A Note on the Splitting Property in Strongly Dense Posets of Size $\aleph_0$

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Abstract

We show that it is not true that every countable infinite strongly dense poset has the splitting property, so answering a question of R. Ahlswede, P.L. Erdös and N. Graham. [1]

[1]We thank Péter L. Erdös and Niall Graham for discussing the problem, and Mary-Ellen Rudin and Jean-Marc Vanden-Broeck for listening to the proof.

1 Introduction

Definition 1.1. Suppose that $\mathcal{P} = \langle P, \leq_P \rangle$ is a partially ordered set, and $H \subseteq P$. We define

(1)

$$\mathcal{D}(H) \stackrel{\text{def}}{=} \{x \in P : (\exists s \in H)(x \leq s)\}$$

and

$$\mathcal{U}(H) \stackrel{\text{def}}{=} \{x \in P : (\exists s \in H)(x \geq s)\}.$$

(2) H is called an *antichain* or a *Sperner system* of P if no two elements of H are comparable.

(3) An antichain A of P is *maximal* iff

$$[P \supseteq A' \supseteq A \ \& \ A' \text{ an antichain of } P] \Longrightarrow A' = A.$$

(4) An antichain A of P is said to *split* iff there is a partition $\{D, U\}$ of A such that $P = \mathcal{D}(D) \cup \mathcal{U}(U)$.

(5) P is said to have the *splitting property* iff every maximal antichain of P splits.

Notation 1.2. In a poset $\mathcal{P} = \langle P, \leq_P \rangle$:

(1) For $x, y \in P$ we let $x <_P y$ iff $x \leq_P y$ and x and y are not equal.

(2) For $x, y \in P$

$$(x, y) \stackrel{\text{def}}{=} \{z \in P : x <_P z <_P y\},$$

and sets of the form (x, y) for $x <_P y \in P$ are called open intervals of P.

(3) If $x, y \in P$ are incomparable, i.e. neither $x \leq_P y$ nor $y \leq_P x$, we denote this by $x >< y$.

Definition 1.3. A poset $\mathcal{P} = \langle P, \leq_P \rangle$ is said to be *strongly dense* iff for every non-empty open interval (x, y) of P, there are $z_1, z_2 \in (x, y)$ such that $z_1 >< z_2$.

In [1], it was shown that every finite strongly dense poset has the splitting property. It was conjectured that the same is true for countably infinite strongly dense posets. We give an example showing that the conjecture is not true. In §2 we give a formal construction of the example and the proof that it is as required. The idea is however very simple: take the binary tree ${}^{<\omega}2$. Consider the maximal antichain $A = \{\langle 0\rangle, \langle 1\rangle\}$ in the tree: it does not split. But the binary tree is not strongly dense. To remedy this, we replace every node of the tree by a copy of ${}^{<\omega}2$, and continue ω many times, extending the partial order in the natural way as we go. Taking the union at the end, we arrive at a strongly dense partial order, and the version of A on the second level of the construction can be used to exemplify a maximal antichain which does not split.

The splitting property for arbitrary posets is further discussed in [2] where a sufficient condition for the splitting is given. One can consult [2] or [1] for more on the history of the problem and further references.

2 The example

Notation 2.1. (1) ${}^{<\omega}2 \stackrel{\text{def}}{=} \bigcup_{n<\omega}\{s : s : n \to \{0,1\}$ is a function $\}$, ordered by by being an initial segment.

(2) For $s,t \in {}^{<\omega}2$ we let $s \trianglelefteq t$ iff s is an initial segment of t.

(3) $\langle\rangle$ stands for the empty sequence.

Theorem 2.2. There is a countably infinite strongly dense poset which does not have the splitting property.

(Moreover, the poset $\langle C, \leq\rangle$ has the property described in the statement of Claim 2.5(3) below).

Proof. We start by defining the poset:

Definition 2.3. (1) By induction on $n < \omega$ we define C_n:

$\underline{n = 0}$.

$$C_0 \stackrel{\text{def}}{=} \{\langle \emptyset, s, 0\rangle : s \in {}^{<\omega}2\}$$

ordered by $\leq_0$ given by

$$\langle \emptyset, s, 0\rangle \leq_0 \langle \emptyset, t, 0\rangle \iff s \trianglelefteq t.$$

$\underline{n+1}$. Given C_n and a partial order $\leq_n$ on C_n. We define:

$$C_{n+1} \stackrel{\text{def}}{=} C_n \cup \{\langle x, s, n+1\rangle : s \in {}^{<\omega}2 \setminus \{\langle\rangle\} \ \&\ x \in C_n\},$$

ordered by letting $\langle a_1, b_1, c_1\rangle \leq_{n+1} \langle a_2, b_2, c_2\rangle$ iff one of the following occurs:

(i) $c_1, c_2 \leq n$ and $\langle a_1, b_1, c_1\rangle \leq_n \langle a_2, b_2, c_2\rangle$,

(ii) $c_1 \leq n \ \&\ c_2 = n+1 \ \&\ \langle a_1, b_1, c_1\rangle \leq_n a_2$,

(iii) $c_1 = n+1 \ \&\ c_2 \leq n \ \&\ a_1 <_n \langle a_2, b_2, c_2\rangle$,

(iv) $c_1 = c_2 = n+1 \ \&\ a_1 = a_2 \ \&\ b_1 \trianglelefteq b_2$,

(v) $c_1 = c_2 = n+1 \ \&\ a_1 <_n a_2$.

(2) We let $C \stackrel{\text{def}}{=} \bigcup_{n<\omega} C_n$, ordered by

$$x \leq y \iff x \leq_n y \text{ for the minimal } n \text{ such that } x, y \in C_n.$$

Remark 2.4. Observe that in Definition 2.3(2), for $x, y \in C$ we have

$$x \leq y \iff x \leq_n y \text{ for any } n \text{ such that } x, y \in C_n,$$

because of the item (i) in Definition 2.3(1). $\blacksquare_{2.4}$

Claim 2.5. (1) C is countable.

(2) $\leq$ is a partial order on C.

(3) $\langle C, \leq\rangle$ is strongly dense, moreover for every $x, y \in P$

$$x < y \Longrightarrow (\exists Z \subseteq (x, y))\,[|Z| = \aleph_0 \ \&\ z_1 \neq z_2 \Longrightarrow z_1 >< z_2].$$

Proof of the Claim. (1) Obvious, proving by induction on n that C_n is countable.

(2) The only part which needs checking is that $\leq$ is transitive. As $\leq = \bigcup_{n<\omega} \leq_n$, it suffices to show that every $\leq_n$ is transitive. We do this by induction on n.

<u>$n = 0$</u>. Obvious.

<u>$n + 1$</u>. Given $\langle a_l, b_l, c_l\rangle$ for $l \in \{0, 1, 2\}$ such that

$$\langle a_l, b_l, c_l\rangle \leq_{n+1} \langle a_{l+1}, b_{l+1}, c_{l+1}\rangle \text{ for } l \in \{0, 1\}.$$

We shall prove that $\langle a_0, b_0, c_0\rangle \leq_{n+1} \langle a_2, b_2, c_2\rangle$ by considering several cases.

<u>Case 1</u>. $c_l \leq n$ for $l \in \{0, 1, 2\}$.

Follows by the induction hypothesis.

<u>Case 2</u>. $c_0 \leq n$ and $c_1 = c_2 = n + 1$.

Hence we have $\langle a_0, b_0, c_0\rangle \leq_n a_1 \leq_n a_2$ (by cases (ii), (iv), (v) of Definition 2.3(1)), so the conclusion follows by case (ii) of Definition 2.3(1).

<u>Case 3</u>. $c_0, c_2 \leq n$ and $c_1 = n + 1$.

Hence $\langle a_0, b_0, c_0\rangle \leq_n a_1 <_n \langle a_2, b_2, c_2\rangle$, (by cases (ii) and (iii) of Definition 2.3(1)), so the conclusion follows by the induction hypothesis.

<u>Case 4</u>. $c_0 = n + 1$ and $c_1, c_2 \leq n$.

Hence $a_0 <_n \langle a_1, b_1, c_1\rangle \leq_n \langle a_2, b_2, c_2\rangle$, (by (ii), (iv) and (v) of Definition 2.3(1)), so the conclusion follows by case (iii) of Definition 2.3(1).

<u>Case 5</u>. $c_0 = c_2 = n + 1$ and $c_1 \leq n$.

Hence $a_0 <_n \langle a_1, b_1, c_1\rangle \leq_n \langle a_2, b_2, c_2\rangle$, (by (iii) and (ii) of Definition 2.3(1)), so the conclusion follows by case (v) of Definition 2.3(1).

<u>Case 6</u>. $c_0, c_1 \leq n$ and $c_2 = n + 1$.

Hence $\langle a_0, b_0, c_0\rangle \leq_n \langle a_1, b_1, c_1\rangle \leq_n a_3$, (by (i) and (ii) of Definition 2.3(1)), so the conclusion follows by case (ii) of Definition 2.3(1).

<u>Case 7</u>. $c_0 = c_1 = c_2 = n + 1$.

By case (iv) and (v) of Definition 2.3(1), one of the following occurs:

(α) $a_0 = a_2 = a_3$ and $b_0 \trianglelefteq b_1 \trianglelefteq b_2$, or

(β) $a_0 = a_1 <_n a_2$ or

(γ) $a_0 <_n a_1 = a_2$ or

(δ) $a_0 <_n a_1 <_n a_2$.

In any case, the conclusion follws from cases (iv) and (v) of Definition 2.3(1).

Case 8. $c_0 = c_1 = n+1$ and $c_2 \leq n$.

Hence $a_0 \leq_n a_1 <_n \langle a_2, b_2, c_2\rangle$ (by cases (iv), (v) and (iii) of Definition 2.3(1)), so the conclusion follows by case (v) of Definition 2.3(1).

(3) Let $x = \langle a_0, b_0, c_0\rangle \neq y = \langle a_1, b_1, c_1\rangle \in C$ be such that $x \leq y$. Let n be the first such that $x, y \in C_n$, hence at least one among c_0, c_1 is n. We proceed by induction on n.

$n = 0$. Hence $a_0 = a_2 = \emptyset$ and $c_0 \leq c_1 = 0$, while $b_0 \trianglelefteq b_1$ and $b_0 \neq b_1$. For all $s \in {}^{<\omega}2 \setminus \{\langle\rangle\}$ we have

$$x < \langle x, s, 1\rangle < y,$$

by rules (ii) and (iii) of Definition 2.3(1). If $s, t \in {}^{<\omega}2 \setminus \{\langle\rangle\}$ are incomparable in ${}^{<\omega}2$ we have, by rules (iv) and (v) of Definition 2.3(1), that

$$\langle x, s, 1\rangle >< \langle x, t, 1\rangle,$$

hence setting $Z \stackrel{\text{def}}{=} \{\langle x, s, 1\rangle : s \in {}^{<\omega}2 \setminus \{\langle\rangle\}\}$ will satisfy the requirement.

$n+1$. Similarly to the discusion of transitivity, we consider sevaral possibilities. We include the complete discussion for the sake of completeness.

Case 1. $c_0 \leq n$ and $c_1 = n+1$ and $x <_n a_1$.

By the induction hypothesis, there is an infinite antichain Z of $\langle C, \leq\rangle$ such that for all $z \in Z$ we have $x < z < a_1$. By rule (ii) of Definition 2.3(1), we have that $a_1 \leq y$, so we finish by transitivity.

Case 2. $c_0 \leq n$ and $c_1 = n+1$ and $a_1 = x$.

For $s \in {}^{<\omega}2 \setminus \{\langle\rangle\}$ consider $\langle x, s, n+2\rangle$, which is an element of C_{n+2}. For any such s we have

$$x \leq_{n+2} \langle x, s, n+2\rangle \leq_{n+2} y,$$

by rules (ii) and (iii) of Definition 2.3(1) respectively. We finish by transitivity, notcing that for $s, t \in {}^{<\omega}2 \setminus \{\langle\rangle\}$ which are incomparable in ${}^{<\omega}2$ we have that $\langle x, s, n+2\rangle$ and $\langle x, t, n+2\rangle$ are incomparable in $\mathcal{C}$.

Case 3. $c_0 = n+1$ and $c_1 \leq n$ and $a_1 <_n y$.

For $s \in {}^{<\omega}2$ such that $b_1 \trianglelefteq s$ we have that $x \leq \langle a_1, s, c_1 \rangle$, by rule (iv) of Definition 2.3(1), and we have $\langle a_1, s, c_1 \rangle \leq y$ by rule (iii). Again for s, t incomparable in ${}^{<\omega}2$, we have that $\langle a_1, s, c_1 \rangle$ and $\langle a_1, t, c_1 \rangle$ are incomparable.

Case 3. $c_0 = n+1 = c_1$.

Hence for all $s \in {}^{<\omega}2 \setminus \{\langle\rangle\}$ we have that $x \leq \langle x, s, n+2 \rangle \leq y$ by rules (ii) and (iii) respectively.

Noticing that we have covered all the possibilities, we finish the proof of the Claim. $\blacksquare_{2.5}$

Claim 2.6. The set

$$A \stackrel{\text{def}}{=} \{x \stackrel{\text{def}}{=} \langle\langle \emptyset, \langle\rangle, 0 \rangle, \langle 0 \rangle, 1 \rangle \,, y \stackrel{\text{def}}{=} \langle\langle \emptyset, \langle\rangle, 0 \rangle, \langle 1 \rangle, 1 \rangle\}$$

is a maximal antichain in $\langle C, \leq \rangle$.

Proof of the Claim. A is obviously an antichain. Given $z = \langle a, b, c \rangle \in C$.

If $c = 0$, then $a = \emptyset$ and if $b = \langle\rangle$, then $x < z$ by rule (ii). Otherwise, $x \leq z$ by rule (iii).

By induction on $c \geq 1$ we show that $x \leq z$ or $y \leq z$ (or both).

Suppose $\underline{c = 1}$. If $a = \langle \emptyset, \langle\rangle, 0 \rangle$, the conclusion follows by case (iv) of Definition 2.3(1). Otherwise, we use case (v) of Definition 2.3(1).

For $c = n+1$, we know by the induction hypothesis that $x \leq a$ or $y \leq a$. We finish by rule (v). $\blacksquare_{2.6}$

Claim 2.7. A from Claim 2.6 does not split.

Proof of the Claim. As $\langle\langle \emptyset, \langle\rangle, 0 \rangle, \langle 0, 1 \rangle, 1 \rangle \in \mathcal{U}(x) \setminus [\mathcal{U}(y) \cup \mathcal{D}(x)]$ and $\langle\langle \emptyset, \langle\rangle, 0 \rangle, \langle 1, 1 \rangle, 1 \rangle \in \mathcal{U}(y) \setminus [\mathcal{U}(x) \cup \mathcal{D}(y)]$, and $\langle \emptyset, \langle\rangle, 0 \rangle \notin \mathcal{U}(x) \cup \mathcal{U}(y)$. $\blacksquare_{2.7}$

$\blacksquare_{2.2}$

Remark 2.8. The above generalizes to an induction longer than ω steps, in the obvious manner.$\blacksquare_{2.8}$

References

[1] R. Ahlswede, P. L. Erdös and N. Graham, *A Splitting Property of Maximal Antichains*, Combinatorica 15 (4) 1995, pp. 475-480.

[2] P. L. Erdös, *Splitting Property in Infinite Posets*, to appear in Discrete Appl. Math.

Sadržaj

Parcijalno uredjenje $(P, \leq)$ se zove jako gustim, ako u svakom nepraznom intervalu (a, b) u P, postoje dva neuporediva elementa. Podskup A od P se zove Spernerov sistem, ako su svaka dva neindentična elementa u A neuporediva. Ako je A maksimalan, u odnosu na $\subseteq$, Spernerov sistem, kažemo da je A podijeljen ako $A = D \bigcup U$ gdje su D i U disjunktni, i svaki element od P je ili $\leq$ nego neki element od D, ili $\geq$ nego neki element od U. U svom članku [1], R. Alshwede, P.L. Erdös and N. Graham, su pokazali da svako konačno jako gusto parcijalno uredjenje, ima osobinu da su svi njegovi maksimalni Spernerovi sistemi podijeljeni. U istom članku autori su dali hipotezu da je isto tačno i za beskonačna jako gusta prebrojiva uredjenja. U ovom članku, mi smo pokazali da hipoteza iz [1] nije tačna, konstruišući beskonačno jako gusto parcijalno uredjenje koje ima maksimalni Spernerov sistem koji je nepodijeljen.