

Conceptually-driven and visually-rich tasks in texts and teaching practice: the case of infinite series

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The study we report here examines parts of what Chevallard calls the *institutional* dimension of the students' learning experience of a relatively under-researched, yet crucial, concept in Analysis, the concept of infinite series. In particular we examine how the concept is introduced to students in texts and in teaching practice. To this purpose we employ Duval's Theory of Registers of Semiotic Representation towards the analysis of twenty-two texts used in Canada and UK post-compulsory courses. We also draw on interviews with in-service teachers and university lecturers in order to discuss briefly teaching practice and some of their teaching suggestions. Our analysis of the texts highlights that the presentation of the concept is largely a-historical, with few graphical representations, few opportunities to work across different registers (algebraic, graphical, verbal), few applications or intra-mathematical references to the concept's significance and few conceptually-driven tasks that go beyond practising with the application of convergence tests and prepare students for the complex topics in which the concept of series is implicated. Our preliminary analysis of the teacher interviews suggests that pedagogical practice often reflects the tendencies in the texts. Furthermore the interviews with the university lecturers point at the pedagogical potential of: illustrative examples and evocative visual representations in teaching; and, student engagement with systematic guesswork and writing explanatory accounts of their choices and applications of convergence tests.

Keywords: infinite series, Calculus, visualisation, conceptual understanding, tasks, texts, theory of registers of semiotic representation, anthropological framework, institutional dimension of learning, post-compulsory education

1. Infinite series: A significant, yet under-studied, concept

*Infinite series*¹ is a complex, often counter-intuitive but significant mathematical concept with a wide range of applications within mathematics and in science. The idea of infinite summation has one of its most immediate applications in the calculation of an area under a curve in a given interval through approximation – through the summing of increasingly thinner rectangular areas. The concept of the Riemann sum and of the definite integral originates in this calculation which is an essential part of students' introduction to integration. Another mathematical application of infinite series (thereafter called *series*) is in the writing of numbers with infinite decimals ($0.333... = 0.3 + 0.03 + 0.003 + ...$). The concept of series is also used in the definition of power series, as well as Fourier series, is key to the study of families of polynomials (e.g. Taylor polynomials) and is implicated in the methods for solving several types of differential equations. In the sciences the concept is used in the study of waves as well as for modelling situations such as the distribution of medicines or pollutants. In sum, given the diverse applications of the concept in different sciences (Physics, Economics, Biology, Medicine etc.) and within mathematics, a study of its teaching and learning is of relevance to the training of future professionals in multiple fields. It is therefore quite surprising that this study has been rather rare and often appearing only implicitly in works about sequences or convergence.

Our motivation for focusing on the concept of series originates in some of the findings reported in the doctoral thesis of González-Martín [1]. For the research reported in the thesis González-Martín constructed an alternative introduction to the concept of improper integral²

¹ $\sum_{n=1}^{\infty} a_n$ where a_n is a sequence of real numbers and $n \in \mathbb{N}$.

² To define the Riemann integral of a given function within an interval we need the interval to be closed and the function to be bounded within this interval. When one of these two conditions is not fulfilled, we define the improper integral as a generalisation of the Riemann integral.

grounded in the participating university students' prior knowledge of definite integrals and series. During the study it became evident that the students' understanding of series was very fragile, especially with regard to the association between the general notion of series and improper integrals. For instance, when they were asked whether the following implication was true:

$\sum_{n=1}^{\infty} f(n) < \infty \Rightarrow \int_1^{\infty} f(x)dx < \infty$, some of the obtained answers were: *'it's true [...] since the addition*

of the terms of a series is a part of the integral. So, if a part converges, therefore the integral converges', *'it's true [...], since the integral is the limit of the sum of the partitions'*, or *'in the integral, the series is implicit'*. Further consideration of the students' difficulties with the concept of series during the study led to the following conjectures: in these early stages of their studies students have not mastered yet the skill to work efficiently with the convergence tests; students find it hard to establish connections between tests for convergence for series and those for improper integrals; and, there is little visual imagery associated with the concept of series. In a different study Nardi [2, 3] reported students' logical and conceptual difficulties in dealing with the tests of convergence: for example, the students tended to 'deform' statements about convergence, such as the Limit Comparison Test, into statements about divergence; they also appeared to resist the idea that, in certain occasions, convergence tests can be inconclusive. It is in the light of such findings in our previous research that we decided to work on a systematic investigation of learning and teaching issues regarding the concept of series. In the next section we summarise findings from some of the works done in this area so far.

2. Studies on the teaching and learning of the concept of infinite series

As mentioned above there is little research that focuses on the concept of series – the concept usually appears implicitly in works about sequences or convergence, e.g. [4, 5], and some works

also stress how difficulties with this concept are likely to impact on students' understanding of the concept of integral [6] and, in particular, the concept of improper integral [1]. For the purposes of this paper, we refer to a selection of key works in this area and we structure our discussion around three themes: the presentation of the concept in texts; some difficulties that students experience in learning the concept; and, recommendations regarding its teaching.

Two of the early studies on the understanding of convergence in which the concept of series appears implicitly are by Robert [4] and Boschet [5]. In one of the pioneer works in this area Robert analysed the different, and often inadequate, mental and written representations of convergence of sequences held by university students in France. Her analyses attributed this inadequacy partly to the limited nature of the exercises used in teaching. Boschet's results, based on an analysis of the presentation of numerical sequences in undergraduate courses – mostly in texts and students' and teachers' notes – were similar. She also pointed out that traditional teaching usually includes very few examples of graphic representation of convergence (with the existing ones fostering mostly static representations). Finally, she also observed that sequences are not seen as a particular case of function – an observation also made a few years later by Mamona [7]. Another study, around the same time, that offered an analysis of texts used in undergraduate courses, and focused on the concept of series and its links with improper integrals, was by Fay & Webster [8]. They describe how the theorems and tests for convergence of series and improper integrals are presented in the texts and conclude that little or no attention is given to highlighting the connections between improper integrals and series, with the exception of the integral test for the convergence of series. In the absence of explicit pointers at these connections it is of little surprise, they stress, that the students often miss the correspondence between the tests of convergence for integrals and those for series. González -Martín's more recent results [1]

corroborated this observation: the students participating in his study too could not make these connections.

With regard to the difficulties that students have with learning the concept of series – and beyond the difficulties relating to the notions of sequence, limit, and convergence, often reported in research – one of the few works with a specific focus on the concept is that of Kidron [9]. She identified three main difficulties with the concept, all germane to the learning of most Calculus concepts. The first is the process-object duality of the concept [10, 11], especially in terms of the subtle distinction between a mathematical concept and the practical steps (or algorithms) that allow employing this concept in applications. The second concerns the notion of potential vs. actual infinity [12] implicit in the concept, especially in terms of the difficulty to conceive the process of adding infinite terms as a whole, while having to engage with a process that has to be performed step by step. And the third concerns the discrepancies between the concept's formal definition and students' concept image [13] of series, especially in terms of the potential gap between the definitions given to the students and the tasks (problems, exercises, applications etc.) which allow building a personal rapport with the concept. Kidron also highlighted that the reading of the equality $S = a_0 + a_1 + \dots + a_n + \dots$ from right to left, or from left to right are cognitively different and pointed out the difficulties which are evident in the students' use of symbolic notation. Her observations are in agreement with Mamona [7] who also highlighted students' confusion between sequences and series as well as their resistance to see sequences as a type of function.

Finally, with regard to teaching recommendations, some authors have argued that the use of visual reasoning could be advantageous for students [14-16], especially for the purpose of assisting students' meaning bestowing to the concept. Alcock & Simpson [14] propose that the

use of visual reasoning could help students establish links between the formal and the visual representations of the concept. Bagni [15, 17, 18] has also suggested that the use of historical examples could help students overcome misconceptions such as ‘infinite terms implies infinite sum’. A *didactic transposition* [19] regarding the teaching of series, Bagni claims, must consider students’ possible reactions to the concept as these might be similar to those of mathematicians in the past. He distinguishes two levels in the conceptualisation of the concept, the *operational* and the *structural*, and he observes that this distinction is not usually taken into account in traditional teaching. The problem of passing from the finite to the infinite has a cultural root, he observes. It is therefore important that students approaching this problem in the context of learning Calculus, and particularly series, become aware of these historical and cultural roots. Agreeing with Bagni, Codes & Sierra [16] developed some activities making use both of visualisation with the help of a computer and of historical references. They constructed some activities in an electronic environment to introduce the basic notions of numerical sequences before introducing series as the limit of a certain type of sequence. Their activities are based on Oresme’s work and, with the assistance of the electronic environment, they also emphasise graphical representations. Their results indicate that the use of the computer and graphical representations helped some students visualise and better understand some of the processes into play as well as reason more effectively in some of the paper-and-pencil questions. Therefore, it seems that the representations generated in the electronic environment improved or extended the students’ own representations of an infinite series.

The study we report in this paper aims to examine the ways in which the concept of series is introduced to students in texts and in teaching practice – both parts of what Chevallard [20] calls the *institutional* dimension of the students’ learning experience; also whether these ways take into account the student difficulties and the teaching recommendations we summarised in

this section. In particular, in this paper, we present the analysis of twenty-two texts used in Canada and the UK in post-compulsory – post-secondary, foundation and university – courses. We also draw on interviews with in-service teachers and university lecturers in order to discuss briefly teaching practice and some of their teaching suggestions.

3. A study of texts introducing the concept of series: Focus, theoretical background, context

The foci of our text analysis are: *visualisation*; types of *problems/exercises/applications*; and, *historical references*. The paper concerns mostly the first two as our analysis yielded little with regard to the third – a small number of historical anecdotes (such as Zeno’s paradox) and portraits of mathematicians, all essentially unconnected to the main flow of the text. We are also particularly interested in whether texts take into account the student difficulties with the concept identified by some of the research we reviewed in the previous section. Below we outline some of the theoretical underpinnings of our choice to focus the analysis on *visualisation* and *problems/exercises/applications*.

For the former, we consider the type of learning activity fostered by the texts under examination from the perspective of Duval’s *Theory of Registers of Semiotic Representation* [21]. We do so through considering the use of different registers of semiotic representation as necessary for the construction of sufficient mental images of mathematical concepts.

For the latter, we take into account the fact that students’ construction of series depends on how the concept is considered by the ‘institution’ in which they learn. For this part, we draw on Chevallard’s anthropological approach [20], which offers a way of analysing the practices in an institution (in our case, the tasks available to the students in the texts) that aim to generate the learning of a particular mathematical ‘object’ (in our case, the concept of series). We employ Chevallard’s theory in order to examine the types of *problems/exercises/applications* in the texts.

In the following we briefly introduce the parts of Duval's and Chevallard's theories that are relevant to the analysis we present in this paper.

According to Duval's *Theory of Registers of Semiotic Representation* [21] a mathematical object can only be accessed through its different representations; and, communication about and conceptualisation of the object can take place within different registers (e.g. algebraic, graphic, verbal). Each representation features some of the object's key characteristics, so to have a comprehensive understanding of the object, the coordination of different representations is necessary. The use of different representations is likely to foster a varied and diverse understanding of these characteristics, and thus of the mathematical object. Emphatically it is not sufficient to simply use different representations in a disconnected manner; the learner must also be able to make connections between them, what Duval calls *coordination* of registers. Duval identifies three activities related to *semiosis*, namely *an individual's* attribution of meaning to representations of a new concept: the formation of a representation within a register, treatment of this representation within that same register and the conversion to another representation in another register. It is with these three activities in mind that we examine the texts and ask whether they introduce the notion of series using different registers, mainly the algebraic and the graphic registers, and whether they encourage the students to engage with the three activities outlined above. The choice of these two registers, particularly the graphic register, was also motivated by findings in previous research outlined earlier in this paper.

Our perspective on the *problems, exercises and applications* present in the analysed texts originates in Chevallard's [20] anthropological theory, particularly in the ways in which this theory aims to examine how an institution – a secondary school, a university department etc. – chooses to introduce a new mathematical object in texts, in teaching etc. (in our case: *problems/exercises/applications* in texts related to the concept of series). This theory recognises

that mathematical objects are not absolute objects but entities whose meanings arise from the practices within a given institution and the practices in which individuals are invited to engage and adopt. Chevallard describes these practices in terms of *tasks*, in terms of *techniques* used to solve these tasks, in terms of *technologies*, the discourses which explain and justify the techniques and in terms of *theories* which organise and justify the technologies. In this sense, to understand the meaning that an institution attributes to ‘knowing/understanding an object’ is to identify the *tasks*, *techniques*, *technologies* and *theories* which bring this object into play. In our case a task can be one of the series-related *problems*, *exercises* and *applications* in the texts; a technique can be the use of one of the tests typically employed to determine the convergence of a series (e.g. Comparison Test etc.); a technology can be the proof of a convergence test or the theory behind the necessary steps to solve a series-related problem; and, a theory is the set of the fundamental theoretical principles of the mathematical domain of Analysis. This paper focuses mainly on our examination of the series-related *tasks in the texts* and on some inferences from this examination concerning an *institution’s learning priorities* with regard to this concept.

In the light of above outlined perspectives our analysis of a selection of Canadian and UK texts used in post-compulsory courses aimed to address the following questions:

- Do texts support students’ overcoming of key misconceptions typically associated with the learning of the concept of series, such as ‘infinitely many addends, infinitely great sum’? If so, how?
- Do texts instil an algorithmic and procedural approach to the concept? If so, how?
- Do texts use graphical or any other visual representations in order to enrich students’ visual understanding of the concept? If so, how?
- Which representations and task types are privileged in texts?
- Do texts contextualise the concept in terms of its *raison-d’être* in mathematics and its many applications in mathematical and other situations? If so, how?
- Overall is there a clear and visible evolution in the way in which texts introduce series?
- Overall do texts take into account research results concerning the learning of series?

With regard to the choice of texts we aimed to analyse mainstream texts used to introduce the concept to Canadian and UK students in post-compulsory education. The concept of series is first introduced to different audiences in the two countries, so we first outline these differences. In Québec, site of the Canadian part of the study³, the concept of series is firstly introduced during what is called *collégial* or *cégep* studies (17-18 year-old students). In Québec, compulsory education is divided into primary (6-11 years of age) and secondary (12-16 years of age). Students then can proceed to professional training or prepare for university studies (two years of *collégial* studies). There are different orientations of *collégial* studies, according to the student's field of subsequent university specialisation (technology, sciences, humanities etc.). For students who want to pursue technological or scientific studies, the concept of series is first presented in their *collégial* studies (during the first year), and is later developed further in their undergraduate studies. We have chosen to analyse how the concept of series is introduced to students in these *collégial* courses (in the technological and scientific orientation) in a selection of texts of, roughly, the last twenty years. In many cases, some texts are just a newer edition of the original texts, but the chapter about series has not undergone substantial changes. In these cases, we have just considered the earlier edition. Our final sample (Table 1, upper part) includes seventeen texts appearing in the official programs of several post-compulsory establishments in Montréal. For our research, we have only considered the Montréal area, on the grounds that other areas often follow what is done there. In Table 1 (upper part) we name the 17 Canadian texts Text *A* to *Q*, taken in chronological order (from 1992 to 2008).

[Insert Table 1 about here]

³ We note that Canada does not have a Ministry of Education. Syllabi and other curricular aspects of compulsory education are the responsibility of educational authorities in each province, one of which is Québec

Analogously, in the UK, we have identified – with the help of lecturers teaching the topic to undergraduates in mathematics, science and engineering – texts, both from applied and pure mathematics, typically recommended to university or foundation students following the introduction of the concept in the lectures. Students' first informal encounter with the concept is during their A level mathematics studies (16-18 years of age), usually in the context of the geometric series. We note that books are not the only, and not always the dominant, resource that UK university students use – a more comprehensive analysis of curriculum content would have to take into consideration lecture notes, exercise sheets, etc., particularly those in which the concept is introduced for the first time. As the role of the texts in UK is slightly different from that in Québec, and the audiences of these texts are also slightly different, we use the UK sample in order to reinforce some of the discussion emerging from the analysis of the Québec sample. Our UK sample consists of five texts, four appearing in the reading lists of university courses and one used in foundation courses. In Table 1 (lower part) we name the 5 UK texts Text *R* to *V*, taken in chronological order (from 1967 to 2006).

In accordance with the questions listed above we have recorded, amongst other, in a spreadsheet the following quantitative and qualitative information on each text – see also 44-46]:

- The number of pages that the text dedicates to the concept of series.
- The number and type of figural representations (e.g. graphs, drawings etc.) and the ratio of representation per page and the role of these representations in the text.
- The number and type of tasks (problems, exercises, applications) involving the concept of series and the respective ratio per page.

Across the above we also aimed to identify:

- Semiotic activity explicitly requested from the student.
- Reasons given to justify the introduction of the new concept.

In what follows we present our findings in two sections, one on *visualisation* and one on *tasks* (problems, exercises and applications). We then discuss the subsequent phase of our work which focuses on an analogous study of teaching practices.

4. Data and findings: The presence and role of *visualisation* in the texts

Below we discuss the presence and role of visualisation in the analysed texts in the light of the following classification:

- A non-conceptual (NC) image does not relate to a mathematical concept (e.g. is a portrait or a photograph). Its use is merely decorative.
- A bland-conceptual (BC) image is an illustration of a mathematical concept, but is not an integral part of an argument or an explanation (e.g. the graph of a function in a margin, to remind the student of its shape).
- A conceptual (C) image is used to explain a concept, or to illustrate one step of a proof. It might be part of the proving process and it is explicitly intended to help the student understand a notion or a mathematical argument.

This classification emerged from the analytical comments on each text recorded in the spreadsheet, from our particular interest in examining the texts from Duval's [21] 'semiotic activity' perspective and from previous classifications employed in the literature such as the one proposed by Elia & Philippou [47] for analysing the role of visual representations in problems: *decorative* (does not provide any information to solve the problem), *representational* (represents a part of the content of the problem, but it is not essential to solve the problem), *organisational* (helps to organise the steps for the solution, but it is not essential to solve the problem) and *informational* (the solution of the problem is based on this visual representation; some of the necessary data is in this image, so the image is essential to solving the problem). This classification is appropriate for examining the role of images in the statement and solution of a problem or exercise. However our aims are a little wider and include analysing the role of images used also in the theoretical parts of the text to define or explain the concept. Therefore we

adapted these categories to the NC, BC and C ones listed above that serve our purpose more precisely.

Most of the seventeen Canadian texts listed in Table 1 (upper part) concentrate on integration and sequences and series, and they have a separate volume for continuity and derivatives. Unsurprisingly then these seventeen texts allocate, on average, a substantial 11% of the space to the concept of series. In this section we analyse the parts of the texts in which the concept of series is introduced and discussed.

Table 2 shows the number of graphs and of other images in texts A-Q. We count graphs separately from other images in the sense of Duval [21] who distinguishes between the graphical and the figural register. The former refers to any representation that deploys the Cartesian rules of representation (e.g. function graphs, the real line etc.) and the latter to those not deploying these rules (e.g. shapes, diagrams). The first row of numbers in the cells of Table 2 indicates the number of pages used for series in each text, the second shows the number of graphs (denoted g)/other images (denoted i) and the third row indicates the average of visual representations per page:

[Insert Table 2 about here]

As evident in the above the number of visual representations is small, especially when considering the overall space allocated to the topic of series in the text. We can also see that the ratio does not necessarily increase with the most recent texts, from text A to Q.

Table 3 shows the use of the three different kinds of visual representation in each text (NC-non conceptual, BC-bland conceptual, C-conceptual). The first row of numbers shows the number of NC graphs and other images, the second shows the number of BC graphs and other images, and the third shows the number of C graphs and other images (in bold):

[Insert Table 3 about here]

As evident in Table 3 there are few Conceptual visual representations of the concept of series in these texts, even though several other representations are possible – see, for example, Figure 1.

[Insert Figure 1 about here]

A closer look at the Conceptual graphical representations used in the texts reveals that almost all the texts that use them, do so in order to illustrate the Integral Test, stating when the behaviour of a series coincides with that of the associated improper integral. (Usually the images illustrate the connection between summation and definite integral before generalising to the connection between series and the improper integral). We note that this applies also for the two texts with the highest number of Conceptual graphs, texts N and P, which use variations of the graphic image used to illustrate the Integral Test in other parts of the chapter for series:

[Insert Figure 2 about here]

[Insert Figure 3 about here]

The graphical representations used to illustrate the Integral Test, such as the ones in Figures 2 and 3, are typically the first occasions in which a graphical representation related to the concept of series appears in the text. Also typically, these representations are not accompanied by an account that aims to link the representation with the algebraic and other symbolic representations of the concept used in the text. The authors of the texts appear to take for granted that the students will instantly establish this connection and, for example, will interpret the rectangles appearing under a curve as representing the terms of the sum within a series. However, as suggested by findings in our previous research [1], most often this is not the case. Furthermore no semiotic activity [21] is explicitly asked of the students with regard to the graphic or the figural register in any of the seventeen texts examined above. For example, the student is never explicitly asked to produce a visual image of series, or to convert from one register to another.

Similar observations emerge from the analysis of the five UK texts (Table 1 lower part). We found nine visual representations, all of them graphical, in these texts. As in the Canadian texts we looked in the ‘theoretical’ sections, namely those where the concept is introduced and discussed. The distribution of the nine visual representations in the five texts is as follows: four in text R three in text S and two in text V (0.19, 0.16 and 0.2 representations per page respectively). They are mainly visual representations of the series terms or the partial sums as points on the real line (Figures 4 and 5) or areas of rectangles (Figures 6, 7 and 8).

In particular, Figure 4 features some of the n^{th} partial sums (s_n) of the series $x_1 - x_2 + x_3 - x_4 + \dots$ where $\{x_n\}$ is a monotone decreasing to zero sequence in the proof of the ‘Leibniz Test for Real Series’ (Text V, p. A70). According to the Leibniz Test this series converges and the illustration of Figure 4 supports the claim in the proof that:

$s_1 \geq s_3 \geq s_5 \geq \dots \geq s_6 \geq s_4 \geq s_2$. This representation also appears in some of the Canadian texts (texts G, K, N, and O, and variations of it also appear in texts F, H, J, M, and P).

[Insert Figure 4 about here]

Figure 5 features the terms and the partial sums of the series: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 1$. Through this picture, not only the order of the terms is illustrated but the infinite sum being 1 is also evident and, as suggested in Text R, this is ‘an infinite sum which can always be remembered from the picture’ (p. 391). This image also appears in some of the Canadian texts.

[Insert Figure 5 about here]

Figure 6 features ‘... a graphical argument. Each term of the series represents the area of the rectangle with base equal to the unity and height equal to the magnitude of the term’ (Text S,

p.72). Whereas Figure 7 visualises the symbolic expression: $f(n+1) < \int_n^{n+1} f < f(n)$ for monotonic functions in the proof of the ‘Integral Test’ (Text R, p. 396).

[Insert Figure 6 about here]

[Insert Figure 7 about here]

Similar representations connected with the Integral Test we found in all three books. We characterised all nine visual representations as Conceptual as they were an integrated part of a proof and supported intermediate steps of the argument. As in the Canadian texts, there was little explanatory account linking them to the rest, algebraic and other symbolic, representations of the concept used in the text. But there was some. For example: the figure ‘illustrates relationships’ (Text R, p. 399); ‘convergence follows from’ the figure (Text V, p. 670); ‘we conclude’ (Text V, p. A70) or ‘is clear’ (Text R, p. 401) from the figure. Also, the infinite sum ‘can always be remembered from the picture’ (Text R, p. 391) and the proof may ‘easily’ be seen or obtained by a ‘graphical argument’ (Text S, p. 72, 77). However, in only one of these cases, in Text S, there is a detailed interpretation of the relationship between the area of the rectangles, the elements of the sum, the area under the curve and the integral (Figure 8).

[Insert Figure 8 about here]

In some of the texts we found the following rather evocative representation of the proof of the divergence of the harmonic series (via grouping of the terms) – we return to the pedagogical potential of using the example of the harmonic series more generally in the final section of the paper.

[Insert Figure 9 about here]

The scarcity, as well as the often disconnected presence of the visual representations used in the texts we examined led us to the conclusion that the texts somewhat miss out on

opportunities to encourage the – largely supported by the relevant research (e.g. [14]) – dynamic interplay between the verbal, algebraic and the graphic registers in their introductions to the concept of series. In this sense, namely the limited facilitation of what Duval terms ‘conversion’ and ‘coordination’, they do not seem to explicitly support students’ semiotic activity with regard to this newly introduced concept.

A case in which the above is particularly pertinent is that of worked examples, especially those present in the sections where the concept is introduced and discussed. We examined which registers are privileged (algebraic, graphic, verbal) as well as how the texts establish cross-register links in the presentation of these examples. Our observations on these examples echo those made earlier in this section. For instance, of the Canadian texts none offers any example exclusively in the graphic register; only eight texts offer mixed-register examples (namely using at the same time the visual and the algebraic registers – see, for instance, Figure 10); and, on average there is a proportion of about 2 mixed-register examples for every 100 purely algebraic examples. Again, in the mixed-register examples, there is no commentary that facilitates the coordination of and fluency across different registers, conveying the impression that the students will be able to interpret the visual information and establish its links with the algebraic and symbolic expressions on their own.

[Insert Figure 10 about here]

5. Data and findings: The presence and role of tasks in the texts

In this section we discuss the types of tasks that students are invited to engage with in the light of the following classification:

- Applications, namely tasks with a context

- Tasks asking the students to determine whether a series is convergent or divergent using the convergence tests (**Type I**)
- Tasks asking the students to calculate the sum of a series (**Type II**)
- Tasks asking the students to both determine whether a series is convergent and calculate the sum (**Type III**)

This classification emerged from the analytical comments on each text recorded in the spreadsheet and from our particular interest in examining the texts from Chevallard's [20] perspective on how particular types of meaning emerge from particular types of practices that students are invited to engage with.

With regard to applications, six of the Canadian texts offer none. Again this is not necessarily a characteristic of older texts as the six texts are A, B, H, J, M and Q, ranging from year of publication 1992 to 2008. Table 4 shows for each text: in the first row of numbers the total number of tasks, including applications, exercises and problems (we have considered the smallest unit, so one exercise asking to decide the convergence of six series will count as six different tasks); in the second row the number of these which are applications; and, in the third row the percentage of applications over the total number of tasks. The average ratio of applications in the sample is about 2 applications out of every 100 tasks.

[Insert Table 4 about here]

With regard to Type I-III tasks Table 5 shows: in the first row of numbers, the total number of tasks; in the second, Type I tasks; in the third, Type II tasks; and, in the fourth, Type III tasks. Finally, the last row gives the percentage over the total number of tasks that these three Types taken together represent:

[Insert Table 5 about here]

As the high percentages in Table 5 suggest (with an average in the seventeen texts of about 77% of tasks being of Type I, II or III), the majority of the tasks are Type I and II, namely,

in most cases the students are asked to determine the convergence of a series or calculate the sum. Most of these questions can be tackled via the application of one of the convergence tests. More conceptually-driven tasks, namely tasks that encourage explicitly student engagement with Duval's [21] three kinds of semiotic activity listed earlier, are rare in these texts. In Chevallard's [20] terms the meaning ascribed to 'knowing/understanding an object', in our case the concept of series, implied by the choice of tasks in the texts is more or less equated with 'being able to apply the convergence tests correctly in order to determine the convergence of a series and to calculate its sum'. If more than three fourths, on average, of the tasks in the texts concentrate on the application of convergence tests (even if some of these tasks require a relatively sophisticated application) valuable space is taken away from tasks of a different kind; tasks, for example, that highlight aspects of the concept that are significant to the students' introduction to other topics such as integration and power series. In this sense, the *techniques* [20], that the students are encouraged to engage with and apply, concern mostly the application of the convergence criteria. Therefore the theoretical knowledge that the students are invited to be engaged with (in Chevallard's [20] terms, the relevant *technology*) is more pronouncedly linked with the use of criteria to decide convergence than with constructing a conceptually rich image of the concept of series (and the notion of convergence). For these texts, knowing the mathematical object 'concept of series' seems to be identified with knowing how to apply some of the convergence criteria. Again, similarly to our observation in the visualisation section, the texts seem to miss out on opportunities to engage students with a more epistemologically rich approach to the new concept (and also to address some of the potential difficulties that students may have with the concept, identified in the literature).

Analogous discussion emerged from our examination of the UK texts. Here we describe briefly the approach to problems and exercises in each separately. As evident in these individual

text profiles the approaches are quite distinct and vary according to the aims and intended uses of each text.

Text U is used mainly in A/AS level and Foundation courses in mathematics. Its emphasis is mostly on specific cases of sums (e.g. geometric and arithmetic progression). Series and the convergence of series are considered only in the context of the geometric progression (and not through the convergence tests). To this aim the exercises are mainly applications of the formulae of the arithmetic and geometric progressions. In this sense they are what we called above Type II tasks. Only two exercises (over the total 119) are applications of the geometric progression in a financial context (a mortgage and a bank loan problem). In Text U there are also two applications in the section of the text introducing and discussing the concept: the first one is an extra-mathematical application concerning a problem the solution of which needs the modelling in a geometric progression and then the algebraic manipulation of the related formulae. The second application provides a context for a materially-based calculation of $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ (a piece of string cut in half, and then in half etc.).

Text V is used mainly for engineering mathematics courses in which a course in elementary Calculus is prerequisite. The reference to the series is intended as a supportive complement to a presentation that focuses mostly on intra-mathematical applications of the notion (e.g. the power series). In the section of the text that introduces the concept of series and the convergence tests there are fifteen tasks. Nine of these are of Type I. Of the remaining six (all on p.672): four are theoretical (e.g. Problem 28 asks for an example that demonstrates how one version of the Ratio Test presented earlier in the text is more general than another); one is a CAS (Computer Algebra System) project that includes a request to use the program that the students are expected to write in the first part of the project to ‘experiment with the rapidity of

convergence of series of [their] choice' (p.672); and, another one is a team project in which students are asked to 'write a short report on the basic concepts and properties of series of numbers, explaining in each case whether or not they carry over from real series (discussed in Calculus) to complex series, with reasons given'. In these six there is an explicit focus on semiotic activity that is quite distinct from the Type I-III tasks dominating most of the analysed texts.

Text S is mainly used for courses in mathematics and is seen as suitable for undergraduate students of the physical sciences. In the chapter 'Convergence of Infinite Series' of the seventeen tasks, fifteen are Type I-III and two are more sophisticated Type I tasks. For example, Problem 15 on p.91 involves some engagement at the level of theory with a newly introduced test stated early in the problem.

Texts R and T are mainly used in first year undergraduate courses in mathematics. Their approaches to tasks vary considerably. Of the fifteen tasks in Text T eleven are Type I; one is Type II; and, three ask for proofs of theorems (e.g. the proof for the Sum and Scalar Product rules for series). In contrast, in Text R, of the forty two tasks only twenty are Type I; one requests the calculation of a shaded area defined by a function graph in a given figure (incidentally also the one and only image used across all the task sections in the five UK texts). The remaining twenty one tasks ask theoretical questions. Some of these are explicitly conceptually driven, namely encouraging explicitly semiotic activity in the sense of Duval [21] we described earlier: some are

geared towards a generalisation (e.g. from the $\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$ to $\sum_{n=1}^{\infty} \frac{a^n n!}{n^n}$); some establish the

connection between two theorems; some encourage reflection on the convergence tests and their proofs (e.g. via expressions such as 'another kind of 'comparison test'' in Problem 6 on p.407 or 'alternative proof' of a convergence test in Problem 9 on p.408). Also, we note that in Text R,

although we found no extra-mathematical applications (neither in the theoretical nor the exercise part), we nevertheless identified a tendency for intra-mathematical connections (namely connections not necessarily between different disciplines but between mathematical topics). So, for example, in Exercise 22 on p.411, the non-convergence of the *harmonic series* ($\sum_{n=1}^{\infty} \frac{1}{n}$) is connected to the discussion of the infinite number of positive rational numbers.

Despite this small number of exceptions, particularly in Texts R and V – and bearing in mind the diversity of aims and target audiences of the texts – there is an overwhelming similarity across the 22 analysed texts: the predominance of Type I-III tasks. In the discussion that follows we examine some implications of this in the light of the Duval [21] and Chevallard [20] perspectives that guided our text analysis.

6. Introducing the concept of series within the confines of the algebraic register

The analysis of the Canadian texts suggests that, even though the concept enjoys substantial coverage in most texts used in post-compulsory education to prepare students for undergraduate studies, its presentation is largely a-historical and decontextualised, dominantly in the algebraic register and with few graphical representations and applications. Particularly few texts offer applications of the concept in other disciplines (e.g. Medicine, Economics etc.); and it is not always the more recent texts that present more applications. Finally, there are hardly any historical references in these texts – with the exception of a few, but usually out of context, references – and no attempt is made to present the concept's evolution in the history of mathematics. In sum the Canadian texts do not appear to take into account research results regarding the pedagogical potential of introducing concepts in multiple registers and with a

historical-epistemological perspective. The algebraic register is privileged and, overall, the introduction to this concept remains quite formal.

The analysis of the UK texts suggests similar results. Texts allow little space for visual representations and hardly require of the students a coordination of the algebraic and graphic registers. None of the UK texts makes any historical reference and the applications shown are rare. Usually the concept is introduced without explaining what it is useful for (with the exception of one text that introduces the concept in terms of its relevance to other mathematical topics). These results seem to suggest that the elements detected in the Québec texts are characteristics that transcend the boundaries of this province, and, given the slightly different audiences of the Canadian and UK texts, are perhaps independent of the audience that the text addresses.

We recognise that the number of UK texts we examined is very small and that their selection is slightly opportunistic and largely based on informal recommendations indicated to us by a small number of colleagues with teaching experience in this area. Therefore generalisations from these results need to be made with caution, if at all, and only bearing also in mind the different roles that the texts play in the UK and Canadian context. In the UK the texts are recommended to the students by the lecturer who also gives them tasks in the form of problem sheets or implicitly in the lecture notes. In the Canadian post-compulsory context the text operates more or less as a textbook and engagement with the text represents a quite substantial part of the students' learning experience.

The fact that there is little – and rather diverse – research developed around the concept of series may not encourage text authors and editors to use other than the algorithmic and algebraic approaches so far privileged in the texts. We have seen that, even if texts give a relatively substantial space to content about series, the approach that they use seems to be ‘traditional’, in

the sense that the register used is almost exclusively algebraic. It seems that Boschet's [5] remarks about the absence of visual representations of convergence still hold for concepts like series, more than two decades later, despite recommendations in the literature [14-16] regarding the pedagogical potential of these representations. The same more or less applies with regard to recommendations for the integration of historical examples [15, 17, 18].

In addition to that, and when the texts provide a *raison-d'être* for it, series is typically introduced as a mathematical object that fulfils mathematical needs only (integration, Taylor series, ...) without any, or with little, suggestion of its applications outside mathematics. Furthermore the texts do not foster engagement with this concept in ways that would require the coordination of multiple registers (algebraic, graphical, verbal). In Duval's sense the students' semiotic activity is thus likely to be limited and within the confines of the privileged register. From Chevallard's point of view students will learn about a concept only in the ways that are implicit or explicit in the tasks they are invited to engage with. So, for instance, if the students engage with the concept of series mostly in the context of tasks that require the application of convergence tests, it might not be surprising that many are unable to interpret a graph of a series; or, that they are unclear about aspects of the concept that are highly relevant to other key concepts such as improper integrals and power series [1]; or they have not observed that the Σ notation represents both the process of addition and the result of this process [9]; or that they have not discerned the possibility offered by the concept to add all the terms without having to add them one by one (potential vs actual infinity [9]).

Another observation that emerged in the course of our text analyses was that of the longevity of what we called above a 'traditional' presentation in the texts. Texts may have

become more colourful, sometimes even with higher production values, but not necessarily with conceptually richer presentations.

Overall it seems that generally, despite meticulous presentations of the concept of series, the texts that we examined do not take the opportunity to address the difficulties with the concept identified in previous research. In another currently ongoing part of our work we have started to examine whether this is also the case in another significant part of *the institutional* dimension of the students' learning, teaching practice. To this purpose in the following we report briefly on interviews with post-compulsory teachers (in Canada) and university lecturers (in the UK).

7. From texts to teaching practice: a similar picture?

As part of our currently ongoing examination of teaching practice González-Martín [48] interviewed five teachers from different⁴ post-compulsory establishments in Montréal in order to explore: their subject matter content knowledge and pedagogical content knowledge [49] about the concept of series; and, how their teaching practices are related to the presentation of the concept in the texts. In the 45 minute interviews the teachers also had the opportunity to reflect on the effectiveness of these practices. The interviews focused on the following:

- the interviewees' teaching experience at the *cégep* level, particularly on the concept of series;
- their accounts of the ways in which the concept is presented to the students, their preferred text for this presentation and their comments on the adequacy of this text, from a teacher and from a learner perspective;
- their accounts of how they address the apparent paradox inherent in the concept of series about trying to add infinite number of terms, what examples and applications they offer; and, in what registers (algebraic, graphic etc.) they discuss the concept of series;
- their views on visualisation with regard to the concept of series and the tasks they typically invite the students to engage with;

⁴ As approaches to the presentation of curriculum content are decided by teachers in agreement with the mathematics department in their institution, teachers from five different institutions were selected in order to access as great a range of approaches as possible.

- their own perceptions of the concept, its key characteristics, its importance in mathematics, the difficulties that its learning may involve, the ways in which they help students overcome these difficulties and, finally, what ‘ideal’ teaching of the concept might involve.

One conjecture emerging from the preliminary analyses of these interviews is that these teachers’ practices seem to reflect the tendencies we identified in the texts. Overall the reported teaching appears to be confined to the application of convergence tests and to presenting series as a prerequisite tool for introducing Taylor polynomials. The teachers’ own perceptions of series are generally in the formal-algebraic register and their teaching follows closely the presentation in the texts they choose to use in their lessons. As a result they offer few opportunities to the students to work across different registers and they rarely discuss the applications of series within mathematics and in other disciplines. Most of the teaching revolves around the learning and application of convergence tests and, on the basis of their accounts in these interviews, these teachers seem themselves unaware of the different representations and applications of series. This lack of awareness seems to have a significant effect on both their subject matter content knowledge and pedagogical content knowledge [49] about series. For example, when the teachers in the interviews articulated a more conceptually-driven interest in the concept, we asked them to discuss the tasks that they would use in order to foster that interest in their lessons. Their responses did not exactly match this intention as their offers were mostly single-register, algorithmic tasks. There seems to be a pertinent teacher education issue implicit in the above that needs to be addressed.

Some suggestions on how the introduction of the concept can move beyond the mostly algorithmic, single-register approach in some texts are available in the literature but were also made to us in interviews with 21 lecturers based in UK mathematics departments (reported also in: [3, 45, 46, 50]). The focus of these suggestions is on finding ways to reflect on the epistemological richness and complexity of the concept as well as addressing students’ reported

difficulties with the concept. Below we summarise some of these suggestions. We highlight in bold type the substance of the views expressed by the interviewed lecturers.

8. Towards a richer introduction of the concept of series: University lecturers' perspectives

The interviewed lecturers⁵ stated their scepticism about engagement with tasks that ask students to determine the convergence of a series without much explanation on why they chose a particular convergence test. Presenting a **convincing account of their choice**, as well as **writing out the entire formulation of the test** they are using, might help students realise what the test's conditions are and what makes the use of this particular test pertinent in the case of the series they are examining. In fact it might also assist with the difficulty students have at this stage in dealing with theorems that have several conditions.

The lecturers stressed that the students need to **engage with as many examples of using the convergence tests as possible** in order to start building a sense of the scope of each test. They named this lack of illustrative examples as one source of the students' difficulties with tackling problems of convergence. Another source is students' unease with *if and only if* statements and other logical difficulties of this ilk – such as the students' unease with accepting the inconclusiveness of convergence tests (also briefly mentioned in the literature review earlier in this paper).

⁵ This brief summary reports data available on p.64-8, 189, 190, 200, 214, 248-9 in [3]. The interviews in which the data reported here originate were lengthy (approximately four-hour) interviews on a range of issues concerning the teaching and learning of mathematics at university level. There were 11 such interviews and were conducted in groups of 3-5 university lecturers (several participated in more than one interview). Chapter 2 in [3] gives an extensive methodological account of the study. The interviewees' references to the concept of series were 'scattered' across the dataset and were made in the context of several discussions (e.g. on students' uses of theorems, visualisation, learning issues in Calculus etc.). Here we present a synthesis of these references.

According to the interviewees learning about the concept of series gives students an opportunity to **rethink some of their often longstanding perceptions of mathematics as a merely calculational activity**. One example of this was offered in the context of discussing students' typical responses to the task 'Prove that $b_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n}$ converges to 2'. The lecturers reported that some students had calculated the sum first and then considered the convergence of the *sequence* b_n , namely a more familiar object to them than the series. In one of the interviews the following thought was put forward: this approach does indeed give an opportunity for some basic, and gratifying, algebraic manipulation. It also draws on approaches (such as the geometric progression) that the students have seen before. However it is not an approach that will prove effective in the longer term in Analysis. Students need to become accustomed to treating series as entities on their own – not convert them into a number or a function or another more familiar object – and develop a set of techniques in tackling problems of convergence. In this sense their otherwise sensible approach to this task somewhat arrests the development of this treatment and these techniques.

The interviewees described some of these techniques as exercises in trial and error and in **developing the skill to make appropriate, clever choices when deciding, for example, which convergence test to use**. The approach was described as **guess-then-prove**. Students are encouraged to experiment with a series until they form a relatively solid belief about its convergence or divergence. They can do this via collecting tabular evidence or working in an electronic environment to examine, for example, values for large n . Then they are invited to discuss the choice of an appropriate test as well as an appropriate choice of, say, a series to compare with, in case they are using one of the comparison tests. This experimentation and guess-work, the lecturers stressed, is crucial as students' immediate intuition about convergence

or divergence at this stage may not be that rich. Furthermore this more experimental type of work helps students starting to **move away from the idea of a mathematical task being always about something to be worked out, achieving a number as a result**, etc. Reflecting on their students' prior experiences, the lecturers described how students often come with earlier mathematical experiences which do not emphasise acts of choice such as the ones involved, for example, in determining which convergence test to use. Not only that, but students are normally protected, even turned away, from unsuccessful choices and thus miss the **opportunity to develop a sense of how things work in mathematics through looking at things that do not work** – for example, through **being allowed to find out why particular convergence tests may not work in particular cases**.

What is pertinent in the above recommendations put forward by the interviewed lecturers is that they have the potential to transform what we called Type I-III tasks in the previous sections into more conceptually-driven engagement with the concept of series and the notion of convergence. Furthermore, and in tandem with our observations in earlier sections of the paper, the lecturers were generally supportive of the idea to **involve more visualisation** in the teaching of the concept of series, as well as **evocative verbal descriptions that link algebraic and graphical illustrations of the concept**. However they also stressed that **there is not always a good, intuitive picture of everything in mathematics** and they quoted the statement 'a series does not converge' as one of those that are hard to represent pictorially. Similarly they stressed that **verbal descriptions of mathematical ideas are often notoriously difficult** – for example, we often read a mathematical utterance backwards when we try to express it in words – and should therefore be presented to students with extreme caution. So, in sum, the interviewed lecturers' recommendations favoured a conceptually-driven orientation of the tasks and a cautious employment of visual and verbal representations of the concept.

We close this paper with an elaboration on some of the above recommendations as put forward by one interviewed lecturer⁶. With regard to the potency of visualisation the lecturer stresses that diagrams are ‘a good heuristic’. A way to stress this potency to the students, he claims, is to acknowledge the presence of visualisation in their written work both with encouraging verbal commentary and with some, if not full, credit. At the heart of his argument seems to be that **visual representations** can be persuasive and **can help students overcome limited or misleading perceptions concerning series**. He quotes two of these perceptions: ‘if the terms of the series get smaller then the series converges’ and ‘confusing the series $\sum x_n$ with the sequence $\{x_n\}$, particularly as students need to understand partial sums S_n of the terms of $\{x_n\}$ as a sequence itself’.

The lecturer offers two explanations for the origins of the first of these two student perceptions.

First, students’ previous experience with series may consist entirely of working with the

geometric series $\sum_{n=0}^{\infty} x^n$ (which, for $0 < x < 1$, is convergent and the terms of the sequence indeed

become smaller). Second, their previous experience and practice is often limited to a very small number of examples, almost always of convergent series. As convergence offers the opportunity to formulate questions with a number as an answer to arrive at – and this sense of mathematical closure is often given priority in school mathematics – students may arrive at university with divergence representing a situation they have never faced.

Evident in the above is the interviewee’s belief in **the role of key examples in altering student perceptions** such as the above. The interviewee elaborates the harmonic series as one of

⁶ In addition to extracting from those 11 interviews Nardi and Biza also conducted a lengthy interview (about one hour) with one university lecturer with about sixteen years of teaching experience in applied and pure mathematics as well as other disciplines. The list of topics discussed with him resembled substantially the list of topics discussed with the *cégep* teachers in Canada and was adapted to suit the university-level experience of the interviewee. For a more elaborate account of this interview see [45, 46, 50].

these key examples: under the influence of above mentioned perception students start off absolutely convinced of its convergence. But the series is divergent. He recalls an illustration of the divergence of the harmonic series that he uses often (Figure 11). He also stresses the effectiveness of offering students ‘visual portrayals’ of convergence as well as divergence and mentions two: a ‘squares and rectangles’ one useful for representing, for example, the geometric series for powers of $\frac{1}{2}$; and a ‘staircase’ one in which the height of the staircase represents the sum of the series (convergence) and which, with a slight modification in the size of the staircase steps, becomes a ‘dramatic’ demonstration of divergence.

[Insert Figure 11 about here]

9. Introducing the concept of series: Ways forward

In the previous section we discussed briefly suggestions made by experienced lecturers concerning the introduction to the concept of series in terms of how this practice can address student difficulties with the concept and its uses as well as illustrate the concept’s richness and epistemological complexity. These suggestions involve benefiting from potent examples (e.g. the harmonic series) and highly evocative visual representations (such as the ones in Figure 11); encouraging students to provide explanatory accounts of their choices; encouraging students to engage with systematic guesswork that helps to build their intuition about convergence and develop the skill to make appropriate choices of convergence test. In our discussion we discerned the potential that these recommendations may have to transform Type I-III tasks into more conceptually-driven engagement with the concept of series and the notion of convergence; and, to support students’ modifying limited or misleading perceptions concerning series.

We see the concept of series as deeply intertwined with other topics. We hope that our study of the students’ first encounters with this concept – in this paper focusing mostly on texts

and to some extent on pedagogical practice and suggestions – will provide insights that are germane to the learning and teaching of other topics and concepts in Calculus.

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References

- [1] A.S. González-Martín, La generalización de la integral definida desde las perspectivas numérica, gráfica y simbólica utilizando entornos informáticos. Problemas de enseñanza y de aprendizaje, Ph.D. diss., University of La Laguna, Spain. ISBN: 84-7756-679-8, 2006.
- [2] E. Nardi, and P. Iannone, On convergence of a series: The unbearable inconclusiveness of the Limit Comparison Test, in Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education, (PME) M. van den Heuvel-Panhuizen ed., PME, Utrecht, The Netherlands, 2001, pp. 399-406.
- [3] E. Nardi, Amongst mathematicians: Teaching and learning mathematics at university level, Springer, New York, 2008.
- [4] A. Robert, L'Acquisition de la notion de convergence des suites numériques dans l'Enseignement Supérieur, in Recherches en Didactique des Mathématiques, 1982, pp. 307-341.
- [5] F. Boschet, Les suites numériques comme objet d'enseignement (premier cycle de l'enseignement supérieur français), in Recherches en Didactique des Mathématiques, 1983, pp. 141-163.
- [6] J. Bezuidenhout, and A. Olivier, Students' conceptions of the integral, in Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education (PME), T. Nakahara and M. Koyama eds., PME, Hiroshima, Japan, 2000, pp. 73-80.
- [7] J. Mamona, Sequences and series –Sequences and functions: Students' confusions, in International Journal of Mathematical Education in Science and Technology, 1990, pp. 333-337.
- [8] T. Fay, and P. Webster, Infinite series and improper integrals: a dual approach, in Mathematics and computer education, 1985, pp. 185-191.
- [9] I. Kidron, Concept definition, concept image, and the notion of infinite sum in old and new environments, in Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education (PME), A. D. Cokburn and E. Nardi eds., PME, Norwich, United Kingdom, 2002, pp. 209-216.
- [10] A. Sfard, On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin, in Educational Studies in Mathematics, 1991, pp. 1-36.
- [11] E. Gray, and D.O. Tall, Duality, Ambiguity and Flexibility: A Proceptual View of Simple Arithmetic, in The Journal for Research in Mathematics Education, 1994, pp. 115-141.
- [12] E. Fischbein, Intuition in Science and Mathematics: An Educational Approach, Reidel, Dordrecht, The Netherlands, 1987.

- [13] D.O. Tall, and S. Vinner, Concept Image and Concept Definition in Mathematics with particular reference to Limits and Continuity, in *Educational Studies in Mathematics*, 1981, pp. 151–169.
- [14] L.J. Alcock, and A.P. Simpson, Convergence of sequences and series: Interactions between visual reasoning and the learner's beliefs about their own role, in *Educational Studies in Mathematics*, 2004, pp. 1-32.
- [15] G. Bagni, Difficulties with series in history and in the classroom, in *History in mathematics education: the ICMI study*, J. F. J. Maanen ed., Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000, pp. 82-86.
- [16] M. Codes, and M. Sierra, Actividad rectángulos: Un ejemplo de aplicación de metodologías activas en el aula universitaria de matemáticas, in *Actas de las IV Jornadas internacionales de Innovación Universitaria*, Ed. Universidad Europea de Madrid, 2007.
- [17] G. Bagni, Infinite series from history to mathematics education, in *International Journal for Mathematics Teaching and Learning*, 2005. Retrieved April 2010, from <http://www.cimt.plymouth.ac.uk/journal/bagni.pdf>.
- [18] ---, Mathematics education and historical references: Guido Grandi's infinite series, in *Normat - Nordisk Matematisk Tidsskrift*, 2005, pp. 173-185.
- [19] Y. Chevallard, *La transposition didactique*, La Pensée Sauvage Éditions, Grenoble, France, 1985.
- [20] ---, L'analyse des pratiques enseignantes en théorie anthropologique du didactique, in *Recherches en Didactique des Mathématiques*, 1999, pp. 221-266.
- [21] R. Duval, *Sémiosis et Pensée Humaine. Registres sémiotiques et apprentissages intellectuels*, Peter Lang, Bern, 1995.
- [22] S. Robert, *Calcul différentiel et intégral*, Cégep de Saint-Jean-sur-Richelieu, Québec, 1992.
- [23] F. Ayres, and E. Mendelson, *Calcul différentiel et intégral*, Chenelière/McGraw-Hill, Québec, 1993.
- [24] G. Charron, and P. Parent, *Calcul différentiel et intégral II*, Études Vivantes, Québec, 1993.
- [25] G. Beaudoin, and J. Laforest, *Calcul 2*, Les éditions BL, Montréal, Canada, 1994.
- [26] S. Wild, *Calcul différentiel et intégral II*, Études vivantes, Québec, 1995.
- [27] E.W. Swokowski, *Analyse*, De Boeck, Bruxelles, 1995.
- [28] H. Anton, *Calcul intégral*, Reynald Goulet Inc, Québec, 1996.
- [29] B. Massé, *Calcul intégral*, Saint-Laurent: Éditions du renouveau pédagogique., Éditions du renouveau pédagogique, Saint-Laurent, Québec, 1997.
- [30] G. Charron, and P. Parent, *Calcul intégral*, Études Vivantes, Québec, 1997.
- [31] G. Ouellet, *Calcul 2 : Introduction au calcul intégral*, Le Griffon D'argile, Québec, 2000.
- [32] D. Hughes-Hallet, A.M. Gleason, M. Beaudoin, P. Charlebois, M. Labrie, and B. Fraser, , *Calcul intégral*, Chenelière/McGraw-Hill, Montréal-Toronto, 2001.
- [33] G.L. Bradley, K.J. Smith, A. Franco, and B. Marcheterre, *Calcul intégral*, ERPI, Québec, 2002.
- [34] S. Dominguez, and J.P. Rouquès, *Leçon particulière sur le cours de math: Analyse 1ère année*, Ellipses, Paris, 2002.
- [35] G.B. Thomas, R.L. Finney, M.D. Weir, and F.R. Giordano, *Calcul intégral*, Beauchemin, Québec, 2002.
- [36] G. Charron, and P. Parent, *Calcul intégral*, Beauchemin, Québec, 2004.

- [37] A. Ross, Calcul intégral pour les sciences de la nature, Le Griffon D'argile, Québec, 2006.
- [38] L. Amyotte, Calcul intégral, ERPI, Québec, 2008.
- [39] M. Spivak, Calculus, W. A. Benjamin, Inc., New York, 1967.
- [40] G. Stephenson, Mathematical methods for Science students, Longman Scientific & Technical, Essex, UK, 1973.
- [41] R. Haggarty, Fundamentals of Mathematical Analysis, Addison-Wesley Publishing Company, Wokingham, UK, 1989.
- [42] L. Bostock, and S. Chandler, Core maths for advanced level, Nelson Thornes Ltd, Cheltenham, UK, 2000.
- [43] E. Kreyszig, Advanced engineering mathematics, John Wiley & Sons, Inc, Hoboken, NJ, 2006.
- [44] A.S. González-Martín, R. Seffah and E. Nardi, The concept of series in the textbooks: A meaningful introduction?, in Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education (PME), M. Tzekaki, M. Kaldrimidou and C. Sakonidis eds., PME, Thessaloniki, Greece, 2009, pp. 105-112.
- [45] E. Nardi, I. Biza and A.S. González-Martín, Introducing the concept of infinite series: The role of visualisation and exemplification, in Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education (PME), M. Tzekaki, M. Kaldrimidou and C. Sakonidis eds., PME, Thessaloniki, Greece, 2009, pp. 193-200.
- [46] E. Nardi, I. Biza and A.S. González-Martín, Introducing the concept of infinite sum: Preliminary analyses of curriculum content, in Proceedings of the Conference of the British Society for Research into the Learning of Mathematics, M. Joubert ed., London, 2008, pp. 84-89.
- [47] I. Elia, and G. Philippou, The functions of pictures in problem solving, in Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education (PME), M. J. Høines and A. B. Fuglestad eds., PME, Bergen, Norway, 2004, pp. 327-334.
- [48] A.S. González-Martín, The concept of series: teachers' conceptions and teachers practices, in Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education (PME), PME, Belo Horizonte, Brazil, 2010, vol. 3, pp. 33-40.
- [49] L.S. Shulman, Those who understand: Knowledge growth in teaching, in Educational Researcher, 1986, pp. 4-14.
- [50] I. Biza, E. Nardi, and A.S. González-Martín, Introducing the concept of infinite series: preliminary analyses of curriculum content and pedagogical practice, in Research in Mathematics Education, 2009, pp. 185 - 186.

List of Tables

Table 1. The texts in our sample, A-Q (Canada) and R-V (UK) , in chronological order.

<i>Canadian Texts</i>									
A: Robert (1992). Calcul différentiel et intégral [22]									
B: Ayres & Mendelson (1993). Calcul différentiel et intégral [23]									
C: Charron & Parent (1993). Calcul différentiel et intégral II [24]									
D : Beaudoin & Laforest (1994). Calcul 2 [25]									
E : Wild (1995). Calcul différentiel et intégral II [26]									
F: Swokowski (1995). Analyse [27]									
G: Anton (1996). Calcul intégral [28]									
H: Massé (1997). Calcul intégral [29]									
I: Charron & Parent (1997). Calcul intégral [30]									
J : Ouellet (2000). Calcul 2 : Introduction au calcul intégral [31]									
K : Hughes-Hallet et al (2001). Calcul intégral [32]									
L : Bradley et al (2002). Calcul intégral [33]									
M : Dominguez & Rouquès (2002). Leçon particulière sur le cours de math : Analyse 1 ^{ère} année [34]									
N : Thomas et al (2002). Calcul intégral [35]									
O: Charron & Parent (2004). Calcul intégral [36]									
P: Ross (2006). Calcul intégral pour les sciences de la nature [37]									
Q : Amyotte (2008). Calcul intégral [38]									
<i>UK texts</i>									
R : Spivak (1967). Calculus [39]									
S: Stephenson (1973). Mathematical methods for Science students [40]									
T: Haggarty (1989). Fundamentals of Mathematical Analysis [41]									
U: Bostock & Chandler (2000). Core maths for advanced level [42]									
V: Kreyszig (2006). Advanced engineering mathematics [43]									

Table 2. Visual representations related to the concept of series in the Canadian texts.

<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>	<i>F</i>	<i>G</i>	<i>H</i>	<i>I</i>
21.5p	20.5p	38.5p	59p	37.5p	40p	64p	51p	41p
0g 2i	1g 2i	2g 1i	2g 1i	1g 2i	2g 2i	2g 1i	3g 8i	2g 2i
0.09	0.15	0.08	0.05	0.08	0.10	0.05	0.22	0.10
<i>J</i>	<i>K</i>	<i>L</i>	<i>M</i>	<i>N</i>	<i>O</i>	<i>P</i>	<i>Q</i>	
37.5p	13.5p	53.5p	28p	49p	54.5p	64p	50p	
1g 2i	4g 1i	7g 4i	2g 3i	5g 10i	2g 10i	19g 11i	2g 11i	
0.08	0.37	0.21	0.18	0.31	0.22	0.47	0.26	

Table 3. Non-Conceptual (NC), Bland-Conceptual (BC) and Conceptual (C) visual representations related to the concept of series in the Canadian texts.

	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>	<i>F</i>	<i>G</i>	<i>H</i>	<i>I</i>
NC	-	-	-	-	-	-	-	0g 7i	-
BC	0g 2i	0g 2i	-	0g 1i	0g 2i	0g 2i	-	1g 1i	0g 1i
C	-	1g 0i	2g 1i	2g 0i	1g 0i	2g 0i	2g 1i	2g 0i	2g 1i
	<i>J</i>	<i>K</i>	<i>L</i>	<i>M</i>	<i>N</i>	<i>O</i>	<i>P</i>	<i>Q</i>	
NC	0g 1i	-	0g 1i	-	0g 4i	0g 7i	0g 1i	0g 8i	
BC	0g 1i	0g 1i	5g 0i	0g 3i	0g 3i	0g 2i	14g 2i	-	
C	1g 0i	4g 0i	2g 3i	2g 0i	5g 3i	2g 1i	5g 8i	2g 3i	

Table 4. The presence of applications in the Canadian texts.

<i>Total</i>	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>	<i>F</i>	<i>G</i>	<i>H</i>	<i>I</i>
49	154	272	132	92	228	320	75	166	
<i>Applications</i>	0	0	6	3	1	7	3	0	2
%	0%	0%	2.2%	2.3%	1.1%	3.1%	0.9%	0%	1.2%
<i>Total</i>	<i>J</i>	<i>K</i>	<i>L</i>	<i>M</i>	<i>N</i>	<i>O</i>	<i>P</i>	<i>Q</i>	
104	45	293	0	217	167	173	51		
<i>Applications</i>	0	11	11	0	2	2	5	0	
%	0%	24.4%	3.8%	0%	0.9%	1.2%	2.9%	0%	

Table 5. Type I, II and III tasks in the Canadian texts.

<i>Total</i>	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>	<i>F</i>	<i>G</i>	<i>H</i>	<i>I</i>
49	154	272	132	92	228	320	75	166	
<i>I</i>	30	129	139	117	60	152	165	47	82
<i>II</i>	3	10	32	4	16	20	57	16	18
<i>III</i>	0	3	48	3	0	22	18	8	27
<i>I-III %</i>	67.3%	92.2%	80.5%	93.9%	82.6%	85.1%	75%	94.7%	76.5%
<i>Total</i>	<i>J</i>	<i>K</i>	<i>L</i>	<i>M</i>	<i>N</i>	<i>O</i>	<i>P</i>	<i>Q</i>	
104	45	293	0	217	167	173	51		
<i>I</i>	88	10	206	0	145	78	112	34	
<i>II</i>	10	16	27	0	27	30	11	10	
<i>III</i>	1	3	20	0	10	24	4	0	
<i>I-III %</i>	95.2%	64.4%	86.3%	0%	83.9%	79%	73.4%	86.3%	

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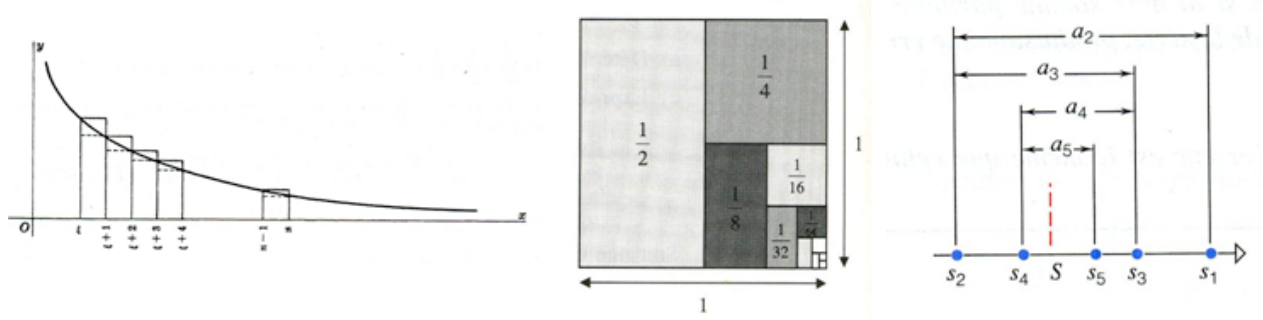


Figure 1. Three Conceptual visual representations related to the concept of series (graphical and figural) in Texts B, C and G.

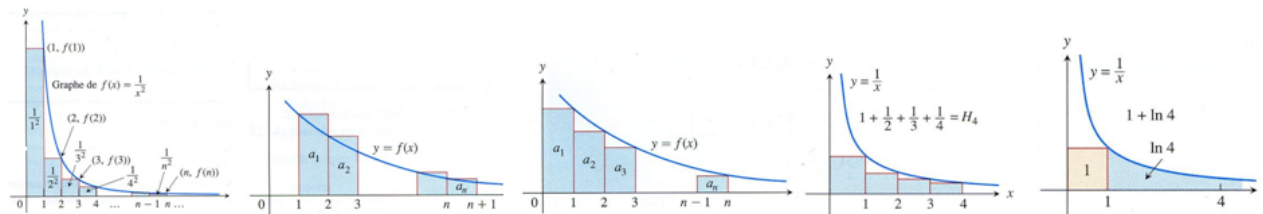


Figure 2. Text N, Conceptual graphs related to the Integral Test.

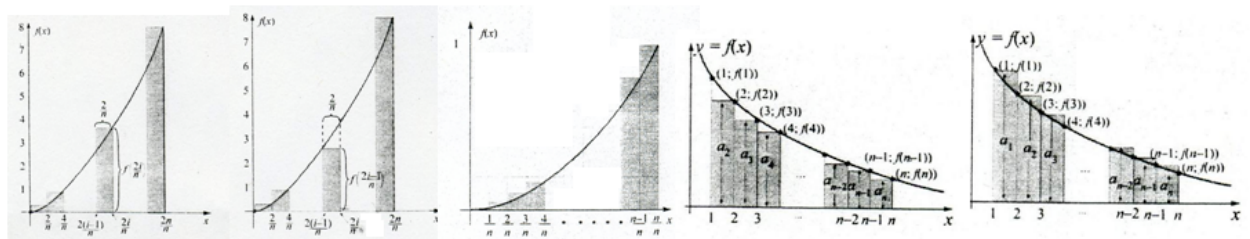


Figure 3. Text P, Conceptual graphs related to the Integral Test.

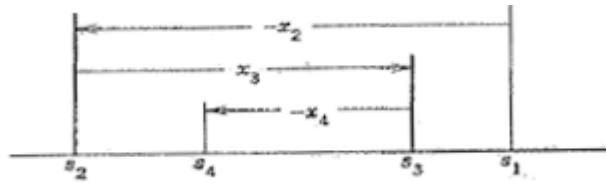


Figure 4. Text V, p. A70.

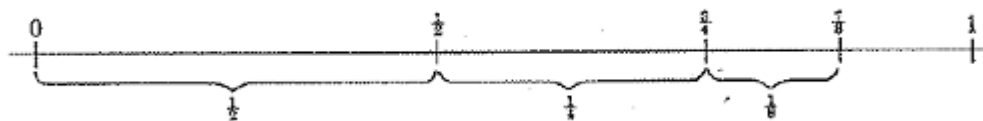


Figure 5. Text R, p. 391.

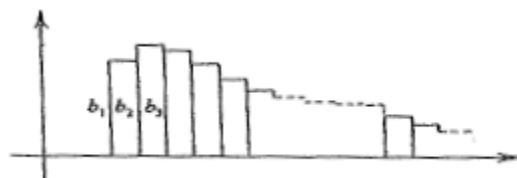


Figure 6. Text S, p.72.

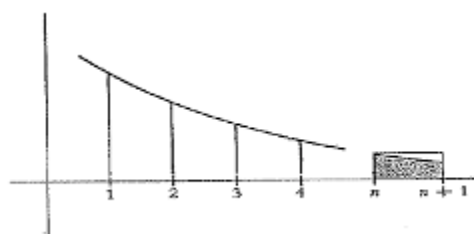
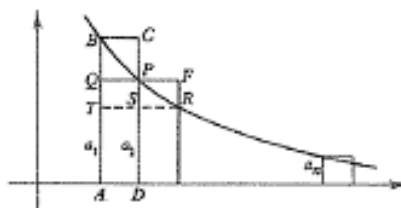


Figure 7. Text R, p.396.

$$\text{area } ABCD = a_1.$$


The area under the curve $f(x)$ between A and B is $\int_a^b f(x) dx$.
Consequently

$$a_1 - \int_1^2 f(x) dx = \text{area } BCP < \text{area } BCPQ. \quad (43)$$

$$q_2 - \int_0^2 f(x) dx = \text{area } PFR < \text{area } PFRS = \text{area } QPST. \quad (44)$$
$$0 < (a_1 + a_2 + a_3 + \dots + a_n) - \int_1^{n+1} f(x) dx < \text{area } ABCD = f(1), \quad (45)$$

Figure 8. Interpretive commentary in Text S, p. 77.

$$1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{9} + \dots + \frac{1}{16}}_{\geq \frac{1}{2}} + \dots$$

Figure 9. A Conceptual representation of the divergence of $\sum_{n=1}^{\infty} \frac{1}{n}$ from Text R, p.390

Exemple Évaluons $\sum_{i=1}^{+\infty} \frac{1}{2^n}$.

En énumérant les termes de cette somme, nous trouvons:

$$\sum_{i=1}^{+\infty} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots + \frac{1}{2^n} + \dots$$

Nous pouvons considérer la somme des termes du membre de droite comme équivalente à l'aire d'un carré de côté de longueur 1, subdivisé comme dans la représentation ci-contre.

Carré d'aire égale à 1 u^2 .

Ainsi $\sum_{i=1}^{+\infty} \frac{1}{2^i} = 1$ (cette série est convergente).

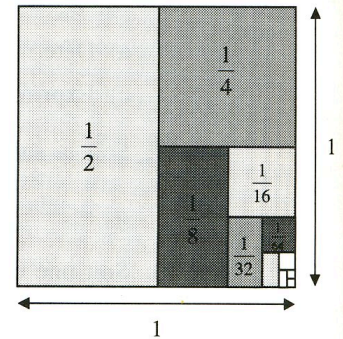


Figure 10: Example of a mixed-register example in Text C

