Semisimple strata for p-adic classical groups

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Abstract

Let F_0 be a non-archimedean local field, of residual characteristic different from 2, and let G be a unitary, symplectic or orthogonal group defined over F_0. In this paper, we prove some fundamental results towards the classification of the representations of G via *types* ([8]). In particular, we show that any positive level supercuspidal representation of G contains a *semisimple skew stratum*, that is, a special character of a certain compact open subgroup of G. The intertwining of such a stratum has been calculated in [19].

Résumé

Soit F_0 un corps local non-archimédien, de caractéristique résiduelle différente de 2, et soit G un groupe unitaire, symplectique ou orthogonal défini sur F_0. Dans cet article, nous démontrons des résultats fondamentals pour la classification des représentations de G par les *types* ([8]). En particulier, nous démontrons que toute représentation supercuspidale de G de niveau strictement positif contienne une *strate gauche semisimple*, c'est-à-dire, un caractère particulier d'un certain sous-groupe ouvert compact de G. L'entrelacement d'une telle strate a été calculé dans [19].

1 Introduction

Let F be a non-archimedean local field of residual characteristic different from 2, equipped with a galois involution with fixed field F_0 (here, we allow the possibility $F_0 = F$). Let V be an N-dimensional vector space over F and let h be a nondegenerate ϵ-hermitian form on V. We put $A = \mathrm{End}_F V$ and let $^-$ be the adjoint involution on A induced by h. Put $\widetilde{G} = \mathrm{Aut}_F V$ and let σ be the involution of $\widetilde{G}$ given by $g \mapsto \overline{g}^{-1}$, for $g \in \widetilde{G}$; σ also acts on the Lie algebra A via the differential, $x \mapsto -\overline{x}$. Finally, we put $G = \widetilde{G}^{\sigma}$, the

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fixed points of σ in $\widetilde{G}$, a unitary group defined over F_0 (possibly symplectic or orthogonal) and $A_- = A^\sigma$.

We are seeking a classification of the representations of G via the theory of *types* ([8]). Let π be an irreducible smooth complex representation of G. The representations of level zero of any connected reductive group have been classified by Morris [15] and Moy & Prasad [17] so we will only consider positive level representations here.

A basic result of Moy & Prasad [16] states that π contains an *unrefined minimal K-type*, that is, a certain character of a compact open subgroup of G. In this paper we both refine and make explicit these constructions.

Let Λ be a self-dual lattice sequence in V (see §2.1). Associated to Λ, we have a parahoric subgroup P, equipped with a filtration by normal open subgroups P^{n+1}, $n \geq 0$. The characters of P^n trivial on P^{n+1} are parametrized by *skew strata* $[\Lambda, n, n-1, b]$, for certain $b \in A_-$, and associated to each skew stratum is a characteristic polynomial $\varphi(X)$. We call the stratum *fundamental* if $\varphi(X) \neq X^N$.

In [18], Pan & Yu show that an unrefined minimal K-type is precisely a fundamental skew stratum (cf. also Morris [13]). In fact we can also deduce that π contains some fundamental skew stratum from our results here , using the notion of "optimal points" from [16].

We prove the following two results:

1. If π contains a skew stratum whose characteristic polynomial has a factor which is not fixed (upto sign) by σ (we call such a stratum *G-split*) then π is not supercuspidal (cf. [11], [7], [3], [12]).

2. Otherwise π contains a "refined" fundamental stratum, called a *semisimple* skew stratum.

The notion of semisimple here was proposed by the author in [19], [20] and generalizes that of a *simple stratum* (for $\widetilde{G}$) from [6]; it is an orthogonal direct sum of simple or null skew strata which have coprime characteristic polynomials.

In particular, these results imply that any positive level irreducible supercuspidal representation of G contains a semisimple skew stratum. Moreover the intertwining of such a stratum is computable and conforms to the general philosophy of reducing to a smaller reductive group (see [19]). It makes sense, therefore, to think of this as the first step of an iterative process leading, eventually, to a full classification of the irreducible representations of classical groups. Furthermore, all of this strictly parallels the constructions for $\widetilde{G}$ in [6], both formally and by explicit transfer.

The first result is proved using the method of *covers* ([8]), following the techniques of [7] §3 (see also [3] §2). In particular, the idea of using covers here is due to Bushnell. The spirit of the proof is also the same as the very general result of [12]; however, the language used there is very different and a comparison of the notions of "split" has not been done.

Part of this work formed a section of my doctoral thesis, although the proofs have changed considerably since then. I would like to thank my supervisor, Colin Bushnell, for setting me on this project and for his support and encouragement. Thanks also to Gopal Prasad for some very useful discussions. Particular thanks are due to Paul Broussous, for many explanations and conversations; indeed, this paper owes a great debt to [3].

2 Preliminaries and statement of results

2.1 Lattice sequences

Let F be a non-archimedean local field equipped with a galois involution $^-$ with fixed field F_0; we allow the possibility $F = F_0$. Let $\mathfrak{o}_F$ be the ring of integers of F, $\mathfrak{p}_F$ its maximal ideal and $k_F = \mathfrak{o}_F/\mathfrak{p}_F$ the residue field, of characteristic different from 2. We denote $\mathfrak{o}_0$, $\mathfrak{p}_0$, k_0 the same objects in F_0, and will use similar notation for any non-archimedean local field. We fix a uniformizer ϖ_F of F such that $\overline{\varpi_F} = -\varpi_F$ if F/F_0 is ramified, $\overline{\varpi_F} = \varpi_F$ otherwise. We put $\varpi_0 = \varpi_F^2$ if F/F_0 is ramified, $\varpi_0 = \varpi_F$ otherwise; so ϖ_0 is a uniformizer of F_0.

Let V be an N-dimensional vector space over F, equipped with a non-degenerate ϵ-hermitian form, with $\epsilon = \pm 1$. We put $A = \mathrm{End}_F V$ and denote by $^-$ the adjoint (anti-)involution on A induced by h. Set also $\widetilde{G} = \mathrm{Aut}_F V$ and let σ be the involution given by $g \mapsto \overline{g}^{-1}$, for $g \in \widetilde{G}$. We also have an action of σ on the Lie algebra A given by $a \mapsto -\overline{a}$, for $a \in A$ (this is the differential of the action on $\widetilde{G}$). We put $\Sigma = \{1, \sigma\}$, where 1 acts as the identity on both $\widetilde{G}$ and A.

We put $G = \widetilde{G}^\Sigma = \{g \in \widetilde{G} : h(gv, gw) = h(v, w) \text{ for all } v, w \in V\}$, a unitary, symplectic or orthogonal group over F_0, and $A_- = A^\Sigma \simeq \mathrm{Lie}\, G$.

Recall from [7] §2, that an $\mathfrak{o}_F$*-lattice sequence* in V is a function Λ from $\mathbb{Z}$ to the set of $\mathfrak{o}_F$-lattices in V such that

(i) $\Lambda(k) \subset \Lambda(j)$, for $k \geq j$;

(ii) there exists a positive integer $e = e(\Lambda|\mathfrak{o}_F)$, called the $\mathfrak{o}_F$*-period* of Λ, such that $\varpi_F \Lambda(k) = \Lambda(k + e)$, for all $k \in \mathbb{Z}$.

An $\mathfrak{o}_F$-lattice sequence Λ is called *strict* if $\Lambda(k) \neq \Lambda(k+1)$, for all $k \in \mathbb{Z}$. Note also that an $\mathfrak{o}_F$-lattice sequence is certainly an $\mathfrak{o}_0$-lattice sequence.

We also recall the definition of the direct sum of two lattice sequences: if $V = V_1 \oplus V_2$ and, for $i = 1, 2$, Λ^i is an $\mathfrak{o}_F$-lattice sequence in V_i of $\mathfrak{o}_F$-period

e, then the direct sum $\Lambda = \Lambda^1 \oplus \Lambda^2$ is given by

$$\Lambda(k) = \Lambda^1(k) \oplus \Lambda^2(k), \qquad \text{for all } k \in \mathbb{Z}.$$

For L an $\mathfrak{o}_F$-lattice in V, we put $L^\# = \{v \in V : h(v, L) \subset \mathfrak{p}_F\}$. Then we call an $\mathfrak{o}_F$-lattice sequence Λ *self-dual* if there exists $d \in \mathbb{Z}$ such that $\Lambda(k)^\# = \Lambda(d-k)$, for all $k \in \mathbb{Z}$. If $\Lambda = \Lambda^1 \oplus \Lambda^2$ is a direct sum of self-dual $\mathfrak{o}_F$-lattice sequences such that $V = V_1 \perp V_2$, then we write $\Lambda = \Lambda^1 \perp \Lambda^2$.

Given an $\mathfrak{o}_F$-lattice sequence Λ in V, there are two operations we can apply to it. First, we have "translation": for $k_0 \in \mathbb{Z}$ we define Λ' by $\Lambda'(k) = \Lambda(k+k_0)$, for $k \in \mathbb{Z}$. Second, we have "normalization": for $m \in \mathbb{N}$, we define Λ' by $\Lambda'(k) = \Lambda([\frac{k}{m}])$, for $k \in \mathbb{Z}$, where $[x]$ denotes the greatest integer not greater than x. These two operations do not change the associated objects which we describe below, except upto a renormalization of the index. In particular, we may assume, where necessary, that a self-dual $\mathfrak{o}_F$-lattice sequence Λ in V is normalized such that $\Lambda(k)^\# = \Lambda(1-k)$.

Associated to an $\mathfrak{o}_F$-lattice sequence Λ in V, we have a decreasing filtration $\{\mathfrak{a}_n(\Lambda) : n \in \mathbb{Z}\}$ of A by $\mathfrak{o}_F$-lattices, given by

$$\mathfrak{a}_n = \mathfrak{a}_n(\Lambda) = \{x \in A : a\Lambda(k) \subset \Lambda(k+n) \text{ for all } k \in \mathbb{Z}\}.$$

Moreover, $\mathfrak{a}_0$ is a hereditary $\mathfrak{o}_F$-order in A and $\mathfrak{a}_1$ is its Jacobson radical. If Λ is self-dual, then each $\mathfrak{a}_n(\Lambda)$ is fixed by σ and we put $\mathfrak{a}_n^- = \mathfrak{a}_n^-(\Lambda) = \mathfrak{a}_n(\Lambda)^\Sigma = \mathfrak{a}_n(\Lambda) \cap A_-$, which gives a filtration of A_- by $\mathfrak{o}_F$-lattices.

The filtration on A gives rise to a valuation ν_Λ on A, by

$$\nu_\Lambda(x) = \sup\{k \in \mathbb{Z} : x \in \mathfrak{a}_n(\Lambda)\},$$

with the understanding that $\nu_\Lambda(0) = +\infty$. If Λ is self-dual then ν_Λ is fixed by σ.

Given an $\mathfrak{o}_F$-lattice sequence Λ, we also put

$$\begin{aligned} U &= U(\Lambda) = \mathfrak{a}_0(\Lambda)^\times; \\ U_n &= U_n(\Lambda) = 1 + \mathfrak{a}_n(\Lambda), \qquad \text{for } n \geq 1. \end{aligned}$$

Then U is a compact open subgroup of $\widetilde{G}$ and $\{U_n : n \geq 1\}$ is a filtration by normal subgroups. Moreover, for all $n \geq 1$, we have an isomorphism

$$\mathfrak{a}_n/\mathfrak{a}_{n+1} \xrightarrow{\sim} U_n/U_{n+1} \tag{2.1}$$

induced by $x \mapsto 1 + x$. If Λ is self-dual, then U, U_n are fixed by σ and we put

$$\begin{aligned} P &= P(\Lambda) = U(\Lambda)^\Sigma = U(\Lambda) \cap G; \\ P_n &= P_n(\Lambda) = U_n(\Lambda)^\Sigma = U_n(\Lambda) \cap G, \qquad \text{for } n \geq 1. \end{aligned}$$

As before, P is a compact open subgroup of G, with a filtration by normal subgroups P_n, and the isomorphism (2.1) induces an isomorphism ([14] (2.1.4)(b))

$$\mathfrak{a}_n^-/\mathfrak{a}_{n+1}^- \xrightarrow{\sim} P_n/P_{n+1}. \tag{2.2}$$

We define the normalizer $\mathfrak{K}(\Lambda)$ to be

$$\mathfrak{K}(\Lambda) = \bigcap_{n\in\mathbb{Z}} N_{\widetilde{G}}(U_n(\Lambda)),$$

where $N_{\widetilde{G}}$ denotes the normalizer in $\widetilde{G}$. An element $x \in A$ is called Λ-*invertible* if $x \in \mathfrak{K}(\Lambda)$; equivalently, if $x\Lambda(k) = \Lambda(k + \nu_\Lambda(x))$, for all $k \in \mathbb{Z}$. Finally, note that if Λ is self-dual, then we have $\mathfrak{K}(\Lambda) \cap G = P(\Lambda)$.

Now we turn our attention to the characters of the groups U_n, P_n. We fix ψ_0 an additive character of F_0 with conductor $\mathfrak{p}_0$ and put $\psi_F = \psi_0 \circ \mathrm{tr}_{F/F_0}$, where tr denotes trace. Since F/F_0 is at worst tamely ramified, ψ_F has conductor $\mathfrak{p}_F$. We also set $\psi_A = \psi_F \circ \mathrm{tr}_{A/F}$.

For S an $\mathfrak{o}_F$-lattice in A, we put

$$S^* = \{x \in A : \psi_A(xS) = 1\}.$$

Then, for Λ a lattice sequence, we have $\mathfrak{a}_n(\Lambda)^* = \mathfrak{a}_{1-n}(\Lambda)$, by [7] (2.10). If S is fixed by σ, then, putting $S_- = S \cap A_-$, we have

$$S^* \cap A_- = \{x \in A_- : \psi_A(xS_-) = 1\}. \tag{2.3}$$

Let $\widehat{\ }$ denote the Pontrjagin dual. Then, for Λ an $\mathfrak{o}_F$-lattice sequence in V and $n \geq 1$, we obtain a $\mathfrak{K}(\Lambda)$-equivariant isomorphism

$$\begin{aligned} \mathfrak{a}_{-n}/\mathfrak{a}_{1-n} &\xrightarrow{\sim} (U_n/U_{n+1})\widehat{\ }, \\ b + \mathfrak{a}_{1-n} &\mapsto (\psi_b : x \mapsto \psi_A(b(x-1)), \text{ for } x \in U_n). \end{aligned} \tag{2.4}$$

Moreover, if Λ is self-dual then (by [16] (4.19)) this restricts to a $P(\Lambda)$-equivariant isomorphism

$$\begin{aligned} \mathfrak{a}_{-n}^-/\mathfrak{a}_{1-n}^- &\xrightarrow{\sim} (P_n/P_{n+1})\widehat{\ }, \\ b + \mathfrak{a}_{1-n}^- &\mapsto (\psi_b^- : x \mapsto \psi_A(b(x-1)), \text{ for } x \in P_n). \end{aligned} \tag{2.5}$$

2.2 Strata

Definition 2.6 ([6] (1.5), [7] (3.1)) (i) A *stratum* in A is a 4-tuple $[\Lambda, n, n-1, b]$, where Λ is an $\mathfrak{o}_F$-lattice sequence, $n \geq 1$ is an integer and $b \in \mathfrak{a}_{-n}(\Lambda)$.

(ii) Two strata $[\Lambda, n, n-1, b_i]$, $i = 1, 2$, are called *equivalent* if $b_1 - b_2 \in \mathfrak{a}_{1-n}(\Lambda)$.

(iii) A stratum $[\Lambda, n, n-1, b]$ is called *skew* if Λ is self-dual and $b \in A_-$.

Then, by (2.4), an equivalence class of strata corresponds to a character of $U_n(\Lambda)$ and, by (2.5), an equivalence class of skew strata corresponds to a character of $P_n(\Lambda)$.

Let $[\Lambda, n, n-1, b]$ be a stratum in A. Put $y_b = \varpi_F^{n/g} b^{e/g} \in \mathfrak{a}_0(\Lambda)$, where $e = e(\Lambda)$ and $g = (n, e)$. Let $\Phi(X) \in \mathfrak{o}_F(X)$ be the characteristic polynomial of y_b. Then we define the *characteristic polynomial* $\varphi_b(X) \in k_F[X]$ of the stratum to be the reduction modulo $\mathfrak{p}_F$ of $\Phi(X)$. Note that this depends only on the equivalence class of the stratum and is, moreover, an intertwining invariant.

Definition 2.7 ([6] (2.3)) (i) A stratum $[\Lambda, n, n-1, b]$ in A is called *fundamental* if $\varphi_b(X) \neq X^N$.

(ii) A stratum $[\Lambda, n, n-1, b]$ in A is called *split* if $\varphi_b(X)$ has two coprime factors.

Let $f(X)$ be a polynomial with coefficients in F or k_F, written $f(X) = \sum_{i=0}^{n} a_i X^i$. We define the polynomial $\overline{f}(X)$ by $\overline{f}(X) = \sum_{i=0}^{n} \overline{a}_i X^i$.

Now suppose that $[\Lambda, n, n-1, b]$ is a skew stratum in A. Then we have $y_b = \eta \overline{y}_b$, for $\eta = \pm$ a sign (precisely, $\eta = (-)^{e/g}$ if F/F_0 is unramified, $\eta = (-)^{n/g}(-)^{e/g}$ otherwise), and thus $\Phi(X) = \overline{\Phi}(\eta X)$ and $\varphi_b(X) = \overline{\varphi}_b(\eta X)$. Then, if we have a factorization $\Phi(X) = \Phi_1(X)\Phi_2(X)$, we have $\Phi(X) = \overline{\Phi}(\eta X) = \overline{\Phi}_1(\eta X)\overline{\Phi}_2(\eta X)$ so $\overline{\Phi}_1(\eta X)$ is also a factor of $\Phi(X)$. The same applies to $\varphi_b(X)$.

Definition 2.8 We say that the skew stratum $[\Lambda, n, n-1, b]$ is *G-split* if $\varphi_b(X)$ has an irreducible factor $\psi(X)$ such that $(\psi(X), \overline{\psi}(\eta X)) = 1$.

Note that a G-split stratum is necessarily fundamental, since we have $\psi(X) \neq X$. Further, a G-split stratum is split, since, by the argument above, $\overline{\psi}(\eta X)$ is also a factor of $\varphi_b(X)$.

Definition 2.9 ([6] (1.5.5). [7] (5.1)) A stratum $[\Lambda, n, n-1, b]$ in A is called *simple* if

(i) the algebra $E = F[b]$ is a field;

(ii) Λ is an $\mathfrak{o}_E$-lattice chain;

(iii) $\nu_\Lambda(b) = -n$;

(iv) b is *minimal*, that is, writing $e = e(E|F)$ for the ramification index and $\nu = \nu_E(b)$ for the normalized valuation of b in E, we have

(a) $\gcd(\nu, e) = 1$;

(b) $\varpi_F^{-\nu} b^e + \mathfrak{p}_E$ generates the residue field extension k_E/k_F.

Simple strata play an important role in the construction and classification of the representations of $\widetilde{G}$ because the associated characters have a "nice" intertwining formula.

Definition 2.10 (cf. [19] (3.8)) A skew stratum $[\Lambda, n, n-1, b]$ in A is called *semisimple* if either it is simple or we have a non-trivial splitting $V = V_0 \perp \cdots \perp V_r$ such that all the following hold:

(i) $\Lambda = \Lambda^0 \perp \cdots \perp \Lambda^r$, where $\Lambda^i(k) = \Lambda(k) \cap V_i$, for $i = 0, \ldots, r$;

(ii) $b = b_0 + \cdots + b_r$, where $b_i = b|_{V_i}$, for $i = 0, \ldots, r$;

(iii) the polynomials $\varphi_{b_i}(X)$ are pairwise coprime;

(iv) the strata $[\Lambda^i, n, n-1, b_i]$ in $\mathrm{End}_F(V_i)$ are simple, with the possible exception that $b_0 = 0$.

2.3 The Theorems

Let π be a smooth representation of G. We say that π *contains* a skew stratum $[\Lambda, n, n-1, b]$ if it contains the associated character ψ_b^- of $P_n(\Lambda)$.

Theorem 2.11 *Let π be a smooth representation of G of positive level, that is, π has no fixed vector under $P_1(\Lambda)$, for Λ any self-dual lattice sequence in V. Then π contains some fundamental skew stratum $[\Lambda, n, n-1, b]$. Moreover, putting $e = e(\Lambda|\mathfrak{o}_F)$, $g = (n, e)$, we have $e/g \le N$.*

Proof The first assertion is given by [16] (5.2), where they call a fundamental stratum an "unrefined minimal K-type" (see [18], especially §5, for a translation into lattice-theoretic language). We remark that we could also deduce this from the results of §4 (see (4.3)).

For the second assertion, we put

$$S = \left\{(\Lambda', n') : \Lambda' \text{ is strict and } (b + \mathfrak{a}_{1-n}) \cap \mathfrak{a}'_{-n'} \neq \emptyset\right\},$$

where $\mathfrak{a}'_{-n'} = \mathfrak{a}(\Lambda')_{-n'}$. This is clearly non-empty and, for $(\Lambda', n') \in S$, Λ' is strict so $e' = e(\Lambda'|\mathfrak{o}_F) \le N$. Moreover, as in [9] (5.4) (or, more generally, [16] (6.4)), we have $n'/e' \ge n/e$ for all $(\Lambda', n') \in S$.

We choose $(\Lambda', n') \in S$ with n'/e' minimal and $b' \in (b + \mathfrak{a}_{1-n}) \cap \mathfrak{a}'_{-n'}$. If $[\Lambda', n', n'-1, b']$ is not fundamental then, by [5] Theorem 1, there exists (Λ'', n'') with Λ'' strict, $b' + \mathfrak{a}'_{1-n'} \subset \mathfrak{a}''_{-n''}$ and $n''/e'' < n'/e'$. But then $(\Lambda'', n'') \in S$, contradicting the minimality of n'/e'. Hence $[\Lambda', n', n'-1, b']$ is fundamental so, again as in [9] (5.4), we have $n'/e' = n/e$ and the result follows. ∎

Theorem 2.12 *Let π be a smooth representation of G which contains a G-split skew stratum. Then π is not supercuspidal.*

We will prove this in §3, where we construct a non-trivial Jacquet module for π.

Theorem 2.13 *Let π be a smooth representation of G which contains a non-G-split fundamental skew stratum. Then π contains a semisimple skew stratum.*

This is an easy consequence of the following proposition, which we prove in §4:

Proposition 2.14 *Let $[\Lambda, n, n-1, b]$ be a non-G-split fundamental skew stratum in A. Then there exists a semisimple skew stratum $[\Lambda', n', n'-1, \beta]$ in A such that*

$$b + \mathfrak{a}_{1-n}^{-}(\Lambda) \subset \beta + \mathfrak{a}_{1-n'}^{-}(\Lambda')$$

and $n/e(\Lambda|\mathfrak{o}_F) = n'/e(\Lambda'|\mathfrak{o}_F)$.

In particular, these three theorems imply that any positive level supercuspidal representation of G contains a semisimple skew stratum $[\Lambda, n, n-1, \beta]$ such that $n/e(\Lambda|\mathfrak{o}_F)$ has denominator at most N (when written in its lowest terms). We also remark that these strata have a "nice" intertwining formula (see [19] (3.17)).

3 G-split strata

3.1 Intertwining

Let $[\Lambda, n, n-1, b]$ be a G-split skew stratum in A and put $y_b = \varpi_F^{n/g} b^{e/g} \in \mathfrak{a}_0(\Lambda)$, where $e = e(\Lambda)$ and $g = (n, e)$. Let $\Phi(X) \in \mathfrak{o}_F(X)$ be the characteristic polynomial of y_b and $\varphi_b(X)$ its reduction modulo $\mathfrak{p}_F$. Let $\psi(X)$ be a monic irreducible factor of $\varphi_b(X)$ such that $\psi(X) \neq \overline{\psi}(\eta X)$ and write $\varphi_b(X) = \psi(X)^s \overline{\psi}(\eta X)^s \theta(X)$, with $\theta(X)$ coprime to $\psi(X)$ and $\overline{\psi}(\eta X)$, $\theta(X) = \overline{\theta}(\eta X)$. (Note that we may have $\theta(X) = 1$ here.) By Hensel's Lemma, there exist coprime polynomials $\Psi(X)$, $\Theta(X) \in \mathfrak{o}_F[X]$, whose reductions modulo $\mathfrak{p}_F$ are $\psi(X)^s$, $\theta(X)$ respectively, such that $\Phi(X) = \Psi(X)\overline{\Psi}(\eta X)\Theta(X)$.

We put $V_1 = \ker \Psi(y_b)$, $V_{-1} = \ker \overline{\Psi}(\eta y_b)$ and $V_0 = \ker \Theta(y_b)$. These spaces are preserved by b and we have

$$V = V_0 \perp (V_1 \oplus V_{-1})$$

and V_1, V_{-1} are totally isotropic and in duality with respect to h. (Note that, if $\theta(X)=1$ then $V_0=0$.) For $i=-1,0,1$, we define lattice sequences Λ^i in V_i by $\Lambda^i(k)=\Lambda(k)\cap V_i$, for $k\in\mathbb{Z}$. Then, as in [11] (3.5)(3.6), we have

$$\Lambda(k)=\Lambda^1(k)\oplus\Lambda^0(k)\oplus\Lambda^{-1}(k), \qquad \text{for all } k\in\mathbb{Z},$$

and, putting $b_i=b|_{V_i}$ for $-1\le i\le 1$, we have that, for $i=1,-1$, b_i is Λ^i-invertible and $\nu_{\Lambda^i}(b_i)=-n$. Indeed, in this situation, $[\Lambda^i,n,n-1,b_i]$ is non-split fundamental, for $i=1,-1$.

Writing $A^{ij}=\mathrm{Hom}\,(V^j,V^i)$, by [6] (2.9) we have

$$\begin{array}{ll} \mathfrak{a}_k(\Lambda)=\bigoplus_{-1\le i,j\le 1}\mathfrak{a}_k(\Lambda)\cap A^{ij}, & \text{for all } k\in\mathbb{Z}, \\ \mathfrak{a}_k(\Lambda)\cap A^{ii}=\mathfrak{a}_k(\Lambda^i), & \text{for } -1\le i\le 1,\ k\in\mathbb{Z}. \end{array}$$

We will abbreviate $\mathfrak{a}_k^{ij}=\mathfrak{a}_k(\Lambda)\cap A^{ij}$, and in the block description

$$A=\begin{pmatrix} A^{-1,-1} & A^{-1,0} & A^{-1,1} \\ A^{0,-1} & A^{0,0} & A^{0,1} \\ A^{1,-1} & A^{1,0} & A^{1,1} \end{pmatrix},$$

we will usually omit the superscript ij.

We define $\mathfrak{o}_F$-lattices in A by

$$_q\mathfrak{h}_1=\begin{pmatrix} \mathfrak{a}_n & \mathfrak{a}_{q+1} & \mathfrak{a}_{q+1} \\ \mathfrak{a}_n & \mathfrak{a}_n & \mathfrak{a}_{q+1} \\ \mathfrak{a}_n & \mathfrak{a}_n & \mathfrak{a}_n \end{pmatrix}, \qquad {}_q\mathfrak{h}_2=\begin{pmatrix} \mathfrak{a}_{n+1} & \mathfrak{a}_{q+1} & \mathfrak{a}_{q+1} \\ \mathfrak{a}_n & \mathfrak{a}_{n+1} & \mathfrak{a}_{q+1} \\ \mathfrak{a}_n & \mathfrak{a}_n & \mathfrak{a}_{n+1} \end{pmatrix},$$

for $0\le q\le n$. We put $_qH_j=1+{}_q\mathfrak{h}_j$, for $j=1,2$, $0\le q\le n$; we abbreviate $_0H_j=H_j$ and $_0\mathfrak{h}_j=\mathfrak{h}_j$, for $j=1,2$. The sets $_qH_j$ are compact open subgroups of $\widetilde{G}$, for $j=1,2$, $0\le q\le n$, and the map $x\mapsto 1+x$ induces isomorphisms of groups

$$_q\mathfrak{h}_1/{}_q\mathfrak{h}_2\to{}_qH_1/{}_qH_2.$$

For each $q=0,\ldots,n$, we define a character ψ_b of $_qH_1$, trivial on $_qH_2$, by $\psi_b(1+x)=\psi_A(bx)$, for $x\in{}_q\mathfrak{h}_1$.

We write $M=(A^{-1,-1})^\times\times(A^{0,0})^\times\times(A^{1,1})^\times$; this is a Levi subgroup of $\widetilde{G}$. We also put $A_u=A^{-1,0}\oplus A^{-1,1}\oplus A^{0,1}$, $A_l=A^{0,-1}\oplus A^{1,-1}\oplus A^{1,0}$ and $N_u=1+A_u$, $N_l=1+A_l$.

Let K be a compact open subgroup of $\widetilde{G}$ and let ψ be a character of K. Then the $\widetilde{G}$-intertwining of ψ is defined to be

$$I_{\widetilde{G}}(\psi|K)=\left\{g\in\widetilde{G}:\psi(gkg^{-1})=\psi(k),\ \text{for all } k\in K\cap g^{-1}Kg\right\}.$$

Proposition 3.1 (cf. [7] Theorem (3.7), [3] (2.3.2)) *The $\widetilde{G}$-intertwining of the character $\psi_b|H_1$ satisfies*

$$I_{\widetilde{G}}(\psi_b|H_1)\subset H_1\cdot M\cdot H_1.$$

Proof This follows by iterating [7] (3.7), having observed (see [3] (2.3.2)), that we may indeed apply it in this, slightly more general, situation. Explicitly, we put

$$H_1' = \begin{pmatrix} 1+\mathfrak{a}_n & \mathfrak{a}_1 & \mathfrak{a}_1 \\ \mathfrak{a}_n & 1+\mathfrak{a}_n & \mathfrak{a}_n \\ \mathfrak{a}_n & \mathfrak{a}_n & 1+\mathfrak{a}_n \end{pmatrix}, \quad \text{and } M' = \begin{pmatrix} A^{-1,-1} & 0 & 0 \\ 0 & A^{0,0} & A^{0,1} \\ 0 & A^{1,0} & A^{1,1} \end{pmatrix}^{\times}.$$

By [7] (3.7), we have $I_{\widetilde{G}}(\psi_b|H_1) \subset I_{\widetilde{G}}(\psi_b|H_1') \subset H_1'M'H_1'$. Since $H_1' \subset H_1$ normalizes $\psi_b|H_1$, we in fact have $I_{\widetilde{G}}(\psi_b|H_1) \subset H_1' I_{M'}(\psi_b|H_1 \cap M')H_1'$. But, again by [7] (3.7), we have $I_{M'}(\psi_b|H_1 \cap M') \subset (H_1 \cap M')M(H_1 \cap M')$ and the result follows. ■

We observe now that all the groups ${}_qH_j$ are fixed by σ. We put ${}_qH_j^- = {}_qH_j \cap G$ and ${}_q\mathfrak{h}_j^- = {}_q\mathfrak{h}_j \cap A_-$, for $j = 1, 2$, $0 \le q \le n$, and $M^- = M \cap G$, $N_u^- = N_u \cap G$, $N_l^- = N_l \cap G$. Note that $P_u^- = M^-N_u^-$ is a maximal parabolic subgroup of G, with Levi component M^- and unipotent radical N_u^-, and $P_l^- = M^-N_l^-$ is the opposite parabolic.

We write ψ_b^- for the restriction $\psi_b|_q H_1^-$.

Proposition 3.2 *We have* $I_G(\psi_b^-|H_1^-) \subset H_1^- \cdot M^- \cdot H_1^-$.

Proof By [20] (2.5), we have $I_G(\psi_b^-|H_1^-) = I_{\widetilde{G}}(\psi_b|H_1) \cap G$ and, by [19] (2.3) (see also ibid.(4.15)), we have $H_1MH_1 \cap G = H_1^-M^-H_1^-$ so the result follows from (3.1). ■

3.2 Covers

We continue in the situation of §3.1. For $-1 \le i \le 1$, we put $G_i = (A^{ii})^\times$ and suppose that we are given:

(i) a subgroup K_1 of $U(\Lambda^1)$ containing $H_1^- \cap G_1$ and an irreducible representation ρ_1 of K_1 whose restriction to $H_1^- \cap G_1$ is a multiple of ψ_{b_1};

(ii) a subgroup K_0^- of $P(\Lambda^0)$ containing $H_1^- \cap G_0$ and an irreducible representation ρ_0^- of K_0^- whose restriction to $H_1^- \cap G_0$ is a multiple of $\psi_{b_0}^-$.

We think of K_1 embedded in G as $\left\{ \begin{pmatrix} \overline{k}^{-1} & & \\ & 1 & \\ & & k \end{pmatrix} : k \in K_1 \right\}$.

Corollary 3.3 (cf. [7] (3.9)) *(i) The set* $K^- = (K_1 \times K_0^-).H_1^-$ *is a group.*

(ii) *There is a unique irreducible representation* ρ_- *of* K^- *which is trivial on* $K^- \cap N_u^-$, $K^- \cap N_l^-$ *and whose restriction to* $K_1 \times K_0^-$ *is* $\rho_1 \otimes \rho_0^-$.

(iii) *The pair* (K^-, ρ_-) *is a* G*-cover of* $(K_1 \times K_0^-, \rho_1 \otimes \rho_0^-)$.

Proof This is identical to [7] (3.9), except we take the element ζ to be

$$\zeta = \begin{pmatrix} \varpi_F & & \\ & 1 & \\ & & \overline{\varpi_F}^{-1} \end{pmatrix}.$$

∎

3.3 Jacquet modules

We again continue with the notation of §3.1.

Lemma 3.4 (cf. [3] (2.3.9)) *Let* $1 \leq q < n$ *and put* $N_{l,q}^- = N_l^- \cap P_{n-q}(\Lambda)$. *Then the group* $N_{l,q}^-$ *normalizes* ${}_qH_1^-$ *and acts transitively by conjugation on the set of characters of* ${}_{q-1}H_1^-$ *agreeing with* ψ_b^- *on the subgroup* ${}_qH_1^-$.

Proof By [7] (3.7) Lemma 4, the map $y \mapsto yb - by$, $y \in A$, induces an isomorphism $\mathfrak{a}_{n-q} \cap A_l \to \mathfrak{a}_{-q} \cap A_l$. Moreover, this map preserves A_- and hence restricts to an isomorphism, $\mathfrak{a}_{n-q}^- \cap A_l \to \mathfrak{a}_{-q}^- \cap A_l$, and the result follows. ∎

Proposition 3.5 (cf. [3] (2.4.4)) *Let* $(\pi, \mathcal{V})$ *be a smooth representation of* G *containing the* G*-split skew stratum* $[\Lambda, n, n-1, b]$. *Then it contains the character* $\psi_b^- | H_1^-$ *also.*

Proof Given the previous lemma, this is identical to [3] (2.4.4). ∎

Theorem 3.6 (cf. [3] (2.4.2)) *Let* $(\pi, \mathcal{V})$ *be a smooth representation of* G *containing the* G*-split skew stratum* $[\Lambda, n, n-1, b]$. *Then* π *is not supercuspidal.*

Proof By the previous proposition, $(\pi, \mathcal{V})$ contains the character $\psi_b^- | H_1^-$ and, by the corollary above, (H_1^-, ψ_b^-) is a G-cover of the pair $(H_1^- \cap M, \psi_b^- | H_1^- \cap M)$. Then, by [8] (7.9), we have an isomorphism of $\mathbb{C}$-spaces

$$\mathcal{V}^{\psi_b^-} \to \mathcal{V}_u^{\psi_b^- | H_1^- \cap M},$$

where $\mathcal{V}_u$ is the Jacquet module of $\mathcal{V}$ attached to P_u^- and $\mathcal{V}^{\psi_b^-}$ is the ψ_b^--isotypic component. In particular, $\mathcal{V}_u \neq 0$ so π is not supercuspidal. ∎

Remarks 3.7 (i) To obtain this, we could have appealed to the very general result of [12]. However, the language used there is very different and a translation into the lattice-theoretic language used here has not yet been done.

(ii) Thanks to [8] (8.3), Corollary 3.3 above gives a method for constructing types for certain (non-supercuspidal) components of the Bernstein spectrum, assuming we have all the supercuspidal types (cf. [1], [2]).

4 Semisimple strata

We now turn to the proof of Proposition 2.14. We adapt the notation of [10] to our situation.

Let Λ be a self-dual $\mathfrak{o}_F$-lattice chain, normalized so that $\Lambda(k)^{\#} = \Lambda(1-k)$ for $k \in \mathbb{Z}$, and we put $e_0 = e(\Lambda|\mathfrak{o}_0)$. We consider the quotients

$$\widetilde{\Lambda}(k) = \Lambda(k)/\Lambda(k+1), \qquad k \in \mathbb{Z}.$$

Multiplication by ϖ_0 allows us to identify $\widetilde{\Lambda}(k)$ with $\widetilde{\Lambda}(k+e_0)$, for $k \in \mathbb{Z}$. Given $k \in \mathbb{Z}$, let $\widetilde{k}$ denote its image in $\mathbb{Z}/e_0\mathbb{Z}$. We put

$$\widetilde{\Lambda} = \sum_{\widetilde{k}\in\mathbb{Z}/e_0\mathbb{Z}} \widetilde{\Lambda}(\widetilde{k}),$$

a vector space over k_F, hence over k_0. We consider $\mathrm{End}_{k_F}(\widetilde{\Lambda})$; we have

$$\begin{aligned} \mathrm{End}_{k_F}(\widetilde{\Lambda}) &= \sum_{\widetilde{j}\in\mathbb{Z}/e_0\mathbb{Z}} \mathrm{End}(\widetilde{\Lambda})_{\widetilde{j}}, \qquad \text{where} \\ \mathrm{End}(\widetilde{\Lambda})_{\widetilde{j}} &= \sum_{\widetilde{k}\in\mathbb{Z}/e_0\mathbb{Z}} \mathrm{Hom}_{k_F}(\widetilde{\Lambda}(\widetilde{k}), \widetilde{\Lambda}(\widetilde{k}+\widetilde{j})). \end{aligned}$$

Further, we have $\mathrm{End}(\widetilde{\Lambda})_{\widetilde{i}}\mathrm{End}(\widetilde{\Lambda})_{\widetilde{j}} \subset \mathrm{End}(\widetilde{\Lambda})_{\widetilde{i}+\widetilde{j}}$, for $\widetilde{i}, \widetilde{j} \in \mathbb{Z}/e_0\mathbb{Z}$. Altogether, we have the structure of a $\mathbb{Z}/e_0\mathbb{Z}$-graded algebra on $\mathrm{End}_{k_F}(\widetilde{\Lambda})$.

Given an element $b \in \mathfrak{a}_{-n}(\Lambda)$, we obtain, by reduction, maps $\widetilde{b}_{\widetilde{i}} : \widetilde{\Lambda}(\widetilde{i}) \to \widetilde{\Lambda}(\widetilde{i}-\widetilde{n})$, for each $\widetilde{i} \in \mathbb{Z}/e_0\mathbb{Z}$, and hence a map

$$\widetilde{b} = \sum_{\widetilde{i}\in\mathbb{Z}/e_0\mathbb{Z}} \widetilde{b}_{\widetilde{i}} \in \mathrm{End}(\widetilde{\Lambda})_{\widetilde{-n}}.$$

We now describe the duality on $\widetilde{\Lambda}$ and $\mathrm{End}_{k_F}(\widetilde{\Lambda})$ induced by h. We have, for $k \in \mathbb{Z}$, a well-defined pairing

$$\begin{aligned} \widetilde{h}_k : \widetilde{\Lambda}(k) \times \widetilde{\Lambda}(-k) &\to k_F, \\ (v + \Lambda(k+1), v' + \Lambda(1-k)) &\mapsto h(v,v') + \mathfrak{p}_F. \end{aligned}$$

This pairing is ϵ-hermitian (or ϵ-bilinear if $k_F = k_0$) and is, moreover, nondegenerate: for $v \in \Lambda(k)$, $\widetilde{h}_k(v+\Lambda(k+1), \widetilde{\Lambda}(-k)) = 0$ implies $v \in \Lambda(-k)^\# = \Lambda(k+1)$. Multiplication by ϖ_0 transforms $\widetilde{h}_k$ into $\widetilde{h}_{k+e_0}$, for all $k \in \mathbb{Z}$, and hence we obtain nondegenerate ϵ-hermitian pairings

$$\widetilde{h}_{\widetilde{k}} : \widetilde{\Lambda}(\widetilde{k}) \times \widetilde{\Lambda}(\widetilde{-k}) \to k_F.$$

Putting these all together, we have a nondegenerate pairing

$$\widetilde{h} : \widetilde{\Lambda} \times \widetilde{\Lambda} \to k_F.$$

The pairing $\widetilde{h}$ induces an adjoint involution on $\mathrm{End}_{k_F}(\widetilde{\Lambda})$, which we denote $^{-}$. Then, for $b \in A$ we have $\overline{\widetilde{b}} = \widetilde{\overline{b}}$. In particular, if we put $\mathrm{End}^-_{k_F}(\widetilde{\Lambda}) = \{\varrho \in \mathrm{End}_{k_F}(\widetilde{\Lambda}) : \varrho + \overline{\varrho} = 0\}$, then the reduction map $^{\sim}$ sends A_- onto $\mathrm{End}^-_{k_F}(\widetilde{\Lambda})$.

Definition 4.1 Let Λ be a self-dual $\mathfrak{o}_F$-lattice sequence. We call a self-dual $\mathfrak{o}_F$-lattice sequence Λ' a *refinement* of Λ if there exists $m \in \mathbb{N}$ *odd* such that $\Lambda(k) = \Lambda'(mk)$ for all $k \in \mathbb{Z}$.

Note that a k_F-subspace $\mathcal{V}$ of $\widetilde{\Lambda}(\widetilde{k})$ corresponds to a unique $\mathfrak{o}_F$-lattice L such that $\Lambda(k) \supset L \supset \Lambda(k+1)$ (more precisely, to the set of lattices $\{\varpi_0{}^i L : i \in \mathbb{Z}\}$). Moreover, if we put $\mathcal{V}^\perp = \{\widetilde{v} \in \widetilde{\Lambda}(-\widetilde{k}) : \widetilde{h}(\mathcal{V}, \widetilde{v}) = 0\}$, then $\mathcal{V}^\perp$ corresponds to the lattice $L^\#$. In particular, a refinement Λ' of Λ corresponds to flags of k_F-subspaces

$$\widetilde{\Lambda}(\widetilde{k}) = \mathcal{W}^0_{\widetilde{k}} \supset \mathcal{W}^1_{\widetilde{k}} \supset \cdots \mathcal{W}^m_{\widetilde{k}} = 0, \qquad \text{for } \widetilde{k} \in \mathbb{Z}/e_0\mathbb{Z},$$

with $(\mathcal{W}^i_{\widetilde{k}})^\perp = \mathcal{W}^{m-i}_{-\widetilde{k}}$ and $\widetilde{\varpi_F}\mathcal{W}^i_{\widetilde{k}} = \mathcal{W}^i_{\widetilde{k}+\widetilde{e}}$, for $\widetilde{k} \in \mathbb{Z}/e_0\mathbb{Z}$, $0 \le i \le m$, where $e = e(\Lambda|\mathfrak{o}_F)$.

Proposition 4.2 *With notation as above, let $b \in \mathfrak{a}^-_{-n}(\Lambda) \setminus \mathfrak{a}^-_{1-n}(\Lambda)$. Then there exists a refinement Λ' of Λ and $n' \in \mathbb{Z}$ such that*

(i) $n'/e(\Lambda'|\mathfrak{o}_F) = n/e(\Lambda|\mathfrak{o}_F)$;

(ii) $\mathfrak{a}_{1-n}(\Lambda) \subset \mathfrak{a}_{1-n'}(\Lambda')$;

(iii) $b \in \mathfrak{a}^-_{-n'}(\Lambda')$ and the reduced map $\widetilde{b}' \in \mathrm{End}_{k_F}(\widetilde{\Lambda}')$ is semisimple.

Proof Consider the reduced map $\widetilde{b} \in \mathrm{End}_{k_F}(\widetilde{\Lambda})$ and let $\widetilde{b} = \widetilde{b}_{ss} + \widetilde{b}_{np}$ be its Jordan decomposition. We have $\widetilde{b}^m_{np} = \widetilde{0}$, for some $m \in \mathbb{Z}$, which we may (and do) assume odd, $m = 2s-1$. Then, following [18] (5.5), we put

$$\begin{aligned}
\mathcal{V}^i_{\widetilde{k}} &= (\widetilde{b}_{np})^i \widetilde{\Lambda}(\widetilde{k} + \widetilde{in}) \subset \widetilde{\Lambda}(\widetilde{k}) && \text{for } \widetilde{k} \in \mathbb{Z}/e_0\mathbb{Z},\ 0 \le i \le m; \\
\mathcal{W}^i_{\widetilde{k}} &= \textstyle\bigcap_{q-p=2(k-1-i)} (\mathcal{V}^p_{\widetilde{k}} + (\mathcal{V}^q_{-\widetilde{k}})^\perp) && \text{for } \widetilde{k} \in \mathbb{Z}/e_0\mathbb{Z},\ 0 \le i \le s-1; \\
\mathcal{W}^i_{\widetilde{k}} &= (\mathcal{W}^{m-i}_{-\widetilde{k}})^\perp && \text{for } \widetilde{k} \in \mathbb{Z}/e_0\mathbb{Z},\ s \le i \le 2s-1.
\end{aligned}$$

Thus we have flags of k_F-subspaces $\widetilde{\Lambda}(\widetilde{k}) = \mathcal{W}^0_{\widetilde{k}} \supset \cdots \mathcal{W}^m_{\widetilde{k}} = 0$, which give rise to a refinement Λ' of Λ, with

$$\Lambda'(km+i)/\Lambda(k+1) = \mathcal{W}^i_{\widetilde{k}}, \qquad \text{for } k \in \mathbb{Z},\ 0 \le i \le m-1.$$

We put $n' = nm$ and (i) is clear since $e(\Lambda'|\mathfrak{o}_F) = me(\Lambda|\mathfrak{o}_F)$.

For $a \in \mathfrak{a}_{1-n}(\Lambda)$, we have $a\Lambda'(km+i) \subset a\Lambda(k) \subset \Lambda(k-n+1) = \Lambda'(km+(m-mn))$; but $m-i-mn \ge 1-n'$ so $a \in \mathfrak{a}_{1-n'}(\Lambda')$ as required. Also, since $\widetilde{b}\mathcal{W}^i_{\widetilde{k}} \subset \mathcal{W}^i_{\widetilde{k}-\widetilde{n}}$, we have that $b\Lambda'(km+i) \subset \Lambda'((k-n)m+i)$, that is $b \in \mathfrak{a}_{-n'}(\Lambda')$.

Finally, we have $\widetilde{\Lambda}'(\widetilde{k}'\widetilde{m}'+\widetilde{i}') \simeq \mathcal{W}^i_{\widetilde{k}}/\mathcal{W}^{i+1}_{\widetilde{k}}$ (where $\widetilde{k}'$ denotes the image of k in $\mathbb{Z}/e_0m\mathbb{Z}$) so we may think of $\widetilde{b}'$ as a further reduction of $\widetilde{b}$. Now $\widetilde{b}_{np}$ reduces to $\widetilde{0}'$ so $\widetilde{b}'$ is semisimple as required. ■

Remark 4.3 If, in the situation of (4.2), $\widetilde{b}$ is nilpotent, we have

$$b + \mathfrak{a}_{1-n}(\Lambda) \subset \mathfrak{a}_{1-n'}(\Lambda')$$

and $(n'-1)/e(\Lambda') < n/e(\Lambda)$. However, we cannot use this directly to deduce, as in [5], [10], [13], that a smooth representation π of G contains a fundamental skew stratum, since $e(\Lambda')$ is not bounded. For this, we must use the notion of an "optimal point" from [16] §6.

We adopt the notation of [16], noting that the lattice sequence Λ' corresponds to a rational point $x_{\Lambda'}$ in the building of $\widetilde{G}$ (see [4]) which is fixed by Σ. Then $\mathfrak{a}_m(\Lambda')$ corresponds to $\mathfrak{g}^*_{x_{\Lambda'},m/e(\Lambda')}$, when we have identified the Lie algebra $A = \mathfrak{g}$ with its dual.

We choose a self-dual basis for Λ' as in [14] (1.7); this gives rise to a maximal torus in $\widetilde{G}$ fixed by Σ and a simplex S in the apartment determined by this torus which contains $x_{\Lambda'}$ and is also fixed by Σ (and maximal for this property – it is of codimension at most 2). Let C be a chamber containing S in its closure and let Φ be the set of affine roots which take values (strictly) between 0 and 1 on C. Then $\Phi \cap {}^\sigma\Phi$ is the set of affine roots which take values between 0 and 1 on S.

For $\Xi \subset \Phi \cap {}^\sigma\Phi$ fixed by Σ, the optimal point x_Ξ (which may, and will, be taken as a barycentre of a face of C – that is, a point corresponding to a strict lattice sequence, period at most N) is in the closure of S and, moreover, ${}^\sigma x_\Xi$ is also an optimal point for Ξ in the closure of S. Hence $y_\Xi = \frac{1}{2}(x_\Xi + {}^\sigma x_\Xi)$ is also an optimal point, fixed by Σ, and the corresponding lattice sequence Λ_Ξ is self-dual and of period at most $2N$.

Now let $\Xi \subset \Phi \cap {}^\sigma\Phi$ be the set of affine roots ψ such that $0 < \psi(x_{\Lambda'}) \le \frac{1-n'}{e(\Lambda')} - [\frac{1-n'}{e(\Lambda')}]$. This is fixed by Σ and, by the definition of optimal point, there exists $n_\Xi \in \mathbb{Z}$ such that $\mathfrak{a}_{1-n'}(\Lambda') \subset \mathfrak{a}_{-n_\Xi}(\Lambda_\Xi)$ and $n_\Xi/e(\Lambda_\Xi) < (n'-1)/e(\Lambda')$.

We now complete the proof of Proposition 2.14.

Theorem 4.4 *Let $[\Lambda, n, n-1, b]$ be a non-G-split fundamental skew stratum in A. Then there exists a semisimple skew stratum $[\Lambda', n', n'-1, \beta]$ in A, with Λ' a refinement of Λ, such that*

(i) $n'/e(\Lambda'|\mathfrak{o}_F) = n/e(\Lambda|\mathfrak{o}_F)$;

(ii) $\mathfrak{a}_{1-n}(\Lambda) \subset \mathfrak{a}_{1-n'}(\Lambda')$;

(iii) $b \in \beta + \mathfrak{a}_{1-n'}^{-}(\Lambda')$.

Proof From (4.2) we obtain Λ', n' such that (i) and (ii) are satisfied and, since the stratum is fundamental, the reduction $\widetilde{b'} \in \mathrm{End}_{k_F}(\widetilde{\Lambda'})$ is non-zero semisimple.

Put $y = b^{e'/g'}\varpi_F^{n'/g'}$, where $e' = e(\Lambda'|\mathfrak{o}_F)$ and $g' = (e', n')$; so $y = \eta\overline{y}$, for $\eta = \pm$ a sign. Let $\Phi(X) \in \mathfrak{o}_F[X]$ be the characteristic polynomial of y and let $\varphi_b(X) \in k_F[X]$ be the characteristic polynomial of the stratum $[\Lambda', n', n'-1, b]$. Since the stratum is non-G-split, we have

$$\varphi_b(X) = \prod_{i=1}^{r} \phi_i(X)^{s_i} \cdot X^M,$$

where the $\phi_i(X)$ are monic, irreducible, pairwise coprime, $\phi_i(X) = \overline{\phi}_i(\eta X)$ and $\sum_{i=1}^{r} s_i + M = N$. (Note that we may have $M = 0$ here.) By Hensel's Lemma, we may lift this to

$$\Phi(X) = \prod_{i=1}^{r} \Phi_i(X) \cdot \Theta(X),$$

where the $\Phi_i(X)$ are monic, pairwise coprime, $\Phi_i(X) = \overline{\Phi}_i(\eta X)$ and reduce modulo $\mathfrak{p}_F$ to $\phi_i(X)^{s_i}$, and $\Theta(X)$ reduces modulo $\mathfrak{p}_F$ to X^M.

Put

$$\begin{aligned} V_i &= \ker \Phi_i(y), \qquad \text{for } i = 1, \dots, r; \\ V_0 &= \ker \Theta(y). \end{aligned}$$

Then, as in [11] (3.4), (3.5), we have $V = V_0 \perp \cdots \perp V_r$, this decomposition is fixed by b and, putting $\Lambda^i(k) = \Lambda(k) \cap V_i$ for $i = 0, \dots, r$, we have

$$\Lambda(k) = \bigoplus_{i=0}^{r} \Lambda^i(k), \qquad \text{for } k \in \mathbb{Z}.$$

Hence we obtain skew strata $[\Lambda^i, n', n'-1, b_i]$ in $A^i = \mathrm{End}V^i$, where $b_i = b|_{V^i}$, and the stratum $[\Lambda', n', n'-1, b]$ is the sum of these strata. Moreover, for $i = 1, ..., r$, the stratum is non-split fundamental, while for $i = 0$ it is non-fundamental.

We treat first the case $i = 0$. The reduction of b_0 in $\mathrm{End}_{k_F}(\widetilde{\Lambda^0})$ is semisimple but also nilpotent, since the stratum is non-fundamental. Hence it is 0 and we put $\beta_0 = 0$.

Now let $1 \leq i \leq r$. The skew stratum $[\Lambda^i, n', n'-1, b_i]$ is non-split fundamental and the reduction of b_i in $\mathrm{End}_{k_F}(\widetilde{\Lambda^i})$ is semisimple; hence, by [6] (2.5.8), the stratum is equivalent to a simple stratum $[\Lambda^i, n', n'-1, \alpha_i]$. Now $\alpha_i + \overline{\alpha}_i \in \mathfrak{a}_{1-n'}(\Lambda^i)$ so, by [20] (1.10), the stratum is equivalent to a simple skew stratum $[\Lambda^i, n', n'-1, \beta_i]$.

We now put $\beta = \sum_{i=0}^{r} \beta_i$; then $[\Lambda', n', n'-1, \beta]$ is semisimple skew and is equivalent to $[\Lambda', n', n'-1, b]$, as required. ■

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