamma\in E_\beta\setminus\alpha^*$. As $\gamma\in A$, we have $\mathrm{cf}(\gamma)=\lambda$, hence $\gamma\in\mathrm{nacc}(E_\beta)$, a contradiction.
(2) The statement of Claim **??** is true even when $\kappa=\aleph_1$, but if we assume that $\lambda\geq 2^{\aleph_0}$. Namely, under these assumptions, we have that $\theta=\aleph_0$, so there is $W\subseteq\{w\subseteq\omega_1:\mathrm{otp}(w)=\omega\}$ such that $|W|\leq\lambda$, and for every club C of ω_1, for some $w\in W$, we have $w\subseteq C$. Now we can just repeat the proof of Claim **??**, using W we have just defined. $\bigstar_{??}$

2 A negation of guessing

Theorem 2.1. Assume that there is a supercompact cardinal. Then

(1) It is consistent that there is λ a strong limit singular of cofinality $\aleph_0$, such that $2^\lambda>\lambda^+$ and

(∗) There is a function $f:\lambda^+\to\omega$ such that for every $\mathcal{P}\subseteq[\lambda^+]^{\aleph_0}$ of cardinality $<2^\lambda$, for some $X\in[\lambda^+]^{\lambda^+}$ we have
(i) $(\forall\zeta<\omega)[|X\cap f^{-1}(\{\zeta\})|=\lambda^+]$,
(ii) If $a\in\mathcal{P}$, then $\sup(\mathrm{Rang}(f\restriction(a\cap X))<\omega$.

(2) Moreover, in (1) we can replace $\aleph_0$ by any regular $\kappa<\lambda$.

Remark 2.2. So the theorem basically states that no $\mathcal{P}$ as above provides a guessing.

Proof. (1) We start with a universe in which λ is a supercompact cardinal and $2^\lambda=\lambda^+$ holds. We extend the universe by Laver's forcing ([**?**]), which makes the supercompactness of λ indestructible by any extension by a $(<\lambda)$-directed-closed forcing. This forcing will preserve the fact that $2^\lambda=\lambda^+$. Let us call the so obtained universe V.

Now choose μ such that $\mu=\mu^\lambda>\lambda^+$. By [**?**], there is a $(<\lambda)$-directed-closed λ^{++}-cc forcing notion P, not collapsing λ^+ of size μ adding μ unbounded subsets $A_\alpha\,(\alpha<\mu)$ to λ^+ such that

$(**)$ $\alpha \neq \beta < \mu \Longrightarrow |A_\alpha \cap A_\beta| < \lambda$.

In particular, in V^P we have $\lambda^+ < 2^\lambda = \mu$ ([?], 6.1.), while λ is supercompact. In V^P, let Q be Prikry's forcing which does not collapse cardinals and makes λ singular with $\mathrm{cf}(\lambda) = \aleph_0$, [?]. As this forcing does not add bounded subsets to λ, in the extension λ is a strong limit singular and clearly satisfies $2^\lambda = \mu$. In $V^{P * \underset{\sim}{Q}}$ we have $(**)$. We now work in $V^{P * \underset{\sim}{Q}}$.

Let $\lambda = \sum_{\zeta < \omega} \lambda_\zeta$ where each $\lambda_\zeta < \lambda$ is regular. Let χ be large enough regular and $M \prec (\mathcal{H}(\chi), \in)$ with $||M|| = \lambda^+$ such that $\lambda^+ \subseteq M$ and $\langle A_\alpha : \alpha < \mu \rangle, \langle \lambda_\zeta : \zeta < \omega \rangle \in M$. We list $\bigcup_{\zeta < \omega}([\lambda^+]^{\lambda_\zeta} \cap M)$ as $\{b_i : i < \lambda^+\}$.

We define $f : \lambda^+ \to \omega$ by $f(i) = \zeta$ iff $|b_i| = \lambda_\zeta$. For $\alpha < \mu$, let $X_\alpha \stackrel{\text{def}}{=} \{i : b_i \subseteq A_\alpha\}$.

Now suppose that $\mathcal{P} \subseteq [\lambda^+]^{\aleph_0}$ is of cardinality $< 2^\lambda \leq \mu$, we shall look for X as required in $(*)$.

If $\alpha < \mu$ is such that X_α fails to serve as X, then at least one of the following two cases must hold:

Case 1. For some $\zeta < \omega$ we have $|\{i : b_i \subseteq A_\alpha \ \& \ |b_i| = \lambda_\zeta\}| < \lambda^+$, or

Case 2. For some $a \in \mathcal{P}$ we have $\sup(\mathrm{Rang}(f \restriction (a \cap X_\alpha))) = \omega$.

Considering the second case, we shall show that for any $a \in \mathcal{P}$, there are $< \lambda$ ordinals α such that the second case holds for X_α, a. Fix an $a \in \mathcal{P}$. If $\alpha < \mu$ is such that Case 2 holds for X_α, a, then

$$\sup(\{\zeta : (\exists i \in a)[b_i \subseteq A_\alpha \ \& \ |b_i| = \lambda_\zeta]\} = \omega.$$

For $\zeta < \omega$ and $\alpha < \mu$ let $B^\alpha_\zeta \stackrel{\text{def}}{=} \{i \in a : b_i \subseteq A_\alpha \ \& \ |b_i| = \lambda_\zeta\}$. Notice that if $\alpha \neq \beta < \mu$ we have that for some $\zeta_{\alpha,\beta}$ the intersection $A_\alpha \cap A_\beta$ has size $< \lambda_{\zeta_{\alpha,\beta}}$, hence for all $\zeta \geq \zeta_{\alpha,\beta}$ we have $B^\alpha_\zeta \cap B^\beta_\zeta = \emptyset$.

Let $A \stackrel{\text{def}}{=} \{\alpha : \text{ Case 2 holds for } a, \alpha\}$. For every $\alpha \in A$, let

$$\bar{s_\alpha} \stackrel{\text{def}}{=} \langle \, B^\alpha_\zeta : \zeta < \omega \rangle,$$

hence $\alpha \neq \beta \Longrightarrow \bar{s_\alpha} \neq \bar{s_\beta}$. Hence $|A| \leq 2^{\aleph_0} < \lambda$.

Now note that if $\alpha < \mu$, then $A_\alpha \in [\lambda^+]^{\lambda^+}$. For $\gamma_0 < \lambda^+$ for some $\gamma_1 < \lambda^+$ we have $|A_\alpha \cap \gamma_1 \setminus \gamma_0| = \lambda$. In M we have a sequence $\langle c_\xi : \xi < \omega \rangle$ such that $\cup_\xi c_\xi = \gamma_1 \setminus \gamma_0$ and $|c_\xi| = \lambda_\xi$. For every $\zeta, \varepsilon < \omega$, for some large enough $\xi < \omega$

we have $|A_\alpha \cap c_\xi| \geq \lambda_\varepsilon$. But $[A_\alpha \cap c_\xi]^{\lambda_\zeta} \subseteq \mathcal{P}(c_\xi) \subseteq M$ (as $\lambda^{<\lambda} < \lambda$), so for some i we have $b_i \subseteq A_\alpha \cap [\gamma_0, \gamma_1)$ and $|b_i| = \lambda_\zeta$. Hence

$$|\{i :\ b_i \subseteq A_\alpha \ \&\ |b_i| = \lambda_\zeta\}| = \lambda^+,$$

so Case 1 does not happen for this (any) α.

As we can find $\alpha < \mu$ such that Case 2 does not happen, we are finished.

(2) Use Magidor's forcing from [**?**] in place of Prikry's forcing in (1). $\bigstar_{??}$

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