ON SHARPNESS IN LOCAL CONVERSE THEOREMS FOR CLASSICAL GROUPS AND G_2

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Abstract. We prove various results about the Local Converse Problem for split reductive groups G over a non-archimedean local field F of characteristic 0 and residual characteristic p. In particular, we prove that when G is a symplectic or special orthogonal group, or the exceptional group G_2, and p is large enough, then the optimal standard Local Converse Theorem for $\mathrm{G}(F)$ requires twisting by representations of $\mathrm{GL}_r(F)$ with r up to half the dimension of the standard representation of the dual group of G. However, if we restrict to generic supercuspidal representations of $\mathrm{G}(F)$ then it can be improved when $\mathrm{G} = \mathrm{SO}_{2N}$; we conjecture that the same is true for symplectic and odd special orthogonal groups. We also consider the possibility of using non-standard representations of the dual group to distinguish representations, giving counterexamples to possible improvements for general linear groups, G_2 and SO_{2N}.

1. Introduction

Let F be a non-archimedean local field of characteristic 0 and residual characteristic p, and let $G = \mathrm{G}(F)$ be the group of points of a connected reductive group over F. Given some set $\mathcal{S}$ of (equivalence classes of) irreducible (complex) representations of G, a natural question to ask is:

(Q) Which invariants can we use to distinguish two elements of $\mathcal{S}$?

In the context of the local Langlands correspondence, and in this paper, the invariants we consider are *twisted γ-factors*, which are certain functions of a complex variable.

For the group $\mathrm{G} = \mathrm{GL}_N$ this question is now well-understood. We fix a non-trivial additive character ψ of F and let $\mathcal{S} = \mathrm{Irr}^{\mathrm{gen}}(\mathrm{GL}_N(F))$ be the set of irreducible generic representations of $\mathrm{GL}_N(F)$. Then the *Local Converse Theorem for* GL_N [JL18, Cha19] says that one possible answer to (Q) is:

(A) $\left\{\gamma(s, \pi \times \tau, \psi) \,:\, \tau \in \mathrm{Irr}^{\mathrm{gen}}(\mathrm{GL}_r(F)), r \leq \left\lfloor \frac{N}{2} \right\rfloor\right\}$,

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where the γ-factors are those defined via Rankin–Selberg convolution [JPSS83] or by the Langlands–Shahidi method [Sha84]. Moreover, if $p > N$ then this answer is optimal in the sense that the bound on r cannot be reduced, and it remains optimal if we replace $\mathcal{S}$ by the set of irreducible supercuspidal representations of $\mathrm{GL}_N(F)$ (see [ALST18, Adr23]). Interestingly, if we look only at the set of *simple* (or *epipelagic*) supercuspidal representations (those of depth $\frac{1}{N}$) then one can do better: only twists by characters of $\mathrm{GL}_1(F)$ are required (see [BH14, Proposition 2.2], and also [LS25] for further discussion of this).

For other groups G, analogous γ-factors have also been defined; for example, for split classical groups over p-adic fields. However, the situation is not necessarily as clean as in the case of general linear groups: indeed, in full generality it *should not even be possible* to distinguish irreducible representations using only γ-factors, as we now explain. We will assume that the group G is split and we fix a Whittaker datum for $G = \mathrm{G}(F)$, so that *generic* always means relative to this fixed datum.

Write ${}^\vee\mathrm{G}(\mathbb{C})$ for the complex dual group of G and consider $\varrho : {}^\vee\mathrm{G}(\mathbb{C}) \to \mathrm{GL}_M(\mathbb{C})$ an algebraic representation, for some integer M. To each $\pi \in \mathrm{Irr}^{\mathrm{gen}}(G)$, if we have a Langlands correspondence for G then there is a corresponding Langlands parameter φ_π and we should obtain a *functorial lifting* ${}_\varrho\Pi \in \mathrm{Irr}^{\mathrm{gen}}(\mathrm{GL}_M(F))$ which corresponds to the Langlands parameter $\varrho\circ\varphi_\pi$. Attached to the pair (π, ϱ) is a conjectural gamma factor $\gamma(s, \pi, \varrho, \psi)$, predicted by the Langlands program. Then an equality of twisted γ-factors for $\pi_1, \pi_2 \in \mathrm{Irr}^{\mathrm{gen}}(G)$ (relative to ϱ, when these have been defined) would only guarantee ${}_\varrho\Pi_1 \cong {}_\varrho\Pi_2$, but not necessarily $\pi_1 \cong \pi_2$. This phenomenon is equivalent to the question of whether, given two generic Langlands parameters φ_1, φ_2 for G, having $\varrho \circ \varphi_1$ conjugate to $\varrho \circ \varphi_2$ over $\mathrm{GL}_M(\mathbb{C})$, implies that φ_1, φ_2 are ${}^\vee\mathrm{G}(\mathbb{C})$-conjugate. There are some groups and representations for which this is true – for example when G is Sp_{2N} or SO_{2N+1} and ϱ is the standard representation – but that is not always the case.

This question, as we allow ϱ to vary over all possible algebraic representations of ${}^\vee\mathrm{G}(\mathbb{C})$, is encapsulated in the notion of acceptability. First, we say that a morphism $WD_F \to {}^\vee\mathrm{G}(\mathbb{C})$ of the Weil-Deligne group is *generic* if its adjoint L-function is holomorphic at $s = 1$; when the local Langlands correspondence is known, generic parameters correspond precisely to L-packets that have a generic element (see [Mat24, §2.2] and [GI16, Proposition B.1]). We say that two morphisms $\varphi, \varphi' : WD_F \to {}^\vee\mathrm{G}(\mathbb{C})$ are *locally conjugate* if, for all algebraic representations ϱ of ${}^\vee\mathrm{G}(\mathbb{C})$, we have $\varrho\circ\varphi \cong \varrho\circ\varphi'$; equivalently (see [CG24]), $\varphi(w)$ and $\varphi'(w)$ are conjugate in ${}^\vee\mathrm{G}(\mathbb{C})$ for each $w \in WD_F$. Finally, we say that ${}^\vee\mathrm{G}(\mathbb{C})$ is *generically WD_F-acceptable* if any pair of locally conjugate generic morphisms $WD_F \to {}^\vee\mathrm{G}(\mathbb{C})$ are in fact conjugate (i.e. there is a $g \in {}^\vee\mathrm{G}(\mathbb{C})$ such that $\varphi' = \mathrm{Ad}(g) \circ \varphi$). Then, the most general version of the local converse problem on the automorphic side, when transferred to the Galois side, requires a given dual group ${}^\vee\mathrm{G}(\mathbb{C})$ to be generically WD_F-acceptable.

On the other hand, there is a more general definition of acceptability, which we now recall (following [Lar94, CG24, Mat24]). We say that ${}^\vee\mathrm{G}(\mathbb{C})$ is *acceptable* if, for any *finite* group Γ, any two locally conjugate group homomorphisms $\varphi, \varphi' : \Gamma \to {}^\vee\mathrm{G}(\mathbb{C})$ are automatically conjugate. Matringe [Mat24] has shown that if ${}^\vee\mathrm{G}(\mathbb{C})$ is acceptable, then it is also generically WD_F-acceptable. However, it is unclear whether the reverse implication is true. For example, it is known that $\mathrm{SO}_{2N}(\mathbb{C})$ is unacceptable when $N \geq 3$ (see [Yu22]) but unknown whether or not it is in general generically WD_F-unacceptable. In [Mat24], it is shown that $\mathrm{SO}_{2N}(\mathbb{C})$ is generically WD_F-unacceptable when $q \equiv 1 \pmod 4$; we will show that the same is true when $q \equiv 3 \pmod 4$.

We consider now the algebraic representation ring $\mathrm{R}[{}^\vee\mathrm{G}]$ of ${}^\vee\mathrm{G}(\mathbb{C})$. It is generated by finitely many representations (see for example [FH91, Section 23]). For example, in the case of $\mathrm{GL}_N(\mathbb{C})$ these are the exterior powers Λ^i of the standard representation, for $i = 1, \ldots N$. If ${}^\vee\mathrm{G}(\mathbb{C})$ is simply connected, these are the *fundamental representations*, which are the irreducible finite-dimensional representations whose highest weight is a fundamental weight; see later for some other examples. Thus, WD_F-acceptability of ${}^\vee\mathrm{G}(\mathbb{C})$ can be detected by only testing on a finite number of representations: if $\varrho \circ \varphi \cong \varrho \circ \varphi'$ for these (finitely many) representations ϱ then it is also true for all algebraic representations ρ.

Thus, when there is a *suitable local Langlands correspondence for* G (by which we mean one satisfying [KT, Conjectures 6.1,6.5] in the formulation of Kaletha–Taïbi, so it sends tempered representations to parameters with bounded image ([KT, Conjecture 6.1, §5.3]), and there is unicity of generic representations in a tempered L-packet) we get the following result immediately from the Local Converse Theorem for GL_N. To state it, we *define* the twisted local factors by

$$\gamma(s, \pi \rtimes \tau, \varrho, \psi) := \gamma(s, (\varrho \circ \varphi_\pi) \otimes \varphi_\tau, \psi),$$

for ϱ an algebraic representation of ${}^\vee\mathrm{G}(\mathbb{C})$ and $\tau \in \mathrm{Irr}^{\mathrm{gen}}(\mathrm{GL}_r(F))$, though of course one would hope to have a direct definition of these γ-factors on the automorphic side which matches with this.

The Local Converse Theorem. *Let* G *be a split connected reductive group over F such that the dual group ${}^\vee\mathrm{G}(\mathbb{C})$ is generically WD_F-acceptable, and for which there is a suitable local Langlands correspondence. Let ϱ_i, for $i \in I$, be generators of* $\mathrm{R}[{}^\vee\mathrm{G}]$ *(which can be taken to be finite in number), and put $K = \max_{i\in I} \dim(\varrho_i)$. Suppose that π_1, π_2 are irreducible generic tempered representations of* $\mathrm{G}(F)$ *such that*

$$\gamma(s, \pi_1 \rtimes \tau, \varrho_i, \psi) = \gamma(s, \pi_2 \rtimes \tau, \varrho_i, \psi)$$

for all $i \in I$ and all $\tau \in \mathrm{Irr}^{\mathrm{gen}}(\mathrm{GL}_r(F))$ with $r \leq \left\lfloor \frac{K}{2} \right\rfloor$. Then $\pi_1 \cong \pi_2$.

Note that this version of the Local Converse Theorem was proven in [Mat24] in the case that ${}^\vee\mathrm{G}$ is simply connected; in that case, the fundamental representations generate $\mathrm{R}[{}^\vee\mathrm{G}]$.

Thus, in the situation of this Local Converse Theorem, all these twisted γ-factors together provide an answer to (Q), but this is almost certainly suboptimal: indeed, in the case of $\mathrm{GL}_N(F)$ one does not need any of the fundamental representations other than the standard one, though one could also wonder if we could choose to cut down in a different way (see later in the introduction). This paper, then, is about the possible suboptimality of this Local Converse Theorem.

Since most of our results concern groups whose dual groups have a standard representation ϱ_{std} (faithful of smallest dimension), it is convenient to make the following definition.

Definition 1.1. *Let* G *be a split connected reductive group over* F*, whose dual group has a standard representation* ϱ_{std}*. Let* π_1, π_2 *be irreducible generic representations of* $\mathrm{G}(F)$ *and let* $m \geq 1$ *be an integer. We say that* π_1, π_2 *are* γ-equivalent to level m *if*

$$\gamma(s, \pi_1 \rtimes \tau, \varrho_{\mathrm{std}}, \psi) = \gamma(s, \pi_2 \rtimes \tau, \varrho_{\mathrm{std}}, \psi)$$

for all irreducible supercuspidal representations τ *of* $\mathrm{GL}_r(F)$*, with* $1 \leq r \leq m$*. We use the same terminology for Langlands parameters also.*

Note that, by multiplicativity of γ-factors, if π_1, π_2 are γ-equivalent to level m then we in fact have $\gamma(s, \pi_1 \rtimes \tau, \varrho_{\mathrm{std}}, \psi) = \gamma(s, \pi_2 \rtimes \tau, \varrho_{\mathrm{std}}, \psi)$ for all *generic* irreducible representations τ of $\mathrm{GL}_r(F)$, with $1 \leq r \leq m$.

1.1. **The classical groups.** From now on, we assume that the residual characteristic of F is odd. Let G_N be the group of points of a split classical group of rank N – i.e. $\mathrm{Sp}_{2N}(F), \mathrm{SO}_{2N}(F)$, or $\mathrm{SO}_{2N+1}(F)$. The main theorem that has already been proven in the literature, which we will refer to as the *the standard local converse theorem*, says the following, where we say that irreducible representations π_1, π_2 of G_N are *outer conjugate* if either they are conjugate (when G_N is $\mathrm{Sp}_{2N}(F)$ or $\mathrm{SO}_{2N+1}(F)$) or there is an element $g \in \mathrm{O}_{2N}(F)$ such that ${}^g\pi_1 \cong \pi_2$ (when $G_N = \mathrm{SO}_{2N}(F)$).

The Standard Local Converse Theorem for G_N (see [JS03, Zha18, HL25])**.** *Suppose that* π_1, π_2 *are irreducible generic representations of* G_N*, with the same central character, which are* γ*-equivalent to level* N*. Then* π_1 *and* π_2 *are outer conjugate.*

Thus, in the setting of the general Local Converse Theorem, this says that we *only* need to consider the standard representation of the dual group. Our first result is that we may remove the condition on the central characters of π_1, π_2 (see Theorem 3.2). Our subsequent focus is on the possible suboptimality of the Standard Local Converse Theorem for G_N (namely, can we reduce the number of GL twists while considering only $\varrho = \varrho_{\mathrm{std}}$), although in principle one could also consider generators of $R[{}^\vee\mathrm{G}]$ other than the standard representation. We first prove the following theorem, which shows that the Standard Local Converse Theorem for G_N is optimal in the tame setting, for generic, tempered, irreducible representations of G_N.

Theorem 1.2 (see Corollary 3.6)**.** *Suppose $p > N$. Then there are generic irreducible tempered (non-cuspidal) representations π, π' of G_N, which are γ-equivalent to level $N-1$ but not outer conjugate.*

However, the situation changes if one assumes that π_1, π_2 are also supercuspidal. We first state the main results in the setting of SO_{2N} since we have a clean statement there, and then we discuss the results for a general classical group.

Theorem 1.3 (Theorem 3.16 and Corollary 3.9)**.** *Suppose $p > N$.*

(i) *If N is even then the standard local converse theorem for SO_{2N} is optimal.*
(ii) *If N is odd and π_1, π_2 are irreducible generic supercuspidal representations of $\mathrm{SO}_{2N}(F)$ which are γ-equivalent to level $N-1$, then π_1, π_2 are outer conjugate. Moreover, this new bound is optimal.*

We are able to prove the same results for the other classical groups *conditional on a conjecture* which we now explain. First, we recall that a discrete Langlands parameter for a classical group is a multiplicity-free direct sum of twists of irreducible self-dual representations of W_F by irreducible representations of $\mathrm{SL}_2(\mathbb{C})$ (see the proof of Lemma 3.14 for a more precise statement). It therefore becomes natural to understand if there is any possible improvement in the standard local converse theorem for GL_N when one restricts to the setting of self-dual supercuspidal representations. While the standard local converse theorem for GL_N asserts that the bound $[N/2]$ is optimal for general supercuspidal representations of GL_N, we make the following conjecture.

Conjecture 1.4. *(see Conjecture 3.10) Suppose $p > N$ and N is odd. Let π, π' be self-dual supercuspidal representations of $\mathrm{GL}_{2N}(F)$ of the same parity which are γ-equivalent to level $N-1$. Then $\pi \cong \pi'$.*

We have no evidence as to whether the assumptions on p or N are needed, or even whether the above bound is optimal; at best, it can be lowered by 1 more, as the following shows:

Theorem 1.5. *(see Theorem 3.12) Suppose $p > N$, set $M = 2\lfloor \frac{N}{2} \rfloor$, and fix a parity. There exist inequivalent self-dual supercuspidal representations π, π' of $\mathrm{GL}_{2N}(F)$ of this parity which are γ-equivalent to level $M-2$.*

Assuming Conjecture 1.4, we also have the analogue of Theorem 1.3 for the symplectic and odd special orthogonal groups. We refer to Theorem 3.13 and Corollary 3.9 for the statement.

We end this section by considering the question of acceptability. In Example 3.17, we show that SO_6 is generically WD_F-unacceptable when $q \equiv 3 \pmod 4$. To be more specific, the representation ring $R[\mathrm{SO}_6(\mathbb{C})]$ is generated by $\wedge^1, \wedge^2, \wedge^3_+, \wedge^3_-$, where $\wedge^3 = \wedge^3_+ \oplus \wedge^3_-$, and we give an example of inequivalent generic (supercuspidal) parameters $\varphi_1, \varphi_2 : W_F \to \mathrm{SO}_6(\mathbb{C})$ which are conjugate under $\mathrm{O}_6(\mathbb{C})$ (so that their compositions with $\wedge^r$ are equivalent, for any r) and, moreover $\wedge^3_+ \circ \varphi_1 \cong \wedge^3_- \circ \varphi_2$. In particular, this means that all their twisted γ-factors will be equal, whatever representation of the dual group we take. As we comment in Remark 3.19, the same method could be used to give discrete parameter counterexamples for SO_{2N} whenever $N \geq 3$ is odd.

Moreover, we are able to leverage this example to show that $\mathrm{SO}_{2N}(\mathbb{C})$ is generically WD_F-unacceptable when $q \equiv 3 \pmod 4$ (see Corollary 3.18); this complements Matringe's result [Mat24, Proposition 7.3] that $\mathrm{SO}_{2N}(\mathbb{C})$ is generically WD_F-unacceptable when $q \equiv 1 \pmod 4$. Note, however, that the parameters demonstrating these results are *not* discrete.

1.2. **The exceptional group G_2.** We also investigate the exceptional split group of type G_2, when $p > 3$. In this case there are two fundamental representations of $\mathrm{G}_2(\mathbb{C})$: the standard 7-dimensional, and the adjoint, which is 14-dimensional. We begin by proving an expected result, the *standard local converse theorem for* G_2, which says that the family of GL_i-twisted gamma factors, $i = 1, 2, 3$, using only the standard representation of the dual group, determines an irreducible generic representation of G_2. We then prove that one cannot lower this bound of twisting to $i = 1, 2$, and that even if we include the adjoint representation of the dual group, we still do not obtain a local converse theorem. To summarize, we prove the following theorem (see Theorem 4.7).

Theorem 1.6. *Suppose $p > 3$.*

(i) *Let φ_1, φ_2 be two generic Langlands parameters for $\mathrm{G}_2(F)$. If φ_1, φ_2 are γ-equivalent to level 3, then $\varphi_1 \cong \varphi_2$.*

(ii) *There are inequivalent generic supercuspidal representations π_1, π_2 of $\mathrm{G}_2(F)$ such that*

$$\gamma(s, \pi_1 \rtimes \tau, \varrho, \psi) := \gamma(s, \pi_2 \rtimes \tau, \varrho, \psi),$$

for ϱ either fundamental representation of $\mathrm{G}_2(\mathbb{C})$, and all supercuspidal representations of $\mathrm{GL}_r(F)$, with $1 \leq r \leq 2$; in particular, they are γ-equivalent to level 2.

1.3. **The general linear groups GL_N.** Finally, we investigate whether one can improve upon the Local Converse Theorem for GL_N in ways other than by cutting out the non-standard fundamental representations. In fact, in [YZ22, Conjecture 1.2], a hint was provided along these lines. The authors state a conjecture, attributed to Ramakrishnan [Ram94], that for unitarizable supercuspidal representations of $\mathrm{GL}_N(F)$ one only needs to consider the GL_1-twisted gamma factors while allowing all fundamental representations. Namely, the gamma factors

$$\gamma(s, \pi \rtimes \eta, \wedge^i, \psi), \text{ for } i = 1, \ldots, \left\lfloor \tfrac{N}{2} \right\rfloor \text{ and } \eta \text{ a character of } F^\times.$$

(They also assume that the central character is fixed, though that is unnecessary since the central character is determined by the standard twisted γ-factors $\gamma(s, \pi \times \eta, \psi)$;

see, for example, [JNS15, Corollary 2.7].) In Section 5, we prove that this conjecture is false in general:

Theorem 1.7. *Suppose p is odd. Then there are inequivalent irreducible parameters $\varphi_1, \varphi_2 : W_F \to \mathrm{GL}_4(\mathbb{C})$, corresponding to unitarizable supercuspidal representations of $\mathrm{GL}_4(F)$, such that*

$$\gamma(s, (\wedge^i \circ \varphi_1) \otimes \eta, \psi) = \gamma(s, (\wedge^i \circ \varphi_2) \otimes \eta, \psi),$$

for all characters η of W_F and $i = 1, \ldots, 4$.

Thus twisting only by characters of $F^\times$ cannot be sufficient to distinguish supercuspidal representations, even if all fundamental representations of the dual group are considered.

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2. Representations of the Weil group and γ-factors

We recall the parametrization of tame Langlands parameters and their associated local factors. Let F be a non-archimedean local field of characteristic zero, with ring of integers $\mathcal{O}_F$, maximal ideal $\mathcal{P}_F$, residue field $k_F \simeq \mathbb{F}_q$, where $q = q_F = p^f$ is a power of the residual characteristic p, and μ'_F the group of roots of unity of order prime to p. We use analogous notation for any extension E of F. We also fix a non-trivial additive character ψ_F of F of level one: that is, it is trivial on $\mathcal{P}_F$ but not on $\mathcal{O}_F$. For any finite extension E/F, write $e_{E/F}$ and $f_{E/F}$ for the ramification index and residue degree of E/F respectively. We also define an additive character ψ_E of E via $\psi_E = \psi_F \circ \mathrm{tr}_{E/F}$; if E/F is tamely ramified (that is, $p \nmid e_{E/F}$) then ψ_E is also of level one.

Let $\overline{F}$ be a fixed algebraic closure of F. We normalize the additive valuation val on F to have image $\mathbb{Z} \cup \{\infty\}$ and extend this to an additive valuation on $\overline{F}$; thus, for E/F any finite extension, we have $\mathrm{val}(E^\times) = \frac{1}{e_{E/F}}\mathbb{Z}$. For r a real number, we write

$$\mathcal{P}_E^r = \{x \in E : \mathrm{val}(x) \geq r\}, \qquad \mathcal{P}_E^{r+} = \{x \in E : \mathrm{val}(x) > r\}$$

so, for example, $\mathcal{P}_E^{1/e_{E/F}} = \mathcal{P}_E = \mathcal{P}_E^{0+}$. We put $U_E = U_E^0 = \mathcal{O}_E^\times$, with filtration subgroups $U_E^r = 1 + \mathcal{P}_E^r$, for $r > 0$, and $U_E^{r+} = 1 + \mathcal{P}_E^{r+}$, for $r \geq 0$.

We also write $|\cdot|$ for the normalized absolute value on $\overline{F}$, given by

$$|x| = q^{-\mathrm{val}(x)}, \qquad \text{for } x \in \overline{F}^\times.$$

Note that if E/F is a finite extension then the usual normalization of the absolute value on E is given by $x \mapsto |x|^{[E:F]}$, for $x \in E$.

Let W_F be the Weil group of F. Implicit in the definition of the Weil group are compatible choices of isomorphisms $E^\times \simeq W_E^{\mathrm{ab}}$, for each finite extension E/F, via which we will identify the quasi-characters of $E^\times$ and of W_E. Moreover, if $F \subseteq E \subseteq K$ and χ is a quasi-character of $E^\times$, let $\chi_K := \chi \circ \mathrm{N}_{K/E}$. When χ is viewed as a quasi-character of W_E, the quasi-character χ_K is simply the restriction of χ to the subgroup W_K. For E/F a finite extension, write $\mathrm{Ind}_{E/F} = \mathrm{Ind}_{W_E}^{W_F}$ for the induction functor.

2.1. **Admissible pairs.** Let $N > 1$ be an integer not divisible by p. In this situation, there is a nice parametrization of the irreducible N-dimensional representations of W_F in terms of admissible quasi-characters, introduced by Howe.

Definition 2.1 (Howe [How77]). *An* admissible pair *of degree N is a pair $(E/F, \chi)$ where E/F is a (tamely ramified) extension of degree N and χ is a quasi-character of $E^\times$ such that*

(i) *χ does not come via the norm from a proper subfield of E containing F; and*
(ii) *if the restriction $\chi|_{U_E^{0+}}$ comes via the norm from a subfield $F \subseteq L \subseteq E$, then E/L is unramified.*

Two admissible pairs $(E_1/F, \chi_1)$ and $(E_2/F, \chi_2)$ are said to be conjugate *if there is an F-isomorphism from E_1 to E_2 which takes χ_1 to χ_2.*

In this situation we also say that χ is an *admissible quasi-character* of $E^\times$ (relative to F). For any quasi-character of $E^\times$, we write

$$\rho_\chi := \mathrm{Ind}_{E/F}\, \chi.$$

Theorem 2.2 (Moy [Moy86, Theorem 2.2.2]). *If $(E/F, \chi)$ is an admissible pair of degree N, then ρ_χ is an irreducible N-dimensional representation of W_F. Furthermore, two admissible pairs induce to equivalent representations if and only if they are conjugate, and each irreducible N-dimensional representation of W_F is induced from an admissible pair.*

Let E/F be a (tamely ramified) extension of degree N, and χ a *ramified* quasi-character of $E^\times$; that is, χ is non-trivial on U_E. Let $d(\chi) \in \frac{1}{e_{E/F}}\mathbb{Z}$ be the *normalized depth* of χ: that is, χ is trivial on $U_E^{d(\chi)+}$ but not on $U_E^{d(\chi)}$. (The *conductoral exponent* of χ is then $e_{E/F}d(\chi) + 1$.) There is then $c_\chi \in \mathcal{P}_E^{-d(\chi)}$, well-defined modulo $\mathcal{P}_E^{-d(\chi)/2}$ such that

$$\chi(1 + x) = \psi_E(c_\chi x), \qquad \text{for } x \in \mathcal{P}_E^{(d(\chi)/2)+}.$$

We say that c_χ *represents* χ. We extend this to unramified quasi-characters χ of $E^\times$ by setting $d(\chi) = 0$ and $c_\chi = 0$ (which we will also take for quasi-characters of depth 0).

Note that if K/E is tamely ramified then $d(\chi_K) = d(\chi)$, since $\mathrm{N}_{K/E}(U_K^r) = U_E^r$, for any $r > 0$. Moreover χ_K is also represented by c_χ: that is, we can choose $c_{\chi_K} = c_\chi$. We also make the following simple observation.

Lemma 2.3. *Let E/F and L/F be finite tamely ramified extensions and let χ, η be quasi-characters of $E^\times, L^\times$ respectively, represented by c_χ and c_η respectively. We consider the quasi-character $\chi_{EL}\eta_{EL}$ of the compositum $(EL)^\times$.*

(i) *We have $d(\chi_{EL}\eta_{EL}) \leq \max\{d(\chi), d(\eta)\}$.*
(ii) *If $d(\chi) \neq d(\eta)$ then $\chi_{EL}\eta_{EL}$ is ramified of normalized depth $d(\chi_{EL}\eta_{EL}) = \max\{d(\chi), d(\eta)\}$, and is represented by $c_\chi + c_\eta$.*

There are certain situations in which we can tell directly from a representative for a quasi-character χ of $E^\times$ that it is admissible over F. For $d < 0$ a rational number, we say that $\beta \in E$ is *quasi-minimal (over F) of depth d* if $\mathrm{val}(\beta) = -d$ and

- every element of the coset $\beta + \mathcal{P}_E^{-d/2}$ generates the (tame) extension E/F.

Note that β is called *minimal* in [BK93, (1.4.14)] if $\mathrm{val}(\beta) = -d$ and every element of the coset $\beta + \mathcal{P}_E^{-d+}$ generates E/F. We will use the notion especially in the *minimal totally ramified* case, that is when E/F is totally ramified of degree N and d has least denominator N.

If β is quasi-minimal then the character $1 + x \mapsto \psi_E(\beta x)$ of $U_E^{(d/2)+}$ does not come via the norm from a proper subfield of E containing F. In particular, any quasi-character represented by a quasi-minimal element is admissible; this generalizes the remark that any *generic* quasi-character (in the sense of Kutzko [Kut80], see [Moy86, Definition 2.2.3]) is admissible.

2.2. **Local factors.** As above, let E/F be an extension of degree N not divisible by p, let χ be a ramified quasi-character of $E^\times$ of depth $d(\chi) > 0$, and set $J = U_E^{d(\chi)/2}$ and $H = U_E^{(d(\chi)/2)+}$. We define the Gauss sum

$$G(\chi, \psi_F) = [J : H]^{-\frac{1}{2}} \sum_{x \in J/H} \chi^{-1}(x)\psi_F(c_\chi(x-1)).$$

This collapses to 1 if $J = H$, which happens precisely when the integer $e_{E/F}d(\chi)$ is odd.

Associated to any quasi-character χ of $E^\times$ we also have local factors $L(s,\chi)$, $\epsilon(s,\chi,\psi_E)$ and $\gamma(s,\chi,\psi_E)$, by Tate's thesis [Tat67] (see also the account in [BH06, §23]). When χ is ramified we have $L(s,\chi) = 1$ and $\epsilon(s,\chi,\psi_E) = \gamma(s,\chi,\psi_E)$. Denote by $\epsilon(\chi,\psi_E)$ the value of the ϵ-factor at $s = 0$. Note that $\epsilon(s,\chi,\psi_E) = \epsilon(\chi|\cdot|^{Ns},\psi_E)$; the extra factor N appears here because of our normalization of the absolute value.

Proposition 2.4 (Moy, [Moy86, (2.3.17)])**.** *For χ a ramified quasi-character of $E^\times$, we have*

$$\epsilon(\chi,\psi_E) = \chi^{-1}(c_\chi)\psi_E(c_\chi)|c_\chi|^{N/2}G(\chi,\psi_E).$$

As an immediate corollary, we get the following simple result.

Corollary 2.5 (cf. [Adr23, Lemma 4.3]). *Suppose χ_1, χ_2 are ramified quasi-characters of $E^\times$ of normalized depth $d \geq 0$, and that χ_1, χ_2 coincide on $U_E^{d/2}$ so that they can be represented by a common element c_χ. Then*

$$\gamma(s, \chi_1, \psi_E) = \left(\frac{\chi_1(c_\chi)}{\chi_2(c_\chi)}\right) \gamma(s, \chi_2, \psi_E).$$

More generally, for ρ any semisimple finite-dimensional complex representation of the Weil group W_F, there are local factors $L(s, \rho)$, $\epsilon(s, \rho, \psi_F)$ and $\gamma(s, \rho, \psi_F)$ (see, for example, [BH06, Theorem 29.4]). We have $L(s, \rho) = 1$ when ρ is irreducible of dimension $N > 1$ so the ϵ and γ factors again coincide. Moreover, the local factors are multiplicative: $\gamma(s, \rho_1 \oplus \rho_2, \psi_F) = \gamma(s, \rho_1, \psi_F)\gamma(s, \rho_2, \psi_F)$, and similarly for L and ϵ.

We end this section by recalling the following results on the tensor product of such representations. For K/F a finite extension, we write $\lambda_{K/F}(\psi_F)$ for the Langlands constant associated to K/F and ψ_F (see [BH06, Theorem 29.4 and §34.3]).

Lemma 2.6 ([ALST18, Lemmas 2.4, 2.5]). *Let E and L be tamely ramified field extensions of F, and let χ and η be quasi-characters of $E^\times$ and $L^\times$ respectively. For $g \in W_F$ we write $E_g = g(E)$ and $K_g = E_g L$, and set $\theta_g = ({}^g\chi)_{K_g} \eta_{K_g}$. Then*

$$\rho_\chi \otimes \rho_\eta \cong \bigoplus_{g \in W_L \backslash W_F / W_E} \operatorname{Ind}_{K_g/F} \theta_g$$

and

$$\gamma(s, \rho_\chi \otimes \rho_\eta, \psi_F) = \prod_{g \in W_L \backslash W_F / W_E} \lambda_{K_g/F}(\psi_F) \gamma(s, \theta_g, \psi_{K_g}).$$

2.3. **Totally ramified quasi-characters.** In this subsection, we give the main technical results on equalities of γ-factors for representations of W_F which are at the heart of our sharpness examples. We continue with E/F an extension of degree N not divisible by p and let χ be a ramified quasi-character of $E^\times$ of depth $d(\chi) > 0$. We say that χ is *totally ramified* if E/F is totally ramified, and that χ is *minimal totally ramified* if also it is represented by a minimal element (equivalently, $d(\chi)$ has least denominator N).

For L/F a finite extension of degree r and $\alpha \in L$, we write $f_\alpha(X) \in F[X]$ for the characteristic polynomial of α acting by multiplication on L, so that

$$f_\alpha(X) = \prod_{\sigma \in \mathrm{Gal}(L/F)} (X - \sigma(\alpha)),$$

where $\mathrm{Gal}(L/F)$ is the set of F-embeddings of L into $\overline{F}$. Of course, this polynomial f_α depends on the field L as well as α but the field will be clear from context.

Proposition 2.7. *For i in some finite set I, let $(E_i/F, \chi_i), (E_i/F, \chi_i')$ be totally ramified admissible pairs of degree N_i and depth d_i, such that χ_i, χ_i' coincide on $U_{E_i}^{d_i/2}$; choose β_i which represents both. Suppose (L, η) is an admissible pair of degree r,* *represented by α, with depth distinct from all d_i, and set $\tau = \mathrm{Ind}_{L/F}\,\eta$. Then the following are equivalent:*

(i) $\gamma(s, (\bigoplus_{i\in I} \rho_{\chi_i}) \otimes \tau, \psi_F) = \gamma(s, (\bigoplus_{i\in I} \rho_{\chi'_i}) \otimes \tau, \psi_F)$;

(ii) $\prod_{i\in I} \chi_i((-1)^r f_\alpha(-\beta_i)) = \prod_{i\in I} \chi'_i((-1)^r f_\alpha(-\beta_i))$.

Note that the condition on the depth of η in Proposition 2.7 is implied if r is not divisible by any least denominator of d_i, since $d(\eta) \in \frac{1}{r}\mathbb{Z}$.

Proof. The ideas of the proof are essentially in the proof of [Adr23, Theorem 3.2]. For $g \in W_F$ and $i \in I$, we write $E_{i,g} = g(E_i)$ and $K_{i,g} = E_{i,g}L$, and set $\theta_{i,g} = ({}^g\chi_i)_{K_{i,g}}\eta_{K_{i,g}}$ and $\theta'_{i,g} = ({}^g\chi'_i)_{K_{i,g}}\eta_{K_{i,g}}$. By Lemma 2.3(ii), the quasi-characters $\theta_{i,g}$ and $\theta'_{i,g}$ are represented by $g(\beta_i) + \alpha$ and have depth $d_{i,g} := \max\{d_i, d(\eta)\}$. Thus, by Lemma 2.6 and multiplicativity, we have

$$\gamma(s, (\bigoplus_{i\in I} \rho_{\chi_i}) \otimes \tau, \psi_F) = \prod_{i\in I} \prod_{g\in W_L\backslash W_F/W_{E_i}} \lambda_{K_{i,g}/F}(\psi_F)\gamma(s, \theta_{i,g}, \psi_{K_{i,g}}),$$

Thus the equality in (i) is equivalent to

$$\prod_{i\in I} \prod_{g\in W_L\backslash W_F/W_{E_i}} \gamma(s, \theta_{i,g}, \psi_{K_{i,g}}) = \prod_{i\in I} \prod_{g\in W_L\backslash W_F/W_{E_i}} \gamma(s, \theta'_{i,g}, \psi_{K_{i,g}}).$$

Moreoever, $\theta_{i,g}$ and $\theta'_{i,g}$ coincide on $U_{K_{i,g}}^{d_{i,g}/2}$ since $\mathrm{N}_{K_{i,g}/E_{i,g}}(U_{K_{i,g}}^{d_{i,g}/2}) \subseteq U_{E_{i,g}}^{d_i/2}$, where the quasi-characters ${}^g\chi_i$ and ${}^g\chi'_i$ coincide by hypothesis. Thus it follows from Corollary 2.5 that this is equivalent to

$$\prod_{i\in I} \prod_{g\in W_L\backslash W_F/W_{E_i}} \theta_{i,g}(g(\beta_i) + \alpha) = \prod_{i\in I} \prod_{g\in W_L\backslash W_F/W_{E_i}} \theta'_{i,g}(g(\beta_i) + \alpha),$$

which is in turn equivalent to

$$\prod_{i\in I} \prod_{g\in W_L\backslash W_F/W_{E_i}} ({}^g\chi_i)_{K_{i,g}}(g(\beta_i) + \alpha) = \prod_{i\in I} \prod_{g\in W_L\backslash W_F/W_{E_i}} ({}^g\chi'_i)_{K_{i,g}}(g(\beta_i) + \alpha).$$

Thus, to complete the proof, it suffices to show that

$$\prod_{g\in W_L\backslash W_F/W_{E_i}} ({}^g\chi_i)_{K_{i,g}}(g(\beta_i) + \alpha) = \chi_i((-1)^r f_\alpha(-\beta_i)),$$

for which we drop the subscripts i. As in [Adr23, after (4.3)], we have

$$\prod_{g\in W_L\backslash W_F/W_E} ({}^g\chi)_{K_g}(g(\beta) + \alpha) = \chi\left(\prod_{g\in W_L\backslash W_F/W_E} \prod_{\sigma\in \mathrm{Gal}(K_g/E_g)} (\beta + g^{-1}\sigma(\alpha)) \right).$$

But, as also explained in [Adr23], this further simplifies to

$$\chi\left(\prod_{\sigma\in\mathrm{Gal}(L/F)}(\beta+\sigma(\alpha))\right)=\chi((-1)^r f_\alpha(-\beta)).$$

□

We will apply this lemma to the case where all $N_i = N$ are equal and all depths d_i are equal, since in that case one can further simplify. To this end, we introduce some further notation. Let $(E/F,\chi)$ be a totally ramified admissible pair of degree N and depth d represented by β, and let M be the least denominator of d. For L/F an extension of degree less than M and $\alpha\in L$ we set

$$u^+_\alpha(X)=\mathrm{N}_{L/F}(-\alpha)^{-1}f_\alpha(-X),\qquad u^-_\alpha(X)=(-X)^{-r}f_\alpha(-X).$$

Since $u^\pm_\alpha(X)$ are Laurent polynomials with coefficients in F, we have $u^\pm_\alpha(\beta)\in E$. Moreover, since $\mathrm{val}(\alpha)\neq\mathrm{val}(\beta)$, we see that

$$\begin{cases} u^+_\alpha(\beta)\in U_E^{0+} & \text{if } \mathrm{val}(\alpha)<\mathrm{val}(\beta),\\ u^-_\alpha(\beta)\in U_E^{0+} & \text{if } \mathrm{val}(\alpha)>\mathrm{val}(\beta).\end{cases}$$

Now put

$$t^+_{\beta,L}=\inf\left\{\mathrm{val}\left(1-u^+_\alpha(\beta)\right)\right) \,:\, \alpha\in L,\ \mathrm{val}(\alpha)<\mathrm{val}(\beta)\right\},$$

$$t^-_{\beta,L}=\inf\left\{\mathrm{val}\left(1-u^-_\alpha(\beta)\right)\right) \,:\, \alpha\in L,\ \mathrm{val}(\beta)<\mathrm{val}(\alpha)<0\right\},$$

where we interpret $t^-_{\beta,L}=\infty$ if the set in its definition is empty. Since $\mathrm{val}(E)=\frac{1}{N}\mathbb{Z}$, we have $t^\pm_{\beta,L}\geq\frac{1}{N}$.

Now, for $r\leq M$ an integer, we set

$$t_\beta(r)=\min\left\{t^+_{\beta,L},t^-_{\beta,L} \,:\, [L:F]<r\right\}.$$

Again, we have $t_\beta(r)\geq\frac{1}{N}$. We will be particularly interested in cases when this inequality is strict. As a simple case, we have

$$t_\beta(2)=\min\left\{\left|d-\tfrac{n}{M}\right| \,:\, n\in\mathbb{Z}\right\},\tag{2.8}$$

since we only need to consider $\alpha\in F$, where $u^\pm_\alpha(X)=1+(\alpha^{-1}X)^{\pm1}$. More generally, we have the following, which will need later:

Lemma 2.9. *Suppose E/F is totally ramified of degree N and $d=\frac{m}{M}$ has least denominator M. Suppose $\beta\in E$ has valuation $-d$ and let r be the least integer such that $rm\equiv\pm1 \pmod{M}$. If $r>1$, then $t_\beta(r)>\frac{1}{M}$.*

Proof. It is enough to prove that, for any integer $s<r$, any extension L/F of degree s, and any $\alpha\in L$, we have $u^\pm_\alpha(\beta)\in U_E^{(1/M)+}$. Now $u^\pm_\alpha(X)$ is a polynomial of degree s in $X^{\pm1}$ with coefficients in F and constant term 1. Thus, for each $1\leq i\leq s$, the valuation of the monomial of degree i when evaluated at β lies in $\pm\frac{mi}{M}+\mathbb{Z}$, and it

is greater than 0 by construction. In particular, these all have distinct valuations (as $s < M$) so the valuation of $u_\alpha^{\pm}(\beta) - 1$ is exactly the minimum of the valuations of these monomials. Since none of these monomials have valuation in $\frac{1}{M} + \mathbb{Z}$ (by the minimality of r), they are each at least $\frac{2}{M}$. Thus the valuation of $u_\alpha^{\pm}(\beta) - 1$ is at least $\frac{2}{M}$, so strictly greater than $\frac{1}{M}$, so $u_\alpha^{\pm}(\beta) \in U_E^{(1/M)+}$. □

The following result is at the root of all our sharpness examples, and generalizes the essence of the proof of sharpness in the local converse theorem for GL_N in [ALST18, Adr23].

Proposition 2.10. *Let $d > 0$ be a rational number with least denominator M and let $r \leq \min\{p, M\}$ be a natural number. For $i \in I$, let $(E_i/F, \chi_i), (E_i/F, \chi_i')$ be totally ramified admissible pairs of degree N and depth d, such that χ_i, χ_i' coincide on $U_{E_i}^{d/2}$; choose β_i which represents both. Set $t_i = t_{\beta_i}(r)$, for $i \in I$, and suppose that:*

(i) *χ_i, χ_i' coincide on restriction to $U_{E_i}^{t_i}$, for each $i \in I$;*

(ii) *$\prod_{i\in I} \chi_i(\beta_i) = \prod_{i\in I} \chi_i'(\beta_i)$;*

(iii) *$\prod_{i\in I} \chi_i$ and $\prod_{i\in I} \chi_i'$ coincide on restriction to $F^\times$.*

Then, for τ any irreducible representation of W_F of dimension less than r, we have

$$\gamma(s, (\bigoplus_{i\in I} \rho_{\chi_i}) \otimes \tau, \psi_F) = \gamma(s, (\bigoplus_{i\in I} \rho_{\chi_i'}) \otimes \tau, \psi_F).$$

Note that if d is chosen to have least denominator N then we are considering minimal totally ramified admissible pairs in Proposition 2.10.

Proof. Suppose τ is an irreducible representation of W_F of dimension less than r. Since $p \geq r > \dim(\tau)$, we have $p \nmid \dim(\tau)$ so Theorem 2.2 implies that $\tau \cong \mathrm{Ind}_{L/F}\,\eta$, for some admissible pair $(L/F, \eta)$. Write $d(\eta)$ for its normalized depth and $\alpha \in L$ for some element representing η. Since $\dim(\tau) < r \leq M$ we have $d(\eta) \neq d$ so, by Proposition 2.7, we need only check that

$$\prod_{i\in I} \chi_i((-1)^r f_\alpha(-\beta_i)) = \prod_{i\in I} \chi_i'((-1)^r f_\alpha(-\beta_i)).$$

If $d(\eta) < d$ then either $\alpha = 0$ or $0 > \mathrm{val}(\alpha) > \mathrm{val}(\beta_i)$ for each $i \in I$. In the former case, $f_\alpha(X) = X^r$ so the equality follows from (ii), while in the latter case we note that $(-1)^r f_\alpha(-\beta_i) = \beta_i^r u_\alpha^-(\beta_i)$ and the equality follows from (ii) together with (i).

On the other hand, if $d(\eta) > d$ then $\mathrm{val}(\alpha) < \mathrm{val}(\beta_i)$ for each $i \in I$. This time we write $(-1)^r f_\alpha(-\beta_i) = \mathrm{N}_{L/F}(\alpha) u_\alpha^+(\beta_i)$ so the equality follows from (iii) together with (i). □

2.4. **Self-dual representations and determinants.** We recall some results on self-duality of representations of W_F and determinants. We assume throughout that this subsection that $p \neq 2$. Firstly, we have the following result due to Moy.

Proposition 2.11 ([Moy84, Theorem 1]). *Let $(E/F, \chi)$ be an admissible pair with $E \neq F$. Then the irreducible representation ρ_χ is self-dual if and only if there is an intermediate field $E \supset L \supseteq F$ such that E/L is quadratic and, writing σ for the non-trivial element of $\mathrm{Gal}(E/L)$, we have ${}^\sigma\chi = \chi^{-1}$. In that case,*

(i) *ρ_χ is orthogonal if and only if $\chi|_{L^\times}$ is trivial;*
(ii) *ρ_χ is symplectic if and only if $\chi|_{L^\times}$ is non-trivial.*

We call an admissible pair $(E/F, \chi)$ as in Proposition 2.11 *self-dual*, and also call it symplectic/orthogonal according to the parity of ρ_χ.

We note that, since $p \neq 2$, the only odd-dimensional self-dual irreducible representations of W_F are the characters of order dividing 2 (all orthogonal). We also observe that there are many even-dimensional self-dual irreducible representations of W_F of both types. Indeed, suppose E/F is an extension of degree $2N$ with an involution σ whose fixed field is L. For $d \geq 0$ we write $\overline{U}_E^{d+} = \{x \in U_E^{d+} : \mathrm{N}_{E/L}(x) = 1\}$.

Let $\beta \in E$ be quasi-minimal of depth $d > 0$ such that $\sigma(\beta) = -\beta$, so that every quasi-character of $E^\times$ represented by β is admissible. We write Ξ_β for the set of characters of the pro-p-group U_E^{0+} which are represented by β, so their restrictions to $U_E^{(d/2)+}$ are all equal; write ψ_β for this restriction. Then σ acts on Ξ_β by $(\sigma\chi)(x) = \chi(\sigma(x)^{-1})$, and the Glauberman correspondence gives a bijection between the set of fixed points Ξ_β^σ and the set of characters of $\overline{U}_E^{0+}$ whose restriction to $\overline{U}_E^{(d/2)+}$ is $\psi_\beta|_{\overline{U}_E^{(d/2)+}}$. Thus Ξ_β^σ has size $[\overline{U}_E^{0+} : \overline{U}_E^{(d/2)+}]$, which is always non-zero, and bigger than 1 (so at least p) as soon as $d \geq \frac{2}{e_{E/F}}$.

Notice that each $\chi \in \Xi_\beta^\sigma$ restricts trivially to U_L^{0+} (since $p \neq 2$) so we can extend χ to $L^\times U_E^{0+}$ by requiring that $\chi|_{L^\times}$ is either trivial or the class field character $\omega_{E/L}$. There are then exactly $[E^\times : L^\times U_E^{0+}]$ further extensions to a character χ of $E^\times$, and we have ${}^\sigma\chi = \chi^{-1}$ for all such extensions since $\mathrm{N}_{E/L}(E^\times) \subset L^\times U_E^{0+}$. When E/L is ramified, there are exactly two such extensions, and their quotient is the unramified character of $E^\times$ of order 2; that is, they differ only by a sign on a uniformizer of E. We summarize this discussion:

Lemma 2.12. *Let E/F be an extension of degree $2N$ with involution σ and let $\beta \in E$ be quasi-minimal of depth $d > 0$ such that $\sigma(\beta) = -\beta$. Suppose $d > \frac{2}{e_{E/F}}$.*

(i) *There are at least p characters χ of U_E^{0+} which are represented by β, coincide on $U_E^{(1/e_{E/F})+}$ and such that $\chi^\sigma = \chi^{-1}$.*
(ii) *For each character as in (i), and each choice of parity (symplectic/orthogonal) there are at least two extensions to an admissible character χ such that ρ_χ is irreducible self-dual of that parity.*

We now turn to determinants. For E/F a finite extension, we define the character $\varkappa_{E/F}$ of W_F (which we also identify as a character of $F^\times$) by

$$\varkappa_{E/F} = \det(\mathrm{Ind}_{E/F} \mathbf{1}_{E^\times}),$$

where $\mathbf{1}_{E^\times}$ denotes the trivial character. According to [BH06, §29.2] this character has order dividing 2, and when E/F is quadratic it is precisely the associated quadratic class field character trivial on $\mathrm{N}_{E/F}(E^\times)$ (see [BH06, 34.3 Proposition]). More generally, if $E \supseteq L \supseteq F$ an intermediate field then, by [BF83, (10.1.3)], we have

$$\varkappa_{E/F} = \varkappa_{L/F}^{[E:L]} \cdot \varkappa_{E/L}|_{F^\times} \tag{2.13}$$

For $(E/F, \chi)$ an admissible pair, we write $\omega_\chi = \det \rho_\chi$ for the determinant of the induced representation. We then have, as a special case of [BH06, 29.2 Proposition]:

Lemma 2.14. *For $(E/F, \chi)$ an admissible pair we have*

$$\omega_\chi = \varkappa_{E/F} \cdot \chi|_{F^\times}.$$

In particular, ρ_χ has trivial determinant if and only if $\chi|_{F^\times} = \varkappa_{E/F}$.

We can be more specific about the value of the character $\varkappa_{E/F}$, from the calculations in [BF83, §10].

Lemma 2.15 ([BF83, Propositions 10.1.5, 10.1.6])**.** *Let E/F be a tame extension of degree N.*

(i) *If E/F is unramified then $\varkappa_{E/F}$ is trivial if N is odd and unramified nontrivial if N is even.*
(ii) *If E/F is totally ramified then:*
  (a) *if N is odd then $\varkappa_{E/F}$ is unramified, and is trivial if and only if the Jacobi symbol $\left(\frac{q}{N}\right) = 1$;*
  (b) *if N is even then there is a unique intermediate field $E \supset L \supseteq F$ with E/L quadratic, and then $\varkappa_{E/F} = \varkappa_{E/L}|_{F^\times}$ is a non-trivial ramified character.*

Corollary 2.16. *Suppose E/F is tamely ramified of even degree $2N$. If $\varkappa_{E/F}$ is trivial then both $e_{E/F}$ and $f_{E/F}$ are even. In particular, if N is odd then $\varkappa_{E/F}$ is non-trivial.*

Proof. We prove the contrapositive. Let K/F be the maximal unramified subextension in E. If E/K has odd degree then K/F has even degree so q_K is a square and $\varkappa_{E/K}$ is trivial by Lemma 2.15(ii)(a). But then (2.13) implies $\varkappa_{E/F} = \varkappa_{K/F}$, which is non-trivial unramified by Lemma 2.15(i).

If K/F has odd degree, so E/K has even degree, then an element of μ'_F is a square if and only if it is a square in μ'_K. But $\varkappa_{E/K}$ is the non-trivial quadratic character on μ'_K, by Lemma 2.15(ii)(b), so $\varkappa_{E/F} = \varkappa_{E/K}|_{F^\times}$ is then also a non-trivial ramified quadratic character. □

As an immediate corollary of this, with Proposition 2.11(i) and Lemma 2.14, we have:

Corollary 2.17. *Let $(E/F,\chi)$ be an admissible pair of degree $2N$ such that the irreducible representation ρ_χ is orthogonal. If N is odd then ω_χ is non-trivial.*

On the other hand, we can find totally ramified orthogonal admissible pairs $(E/F,\chi)$ of degree $2N$ whose determinant is either of the two ramified quadratic characters, regardless of the parity of N. Indeed, suppose ω is a ramified quadratic character of $F^\times$ and choose a uniformizer ϖ_F such that $\omega(\varpi_F)=1$. We put $L=F(\sqrt[N]{\varpi_F})$ and set $\varpi_L=-\sqrt[N]{\varpi}_F$, and then take $E=L(\sqrt{\varpi_L})$. Then $\varkappa_{E/L}$ is trivial on $\mathrm{N}_{E/L}(\sqrt{\varpi_L})=-\varpi_L$ so is also trivial on $\varpi_F=(-\varpi_L)^N$. Since its restriction to $F^\times$ is a non-trivial ramified quadratic character, by Lemma 2.15(ii)(b), we must have $\varkappa_{E/L}|_{F^\times}=\omega$. Now if $(E/F,\chi)$ is any orthogonal admissible pair, we have $\omega_\chi=\varkappa_{E/F}=\varkappa_{E/L}|_{F^\times}=\omega$.

3. Classical Groups

In this section, we prove the results about classical groups explained in the introduction. We recall first the local Langlands correspondence for split classical groups and observe how the local converse theorem follows from it and the unicity of generic representations in a packet. We assume that p is odd and, for $N\geq 1$, write G_N for one of the groups $\mathrm{Sp}_{2N}(F),\mathrm{SO}_{2N}(F),\mathrm{SO}_{2N+1}(F)$; in the case of $\mathrm{SO}_{2N}(F)$ we further assume $N\geq 2$. We also set $G_N^+=\mathrm{O}_{2N}(F)$ when $G_N=\mathrm{SO}_{2N}(F)$, and $G_N^+=G_N$ otherwise. We also fix a Whittaker datum ξ for G_N, so that *generic* will always mean relative to this fixed datum; if we wish to specify the datum then we will write *ξ-generic*.

We write N^* for the dimension of the standard representation of the dual group ${}^\vee\mathrm{G}_N(\mathbb{C})$ of G_N, so that

$$N^*=\begin{cases}2N & \text{if } G_N=\mathrm{SO}_{2N}(F) \text{ or } \mathrm{SO}_{2N+1}(F),\\ 2N+1 & \text{if } G_N=\mathrm{Sp}_{2N}(F),\end{cases}$$

and write $\varrho_{\mathrm{std}}:{}^\vee\mathrm{G}_N(\mathbb{C})\hookrightarrow\mathrm{GL}_{N^*}(\mathbb{C})$ for the standard representation.

We say that parameters $\varphi_1,\varphi_2:WD_F\to{}^\vee\mathrm{G}_N(\mathbb{C})$ are *outer equivalent* if $\varrho_{\mathrm{std}}\circ\varphi_1\cong\varrho_{\mathrm{std}}\circ\varphi_2$. Note that this is, a priori, weaker than saying that they are locally conjugate (in the sense of the introduction); in fact, in the case of symplectic and odd orthogonal groups, it is the same as saying that φ_1,φ_2 are conjugate, while for even orthogonal groups it is the same as saying they are conjugate in the full orthogonal group $\mathrm{O}_{2N}(\mathbb{C})$. We write $\Phi(G_N)$ for the set of (conjugacy classes of) Langlands parameters for G_N, and $\overline{\Phi}(G_N)$ for the set of outer equivalence classes.

Similarly, as in the introduction, we say that irreducible representations π_1,π_2 of G_N are *outer conjugate* if there is an element $g\in G_N^+$ such that ${}^g\pi_1\cong\pi_2$, and we write $\overline{\mathrm{Irr}}(G_N)$ for the set outer conjugacy classes. Then, by the main local result of

Arthur in [Art13], we have a canonical surjective map on the set of outer conjugacy classes of tempered irreducible representations

$$\begin{aligned} \overline{\mathrm{Irr}}^{\mathrm{temp}}(G_N) &\to \overline{\Phi}^{\mathrm{temp}}(G_N) \\ [\pi] &\mapsto [\varphi_\pi] \end{aligned}$$

with finite fibres, where we write φ_π for some representative of the class.

On the other hand, if π is a generic irreducible representation of G_N then Cogdell–Kim–Piatetski-Shapiro–Shahidi constructed its *transfer* Π to $\mathrm{GL}_{N^*}(F)$ (see [CKPSS04, Proposition 7.2]), which is a self-dual irreducible representation such that

$$\gamma(s, \Pi \times \tau, \psi) = \gamma(s, \pi \times \tau, \psi),$$

for all irreducible supercuspidal representations τ of $\mathrm{GL}_r(F)$ with $r \geq 1$. Moreover, by a result of Henniart (see [AHKO25, Appendix B]), this transfer coincides with the Arthur transfer whenever π is supercuspidal generic: that is, the Langlands parameter of $\mathrm{GL}_{N^*}(F)$ corresponding to Π is $\varrho_{\mathrm{std}} \circ \varphi_\pi$ (which is independent of the choice of representative φ_π). It also follows from this that the local Langlands correspondence for G_N preserves twisted γ-factors when π is supercuspidal generic:

$$\gamma(s, \pi \times \tau, \psi) = \gamma(s, \varphi_\pi \times \tau, \psi),$$

where the γ-factor on the RHS is $\gamma(s, (\varrho_{\mathrm{std}} \circ \varphi_\pi) \otimes \tau, \psi)$.

Lemma 3.1. *Let π_1, π_2 be generic supercuspidal representations of G_N, with transfers Π_1, Π_2 to $\mathrm{GL}_{N^*}(F)$. If $\Pi_1 \cong \Pi_2$ then π_1, π_2 are outer conjugate.*

Proof. Since $\Pi_1 \cong \Pi_2$, the local Langlands correspondence for $\mathrm{GL}_{N^*}(F)$ implies that $\varrho_{\mathrm{std}} \circ \varphi_{\pi_1} \cong \varrho_{\mathrm{std}} \circ \varphi_{\pi_2}$. Thus $\varphi_{\pi_1}, \varphi_{\pi_2}$ are outer equivalent. Now a result of Varma [Var17, Corollary 6.16] says that there is a unique outer conjugacy class of generic irreducible representations of G_N corresponding to $[\varphi_{\pi_1}]$, so π_1, π_2 are outer conjugate. □

Up to outer conjugation, the Local Converse Theorem for G_N is now straightforward for supercuspidal representations, and follows for all generic irreducible representations by standard techniques.

Theorem 3.2. *Let π_1, π_2 be generic irreducible representations of G_N. If*

$$\gamma(s, \pi_1 \times \tau, \psi) = \gamma(s, \pi_2 \times \tau, \psi)$$

for all generic irreducible representations τ of $\mathrm{GL}_r(F)$, for $1 \leq r \leq N$, then π_1, π_2 are outer conjugate.

Remark 3.3. (i) *We note that our version of the local converse theorem removes the assumption in Zhang* [Zha18] *and Jo* [Jo25, Theorem A] *for symplectic groups, and in Hazeltine–Liu* [HL25, Theorem 1.2] *(for even special orthogonal groups), that the central characters of π_1, π_2 are equal.*

(ii) *Another way of expressing Theorem 3.2 is that the map on irreducible ξ-generic representations*

$$\overline{\mathrm{Irr}}^{\xi\text{-gen}} \to \overline{\Phi}^{\mathrm{gen}}$$

induced by the local Langlands correspondence is injective*, where a parameter φ is said to be generic if its adjoint L-function $L(s,\varphi,\mathrm{Ad})$ is holomorphic at $s=1$ (see* [Mat24, §2.2] *and* [GI16, Proposition B.1]*). Moreover, as pointed out to us by the referee, it is surjective by Cunningham–Dijols–Fiori–Zhang* [CDFZ25, Theorem 0.2(2,3)]. *One can also compare with Jantzen–Liu* [JL24, Theorem 6.20]*, which proves the surjectivity of the map*

$$\bigcup_{\xi} \overline{\mathrm{Irr}}^{\xi\text{-gen}} \to \overline{\Phi}^{\mathrm{gen}},$$

where the union is taken over all (conjugacy classes of) Whittaker data.

Proof. Firstly, let π_1, π_2 be generic supercuspidal representations of G_N satisfying the hypotheses of the theorem and let Π_1, Π_2 be their transfers to $\mathrm{GL}_{N^*}(F)$. Then, by the definition of transfer, $\gamma(s, \Pi_1 \times \tau, \psi) = \gamma(s, \Pi_2 \times \tau, \psi)$, for all generic irreducible representations τ of $\mathrm{GL}_r(F)$ with $1 \leq r \leq N = \lfloor \frac{N^*}{2} \rfloor$, so the Local Converse Theorem for $\mathrm{GL}_{N^*}(F)$ implies that $\Pi_1 \cong \Pi_2$. The result now follows from Lemma 3.1.

The reduction to the supercuspidal case is now done as in Jiang–Soudry for odd special orthogonal groups (see [JS03, Theorem 5.1]), Jo for symplectic groups (see [Jo25, Proposition 2.10]), and Hazeltine–Liu for even special orthogonal groups (see [HL25, Theorem 4.3]), noting that the proofs in the latter two cases only require equality of central characters because their versions of the converse theorem for supercuspidal representations also requires this. □

In the following sections, we will discuss the sharpness of Theorem 3.2 (i.e. whether the condition $r \leq N$ can be improved) and the possibility of refining it to remove the outer conjugation (in the case of special orthogonal groups). For ease of exposition, we introduce the following definition.

Definition 3.4. *Let π_1, π_2 be irreducible generic representations of G_N or of $\mathrm{GL}_N(F)$ and let $m \geq 1$ be an integer. We say that π_1, π_2 are* γ-equivalent to level m *if*

$$\gamma(s, \pi_1 \times \tau, \psi) = \gamma(s, \pi_2 \times \tau, \psi)$$

for all irreducible supercuspidal representations τ of $\mathrm{GL}_r(F)$, with $1 \leq r \leq m$. We use the same terminology for Langlands parameters also.

Note again that, by multiplicativity of γ-factors, if π_1, π_2 are γ-equivalent to level m then we in fact have $\gamma(s, \pi_1 \times \tau, \psi) = \gamma(s, \pi_2 \times \tau, \psi)$ for all *generic* irreducible representations τ of $\mathrm{GL}_r(F)$, with $1 \leq r \leq m$. Thus the Local Converse Theorem 3.2 says that if generic irreducible representations of G_N are γ-equivalent to level N then they are outer conjugate.

3.1. **Sharpness Results.** Our first sharpness result concerns *non-supercuspidal* representations. For this, recall that we have a map

$$\iota_{\mathrm{L}} : \mathrm{GL}_N(\mathbb{C}) \hookrightarrow {}^{\vee}\mathrm{G}_N(\mathbb{C})$$

by identifying $\mathrm{GL}_N(\mathbb{C})$ with a Siegel Levi subgroup ${}^{\vee}\mathrm{L}(\mathbb{C})$ of ${}^{\vee}\mathrm{G}_N(\mathbb{C})$. Note that, although this embedding is only important up to conjugacy, there are two non-conjugate choices in the case of $\mathrm{SO}_{2N}(F)$; we fix one arbitrarily.

For $(E/F, \chi)$ an admissible pair of degree N and $\rho_\chi = \mathrm{Ind}_{E/F}\,\chi$, we thus obtain a parameter $\varphi_\chi = \iota_{\mathrm{L}} \circ \rho_\chi$ for G_N. Note that, when we compose this with the standard representation, we get

$$\varrho_{\mathrm{std}} \circ \varphi_\chi \cong \begin{cases} \rho_\chi \oplus \rho_{\chi^{-1}} & \text{if } G_N = \mathrm{SO}_{2N}(F) \text{ or } \mathrm{SO}_{2N+1}(F), \\ \rho_\chi \oplus \mathbf{1} \oplus \rho_{\chi^{-1}} & \text{if } G_N = \mathrm{Sp}_{2N}(F). \end{cases}$$

Note then that, if $(E'/F, \chi')$ is another admissible pair of degree N such that $\varphi_\chi, \varphi_{\chi'}$ define equivalent Langlands parameters for G_N, then either $\rho_{\chi'} \cong \rho_\chi$ or $\rho_{\chi'} \cong \rho_{\chi^{-1}}$, so $(E'/F, \chi')$ is conjugate to either $(E/F, \chi)$ or $(E/F, \chi^{-1})$.

Theorem 3.5. *Suppose $p > N$. Let $(E/F, \chi), (E/F, \chi')$ be minimal totally ramified admissible pairs of degree N such that χ, χ' coincide on U_E. Then $\varphi_\chi, \varphi_{\chi'}$ are γ-equivalent to level $N-1$.*

There are infinitely many minimal totally ramified admissible pairs of degree N with a fixed restriction to U_E (by choosing the value on a uniformizer), so also infinitely many up to equivalence and duality, even if we also require the image to be bounded and the induced representation not to be self-dual. Thus the local Langlands correspondence for G_N immediately gives us:

Corollary 3.6. *Suppose $p > N$. Then there are generic irreducible tempered representations π, π' of G_N, which are γ-equivalent to level $N-1$ but not outer conjugate.*

Proof of Theorem 3.5. By definition, we have

$$\gamma(s, \varphi_\chi \times \tau, \psi) = \gamma(s, (\varrho_{\mathrm{std}} \circ \varphi_\chi) \otimes \tau, \psi),$$

to which we seek to apply Proposition 2.10. In the symplectic case, the component $\mathbf{1}$ in $\varrho_{\mathrm{std}} \circ \varphi_\chi$ is the same for χ and χ' so, by multiplicativity of γ-factors, in all cases we need only prove

$$\gamma(s, (\rho_\chi \oplus \rho_{\chi^{-1}}) \otimes \tau, \psi) = \gamma(s, (\rho_{\chi'} \oplus \rho_{\chi'^{-1}}) \otimes \tau, \psi). \tag{3.7}$$

Condition (i) of Proposition 2.10 is immediate from the hypotheses, while (iii) is also immediate as $\chi\chi^{-1} = \chi'\chi'^{-1}$ are trivial, so we only need to verify (ii). If χ, χ' are represented by β then χ^{-1}, χ'^{-1} are represented by $-\beta$ so the condition is

$$\chi(\beta)\chi^{-1}(-\beta) = \chi'(\beta)\chi'^{-1}(-\beta).$$

This simply says $\chi(-1) = \chi'(-1)$, which is true by hypothesis. Thus Proposition 2.10 says that (3.7) holds for all irreducible representations τ of W_F of dimension less than N, as required. $\square$

Now we turn to the case of supercuspidal representations, for which we assume that $N \geq 2$. We will define some Langlands parameters φ via their composition with ϱ_{std}; as previously discussed, this only determines φ up to outer equivalence but this will be sufficient for our purposes here. We write $M = 2\lfloor \frac{N}{2} \rfloor$, so that M is the largest even integer no greater than N. We also fix arbitrarily a self-dual admissible pair $(E_0/F, \chi_0)$ of degree 2 such that E_0/F is unramified and ρ_{χ_0} is of the same parity as ${}^\vee\mathrm{G}_N$ (which imposes a constraint on $\chi_0|_{F^\times}$ according to Proposition 2.11).

We will consider a pair of self-dual minimal totally ramified admissible pairs $(E_1/F, \chi_1), (E_2/F, \chi_2)$ of degree $2M$ which induce to give self-dual irreducible representations $\rho_{\chi_1}, \rho_{\chi_2}$ of the same parity as ${}^\vee\mathrm{G}_N$ such that

$$\omega_{\chi_1}\omega_{\chi_2} = \begin{cases} \mathbf{1} & \text{if } N \text{ is even,} \\ \omega_{\chi_0} & \text{if } N \text{ is odd,} \end{cases}$$

where we recall that $\omega_\chi = \det \rho_\chi$, for $(E/F, \chi)$ an admissible quasi-character. Note that this condition is vacuous when ${}^\vee\mathrm{G}_N$ is symplectic, and we can always achieve it when ${}^\vee\mathrm{G}_M$ is orthogonal by the remarks after Corollary 2.17.

Associated to this pair, we define a Langlands parameter φ_{χ_1,χ_2} for G_N (up to outer equivalence) by

$$\varrho_{\mathrm{std}} \circ \varphi_{\chi_1,\chi_2} = \begin{cases} \rho_{\chi_1} \oplus \rho_{\chi_2} & \text{if } G_N \text{ is orthogonal and } N \text{ is even,} \\ \rho_{\chi_0} \oplus \rho_{\chi_1} \oplus \rho_{\chi_2} & \text{if } G_N \text{ is orthogonal and } N \text{ is odd,} \\ \mathbf{1} \oplus \rho_{\chi_1} \oplus \rho_{\chi_2} & \text{if } G_N \text{ is symplectic and } N \text{ is even,} \\ \mathbf{1} \oplus \rho_{\chi_0} \oplus \rho_{\chi_1} \oplus \rho_{\chi_2} & \text{if } G_N \text{ is symplectic and } N \text{ is odd.} \end{cases}$$

Recall also from Lemma 2.12 that there are self-dual minimal totally ramified admissible pairs $(E_1/F, \chi_1'), (E_2/F, \chi_2')$ of the same parity as ${}^\vee\mathrm{G}_N$ such that χ_i, χ_i' differ only (by a sign) on a uniformizer. Notice also that, if β_i represents χ_i, then $\chi_i(\beta_i) = -\chi_i'(\beta_i)$, since $\mathrm{val}(\beta_i)$ has least denominator $2M$ so is an odd power of a uniformizer. Then we have:

Theorem 3.8. *Suppose $p > N$ and set $M = 2\lfloor \frac{N}{2} \rfloor$. With the notation as above, suppose that $(E_1/F, \chi_1)$ is equivalent to neither $(E_2/F, \chi_2)$ nor $(E_2/F, \chi_2')$. Then*

(i) *φ_{χ_1,χ_2} and $\varphi_{\chi_1',\chi_2'}$ are γ-equivalent to level $M-1$;*
(ii) *$\varrho_{\mathrm{std}} \circ \varphi_{\chi_1,\chi_2} \not\cong \varrho_{\mathrm{std}} \circ \varphi_{\chi_1',\chi_2'}$.*

Notice that we can easily find examples such that $(E_1/F, \chi_1)$ is equivalent to neither $(E_2/F, \chi_2)$ nor $(E_2/F, \chi_2')$, for example by requiring that χ_1, χ_2 have different depths. Again, the local Langlands correspondence for G_N immediately gives us:

Corollary 3.9. *Suppose $p > N$ and set $M = 2\lfloor \frac{N}{2} \rfloor$. Then there are generic supercuspidal representations π, π' of G_N, which are γ-equivalent to level $M-1$ but not outer conjugate.*

Proof of Theorem 3.8. By multiplicativity, for (i) we need to prove

$$\gamma(s, (\rho_{\chi_1} \oplus \rho_{\chi_2}) \otimes \tau, \psi) = \gamma(s, (\rho_{\chi_1'} \oplus \rho_{\chi_2'}) \otimes \tau, \psi).$$

But this follows exactly from Proposition 2.10, since the conditions there are clearly satisfied.

For (ii), we just need to verify that ρ_{χ_1} is inequivalent to $\rho_{\chi_1'}$ (since it is inequivalent to $\rho_{\chi_2'}$ by hypothesis); that is, we need to check that the admissible pairs $(E_1/F, \chi_1)$ and $(E_1/F, \chi_1')$ are inequivalent. But the characters χ_1, χ_1' agree, and do not factor through any norm, on restriction to $U_{E_1}^{0+}$, so the only F-automorphism of E_1 which conjugates this restriction is the identity. But $\chi_1 \neq \chi_2$ so we are done. □

3.2. **Self-dual supercuspidal representations of** $\mathrm{GL}_{2N}(F)$**.** When N is odd, Theorem 3.8 does not quite close the gap with Theorem 3.2; that is, it may be that twisting only by representations of dimension up to $N-1$ *is sufficient* to determine the Langlands parameter of a supercuspidal representation. This is very closely related to the question of whether, for self-dual supercuspidal representations of $\mathrm{GL}_{2N}(F)$, twisting only by representations of $\mathrm{GL}_r(F)$ with $r \leq N-1$ is sufficient to distinguish them.

Let us explain. Write a supercuspidal Langlands parameter φ for G_N as a direct sum $\varphi_1 \oplus \cdots \oplus \varphi_n$ of irreducible, self-dual representations of W_F. If $N > 1$ is odd, then none of the φ_i can be N-dimensional, since there is no self-dual irreducible representation of W_F of odd dimension when $p \neq 2$. Thus, as long as φ is not irreducible, twisting by representations of dimensions $1, ..., N-1$ is all that is needed in order to distinguish a parameter via twisted gamma factors. In searching for a sharpness example to try to show that N is sharp, therefore, we were forced to consider irreducible parameters of G_N. We could not find such a sharpness example, at least when the extension E/F is totally ramified. Only on this basis, we felt that we could make the following conjecture:

Conjecture 3.10. *Suppose $p > N$ and suppose that N is odd. Let π, π' be self-dual supercuspidal representations of* $\mathrm{GL}_{2N}(F)$ *of the same parity which are γ-equivalent to level $N-1$. Then $\pi \cong \pi'$.*

Remark 3.11. *We have no evidence as to whether the assumption that $p > N$ or the assumption that N be odd are needed.*

The following theorem shows that Conjecture 3.10 is almost best possible: the bound can be improved at most to $N-2$. We have no evidence as to whether this improvement in bound is possible. The theorem also shows that, in the case that N is even, the same statement as in Conjecture 3.10 would be the best possible.

Theorem 3.12. *Suppose $p > N$, set $M = 2\lfloor \frac{N}{2} \rfloor$, and fix a parity. There exist inequivalent self-dual supercuspidal representations π, π' of $\mathrm{GL}_{2N}(F)$ of this parity which are γ-equivalent to level $M-2$.*

Proof. As in other results, by the local Langlands correspondence we translate to a question about self-dual irreducible representations of W_F. Let L/F be a totally ramified extension of degree N, and E/L a ramified quadratic extension with galois involution σ. We set

$$m = \begin{cases} N+1 & \text{if } N \text{ is even,} \\ \frac{3N+(-1)^{(N+1)/2}}{2} & \text{if } N \text{ is odd.} \end{cases}$$

Note that m is odd and $d = \frac{m}{2N}$ has least denominator $2N$. Now an easy exercise in congruences shows that the least integer r such that $rm \equiv \pm 1 \pmod{2N}$ is $r = M-1$, so that Lemma 2.9 implies that $t_\beta(M-1) > \frac{1}{2N}$.

Now let $\beta \in E$ be an element of valuation $-d$ such that $\sigma(\beta) = -\beta$, and let $(E/F, \chi)$ and $(E/F, \chi')$ be self-dual admissible characters of the required parity which are represented by β, coincide on $U_E^{(1/2N)+}$, and such that $\chi(\beta) = \chi'(\beta)$; these exist by Lemma 2.12, noting that since $\beta^2 \in L^\times$ we must have $\chi(\beta^2) = \chi'(\beta^2)$ and both choices of square root are permitted. Then Proposition 2.10 gives the result. □

3.3. **A (conjectural) improved bound for classical groups when N is odd.** When N is odd, Conjecture 3.10 implies an improved bound for the Local Converse Theorem for the classical group G_N, which would then mean that our sharpness result in Corollary 3.9 is optimal.

Theorem 3.13. *Suppose N is odd and $p > N$. Suppose also that Conjecture 3.10 is true for $\mathrm{GL}_{2N}(F)$. Let π_1, π_2 be irreducible generic supercuspidal representations of G_N which are γ-equivalent to level $N-1$. Then π_1, π_2 are outer conjugate.*

The proof of this relies on the following (unconditional) lemma.

Lemma 3.14. *Suppose N is odd and $p > N$. Let π_1 be an irreducible generic supercuspidal representation of G_N such that largest irreducible component of the restriction to W_F of $\varrho_{\mathrm{std}} \circ \varphi_{\pi_1}$ has dimension strictly less than $2N$. Let π_2 be an irreducible generic supercuspidal representation of G_N such that π_1, π_2 are γ-equivalent to level $N-1$. Then π_1, π_2 are outer conjugate.*

Proof. Let π_1, π_2 be irreducible generic supercuspidal representations of G_N satisfying the hypotheses, so that

$$\gamma(s, \pi_1 \times \tau, \psi) = \gamma(s, \pi_2 \times \tau, \psi),$$

for all irreducible supercuspidal representations τ of GL_r with $r = 1, \ldots, N-1$. We abbreviate $\varphi_i^{\mathrm{GL}} = \varrho_{\mathrm{std}} \circ \varphi_{\pi_i}$ and let Π_1, Π_2 be the transfers to $\mathrm{GL}_{N^*}(F)$. Then we can write

$$\varphi_i^{\mathrm{GL}} \cong \bigoplus_{j=1}^{t_i} \varphi_i^{(j)} \otimes \left(\mathrm{st}_{a_i^{(j)}} \oplus \mathrm{st}_{a_i^{(j)}-2} \oplus \cdots\right),$$

where st_a is the unique a-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ and the $\varphi_i^{(j)}$ are distinct self-dual irreducible representations of the Weil group W_F. Then, writing φ_τ for the Langlands parameter corresponding to τ, we have

$$\gamma(s, \Pi_i \times \tau, \psi) = \gamma(s, \varphi_i^{\mathrm{GL}} \otimes \varphi_\tau, \psi) = \prod_{j=1}^{t_i} \prod_{k=\frac{1-a_i^{(j)}}{2}}^{\frac{a_i^{(j)}-1}{2}} \gamma(s+k, \varphi_i^{(j)} \otimes \varphi_\tau, \psi). \tag{3.15}$$

By Lemma 3.1, we need only show that φ_1^{GL} is equivalent to φ_2^{GL}. For contradiction, we suppose not. Then there are an irreducible self-dual representation φ of W_F and natural number a such that $\varphi \otimes \mathrm{st}_a$ appears in φ_1^{GL} but not φ_2^{GL}, and we can choose the maximal such a.

Suppose first that there is such φ with $\dim(\varphi) \leq N^*/2$, so that $\dim(\varphi) \leq N$. As $N \geq 3$ is odd while $\dim(\varphi)$ is 1 or even, we see that in fact $\dim(\varphi) \leq N-1$. But then, writing π_φ for the corresponding (self-dual) irreducible representation a general linear group, [JNS15, Proposition 2.9] implies that $\gamma(s, \Pi_1 \times \pi_\varphi^\vee, \psi)$ has a real pole at $s = \frac{a+1}{2}$, while $\gamma(s, \Pi_2 \times \pi_\varphi^\vee, \psi)$ does not, a contradiction.

Thus the only such φ must have $\dim(\varphi) > N^*/2$. Then $\varphi \otimes \mathrm{st}_1$ is in fact the *unique* component of φ_1^{GL} which does not appear in φ_2^{GL} (since every other such component would have dimension less than $N^*/2$). Moreover, there is also a unique component $\varphi' \otimes \mathrm{st}_1$ of φ_2^{GL} which does not appear in φ_1^{GL} (again, if there were two then at least one would have dimension less than $N^*/2$, which is impossible by the previous paragraph). But then, since all other terms in (3.15) are identical for $i = 1, 2$, we have that $\dim(\varphi) = \dim(\varphi') < 2N$ (by hypothesis) and they are γ-equivalent to level $N-1$. Then Theorem 3.2 implies that φ, φ' are equivalent, which is absurd. ☐

Proof of Theorem 3.13. By Lemma 3.14, we need only consider the case where $\varrho_{\mathrm{std}} \circ \varphi_{\pi_1}$ and $\varrho_{\mathrm{std}} \circ \varphi_{\pi_2}$ are irreducible of dimension $2N$ or the sum of a quadratic character and an irreducible representation of dimension $2N$. As in the proof of Lemma 3.14, in the latter case we see that the quadratic character is the same so, by multiplicativity of γ-factors, we reduce to the first case. But then these are the Langlands parameters of a pair of self-dual supercuspidal representations of GL_{2N} of the same parity, so Conjecture 3.10 implies they are equivalent. Thus $\varphi_{\pi_1}, \varphi_{\pi_2}$ are outer equivalent, so π_1, π_2 are outer conjugate. ☐

In fact, we can prove this improved Local Converse Theorem in the case of even special orthogonal groups.

Theorem 3.16. *Suppose N is odd and $p > N$. Let π_1, π_2 be irreducible generic supercuspidal representations of $\mathrm{SO}_{2N}(F)$ which are γ-equivalent to level $N-1$. Then π_1, π_2 are outer conjugate.*

Proof. By Lemma 3.14, we need only consider the case where $\varrho_{\mathrm{std}} \circ \varphi_{\pi_1}$ is irreducible. But, by Corollary 2.17, there are no irreducible orthogonal representations of W_F of dimension $2N$ and determinant 1 when N is odd, so there is nothing to prove. □

3.4. **Generic WD_F-unacceptability of SO_6.** In the case of even special orthogonal groups, twisted standard γ-factors cannot distinguish between Langlands parameters φ, φ' which are outer equivalent. One might hope that we can use different algebraic representations of the dual group to help distinguish them. The representation ring of $\mathrm{SO}_{2N}(\mathbb{C})$ is generated by the fundamental representations $\wedge^1, \cdots, \wedge^{N-1}, \wedge^N_+, \wedge^N_-$, where $\wedge^r$ denotes the r^{th} exterior power of ϱ_{std} and $\wedge^N_\pm$ are described in [YZ24] (see also below in the case $N = 3$). We cannot hope to use the exterior powers to distinguish φ from φ' since

$$\wedge^i \circ \varphi = \wedge^i \circ \varrho_{\mathrm{std}} \circ \varphi$$

and $\varrho_{\mathrm{std}} \circ \varphi \cong \varrho_{\mathrm{std}} \circ \varphi'$. Thus it is only using the representations $\wedge^N_+, \wedge^N_-$ that we can hope to distinguish them. Since $\wedge^N_+$ is obtained by composing $\wedge^N_-$ with the outer automorphism, we get that $\wedge^N_+ \circ \varphi' \cong \wedge^N_- \circ \varphi$; thus we can use these to distinguish φ from φ' precisely when $\wedge^N_+ \circ \varphi \ncong \wedge^N_- \circ \varphi$.

In the case of $\mathrm{SO}_4(F)$ it is shown in [YZ24] that this is always the case, so that twisted γ-factors can be used to distinguish outer equivalent parameters. However, this seems unlikely to be true in general as $\mathrm{SO}_{2N}(\mathbb{C})$ is not acceptable for $N \geq 3$. The following example shows that, for $\mathrm{SO}_6(F)$ when $q \equiv 3 \pmod 4$, there is indeed a parameter φ which is not conjugate to its outer twist such that $\wedge^N_+ \circ \varphi \cong \wedge^N_- \circ \varphi$. We note that in [Mat24, Proposition 7.3], an example for $\mathrm{SO}_{2N}(F)$ is given when $q \equiv 1 \pmod 4$; one difference is that our example is a discrete parameter (indeed the representations in the corresponding L-packet are all supercuspidal) while that in [Mat24] is not.

Example 3.17. *Let L/F be the biquadratic extension of F, with E_1, E_2, E_3 the quadratic subfields and E_1/F unramified. Let $\zeta \in \mu'_{E_1}$ be a generator, so that ζ has order $q^2 - 1$, and let ϖ_F be a fixed uniformizer of F. Define a complex character χ_1 of $E_1^\times$ by*

$$\chi_1(\varpi_F^n \zeta^k u_1) = \mathrm{i}^k, \qquad \text{for } u_1 \in U_{E_1}^{0+},$$

where i is a fixed complex square root of -1. Since $q \equiv 3 \pmod 4$, this is an orthogonal admissible character of order 4. We pick orthogonal admissible ramified characters χ_2, χ_3, of E_2, E_3 respectively, of distinct positive depths. Define φ up to outer equivalence by

$$\varrho_{\mathrm{std}} \circ \varphi = \bigoplus_{i=1}^{3} \rho_{\chi_i}.$$

This does define a parameter into $\mathrm{SO}_6(\mathbb{C})$ *since* $\det \rho_{\chi_i} = \omega_{E_i/F}$*, the local class field character, and the product of the three class field characters is trivial. Then we claim that* $\wedge^3_+ \circ \varphi \cong \wedge^3_- \circ \varphi$*, and that* φ *is not equivalent to its own outer conjugate.*

Proof. We first explain why our parameter φ is not equivalent to its own outer conjugate. Suppose $g \in \mathrm{O}(6, \mathbb{C})$ is such that there exists $h \in \mathrm{SO}(6, \mathbb{C})$ such that ${}^g\varphi = {}^h\varphi$. Then $\varrho_{\mathrm{std}}(h^{-1}g)$ centralizes $\varrho_{\mathrm{std}} \circ \varphi = \rho_{\chi_1} \oplus \rho_{\chi_2} \oplus \rho_{\chi_3}$ so, by Schur's Lemma, $\varrho_{\mathrm{std}}(h^{-1}g)$ lies in $\{(\pm \mathrm{I}_2, \pm \mathrm{I}_2, \pm \mathrm{I}_2)\}$, where I_2 is the 2-by-2 identity matrix. But the determinant of such a matrix is equal to one, and thus g also lies in $\mathrm{SO}(6, \mathbb{C})$. In particular, we conclude that φ is not equivalent to its own outer conjugate.

We now compute $\wedge^3 \circ \varphi$, which is the sum of the two $\wedge^3_\pm \circ \varphi$. Note that we are really computing $\varrho_{\mathrm{std}} \circ \wedge^3 \circ \varphi$, but we will omit the ϱ_{std} to simplify notation. We abbreviate $\rho_i = \rho_{\chi_i}$ and $\omega_i = \omega_{E_i/F}$. Since $\wedge^2 \rho_i = \det \rho_i = \omega_i$, we have

$$\wedge^3 \circ \varphi \cong \rho_1 \otimes \rho_2 \otimes \rho_3 \oplus \bigoplus_{i \neq j} \rho_i \otimes \omega_j.$$

Moreover, applying Lemma 2.6 twice and using that the χ_i are self-dual, we have

$$\rho_1 \otimes \rho_2 \otimes \rho_3 \cong \mathrm{Ind}_{L/F}\left(\prod_{i=1}^{3} \chi_{i,L}\right) \oplus \mathrm{Ind}_{L/F}\left(\prod_{i=1}^{3} \chi_{i,L}^{-1}\right),$$

where we recall that $\chi_{i,L} = \chi_i \circ \mathrm{N}_{L/E_i}$. Moreover, in our situation, these two 4-dimensional representations are irreducible and isomorphic. Indeed, $\chi_{1,L}$ is a quadratic character because $\mathrm{N}_{L/E_1}(L^\times) = \varpi_F^{\mathbb{Z}} \langle \zeta^2 \rangle U_{E_1}^{0+}$, while the characters $\chi_{2,L}$, $\chi_{3,L}$ and $\chi_{2,L}\chi_{3,L}$ are all non-quadratic (since they have positive depth), so each 4-dimensional piece restricts to W_L to a sum of distinct characters and is thus irreducible; but both restrict to the sum of the same four characters, so they are isomorphic.

Moreover, for $i \neq j$, we have $\omega_j \circ \mathrm{N}_{E_i/F} = \omega_{L/E_i}$ so

$$\rho_i \otimes \omega_j \cong \mathrm{Ind}_{E_i/F}(\chi_i \omega_{L/E_i}),$$

which is independent of j. Thus we have a decomposition

$$\wedge^3 \circ \varphi \cong \left(\mathrm{Ind}_{L/F}\left(\prod_{i=1}^{3} \chi_{i,L}\right) \oplus \mathrm{Ind}_{L/F} \bigoplus_{i=1}^{3} \mathrm{Ind}_{E_i/F}(\chi_i \omega_{L/E_i})\right)^{\oplus 2},$$

in which each representation appearing is irreducible and is determined by the characters appearing in its restriction to W_L. Thus, to verify that $\wedge^3_+ \circ \varphi \cong \wedge^3_- \circ \varphi$, we need only check that they have the same restrictions to W_L.

We now use the general description of $\wedge^n_\pm$ from [YZ24] to make these explicit. We write $(W, \langle \cdot, \cdot \rangle)$ for the 6-dimensional complex orthogonal space, with Witt basis $e_1, e_2, e_3, e_{-3}, e_{-2}, e_{-1}$, so that

$$\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } j = -i, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathrm{SO}_6(\mathbb{C})$ is (identified with) the group of automorphisms of W which preserve the symmetric form $\langle\cdot,\cdot\rangle$ and have determinant 1. We define an involution τ on $\wedge^3 W$ as follows: for a simple tensor $x = e_i \wedge e_j \wedge e_k$, we let $\tau(x)$ be the unique simple tensor y such that

$$e_{-i} \wedge e_{-j} \wedge e_{-k} \wedge y = e_1 \wedge e_2 \wedge e_3 \wedge e_{-3} \wedge e_{-2} \wedge e_{-1}.$$

Then $\wedge^3_\pm(W)$ is the (±1)-eigenspace of τ on W and we can write down bases for them: for (i,j,k) any cyclic permutation of $(1,2,3)$, we write

$$\begin{aligned} u_i^\pm &= e_{\pm i} \wedge e_{\mp j} \wedge e_{\mp k}, \\ v_i^\pm &= e_i \wedge e_j \wedge e_{-j} \pm e_i \wedge e_k \wedge e_{-k}, \\ v_{-i}^\pm &= e_{-i} \wedge e_j \wedge e_{-j} \mp e_{-i} \wedge e_k \wedge e_{-k}; \end{aligned}$$

then $\wedge^3_\pm(W)$ has basis

$$\left\{ e_{\pm1} \wedge e_{\pm2} \wedge e_{\pm3}, u_i^\pm, v_i^\pm, v_{-i}^\pm : i = 1,2,3 \right\}.$$

Now we realise our parameter φ with W_{E_i} acting on e_i via the character χ_i (so on e_{-i} via the character χ_i^{-1}), and check the action of W_L: since $\chi_{1,L} = \chi_{1,L}^{-1}$, the characters appearing in $\mathrm{Res}_{W_L}^{W_F} \wedge^3_+ \circ \varphi$ are the same as those in $\mathrm{Res}_{W_L}^{W_F} \wedge^3_- \circ \varphi$, and we are done.

Thus, we have produced a parameter φ for $\mathrm{SO}_6(F)$ that is not isomorphic to its own outer conjugate, and which satisfies $\wedge^3_+ \circ \varphi = \wedge^3_- \circ \varphi$, completing the proof. $\square$

It is now not hard to use this example to deduce the following:

Corollary 3.18. *The group* $\mathrm{SO}_{2N}(\mathbb{C})$ *is generically* WD_F*-unacceptable.*

We thank Nadir Matringe for pointing out how this should be possible, using the ideas in Yu [Yu22]. We also note that Matringe proved this (in a slightly different way) when $q \equiv 1 \pmod 4$ in [Mat24, Proposition 7.3].

Proof. By [Mat24, Proposition 7.3], we need only treat the case $q \equiv 3 \pmod 4$ so we let $\varphi : W_F \to \mathrm{SO}_6(\mathbb{C})$ be the Langlands parameter constructed in Example 3.17. Then there is a finite Galois extension K/F such that φ is trivial on W_K, so φ factors through the finite group $\Gamma := \mathrm{Gal}(K/F)$. Now let L/F be an unramified extension of degree $M > 2N$ such that M is prime to the degree of K/F, and set $E = KL$. Then, since the degrees of the extensions are coprime, we have

$$\mathrm{Gal}(E/F) \simeq \mathrm{Gal}(K/F) \times \mathrm{Gal}(L/F) \simeq \Gamma \times \mathbb{Z}/M\mathbb{Z}.$$

Thus $\Gamma \times \mathbb{Z}/M\mathbb{Z}$ is isomorphic to the quotient W_F/W_E of W_F and we can now apply the method in [Yu22, Lemma 2.3] to construct a parameter with image $\mathrm{SO}_{2N}(\mathbb{C})$ as follows. We pick a primitive M^{th} root of unity ζ in $\mathbb{C}$ and define φ' to be the inflation to W_F of the map

$$\Gamma \times \mathbb{Z}/M\mathbb{Z} \to \mathrm{SO}_{2N}(\mathbb{C}), \quad (\gamma, a) \mapsto \mathrm{diag}(\zeta^a, \ldots, \zeta^{(N-3)a}, \varphi(\gamma), \zeta^{(3-N)a}, \ldots, \zeta^{-a}),$$

for $\gamma \in \Gamma$ and $a \in \mathbb{Z}/M\mathbb{Z}$.

Now take $\varphi_1 = \varphi$ and let φ_2 be its outer conjugate, from which we construct morphisms φ_1', φ_2' (with the same choice ζ).Then φ_1', φ_2' are clearly also outer equivalent. If they were conjugate by some $g \in \mathrm{SO}_{2N}(\mathbb{C})$ then we would have

$$\varphi_1'(\gamma, a) = \mathrm{Ad}(g) \circ \varphi_2'(\gamma, a), \quad \text{for all } \gamma \in \Gamma, a \in \mathbb{Z}/M\mathbb{Z}.$$

By considering the element $(\gamma, a) = (1, 1)$, we see that g is in the centralizer

$$Z_{\mathrm{SO}_{2N}(\mathbb{C})}(\mathrm{diag}(\zeta, \dots, \zeta^{N-3}, \mathrm{Id}_6, \zeta^{3-N}, \dots, \zeta^{-1}) \simeq \mathrm{GL}_1(\mathbb{C}) \times \cdots \times \mathrm{GL}_1(\mathbb{C}) \times \mathrm{SO}_6(\mathbb{C}).$$

But then taking the elements $(\gamma, 0)$, for all $\gamma \in \Gamma$, we would get that φ_1, φ_2 are conjugate in $\mathrm{SO}_6(\mathbb{C})$, a contradiction.

□

Remark 3.19. *As remarked above, one difference between our example for* $\mathrm{SO}_6(\mathbb{C})$ *(when* $q \equiv 3 \pmod 4$*) and the example of Matringe (when* $q \equiv 1 \pmod 4$*) is that ours is a discrete parameter whilst his is not. The parameter constructed in Corollary 3.18 is also not discrete.*

However, the only thing we have really used in our $\mathrm{SO}_6(\mathbb{C})$ *example is the fact that the character* $\chi_{1,L} = \chi_1 \circ \mathrm{N}_{L/E_1}$ *is quadratic. By a similar (but notationally more complicated!) method, one can find a parameter* φ *for* $\mathrm{SO}_{2(2N+1)}(F)$ *such that* $\wedge_+^{2N+1} \circ \varphi \cong \wedge_-^{2N+1} \circ \varphi$*, so that one also sees the non-acceptability of* $\mathrm{SO}_{2(2N+1)}(\mathbb{C})$ *manifested by a Langlands parameter. One need only add to* ρ_{χ_1} *a sum of (pairwise inequivalent) irreducible* 2*-dimensional orthogonal representations of distinct positive depths induced from ramified quadratic extensions (necessarily an odd number from each of the two quadratic ramified extensions, so that the parameter has determinant* 1*).*

Thus it is possible to find non-conjugate discrete parameters which are outer-equivalent for $\mathrm{SO}_{2N}(F)$ *whenever* N *is odd and* $q \equiv 3 \pmod 4$*. We do not give details here, nor investigate the situation for general* $\mathrm{SO}_{2N}(F)$ *further, leaving this to future work, or others.*

4. The exceptional group G_2

Now we turn to the case of the exceptional group G_2. Since there is not yet a direct definition of twisted γ-factors on the automorphic side, we will follow Gan–Savin [GS23b], where they *define* these γ-factors to be those of the corresponding Langlands parameter, via their local Langlands correspondence (see [GS23a]).

4.1. **The converse theorem for parameters for** $\mathrm{G}_2(F)$**.** We consider first a Langlands parameter $\varphi : WD_F \to \mathrm{G}_2(\mathbb{C})$ for G_2 and the standard embedding $\varrho_{\mathrm{std}} : \mathrm{G}_2(\mathbb{C}) \hookrightarrow \mathrm{GL}_7(\mathbb{C})$. Then, for $\tau : W_F \to \mathrm{GL}_r(\mathbb{C})$ an irreducible representation, by definition we have

$$\gamma(s, \varphi \times \tau, \psi) = \gamma(s, (\varrho_{\mathrm{std}} \circ \varphi) \otimes \tau, \psi).$$

As for classical groups, we make the following definition.

Definition 4.1. *Let φ_1, φ_2 be Langlands parameters for $\mathrm{G}_2(F)$ and let $m \geq 1$ be an integer. We say that φ_1, φ_2 are* γ-equivalent to level m *if*

$$\gamma(s, \varphi_1 \times \tau, \psi) = \gamma(s, \varphi_2 \times \tau, \psi)$$

for all irreducible irreducible r-dimensional representations τ of W_F, with $1 \leq r \leq m$.

The following theorem seems to be implicit in [GS23b], and see also [Mat24, Theorem 8.2(b)].

Theorem 4.2. *Let φ_1, φ_2 be two Langlands parameters for $\mathrm{G}_2(F)$. If φ_1, φ_2 are γ-equivalent to level 3, then $\varphi_1 \cong \varphi_2$.*

We will see in the next subsection how our results for classical groups can be used to show that this is the best possible, at least when $p > 3$.

Proof. By [Mat24, Theorem 8.2(b)], the hypotheses imply that $\varrho_{\mathrm{std}} \circ \varphi_1, \varrho_{\mathrm{std}} \circ \varphi_2$ are conjugate under the normalizer in $\mathrm{GL}_7(\mathbb{C})$ of $\varrho_{\mathrm{std}}(\mathrm{G}_2(\mathbb{C}))$; since this normalizer is $\mathbb{C}^\times \varrho_{\mathrm{std}}(\mathrm{G}_2(\mathbb{C}))$ (see the proof in *loc. cit.*), we deduce that they are in fact conjugate under $\varrho_{\mathrm{std}}(\mathrm{G}_2(\mathbb{C}))$, so that the parameters φ_1, φ_2 are $\mathrm{G}_2(\mathbb{C})$-conjugate, as required. □

Remark 4.3. *As the referee has pointed out to us, Theorem 4.2 also follows from the fact that ϱ_{std} factors through the inclusion $\mathrm{G}_2(\mathbb{C}) \hookrightarrow \mathrm{SO}_7(\mathbb{C})$: thus we get an injective map $\Phi(\mathrm{G}_2) \hookrightarrow \Phi(\mathrm{Sp}_6)$ (see* [GS23a, Lemma 2.3(ii)]*) and the result follows from the local converse theorem for Sp_6.*

4.2. **On the optimal bound for parameters for $\mathrm{G}_2(F)$.** For the points $\mathrm{G}(F)$ of a reductive group over F, we write $\Phi_{\mathrm{ds}}(\mathrm{G})$ for the set of discrete Langlands parameters for $\mathrm{G}(F)$. By [GS23a, Lemma 2.4(i)], the inclusion map $\iota : \mathrm{SL}_3(\mathbb{C}) \to \mathrm{G}_2(\mathbb{C})$ induces a map

$$\Phi_{\mathrm{ds}}(\mathrm{PGL}_3) \to \Phi_{\mathrm{ds}}(\mathrm{G}_2) \tag{4.4}$$

whose fibres consist of orbits under the outer automorphism of $\mathrm{SL}_3(\mathbb{C})$. In particular, for $\rho : W_F \to \mathrm{SL}(3, \mathbb{C})$ a parameter for $\mathrm{PGL}_3(F)$, we have

$$\varrho_{\mathrm{std}} \circ \iota \circ \rho = \rho \oplus \mathbf{1} \oplus \rho^\vee.$$

We also recall that an admissible pair $(L/F, \chi)$ gives rise to a parameter for $\mathrm{PGL}_3(F)$ if and only if the induced representation $\rho_\chi = \mathrm{Ind}_{L/F}\, \chi$ has determinant 1; equivalently, $\chi|_{F^\times} = \varkappa_{L/F}$, where $\varkappa_{L/F} = \det(\mathrm{Ind}_{L/F} \mathbf{1}_{L^\times})$.

Theorem 4.5. *Suppose $p > 3$. There exist inequivalent discrete Langlands parameters for $\mathrm{G}_2(F)$ which are γ-equivalent to level 2.*

Proof. We make use of our result on symplectic groups in Theorem 3.5. More precisely, suppose we have found minimal totally ramified admissible pairs $(E/F, \chi_1)$ and $(E/F, \chi_2)$, with E/F a cubic extension, such that

(i) $\chi_i|_{F^\times} = \varkappa_{E/F}$;

(ii) $\chi_1|_{U_E} = \chi_2|_{U_E}$; and
(iii) $(E/F, \chi_1)$ is not equivalent to $(E/F, \chi_2^{\pm 1})$.

Then, writing $\rho_i = \mathrm{Ind}_{E/F}\,\chi_i$, we get a parameter $\varphi_i = \iota \circ \rho_i$ for $\mathrm{G}_2(F)$ and a parameter $\phi_i = \iota_L \circ \rho_i$ for $\mathrm{Sp}_6(F)$ (in the notation of §3.1) such that $\varrho_{\mathrm{std}} \circ \varphi_i \simeq \varrho_{\mathrm{std}} \circ \phi_i$ (where we are abusing notation by using ϱ_{std} denote the standard representation of both $\mathrm{G}_2(\mathbb{C})$ and $\mathrm{SO}_7(\mathbb{C})$). Then Theorem 3.5 implies that ϕ_1, ϕ_2 are γ-equivalent to level 2, so the same is true of φ_1, φ_2. Moreover, the final condition (iii) ensures that φ_1, φ_2 are inequivalent as parameters for $\mathrm{G}_2(F)$.

Now we explain how to find admissible pairs satisfying these conditions. We take E/F to be totally ramified of degree 3, with uniformizer ϖ_E such that $\varpi_E^3 = \varpi_F$ lies in F. Since $p > 3$, this is a tamely ramified extension and, by Lemma 2.15, the character $\varkappa_{E/F}$ is unramified of order dividing 2. We choose any non-trivial character of U_E^{0+} which is trivial on $U_E^{(1/3)+}$ (for example, represented by ϖ_E^{-1}); since $U_F^1 \subset U_E^{(1/3)+}$, this is trivial on U_F^1 so we can extend it to a character χ of $F^\times U_E^{0+}$ by requiring $\chi|_{F^\times} = \varkappa_{E/F}$.

Finally, we extend χ to two distinct characters χ_1, χ_2 of $E^\times$. Any such extension must send ϖ_E to a cube root of $\varkappa_{E/F}(\varpi_F) = \pm 1$. So we set $\chi_1(\varpi_E) = \varkappa_{E/F}(\varpi_F)$ and $\chi_2(\varpi_E) = \zeta \varkappa_{E/F}(\varpi_F)$, where $\zeta \in \mathbb{C}^\times$ is a primitive cube root of unity.

It remains only to check the final condition (iii). But if $\sigma \in \mathrm{Aut}(E/F)$ then $\sigma(\varpi_E)$ differs from ϖ_E by a cube root of unity in U_F, on which $\varkappa_{E/F}$ is trivial; hence we deduce ${}^\sigma\chi_1(\varpi_E) = \chi_1(\varpi_E) \neq \chi_2(\varpi_E)^{\pm 1}$. □

Remark 4.6. *The representations ρ_1, ρ_2 in the proof of Theorem 4.5 have the further property that $\rho_1 \otimes \rho_1^\vee \simeq \rho_2 \otimes \rho_2^\vee$. Indeed, if E/F is Galois, with automorphism group generated by σ, then*

$$\rho_i \otimes \rho_i^\vee \simeq \bigoplus_{j=0}^{2} \mathrm{Ind}_{E/F}\, {}^{\sigma^j}\chi_i \chi_i^{-1};$$

but, for fixed j, the characters ${}^{\sigma^j}\chi_i\chi_i^{-1}$ certainly coincide on U_E by construction, while we have seen that ${}^\sigma\chi_i(\varpi_E) = \chi_i(\varpi_E)$ so that ${}^{\sigma^j}\chi_i\chi_i^{-1}$ are both trivial on ϖ_E. On the other hand, if E/F is not Galois then, writing L/F for the unramified quadratic extension and EL for the compositum of E and L, we have

$$\rho_i \otimes \rho_i^\vee \simeq \mathrm{Ind}_{E/F}\, \mathbf{1} \oplus \mathrm{Ind}_{EL/F}\left({}^\sigma\chi_{i,EL}\chi_{i,EL}^{-1}\right),$$

where $\chi_{i,EL} = \chi_i \circ \mathrm{N}_{EL/E}$ and ${}^\sigma\chi_{i,EL} = {}^\sigma\chi_i \circ \mathrm{N}_{EL/\sigma(E)}$. Again, the characters ${}^\sigma\chi_{i,EL}\chi_{i,EL}^{-1}$ coincide on U_{EL} by construction, and it is easy to check they are trivial on the uniformizer ϖ_E of EL.

Now

$$\wedge^2 \circ \varrho_{\mathrm{std}} \circ \varphi_i \simeq \wedge^2 \left(\rho_i \oplus \mathbf{1} \oplus \rho_i^\vee\right) \simeq \rho_i^{\oplus 2} \oplus \left(\rho_i \otimes \rho_i^\vee\right) \oplus (\rho_i^\vee)^{\oplus 2},$$

where we have used that $\wedge^2\rho_i \simeq \rho_i^\vee$, since ρ_i is 3-dimensional and has trivial determinant. In particular, from Theorem 4.5 and the isomorphism $\rho_1 \otimes \rho_1^\vee \simeq \rho_2 \otimes \rho_2^\vee$, we *deduce that*

$$\gamma(s, (\wedge^2 \circ \varrho_{\mathrm{std}} \circ \varphi_1) \otimes \tau, \psi) = \gamma(s, (\wedge^2 \circ \varrho_{\mathrm{std}} \circ \varphi_2) \otimes \tau, \psi),$$

for all irreducible r-dimensional representations of W_F with $1 \leq r \leq 2$. Moreover, since $\wedge^2 \circ \varrho_{\mathrm{std}} \simeq \varrho_{\mathrm{std}} \oplus \mathrm{Ad}$ is the sum of the two fundamental representations of $\mathrm{G}_2(\mathbb{C})$, multiplicativity of γ-factors implies that

$$\gamma(s, (\varrho \circ \varphi_1) \otimes \tau, \psi) = \gamma(s, (\varrho \circ \varphi_2) \otimes \tau, \psi),$$

for all fundamental representations ϱ of $\mathrm{G}_2(\mathbb{C})$ and all irreducible r-dimensional representations of W_F with $1 \leq r \leq 2$.

Thus the necessity of twisting by 3-dimensional representations of W_F to distinguish parameters is not an artefact of only considering the standard representation of $\mathrm{G}_2(\mathbb{C})$.

4.3. **The optimal local converse theorem for G_2.** Finally, we interpret the results of the previous sections in terms of irreducible representations of $\mathrm{G}_2(F)$. We have, from [GS23a] (see also [HKT23] and [AX23]) a local Langlands correspondence for $\mathrm{G}_2(F)$; for π an irreducible representation of $\mathrm{G}_2(F)$, we write $\varphi_\pi : WD_F \to \mathrm{G}_2(\mathbb{C})$ for the corresponding Langlands parameter, and by Π_π the L-packet of π, which contains a unique generic representation. For τ an irreducible representation of $\mathrm{GL}_m(F)$ with Langlands parameter φ_τ, as in [GS23b] we *define*

$$\gamma(s, \pi \rtimes \tau, \varrho, \psi) := \gamma(s, (\varrho \circ \varphi_\pi) \otimes \varphi_\tau, \psi),$$

for ϱ any algebraic representation of $\mathrm{G}_2(\mathbb{C})$.

Writing $\mathrm{Irr}^{\mathrm{gen}}_{\mathrm{sc}}(\mathrm{G}(F))$ for the set of equivalence classes of irreducible generic supercuspidal representations of the points $\mathrm{G}(F)$ of a reductive group, the map in (4.4) corresponds to a map

$$\mathrm{Irr}^{\mathrm{gen}}_{\mathrm{sc}}(\mathrm{PGL}_3) \to \mathrm{Irr}^{\mathrm{gen}}_{\mathrm{sc}}(\mathrm{G}_2),$$

which is given by an exceptional theta correspondence, and is in fact part of how the Langlands correspondence is defined (see also [SW11, Theorem 1.8]). Since the parameters used in the construction in the proof of Theorem 4.5 are indeed for supercuspidal representations of $\mathrm{PGL}_3(F)$, the previous results of this section (also using Remark 4.6) translate to give:

Theorem 4.7. (i) *Let π_1, π_2 be irreducible generic representations of $\mathrm{G}_2(F)$ which are γ-equivalent to level 3. Then $\pi_1 \simeq \pi_2$.*

(ii) *Suppose $p > 3$. Then there are inequivalent generic supercuspidal representations π_1, π_2 of $\mathrm{G}_2(F)$ such that*

$$\gamma(s, \pi_1 \rtimes \tau, \varrho, \psi) := \gamma(s, \pi_2 \rtimes \tau, \varrho, \psi),$$

for ϱ either fundamental representation of $\mathrm{G}_2(\mathbb{C})$, and all supercuspidal representations of $\mathrm{GL}_r(F)$, with $1 \leq r \leq 2$; in particular, they are γ-equivalent to level 2.

5. Ramakrishnan conjecture

In this final section, we show that we also cannot use only γ-functions of character twists to determine a supercuspidal representation of $\mathrm{GL}_N(F)$. We recall that, in [YZ22, Conjecture 1.2], it is conjectured that if π_1, π_2 are unitarizable supercuspidal representations of $\mathrm{GL}_N(F)$ such that

$$\gamma(s, \pi_1 \rtimes \eta, \wedge^i, \psi) = \gamma(s, \pi_2 \rtimes \eta, \wedge^i, \psi),$$

for $i = 1, \dots, \left\lfloor \frac{N}{2} \right\rfloor$ and η any character of $F^\times$, then $\pi_1 \cong \pi_2$. This is of course true for $N = 2, 3$ since it is then simply the Local Converse Theorem for $\mathrm{GL}_N(F)$. On the other hand, for $N \geq 6$ the requisite γ-functions have not yet been defined on the automorphic side of the Langlands correspondence so we interpret these γ-factors on the Galois side as for G_2 and as in the introduction.

We show here that this conjecture is false already for $\mathrm{GL}_4(F)$, when p is odd. Indeed, we give examples of inequivalent irreducible 4-dimensional representations φ_1, φ_2 of W_F, corresponding to irreducible unitarizable supercuspidal representations of $\mathrm{GL}_4(F)$, such that

$$\gamma(s, (\wedge^i \circ \varphi_1) \otimes \eta, \psi) = \gamma(s, (\wedge^i \circ \varphi_2) \otimes \eta, \psi), \tag{5.1}$$

for $i = 1, \dots, 4$ and all characters η of W_F.

First, we remark that if a supercuspidal representation of $\mathrm{GL}_N(F)$ has unitary central character, then it is unitarizable (see [KT, p.4]). We also remark that if $(E/F, \chi)$ is an admissible pair, then its corresponding supercuspidal representation π_χ via LLC has unitary central character if and only if $\chi|_{F^\times}$ is unitary. Note that $\chi|_{\mathcal{O}_F^\times}$ is automatically unitary since $\mathcal{O}_F^\times$ is compact.

So suppose p is odd and let $(E/F, \chi)$ be any totally ramified unitary admissible pair of degree 4 whose depth d has least denominator 2. For example, we could choose β to be the quasi-minimal element $\varpi_E^{-10} + \varpi_E^{-7}$, where ϖ_E is a uniformizer of E, and take χ to be any unitary quasi-character of $E^\times$ represented by β. Then we take $(E/F, \chi')$ to be the admissible pair such that $\chi^{-1}\chi'$ is unramified of order 2. Then χ, χ' coincide on $F^\times$, and also on β since $4\,\mathrm{val}(\beta)$ is an even integer.

Now note that

$$\rho_\chi \cong \rho_{\chi'} \otimes \eta_8,$$

where η_n denotes an unramified character of W_F of order n (the choice is unimportant), since $\eta_8 \circ \mathrm{N}_{E/F}$ is the unramified quadratic character of $E^\times$. In particular, since also $\rho_\chi \cong \rho_\chi \otimes \eta_4$, we have $\wedge^2 \circ \rho_\chi \cong \wedge^2 \circ \rho_{\chi'}$, and similarly, $\omega_\chi = \wedge^4 \circ \rho_\chi = \wedge^4 \circ \rho_\chi = \omega_{\chi'}$. On the other hand, the pairing $\wedge^3 \circ \rho_\chi \times \wedge^1 \circ \rho_\chi \to \wedge^4 \circ \rho_\chi$ gives us $\wedge^3 \circ \rho_\chi \cong \rho_\chi^\vee \otimes \omega_\chi$.

Thus, putting $\varphi_1 = \rho_\chi$ and $\varphi_2 = \rho_{\chi'}$, to prove (5.1) we are reduced to verifying

$$\gamma(s, \rho_\chi \otimes \eta, \psi) = \gamma(s, \rho_{\chi'} \otimes \eta, \psi) \quad \text{and} \quad \gamma(s, \rho_{\chi^{-1}} \otimes \eta, \psi) = \gamma(s, \rho_{\chi'^{-1}} \otimes \eta, \psi),$$

for all characters η of W_F. But these follow immediately from Proposition 2.10.

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