

On Modular Homology of Simplicial Complexes: Shellability

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For a simplicial complex Δ and coefficient domain F let $F\Delta$ be the F -module with basis Δ . We investigate the *inclusion map* given by $\partial: \tau \mapsto \sigma_1 + \sigma_2 + \sigma_3 + \cdots + \sigma_k$ which assigns to every face τ the sum of the co-dimension 1 faces contained in τ . When the coefficient domain is a field of characteristic $p > 0$ this map gives rise to homological sequences. We investigate this modular homology for shellable complexes. © 2001 Academic Press

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1. INTRODUCTION

The standard homology for a simplicial complex Δ is concerned with the \mathbb{Z} -module $\mathbb{Z}\Delta$ with basis Δ and the boundary map

$$\tau \mapsto \sigma_1 - \sigma_2 + \sigma_3 - \cdots \pm \sigma_k$$

which assigns to the face τ the alternating sum of the co-dimension 1 faces of τ . This defines a homological sequence over \mathbb{Z} and hence over any domain with identity.

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The purpose of this paper is to investigate the same module with respect to a different homomorphism. This is the *inclusion map* $\partial: \mathbb{Z}\Delta \rightarrow \mathbb{Z}\Delta$ given by

$$\partial: \tau \mapsto \sigma_1 + \sigma_2 + \sigma_3 + \cdots + \sigma_k.$$

Clearly, $\partial^2 \neq 0$. However, when coefficients are taken modulo an integer p then a simple calculation shows that $\partial^p = 0$. One may attempt therefore to build a generalized *modular homology theory* for simplicial complexes, in particular when $p > 0$ is a prime. Note, for $p = 2$ the inclusion map agrees with the usual boundary operator and any result here then corresponds to its counterpart in ordinary homology theory with coefficients in $\mathbb{Z}/2$. The purpose of this paper is to begin this investigation by analysing the modular homology of shellable complexes for arbitrary $p \geq 2$.

In [1, 9, 10] we have determined the modular homology of the simplex (or equivalently, of the Boolean algebra) and finite projective spaces (which may be considered as q -simplices) were dealt with in [11]. One consequence of [9] in particular is that modular homology is *not* homotopy invariant and hence does not coincide with the standard simplicial homology. See however the comment at the end of this section.

The standard homology of simplicial complexes has important applications to the theory of posets and this connection comes via the *order homology of a poset*, see for instance [3, 4, 15]. The principal feature here is that the homological sequences attached to a complex or to a poset often have vanishing homologies in all but one position. It will be very useful to have a name for this property and so we shall call a homological sequence *almost exact* if at most one of its homologies is non-trivial.

Our observations in [9–11] have shown that many combinatorial standard objects give rise to almost exact modular homology. The wish to examine this phenomenon further is guiding also this investigation. In Section 3 we have a complete description of the modular homology for a part of the boundary of a simplex. Sections 4 analyses how the modular homology behaves under gluing and in Section 5 we apply these results to shellable complexes.

For instance, Corollary 5.2 shows that every shellable complex with h -vector of the form $(h_0, \dots, h_k, 0, \dots, 0)$ has some almost exact p -modular homological sequence if $k \leq p$. So almost exactness turns out to be quite a common property, just as in the standard theory of shellable complexes and Cohen–Macaulay posets.

In Proposition 5.4 we show that the initial part of every p -homological sequence of a shellable complex is exact. We use this in Corollary 5.5 to compute the Brauer character on the kernels of the inclusion map for automorphisms of shellable complexes. Corollary 5.6 gives the p -rank of incidence matrices in shellable complexes which has counterparts in many

branches of combinatorics, see for instance the well-known rank formula of Frankl and Wilson [6, 17]. We prove that if Δ is a shellable $(n-1)$ -dimensional complex and if $t+s < n$ and $0 < t-s < p$ then the p -rank of the incidence matrix of s -faces of Δ versus its t -faces is equal to $f_s - f_{t-p} + f_{s-p} - f_{t-2p} + f_{s-2p} - \dots$ where f_i is the number of $(i-1)$ -dimensional faces of Δ . This result extends further and determines, as in [9], the p -rank of the orbit inclusion matrix of any subgroup of $\text{Aut}(\Delta)$ whose order is co-prime to p .

We conclude with some historical comments to throw some light onto the connection between ordinary and modular homology. The latter was first investigated in 1947 by W. Mayer [8]. Mayer's set-up deals with much more complicated modules: For these every face is composed of vertices each carrying their own *weight*. Consequently the usual modul attached to the complex appears only as a relatively small submodule in Mayer's module. The operator, on the other hand, when restricted to the usual module, is exactly the same as the inclusion map defined here.

This enlargement of the modules has the maybe desirable consequence that the Mayer homology is homotopy invariant. Ultimately, however, it also causes its demise: In a deep paper of 1949 Spanier [16] shows that homotopy invariance forces the Mayer homology to coincide with ordinary homology, when coefficients are taken in \mathbb{Z}/p .

The isomorphism between the two kinds of homologies is quite non-trivial, essentially based on the axiomatics of ordinary homology theory. If one accepts that there is some relation between the Mayer homology and the modular homology considered here, then this isomorphism may point towards connections between ordinary and modular homology, and these remain to be investigated. If one is prepared to work without homotopy invariance then Mayer's original question is again wide open. Other recent papers on nilpotent homomorphisms include Dubois-Violette [5] and Kapranov [7].

2. PREREQUISITES

2.1. Shellable Complexes

Let Ω be a set and denote by $2^{[\Omega]}$ the collection of all finite subsets of Ω . A well-known notion is that of an *order ideal* or *simplicial complex* in $2^{[\Omega]}$: these are subsets $\Delta \subseteq 2^{[\Omega]}$ such that $\tau \in \Delta$ and $\sigma \subseteq \tau$ implies $\sigma \in \Delta$.

So let Δ be a simplicial complex, or just *complex*. The elements of Δ are the *faces* of Δ and the *dimension* of the face σ is $\dim \sigma := |\sigma| - 1$. The *dimension* $\dim \Delta$ of Δ is the maximum of $\{\dim \sigma : \sigma \in \Delta\}$. We call Δ *pure* if all maximal faces have the same dimension. Maximal faces are also called

facets. If $\Delta = 2^{[\Omega]}$ then Δ is the *simplex* over Ω and if $|\Omega| = n$ then this simplex is denoted by Σ^n .

For the remainder suppose that Δ is finite and has dimension $n - 1$. Then the *face enumerator* of Δ is

$$f_{\Delta}(x) := x^n + f_1 x^{n-1} + \cdots + f_n$$

where f_k is the number of faces of cardinality k .

If $\sigma_1, \sigma_2, \dots, \sigma_t$ are elements of Δ then $[\sigma_1, \sigma_2, \dots, \sigma_t]$ denotes the subcomplex $2^{\sigma_1} \cup 2^{\sigma_2} \cup \cdots \cup 2^{\sigma_t}$ of Δ . If $t > 1$, all σ_i are pairwise distinct, there is some n with $|\sigma_i| = n - 1$ for all i and if $|\sigma_1 \cup \sigma_2 \cup \cdots \cup \sigma_t| = n$ then $[\sigma_1, \sigma_2, \dots, \sigma_t]$ is denoted by Σ_t^{n-1} . This complex is determined up to isomorphism by n and t only and we will write $\Delta = \Sigma_t^{n-1}$ to indicate that Δ is isomorphic to Σ_t^{n-1} . It will be convenient to extend this definition and put $\Sigma_1^{n-1} := \Sigma^{n-1}$.

A finite pure complex Δ of dimension $n - 1$ is called *shellable* if its maximal faces can be arranged as $\sigma_1, \sigma_2, \dots, \sigma_m$ in such a way that for every $j \leq m$ there is a t_j for which

$$[\sigma_1, \dots, \sigma_{j-1}] \cap [\sigma_j] = \Sigma_{t_j}^{n-1}.$$

In geometric language Δ therefore is shellable if and only if its maximal faces can be ordered in such a way that for each j the j th facet σ_j intersects the union of the preceding facets along a part of its boundary which is a union of facets of σ_j .

The arrangement $\sigma_1, \sigma_2, \dots, \sigma_m$ is a *shelling* of Δ and the polynomial

$$h_{\Delta}(x) := x^n + x^{n-t_2} + x^{n-t_3} + \cdots + x^{n-t_m}$$

is the *shelling polynomial* of Δ . It is known that $h_{\Delta}(1+x) = f_{\Delta}(x)$ (see [4, 18]) and so the shelling polynomial indeed is independent of the shelling. The sequence (h_0, h_1, \dots, h_n) defined by

$$h_0 x^n + h_1 x^{n-1} + \cdots + h_n = h_{\Delta}(x)$$

is called the *h-vector* and (f_0, f_1, \dots, f_n) is called the *f-vector* of Δ . We consider a special case of shellability.

DEFINITION 1. Let Δ be a shellable $(n - 1)$ -dimensional complex and $k \leq n$ a positive integer. Then Δ is *k-shellable* if x^{n-k} divides $h_{\Delta}(x)$, or equivalently, if its *h-vector* is of the form $(h_0, h_1, \dots, h_k, 0, \dots, 0)$.

For instance, when $k < n$ note that Σ_{k+1}^{n-1} has shelling polynomial $x^n + x^{n-1} + \cdots + x^{n-k}$ and so is *k-shellable*. Similarly, if Δ is 1-shellable then $h_{\Delta}(x) = x^n + (m - 1)x^{n-1}$ where m is the number of facets. In the same

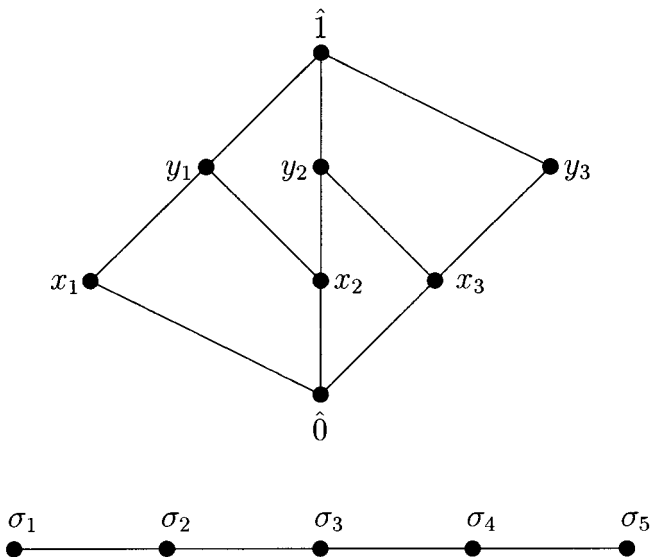


FIG. 1. A 1-shellable poset P and the graph of $\Delta(P)$.

fashion, using the relation $h_{\Delta}(1+x) = f_{\Delta}(x)$ above, we see that k -shellability for a shellable complex is determined entirely by its f -vector.

A fundamental construction due to Alexandrov assigns to any poset P its *order complex* $\Delta(P)$: If Ω denotes the set of elements of P then $\Delta \subseteq 2^{[\Omega]}$ is the collection of all finite linearly ordered subsets of Ω . We will use this to illustrate some k -shellable complexes in the examples below. It will also be useful to depict a shellable complex by a graph. The vertices of this graph are the facets of the complex with any two vertices connected by an edge if and only if the corresponding facets have a face of co-dimension 1 in common.

EXAMPLE. Let P be the poset in Fig. 1. One can see that P has 5 maximal chains: $\sigma_1 := (\hat{0} < x_1 < y_1 < \hat{1})$, $\sigma_2 := (\hat{0} < x_2 < y_1 < \hat{1})$, $\sigma_3 := (\hat{0} < x_2 < y_2 < \hat{1})$, $\sigma_4 := (\hat{0} < x_3 < y_2 < \hat{1})$, $\sigma_5 := (\hat{0} < x_3 < y_3 < \hat{1})$. Hence $\Delta(P)$ has 5 facets of dimension 3 and its graph is $\Delta(P)$. By inspection one can see that Δ is 1-shellable.

2.2. Modular Homology

Let F be a field and k a non-negative integer. Let then M_k denote the F -vector space with k -element subsets of Ω as basis and put $M := \bigoplus_{0 \leq k} M_k$. We refer to F also as the *coefficient domain* of M . The *inclusion map* is the linear map $\partial: M_k \rightarrow M_{k-1}$ defined on a basis by mapping each

k -element subset of Ω unto the sum of all its $(k - 1)$ -element subsets. From this we obtain the sequence

$$\mathcal{M}: 0 \xleftarrow{\partial} M_0 \xleftarrow{\partial} M_1 \xleftarrow{\partial} \dots \xleftarrow{\partial} M_{k-1} \xleftarrow{\partial} M_k \xleftarrow{\partial} \dots$$

If $\Delta \subseteq 2^{\lfloor \Omega \rfloor}$ denote by M^Δ the subspace of M with basis Δ and let $M_k^\Delta := M^\Delta \cap M_k$. We are interested in the situation when ∂ restricts to maps $M_k^\Delta \rightarrow M_{k-1}^\Delta$ for all k : Note that this is the case precisely when Δ is an order ideal in $\Delta \subseteq 2^{\lfloor \Omega \rfloor}$, or equivalently, a simplicial complex. Thus we can attach to the simplicial complex Δ the sequence

$$\mathcal{M}^\Delta: 0 \xleftarrow{\partial} M_0^\Delta \xleftarrow{\partial} M_1^\Delta \xleftarrow{\partial} \dots \xleftarrow{\partial} M_{k-1}^\Delta \xleftarrow{\partial} M_k^\Delta \xleftarrow{\partial} \dots$$

of submodules of M .

Throughout we suppose that p is a fixed prime and that the coefficient domain F is a field of characteristic p . Let Δ be a complex and \mathcal{M}^Δ the sequence associated to Δ . For any j and $0 < i < p$ consider the sequence

$$\dots \leftarrow M_{j-p}^\Delta \leftarrow M_{j-i}^\Delta \leftarrow M_j^\Delta \leftarrow M_{j+p-i}^\Delta \leftarrow M_{j+p}^\Delta \leftarrow \dots$$

in which each arrow is the appropriate power of ∂ . This sequence is determined uniquely by any arrow $M_l^\Delta \leftarrow M_r^\Delta$ in it, and so is denoted by $\mathcal{M}_{(l,r)}^\Delta$. The unique arrow $M_a^\Delta \leftarrow M_b^\Delta$ in it for which $0 \leq a + b < p$ is the *initial arrow*. We regard M_b^Δ as the *0-position* of $\mathcal{M}_{(l,r)}^\Delta$ and while a may be negative b is always positive. The position of any other module in $\mathcal{M}_{(l,r)}^\Delta$ will be counted from this 0-position and (a, b) is referred to as the *type* of $\mathcal{M}_{(l,r)}^\Delta$.

As F has characteristic $p > 0$ it follows immediately that $\partial^p = 0$. In particular, the composition of any two consecutive arrows in $\mathcal{M}_{(l,r)}^\Delta$ is 0 and so this sequence is homological. The homology module at $M_{j-i}^\Delta \leftarrow M_j^\Delta \leftarrow M_{j+p-i}^\Delta$ is denoted by

$$H_{j,i}^A := (\text{Ker } \partial^i \cap M_j^A) / \partial^{p-i}(M_{j+p-i}^A).$$

If $\mathcal{M}_{(l,r)}^\Delta$ has at most one non-vanishing homology then it is said to be *almost exact* and the only non-trivial homology then is denoted by $H_{(l,r)}^A$. Its dimension is the Betti number

$$\beta_{(l,r)}^A := \dim(H_{(l,r)}^A)$$

and we say that $\beta_{(l,r)}^A$ or $H_{(l,r)}^A$ is *at position* d if the non-trivial homology occurs at position d . It is useful to allow the possibility $d = \infty$ so that an almost exact sequence $\mathcal{M}_{(l,r)}^\Delta$ is exact iff either $d = \infty$ or $\beta_{(l,r)}^A = 0$. Finally, if $\mathcal{M}_{(l,r)}^\Delta$ is almost exact for every choice of l and r , then \mathcal{M}^Δ is *almost p -exact*.

We reformulate a result from [1, 9] about the p -modular homology of the standard simplex Σ^n . For this important object we shall throughout use the notation

$$\mathcal{M}^{\Sigma^n} =: \mathcal{M}^n, \quad \mathcal{M}_{(l,r)}^{\Sigma^n} =: \mathcal{M}_{(l,r)}^n, \quad H_{(l,r)}^{\Sigma^n} =: H_{(l,r)}^n, \quad H_{j,i}^{\Sigma^n} =: H_{j,i}^n.$$

THEOREM 2.1 *The sequence \mathcal{M}^n is almost p -exact. For any l, r with $0 < r - l < p$ the Betti number of $\mathcal{M}_{(l,r)}^n$ is*

$$\beta_{(l,r)}^n := \left| \sum_{t=-\infty}^{+\infty} \binom{n}{l-pt} - \binom{n}{r-pt} \right|$$

at position

$$d_{(l,r)}^n := \begin{cases} \left\lfloor \frac{n-a-b}{p} \right\rfloor & \text{if } n-a-b \not\equiv 0 \pmod{p}, \\ \infty & \text{if } n-a-b \equiv 0 \pmod{p} \end{cases}$$

where (a, b) is the type of $\mathcal{M}_{(l,r)}^n$. Moreover, if $p > 2$ then $\mathcal{M}_{(l,r)}^n$ is exact if and only if $d_{(l,r)}^n = \infty$.

Remarks. (1) Note therefore that for $p > 2$ both the position of the non-trivial homology in a sequence and the exactness of the sequence are completely determined by n, p and $a + b$. Non-trivial Betti numbers on the other hand depend also on the type of the sequence if $p > 3$ (for $p = 3$ all Betti numbers are 0 or 1). This makes the case of $p = 2$ special.

(2) The parameters j and i of the non-trivial homology $H_{j,i}^n = H_{(l,r)}^n$ are completely determined by l, r, n and p : these are the solutions of the inequality $n - p < 2j - i < n$ which refers to *dimensions* and which in [1] is called the *middle term condition*. Note that the *position* of the homology depends only from $l + r, n$ and p , and for the remainder we will mostly use positions instead of dimensions to describe homologies. Nevertheless, the middle term condition will be crucial for the proofs, and so these two ways of describing homologies are complementary.

(3) The structure of $H_{(l,r)}^n$ as a $\text{Sym}(n)$ -module is rather interesting in its own right and has been determined in [1].

3. THE MODULAR HOMOLOGY OF Σ_k^{n-1}

The purpose of this section is to give a complete description of the p -modular homology of the complex Σ_k^{n-1} for any prime $p > 2$. (The case $p = 2$ is trivial: since 2-modular homology is just the standard simplicial

homology over the field of characteristic 2, we immediately know that for $k < n$ the complex Σ_k^{n-1} has only trivial homologies, and Σ_n^{n-1} is almost exact with non-trivial 1-dimensional homology at the top.)

Before we state the result we give an outline to explain how this homology can be described. If A denotes Σ_k^{n-1} then the inclusion $A \subset \Sigma^n$ induces a natural embedding $M_j^A \hookrightarrow M_j^n$. It is a simple matter to verify that

$$M_j^n / M_j^A \simeq M_{j-k}^{n-k}$$

and this leads us to consider the sequence $\mathcal{M}_{(l,r)}^n$ corresponding to Σ^n and the shifted sequence $\mathcal{M}_{(l-k, r-k)}^{n-k}$ corresponding to Σ^{n-k} . Their modular homology is known from Theorem 2.1 and so we are able to determine the homology of \mathcal{M}^A by the usual long sequence arguments. To state the result we need one additional invariant of a sequence: If Δ is a complex of dimension $n-1$ then the *weight* of $\mathcal{M}_{(l,r)}^A$ is the integer $0 < w \leq p$ with $w \equiv r + l - n \pmod p$.

THEOREM 3.1. *Suppose that $p > 2$ and let A denote the complex Σ_k^{n-1} for some $n \geq k > 1$. For given (l, r) put $d := d_{(l,r)}^n$, $u := d_{(l,r)}^{n+k}$ and let w denote the weight of $\mathcal{M}_{(l,r)}^n$.*

(i) *Suppose that $w < p$.*

- *If $k \equiv w \pmod p$ then $\mathcal{M}_{(l,r)}^A$ and $\mathcal{M}_{(l,r)}^n$ have the same homologies.*
- *If $1 \leq k < p + w < 2p$ and $k \not\equiv w \pmod p$ then $\mathcal{M}_{(l,r)}^A$ is almost exact with non-trivial homology*

$$H_{(l,r)}^A \simeq \begin{cases} H_{(l,r)}^n / H_{(l-k, r-k)}^{n-k} & \text{if } 1 \leq k < w \\ H_{(l,r)}^n \oplus H_{(l-k, r-k)}^{n-k} & \text{if } w < k < p + w \end{cases}$$

in the same position d as $\mathcal{M}_{(l,r)}^n$.

- *If $k > p + w$ and $k \not\equiv w \pmod p$ then $\mathcal{M}_{(l,r)}^A$ has precisely two non-trivial homologies at positions d and $u - 1$, isomorphic to $H_{(l,r)}^n$ and $H_{(l-k, r-k)}^{n-k}$, respectively.*

(ii) *Suppose that $w = p$. Then $\mathcal{M}_{(l,r)}^A$ is almost exact with non-trivial homology $H_{(l,r)}^A \simeq H_{(l-k, r-k)}^{n-k}$ at position $u - 1$. Moreover, if also $k \equiv 0 \pmod p$ then $\mathcal{M}_{(l,r)}^A$ is exact.*

Remark. In this theorem we have expressed all relevant information in terms of positions. Using Theorem 2.1 it is clear that the homology modules at positions d and $u - 1$ could be written, in terms of dimensions, as $H_{j,i}^A$ and $H_{x,y}^A$ with $n - p < 2j - i < n$ and $n + k - 2p < 2x - y < n + k - p$.

Before proving the theorem we note the following consequence of it:

COROLLARY 3.2. *If A denotes the complex Σ_k^{n-1} for some $n \geq k > 1$ then \mathcal{M}^A is almost p -exact if and only if $k \leq p + 1$. For $k > p + 1$ every $\mathcal{M}_{(l,r)}^A$ has at most two non-trivial homology modules.*

Proof of Theorem 3.1. We fix some l and r such that $0 < r - l < p$ and consider the sequence

$$\mathcal{M}_{(l,r)}^A: \dots \leftarrow M_l^A \leftarrow M_r^A \leftarrow \dots \leftarrow M_{j-i}^A \leftarrow M_j^A \leftarrow M_{j+p-i}^A \leftarrow \dots$$

which passes through the arrow $M_l^A \leftarrow M_r^A$. It will be very convenient to write this sequence as

$$\mathcal{A} := \dots \leftarrow A_{-1} \leftarrow A_0 \leftarrow A_1 \leftarrow \dots \leftarrow A_{t-1} \leftarrow A_t \leftarrow A_{t+1} \leftarrow \dots$$

where t indicates the *position* of a module. For the purpose of later proofs we shall denote the simplex Σ^n by B and write $\mathcal{M}_{(l,r)}^n$ as

$$\mathcal{B} := \dots \leftarrow B_{-1} \leftarrow B_0 \leftarrow B_1 \leftarrow \dots \leftarrow B_{t-1} \leftarrow B_t \leftarrow B_{t+1} \leftarrow \dots$$

Let H_t^A and H_t^n denote the homology modules of \mathcal{A} and \mathcal{B} respectively. As mentioned before, the embedding $A \subseteq \Sigma^n$ yields an embedding $\iota: \mathcal{A} \hookrightarrow \mathcal{B}$ and we have

$$M_j^n / M_j^A \simeq M_{j-k}^{n-k}$$

as can be shown easily. If we denote $\mathcal{M}_{(l-k,r-k)}^{n-k}$ by

$$\mathcal{C} := \dots \leftarrow E_{-1} \leftarrow E_0 \leftarrow E_1 \leftarrow \dots \leftarrow E_{t-s-1} \leftarrow E_{t-s} \leftarrow \dots$$

then we obtain the commutative diagram shown in Fig. 2 in which the vertical arrows yield short exact sequences. Here, evidently, E_{-s} corresponds to A_0 and B_0 . We denote the homology of \mathcal{C} by H_{t-s}^E and note that $H_{u-s}^E = H_{(l-k,r-k)}^{n-k}$ is the unique non-trivial homology of \mathcal{C} .

We have therefore induced maps $\bar{\iota}_t: H_t^A \rightarrow H_t^n$ and $\bar{\theta}_j: H_t^n \rightarrow H_{t-s}^E$ on the homology modules and these lead to the following exact sequence

$$\begin{aligned} \dots &\leftarrow H_{t-s-1}^E \xleftarrow{\bar{\theta}_{t-1}} H_{t-1}^n \xleftarrow{\bar{\iota}_{t-1}} H_{t-1}^A \\ &\leftarrow H_{t-s}^E \xleftarrow{\bar{\theta}_t} H_t^n \xleftarrow{\bar{\iota}_t} H_t^A \\ &\leftarrow H_{t-s+1}^E \xleftarrow{\bar{\theta}_{t+1}} H_{t+1}^n \xleftarrow{\bar{\iota}_{t+1}} H_{t+1}^A \leftarrow \dots \end{aligned}$$

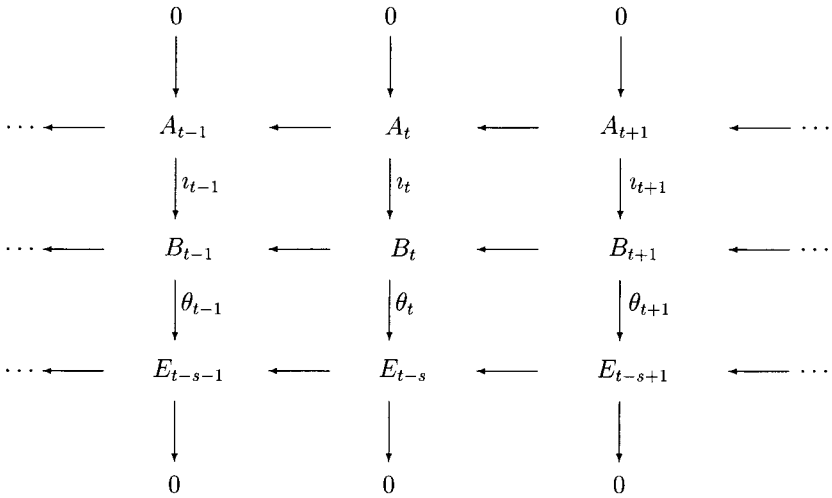


FIG. 2. The basic embedding.

Note that \bar{i}_t is in general not injective, its kernel will be extremely important for us.

We need to determine homologies in the third column while those in the first and the second column are all very well known. So we are in the fortunate position to have to consider only a few cases.

Case 1. When both \mathcal{B} and \mathcal{E} are exact. By Theorem 2.1 this condition is equivalent to $k \equiv w \pmod{p}$ and $w = p$. In this case \mathcal{A} is evidently exact.

Case 2. When \mathcal{B} is exact and \mathcal{E} is almost exact. By Theorem 2.1 this is equivalent to the condition $k \not\equiv w \equiv 0 \pmod{p}$. Since the non-trivial homology of \mathcal{E} is $H_{u-s}^E = H_{(l-k, r-k)}^{n-k}$, we obtain the 2-term exact sequence $0 \xleftarrow{\bar{i}_{u-1}} H_{u-1}^A \leftarrow H_{u-s}^E \leftarrow 0$. Evidently, \mathcal{A} is almost exact with non-trivial homology $H_{u-1}^A \simeq H_{(l-k, r-k)}^{n-k}$. Note that $\bar{i}_{u-1} = 0$.

From now on we suppose that $w < p$ and so the sequence \mathcal{B} is almost exact with non-trivial homology $H_{(l,r)}^n$ at position d .

Case 3. When \mathcal{B} is almost exact and \mathcal{E} is exact. Equivalently, $w < p$ and $k \equiv w \pmod{p}$. Here we have the exact sequence $0 \leftarrow H_d^n \xleftarrow{\bar{i}_d} H_d^A \leftarrow 0$. The map \bar{i}_d is an isomorphism and so the homology of \mathcal{A} coincides with the corresponding homology of the simplex Σ^n .

Case 4. When both \mathcal{B} and \mathcal{E} are almost exact. Equivalently, $w < p$ and $k \not\equiv w \pmod{p}$. Here \mathcal{B} and \mathcal{E} have non-trivial homologies at positions d and $u - s$ respectively. Note that

$$u - d = \left\lfloor \frac{n - a - b + k}{p} \right\rfloor - \left\lfloor \frac{n - a - b}{p} \right\rfloor \geq 0.$$

Again there are 3 possible cases:

Case 4.1. When $u = d + 1$ or equivalently $w < k < w + p < 2p$. Here we have the short exact sequence

$$0 \leftarrow H_d^n \xleftarrow{\bar{i}_d} H_d^A \leftarrow H_{u-s}^E \leftarrow 0$$

and so \mathcal{A} is almost exact with non-trivial homology $H_{(l,r)}^A \simeq H_{(l,r)}^n \oplus H_{(l-k,r-k)}^{n-k}$ at position d . The map \bar{i}_d is a surjection with $\text{Ker } \bar{i}_d = H_{(l-k,r-k)}^{n-k}$.

Case 4.2. When $u \geq d + 2$ or equivalently $k > w + p$ and $k \not\equiv w \pmod{p}$. This is a combination of cases 2 and 3 with two 2-term sequences arising. So the sequence \mathcal{A} is not almost exact but has two non-trivial homologies isomorphic to $H_{(l,r)}^n$ and $H_{(l-k,r-k)}^{n-k}$ at positions d and $u - 1$ respectively.

Case 4.3. When $u = d$ or equivalently $1 \leq k < w < p$. This is the most complicated case since here we have a 4-term exact sequence

$$0 \xleftarrow{\bar{i}_{d-1}} H_{d-1}^A \leftarrow H_{d-s}^E \xleftarrow{\bar{\theta}_d} H_d^n \xleftarrow{\bar{i}_d} H_d^A \leftarrow 0.$$

Clearly $\text{Ker } \bar{i}_d = 0$ but we need some further information to find out if there are two or one non-zero homologies in \mathcal{A} . We will prove now that H_{d-1}^A is trivial and so \mathcal{A} is almost exact with non-trivial homology $H_{(l,r)}^n / H_{(l-k,r-k)}^{n-k}$ at position d . This is equivalent to the next lemma whose proof will complete the proof of Theorem 3.1. ■

LEMMA 3.3. The map $\bar{\theta}_d: H_d^n \rightarrow H_{d-s}^E$ in the diagram above is a surjection.

Proof of the Lemma. Suppose that the non-trivial homology H_d^n of $\mathcal{M}_{(l,r)}^n$ appears at $M_{j-i}^n \leftarrow M_j^n \leftarrow M_{j+p-i}^n$. We need to look at the map θ in more detail. Let $\alpha_1, \alpha_2, \dots, \alpha_k$ be those vertices of Σ^n which are complements of the facets of Σ_k^{n-1} and put $a = \{\alpha_1, \dots, \alpha_k\}$. In [9] we have introduced a \cup -product for elements $f = \sum_{\zeta} f_{\zeta} \zeta$ and $f^* = \sum_{\tau} f_{\tau}^* \tau$ in M by putting $f \cup f^* := \sum_{\zeta, \tau} (f_{\zeta} f_{\tau}^*) (\zeta \cup \tau)$. With this notation it is clear that every $f \in M_j^n$ can be uniquely written as $f = a \cup b + g$ with $b \in M_{j-k}^{n-k}$ and so that θ is given by $\theta(f) = b$.

TABLE I

The 3-Modular Homologies of Σ_k^7

(l, r)	$(1, 2)$			$(1, 3)$			$(2, 3)$		
w	1			2			3		
$\beta(\Sigma^8)$	1			1			0		
d	2			2			∞		

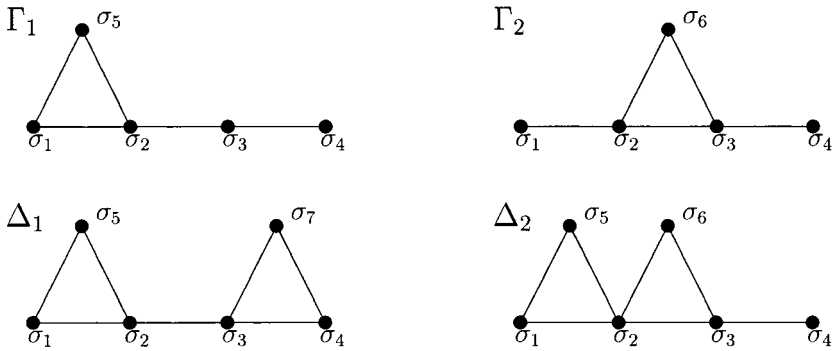
k	$\beta(\mathcal{E}_k)$	u	$\beta(\Sigma_k^7)/t$	$\beta(\mathcal{E}_k)$	u	$\beta(\Sigma_k^7)/t$	$\beta(\mathcal{E}_k)$	u	$\beta(\Sigma_k^7)/t$
1	0	∞	1/2	1	2	0/ ∞	1	2	1/1
2	1	3	2/2	0	∞	1/2	1	2	1/1
3	1	3	2/2	1	3	2/2	0	∞	0/ ∞
4	0	∞	1/2	1	3	2/2	1	3	1/2
5	1	4	1/2, 1/3	0	∞	1/2	1	3	1/2
6	1	4	1/2, 1/3	1	4	1/2, 1/3	0	∞	0/ ∞
7	0	∞	1/2	1	4	1/2, 1/3	1	4	1/3
8	1	5	1/2, 1/4	0	∞	1/2	1	4	1/3

So let b be in the kernel of ∂^i . It follows from Theorem 2.2 of [9] that b can be written as a sum of elements $b_1, b_2, \dots, b_t, \dots$ from the kernel of ∂^i such that each b_t involves no more than $2(j-k) - (i-1)$ different vertices. Since $H_{(l,r)}^n$ is non-trivial it follows from Theorem 2.1 that $2j-i < n$. Hence the number of different vertices in b_t is at most $n-2k$ and so one can choose $2k$ vertices $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_k$ which do not occur in b_t . Now let $f_t := (\alpha_1 - \beta_1) \cup \dots \cup (\alpha_k - \beta_k) \cup b_t$. Then $f := f_1 + f_2 + \dots$ satisfies $\partial^i(f) = 0$ and $\theta(f) = b$. ■

Table I illustrates the theorem for $n=8$ and $p=3$. Here $\beta(\mathcal{E}_k) := \beta_{(l-k, r-k)}^{n-k}$ and t indicates the position of the corresponding homology.

4. GLUING SIMPLICES

Now we consider a more general situation. Let Γ be a complex and suppose that Δ is a complex containing Γ and Σ^n such that $\Delta \cap \Sigma^n = \Sigma_k^{n-1}$. Such a complex is denoted by $\Gamma \overset{k}{\cup} \Sigma^n$ and we think of it as being obtained by gluing an n -simplex onto Γ along k facets. The purpose of this section is to investigate the modular homology of Δ in terms of the homology of Γ . As Σ^n is contractible we see from Theorem 2.1 that the modular homology is not homotopy invariant. Moreover, there are complexes with the same h -vector but different homology modules, see the example below. In particular, as the last component of the h -vector is the Euler characteristic

FIG. 3. The graphs of $\Gamma_1, \Gamma_2, \Delta_1$ and Δ_2 .

(as least for shellable complexes) we see that the Euler characteristic does not determine the modular homology of a complex.

EXAMPLE. Let B be the 7-dimensional complex with vertices $\{x_1, \dots, x_{11}\}$ and facets $\sigma_1 := \{x_1, \dots, x_8\}$, $\sigma_2 := \{x_2, \dots, x_9\}$, $\sigma_3 := \{x_3, \dots, x_{10}\}$ and $\sigma_4 := \{x_4, \dots, x_{11}\}$. Then B is 1-shellable. Now consider the facets $\sigma_5 := \{x_1, x_2, x_4, \dots, x_9\}$, $\sigma_6 := \{x_2, x_3, x_5, \dots, x_{10}\}$, $\sigma_7 := \{x_3, x_4, x_6, \dots, x_{11}\}$ and define

$$\Gamma_1 := B \cup \sigma_5, \quad \Gamma_2 := B \cup \sigma_6, \quad \Delta_1 := \Gamma_1 \cup \sigma_7 \quad \text{and} \quad \Delta_2 := \Gamma_2 \cup \sigma_5.$$

The graphs of these complexes are shown in Fig. 3.

Note that Γ_1 and Γ_2 share the same shelling polynomial $h_{\Gamma_1}(x) = h_{\Gamma_2}(x) = x^8 + 3x^7 + x^6$ and so these are 2-shellable with the same f -vector $(1, 11, 50, 125, 190, 181, 106, 35, 5)$. Similarly, Δ_1 and Δ_2 are 2-shellable with $h_{\Delta_1}(x) = h_{\Delta_2}(x) = x^8 + 3x^7 + 2x^6$ and with f -vector $(1, 11, 51, 131, 205, 201, 121, 41, 6)$.

Tables II and III give the 3- and 5-modular homologies of these complexes.

TABLE II
The 3-Modular Homologies

(l, r)	(a, b)	w	Γ_1, Γ_2	Δ_1, Δ_2
(1, 2)	(-1, 1)	1	exact	$\beta_{5,1} = 1$
(1, 3)	(0, 1)	2	$\beta_{4,1} = 4$	$\beta_{4,1} = 4$
(2, 3)	(0, 2)	3	$\beta_{5,2} = 4$	$\beta_{5,2} = 5$

TABLE III
The 5-Modular Homologies

(l, r)	(a, b)	w	Γ_1, Γ_2	Δ_1	Δ_2
(1, 2)	(1, 2)	5	$\beta_{6,4} = 32$	$\beta_{6,4} = 40$	$\beta_{6,4} = 40$
(1, 3)	(1, 3)	1	$\beta_{3,2} = 13$	$\beta_{3,2} = 5$	$\beta_{3,2} = 8, \beta_{6,3} = 3$
(1, 4)	(-1, 1)	2	$\beta_{4,3} = 73$	$\beta_{4,3} = 73$	$\beta_{4,3} = 73$
(1, 5)	(0, 1)	3	$\beta_{5,4} = 65$	$\beta_{5,4} = 70$	$\beta_{5,4} = 70$
(2, 3)	(-2, 2)	2	$\beta_{3,1} = 45$	$\beta_{3,1} = 45$	$\beta_{3,1} = 45$
(2, 4)	(-1, 2)	3	$\beta_{4,2} = 105$	$\beta_{4,2} = 113$	$\beta_{4,2} = 113$
(2, 5)	(0, 2)	4	$\beta_{5,3} = 97$	$\beta_{5,3} = 110$	$\beta_{5,3} = 110$
(3, 4)	(-1, 3)	4	$\beta_{4,1} = 60$	$\beta_{4,1} = 68$	$\beta_{4,1} = 68$
(3, 5)	(0, 3)	5	$\beta_{5,2} = 52$	$\beta_{5,2} = 65$	$\beta_{5,2} = 65$
(4, 5)	(0, 4)	1	$\beta_{4,4} = 8$	$\beta_{4,4} = 3$	$\beta_{4,4} = 5, s_{,1} = 2$

One can see that all homologies of Γ_1 and Γ_2 are the same while Δ_1 and Δ_2 have different 5-modular homologies for sequences of weight 1. Some of these phenomena will be explained in the next theorems.

The example shows that one can not expect a full description of the homology that occurs in this gluing process. Some information, however, can be derived: As $\Delta = \Gamma \cup \Sigma^n$ we have $\Sigma_k^{n-1} = \Gamma \cap \Sigma^n$ and so standard arguments can be applied. By Theorem 3.1 the homology of Σ_k^{n-1} is known and so the Mayer-Vietoris sequence gives some information about the homology of Δ . Under certain conditions this is enough to determine the homology of Δ entirely:

THEOREM 4.1 (Good Gluing). *Let Γ be some complex and suppose that $\Delta = \Gamma \cup \Sigma^n$ for some $n \geq k > 1$. For $2 < p$ and given (l, r) put $d := d_{(l,r)}^n, u := d_{(l,r)}^{n+k}$ and let w denote the weight of $\mathcal{M}_{(l,r)}^n$.*

- (i) *If $k \equiv w \pmod{p}$ then $\mathcal{M}_{(l,r)}^\Gamma$ and $\mathcal{M}_{(l,r)}^\Delta$ have the same homology.*
- (ii) *If $1 \leq k < w < p$ then $\mathcal{M}_{(l,r)}^\Gamma$ and $\mathcal{M}_{(l,r)}^\Delta$ have the same homology except for a single position $u = d$ in which $H_u^\Delta \simeq H_u^\Gamma \oplus H_{(l-k, r-k)}^{n-k}$.*

In the remaining case one can nevertheless obtain the following information about H^Δ :

THEOREM 4.2 (Bad Gluing). *Let p, Γ and Δ be as above. Suppose that $k \not\equiv w \equiv 0 \pmod{p}$ or that $k > w$ and $k \not\equiv w \pmod{p}$. Then $\mathcal{M}_{(l,r)}^\Gamma$ and $\mathcal{M}_{(l,r)}^\Delta$ have the same homology except in the two neighbouring positions $u - 1$ and u where we have the 5-term exact sequence*

$$0 \leftarrow H_{u-1}^\Delta \leftarrow H_{u-1}^\Gamma \oplus H_{u-1}^n \leftarrow H_{u-1}^n \oplus H_{(l-k, r-k)}^{n-k} \leftarrow H_u^\Delta \leftarrow H_u^\Gamma \leftarrow 0.$$

In particular, the Betti numbers satisfy $\beta_{u-1}^A - \beta_u^A = \beta_{u-1}^\Gamma - \beta_u^\Gamma - \beta_{(l-k, r-k)}^{n-k}$. Furthermore, when $H_{u-1}^\Gamma = 0$ we have $H_{u-1}^A = 0$ and $H_u^A \simeq H_u^\Gamma \oplus H_{(l-k, r-k)}^{n-k}$ just as for Good Gluing.

Remarks. (1) The condition $H_{u-1}^\Gamma = 0$ at the end of Theorem 4.2 will be important later on and we refer to it as the *special case* of the theorem.

(2) Using dimensions instead of positions the sequence above becomes

$$\begin{aligned} 0 \leftarrow H_{x,y}^A \leftarrow H_{x,y}^\Gamma \oplus H_{x,y}^n \leftarrow H_{x,y}^n \oplus H_{(l-k, r-k)}^{n-k} \\ \leftarrow H_{x+p-y, p-y}^A \leftarrow H_{x+p-y, p-y}^\Gamma \leftarrow 0 \end{aligned}$$

where $n + k - 2p < 2x - y < n + k - p$.

Together these theorems cover all congruences for w and k . Their proofs use similar ideas, following those of Theorem 3.1, and are presented together.

Proof of Theorems 4.1 and 4.2. We use the symbols $A, \mathcal{A}, B, \mathcal{B}$ and \mathcal{C} exactly as in the proof of Theorem 3.1. In addition we write $\mathcal{M}_{(l,r)}^\Gamma$ and $\mathcal{M}_{(l,r)}^A$ as

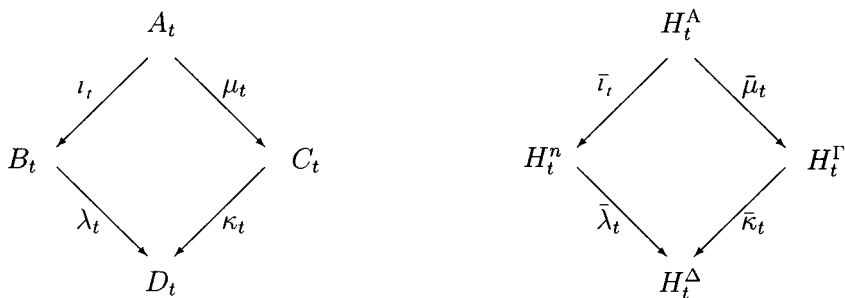
$$\begin{aligned} \mathcal{C}: \dots \leftarrow C_{t-1} \leftarrow C_t \leftarrow C_{t+1} \leftarrow \dots \quad \text{and} \\ \mathcal{D}: \dots \leftarrow D_{t-1} \leftarrow D_t \leftarrow D_{t+1} \leftarrow \dots \end{aligned}$$

where t indicates the position of a module. Since $A \simeq B \cap \Gamma$ we have an exact Mayer–Vietoris sequence

$$0 \leftarrow \mathcal{D} \xleftarrow{\psi} \mathcal{C} \oplus \mathcal{B} \xleftarrow{\phi} \mathcal{A} \leftarrow 0.$$

This sequence of complexes can be written as the commutative diagram shown in Fig. 4.

Here the homomorphisms ϕ_t and ψ_t are defined as follows (see [13, p. 143]). Let $\iota_t, \kappa_t, \lambda_t$, and μ_t be the inclusion mappings as indicated:



Then $\phi_t(a) = (\iota_t(a), -\mu_t(a))$ and $\psi_t(b, c) = \lambda_t(b) + \kappa_t(c)$.

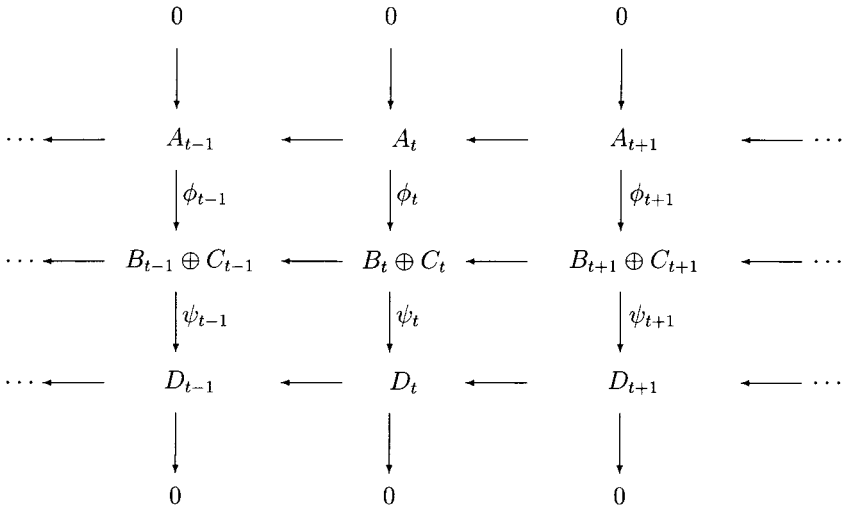


FIG. 4. Mayer-Vietoris Sequence.

So we obtain the exact sequence

$$\begin{aligned}
 \cdots \leftarrow H_{t-1}^A \leftarrow H_{t-1}^\Gamma \oplus H_{t-1}^n \leftarrow H_{t-1}^A \\
 \leftarrow H_1^A \xleftarrow{\bar{\psi}_t} H_t^\Gamma \oplus H_t^n \xleftarrow{\bar{\phi}_t} H_t^A \\
 \leftarrow H_{t+1}^A \leftarrow H_{t+1}^\Gamma \oplus H_{t+1}^n \leftarrow H_{t+1}^A \leftarrow \cdots
 \end{aligned}$$

where $\bar{\phi}_t$ and $\bar{\psi}_t$ are induced by ϕ_t and ψ_t . It is easy to note that $\bar{\phi}_t = (\bar{i}_t, -\bar{\mu}_t)$. Crucial now is the following observation: *The map \bar{i}_t is the same as in the proof of Theorem 3.1. Moreover, $\text{Ker } \bar{\phi}_t = \text{Ker } \bar{i}_t \cap \text{Ker } \bar{\mu}_t$ and so*

- if \bar{i}_t is an injection then $\bar{\phi}_t$ is also an injection;
- if $\bar{\mu}_t = 0$ then $\text{Ker } \bar{\phi}_t = \text{Ker } \bar{i}_t$;
- if $\bar{i}_t = 0$ then $\text{Ker } \bar{\phi}_t = \text{Ker } \bar{\mu}_t$.

We consider now the same 6 cases as in the proof of Theorem 3.1.

Case 1. When $k \equiv w \equiv 0 \pmod{p}$. It follows from Theorem 3.1 that \mathcal{A} and \mathcal{B} are exact and so \mathcal{D} has the same homologies as \mathcal{C} .

Case 2. When $k \equiv w \not\equiv 0 \pmod{p}$. Here both \mathcal{A} and \mathcal{B} are almost exact with the same non-trivial homology $H_d^A \simeq H_d^B = H_{(l,r)}^n$. We have the 5-term sequence

$$0 \leftarrow H_d^A \leftarrow H_d^\Gamma \oplus H_{(l,r)}^n \xleftarrow{\bar{\phi}_d} H_d^A \leftarrow H_{d+1}^A \leftarrow H_{d+1}^\Gamma \leftarrow 0,$$

and we know from Theorem 3.1 that $\text{Ker } \bar{i}_d = 0$. Hence also $\text{Ker } \bar{\phi}_d = 0$ and so the sequence above is in fact the short exact sequence

$$0 \leftarrow H_d^A \leftarrow H_d^\Gamma \oplus H_{(l,r)}^n \leftarrow H_{(l,r)}^n \leftarrow 0.$$

So again the homologies of Γ and Δ are the same and we have proved the next general fact: *If $k \equiv w \pmod{p}$ then k -gluing of a simplex to an $(n - 1)$ -dimensional complex does not change the p -modular homologies in any sequence of weight w .*

Case 3. When $w = p$ but $k \not\equiv w \pmod{p}$. Now \mathcal{B} is exact and \mathcal{A} is almost exact with non-trivial homology $H_{u-1}^A \simeq H_{u-s}^E = H_{(l-k,r-k)}^{n-k}$. So we have the 5-term exact sequence

$$0 \leftarrow H_{u-1}^A \leftarrow H_{u-1}^\Gamma \xleftarrow{\bar{\phi}_{u-1}} H_{u-1}^A \leftarrow H_u^A \leftarrow H_u^\Gamma \leftarrow 0.$$

Moreover, we know that $\bar{i}_{u-1} = 0$ and so $\text{Ker } \bar{i}_{u-1} = H_{u-1}^A \simeq H_{(l-k,r-k)}^{n-k}$. Unfortunately, in this case of Bad Gluing we have no additional information for arbitrary Γ . Nevertheless, in the particular case when $H_{u-1}^\Gamma = 0$ we obtain the short exact sequence

$$0 \leftarrow H_{(l-k,r-k)}^{n-k} \leftarrow H_u^A \leftarrow H_u^\Gamma \leftarrow 0,$$

and so $H_{u-1}^A = 0$ and $H_u^A = H_u^\Gamma \oplus H_{(l-k,r-k)}^{n-k}$.

Case 4. When $w < p$ and $k \not\equiv w \pmod{p}$. Here we assume that \mathcal{B} is almost exact (with non-trivial homology H_d^A at the center of the diagram above) and also \mathcal{C} is almost exact with non-trivial homology $H_{u-s}^E := H_{(l-k,r-k)}^{n-k}$ at the distance $u - d > 0$ from H_d^A . Again there are the same 3 subcases as in the proof of Theorem 3.1, and we look at the last Case 4.3 first:

Case 4.3. When $1 \leq k < w \leq p - 1$ or equivalently $u = d$. Here \mathcal{A} is almost exact with $H_d^A = H_{(l,r)}^n / H_{(l-k,r-k)}^{n-k}$ and $\text{Ker } \bar{\phi}_d = \text{Ker } \bar{i}_d = 0$. (Note that if $p = 3$ then \mathcal{A} will be exact.) So from the sequence

$$0 \leftarrow H_d^A \leftarrow H_d^\Gamma \oplus H_{(l,r)}^n \leftarrow H_{(l,r)}^n / H_{(l,r)}^{n-k} \leftarrow 0,$$

it follows that $H_d^A = H_d^\Gamma \oplus H_{(l-k,r-k)}^{n-k}$. This is an instance of Good Gluing.

Case 4.1. When $w < k < p + w < 2p$ or equivalently $u = d + 1$. Now \mathcal{A} is almost exact with homology $H_d^A = H_{(l,r)}^n \oplus H_{(l-k,r-k)}^{n-k}$. The map \bar{i}_d is a surjection and so in general we only know that $\text{Ker } \bar{\phi}_d = \text{Ker } \bar{i}_d \cap \text{Ker } \bar{\mu}_d \subseteq \text{Ker } \bar{i}_d = H_{(l-k,r-k)}^{n-k}$. The corresponding 5-term exact sequence is

$$0 \leftarrow H_d^A \leftarrow H_d^\Gamma \oplus H_{(l,r)}^n \xleftarrow{\bar{\phi}_d} H_d^A \leftarrow H_u^A \leftarrow H_u^\Gamma \leftarrow 0.$$

If we suppose that $H_{u-1}^\Gamma = 0$ then $\text{Ker } \bar{\phi}_d = \text{Ker } \bar{i}_d = H_{(l-k, r-k)}^{n-k}$ and the sequence above can be decomposed into the 4-term sequence

$$0 \leftarrow H_d^A \leftarrow H_d^\Gamma \oplus H_{(l,r)}^n \leftarrow H_{(l-k, r-k)}^n \oplus H_{(l-k, r-k)}^{n-k} \leftarrow H_{(l-k, r-k)}^{n-k} \leftarrow 0$$

and the 3-term sequence

$$0 \leftarrow H_{(l-k, r-k)}^{n-k} \leftarrow H_u^A \leftarrow H_u^\Gamma \leftarrow 0.$$

Hence in this case $H_d^A = H_d^\Gamma = 0$ and $H_u^A = H_u^\Gamma \oplus H_{(l-k, r-k)}^{n-k}$.

Case 4.2. When $k > w + p$ and $k \not\equiv w \pmod{p}$ or equivalently $u \geq d + 2$. First we look at the more complicated case of $u = d + 2$. Here there are two neighbouring homologies in \mathcal{A} , namely $H_d^A = H_{(r,l)}^n$ and $H_{d+1}^A = H_{(r-k, l-k)}^{n-k}$. So one needs to consider the 8-term exact sequence

$$\begin{aligned} 0 \leftarrow H_d^A \leftarrow H_d^\Gamma \oplus H_{(l,r)}^n &\xleftarrow{\bar{\phi}_d} H_d^A \\ \leftarrow H_{d+1}^A \leftarrow H_{d+1}^\Gamma &\xleftarrow{\bar{\phi}_{d+1}} H_{d+1}^A \leftarrow H_u^A \leftarrow H_u^\Gamma \leftarrow 0. \end{aligned}$$

Here \bar{i}_d is an isomorphism (so that $\text{Ker } \bar{\phi}_d = 0$) and \bar{i}_{d+1} is a surjection with kernel $H_{(l-k, r-k)}^{n-k}$. So $H_d^A = H_d^\Gamma$ and we obtain the same 5-term exact sequence as in the Case 4.1. Hence this case is just the composition of Cases 4.1 and 4.3. The same holds also for $u > d + 2$.

Collecting together Cases 1, 2, and 4.1, we obtain the statement of Theorem 4.1; the others prove Theorem 4.2. ■

5. THE HOMOLOGY OF k -SHELLABLE COMPLEXES

We shall now apply our results to the complexes we have considered in Section 2.1.

THEOREM 5.1. *Let Δ be a k -shellable $(n - 1)$ -dimensional complex with h -vector $(h_0, \dots, h_k, 0, \dots, 0)$. Let $\mathcal{M}_{(l,r)}^A$ be a fixed sequence and suppose that its weight is at least k . Then it is almost p -exact with homology*

$$H_{(l,r)}^A \simeq \bigoplus_{j=0}^k (H_{(l-j, r-j)}^{n-j})^{h_j}$$

of dimension

$$\beta_{(l,r)}^A = \sum_{j=0}^k h_j \beta_{(l-j, r-j)}^{n-j} = \left| \sum_{t=-\infty}^{+\infty} f_{l-pt} - f_{r-pt} \right|.$$

The position of $H_{(l,r)}^A$ is obtained as the minimum among $d_{(l,r)}^n$ and $d_{(l,r)}^{n+1}$.

Proof. Fix a shelling $\sigma_1, \dots, \sigma_m$ of Δ and define complexes $\Delta_1, \dots, \Delta_m$ by the rule

$$\Delta_1 = [\sigma_1] \simeq \Sigma^n \quad \text{and} \quad \Delta_{t+1} = \Delta_t \overset{k_t}{\cup} [\sigma_{t+1}] \quad \text{for } 1 < t < m$$

where $k_t \leq k$ and where we suppose that σ_{t+1} is glued to Δ_t in such a way that $\Delta_m = \Delta$. Now we can use induction: If $k \leq w < p$ we have only cases of Good Gluing and the result follows immediately from Theorem 4.1. Suppose that $k \leq w = p$. This is Case 3 in the proof of the Bad Gluing Theorem 4.2. Nevertheless, all homologies of Δ_1 are trivial, and so the *special case* of the theorem applies to Δ_2 . By induction the special case now applies to all Δ_t 's. The second equality for the Betti number is the Euler characteristic of $\mathcal{M}_{(l,r)}^A$. From the inductive process above it follows at the same time that the position of $H_{(l,r)}^A$ is the minimum among $d_{(l,r)}^n, d_{(l,r)}^{n+1}, \dots, d_{(l,r)}^{n+k}$. In this sequence, by definition, no two consecutive terms can be ∞ and so only the first two terms are relevant. ■

COROLLARY 5.2. *Every k -shellable complex with $k \leq p$ has almost exact p -homological sequences.*

The homology of 1-shellable complexes can be fully determined. In view of the last part of Theorem 5.1 it is useful to call $d := \min\{d_{(l,r)}^n, d_{(l,r)}^{n+1}\}$ the *middle position* or just the *middle* of $\mathcal{M}_{(l,r)}^A$.

COROLLARY 5.3. *Let Δ be a 1-shellable $(n-1)$ -dimensional complex with m facets and let (l, r) be given. Then $\mathcal{M}_{(l,r)}^A$ is almost p -exact with homology*

$$H_{(l,r)}^A \simeq H_{(l,r)}^n \oplus (H_{(l-1,r-1)}^{n-1})^{m-1}$$

in the middle.

The next general result follows by induction and from Theorems 4.1 and 4.2. Here one only needs to verify that each bad gluing is covered by the special case in Theorem 4.2.

PROPOSITION 5.4. *Let Δ be a shellable $(n-1)$ -dimensional complex and let (l, r) be given. Then the initial part of $\mathcal{M}_{(l,r)}^A$ is exact up to its middle position, that is,*

$$\dots \leftarrow M_{d-3}^A \leftarrow M_{d-2}^A \leftarrow M_{d-1}^A \leftarrow M_d^A$$

is exact.

Written in terms of dimensions instead this proposition becomes

PROPOSITION 5.4'. *Let Δ be a shellable $(n-1)$ -dimensional complex. If $0 < i < p$ and $2j-i < n$ then*

$$\cdots \leftarrow M_{j-2p}^A \leftarrow M_{j-i-p}^A \leftarrow M_{j-p}^A \leftarrow M_{j-i}^A \leftarrow M_j^A$$

is exact.

From this proposition one can also compute the Brauer character of the automorphism group $\text{Aut}(\Delta)$ of Δ in its action on the kernel of ∂^i for all $i < p$, see also our paper [11]. If $g \in \text{Aut}(\Delta)$ then g permutes the faces of Δ and the number of k -vertex faces fixed by g is denoted $\text{fix}(g, k)$.

COROLLARY 5.5. *Let Δ be a shellable $(n-1)$ -dimensional complex, let K be the kernel of ∂^i on M_j^A , and let $G \subseteq \text{Aut}(\Delta)$. If $0 < i < p$ and $2j-i < n$ then*

$$\chi(g, K) = \pm \sum_{t \geq 0} \text{fix}(g, j-tp) - \text{fix}(g, j-i-tp).$$

The \pm sign is not an ambiguity but can be decided from the condition $\chi(1, K) > 0$, see also the next corollary. The same sequences can be used too to determined the p -rank of the incidence matrices attached to the complex. The following generalizes the rank formula of Wilson in [17].

COROLLARY 5.6 *Let Δ be a shellable $(n-1)$ -dimensional complex with f -vector (f_0, f_1, \dots, f_n) . If $0 < i < p$ and $2j-i < n$ then $f_{j-i} - f_{j-p} + f_{j-p-i} - f_{j-2p} + \cdots$ is the p -rank of $\partial^i: M_j^A \rightarrow M_{j-i}^A$.*

As in [9, 10] one can use the exactness of the sequences also to determine the p -rank for the orbit inclusion matrix for any group whose order is not divisible by p .

Remark. Using the results of this paper it is possible to determine the modular homology of finite buildings. This is the subject of a forthcoming paper [12].

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