Semisimple characters for p-adic classical groups

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Abstract

Let G be a unitary, symplectic or orthogonal group over a non-archimedean local field of residual characteristic different from 2, considered as the fixed point subgroup in a general linear group $\widetilde{G}$ of an involution. Following [7] and [13], we generalize the notion of a semisimple character for $\widetilde{G}$ and for G. In particular, following the formalism of [4], we show that these semisimple characters have certain functorial properties. Finally, we show that any positive level supercuspidal representation of G contains a semisimple character.

Introduction

Let F be a non-archimedean local field and let $\widetilde{G} = GL(N, F)$. One of the main ingredients in the description of the admissible dual of $\widetilde{G}$ by Bushnell and Kutzko ([6], [7]) is the notion of *simple characters*: these are arithmetically defined characters of certain compact open subgroups of $\widetilde{G}$. To obtain all the irreducible supercuspidal representations of $\widetilde{G}$ in [6], there are three main steps: first, to show that these simple characters have some rather remarkable properties of functoriality (and it turns out that they even have such properties when the dimension N is allowed to vary (see [6], [10]) and similarly for the base field F (see [4], [5] and sequels); second, to show that any irreducible supercuspidal representation of $\widetilde{G}$ contains a simple character θ of a group denoted H^1; and finally, to find the representations of the normalizer in $\widetilde{G}$ of θ which contain θ.

The purpose of this paper is to prove results analogous to the first two steps for unitary, symplectic and orthogonal groups G, in the case where the residual characteristic of F is not 2. To do this, we must first generalize the notion of simple characters to what we call *semisimple characters* of $\widetilde{G}$ and G. (There is a definition of semisimple characters in [13] but, as is remarked there, it is not sufficiently general.) We calculate the intertwining of these characters and demonstrate some functorial properties. Finally, we show that any irreducible supercuspidal representation of G contains a semisimple character of a group denoted H^1_-.

Now we give a more detailed description of the results obtained. As above, let F be a non-archimedean local field of residual characteristic different from 2, equipped with a galois involution with fixed field F_0 (where we allow the possibility $F_0 = F$). Let V be an N-dimensional F-vector space, $\widetilde{G} = \mathrm{Aut}_F(V) \simeq GL(N, F)$ and let G be the group of fixed points in $\widetilde{G}$ of an involution σ defined by a nondegenerate hermitian or skew-hermitian form on V; so G is a unitary, symplectic or orthogonal group defined over F_0. We also set $A_- = \mathrm{Lie}\, G \subset A = \mathrm{End}_F(V) \simeq \mathbb{M}(N, F)$.

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The simple characters for $\widetilde{G}$ are parametrized by triples (β, Λ, m) consisting of: an element $\beta \in A$ which generates a field extension E over F, with the technical condition $k_F(\beta) < 0$; an $\mathfrak{o}_E$-*lattice sequence* Λ of period $e(\Lambda)$ in V, where $\mathfrak{o}_E$ is the ring of integers of E; and an integer m with $0 \leq m < k_F(\beta)e(\Lambda)$. (See §1.2 and §2.1 for some explanations of the terms here.) To such a triple is associated a compact open subgroup $H^{m+1} = H^{m+1}(\beta, \Lambda)$ of $\widetilde{G}$ and a finite set $\mathcal{C}(\Lambda, m, \beta)$ of simple characters. If (β, Λ', m') is another such triple with $\lfloor \frac{m}{e(\Lambda)} \rfloor = \lfloor \frac{m'}{e(\Lambda')} \rfloor$, the functoriality properties mentioned above give a canonical bijection between the sets $\mathcal{C}(\Lambda, m, \beta)$ and $\mathcal{C}(\Lambda', m', \beta)$. (Here, $\lfloor q \rfloor$ denotes the greatest integer less than or equal to q.)

If we have a *self-dual* triple, that is $\beta \in A_-$ and Λ is a self-dual lattice sequence, then the group H^{m+1} and the set of simple characters are fixed by the involution σ and we can define the set $\mathcal{C}_-(\Lambda, m, \beta)$ of simple characters for G to be obtained by restricting to $H_-^{m+1} := H^{m+1} \cap G$ the simple characters in $\mathcal{C}(\Lambda, m, \beta)$. Equivalently, and often more usefully, they are the transfers of those simple characters fixed by σ under the Glauberman correspondence (note that H^{m+1} is a pro-p group, with $p \neq 2$). The intertwining of simple characters for G can be calculated by intersection, from the intertwining in $\widetilde{G}$ (cf. [13], [12]). Moreover, if (β, Λ', m') is another such triple with $\lfloor \frac{m}{e(\Lambda)} \rfloor = \lfloor \frac{m'}{e(\Lambda')} \rfloor$, the canonical bijection above commutes with action of σ (Proposition 2.12) and so induces a bijection between $\mathcal{C}_-(\Lambda, m, \beta)$ and $\mathcal{C}_-(\Lambda', m', \beta)$.

In [13], the notion of simple character is generalized to that of semisimple character, and in §3 we generalize it further (cf. [13] §5 Remark 2). We take now a triple (β, Λ, m) where β generates a sum of fields $E = \oplus_i E_i$; this gives us decompositions $\beta = \sum_i \beta_i$ (with $E_i = F[\beta_i]$) and $V = \oplus_i V^i$, with each V^i an $E_i = F[\beta_i]$-vector space, and we also require that Λ decompose as a direct sum $\oplus_i \Lambda^i$, with each Λ^i an $\mathfrak{o}_{E_i}$-lattice chain. (Again, there is a technical condition which can be written $k_0(\beta, \Lambda) < 0$.)

For each i, the triple (β_i, Λ_i, m) determines a compact open subgroup $H^{m+1}(\beta_i, \Lambda_i)$ and a set of simple characters as above. Analogously to the definitions in the simple case, we can then define a compact open subgroup $H^{m+1}(\beta, \Lambda)$, whose restriction to the Levi subgroup M of $\widetilde{G}$ determined by the decomposition $V = \oplus_i V^i$ is the product of the groups $H^{m+1}(\beta_i, \Lambda_i)$. The set of semisimple characters $\mathcal{C}(\Lambda, m, \beta)$ for $\widetilde{G}$ then consists of characters of $H^{m+1}(\beta, \Lambda)$ which restrict to simple characters on each $H^{m+1}(\beta_i, \Lambda_i)$ and are trivial elsewhere (there must also be some compatibility conditions between these simple characters).

In §3.3 we calculate the intertwining of such a semisimple character and in §3.5 we show that semisimple characters possess the same transfer properties as simple characters. Finally, in §3.6, we consider the situation when, for each i, the triple (β_i, Λ_i, m) is self-dual (note that, in this situation, $M \cap G$ is *not* a Levi subgroup of G); using Glauberman's correspondence again, all the results pass over to G, as in the simple case, and we get a set $\mathcal{C}_-(\Lambda, m, \beta)$ of semisimple characters for G.

Our main result is Theorem 5.1:

> Any positive-level irreducible supercuspidal representation of G contains a semisimple character $\theta \in \mathcal{C}_-(\Lambda, 0, \beta)$, for some self-dual semisimple triple $(\beta, \Lambda, 0)$.

The proof is very much along the lines of [6] (8.1.5), though the geometry causes some extra complications. We know already, from [14] §1.3, that such a representation π contains a *semisimple stratum*, that is, a semisimple character in some $\mathcal{C}_-(\Lambda, n-1, \beta)$, where $n = -\nu_\Lambda(\beta)$. The idea is to "refine" this character, that is, to find a related semisimple character of lower level which is also contained in π. We illustrate the first step of the process here.

The representation π must contain a character of the group $H_-^{n-1}(\beta,\Lambda)$; comparing this to the semisimple characters of $H_-^{n-1}(\beta,\Lambda)$ gives rise to a *derived stratum* in the centralizer of β. There are three possibilities now: if the stratum is non-fundamental (see Definition 1.2), by changing Λ, we can obtain a semisimple character of lower level; if the stratum is G-split (see Definition 1.3), we show (Theorem 4.9) that π has a non-zero Jacquet module, which contradicts the supercuspidality of π; otherwise, by replacing both β and Λ we obtain a semisimple character of lower level.

To prove the theorem, we iterate this, noting that we can bound the denominator of the level of the semisimple characters we consider so the process will terminate.

1 Preliminaries

We refer the reader to [6], [7], [12], [13] for more details on the results recalled in this section.

1.1 Notations

Let F be a non-archimedean local field equipped with a galois involution $\bar{\ }$ with fixed field F_0; we allow the possibility $F = F_0$. Let $\mathfrak{o}_F$ be the ring of integers of F, $\mathfrak{p}_F$ its maximal ideal and $k_F = \mathfrak{o}_F/\mathfrak{p}_F$ the residue field; we assume throughout that the residual characteristic $p := \operatorname{char} k_F$ is not 2. We denote by $\mathfrak{o}_0$, $\mathfrak{p}_0$, k_0 the same objects in F_0, and will use similar notation for any non-archimedean local field. We fix a uniformizer ϖ_F of F such that $\overline{\varpi_F} = -\varpi_F$ if F/F_0 is ramified, $\overline{\varpi_F} = \varpi_F$ otherwise. We put $\varpi_0 = \varpi_F^2$ if F/F_0 is ramified, $\varpi_0 = \varpi_F$ otherwise; so ϖ_0 is a uniformizer of F_0.

Let V be an N-dimensional vector space over F, equipped with a nondegenerate ϵ-hermitian form, with $\epsilon = \pm 1$. We put $A = \operatorname{End}_F V$ and denote by $\bar{\ }$ the adjoint (anti-)involution on A induced by h. Set also $\widetilde{G} = \operatorname{Aut}_F V$ and let σ be the involution given by $g \mapsto \overline{g}^{-1}$, for $g \in \widetilde{G}$. We also have an action of σ on the Lie algebra A given by $a \mapsto -\overline{a}$, for $a \in A$ (this is the differential of the action on $\widetilde{G}$). We put $\Sigma = \{1, \sigma\}$, where 1 acts as the identity on both $\widetilde{G}$ and A.

We put $G = \widetilde{G}^\Sigma = \{g \in \widetilde{G} : h(gv, gw) = h(v, w) \text{ for all } v, w \in V\}$, a unitary, symplectic or orthogonal group over F_0, and $A_- = A^\Sigma \simeq \operatorname{Lie} G$. In general, for S a subset of A, we will write S_- or S^- for $S \cap A_-$, and, for H a subgroup of $\widetilde{G}$, we will write H_- or H^- for $H \cap G$.

Let ψ_0 be a character of the additive group of F_0, with conductor $\mathfrak{p}_0$. Then we put $\psi_F = \psi_0 \circ \operatorname{tr}_{F/F_0}$; since $p \neq 2$, F/F_0 is at worst tamely ramified so ψ_F is a character of the additive group of F with conductor $\mathfrak{p}_F$. For S an $\mathfrak{o}_F$-lattice in A, we put $S^* = \{x \in A : \psi_A(xS) = 1\}$. If S is fixed by σ, then we have $S^* \cap A_- = \{x \in A_- : \psi_A(xS_-) = 1\}$.

We will also frequently have a decomposition $V = \bigoplus_{i=1}^l V^i$. Then, for $1 \leq i \leq l$, we denote by $\mathbf{1}^i$ the projection $V \to V^i$ with kernel $\bigoplus_{j\neq i} V^j$ and put $A^{ij} = \mathbf{1}^i A \mathbf{1}^j = \operatorname{Hom}_F(V^j, V^i)$. Then we will use the block decomposition

$$A = \begin{pmatrix} A^{11} & \cdots & A^{1l} \\ \vdots & \ddots & \vdots \\ A^{l1} & \cdots & A^{ll} \end{pmatrix}.$$

We also put $A_u = \bigoplus_{1\leq i<j\leq l} A^{ij}$, $A_l = \bigoplus_{1\leq j<i\leq l} A^{ij}$ and $\mathcal{M} = \bigoplus_{i=1}^l A^{ii}$; and $N_u = 1 + A_u$, $N_l = 1 + A_l$, $M = \mathcal{M}^\times$, $P_u = MN_u$, $P_l = MN_l$.

Finally, for $r \in \mathbb{R}$, we denote: by $\lceil r \rceil$ the smallest integer greater than or equal to r; by $r+$ the smallest integer *strictly* greater than r; by $\lfloor r \rfloor$ the greatest integer less than or equal to r; and by $r-$ the greatest integer *strictly* less than r.

1.2 Strata

Recall, from [7] §2, that an $\mathfrak{o}_F$-*lattice sequence* in V is a function Λ from $\mathbb{Z}$ to the set of $\mathfrak{o}_F$-lattices in V such that

(i) $\Lambda(k) \subset \Lambda(j)$, for $k \geq j$;

(ii) there exists a positive integer $e = e(\Lambda|\mathfrak{o}_F)$, called the $\mathfrak{o}_F$-period of Λ, such that $\varpi_F \Lambda(k) = \Lambda(k+e)$, for all $k \in \mathbb{Z}$.

A lattice sequence is called *strict* if $\Lambda(k) \neq \Lambda(j)$, for $k \neq j$ (so Λ is really just an $\mathfrak{o}_F$-lattice chain – see e.g. [6] (1.1.1)).

For L an $\mathfrak{o}_F$-lattice in V, we put $L^\# = \{v \in V : h(v, L) \subset \mathfrak{p}_F\}$. Then we call an $\mathfrak{o}_F$-lattice sequence Λ *self-dual* if there exists $d \in \mathbb{Z}$ such that $\Lambda(k)^\# = \Lambda(d-k)$, for all $k \in \mathbb{Z}$. Without changing any of the objects associated to a self-dual $\mathfrak{o}_F$-lattice sequence Λ (except for a scale of the indices), **we may (and do) normalize all self-dual lattice sequences $\boldsymbol{\Lambda}$ so that $\boldsymbol{d = 1}$** (see [14] §2).

There is also a well-defined notion of the direct sum of lattice sequences (see [7] §2 for the definition and some properties). The direct sum of self-dual lattice sequences is itself self-dual, by the assumption $d = 1$.

Associated to an $\mathfrak{o}_F$-lattice sequence Λ in V, we have a decreasing filtration $\{\mathfrak{a}_n(\Lambda) : n \in \mathbb{Z}\}$ of A by $\mathfrak{o}_F$-lattices; $\mathfrak{a}_0$ is a hereditary $\mathfrak{o}_F$-order in A and $\mathfrak{a}_1$ is its Jacobson radical. As in [7], we will allow the indices in the filtration to be real numbers, by putting $\mathfrak{a}_n = \mathfrak{a}_{\lceil n \rceil}$, for $n \in \mathbb{R}$. Note also that, for $n \in \mathbb{Z}$, the integers $\lfloor \frac{n}{2} \rfloor + 1$ and $\lfloor \frac{n+1}{2} \rfloor$ often appear in [6], [7] etc.; with the notation as here, we have

$$\mathfrak{a}_{\lfloor \frac{n}{2} \rfloor + 1} = \mathfrak{a}_{\frac{n}{2}+}, \qquad \text{and} \qquad \mathfrak{a}_{\lfloor \frac{n+1}{2} \rfloor} = \mathfrak{a}_{\frac{n}{2}}.$$

The filtration on A also gives rise to a valuation ν_Λ on A, with $\nu_\Lambda(0) = +\infty$.

If Λ is self-dual, then each $\mathfrak{a}_n(\Lambda)$ is fixed by σ and $\mathfrak{a}_n^- = \mathfrak{a}_n^-(\Lambda) = \mathfrak{a}_n(\Lambda) \cap A_-$ gives a filtration of A_- by $\mathfrak{o}_F$-lattices. Moreover, ν_Λ is fixed by σ.

Given an $\mathfrak{o}_F$-lattice sequence Λ, we also put $U = U(\Lambda) = \mathfrak{a}_0(\Lambda)^\times$, a compact open subgroup of $\widetilde{G}$, and $U_n = U_n(\Lambda) = 1 + \mathfrak{a}_n(\Lambda)$, for $n > 0$, a filtration of $U(\Lambda)$ by normal subgroups. For $n > 0$, we have a group isomorphism $\mathfrak{a}_n/\mathfrak{a}_{n+} \xrightarrow{\sim} U_n/U_{n+}$ induced by $x \mapsto 1 + x$.

If Λ is self-dual, then U, U_n are fixed by σ and we put $P = P(\Lambda) = U(\Lambda)^\Sigma = U(\Lambda) \cap G$, a compact open subgroup of G, with a filtration of $P(\Lambda)$ by normal subgroups $P_n = P_n(\Lambda) = U_n(\Lambda)^\Sigma = U_n(\Lambda) \cap G$, for $n \geq 1$. As before, for $n \geq 1$ we have a group isomorphism $\mathfrak{a}_n^-/\mathfrak{a}_{n+}^- \xrightarrow{\sim} P_n/P_{n+}$. We also have, for $n > 0$, a bijection $\mathfrak{a}_n^- \to P_n$ given by the Cayley map $x \mapsto C(x) = (1 + \frac{x}{2})(1 - \frac{x}{2})^{-1}$, which is equivariant under conjugation by P.

We define the normalizer $\mathfrak{K}(\Lambda)$ to be $\mathfrak{K}(\Lambda) = \bigcap_{n \geq 0} N_{\widetilde{G}}(U_n(\Lambda))$, where $N_{\widetilde{G}}$ denotes the normalizer in $\widetilde{G}$. Note that, if $x \in \mathfrak{K}(\Lambda)$, then $\nu_\Lambda(x) = -\nu_\Lambda(x^{-1})$ (see [7] (3.4)). On the other hand, if $x \in G$ and Λ is self-dual then $\nu_\Lambda(x) = \nu_\Lambda(\overline{x}^{-1}) = \nu_\Lambda(x^{-1})$, since ν_Λ is fixed by σ (acting on A). Thus, if $x \in \mathfrak{K}(\Lambda) \cap G$, we have $\nu_\Lambda(x) = 0$, whence $\mathfrak{K}(\Lambda) \cap G = P(\Lambda)$.

Definition 1.1 ([6] (1.5), [7] (3.1)). (i) A *stratum* in A is a 4-tuple $[\Lambda, n, r, b]$, where Λ is an $\mathfrak{o}_F$-lattice sequence, $n \in \mathbb{Z}$ and $r \in \mathbb{R}$ with $n \geq r \geq 0$ and $b \in \mathfrak{a}_{-n}(\Lambda)$.

(ii) Two strata $[\Lambda, n, r, b_i]$, $i = 1, 2$, are called *equivalent* if $b_1 - b_2 \in \mathfrak{a}_{-r}(\Lambda)$.

(iii) A stratum $[\Lambda, n, r, b]$ is called *skew* if Λ is self-dual and $b \in A_-$.

(iv) A stratum $[\Lambda, n, r, b]$ is called *null* if $n = r$ and $b = 0$.

Then, for $n \geq r \geq \frac{n}{2} > 0$, an equivalence class of strata corresponds to a character of $U_{r+}(\Lambda)$, by

$$[\Lambda, n, r, b] \mapsto (\psi_b : x \mapsto \psi_A(b(x-1)), \text{ for } x \in U_{r+}),$$

and an equivalence class of skew strata corresponds to a character of $P_{r+}(\Lambda)$, by

$$[\Lambda, n, r, b] \mapsto \psi_b^- := \psi_b|_{P_{r+}}.$$

Let $[\Lambda, n, r, b]$ be a stratum in A. Put $y_b = \varpi_F^{n/g} b^{e/g} \in \mathfrak{a}_0(\Lambda)$, where $e = e(\Lambda)$ and $g = (n, e)$. Let $\Phi(X) \in \mathfrak{o}_F(X)$ be the characteristic polynomial of y_b. Then we define the *characteristic polynomial* $\varphi_b(X) \in k_F[X]$ of the stratum to be the reduction modulo $\mathfrak{p}_F$ of $\Phi(X)$. Note that this depends only on the equivalence class of the stratum $[\Lambda, n, n-1, b]$.

Definition 1.2. We say that the stratum $[\Lambda, n, n-1, b]$ is *fundamental* if $\varphi_b(X) \neq X^N$.

Now suppose that $[\Lambda, n, r, b]$ is a skew stratum in A. Then we have $y_b = \eta \overline{y}_b$, for $\eta = \pm$ a sign (precisely, $\eta = (-)^{e/g}$ if F/F_0 is unramified, $\eta = (-)^{n/g}(-)^{e/g}$ otherwise), and thus $\Phi(X) = \overline{\Phi}(\eta X)$ and $\varphi_b(X) = \overline{\varphi}_b(\eta X)$.

Definition 1.3. We say that the skew stratum $[\Lambda, n, n-1, b]$ is *G-split* if $\varphi_b(X)$ has an irreducible factor $\psi(X)$ such that $(\psi(X), \overline{\psi}(\eta X)) = 1$.

Definition 1.4 ([6] (1.5.5). [7] (5.1)). A stratum $[\Lambda, n, r, \beta]$ in A is called *pure* if

(i) the algebra $E = F[\beta]$ is a field;

(ii) Λ is an $\mathfrak{o}_E$-lattice chain (we usually write $\Lambda_{\mathfrak{o}_E}$ when we are thinking of it as such);

(iii) $\nu_\Lambda(\beta) = -n$;

Let $[\Lambda, n, r, \beta]$ be a pure stratum and $E = F[\beta]$. We put $B = B_\beta = C_A(E)$, the A-centralizer of E, and $\mathfrak{b}_k = \mathfrak{a}_k \cap B$, for $k \in \mathbb{R}$. We also let a_β denote the adjoint map (with kernel B) $x \mapsto \beta x - x\beta$, $x \in A$. For $k \in \mathbb{R}$, we put $\mathfrak{n}_k = \mathfrak{n}_k(\beta, \Lambda) = \{x \in \mathfrak{a}_0 : a_\beta(x) \in \mathfrak{a}_k\}$. Then we define

$$k_0(\beta, \Lambda) = \max\left\{-n, \max\left\{k \in \mathbb{R} : \mathfrak{n}_k \not\subset \mathfrak{b}_0 + \mathfrak{a}_1\right\}\right\}.$$

Note that, in the case $E = F$, this is not the same definition as in [6] (1.4.5) ($k_0(\beta, \Lambda) = -\infty$ there). If $e(\Lambda|\mathfrak{o}_E)$ denotes the $\mathfrak{o}_E$-period of Λ, then $k_0(\beta, \Lambda)/e(\Lambda|\mathfrak{o}_E)$ is an integer independent of the choice of Λ; we denote it $k_F(\beta)$. (See [7] and [12] for more details.)

Definition 1.5 ([6] (1.5.5), [7] (5.1)). A stratum $[\Lambda, n, r, \beta]$ in A is called *simple* if, either it is null, or it is pure and $k_0(\beta, \Lambda) < -r$.

We remark that this is not quite the usual definition of simple strata since we call a null stratum simple (but see the remarks following [6] (5.5.10)). In particular, any stratum $[\Lambda, n, n, \beta]$ is equivalent to the null stratum $[\Lambda, n, n, 0]$ so we may use the null stratum as the initial step in inductive proofs "along $k_0(\beta, \Lambda)$".

Finally, for $\beta \in A$ such that $E = F[\beta]$ is a field, we put $B = C_A(E)$ and recall the notion of a tame corestriction $s = s_\beta$ on A relative to E/F ([6] (1.3.3)): it is a (B, B)-bimodule homomorphism $s : A \to B$ such that $s(\mathfrak{a}_0(\Lambda)) = \mathfrak{b}_0(\Lambda)$ for all $\mathfrak{o}_E$-lattice sequences Λ. It is unique upto multiplication by a unit $u \in \mathfrak{o}_E^\times$ and we have $s(\mathfrak{a}_n) = \mathfrak{b}_n$. If, moreover, $\beta \in A_-$, then there exists a tame corestriction s which commutes with the involution $^-$ on A (see [11] (2.1.1)); it is unique upto multiplication by a unit $u \in \mathfrak{o}_E^\times$ such that $u\overline{u} = 1$. Then we have $s(\mathfrak{a}_n^-) = \mathfrak{b}_n^-$.

2 Simple characters

In this section we recall some properties of *simple characters* for $\widetilde{G}$ and G (see [6], [7], [13]). Many of these are only available for strict lattice sequences and we will require them for general lattice sequences, which is the main purpose of this section. We remark that, for $\widetilde{G}$, these results (and much more) have also been obtained by Secherre in [10].

We begin with a general intertwining lemma which will prove useful in extending results known for strict lattice sequences to the general case. Let M be a Levi subgroup of $\widetilde{G}$ and P_u a parabolic subgroup with Levi component M. Let N_u be the unipotent radical of P_u and let N_l be the unipotent radical of the opposite parabolic P_l. Recall that a subgroup H of $\widetilde{G}$ is said to have an *Iwahori decomposition with respect to* (M, P_u) if

$$H = (H \cap N_l)(H \cap M)(H \cap N_u).$$

We also recall that, given subgroups H_1, H_2 of $\widetilde{G}$ and representations ρ_1, ρ_2 of H_1, H_2 respectively, the *intertwining in* $\widetilde{G}$ *of* ρ_1 *with* ρ_2 is

$$I_{\widetilde{G}}(\rho_1|_{H_1}, \rho_2|_{H_2}) = \{g \in \widetilde{G} : \mathrm{Hom}_{^gH_1 \cap H_2}(^g\rho_1, \rho_2) \neq 0\},$$

where $^g\rho_1$ is the representation of $^gH_1 = gH_1g^{-1}$ given by $^g\rho_1(x) = \rho_1(g^{-1}xg)$, for $x \in {^gH_1}$. Notice that, if ρ_1, ρ_2 are characters, then g intertwines ρ_1 with ρ_2 if and only if

$$\rho_1(g^{-1}xg) = \rho_2(x), \qquad \text{for all } x \in gH_1g^{-1} \cap H_2.$$

We use analogous notation for the intertwining of representations of subgroups of G, M etc.

Lemma 2.1. *For $i = 1, 2$, let H_i be a subgroup of $\widetilde{G}$ with an Iwahori decomposition with respect to (M, P_u) and let ξ_i be a character of H_i which is trivial on N_l and N_u. Then*

$$I_{\widetilde{G}}(\xi_1|_{H_1}, \xi_2|_{H_2}) \cap M = I_M(\xi_1|_{H_1 \cap M}, \xi_2|_{H_2 \cap M}).$$

Proof We certainly have the containment $\subset$. For the converse, we take $m \in I_M(\xi_1|_{H_1 \cap M}, \xi_2|_{H_2 \cap M})$ and $h_1 = mh_2m^{-1} \in H_1 \cap mH_2m^{-1}$; by the Iwahori decomposition, we have

$$h_{1,l}h_{1,M}h_{1,u} = h_1 = mh_2m^{-1} = (mh_{2,l}m^{-1})(mh_{2,M}m^{-1})(mh_{2,u}m^{-1}),$$

where $h_{i,l} \in H_i \cap N_l$, $h_{i,M} \in H_i \cap M$, $h_{i,u} \in H_i \cap N_u$. But, by uniqueness of Iwahori decomposition, we have $h_{1,l} = mh_{2,l}m^{-1}$, etc.. In particular, $H_1 \cap mH_2m^{-1}$ has an Iwahori decomposition. The assertion of the lemma is now trivial. ∎

Corollary 2.2. *Let H be a subgroup of $\widetilde{G}$ with an Iwahori decomposition with respect to (M, P_u) and let ξ be a character of H which is trivial on N_l and N_u. Let $m \in M$ be such that m normalizes H. Then m normalizes ξ if and only if it normalizes $\xi|_{H\cap M}$.*

2.1 Lattice sequences

Let $[\Lambda, n, m, \beta]$ be a simple stratum in A. When Λ is strict, the set of simple characters $\mathcal{C}(\Lambda, m, \beta)$ is defined in [6] (3.2) – the elements are certain arithmetically defined characters of the group $H^{m+}(\beta, \Lambda)$. (This group is defined in [6] (3.1) and denoted $H^{m+1}(\beta, \Lambda)$ there – note that, for m real, we have $\mathcal{C}(\Lambda, m, \beta) = \mathcal{C}(\Lambda, \lfloor m \rfloor, \beta)$.) Moreover, a large number of properties of these characters are described in [6] §3. The definitions are extended to the case when Λ is not strict in [7] §5 and certain of the properties are established (see also [10]). However, for our purposes, we require more of these; in particular, we calculate the intertwining.

It is convenient here to express our results in terms of *ps-characters*, whose definition we recall ([4] §8). First, a *simple pair* is a pair (k, β) consisting of a nonzero element β generating a field extension E of F and a positive integer $k < k_F(\beta)$. Then, if we are given

$$\begin{cases} \text{(i)} & V' \text{ a finite dimensional } E\text{-vector space,} \\ \text{(ii)} & \mathfrak{B}' \text{ a hereditary } \mathfrak{o}_E\text{-order in } \mathrm{End}_E V', \\ \text{(iii)} & m' \text{ a real number such that } \lfloor m'/e(\mathfrak{B}'|\mathfrak{o}_E) \rfloor = k, \end{cases} \tag{2.3}$$

we obtain a stratum $[\mathfrak{A}', n', m', \beta]$ in $A' = \mathrm{End}_F V'$ as follows: $\mathfrak{A}'$ is the hereditary $\mathfrak{o}_F$-order defined by the same lattice chain as $\mathfrak{B}'$ and the integer $n' = -\nu_E(\beta)e(\mathfrak{B}'|\mathfrak{o}_E)$, where ν_E is the normalized valuation on E. The condition on k means precisely that this stratum is simple, for any choice of $(V', \mathfrak{B}', m')$ as in (2.3). Moreover, given two triples $(V'_i, \mathfrak{B}'_i, m'_i)$, $i = 1, 2$, as in (2.3), we have, by [6] (3.6.14), a canonical bijection

$$\tau_{\mathfrak{A}'_1,\mathfrak{A}'_2,\beta} : \mathcal{C}(\mathfrak{A}'_1, m'_1, \beta) \xrightarrow{\sim} \mathcal{C}(\mathfrak{A}'_2, m'_2, \beta). \tag{2.4}$$

Recall ([6] (3.6.1)) that if $V'_1 = V'_2 = V'$ and $\theta'_1 \in \mathcal{C}(\mathfrak{A}'_1, m'_1, \beta)$ then $\tau_{\mathfrak{A}'_1,\mathfrak{A}'_2,\beta}(\theta'_1)$ is the unique simple character $\theta'_2 \in \mathcal{C}(\mathfrak{A}'_2, m'_2, \beta)$ such that $1 \in \widetilde{G}'$ intertwines θ'_1 with θ'_2, where $\widetilde{G}' = \mathrm{Aut}_F V'$.

A *ps-character* attached to a simple pair (k, β) is a simple-character-valued function Θ which attaches to each triple $(V', \mathfrak{B}', m')$ as in (2.3), a simple character $\Theta(\mathfrak{A}') \in \mathcal{C}(\mathfrak{A}', m', \beta)$ (called the *realization of* Θ *on* $\mathfrak{A}'$ *of level* m) subject to the condition that, given two realizations $\Theta(\mathfrak{A}'_i)$, $i = 1, 2$, we have $\Theta(\mathfrak{A}'_2) = \tau_{\mathfrak{A}'_1,\mathfrak{A}'_2,\beta}(\Theta(\mathfrak{A}'_1))$. Thus a ps-character is completely determined by any one of its realizations.

Now we put ourselves in the following situation (cf. [7] (5.2)): let $[\Lambda, n, m, \beta]$ be a simple stratum in A, with $E = F[\beta]$ and $e = e(\Lambda|\mathfrak{o}_E)$. Let V_0 be a finite dimensional E-vector space and let Λ^0 be a strict $\mathfrak{o}_E$-lattice sequence in V_0 of $\mathfrak{o}_E$-period e. We put

$$\begin{aligned} V' &= V \oplus V_0; \\ \Lambda' &= \Lambda \oplus \Lambda^0. \end{aligned}$$

We also put $A' = \mathrm{End}_F V'$ and $\widetilde{G}' = \mathrm{Aut}_F V'$, $\widetilde{G}_0 = \mathrm{Aut}_F V_0$. Then $M = \widetilde{G}_0 \times \widetilde{G}$ is a Levi subgroup of $\widetilde{G}'$ and we put $N_l = 1 + \mathrm{Hom}_F(V, V_0)$, $N_u = 1 + \mathrm{Hom}_F(V_0, V)$. We also denote by $\mathbf{1}_V$ the projection onto V with kernel V_0.

Now Λ' is a strict $\mathfrak{o}_E$-lattice chain in V of $\mathfrak{o}_E$-period e and $[\Lambda', n, m, \beta]$ is a simple stratum in A'. Hence we have the set of simple characters $\mathcal{C}(\Lambda', m, \beta)$ of $H^{m+}(\beta, \Lambda')$. Then, by definition, $H^{m+}(\beta, \Lambda) = H^{m+}(\beta, \Lambda') \cap \widetilde{G}$ and $\mathcal{C}(\Lambda, m, \beta)$ is the set of restrictions $\theta'|_{H^{m+}(\beta,\Lambda)}$, for $\theta' \in \mathcal{C}(\Lambda', m, \beta)$. We remark that, from [7] (5.5), this is independent of the choice of Λ^0; indeed it depends only on the ps-character determined by θ', and may be thought of as the realization of this ps-character on Λ.

Let $r = -k_0(\beta, \Lambda) = -k_0(\beta, \Lambda')$. We put

$$\mathfrak{M}'_m = \mathfrak{M}_m(\beta, \Lambda') = \mathfrak{a}_{r-m}(\Lambda') \cap \mathfrak{n}_{-m}(\beta, \Lambda') + \mathfrak{J}^{\frac{r}{2}}(\beta, \Lambda'),$$

where $\mathfrak{J}^{\frac{r}{2}}(\beta, \Lambda') = \mathfrak{J}^{\lfloor \frac{r+1}{2} \rfloor}(\beta, \Lambda')$ is defined in [6] (3.1). Then, for $\theta' \in \mathcal{C}(\Lambda', m, \beta)$, we have

$$I_{\widetilde{G}'}(\theta'|H^{m+}(\beta, \Lambda')) = (1 + \mathfrak{M}'_m) B'^{\times} (1 + \mathfrak{M}'_m),$$

where $B' = End_E V'$, by [6] (3.3.2).

Proposition 2.5. *Let $\theta \in \mathcal{C}(\Lambda, m, \beta)$ so that $\theta = \theta'|_{H^{m+}(\beta,\Lambda)}$, for some $\theta' \in \mathcal{C}(\Lambda', m, \beta)$. Then*

$$I_{\widetilde{G}}(\theta|H^{m+}(\beta, \Lambda)) = (1 + \mathfrak{M}_m) B^{\times} (1 + \mathfrak{M}_m),$$

where $\mathfrak{M}_m = \mathfrak{M}_m(\beta, \Lambda) = \mathfrak{M}'_m \cap A = \mathbf{1}_V \mathfrak{M}'_m \mathbf{1}_V$ and $B = \mathrm{End}_E V$.

Proof We remark first that, by [7] (5.6), $\mathfrak{M}_m$ is independent of the choice of Λ^0. We abbreviate $H' = H^{m+}(\beta, \Lambda')$ and $H = H^{m+}(\beta, \Lambda)$.

By Lemma 2.1, we have $I_M(\theta'|H' \cap M) = I_M(\theta'|H') = (1 + \mathfrak{M}'_m) B'^{\times} (1 + \mathfrak{M}'_m) \cap M$ and this is precisely $(1 + \mathfrak{M}'_m) \cap M \cdot B' \cap M \cdot (1 + \mathfrak{M}'_m) \cap M$ by [12] (1.3) (cf. *op. cit.* (3.15)). But then

$$\begin{aligned} I_{\widetilde{G}}(\theta|H) &= I_{\widetilde{G}}(\theta'|H' \cap M) \\ &= \big((1 + \mathfrak{M}'_m) \cap M \cdot B' \cap M \cdot (1 + \mathfrak{M}'_m) \cap M\big) \cap \widetilde{G} \\ &= (1 + \mathfrak{M}_m) B^{\times} (1 + \mathfrak{M}_m) \end{aligned}$$

as required. ∎

Lemma 2.6. *Let Θ be a ps-character attached to the simple pair (k, β). For $i = 1, 2$, let Λ^i be an $\mathfrak{o}_E$-lattice sequence of $\mathfrak{o}_E$-period e in a finite dimensional E-vector space V and let $m_i \in \mathbb{R}$ be such that $\lfloor m_i/e \rfloor = k$. Let $\theta_i = \Theta(\Lambda^i)$ be the realization of Θ on Λ^i of level m_i. Then we have*

$$1 \in I_{\widetilde{G}}(\theta_1, \theta_2).$$

Moreover, θ_2 is the unique simple character in $\mathcal{C}(\Lambda^2, m_2, \beta)$ such that $1 \in I_{\widetilde{G}}(\theta_1, \theta_2)$.

Proof As above, let V_0 be a finite dimensional E-vector space and let Λ^0 be a strict $\mathfrak{o}_E$-lattice sequence in V_0 of $\mathfrak{o}_E$-period e. We put $V' = V \oplus V_0$, $\Lambda'_1 = \Lambda^1 \oplus \Lambda^0$, $\Lambda'_2 = \Lambda^2 \oplus \Lambda^0$ and also $A' = \mathrm{End}_F V'$ and $\widetilde{G}' = \mathrm{Aut}_F V'$.

For $i = 1, 2$, let $\theta'_i = \Theta(\Lambda'_i)$ be the realization of Θ on the strict lattice sequence Λ'_i of level m_i, so we have $H^{m_i+}(\beta, \Lambda'_i) \cap \widetilde{G} = H^{m_i+}(\beta, \Lambda^i)$ and $\theta'_i|_{H^{m_i+}(\beta,\Lambda^i)} = \theta_i$. Then the first assertion follows immediately from Lemma 2.1.

For the final assertion, suppose $\theta'_2 = \Theta'(\Lambda^2)$ and $1 \in I_{\widetilde{G}}(\theta_1, \theta'_2)$. Consider $V_0 = V \oplus \cdots \oplus V$ (e times) and, for $i = 1, 2$, the strict lattice sequence Λ_0^i given by

$$\Lambda_0^i(k) = \Lambda^i(k) \oplus \Lambda^i(k+1) \oplus \cdots \oplus \Lambda^i(k+e-1), \qquad \text{for } k \in \mathbb{Z}.$$

Put $\theta_0 = \Theta(\Lambda_0^1)$ and $\theta_0' = \Theta'(\Lambda_0^2)$. Writing $M = \widetilde{G} \times \cdots \times \widetilde{G} \subset \widetilde{G}_0 = \mathrm{Aut}_F V_0$, we have $\theta_0|_M = \theta_1 \otimes \cdots \otimes \theta_1$ and $\theta_0'|_M = \theta_2' \otimes \cdots \otimes \theta_2'$; in particular, $1 \in I_M(\theta_0|_M, \theta_0'|_M)$. But, by [7] (5.2) Proposition, θ_0, θ_0' restrict trivially to the unipotent radical of any parabolic subgroup with Levi factor M so, by Lemma 2.1, we have $1 \in I_{\widetilde{G}_0}(\theta_0, \theta_0')$. But each Λ_0^i is strict so, by [6] (3.6.1), the characters θ_0, θ_0' correspond under the canonical bijection $\tau_{\Lambda_0^1, \Lambda_0^2, \beta}$ in (2.4). Hence $\theta_0' = \Theta(\Lambda_0^2)$ also and, since a ps-character is determined by one of its realizations, we have $\Theta = \Theta'$, as required. ■

We will sometimes use $\tau_{\Lambda^1, \Lambda^2, \beta}$ to denote the correspondence $\mathcal{C}(\Lambda^1, m_1, \beta) \xrightarrow{\sim} \mathcal{C}(\Lambda^2, m_2, \beta)$ given by the ps-characters.

Recall that, if we have $[\Lambda, n, r, \beta]$ pure, with Λ strict and $r = -k_0(\beta, \Lambda)$, and $[\Lambda, n, r, \gamma]$ is simple and equivalent to $[\Lambda, n, r, \beta]$, then the map

$$\begin{array}{rcl} \mathcal{C}(\Lambda, m, \beta) & \to & \mathcal{C}(\Lambda, m, \gamma) \\ \theta & \mapsto & \theta \psi_{\gamma - \beta} \end{array}$$

is bijective for $\frac{r}{2} \le m < r$, by [6] (3.3.18). Note that this then clearly holds also when Λ is not necessarily strict.

In fact, there will be several occasions when we will need to be careful in the way we choose a γ as above. To describe this, we recall the notion of a "generalized (W, E)-decomposition" from [7] §5.3 (see also [6] §1.2).

Let E/F be a field extension in A and put $B = \mathrm{End}_E V$. We write $A(E) = \mathrm{End}_F E$ and $\mathfrak{A}(E)$ for the unique hereditary order in $A(E)$ normalized by $E^\times$. Let W be the F-span of an E-basis of V. Then the isomorphism $E \otimes_F W \to V$ induces an isomorphism of $(A(E), B)$-bimodules.

$$A(E) \otimes_E B \simeq A.$$

In particular, the choice of W also induces an embedding of algebras $\iota_W : A(E) \hookrightarrow A$ extending the embedding of E in A.

Now let Λ be an $\mathfrak{o}_E$-lattice sequence in V, which we may also view as an $\mathfrak{o}_F$-lattice sequence. We put $\mathfrak{b}_n(\Lambda) = \mathfrak{a}_n(\Lambda) \cap B$, for $n \in \mathbb{R}$. We say that W is in *general position relative to* Λ *over* E if W has an F basis $w_1, ..., w_m$ such that, for each $k \in \mathbb{Z}$, there are integers $k(i)$, $1 \le i \le m$, such that

$$\Lambda(k) = \mathfrak{p}_E^{k(1)} w_1 \oplus \cdots \oplus \mathfrak{p}_E^{k(m)} w_m.$$

(That is, W is the F-span of an E-basis of V which is a *splitting* of Λ – see §3.1.) Then [7] 5.3 Lemma says that, for such W, we have isomorphisms

$$\mathfrak{A}(E) \otimes_{\mathfrak{o}_E} \mathfrak{b}_n(\Lambda) \simeq \mathfrak{a}_n(\Lambda), \qquad n \in \mathbb{R},$$

of $(\mathfrak{A}(E), \mathfrak{b}_0(\Lambda))$-bimodules.

Now suppose $[\Lambda, n, r, \beta]$ is a pure stratum with $r = -k_0(\beta, \Lambda)$ and $E = F[\beta]$. Then [7] 5.3 Corollary states that, for W in general position relative to Λ over E_β, $[\Lambda, n, r, \beta]$ is equivalent to a simple stratum $[\Lambda, n, r, \gamma]$ with $\gamma \in \iota_W(\mathfrak{K}(\mathfrak{A}(E)))$.

Now, for $i = 1, 2$, let $[\Lambda^i, n, r, \beta]$ be a pure stratum in $A^i = \mathrm{End}_F V_i$, with $e = e(\Lambda^i | \mathfrak{o}_{E_\beta})$ and $r = -k_0(\beta, \Lambda^i)$. Put $V = V_1 \oplus V_2$, $A = \mathrm{End}_F V$ and $\Lambda = \Lambda^1 \oplus \Lambda^2$; then $[\Lambda, n, r, \beta]$ is a pure stratum in A. For $i = 1, 2$, let W_i be in general position relative to Λ^i over E_β and put $W = W_1 \oplus W_2$, which is in general position relative to Λ over E_β. Then there exists $\gamma \in \iota_W(\mathfrak{K}(\mathfrak{A}(E))) \subset A^1 \oplus A^2$ such that $[\Lambda, n, r, \gamma]$ is simple and equivalent to $[\Lambda, n, r, \beta]$. In particular, we can regard γ as an element of A, A^1 or A^2 and, for $i = 1, 2$, $[\Lambda^i, n, r, \gamma]$ is a simple stratum equivalent to $[\Lambda^i, n, r, \beta]$. Then, for $\theta \in \mathcal{C}(\Lambda, m, \beta)$ with $\frac{r}{2} \le m < r$ and $i = 1, 2$, we have

$$\tau_{\Lambda,\Lambda^i,\beta}(\theta)\psi_{\gamma-\beta} = \tau_{\Lambda,\Lambda^i,\gamma}(\theta\psi_{\gamma-\beta}), \tag{2.7}$$

since the transfer maps are simply restriction. In particular, we obtain that, for $\theta \in \mathcal{C}(\Lambda^1, m, \beta)$, we have $\tau_{\Lambda^1,\Lambda^2,\beta}(\theta)\psi_{\gamma-\beta} = \tau_{\Lambda^1,\Lambda^2,\gamma}(\theta\psi_{\gamma-\beta})$.

2.2 The orders $\mathfrak{H}$ and $\mathfrak{J}$

Let $[\Lambda, n, 0, \beta]$ be a simple stratum in A. If Λ is strict then the orders $\mathfrak{H}(\beta, \Lambda)$ and $\mathfrak{J}(\beta, \Lambda)$ are defined in [6] (3.1); if Λ is not strict, they are defined in [7] by restriction from a larger space, as in the previous section. However, it would be possible to make the definitions directly as in [6] (3.1). In this section we show that these two definitions coincide. In fact, this will follow almost immediately from the following two lemmas. We suppose $V = V_1 \oplus V_2$ and use our standard block notation.

Lemma 2.8. *Let X, Y be $\mathfrak{o}_F$-lattices in A such that $\mathbf{1}^i X \mathbf{1}^j \subset X$ and $\mathbf{1}^i Y \mathbf{1}^j \subset Y$, for $i, j = 1, 2$. Then, for $i, j = 1, 2$,*

(i) $\mathbf{1}^i(X + Y)\mathbf{1}^j = \mathbf{1}^i X \mathbf{1}^j + \mathbf{1}^i Y \mathbf{1}^j$;

(ii) $\mathbf{1}^i(X \cap Y)\mathbf{1}^j = \mathbf{1}^i X \mathbf{1}^j \cap \mathbf{1}^i Y \mathbf{1}^j$.

Proof (i) is clear while for (ii) we have $\mathbf{1}^i(X \cap Y)\mathbf{1}^j \subset \mathbf{1}^i X \mathbf{1}^j \cap \mathbf{1}^i Y \mathbf{1}^j \subset X \cap Y$; then, applying $\mathbf{1}^i$ on the left and $\mathbf{1}^j$ on the right, we have the required equality. ■

Lemma 2.9. *Let X be as in the previous lemma. Define $(\mathbf{1}^i X \mathbf{1}^i)^* = \{a \in A^i : \mathrm{tr}_{A^i/F}(a\mathbf{1}^i X \mathbf{1}^i) \subset \mathfrak{p}_F\}$. Then*

$$(\mathbf{1}^i X \mathbf{1}^i)^* = \mathbf{1}^i X^* \mathbf{1}^i.$$

Proof Straightforward properties of trace. ■

Let now $[\Lambda, n, 0, \beta]$ be a simple stratum in A with $r = -k_0(\beta, \Lambda)$ and $e = e(\Lambda|\mathfrak{o}_E)$, where $E = F[\beta]$. Let V_0 be a finite dimensional E-vector space and let Λ^0 be a strict *regular* $\mathfrak{o}_E$-lattice sequence in V_0 of $\mathfrak{o}_E$-period e, where regular means that $\dim_{k_E} \Lambda^0(i)/\Lambda^0(i+1)$ is independent of i (so that the associated hereditary order $\mathfrak{b}_{\beta,0} = \mathfrak{a}_0 \cap B_\beta$ is principal, where B_β is the centralizer of β). Note that, since Λ^0 is regular, the valuation map $\nu_{\Lambda^0} : \mathfrak{K}(\Lambda^0_{\mathfrak{o}_E}) \to \mathbb{Z}$ is surjective.

We put $V' = V \oplus V_0$, $\Lambda' = \Lambda \oplus \Lambda^0$. Using a generalized (W, E)-decomposition as above, we choose γ such that $[\Lambda, n, r, \gamma]$ is a simple stratum equivalent to $[\Lambda, n, r, \beta]$ and $[\Lambda', n, r, \gamma]$ is simple and equivalent to $[\Lambda', n, r, \beta]$. Now Lemma 2.8 implies immediately that we have $\mathfrak{H}(\beta, \Lambda) = \mathfrak{b}_{\beta,0} + \mathfrak{H}^{\frac{r}{2}+}(\gamma, \Lambda)$ and $\mathfrak{J}(\beta, \Lambda) = \mathfrak{b}_{\beta,0} + \mathfrak{J}^{\frac{r}{2}}(\gamma, \Lambda)$.

We also observe that, since Λ^0 is regular, we have $\mathfrak{K}(\Lambda_{\mathfrak{o}_E}) = \mathbf{1}_V\mathfrak{K}(\Lambda'_{\mathfrak{o}_E})\mathbf{1}_V$: the containment $\supset$ is clear; conversely, for $x \in \mathfrak{K}(\Lambda_{\mathfrak{o}_E})$, there exists $x_0 \in \mathfrak{K}(\Lambda^0_{\mathfrak{o}_E})$ such that $\nu_\Lambda(x) = \nu_{\Lambda^0}(x_0)$ and then

$$x = \mathbf{1}_V \begin{pmatrix} x & 0 \\ 0 & x_0 \end{pmatrix} \mathbf{1}_V \in \mathbf{1}_V\mathfrak{K}(\Lambda'_{\mathfrak{o}_E})\mathbf{1}_V.$$

Now the main results of [6] (3.1) follow easily for an arbitrary lattice sequence. In particular we have

Lemma 2.10 (cf. [6] (3.1.9–13)). (i) *For* $0 \le t \le r$, $\mathfrak{H}^{\frac{t}{2}+}(\beta,\Lambda)$ *is a bimodule over the ring* $\mathfrak{n}_{-t}(\beta,\Lambda)$, *as is* $\mathfrak{J}^{\frac{t}{2}}(\beta,\Lambda)$.

(ii) *For* $k \in \mathbb{R}$, $\mathfrak{H}^k(\beta,\Lambda) \subset \mathfrak{J}^k(\beta,\Lambda)$ *are invariant under conjugation by* $\mathfrak{K}(\Lambda_{\mathfrak{o}_E})$.

(iii) *For* $k, l > 0$, $\mathfrak{J}^k(\beta,\Lambda)\mathfrak{J}^l(\beta,\Lambda) \subset \mathfrak{H}^{k+l}(\beta,\Lambda)$.

(iv) *For* $k > 0$, $\mathfrak{H}^k(\beta,\Lambda)$ *is a two-sided ideal of* $\mathfrak{J}(\beta,\Lambda)$.

2.3 Simple characters for G

We say that a simple pair (k,β) is *skew* if the galois involution $^-$ can be extended to $E = F[\beta]$ in such a way that $\overline{\beta} = -\beta$. Then, if we are given

$$\begin{cases} \text{(i)} & V' \text{ a finite dimensional } E\text{-vector space equipped with an } \epsilon\text{-hermitian} \\ & \text{form } f' : V' \times V' \to E, \\ \text{(ii)} & \mathfrak{B}' \text{ a hereditary } \mathfrak{o}_E\text{-order in } \mathrm{End}_E V' \text{ fixed by the involution } ^- \\ & \text{induced by } f', \\ \text{(iii)} & m' \text{ a real number such that } \lfloor m'/e(\mathfrak{B}'|\mathfrak{o}_E)\rfloor = k, \end{cases} \tag{2.11}$$

we obtain a *skew* stratum $[\mathfrak{A}', n', m', \beta]$ in $A' = \mathrm{End}_F V'$ as follows: we choose an F_0-linear form λ_0 on E_0 such that

$$\{e \in E_0 \ ; \ \lambda_0(e\mathfrak{o}_{E_0}) \subset \mathfrak{p}_{F_0}\} = \mathfrak{p}_{E_0}$$

(as in [3] §5), and let λ be the F-linear form on E given either by extending linearly (if $F \neq F_0$) or by composing with tr_{E/E_0} (if $F = F_0$); then V', as an F-vector space, is equipped with the ϵ-hermitian form $h' = \lambda \circ f'$. Note that the duality induced by h' is independent of the choice of λ_0.

We say that a ps-character Θ attached to a skew simple pair (k,β) is *self-dual* if there exists a triple as in (2.11) such that the realization $\Theta(\mathfrak{A}') \in \mathcal{C}(\mathfrak{A}', m', \beta)$ is fixed by $\sigma : x \mapsto \overline{x}^{-1}$, $x \in \widetilde{G}'$, where $\widetilde{G}' = \mathrm{Aut}_F V'$.

Proposition 2.12. *Let* (k,β) *be a skew simple pair and, for* $i = 1, 2$, *let* V'_i *be an* E*-vector space as in (2.11)(i),* Λ'_i *be a self-dual* $\mathfrak{o}_E$*-lattice sequence of* $\mathfrak{o}_E$*-period* e'_i *in* V'_i *and* $m'_i \in \mathbb{R}$ *be such that* $\lfloor m'_i/e'_i\rfloor = k$. *Then the canonical bijection* $\tau_{\Lambda'_1,\Lambda'_2,\beta}$ *commutes with* σ.

Proof Let Λ'_0 be a strict self-dual $\mathfrak{o}_E$-lattice sequence of $\mathfrak{o}_E$-period $e'_0 = lcm(e'_1, e'_2)$ in an E-vector space V'_0 as in (2.11)(i) and let $m'_0 = ke'_0$. We consider the E-vector space $V' = V'_0 \perp V'_1 \perp V'_2$ equipped with the form $f' = f'_0 \perp f'_1 \perp f'_2$, that is

$$f'(v_0 + v_1 + v_2, w_0 + w_1 + w_2) = f'_0(v_0, w_0) + f'_1(v_1, w_1) + f'_2(v_2, w_2), \quad \text{for } v_i, w_i \in V'_i.$$

Set $\Lambda = \Lambda'_0 \perp \Lambda'_1 \perp \Lambda'_2$, a strict self-dual $\mathfrak{o}_E$-lattice sequence of $\mathfrak{o}_E$-period $e' = e'_0$ in V', and $m' = m'_0$. Then we have $\tau_{\Lambda'_1,\Lambda'_2,\beta} = \tau_{\Lambda',\Lambda'_2,\beta} \circ \tau^{-1}_{\Lambda',\Lambda'_1,\beta}$ so we need only check that $\tau_{\Lambda',\Lambda'_1,\beta}$ commutes with σ, by symmetry.

Let $\theta \in \mathcal{C}(\Lambda', m', \beta)$, $\theta_1 \in \mathcal{C}(\Lambda'_1, m'_1, \beta)$ be such that $\theta_1 = \tau_{\Lambda',\Lambda'_1,\beta}(\theta)$. Then, by the definition of $\mathcal{C}(\Lambda'_1, m'_1, \beta)$ (cf. [7] (5.5)), we have $\theta_1 = \theta|_{\widetilde{G}_1}$, where $\widetilde{G}_1 = \mathrm{Aut}_F V'_1$. But then $\theta_1^\sigma = \theta^\sigma|_{\widetilde{G}_1}$ so, again by definition, $\theta_1^\sigma = \tau_{\Lambda',\Lambda'_1,\beta}(\theta^\sigma)$ as required. ■

Corollary 2.13. *Let Θ be a self-dual ps-character attached to the skew simple pair (k, β). Let V be an E-vector space as in (2.11)(i), Λ be a self-dual $\mathfrak{o}_E$-lattice sequence of $\mathfrak{o}_E$-period e in V and $m \in \mathbb{R}$ be such that $\lfloor m/e \rfloor = k$. Then the realization $\theta = \Theta(\Lambda)$ on Λ of level m is fixed by σ.*

In particular, a ps-character is self-dual if and only if *every* realization of it is fixed by σ.

3 Semisimple characters

In [13], the author defined semisimple characters for split semisimple strata. Here we extend this definition to the "relatively split" case of [7] §6. We lay down the groundwork in §3.1 and define the relevant groups and semisimple characters in §3.2. The main results are then the calculation of the intertwining of semisimple character (Theorem 3.22) and the transfer property (Proposition 3.26). In §3.6, we let the involution σ act and obtain all the analogous results for our classical group G.

3.1 Preparation

Let $[\Lambda, n, r, \beta]$ be a stratum in A and suppose we have a decomposition $V = \bigoplus_{i=1}^{l} V^i$. Let Λ^i be the lattice sequence in V^i given by $\Lambda^i(k) = \Lambda(k) \cap V^i$ and put $\beta_i = \mathbf{1}^i \beta \mathbf{1}^i$, where $\mathbf{1}^i$ is the projection onto V^i with kernel $\bigoplus_{j \neq i} V^j$. We say that $V = \bigoplus_{i=1}^{l} V^i$ is a *splitting* for the stratum $[\Lambda, n, r, \beta]$ if we have $\Lambda(k) = \bigoplus_{i=1}^{l} \Lambda^i(k)$, for all $k \in \mathbb{Z}$, and $\beta = \sum_{i=1}^{l} \beta_i$. Similarly, we say that a basis $v_1, ..., v_N$ for V is a *splitting* of Λ (respectively the stratum) if $V = \bigoplus_{i=1}^{N} F v_i$ is a splitting for it.

Whenever we have such a splitting, we will use the block notation $A^{ij} = \mathrm{Hom}_F(V^j, V^i)$ as in §1.1. In particular, $\mathcal{M} = \bigoplus_{i=1}^{l} A^{ii}$

Definition 3.1 (cf. [7] (3.6)). A stratum $[\Lambda, n, n-1, \beta]$ in A is called *split* if there exists a splitting $V = \bigoplus_{i=1}^{l} V^i$ such that the characteristic polynomials $\phi_i(X)$ of $[\Lambda^i, n, n-1, \beta_i]$, $i = 1, \ldots, l$ are pairwise coprime.

Definition 3.2. A stratum $[\Lambda, n, r-, \beta]$ in A is called *semisimple* if either it is null or $\nu_\Lambda(\beta) = -n$ and there exists a splitting $V = \bigoplus_{i=1}^{l} V^i$ for the stratum such that

(i) for $1 \leq i \leq l$, $[\Lambda^i, q_i, r-, \beta_i]$ is a simple or null stratum, where $q_i = r-$ if $\beta_i = 0$, $q_i = -\nu_{\Lambda^i}(\beta_i)$ otherwise;

(ii) for $1 \leq i, j \leq l$, $i \neq j$, $[\Lambda^i \oplus \Lambda^j, q, r-, \beta_i + \beta_j]$ is not equivalent to a simple or null stratum, with $q = \max\{q_i, q_j\}$.

Remarks 3.3. (i) A simple stratum is semisimple, with the trivial splitting. We will also consider null strata $[\Lambda, n, n, 0]$ as a special case of simple strata, as in §1.2.

(ii) A non-simple semisimple stratum $[\Lambda, n, n-1, \beta]$ is certainly split (by a coarsening of the same splitting), by [6] (2.5.8).
(iii) If $[\Lambda, n, r-, \beta]$ is a semisimple stratum then the associated splitting $V = \bigoplus_{i=1}^{l} V^i$ is determined (upto order) by β: for $i = 1, .., l$, let $\Psi_i(X)$ denote the minimal polynomial of β_i, which is irreducible since $F[\beta_i]$ is a field; then the $\Psi_i(X)$ are distinct (so pairwise coprime), by condition (ii) of Definition 3.2 so the minimal polynomial of β is $\prod_{i=1}^{l} \Psi_i(X)$ and $V^i = \ker \Psi_i(\beta)$.

Note that any stratum satisfying condition (i) of Definition 3.2 is clearly equivalent to a semisimple stratum, by coarsening the splitting suitably. In particular, for $1 \le i \le l$, let $[\Lambda^i, q_i, r, \gamma_i']$ be a simple or null stratum equivalent to $[\Lambda^i, q_i, r, \beta_i]$ and put $\gamma' = \sum_{i=1}^{l} \gamma_i'$. Then $[\Lambda, n, r, \gamma']$ satisfies (i) and hence is equivalent to a semisimple stratum $[\Lambda, n, r, \gamma]$, with splitting $V = \bigoplus_{j=1}^{m} V^{I_j}$, where $\{1, \dots, l\} = \bigcup_{j=1}^{m} I_j$ and $V^{I_j} = \bigoplus_{i \in I_j} V^i$. This allows us to proceed by induction along r for semisimple strata. (Note that, although we allow real values of r, only integer values really play a role.)
In fact, we will have to be a little more careful in the way in which we choose γ. Recall that, given $[\Lambda, n, 0, \beta]$ a simple stratum with $r = -k_0(\beta, \Lambda)$ and $E = F[\beta]$, [7] 5.3 Corollary states that, for W in general position relative to Λ over E, $[\Lambda, n, r, \beta]$ is equivalent to a simple stratum $[\Lambda, n, r, \gamma]$ with $\gamma \in \iota_W(\mathfrak{K}(\mathfrak{A}(E)))$. We show now that we have a similar result for semisimple strata.

Proposition 3.4. *Let $[\Lambda, n, r-, \beta]$ be a semisimple stratum in A, split by $V = \oplus_{i=1}^{l} V^i$ and such that $[\Lambda, n, r, \beta]$ is equivalent to a simple stratum $[\Lambda, n, r, \gamma]$. Put $E_i = F[\beta_i]$; then, given W^i in general position relative to Λ^i over E_i, for $1 \le i \le l$, we may choose $\gamma \in \prod \iota_{W^i}(\mathfrak{K}(\mathfrak{A}(E_i))) \subset \mathcal{M}$.*

Proof We show, by induction, that $[\Lambda, n, t, \beta]$ is equivalent to a simple stratum $[\Lambda, n, t, \gamma^{(t)}]$ as required, $n - 1 \ge t \ge r$, $t \in \mathbb{Z}$.

Let $\psi_i(X) = \varphi_i(X)^{d_i}$ be the characteristic polynomial of the stratum $[\mathfrak{A}(E_i), n_i, n_i - 1, \beta_i]$, where $n_i = n/e(\Lambda^i|\mathfrak{o}_{E_i})$. Then the characteristic polynomial of $[\Lambda^i, n, n-1, \beta_i]$ is $\psi_i(X)^{\delta_i}$, where $\delta_i = \dim_{E_i} V^i$, so the characteristic polynomial of $[\Lambda, n, n-1, \beta]$ is $\prod_{i=1}^{l} \psi_i(X)^{\delta_i}$. This stratum is equivalent to a simple stratum so it is non-split; hence $\varphi_i(X) = \varphi(X)$, for $1 \le i \le l$.
Now we choose $\Phi(X) \in \mathfrak{o}_F[X]$ such that $\varphi(X) = \Phi(X) \pmod{\mathfrak{p}_F}$. Then, by [6] (2.5.11), we can find simple strata $[\mathfrak{A}(E_i), n_i, n_i - 1, \gamma_i]$ equivalent to $[\mathfrak{A}(E_i), n_i, n_i - 1, \beta_i]$ such that the minimal polynomial of γ_i is $\Phi((\varpi_F^{-n/g} X)^{e/g})$. Then we put $\gamma^{(n-1)} = \sum_{i=1}^{l} \iota_{W^i}(\gamma_i)$.

Now suppose we have found $[\Lambda, n, t, \gamma^{(t)}]$ equivalent to $[\Lambda, n, t, \beta]$ as required. We will omit the superscript $^{(t)}$ and put $E_\gamma = F[\gamma]$. We choose tame corestrictions s_γ on A, $A(E_i)$ which are compatible with the (W, E)-decompositions (cf. [6] (1.3.9), (2.2.8)).
We have $[\mathfrak{A}(E_i), n_i, t_i, \gamma] \sim [\mathfrak{A}(E_i), n_i, t_i, \beta_i]$, with $e_i = e(\Lambda^i|\mathfrak{o}_{E_i})$, $t_i = \lfloor \frac{t}{e_i} \rfloor$. Let $[\mathfrak{A}(E_i), n_i, \lfloor \frac{t-1}{e_i} \rfloor, \xi_i]$ be a simple stratum equivalent to $[\mathfrak{A}(E_i), n_i, \lfloor \frac{t-1}{e_i} \rfloor, \beta_i]$. By [6] (2.4.1), $[\mathfrak{A}(E_i)_{\mathfrak{o}_{E_\gamma}}, t_i, \lfloor \frac{t-1}{e_i} \rfloor, s_\gamma(\xi_i - \gamma)]$ is equivalent to a simple or null stratum in $A(E_i)$, where s_γ is a tame corestriction relative to E_γ/F. Let $\psi_i(X) = \varphi_i(X)^{d_i}$ be the characteristic polynomial of this stratum. Then, as above, the characteristic polynomial of $[\Lambda_{\mathfrak{o}_{E_\gamma}}, t, t-1, s_\gamma(\xi - \gamma)]$ is $\prod_{i=1}^{l} \psi_i(X)^{\delta_i}$, where $\xi = \sum_{i=1}^{l} \iota_{W^i}(\xi_i)$.
Now let $[\Lambda, n, t-1, \beta']$ be a simple stratum equivalent to $[\Lambda, n, t-1, \beta]$; then, if $b' = \beta' - \gamma = (\beta' - \beta) + (\beta - \gamma)$, we have $b' \equiv \xi - \gamma \pmod{\mathfrak{a}_{1-t}}$ so

$$[\Lambda_{\mathfrak{o}_{E_\gamma}}, t, t-1, s_\gamma(b')] \sim [\Lambda_{\mathfrak{o}_{E_\gamma}}, t, t-1, s_\gamma(\xi - \gamma)].$$

The former stratum is equivalent to a null or simple stratum, by [6] (2.4.1), so we deduce that $\varphi_i(X) = \varphi(X)$, for $1 \le i \le l$.

First suppose $\varphi(X) = X$, i.e. the strata above are equivalent to null strata. Then, by [6] (2.2.1), for $1 \le i \le l$ there exists a conjugate γ_i' of γ by the group $U^1(\mathfrak{A}(E_i))$ such that $[\mathfrak{A}(E_i), n_i, \lfloor \frac{t-1}{e_i} \rfloor, \gamma_i']$ is equivalent to $[\mathfrak{A}(E_i), n_i, \lfloor \frac{t-1}{e_i} \rfloor, \beta_i]$ and we put $\gamma^{(t-1)} = \sum_{i=1}^{l} \iota_{W^i}(\gamma_i')$.

So we may assume $\varphi(X) \neq X$; in particular, $\lfloor \frac{t-1}{e_i} \rfloor = t_i - 1$. Choose $\Phi(X) \in \mathfrak{o}_{E_\gamma}[X]$ such that $\varphi(X) = \Phi(X) \pmod{\mathfrak{p}_{E_\gamma}}$ and choose $c_i \in B_\gamma(E_i) = \mathrm{End}_{E_\gamma} E_i$ such that

$$[\mathfrak{A}(E_i)_{\mathfrak{o}_{E_\gamma}}, t_i, t_i - 1, s_\gamma(\xi_i - \gamma)] \sim [\mathfrak{A}(E_i)_{\mathfrak{o}_{E_\gamma}}, t_i, t_i - 1, c_i]$$

and c_i has minimal polynomial $\Phi((\varpi_{E_\gamma}^{-t/g} X)^{e/g})$, where $g = (e, t)$ and ϖ_{E_γ} is a (fixed) uniformizer in E_γ.

We put $E_i' = E_\gamma[c_i]$; these are all isomorphic to $E' := E_\gamma[c]$, where c has minimal polynomial $\Phi((\varpi_{E_\gamma}^{-t/g} X)^{e/g})$. Consider the simple stratum $[\mathfrak{A}(E'), n', t', \gamma]$ in $A(E')$ and the derived stratum $[\mathfrak{A}(E')_{\mathfrak{o}_{E_\gamma}}, t', t' - 1, c]$, where $e' = e(\Lambda^i | \mathfrak{o}_{E_i'})$ and $n' = n/e'$, $t' = t/e'$. Choose $b \in A(E')$ such that $\nu_{\mathfrak{A}(E')}(b) = -t'$ and $s_\gamma'(b) = c$, where s_γ' is a tame corestriction on $A(E')$ relative to E_γ/F; then the stratum $[\mathfrak{A}(E'), n', t' - 1, \gamma + b]$ is simple, by [6] (2.2.3).

Now let Y_i be in general position relative to $\mathfrak{A}(E_i)$ over $E' \simeq E_i'$ and consider the stratum $[\mathfrak{A}(E_i), n_i, t_i - 1, \iota_{Y_i}(\gamma + b)]$. This is simple and, as in [6] (2.2.8), some conjugate, by the group $U^1(\mathfrak{A}(E_i))$, $[\mathfrak{A}(E_i), n_i, t_i - 1, \gamma_i']$ is equivalent to $[\mathfrak{A}(E_i), n_i, t_i - 1, \xi_i] \sim [\mathfrak{A}(E_i), n_i, t_i - 1, \beta_i]$. Then $\gamma^{(t-1)} = \sum_{i=1}^{l} \iota_{W^i}(\gamma_i')$ is as required. ■

The previous proposition shows, in particular, that we may choose $\gamma \in \mathcal{M}$, and this is the only property of γ which we will use in §§3.2–3.3. Then each decomposition $V^{I_j} = \bigoplus_{i \in I_j} V^i$ is a decomposition of $F[\gamma_j]$-spaces, $1 \le j \le m$. If s_j is a tame corestriction on A^{I_j, I_j} relative to $F[\gamma_j]/F$, then $[\Lambda^{I_j}_{\mathfrak{o}_{E_j}}, r, r-1, s_j(\beta_{I_j} - \gamma_{I_j})]$ is equivalent to a semisimple stratum with splitting $V^{I_j} = \bigoplus_{i \in I_j} V^i$ ((i) comes from [6] (2.4.1) while (ii) follows by [6] (2.2.8)).

We also have a converse to these observations, which follows from [6] (2.2.8), (2.3.12):

Lemma 3.5 (cf. [6] (2.2.8)). *Let $[\Lambda, n, r, \gamma]$ be a semisimple stratum with splitting $V = \bigoplus_{j=1}^{m} V^j$. Put $E_j = F[\gamma_j]$ and let s_j be a tame corestriction on A^{jj} relative to E_j/F. For $1 \le j \le m$, let $b_j \in \mathfrak{a}_{-r}^{jj}$ be such that $[\Lambda^j_{\mathfrak{o}_{E_j}}, r, r-, s_j(b_j)]$ is equivalent to a semisimple stratum and put $b = \sum_{j=1}^{m} b_j$. Then $[\Lambda, n, r-, \gamma + b]$ is equivalent to a semisimple stratum.*

Now let $[\Lambda, n, 0, \beta]$ be a non-null semisimple stratum and put

$$k_0(\beta, \Lambda) = -\min\{r \in \mathbb{Z} : [\Lambda, n, r, \beta] \text{ is not semisimple}\}. \tag{3.6}$$

Note that this is consistent with the definition for simple strata in §1.2. There are two possibilities here:

(i) For some i, $1 \le i \le l$, we have $k_0(\beta, \Lambda) = k_0(\beta_i, \Lambda^i)$. Then, putting $e_i = e(E_i|F)$, we have

$$\frac{k_0(\beta, \Lambda)}{e(\Lambda|\mathfrak{o}_F)} = \frac{k_F(\beta_i)}{e_i}.$$

(ii) There exist i, j, $1 \le i, j \le l$, such that $[\Lambda^i, q, r, \beta_i]$ and $[\Lambda^j, q, r, \beta_j]$ are simple and $[\Lambda^i \oplus \Lambda^j, q, r, \beta_i + \beta_j]$ is equivalent to a simple stratum $[\Lambda^i \oplus \Lambda^j, q, r, \gamma]$, where $r = -k_0(\beta, \Lambda)$ and $q =$

$q_i = q_j$ are as in Definition 3.2. Then, by [6] (2.4.1), there exist $c_i, c_j \in E_\gamma = F[\gamma]$ such that $[\Lambda^i_{\mathfrak{o}_{E_\gamma}}, r, r-1, s_\gamma(\gamma - \beta_i)] \sim [\Lambda^i_{\mathfrak{o}_{E_\gamma}}, r, r-1, c_i]$, and likewise for j. Since, by definition of $k_0(\beta,\Lambda)$, $[\Lambda^i \oplus \Lambda^j, q, r-1, \beta_i + \beta_j]$ is not equivalent to a simple stratum, the derived strata above do not have the same characteristic polynomial, by [6] (2.2.8). In particular, at least one (say for i) is fundamental and we have $r = -\nu_{\Lambda^i_{\mathfrak{o}_{E_\gamma}}}(c_i)$. Then, putting $e_i = e(E_i|F) = e(E_\gamma|F)$, we have

$$\frac{k_0(\beta,\Lambda)}{e(\Lambda|\mathfrak{o}_F)} = -\frac{\nu_{E_\gamma}(c_i)}{e_i}.$$

Now let $[\Lambda', n', 0, \beta]$ be another semisimple stratum (with splitting $V = \bigoplus_{i=1}^l V^i$). Then, whichever of the two cases above occurs, we have

$$\frac{k_0(\beta,\Lambda)}{e(\Lambda|\mathfrak{o}_F)} = \frac{k_0(\beta,\Lambda')}{e(\Lambda'|\mathfrak{o}_F)}.$$

(For case (ii), note that there is an E_i-basis of V^i which is a splitting of both Λ^i and Λ'^i; taking W^i to be the F-linear span of this basis – so that it is in general position relative to both Λ^i and Λ'^i – and choosing W^j similarly, we may use Proposition 3.4 to choose the same γ for Λ and Λ'.)

3.2 Definitions

We continue in the situation above, so $[\Lambda, n, r-, \beta]$ is a semisimple stratum with splitting $V = \bigoplus_{i=1}^l V^i$ and $[\Lambda, n, r, \beta]$ is equivalent to the semisimple stratum $[\Lambda, n, r, \gamma]$, with $\gamma \in \mathcal{M}$; we write $b = \beta - \gamma$. Let B_β denote the A-centralizer of β; we have $B_\beta = \bigoplus_{i=1}^l B^{ii}_{\beta_i}$. We consider the adjoint map $a_\beta : x \mapsto \beta x - x\beta$, for $x \in A$. Note that the restriction of a_β to A^{ij} is certainly bijective for $i \neq j$. For $k \in \mathbb{Z}$ we put

$$\mathfrak{n}_k(\beta,\Lambda) = \{a \in \mathfrak{a}_0 : a_\beta(x) \in \mathfrak{a}_k\}.$$

We clearly have $\mathfrak{n}_k(\beta,\Lambda)^{ii} = \mathfrak{n}_k(\beta_i,\Lambda^i)$, for $1 \leq i \leq l$.

Lemma 3.7. *For $k \leq r$ we have*

(i) *for $i \neq j$, $\mathfrak{n}_{-k}(\beta,\Lambda)^{ij} \subset \mathfrak{a}_{r-k}$;*

(ii) $\mathfrak{n}_{-k}(\beta,\Lambda) = \mathfrak{b}_{\beta,0} + \mathfrak{n}_{-k}(\beta,\Lambda) \cap \mathfrak{a}_{r-k}$.

Proof We note first that we have $\mathfrak{n}_{-k}(\beta,\Lambda) \cap \mathfrak{a}_{r-k} = \mathfrak{n}_{-k}(\gamma,\Lambda) \cap \mathfrak{a}_{r-k}$, since, for $x \in \mathfrak{a}_{-r-k}$, $a_\beta(x) \equiv a_\gamma(x) \pmod{\mathfrak{a}_{-k}}$. Also, (ii) holds in the simple case by [6] (1.4.9) (see also [12] §4) and hence follows immediately from (i) in the general case.

Let us fix $i \neq j$. We put $q = \max\{q_i, q_j\}$ and let $t \in \mathbb{Z}$ with $r \leq t \leq q$ be minimal such that $[\Lambda^i \oplus \Lambda^j, q, t, \beta_i + \beta_j]$ is equivalent to a null or simple stratum, say $[\Lambda^i \oplus \Lambda^j, q, t, \zeta]$. Put $E = F[\zeta]$ and let s_ζ be a tame corestriction on $\mathrm{End}_F(V^i \oplus V^j)$ relative to E/F. Put $b = \beta_i + \beta_j - \zeta$; then, by the minimality of t and [6] (2.2.8), the derived stratum $[\Lambda^i_{\mathfrak{o}_E} \oplus \Lambda^j_{\mathfrak{o}_E}, t, t-1, s_\zeta(b)]$ is split. We put $s = -k_0(\zeta, \Lambda^i \oplus \Lambda^j)$.

Now let $y \in \mathfrak{a}^{ij}_{-k}$; then $s_\zeta(y) \in \mathfrak{b}^{ij}_{\zeta,-k}$ so, by [7] (3.7) Lemma 4, there exists $z \in \mathfrak{b}_{\zeta,t-k}$ such that $a_{s_\zeta(b)}(z) = s_\zeta(y)$. Then $y - a_b(z) \in \ker s_\zeta \cap \mathfrak{a}_{-k}$ so, by [6] (1.4.10), there exists $x \in \mathfrak{n}_{-k}(\zeta, \Lambda^i \oplus \Lambda^j) \cap$

$\mathfrak{a}_{s-k}$ such that $a_\zeta(x) = y - a_b(z)$. Then $a_\beta(x+z) \equiv y \pmod{\mathfrak{a}_{1-k}}$ and, since $x+z \in \mathfrak{a}_{r-k}$, it follows easily that the image $a_\beta\left(\mathfrak{a}^{ij}_{s-k}\right)$ contains $\mathfrak{a}^{ij}_{-k}$. Since a_β is bijective on A^{ij}, we get an isomorphism

$$\mathfrak{n}_{-k}(\beta,\Lambda) \cap \mathfrak{a}^{ij}_{s-k} \xrightarrow{\sim} \mathfrak{a}^{ij}_{-k}. \tag{3.8}$$

Since $s > r$, we have $\mathfrak{n}_{-k}(\beta,\Lambda)^{ij} \subset \mathfrak{a}_{s-k} \subset \mathfrak{a}_{r-k}$, as required. ■

Now we define the orders $\mathfrak{H}(\beta,\Lambda) \subset \mathfrak{J}(\beta,\Lambda)$ inductively by

$$\begin{aligned} \mathfrak{H}(\beta,\Lambda) &= \mathfrak{b}_{\beta,0} + \mathfrak{H}^{\frac{r}{2}+}(\gamma,\Lambda), \\ \mathfrak{J}(\beta,\Lambda) &= \mathfrak{b}_{\beta,0} + \mathfrak{J}^{\frac{r}{2}}(\gamma,\Lambda), \end{aligned}$$

with $\mathfrak{H}(0,\Lambda) = \mathfrak{J}(0,\Lambda) = \mathfrak{a}_0(\Lambda)$. Note that this is consistent with the definitions of [6] §3.1 in the simple case, by *op. cit.* (3.1.9)(v), (3.1.10)(v). Moreover, as in the Remark following *loc. cit.*, to check that this definition of $\mathfrak{H}(\beta,\Lambda)$ is independent of the choice of $\gamma \in \mathcal{M}$, we need only prove:

Lemma 3.9 (cf. [6] (3.1.9)(v)). *Let $[\Lambda, n, r-, \beta']$ be a semisimple stratum equivalent to $[\Lambda, n, r-, \beta]$ and with the same splitting. Then*

$$\mathfrak{H}^{\frac{r}{2}}(\beta',\Lambda) = \mathfrak{H}^{\frac{r}{2}}(\beta,\Lambda).$$

Proof We assume in this proof that $\mathfrak{H}(\beta',\Lambda)$ has been defined relative to the same semisimple stratum $[\Lambda, n, r, \gamma]$; then the only possible difference between the two orders must lie in $\mathcal{M}$, since B_β and $B_{\beta'}$ are both contained in $\mathcal{M}$. But, for each i, $[\Lambda^i, n, r-, \beta'_i]$ and $[\Lambda^i, n, r-, \beta_i]$ are equivalent simple strata so

$$\mathfrak{H}^{\frac{r}{2}}(\beta',\Lambda)^{ii} = \mathfrak{H}^{\frac{r}{2}}(\beta'_i,\Lambda^i) = \mathfrak{H}^{\frac{r}{2}}(\beta_i,\Lambda^i) = \mathfrak{H}^{\frac{r}{2}}(\beta,\Lambda)^{ii},$$

by the simple case [6] (3.1.9)(v). ■

Similarly, $\mathfrak{J}(\beta,\Lambda)$ is independent of the choice of $\gamma \in \mathcal{M}$.

Lemma 3.10 (cf. [6] (3.1.10)). *For $0 \le k \le r$, we have*

(i) $\mathfrak{n}_{-\frac{k}{2}}(\beta,\Lambda) \cap \mathfrak{a}_{r-\frac{k}{2}} \subset \mathfrak{J}^{r-\frac{k}{2}}(\beta,\Lambda)$;

(ii) $\mathfrak{J}^{\frac{k}{2}}(\beta,\Lambda)$ *is an* $\mathfrak{n}_{-k}(\beta,\Lambda)$*-bimodule.*

Proof We proceed by induction on r, with the simple case given by [6] (3.1.10). By Lemma 3.7(ii),

$$\begin{aligned} \mathfrak{n}_{-\frac{k}{2}}(\beta,\Lambda) \cap \mathfrak{a}_{r-\frac{k}{2}} &= \mathfrak{n}_{-\frac{k}{2}}(\gamma,\Lambda) \cap \mathfrak{a}_{r-\frac{k}{2}} \\ &= \mathfrak{b}_{\gamma,r-\frac{k}{2}} + \mathfrak{n}_{-\frac{k}{2}}(\gamma,\Lambda) \cap \mathfrak{a}_{r-\frac{k}{2}+}. \end{aligned}$$

Now, since $r - \frac{k}{2} \ge \frac{r}{2}$, we have that $\mathfrak{J}^{r-\frac{k}{2}}(\beta,\Lambda) = \mathfrak{J}^{r-\frac{k}{2}}(\gamma,\Lambda)$ and also $\mathfrak{b}_{\gamma,r-\frac{k}{2}} \subset \mathfrak{J}^{r-\frac{k}{2}}(\gamma,\Lambda)$, while $\mathfrak{n}_{-\frac{k}{2}}(\gamma,\Lambda) \cap \mathfrak{a}_{r-\frac{k}{2}+} \subset \mathfrak{J}^{r-\frac{k}{2}+}(\gamma,\Lambda) \subset \mathfrak{J}^{r-\frac{k}{2}}(\gamma,\Lambda)$, by induction, so (i) follows.

We have that $\mathfrak{J}^{\frac{k}{2}}(\beta,\Lambda) = \mathfrak{b}_{\beta,\frac{k}{2}} + \mathfrak{J}^{\frac{r}{2}}(\gamma,\Lambda)$ and also $\mathfrak{n}_{-k}(\beta,\Lambda) \subset \mathfrak{n}_{-r}(\beta,\Lambda) = \mathfrak{n}_{-r}(\gamma,\Lambda)$ so that $\mathfrak{n}_{-k}(\beta,\Lambda)\mathfrak{J}^{\frac{r}{2}}(\gamma,\Lambda) \subset \mathfrak{J}^{\frac{r}{2}}(\gamma,\Lambda)$ by induction. Also, $\mathfrak{n}_{-k}(\beta,\Lambda) = \mathfrak{b}_{\beta,0} + \mathfrak{n}_{-k}(\beta,\Lambda) \cap \mathfrak{a}_{r-k}$ by Lemma 3.7(ii). The result now follows from (i) since

$$\begin{aligned}(\mathfrak{n}_{-k}(\beta,\Lambda) \cap \mathfrak{a}_{r-k})\mathfrak{b}_{\beta,\frac{k}{2}} &\subset \mathfrak{n}_{-\frac{k}{2}}(\beta,\Lambda) \cap \mathfrak{a}_{r-\frac{k}{2}} \\ &\subset \mathfrak{J}^{r-\frac{k}{2}}(\beta,\Lambda) \subset \mathfrak{J}^{\frac{k}{2}}(\beta,\Lambda).\end{aligned}$$

■

Similarly, $\mathfrak{n}_{-\frac{k}{2}}(\beta,\Lambda) \cap \mathfrak{a}_{r-\frac{k}{2}} \subset \mathfrak{J}^{r-\frac{k}{2}}(\beta,\Lambda)$ and $\mathfrak{H}^{\frac{k}{2}+}(\beta,\Lambda)$ is an $\mathfrak{n}_{-k}(\beta,\Lambda)$-bimodule, for $0 \le k \le r$.

Lemma 3.11 (cf. [6] (3.1.13)). (i) *For $k < r$, we have*

$$(\mathfrak{n}_{-k}(\beta,\Lambda) \cap \mathfrak{a}_{r-k})\mathfrak{J}^{\frac{k}{2}}(\beta,\Lambda) \subset \mathfrak{H}^{\frac{k}{2}+}(\beta,\Lambda).$$

(ii) *For $k > 0$, we have $\mathfrak{b}_{\beta,1}\mathfrak{J}^k(\beta,\Lambda) \subset \mathfrak{H}^{k+1}(\beta,\Lambda)$.*

(iii) *For $k, l > 1$, $\mathfrak{J}^k(\beta,\Lambda)\mathfrak{J}^l(\beta,\Lambda) \subset \mathfrak{H}^{k+l}(\beta,\Lambda)$.*

(iv) *For $k > 0$, $\mathfrak{H}^k(\beta,\Lambda)$ is a two-sided ideal of $\mathfrak{J}(\beta,\Lambda)$.*

Proof The simple case is given by [6] (3.1.13). We have $\mathfrak{J}^{\frac{k}{2}} = \mathfrak{b}_{\beta,\frac{k}{2}} + \mathfrak{J}^{\frac{r}{2}}(\gamma,\Lambda)$. Now

$$\begin{aligned}(\mathfrak{n}_{-k}(\beta,\Lambda) \cap \mathfrak{a}_{r-k})\mathfrak{b}_{\beta,\frac{k}{2}} &\subset \mathfrak{n}_{-\frac{k}{2}}(\beta,\Lambda) \cap \mathfrak{a}_{r-\frac{k}{2}} \\ &\subset \mathfrak{H}^{r-\frac{k}{2}}(\beta,\Lambda) \subset \mathfrak{H}^{\frac{k}{2}+}(\beta,\Lambda),\end{aligned}$$

since $k < r$. On the other hand,

$$(\mathfrak{n}_{-k}(\beta,\Lambda) \cap \mathfrak{a}_{r-k})\mathfrak{J}^{\frac{r}{2}}(\beta,\Lambda) = (\mathfrak{n}_{-k}(\gamma,\Lambda) \cap \mathfrak{a}_{r-k})\mathfrak{J}^{\frac{r}{2}}(\gamma,\Lambda) \subset \mathfrak{H}^{\frac{r}{2}+}(\gamma,\Lambda)$$

by induction.
For (ii), it suffices to show that $\mathfrak{b}_{\beta,1}\mathfrak{J}^{\frac{r}{2}}(\beta,\Lambda) \subset \mathfrak{H}(\beta,\Lambda)$, which is immediate from (i), with $k = r-1$. The remaining assertions follow, as in [6] (3.1.13). ■

Now, for $m \ge -1$, we put $H^{m+1}(\beta,\Lambda) = \mathfrak{H}(\beta,\Lambda) \cap U^{m+1}(\Lambda)$ and similarly for $J^{m+1}(\beta,\Lambda)$. We will usually write $J(\beta,\Lambda) = J^0(\beta,\Lambda)$. Note also that $H^{m+1}(\beta,\Lambda) \cap \widetilde{G}_i = H^{m+1}(\beta_i,\Lambda^i)$, for $1 \le i \le l$,

Corollary 3.12 (cf. [6] (3.1.15)). *Abbreviating $H^m = H^m(\beta,\Lambda)$ and likewise for J^m, we have:*

(i) *for $0 < m \le \frac{r}{2}+$ and $0 < l \le \frac{r}{2}$,*

$$H^m = (U^m(\Lambda) \cap B_\beta) \cdot H^{\frac{r}{2}+}, \qquad J^l = (U^l(\Lambda) \cap B_\beta) \cdot J^{\frac{r}{2}};$$

(ii) *for $m \ge 0$, $H^m \subset J^m$ and, for $m > 0$, $H^m \triangleleft J$;*

(iii) *for $k, l > 0$, $[J^k, J^l] \subset H^{k+l}$.*

We also remark that $H(\beta,\Lambda)$ and $J(\beta,\Lambda)$ have Iwahori decompositions with respect to M.

Definition 3.13 (cf. [6] (3.2.1), (3.2.3), [13] (3.11))). For $0 \le m < r$, the set $\mathcal{C}(\Lambda, m, \beta)$ of *semisimple characters* of $H^{m+}(\beta, \Lambda)$ is the set of characters θ such that

(i) $\theta|_{H^{m+}(\beta,\Lambda)\cap \widetilde{G}_i} \in \mathcal{C}(\Lambda^i, m, \beta_i)$, for $1 \le i \le l$;

(ii) if $m' = \max\{m, \frac{r}{2}\}$, the restriction $\theta|_{H^{m'+}(\beta,\Lambda)}$ is of the form $\theta_0\psi_b$ for some $\theta_0 \in \mathcal{C}(\Lambda, m', \gamma)$, where $b = \beta - \gamma$.

Note that this too is consistent with the definitions of [6] §3.2, by [6] (3.3.20).

Remarks 3.14. (i) If $m \ge \frac{r}{2}$ then we have $H^{m+}(\beta, \Lambda) = H^{m+}(\gamma, \Lambda)$ and condition (i) of the definition is implied by (ii). In particular we have a bijection

$$\begin{array}{rcl} \mathcal{C}(\Lambda, m, \gamma) & \to & \mathcal{C}(\Lambda, m, \beta); \\ \theta & \mapsto & \theta\psi_b. \end{array}$$

(ii) Suppose $[\Lambda, n, r, \gamma']$ is another semisimple stratum which is equivalent to $[\Lambda, n, r, \beta]$, with $\gamma' \in \mathcal{M}$. Then we could define the set $\mathcal{C}(\Lambda, m, \beta)$ with respect to this stratum also. However, these definitions coincide: this follows as in [6] (3.3.20) from the fact that, if $[\Lambda, n, r-, \beta']$ is a semisimple stratum equivalent to $[\Lambda, n, r-, \beta]$ and with the same splitting, then

$$\mathcal{C}\left(\Lambda, \tfrac{r}{2}-, \beta'\right) = \mathcal{C}\left(\Lambda, \tfrac{r}{2}-, \beta\right).\psi_{\beta'-\beta}$$

(cf. [6] (3.3.20)(ii)). To prove this, we assume that $\mathcal{C}\left(\Lambda, \frac{r}{2}-, \beta'\right)$ has been defined relative to the same semisimple stratum $[\Lambda, n, r, \gamma]$; then, given $\theta \in \mathcal{C}\left(\Lambda, \frac{r}{2}-, \beta\right)$, the character $\theta\psi_{\beta'-\beta}$ of $H^{\frac{r}{2}}(\beta, \Lambda) = H^{\frac{r}{2}}(\beta', \Lambda)$ certainly satisfies condition (ii) of Definition 3.13, while condition (i) comes from the simple case [6] (3.3.20)(i).

(iii) The set $\mathcal{C}(\Lambda, m, \beta)$ is indeed independent of r since, if $r < -k_0(\beta, \Lambda)$, we may take $\gamma = \beta$ in the definitions.

Lemma 3.15. (i) *Let $0 \le m < r$ and $\theta_0 \in \mathcal{C}(\Lambda, \frac{r}{2}, \gamma)$. For $1 \le i \le l$, let $\theta_i \in \mathcal{C}(\Lambda^i, m, \beta_i)$ be such that θ_i agrees with $\theta_0\psi_b$ on $H^{\frac{r}{2}+}(\beta, \Lambda) \cap \widetilde{G}_i$. Then there exists a unique character θ of $H^{m+} = H^{m+}(\beta, \Lambda)$ such that $\theta|_{H^{m+}\cap M} = \bigotimes_{i=1}^{l} \theta_i$ and $\theta|_{H^{\frac{r}{2}+}(\beta,\Lambda)} = \theta_0\psi_b$. Moreover, θ is trivial on N_l and N_u.*

(ii) *Let $\theta_0 \in \mathcal{C}(\Lambda, \frac{r}{2}, \gamma)$. Then $J(\beta, \Lambda)$ normalizes $\theta_0\psi_b|_{H^{\frac{r}{2}+}(\gamma,\Lambda)}$.*

(iii) *Let $0 \le m \le r-1$ and $\theta \in \mathcal{C}(\Lambda, m, \beta)$. Then θ is normalized by $J(\beta, \Lambda)$.*

Proof We proceed by induction, the case of a simple stratum being given by [6] (3.3.1); so we assume the results hold for γ, β_i.

(i) The characters $\theta_0\psi_b$ and $\bigotimes_{i=1}^{l} \theta_i$ certainly agree where they are both defined. Now $H^{m+} \cap M$ normalizes $H^{\frac{r}{2}+}(\beta, \Lambda)$ since $\theta_0\psi_b|_{H^{\frac{r}{2}+}(\beta,\Lambda)\cap\widetilde{G}_i} \in \mathcal{C}(\Lambda^i, \frac{r}{2}, \beta_i)$, for $1 \le i \le l$. But $\theta_0\psi_b$ is trivial on N_l and N_u and hence $H^{m+} \cap M$ normalizes the pair $(H^{\frac{r}{2}+1}(\gamma, \Lambda), \theta_0\psi_b)$, by Corollary 2.2. The assertions are now clear.

(ii) For $j \in J^{\frac{r}{2}}(\gamma, \Lambda)$, this is implied by (iii) for γ and, as in (i), $J \cap M$ normalizes the pair $(H^{\frac{r}{2}+}(\gamma, \Lambda), \theta_0\psi_b)$.

(iii) For $m \ge \frac{r}{2}$, this follows from (ii) so assume $m < \frac{r}{2}$; then, for $j \in J(\beta, \Lambda)$, $h \in H^{\frac{r}{2}+}(\beta, \Lambda)$ we have $\theta(jhj^{-1}) = \theta(h)$. Also, for $j \in J(\beta, \Lambda) \cap N_l$, $h \in H^{m+}(\beta, \Lambda) \cap M$, we have $\theta([j, h]) = 1$ since $[j, h] \in H^{m+}(\beta, \Lambda) \cap N_l \subset \ker\theta$, and likewise for N_u. Hence we need only check that $J(\beta, \Lambda) \cap M$ normalizes $\bigotimes_{i=1}^{l} \theta_i$, which follows from the simple case. ■

For $m \leq r$, we put

$$\mathfrak{m}_m(\beta,\Lambda) = \mathfrak{n}_{-m}(\beta,\Lambda) \cap \mathfrak{a}_{r-m} + \mathfrak{J}^{\frac{r}{2}}(\beta,\Lambda).$$

Note that for $m \leq \frac{r}{2}$ we have $\mathfrak{m}_m(\beta,\Lambda) = \mathfrak{J}^{\frac{r}{2}}(\beta,\Lambda)$. On the other hand, for $m > \frac{r}{2}$, $\mathfrak{n}_{-m}(\beta,\Lambda) \cap \mathfrak{a}_{r-m} = \mathfrak{n}_{-m}(\gamma,\Lambda) \cap \mathfrak{a}_{r-m} = \mathfrak{b}_{\gamma,r-m} + \mathfrak{n}_{-m}(\beta,\Lambda) \cap \mathfrak{a}_{r-m+}$, by Lemma 3.7(ii); hence $\mathfrak{m}_m(\beta,\Lambda) = \mathfrak{b}_{\gamma,r-m} + \mathfrak{m}_m(\gamma,\Lambda)$.
We note that, if $[\Lambda, n, r-, \beta]$ is simple, then $\mathfrak{m}_m(\beta,\Lambda)$ is not the same as the lattice $\mathfrak{M}_m(\beta,\Lambda)$ defined in [7] (5.6), unless $\lceil r \rceil = -k_0(\beta,\Lambda)$; otherwise, we have $\mathfrak{m}_m(\beta,\Lambda) = \mathfrak{b}_{\beta,r-m} + \mathfrak{M}_m(\beta,\Lambda)$ so $\mathfrak{m}_m(\beta,\Lambda)$ does in fact depend on r, though only in a rather trivial way.

For $0 \leq m < r$, we put $\Gamma_m(\beta,\Lambda) = 1 + \mathfrak{m}_m(\beta,\Lambda)$ and we also put $\Gamma_r(\beta,\Lambda) = \mathfrak{m}_r(\beta,\Lambda)^\times$. Hence we have, for $m > \frac{r}{2}$, $\Gamma_m(\beta,\Lambda) = (U_{r-m}(\Lambda) \cap B_\gamma)\Gamma_m(\gamma,\Lambda)$, with the first factor normalizing the second.

Lemma 3.16. *Let $0 \leq m < r$ and $\theta \in \mathcal{C}(\Lambda, m, \beta)$. Then θ is normalized by $\Gamma_m(\beta,\Lambda)$.*

Proof For $m \leq \frac{r}{2}$ this is weaker than Lemma 3.15 so suppose $m > \frac{r}{2}$ so that $\Gamma_m(\beta,\Lambda) = (U_{r-m}(\Lambda) \cap B_\gamma)\Gamma_m(\gamma,\Lambda)$ and $\theta = \theta_0\psi_b$. Then $\Gamma_m(\beta,\Lambda)$ normalizes θ_0, by induction and Lemma 3.15. But we have $\Gamma_m(\beta,\Lambda) \subset U_{r-m}(\Lambda)$ so $\Gamma_m(\beta,\Lambda)$ clearly normalizes ψ_b also. ■

3.3 Intertwining

In this section we calculate the intertwining of semisimple characters. We remark that we certainly have $B_\beta^\times \subset I_M(\theta|_{H^{m+}(\beta,\Lambda)\cap M})$, by the simple case [6] (3.3.2), and hence $B_\beta^\times \subset I_{\widetilde{G}}(\theta)$ by Lemma 2.1. Then, by Lemma 3.16, we certainly have

$$I_{\widetilde{G}}(\theta) \supset \Gamma_m(\beta,\Lambda) B_\beta^\times \Gamma_m(\beta,\Lambda).$$

We will show that we in fact have equality here. First we need some exact sequences.

Lemma 3.17 (cf. [7] (6.3) Lemma). *Let $0 \leq m < r$. The sequence*

$$0 \to \mathfrak{b}_{\beta,r-m} \to \mathfrak{m}_m(\beta,\Lambda) \xrightarrow{a_\beta} (\mathfrak{H}^{m+}(\beta,\Lambda))^* \cap \operatorname{im} a_\beta \to 0$$

is exact. Moreover, if $h \in B_\beta^\times$ and $0 \to \mathfrak{l}_1 \to \mathfrak{l}_2 \to \mathfrak{l}_3 \to 0$ denotes the above sequence, then the sequence

$$0 \to h^{-1}\mathfrak{l}_1^{ij}h + \mathfrak{l}_1^{ij} \to h^{-1}\mathfrak{l}_2^{ij}h + \mathfrak{l}_2^{ij} \to h^{-1}\mathfrak{l}_3^{ij}h + \mathfrak{l}_3^{ij} \to 0$$

is also exact, for any $1 \leq i, j \leq l$.

Proof In the (i,i)-blocks, the sequences are exact by the simple case, [7] (6.3) Lemma (recall that $\mathfrak{m}_m(\beta,\Lambda)^{ii} = \mathfrak{b}_{\beta_i,r-m} + \mathfrak{M}_m(\beta_i,\Lambda^i)$), while in the (i,j)-blocks, $i \neq j$, the exactness of the first sequence says that a_β induces an isomorphism

$$\mathfrak{m}_m(\beta,\Lambda)^{ij} \xrightarrow{\sim} (\mathfrak{H}^{m+}(\beta,\Lambda))^{*ij}.$$

We put $q = \max\{q_i, q_j\}$ and let $r \leq t \leq q$ be minimal such that $[\Lambda^i \oplus \Lambda^j, q, t, \beta_i + \beta_j]$ is equivalent to a null or simple stratum, say $[\Lambda^i \oplus \Lambda^j, q, t, \zeta]$. Put $E = F[\zeta]$ and let s_ζ be a tame corestriction on $\operatorname{End}_F(V^i \oplus V^j)$ relative to E/F. Also put $b = \beta_i + \beta_j - \zeta$ and $s = -k_0(\zeta,\Lambda)$. The derived stratum $[\Lambda^i_{\mathfrak{o}_E} \oplus \Lambda^j_{\mathfrak{o}_E}, t, t-1, s_\zeta(b)]$ is split.

We have

$$\mathfrak{m}_m(\beta,\Lambda)^{ij} = \mathfrak{n}_{-m}(\zeta,\Lambda^i\oplus\Lambda^j)\cap\mathfrak{a}^{ij}_{s-m} + \mathfrak{J}^{\frac{s}{2}}(\zeta,\Lambda^i\oplus\Lambda^j)^{ij},$$
$$(\mathfrak{H}^{m+}(\beta,\Lambda))^{*ij} = (\mathfrak{H}^{m'+}(\zeta,\Lambda^i\oplus\Lambda^j))^{*ij},$$

where $m' = \max\{m, \frac{s}{2}\}$. Then

$$a_\zeta(\mathfrak{m}_m(\beta,\Lambda)^{ij}) \subset \mathfrak{a}^{ij}_{-m} + a_\zeta(\mathfrak{J}^{\frac{s}{2}}(\zeta,\Lambda^i\oplus\Lambda^j)^{ij}),$$

which is contained in $(\mathfrak{H}^{m+}(\beta,\Lambda))^{*ij}$ by the simple case, and

$$a_b(\mathfrak{m}_m(\beta,\Lambda)^{ij}) \subset \mathfrak{a}^{ij}_{s-t-m} + \mathfrak{a}^{ij}_{\frac{s}{2}-t} \subset \mathfrak{a}^{ij}_{-m'}.$$

Hence we have $a_\beta(\mathfrak{m}_m(\beta,\Lambda)^{ij}) \subset (\mathfrak{H}^{m+}(\beta,\Lambda))^{*ij}$ so we need only check surjectivity.

Let $y \in (\mathfrak{H}^{m+}(\beta,\Lambda))^{*ij}$; then $s_\zeta(y) \in \mathfrak{b}^{ij}_{\zeta,-m'}$ so, by [7] (3.7) Lemma 4, there exists $z \in \mathfrak{b}^{ij}_{\zeta,t-m'}$ such that $s_\zeta(y) = a_{s_\zeta(b)}(z)$. Then $y - a_b(z) \in (\mathfrak{H}^{m+}(\beta,\Lambda))^{*ij} \cap \ker s_\zeta$ so, by the simple case, there exists $x \in \mathfrak{m}_{m'}(\beta,\Lambda)^{ij} = \mathfrak{J}^{\frac{s}{2}}(\zeta,\Lambda^i\oplus\Lambda^j)^{ij}$ such that $a_\zeta(x) = y - a_b(z)$. Then $a_\beta(x+z) = y + a_b(x)$ and we have $a_b(x) \in \mathfrak{a}^{ij}_{\frac{s}{2}-t} \subset \mathfrak{a}^{ij}_{-\frac{s}{2}}$. Then, by (3.8) with $k = \frac{s}{2}$, there exists $v \in \mathfrak{n}_{-\frac{s}{2}}(\beta,\Lambda)\cap\mathfrak{a}^{ij}_{\frac{s}{2}} \subset \mathfrak{J}^{\frac{s}{2}}(\zeta,\Lambda^i\oplus\Lambda^j)^{ij}$ such that $a_\beta(z) = a_b(x)$. Then we have $y = a_\beta(x+z-v)$.
The exactness of the second sequence is now clear, since all the $\mathfrak{l}_k$ are $\mathfrak{b}_{\beta,0}$-modules and the canonical projections $\mathbf{1}^i$ lie in $\mathfrak{b}_{\beta,0}$. ■

Similarly, we have an exact sequence

$$0 \to \mathfrak{b}_{\beta,m+} \to \mathfrak{H}^{m+}(\beta,\Lambda) \xrightarrow{a_\beta} (\mathfrak{m}_m(\beta,\Lambda))^* \cap \operatorname{im} a_\beta \to 0.$$

In particular, with $m = \frac{r}{2}$ we have

$$0 \to \mathfrak{b}_{\beta,\frac{r}{2}+} \to \mathfrak{H}^{\frac{r}{2}+}(\beta,\Lambda) \xrightarrow{a_\beta} (\mathfrak{J}^{\frac{r}{2}}(\beta,\Lambda))^* \cap \operatorname{im} a_\beta \to 0. \tag{3.18}$$

Lemma 3.19. *Let $0 \le m < r$, $g \in \Gamma_m(\beta,\Lambda)$, $h \in H^{\frac{m}{2}+}$ and $\theta \in \mathcal{C}(\Lambda,\frac{r}{2},\beta)$. Then the commutator $[g,h] \in H^{\frac{r}{2}+}(\beta,\Lambda)$ and*

$$\theta[g,h] = \psi_{g^{-1}\beta g-\beta}(h).$$

Proof We proceed by induction, the simple case being given by [6] (3.2.11). We suppose first that $m > \frac{r}{2}$ so that we can write $g = ug'$ with $u \in (U_{r-m}(\Lambda)\cap B_\gamma)$, $g' \in \Gamma_m(\gamma,\Lambda)$. Also, if $[\Lambda,n,r,\gamma]$ is a semisimple stratum equivalent to $[\Lambda,n,r,b]$, then $\theta = \theta_0\psi_b$, for some $\theta_0 \in \mathcal{C}(\Lambda,\frac{r}{2},\gamma)$ and $b = \beta-\gamma$. Then

$$\begin{aligned} \theta_0[g,h] = \theta_0[ug',h] &= \theta_0[g',h] && \text{since } u \text{ normalizes } \theta_0 \\ &= \psi_{g'^{-1}\gamma g'-\gamma}(h) && \text{by induction} \\ &= \psi_{g^{-1}\gamma g-\gamma}(h) && \text{since } u \text{ commutes with } \gamma. \end{aligned}$$

We easily see that $\psi_b[g,h] = \psi_{g^{-1}bg-b}(h)$ so the result holds for $m > \frac{r}{2}$.
If $m \le \frac{r}{2}$ then $g \in \Gamma_m = J^{\frac{r}{2}}$ normalizes θ. Then the result follows from the fact that $a_\beta(\mathfrak{m}_m(\beta,\Lambda)) \subset (\mathfrak{H}^{\frac{r}{2}+}(\beta,\Lambda))^*$, from Lemma 3.17. ■

Corollary 3.20. *Let $0 \le m < r$, $g \in \Gamma_m(\beta,\Lambda)$ and $\theta \in \mathcal{C}(\Lambda, \frac{m}{2}, \beta)$. Then g normalizes $H^{\frac{m}{2}+}(\beta,\Lambda)$ and*

$$\theta^g = \theta\psi_{g^{-1}\beta g-\beta}.$$

Proof $\mathfrak{H}^{\frac{m}{2}+}$ is an $\mathfrak{n}_{-m}$-bimodule so g certainly normalizes $H^{\frac{m}{2}+}$ and, for $h \in H^{\frac{m}{2}+}$, $\theta^g(h) = \theta(h)\theta([g,h])$. But $[g,h] \in H^{\frac{r}{2}+}$ so $\theta([g,h])$ depends only on the restriction $\theta|H^{\frac{r}{2}+}$ and, by Lemma 3.19, $\theta([g,h]) = \psi_{g^{-1}\beta g-\beta}(h)$, as required. ∎

Corollary 3.21. *Let $0 \le m < r$, $g \in \Gamma_{m+}(\beta,\Lambda)B_\beta^\times\Gamma_{m+}(\beta,\Lambda)$ and $\theta \in \mathcal{C}(\Lambda, m, \beta)$. Then, as characters of $H^{m+}(\beta,\Lambda) \cap g^{-1}H^{m+}(\beta,\Lambda)g$, we have*

$$\theta^g = \theta\psi_{g^{-1}\beta g-\beta}.$$

Proof For $g \in \Gamma_{m+}$, this is given by Corollary 3.20. Now consider $g = ybh$, with $y, h \in \Gamma_{m+}$, $b \in B_\beta^\times$. Then, since Γ_{m+} normalizes H^{m+}, for any $x \in H^{m+} \cap g^{-1}H^{m+}g$, we also have $x \in h^{-1}b^{-1}H^{m+}bh$ and $x \in h^{-1}H^{m+}h$. Hence we have

$$\begin{aligned}\theta^g(x) = \theta^y(bhxh^{-1}b^{-1}) &= \theta(bhxh^{-1}b^{-1})\psi_{y^{-1}\beta y-\beta}(bhxh^{-1}b^{-1}) \\ &= \theta(hxh^{-1})\psi_{g^{-1}\beta g-h^{-1}\beta h}(x) \\ &= \theta(x)\psi_{h^{-1}\beta h-\beta}(x)\psi_{g^{-1}\beta g-h^{-1}\beta h}(x) \\ &= \theta(x)\psi_{g^{-1}\beta g-\beta}(x)\end{aligned}$$

as required. ∎

Theorem 3.22 (cf. [6] (3.3.2), [7] (6.4)). *Let $0 \le m < r$ and let $\theta \in \mathcal{C}(\Lambda, m, \beta)$. Then we have*

$$I_{\widetilde{G}}(\theta) = \Gamma_m(\beta,\Lambda)B_\beta^\times\Gamma_m(\beta,\Lambda).$$

Proof We proceed by induction on r, the simple case being given by [6] (3.3.2). Further, the result for $m < \frac{r}{2}$ follows from the case $m = \lfloor\frac{r}{2}\rfloor$ since then $\mathfrak{m}_m(\beta,\Lambda) = \mathfrak{m}_{\lfloor\frac{r}{2}\rfloor}(\beta,\Lambda)$. So we may assume $m \ge \lfloor\frac{r}{2}\rfloor$. By Lemma 3.16, Lemma 2.1 and the simple case, we need only show that $I_{\widetilde{G}}(\theta) \subset \Gamma_m M \Gamma_m$.
Since the result for all m follows from that for integral m, we suppose $m \in \mathbb{Z}$ (in particular, we will write $m+1$ for $m+$, etc.). We proceed by induction on m, beginning with the case "$m = r$". That is, we show

$$I_{\widetilde{G}}(\theta|_{H^{r+1}(\beta,\Lambda)}) = \Gamma_r(\beta,\Lambda)B_\beta^\times\Gamma_r(\beta,\Lambda).$$

But this is immediate from induction on r, since $\theta|_{H^{r+1}(\beta,\Lambda)} \in \mathcal{C}(\Lambda, r, \beta)$ and $B_\beta^\times \subset (U_0(\Lambda) \cap B_\beta)M(U_0(\Lambda) \cap B_\beta)$. So we suppose $m < r$ and we have the result for $m+1$. Let $g \in I_{\widetilde{G}}(\theta)$; then g certainly intertwines the restriction of θ to $H^{m+2}(\beta,\Lambda)$ so we can write $g = yhy'$, with $y, y' \in \Gamma_{m+1}$ and $h \in B_\beta^\times$. Also, by Lemma 3.21, we have

$$\theta^g = \theta\psi_{g^{-1}\beta g-\beta} \qquad \text{as characters of } H^{m+1}(\beta,\Lambda) \cap g^{-1}H^{m+1}(\beta,\Lambda)g.$$

Now we proceed by induction on l, the case $l = 1$ being the simple case. Let $V = W^1 \oplus W^2$ be a coarsening of the splitting $V = \bigoplus_{i=1}^l V^i$. We now use our block notation with respect this new splitting (so A^{21} denotes $\mathrm{Hom}_F(W^1, W^2)$, etc.), but denote the Levi subgroup and unipotent radicals M', N'_l, N'_u. By induction, we assume that the result holds for $\theta|_{\widetilde{G}^i} \in \mathcal{C}(\Lambda^i, m, \beta_i)$, $i = 1, 2$. Consider the restriction of θ to the group

$$K_l = 1 + \mathfrak{k}_l, \qquad \mathfrak{k}_l = \begin{pmatrix} \mathfrak{H}^{m+2} & \mathfrak{H}^{m+1} \\ \mathfrak{H}^{m+2} & \mathfrak{H}^{m+2} \end{pmatrix}.$$

Then g intertwines θ on K_l so we have

$$g^{-1}(\beta + \mathfrak{k}_l^*)g \cap (\beta + \mathfrak{k}_l^*) \neq \emptyset.$$

Write $y = n_y m_y l_y$, with $n_y = 1 + y_n \in N'_u \cap \Gamma_{m+1}$, $m_y \in M' \cap \Gamma_{m+1}$ and $l_y = 1 + y_l \in N'_l \cap \Gamma_{m+1}$; likewise, $y' = l'_y m'_y n'_y$. Now $a_\beta(y_n) \in \mathfrak{k}_l^*$ so $n_y^{-1}(\beta + \mathfrak{k}_l^*)n_y = \beta + \mathfrak{k}_l^*$, and likewise for n'_y. The same is also true for m_y, m'_y since they normalize $\theta|_{K_l}$ by Corollary 2.2. Hence $g' = l_y h l'_y$ intertwines the coset $\beta + \mathfrak{k}_l^*$, that is

$$h^{-1}a_\beta(y_l)h + a_\beta(y'_l) \equiv 0 \pmod{h^{-1}\mathfrak{k}_l^* h + \mathfrak{k}_l^*}.$$

This is certainly satisfied in all blocks except possibly the (2,1)-block, where we have

$$h^{-1}a_\beta(y_l)h + a_\beta(y'_l) \equiv 0 \pmod{(h^{-1}\mathfrak{k}_l^* h + \mathfrak{k}_l^*) \cap A^{21}}.$$

By Lemma 3.17, there exist z_l, $z'_l \in \mathfrak{m}_m^{21}$ such that

$$a_\beta(h^{-1}y_l h + y'_l) = a_\beta(h^{-1}z_l h + z'_l).$$

Then, by the injectivity of a_β on A^{21}, we have $g' = (1 + z_l)h(1 + z'_l)$.

Now the fact that Γ_{m+1} normalizes Γ_m implies that, absorbing factors into Γ_m, we may assume $g = n_y m_y h m'_y n'_y$. Similarly, by considering the restriction of θ to

$$K_u = \begin{pmatrix} 1 + \mathfrak{H}^{m+2} & \mathfrak{H}^{m+2} \\ \mathfrak{H}^{m+1} & 1 + \mathfrak{H}^{m+2} \end{pmatrix},$$

(that is, reversing the roles of the $(1,2)$- and $(2,1)$-blocks) we reduce to the case $g = m_y h m'_y \in M'$ so the result holds by the inductive hypothesis. ∎

3.4 Heisenberg extension

We continue with the notation of the previous section, so $[\Lambda, n, r-, \beta]$ is a semisimple stratum, $[\Lambda, n, r, \gamma]$ is a semisimple stratum equivalent to $[\Lambda, n, r, \beta]$ and we put $b = \beta - \gamma$.

Lemma 3.23 (cf. [6] (3.2.8)). *Let $0 \le m < r$, $\theta \in \mathcal{C}(\Lambda, m, \beta)$ and let $j \in J^k(\beta, \Lambda)$, $j' \in J^l(\beta, \Lambda)$ with $k + l > m$. Then $[j, j'] \in H^{m+}(\beta, \Lambda)$ and*

$$\theta[j, j'] = \psi_{j^{-1}\beta j - \beta}(j').$$

Proof The first assertion is Corollary 3.12(iv). We proceed by induction on r but first reduce to the case $k, l \geq \frac{r}{2}$. Put $k' = \max\{k, \frac{r}{2}\}$, $l' = \max\{l, \frac{r}{2}\}$; then we may write $j = uh$, with $u \in U_k(\Lambda) \cap B_\beta^\times$ and $h \in J^{k'}(\beta, \Lambda)$, and likewise $j' = u'h'$. Then

$$\theta[j, j'] = \theta[u, hj'h^{-1}]\theta[h, h']\theta[h'hh'^{-1}, u'].$$

Let $\widetilde{\theta} \in \mathcal{C}(\Lambda, k-, \beta)$ extend θ; then $u \in H^k(\beta, \Lambda)$ and $\theta[u, hj'h^{-1}] = \widetilde{\theta}[u, hj'h^{-1}] = 1$, since $J(\beta, \Lambda)$ normalizes $\widetilde{\theta}$. Similarly, $\theta[h'hh'^{-1}, u'] = 1$ so we have

$$\theta[j, j'] = \theta[h, h'].$$

On the other hand,

$$\psi_{j^{-1}\beta j - \beta}(j') = \psi_{h^{-1}\beta h - \beta}(u'h'),$$

since u commutes with β. We write $h = 1 + x$, $u' = 1 + y$, with $x \in \mathfrak{J}^{k'}(\beta, \Lambda)$, $y \in \mathfrak{b}_{\beta,l}$. Then

$$\psi_{h^{-1}\beta h - \beta}(u') = \psi_F \circ \mathrm{tr}(a_\beta(x)y - x(1+x)^{-1}a_\beta(x)y),$$

where tr is $\mathrm{tr}_{A/F}$. Now $\psi_F \circ \mathrm{tr}(a_\beta(x)y) = \psi_F \circ \mathrm{tr}(-xa_\beta(y)) = 1$ as y commutes with β. Also

$$\psi_F \circ \mathrm{tr}(x(1+x)^{-1}a_\beta(x)y) = \psi_F \circ \mathrm{tr}(a_\beta(x)yx(1+x)^{-1})$$

and $yx(1+x)^{-1} \in \mathfrak{b}_{\beta,l}\mathfrak{J}^{k'}(\beta, \Lambda) \subset \mathfrak{H}^{\frac{r}{2}+}(\beta, \Lambda)$. But $a_\beta(x) \in (\mathfrak{H}^{\frac{r}{2}+}(\beta, \Lambda))^*$, by Lemma 3.17, so altogether we have

$$\psi_{j^{-1}\beta j - \beta}(j') = \psi_{h^{-1}\beta h - \beta}(h')$$

and we have reduced to the case $k, l \geq \frac{r}{2}$. Indeed, we may (and do) assume $k = l = \frac{r}{2}$ so $[j, j'] \in H^r(\beta, \Lambda)$.

We have

$$\theta[j, j'] = \theta_0[j, j']\psi_b[j, j'],$$

for some $\theta_0 \in \mathcal{C}(\Lambda, r-, \gamma)$, and $\theta_0[j, j'] = \psi_{j^{-1}\gamma j - \gamma}(j')$ by induction. But it is straightforward that $\psi_b[j, j'] = \psi_{j^{-1}bj - b}(j')$ so the result follows. ■

Proposition 3.24 (cf. [6] (3.4.1)). *Let $0 < m \leq r$ and let $\theta \in \mathcal{C}(\Lambda, m-, \beta)$. The pairing*

$$\boldsymbol{k}_\theta : (g, g') \mapsto \theta[g, g'], \qquad g, g' \in J^m(\beta, \Lambda)$$

induces a nondegenerate alternating bilinear form

$$J^m(\beta, \Lambda)/H^m(\beta, \Lambda) \times J^m(\beta, \Lambda)/H^m(\beta, \Lambda) \to \mathbb{C}^\times.$$

Proof As in [6] (3.4.1), we need only show that

$$\theta[g, g'] = 1 \quad \forall g' \in J^m(\beta, \Lambda) \qquad \Longleftrightarrow \qquad g \in H^m(\beta, \Lambda),$$

the implication $\Leftarrow$ being immediate, from Lemma 3.15(iii).

We proceed as usual by induction, the simple case being [6] (3.4.1). Suppose first $m > \frac{r}{2}$; then $\theta = \theta_0\psi_b$, for some $\theta_0 \in \mathcal{C}(\Lambda, m-, \gamma)$, and, for $g, g' \in J^m(\beta, \Lambda)$, we have $[g, g'] \in U^{r+}(\Lambda) \subset \ker \psi_b$.

Hence $\boldsymbol{k}_\theta = \boldsymbol{k}_{\theta_0}$ and the result follows from induction, since $J^m(\beta,\Lambda) = J^m(\gamma,\Lambda)$, and likewise for H^m.

Now suppose $m \le \frac{r}{2}$ and write $g = 1+x$, $g' = 1+y$, with $x, y \in \mathfrak{J}^m(\beta,\Lambda)$. By Lemma 3.23, we have

$$\theta[g,g'] = \psi_{g^{-1}\beta g-\beta}(g') = \psi_F \circ \mathrm{tr}((1+x)^{-1}a_\beta(x)y) = 1,$$

where tr is $\mathrm{tr}_{A/F}$. This is true for all $y \in \mathfrak{J}^m(\beta,\Lambda)$ so we have $(1+x)^{-1}a_\beta(x) \in (\mathfrak{J}^m(\beta,\Lambda))^* \subset (\mathfrak{J}^{\frac{r}{2}}(\beta,\Lambda))^*$. Then $a_\beta(x) \in (\mathfrak{J}^{\frac{r}{2}}(\beta,\Lambda))^* \cap \mathrm{im}\, a_\beta$ so, by (3.18), we have $x \in (B_\beta + \mathfrak{H}^{\frac{r}{2}+}(\beta,\Lambda)) \cap \mathfrak{J}^m(\beta,\Lambda) = \mathfrak{H}^m(\beta,\Lambda)$ as required. ■

Corollary 3.25. *Let $0 \le m < r$ and let $\theta \in \mathcal{C}(\Lambda,m,\beta)$. Then there exists a unique irreducible representation η of $J^{m+}(\beta,\Lambda)$ which contains θ. Moreover, $\dim\eta = (J^{m+} : H^{m+})^{\frac{1}{2}}$ and $I_{\widetilde{G}}(\eta) = \Gamma_m B_\beta^\times \Gamma_m$.*

3.5 Transfer property

In this section we extend the transfer property [6] (3.6.1) to semisimple strata. We continue with a semisimple stratum $[\Lambda,n,r-,\beta]$ with splitting $V = \bigoplus_{i=1}^l V^i$ and let $\theta \in \mathcal{C}(\Lambda,m,\beta)$, $0 \le m < r$. We assume moreover that $r = -k_0(\beta,\Lambda)$ (see (3.6)) and let $[\Lambda,n,r,\gamma]$ be a semisimple stratum equivalent to $[\Lambda,n,r,\beta]$ chosen as in Proposition 3.4. Put $m_0 = \max\{m, \frac{r}{2}\}$; then we have

$$\begin{cases} \theta|_{\widetilde{G}_i} = \theta_i, & \text{for some } \theta_i \in \mathcal{C}(\Lambda^i,m,\beta_i), \\ \theta|_{H^{m_0+}(\beta,\Lambda)} = \theta_0\psi_{\gamma-\beta}, & \text{for some } \theta_0 \in \mathcal{C}(\Lambda,m_0,\gamma). \end{cases}$$

Now let $[\Lambda',n',r'-,\beta]$ be another semisimple stratum, $r' = -k_0(\beta,\Lambda')$. Let $m' \in \mathbb{Z}$ be such that $\lfloor \frac{m'}{e(\Lambda'|\mathfrak{o}_F)} \rfloor = \lfloor \frac{m}{e(\Lambda|\mathfrak{o}_F)} \rfloor$ and $\theta' \in \mathcal{C}(\Lambda',m',\beta')$. For each i, there is an E_i-basis of V^i which is a splitting of both Λ^i and Λ'^i. Taking W^i to be the F-linear span of this basis (so that it is in general position relative to both Λ^i and Λ'^i), we may use Proposition 3.4 to choose γ as above in $\prod \iota_{W^i}(\mathfrak{K}(\mathfrak{A}(E_i)))$ (with notation as in 3.4). In particular, we may assume that $[\Lambda',n',r',\gamma]$ is a semisimple stratum equivalent to $[\Lambda',n',r',\beta]$. Put $m_0' = \max\{m', \frac{r'}{2}\}$ and define θ_i' and θ_0' as above; note that we have $\lfloor \frac{m_0'}{e(\Lambda'|\mathfrak{o}_F)} \rfloor = \lfloor \frac{m_0}{e(\Lambda|\mathfrak{o}_F)} \rfloor$.

Proposition 3.26 (cf. [6] (3.6.1)). *There exists a canonical bijection*

$$\tau_{\Lambda,\Lambda',\beta} : \mathcal{C}(\Lambda,m,\beta) \to \mathcal{C}(\Lambda',m',\beta')$$

such that, for $\theta \in \mathcal{C}(\Lambda,m,\beta)$, $\theta' := \tau_{\Lambda,\Lambda',\beta}(\theta)$ is the unique simple character in $\mathcal{C}(\Lambda',m',\beta')$ such that $B_\beta^\times \cap I_{\widetilde{G}}(\theta,\theta') \ne \emptyset$. Moreover $B_\beta^\times \subset I_{\widetilde{G}}(\theta,\theta')$.

Proof Let $\theta \in \mathcal{C}(\Lambda,m,\beta)$ be as above and suppose $\theta' \in \mathcal{C}(\Lambda',m',\beta')$ is such that $b \in I_{\widetilde{G}}(\theta,\theta') \cap B_\beta^\times$. Then $b_i \in I_{\widetilde{G}_i}(\theta_i,\theta_i') \cap B_{\beta_i}^\times$ so we have $\theta_i' = \tau_{\Lambda^i,\Lambda'^i,\beta_i}(\theta_i)$ by the simple case. But θ' is trivial on N_u, N_l and hence it is clearly uniquely determined.

So we need only show the existence of such a θ'. We proceed by induction on $k_0(\beta,\Lambda) = -r$, the simple case being given by Lemma 2.6 (see also [6](3.6.1)). Consider $\tau_{\Lambda,\Lambda',\gamma}(\theta_0)\psi_{\gamma-\beta}$ as a character of $H^{m_0'+}(\beta,\Lambda')$. Then, by (2.7) and putting $H = H^{m_0+}(\beta,\Lambda') \cap \widetilde{G}_i$ and $H' = H^{m_0'+}(\beta,\Lambda') \cap \widetilde{G}_i$, we have

$$\begin{aligned} \tau_{\Lambda^i,\Lambda'^i,\beta_i}(\theta_i)|_{H'} = \tau_{\Lambda^i,\Lambda'^i,\beta_i}(\theta_i|_H) &= \tau_{\Lambda^i,\Lambda'^i,\gamma_i}(\theta_0|_H)\psi_{\gamma_i-\beta_i} \\ &= \tau_{\Lambda,\Lambda',\gamma}(\theta_0)\psi_{\gamma-\beta}|_{H'}. \end{aligned}$$

Then, by Lemma 3.15, there exists $\theta' \in \mathcal{C}(\Lambda', m', \beta)$ such that $\theta'|_{H^{m'_0+}(\beta,\Lambda')} = \tau_{\Lambda,\Lambda',\gamma}(\theta_0)\psi_{\gamma-\beta}$ and $\theta'|_{\widetilde{G}_i} = \tau_{\Lambda^i,\Lambda'^i,\beta_i}(\theta_i)$. Finally, we have $B_\beta^\times \subset I_{\widetilde{G}}(\theta, \theta')$ by Lemma 2.1. ■

We remark that the above result holds also without the assumption $r = -k_0(\beta, \Lambda)$, since the set $\mathcal{C}(\Lambda, m, \beta)$ does not depend on r.

3.6 Semisimple characters for G

Finally, in this section we describe the situation for the group G. As for simple characters, the semisimple characters of G will be obtained by transfer from those for $\widetilde{G}$.

Let $[\Lambda, n, r-, \beta]$ be a semisimple stratum in A which, in addition, is skew – that, is, $\beta \in A_-$ and the decomposition $V = \bigoplus_{i=1}^l V^i$ is orthogonal with respect to the form h. Let $[\Lambda, n, r, \gamma]$ be a semisimple stratum equivalent to $[\Lambda, n, r, \beta]$, with $\gamma \in M$. Then, by [13] (1.10), we may in fact suppose that $\gamma \in M_-$ so the stratum is skew. In particular, we see that the groups $H^{m+}(\beta, \Lambda)$, $J^{m+}(\beta, \Lambda)$, $\Gamma_m(\beta, \Lambda)$ are fixed by Σ and that Σ acts on the set $\mathcal{C}(\Lambda, m, \beta)$ of semisimple characters. We put $H_-^{m+}(\beta, \Lambda) = H^{m+}(\beta, \Lambda)^\Sigma$, and likewise for $J_-^{m+}(\beta, \Lambda)$ and $\Gamma_m^-(\beta, \Lambda)$. Then Glauberman's correspondence ([9] – see [13] §2 for this situation) gives a bijection $\mathbf{g}$ between the irreducible representations of $H^{m+}(\beta, \Lambda)$ fixed by Σ and the irreducible representations of $H_-^{m+}(\beta, \Lambda)$. We set

$$\mathcal{C}_-(\Lambda, m, \beta) = \{\mathbf{g}(\theta) : \theta \in \mathcal{C}^\Sigma(\Lambda, m, \beta)\},$$

where $\mathcal{C}^\Sigma(\Lambda, m, \beta)$ denotes the semisimple characters fixed by Σ, and $\theta_- \in \mathcal{C}_-(\Lambda, m, \beta)$ is called a *skew semisimple character*. Note that, since θ is a character, $\mathbf{g}(\theta)$ is in fact just the restriction of θ.

Proposition 3.27. *Let $\theta_- \in \mathcal{C}_-(\Lambda, m, \beta)$; then $I_G(\theta_-) = \Gamma_m^-(\beta, \Lambda) \cdot B_\beta \cap G \cdot \Gamma_m^-(\beta, \Lambda)$.*

Proof We have $\theta_- = \mathbf{g}(\theta)$, for some $\theta \in \mathcal{C}^\Sigma(\Lambda, m, \beta)$ and, by [13] (2.5), $I_G(\theta_-) = i_{\widetilde{G}}(\theta)^\Sigma$. Now the result follows from Theorem 3.22 and [12] (2.3), if we can show that, for $b \in B_\beta^\times$,

$$\Gamma_m b \Gamma_m \cap B_\beta^\times = \Gamma_m \cap B_\beta^\times \cdot b \cdot \Gamma_m \cap B_\beta^\times.$$

Note that the containment $\supset$ is clear.

Put $m' = \min\{r - m, \frac{r}{2}\}$; then we have $\mathfrak{b}_{m'} \subset \mathfrak{m}_m \subset \mathfrak{a}_{m'}$ so $\Gamma_m \cap B_\beta^\times = U_{m'}(\Lambda) \cap B_\beta^\times$ and $\Gamma_m \subset U_{m'}(\Lambda)$. Then, by [6] (1.6.1), we have

$$\Gamma_m b \Gamma_m \cap B_\beta^\times \subset U_{m'}(\Lambda) b U_{m'}(\Lambda) \cap B_\beta^\times = U_{m'}(\Lambda) \cap B_\beta^\times \cdot b \cdot U_{m'}(\Lambda) \cap B_\beta^\times$$

as required. ■

Proposition 3.28. *Let $\theta_- \in \mathcal{C}_-(\Lambda, m-, \beta)$. Then the pairing*

$$\boldsymbol{k}_{\theta_-} : (g, g') \mapsto \theta[g, g'], \qquad g, g' \in J_-^m(\beta, \Lambda)$$

induces a nondegenerate alternating bilinear form

$$J_-^m(\beta, \Lambda)/H_-^m(\beta, \Lambda) \times J_-^m(\beta, \Lambda)/H_-^m(\beta, \Lambda) \to \mathbb{C}^\times.$$

Proof We have $\theta_- = \mathbf{g}(\theta)$, for some $\theta \in \mathcal{C}^\Sigma(\Lambda, m-, \beta)$ and we consider the form $\boldsymbol{k}_\theta : J^m/H^m \times J^m/H^m \to \mathbb{C}^\times$. Now σ acts linearly on the k_F-space J^m/H^m and, moreover, preserves $\boldsymbol{k}_\theta$. The result is now immediate from Proposition 3.24, since the homomorphism $J^m_- \hookrightarrow J^m$ induces an isomorphism $J^m_-/H^m_- \simeq (J^m/H^m)^\Sigma$ and $\boldsymbol{k}_{\theta_-}$ corresponds to the restriction of $\boldsymbol{k}_\theta$ to $(J^m/H^m)^\Sigma$. ∎

Corollary 3.29. *Let $\theta \in \mathcal{C}^\Sigma(\Lambda, m, \beta)$ and put $\theta_- = \mathbf{g}(\theta) \in \mathcal{C}_-(\Lambda, m, \beta)$. Then there exists a unique irreducible representation η_- of $J^{m+}_-(\beta, \Lambda)$ which contains θ_-, $\dim \eta_- = (J^{m+}_- : H^{m+}_-)^{\frac{1}{2}}$ and $I_G(\eta_-) = \Gamma^-_m \cdot B_\beta \cap G \cdot \Gamma^-_m$. Moreover, if η is the irreducible representation of $J^{m+}(\beta, \Lambda)$ containing θ, then we have $\eta_- = \mathbf{g}(\eta)$.*

Proof The first assertions are immediate from the previous proposition and, moreover, $\dim \eta_- = (J^{m+}_-(\beta, \Lambda) : H^{m+}_-(\beta, \Lambda))^{\frac{1}{2}}$. Now the restriction $\eta|_{H^{m+}_-(\beta,\Lambda)}$ is a multiple of θ_- so $\eta|_{J^{m+}_-(\beta,\Lambda)}$ is a multiple of η_-. In particular, $\eta_- = \mathbf{g}(\eta)$. ∎

Remark 3.30. Let $[\Lambda, n, r-, \beta]$ be a skew semisimple stratum in A, with associated splitting $V = \bigoplus_{i=1}^l V^i$, and put $E_i = F[\beta_i]$ as usual; we also denote $E_{i,0}$ the fixed field of the involution in E_i. Suppose that we have $\sum_{i=1}^l [E_i; F] = N$; then $J_-(\beta, \Lambda)/J^1_-(\beta, \Lambda) \simeq \prod_{i=1}^l N_1(k_i)$ is a product of cyclic groups, where k_i is the residue field of E_i and $N_1(k_i)$ denotes the elements $x \in k_i$ such that $N_{k_i/k_{i,0}} x = 1$, where $k_{i,0}$ is the residue field of $E_{i,0}$. In particular, there exists an extension of η_- to a representation κ_- of $J_-(\beta, \Lambda)$, and any extension takes the form $\kappa_- \otimes \chi$, for χ the inflation of a character of J_-/J^1_-. We have $I_G(\kappa_-) \subset I_G(\eta_-) = J^1_- \cdot B_\beta \cap G \cdot J^1_- = J_-$ so the induced representation

$$\pi = \operatorname{Ind}_{J_-}^G \kappa_-$$

is irreducible and supercuspidal (since J_- is compact) and (J_-, κ_-) is a $[G, \pi]_G$-type.

In general, to obtain a supercuspidal representation we will have to extend η_- to a representation of J_- and twist by the inflation of a cuspidal representation of J_-/J^1_-. In order to control the intertwining of this representation (cf. [6] (5.3), [15] (SC3)), we will need the following result:

Proposition 3.31 (cf. [6] (5.1.8), [13] (4.3)). *Let $[\Lambda, n, 0, \beta]$ be a skew semisimple stratum in A, let $\theta_- \in \mathcal{C}_-(\Lambda, 0, \beta)$ and let η_- be as in Corollary 3.29. Then, for $g \in G$, we have*

$$\dim I_g(\eta_-, \eta_-) = \begin{cases} 1 & \textit{if } g \in J^1_- \cdot B_\beta \cap G \cdot J^1_-; \\ 0 & \textit{otherwise.} \end{cases}$$

Proof As in [6] (5.1.8), the result will follow if we can show that, for $y \in B_\beta \cap G$, $J^1_- y J^1_-$ is the union of $(J^1_- : H^1_-)$ distinct (H^1_-, H^1_-)-double cosets, where $(K_1 : K_2)$ denotes group index. As in [6] (5.1.9), it is then enough to prove that $(J^1_- : J^1_- \cap (J^1_-)^y) = (H^1_- : H^1_- \cap (H^1_-)^y)$.

Now we have $J^1_- = C(\mathfrak{J}^1_-)$ and $H^1_- = C(\mathfrak{H}^1_-)$, where C denotes the Cayley transform; also $J^1_- \cap (J^1_-)^y = C(\mathfrak{J}^1_- \cap (\mathfrak{J}^1_-)^y)$ and likewise for H^1_-. Then, as in [6] (5.1.10), the result will follow if we can show that the following sequence is exact:

$$0 \to \mathfrak{b}^-_{\beta,1} + (\mathfrak{b}^-_{\beta,1})^y \to \mathfrak{J}^1_- \cap (\mathfrak{J}^1_-)^y \xrightarrow{a_\beta} (\mathfrak{H}^1_-)^* \cap ((\mathfrak{H}^1_-)^*)^y \xrightarrow{s} \mathfrak{b}^-_{\beta,0} + (\mathfrak{b}^-_{\beta,0})^y \to 0,$$

where s is a *tame corestriction* on A given by

$$s\left(\sum_{i,j=1}^l a_{ij}\right) = \sum_{i=1}^l s_i(a_{ii}), \quad a_{ij} \in A^{ij},$$

for s_i a tame corestriction on A^{ii} relative to E_i/F which commutes with the involution, $1 \le i \le l$. Exactness comes from Lemma 3.17 at all places except the final one, where it follows from the simple case [7] (6.3) Lemma. ■

We end this section by looking at the transfer of skew semisimple characters.

Proposition 3.32. *Let $[\Lambda, n, 0, \beta]$, $[\Lambda', n', 0, \beta]$ be skew semisimple strata in A and let m, m' be such that $0 \le m < k_0(\beta, \Lambda)$ and $\lfloor \frac{m'}{e(\Lambda'|\mathfrak{o}_F)} \rfloor = \lfloor \frac{m}{e(\Lambda|\mathfrak{o}_F)} \rfloor$. Then the bijection $\tau_{\Lambda,\Lambda',\beta}$ given by Proposition 3.26 commutes with σ.*

Proof Given $\theta \in \mathcal{C}(\Lambda, m, \beta)$ with $\theta|_{\widetilde{G}_i} = \theta_i \in \mathcal{C}(\Lambda^i, m, \beta_i)$, the transfer $\theta' := \tau_{\Lambda,\Lambda',\beta}(\theta)$ is the character of $H^{m'+}(\beta, \Lambda')$ which is trivial on N_u, N_l and such that $\theta'|_{\widetilde{G}_i} = \tau_{\Lambda^i,\Lambda'^i,\beta_i}(\theta_i)$ so the result follows immediately from the simple case in Proposition 2.12. ■

In particular, this implies that we have a canonical bijection

$$\tau_{\Lambda,\Lambda',\beta} : \mathcal{C}_-(\Lambda, m, \beta) \to \mathcal{C}_-(\Lambda', m', \beta)$$

and, for $\theta_- \in \mathcal{C}_-(\Lambda, m, \beta)$, $\theta'_- := \tau_{\Lambda,\Lambda',\beta}(\theta_-)$ is the unique simple character such that $1 \in I_G(\theta_-, \theta'_-)$, by [13] (2.5).

4 Relatively G-split strata

In this section we look at the "relatively G-split" case and construct a non-zero Jacquet module (cf. [7] §6 and [14] §3). This will be crucial to the refinement process in §5.

4.1 Definition and intertwining

Let $[\Lambda, n, m, \beta]$ be a skew semisimple stratum in A, with associated splitting $V = \bigoplus_{i=1}^{l} V^i$. In this section we will assume that m is an integer. We have B_β the A-centralizer of β and let B_i be the A^{ii}-centralizer of β_i, $1 \le i \le l$, so that $B_\beta = \bigoplus_{i=1}^{l} B_i$. We put $E_i = F[\beta_i]$ and $\mathfrak{o}_i = \mathfrak{o}_{E_i}$, for $1 \le i \le l$, and let s_i be a tame corestriction on A^{ii} relative to E_i/F. We suppose we are given a decomposition of E_1-vector spaces $V^1 = V_0^1 \perp (V_1^1 \oplus V_{-1}^1)$, where V_1^1 and V_{-1}^1 are totally isotropic with respect to h, such that $\Lambda^1(k) = \bigoplus_{-1\le j\le 1}(\Lambda^1(k) \cap V_j^1)$. We will write $A^{11} = \bigoplus_{-1\le j,k\le 1} A_{jk}^{11}$ for the corresponding decomposition of A^{11}.

Let $b_i \in A^{ii} \cap \mathfrak{a}_{-m}$, for $1 \le i \le l$, where $b_1 = \sum_{-1\le j\le 1} b_{1,j}$ with $b_{1,j} \in A_{jj}^{11}$. We suppose that the derived stratum $[\Lambda_{\mathfrak{o}_1}^1, m, m-1, s_1(b_1)]$ in B_1 is G-split (see Definition 1.3) by the decomposition $V^1 = V_0^1 \perp (V_1^1 \oplus V_{-1}^1)$ (cf. [14] §3.1). That is, the stratum is split by this decomposition, the stratum $[\Lambda_{\mathfrak{o}_1}^{1,1}, m, m-1, s_1(b_{1,1})]$ has characteristic polynomial of the form $\psi(X)^d$ and the stratum $[\Lambda_{\mathfrak{o}_1}^{1,-1}, m, m-1, s_1(b_{1,-1})]$ has characteristic polynomial $\overline{\psi}(\eta X)^d$, where $\eta = (-1)^{m/g}(-1)^{e/g}$ with $e = e(\Lambda^1|\mathfrak{o}_1)$ and $g = (m, e)$; in particular, $\psi(X)$ is coprime to $\overline{\psi}(\eta X)$.

We now consider the decomposition

$$V = V_{-1}^1 \oplus \left(V_0^1 \oplus \bigoplus_{i=1}^{l} V^i \right) \oplus V_1^1$$

and we will abbreviate $V_{-1}^1 = V_{-1}$, $V_1^1 = V_1$ and $V_0^1 \oplus \bigoplus_{i=1}^l V^i = V_0$. We will consider the block picture $A = \bigoplus_{-1\le j,k\le 1} A_{jk}$ with respect to this splitting and we put $A_u = \bigoplus_{-1\le j<k\le 1} A_{jk}$, $A_l = \bigoplus_{-1\le k<j\le 1} A_{jk}$, as usual. We also put $\mathcal{M} = \bigoplus_{-1\le j\le 1} A_{jj}$, $M = \mathcal{M}^\times$, $N_u = 1 + A_u$, $N_l = 1 + A_l$, $P_u = MN_u$, $P_l = MN_l$, and we write $b_0 = b_{1,0} + \sum_{i=2}^l b_i \in A_{0,0}$. We will also retain the block notation $A^{ii} = \mathrm{End}_F V^i$, for $1 \le i \le l$, alongside this new block decomposition.

We look at the lattice

$$\mathfrak{k} = \begin{pmatrix} \mathfrak{H}^m & \mathfrak{b}_1 + \mathfrak{m}_{m-1} & \mathfrak{b}_1 + \mathfrak{m}_{m-1} \\ \mathfrak{H}^m & \mathfrak{H}^m & \mathfrak{b}_1 + \mathfrak{m}_{m-1} \\ \mathfrak{H}^m & \mathfrak{H}^m & \mathfrak{H}^m \end{pmatrix},$$

where $\mathfrak{b}_1 = \mathfrak{b}_{\beta,1} = \mathfrak{a}_1 \cap B_\beta$ and $\mathfrak{m}_{m-1} = \mathfrak{m}_{m-1}(\beta,\Lambda)$. Since $\Gamma_{m-1} = 1 + \mathfrak{m}_{m-1}$ and $1 + \mathfrak{b}_1$ normalize $H^m(\beta,\Lambda)$, $K = 1 + \mathfrak{k}$ is a compact open subgroup of $\widetilde{G}$ which is, moreover, fixed by Σ.

Proposition 4.1 (cf. [7] (6.1)). *Let $\theta \in \mathcal{C}^\Sigma(\Lambda, m-1, \beta)$. There exists a unique character ϑ of K which extends θ and is trivial on $K \cap N_u$. Moreover, ϑ is fixed by σ.*

Proof Uniqueness is clear, while existence is because $K \cap N_u$ normalizes θ. The final statement is clear, since $K \cap N_u$ is fixed by σ. ■

Note that, if $0 \le k \le m-1$ and $\widetilde{\theta} \in \mathcal{C}(\Lambda, k, \beta)$ extends θ, then $\widetilde{\theta}$ is trivial on $H^{k+1} \cap N_u$ (and on $H^{k+1} \cap N_l$), by [7] (5.2) Proposition and Lemma 3.15(i), so ϑ and $\widetilde{\theta}$ agree on $H^{k+1} \cap K$.

We now consider the character $\xi = \vartheta\psi_b$ of the group K, where $b = b_{-1,1} + b_0 + b_{1,1}$ and ψ_b denotes the extension of $\psi_b|_{H^m}$ to K which is trivial on $K \cap N_u$.

Theorem 4.2 (cf. [7] (6.2)). *Suppose $g \in N_u$ intertwines the character ξ of K. Then $g \in K \cap N_u$.*

Before giving the proof of this theorem, we observe the following easy consequence:

Corollary 4.3. *Let ξ_- be the restriction of ξ to the group $K_- = K \cap G$. Suppose $g \in N_u^-$ intertwines ξ_-; then $g \in K_- \cap N_u^-$.*

Proof We have $I_G(\xi_-) = I_{\widetilde{G}}(\xi) \cap G$, by [13] (2.5), so the result is immediate. ■

Now we will prove Theorem 4.2 so we suppose $g \in I_{\widetilde{G}}(\xi|K)$. Then g certainly intertwines $\xi|_{H^{m+1}} = \theta|_{H^{m+1}} \in \mathcal{C}(\Lambda, m, \beta)$ so we have $g \in \Gamma_m B_\beta^\times \Gamma_m$. We will first show

$$g \in \Gamma M \Gamma, \tag{4.4}$$

where $\Gamma = 1 + \mathfrak{b}_1 + \mathfrak{m}_{m-1}$. Note also that Γ normalizes the pair $(H^m, \xi|_{H^m})$.

We write $g = (1+x)t(1+y)^{-1}$, with $x, y \in \mathfrak{m}_m$, and $t \in B_\beta^\times$. By Lemma 3.21, we have

$$\theta^{1+x} = \theta\psi_{(1+x)^{-1}\beta(1+x)-\beta}$$

as characters of $H^m(\beta,\Lambda)$. But

$$(1+x)^{-1}\beta(1+x) - \beta = a_\beta(x) - (1+x)^{-1}x a_\beta(x)$$

and $(1+x)^{-1}xa_\beta(x) \in (\mathfrak{H}^{m+1})^*$, by Lemma 3.17 and Lemmas 3.10, 3.11, so we have

$$\theta^{1+x} = \theta\psi_{a_\beta(x)}.$$

Now t intertwines the restrictions of ξ^{1+x}, ξ^{1+y} to $H^m(\beta,\Lambda)$ so it intertwines their restrictions to $H^m(\beta,\Lambda)\cap B_\beta^\times$. Now $\psi_{a_\beta(x)}$, $\psi_{a_\beta(y)}$ restrict trivially here, whilst $(1+x)$, $(1+y)$ fix the characters ψ_b, so t intertwines $\theta\psi_b$ on $H^m(\beta,\Lambda)\cap B_\beta^\times$. But $\theta \in \mathcal{C}(\Lambda, m-1,\beta)$ so $\theta|_{H^m(\beta,\Lambda)\cap B_\beta^\times}$ is intertwined by all of $B_\beta^\times$, by Theorem 3.22, and t intertwines the character $\psi_b|_{H^m(\beta,\Lambda)\cap B_\beta^\times}$.

We write $t = \sum_{i=1}^l t_i$, with $t_i \in B_i^\times$, for $1 \le i \le l$, and look at the character $\psi_b|_{H^m(\beta,\Lambda)\cap B_1^\times} = \psi_{b_1}$. There exists a character ψ_{B_1} of B_1, of the form $\psi_{E_1}\circ \mathrm{tr}_{B_1/E_1}$, such that $\psi_a|_{B_1} = \psi_{B_1,s_1(a)}$ for any $a \in A^{11}$. In particular, $\psi_{b_1}|_{B_1} = \psi_{B_1,s_1(b_1)}$ so [12] (4.14) shows that

$$t_1 \in U_1(\Lambda^1_{\mathfrak{o}_1})\cdot B_1^\times \cap M \cdot U_1(\Lambda^1_{\mathfrak{o}_1}).$$

Then, since $B_i^\times \subset M$ for $2 \le i \le l$, we have

$$t \in (1+\mathfrak{b}_1)\cdot B_\beta^\times \cap M\cdot(1+\mathfrak{b}_1).$$

As $(1+\mathfrak{b}_1)$ normalizes Γ_m, we may absorb these factors into Γ and assume $g=(1+x)t(1+y)$ with $x,y \in \mathfrak{m}_m$ and $t \in \mathfrak{b}_\beta^\times \cap M$. We write $1+x = (1+x_l)(1+x_m)(1+x_u)$, with $x_l \in A_l$, $x_m \in \mathcal{M}$, $x_u \in A_u$, and, likewise, $1+y = (1+y_u)(1+y_m)(1+y_l)$.

We now consider the restriction of ξ to the group

$$K_l = 1+\mathfrak{k}_l, \qquad \mathfrak{k}_l = \begin{pmatrix} \mathfrak{H}^{m+1} & \mathfrak{H}^m & \mathfrak{H}^m \\ \mathfrak{H}^{m+1} & \mathfrak{H}^{m+1} & \mathfrak{H}^m \\ \mathfrak{H}^{m+1} & \mathfrak{H}^{m+1} & \mathfrak{H}^{m+1} \end{pmatrix}.$$

Then, as in the proof of Theorem 3.22, we find that we may assume $x_l, y_l \in \mathfrak{m}_{m-1}$. (Note that the projections $\mathbf{1}_i : V \to V_i$ lie in B_β so the exact sequences of Lemma 3.17 are still exact.) Repeating with K_u, again as in the proof of Theorem 3.22, we get $x_u, y_u \in \mathfrak{m}_{m-1}$ also so, absorbing these terms into Γ, we get $g \in \Gamma_m \cap M \cdot B_\beta^\times \cap M \cdot \Gamma_m \cap M \subset M$ as required.

So we have proved (4.4). Now let $g \in N_u$ intertwine ξ; then $g = \gamma m \gamma'$, for some $m \in M$, $\gamma,\gamma' \in \Gamma$. The group Γ has Iwahori decomposition $\Gamma = \Gamma\cap N_l \cdot \Gamma \cap M \cdot \Gamma \cap N_u$ so we can write $\gamma = \gamma_u\gamma_M\gamma_l$, $\gamma' = \gamma'_l\gamma'_M\gamma'_u$. Thus, for a certain $m' \in M$, we have $\gamma_u^{-1} g \gamma_u'^{-1} = \gamma_l m' \gamma'_l \in N_u \cap P_l = \{1\}$. Hence $g = \gamma_u\gamma'_u \in \Gamma\cap N_u = K \cap N_u$ as required. ∎

4.2 Covers

For $-1 \le i \le 1$, we put $G_i = G \cap \mathrm{Aut}_F(V^i)$ and suppose that we are given:

(i) a subgroup K_1 of $U(\Lambda^1)$ containing and normalizing $H_-^m \cap G_1$ and an irreducible representation ρ_1 of K_1 whose restriction to $K_1 \cap K_-$ is a multiple of ξ_-;

(ii) a subgroup K_0^- of $P(\Lambda^0)$ containing and normalizing $H_-^m \cap G_0$ and an irreducible representation ρ_0^- of K_0^- whose restriction to $K_0^- \cap K_-$ is a multiple of ξ_-.

We think of K_1 embedded in G as $\left\{ \begin{pmatrix} k^\sigma & & \\ & 1 & \\ & & k \end{pmatrix} : k \in K_1 \right\}$.

Proposition 4.5 (cf. [7] (6.6)). (i) *The set* $\widetilde{K}_- = (K_1 \times K_0^-).K_-$ *is a group.*

(ii) *There is a unique irreducible representation* ρ_- *of* $\widetilde{K}_-$ *which is trivial on* $K_- \cap N_u^-$, $K_- \cap N_l^-$ *and whose restriction to* $K_1 \times K_0^-$ *is* $\rho_1 \otimes \rho_0^-$.

(iii) *The pair* $(\widetilde{K}_-, \rho_-)$ *is a* G*-cover of* $(\widetilde{K}_- \cap M, \rho_1 \otimes \rho_0^-)$.

Proof This is identical to [7] (6.6), except we take the element ζ to be

$$\zeta = \begin{pmatrix} \varpi_F & & \\ & 1 & \\ & & \varpi_F^\sigma \end{pmatrix}.$$

∎

4.3 Jacquet modules

Proposition 4.6. *Let* $(\pi, \mathcal{V})$ *be a smooth representation of* G *which contains* $\theta_-\psi_b^-$ *on* H_-^m. *Then* π *contains the character* ξ_- *of* K_- *also.*

Proof We first suppose $m \geq 2$ and put, for $q \in \mathbb{Z}$,

$$\mathfrak{k}_q = \begin{cases} \mathfrak{H}^m + \mathfrak{H}^{q+1} \cap A_u & \text{for } \lfloor \frac{m+1}{2} \rfloor \leq q \leq m-1; \\ \mathfrak{H}^m + \mathfrak{m}_{m-(q+1)} \cap A_u & \text{for } 0 \leq q \leq \lfloor \frac{m}{2} \rfloor - 1; \\ \mathfrak{H}^m + (\mathfrak{b}_1 + \mathfrak{m}_{m-1}) \cap A_u & \text{for } q = -1. \end{cases}$$

Then we put $K_q = 1 + \mathfrak{k}_q$ and $K_q^- = K_q \cap G$. Note, in particular, that $K_{-1} = K$.
We also put

$$\Xi_q^- = \begin{cases} \Gamma_q^- \cap N_l & \text{for } \lfloor \frac{m+1}{2} \rfloor \leq q \leq m-1; \\ H_-^m \cap N_l & \text{for } 1 \leq q \leq \lfloor \frac{m}{2} \rfloor - 1; \\ (1 + \mathfrak{b}_{m-1}) \cap N_l^- & \text{for } q = 0. \end{cases}$$

Lemma 4.7. *For* $\lfloor \frac{m+1}{2} \rfloor < q \leq m-1$ *or* $0 \leq q \leq \lfloor \frac{m}{2} \rfloor - 1$, Ξ_q^- *acts transitively on the characters of* K_{q-1}^- *which restrict to* ξ_- *on* K_q^-.

Proof We treat first the case $\lfloor \frac{m+1}{2} \rfloor < q \leq m-1$. The quotient K_{q-1}^-/K_q^- is abelian so any character of K_{q-1}^- which restricts to ξ_- on K_q^- is given by $\xi_-\psi_c^-$, for some $c \in \mathfrak{k}_{q,-}^*/\mathfrak{k}_{q-1,-}^*$, that is, in $(\mathfrak{H}^{q+1})^* \cap A_l^- / (\mathfrak{H}^q)^* \cap A_l^-$.
We have $\mathfrak{m}_q \mathfrak{H}^q \subset \mathfrak{H}^{m+1}$, by Proposition 2.12, so $(\psi_b^-)^{C(x)} = \psi_b^-$, for $x \in \mathfrak{m}_q \cap A_l^-$, and, for $k \in K_{q-1}^-$, we have $[C(x), k] \in H_-^{m+1}$. Let $\widetilde{\theta}_- \in \mathcal{C}_-(\Lambda, q, \beta)$ extend θ_-; then, by Lemma 3.21,

$$\theta_-[C(x), k] = \widetilde{\theta}_-^{C(x)}(k)\widetilde{\theta}_-(k^{-1}) = \psi_{C(x)^{-1}\beta C(x) - \beta}^-(k).$$

Now $C(x)^{-1}\beta C(x) - \beta \equiv a_\beta(x) \pmod{(\mathfrak{H}^q)^* \cap A_l^-}$ so, altogether, we have

$$\xi_-^{C(x)} = \xi_-\psi_{a_\beta(x)}^-.$$

But, by Lemma 3.17, $a_\beta(\mathfrak{m}_q \cap A_l^-) = (\mathfrak{H}^{q+1})^* \cap A_l^-$ and the result follows.

For $1 \le q \le \lfloor \frac{m}{2} \rfloor - 1$, the proof is almost identical, with the roles of $\mathfrak{m}$ and $\mathfrak{H}$ reversed.

Suppose then that $q = 0$. We have $K_{-1} = K_0 \cdot (1 + \mathfrak{b}_1) \cap N_u^-$, with the second factor normalizing the first and normalizing ξ_- on it. Hence any character of K_{-1} which restricts to ξ_- on K_0 takes the form $\xi_- \psi_c^-$, for ψ_c^- some character of $(1 + \mathfrak{b}_1) \cap N_u^-$ trivial on $(1 + \mathfrak{b}_2) \cap N_u^-$, that is, $c \in \mathfrak{b}_{-1} \cap A_l^- / \mathfrak{b}_0 \cap A_l^-$. Now $(1 + \mathfrak{b}_{m-1}) \cap N_l^-$ normalizes ξ_- on K_0^- while, on $(1 + \mathfrak{b}_1) \cap N_u^-$, we have

$$\xi_- = \psi_b^- = \psi_{B_1, s_1(b_1)}^-,$$

where s_1 is a tame corestriction on A^{11} relative to E_1/F which commutes with the involution. (Note that $B_\beta \cap A_u \subset B_1$ and, likewise $B_\beta \cap A_l \subset B_1$.) We put $\delta = s_1(b_1) \in \mathfrak{b}_{-m}^-$. Then for $x \in \mathfrak{b}_{m-1} \cap A_l^-$, we have

$$(\psi_{B_1,\delta}^-)^{C(x)} = \psi_{B_1,\delta}^- \psi_{B_1, a_\delta(x)}^-.$$

Then the result follows since $a_\delta(\mathfrak{b}_{m-1} \cap A_l^-) = \mathfrak{b}_1 \cap A_l^-$ as in [14] (3.4). ■

Lemma 4.8. *$\Xi^-_{\lfloor \frac{m+1}{2} \rfloor}$ acts transitively on the characters of $K^-_{\lfloor \frac{m}{2} \rfloor - 1}$ which restrict to ξ on $K^-_{\lfloor \frac{m+1}{2} \rfloor}$.*

Proof The proof is almost identical to the first case of the previous lemma, but we must use Lemma 3.23 in place of Lemma 3.21. ■

The proof of Proposition 4.6 in the case $m \ge 2$ is now immediate since, by successively conjugating by a suitable element of Ξ_q^-, we see that π contains $\xi_- | K_{q-1}^-$ (where $q - 1$ should be replaced by $\lfloor \frac{m}{2} \rfloor - 1$ in the case $q = \lfloor \frac{m+1}{2} \rfloor$). The proof in the case $m = 1$ is similar but easier, requiring only Lemma 4.8. ■

Theorem 4.9. *Let $(\pi, \mathcal{V})$ be a smooth representation of G containing the character $\theta_- \psi_b^-$ of H_-^m. Then π is not supercuspidal.*

Proof By Proposition 4.6, $(\pi, \mathcal{V})$ contains the character $\xi_- | K_-$ and, by Proposition 4.5, the pair (K_-, ξ_-) is a G-cover of $(K_- \cap M, \xi_- | K_- \cap M)$. Then, by [8] (7.9), the $\xi_-|_{K_- \cap M}$-isotypic component $\mathcal{V}_u^{\xi_-|_{K_- \cap M}}$ of the Jacquet module of $\mathcal{V}$ with respect to $P_u \cap G$ is non-zero. ■

5 Supercuspidal Representations

This section is devoted to the proof of our main theorem, that every positive-level irreducible supercuspidal representation of G contains a skew semisimple character. The proof is very much along the lines of [6] (8.1.5), though there are added geometric complications. In particular, our Lemma 5.4 (whose proof is given in §5.2) would be relatively straightforward in the case of $\widetilde{G}$.

5.1 The main theorem

Theorem 5.1 (cf. [6] (8.1)). *Let π be a positive-level irreducible supercuspidal representation of G. Then π contains a skew semisimple character $\theta_- \in \mathcal{C}_-(\Lambda, 0, \beta)$, for some skew semisimple stratum $[\Lambda, n, 0, \beta]$.*

Proof We know from [14] §2.3 that π contains some skew semisimple stratum $[\Lambda, n, n-1, \beta]$ in A. We consider pairs $([\Lambda, n, m, \beta], \theta_-)$ consisting of a skew semisimple stratum with $m \in \mathbb{Z}$ and a semisimple character $\theta_- \in \mathcal{C}_-(\Lambda, m, \beta)$ such that $\pi | H_-^{m+1}(\beta, \Lambda)$ contains θ_-.

If $m = 0$ then we are done, so assume $m \geq 1$ for all such pairs. Then π contains some irreducible representation ϑ of $H_-^m(\beta, \Lambda)$ such that $\vartheta | H_-^{m+}(\beta, \Lambda)$ contains θ_-. However, θ_- extends to an abelian character $\widetilde{\theta}_- \in \mathcal{C}_-(\Lambda, m-1, \beta)$ of $H_-^m(\beta, \Lambda)$ and the quotient H_-^m / H_-^{m+} is abelian. Hence ϑ is one-dimensional and we may write

$$\vartheta = \widetilde{\theta}_- \psi_c^- | H_-^m(\beta, \Lambda) \qquad \text{for some } c \in \mathfrak{a}_{-m}^-.$$

Lemma 5.2 (cf. [6] (8.1.12)). *Let $x \in \mathfrak{m}_m^-$. Then, $C(x)$ normalizes $H_-^m(\beta, \Lambda)$ and*

$$\vartheta^{C(x)} = \vartheta \psi_{a_\beta(x)}^-.$$

Proof Since $m < r$, we have $\mathfrak{m}_m \subset \alpha_1$; as $c \in \mathfrak{a}_m$, we surely have $\psi_c^{C(x)} = \psi_c$ on H^m. On the other hand, by Lemma 3.21, we have

$$\widetilde{\theta}_-^{C(x)} = \widetilde{\theta}_- \psi_{C(x)^{-1}\beta C(x) - \beta}^-;$$

a simple computation now shows that

$$C(x)^{-1}\beta C(x) - \beta \equiv a_\beta(x) \pmod{(\mathfrak{H}^{m+})^*},$$

and the lemma follows. ■

We return to the proof of Theorem 5.1. Let $V = \bigoplus_{i=1}^l V^i$ be the decomposition associated to the skew semisimple stratum $[\Lambda, n, m, \beta]$. For $i \neq j$, we have $c_{ij} \in \mathfrak{a}_{-m}^- \cap A^{ij}$. Hence, by (3.8), there exists $x \in \mathfrak{n}_{-m} \cap \mathfrak{a}_{r-m}^-$ such that $c_{ij} = -a_\beta(x)_{ij}$, for $i \neq j$. By Lemma 5.2, $\vartheta^{C(x)} = \widetilde{\theta}_- \psi_{c'}^-$; this character certainly occurs in π and we have $s(c') = s(c)$ and $c'_{ij} = 0$, for $i \neq j$. Hence we may, and will, assume that $c \in \mathcal{M}$.

Put $E_i = F[\beta_i]$, $\mathfrak{o}_i = \mathfrak{o}_{E_i}$, and let s_i be a tame corestriction on A^{ii} relative to E_i/F. We consider the derived stratum

$$\bigoplus_{i=1}^{l} [\Lambda_{\mathfrak{o}_i}^{(i)}, m, m-1, s_i(c_i)].$$

We will write $E = \oplus_{i=1}^l E_i$, $\mathfrak{o}_E = \oplus_{i=1}^l \mathfrak{o}_i$ and we will abuse notation by calling an $\mathfrak{o}_F$-lattice sequence Λ' in V an *$\mathfrak{o}_E$-lattice sequence* if it is of the form $\Lambda' = \oplus_{i=1}^l \Lambda'^{(i)}$, with $\Lambda'^{(i)}$ an $\mathfrak{o}_i$-lattice sequence in V^i. We also let $s : \mathcal{M} \to B$ be the map $s = \oplus_{i=1}^l s_i$ and we will call it a *tame corestriction* on A relative to E/F. Altogether, we will write the derived stratum above as

$$[\Lambda_{\mathfrak{o}_E}, m, m-1, s(c)]. \tag{5.3}$$

The following lemma, whose proof we defer to §5.2 will be crucial.

Lemma 5.4. *Let $\widetilde{\theta} \in \mathcal{C}_-(\Lambda, m-1, \beta)$ and $c \in \mathfrak{a}_{-m}^- \cap \mathcal{M}$ be such that π contains $\vartheta := \widetilde{\theta}_- \psi_c^-$ on $H_-^m(\beta, \Lambda)$. Suppose we have an $\mathfrak{o}_E$-lattice sequence Λ', an integer m' and $\alpha' \in \mathfrak{b}_{-m}^-$ such that*

$$s(c) + \mathfrak{b}_{-m+} \subset \alpha' + \mathfrak{b}'_{-m'+}$$

Then there exist $\widetilde{\theta}' \in \mathcal{C}_-(\Lambda', m'-1, \beta)$ and $c' \in \mathfrak{a}'^-_{-m'} \cap \mathcal{M}$ such that $s(c') = \alpha'$ and π contains the character $\vartheta' := \theta'_- \psi_{c'}^-$ of $H_-^{m'}(\beta, \Lambda')$. Moreover, if $\alpha' = 0$ then we may take $c' = 0$.

Returning to the proof of Theorem 5.1, we first show that we may assume that the derived stratum in (5.3) is fundamental, in the sense that at least one of the strata $[\Lambda^{(i)}_{\mathfrak{o}_i}, m, m-1, s_i(c_i)]$ is fundamental.

Proposition 5.5. *Let $[\Lambda, n, m, \beta]$ be a semisimple skew stratum, $\widetilde{\theta}_- \in \mathcal{C}_-(\Lambda, m-1, \beta)$ and $c \in \mathfrak{a}^-_{-m} \cap \mathcal{M}$ be such that π contains $\widetilde{\theta}_-\psi^-_c$. Then there exist $[\Lambda', n', m', \beta]$, a skew semisimple $\mathfrak{o}_E$-stratum, $\widetilde{\theta}'_- \in \mathcal{C}_-(\Lambda', m'-1, \beta)$ and $c \in {\mathfrak{a}'_{-m'}}^- \cap \mathcal{M}$ such that $m'/e(\Lambda'|\mathfrak{o}_F) \leq m/e(\Lambda|\mathfrak{o}_F)$ and π contains $\widetilde{\theta}'_-\psi^-_c$ and the derived stratum $[\Lambda'_{\mathfrak{o}_E}, m', m'-1, s(c)]$ is fundamental. In particular, $e'/(m', e') \leq N$, where $e' = e(\Lambda'|\mathfrak{o}_F)$.*

Proof We first observe that the final assertion follows from the fact that one $[\Lambda'^{(i)}_{\mathfrak{o}_i}, m', m'-1, s_i(c_i)]$ is fundamental (cf. [14] (2.11)).
As above, π contains the character $\vartheta = \widetilde{\theta}_-\psi^-_c$, where $\widetilde{\theta}_- \in \mathcal{C}_-(\Lambda, m-1, \beta)$ extends θ_- and $c \in \mathfrak{a}^-_{-m}$. If $[\Lambda_{\mathfrak{o}_E}, m, m-1, s(c)]$ is fundamental, we are done so we assume it is not. Then, by [14] (4.3), for $i = 1, \ldots, l$, there exist a self-dual $\mathfrak{o}_i$-lattice sequence $\Lambda'^{(i)}_{\mathfrak{o}_i}$, with $e(\Lambda'^{(i)}|\mathfrak{o}_i) \leq 2(2\dim_{E_i} V^i)$, and $m'_i \in \mathbb{Z}$ such that

$$\begin{aligned} s_i(c_i) + \mathfrak{b}^{ii}_{-m+} &\subset \mathfrak{b}'^{ii}_{-m'_i}, \\ m'_i/e(\Lambda'^{(i)}|\mathfrak{o}_i) &< m/e(\Lambda^{(i)}|\mathfrak{o}_i). \end{aligned}$$

[The extra factor 2 compared to *loc. cit.* in the bound for the period of $\Lambda'^{(i)}$ comes from the assumption that "$d = 1$" for the duality on on all self-dual lattice sequences (see §1.2). In any case, the period $e(\Lambda'^{(i)}|\mathfrak{o}_F) \leq 4\dim_{E_i} V^i e(E_i/F) \leq 4N$.]
We put $\Lambda' = \oplus_{i=1}^l \Lambda'^{(i)}$, $e' = e(\Lambda'|\mathfrak{o}_F)$, $n' = -\nu_{\Lambda'}(\beta)$ and $m' = e' \sup\{m'_i/e(\Lambda'^{(i)}|\mathfrak{o}_F)\}$ so that $[\Lambda', n', m', \beta]$ is a skew semisimple $\mathfrak{o}_E$-stratum. By Lemma 5.4, there exists $\theta'_- \in \mathcal{C}_-(\Lambda', m', \beta)$ contained in π, and we also have $m'/e' < m/e$. As above, there now exist $\widetilde{\theta}'_- \in \mathcal{C}_-(\Lambda', m'-1, \beta)$ and $c \in {\mathfrak{a}'_{-m'}}^- \cap \mathcal{M}$ such that π contains $\widetilde{\theta}'_-\psi^-_c$.
The result now follows by iterating the above process, which will end either with $m' = 0$, contradicting the assumption on π, or with a fundamental derived stratum as required. Note that the iteration will terminate since the rational m/e is strictly decreasing each time, while its denominator is bounded by, for example, $(4N)!$. ■

Hence we may take our pair $([\Lambda, n, m, \beta], \theta_-)$, with θ_- contained in π, such that π contains $\vartheta = \widetilde{\theta}_-\psi^-_c$, for some $c \in \mathfrak{a}^-_{-m}$, with $[\Lambda_{\mathfrak{o}_E}, m, m-1, s(c)]$ fundamental. Moreover, since $e/(m, e)$ is bounded, where $e = e(\Lambda|\mathfrak{o}_F)$, we may take such a pair with m/e minimal.

Now suppose that our fundamental stratum $[\Lambda_{\mathfrak{o}_E}, m, m-1, s(c)]$ is non-G-split, in the sense that none of the $[\Lambda_{\mathfrak{o}_i}, m, m-1, s_i(c_i)]$ are G-split. Then, by [14] (4.4), for $i = 1, \ldots, l$, there exists a semisimple skew stratum $[\Lambda'^{(i)}_{\mathfrak{o}_i}, m'_i, m'_i - 1, \alpha'_i]$ in B^{ii} such that

$$\begin{gathered} s_i(c_i) + \mathfrak{b}^{ii}_{-m+} \subset \alpha'_i + \mathfrak{b}'^{ii}_{-m'_i+}, \\ m/e(\Lambda^{(i)}|\mathfrak{o}_i) = m'_i/e(\Lambda'^{(i)}|\mathfrak{o}_i), \\ \Lambda'^{(i)} \text{ is a refinement of } \Lambda^{(i)}. \end{gathered}$$

As above, we put $\Lambda' = \oplus_{i=1}^l \Lambda'^{(i)}$, $e' = e(\Lambda'|\mathfrak{o}_F)$ and $m' = m'_i e'/e(\Lambda'^{(i)}|\mathfrak{o}_F)$. Note that the final two conditions imply that $\mathfrak{a}'_{-m'} \cap \mathcal{M} \subset \mathfrak{a}_{-m} \cap \mathcal{M}$, so that $\alpha' \in \mathfrak{b}'_{-m'} \subset \mathfrak{b}_{-m}$.

Now we may apply Lemma 5.4 once again to conclude that there exist $\widetilde{\theta}' \in \mathcal{C}_-(\Lambda', m'-1, \beta)$ and $c' \in \mathfrak{a}'_{-m'}$ such that $s(c') = \alpha'$ and π contains the character $\vartheta' := \theta'_-\psi^-_{c'}$ of $H^{m'}_-(\beta, \Lambda')$. However, $[\Lambda', n', m'-1, \beta + c']$ is equivalent to a semisimple stratum $[\Lambda', n', m'-1, \beta']$, by Lemma 3.5, and $\widetilde{\theta'_-}\psi^-_{c'} \in \mathcal{C}_-(\Lambda', m'-1, \beta')$. But we have $(m'-1)/e(\Lambda'|\mathfrak{o}_F) < m/e(\Lambda|\mathfrak{o}_F)$ and (using Proposition 5.5 if necessary to obtain a fundamental derived stratum) this contradicts the minimality of m/e.

Hence we must have that the derived stratum $[\Lambda_{\mathfrak{o}_E}, m, m-1, s(c)]$ is G-split; we suppose, without loss of generality, that $[\Lambda'_{\mathfrak{o}_1}, m', m'-1, s_1(c_1)]$ is G-split. As in [14] §2.1, this gives rise to a decomposition of E_1-vector spaces $V^1 = V^1_0 \perp (V^1_1 \oplus V^1_{-1})$, where V^1_1 and V^1_{-1} are totally isotropic with respect to h, such that $\Lambda(k) = \oplus_{-1\le j\le 1}(\Lambda(k) \cap V^1_j)$ and $s(c)V^1_j \subset V^1_j$, for $j = -1, 0, 1$. Note, however, that we do not necessarily have $c_1 V^1_j \subset V^1_j$, for $j = -1, 0, 1$. We show that we may in fact assume that this is the case.

For $j, k \in \{-1, 0, 1\}$, we write $A^{(1)}_{jk}$ for the space $\mathrm{Hom}_F(V^1_k, V^1_j)$, and, for $a \in A^{(11)}$, we will write $a = \sum_{j,k} a_{jk}$, with $a_{jk} \in A^{(1)}_{jk}$. For $j \neq k$ we have $s_1(c_1)_{jk} = 0$ so $(c_1)_{jk} \in \mathfrak{a}^-_{-m} \cap a_\beta(A^{(1)}_{jk})$. In particular, by (3.8), there exists $x \in \mathfrak{m}^{(11)-}_m$ such that $(c_1)_{jk} = -a_\beta(x)_{jk}$. But, by Lemma 5.2, $C(x)$ normalizes H^m_- and we have $\vartheta^{C(x)} = \widetilde{\theta}_-\psi^-_{c'}$, where $c' = c + a_\beta(x)$. This character certainly occurs in π and we have $s(c') = s(c)$ and $c'V^1_j \subset V^1_j$, for $j = -1, 0, 1$, as required.

We are now in the situation of §4 and, by Theorem 4.9, π is not supercuspidal, a contradiction. This completes the proof of Theorem 5.1. ■

Remark 5.6. We have shown that any positive-level irreducible supercuspidal representation π of G contains a skew semisimple character $\theta_- \in \mathcal{C}_-(\Lambda, 0, \beta)$. Then π certainly also contains the Heisenberg representation η_- of $J^1_-(\beta, \Lambda)$.

5.2 Proof of Lemma 5.4

We are now left to prove Lemma 5.4, for which we need some preliminary lemmas. All notation will be as in the previous section, so that $[\Lambda, n, m, \beta]$ is a skew semisimple stratum with associated decomposition $V = \bigoplus_{i=1}^l V^i$ and π is an irreducible positive-level supercuspidal representation of G containing $\vartheta = \widetilde{\theta}_-\psi^-_c$, for some $\widetilde{\theta}_- \in \mathcal{C}_-(\Lambda, m-1, \beta)$ and $c \in \mathfrak{a}^-_{-m} \cap \mathcal{M}$. For $t > 0$, we put

$$\mathfrak{k}^t_1(\Lambda) = \bigoplus_{i\neq j} \mathfrak{a}^{ij}_{\frac{t}{2}+} \oplus \bigoplus_i \mathfrak{a}^{ii}_t, \qquad K^t_1(\Lambda) = 1 + \mathfrak{k}^t_1(\Lambda),$$
$$\mathfrak{k}^t_2(\Lambda) = \bigoplus_{i\neq j} \mathfrak{a}^{ij}_{\frac{t}{2}} \oplus \bigoplus_i \mathfrak{a}^{ii}_t, \qquad K^t_2(\Lambda) = 1 + \mathfrak{k}^t_2(\Lambda),$$

and $K^t_{i,-} = K^t_{i,-}(\Lambda) = K^t_i(\Lambda) \cap G$, for $i = 1, 2$. We also put $H^t_i = H(\beta, \Lambda) \cap K^t_i$, for $i = 1, 2$, and define J^t_i, $H^t_{i,-}$ and $J^t_{i,-}$ similarly. We observe that we have $H^m_1 = H^m_2$, and similarly for J.

Let $\vartheta = \widetilde{\theta}_-\psi^-_c$, as in the previous section, and let $\widetilde{\vartheta}$ denote the extension of ϑ to $H^m_{1,-}$ which is trivial on the unipotent parts – that is, $\widetilde{\vartheta} = \widetilde{\theta}_-\psi^-_c$, where $\widetilde{\theta}_-$ is now extended to a simple character (also denoted $\widetilde{\theta}_-$) of $H^{\frac{m}{2}+}$.

Lemma 5.7. *π contains $\widetilde{\vartheta}$.*

Proof We show that π contains the character $\widetilde{\theta}_-\psi^-_{c'}$ of $H^{\frac{m}{2}+}_-$, for some $c' \in \mathfrak{a}^-_{-m} \cap \mathcal{M}$ such that $c \equiv c' \pmod{(\mathfrak{H}^m)^*}$. Then certainly $\widetilde{\vartheta} = \widetilde{\theta}_-\psi^-_{c'}|H^m_{1,-}$ is contained in π.

Since $H_-^{\frac{m}{2}+}/H_-^m$ is abelian and ϑ extends to a character $\widetilde{\theta}_-\psi_c^-$ of $H_-^{\frac{m}{2}+}$, π certainly contains some character of $H_-^{\frac{m}{2}+}$ extending ϑ, of the form

$$\vartheta_1 = \widetilde{\theta}_-\psi_{c_1}^-, \qquad \text{for some } c_1 \in \mathfrak{a}_{-m} \text{ with } c_1 \equiv c \pmod{(\mathfrak{H}^m)^*}$$

Now, since $c \in \mathcal{M}$, for $i \neq j$ we have $(c_1)_{ij} \in (\mathfrak{H}^m)^*$. By Lemma 3.17, there exists $x \in \mathfrak{m}_{m-1}$ such that $a_\beta(x)_{ij} = -(c_1)_{ij}$, for $i \neq j$. By Corollary 3.20, $C(x)$ normalizes $H_-^{\frac{m}{2}+}$ and we have

$$\vartheta_2 := \vartheta_1^{C(x)} = \widetilde{\theta}_-\psi_{c_2}^-, \quad \text{where} \quad c_2 = c_1 + C(x)^{-1}(\beta + c_1)C(x) - (\beta + c_1)$$

A simple calculation shows that $c_2 \equiv c_1 + a_\beta(x) \pmod{(\mathfrak{H}^{m-1})^*}$ so $(c_2)_{ij} \in (\mathfrak{H}^{m-1})^*$, for $i \neq j$. Since ϑ_2 certainly occurs in π, we can iterate this process until, with $m' = \lfloor \frac{m+1}{2} \rfloor$, we get $\vartheta_{m'} = \widetilde{\theta}_-\psi_{c_{m'}}^-$ in π, with $(c_{m'})_{ij} \in (\mathfrak{H}^{\frac{m}{2}+})^*$, for $i \neq j$. Now, putting $c' = \sum_{i=1}^l (c_{m'})_{ii} \in \mathcal{M}$, we see that π contains $\vartheta_{m'} = \widetilde{\theta}_-\psi_{c'}^-$, as required. ■

Proposition 5.8 (cf. [6] (8.1.7)). *Let $\mathfrak{G}$ be an open subgroup of $K_{2,-}^m$ and ρ an irreducible representation of $\mathfrak{G}$ such that $\rho|H_{1,-}^m(\beta,\Lambda) \cap \mathfrak{G}$ contains $\widetilde{\vartheta}|H_{1,-}^m(\beta,\Lambda) \cap \mathfrak{G}$. Then $\pi|\mathfrak{G}$ contains ρ.*

Proof We know, from Lemma 5.7, that π contains $\widetilde{\vartheta}$. Now the proof is identical to that of [6] (8.1.7), given the following lemma:

Lemma 5.9 (cf. [6] (8.1.8)). *There is a unique irreducible representation τ of K_-^m such that $\tau|H_{1,-}^m(\beta,\Lambda)$ contains $\widetilde{\vartheta}$.*

Proof By Corollary 3.12(iii), the commutator $[J_1^m, J_1^m] \subset H^{m+}$, so the pairing

$$\begin{gathered} \boldsymbol{k}_{\widetilde{\vartheta}} : J_{1,-}^m/H_{1,-}^m \times J_{1,-}^m/H_{1,-}^m \to \mathbb{C}^\times, \\ \boldsymbol{k}_{\widetilde{\vartheta}}(x,y) = \widetilde{\vartheta}[x,y], \end{gathered}$$

depends only on $\widetilde{\vartheta}|H_-^{m+} = \theta_-$. By Proposition 3.28, $\boldsymbol{k}_{\widetilde{\theta}_-}$ on $J^{\frac{m}{2}+}/H^{\frac{m}{2}+}$ is nondegenerate and since, from the simple case, $\boldsymbol{k}_{\widetilde{\theta}_-}$ is nondegenerate on $J^{\frac{m}{2}+} \cap M/H^{\frac{m}{2}+} \cap M$, it follows that $\boldsymbol{k}_{\widetilde{\vartheta}}$ is also nondegenerate. Hence there is a unique irreducible representation μ of $J_{1,-}^m$ containing $\widetilde{\vartheta}$.
The result now follows once we have shown that the intertwining of μ in $K_{2,-}^m$ is contained in $J_{1,-}^m$, for then $\tau := \mathrm{Ind}_{J_{1,-}^m}^{K_{2,-}^m} \mu$ is irreducible and is as required.

Since the restriction of μ to $H_{2,-}^{m+}$ is actually a multiple of $\widetilde{\theta}_-|H_{2,-}^{m+}$, it is enough to show that the intertwining of $\widetilde{\theta}_-|H_{2,-}^{m+}$ in $K_{2,-}^m$ is contained in $J_{1,-}^m = J_{2,-}^m$. Moreover, using Glauberman's correspondence as usual, we have $\widetilde{\theta}_- = \mathbf{g}(\widetilde{\theta})$, for some $\widetilde{\theta} \in \mathcal{C}^\Sigma(\Lambda, \frac{m}{2}, \beta)$ and it is enough to show that the intertwining in K_2^m of the character $\widetilde{\theta}|H_2^{m+}$ is contained in J.
So let $g \in K_2^m$ intertwine $\widetilde{\theta}|H_2^{m+}$; then certainly g intertwines the simple character $\widetilde{\theta}|H^{m+}$ so, by Theorem 3.22, we have $g \in \Gamma_m B_\beta^\times \Gamma_m \cap K_2^m$. Since $\Gamma_m K_2^m \Gamma_m \cap B_\beta^\times \subset J$ normalizes $\widetilde{\theta}$, we need only consider $g \in \Gamma_m \cap K_2^m$. We write $g = 1 + x$, with $x \in \mathfrak{m}_m$, and use the Iwahori decomposition $g = l_g m_g u_g$, with $l_g \in N_l$, $m_g \in M$, $u_g \in N_u$. We also write $l_g = 1 + x_l$, $m_g = 1 + x_m$ and $u_g = 1 + x_u$, with $x_l \in \mathfrak{m}_m \cap A_l$, $x_m \in \mathfrak{m}_m \cap \mathcal{M}$ and $x_u \in \mathfrak{m}_m \cap A_u$.

We proceed now in a very similar way to the proofs of Theorems 3.22 and 4.2. Set

$$\mathfrak{k}_l = \mathfrak{H}^{m+} + \mathfrak{H}^{\frac{m}{2}+} \cap A_u, \qquad K_l = 1 + \mathfrak{k}_l.$$

g certainly intertwines $\widetilde{\theta}$ on K_l so, by Corollary 3.20, we have

$$g^{-1}(\beta + \mathfrak{k}_l^*)g \cap (\beta + \mathfrak{k}_l^*) \neq \emptyset.$$

Now $a_\beta(x_m)$ and $a_\beta(x_u) \in \mathfrak{k}_l^*$, by Lemma 3.17 so we see that this implies $a_\beta(x_l) \in \mathfrak{k}_l^*$ also. This is all really happening in A_l, where $\mathfrak{k}_l^* \cap A_l = (\mathfrak{H}^{\frac{m}{2}+})^* \cap A_l$ so, again by Lemma 3.17, we find that there exists $y_l \in \mathfrak{m}_{\frac{m}{2}}$ such that $a_\beta(y_l) = \alpha(x_l)$. Then, by injectivity of a_β on A_l, we see that $x_l = y_l$ so that $g_l \in \Gamma_{\frac{m}{2}}$. But $\frac{m}{2} < \frac{r}{2}$ so $\Gamma_{\frac{m}{2}} \subset J$ and g_l normalizes $\widetilde{\theta}$.

Hence we may assume $g = g_m g_u$ and, repeating with $\mathfrak{k}_u = \mathfrak{H}^{m+} + \mathfrak{H}^{\frac{m}{2}+} \cap A_l$, we see we may assume $g \in \Gamma_m \cap M$. Now the result follows from the simple case in the proof of [6] (8.1.8). ■

This also completes the proof of Proposition 5.8. ■

Now we are in a position to prove Lemma 5.4 with the additional hypothesis:

$$(H) \qquad U^{m'}(\Lambda') \subset K_2^m(\Lambda)$$

From the simple case (see [6] §1.3), we may choose $c' \in \mathfrak{a}'^-_{-m'} \cap \mathfrak{a}^-_{-m} \cap \mathcal{M}$ such that $s(c') = \alpha'$. (Note that we may take $c' = 0$ if $\alpha' = 0$). Put $\delta = c' - c$; then, since $s(c) + \mathfrak{b}_{-m+} \subset s(c') + \mathfrak{b}'_{-m'+}$, we have

$$\delta \in \left(\mathfrak{a}'^-_{1-m'} + a_\beta(A)\right) \cap \mathfrak{a}^-_{-m} \cap \mathcal{M} = \mathfrak{a}'^-_{1-m'} \cap \mathfrak{a}^-_{-m} \cap \mathcal{M} + a_\beta(A) \cap \mathfrak{a}^-_{-m} \cap \mathcal{M},$$

where the equality of lattices is from the simple case [6] (8.1.13). Then, again by the simple case [6] (1.4.10), there exists $x \in \mathfrak{n}_m \cap \mathfrak{a}^-_{r-m}$ such that

$$\delta - a_\beta(x) \in \mathfrak{a}'^-_{1-m'} \cap \mathfrak{a}^-_{-m} \cap \mathcal{M}.$$

$C(x)$ normalizes $H^m_-(\beta, \Lambda)$ and $\vartheta^{C(x)} = \vartheta\psi^-_{a_\beta(x)}$, by Lemma 5.2. Now, from the transfer property, there exists $\theta'_- \in \mathcal{C}_-(\Lambda', m'-1, \beta)$ such that θ_- and θ'_- agree where they are both defined. Then, on $H^m_-(\beta, \Lambda) \cap H^{m'}_-(\beta, \Lambda')$, we have

$$\vartheta^{C(x)} = \theta_- \psi^-_c \psi_\delta = \theta'_- \psi^-_{c'}.$$

Moreover, since $C(x) \in M$, $\widetilde{\vartheta}^{C(x)}$ and $\theta'_- \psi^-_{c'}$ are trivial on the unipotent parts N_u and N_l so that $\widetilde{\vartheta}^{C(x)}$ and $\theta'_- \psi^-_{c'}$ agree where they are both defined. Finally, since $H^{m'}(\beta, \Lambda') \subset U^{m'}(\Lambda')$ which, by hypothesis (H), is included in $K_2^m(\Lambda)$, by Proposition 5.8 we have that π contains $\vartheta' := \theta'_- \psi^-_{c'}$, as required.

For the general case we need some additional technical lemmas, which use the description of the building in terms of lattice functions from [2]. We refer the reader to *op. cit.* §§0–7 for more details. The idea is that, if (H) is not satisfied, we can pass more gradually from Λ to Λ', via some intermediate sequences.

For Λ, Λ' two lattice sequences, there exists a common splitting – that is, an F-basis $\mathcal{B}$ of V which splits them – which we fix. Choosing such a basis is the same thing as choosing an apartment in the (extended) affine building $\mathcal{I}^1(\widetilde{G},F)$ of $\widetilde{G}$. Indeed, by [2] Propositions 1.4, 2.4, there is a (unique upto translation) G-set isomorphism between $\mathcal{I}^1(\widetilde{G},F)$ and the set of lattice functions on V, of which, by *op. cit.* §7, the lattice sequences are the subset of "rational points" (i.e. barycentres of vertices with rational weights). By transferring the structure from the building, we then have an affine structure on the set of lattice sequences.

The choice of a basis of V (or apartment in $\mathcal{I}^1(\widetilde{G},F)$) gives us a set of roots $\{\alpha_{ij}\}$, $(1 \le i,j \le N$, $i \ne j)$ by $\alpha_{ij}(\mathrm{diag}(u_1,...,u_N)) = u_j u_i^{-1}$. We also set α_{ii} to be the "zero root", for each i. Then, if $x \in \mathcal{I}^1(\widetilde{G},F)$ corresponds to Λ, [2] Corollaries 4.5, 4.6 say that, for $r \in \mathbb{R}$, $\mathfrak{a}_r(\Lambda)$ is the set of matrices $(y_{ij}) \in M(N,F) \simeq A$ satisfying

$$y_{ij} \in \mathfrak{p}_F^{\left\lceil\left(\frac{r}{e}-\alpha_{ij}(x)\right)\right\rceil}, \quad i,j = 1...N, \tag{5.10}$$

where $e = e(\Lambda)$. This does not depend on the choice of the identification between $\mathcal{I}^1(\widetilde{G},F)$ and the set of lattice functions.

Lemma 5.11. *Let $[\Lambda, m, m-1, b]$, $[\Lambda', m', m'-1, b']$ be strata in A and put $e = e(\Lambda)$, $e' = e(\Lambda')$. Suppose that*

$$b + \mathfrak{a}_{-m+}(\Lambda) \subset b' + \mathfrak{a}_{-m'+}(\Lambda'), \qquad b' \in \mathfrak{a}_{-m}(\Lambda).$$

Let t be a rational number and put

$$\Lambda_t = (1-t)\Lambda + t\Lambda', \qquad e_t = e(\Lambda), \qquad m_t = e_t\left((1-t)\frac{m}{e} + t\frac{m'}{e'}\right).$$

Then $b, b' \in \mathfrak{a}_{-m_t}(\Lambda_t)$, for all $0 \le t \le 1$, and, for $0 \le s \le t \le 1$ rational numbers,

$$\mathfrak{a}_{-m_s+}(\Lambda_s) \subset \mathfrak{a}_{-m_t+}(\Lambda_t)$$

Note that, in this lemma, we are just looking at the line segment $[\Lambda, \Lambda']$ in the building.

Proof For $r \in \mathbb{R}$, we abbreviate $\mathfrak{a}_r = \mathfrak{a}_r(\Lambda)$ and $\mathfrak{a}'_r = \mathfrak{a}_r(\Lambda')$. We note that, since $b' \in \mathfrak{a}_{-m} \cap \mathfrak{a}'_{-m'}$, the same is also true for b and $\mathfrak{a}_{-m+} \subset \mathfrak{a}'_{-m'+}$.

We let x, x' be the points in the building corresponding to Λ, Λ' respectively and, for $0 \le t \le 1$, let x_t be the point corresponding to Λ_t. We also put $a^t_{ij} = \alpha_{ij}(x_t)$, for $1 \le i,j \le N$ and $0 \le t \le 1$, and we will use the description (5.10).

Since $b \in \mathfrak{a}_{-m} \cap \mathfrak{a}'_{-m'}$, we have, for $1 \le i,j \le N$,

$$v_F(b_{ij}) \ge -\tfrac{m}{e} - a^0_{ij}, \qquad v_F(b_{ij}) \ge -\tfrac{m'}{e'} - a^1_{ij}.$$

But then $v_F(b_{ij}) \ge (1-t)(-\frac{m}{e} - a^0_{ij}) + t(-\frac{m'}{e'} - a^1_{ij}) = -\frac{m_t}{e_t} - a^t_{ij}$. Since $v_F(b_{ij})$ is an integer, this means that $b \in \mathfrak{a}_{-m_t}(\Lambda_t)$, as required. The same is true for b'.

For the last assertion, we observe that, since m_t is an integer and every a^t_{ij} can be written as a fraction with denominator e_t, $\mathfrak{a}_{-m_t+}(\Lambda_t)$ is the set of matrices $(y_{ij}) \in M(N,F) \simeq A$ satisfying

$$y_{ij} \in \mathfrak{p}_F^{\left(-\frac{m_t}{e_t}-a^t_{ij}\right)+}, \quad i,j = 1...N.$$

Since $\mathfrak{a}_{-m+} \subset \mathfrak{a}'_{-m'+}$, we have

$$\left(-\tfrac{m}{e} - a_{ij}^0\right) + \geq \left(-\tfrac{m'}{e'} - a_{ij}^1\right) +$$

But we certainly have that $(-\frac{m_t}{e_t} - a_{ij}^t)$ lies in the interval whose endpoints are $(-\frac{m}{e} - a_{ij}^0)$ and $(-\frac{m'}{e'} - a_{ij}^1)$, and the result follows easily. ■

Now, if x, x' are points in the building corresponding to Λ, Λ' respectively, we put

$$|\Lambda - \Lambda'| := \max{}_{1\leq i,j\leq N} |\alpha_{ij}(x) - \alpha_{ij}(x')|.$$

Again, this does not depend on the choice of the identification between $\mathcal{I}^1(\widetilde{G}, F)$ and the set of lattice functions.

Lemma 5.12. *Let $k > 0$. Then there exists $\delta > 0$ such that, for all lattice sequences Λ, Λ' split by $\mathcal{B}$ with $|\Lambda - \Lambda'| < \delta$, we have $\mathfrak{a}_{ke}(\Lambda) \subset \mathfrak{a}_{\frac{ke'}{2}+}(\Lambda')$, where $e = e(\Lambda)$ and $e' = e(\Lambda')$.*

Proof We just take $\delta = \frac{k}{2}$ and the result is clear from (5.10). ■

We can now complete the proof of Lemma 5.4 in the general case (i.e. without the hypothesis (H)). We put $\alpha = s(c)$, $k = \min\{\frac{m}{e}, \frac{m'}{e'}\}$ and choose δ as in Lemma 5.12. Let q be an integer greater than $\frac{1}{\delta}|\Lambda - \Lambda'|$ and, for each integer t with $0 \leq t \leq q$, let Λ_t be the lattice sequence $(1 - \frac{t}{q})\Lambda + \frac{t}{q}\Lambda'$. (This was denoted $\Lambda_{t/q}$ in Lemma 5.11.). Note that each Λ_t is an $\mathfrak{o}_E$-lattice sequence, since Λ and Λ' are. By Lemma 5.11 applied to these $\mathfrak{o}_E$-lattice sequences, we have that, for $0 \leq t < q$,

(i) $\alpha \in \mathfrak{b}^-_{-m_t}(\Lambda_t)$;

(ii) $\mathfrak{b}_{-m_t+}(\Lambda_t) \subset \mathfrak{b}_{-m_{t+1}+}(\Lambda_{t+1})$.

Moreover, by Lemma 5.12, we also have

(iii) $U^{m_{t+1}}(\Lambda_{t+1}) \subset U^{\frac{m_t}{2}+}(\Lambda_t)$.

Taking the dual of (ii), we see that $\mathfrak{b}_{m_{t+1}}(\Lambda_{t+1}) \subset \mathfrak{b}_{m_t}(\Lambda_t)$. Now, by choosing a common "generalized (W, E)-decomposition" for Λ_t, Λ_{t+1}, we see (as in [6] top of page 272) that we also have $\mathfrak{a}_{m_{t+1}}(\Lambda_{t+1}) \cap \mathcal{M} \subset \mathfrak{a}_{m_t}(\Lambda_t) \cap \mathcal{M}$. In particular, together with (iii) this implies

$$U^{m_{t+1}}(\Lambda_{t+1}) \subset K_2^{m_t}(\Lambda_t)$$

The result now follows by repeated application of the case where (H) is satisfied. ■

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