INDUCTION FOR EXTENDED AFFINE TYPE A SOERGEL BIMODULES: FIRST STEPS

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Abstract. In this paper we take the first steps towards the categorification of the Zelevinsky tensor product of finite dimensional representations of extended affine type A Hecke algebras.

Contents

1. Introduction

This paper is part of our ongoing study of the birepresentation theory of extended affine type A_{n-1} Soergel bimodules in characteristic zero, which we started in [MMV]. The monoidal category of these Soergel bimodules, denoted $\widehat{\mathcal{S}}_n^{\mathrm{ext}}$ in this paper, was studied both algebraically and diagrammatically in [MaTh] and [Eli2], and categorifies the extended affine type A Hecke algebra $\widehat{H}_n^{\mathrm{ext}}$. This algebra is infinite dimensional, which on the categorical level corresponds to the fact that $\widehat{\mathcal{S}}_n^{\mathrm{ext}}$ is not finitary but *wide finitary*, in the terminology of [Macph]. Whereas finitary birepresentation theory of finitary monoidal categories/bicategories, such as Soergel bimodules of finite Coxeter type, is fairly well developed, see e.g. [MMMTZ1] and [MMMTZ2] and references therein, birepresentation theory of wide finitary monoidal categories/bicategories is still in its infancy. Some fundamental results in finitary birepresentation theory were generalized to the wide finitary setting by Macpherson in [Macph], but his framework does not cover triangulated birepresentations, which play a prominent role in the birepresentation theory of $\widehat{\mathcal{S}}_n^{\mathrm{ext}}$. For example, the evaluation birepresentations of $\widehat{\mathcal{S}}_n^{\mathrm{ext}}$ in [MMV] are triangulated. In this paper, we continue our study of the birepresentation theory of $\widehat{\mathcal{S}}_n^{\mathrm{ext}}$, taking a first step towards the categorification of parabolic induction. As the reader will see, this also involves triangulated birepresentations.

Parabolic induction and restriction play an important role in the finite dimensional representation theory of $\widehat{H}_n^{\mathrm{ext}}$. In particular, for any integers $1 \leq k < n$, there is a well-known embedding of algebras

$$\psi_{k,n-k}\colon \widehat{H}_k^{\mathrm{ext}} \otimes \widehat{H}_{n-k}^{\mathrm{ext}} \to \widehat{H}_n^{\mathrm{ext}}, \tag{1}$$

which is the analog of the embedding of finite symmetric groups $\mathfrak{S}_k \times \mathfrak{S}_{n-k} \to \mathfrak{S}_n$. Given two finite dimensional representations M_1 and M_2 of $\widehat{H}_k^{\mathrm{ext}}$ and $\widehat{H}_{n-k}^{\mathrm{ext}}$, respectively, one can use $\psi_{k,n-k}$ to define their *Zelevinsky tensor product*

$$M_1 \odot M_2 := \mathrm{Ind}_{\widehat{H}_k^{\mathrm{ext}} \otimes \widehat{H}_{n-k}^{\mathrm{ext}}}^{\widehat{H}_n^{\mathrm{ext}}} M_1 \otimes M_2, \tag{2}$$

which is a finite dimensional $\widehat{H}_n^{\mathrm{ext}}$-representation. Zelevinsky [Zel] classified the finite dimensional irreducible representations of the extended affine type A Hecke algebra in terms of combinatorial objects called multisegments. Leclerc, Nazarov and Thibon [LNT] gave a necessary and sufficient condition for irreducibility of $M_1 \odot M_2$, based on the multisegments corresponding to two finite dimensional irreducible representations M_1 and M_2.

Now, let $k_1, k_2, k_3 \in \mathbb{Z}_{\geq 1}$ such that $k_1 + k_2 + k_3 = n$ (assuming that $n \geq 3$, of course). Then one can show that

$$\psi_{k_1+k_2,k_3}(\psi_{k_1,k_2} \otimes \mathrm{id}_{\widehat{H}_{k_3}^{\mathrm{ext}}}) = \psi_{k_1,k_2+k_3}(\mathrm{id}_{\widehat{H}_{k_1}^{\mathrm{ext}}} \otimes \psi_{k_2,k_3}), \tag{3}$$

which implies that there is a canonical isomorphism

$$(M_1 \odot M_2) \odot M_3 \cong M_1 \odot (M_2 \odot M_3)\,, \tag{4}$$

with M_i being a finite dimensional representation of $\widehat{H}_{k_i}^{\mathrm{ext}}$ for $i = 1, 2, 3$. If we define the embedding of algebras

$$\psi_{k_1,k_2,k_3} \colon \widehat{H}_{k_1}^{\mathrm{ext}} \otimes \widehat{H}_{k_2}^{\mathrm{ext}} \otimes \widehat{H}_{k_3}^{\mathrm{ext}} \to \widehat{H}_{n}^{\mathrm{ext}}$$

as either one of the two composite maps in (3), then $M_1 \odot M_2 \odot M_3$ can also be defined directly as

$$\mathrm{Ind}_{\widehat{H}_{k_1}^{\mathrm{ext}} \otimes \widehat{H}_{k_2}^{\mathrm{ext}} \otimes \widehat{H}_{k_3}^{\mathrm{ext}}}^{\widehat{H}_{n}^{\mathrm{ext}}} M_1 \otimes M_2 \otimes M_3,$$

which by the above is canonically isomorphic to either one of the two isomorphic representations in (4).

More generally, let $k_1, \ldots, k_m \in \mathbb{Z}_{\geq 1}$ such that $1 \leq m \leq n$ and $k_1 + \ldots + k_m = n$. Then there is an embedding of algebras

$$\psi_{k_1,\ldots,k_m} \colon \widehat{H}_{k_1}^{\mathrm{ext}} \otimes \cdots \otimes \widehat{H}_{k_m}^{\mathrm{ext}} \to \widehat{H}_{n}^{\mathrm{ext}}, \tag{5}$$

which can be used to define the corresponding Zelevinsky tensor product

$$M_1 \odot \cdots \odot M_m,$$

with M_i being a finite dimensional representation of $\widehat{H}_{k_i}^{\mathrm{ext}}$ for $i = 1, \ldots, m$, and this tensor product is associative up to canonical isomorphism. For more information on the Zelevinsky tensor product and its role in representation theory, see [LNT] and references therein.

In this paper, we initiate the categorification of the Zelevinsky tensor product. Specifically, we define a linear, monoidal functor

$$\Psi_{k,n-k} \colon \widehat{\mathcal{S}}_{k}^{\mathrm{ext}} \boxtimes \widehat{\mathcal{S}}_{n-k}^{\mathrm{ext}} \to K^b(\widehat{\mathcal{S}}_{n}^{\mathrm{ext}}) \tag{6}$$

which categorifies the embedding $\psi_{k,n-k}$ in (1). Here $K^b(\widehat{\mathcal{S}}_{n}^{\mathrm{ext}})$ denotes the homotopy category of bounded complexes in $\widehat{\mathcal{S}}_{n}^{\mathrm{ext}}$. Before continuing, we should note that $\psi_{k,n-k}$ is usually defined in terms of the Bernstein presentation of the extended affine type A Hecke algebra. Unfortunately, there is currently no categorification of $\widehat{H}_{n}^{\mathrm{ext}}$ based on that presentation. The decategorification of $\widehat{\mathcal{S}}_{n}^{\mathrm{ext}}$ is naturally associated to the Kazhdan-Lusztig presentation of $\widehat{H}_{n}^{\mathrm{ext}}$, so we define $\psi_{k,n-k}$ in terms of that presentation and use it as the starting point for the definition of $\Psi_{k,n-k}$ in this paper. Note that in [StWe], the authors use the categorification of parabolic induction for finite type A Soergel bimodules, where the analog of $\Psi_{k,n-k}$ is much easier to define and does not involve Rouquier complexes.

The way to categorify the Zelevinsky tensor product using $\Psi_{k,n-k}$ is a bit roundabout, because we do not know how to make sense of something like

$$\widehat{\mathcal{S}}_{n}^{\mathrm{ext}} \boxtimes_{\widehat{\mathcal{S}}_{k}^{\mathrm{ext}} \boxtimes \widehat{\mathcal{S}}_{n-k}^{\mathrm{ext}}} \mathbf{M}_1 \boxtimes \mathbf{M}_2,$$

where $\mathbf{M}_1$ and $\mathbf{M}_2$ are finitary birepresentations of $\widehat{\mathcal{S}}_{k}^{\mathrm{ext}}$ and $\widehat{\mathcal{S}}_{n-k}^{\mathrm{ext}}$, respectively. This would require an analog of the balanced box tensor product for module categories over finite tensor categories, defined in [DaNi, Section 2.7], which does not (yet) exist in our setting. Therefore, we follow a different approach, explained below. We first sketch the general idea and then point out some technical hurdles. Recall that, for a fiat bicategory $\mathcal{C}$ (e.g. a linear additive pivotal category satisfying certain finiteness conditions), there is a correspondence between transitive finitary birepresentations of $\mathcal{C}$ and algebra 1-morphisms in (the abelianization of) $\mathcal{C}$, see [MMMT, Corollary 4.8]. Note that in that same paper, and subsequent ones like [MMMTZ2], the correspondence was first formulated with coalgebra 1-morphisms, instead of algebra 1-morphisms. For technical reasons, we prefer algebra 1-morphisms in this paper, see the comments below. Moreover, the bicategories in this paper have only one object and are hence monoidal categories, thus we speak about algebra objects. Any monoidal functor $\mathcal{C} \to \mathcal{D}$ between monoidal categories maps algebra objects to algebra objects and can, therefore, be used to induce birepresentations of $\mathcal{C}$ to birepresentations of $\mathcal{D}$, which is how we would like to go about categorifying the Zelevinsky tensor product. Unfortunately, $\widehat{\mathbb{S}}_n^{\mathrm{ext}}$ is not finitary but wide finitary, and $\Psi_{k,n-k}$ does not take values in $\widehat{\mathbb{S}}_n^{\mathrm{ext}}$ but in $K^b(\widehat{\mathbb{S}}_n^{\mathrm{ext}})$.

In [Macph], a correspondence between wide finitary birepresentations and (co)algebra objects in some completion is given, but there is currently no analog for triangulated birepresentations. In the examples of finitary or triangulated birepresentations of $\widehat{\mathbb{S}}_n^{\mathrm{ext}}$ that we fully understand, the algebra objects are typically given by countable coproducts of tensor products of Soergel bimodules and Rouquier complexes. Hence we introduce certain cocompletions of additive and triangulated monoidal categories/bicategories containing these kinds of algebra objects/1-morphisms in Section 5.2, which are smaller than the completions considered in [Macph]. To explain our preference for algebra objects over coalgebra objects in this paper, note that algebra objects typically belong to cocompletions whereas coalgebra objects typically belong to completions. As can be seen from our calculations in Section 5.3, cocompletions are easier to work with in practice. In Section 5.3 we work out the explicit example of the categorified Zelevinsky tensor product $\mathbf{W} := \mathbf{V} \boxdot \mathbf{V}$ of the trivial finitary birepresentation $\mathbf{V}$ of $\widehat{\mathbb{S}}_1^{\mathrm{ext}}$ with itself. Its construction is technically quite involved. We first define and analyse a wide finitary $\widehat{\mathbb{S}}_2^{\mathrm{ext}}$-birepresentation $\mathbf{U}$ corresponding to the algebra object $\mathrm{Y} := \Psi_{1,1}(\mathrm{X} \boxtimes \mathrm{X}))$ in the cocompletion of $K^b(\widehat{\mathbb{S}}_2^{\mathrm{ext}})$, where X is the algebra object in the cocompletion of $\widehat{\mathbb{S}}_1^{\mathrm{ext}}$ corresponding to $\mathbf{V}$. Following the terminology introduced in [MMV], we call $\mathbf{U}$ a *wide finitary cover* of $\mathbf{W}$. To obtain a triangulated birepresentation, we then observe that Y induces a right action of $\mathbb{Z}^2$ on $K^b(\widehat{\mathbb{S}}_2^{\mathrm{ext}})$ which commutes with the left action of $\widehat{\mathbb{S}}_2^{\mathrm{ext}}$, so the orbit category Ω inherits the structure of a $\widehat{\mathbb{S}}_2^{\mathrm{ext}}$ birepresentation. We prove that $\mathbf{U}$ is equivalent to the full subcategory of Ω whose objects can be identified with those of $\widehat{\mathbb{S}}_2^{\mathrm{ext}}$ under the usual embedding into $K^b(\widehat{\mathbb{S}}_2^{\mathrm{ext}})$. In general, the orbit category of a triangulated category under a group action does not inherit a natural triangulated structure, but in this case we can relate it (via a technical detour which we will explain in Section 5.3.2) to a triangulated orbit category using a dg-enhancement as in [FKQ, Theorem/Definition 1.1]. By construction, this triangulated orbit category is a $\widehat{\mathbb{S}}_2^{\mathrm{ext}}$ birepresentation and we conjecture that its triangulated Grothendieck group is isomorphic to W, see Conjecture 5.29. It seems likely that this approach can be generalized, but we do not develop a general theory of these kinds of birepresentations and the corresponding algebra objects/1-morphisms in this paper, leaving that for future work.

Something else beyond the scope of this paper is the generalization of $\Psi_{k,n-k}$ to a linear monoidal functor

$$\Psi_{k_1,\dots,k_m} \colon \widehat{\mathbb{S}}^{\mathrm{ext}}_{k_1} \boxtimes \cdots \boxtimes \widehat{\mathbb{S}}^{\mathrm{ext}}_{k_m} \to K^b(\widehat{\mathbb{S}}^{\mathrm{ext}}_n)$$

for $k_1,\dots,k_m \in \mathbb{Z}_{\geq 1}$ such that $1 \leq m \leq n$ and $k_1 + \ldots + k_m = n$, categorifying (5). Conceptually it is clear how to define such a generalization, but we postpone the lengthy technical details to a future paper. Finally, to prove the categorical analog of (3), i.e., the existence of a natural isomorphism

$$\Psi_{k_1+k_2,k_3}(\Psi_{k_1,k_2} \boxtimes \mathrm{Id}_{\widehat{\mathbb{S}}^{\mathrm{ext}}_{k_3}}) \cong \Psi_{k_1,k_2+k_3}(\mathrm{Id}_{\widehat{\mathbb{S}}^{\mathrm{ext}}_{k_1}} \boxtimes \Psi_{k_2,k_3})$$

for $1 \leq k_1, k_2, k_3 \leq n$ such that $k_1 + k_2 + k_3 = n$, we would first need to extend $\Psi_{k,n-k}$ to a monoidal functor (with the same notation)

$$\Psi_{k,n-k} \colon K^b(\widehat{\mathbb{S}}^{\mathrm{ext}}_k) \boxtimes K^b(\widehat{\mathbb{S}}^{\mathrm{ext}}_{n-k}) \to K^b(\widehat{\mathbb{S}}^{\mathrm{ext}}_n).$$

Proving the existence of such an extension is a non-trivial problem and is related to similar extension problems in other contexts, see [ALELR, Conjecture 1.2], [Eli2, Section 1.6], [ElHo] and [MMV, Section 1]. The solution of this problem is also a topic for future research.

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2. Parabolic embedding: decategorified story

Throughout the paper, let $n \in \mathbb{Z}_{\geq 1}$. For $n = 1$, the *affine Weyl group* $\widehat{\mathfrak{S}}_1$ of type $\widehat{A}_0$ is the trivial group. Similarly, the finite Weyl group $\mathfrak{S}_1$ of type A_0 is the trivial group.

For $n \geq 2$, let $\widehat{I} := \mathbb{Z}/n\mathbb{Z}$ and $I := \{1,\dots,n-1\}$. By a slight abuse of notation, we will often identify $\widehat{I}$ with the set of representatives $\{0,1,\dots,n-1\}$ and consider I as a subset of $\widehat{I}$.

For $n = 2$, the *affine Weyl group* $\widehat{\mathfrak{S}}_2$ of type $\widehat{A}_1$ is generated by the simple reflections s_0, s_1, subject to the relations

$$s_i^2 = 1,$$

for $i \in \widehat{I}$.

For $n > 2$, the *affine Weyl group* $\widehat{\mathfrak{S}}_n$ of type $\widehat{A}_{n-1}$ is generated by the simple reflections $s_i,\ i \in \widehat{I}$, subject to the relations

$$s_i^2 = 1, \qquad s_i s_j = s_j s_i \ \text{if} \ |i-j| > 1, \qquad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},$$

for $i \in \widehat{I}$.

A *reduced expression (rex)* for an element $w \in \widehat{\mathfrak{S}}_n$ is a finite word $(s_{i_1},\dots,s_{i_m})$ of simple reflections, for some $m \in \mathbb{Z}_{\geq 0}$, such that $w = s_{i_1} \cdots s_{i_m}$ and there is no word consisting of fewer simple reflections with that property. All rexes for w contain the same number of simple reflections, which is called the length of w and denoted by $\ell(w)$. By definition, the rex for the neutral element is the empty word of length zero (i.e. $m=0$).

The *extended* affine Weyl group $\widehat{\mathfrak{S}}_n^{\mathrm{ext}}$, using the weight lattice of GL_n, is the semidirect product

$$\langle\rho\rangle \ltimes \widehat{\mathfrak{S}}_n,$$

where $\langle\rho\rangle$ is an infinite cyclic group generated by ρ and

$$\rho s_i \rho^{-1} = s_{i+1},$$

for $i\in\widehat{I}$. In particular, when $n=1$, the extended affine Weyl group $\widehat{\mathfrak{S}}_1^{\mathrm{ext}}$ is just the infinite cyclic group $\langle\rho\rangle$.

The finite Weyl group of type A_{n-1} is the symmetric group on n letters, $\mathfrak{S}_n$, corresponding to the subgroup of $\widehat{\mathfrak{S}}_n$ generated by $s_i,\ i\in I$.

2.1. **Hecke algebras.** Let q be a formal parameter. For $n=1$, the *extended affine Hecke algebra* $\widehat{H}_1^{\mathrm{ext}}$ is simply the group algebra of $\langle\rho\rangle$ over $\mathbb{Z}[q,q^{-1}]$.

For $n=2$, the *extended affine Hecke algebra* $\widehat{H}_n^{ext}$ is the $\mathbb{Z}[q,q^{-1}]$-algebra generated by T_0, T_1, and $\rho^{\pm 1}$, subject to the relations

$$(T_i+q)(T_i-q^{-1})=0, \qquad \rho\rho^{-1}=1=\rho^{-1}\rho, \qquad \rho T_i\rho^{-1}=T_{i+1}, \tag{7}$$

for $i\in\widehat{I}$.

For $n>2$, the *extended affine Hecke algebra* $\widehat{H}_n^{ext}$ is the $\mathbb{Z}[q,q^{-1}]$-algebra generated by $T_i,\ i\in\widehat{I}$, and $\rho^{\pm 1}$, subject to the relations

$$(T_i+q)(T_i-q^{-1})=0, \qquad T_iT_j=T_jT_i \ \text{ if } \ |i-j|>1, \qquad T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1}, \tag{8}$$

$$\rho\rho^{-1}=1=\rho^{-1}\rho, \qquad \rho T_i\rho^{-1}=T_{i+1}, \tag{9}$$

for $i,j\in\widehat{I}$. Note that T_i is invertible for every $i\in\widehat{I}$:

$$T_i^{-1}=T_i+q-q^{-1}.$$

As is well-known, $\widehat{H}_n^{ext}$ is a q-deformation of the group algebra $\mathbb{Z}[\widehat{\mathfrak{S}}_n^{\mathrm{ext}}]$ with the *standard basis* given by $\{\rho^m T_w \mid m\in\mathbb{Z}, w\in\widehat{\mathfrak{S}}_n\}$, where $T_w := T_{i_1}\cdots T_{i_\ell}$ for any *reduced expression* (rex) $s_{i_1}\cdots s_{i_\ell}$ of w.

Another presentation of $\widehat{H}_n^{ext}$ is given in terms of the *Kazhdan–Lusztig generators* $b_i := T_i + q$, for $i\in\widehat{I}$, and $\rho^{\pm 1}$, subject to the relations

$$b_i^2=[2]b_i, \qquad \rho\rho^{-1}=1=\rho^{-1}\rho, \qquad \rho b_i\rho^{-1}=b_{i+1}, \tag{10}$$

for $n=2$, and

$$b_i^2=[2]b_i, \qquad b_ib_j=b_jb_i \ \text{ if } \ |i-j|>1, \qquad b_ib_{i+1}b_i+b_{i+1}=b_{i+1}b_ib_{i+1}+b_i, \tag{11}$$

$$\rho\rho^{-1}=1=\rho^{-1}\rho, \qquad \rho b_i\rho^{-1}=b_{i+1}, \tag{12}$$

for $n > 2$, where $i \in \widehat{I}$ and $[2] := q + q^{-1}$. Note that $T_i = b_i - q$ and $T_i^{-1} = b_i - q^{-1}$, for every $i \in \widehat{I}$. The *Kazhdan–Lusztig basis* is given by $\{\rho^m b_w \mid m \in \mathbb{Z}, w \in \widehat{\mathfrak{S}}_n\}$, where the definition of b_w requires the choice of a rex for w but is independent of that choice.

For $n = 1$, the (non-extended) *affine Hecke algebra* $\widehat{H}_1$ is the trivial one-dimensional algebra isomorphic to $\mathbb{Z}[q, q^{-1}]$. For $n \geq 2$, the (non-extended) *affine Hecke algebra* $\widehat{H}_n$ is the subalgebra of $\widehat{H}_n^{ext}$ generated by either T_i or b_i, for $i \in \widehat{I}$.

For $n = 1$, the *finite Hecke algebra* H_1 is also the trivial one-dimensional algebra isomorphic to $\mathbb{Z}[q, q^{-1}]$. For $n \geq 2$, the *finite Hecke algebra* H_n is the $\mathbb{Z}[q, q^{-1}]$-subalgebra of $\widehat{H}_n$ generated by either T_i or b_i, for $i \in I$.

There is a third presentation of $\widehat{H}_n^{ext}$, called the *Bernstein presentation*. In that presentation, $\widehat{H}_n^{ext}$ is defined as a twisted tensor product of H_n and the algebra of Laurent polynomials in n indeterminates with coefficients in $\mathbb{Z}[q, q^{-1}]$. There are several possible choices for defining the commutation relations. The one we use here has indeterminates $y_1, \ldots, y_n$, with relations given by

$$T_i^{-1} y_i T_i^{-1} = y_{i+1}, \tag{13}$$

for $i \in I$.

The relation between this Bernstein presentation and our first presentation of $\widehat{H}_n^{ext}$ is given by

$$y_1 = \rho T_{n-1} \cdots T_2 T_1, \tag{14}$$

$$y_i = T_{i-1}^{-1} \cdots T_2^{-1} T_1^{-1} \rho T_{n-1} \cdots T_{i+1} T_i, \quad i = 2, \ldots, n-1. \tag{15}$$

Remark 2.1. Some remarks about the various conventions in the literature are in order. We try to follow conventions close to those in [Eli2]. Our presentation of the extended affine Hecke algebra in Section 2.1 agrees with [Eli2], as does the relation between the standard generators and the Kazhdan–Lusztig generators.

The *standard trace* $\epsilon\colon \widehat{H}_n^{\text{ext}} \to \mathbb{Z}[q, q^{-1}]$, which also plays an important role in this paper, is the $\mathbb{Z}[q, q^{-1}]$-linear map defined by

$$\epsilon(\rho^r T_w) = \delta_{r,0} \delta_{w,e}, \tag{16}$$

for $r \in \mathbb{Z}$ and $w \in \widehat{\mathfrak{S}}_n$, where $\delta_{-,-}$ is the Kronecker delta. The trace induces a q-sesquilinear form $(-,-)$ on $\widehat{H}_n^{\text{ext}}$ defined by

$$(x, y) := \epsilon(\omega(x) y), \tag{17}$$

where ω is the q-antilinear antiinvolution on $\widehat{H}_n^{\text{ext}}$ defined by $\omega(\rho) = \rho^{-1}$ and $\omega(T_w) = T_w^{-1}$, for $w \in \widehat{\mathfrak{S}}_n$. By definition, q-sesquilinear means that $(-,-)$ is $\mathbb{Z}$-bilinear and satisfies $(qx, y) = q^{-1}(x, y) = (x, q^{-1} y)$, for all $x, y \in \widehat{H}_n^{\text{ext}}$. The above definitions imply that $\omega(b_w) = b_{w^{-1}}$ and that the Kazhdan-Lusztig basis is *asymptotically orthonormal* w.r.t. $(-,-)$, see e.g. [EMTW, Theorem 3.21] for non-extended affine type A.

Theorem 2.2. *For all $k,l\in\mathbb{Z}$ and $u,v\in\widehat{\mathfrak{S}}_n$, we have $\left(\rho^k b_u,\rho^l b_v\right)=\delta_{k,l}\,(b_u,b_v)$ and*

$$(b_u,b_v)\in\begin{cases}1+q\mathbb{Z}[q], & \text{if } u=v;\\ q\mathbb{Z}[q], & \text{else.}\end{cases}$$

2.2. **The embedding.** Let $1\leq k\leq n-1$. There are two unital embeddings of $\mathbb{Z}[q,q^{-1}]$-algebras $\psi_L\colon \widehat{H}_k^{\mathrm{ext}}\to\widehat{H}_n^{\mathrm{ext}}$ (the left embedding) and $\psi_R\colon \widehat{H}_{n-k}^{\mathrm{ext}}\to\widehat{H}_n^{\mathrm{ext}}$ (the right embedding), which are easily defined in terms of the Bernstein presentation:

$$\psi_L\colon\begin{cases}T_i\mapsto T_i, & i=1,\dots,k-1,\\ y_i\mapsto y_i, & i=1,\dots,k,\end{cases}\qquad \psi_R\colon\begin{cases}T_j\mapsto T_{k+j}, & j=1,\dots,n-k-1,\\ y_j\mapsto y_{k+j}, & j=1,\dots,n-k.\end{cases}$$

The two embeddings give rise to a $\mathbb{Z}[q,q^{-1}]$-linear map

$$\psi_{k,n-k}\colon \widehat{H}_k^{\mathrm{ext}}\otimes\widehat{H}_{n-k}^{\mathrm{ext}}\to\widehat{H}_n^{\mathrm{ext}},$$

defined by

$$\psi_{k,n-k}(a\otimes b):=\psi_L(a)\psi_R(b).$$

By definition, $\psi_{k,n-k}(a\otimes 1)=\psi_L(a)$ and $\psi_{k,n-k}(1\otimes b)=\psi_R(b)$, where we denote the identity element of the various Hecke algebras by the same symbol 1.

It is easy to see that this map is a homomorphism of algebras. Note that ψ_L and ψ_R form a *commuting pair of algebra homomorphisms*, in the sense that

$$\psi_L(a)\psi_R(b)=\psi_R(b)\psi_L(a)$$

holds for all $a\in\widehat{H}_k^{\mathrm{ext}}, b\in\widehat{H}_{n-k}^{\mathrm{ext}}$, because the y_m for $1\leq m\leq n$ all commute with each other and we have $T_iT_j=T_jT_i$, $T_iy_j=y_jT_i$ and $T_jy_i=y_iT_j$ for all $1\leq i\leq k-1$ and $k+1\leq j\leq n-1$. Therefore, for all $a_1,a_2\in\widehat{H}_k^{\mathrm{ext}}$ and $b_1,b_2\in\widehat{H}_{n-k}^{\mathrm{ext}}$, we have

$$\psi_L(a_1)\psi_L(a_2)\psi_R(b_1)\psi_R(b_2)=\psi_L(a_1)\psi_R(b_1)\psi_L(a_2)\psi_R(b_2),$$

which means that

$$\psi_{k,n-k}(a_1a_2,b_1b_2)=\psi_{k,n-k}(a_1,b_1)\psi_{k,n-k}(a_2,b_2).$$

Using (14) to translate to the presentation in terms of the standard generators and ρ given in (8) and (9), yields

$$\psi_L(\rho_L)=y_1T_1^{-1}T_2^{-1}\cdots T_{k-1}^{-1}=\rho T_{n-1}\cdots T_k,\tag{18}$$

$$\psi_L(T_0)=\psi_L(\rho_L^{-1}T_1\rho_L)=T_k^{-1}\cdots T_{n-1}^{-1}T_0T_{n-1}\cdots T_k,\tag{19}$$

and

$$\psi_R(\rho_R)=y_{k+1}T_{k+1}^{-1}T_{k+2}^{-1}\cdots T_{n-k-1}^{-1}=T_k^{-1}\cdots T_1^{-1}\rho,\tag{20}$$

$$\psi_R(T_0)=\psi_R(\rho_R^{-1}T_1\rho_R)=T_0\cdots T_{k-1}T_kT_{k-1}^{-1}\cdots T_0^{-1},\tag{21}$$

where we use the notation ρ_L and ρ_R for the twist generators of $\widehat{H}_k^{\text{ext}}$ and $\widehat{H}_{n-k}^{\text{ext}}$, with ρ being used for the generator of $\widehat{H}_n^{\text{ext}}$. The underscript letters L and R are meant to signify "left" and "right" to match their position in the tensor product $\widehat{H}_k^{\text{ext}} \otimes \widehat{H}_{n-k}^{\text{ext}}$.

Since $\psi_L(\rho_L)$ and $\psi_R(\rho_R)$ commute, we have

$$\psi_{k,n-k}(\rho_L, \rho_R) = T_k^{-1} \cdots T_1^{-1} \rho\rho T_{n-1} \cdots T_k = \rho T_{n-1} \cdots T_{k+1} T_{k-1}^{-1} \cdots T_1^{-1} \rho. \tag{22}$$

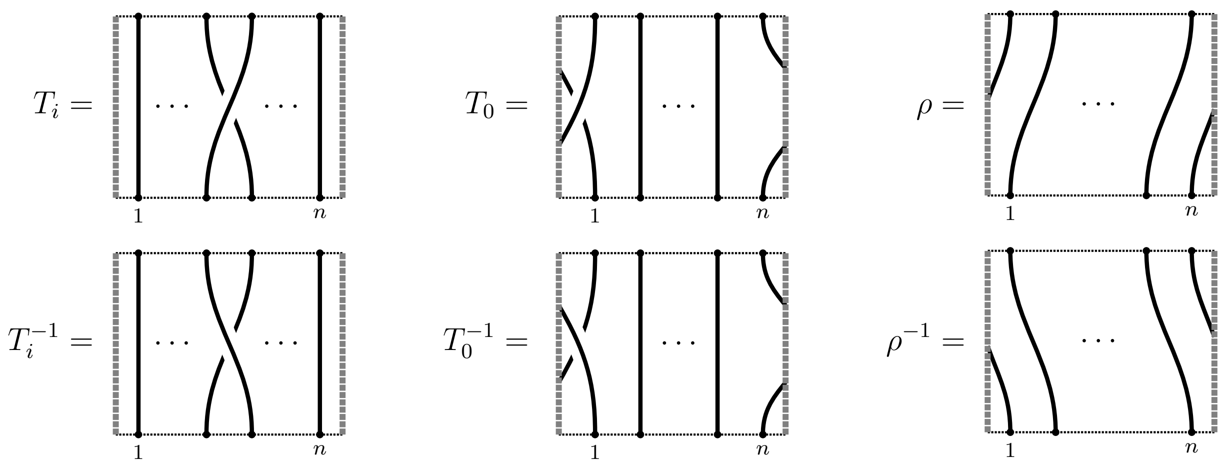
The image of the Kazhdan–Lusztig generators under ψ_L and ψ_R is

$$\psi_L(b_i) = b_i, \quad i = 1, \dots, k-1,$$
$$\psi_R(b_j) = b_{k+j}, \quad j = 1, \dots, n-k-1.$$

Using (19) and (21) it follows at once that

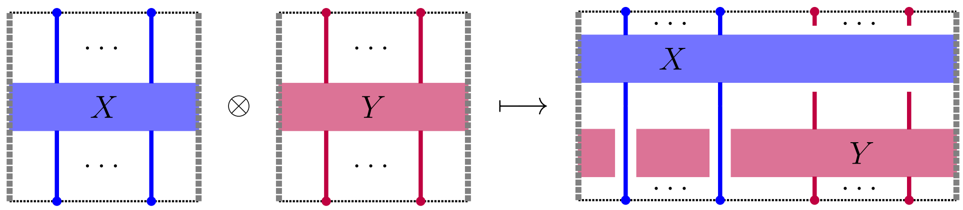
$$\psi_L(b_0) = T_k^{-1} \cdots T_{n-1}^{-1} b_0 T_{n-1} \cdots T_k,$$
$$\psi_R(b_0) = T_0 \cdots T_{k-1} b_k T_{k-1}^{-1} \cdots T_0^{-1}.$$

2.3. **Diagrammatics.** As is well known, the various Hecke algebras described above also have a diagrammatic incarnation, since all of them can be defined as a quotient of the group algebra of the corresponding braid group. In this incarnation, the standard generators correspond to crossings between two neighboring strands. In the affine case, we have to use braid diagrams on a cylinder, which we depict as a rectangle where the two vertical boundary components are identified. In this diagrammatic presentation for $\widehat{H}_n^{\text{ext}}$ we have

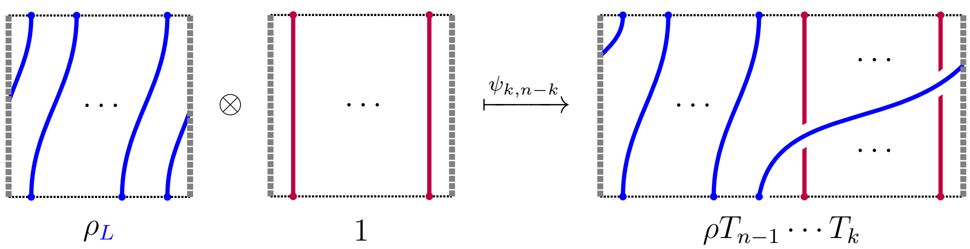


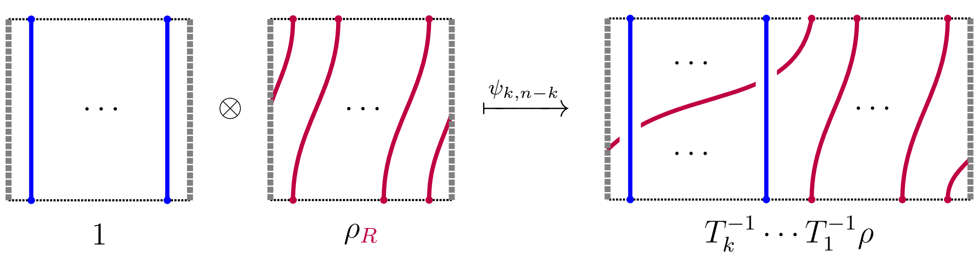
where $i = 1, \dots, n-1$. Note that we don't draw all the vertical lines and that the boundary points at the top, resp. at the bottom, of the braid diagram are cyclically ordered. In our planar presentation we always order them from left to right starting with 1 and ending with n. In our conventions, the product XY consists of gluing the diagram of X atop the one of Y.

The embedding $\psi_{k,n-k}$ can also be described diagrammatically in a natural way. The diagram for $\psi_{k,n-k}(X \otimes Y)$ consists of placing the cyclinder with the diagram for $Y \in \widehat{H}_{n-k}^{\text{ext}}$ inside the cylinder with the diagram for $X \in \widehat{H}_k^{\rm ext}$ and then projecting the outer cylinder over the inner in such a way that the boundary points of the diagram for X are sent to themselves, while the boundary points of the diagram for Y are sent to the points labelled $k+1, \dots, n$. This is easily visualized using colored diagrams: we use blue for the diagrams of the elements of $\widehat{H}_k^{\rm ext}$ and purple for the ones of $\widehat{H}_{n-k}^{\rm ext}$. In this convention, the cylinder with a purple diagram goes inside the one with blue diagram.



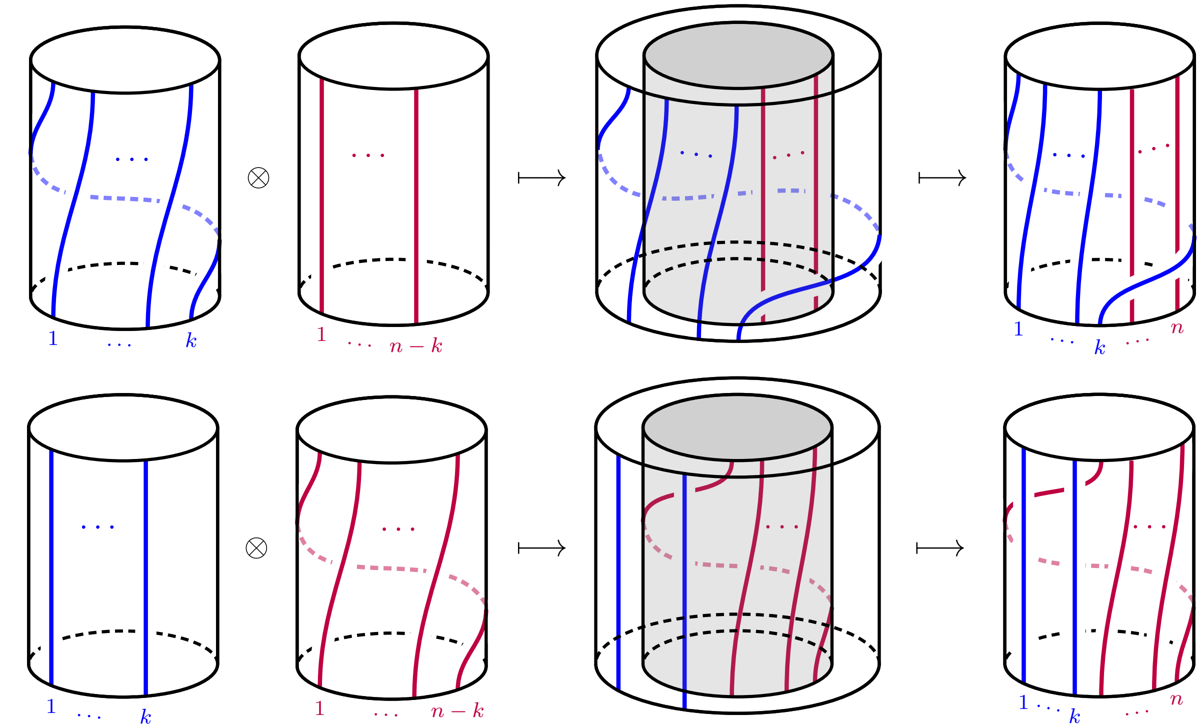
Example 2.3. *The diagrams for $\psi_L(\rho_L) = \psi_{k,n-k}(\rho_L \otimes 1)$ and $\psi_R(\rho_R) = \psi_{k,n-k}(1 \otimes \rho_R)$:*





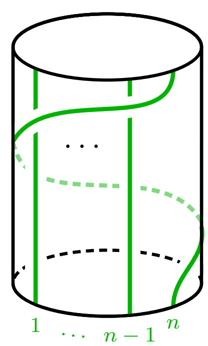
Example 2.4 (Diagrams on a cylinder)**.** *To visualize this product, it might help to draw the inclusions from Example 2.3 using diagrams on a cylinder. For the curious reader we have also*

drawn the diagram for the Bernstein generator y_1.



The Bernstein generator y_1 can be drawn as

$y_1 =$



Note that the equation $\psi_{k,n-k}(\rho_L \otimes 1)\psi_{k,n-k}(1 \otimes \rho_R) = \psi_{k,n-k}(1 \otimes \rho_R)\psi_{k,n-k}(\rho_L \otimes 1)$ translates to "blue and purple sliding past each other" in the corresponding compositions of the diagrams above.

3. Rouquier-Soergel calculus

3.1. **Soergel calculus in extended affine type A and Rouquier complexes.** In this section, we initially revisit the definition of the diagrammatic Soergel category for the extended affine type A as presented in [MaTh] (see also [MMV] and [Eli2]).

Let $\Bbbk$ be an arbitrary field. Our conventions for graded ($\Bbbk$-linear additive) categories and shifts are as in [MMV, §3.1], which match those of [EMTW, Section 4.1]. Since those conventions are important, let us briefly recall them. Throughout this paper, graded will always mean $\mathbb{Z}$-graded. For any $t \in \mathbb{Z}$, the grading shift $\langle t \rangle$ of a graded vector space $M \cong \oplus_{a\in\mathbb{Z}} M_a$ is defined by $(M\langle t\rangle)_a := M_{a+t}$ for all $a \in \mathbb{Z}$. This implies that $\mathrm{hom}(M\langle r\rangle, N\langle t\rangle)$ consists of all homogeneous linear maps between M and N of degree $t-r$. The graded morphism space of all morphisms from M to N is defined as $\mathrm{Hom}(M,N) := \oplus_{t\in\mathbb{Z}}\mathrm{hom}(M, N\langle t\rangle)$. In the categories of this paper, the objects need not be vector spaces, in which case the grading shifts are formal, but the morphism spaces are always vector spaces consisting of (equivalence classes of) linear combinations of diagrams. Each diagram has some degree $s \in \mathbb{Z}$ and can therefore be seen as a morphism between two objects with shifts $M\langle r\rangle$ and $N\langle t\rangle$, such that $s = t - r$, or as a homogeneous morphism between M and N of degree $s = t - r$. To distinguish a non-graded category $\mathcal{C}$ with shift and lower case morphism spaces of the form $\mathrm{hom}(M\langle r\rangle, N\langle t\rangle)$ from the associated graded category with morphism spaces of the form $\mathrm{Hom}(M,N)$, we denote the latter by $\mathcal{C}^*$.

Recall that the *additive envelope* of a $\Bbbk$-linear category $\mathcal{C}$ is the additive, linear category $\mathcal{C}_\oplus$ whose objects are formal direct sums of objects in $\mathcal{C}$ and whose morphisms are matrices of morphisms in $\mathcal{C}$, such that composition is given by matrix multiplication. The *Karoubi envelope* of an additive, linear category is a formal enhancement which results in an idempotent complete category we denote $\mathcal{C}_{\oplus,e}$. For more information on these formal constructions, see e.g. [EMTW, Sections 11.2.2 – 11.2.4].

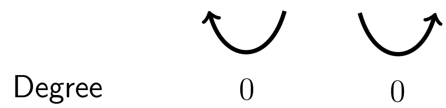
3.1.1. *Soergel calculus in extended affine type A.* The *diagrammatic Bott-Samelson category* of extended type $\widehat{A}_{n-1}$, denoted $\widehat{\mathcal{BS}}_n^{\mathrm{ext}}$, is the $\mathbb{Z}$-graded, $\mathbb{R}$-linear, pivotal category whose objects are grading shifts of finite words in the alphabet $\widehat{I}\cup\{\pm\}$, and whose vector spaces of morphisms are defined below in terms of generating diagrams and relations. Recall that a pivotal category is a monoidal category with duals satisfying certain conditions, see [EGNO, Definition 4.7.8]. We will often identify that alphabet with the corresponding elementary Soergel bimodules $\{\mathrm{B}_i, i \in \widehat{I}\} \cup \{\mathrm{B}_\rho^{\pm 1}\}$, which is justified by [MaTh, Theorem 2.10]. These are bimodules over a polynomial algebra R, defined below (47). Under this identification, a word in the alphabet corresponds to a tensor product of B_i and B_ρ, which is called a *Bott-Samelson bimodule*. The empty word is the identity object, which is identified with R. The tensor product is taken over R, but we will often suppress $\otimes_R$ in our notation, e.g., we will simply write $\mathrm{B}_1\mathrm{B}_2$ for $\mathrm{B}_1 \otimes_R \mathrm{B}_2$. The diagrams then correspond to R-R-bimodule maps, but we will not use those maps explicitly in this paper. As explained above, the diagrams can be seen as morphisms between degree-shifted objects or as homogeneous morphisms between unshifted objects. In this definition, we take the former point of view. As explained above, the graded version of $\widehat{\mathcal{BS}}_n^{\mathrm{ext}}$ will be denoted by $\widehat{\mathcal{BS}}_n^{\mathrm{ext},*}$.

The identity morphism on $i \in \widehat{I}$ is given by a vertical non-oriented strand colored by i, whereas the identity morphism on $+/-$ is given by a vertical upward/downward oriented black strand. As usual, we will color the unoriented strands to facilitate the reading of the diagrams. When there are too many different colors in a diagram, the colors are sometimes indicated by labels next to the strands. We say that two colors $i, j \in \widehat{I}$ are *adjacent* if $i \equiv j \pm 1 \bmod n$ and that they are *distant* otherwise.

The number of different types of diagram increases with n, so below we first give the generating diagrams for $n \geq 1$, then the additional ones for $n \geq 2$ and, finally, the additional ones that only show up for $n \geq 3$.

- For $n \geq 1$,

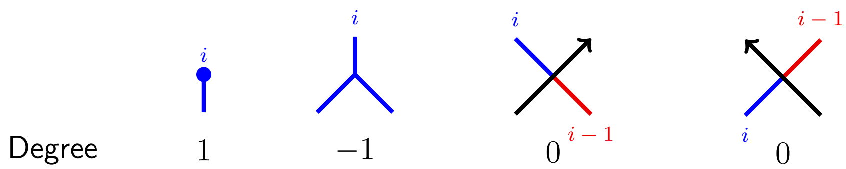
(23)



  and the corresponding oriented black caps.

- For $n \geq 2$, with $i \in \widehat{I}$,

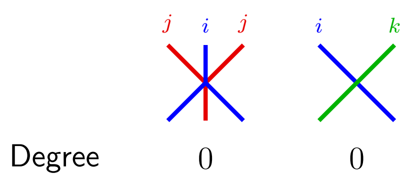
(24)



  and the diagrams obtained from these by a rotation of 180 degrees (which have the same degrees).

- For $n \geq 3$, with $i, j, k \in \widehat{I}$ such that $|i - j| = 1$ and $|i - k| > 1$,

(25)

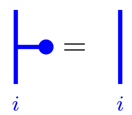


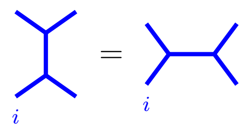
  and the diagrams obtained from these by a rotation of 180 degrees (which have the same degrees).

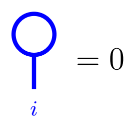
We read diagrams from bottom to top as morphisms, i.e., their source is at the bottom and their target at the top.

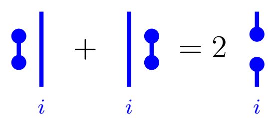
Diagrams can be stacked vertically (composition of morphisms) and juxtaposed horizontally (monoidal product of morphisms), while adding the degrees, and are subject to the relations below. The list contains all relations for all $n \geq 1$, but for $n = 1$ and $n = 2$ only the relations involving the respective generators should be considered. For $n = 2$, there is an additional subtlety in (37), as indicated. We also assume isotopy invariance and cyclicity, meaning that closed parts of the diagrams can be moved around freely in the plane as long as they do not cross any other strands and the boundary is fixed, and all diagrams can be bent and rotated, and the bent and rotated versions of the relations also hold.

- Relations involving only one color:

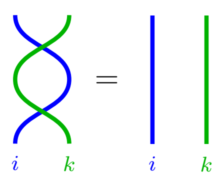
(26)


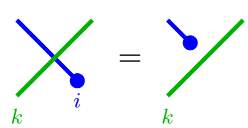
(27)


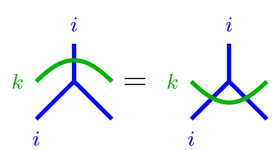
(28)


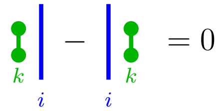
(29)


- Relations involving two distant colors:

(30)


(31)


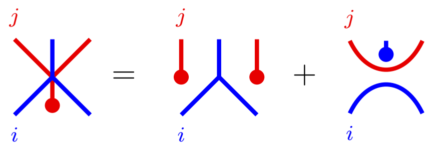
(32)


(33)


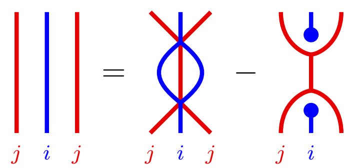
Note that (33) follows from (31).

- Relations involving two adjacent colors:

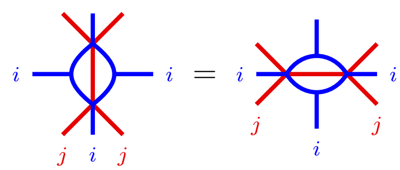
(34)



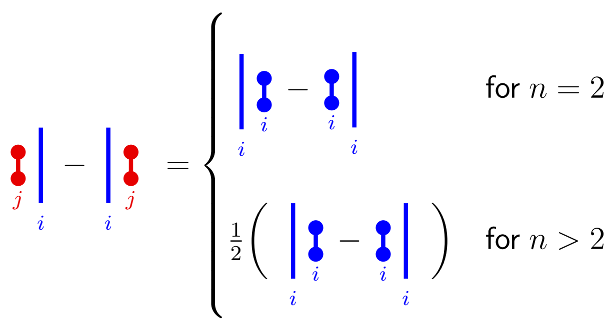
(35)



(36)

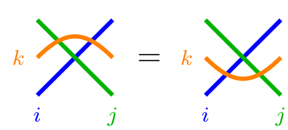


(37)



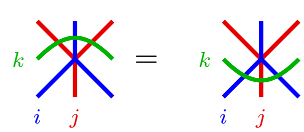
- Relation involving three distant colors:

(38)

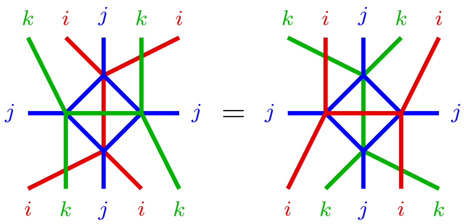


- Relation involving two adjacent colors and one distant from the other two:

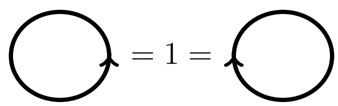
(39)



- Relation involving three colors such that i is adjacent to j, and j is adjacent to k:

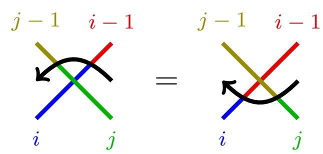
(40) 

- Relations involving only oriented strands:

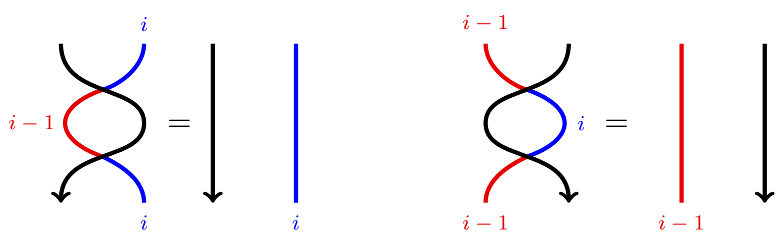
(41) 

(42) 

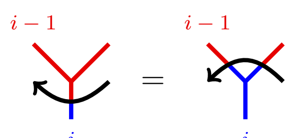
- Relation involving oriented strands and two distant colored strands:

(43) 

- Relations involving oriented strands and two adjacent colored strands:

(44) 

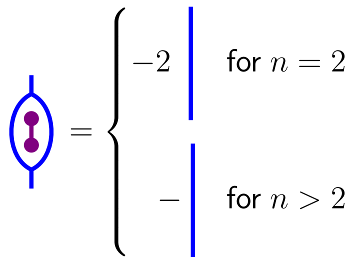
(45) 

(46) 

- Relations involving oriented strands and three adjacent colored strands:

(47)

Remark 3.1. Further relations that are consequences of (37) usually have two cases too, e.g.



and similarly with the two colors switched.

Note that the empty word is the identity object in $\widehat{\mathcal{BS}}_n^{\mathrm{ext}}$ and its endomorphisms are the closed diagrams, which by the relations above are equal to polynomials in the colored dumbbells

As each dumbbell has degree 2, the degree of any polynomial in these dumbbells, as a morphism in $\widehat{\mathcal{BS}}_n^{\mathrm{ext}}$, is twice its polynomial degree. From now on, we denote this polynomial algebra by R.

Note further that, by relations (29), (37) and (33), the morphism $\boxed{\mathrm{y}}$, defined in [MaTh] by

(48) $$\boxed{\mathrm{y}} := \sum_{i=0}^{n-1}$$  i ,

is central, in the sense that it can be slid through all diagrams (i.e. it commutes horizontally with all morphisms), because it is equal to the sum of all simple roots.

Definition 3.2. The *extended Soergel category* $\widehat{\mathcal{S}}_n^{\mathrm{ext}}$ is the Karoubi envelope of the additive envelope of $\widehat{\mathcal{BS}}_n^{\mathrm{ext}}$.

Note that in our definition shifts are already part of $\widehat{\mathcal{BS}}_n^{\mathrm{ext}}$ and recall that the idempotent complete category $\widehat{\mathcal{S}}_n^{\mathrm{ext}}$ is Krull-Schmidt, see e.g. [EMTW, Section 11.2.3]. It therefore makes sense to consider the split Grothendieck group $[\widehat{\mathcal{S}}_n^{\mathrm{ext}}]_\oplus$, which has a natural $\mathbb{Z}[q,q^{-1}]$-module structure defined by $[\mathrm{X}\langle 1\rangle] =: q[\mathrm{X}]$, for every object $\mathrm{X} \in \widehat{\mathcal{S}}_n^{\mathrm{ext}}$. It also has a natural algebra structure, inherited from the monoidal structure on $\widehat{\mathcal{S}}_n^{\mathrm{ext}}$.

Finally, let us recall the *Soergel categorification theorem*, see e.g. [MaTh, Theorem 2.5] for extended affine type A (which was based on previous results by Härterich) and [EMTW, Theorem 11.1] for general Coxeter type. For every $w \in \widehat{\mathfrak{S}}_n$, choose a rex $\underline{w} = (s_{i_1}, \ldots, s_{i_l})$ and define the Bott-Samelson bimodule

$$\mathrm{BS}(\underline{w}) := \mathrm{B}_{i_1} \cdots \mathrm{B}_{i_l} \in \widehat{\mathcal{BS}}_n^{\mathrm{ext}}.$$

There is an essentially unique indecomposable summand $\mathrm{B}_w \in \widehat{\mathcal{S}}_n^{\mathrm{ext}}$ of $\mathrm{BS}(\underline{w})$ that is not a summand of $\mathrm{BS}(\underline{u})$ for any $u \prec w$ in $\widehat{\mathfrak{S}}_n$, where $\prec$ is the Bruhat order. The isomorphism class of B_w does not depend on the choice of rex for w. More generally, for any $k \in \mathbb{Z}$ and any $w \in \widehat{\mathfrak{S}}_n$, the object $\mathrm{B}_\rho^k \mathrm{B}_w$ is indecomposable in $\widehat{\mathcal{S}}_n^{\mathrm{ext}}$ and the set of these indecomposables is complete and irredundant, meaning that they are all mutually non-isomorphic and every indecomposable in $\widehat{\mathcal{S}}_n^{\mathrm{ext}}$ is isomorphic to one of them up to grading shift. Recall further that $\widehat{H}_n^{\mathrm{ext}}$ has a q-sesquilinear form $(-,-)$, see (17). Similarly, $[\widehat{\mathcal{S}}_n^{\mathrm{ext}}]_\oplus$ has a q-sesquilinear form $\langle -,- \rangle$, called the *graded Euler form*. To define it, one has to recall that the morphism spaces in $\widehat{\mathcal{S}}_n^{\mathrm{ext},*}$ are free as left and as right graded R-modules and that their graded rank is finite and does not depend on whether we consider them as left or as right R-modules. By definition, the value of $\langle [X], [Y] \rangle$, for $X, Y \in \widehat{\mathcal{S}}_n^{\mathrm{ext}}$, is equal to the graded rank of $\mathrm{Hom}_{\widehat{\mathcal{S}}_n^{\mathrm{ext},*}}(X, Y)$. The following theorem combines Soergel's categorification theorem and Soergel's conjecture (which has been proved, see [MaTh] and references therein).

Theorem 3.3. *The $\mathbb{Z}[q, q^{-1}]$-linear map*

$$\gamma \colon \widehat{H}_n^{\mathrm{ext}} \to [\widehat{\mathcal{S}}_n^{\mathrm{ext}}]_\oplus$$
$$\gamma(\rho^k b_w) = [\mathrm{B}_{\rho^k} \mathrm{B}_w] \quad (k \in \mathbb{Z}, w \in \widehat{\mathfrak{S}}_n)$$

is an isomorphism of $\mathbb{Z}[q, q^{-1}]$-algebras intertwining the two q-sesquilinear forms.

This intertwining property goes under the name of *Soergel's hom formula* and will be used several times in Section 5.3, so let us state it explicitly here:

$$(\rho^k b_u, \rho^l b_v) = \langle [\mathrm{B}_{\rho^k} \mathrm{B}_u], [\mathrm{B}_{\rho^l} \mathrm{B}_v] \rangle = \underline{\mathrm{rk}} \left(\mathrm{Hom}_{\widehat{\mathcal{S}}_n^{\mathrm{ext},*}}(\mathrm{B}_{\rho^k} \mathrm{B}_u, \mathrm{B}_{\rho^l} \mathrm{B}_v) \right) \tag{49}$$

for all $k, l \in \mathbb{Z}$ and all $u, v \in \widehat{\mathfrak{S}}_n$, where $\underline{\mathrm{rk}}$ denotes the graded rank.

3.1.2. *Rouquier complexes.* We give a summary of the material in [MMV, §4.1] that is needed in the sequel. In particular, we introduced a diagrammatic calculus for certain morphisms between tensor products of Rouquier complexes and Bott–Samelson bimodules in finite type A. In the summary below, we recall the basic features of that calculus, generalizing it to extended affine type A as well. Note that it only describes part of the morphism spaces in $K^b(\widehat{\mathcal{S}}_n^{\mathrm{ext}})$, which are not yet understood in full generality. Describing the homotopy categories of Rouquier complexes in terms of generators and relations is a hard problem, even in finite type A, see for example [LiWi].

Throughout the rest of the paper, if $\mathcal{C}$ a $\Bbbk$-linear, additive category, for any field $\Bbbk$, we write $C^b(\mathcal{C})$ for the category of bounded complexes in $\mathcal{C}$ and $K^b(\mathcal{C})$ for its homotopy category. If $\mathcal{C}$ is also monoidal, then the usual monoidal product of chain complexes equips $K^b(\mathcal{C})$ with a monoidal structure as well. If $\mathcal{C}$ is also graded, then $K^b(\mathcal{C})$ is bigraded and we denote the shift inherited from $\mathcal{C}$ by $\langle\cdot\rangle$ and the homological shift by $[\cdot]$.

For $n \geq 2$, the Rouquier complexes $\mathrm{T}_i^{\pm 1} \in K^b(\widehat{\mathbb{S}}_n^{\mathrm{ext}})$ for $i \in \widehat{I}$ are defined by

(50) $$\mathrm{T}_i := \mathrm{B}_i \xrightarrow{\ \bullet\ } R\langle 1\rangle, \qquad \mathrm{T}_i^{-1} := R\langle -1\rangle \xrightarrow{\ \bullet\ } \mathrm{B}_i,$$

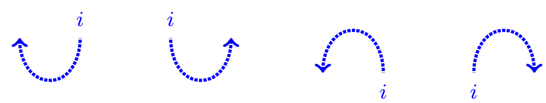
with B_i placed in homological degree zero in both complexes.

The identity morphisms of $\mathrm{T}_i^{\pm 1}$ are depicted by

$\mathrm{Id}_{\mathrm{T}_i} :=$  and $\mathrm{Id}_{\mathrm{T}_i^{-1}} :=$ 

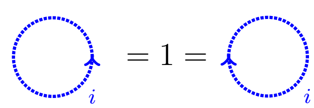
All generators in (51), (55) and (56) below have degree zero with respect to both gradings.

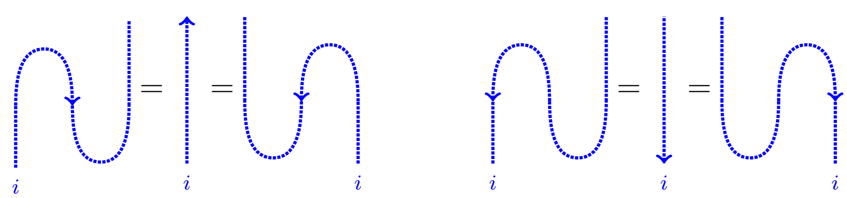
- The *1-color generators* are the cups and caps:

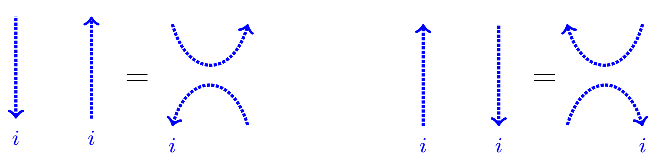
(51) 

The following lemma recalls the content of [MMV, Lemma 4.4].

Lemma 3.4. *For any $i \in \widehat{I}$, we have the following relations between morphisms in $K^b(\widehat{\mathbb{S}}_n^{\mathrm{ext}})$:*

(52) 

(53) 

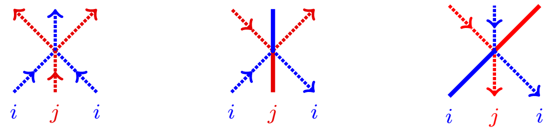
(54) 

For the diagrams and relations given in the rest of this section there are also similar diagrams and relations involving inverses of Rouquier complexes.

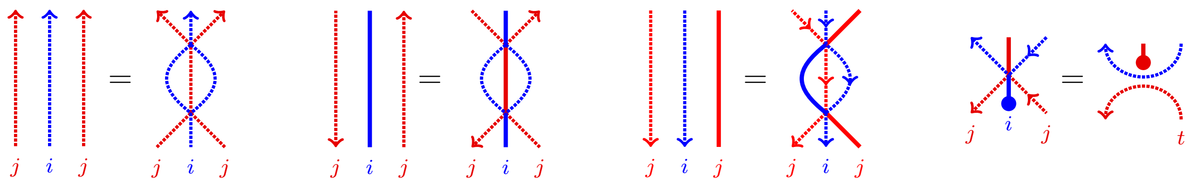
For $n > 2$, there are additional diagrams and relations.

• The *six-valent vertices* for i and j adjacent:

(55)



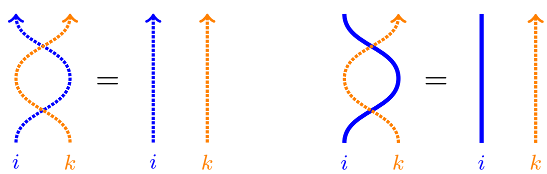
Note that a solid strand corresponds to the identity on a Soergel bimodule, whereas a dashed strand corresponds to the identify morphism on a Rouquier complex, and all morphisms are defined in the homotopy category. These satisfy



• The *crossings* for i and k distant:

(56)

These satisfy



3.2. **More diagrammatic shortcuts for Rouquier complexes.** We now develop some new diagrammatics to handle morphisms between tensor products of Soergel bimodules and Rouquier complexes.

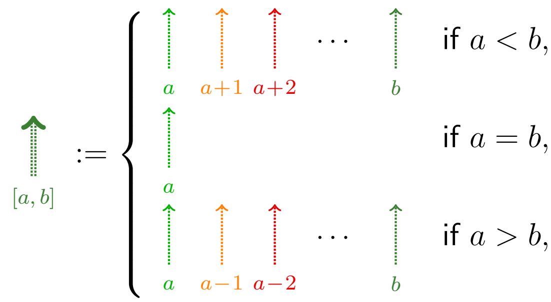
For any $a, b \in \mathbb{Z}$ define

$$\mathrm{T}_{[a,b]} := \begin{cases} \mathrm{T}_a \mathrm{T}_{a+1} \mathrm{T}_{a+2} \dots \mathrm{T}_b & \text{if } a < b, \\ \mathrm{T}_a & \text{if } a = b, \\ \mathrm{T}_a \mathrm{T}_{a-1} \mathrm{T}_{a-2} \dots \mathrm{T}_b & \text{if } a > b, \end{cases}$$

where the indices on the right-hand side can be shifted modulo n so that they belong to $\widehat{I} = \{0, 1, \dots, n-1\}$. For example, we have $\mathrm{T}_{[-2,2]} = \mathrm{T}_{n-2}\mathrm{T}_{n-1}\mathrm{T}_0\mathrm{T}_1\mathrm{T}_2$. Of course, the above notation is not unique, since $\mathrm{T}_{[a,b]} = \mathrm{T}_{[a+kn,b+kn]}$ for any $a, b, k \in \mathbb{Z}$, but we trust that this does not cause any confusion. Note that in general $\mathrm{T}_{[a,b]} \neq \mathrm{T}_{[a+kn,b+ln]}$ if $k \neq l$, e.g., $\mathrm{T}_{[0,1]} = \mathrm{T}_0\mathrm{T}_1$ while $\mathrm{T}_{[n,1]} = \mathrm{T}_0\mathrm{T}_{n-1}\cdots\mathrm{T}_1$, which are different when $n > 2$. Observe that it is important that the indices a, b in $\mathrm{T}_{[a,b]}$ be integers and not residue classes modulo

n, because that allows for products of T_i whose length is greater than n, e.g., $\mathrm{T}_{[0,2n-1]} = \mathrm{T}_0\mathrm{T}_1 \cdots \mathrm{T}_{n-1}\mathrm{T}_0\mathrm{T}_1 \cdots \mathrm{T}_{n-1}$.

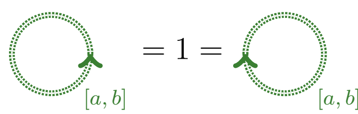
Using the same conventions, we introduce the identity morphism on $\mathrm{T}_{[a,b]}$ by



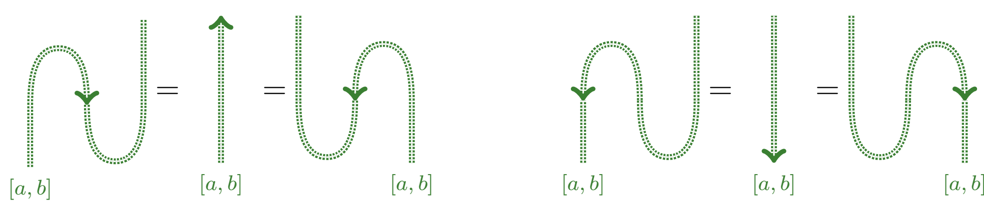
The unit and counit of the adjunction between $\mathrm{T}_{[a,b]}$ and its inverse are drawn as a cup and a cap, respectively, and these satisfy the usual isotopy and invertibility relations (cf. [MMV, Lemma 4.4, Lemma 4.15]). Cups and caps have degree zero, as do the generators in (60), (64), (67), (70), (62), (76) and (79).

Lemma 3.5. *For any $a, b \in \mathbb{Z}$, we have the following relations between morphisms in $K^b(\mathcal{S}_n)$:*

(57)



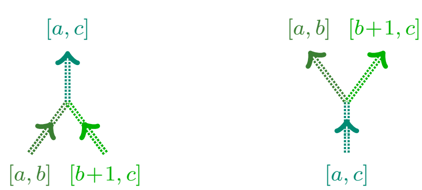
(58)



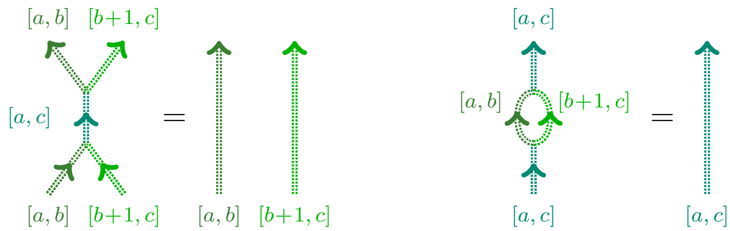
(59)

It is useful to introduce *mergers and splitters*

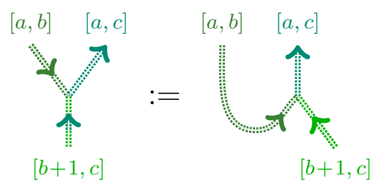
(60)



for $c > b \geq a$, realizing the equalities of chain complexes $\mathrm{T}_{[a,b]}\mathrm{T}_{[b+1,c]} = \mathrm{T}_{[a,c]}$ (and analogously for their inverses). By definition, they satisfy

(61)


Of course, similar mergers and splitters can be defined for $a \geq b > c$, where $[a, c]$ is subdivided into $[a, b]$ and $[b-1, c]$. Further, one can use cups and caps to define generators like

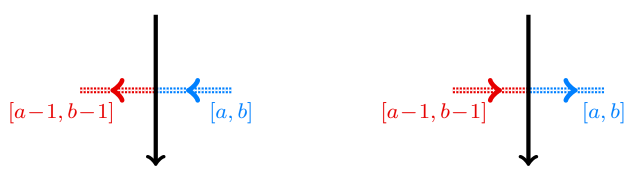


satisfying the obvious relations in $K^b(\mathcal{S}_n)$.

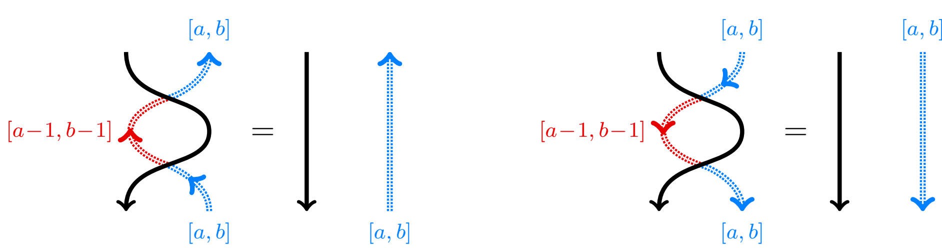
Lemma 3.6. *The following equations hold in* $K^b(\widehat{\mathcal{S}}_n^{\mathrm{ext}})$*:*

Proof. An easy computation using [MMV, Lemma 4.6]. □

The chain complexes $B_\rho^{-1}\mathrm{T}_{[a,b]}$ and $\mathrm{T}_{[a-1,b-1]}B_\rho^{-1}$ are isomorphic. This is an easy consequence of the isomorphism of the chain complexes $B_\rho^{-1}\mathrm{T}_k \cong \mathrm{T}_{k-1}B_\rho^{-1}$, which follows from (44). This homotopy equivalence and its variants are represented by the diagrams

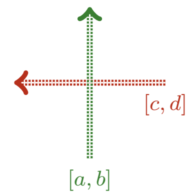
(62)


satisfying the relations (which are easy consequences of (44))

(63)


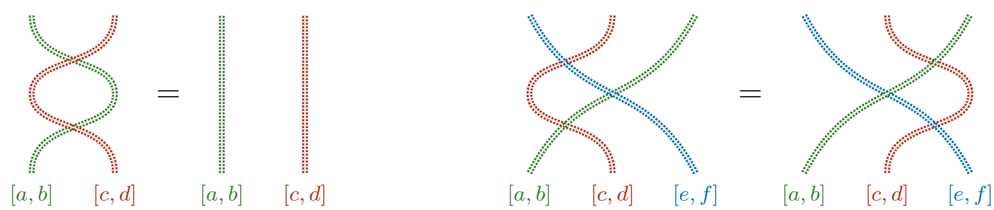
We assume that $n > 2$ until the end of this section. We say that $[a, b]$ and $[c, d]$ are *distant* if every index appearing in $\mathrm{T}_{[a,b]}$ is distant from every index in $\mathrm{T}_{[c,d]}$. In this case the chain complexes $\mathrm{T}_{[a,b]}\mathrm{T}^{\pm 1}_{[c,d]}$ and $\mathrm{T}^{\pm 1}_{[c,d]}\mathrm{T}_{[a,b]}$ are isomorphic, with the isomorphism being given by diagrams like

(64)



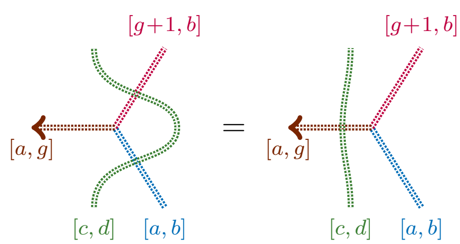
which satisfy the relations in $K^b(\widehat{\mathbb{S}}^{\text{ext}})$ given below, for mutually distant $[a, b]$, $[c, d]$ and $[e, f]$ and with any of the possible orientations

(65)



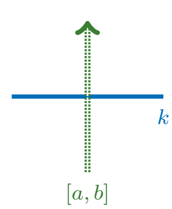
and (together with its variants)

(66)



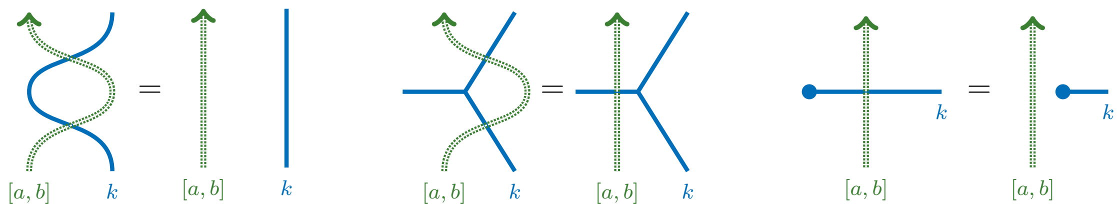
When $k = [k, k]$ is distant from $[a, b]$, the chain complexes $\mathrm{B}_k\mathrm{T}_{[a,b]}$ and $\mathrm{T}_{[a,b]}\mathrm{B}_k$ are homotopy equivalent and the homotopy equivalence can be represented by the diagram by the diagram

(67)



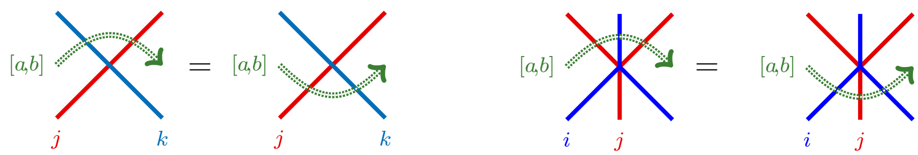
satisfying the relations

(68)



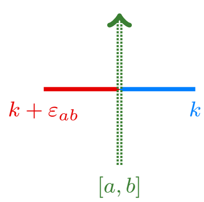
and its variants. If $[a,b]$ is distant from i, j and k, and j is adjacent to i but distant from k, we also have

(69)



For $0 < a, b < n$ the chain complex $\mathrm{T}_{[a,b]}B_k$ is homotopy equivalent to $B_{k-1}\mathrm{T}_{[a,b]}$ when $b < k \leq a$, and homotopy equivalent to $B_{k+1}\mathrm{T}_{[a,b]}$ when $a \leq k < b$ (note that $a \neq b$). This is proved in the same way as [MMV, Lemma 4.11]. The homotopy equivalences are realized by the following family of diagrams

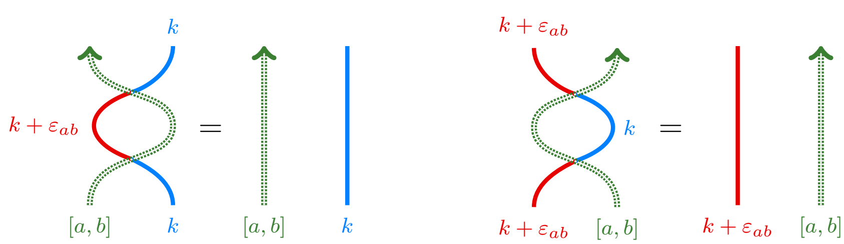
(70)



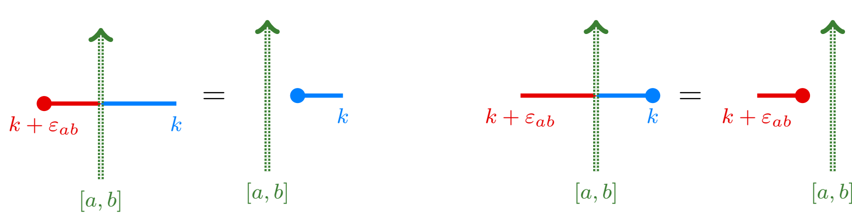
, where $\varepsilon_{ab} = \dfrac{b-a}{|b-a|}$ and k between a and $b - \varepsilon_{ab}$.

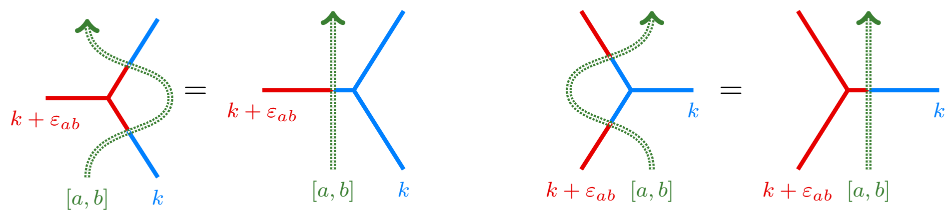
satisfying the relations

(71)



(72)



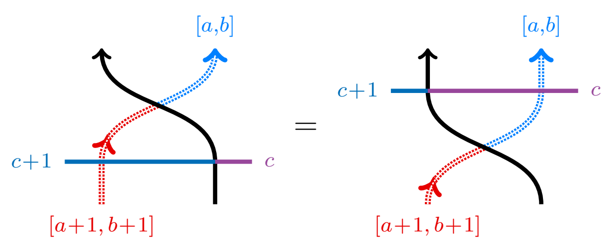
(73) 

and their variants. Relations (71) are immediate, while the relations in the remaining two lines are proved as in [MMV, Lemmas 4.18 and 4.21], respectively.

To prove the next lemma we will use the *hom & dot trick*. Due to its extensive use in the next section, we will briefly explain it in the following remark.

Remark 3.7 (The hom & dot trick)**.** Let A and B be two non-zero diagrams in a given morphism space. If the latter space is one-dimensional, then $A = \lambda B$, where λ is a non-zero scalar. To prove that $\lambda = 1$ we will often attach dots to certain endpoints of A and B and simplify the resulting diagrams so that they can be easily compared.

Lemma 3.8. *For $a, b, c \in \mathbb{Z}$ such that c is distant from $[a, b]$, the following relations hold in $K^b(\widehat{\mathbb{S}}_n^{\mathrm{ext}})$:*

(74) 

Proof. We use the hom & dot trick. By using cups and caps, both diagrams can be deformed to morphisms of degree zero from B_{c+1} to $\mathrm{B}_\rho \mathrm{T}_{[a,b]} \mathrm{B}_c \mathrm{B}_\rho^{-1} \mathrm{T}_{[a+1,b+1]}^{-1}$, which is homotopy equivalent to B_{c+1} because

$$\mathrm{B}_\rho \mathrm{T}_{[a,b]} \mathrm{B}_c \mathrm{B}_\rho^{-1} \mathrm{T}_{[a+1,b+1]}^{-1} \simeq \mathrm{T}_{[a+1,b+1]} \mathrm{B}_\rho \mathrm{B}_c \mathrm{B}_\rho^{-1} \mathrm{T}_{[a+1,b+1]}^{-1} \simeq \mathrm{T}_{[a+1,b+1]} \mathrm{B}_{c+1} \mathrm{T}_{[a+1,b+1]}^{-1} \simeq \mathrm{B}_{c+1}.$$

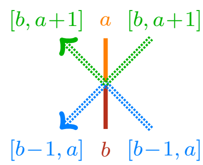
Soergel's hom formula in (49) implies that the space of degree-preserving endomorphisms of B_{c+1} (either in $\widehat{\mathbb{S}}_n^{\mathrm{ext}}$ or in $K^b(\widehat{\mathbb{S}}_n^{\mathrm{ext}})$) is one-dimensional, so the two deformed diagrams are non-zero multiples of each other, which implies that the same holds for the two diagrams in (74). Attaching a dot to one of the free ends of the non-oriented strand we see that these morphisms are equal. □

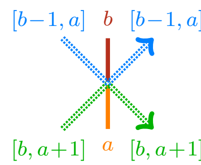
Of course, there are other variants of (74) with, for example, other orientations, or involving one oriented black strand and two thick oriented strands. Note that we can even relax the condition that c is distant from $[a, b]$. We leave the details to the reader.

Until the end of this subsection, assume that $a, b \in \mathbb{Z}$ such that $a < b$. By applying [MMV, Lemma 4.8] recursively, one obtains a homotopy equivalence

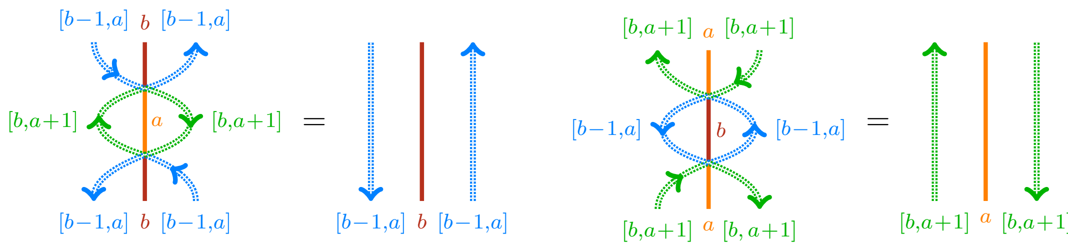
$$\mathrm{T}_{[b-1,a]}^{-1}\mathrm{B}_b\mathrm{T}_{[b-1,a]} \simeq \mathrm{T}_{[b,a+1]}\mathrm{B}_a\mathrm{T}_{[b,a+1]}^{-1}. \tag{75}$$

The corresponding isomorphism in $K^b(\widehat{\mathbb{S}}_n^{\mathrm{ext}})$ and its inverse are represented by the diagrams

(76) 

and 

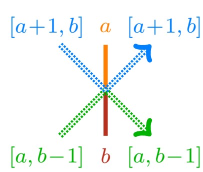
satisfying the relations

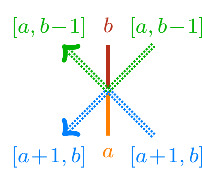
(77) 

in $K^b(\widehat{\mathbb{S}}^{\mathrm{ext}})$. There is a similar homotopy equivalence

$$\mathrm{T}_{[a,b-1]}\mathrm{B}_b\mathrm{T}_{[a,b-1]}^{-1} \simeq \mathrm{T}_{[a+1,b]}^{-1}\mathrm{B}_a\mathrm{T}_{[a+1,b]}, \tag{78}$$

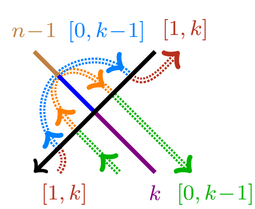
and the corresponding isomorphism in $K^b(\widehat{\mathbb{S}}_n^{\mathrm{ext}})$ and its inverse are represented by the diagrams

(79) 

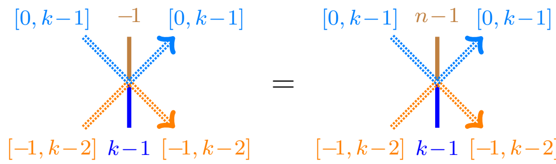
and 

satisfying relations analogous to those in (77).

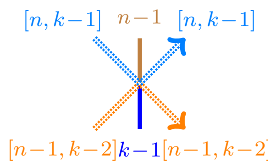
Remark 3.9. One needs to be careful with the use of the colors of the strands in the diagrams in (76) and (79). For example, the definition of Ψ_R in Section 4.1.2 contains the diagram



for $0 < k < n$. To compare the labels of this diagram with the conventions in (79), one needs to use the equality



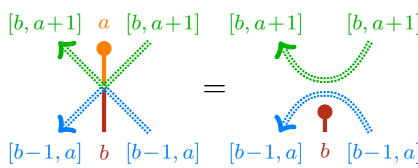
However, there is no homotopy equivalence given by a diagram of the form



because that would imply that $a = n-1$ and $b = k-1$ in (79), violating our assumption that $a < b$.

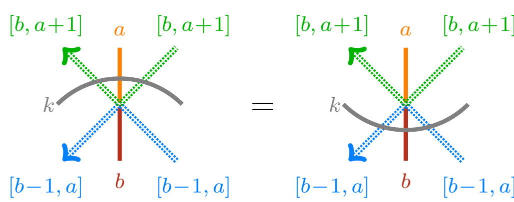
Proceeding as in the proof of [MMV, Lemma 4.25], one can show that the following holds in $K^b(\widehat{\mathbb{S}}^{\text{ext}})$:

(80)



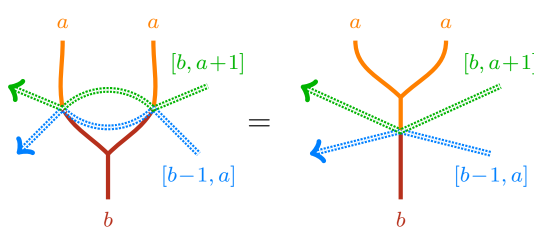
For k distant from $[a, b]$, it is easy to see that

(81)



Lemma 3.10. *We have the following in* $K^b(\widehat{\mathbb{S}}^{\text{ext}})$*:*

(82)



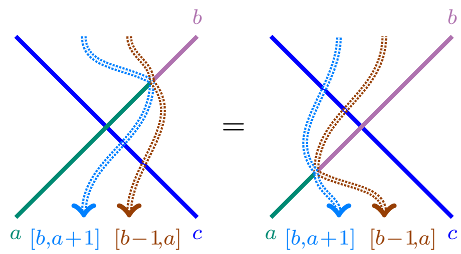
Proof. We use the hom & dot trick. By using cups and caps, both diagrams are isotopic to diagrams realizing morphisms from $\mathrm{T}^{-1}_{[b-1,a]}\mathrm{B}_b\mathrm{T}_{[b-1,a]}$ to $\mathrm{T}_{[b,a+1]}\mathrm{B}_a\mathrm{B}_a\mathrm{T}^{-1}_{[b,a+1]}$. Gluing the second diagram in (76) at the bottom results in two morphisms of degree -1 from $\mathrm{T}_{[b,a+1]}\mathrm{B}_a\mathrm{T}^{-1}_{[b,a+1]}$ to $\mathrm{T}_{[b,a+1]}\mathrm{B}_a\mathrm{B}_a\mathrm{T}^{-1}_{[b,a+1]}$. By biadjointness and the invertibility of $\mathrm{T}_{[b,a+1]}$ in $K^b(\widehat{\mathbb{S}}^{\mathrm{ext}}_n)$, there is a canonical isomorphism

$$\begin{aligned}\hom_{K^b(\widehat{\mathbb{S}}^{\mathrm{ext}}_n)}\left(\mathrm{T}_{[b,a+1]}\mathrm{B}_a\mathrm{T}^{-1}_{[b,a+1]}, \mathrm{T}_{[b,a+1]}\mathrm{B}_a\mathrm{B}_a\mathrm{T}^{-1}_{[b,a+1]}\langle -1\rangle\right) &\cong \hom_{K^b(\widehat{\mathbb{S}}^{\mathrm{ext}}_n)}\left(\mathrm{B}_a, \mathrm{B}_a\mathrm{B}_a\langle -1\rangle\right)\\ &\cong \hom_{\widehat{\mathbb{S}}^{\mathrm{ext}}_n}\left(\mathrm{B}_a, \mathrm{B}_a\mathrm{B}_a\langle -1\rangle\right) \cong \mathbb{R}.\end{aligned}$$

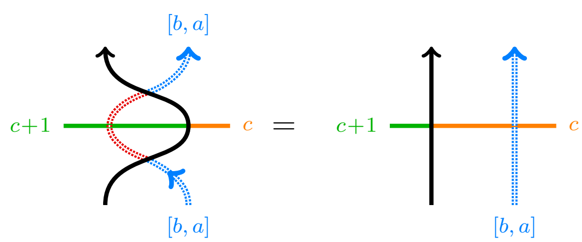
The latter isomorphism follows from Soergel's hom formula in (49) and the fact that $\mathrm{B}_a\mathrm{B}_a \cong \mathrm{B}_a\langle 1\rangle \oplus \mathrm{B}_a\langle -1\rangle$. By attaching dots to the free top ends labeled a in (82) and using (80) and (27), one sees that they are equal. □

Lemma 3.11. *For c distant from $[b,a]$, we have*

(83)



(84)



in $K^b(\widehat{\mathbb{S}}^{\mathrm{ext}})$.

Proof. We use the hom & dot trick again and only prove (83), (84) being similar. As in the proof of Lemma 3.8, first apply cups and caps to get two degree preserving morphisms from $\mathrm{T}_{[b-1,a]}\mathrm{T}_{[b,a+1]}\mathrm{B}_c\mathrm{B}_a\mathrm{T}^{-1}_{[b,a+1]}\mathrm{T}^{-1}_{[b-1,a]}$ to $\mathrm{B}_b\mathrm{B}_c$, then use the homotopy equivalences

$$\begin{aligned}\mathrm{T}_{[b-1,a]}\mathrm{T}_{[b,a+1]}\mathrm{B}_c\mathrm{B}_a\mathrm{T}^{-1}_{[b,a+1]}\mathrm{T}^{-1}_{[b-1,a]} &\simeq \mathrm{T}_{[b-1,a]}\mathrm{T}_{[b,a+1]}\mathrm{B}_a\mathrm{T}^{-1}_{[b,a+1]}\mathrm{T}^{-1}_{[b-1,a]}\mathrm{B}_c\\ &\simeq \mathrm{T}_{[b-1,a]}\mathrm{T}^{-1}_{[b-1,a]}\mathrm{B}_b\mathrm{T}_{[b-1,a]}\mathrm{T}^{-1}_{[b-1,a]}\mathrm{B}_c\\ &\simeq \mathrm{B}_b\mathrm{B}_c\end{aligned}$$

and the fact that $\mathrm{B}_b\mathrm{B}_c \cong \mathrm{B}_{bc}$, which implies that $\hom_{\widehat{\mathbb{S}}^{\mathrm{ext}}_n}(\mathrm{B}_{bc}, \mathrm{B}_{bc})$ is one-dimensional by Soergel's hom formula in (49). We attach dots to the upper boundary points labeled b and c of the two diagrams in (83) and show that the resulting diagrams are equal using (80) and (27). □

4. Parabolic embedding: categorified story

4.1. **The categorical embedding.** Let $k, n \in \mathbb{Z}$ such that $0 < k < n$. In this section we are going to define a monoidal, $\mathbb{R}$-linear functor

$$\Psi_{k,n-k} \colon \widehat{\mathbb{S}}_k^{\mathrm{ext}} \boxtimes \widehat{\mathbb{S}}_{n-k}^{\mathrm{ext}} \to K^b(\widehat{\mathbb{S}}_n^{\mathrm{ext}})$$

categorifying the embedding $\psi_{k,n-k}$ of algebras from Section 2.2, but before giving the definition in Section 4.1.2, we first explain the general idea behind it.

4.1.1. *Symmetric pairs of monoidal functors.* The content of this subsection must be known to experts, but we do not know of any reference in the literature for what we call a *symmetric pair of monoidal functors* in Definition 4.2. We thank Robert Laugwitz for pointing out to us that our definition can be reformulated in terms of the Drinfeld centralizers of the monoidal functors, as we will briefly explain in Remark 4.3. Since we claim no originality in this subsection and it is only meant to help the reader understand our construction of $\Psi_{k,n-k}$, we have not endeavored to formulate the abstract notions and results in the most general setting.

As for the terminology, we call a monoidal category or functor *strict monoidal* if the coherers of the monoidal structure are identities, otherwise we simply call it *monoidal*. However, we should warn the reader that we treat canonical natural isomorphisms, such as the unitors and associators of $\times, \otimes$ and $\oplus$, as identities. With this harmless simplification, the categories $\widehat{\mathbb{S}}_k^{\mathrm{ext}} \boxtimes \widehat{\mathbb{S}}_{n-k}^{\mathrm{ext}}$ and $K^b(\widehat{\mathbb{S}}_n^{\mathrm{ext}})$ are considered to be strict monoidal. However, our monoidal functor $\Psi_{k,n-k}$ is definitely not strict, as will see in Section 4.2.3.

In this more abstract subsection, we work over an arbitrary field $\Bbbk$. Recall that, for $\Bbbk$-linear categories $\mathcal{C}$ and $\mathcal{D}$, their tensor product $\mathcal{C} \diamond \mathcal{D}$ is the $\Bbbk$-linear category with $\mathrm{Ob}(\mathcal{C} \diamond \mathcal{D}) = \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{D})$ and morphism spaces $\mathrm{Hom}_{\mathcal{C}\diamond\mathcal{D}}\big((\mathtt{i},\mathtt{j}),(\mathtt{k},\mathtt{l})\big) = \mathrm{Hom}_{\mathcal{C}}(\mathtt{i},\mathtt{j}) \otimes \mathrm{Hom}_{\mathcal{D}}(\mathtt{k},\mathtt{l})$, where $\otimes = \otimes_{\Bbbk}$. Further, we define their box tensor product $\mathcal{C} \boxtimes \mathcal{D}$ as the Karoubi envelope of the additive envelope of $\mathcal{C} \diamond \mathcal{D}$.

Proposition 4.1. *Let $\mathcal{E}$ be an idempotent complete additive $\Bbbk$-linear category. Then any $\Bbbk$-linear functor $\mathcal{C} \diamond \mathcal{D} \to \mathcal{E}$ extends in an essentially unique way to a $\Bbbk$-linear (and hence additive) functor $\mathcal{C}_{\oplus,e} \boxtimes \mathcal{D}_{\oplus,e} \to \mathcal{E}$.*

Proof. Let $\Psi \colon \mathcal{C} \diamond \mathcal{D} \to \mathcal{E}$ be a $\Bbbk$-linear functor. By the universal property of additive envelopes, this uniquely determines a functor $(\mathcal{C} \diamond \mathcal{D})_{\oplus} \to \mathcal{E}$. Since the natural inclusion $(\mathcal{C} \diamond \mathcal{D})_{\oplus} \hookrightarrow (\mathcal{C}_{\oplus} \diamond \mathcal{D}_{\oplus})_{\oplus}$ is an equivalence, precomposing with an inverse equivalence yields a $\Bbbk$-linear (additive) functor $(\mathcal{C}_{\oplus} \diamond \mathcal{D}_{\oplus})_{\oplus} \to \mathcal{E}$ and, since $\mathcal{E}$ is idempotent complete, also a $\Bbbk$-linear (additive) functor $\Psi' \colon (\mathcal{C}_{\oplus} \diamond \mathcal{D}_{\oplus})_{\oplus,e} \to \mathcal{E}$.

Now any pair e_1 and e_2 of idempotents in $\mathcal{C}_{\oplus}$ and $\mathcal{D}_{\oplus}$ gives rise to an idempotent $e_1 \otimes e_2$ in $(\mathcal{C}_{\oplus} \diamond \mathcal{D}_{\oplus})_{\oplus}$, so we obtain an embedding $\mathcal{C}_{\oplus,e} \diamond \mathcal{D}_{\oplus,e} \hookrightarrow (\mathcal{C}_{\oplus} \diamond \mathcal{D}_{\oplus})_{\oplus,e}$. Thus we can restrict our functor Ψ' to a $\Bbbk$-linear functor $\mathcal{C}_{\oplus,e} \diamond \mathcal{D}_{\oplus,e} \to \mathcal{E}$.

Using the universal property of the additive envelope again, we thus obtain a $\Bbbk$-linear functor

$$\Psi'' \colon (\mathcal{C}_{\oplus,e} \diamond \mathcal{D}_{\oplus,e})_{\oplus} \to \mathcal{E}.$$

Noting that $(\mathcal{C}_{\oplus,e} \diamond \mathcal{D}_{\oplus,e})_{\oplus,e} = \mathcal{C}_{\oplus,e} \boxtimes \mathcal{D}_{\oplus,e}$ by definition, completes the proof. □

Thus, to define $\Psi_{k,n-k} \colon \widehat{\mathbb{S}}_k^{\mathrm{ext}} \boxtimes \widehat{\mathbb{S}}_{n-k}^{\mathrm{ext}} \to K^b(\widehat{\mathbb{S}}_n^{\mathrm{ext}})$, we only need to define its restriction

$$\Psi_{k,n-k} \colon \widehat{\mathcal{B}\mathbb{S}}_k^{\mathrm{ext}} \diamond \widehat{\mathcal{B}\mathbb{S}}_{n-k}^{\mathrm{ext}} \to K^b(\widehat{\mathbb{S}}_n^{\mathrm{ext}}),$$

which is exactly what we will do in Section 4.1.2.

Suppose, additionally, that $\mathcal{C}$ and $\mathcal{D}$ have strict monoidal products $\circ_{\mathcal{C}}$ and $\circ_{\mathcal{D}}$, respectively, and strict identity objects, both denoted by $\mathbb{1}$. Then the coordinatewise monoidal product defines a strict monoidal structure on $\mathcal{C} \diamond \mathcal{D}$ (taking into account the aforementioned simplifications), and $\mathcal{C}$ and $\mathcal{D}$ can be embedded as commuting monoidal, $\Bbbk$-linear subcategories of $\mathcal{C} \diamond \mathcal{D}$ by tensoring objects and morphisms on the right, resp. on the left, with $\mathbb{1}$ and $\mathrm{id}_{\mathbb{1}}$, respectively. Another way to phrase this, is to say that the embeddings of $\mathcal{C}$ and $\mathcal{D}$ into $\mathcal{C} \diamond \mathcal{D}$ form a *commuting pair of strict monoidal, $\Bbbk$-linear functors*, although we could not find this terminology in the literature.

Suppose that $\mathcal{E}$ is also strict monoidal with monoidal product $\circ_{\mathcal{E}}$ and identity object $\mathbb{1}$, and let $\Psi_L \colon \mathcal{C} \to \mathcal{E}$ and $\Psi_R \colon \mathcal{D} \to \mathcal{E}$ be two strict monoidal, $\Bbbk$-linear functors. Then we can define the $\Bbbk$-linear functor $\Psi \colon \mathcal{C} \diamond \mathcal{D} \to \mathcal{E}$ by $\Psi := \Psi_L \circ_{\mathcal{E}} \Psi_R$, where

$$\Psi(X \diamond Y) := \Psi_L(X) \circ_{\mathcal{E}} \Psi_R(Y) \quad \text{and} \quad \Psi(f \diamond g) := \Psi_L(f) \circ_{\mathcal{E}} \Psi_R(g), \tag{85}$$

for any objects $X \in \mathcal{C}, Y \in \mathcal{D}$ and any morphisms $f \in \mathcal{C}, g \in \mathcal{D}$. By definition, $\Psi(X \diamond \mathbb{1}) = \Psi_L(X)$ and $\Psi(\mathbb{1} \diamond Y) = \Psi_R(Y)$ for objects $X \in \mathcal{C}, Y \in \mathcal{D}$, and $\Psi(f \diamond \mathrm{id}_{\mathbb{1}}) = \Psi_L(f)$ and $\Psi(\mathrm{id}_{\mathbb{1}} \diamond g) = \Psi_R(g)$ for morphisms $f \in \mathcal{C}, g \in \mathcal{D}$. Of course, we could equally well define Ψ as $\Psi_R \circ_{\mathcal{E}} \Psi_L$, but the latter functor need not be naturally isomorphic to $\Psi_L \circ_{\mathcal{E}} \Psi_R$ in general.

Definition 4.2. We say that (Ψ_L, Ψ_R) form a *symmetric pair of monoidal functors* if there is a natural isomorphism

$$\zeta \colon \Psi_L \circ_{\mathcal{E}} \Psi_R \xrightarrow{\cong} \Psi_R \circ_{\mathcal{E}} \Psi_L$$

satisfying the *hexagon identities*, given by the commutativity of the diagrams

$$\begin{array}{ccc} \Psi_L(X_1 \circ_{\mathcal{C}} X_2) \circ_{\mathcal{E}} \Psi_R(Y) & \xrightarrow{\zeta_{(X_1 \circ_{\mathcal{C}} X_2) \diamond Y}} & \Psi_R(Y) \circ_{\mathcal{E}} \Psi_L(X_1 \circ_{\mathcal{C}} X_2) \\ \| & & \| \\ \Psi_L(X_1) \circ_{\mathcal{E}} \Psi_L(X_2) \circ_{\mathcal{E}} \Psi_R(Y) & & \| \\ \downarrow{\scriptstyle \mathrm{id}_{\Psi_L(X_1)} \circ_{\mathcal{E}} \zeta_{X_2 \diamond Y}} & & \| \\ \Psi_L(X_1) \circ_{\mathcal{E}} \Psi_R(Y) \circ_{\mathcal{E}} \Psi_L(X_2) & \xrightarrow{\zeta_{X_1 \diamond Y} \circ_{\mathcal{E}} \mathrm{id}_{\Psi_L(X_2)}} & \Psi_R(Y) \circ_{\mathcal{E}} \Psi_L(X_1) \circ_{\mathcal{E}} \Psi_L(X_2) \end{array} \tag{86}$$

and

$$
\begin{array}{ccc}
\Psi_L(X) \circ_{\mathcal{E}} \Psi_R(Y_1 \circ_{\mathcal{D}} Y_2) & \xrightarrow{\zeta_{X \diamond (Y_1 \circ_{\mathcal{D}} Y_2)}} & \Psi_R(Y_1 \circ_{\mathcal{D}} Y_2) \circ_{\mathcal{E}} \Psi_L(X) \\
\| & & \| \\
\Psi_L(X) \circ_{\mathcal{E}} \Psi_R(Y_1) \circ_{\mathcal{E}} \Psi_R(Y_2) & & \| \\
\downarrow{\scriptstyle \zeta_{X \diamond Y_1} \circ_{\mathcal{E}} \mathrm{id}_{\Psi_R(Y_2)}} & & \| \\
\Psi_R(Y_1) \circ_{\mathcal{E}} \Psi_L(X) \circ_{\mathcal{E}} \Psi_R(Y_2) & \xrightarrow{\mathrm{id}_{\Psi_R(Y_1)} \circ_{\mathcal{E}} \zeta_{X \diamond Y_2}} & \Psi_R(Y_1) \circ_{\mathcal{E}} \Psi_R(Y_2) \circ_{\mathcal{E}} \Psi_L(X)
\end{array}
\tag{87}
$$

for all $X, X_1, X_2 \in \mathcal{C}$ and $Y, Y_1, Y_2 \in \mathcal{D}$.

If we write $\Psi(X,Y) := \Psi(X \diamond Y)$ and $\zeta_{X,Y} := \zeta_{X \diamond Y}$, denote composition of morphisms in $\mathcal{E}$ by $\cdot$ and suppress identity morphisms, then the commutativity of the diagrams in (86) and (87) translates to the equations

$$
\zeta_{X_1 \circ_{\mathcal{C}} X_2, Y} = \zeta_{X_1,Y} \cdot \zeta_{X_2,Y} \quad \text{and} \quad \zeta_{X, Y_1 \circ_{\mathcal{D}} Y_2} = \zeta_{X,Y_2} \cdot \zeta_{X,Y_1}, \tag{88}
$$

respectively. These are very similar to the hexagon identities for a braiding, which is symmetric because $\zeta^{-1}_{X,Y}$ defines the braiding for (Ψ_R, Ψ_L). This justifies our choice of terminology in Definition 4.2. Note that the commutative diagrams in (86) and (87) are actually pentagons in our case, because we have omitted the sixth side of the usual hexagons, which in our case is an identity, due to our assumption that the underlying monoidal categories are strict. In fact, those pentagons could be reduced to triangles, since two of the remaining five sides are also identities, but we have decided to include them for clarity.

Note that the embeddings of $\mathcal{C}$ and $\mathcal{D}$ into $\mathcal{C} \diamond \mathcal{D}$ form a symmetric pair of monoidal, $\Bbbk$-linear functors with trivial braiding, in the sense that there is an equality $\Psi_L(X) \circ_{\mathcal{E}} \Psi_R(Y) = \Psi_R(Y) \circ_{\mathcal{E}} \Psi_L(X)$ and $\zeta_{X,Y} = \mathrm{id}_{\Psi(X,Y)}$ for all objects $X \in \mathcal{C}, Y \in \mathcal{D}$.

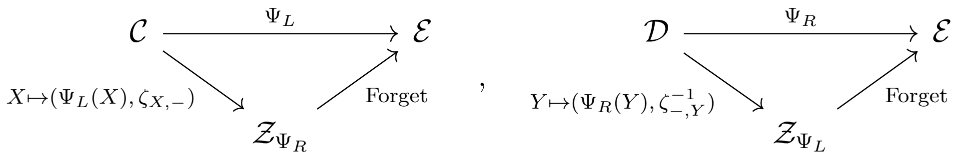
Before we go on, let us explain the aforementioned relation with Drinfeld centralizers.

Remark 4.3. Let $\mathcal{A}$ and $\mathcal{B}$ be two strict monoidal, $\Bbbk$-linear categories and $F\colon \mathcal{A} \to \mathcal{B}$ a strict monoidal, $\Bbbk$-linear functor between them. The *Drinfeld centralizer of F*, denoted $\mathcal{Z}_F = \mathcal{Z}_F(\mathcal{A}, \mathcal{B})$, is the $\Bbbk$-linear category whose objects are pairs (B, β), with $B \in \mathcal{B}$ an object and

$$
\beta\colon F(-) \circ_{\mathcal{B}} B \xrightarrow{\cong} B \circ_{\mathcal{B}} F(-)
$$

a natural isomorphism (called *half-braiding*) satisfying $\beta_{A_1 \circ_{\mathcal{A}} A_2} = \beta_{A_1} \cdot \beta_{A_2}$ for $A_1, A_2 \in \mathcal{A}$ (where we are suppressing two identity morphisms, as before). The morphisms in $\mathcal{Z}_F$ are morphisms in $\mathcal{B}$ which commute with the respective β's. The category $\mathcal{Z}_F$ has an obvious monoidal structure such that the forgetful functor $\mathrm{Forget} : \mathcal{Z}_F \to \mathcal{B}$, defined by $(B, \beta) \mapsto B$, is a monoidal functor. For more information on Drinfeld centralizers, see e.g. [Hoe, Section 1.5] (especially Example 1.5.3) and [ElHo, Section 2.8] and references therein. Note that Hoek uses the term *Drinfeld center*, whereas Elias and Hogancamp use the term *Drinfeld centralizer*, which we find more adequate and therefore adopt here as well.

If (Ψ_L, Ψ_R, ζ) is a symmetric pair of strict monoidal functors, then Ψ_L and Ψ_R factor through each other's Drinfeld centralizers, i.e., both diagrams below commute.



In general, a pair of monoidal functors $F\colon \mathcal{C} \to \mathcal{E}$ and $G\colon \mathcal{D} \to \mathcal{E}$ which factor through each other's Drinfeld centralizers only yields two half-braidings. The factorization of F through $\mathcal{Z}_G$ yields isomorphisms $\beta_{X,Y}$, which satisfy the hexagon identity w.r.t. $X \in \mathcal{C}$, and the factorization of G through $\mathcal{Z}_F$ yields isomorphisms $\beta_{Y,X}$, which satisfy the hexagon identity w.r.t. $Y \in \mathcal{D}$. However, two half-braidings do not necessarily add up to one braiding, which has to satisfy the hexagon identities for both variables at once. In our case they do add up to a braiding, because they satisfy the additional condition $\beta_{Y,X} = \beta^{-1}_{X,Y}$.

Lemma 4.4. *The functor* Ψ *in* (85) *extends to a monoidal functor, which restricts to the strict monoidal functors* Ψ_L *and* Ψ_R*, if and only if* Ψ_L, Ψ_R *form a symmetric pair of monoidal functors.*

Proof. Suppose that Ψ extends to a monoidal functor. Then there exists a natural isomorphism

(89) $$\Psi(X_1 \circ_{\mathcal{C}} X_2, Y_1 \circ_{\mathcal{D}} Y_2) \xrightarrow{\eta_{(X_1,Y_1),(X_2,Y_2)}} \Psi(X_1, Y_1) \circ_{\mathcal{E}} \Psi(X_2, Y_2)$$

for any objects $X_1, X_2 \in \mathcal{C}$ and $Y_1, Y_2 \in \mathcal{D}$. Note that $\Psi(\mathbb{1}, \mathbb{1}) = \Psi_L(\mathbb{1}) \circ_{\mathcal{E}} \Psi_R(\mathbb{1}) = \mathbb{1} \circ_{\mathcal{E}} \mathbb{1} = \mathbb{1}$ holds on the nose, so η is the only coherer that can be non-trivial.

By definition, the object on the l.h.s. of (89) is equal to

(90) $$\Psi_L(X_1) \circ_{\mathcal{E}} \Psi_L(X_2) \circ_{\mathcal{E}} \Psi_R(Y_1) \circ_{\mathcal{E}} \Psi_R(Y_2)$$

while the object on the r.h.s. is equal to

(91) $$\Psi_L(X_1) \circ_{\mathcal{E}} \Psi_R(Y_1) \circ_{\mathcal{E}} \Psi_L(X_2) \circ_{\mathcal{E}} \Psi_R(Y_2),$$

hence $\eta_{(X_1,Y_1),(X_2,Y_2)}$ in (89) is completely determined by the natural isomorphism

(92) $$\Psi_L(X_2) \circ_{\mathcal{E}} \Psi_R(Y_1) \xrightarrow{\zeta_{X_2,Y_1}} \Psi_R(Y_1) \circ_{\mathcal{E}} \Psi_L(X_2),$$

where $\zeta_{X_2,Y_1} := \eta_{(\mathbb{1},Y_1),(X_2,\mathbb{1})}$. Note that we have swapped the order of X_2 and Y_1 in the subscript of ζ, which conforms to the notation of a braiding.

Let $X_1, X_2, X_3 \in \mathcal{C}$ and $Y_1, Y_2, Y_3 \in \mathcal{D}$. The usual coherence condition for η is

(93) $$\eta_{(X_2,Y_2),(X_3,Y_3)} \cdot \eta_{(X_1,Y_1),(X_2 \circ_{\mathcal{C}} X_3, Y_2 \circ_{\mathcal{D}} Y_3)} = \eta_{(X_1,Y_1),(X_2,Y_2)} \cdot \eta_{(X_1 \circ_{\mathcal{C}} X_2, Y_1 \circ_{\mathcal{D}} Y_2),(X_2,Y_3)},$$

where we have suppressed several identity morphisms again. By the same arguments as above, we obtain the corresponding coherence condition for ζ:

(94) $$\zeta_{X_3,Y_2} \cdot \zeta_{X_2 \circ_{\mathcal{C}} X_3, Y_1} = \zeta_{X_2,Y_1} \cdot \zeta_{X_3, Y_1 \circ_{\mathcal{D}} Y_2}.$$

Note that the coherence conditions in (93) and (94) are equivalent.

The hexagon identities for ζ are obtained from (94) as special cases, by putting $Y_2 = \mathbb{1}$ and $X_2 = \mathbb{1}$, respectively. This proves that Ψ_L, Ψ_R is a symmetric pair of monoidal functors.

Conversely, suppose that Ψ_L, Ψ_R is a symmetric pair of monoidal functors with braiding ζ. Thanks to the interchange law for morphisms in a monoidal category, we have

$$\zeta_{X_3,Y_2} \cdot \zeta_{X_2,Y_1} = \zeta_{X_2,Y_1} \cdot \zeta_{X_3,Y_2}, \tag{95}$$

as can be seen from the corresponding commutative diagram:

$$\begin{array}{ccc} \Psi_L(X_2)\Psi_R(Y_1)\Psi_L(X_3)\Psi_R(Y_2) & \xrightarrow{\zeta_{X_2,Y_1}} & \Psi_R(Y_1)\Psi_L(X_2)\Psi_L(X_3)\Psi_R(Y_2) \\ \downarrow{\scriptstyle \zeta_{X_3,Y_2}} & & \downarrow{\scriptstyle \zeta_{X_3,Y_2}} \\ \Psi_L(X_2)\Psi_R(Y_1)\Psi_R(Y_2)\Psi_L(X_3) & \xrightarrow{\zeta_{X_2,Y_1}} & \Psi_R(Y_1)\Psi_L(X_2)\Psi_R(Y_2)\Psi_L(X_3) \end{array} \tag{96}$$

Multiplying both sides in (95) on the right by ζ_{X_3,Y_1} and using the hexagon identities yields (94), which implies that η satisfies (93). This proves that Ψ extends to a monoidal functor. $\square$

The following corollary shows that our choice to define Ψ as $\Psi_L \circ_{\mathcal{E}} \Psi_R$, instead of $\Psi_R \circ_{\mathcal{E}} \Psi_L$, is not essential if the pair of monoidal functors Ψ_L, Ψ_R is symmetric. The proof uses similar arguments as the proof of Lemma 4.4 and is straightforward, so we leave the details to the reader.

Corollary 4.5. *Suppose that Ψ_L, Ψ_R is a symmetric pair of monoidal functors. Then $\Psi_L \circ_{\mathcal{E}} \Psi_R$ and $\Psi_R \circ_{\mathcal{E}} \Psi_L$ are two naturally isomorphic monoidal functors.*

Remark 4.6. The functor Ψ in Lemma 4.4 is a strict monoidal functor if and only if the braiding ζ is trivial.

4.1.2. *Definition of $\Psi_{k,n-k}$ and main theorem.* As in Section 4.1.1, we first define two strict monoidal $\mathbb{k}$-linear functors

$$\Psi_L \colon \widehat{\mathcal{BS}}_k^{\mathrm{ext}} \to K^b(\widehat{\mathcal{S}}_n^{\mathrm{ext}}) \quad \text{and} \quad \Psi_R \colon \widehat{\mathcal{BS}}_{n-k}^{\mathrm{ext}} \to K^b(\widehat{\mathcal{S}}_n^{\mathrm{ext}})$$

and then define

$$\Psi_{k,n-k} \colon \widehat{\mathcal{BS}}_k^{\mathrm{ext}} \diamond \widehat{\mathcal{BS}}_{n-k}^{\mathrm{ext}} \to K^b(\widehat{\mathcal{S}}_n^{\mathrm{ext}}) \tag{97}$$

by

$$\Psi_{k,n-k}(X \diamond Y) := \Psi_L(X)\Psi_R(Y) \quad \text{and} \quad \Psi_{k,n-k}(f \diamond g) = \Psi_L(f)\Psi_R(g),$$

for any objects $X \in \widehat{\mathcal{BS}}_k^{\mathrm{ext}}, Y \in \widehat{\mathcal{BS}}_{n-k}^{\mathrm{ext}}$ and any morphisms $f \in \widehat{\mathcal{BS}}_k^{\mathrm{ext}}, g \in \widehat{\mathcal{BS}}_{n-k}^{\mathrm{ext}}$. We have omitted the symbol for the monoidal product in $K^b(\widehat{\mathcal{S}}_n^{\mathrm{ext}})$. To avoid cluttering in certain diagrams of the definition below, we have reduced the number of labels. When there are several strands of the same color, we often label just one of them. The labels of the others are understood to be the same.

On objects. On the generating objects, Ψ_L and Ψ_R are defined by

$$\Psi_L\colon \begin{cases} \mathrm{B}_i \mapsto \mathrm{B}_i, \quad i = 1, \dots, k-1, \\ \mathrm{B}_0 \mapsto \mathrm{T}_k^{-1} \cdots \mathrm{T}_{n-1}^{-1} \mathrm{B}_0 \mathrm{T}_{n-1} \cdots \mathrm{T}_k, \\ \mathrm{B}_\rho \mapsto \mathrm{B}_\rho \mathrm{T}_{n-1} \dots \mathrm{T}_k, \end{cases} \qquad \Psi_R\colon \begin{cases} \mathrm{B}_j \mapsto \mathrm{B}_{k+j}, \quad j = 1, \dots, n-k-1, \\ \mathrm{B}_0 \mapsto T_0 \cdots \mathrm{T}_{k-1} \mathrm{B}_k \mathrm{T}_{k-1}^{-1} \cdots \mathrm{T}_0^{-1}, \\ \mathrm{B}_\rho \mapsto \mathrm{T}_k^{-1} \dots \mathrm{T}_1^{-1} \mathrm{B}_\rho. \end{cases}$$

On morphisms.

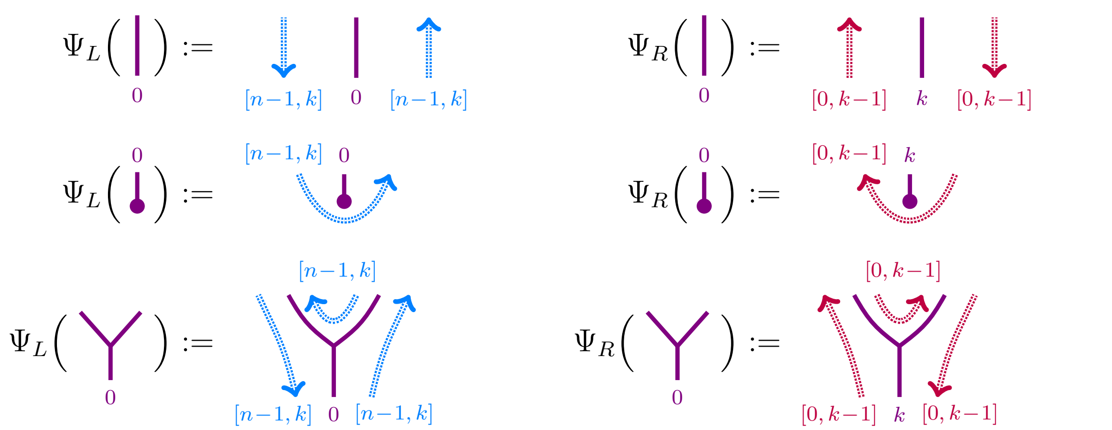
- If $D_{1,\dots,k-1}$ is a morphism of $\widehat{\mathcal{BS}}_k^{\mathrm{ext}}$ in colors $(1, \dots, k-1)$, then

$$\Psi_L(D_{1,\dots,k-1}) := D_{1,\dots,k-1},$$

- If $D'_{1,\dots,k-1}$ is a morphism of $\widehat{\mathcal{BS}}_{n-k}^{\mathrm{ext}}$ in colors $(1, \dots, n-k-1)$, then

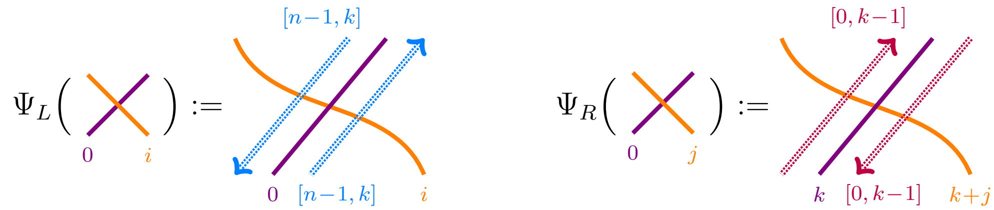
$$\Psi_R(D'_{1,\dots,n-k-1}) := D'_{k+1,\dots,n-1},$$

On other morphisms, Ψ_L and Ψ_R are defined as follows.



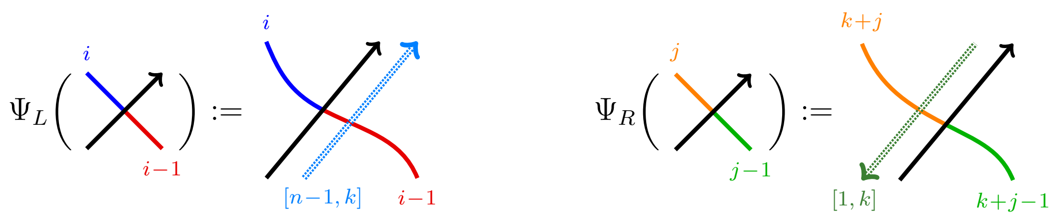
- *On oriented generators:*

• *On generators including strands with distant colors:*

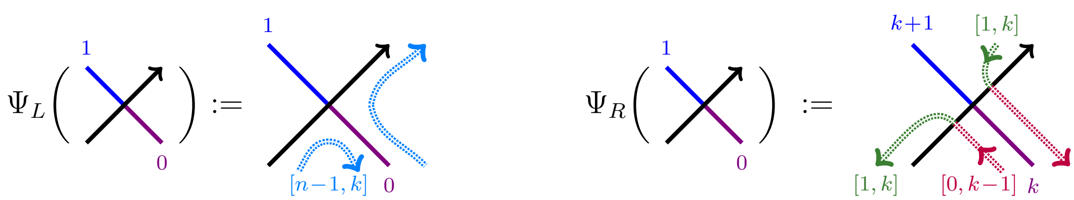


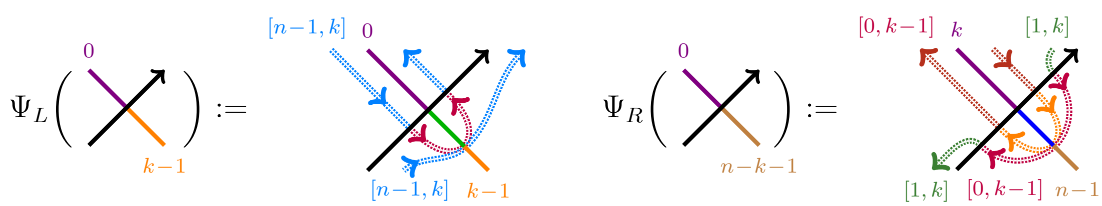
for $1 < i < k-1$ and $1 < j < n-k-1$.

• *On generators including strands with adjacent colors:*

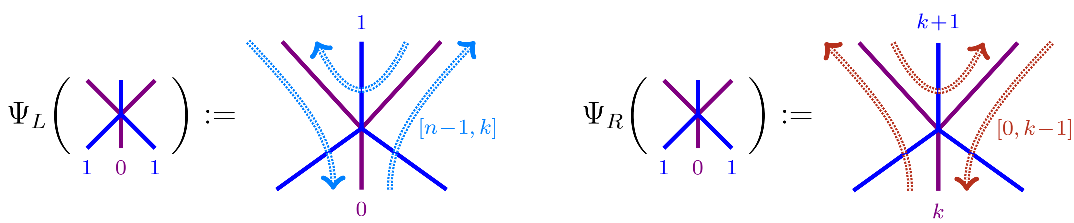


for $i, j \neq 0, 1$, while

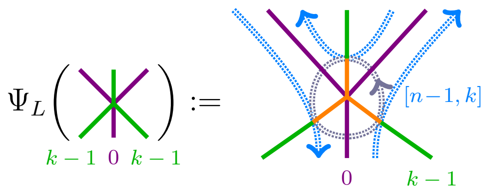


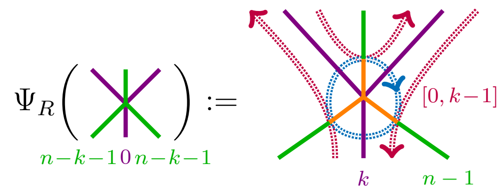


and



for $k > 2$, while





for $k \geq 2$.

The inner labels in the last line are $n-1$ and $[n-2, k-1]$ (left) and $k-1$ and $[-1, k-2]$ (right). This ends the definition of Ψ_L and Ψ_R.

Lemma 4.7. *For any domain X and codomain Y of a generating morphism in $\widehat{\mathcal{BS}}_k^{\mathrm{ext}}$ of degree t, we have*

$$\hom_{K^b(\widehat{\mathcal{S}}_n^{\mathrm{ext}})}(\Psi_L(X), \Psi_L(Y)\langle t\rangle) \cong \hom_{\widehat{\mathcal{BS}}_k^{\mathrm{ext}}}(X, Y\langle t\rangle).$$

Similarly, for any domain Z and codomain W of a generating morphism in $\widehat{\mathcal{BS}}_{n-k}^{\mathrm{ext}}$ of degree t, we have

$$\hom_{K^b(\widehat{\mathcal{S}}_n^{\mathrm{ext}})}(\Psi_R(Z), \Psi_R(W)\langle t\rangle) \cong \hom_{\widehat{\mathcal{BS}}_{n-k}^{\mathrm{ext}}}(Z, W\langle t\rangle).$$

Moreover, all those morphism spaces are one-dimensional.

Proof. The proof is immediate except for two cases in Ψ_L and two cases in Ψ_R:

- a crossing in $\widehat{\mathcal{S}}_k^{\mathrm{ext}}$ involving an oriented strand and unoriented strands labeled 0 and $k-1$, respectively in $\widehat{\mathcal{S}}_{n-k}^{\mathrm{ext}}$ with labels 0 and $n-k-1$;
- a six-valent vertex in $\widehat{\mathcal{S}}_k^{\mathrm{ext}}$ involving strands labeled 0 and $k-1$, respectively in $\widehat{\mathcal{S}}_{n-k}^{\mathrm{ext}}$ with labels 0 and $n-k-1$.

We only prove the second part of each case, the first being similar.

For a crossing in $\widehat{\mathcal{S}}_{n-k}^{\mathrm{ext}}$ involving an oriented strand and unoriented strands labeled 0 and $n-k-1$ we compute

$$\begin{aligned}
&\hom_{K^b(\widehat{\mathcal{S}}_n^{\mathrm{ext}})}(\Psi_R(\mathrm{B}_\rho \mathrm{B}_{n-k-1}), \Psi_R(\mathrm{B}_0\mathrm{B}_\rho)) \\
&\quad\cong \hom_{K^b(\widehat{\mathcal{S}}_n^{\mathrm{ext}})}(\mathrm{T}_{[1,k]}^{-1}\mathrm{B}_\rho\mathrm{B}_{n-1}, \mathrm{T}_{[0,k-1]}\mathrm{B}_k\mathrm{T}_{[0,k-1]}^{-1}\mathrm{T}_{[1,k]}^{-1}\mathrm{B}_\rho) \\
&\quad\cong \hom_{K^b(\widehat{\mathcal{S}}_n^{\mathrm{ext}})}(\mathrm{T}_{[1,k]}^{-1}\mathrm{B}_0, \mathrm{T}_{[0,k-1]}\mathrm{B}_k\mathrm{T}_{[0,k-1]}^{-1}\mathrm{T}_{[1,k]}^{-1}) \\
&\quad\cong \hom_{K^b(\widehat{\mathcal{S}}_n^{\mathrm{ext}})}(\mathrm{B}_0, \mathrm{T}_{[1,k]}\mathrm{T}_{[0,k-1]}\mathrm{B}_k\mathrm{T}_{[0,k-1]}^{-1}\mathrm{T}_{[1,k]}^{-1}) \\
&\quad\overset{(78)}{\cong} \hom_{K^b(\widehat{\mathcal{S}}_n^{\mathrm{ext}})}(\mathrm{B}_0, \mathrm{B}_0) \cong \hom_{\widehat{\mathcal{S}}_n^{\mathrm{ext}}}(\mathrm{B}_0, \mathrm{B}_0) \cong \hom_{\widehat{\mathcal{S}}_{n-k}^{\mathrm{ext}}}(\mathrm{B}_0, \mathrm{B}_0) \\
&\quad\cong \hom_{\widehat{\mathcal{S}}_{n-k}^{\mathrm{ext}}}(\mathrm{B}_\rho\mathrm{B}_{n-k-1}, \mathrm{B}_0\mathrm{B}_\rho).
\end{aligned}$$

For a six-valent vertex in $\widehat{\mathcal{S}}^{\mathrm{ext}}_{n-k}$ involving strands labeled 0 and $n-k-1$ we compute

$$\begin{aligned}
&\hom_{K^b(\widehat{\mathcal{S}}^{\mathrm{ext}}_n)}(\Psi_R(\mathrm{B}_{n-k-1}\mathrm{B}_0\mathrm{B}_{n-k-1}), \Psi_R(\mathrm{B}_0\mathrm{B}_{n-k-1}\mathrm{B}_0))\\
\cong\ &\hom_{K^b(\widehat{\mathcal{S}}^{\mathrm{ext}}_n)}(\mathrm{B}_{n-1}\mathrm{T}_{[0,k-1]}\mathrm{B}_k\mathrm{T}^{-1}_{[0,k-1]}\mathrm{B}_{n-1}, \mathrm{T}_{[0,k-1]}\mathrm{B}_k\mathrm{T}^{-1}_{[0,k-1]}\mathrm{B}_{n-1}\mathrm{T}_{[0,k-1]}\mathrm{B}_k\mathrm{T}^{-1}_{[0,k-1]})\\
\cong\ &\hom_{K^b(\widehat{\mathcal{S}}^{\mathrm{ext}}_n)}(\mathrm{T}^{-1}_{[0,k-1]}\mathrm{B}_{n-1}\mathrm{T}_{[0,k-1]}\mathrm{B}_k\mathrm{T}^{-1}_{[0,k-1]}\mathrm{B}_{n-1}\mathrm{T}_{[0,k-1]}\mathrm{B}_k\mathrm{T}^{-1}_{[0,k-1]}\mathrm{B}_{n-1}\mathrm{T}_{[0,k-1]}\mathrm{B}_k, \mathbb{1})\\
\overset{(78)}{\cong}\ &\hom_{K^b(\widehat{\mathcal{S}}^{\mathrm{ext}}_n)}(\mathrm{T}^{-1}_{[-1,k-2]}\mathrm{B}_{k-1}\mathrm{T}_{[-1,k-2]}\mathrm{B}_k\mathrm{T}^{-1}_{[-1,k-2]}\mathrm{B}_{k-1}\mathrm{T}_{[-1,k-2]}\mathrm{B}_k\mathrm{T}^{-1}_{[-1,k-2]}\mathrm{B}_{k-1}\mathrm{T}_{[-1,k-2]}\mathrm{B}_k, \mathbb{1})\\
\cong\ &\hom_{K^b(\widehat{\mathcal{S}}^{\mathrm{ext}}_n)}(\mathrm{B}_{k-1}\mathrm{B}_k\mathrm{B}_{k-1}\mathrm{B}_k\mathrm{B}_{k-1}\mathrm{B}_k, \mathbb{1}) \cong \hom_{\widehat{\mathcal{S}}^{\mathrm{ext}}_n}(\mathrm{B}_{k-1}\mathrm{B}_k\mathrm{B}_{k-1}, \mathrm{B}_k\mathrm{B}_{k-1}\mathrm{B}_k)\\
\cong\ &\hom_{\widehat{\mathcal{S}}^{\mathrm{ext}}_{n-k}}(\mathrm{B}_{n-k-1}\mathrm{B}_0\mathrm{B}_{n-k-1}, \mathrm{B}_0\mathrm{B}_{n-k-1}\mathrm{B}_0). \qquad \square
\end{aligned}$$

4.2. **Proof of the main theorem.**

Lemma 4.8. *The $\mathbb{R}$-linear, strict monoidal functors Ψ_L and Ψ_R from Section 4.1.2*

a) are well-defined;

b) form a symmetric pair.

Theorem 4.9. *The symmetric pair Ψ_L, Ψ_R gives rise to an $\mathbb{R}$-linear, monoidal functor*

$$\Psi_{k,n-k} \colon \widehat{\mathcal{S}}^{\mathrm{ext}}_k \boxtimes \widehat{\mathcal{S}}^{\mathrm{ext}}_{n-k} \to K^b(\widehat{\mathcal{S}}^{\mathrm{ext}}_n).$$

Proof. By Lemma 4.4 and Lemma 4.8, the $\mathbb{R}$-linear functor in (97)

$$\Psi_{k,n-k} \colon \widehat{\mathcal{BS}}^{\mathrm{ext}}_k \diamond \widehat{\mathcal{BS}}^{\mathrm{ext}}_{n-k} \to K^b(\widehat{\mathcal{S}}^{\mathrm{ext}}_n)$$

extends to a monoidal functor, which in turn gives rise to an $\mathbb{R}$-linear, monoidal functor

$$\Psi_{k,n-k} \colon \widehat{\mathcal{S}}^{\mathrm{ext}}_k \boxtimes \widehat{\mathcal{S}}^{\mathrm{ext}}_{n-k} \to K^b(\widehat{\mathcal{S}}^{\mathrm{ext}}_n),$$

as explained in Section 4.1.1. $\square$

It remains to prove Lemma 4.8, which will be done in Section 4.2.1 and Section 4.2.3.

4.2.1. *Well-definedness.* The proofs that Ψ_L and Ψ_R preserve all diagrammatic relations are very similar, so we only prove that Ψ_L is well-defined. Because of the two cases in (37), the proof that Ψ_L is well-defined contains four cases: $1 = k = n-1$, $1 = k < n-1$, $2 = k < n$ and $2 < k < n$. However, the first two cases are proved in the same way as the corresponding part in the fourth case. The third case is also almost the same as the corresponding part of the fourth case, except for the proof that Ψ_L preserves (37), which we therefore prove separately at the end of this subsection.

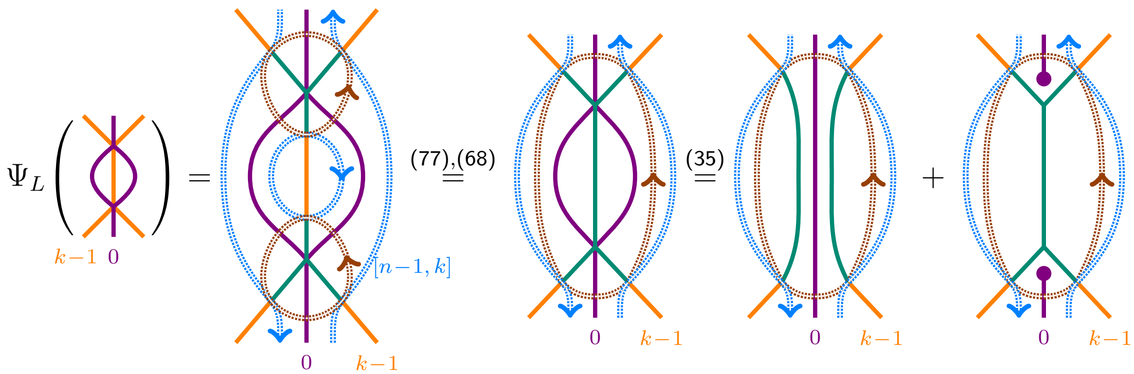
Suppose that $k > 2$. Since Ψ_L is the identity on diagrams only involving colors different from 0 and $k-1$ it preserves the relations involving only those diagrams.

(a1) Relations (26) to (29) for $i = 0$ follow by isotopy and similar arguments as in the first bullet point in the proof of [MMV, Theorem 5.4].

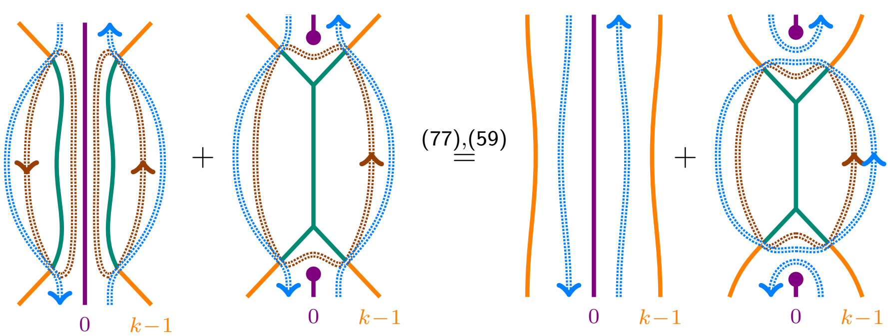
(a2) Relations (30) to (33) for $i = 0$ or $k = 0$ are verified using the relations in (68) and the usual Soergel calculus.

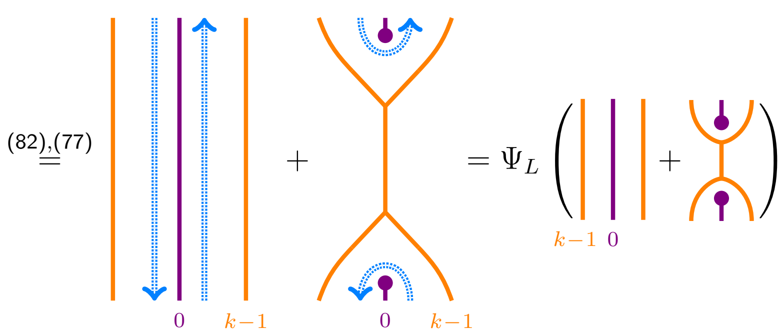
(a3) Relations (34) to (37) for $\{i, j\} = \{0, 1\}$ are checked using similar arguments as in the third bullet point in the proof of [MMV, Theorem 5.4].

(a4) Relations (34) to (37) for $\{i, j\} = \{0, k-1\}$ are more involved and we give a sample computation for relation (35), with the proof of the other relations being similar. We have



where the missing labels in the inner part of the diagram are $n-1$ and $[n-2, k-1]$. Using (59) twice on the strands labeled $[n-2, k-1]$ on the first term, and (68) on the second gives



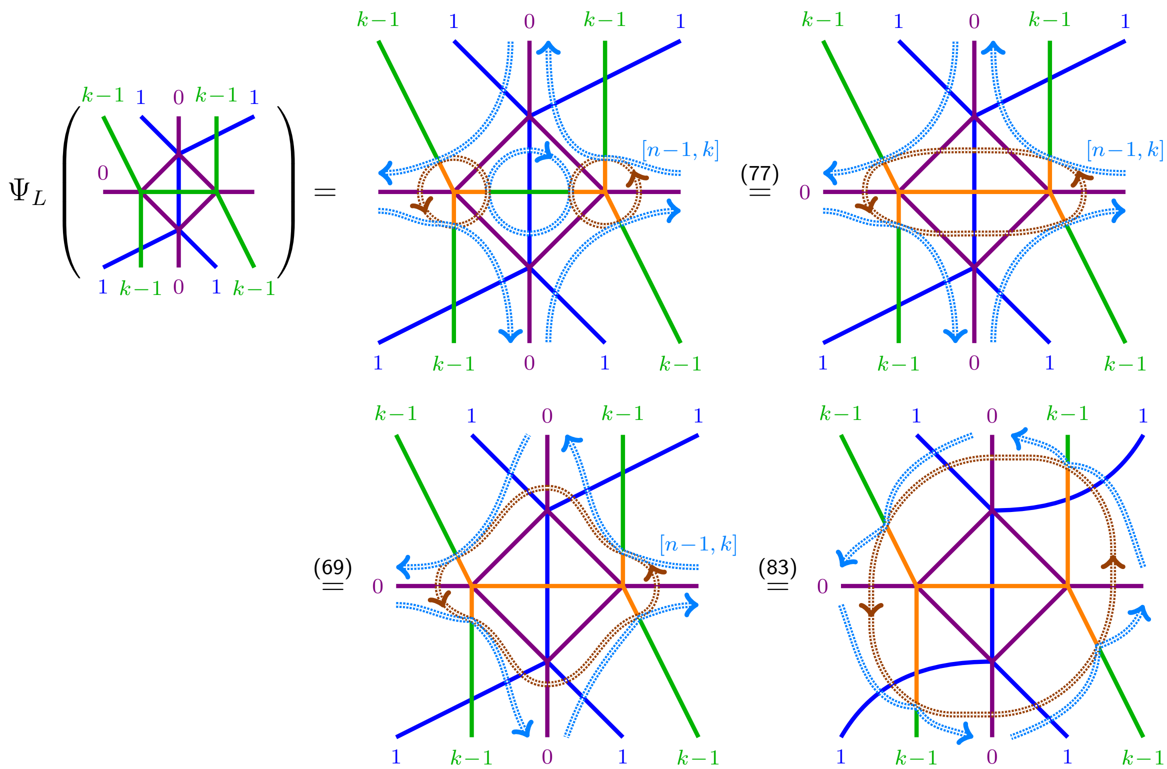


(a5) Relation (38) for one of the labels being 0 follows easily.

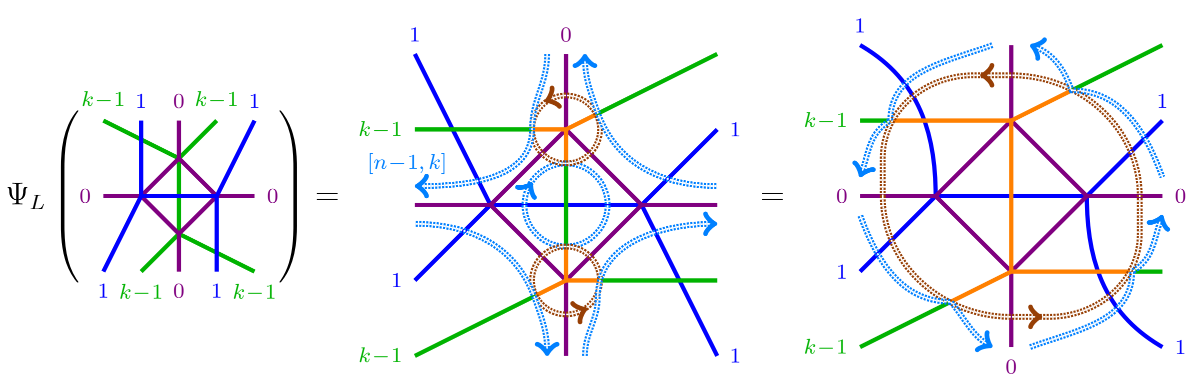
(a6) The relevant cases for relation (39) are $\{i,j\} = \{0,1\}$, $\{i,j\} = \{0,k-1\}$ or $k=0$ and all are straightforward.

(a7) The relevant cases for relation (40) are $(i,j,k) = (0,1,2)$, $(i,j,k) = (k-1,0,1)$ and $(i,j,k) = (k-2,k-1,0)$. Using arguments similar to the seventh bullet point in the proof of [MMV, Theorem 5.4] one verifies the first case. We provide the proof of the second and the third is similar.

Applying Ψ_L to the left-hand side of (40) yields (the unlabeled strands of the diagrams on the right-hand side have labels $[n-2,k-1]$, $k-1$)



A similar procedure gives

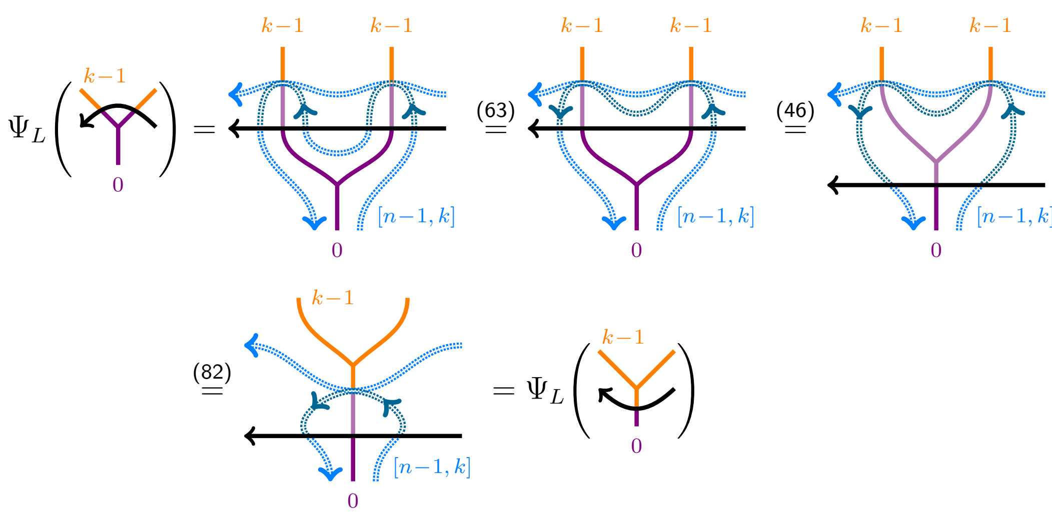


Relation (40) for labels $(k-1,0,1)$ implies that they are equal.

(a8) Relations (41) and (42) are an easy consequence of (57) and (59).

(a9) Relation (43) is immediate if $0 \notin \{i-1,i,j-1,j\}$. The remaining cases can be proved using relations (74), (81) and (83).

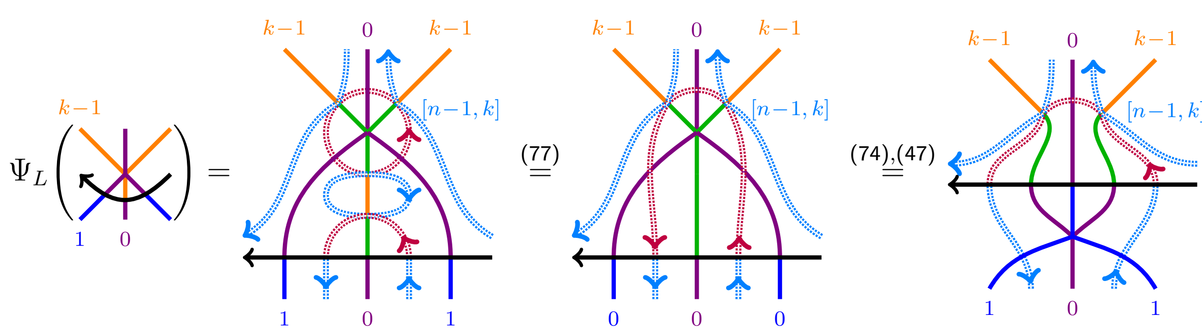
(a10) Checking relation (44) uses only Lemma 3.5 and relations (71) and (63) unless $(i-1,i) = (k-1,0)$. The remaining cases use (77) and (63). Relation (45) is verified using Lemma 3.5 and relations (63) and (80). For (46) the only case that is not immediate is when $i=0$. In this case,

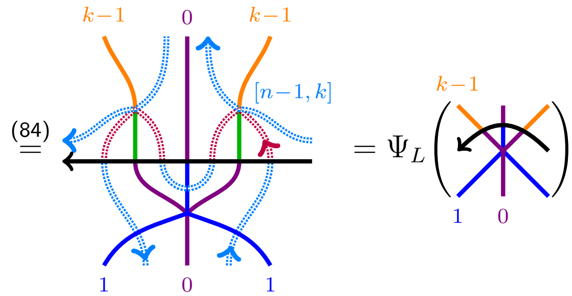


(a11) For relations (47) we need to consider the cases $i-1=0$, $i=0$ and $i+1=0$. The first case uses (68) and (57).

In the remaining cases, we prove the first relation, the second one being similar.

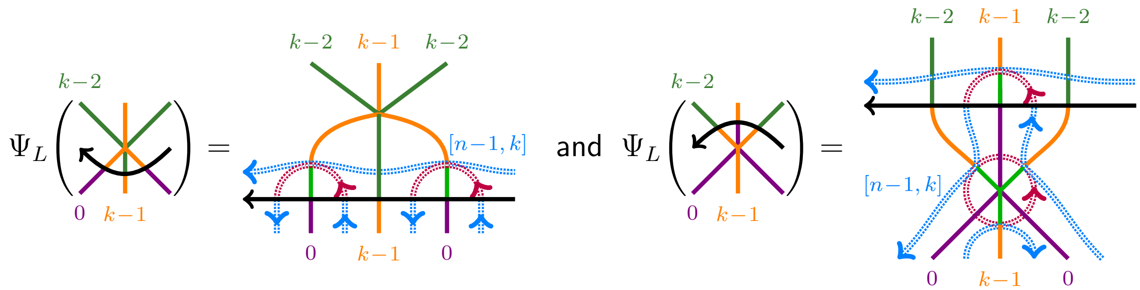
For $i=0$ we compute





For the case $i+1=0$ we use the hom & dot trick. We have that

(98)



define morphisms in

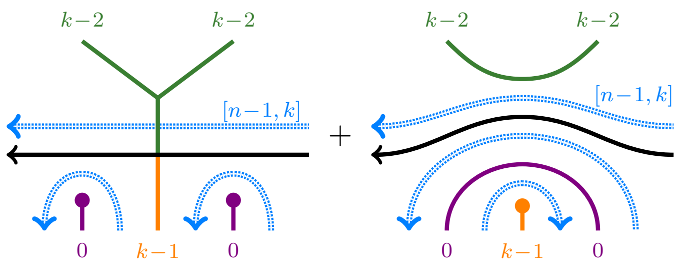
$$\hom_{K^b(\widehat{\mathbb{S}}_n^{\mathrm{ext}})}\left(\mathrm{T}_{[n-1,k]}^{-1}\mathrm{B}_0\mathrm{T}_{[n-1,k]}\mathrm{B}_{k-1}\mathrm{T}_{[n-1,k]}^{-1}\mathrm{B}_0\mathrm{T}_{[n-1,k]},\mathrm{B}_\rho\mathrm{T}_{[n-1,k]}\mathrm{B}_{k-2}\mathrm{B}_{k-1}\mathrm{B}_{k-2}\mathrm{T}_{[n-1,k]}^{-1}\mathrm{B}_\rho^{-1}\right).$$

This morphism space is one-dimensional, as can be seen as follows. By (75) (with $a=k$ and $b=0$) and (68), we have $\mathrm{T}_{[n-1,k]}^{-1}\mathrm{B}_0\mathrm{T}_{[n-1,k]}\simeq\mathrm{T}_{[n,k+1]}\mathrm{B}_k\mathrm{T}_{[n,k+1]}^{-1}$ and $\mathrm{T}_{[n,k+1]}\mathrm{B}_{k-1}\simeq\mathrm{B}_{k-1}\mathrm{T}_{[n,k+1]}$. This implies that the domain is homotopy equivalent to $\mathrm{T}_{[n,k+1]}\mathrm{B}_k\mathrm{B}_{k-1}\mathrm{B}_k\mathrm{T}_{[n,k+1]}^{-1}$. Commuting B_ρ with all factors from left to right shows that the codomain is homotopy equivalent to $\mathrm{T}_{[n,k+1]}\mathrm{B}_{k-1}\mathrm{B}_k\mathrm{B}_{k-1}\mathrm{T}_{[n,k+1]}^{-1}$. By adjunction, the above morphism space is therefore isomorphic to $\hom_{K^b(\widehat{\mathbb{S}}_n^{\mathrm{ext}})}(\mathrm{B}_k\mathrm{B}_{k-1}\mathrm{B}_k,\mathrm{B}_{k-1}\mathrm{B}_k\mathrm{B}_{k-1})$. Moreover, we have

$$\hom_{K^b(\widehat{\mathbb{S}}_n^{\mathrm{ext}})}(\mathrm{B}_k\mathrm{B}_{k-1}\mathrm{B}_k,\mathrm{B}_{k-1}\mathrm{B}_k\mathrm{B}_{k-1})\cong\hom_{\widehat{\mathbb{S}}_n^{\mathrm{ext}}}(\mathrm{B}_k\mathrm{B}_{k-1}\mathrm{B}_k,\mathrm{B}_{k-1}\mathrm{B}_k\mathrm{B}_{k-1})$$

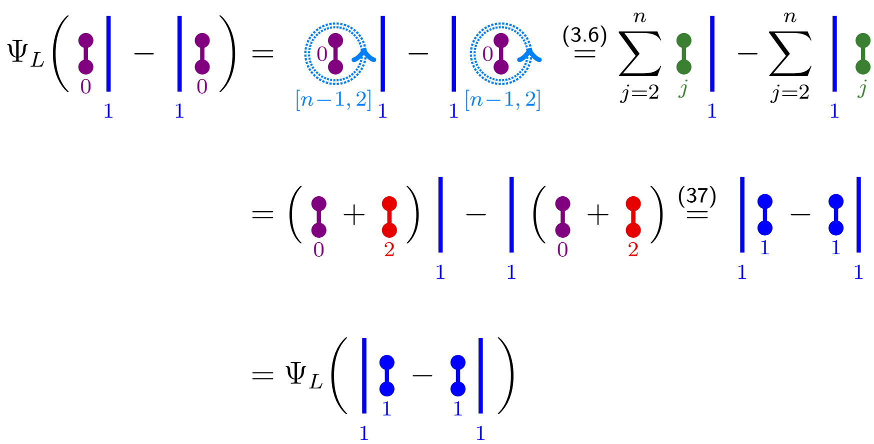
and the latter morphism space is one-dimensional (thanks to Soergel's hom formula in (49)).

We attach a dot to the free end at the top labeled $k-1$ of the diagrams at the right-hand sides of (98). For the left diagram in (98), use (34) to get rid of the 6-valent vertex and obtain two terms which can be further simplified by relations (80), (45), (63) and (77). For the right diagram in (98), first use (80), (45), (63) and (68) to slide the dot closer to the 6-valent vertex, then proceed as in the previous case. In the end, we see that both diagrams in (98) become equal to

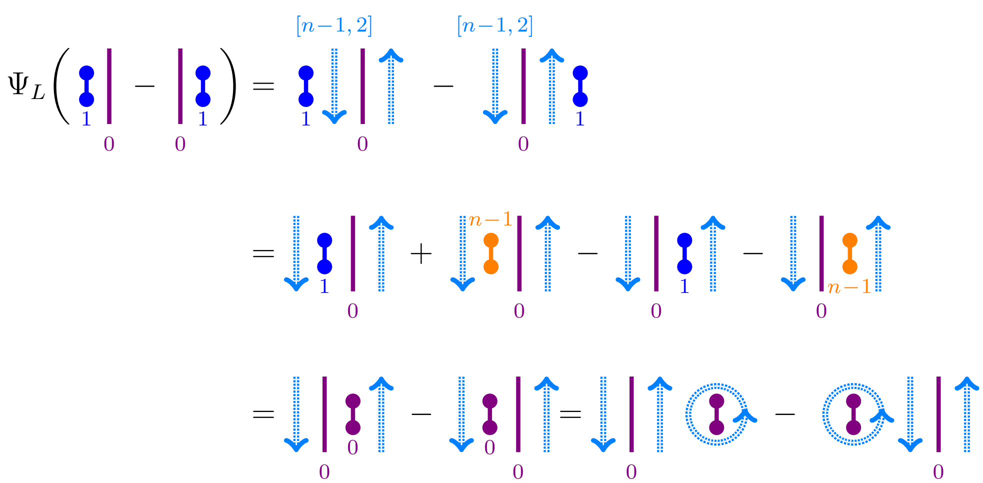


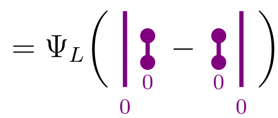
This completes the proof that Ψ_L is well-defined for $k \neq 2$.

Now, let $k = 2$. As already mentioned, it only remains to check (37) in this case. We have

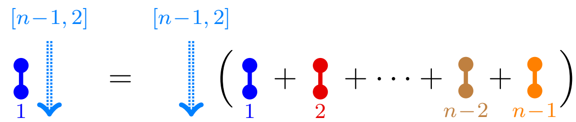


Now, with 0 and 1 switched,





where in the second equality we used (an easy consequence of [MMV, Lemma 4.6])



and in the third equality we used (37) twice, one involving labels 1 and 0 and the other involving colors 0 and $n-1$.

This completes the proof of the last case. □

4.2.2. *Definition of* ζ. We will first show that for every pair of objects $X \in \widehat{\mathcal{BS}}_k^{\mathrm{ext}}, Y \in \widehat{\mathcal{BS}}_{n-k}^{\mathrm{ext}}$, there is an isomorphism in $K^b(\widehat{\mathcal{S}}_n^{\mathrm{ext}})$

$$\zeta_{X,Y} \colon \Psi_L(X)\Psi_R(Y) \xrightarrow{\cong} \Psi_R(Y)\Psi_L(X)$$

and that these isomorphisms satisfy the hexagon identities from Definition 4.2. After that, we will show that ζ is natural in X and Y.

We define $\zeta_{X,Y}$ for pairs of generating objects $X \in \{\mathrm{B}_i, \mathrm{B}_\rho\} \subset \mathrm{Ob}(\widehat{\mathcal{BS}}_k^{\mathrm{ext}})$ and $Y \in \{\mathrm{B}_i, \mathrm{B}_\rho\} \subset \mathrm{Ob}(\widehat{\mathcal{BS}}_{n-k}^{\mathrm{ext}})$ and extend the definition recursively to pairs of monoidal products of generating objects (corresponding to Bott-Samelson bimodules) by the hexagon identities. Concretely, we put

$$\zeta_{X_1X_2,Y_1Y_2} := \zeta_{X_1,Y_1Y_2} \cdot \zeta_{X_2,Y_1Y_2} = \zeta_{X_1,Y_2} \cdot \zeta_{X_1,Y_1} \cdot \zeta_{X_2,Y_2} \cdot \zeta_{X_2,Y_1} \tag{99}$$

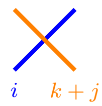
for any objects $X_1, X_2 \in \widehat{\mathcal{BS}}_k^{\mathrm{ext}}$ and $Y_1, Y_2 \in \widehat{\mathcal{BS}}_{n-k}^{\mathrm{ext}}$. Alternatively, we could have put

$$\zeta_{X_1X_2,Y_1Y_2} := \zeta_{X_1X_2,Y_2} \cdot \zeta_{X_1X_2,Y_1} = \zeta_{X_1,Y_2} \cdot \zeta_{X_2,Y_2} \cdot \zeta_{X_1,Y_1} \cdot \zeta_{X_2,Y_1}, \tag{100}$$

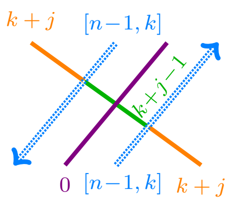
but the expressions in (99) and (100) are equal because $\zeta_{X_1,Y_1} \cdot \zeta_{X_2,Y_2} = \zeta_{X_2,Y_2} \cdot \zeta_{X_1,Y_1}$ thanks to the interchange law, see (95). In particular, the isomorphisms $\zeta_{X,Y}$ satisfy the hexagon identities by definition.

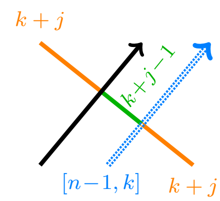
After this explanation, it remains to define $\zeta_{X,Y}$ for pairs of generating objects.

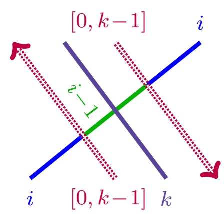
In all cases below, $\zeta_{X,Y}$ is obtained by composing invertible morphisms in $K^b(\widehat{\mathcal{S}}_n^{\mathrm{ext}})$. To see this, one notes that all diagrams appearing are isotopic to a diagram where at each height there is either a merger or a splitter, a four-valent vertex (a crossing), a six-valent one (all involving at least one oriented strand), or just vertical strands. One checks easily that all of them are invertible (this is implied by the various relations in Section 3.1 and Section 3.2), hence so is their composite. Therefore, we only give the diagrams for $\zeta_{X,Y}$ without further explanation.

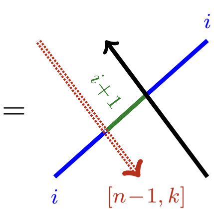
$\zeta_{\mathrm{B}_i,\mathrm{B}_j} =$ 

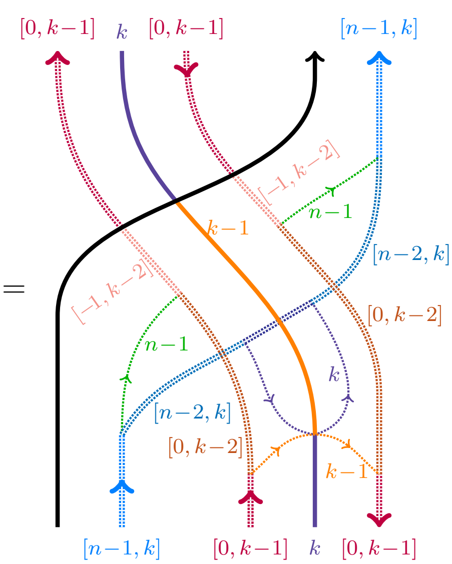
$(i = 1, \ldots, k-1,\ j = 1, \ldots, n-k-1)$

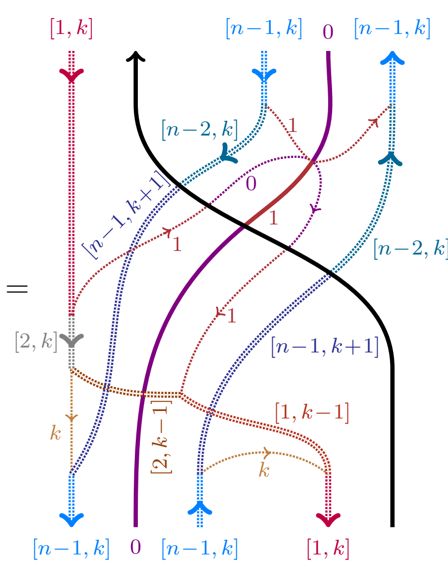
$\zeta_{\mathrm{B}_0,\mathrm{B}_j} =$ 

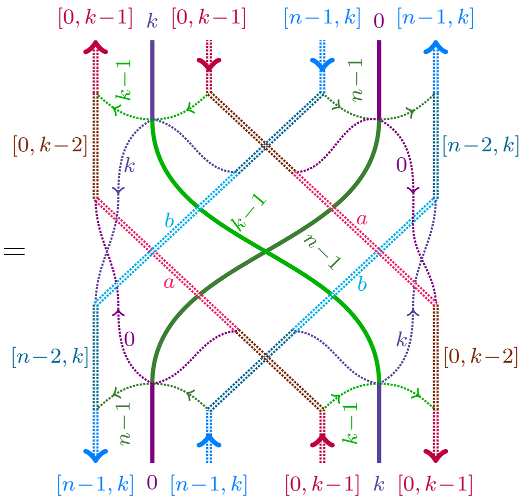
$\zeta_{\mathrm{B}_\rho,\mathrm{B}_j} =$ 

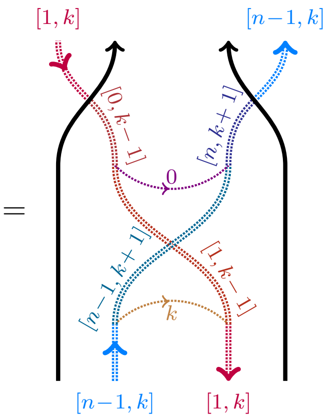
$\zeta_{\mathrm{B}_i,\mathrm{B}_0} =$ 

$\zeta_{\mathrm{B}_i,\mathrm{B}_\rho} =$ 

$\zeta_{\mathrm{B}_\rho,\mathrm{B}_0} =$ 

$\zeta_{\mathrm{B}_0,\mathrm{B}_\rho} =$ 

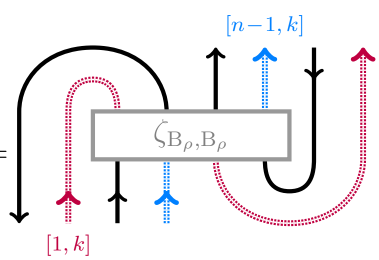
$\zeta_{\mathrm{B}_0,\mathrm{B}_0} =$ 

$\zeta_{\mathrm{B}_\rho,\mathrm{B}_\rho} =$ 

In $\zeta_{\mathrm{B}_\rho,\mathrm{B}_0}$, $\mathrm{T}_{[-1,k-2]} = \mathrm{T}_{n-1}\mathrm{T}_0\mathrm{T}_1 \cdots \mathrm{T}_{k-2}$, and in $\zeta_{\mathrm{B}_0,\mathrm{B}_0}$, $a = [1, k-2]$ and $b = [n-2, k+1]$.

This completes the definition of ζ.

Remark 4.10. Diagrams for ζ involving inverses of B_ρ^{-1} are obtained easily by applying the appropriate adjunctions and taking inverses (if necessary). We have

$\zeta^{-1}_{\mathrm{B}_\rho,\mathrm{B}_\rho^{-1}} =$ 

and similarly for $\zeta^{-1}_{\mathrm{B}_\rho^{-1},\mathrm{B}_\rho}$. A rotation of $\zeta_{\mathrm{B}_\rho,\mathrm{B}_\rho}$ by 180 degrees yields $\zeta_{\mathrm{B}_\rho^{-1},\mathrm{B}_\rho^{-1}}$.

4.2.3. *Naturality of ζ.* To show that ζ is natural, we have to prove commutativity of the diagram

$$\begin{array}{ccc} \Psi_L(X)\Psi_R(Y) & \xrightarrow{\zeta_{X,Y}} & \Psi_R(Y)\Psi_L(X) \\ \Big\downarrow {\scriptstyle \Psi_L(f)\Psi_R(g)} & & \Big\downarrow {\scriptstyle \Psi_R(g)\Psi_L(f)} \\ \Psi_L(W)\Psi_R(Z) & \xrightarrow{\zeta_{W,Z}} & \Psi_R(Z)\Psi_L(W) \end{array} \tag{101}$$

for all objects $X, W \in \widehat{\mathcal{BS}}_k^{\mathrm{ext}}$ and $Y, Z \in \widehat{\mathcal{BS}}_{n-k}^{\mathrm{ext}}$, and all morphisms $f \in \hom_{\widehat{\mathcal{BS}}_k^{\mathrm{ext}}}(X, W)$ and $g \in \hom_{\widehat{\mathcal{BS}}_{n-k}^{\mathrm{ext}}}(Y, Z)$. Thanks to the hexagon identities for ζ and the interchange law for morphisms in $K^b(\widehat{\mathcal{S}}_n^{\mathrm{ext}})$, it suffices to show commutativity in (101) for pairs (f, g) where f is a generating morphism in $\widehat{\mathcal{BS}}_k^{\mathrm{ext}}$ and $g = \mathrm{id}_Y$ for some generating object $Y \in \widehat{\mathcal{BS}}_{n-k}^{\mathrm{ext}}$, or $f = \mathrm{id}_X$ for some generating object $X \in \widehat{\mathcal{BS}}_k^{\mathrm{ext}}$ and g is a generating morphism in $\widehat{\mathcal{BS}}_{n-k}^{\mathrm{ext}}$. Finally, we note that the proof for pairs of the first type and the proof for pairs of the second type are very similar, so we only give the proof for pairs of the second type below. Concretely, those pairs are (id_X, g), where $X \in \{\mathrm{B}_\rho, \mathrm{B}_0, \dots, \mathrm{B}_{k-1}\}$ and g is one of the generators in (23), (24) or (25) with labels in $\{\rho, 0, \dots, n-k-1\}$. The only cases in which commutativity of (101) is not immediate, is when $X \in \{\mathrm{B}_\rho, \mathrm{B}_0\}$ and/or the labels of some strands of g belong to $\{0, \rho\}$.

Note that checking commutativity of (101) always consists of verifying the equality of two diagrams, both belonging to the same morphism space. In all but two cases the dimension of the morphism space in the given degree is one-dimensional (see Lemma 4.11), so we can use the hom & dot trick, explained in Remark 3.7. The remaining two cases are checked directly.

Lemma 4.11. *For each generating object X of $\widehat{\mathcal{BS}}_k^{\mathrm{ext}}$ and each generating morphism $g\colon Y \to Z$ in $\widehat{\mathcal{BS}}_{n-k}^{\mathrm{ext}}$, there are objects $U, V \in \widehat{\mathcal{BS}}_n^{\mathrm{ext}}$ and an isomorphism*

(102) $$\hom_{K^b(\widehat{\mathcal{S}}_n^{\mathrm{ext}})}(\Psi_L(X)\Psi_R(Y), \Psi_R(Z)\Psi_L(X)) \cong \hom_{\widehat{\mathcal{BS}}_n^{\mathrm{ext}}}(U, V).$$

Unless $X = \mathrm{B}_0$ and $g\colon 1 \to \mathrm{B}_0$ is a dot labeled 0 or $g\colon \mathrm{B}_0 \to \mathrm{B}_0\mathrm{B}_0$ is a trivalent vertex labeled 0, the two morphism spaces in (102) *are one-dimensional by Soergel's hom formula in* (49)*.*

Proof. We first note that by invertibility of $\zeta_{X,Z}$ we have $\Psi_R(Z)\Psi_L(X) \cong \Psi_L(X)\Psi_R(Z)$ and we need to compute $\hom_{K^b(\widehat{\mathcal{S}}_n^{\mathrm{ext}})}(\Psi_L(X)\Psi_R(Y), \Psi_L(X)\Psi_R(Z))$. For invertible X, the claim follows from Lemma 4.7 and therefore we have to consider only the cases $X = \mathrm{B}_i$. The case of $i > 0$ is an easy computation in the lines of the proof of Lemma 4.7. For $i = 0$ we proceed case-by-case.

- For $g =$ ,

$$\begin{aligned}
&\hom_{K^b(\widehat{\mathcal{S}}_n^{\mathrm{ext}})}(\Psi_L(\mathrm{B}_0), \Psi_L(\mathrm{B}_0)\Psi_R(\mathrm{B}_0)\langle 1\rangle) \\
&\quad = \hom_{K^b(\widehat{\mathcal{S}}_n^{\mathrm{ext}})}(\mathrm{T}_{[n-1,k]}^{-1}\mathrm{B}_0\mathrm{T}_{[n-1,k]}, \mathrm{T}_{[n-1,k]}^{-1}\mathrm{B}_0\mathrm{T}_{[n-1,k]}\mathrm{T}_{[0,k-1]}\mathrm{B}_k\mathrm{T}_{[0,k-1]}^{-1}\langle 1\rangle) \\
&\quad \cong \hom_{K^b(\widehat{\mathcal{S}}_n^{\mathrm{ext}})}(\mathbb{1}, \mathrm{B}_0\mathrm{T}_{[n-1,k]}\mathrm{T}_{[0,k-1]}\mathrm{B}_k\mathrm{T}_{[0,k-1]}^{-1}\mathrm{T}_{[n-1,k]}^{-1}\mathrm{B}_0\langle 1\rangle) \\
&\quad \cong \hom_{K^b(\widehat{\mathcal{S}}_n^{\mathrm{ext}})}(\mathbb{1}, \mathrm{B}_0\mathrm{T}_{[n-1,k]}\mathrm{T}_{[1,k]}^{-1}\mathrm{B}_0\mathrm{T}_{[1,k]}\mathrm{T}_{[n-1,k]}^{-1}\mathrm{B}_0\langle 1\rangle) \\
&\quad \cong \hom_{K^b(\widehat{\mathcal{S}}_n^{\mathrm{ext}})}(\mathbb{1}, \mathrm{B}_0\mathrm{T}_{[1,k-1]}^{-1}\mathrm{T}_{[n-1,k+1]}\mathrm{B}_0\mathrm{T}_{[n-1,k+1]}^{-1}\mathrm{T}_{[1,k-1]}\mathrm{B}_0\langle 1\rangle) \\
&\quad \cong \hom_{K^b(\widehat{\mathcal{S}}_n^{\mathrm{ext}})}(\mathbb{1}, \mathrm{B}_0\mathrm{T}_1^{-1}\mathrm{T}_0\mathrm{B}_1\mathrm{T}_0^{-1}\mathrm{T}_1\mathrm{B}_0\langle 1\rangle) \\
&\quad \cong \hom_{K^b(\widehat{\mathcal{S}}_n^{\mathrm{ext}})}(\mathbb{1}, \mathrm{B}_1\mathrm{B}_1\mathrm{B}_1\langle 1\rangle) \cong \hom_{K^b(\widehat{\mathcal{S}}_n^{\mathrm{ext}})}(\mathrm{B}_1, \mathrm{B}_1\mathrm{B}_1\langle 1\rangle) \\
&\quad \cong \hom_{\widehat{\mathcal{S}}_n^{\mathrm{ext}}}(\mathrm{B}_1, \mathrm{B}_1\mathrm{B}_1\langle 1\rangle) \cong \mathbb{R}^3.
\end{aligned}$$

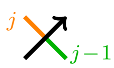
- For $g =$ ,

$$\begin{aligned}
&\hom_{K^b(\widehat{\mathbb{S}}_n^{\mathrm{ext}})}(\Psi_L(\mathrm{B}_0)\Psi_R(\mathrm{B}_0), \Psi_L(\mathrm{B}_0)\Psi_R(\mathrm{B}_0\mathrm{B}_0)\langle -1\rangle)\\
&\qquad = \hom_{K^b(\widehat{\mathbb{S}}_n^{\mathrm{ext}})}(\mathrm{T}_{[n-1,k]}^{-1}\mathrm{B}_0\mathrm{T}_{[n-1,k]}\mathrm{T}_{[0,k-1]}\mathrm{B}_k\mathrm{T}_{[0,k-1]}^{-1},\\
&\qquad\qquad \mathrm{T}_{[n-1,k]}^{-1}\mathrm{B}_0\mathrm{T}_{[n-1,k]}\mathrm{T}_{[0,k-1]}\mathrm{B}_k\mathrm{T}_{[0,k-1]}^{-1}\mathrm{T}_{[0,k-1]}\mathrm{B}_k\mathrm{T}_{[0,k-1]}^{-1}\langle -1\rangle)\\
&\qquad \cong \hom_{K^b(\widehat{\mathbb{S}}_n^{\mathrm{ext}})}(\mathrm{B}_0\mathrm{T}_{[n-1,k]}\mathrm{T}_{[0,k-1]}\mathrm{B}_k, \mathrm{B}_0\mathrm{T}_{[n-1,k]}\mathrm{T}_{[0,k-1]}\mathrm{B}_k\mathrm{B}_k\langle -1\rangle)\\
&\qquad \cong \hom_{K^b(\widehat{\mathbb{S}}_n^{\mathrm{ext}})}(\mathrm{B}_0\mathrm{B}_0, \mathrm{B}_0\mathrm{B}_0\mathrm{B}_0\langle -1\rangle)\\
&\qquad \cong \hom_{\widehat{\mathbb{S}}_n^{\mathrm{ext}}}(\mathrm{B}_0\mathrm{B}_0, \mathrm{B}_0\mathrm{B}_0\mathrm{B}_0\langle -1\rangle) \cong \mathbb{R}^5.
\end{aligned}$$

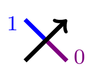
The remaining morphism spaces are one-dimensional. This can be checked using the same type of computation as in the two cases above.

- For $g =$  ,

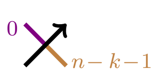
$$\begin{aligned}
&\hom_{K^b(\widehat{\mathbb{S}}_n^{\mathrm{ext}})}(\Psi_L(\mathrm{B}_0)\Psi_R(\mathrm{B}_0\mathrm{B}_j), \Psi_L(\mathrm{B}_0)\Psi_R(\mathrm{B}_j\mathrm{B}_0))\\
&\qquad = \hom_{K^b(\widehat{\mathbb{S}}_n^{\mathrm{ext}})}(\mathrm{T}_{[n-1,k]}^{-1}\mathrm{B}_0\mathrm{T}_{[n-1,k]}\mathrm{T}_{[0,k-1]}\mathrm{B}_k\mathrm{T}_{[0,k-1]}^{-1}\mathrm{B}_{j+k},\\
&\qquad\qquad \mathrm{T}_{[n-1,k]}^{-1}\mathrm{B}_0\mathrm{T}_{[n-1,k]}\mathrm{B}_{k+j}\mathrm{T}_{[0,k-1]}\mathrm{B}_k\mathrm{T}_{[0,k-1]}^{-1})\\
&\qquad \cong \hom_{\widehat{\mathbb{S}}_n^{\mathrm{ext}}}(\mathrm{B}_{n-1}\mathrm{B}_{k-1}\mathrm{B}_{j+k-1}, \mathrm{B}_{n-1}\mathrm{B}_{k-1}\mathrm{B}_{j+k-1}) \cong \mathbb{R}.
\end{aligned}$$

- For $g =$  ,

$$\begin{aligned}
&\hom_{K^b(\widehat{\mathbb{S}}_n^{\mathrm{ext}})}(\Psi_L(\mathrm{B}_0)\Psi_R(\mathrm{B}_\rho\mathrm{B}_{j-1}), \Psi_L(\mathrm{B}_0)\Psi_R(\mathrm{B}_j\mathrm{B}_\rho))\\
&\qquad = \hom_{K^b(\widehat{\mathbb{S}}_n^{\mathrm{ext}})}(\mathrm{T}_{[n-1,k]}^{-1}\mathrm{B}_0\mathrm{T}_{[n-1,k]}\mathrm{T}_{[1,k]}^{-1}\mathrm{B}_\rho\mathrm{B}_{j+k-1}, \mathrm{T}_{[n-1,k]}^{-1}\mathrm{B}_0\mathrm{T}_{[n-1,k]}\mathrm{B}_{j+k}\mathrm{T}_{[1,k]}^{-1}\mathrm{B}_\rho)\\
&\qquad \cong \hom_{\widehat{\mathbb{S}}_n^{\mathrm{ext}}}(\mathrm{B}_0\mathrm{B}_{j+k-1}, \mathrm{B}_0\mathrm{B}_{j+k-1}) \cong \mathbb{R}.
\end{aligned}$$

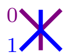
- For $g =$  ,

$$\begin{aligned}
&\hom_{K^b(\widehat{\mathbb{S}}_n^{\mathrm{ext}})}(\Psi_L(\mathrm{B}_0)\Psi_R(\mathrm{B}_\rho\mathrm{B}_0), \Psi_L(\mathrm{B}_0)\Psi_R(\mathrm{B}_1\mathrm{B}_\rho))\\
&\quad = \hom_{K^b(\widehat{\mathbb{S}}_n^{\mathrm{ext}})}(\mathrm{T}_{[n-1,k]}^{-1}\mathrm{B}_0\mathrm{T}_{[n-1,k]}\mathrm{T}_{[1,k]}^{-1}\mathrm{B}_\rho\mathrm{T}_{[0,k-1]}\mathrm{B}_k\mathrm{T}_{[n-1,k]}^{-1}, \mathrm{T}_{[n-1,k]}^{-1}\mathrm{B}_0\mathrm{T}_{[n-1,k]}\mathrm{B}_{1+k}\mathrm{T}_{[1,k]}^{-1}\mathrm{B}_\rho)\\
&\quad \cong \hom_{\widehat{\mathbb{S}}_n^{\mathrm{ext}}}(\mathrm{B}_0\mathrm{B}_k, \mathrm{B}_0\mathrm{B}_k) \cong \mathbb{R}.
\end{aligned}$$

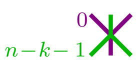
- For $g =$  ,

$$\hom_{K^b(\widehat{\mathbb{S}}_n^{\mathrm{ext}})}(\Psi_L(\mathrm{B}_0)\Psi_R(\mathrm{B}_\rho\mathrm{B}_{n-k-1}), \Psi_L(\mathrm{B}_0)\Psi_R(\mathrm{B}_0\mathrm{B}_\rho))$$

$$= \hom_{K^b(\widehat{\mathbb{S}}_n^{\mathrm{ext}})}(\mathrm{T}^{-1}_{[n-1,k]}\mathrm{B}_0\mathrm{T}_{[n-1,k]}\mathrm{T}^{-1}_{[1,k]}\mathrm{B}_\rho\mathrm{B}_{n-1}, \mathrm{T}^{-1}_{[n-1,k]}\mathrm{B}_0\mathrm{T}_{[n-1,k]}\mathrm{T}_{[0,k-1]}\mathrm{B}_k\mathrm{T}^{-1}_{[0,k-1]}\mathrm{T}^{-1}_{[1,k]}\mathrm{B}_\rho)$$
$$\cong \hom_{\widehat{\mathbb{S}}_n^{\mathrm{ext}}}(\mathrm{B}_{n-1}\mathrm{B}_{k-1}, \mathrm{B}_{n-1}\mathrm{B}_{k-1}) \cong \mathbb{R}.$$

- For $g =$ ,

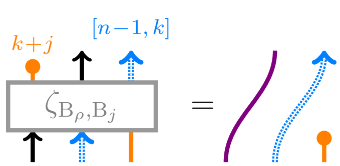
$$\hom_{K^b(\widehat{\mathbb{S}}_n^{\mathrm{ext}})}(\Psi_L(\mathrm{B}_0)\Psi_R(\mathrm{B}_1\mathrm{B}_0\mathrm{B}_1), \Psi_L(\mathrm{B}_0)\Psi_R(\mathrm{B}_0\mathrm{B}_1\mathrm{B}_0))$$
$$= \hom_{K^b(\widehat{\mathbb{S}}_n^{\mathrm{ext}})}(\mathrm{T}^{-1}_{[n-1,k]}\mathrm{B}_0\mathrm{T}_{[n-1,k]}\mathrm{B}_{k+1}\mathrm{T}_{[0,k-1]}\mathrm{B}_k\mathrm{T}^{-1}_{[0,k-1]}\mathrm{B}_{k+1},$$
$$\mathrm{T}^{-1}_{[n-1,k]}\mathrm{B}_0\mathrm{T}_{[n-1,k]}\mathrm{T}_{[0,k-1]}\mathrm{B}_k\mathrm{T}^{-1}_{[0,k-1]}\mathrm{B}_{k+1}\mathrm{T}_{[0,k-1]}\mathrm{B}_k\mathrm{T}^{-1}_{[0,k-1]})$$
$$\cong \hom_{\widehat{\mathbb{S}}_n^{\mathrm{ext}}}(\mathrm{B}_{n-1}\mathrm{B}_k\mathrm{B}_{k-1}\mathrm{B}_k, \mathrm{B}_{n-1}\mathrm{B}_{k-1}\mathrm{B}_k\mathrm{B}_{k-1}) \cong \mathbb{R}.$$

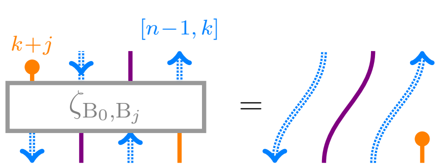
- For $g =$ ,

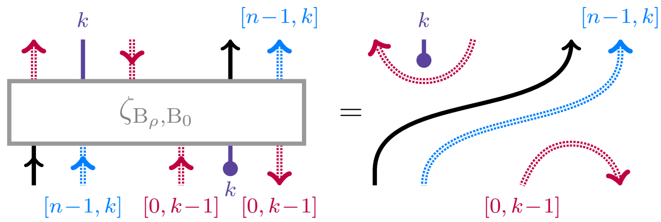
$$\hom_{K^b(\widehat{\mathbb{S}}_n^{\mathrm{ext}})}(\Psi_L(\mathrm{B}_0)\Psi_R(\mathrm{B}_{n-k-1}\mathrm{B}_0\mathrm{B}_{n-k-1}), \Psi_L(\mathrm{B}_0)\Psi_R(\mathrm{B}_0\mathrm{B}_{n-k-1}\mathrm{B}_0))$$
$$= \hom_{K^b(\widehat{\mathbb{S}}_n^{\mathrm{ext}})}(\mathrm{T}^{-1}_{[n-1,k]}\mathrm{B}_0\mathrm{T}_{[n-1,k]}\mathrm{B}_{n-1}\mathrm{T}_{[0,k-1]}\mathrm{B}_k\mathrm{T}^{-1}_{[0,k-1]}\mathrm{B}_{n-1},$$
$$\mathrm{T}^{-1}_{[n-1,k]}\mathrm{B}_0\mathrm{T}_{[n-1,k]}\mathrm{T}_{[0,k-1]}\mathrm{B}_k\mathrm{T}^{-1}_{[0,k-1]}\mathrm{B}_{n-k-1}\mathrm{T}_{[0,k-1]}\mathrm{B}_k\mathrm{T}^{-1}_{[0,k-1]})$$
$$\cong \hom_{\widehat{\mathbb{S}}_n^{\mathrm{ext}}}(\mathrm{B}_{k+1}\mathrm{B}_{k-1}\mathrm{B}_k\mathrm{B}_{k-1}, \mathrm{B}_{k+1}\mathrm{B}_k\mathrm{B}_{k-1}\mathrm{B}_k) \cong \mathbb{R}. \qquad \square$$

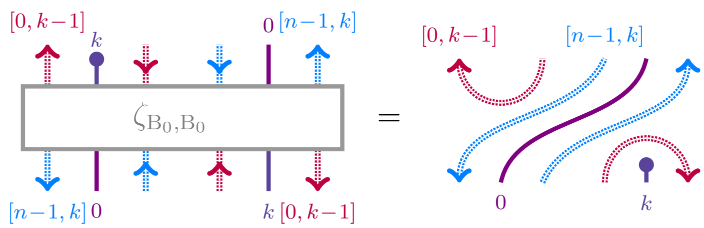
Before checking commutativity of (101), we need a technical lemma. In the following, a rectangular region of a diagram decorated with a $\zeta_{X,Y}$ will be called a *box*.

Lemma 4.12. *The following relations involving boxes and dots hold in* $\mathrm{K}^b(\widehat{\mathbb{S}}_n^{\mathrm{ext}})$*:*

(103) 

(104) 

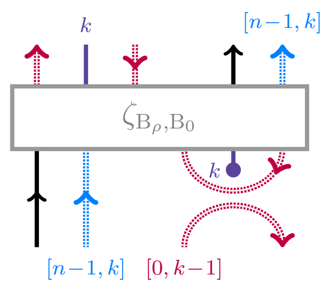
(105) 

(106) 

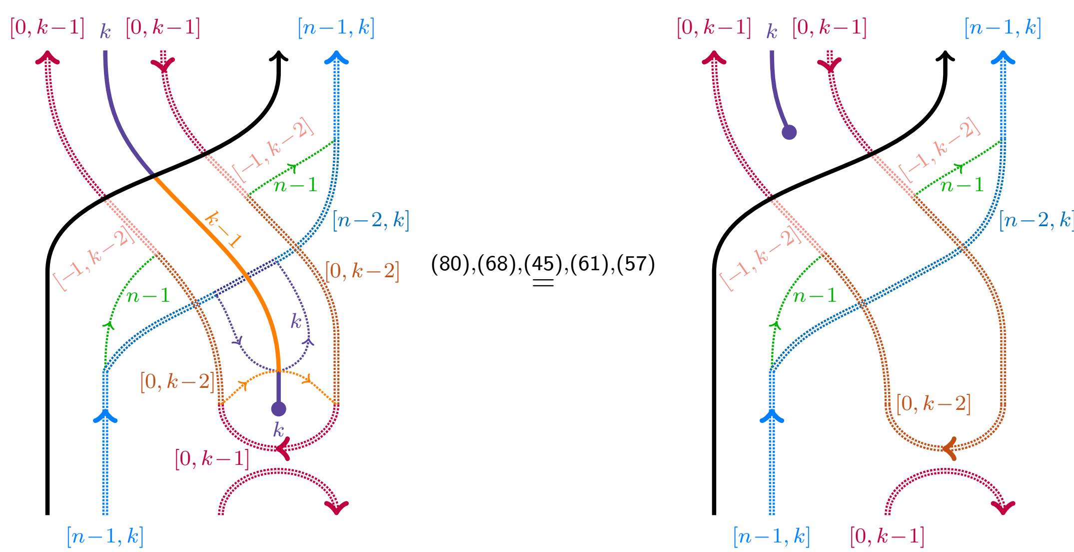
and analogous relations with the dot being attached to any other free end.

Proof. Relations (103) and (104) are immediate.

Proof of (105). By (59) the l.h.s. can be written as

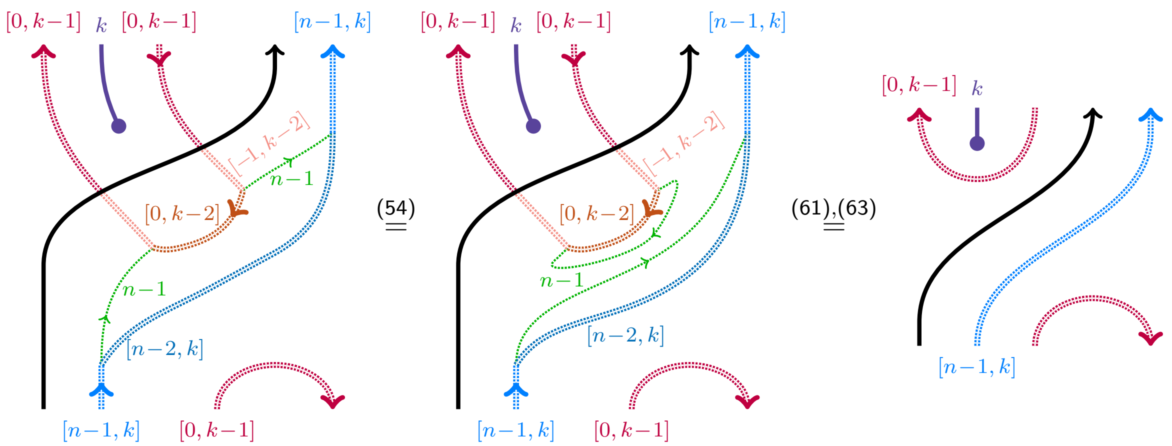


Introducing the diagram of ζ_{B_ρ,B_0} from Section 4.2.3 and computing yields



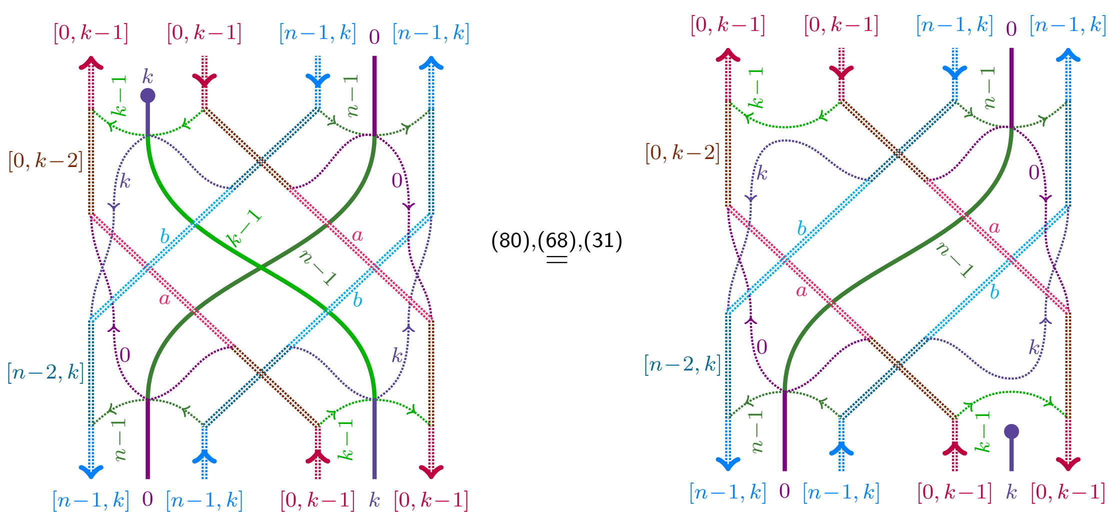
In the equality above, we have used (80) to simplify the only 6-valent vertex appearing (involving labels k and $k-1$, followed by (68) and (45) to slide the dot to the top of the diagram, (61) to simplify the digons involving strands labeled k and $k-1$ that were created by (80), and finally (57) to simplify the loop labeled $k-1$ (created when applying by (61)).

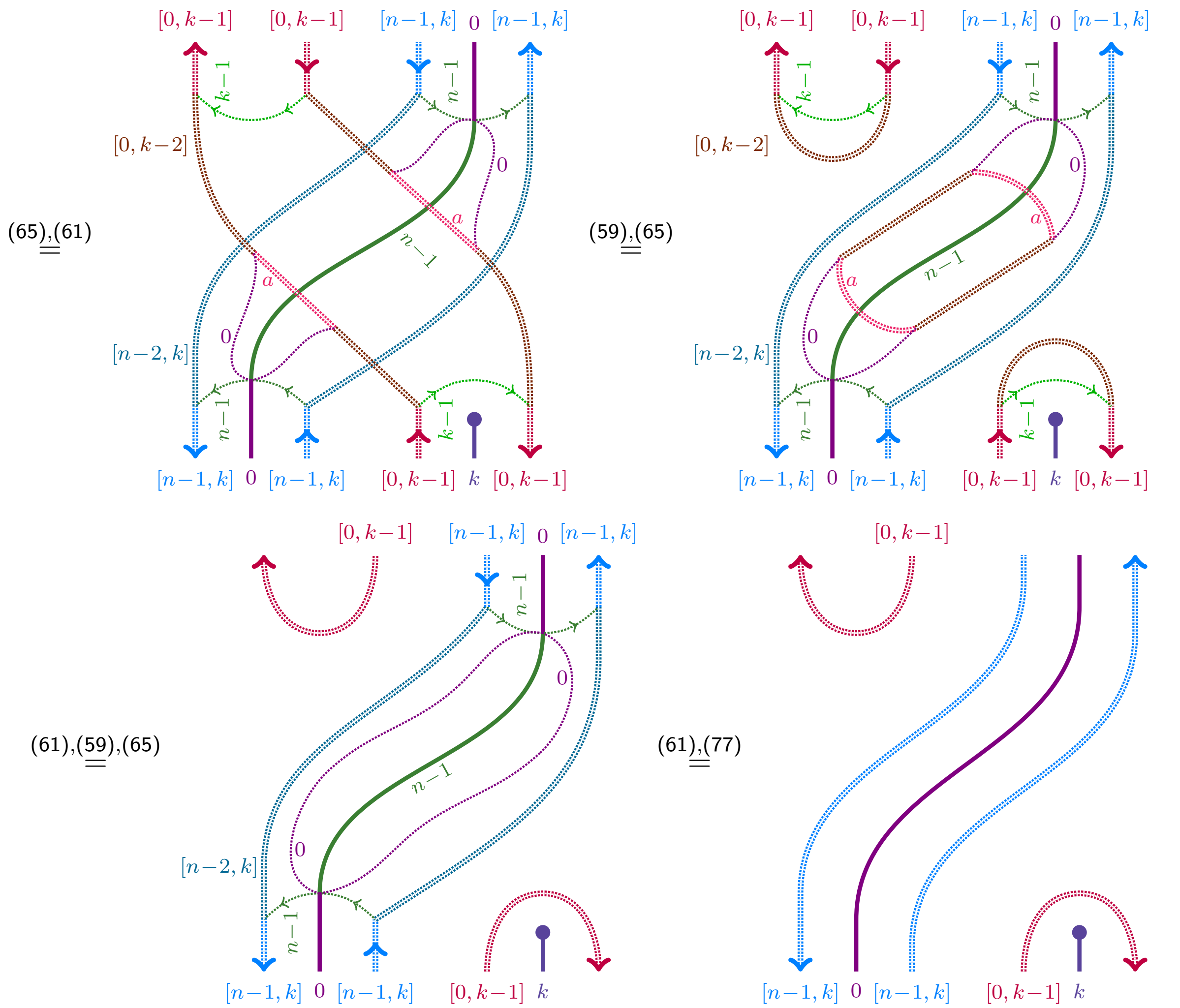
Applying (65) to the strands labeled $[n-2,k]$ and $[0,k-2]$ gives



which is the r.h.s. of (105).

Proof of (106). We introduce the diagram for $\zeta_{\mathrm{B}_0,\mathrm{B}_0}$ in the left-hand side of (106) and compute





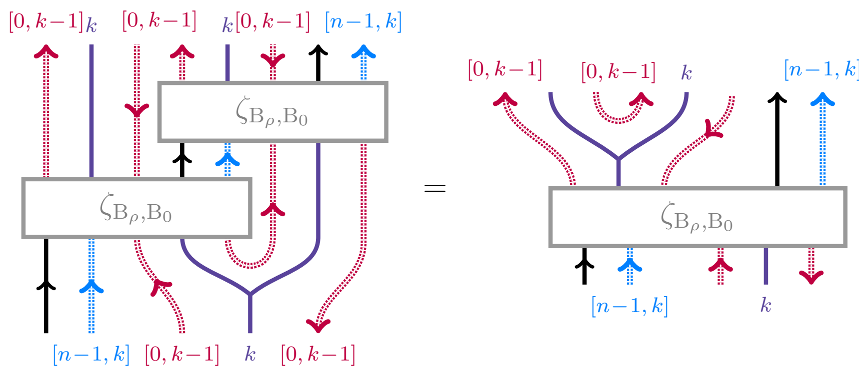
The proof of the analogous relations with the dot being attached at any other free end is similar. □

Now we are ready to prove the naturality of ζ.

(N1) For $X = \mathrm{B}_\rho$ and $g =$  commutativity of (101) follows from (105).

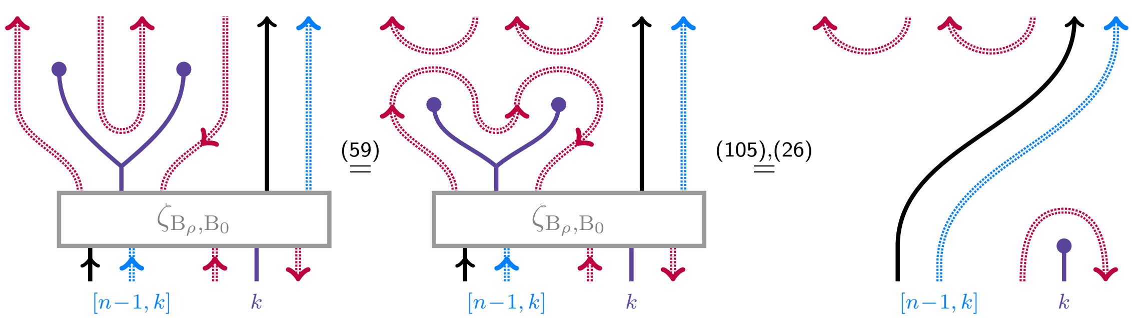
(N2) For $X = B_\rho$ and $g =$  , we have to check

(107)

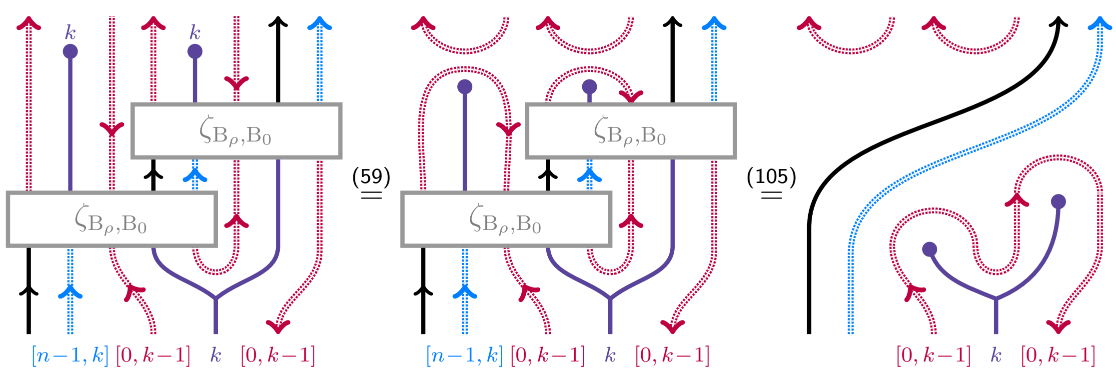


We use the hom & dot trick and attach dots to the upper endpoints labeled k on both diagrams and then use the analogous of (105) with the dot on the top end to pass the dots past the boxes with $\zeta_{\mathrm{B}_\rho,\mathrm{B}_0}$.

The right-hand side of (107) gives



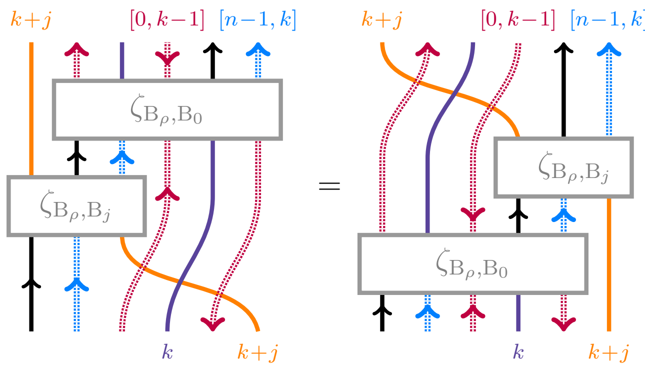
while for the left-hand side we have



which equals the right-hand side by (26).

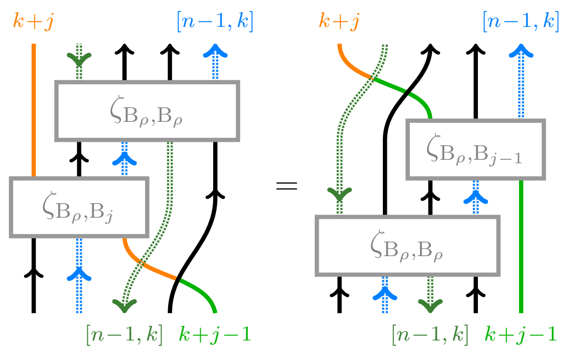
(N3) For the cases $X = \mathrm{B}_\rho$, $g =$  and $X = \mathrm{B}_\rho$, $g =$  we have to check

(108)



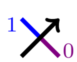
and

(109)

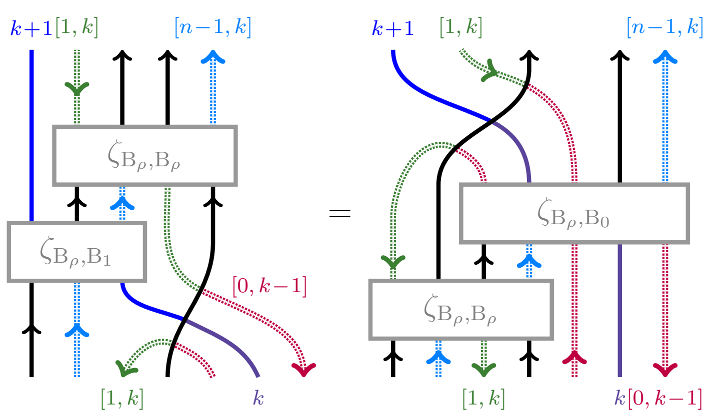


respectively.

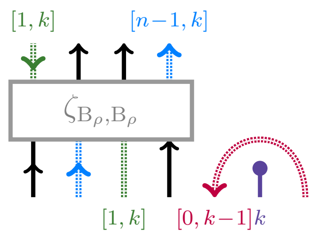
Both equalities can be checked to hold by using the hom & dot trick: attach a dot to the top endpoints labeled $k+1$ and use (103).

(N4) For $X = B_\rho$ and $g =$  we have to check

(110)

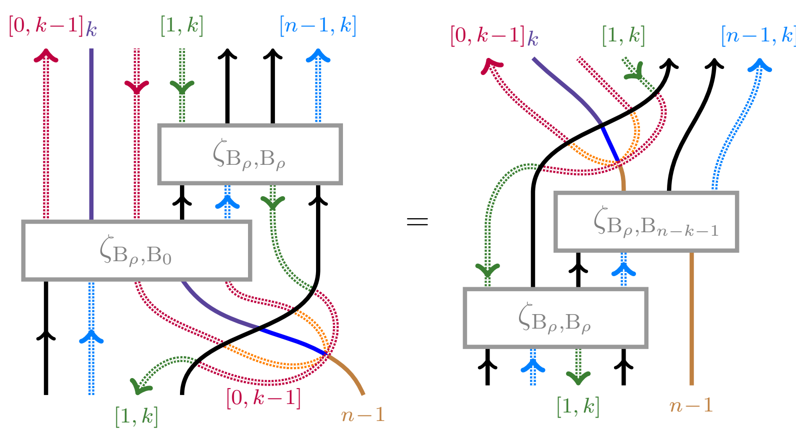


We use the hom & dot trick and attach a dot on the top endpoints labeled $k+1$ of both sides of (110). The left-hand side is simplified using (103) (with $j=1$), followed by (59) on the two strands labeled $[1,k]$ and (63). On the right-hand side, we use (45), followed by (105) and then (59) on the two strands labeled $[1,k]$ and (63). The resulting diagram, in both cases, is

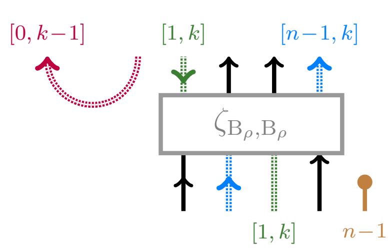


(N5) For $X = B_\rho$ and $g =$ 

we have to check

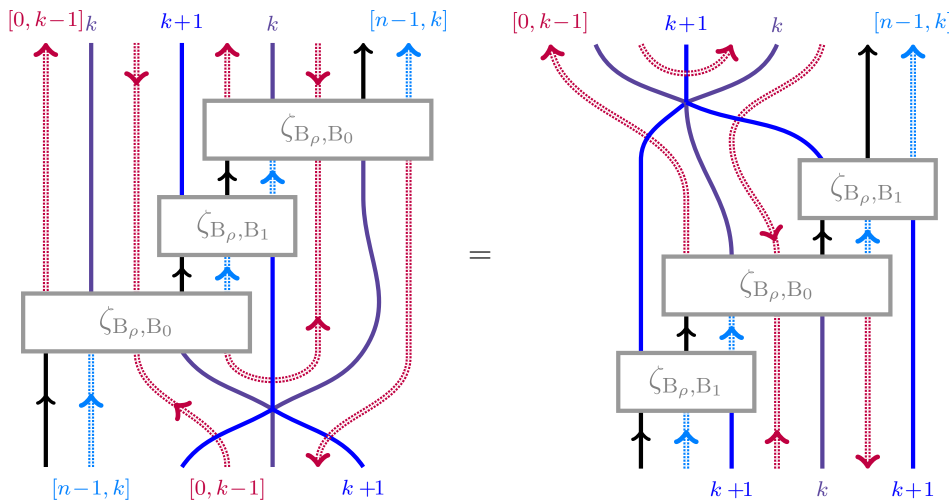
(111) 

We use the hom & dot trick and attach a dot on the top endpoints labeled k of both sides of (111). The left-hand side is simplified using (105), followed by (45), (80), (63) and (57). On the right-hand side, we use (45) followed by (80), (63) and (103). The resulting diagram, in both cases, is

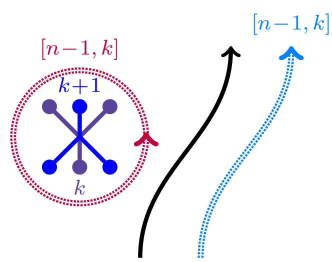


(N6) The cases $X = B_\rho$, $g = {}^{0}_{1}\times$ and $X = B_\rho$, $g = {}^{0}_{n-k-1}\times$ are very similar. We only prove the first case here. We have to check

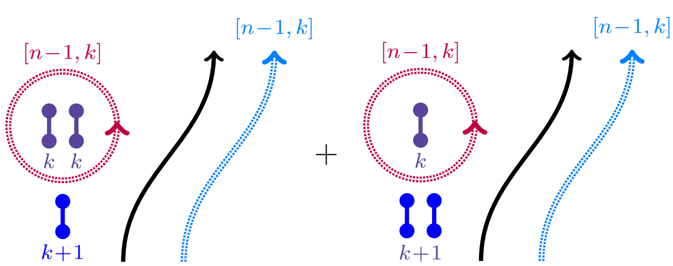
(112)



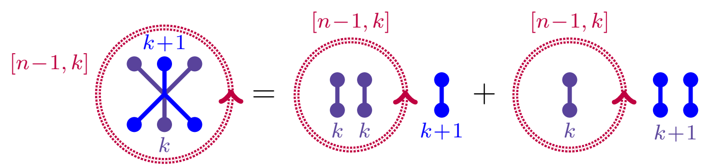
We use the hom & dot trick and attach a dot on all endpoints labeled k and $k+1$ of both sides of (112). On the right-hand side, we use (105), along with (103), to pull all dots from the bottom to the top. The resulting diagram is



On the left-hand side we first use (34) on the six-valent vertex, which results in two terms with dots on the bottom ends. We then use (105) several times and (103) on the resulting diagrams, along with (68) to pull all dots and barbells to to left top. This results in



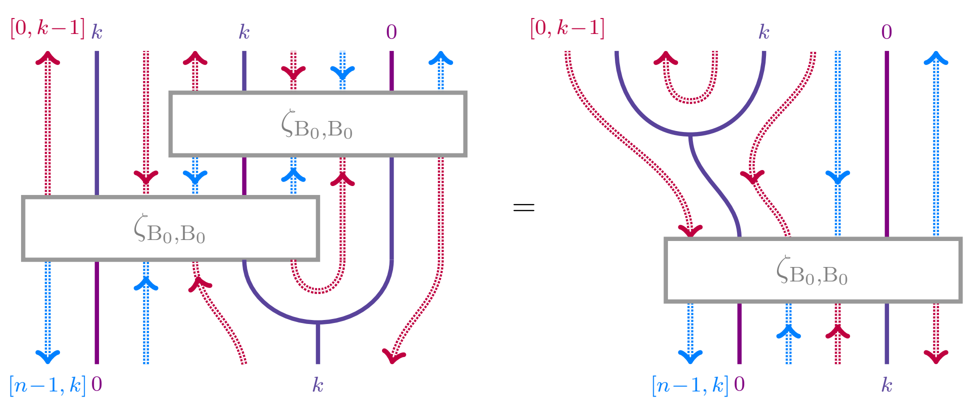
which equals the right-hand side since



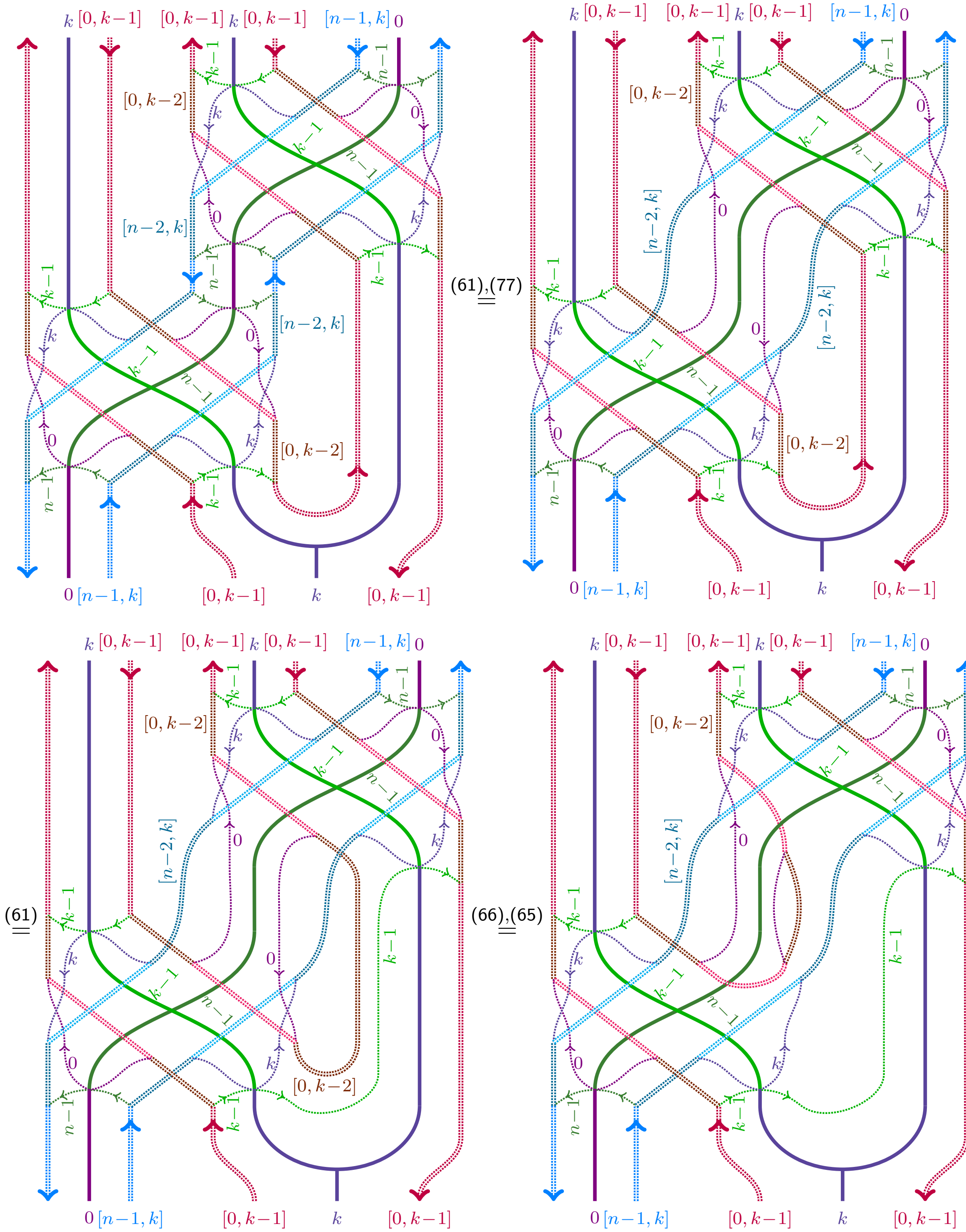
(N7) For $X = \mathrm{B}_0$ and $g = {}_0$ we cannot use the hom & dot trick (see Lemma 4.11). Commutativity of (101) in this case follows from (106) rotated 180 degrees.

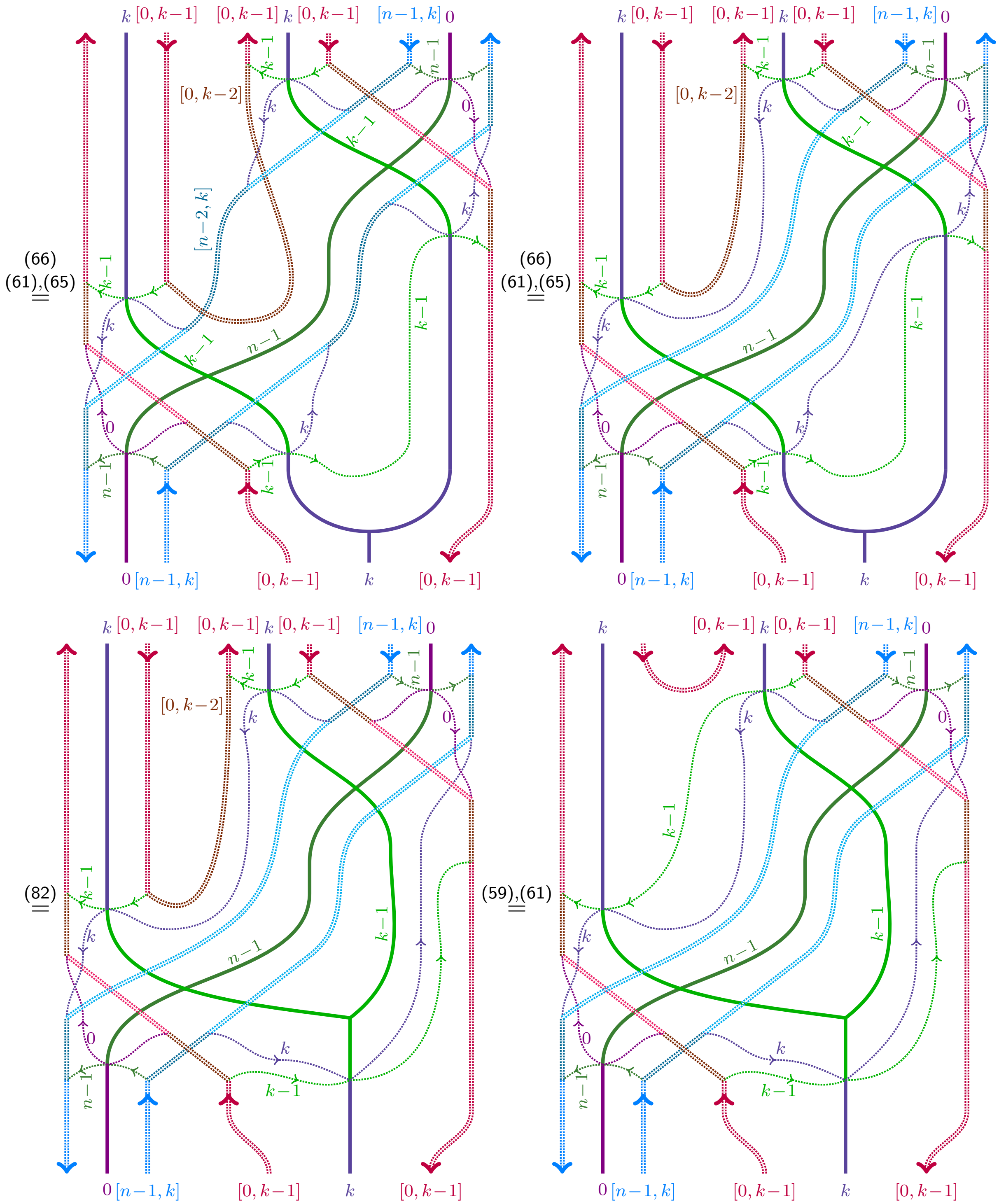
(N8) For $X = B_0$ and $g = {}_0$, we cannot use the hom & dot trick (see Lemma 4.11). We have to check

(113)



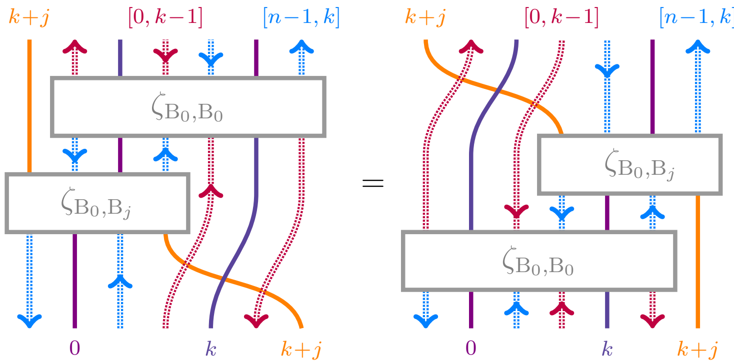
We introduce the diagram for $\zeta_{\mathrm{B}_0,\mathrm{B}_0}$ in the left-hand side of (113) and compute



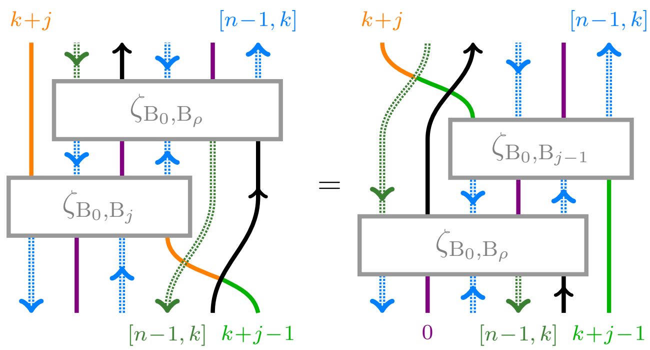


which equals the right-hand side of (113) by (68), (32) and (82).

(N9) For the cases $X = \mathrm{B}_0$, $g =$  and $X = \mathrm{B}_0$, $g =$  we have to check

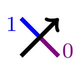
(114)


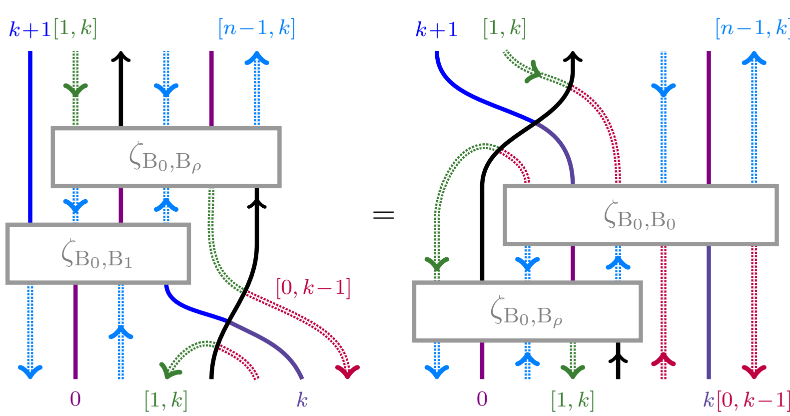
and

(115)


respectively.

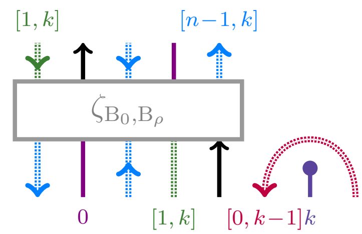
Both equalities can be checked to hold by using the hom & dot trick: attach a dot to the top endpoints labeled $k+1$ and use (104).

(N10) For $X = B_0$ and $g =$  we have to check

(116)


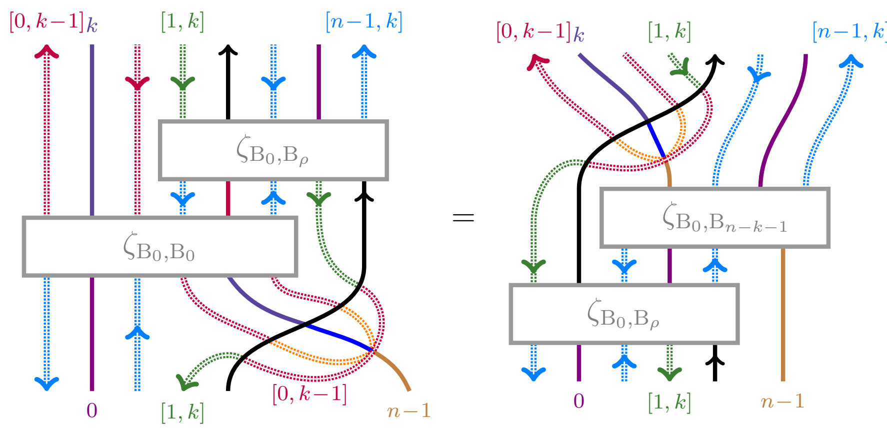
We use the hom & dot trick and attach a dot on the top endpoint labeled $k+1$ of both sides of (116). The left-hand side is simplified using (104) (with $j=k+1$), followed by (45), (59) on the two strands labeled $[1,k]$, and (63). On the right-hand side, we use (45), (59) on the two strands labeled $[1,k]$, (63), and finally (106) to pull the dot across the box $\zeta_{\mathrm{B}_0,\mathrm{B}_0}$ to the bottom right.

The resulting diagram, in both cases, is

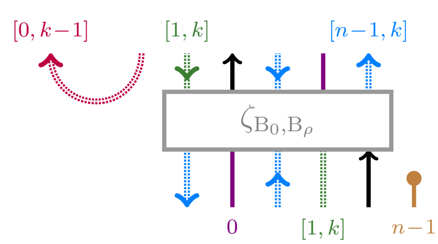


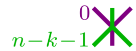
(N11) For $X = B_0$ and $g =$  we have to check

(117)

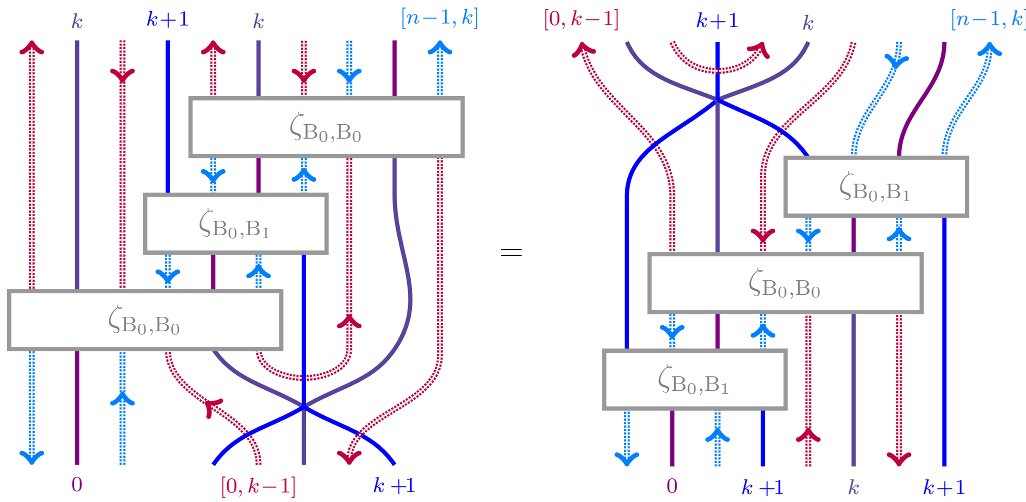


We use the hom & dot trick and attach a dot on the top endpoints labeled k of both sides of (117). The left-hand side is simplified using (106), followed by (45), (80), (63) and (57). On the right-hand side, we use (45) followed by (80), (63) and (104). The resulting diagram, in both cases, is



(N12) The cases $X = B_0$, $g =$  and and $X = B_\rho$, $g =$  are very similar. We only prove the first case here. We have to check

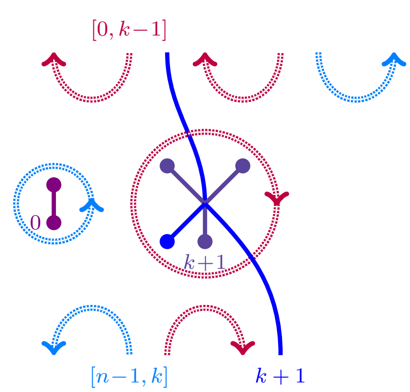
(118)



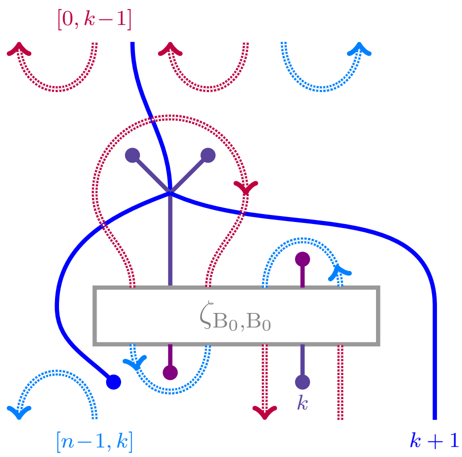
We use the hom & dot trick. At the top of both diagrams we attach a dot at the endpoints labeled 0 and k. At the bottom we attach dots to the endpoints labeled 0, k and the leftmost endpoint labeled $k+1$.

On the resulting diagram at the left-hand side, we use (106) twice and (104) to pull all dots to the bottom. Using (59) and (68) we obtain

(119)



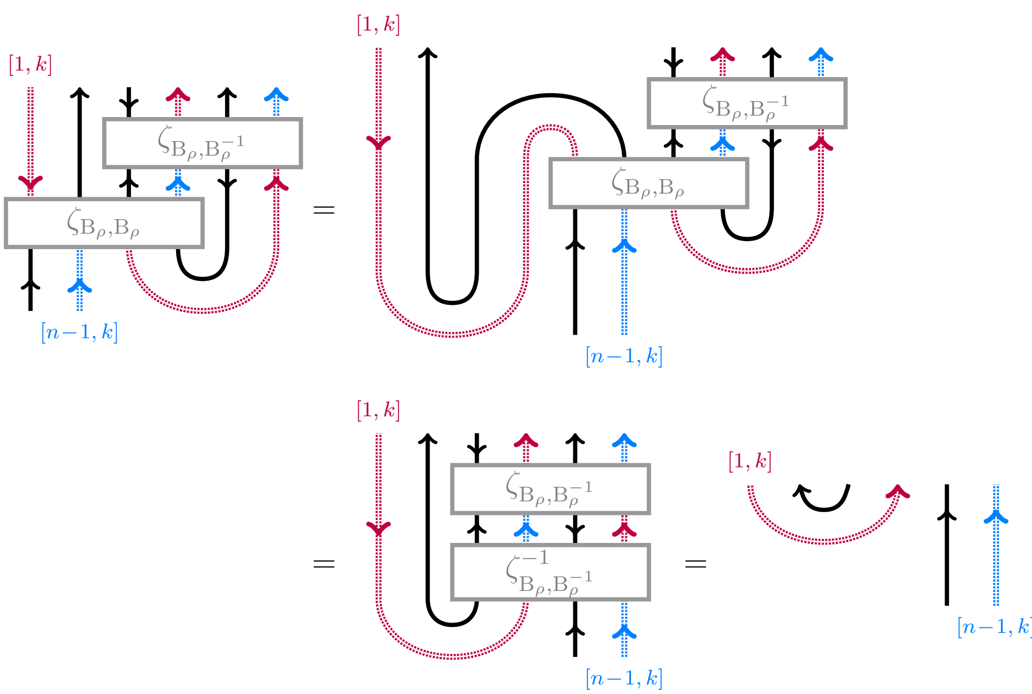
On the right-hand side, we use (a variant of) (104) to pull the dots labeled 0 close to the box $\zeta_{\mathrm{B}_0,\mathrm{B}_0}$ and use (59) to obtain

(120) 

Finally, applying (106) to slide the dot labeled k up through the box, using (68) to slide the dot labeled $k+1$ through the strand labeled $[0,k-1]$, and using (59) on the strands labeled $[0,k-1]$ with endpoints at the bottom gives (119).

This ends the proof of the naturality of ζ. □

Remark 4.13. Naturality for morphisms involving B_ρ^{-1} is easy to check by applying adjunctions (see Remark 4.10). For example, for $X = \mathrm{B}_\rho$ and $g =$  we have



4.2.4. *A conjecture regarding faithfulness.* Note that the functor $\Psi_{k,n-k}$, as defined above, is not faithful.

Lemma 4.14. *Recalling* (48), *we have* $\Psi_{k,n-k}\big(\boxed{y},\varnothing\big)=\boxed{y}=\Psi_{k,n-k}\big(\varnothing,\boxed{y}\big)$.

Proof. An easy computation using the definition of $\Psi_{k,n-k}$ and Lemma 3.6. ☐

Define $\widehat{\mathbb{S}}_k^{\mathrm{ext}}\boxtimes_y\widehat{\mathbb{S}}_{n-k}^{\mathrm{ext}}$ as the usual box tensor product, but with the morphism spaces tensored over the polynomial algebra $\mathbb{R}[\boxed{y}]$.

Conjecture 4.15. *The functor* $\Psi_{k,n-k}$ *descends to a faithful functor* $\widehat{\mathbb{S}}_k^{\mathrm{ext}}\boxtimes_y\widehat{\mathbb{S}}_{n-k}^{\mathrm{ext}}\to K^b(\widehat{\mathbb{S}}_n^{\mathrm{ext}})$.

5. Parabolic induction: an example

5.1. **The decategorified story.** In this section, we will work out an explicit example of an induced triangulated birepresentation $\mathbf{W}$ of $\widehat{\mathbb{S}}_2^{\mathrm{ext}}$ and its wide finitary cover $\mathbf{U}$. Below, we will start with the decategorified story, before we move on to the categorification, after a general intermezzo on completions of additive categories. The categorified story is divided into two parts: one dedicated to $\mathbf{U}$ and another to $\mathbf{W}$. Both birepresentations are defined using a certain algebra object Y in a completion of $K^b(\widehat{\mathbb{S}}_2^{\mathrm{ext}})$, but the construction of $\mathbf{W}$ is much less straightforward than that of $\mathbf{U}$, as the reader will see.

5.1.1. *The Hecke algebra* $\widehat{H}_2^{\mathrm{ext}}$. Recall that $\widehat{\mathfrak{S}}_2$ is generated by two simple reflections s_0 and s_1 which are only subject to the quadratic relations $s_0^2=e=s_1^2$. Any element in $\widehat{\mathfrak{S}}_2$ has therefore a unique rex, which can be identified with an alternating binary sequence. To work with those, we have to introduce some notation. For $m\in\mathbb{Z}_{>0}$ and $i\in\{0,1\}$, let $\widehat{m}_i$ and ${}_i\widehat{m}$ denote the alternating binary sequences of length m ending with i and starting with i, respectively. If $\widehat{m}_j={}_i\widehat{m}$ for certain $m\in\mathbb{Z}$ and $i,j\in\{0,1\}$ (we call this a *parity condition*), then we write ${}_i\widehat{m}_j$ for that alternating sequence to make the computations below easier to read. For example, $\widehat{4}_0={}_1\widehat{4}={}_1\widehat{4}_0=1010$. By convention, we put $\widehat{0}_i={}_i\widehat{0}=\emptyset$, for $i\in\{0,1\}$, which of course corresponds to the neutral element $e\in\widehat{\mathfrak{S}}_2$.

The extended affine symmetric group $\widehat{\mathfrak{S}}_2^{\mathrm{ext}}$ has an extra generator ρ of infinite order, and two extra relations: $\rho s_0\rho^{-1}=s_1$ and $\rho s_1\rho^{-1}=s_0$.

The extended affine Hecke algebra $\widehat{H}_2^{\mathrm{ext}}$ is generated by T_0,T_1 and $\rho^{\pm1}$, subject to only the first relation in (8) and all relations in (9). In this case, it is easy to express the KL basis elements b_w in terms of the regular basis elements T_u explicitly:

$$b_w=\sum_{u\preceq w}q^{\ell(w)-\ell(u)}T_u, \tag{121}$$

where $\preceq$ denotes the Bruhat order on $\widehat{\mathfrak{S}}_2$. Note that $u \preceq w$ holds iff the rex for u is an alternating binary subsequence of the rex for w and it is easy to invert this change of basis:

$$T_w = \sum_{u \preceq w} (-q)^{\ell(w)-\ell(u)} b_u. \tag{122}$$

The multiplication constants w.r.t. the KL basis are also easy to compute in this case and are all equal to 0, 1, 2 or $[2] = q + q^{-1}$. For $m, n \in \mathbb{Z}_{\geq 1}$ (assuming that the parity conditions are met in each line below), we have

$$\begin{aligned}
b_{{}_1\widehat{m}_1} b_{{}_1\widehat{n}_1} &= [2]\left(b_{{}_1\widehat{m+n-1}_1} + b_{{}_1\widehat{m+n-3}_1} + \ldots + b_{{}_1\widehat{|m-n|+3}_1} + b_{{}_1\widehat{|m-n|+1}_1}\right);\\
b_{{}_0\widehat{m}_1} b_{{}_1\widehat{n}_1} &= [2]\left(b_{{}_0\widehat{m+n-1}_1} + b_{{}_0\widehat{m+n-3}_1} + \ldots + b_{{}_0\widehat{|m-n|+3}_1} + b_{{}_0\widehat{|m-n|+1}_1}\right);\\
b_{{}_1\widehat{m}_1} b_{{}_1\widehat{n}_0} &= [2]\left(b_{{}_1\widehat{m+n-1}_0} + b_{{}_1\widehat{m+n-3}_0} + \ldots + b_{{}_1\widehat{|m-n|+3}_0} + b_{{}_1\widehat{|m-n|+1}_0}\right);\\
b_{{}_1\widehat{m}_0} b_{{}_0\widehat{n}_1} &= [2]\left(b_{{}_1\widehat{m+n-1}_1} + b_{{}_1\widehat{m+n-3}_1} + \ldots + b_{{}_1\widehat{|m-n|+3}_1} + b_{{}_1\widehat{|m-n|+1}_1}\right);\\
b_{{}_1\widehat{m}_1} b_{{}_0\widehat{n}_1} &= b_{{}_1\widehat{m+n}_1} + 2b_{{}_1\widehat{m+n-2}_1} + \ldots + 2b_{{}_1\widehat{|m-n|+2}_1} + b_{{}_1\widehat{|m-n|}_1};\\
b_{{}_1\widehat{m}_0} b_{{}_1\widehat{n}_1} &= b_{{}_1\widehat{m+n}_1} + 2b_{{}_1\widehat{m+n-2}_1} + \ldots + 2b_{{}_1\widehat{|m-n|+2}_1} + b_{{}_1\widehat{|m-n|}_1};\\
b_{{}_0\widehat{m}_1} b_{{}_0\widehat{n}_1} &= \begin{cases} b_{{}_0\widehat{2m}_1} + 2b_{{}_0\widehat{2m-2}_1} + \ldots + 2b_{{}_0\widehat{2}_1}, & \text{if } m = n \\ b_{{}_0\widehat{m+n}_1} + 2b_{{}_0\widehat{m+n-2}_1} + \ldots + 2b_{{}_0\widehat{|m-n|+2}_1} + b_{{}_0\widehat{|m-n|}_1}, & \text{if } m \neq n \end{cases};\\
b_{{}_1\widehat{m}_1} b_{{}_0\widehat{n}_0} &= \begin{cases} b_{{}_1\widehat{2m}_0} + 2b_{{}_1\widehat{2m-2}_0} + \ldots + 2b_{{}_1\widehat{2}_0}, & \text{if } m = n \\ b_{{}_1\widehat{m+n}_0} + 2b_{{}_1\widehat{m+n-2}_0} + \ldots + 2b_{{}_1\widehat{|m-n|+2}_0} + b_{{}_1\widehat{|m-n|}_0}, & \text{if } m \neq n \end{cases};
\end{aligned} \tag{123}$$

and the analogous equations with 0 and 1 swapped. These formulas are certainly known to experts, but we couldn't find an explicit reference for all of them in the literature (the first formula can be found in [Lus, Proposition 7.7(a)]). Let us, therefore, briefly explain how we obtained them. There is a close relation between $\widehat{H}_2^{\text{ext}}$ and a two-colored version of the Grothendieck ring of finite-dimensional representations (of type I) of quantum $\mathfrak{sl}_2$, see [Eli1, Section 2.2.1]. In the first four cases above, when the last element of the left alternating binary sequence and the first element of the right alternating binary sequence are equal, this correspondence is given by the bijection

$$b_{{}_i\widehat{m}_j} \longleftrightarrow [2][V_{m-1}],$$

where $[V_{m-1}]$ is the Grothendieck class of the (essentially unique) m-dimensional irreducible representation of quantum $\mathfrak{sl}_2$. Note that the Grothendieck ring over $\mathbb{Z}$ must be tensored with $\mathbb{Z}[q, q^{-1}]$ for multiplication by $[2]$ to make sense. In the first case, for example, we get

$$\begin{gathered}
b_{{}_1\widehat{m}_1} b_{{}_1\widehat{n}_1} \longleftrightarrow \\
[2]^2[V_{m-1}][V_{n-1}] = \\
[2]^2\left([V_{m+n-2}] + [V_{m+n-4}] + \ldots + [V_{|m-n|+2}] + [V_{|m-n|}]\right) \longleftrightarrow
\end{gathered}$$

$$[2]\left(b_{\widehat{{}_1m+n-1}_1} + b_{\widehat{{}_1m+n-3}_1} + \ldots + b_{\widehat{{}_1|m-n|+3}_1} + b_{\widehat{{}_1|m-n|+1}_1}\right),$$

where we have used the well-known Clebsch-Gordan rule for the decomposition of the tensor product of two finite dimensional irreducible representations of quantum $\mathfrak{sl}_2$. It is well-known that $b_i b_{{}_i\widehat{m}_j} = b_{{}_i\widehat{m}_j} b_j = [2] b_{{}_i\widehat{m}_j}$, for $m \in \mathbb{Z}_{\geq 1}$ and $i, j \in \{0, 1\}$ satisfying the parity condition. The remaining cases above can be reduced to the first four cases (sometimes with 0 and 1 swapped) by that basic multiplication rule and a simple trick, assuming (just for the sake of this trick) that we work over $\mathbb{Q}(q)$ for example. Suffice it to give an illustrative example:

$$b_{{}_1\widehat{m}_1} b_{{}_0\widehat{n}_1} = \frac{1}{[2]}(b_{{}_1\widehat{m}_1} b_1) b_{{}_0\widehat{n}_1} = \frac{1}{[2]} b_{{}_1\widehat{m}_1}(b_1 b_{{}_0\widehat{n}_1}) = \frac{1}{[2]} b_{{}_1\widehat{m}_1}(b_{\widehat{{}_1n+1}_1} + b_{\widehat{{}_1n-1}_1}),$$

where we use the equality $b_1 b_{{}_0\widehat{n}_1} = b_{\widehat{{}_1n+1}_1} + b_{\widehat{{}_1n-1}_1}$, which is the special case of the fifth equation in (123) for $m = 1$.

Recall the standard trace $\epsilon\colon \widehat{H}_2^{\mathrm{ext}} \to \mathbb{Z}[q, q^{-1}]$ from (16) and the associated q-sesquilinear form $(-,-)$ on $\widehat{H}_2^{\mathrm{ext}}$ from (17). The formula in (121) implies that

$$\epsilon(b_w) = q^{\ell(w)}$$

for any $w \in \widehat{\mathfrak{S}}_n$. Note further that $\omega(b_{{}_i\widehat{m}_j}) = b_{{}_j\widehat{m}_i}$, for $m \in \mathbb{Z}_{\geq 0}$ and $i, j \in \{0, 1\}$. The following lemma, which is an immediate consequence of the above, is a more detailed version of Theorem 2.2 in this particular case and will be needed in Section 5.3.

Lemma 5.1. *For any $r, s \in \mathbb{Z}$ and $i, j, k, l \in \{0, 1\}$, we have $(\rho^r b_{{}_i\widehat{m}_j}, \rho^s b_{{}_k\widehat{n}_l}) = \delta_{r,s}(b_{{}_i\widehat{m}_j}, b_{{}_k\widehat{n}_l})$ and*

$$(b_{{}_i\widehat{m}_j}, b_{{}_k\widehat{n}_l}) = \begin{cases} q^2 p(q), & \text{if } m = n \ \wedge\ i \neq k \ \wedge\ j \neq l; \\ q^{|m-n|} p'(q) & \text{else}, \end{cases}$$

where $p(q), p'(q) \in \mathbb{Z}_{\geq 0}[q]$ have constant terms equal to two and to one, respectively.

5.1.2. *Two representations of $\widehat{H}_2^{\mathrm{ext}}$.* Let $V = \mathrm{Span}\{v\}$ be the trivial one-dimensional $\widehat{H}_1^{\mathrm{ext}}$-module, defined by

$$\rho(v) = v,$$

and let

$$W := V \odot V,$$

the Zelevinsky tensor product of V with itself. Recall from Section 1 that

$$V \odot V = \mathrm{Ind}_{\widehat{H}_1^{\mathrm{ext}} \otimes \widehat{H}_1^{\mathrm{ext}}}^{\widehat{H}_2^{\mathrm{ext}}}(V \otimes V) = \widehat{H}_2^{\mathrm{ext}} \otimes_{\widehat{H}_1^{\mathrm{ext}} \otimes \widehat{H}_1^{\mathrm{ext}}} (V \otimes V).$$

Recall also that

$$\psi_{1,1}(\rho_L \otimes 1) = \rho T_1, \quad \psi_{1,1}(1 \otimes \rho_R) = T_1^{-1}\rho, \quad \psi_{1,1}(\rho_L \otimes \rho_R) = \rho^2,$$

where $\rho \in \widehat{H}_2^{\mathrm{ext}}$. Thus, for any $r, s \in \mathbb{Z}$, we have

$$\psi_{1,1}(\rho_L^r \otimes \rho_R^s) = (\rho T_1)^r (T_1^{-1}\rho)^s.$$

This implies that W is two-dimensional and that $\{w, w'\}$, where

$$w := 1 \otimes v \otimes v \quad \text{and} \quad w' := \rho \otimes v \otimes v, \tag{124}$$

is a basis of W, on which the action of $\widehat{H}_2^{\mathrm{ext}}$ is given by the matrices

$$[\rho] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad [T_1] = \begin{pmatrix} 0 & 1 \\ 1 & q^{-1} - q \end{pmatrix}, \quad [T_0] = [\rho T_0 \rho^{-1}] = \begin{pmatrix} q^{-1} - q & 1 \\ 1 & 0 \end{pmatrix}. \tag{125}$$

On the KL-basis, the action thus becomes

$$[b_1] = [T_1 + q] = \begin{pmatrix} q & 1 \\ 1 & q^{-1} \end{pmatrix}, \quad [b_0] = [T_0 + q] = \begin{pmatrix} q^{-1} & 1 \\ 1 & q \end{pmatrix}. \tag{126}$$

Using the rules for multiplication w.r.t. the KL-basis, we see that the action matrices satisfy:

$$[b_{\widehat{2r}_0}] = r[b_1][b_0],\ [b_{\widehat{2r+1}_1}] = (2r+1)[b_1],\ [b_{\widehat{2r}_1}] = r[b_0][b_1],\ [b_{\widehat{2r+1}_0}] = (2r+1)[b_0],\ r \in \mathbb{Z}_{\geq 1}.$$

Recall that q is a formal invertible parameter and let $-^{\mathbb{C}(q)} := - \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}(q)$. The following result is well-known, see e.g. [LNT, Theorem 1].

Proposition 5.2. *The $\widehat{H}_2^{\mathrm{ext},\mathbb{C}(q)}$-module $W^{\mathbb{C}(q)}$ is simple.*

There is also an infinite dimensional $\widehat{H}_2^{\mathrm{ext}}$-module U which is closely related to W and whose categorification will be important below. By definition,

$$U := \widehat{H}_2^{\mathrm{ext}}/I,$$

the quotient of the regular left $\widehat{H}_2^{\mathrm{ext}}$-module by the left ideal I generated by

$$\rho^2 - 1 \quad \text{and} \quad b_1\rho - q^{-1}b_1. \tag{127}$$

Remark 5.3. We stress that I is only a left ideal and not a two-sided ideal. For $\rho^2 - 1$ this distinction is irrelevant, because ρ^2 and 1 both belong to the center of $\widehat{H}_2^{\mathrm{ext}}$, but for $b_1\rho - q^{-1}b_1$ it matters. For example, the element $(1 - q^{-2})b_1$ belongs to the two-sided ideal generated by the elements in (127) because

$$(1 - q^{-2})b_1 = (b_1\rho - q^{-1}b_1)(\rho + q^{-1}) - b_1(\rho^2 - 1),$$

but it does not belong to I.

Proposition 5.4. *Let $\pi_U \colon \widehat{H}_2^{\mathrm{ext}} \to U$ be the natural projection and define $u_k := \pi_U(b_{\widehat{k}_1})$ and $u'_k := \pi_U(\rho b_{\widehat{k}_1})$, for all $k \in \mathbb{Z}_{\geq 0}$. Then $\{u_k, u'_k \mid k \in \mathbb{Z}_{\geq 0}\}$ is a basis of U and the action of*

$\widehat{H}_2^{\rm ext}$ *on this basis is determined by*

$$
\begin{aligned}
\rho u_{k-1} = u'_{k-1}, \quad \rho u'_{k-1} = u_{k-1} \\
b_1 u_0 = u_1, \quad b_1 u_{2k} = u_{2k+1} + u_{2k-1}, \quad b_1 u_{2k-1} = [2] u_{2k-1}, \\
b_0 u_0 = q^{-1} u'_1, \quad b_0 u_1 = u_2, \quad b_0 u_{2k} = [2] u_{2k}, \quad b_0 u_{2k+1} = u_{2k+2} + u_{2k}, \\
b_1 u'_0 = q^{-1} u_1, \quad b_1 u'_1 = u'_2, \quad b_1 u'_{2k} = [2] u'_{2k}, \quad b_1 u'_{2k+1} = u'_{2k+2} + u'_{2k}, \\
b_0 u'_0 = u'_1, \quad b_0 u'_{2k} = u'_{2k+1} + u'_{2k-1}, \quad b_0 u'_{2k-1} = [2] u'_{2k-1},
\end{aligned}
\tag{128}
$$

for $k \in \mathbb{Z}_{\geq 1}$.

Proof. To show that $\{u_k, u'_k \mid k \in \mathbb{Z}_{\geq 0}\}$ spans U is easy. The KL-basis of $\widehat{H}_2^{\rm ext}/\langle \rho^2 - 1\rangle$, where $\langle \rho^2 - 1\rangle$ is the (left) ideal generated by $\rho^2 - 1$, is given by $\{\rho^m b_{\widehat{k}_1}, \rho^m b_{\widehat{k}_0} \mid m \in \{0,1\}, k \in \mathbb{Z}_{\geq 0}\}$. The images of those elements under π_U, therefore, span U. By induction, it's easy to show that $b_{\widehat{k}_1}\rho - q^{-1} b_{\widehat{k}_1} \in I$ for all $k \in \mathbb{Z}_{>0}$. For $k = 2$, we have

$$b_{01}\rho - q^{-1} b_{01} = b_0(b_1\rho - q^{-1} b_1) \in I$$

For $k > 2$, induction yields

$$b_{\widehat{k}_1}\rho - q^{-1} b_{\widehat{k}_1} = \begin{cases} b_0(b_{\widehat{k-1}_1}\rho - q^{-1} b_{\widehat{k-1}_1}) - (b_{\widehat{k-2}_1}\rho - q^{-1} b_{\widehat{k-2}_1}) \in I, & \text{if } k \text{ is even;} \\ b_1(b_{\widehat{k-1}_1}\rho - q^{-1} b_{\widehat{k-1}_1}) - (b_{\widehat{k-2}_1}\rho - q^{-1} b_{\widehat{k-2}_1}) \in I, & \text{if } k \text{ is odd.} \end{cases}$$

This shows that $\rho b_{\widehat{k}_0} = b_{\widehat{k}_1}\rho = q^{-1} b_{\widehat{k}_1}$ in U, for all $k \in \mathbb{Z}_{\geq 0}$. This in turn implies that $b_{\widehat{k}_0} = \rho^2 b_{\widehat{k}_0} = q^{-1}\rho b_{\widehat{k}_1}$, for all $k \in \mathbb{Z}_{\geq 0}$, which completes the proof that $\{u_k, u'_k \mid k \in \mathbb{Z}_{\geq 0}\}$ spans U.

It remains to prove that the elements u_k, u'_k, for $k \in \mathbb{Z}_{\geq 0}$, are all linearly independent in U. Suppose that

$$\lambda_{m_1} u_{m_1} + \ldots + \lambda_{m_r} u_{m_r} + \mu_{n_1} u'_{n_1} + \ldots + \mu_{n_s} u'_{n_s} = 0 \text{ in } U,$$

for some $r, s \in \mathbb{Z}_{\geq 0}$ and some $\mu_{m_1}, \ldots, \mu_{m_r}, \nu_{n_1} \ldots, \nu_{n_s} \in \Bbbk$. Then

$$\lambda_{m_1} b_{\widehat{m_1}_1} + \ldots + \lambda_{m_r} b_{\widehat{m_r}_1} + \mu_{n_1}\rho b_{\widehat{n_1}_1} + \ldots + \mu_{n_s}\rho b_{\widehat{n_s}_1} \in I. \tag{129}$$

Since $b_1\rho = \rho b_0$, any non-zero element in I will be a linear combination of KL basis elements containing at least a non-zero multiple of $\rho b_{\widehat{k}_0}$, for some $k \in \mathbb{Z}_{>0}$, or a non-zero multiple of $\rho^2 b_{\widehat{k}_i}$, for some $k \in \mathbb{Z}_{\geq 0}$ and $i \in \{0,1\}$. Either way, linear independence of the KL basis elements in $\widehat{H}_2^{\rm ext}$ implies that (129) can only hold if $\lambda_{m_1} b_{\widehat{m_1}_1} + \ldots + \lambda_{m_r} b_{\widehat{m_r}_1} + \mu_{n_1}\rho b_{\widehat{n_1}_1} + \ldots + \mu_{n_s}\rho b_{\widehat{n_s}_1} = 0$ and, therefore, that $\lambda_{m_i} = \mu_{n_j} = 0$ for all $i = 1, \ldots, r$ and $j = 1, \ldots, s$. This completes the proof that the elements u_k, u'_k, for $k \in \mathbb{Z}_{\geq 0}$, are all linearly independent in U.

Finally, it is easy to determine the action of ρ, b_0 and b_1 on this basis using the KL multiplication rules, because π_U is a morphism of $\widehat{H}_2^{\rm ext}$-modules. ∎

Let $J \subset U$ be the $\widehat{H}_2^{\mathrm{ext}}$-submodule generated by $T_1 u_0 - \rho u_0$ and let $\overline{u} \in U/J$ be the image of $u \in U$ under the natural projection $U \to U/J$. We have

$$\overline{T_1 u_0} = \overline{\rho u_0} = \overline{u_0'},$$
$$\overline{T_1 u_0'} = \overline{T_1^2 u_0} = \overline{(q^{-1} - q) T_1 u_0 + u_0} = \overline{(q^{-1} - q) u_0' + u_0}.$$

Proposition 5.5. *There is a surjective morphism of $\widehat{H}_2^{\mathrm{ext}}$-modules*

$$\pi_W^U \colon U \to W$$

and $\ker(\pi_W^U) = J$.

Proof. Define $\pi_W^U \colon U \to W$ by

$$\pi_W^U(u_0) := w,$$

extending it to the whole of U by the requirement that it be a morphism of $\widehat{H}_2^{\mathrm{ext}}$-modules. Note that the matrices in (126) imply that

$$\pi_W^U(u_0') = \pi_W^U(\rho u_0) = \rho \pi_W^U(u_0) = \rho w = w';$$
$$\pi_W^U(b_1 \rho u_0) = b_1 \pi_W^U(u_0') = b_1 w' = w + q^{-1} w' = q^{-1} b_1 w = q^{-1} \pi_W^U(b_1 u_0),$$

so π_W^U is well-defined.

The fact that $\pi_W^U(u_0) = w$ and $\pi_W^U(u_0') = w'$ proves that π_W^U is surjective.

It remains to show that $\ker(\pi_W^U) = J$. The matrices in (126) show that $\rho T_1 w = w$, which implies that

$$\pi_W^U(\rho T_1 u_0) = \rho T_1 \pi_W^U(u_0) = \rho T_1 w = w = \pi_W^U(u_0),$$

so $J \subseteq \ker(\pi_W^U)$. To prove that this inclusion is an equality we will show that $\dim_{\Bbbk}(U/J) = 2$, which amounts to showing that $\overline{u_k}, \overline{u_k'} \in \mathrm{span}\{\overline{u_0}, \overline{u_0'}\}$ for all $k \in \mathbb{Z}_{\geq 0}$. We will prove this by induction. Recall that $b_i = T_i + q$, for $i \in \{0, 1\}$. Then, for the base step, we have

$$\overline{u_1} = \overline{b_1 u_0} = \overline{T_1 u_0 + q u_0} = \overline{\rho u_0 + q u_0} = \overline{u_0' + q u_0},$$
$$\overline{u_1'} = \overline{\rho u_1} = \overline{\rho u_0' + q \rho u_0} = \overline{u_0 + q u_0'},$$
$$\overline{u_2} = \overline{b_0 u_1} = \overline{b_0 (u_0' + q u_0)} = 2\overline{u_1'} = 2(\overline{u_0 + q u_0'}),$$
$$\overline{u_2'} = \overline{\rho u_2} = 2\overline{u_1} = 2(\overline{u_0' + q u_0}).$$

For the inductive step, let $k > 2$ and assume that that $\overline{u_m}, \overline{u_m'} \in \mathrm{span}\{\overline{u_0}, \overline{u_0'}\}$ for all $0 \leq m < k$. If $k > 2$ is odd, then

$$\overline{u_k} = \overline{b_1 u_{k-1} - u_{k-2}}.$$

By induction, the elements $\overline{u_{k-1}}, \overline{u_{k-2}}$ belong to $\mathrm{span}\{\overline{u_0}, \overline{u_0'}\}$. Since $\overline{b_1 u_0} = \overline{u_1} = \overline{u_0' + q u_0}$ and $\overline{b_1 u_0'} = q^{-1} \overline{u_1} = \overline{q^{-1} u_0' + u_0}$, we conclude that $\overline{u_k} \in \mathrm{span}\{\overline{u_0}, \overline{u_0'}\}$. If $k > 2$ is even, then

$$\overline{u_k} = \overline{b_0 u_{k-1} - u_{k-2}}.$$

By induction again, the elements $\overline{u_{k-1}}, \overline{u_{k-2}}$ belong to $\mathrm{span}\{\overline{u_0}, \overline{u_0'}\}$. Since $\overline{b_0 u_0} = \overline{q^{-1}u_1'} = \overline{q^{-1}u_0 + u_0'}$ and $\overline{b_0 u_0'} = \overline{u_1'} = \overline{u_0 + qu_0'}$, we conclude that $\overline{u_k} \in \mathrm{span}\{\overline{u_0}, \overline{u_0'}\}$. This completes the proof that u_k belongs to $\mathrm{span}\{\overline{u_0}, \overline{u_0'}\}$ for all $k \in \mathbb{Z}_{\geq 0}$. Using similar arguments, one can show that the same also holds for u_k', for all $k \in \mathbb{Z}_{\geq 0}$, which concludes the proof. □

Just to fix notation, let $\pi_W \colon \widehat{H}_2^{\mathrm{ext}} \to W$ be the projection of $\widehat{H}_2^{\mathrm{ext}}$-modules defined by $\pi_W(x) := xw$, for any $x \in \widehat{H}_2^{\mathrm{ext}}$. Then there is a commutative triangle

$$\begin{array}{ccc} \widehat{H}_2^{\mathrm{ext}} & \xrightarrow{\pi_U} & U \\ & \searrow^{\pi_W} & \downarrow{\scriptstyle \pi_W^U} \\ & & W \end{array} \tag{130}$$

Remark 5.6. The infinite dimensional representation U of $\widehat{H}_2^{\mathrm{ext}}$ is the decategorification of a wide finitary birepresentation $\mathbf{U}$ of $\widehat{\mathbb{S}}_2^{\mathrm{ext}}$, which appears naturally as the wide finitary cover of the induced birepresentation $\mathbf{W}$ in Section 5.3. However, we do not know if U also appears naturally in any approach to the representation theory of $\widehat{H}_2^{\mathrm{ext}}$ that does not use categorification. Note that this contrasts with the finitary cover $\widehat{\mathbf{M}}_{r,s}$ ($r, s \in \mathbb{Z}$) of the evaluation birepresentation $\mathbf{M}^{\mathcal{E}v_{r,s}}$ of $\widehat{\mathbb{S}}_n^{\mathrm{ext}}$ in [MMV, Corollary 6.5], which decategorifies to a Graham-Lehrer cell module of $\widehat{H}_n^{\mathrm{ext}}$, see [MMV, Remark 6.6].

5.2. **Completions of linear additive categories.** In this subsection, we lay the groundwork for studying the application of our embedding to the induction of birepresentations. Roughly speaking, birepresentations over an additive bicategory correspond to algebras in some completion of the latter and one can define induction from one bicategory to another by pushing forward the algebras through an appropriate embedding. In our case, the situation is rather delicate as we move between additive and triangulated bicategories, so in this section we carefully study the completions in which our algebras live, both in the additive and triangulated world. Since this section is more general, we will work over an arbitrary field and with bicategories rather than monoidal categories.

5.2.1. *Coproduct completions of additive* $\mathbb{k}$*-linear categories.* Let $\mathcal{C}$ be an additive $\mathbb{k}$-linear category. Recall that the *additive closure* $\mathrm{add}(\mathcal{A})$ of a full subcategory $\mathcal{A}$ of $\mathcal{C}$ is the closure under direct sums and direct summands in $\mathcal{C}$.

We note that $\mathrm{Vect}_{\mathbb{k}}$ is complete and cocomplete with respect to small weighted (and hence, in particular, conical) limits and colimits. This follows from [Rie, Corollary 7.6.4] using [Rie, Example 3.7.5] and the fact that $\mathrm{Vect}_{\mathbb{k}}$ is enriched over itself.

This implies that the category of $\mathbb{k}$-linear functors $\mathcal{C}^\circ = \mathsf{F}un_{\mathbb{k}}(\mathcal{C}^{op}, \mathrm{Vect}_{\mathbb{k}})$, is also (co)complete under weighted (co)limits by [Kel, Section 3.3] with (co)limits being computed object-wise. As usual, we identify an object X in $\mathcal{C}$ with their representables $X^\vee = \mathrm{Hom}_{\mathcal{C}}(-, X)$ under the Yoneda embedding. We denote by $\mathcal{C}^\vee$ the image of $\mathcal{C}$ under this embedding (which is, of course, equivalent to $\mathcal{C}$).

Note that we can realise a (co)product indexed by a set J as the conical (co)limit over the diagram $J \to \mathcal{C}\colon j \mapsto X_j$. This is a special case of the weighted (co)limit of the $\mathrm{Vect}_\Bbbk$-enriched functor $\tilde{J} \to \mathcal{C}$ where $\tilde{J}$ is the free $\mathrm{Vect}_\Bbbk$-enriched category on J and the weight $\tilde{J} \to \mathrm{Vect}_\Bbbk$ is just the constant functor $j \mapsto \Bbbk$. For a family of objects $(X_j)_{j\in J}$ in $\mathcal{C}$, we slightly abuse notation by writing $\coprod_{j\in J} X_j$ for said colimit in $\mathcal{C}^\circ$ rather than the formally correct $\coprod_{j\in J} X_j^\vee$ and likewise for products.

Since $\mathcal{C}$ and hence $\mathcal{C}^\circ$ is additive, in particular has a zero object, there is always a canonical monomorphism

$$\nu_{(X_j)_{j\in J}}\colon \coprod_{j\in J} X_j \to \prod_{j\in J} X_j.$$

Explicitly, denoting by $\pi_i\colon \prod_{j\in J} X_j \to X_i$ and $\iota_i\colon X_i \to \coprod_{j\in J} X_j$ the canonical projections and injections, respectively, ν is defined by the condition that

$$\pi_i \nu \iota_k = \begin{cases} \mathrm{Id}_{X_i} & \text{if } i = k, \\ 0 & \text{otherwise.} \end{cases}$$

We define $\mathcal{C}^\sqcup$ as the full subcategory of $\mathcal{C}^\circ$ where we close $\mathcal{C}^\vee$ under countable coproducts. We then denote by $\mathcal{C}^\diamond = (\mathcal{C}^\sqcup)_e$ its Karoubi envelope. Note that both are again additive $\Bbbk$-linear categories. Moreover, $\mathcal{C}^\diamond$ is closed under countable coproducts. We observe that, by [Bre, Theorem 4.1], $\mathcal{C}^\diamond$ and $\mathcal{C}^\sqcup$ coincide provided $\mathcal{C}$ is Krull-Schmidt.

A $\Bbbk$-linear functor $\mathrm{F}\colon \mathcal{C} \to \mathcal{D}$ between additive $\Bbbk$-linear categories induces $\Bbbk$-linear functors, also denoted by F, from $\mathcal{C}^\sqcup \to \mathcal{D}^\sqcup$, sending an object $\coprod_{j\in J} X_j$ to $\coprod_{j\in J} \mathrm{F}(X_j)$, and hence from $\mathcal{C}^\diamond \to \mathcal{D}^\diamond$.

The next lemma will be crucial for computing morphism spaces in explicit examples.

Lemma 5.7. *Assume $Y = \coprod_{j\in J} Y_j \in \mathcal{C}^\circ$, where each $Y_j \in \mathcal{C}$. Let $X \in \mathcal{C}$ and assume that $\mathrm{Hom}_\mathcal{C}(X, Y_j) = 0$ for all but finitely many $j \in J$. Then $\mathrm{Hom}_{\mathcal{C}^\circ}(X, Y) = \coprod_{j\in J} \mathrm{Hom}_\mathcal{C}(X, Y_j)$.*

Proof. Consider the natural monomorphism

$$\coprod_{j\in J} \mathrm{Hom}_{\mathcal{C}^\circ}(X, Y_j) \xrightarrow{\kappa} \mathrm{Hom}_{\mathcal{C}^\circ}(X, \coprod_{j\in J} Y_j)$$

and compose this with the monomorphism induced by $\nu = \nu_{(Y_j)_{j\in J}}$, i.e.

$$\coprod_{j\in J} \mathrm{Hom}_{\mathcal{C}^\circ}(X, Y_j) \xrightarrow{\kappa} \mathrm{Hom}_{\mathcal{C}^\circ}(X, \coprod_{j\in J} Y_j) \xrightarrow{\nu\circ -} \mathrm{Hom}_{\mathcal{C}^\circ}(X, \prod_{j\in J} Y_j) \cong \prod_{j\in J} \mathrm{Hom}_{\mathcal{C}^\circ}(X, Y_j)$$

The coproduct in the domain and the product in the codomain are both finite and hence isomorphic, so this monomorphism is an isomorphism. In particular, κ is an isomorphism.

The lemma now follows from the fact that $\mathrm{Hom}_{\mathcal{C}^\circ}(X, Y_j) = \mathrm{Hom}_{\mathcal{C}}(X, Y_j)$ by the Yoneda Lemma. □

Given that the (co)product of the morphism spaces in Lemma 5.7 is really finite, we write it as

$$\bigoplus_{j\in J} \mathrm{Hom}_{\mathcal{C}^\circ}(X, Y_j).$$

5.2.2. *Coproduct completions and homotopy categories.* For a $\Bbbk$-linear additive category $\mathcal{C}$, we can consider the category $C^b(\mathcal{C})$ of bounded complexes over $\mathcal{C}$ and its completion $C^b(\mathcal{C})^{\sqcup}$ under countable coproducts, and the Karoubi envelope $C^b(\mathcal{C})^{\diamond}$ of the latter. Similarly, we can consider $C^b(\mathcal{C}^{\sqcup})$ and $C^b(\mathcal{C}^{\diamond})$.

For reference, we record the following lemma, for which we could not find a proof in the literature.

Lemma 5.8. *For an idempotent complete $\Bbbk$-linear additive category $\mathcal{C}$, the category $C^b(\mathcal{C})$ is idempotent complete.*

Proof. Assume $e^\bullet$ is an idempotent endomorphism of an object $(X^\bullet, d^\bullet)$ in $C^b(\mathcal{C})$. In particular, each e^i is an idempotent endormorphism of X^i. Since $\mathcal{C}$ is idempotent complete, X^i splits into a direct sum of object X^i_e and X^i_{1-e}. Since $e^\bullet$ commutes with the differential $d^\bullet$, each of these summands is preserved under the differential and the complex $(X^\bullet, d^\bullet)$ splits into a direct sum of complexes $(X^\bullet_e, e^\bullet d^\bullet e^\bullet)$ and $(X^\bullet, (1-e)^\bullet d^\bullet (1-e)^\bullet)$. □

Note that the Yoneda embedding $\mathcal{C} \xrightarrow{\Upsilon} \mathcal{C}^\diamond$ extends to an embedding $C^b(\mathcal{C}) \hookrightarrow C^b(\mathcal{C}^\diamond)$, which gives rise to an embedding $C^b(\mathcal{C})^\diamond \hookrightarrow C^b(\mathcal{C}^\diamond)^\diamond$. The following lemma is straightforward, noticing that a countable coproduct of complexes, which each have countable coproducts of objects in each component, is again a complex with countable coproducts in each component.

Lemma 5.9. *$C^b(\mathcal{C}^\diamond)$ is closed under countable coproducts, thus $C^b(\mathcal{C}^\diamond) \simeq C^b(\mathcal{C}^\diamond)^\diamond$ and we have an embedding $\iota\colon C^b(\mathcal{C})^\diamond \hookrightarrow C^b(\mathcal{C}^\diamond)$.*

Proof. Let $(X_i^\bullet, d_i^\bullet)_{i\in I}$ be a family of objects in $C^b(\mathcal{C}^\diamond)$ and consider the coproduct $X = \coprod_{i\in I}(X_i^\bullet, d_i^\bullet)$ in $C^b(\mathcal{C}^\diamond)^{\sqcup}$. Now define $Y = (Y^\bullet, d^\bullet)$ in $C^b(\mathcal{C}^\diamond)$ as the complex whose kth component is given by $\coprod_{i\in I} X_i^k$ and whose differential d^k is given by $\coprod_{i\in I} X_i^k \xrightarrow{(d_i^k)_{i\in I}} \coprod_{i\in I} X_i^{k+1}$. Then clearly, $Y \cong X$, so $C^b(\mathcal{C}^\diamond)$ is closed under coproducts and $C^b(\mathcal{C}^\diamond) \simeq C^b(\mathcal{C}^\diamond)^{\sqcup}$. Since by Lemma 5.8, $C^b(\mathcal{C}^\diamond)$ is idempotent complete, $C^b(\mathcal{C}^\diamond) \simeq C^b(\mathcal{C}^\diamond)^\diamond$. □

Denote by $N^b(\mathcal{C}) \subset C^b(\mathcal{C})$ the ideal of nulhomotopic morphisms of complexes, and by $K^b(\mathcal{C}) = C^b(\mathcal{C})/N^b(\mathcal{C})$ the bounded homotopy category.

Observe that the fully faithful functor $C^b(\mathcal{C}) \hookrightarrow C^b(\mathcal{C}^\diamond)$ induced by the Yoneda embedding preserves nullhomotopic maps, and we thus obtain a fully faithful functor $K^b(\mathcal{C}) \hookrightarrow K^b(\mathcal{C}^\diamond)$.

Morover, ι also preserves nullhomotopic morphisms so we also obtain a fully faithful functor $K^b(\mathcal{C})^\diamond \to K^b(\mathcal{C}^\diamond)$, which we again denote by ι.

Corollary 5.10. *(a) We have fully faithful functors* $K^b(\mathcal{C}) \xrightarrow{K^b(\Upsilon)} K^b(\mathcal{C})^\diamond \xrightarrow{\iota} K^b(\mathcal{C}^\diamond)$.

(b) If $\mathcal{D}$ is a $\Bbbk$-linear additive category, a $\Bbbk$-linear functor $\Phi\colon \mathcal{C} \to K^b(\mathcal{D})$ induces a $\Bbbk$-linear functor $\mathcal{C}^\diamond \to K^b(\mathcal{D}^\diamond)$.

Proof. The first part is immediate. For the second part, note that Φ gives rise to a functor $\mathcal{C}^\diamond \to K^b(\mathcal{D})^\diamond$. Composing this with the inclusion from the first part yields the desired functor $\mathcal{C}^\diamond \to K^b(\mathcal{D}^\diamond)$. □

5.2.3. *Wide finitary bicategories.* Here we recall some of the definitions from [Macph]. Remember that a 2-category is a strict bicategory, meaning that all coherers are identities. We will generally be speaking about bicategories, but some of these happen to be 2-categories.

Denote by $\mathfrak{A}_\Bbbk$ the 2-category whose objects are additive $\Bbbk$-linear categories, whose 1-morphisms are $\Bbbk$-linear functors and whose 2-morphisms are natural transformations.

A category $\mathcal{C}$ is *wide finitary* if it is an additive $\Bbbk$-linear Krull–Schmidt category with at most countably many isomorphism classes of indecomposable objects and morphism spaces of at most countable dimension. We denote by $\operatorname{ind}\mathcal{C}$ a set of representatives of isomorphism classes of indecomposable objects in $\mathcal{C}$.

We denote the 2-category whose objects are wide finitary categories, whose 1-morphisms are $\Bbbk$-linear functors, and whose 2-morphisms are natural transformations by $\mathfrak{A}_\Bbbk^{wf}$.

We say that a bicategory $\mathcal{C}$ is *wide finitary* if

- it has finitely many objects;
- $\mathcal{C}(\mathtt{i},\mathtt{j})$ is in $\mathfrak{A}_\Bbbk^{wf}$ for all $\mathtt{i},\mathtt{j} \in \mathcal{C}$;
- horizontal composition is biadditive and $\Bbbk$-linear;
- the identity 1-morphism $\mathbb{1}_\mathtt{i}$ is indecomposable for any $\mathtt{i} \in \mathcal{C}$.

We use $\circ_\mathsf{h}$, resp. $\circ_\mathsf{v}$, for horizontal, respectively vertical, composition in a bicategory.

Recall that if $\mathcal{C}$ is a bicategory, $\mathcal{C}^{\mathsf{co,op}}$ is the bicategory obtained by reversing the direction of both 1- and 2-morphisms. A wide finitary bicategory $\mathcal{C}$ is said to be *wide fiat* if there exists a weak involution $(-)^*\colon \mathcal{C} \to \mathcal{C}^{\mathsf{co,op}}$ such that, for any 1-morphism $\mathrm{F} \in \mathcal{C}(\mathtt{i},\mathtt{j})$, there are natural 2-morphisms $\alpha\colon \mathrm{F}\mathrm{F}^* \to \mathbb{1}_\mathtt{j}, \beta\colon \mathbb{1}_\mathtt{i} \to \mathrm{F}^*\mathrm{F}$ satisfying the usual adjunction triangles, i.e. $(\alpha \circ_0 \mathrm{Id}_\mathrm{F}) \circ_1 (\mathrm{Id}_\mathrm{F} \circ_0 \beta) = \mathrm{Id}_\mathrm{F}$ and $(\beta \circ_0 \mathrm{Id}_{\mathrm{F}^*}) \circ_1 (\mathrm{Id}_{\mathrm{F}^*} \circ_0 \alpha) = \mathrm{Id}_{\mathrm{F}^*}$.

Note that $\widehat{\mathcal{S}}_n^{\mathrm{ext}}$ is a wide fiat bicategory, where the fiat structure is defined by rotating the diagrams in Soergel calculus by 180 degrees and the adjunctions α and β are defined by cups and caps, respectively.

5.2.4. *Completed and triangulated bicategories.* Let $\mathcal{C}$ be a wide finitary bicategory. We define the bicategory $\mathcal{C}^\diamond$ to be the bicategory on the same objects as $\mathcal{C}$, but with morphism categories $\mathcal{C}^\diamond(\mathtt{i},\mathtt{j}) = \mathcal{C}(\mathtt{i},\mathtt{j})^\diamond (= \mathcal{C}(\mathtt{i},\mathtt{j})^\sqcup)$, with horizontal composition given component-wise, i.e.

$$\coprod_{i\in I} \mathrm{X}_i \coprod_{j\in J} \mathrm{Y}_j = \coprod_{\substack{i\in I\\ j\in J}} \mathrm{X}_i \mathrm{Y}_j$$

and similarly for 2-morphisms. Observe that $\mathcal{C}^\diamond$ is locally additive and $\Bbbk$-linear.

Remark 5.11. Note that this naturally embeds into the completion $\mathcal{C}^\circ$ with morphism categories $\mathcal{C}^\circ(\mathtt{i},\mathtt{j}) = \mathcal{C}(\mathtt{i},\mathtt{j})^\circ$ and horizontal composition given by Day convolution.

Likewise, for any locally additive $\Bbbk$-linear (or in particular wide finitary) bicategory $\mathcal{C}$, we can define $K^b(\mathcal{C})$ as having the same objects as $\mathcal{C}$, morphism categories $K^b(\mathcal{C})(\mathtt{i},\mathtt{j}) = K^b(\mathcal{C}(\mathtt{i},\mathtt{j}))$ and horizontal composition induced by taking the total complex of the tensor product of complexes. This is a triangulated bicategory, which fits within the bicategorical version of the framework considered in [LaMi, Section 3.1].

Lemma 5.12. *If $\mathcal{C}$ is wide finitary and all 1-morphisms have finite-dimensional endomorphism rings, then $K^b(\mathcal{C})$ is wide finitary.*

Proof. Since $\mathcal{C}$ is locally idempotent complete, so is $K^b(\mathcal{C})$ by [Sch, Theorem 3.4]. Since endomorphism rings of 1-morphisms in $\mathcal{C}$ are finite-dimensional, the same is true for endomorphism rings of 1-morphisms in $K^b(\mathcal{C})$. Thus $K^b(\mathcal{C})$ is Krull-Schmidt by [Kra, Cor 4.4]. Countability of isomorphism classes of indecomposables is inherited from $\mathcal{C}$. □

Corollary 5.13. *For any $n \in \mathbb{Z}_{\geq 2}$, the monoidal category (one-object bicategory) $K^b(\widehat{\mathbb{S}}_n^{\mathrm{ext}})$ is wide finitary.*

5.2.5. *Birepresentations.* An *additive birepresentation* of a locally additive $\Bbbk$-linear bicategory $\mathcal{C}$ is a $\Bbbk$-linear pseudofunctor from $\mathcal{C}$ to $\mathfrak{A}_\Bbbk$. A *wide finitary birepresentation* of a wide finitary bicategory $\mathcal{C}$ is a $\Bbbk$-linear pseudofunctor from $\mathcal{C}$ to $\mathfrak{A}_\Bbbk^{wf}$.

We can extend an additive or wide finitary birepresentation $\mathbf{M}$ of $\mathcal{C}$ to an additive birepresentation $\mathbf{M}^\diamond$ of $\mathcal{C}^\diamond$ by defining the action of $\mathrm{F} = \coprod_{i\in I} \mathrm{F}_i \in \mathcal{C}^\diamond(\mathtt{i},\mathtt{j})$ on $X = \coprod_{j\in J} X_j \in \mathbf{M}^\diamond(\mathtt{i})$ componentwise, i.e.

$$\mathbf{M}^\diamond(\mathrm{F})(X) = \mathbf{M}^\diamond\left(\coprod_{i\in I} \mathrm{F}_i\right)\left(\coprod_{j\in J} X_j\right) = \coprod_{\substack{i\in I\\ j\in J}} \mathbf{M}(\mathrm{F}_i)(X_j).$$

Similarly, any additive birepresentation of $\mathbf{M}$ gives rise to a *triangulated* birepresentation $K^b(\mathbf{M})$ of $K^b(\mathcal{C})$ where $K^b(\mathbf{M})(\mathtt{i}) = K^b(\mathbf{M}(\mathtt{i}))$.

5.2.6. *Algebras and birepresentations in $\mathcal{C}^\diamond$.* Let $\mathcal{C}$ be a wide finitary bicategory.

Now assume G is an abelian group and $\mathrm{Y} = \coprod_{j\in G} \mathrm{Y}_j \in \mathcal{C}^\diamond(\mathtt{i},\mathtt{i})$ is an algebra 1-morphism with product

$$\mathbf{m}\colon \coprod_{i,j\in G} \mathrm{Y}_i\mathrm{Y}_j \to \coprod_{k\in G} \mathrm{Y}_k$$

determined by component morphisms $\mathbf{m}_{i,j} = \mathbf{m}(\iota_i \circ_{\mathsf{h}} \iota_j)\colon \mathrm{Y}_i\mathrm{Y}_j \to \mathrm{Y}_{i+j}$, where we identify Y_{i+j} with the subobject of $\coprod_{k\in G} \mathrm{Y}_k$ given by $\iota_{i+j}(\mathrm{Y}_{i+j})$, and unit $\mathbf{u}$ determined by a morphism $\mathbb{1}_{\mathtt{i}} \to \mathrm{Y}_0$, where 0 is the neutral element in G.

For each $\mathtt{j} \in \mathcal{C}$, we can consider the category of Y-modules in $\mathcal{C}^\diamond(\mathtt{i},\mathtt{j})$, denoted by $\mathrm{mod}_{\mathcal{C}(\mathtt{i},\mathtt{j})^\diamond}\mathrm{Y}$, which gives rise to an additive birepresentation $\mathbf{mod}_{\mathcal{C}^\diamond}\mathrm{Y}$ of $\mathcal{C}^\diamond$. Note that each $\mathrm{mod}_{\mathcal{C}(\mathtt{i},\mathtt{j})^\diamond}\mathrm{Y}$ is idempotent complete by construction.

By the usual free forgetful adjunction, we have, for $\mathrm{F} \in \mathcal{C}(\mathtt{i},\mathtt{i})$ an isomorphism

$$\begin{array}{rcl} \mathrm{Hom}_{\mathrm{mod}_{\mathcal{C}(\mathtt{i},\mathtt{i})^\diamond}\mathrm{Y}}(\mathrm{FY},\mathrm{Y}) & \cong & \mathrm{Hom}_{\mathcal{C}(\mathtt{i},\mathtt{i})^\diamond}(\mathrm{F},\mathrm{Y}) \\ f & \mapsto & f \circ_{\mathsf{v}} (\mathrm{Id}_{\mathrm{F}} \circ_{\mathsf{h}} \mathbf{u}) \\ \mathbf{m} \circ_{\mathsf{v}} (g \circ_{\mathsf{h}} \mathrm{Id}_{\mathrm{Y}}) & \leftarrow\!\shortmid & g. \end{array} \tag{131}$$

Lemma 5.14. *Let $\mathrm{Y} = \coprod_{j\in G} \mathrm{Y}_j \in \mathcal{C}^\diamond(\mathtt{i},\mathtt{i})$ be an algebra 1-morphism as above, $\mathrm{F} \in \mathcal{C}(\mathtt{i},\mathtt{i})$ and $g \in \mathrm{Hom}_{\mathcal{C}(\mathtt{i},\mathtt{i})^\diamond}(\mathrm{F},\mathrm{Y})$. Assume that g factors over ι_j, i.e. there exists an $g_j \in \mathrm{Hom}_{\mathcal{C}(\mathtt{i},\mathtt{i})}(\mathrm{F},\mathrm{Y}_j)$ such that $g = \iota_j \circ_{\mathsf{v}} g_j$. Then under the isomorphism in* (131)*, g corresponds to a morphism f defined by*

$$f \circ_{\mathsf{v}} (\mathrm{Id}_{\mathrm{F}} \circ_{\mathsf{h}} \iota_k) = \mathbf{m}_{j,k} \circ_{\mathsf{v}} (g_j \circ_{\mathsf{h}} \mathrm{Id}_{\mathrm{Y}_k})\colon \mathrm{FY}_k \to \mathrm{Y}_{j+k}.$$

Proof. This is a direct computation, observing that

$$\begin{aligned} f \circ_{\mathsf{v}} (\mathrm{Id}_{\mathrm{F}} \circ_{\mathsf{h}} \iota_k) &= \mathbf{m} \circ_{\mathsf{v}} (g \circ_{\mathsf{h}} \mathrm{Id}_{\mathrm{Y}}) \circ_{\mathsf{v}} (\mathrm{Id}_{\mathrm{F}} \circ_{\mathsf{h}} \iota_k) \\ &= \mathbf{m} \circ_{\mathsf{v}} ((\iota_j \circ_{\mathsf{v}} g_j) \circ_{\mathsf{h}} \iota_k) \\ &= \mathbf{m} \circ_{\mathsf{v}} (\iota_j \circ_{\mathsf{h}} \iota_k) \circ_{\mathsf{v}} (g_j \circ_{\mathsf{h}} \mathrm{Id}_{\mathrm{Y}_k}) \\ &= \mathbf{m}_{j,k} \circ_{\mathsf{v}} (g_j \circ_{\mathsf{h}} \mathrm{Id}_{\mathrm{Y}_k}) \end{aligned}$$

so the claim follows. □

Lemma 5.15. *Let $\mathcal{C}$ be a wide finitary bicategory and $\mathrm{Y} \in \mathcal{C}(\mathtt{i},\mathtt{i})^\diamond$ an algebra 1-morphism. Assume that for any $\mathrm{G} \in \mathcal{C}(\mathtt{i},\mathtt{i})$, the morphism space $\mathrm{Hom}_{\mathcal{C}(\mathtt{i},\mathtt{i})^\diamond}(\mathrm{G},\mathrm{Y})$ is finite-dimensional. Then the birepresentation $\mathbf{M}_{\mathrm{Y}}$ given by $\mathbf{M}_{\mathrm{Y}}(\mathtt{j}) = \mathrm{add}\{\mathrm{FY} \,|\, \mathrm{F} \in \mathcal{C}(\mathtt{i},\mathtt{j})\}$ (where the additive closure is taken in $\mathrm{mod}_{\mathcal{C}(\mathtt{i},\mathtt{j})^\diamond}\mathrm{Y}$) is wide finitary.*

Proof. By (131), for each $\mathrm{F} \in \mathcal{C}(\mathtt{i},\mathtt{j})$, the endomorphism ring $\mathrm{End}_{\mathrm{mod}_{\mathcal{C}(\mathtt{i},\mathtt{j})^\diamond}\mathrm{Y}}(\mathrm{FY})$ is isomorphic to $\mathrm{Hom}_{\mathcal{C}(\mathtt{i},\mathtt{i})^\diamond}(\mathrm{F}^*\mathrm{F},\mathrm{Y})$, which by assumption is finite-dimensional. Since each $\mathrm{mod}_{\mathcal{C}(\mathtt{i},\mathtt{j})^\diamond}\mathrm{Y}$ is idempotent complete, [Kra, Cor 4.4] implies that each $\mathbf{M}(\mathtt{j})$ is Krull-Schmidt. Moreover, the number of isomorphism classes of indecomposable objects is countable and morphism spaces are even finite-dimensional, so $\mathbf{M}_{\mathrm{Y}}$ is wide finitary. □

5.3. **The categorified story.** In the first part of this section, we will define a graded wide-finitary birepresentation $\mathbf{U}$ of $\widehat{\mathbb{S}}_2^{\mathrm{ext}}$ and show that it categorifies the representation U of $\widehat{H}_2^{\mathrm{ext}}$. In the second part, we will define a triangulated birepresentation $\mathbf{W}$ of $\widehat{\mathbb{S}}_2^{\mathrm{ext}}$ and conjecture that it categorifies the representation W of $\widehat{H}_2^{\mathrm{ext}}$.

5.3.1. *The categorification of U.* Let $\mathbf{V}$ be the rank one finitary 2-representation of $\widehat{\mathbb{S}}_1^{\mathrm{ext}}$ categorifying V. The corresponding algebra object $\mathrm{X} \in \widehat{\mathbb{S}}_1^{\mathrm{ext},\diamond}$ is given by

$$\mathrm{X} := \coprod_{r\in\mathbb{Z}} \mathrm{B}_\rho^r,$$

with multiplication $\mu\colon \mathrm{XX} \to \mathrm{X}$ defined by the identity on $\mathrm{B}_\rho^k \mathrm{B}_\rho^{r-k} = \mathrm{B}_\rho^r$, for all $r, k \in \mathbb{Z}$, and unit $\epsilon\colon \mathrm{I} \to \mathrm{X}$ defined as the identity on $\mathrm{I} = \mathrm{B}_\rho^0$ and zero to B_ρ^r, for all $r \neq 0$. Note that $(\mathrm{B}_\rho)^k \cong \mathrm{B}_{\rho^k}$ and $\hom_{\widehat{\mathbb{S}}_1^{\mathrm{ext}}}(\mathrm{B}_{\rho^k}\langle s\rangle, \mathrm{B}_{\rho^m}\langle t\rangle) \cong \delta_{s,t}\delta_{k,m}\mathbb{R}$, for any $k, m, s, t \in \mathbb{Z}$, where $\delta_{-,-}$ is the Kronecker delta.

Consider $\mathrm{Y} := \Psi_{1,1}(\mathrm{X} \boxtimes \mathrm{X}) \in K^b(\widehat{\mathbb{S}}_2^{\mathrm{ext},\diamond})$. The fact that $\Psi_{1,1}$ is a monoidal functor and the isomorphisms $\mathrm{B}_\rho \boxtimes \mathrm{B}_\rho \cong (\mathrm{B}_\rho \boxtimes \mathrm{I})(\mathrm{I} \boxtimes \mathrm{B}_\rho) \cong (\mathrm{I} \otimes \mathrm{B}_\rho)(\mathrm{B}_\rho \boxtimes \mathrm{I})$ imply that

$$\mathrm{Y} \cong \coprod_{r\in\mathbb{Z}}\coprod_{s\in\mathbb{Z}} (\mathrm{B}_\rho\mathrm{T}_1)^r(\mathrm{T}_1^{-1}\mathrm{B}_\rho)^s \cong \coprod_{r\in\mathbb{Z}}\coprod_{s\in\mathbb{Z}} (\mathrm{T}_1^{-1}\mathrm{B}_\rho)^s(\mathrm{B}_\rho\mathrm{T}_1)^r.$$

Note that, for any $k, l \in \mathbb{Z}$, there are natural isomorphisms

$$(\mathrm{B}_\rho\mathrm{T}_1)^k\,(\mathrm{T}_1^{-1}\mathrm{B}_\rho)^l\,\mathrm{Y} \cong \mathrm{Y} \cong (\mathrm{T}_1^{-1}\mathrm{B}_\rho)^l(\mathrm{B}_\rho\mathrm{T}_1)^k\,\mathrm{Y},$$

because $(\mathrm{B}_\rho\mathrm{T}_1)^k$ and $(\mathrm{T}_1^{-1}\mathrm{B}_\rho)^l$ commute and

$$(\mathrm{B}_\rho\mathrm{T}_1)^k\,(\mathrm{T}_1^{-1}\mathrm{B}_\rho)^l\,\mathrm{Y} \cong \coprod_{r\in\mathbb{Z}}\coprod_{s\in\mathbb{Z}} (\mathrm{B}_\rho\mathrm{T}_1)^{r+k}(\mathrm{T}_1^{-1}\mathrm{B}_\rho)^{s+l} \cong \mathrm{Y}.$$

In particular, we see that

$$\mathrm{B}_\rho\mathrm{T}_1\mathrm{Y} \cong \mathrm{Y}, \quad \mathrm{T}_1^{-1}\mathrm{B}_\rho\mathrm{Y} \cong \mathrm{Y}, \quad \mathrm{B}_\rho^2\mathrm{Y} \cong \mathrm{B}_\rho\mathrm{T}_1\mathrm{T}_1^{-1}\mathrm{B}_\rho\mathrm{Y} \cong \mathrm{Y}. \tag{132}$$

For any $r, s \in \mathbb{Z}$, define

$$\mathrm{Y}^{r,s} := (\mathrm{B}_\rho\mathrm{T}_1)^r(\mathrm{T}_1^{-1}\mathrm{B}_\rho)^s.$$

Lemma 5.16. *Let $i \in \{0, 1\}$, and $k, t \in \mathbb{Z}$, and $m \in \mathbb{Z}_{\geq 0}$. Then*

$$\hom_{K^b(\widehat{\mathbb{S}}_2^{\mathrm{ext}})}\left(\mathrm{B}_{\rho^k}\mathrm{B}_{\widehat{m}\,i}\langle t\rangle, \mathrm{Y}^{r,s}\right) = 0 \tag{133}$$

for all but finitely many $r, s \in \mathbb{Z}$.

Proof. By [MaTh, Theorem 2.5], the hom-space in (133) is zero if $r + s \neq k$. Now, assume that $r + s = k$. We also assume that $r \geq k$, the case $r \leq k$ being similar. Then

$$\mathrm{Y}^{r,s} = (\mathrm{B}_\rho\mathrm{T}_1)^r(\mathrm{T}_1^{-1}\mathrm{B}_\rho)^{k-r} \cong (\mathrm{B}_\rho\mathrm{T}_1)^k(\mathrm{B}_\rho\mathrm{T}_1)^{r-k}(\mathrm{B}_\rho^{-1}\mathrm{T}_1)^{r-k} \cong \mathrm{B}_\rho^k\mathrm{T}_{\widehat{2r-k}\,1} \cong \mathrm{B}_{\rho^k}\mathrm{T}_{\widehat{2r-k}\,1}.$$

By biadjointness of B_{ρ^k} and $\mathrm{B}_{\rho^{-k}}$, we have

$$\hom_{K^b(\widehat{\mathcal{S}}_2^{\mathrm{ext}})}\left(\mathrm{B}_{\rho^k}\mathrm{B}_{\widehat{m}_i}\langle t\rangle, \mathrm{B}_{\rho^k}\mathrm{T}_{\widehat{2r-k}_1}\right) \cong \hom_{K^b(\widehat{\mathcal{S}}_2^{\mathrm{ext}})}\left(\mathrm{B}_{\widehat{m}_i}\langle t\rangle, \mathrm{T}_{\widehat{2r-k}_1}\right),$$

which in turn is isomorphic to

$$\hom_{K^b(\widehat{\mathcal{S}}_2^{\mathrm{ext}})}\left(\mathrm{B}_{\widehat{m}_i}\langle t\rangle, \mathrm{T}^{\min}_{\widehat{2r-k}_1}\right),$$

where $\mathrm{T}^{\min}_{\widehat{2r-k}_1}$ is the minimal complex obtained from $\mathrm{T}_{\widehat{2r-k}_1}$ by Gaussian elimination, which is unique up to homotopy equivalence. By [ElWi1, Theorem 6.9], the degree-zero cochain object of $\mathrm{T}^{\min}_{\widehat{2r-k}_1}$ is isomorphic to $\mathrm{B}_{\widehat{2r-k}_1}$, which implies that

$$\dim_{\mathbb{R}}\left(\hom_{K^b(\widehat{\mathcal{S}}_2^{\mathrm{ext}})}\left(\mathrm{B}_{\widehat{m}_i}\langle t\rangle, \mathrm{T}^{\min}_{\widehat{2r-k}_1}\right)\right) \leq \dim_{\mathbb{R}}\left(\hom_{K^b(\widehat{\mathcal{S}}_2^{\mathrm{ext}})}\left(\mathrm{B}_{\widehat{m}_i}\langle t\rangle, \mathrm{B}_{\widehat{2r-k}_1}\right)\right).$$

For a fixed choice of $m, t \in \mathbb{Z}$ and $m \in \mathbb{Z}_{\geq 0}$, Soergel's hom formula and the equations in (123) imply that the latter morphism space is zero for all but finitely many $r \in \mathbb{Z}$, which proves the lemma. □

The following corollary is an immediate consequence of Lemma 5.7 and Lemma 5.16.

Corollary 5.17. *For any $i \in \{0,1\}$, and $k, t \in \mathbb{Z}$, and $m \in \mathbb{Z}_{\geq 0}$, there is an isomorphism*

$$\hom_{K^b(\widehat{\mathcal{S}}_2^{\mathrm{ext},\diamond})}\left(\mathrm{B}_{\rho^k}\mathrm{B}_{\widehat{m}_i}\langle t\rangle, \mathrm{Y}\right) \cong \bigoplus_{r,s\in\mathbb{Z}} \hom_{K^b(\widehat{\mathcal{S}}_2^{\mathrm{ext}})}\left(\mathrm{B}_{\rho^k}\mathrm{B}_{\widehat{m}_i}\langle t\rangle, \mathrm{Y}^{r,s}\right). \tag{134}$$

We are now going to define a graded wide finitary birepresentation $\mathbf{U}$ of $\widehat{\mathcal{S}}_2^{\mathrm{ext}}$ and the rest of this section after the definition will be dedicated to the proof that it categorifies U.

Definition 5.18. Let

$$\mathbf{U} := \mathbf{M}_{\mathrm{Y}}$$

be the wide finitary birepresentation of $\widehat{\mathcal{S}}_2^{\mathrm{ext}}$ generated by Y, as defined in Lemma 5.15.

Lemma 5.7 and the isomorphism in (131) imply that

$$\hom_{\mathrm{Y}}(\mathrm{AY}, \mathrm{BY}) \cong \hom_{K^b(\widehat{\mathcal{S}}_2^{\mathrm{ext},\diamond})}(\mathrm{A}, \mathrm{BY}) \cong \bigoplus_{r,s\in\mathbb{Z}} \hom_{K^b(\widehat{\mathcal{S}}_2^{\mathrm{ext}})}(\mathrm{A}, \mathrm{BY}^{r,s}), \quad \mathrm{A}, \mathrm{B} \in \widehat{\mathcal{S}}_2^{\mathrm{ext}}, \tag{135}$$

where for simplicity we write $\hom_{\mathrm{Y}}(\mathrm{AY}, \mathrm{BY}) = \hom_{\mathrm{mod}_{K^b(\widehat{\mathcal{S}}_2^{\mathrm{ext},\diamond})}\mathrm{Y}}(\mathrm{AY}, \mathrm{BY})$. The isomorphisms in (135) show that any morphism $f \in \hom_{\mathrm{Y}}(\mathrm{AY}, \mathrm{BY})$ is determined by its components $f^{r,s} \in \hom_{K^b(\widehat{\mathcal{S}}_2^{\mathrm{ext}})}(\mathrm{A}, \mathrm{BY}^{r,s})$, for $r, s \in \mathbb{Z}$. Suppose $f\colon \mathrm{AY} \to BY$ and $g\colon \mathrm{BY} \to \mathrm{CY}$ are such that their only nonzero components under the free forgetful adjunction are $f^{r,s}\colon \mathrm{A} \to \mathrm{BY}^{r,s}$ and $g^{p,q}\colon \mathrm{B} \to \mathrm{CY}^{p,q}$. Then the only nonzero component of $g \circ f\colon \mathrm{AY} \to \mathrm{CY}$ under the free forgetful adjunction, is given by the composition

$$\mathrm{A} \xrightarrow{f^{r,s}} \mathrm{BY}^{r,s} \xrightarrow{g^{p,q}\circ_{\mathsf{h}}\mathrm{Id}_{\mathrm{Y}^{r,s}}} \mathrm{CY}^{p,q}\mathrm{Y}^{r,s} \xrightarrow{\mathrm{Id}_{\mathrm{C}}\circ_{\mathsf{h}}\mathbf{m}} \mathrm{CY}^{p+r,q+s}. \tag{136}$$

For morphisms with multiple nonzero components, this extends additively.

Proposition 5.19. *Let $k, m \in \mathbb{Z}_{\geq 0}$ and $r, s \in \mathbb{Z}$. When $r \in \mathbb{Z}_{>0}$, the morphism spaces*

$$\mathrm{hom}_{K^b(\widehat{\mathcal{S}}_2^{\mathrm{ext}})}(\mathrm{B}_{\widehat{k}_1}\langle t\rangle, \mathrm{B}_{\widehat{m}_1}\mathrm{Y}^{r,s}) \text{ and } \mathrm{hom}_{K^b(\widehat{\mathcal{S}}_2^{\mathrm{ext}})}(\mathrm{B}_{\widehat{k}_0}\langle t\rangle, \mathrm{B}_{\widehat{m}_1}\mathrm{Y}^{r,s})$$

are both zero for all $t \geq 0$.

Proof. We only prove the first case, the proof of the second one being similar. Biadjointness and the isomorphism $\mathrm{B}_{\widehat{m}_1}^* \cong \mathrm{B}_{{}_1\widehat{m}}$ imply that

$$\mathrm{hom}_{K^b(\widehat{\mathcal{S}}_2^{\mathrm{ext}})}(\mathrm{B}_{\widehat{k}_1}\langle t\rangle, \mathrm{B}_{\widehat{m}_1}\mathrm{Y}^{r,s}) \cong \mathrm{hom}_{K^b(\widehat{\mathcal{S}}_2^{\mathrm{ext}})}(\mathrm{B}_{{}_1\widehat{m}}\mathrm{B}_{\widehat{k}_1}\langle t\rangle, \mathrm{Y}^{r,s}).$$

By [MaTh, Theorem 2.5], that hom-space is zero if $r+s \neq 0$, so suppose that $r+s=0$. Then

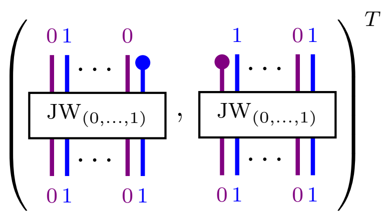
$$\mathrm{Y}^{r,s} \cong (\mathrm{T}_0\mathrm{T}_1)^r.$$

Let $r > 0$. By [AMRW, Section 9.1] (see also [ElWi1, Theorem 6.9] and [EMTW, Theorem 19.47]), the minimal complex $[(\mathrm{T}_0\mathrm{T}_1)^r]^{\min}$ is of the form

$$\underline{\mathrm{B}_{\widehat{2r}_1}} \xrightarrow{\mathrm{d}^0} \mathrm{B}_{\widehat{2r-1}_0}\langle 1\rangle \oplus \mathrm{B}_{\widehat{2r-1}_1}\langle 1\rangle \longrightarrow \ldots, \tag{137}$$

where the underlined cochain object has homological degree zero and the differential d^0 is given by

(138) $\mathrm{d}^0 =$



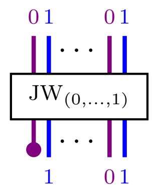
where $\mathrm{JW}_{(0,\ldots,1)}$ is the idempotent in $\mathrm{end}_{\widehat{\mathcal{S}}_2^{\mathrm{ext}}}(\mathrm{B}_0 \cdots \mathrm{B}_1)$ obtained from the Jones-Wenzl projector JW_{2r} by the quantum Satake correspondence, as in [Eli1, Section 5.3.2] (see also [EMTW, Section 9.3]), and T denotes the transpose. Note that Elias and Williamson do not give the multiplicities of the indecomposables in positive homological degree, but in affine type A_1 both multiplicities in homological degree one can be deduced from the corresponding coefficients in (122). On the other hand, the formulas in (123) imply that the decomposition of $\mathrm{B}_{{}_1\widehat{m}}\mathrm{B}_{\widehat{k}_1}\langle t\rangle$ only contains indecomposables of the form $\mathrm{B}_{{}_1\widehat{n}_1}\langle t+v\rangle$, with $0 \leq n \leq k+m$ and $v \in \{-1, 0, 1\}$. The parity conditions for ${}_1\widehat{n}_1$ amount to n being odd. Any non-zero morphism of complexes in $\mathrm{hom}_{K^b(\widehat{\mathcal{S}}_2^{\mathrm{ext}})}(\mathrm{B}_{{}_1\widehat{m}}\mathrm{B}_{\widehat{k}_1}\langle t\rangle, [(\mathrm{T}_0\mathrm{T}_1)^r]^{\min})$ is given by a non-zero morphism in the morphism space $\mathrm{hom}_{\widehat{\mathcal{S}}_2^{\mathrm{ext}}}(\mathrm{B}_{{}_1\widehat{m}}\mathrm{B}_{\widehat{k}_1}\langle t\rangle, \mathrm{B}_{\widehat{2r}_1})$ that is annihilated by postcomposition with the differential d^0. The latter morphism space is the direct sum of morphism spaces whose sources are the indecomposables of the form $\mathrm{B}_{{}_1\widehat{n}_1}\langle t+v\rangle$, with $0 \leq n \leq k+m$ and $v \in \{-1, 0, 1\}$, and Soergel's hom formula in (49) and Lemma 5.1 imply that the graded rank of $\mathrm{Hom}_{\widehat{\mathcal{S}}_2^{\mathrm{ext},*}}(\mathrm{B}_{{}_1\widehat{n}_1}, \mathrm{B}_{\widehat{2r}_1})$ is equal to

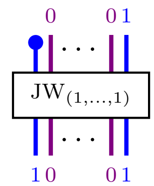
$$\left(b_{{}_1\widehat{n}_1}, b_{{}_0\widehat{2r}_1}\right) = q^{|n-2r|}p'(q), \tag{139}$$

where $p'(q) \in \mathbb{Z}_{\geq 0}[q]$ has constant term equal to one. This implies that the morphism space $\mathrm{hom}_{\widehat{\mathcal{S}}_2^{\mathrm{ext}}}(\mathrm{B}_{{}_1\widehat{n}_1}\langle t+v\rangle, \mathrm{B}_{\widehat{2r}_1})$ is zero unless $t+v = -|n-2r|$. Suppose that $t \geq 0$ and recall that $v \in \{-1,0,1\}$. Since n must be odd, we also see that $|n-2r|>0$, so $t+v=-|n-2r|$ can only hold if $t=0$, $v=-1$ and $|n-2r|=1$. In other words, the only non-zero morphism spaces are

$$\mathrm{hom}_{\widehat{\mathcal{S}}_2^{\mathrm{ext}}}(\mathrm{B}_{{}_1\widehat{2r-1}_1}\langle -1\rangle, \mathrm{B}_{{}_0\widehat{2r}_1}) \quad \text{and} \quad \mathrm{hom}_{\widehat{\mathcal{S}}_2^{\mathrm{ext}}}(\mathrm{B}_{{}_1\widehat{2r+1}_1}\langle -1\rangle, \mathrm{B}_{{}_0\widehat{2r}_1}),$$

which are both one-dimensional and generated by

(140) 

and 

,

respectively. However, neither of these two morphisms extends to a map of complexes whose target is $[(\mathrm{T}_0\mathrm{T}_1)^r]^{\min}$. To see this, note that postcomposition with d^0 adds an extra dot to the rightmost, resp. leftmost, top strand of the morphisms in (140). These morphisms are all non-zero, because even with dots on all strands such morphisms are non-zero, see e.g. [Eli1, Section 5.3.4] and [EMTW, (10.8i) and (10.9)]. This finishes the proof that

$$\mathrm{hom}_{K^b(\widehat{\mathcal{S}}_2^{\mathrm{ext}})}(\mathrm{B}_{\widehat{k}_1}\langle t\rangle, \mathrm{B}_{\widehat{m}_1}\mathrm{Y}^{r,s}) = 0, \quad \text{if } r>0 \text{ and } t\geq 0. \qquad \square$$

Corollary 5.20. *For all $k\in\mathbb{Z}_{\geq 0}$, the objects $\mathrm{B}_{\widehat{k}_1}\mathrm{Y}$ and $\mathrm{B}_\rho\mathrm{B}_{\widehat{k}_1}\mathrm{Y}$ are indecomposable in $\mathbf{U}$.*

Proof. The isomorphisms in (135) show that any non-zero idempotent $e\in\mathrm{hom}_{\mathrm{Y}}(\mathrm{B}_{\widehat{k}_1}\mathrm{Y}, \mathrm{B}_{\widehat{k}_1}\mathrm{Y})$ is completely determined by its components $e^{r,s}\in\mathrm{hom}_{K^b(\widehat{\mathcal{S}}_2^{\mathrm{ext}})}(\mathrm{B}_{\widehat{k}_1}, \mathrm{B}_{\widehat{k}_1}\mathrm{Y}^{r,s})$, for $r,s\in\mathbb{Z}$, and that only finitely many components are non-zero. By [MaTh, Theorem 2.5] and Proposition 5.19, we have $e^{r,s}=0$ unless $r+s=0$ and $r\leq 0$. Furthermore, $e^{0,0}$ must be a multiple of the identity on $\mathrm{B}_{\widehat{k}_1}$, by Soergel's hom formula in (49). By a slight abuse of notation, justified by Lemma 5.14, let us write $e=e^{r_1,-r_1}+\ldots+e^{r_m,-r_m}$, for some $r_i\in\mathbb{Z}_{\leq 0}$ and $m\in\mathbb{Z}_{>0}$. Then $e^2=e$ if and only if

$$\sum_{\substack{1\leq i,j\leq m\\ r_i+r_j=r_k}} e^{r_i,-r_i}\circ e^{r_j,-r_j} = e^{r_k,-r_k}, \text{ for all } 1\leq k\leq m. \tag{141}$$

First, assume that $e^{0,0}=0$ and let $r_{\max}$ be the maximal $r\in\mathbb{Z}_{<0}$ such that $e^{r,-r}\neq 0$. Then the equation in (141) for $r_k=r_{\max}$ has no solution, which shows that $e^2=e$ cannot hold.

Next, assume that $e^{0,0}\neq 0$. Without loss of generality, we may assume that $r_1=0$ and $r_i<0$ for all $2\leq i\leq m$. Then the equations in (141) can only hold if $e^{0,0}=\mathrm{id}_{\mathrm{B}_{\widehat{k}_1}}$ and

$$e^{r_k,-r_k} + \sum_{\substack{i,j\in\{2,\ldots,\hat{k},\ldots,m\}\\ r_i+r_j=r_k}} e^{r_i,-r_i}\circ e^{r_j,-r_j} = 0, \text{ for all } 2\leq k\leq m. \tag{142}$$

In turn, the equations in (142) can only hold if $e^{r_k,-r_k}=0$ for all $2\leq k\leq m$, as can be seen by first considering the maximal $r_k<0$ and then working one's way down until the minimal

$r_k < 0$. This shows that $e^2 = e$ holds if and only if $e = \mathrm{id}_{\mathrm{B}_{\widehat{k}_1}\mathrm{Y}}$, hence $\mathrm{B}_{\widehat{k}_1}\mathrm{Y}$ is indecomposable in $\mathbf{U}$.

Since $\mathrm{B}_\rho^* \cong \mathrm{B}_{\rho^{-1}}$, we have

$$\mathrm{hom}_{\mathrm{Y}}(\mathrm{B}_\rho \mathrm{B}_{\widehat{k}_1}\mathrm{Y}, \mathrm{B}_\rho \mathrm{B}_{\widehat{k}_1}\mathrm{Y}) \cong \mathrm{hom}_{\mathrm{Y}}(\mathrm{B}_{\widehat{k}_1}\mathrm{Y}, \mathrm{B}_{\widehat{k}_1}\mathrm{Y}),$$

which implies that $\mathrm{B}_\rho \mathrm{B}_{\widehat{k}_1}\mathrm{Y}$ is also indecomposable in $\mathbf{U}$, by the arguments above. □

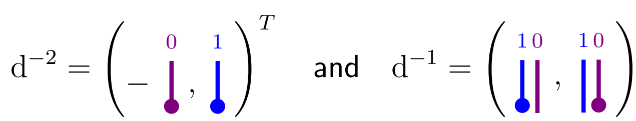
Remark 5.21. When $r < 0$, the morphism space $\mathrm{hom}_{K^b(\widehat{\mathbb{S}}_2^{\mathrm{ext}})}(\mathrm{B}_{\widehat{k}_1}, \mathrm{B}_{\widehat{m}_1}\mathrm{Y}^{r,s})$ can be non-zero. Take $k = m = 2$ and $r = -1$, for example. Then $\mathrm{Y}^{-1,+1} \cong (\mathrm{T}_0\mathrm{T}_1)^{-1} \cong \mathrm{T}_1^{-1}\mathrm{T}_0^{-1}$ and

$$\mathrm{hom}_{K^b(\widehat{\mathbb{S}}_2^{\mathrm{ext}})}(\mathrm{B}_{01}, \mathrm{B}_{01}\mathrm{T}_1^{-1}\mathrm{T}_0^{-1}) \cong \mathrm{hom}_{K^b(\widehat{\mathbb{S}}_2^{\mathrm{ext}})}(\mathrm{B}_{10}\mathrm{B}_{01}, \mathrm{T}_1^{-1}\mathrm{T}_0^{-1})$$

On the one hand, the complex $\mathrm{T}_1^{-1}\mathrm{T}_0^{-1}$ (which is already minimal) is given by

(143) $$R\langle -2\rangle \xrightarrow{d^{-2}} \mathrm{B}_0\langle -1\rangle \oplus \mathrm{B}_1\langle -1\rangle \xrightarrow{d^{-1}} \underline{\mathrm{B}_{10}},$$

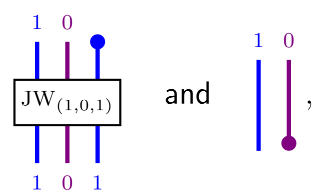
with d^{-2} and d^{-1} given by

(144) 

On the other hand, we have

(145) $$\mathrm{B}_{10}\mathrm{B}_{01} \cong \mathrm{B}_{101}\langle 1\rangle \oplus \mathrm{B}_{101}\langle -1\rangle \oplus \mathrm{B}_1\langle 1\rangle \oplus \mathrm{B}_1\langle -1\rangle.$$

Any non-zero morphism in $\mathrm{hom}_{\widehat{\mathbb{S}}_2^{\mathrm{ext}}}(\mathrm{Z}, \mathrm{B}_{10})$, where Z is one of the four indecomposables in (145), induces a map of complexes $\mathrm{Z} \to \mathrm{T}_1^{-1}\mathrm{T}_0^{-1}$, which can be null-homotopic or not. By Soergel's hom formula in (49), the morphism spaces $\mathrm{hom}_{\widehat{\mathbb{S}}_2^{\mathrm{ext}}}(\mathrm{B}_{101}\langle -1\rangle, \mathrm{B}_{10})$ and $\mathrm{hom}_{\widehat{\mathbb{S}}_2^{\mathrm{ext}}}(\mathrm{B}_1\langle -1\rangle, \mathrm{B}_{10})$ are both one-dimensional, with respective generators given by

(146) 

and the other two possible morphism spaces are both zero. The map of complexes induced by the non-zero morphism in $\mathrm{hom}_{\widehat{\mathbb{S}}_2^{\mathrm{ext}}}(\mathrm{B}_1\langle -1\rangle, \mathrm{B}_{01})$ is null-homotopic, with the homotopy being induced by the identity on $\mathrm{B}_1\langle -1\rangle$. The map of complexes induced by the non-zero morphism in $\mathrm{hom}_{\widehat{\mathbb{S}}_2^{\mathrm{ext}}}(\mathrm{B}_{101}\langle -1\rangle, \mathrm{B}_{10})$ is not null-homotopic, because Soergel's hom formula implies that $\mathrm{hom}_{\widehat{\mathbb{S}}_2^{\mathrm{ext}}}(\mathrm{B}_{101}\langle -1\rangle, \mathrm{B}_0\langle -1\rangle) = \mathrm{hom}_{\widehat{\mathbb{S}}_2^{\mathrm{ext}}}(\mathrm{B}_{101}\langle -1\rangle, \mathrm{B}_1\langle -1\rangle) = 0$, so there are no morphisms which could constitute a homotopy.

Note that this implies that $\mathrm{hom}_{\mathrm{Y}}(\mathrm{B}_{01}\mathrm{Y}, \mathrm{B}_{01}\mathrm{Y})$ contains a non-zero morphism which is not a multiple of an idempotent.

Theorem 5.22. *The following holds in* $\mathbf{U}$.

a) We have

$$\mathrm{B}_\rho \mathrm{B}_{\widehat{k}_0}\mathrm{Y} \cong \mathrm{B}_{\widehat{k}_1}\mathrm{B}_\rho \mathrm{Y} \cong \mathrm{B}_{\widehat{k}_1}\mathrm{Y}\langle -1\rangle, \quad \forall\, k \in \mathbb{Z}_{>0}.$$

b) We have

$$\mathrm{B}_{\widehat{k}_0}\mathrm{Y} \cong \mathrm{B}_\rho \mathrm{B}_{\widehat{k}_1}\mathrm{Y}\langle -1\rangle, \quad \forall\, k \in \mathbb{Z}_{>0}.$$

c) The indecomposables

$$\mathrm{B}_{\widehat{k}_1}\mathrm{Y}\langle s\rangle,\ \mathrm{B}_\rho \mathrm{B}_{\widehat{m}_1}\mathrm{Y}\langle t\rangle, \quad k, m \in \mathbb{Z}_{\geq 0}; s, t \in \mathbb{Z}$$

are pairwise non-isomorphic.

Proof. (a) The isomorphism $\mathrm{B}_1\mathrm{T}_1 \cong \mathrm{B}_1\langle -1\rangle$ in $K^b(\widehat{\mathcal{S}}_2^{\mathrm{ext}})$ and the isomorphism $\mathrm{B}_\rho\mathrm{Y} \cong \mathrm{T}_1\mathrm{Y}$ in $\mathbf{U}$ imply that

$$\mathrm{B}_\rho\mathrm{B}_0\mathrm{Y} \cong \mathrm{B}_1\mathrm{B}_\rho\mathrm{Y} \cong \mathrm{B}_1\mathrm{T}_1\mathrm{Y} \cong \mathrm{B}_1\mathrm{Y}\langle -1\rangle \text{ in } \mathbf{U}.$$

The general result, as stated in the theorem, follows by induction and uniqueness of decomposition into indecomposables in $\mathbf{U}$, because

$$\mathrm{B}_0\mathrm{B}_1 \cong \mathrm{B}_{01} \quad \text{and} \quad \mathrm{B}_{\widehat{k-1}_0}\mathrm{B}_1 \cong \mathrm{B}_{\widehat{k}_1} \oplus \mathrm{B}_{\widehat{k-2}_1} \text{ in } \widehat{\mathcal{S}}_2^{\mathrm{ext}},$$

for all $k \in \mathbb{Z}_{>2}$.

(b) The isomorphism in this case can be obtained from the one in the previous case by tensoring with B_ρ on the left and using that B_ρ^2 acts as the identity on $\mathbf{U}$.

(c) There are three different cases: we have to show that

i) $\mathrm{B}_{\widehat{k}_1}\mathrm{Y}\langle s\rangle \ncong \mathrm{B}_{\widehat{m}_1}\mathrm{Y}\langle t\rangle$, for $k, m \in \mathbb{Z}_{\geq 0}$ with $k \neq m$ and $s, t \in \mathbb{Z}$;

ii) $\mathrm{B}_{\widehat{k}_1}\mathrm{Y}\langle s\rangle \ncong \mathrm{B}_\rho\mathrm{B}_{\widehat{m}_1}\mathrm{Y}\langle t\rangle$, for $k, m \in \mathbb{Z}_{\geq 0}$ and $s, t \in \mathbb{Z}$;

iii) $\mathrm{B}_\rho\mathrm{B}_{\widehat{k}_1}\mathrm{Y}\langle s\rangle \ncong \mathrm{B}_\rho\mathrm{B}_{\widehat{m}_1}\mathrm{Y}\langle t\rangle$, for $k, m \in \mathbb{Z}_{\geq 0}$ with $k \neq m$ and $s, t \in \mathbb{Z}$.

The first case implies the third one, because

$$\mathrm{hom}_{\mathrm{Y}}(\mathrm{B}_\rho\mathrm{B}_{\widehat{m}_1}\mathrm{Y}\langle t\rangle, \mathrm{B}_\rho\mathrm{B}_{\widehat{k}_1}\mathrm{Y}) \cong \mathrm{hom}_{\mathrm{Y}}(\mathrm{B}_{\widehat{m}_1}\mathrm{Y}\langle t\rangle, \mathrm{B}_{\widehat{k}_1}\mathrm{Y})$$

by adjunction, so it suffices to prove the first two cases.

i) Suppose that there is an isomorphism in $\mathbf{U}$

$$f\colon \mathrm{B}_{\widehat{k}_1}\mathrm{Y}\langle t\rangle \to \mathrm{B}_{\widehat{m}_1}\mathrm{Y}$$

for some $k, m \in \mathbb{Z}_{\geq 0}$ with $k \neq m$ and some $t \in \mathbb{Z}$, which we may assume to be non-negative without loss of generality. As before, f is determined by its non-zero components in $K^b(\widehat{\mathcal{S}}_2^{\mathrm{ext}})$

$$f^{r,s}\colon \mathrm{B}_{\widehat{k}_1}\langle t\rangle \to \mathrm{B}_{\widehat{m}_1}\mathrm{Y}^{r,s}$$

for a finite number of integers r, s. By [MaTh, Theorem 2.5] and Proposition 5.19, this implies that $r + s = 0$ and $r \leq 0$. Since $t \geq 0$ and $k \neq m$, both by assumption, Soergel's hom formula

(49) implies that $f^{-s,s}=0$ if $s=0$. Thus, we must have $s>0$ for all non-zero $f^{-s,s}$. Recall that $\mathrm{Y}^{-s,s}\simeq(\mathrm{T}_1^{-1}\mathrm{T}_0^{-1})^s$ and that the minimal complex $[(\mathrm{T}_1^{-1}\mathrm{T}_0^{-1})^s]^{\min}$ is given by

$$\cdots\longrightarrow \mathrm{B}_{\widehat{s-1}_0}\langle -1\rangle\oplus \mathrm{B}_{\widehat{s-1}_1}\langle -1\rangle \xrightarrow{\mathrm{d}^{-1}} \mathrm{B}_{\widehat{s}_0}.$$

By adjunction, we also have

$$\hom_{K^b(\widehat{\mathbb{S}}_2^{\mathrm{ext}})}(\mathrm{B}_{\widehat{k}_1}\langle t\rangle,\mathrm{B}_{\widehat{m}_1}\mathrm{Y}^{-s,s})\cong \hom_{K^b(\widehat{\mathbb{S}}_2^{\mathrm{ext}})}(\mathrm{B}_{{}_1\widehat{m}}\mathrm{B}_{\widehat{k}_1}\langle t\rangle,\mathrm{Y}^{-s,s}),$$

and (123) and Theorem 3.3 imply that the indecomposables in the decomposition of $\mathrm{B}_{{}_1\widehat{m}}\mathrm{B}_{\widehat{k}_1}\langle t\rangle$ are of the form $\mathrm{B}_{{}_1\widehat{n}_1}\langle t'\rangle$, for certain integers $n\geq |k-m|$ and certain $t'\in\{t-1,t,t+1\}$. Since $k\neq m$, we must have $n>0$. Therefore, Soergel's hom formula (49) implies that

$$\hom_{\widehat{\mathbb{S}}_2^{\mathrm{ext}}}(\mathrm{B}_{{}_1\widehat{n}_1}\langle t'\rangle,\mathrm{B}_{\widehat{s}_0})=0$$

for all integers $n,s>0$ and $t'\geq 0$. Since $t'<0$ can only hold if $t\leq 0$, our initial assumption that $t\geq 0$ implies that we must have $t=0$ and $t'=-1$. However, this leads to a contradiction. Since $f^{-s,s}$ is zero unless $s>0$, the inverse of f must have a non-zero component $(f^{-1})^{s,-s}\colon \mathrm{B}_{\widehat{m}_1}\to \mathrm{B}_{\widehat{k}_1}\mathrm{Y}^{s,-s}$ for some $s>0$, which is impossible according to Proposition 5.19. This completes the proof that the $\mathrm{B}_{\widehat{k}_1}\mathrm{Y}\langle t\rangle$ are all pairwise non-isomorphic.

ii) It suffices to show that $\mathrm{B}_\rho\mathrm{B}_{\widehat{k}_1}\mathrm{Y}\not\cong \mathrm{B}_{\widehat{m}_1}\mathrm{Y}\langle t\rangle$, for any $k,m\in\mathbb{Z}_{\geq 0}$ and any $t\in\mathbb{Z}$. First assume that $k=m=0$ and suppose, on the contrary, that there is an isomorphism

$$f\colon \mathrm{B}_\rho\mathrm{Y}\to\mathrm{Y}\langle t\rangle$$

for some $t\in\mathbb{Z}$. By tensoring on the left with B_ρ and using the fact that B_ρ^2 acts as the identity, we see that t has to be zero. Further, f is determined by its non-zero components $f^{r,s}\colon \mathrm{B}_\rho\to\mathrm{Y}^{r,s}$. By [MaTh, Theorem 2.5], such a component can only be non-zero if $r+s=1$, so assume this and assume also that $r\geq 1$. Then $\mathrm{Y}^{r,s}\simeq \mathrm{B}_\rho\mathrm{T}_{\widehat{2r-1}_1}$. By adjunction, we have

$$\hom_{K^b(\widehat{\mathbb{S}}_2^{\mathrm{ext}})}(\mathrm{B}_\rho,\mathrm{B}_\rho\mathrm{T}_{\widehat{2r-1}_1})\cong \hom_{K^b(\widehat{\mathbb{S}}_2^{\mathrm{ext}})}(R,\mathrm{T}_{\widehat{2r-1}_1}). \tag{147}$$

The minimal complex $[\mathrm{T}_{\widehat{2r-1}_1}]^{\min}$ is given by $\mathrm{B}_{\widehat{2r-1}_1}$ in homological degree zero, so Soergel's hom formula in (49) implies that the second morphism space in (147) is zero because $2r-1>0$. The case when $r<1$ can be proved similarly. This implies that $f^{r,s}=0$ for all $r,s\in\mathbb{Z}$, which proves that $\mathrm{B}_\rho\mathrm{Y}$ can not be isomorphic to $\mathrm{Y}\langle t\rangle$ for any $t\in\mathbb{Z}$.

Now assume that $k>0$. Item b) implies that the proof is complete if we can show that $\mathrm{B}_{\widehat{k}_0}\mathrm{Y}$ and $\mathrm{B}_{\widehat{m}_1}\mathrm{Y}\langle t\rangle$ are not isomorphic for any $m\in\mathbb{Z}_{\geq 0}$ and any $t\in\mathbb{Z}$. Note that the case when $m>0$ reduces to the case when $k>0$ by tensoring both complexes on the left with B_ρ and using that B_ρ^2 acts as the identity, so we can indeed assume that $k>0$ without loss of generality. Now, suppose in that case that there exists an isomorphism

$$f\colon \mathrm{B}_{\widehat{k}_0}\mathrm{Y}\langle t\rangle\to \mathrm{B}_{\widehat{m}_1}\mathrm{Y} \tag{148}$$

in $\mathbf{U}$, for some $k,m\in\mathbb{Z}_{\geq 0}$ and $t\in\mathbb{Z}$.

We will first show that $t=1$ must hold. Then we will show that $\mathrm{B}_{\widehat{k}_0}\langle 1\rangle$ must be a direct summand of $\mathrm{B}_{\widehat{m}_1}\mathrm{Y}^{-s,s}$ for some $s>0$. Finally, we will show that that is impossible. To arrive at that contradiction, we will first show that there is only a monomorphism $\mathrm{B}_{\widehat{k}_0}\langle 1\rangle \to \mathrm{B}_{\widehat{m}_1}\mathrm{Y}^{-s,s}$ for $k=1$, but that there is no epimorphism $\mathrm{B}_{\widehat{m}_1}\mathrm{Y}^{-s,s} \to \mathrm{B}_0\langle 1\rangle$. To prove that $k=1$ must hold, we use *localization*, which we briefly recall in the beginning of the relevant paragraph.

Note that

$$\begin{aligned}\hom(\mathrm{B}_{\widehat{k}_0}\mathrm{Y}\langle t\rangle, \mathrm{B}_{\widehat{m}_1}\mathrm{Y}) &\cong \hom(\mathrm{B}_\rho\mathrm{B}_{\widehat{k}_0}\mathrm{Y}\langle t\rangle, \mathrm{B}_\rho\mathrm{B}_{\widehat{m}_1}\mathrm{Y})\\ &\cong \hom(\mathrm{B}_{\widehat{k}_1}\mathrm{Y}\langle t-1\rangle, \mathrm{B}_{\widehat{m}_0}\mathrm{Y}\langle 1\rangle)\\ &\cong \hom(\mathrm{B}_{\widehat{k}_1}\mathrm{Y}, \mathrm{B}_{\widehat{m}_0}\mathrm{Y}\langle 2-t\rangle).\end{aligned}$$

Hence, the inverse of f induces an isomorphism $\mathrm{B}_{\widehat{m}_0}\mathrm{Y}\langle 2-t\rangle \to \mathrm{B}_{\widehat{k}_1}\mathrm{Y}$ in $\mathbf{U}$. Since $t\leq 1$ if and only if $2-t\geq 1$, we can assume that $t\geq 1$ in (148) without loss of generality. Thus, let $t\geq 1$. As before, the isomorphism f is determined by its components

$$f^{r,s}\colon \mathrm{B}_{\widehat{k}_0}\langle t\rangle \to \mathrm{B}_{\widehat{m}_1}\mathrm{Y}^{r,s}.$$

By [MaTh, Theorem 2.5] and Proposition 5.19, we have $f^{r,s}=0$ unless $r+s=0$ and $r\leq 0$. Soergel's hom formula in (49) implies that $f^{0,0}=0$, thus all non-zero components $f^{r,-r}$ satisfy $r<0$ (which is equivalent to saying that $f^{-s,s}\neq 0$ can only hold if $s>0$, of course).

Now, suppose that $s=-r>0$. We claim that $t=1$ must hold. Note that $\mathrm{Y}^{-s,s}\simeq (\mathrm{T}_1^{-1}\mathrm{T}_0^{-1})^s$ and the 0-cochain object of $[(\mathrm{T}_1^{-1}\mathrm{T}_0^{-1})^s]^{\min}$ is $\mathrm{B}_{\widehat{2s}_0}$. By adjunction, we have

$$\hom_{K^b(\widehat{\mathcal{S}}_2^{\mathrm{ext}})}(\mathrm{B}_{\widehat{k}_0}\langle t\rangle, \mathrm{B}_{\widehat{m}_1}\mathrm{Y}^{-s,s}) \cong \hom_{K^b(\widehat{\mathcal{S}}_2^{\mathrm{ext}})}(\mathrm{B}_{1\widehat{m}}\mathrm{B}_{\widehat{k}_0}\langle t\rangle, \mathrm{Y}^{-s,s}),$$

and the indecomposables in the decomposition of $\mathrm{B}_{1\widehat{m}}\mathrm{B}_{\widehat{k}_0}\langle t\rangle$ are all of the form $\mathrm{B}_{\widehat{n}_0}\langle t'\rangle$ for certain $n\geq |k-m|$ and $t'\in\{t-1,t,t+1\}$. Soergel's hom formula in (49) implies that

$$\hom_{\widehat{\mathcal{S}}_2^{\mathrm{ext}}}(\mathrm{B}_{\widehat{n}_0}\langle t'\rangle, \mathrm{B}_{\widehat{2s}_0})$$

is zero for all $t'>0$ (and any $n\geq 0$) and is one-dimensional for $t'=0$ (in which case $n=2s$). This shows that $f^{-s,s}\neq 0$ implies that $t'=0$, which can only hold if $t=1$ since $t\geq 1$ by assumption. This completes the proof of the claim.

Now suppose that $g=f^{-1}$. Then there must be an integer $s>0$ such that the following composite is non-zero:

$$\mathrm{B}_{\widehat{k}_0}\langle 1\rangle \xrightarrow{f^{-s,s}} \mathrm{B}_{\widehat{m}_1}\mathrm{Y}^{-s,s} \xrightarrow{g^{s,-s}\circ\, \mathrm{id}_{\mathrm{Y}^{-s,s}}} \mathrm{B}_{\widehat{k}_0}\langle 1\rangle\mathrm{Y}^{s,-s}\mathrm{Y}^{-s,s} \xrightarrow{\mathrm{id}_{\mathrm{B}_{\widehat{k}_0}\langle 1\rangle}\circ\mu} \mathrm{B}_{\widehat{k}_0}\langle 1\rangle,$$

where we have used the canonical isomorphism $\mathrm{B}_{\widehat{k}_0}\mathrm{Y}^{0,0}\langle 1\rangle \cong \mathrm{B}_{\widehat{k}_0}\langle 1\rangle$ for the final codomain. Soergel's hom formula in (49) implies that that composite is a non-zero multiple of the identity, hence $\mathrm{B}_{\widehat{k}_0}\langle 1\rangle$ is a direct summand of $\mathrm{B}_{\widehat{m}_1}\mathrm{Y}^{-s,s}$. However, that is impossible, as we will show below.

First identify $\widehat{\mathcal{S}}_2^{\mathrm{ext}}$ with the algebraic category of Soergel bimodules over R (the polynomial ring generated by the dumbbells). Let Q be the field of fractions of R and recall the *localization functor* $\mathrm{Loc}\colon \widehat{\mathcal{S}}_2^{\mathrm{ext}} \to \widehat{\mathcal{S}}_{2,Q}^{\mathrm{ext}}$, defined by forgetting the grading of Soergel bimodules and tensoring them on the right with Q (over R), see [EMTW, Section 5.4]. By [EMTW, Lemma 5.20], this is a monoidal functor. For any $u \in \widehat{\mathfrak{S}}_2$, let Q_u be the *standard* Q-Q-bimodule, which is the one-dimensional Q-vector space with left and right Q-actions defined by

$$x \cdot q \cdot y := xqu(y),$$

for $q, x, y \in Q$. For any $u, v \in \widehat{\mathfrak{S}}_2$, we have $Q_u Q_v \cong Q_{uv}$, see e.g. [EMTW, (5.5)], and the morphism spaces between the standard Q-Q-bimodules are very easy to describe (as follows from e.g. [EMTW, Lemma 5.2]):

$$\mathrm{hom}_{Q\text{-}Q-\mathrm{bimod}}(Q_u, Q_v) \cong \begin{cases} Q, & \text{if } u = v; \\ 0, & \text{else.} \end{cases}$$

As indicated in the commutative diagram [EMTW, (5.25)] and proved in [EMTW, Theorem 18.22], for every $w \in \widehat{\mathfrak{S}}_2$, we have

$$B_w \otimes_R Q \cong \bigoplus_{u \preceq w} Q_u,$$

because in affine type A_1 we have $p_{u,w}(1) = 1$ for all $u \preceq w$ and zero else (where $p_{u,w}$ is the Kazhdan-Lusztig polynomial for $u, w \in \widehat{\mathfrak{S}}_2$). Concretely, this means that

$$\mathrm{Loc}\,(B_{\widehat{n}_i}) \cong Q_{\widehat{n}_i} \oplus \bigoplus_{0<l<n} \left(Q_{\widehat{l}_0} \oplus Q_{\widehat{l}_1}\right) \oplus Q,$$

for any $n \in \mathbb{Z}_{\geq 0}$ and $i \in \{0, 1\}$ (note that $Q_{\widehat{0}_i} = Q$). Localization can also be applied to Rouquier complexes and [EMTW, Lemma 5.21] implies that, for every $w \in \widehat{\mathfrak{S}}_2$, we have

$$\mathrm{Loc}(\mathrm{T}_w) \simeq Q_w \quad \text{and} \quad \mathrm{Loc}(\mathrm{T}_w^{-1}) \simeq Q_{w^{-1}}.$$

The results in the previous paragraph imply that $\mathrm{Loc}(\mathrm{B}_{\widehat{k}_0})$ must be a direct summand of $\mathrm{Loc}\,(\mathrm{B}_{\widehat{m}_1}\mathrm{Y}^{-s,s})$, which imposes conditions on k, m and s. On the one hand, we get

$$\mathrm{Loc}(\mathrm{B}_{\widehat{k}_0}) \cong Q_{\widehat{k}_0} \oplus \bigoplus_{0<l<k} \left(Q_{\widehat{l}_0} \oplus Q_{\widehat{l}_1}\right) \oplus Q$$

and, on the other hand, we get

$$\begin{gathered} \mathrm{Loc}\left(\mathrm{B}_{\widehat{m}_1}\mathrm{Y}^{-s,s}\right) \cong \\ \left(Q_{\widehat{m}_1} \oplus \bigoplus_{0<n<m} (Q_{\widehat{n}_0} \oplus Q_{\widehat{n}_1}) \oplus Q\right) Q_{{}_1\widehat{2s}_0} \cong \\ Q_{\widehat{|m-2s|}_0} \oplus \bigoplus_{0<n<m} \left(Q_{\widehat{n+2s}_0} \oplus Q_{\widehat{|n-2s|}_0}\right) \oplus Q_{{}_1\widehat{2s}_0}. \end{gathered}$$

By the Krull-Schmidt property of $\mathrm{add}(\bigoplus_{w \in \widehat{\mathfrak{S}}_2} Q_w)$, where the additive closure is taken in the category of all Q-Q-bimodules, we see that $\mathrm{Loc}(\mathrm{B}_{\widehat{k}_0})$ can only be a direct summand of $\mathrm{Loc}\,(\mathrm{B}_{\widehat{m}_1}\mathrm{Y}^{-s,s})$ if $k = 1$ and $m \geq 2s$.

Thus we have an embedding $f^{-s,s} \colon \mathrm{B}_0\langle 1 \rangle \to \mathrm{B}_{\widehat{m}_1}\mathrm{Y}^{-s,s}$ in $K^b(\widehat{\mathbb{S}}_2^{\mathrm{ext}})$, with $m \geq 2s$. By adjunction, this implies that there is a non-zero morphism $\mathrm{B}_{{}_1\widehat{m}}\mathrm{B}_0\langle 1 \rangle \to \mathrm{Y}^{-s,s}$. Since $[\mathrm{Y}^{-s,s}]^{\min}$

is given by $\mathrm{B}_{{}_1\widehat{2s}_0}$ in homological degree zero, such a non-zero morphism can only exist if $m = 2s$, thanks to Theorem 3.3 and equations (49) and (123). In particular, we have

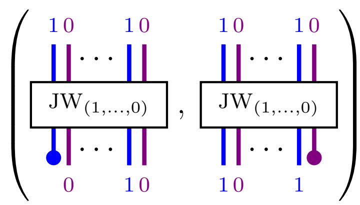
$$\mathrm{B}_{{}_0\widehat{2s}_1}\mathrm{B}_{{}_1\widehat{2s}_0} \cong \left(\mathrm{B}_{\widehat{4s-1}_0} \oplus \cdots \oplus \mathrm{B}_0\right)^{q+q^{-1}},$$

so the embedding $f^{-s,s}\colon \mathrm{B}_0\langle 1\rangle \to \mathrm{B}_{{}_0\widehat{2s}_1}\mathrm{Y}^{-s,s}$ is induced by the identity on $\mathrm{B}_0\langle 1\rangle$. However, we claim that that embedding does not split. Recall that $[\mathrm{Y}^{-s,s}]^{\min} \cong [(\mathrm{T}_1^{-1}\mathrm{T}_0^{-1})^s]^{\min}$ is given by

$$\cdots \longrightarrow \mathrm{B}_{{}_0\widehat{2s-1}_0}\langle -1\rangle \oplus \mathrm{B}_{{}_1\widehat{2s-1}_1}\langle -1\rangle \xrightarrow{\mathrm{d}^{-1}} \underline{\mathrm{B}_{{}_1\widehat{2s}_0}}$$

where

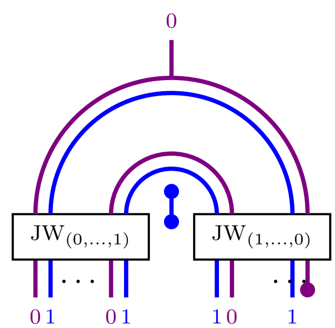
$$\mathrm{d}^{-1} = (\mathrm{d}_{\mathrm{L}}^{-1}, \mathrm{d}_{\mathrm{R}}^{-1}) =$$



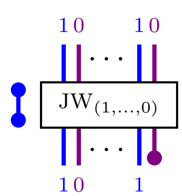
Tensoring this complex on the left with $\mathrm{B}_{{}_0\widehat{2s}_1}$ (over R) yields a complex containing the morphism

$$\mathrm{B}_{{}_0\widehat{2s}_1}\mathrm{B}_{{}_1\widehat{2s-1}_1}\langle -1\rangle \xrightarrow{\mathrm{id}_{\mathrm{B}_{{}_0\widehat{2s}_1}}\circ\, \mathrm{d}_{\mathrm{R}}^{-1}} \underline{\mathrm{B}_{{}_0\widehat{2s}_1}\mathrm{B}_{{}_1\widehat{2s}_0}}$$

When we compose this morphism with the projection of $\mathrm{B}_{{}_0\widehat{2s}_1}\mathrm{B}_{{}_1\widehat{2s}_0}$ onto $\mathrm{B}_0\langle 1\rangle$, we get a non-zero multiple of



It is easy to see that the latter diagram is non-zero. For example, attaching a dot to the top 0-strand and cups to the $2s$ bottom left strands, and using adjunction to straighten the resulting diagram, yields



which is non-zero, as we know. This implies that the projection of $\mathrm{B}_{{}_0\widehat{2s}_1}\mathrm{B}_{{}_1\widehat{2s}_0}$ onto $\mathrm{B}_0\langle 1\rangle$, which is unique up to a non-zero scalar, does not induce a morphism of complexes $\mathrm{B}_{{}_0\widehat{2s}_1}\mathrm{Y}^{-s,s} \to \mathrm{B}_0\langle 1\rangle$, so the embedding $f^{-s,s}$ does not split.

This shows that $\mathrm{B}_{\widehat{k}_0}\mathrm{Y}\langle t\rangle$ and $\mathrm{B}_{\widehat{m}_1}\mathrm{Y}$ are not isomorphic for any $k, m \in \mathbb{Z}_{\geq 0}$ such that $k > 0$ and any $t \in \mathbb{Z}$, which was the last case we had to prove. ☐

Let $[\mathbf{U}]_{\oplus}$ be the split Grothendieck group of $\mathbf{U}$. Since $[\widehat{\mathbb{S}}_2^{\mathrm{ext}}]_{\oplus} \cong \widehat{H}_2^{\mathrm{ext}}$, the free $\mathbb{Z}[q, q^{-1}]$-module $[\mathbf{U}]_{\oplus}$ carries the structure of an $\widehat{H}_2^{\mathrm{ext}}$-module.

Corollary 5.23. *The $\mathbb{Z}[q, q^{-1}]$-linear map $\gamma_U \colon U \to [\mathbf{U}]_{\oplus}$, defined by*

$$\gamma_U(u_k) := \left[\mathrm{B}_{\widehat{k}_1}\mathrm{Y}\right] \quad \textit{and} \quad \gamma_U(u'_k) := \left[\mathrm{B}_\rho \mathrm{B}_{\widehat{k}_1}\mathrm{Y}\right], \quad k \in \mathbb{Z}_{\geq 0},$$

is an isomorphism of $\widehat{H}_2^{\mathrm{ext}}$-modules.

Proof. This corollary is an immediate consequence of Proposition 5.4 and Theorem 5.22. ☐

5.3.2. *A triangulated $\widehat{\mathbb{S}}_2^{\mathrm{ext}}$ birepresentation $\mathbf{W}$.* Unfortunately, there is no obvious triangulated structure on the category $\mathrm{mod}_{K^b(\widehat{\mathbb{S}}_2^{\mathrm{ext},\diamond})}(\mathrm{Y})$, e.g., there is no obvious way to define the cone of a morphism between two objects in that category. This is a well-known problem for categories of modules over algebra objects in triangulated monoidal categories, see e.g. [Bal]. We will therefore use a different strategy to obtain a triangulated birepresentation $\mathbf{W}$.

This strategy is motivated by the observation that horizontal composition on the right with $\mathrm{Y}^{r,s}$, for $(r, s) \in \mathbb{Z}^2$, defines a $\mathbb{Z}^2$-action on $K^b(\widehat{\mathbb{S}}_2^{\mathrm{ext}})$. To be precise, this is a strong but not a strict action, because $\mathrm{Y}^{r,s}\mathrm{Y}^{r',s'}$ is isomorphic to $\mathrm{Y}^{r+r',s+s'}$ but not equal, in general. The non-trivial isomorphism is given by ζ, defined in Section 4.2.2, and the $\mathbb{Z}^2$-action is strong precisely because ζ is natural in both entries and satisfies the hexagon identities.

Using this action, we can define the *orbit category* $\widehat{\mathbf{\Omega}} := K^b(\widehat{\mathbb{S}}_2^{\mathrm{ext}})/\mathrm{Y}^{\mathbb{Z},\mathbb{Z}}$, which has the same objects as $K^b(\widehat{\mathbb{S}}_2^{\mathrm{ext}})$ and its morphism spaces are defined by

$$\mathrm{hom}_{\widehat{\mathbf{\Omega}}}(\mathrm{A}, \mathrm{B}) := \bigoplus_{(r,s)\in\mathbb{Z}^2} \mathrm{hom}_{K^b(\widehat{\mathbb{S}}_2^{\mathrm{ext}})}(\mathrm{A}, \mathrm{B}\mathrm{Y}^{r,s}),$$

for $\mathrm{A}, \mathrm{B} \in K^b(\widehat{\mathbb{S}}_2^{\mathrm{ext}})$. Composition is defined as in (136). By construction, the left regular $K^b(\widehat{\mathbb{S}}_2^{\mathrm{ext}})$ action on $K^b(\widehat{\mathbb{S}}_2^{\mathrm{ext}})$ induces a left $K^b(\widehat{\mathbb{S}}_2^{\mathrm{ext}})$-action on $\widehat{\mathbf{\Omega}}$. The natural functor $\Pi_{\widehat{\mathbf{\Omega}}} \colon K^b(\widehat{\mathbb{S}}_2^{\mathrm{ext}}) \to \widehat{\mathbf{\Omega}}$, which is the identity on objects and sends any morphism $f \colon \mathrm{A} \to \mathrm{B}$ in $K^b(\widehat{\mathbb{S}}_2^{\mathrm{ext}})$ to $f \colon \mathrm{A} \to \mathrm{B}\mathrm{Y}^{0,0}$ in $\widehat{\mathbf{\Omega}}$, is a morphism of wide finitary $\widehat{\mathbb{S}}_2^{\mathrm{ext}}$-bireprentations. In the following proposition, we do not consider any triangulated structures yet.

Proposition 5.24. *There is an equivalence of wide finitary $\widehat{\mathbb{S}}_2^{\mathrm{ext}}$ birepresentations*

$$\mathrm{add}\left\{\mathrm{AY} \colon \mathrm{A} \in K^b(\widehat{\mathbb{S}}_2^{\mathrm{ext}})\right\} \cong \widehat{\mathbf{\Omega}},$$

where the additive closure on the left-hand side is taken in $K^b(\widehat{\mathbb{S}}_2^{\mathrm{ext},\diamond})$.

Proof. This is exactly the content of the free forgetful adjunction in (135) and (136). ☐

Corollary 5.25. *The equivalence in Proposition 5.24 restricts to an equivalence of wide finitary $\widehat{\mathcal{S}}_2^{\mathrm{ext}}$ birepresentations*

$$\mathbf{U} \cong \widehat{\mathbf{\Omega}}',$$

where $\widehat{\mathbf{\Omega}}'$ is the full $\widehat{\mathcal{S}}_2^{\mathrm{ext}}$ subbirepresentation of $\widehat{\mathbf{\Omega}}$ whose objects are those of $\widehat{\mathcal{S}}_2^{\mathrm{ext}}$, which can be identified with the complexes in $K^b(\widehat{\mathcal{S}}_2^{\mathrm{ext}})$ concentrated in homological degree zero via the usual embedding.

In order to construct a triangulated $\widehat{\mathcal{S}}_2^{\mathrm{ext}}$ birepresentation, we proceed in two steps. We consider the $\mathbb{Z}^2$-action on $K^b(\widehat{\mathcal{S}}_2^{\mathrm{ext}})$ as generated by $\mathrm{Y}^{1,1} \cong \mathrm{B}_\rho^2$ and $\mathrm{Y}^{0,1} \cong \mathrm{T}_1^{-1}\mathrm{B}_\rho$. Instead of considering all orbits, we first use results of [Eli2, Section 3.4] to construct a new wide finitary category $\overline{\mathcal{S}}_2^{\mathrm{ext}}$ which already contains a canonical isomorphism $\mathrm{B}_\rho^2 \cong R$, and then apply [FKQ, Theorem/Definition 1.1] to construct a triangulated orbit category from $K^b(\overline{\mathcal{S}}_2^{\mathrm{ext}})$ under the action of the second generator.

Recall from (127) and Proposition 5.5 that W is isomorphic to the quotient of $\widehat{H}_2^{\mathrm{ext}}$ by the left ideal generated by

$$\rho^2 - 1 \quad \text{and} \quad \rho - T_1.$$

The first generator is central, so the ideal it generates is two-sided and

$$\overline{H}_2^{\mathrm{ext}} := \widehat{H}_2^{\mathrm{ext}} / \langle \rho^2 - 1 \rangle \tag{149}$$

is a quotient algebra. Of course, we have

$$W \cong \overline{H}_2^{\mathrm{ext}} / \overline{H}_2^{\mathrm{ext}}(\rho - T_1), \tag{150}$$

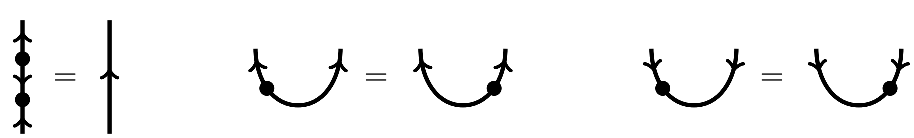
where $\overline{H}_2^{\mathrm{ext}}(\rho - T_1)$ is the left $\overline{H}_2^{\mathrm{ext}}$ ideal generated by $\rho - T_1$. Elias [Eli2, Section 3.4] showed that $\overline{H}_2^{\mathrm{ext}}$ can be categorified by equipping $\widehat{\mathcal{BS}}_2^{\mathrm{ext}}$ with an additional morphism and its "inverse", imposing some additional relations, and defining the new wide finitary monoidal category $\overline{\mathcal{S}}_2^{\mathrm{ext}}$ as the Karoubi envelope of the additive envelope of this "enhanced" $\widehat{\mathcal{BS}}_2^{\mathrm{ext}}$. Concretely, we add the generators

(151)

Degree $\quad 0 \quad\quad 0$

and the relations

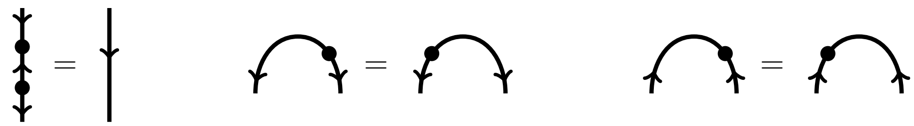
(152)



(153)

It is easy to see that our relations in (152) and (153) are equivalent to [Eli2, (3.22a-c), (3.23)], and that they imply the relations below.

Lemma 5.26. *In $\overline{\mathbb{S}}_2^{\mathrm{ext}}$, we have*

(154) 

By [Eli2, Lemma 3.25 and Theorem 3.28], there is an isomorphism $\mathrm{B}_\rho^2 \cong R$ in $\overline{\mathbb{S}}_2^{\mathrm{ext}}$, the split Grothendieck group of $\overline{\mathbb{S}}_2^{\mathrm{ext}}$ is isomorphic to $\overline{H}_2^{\mathrm{ext}}$ and the natural functor $E\colon \widehat{\mathbb{S}}_2^{\mathrm{ext}} \to \overline{\mathbb{S}}_2^{\mathrm{ext}}$ induces the projection $\widehat{H}_2^{\mathrm{ext}} \cong [\widehat{\mathbb{S}}_2^{\mathrm{ext}}]_\oplus \to \overline{H}_2^{\mathrm{ext}} \cong [\overline{\mathbb{S}}_2^{\mathrm{ext}}]_\oplus$.

Since $\mathrm{Y}^{1,1} = \mathrm{B}_\rho^2$, the above implies that $\mathrm{Y}^{r,s} \cong \mathrm{Y}^{0,s-r}$ in $K^b(\overline{\mathbb{S}}_2^{\mathrm{ext}})$, for any $(r,s) \in \mathbb{Z}^2$. Hence, it suffices to consider a $\mathbb{Z}$-action on $K^b(\overline{\mathbb{S}}_2^{\mathrm{ext}})$, with $s \in \mathbb{Z}$ acting by horizontal composition on the right with $\mathrm{Y}^{0,s} = (\mathrm{T}_1^{-1}\mathrm{B}_\rho)^s$. Let $\overline{\mathbf{\Omega}} := K^b(\overline{\mathbb{S}}_2^{\mathrm{ext}})/\mathrm{Y}^{0,\mathbb{Z}}$ be the corresponding orbit category and let $\overline{\mathbf{\Omega}'}$ be its full $\widehat{\mathbb{S}}_2^{\mathrm{ext}}$ subbirepresentation whose objects are those of $\overline{\mathbb{S}}_2^{\mathrm{ext}}$, which can be identified with the objects in $K^b(\overline{\mathbb{S}}_2^{\mathrm{ext}})$ concentrated in homological degree zero.

Lemma 5.27. *The natural $\mathbb{R}$-linear monoidal functor $E\colon K^b(\widehat{\mathbb{S}}_2^{\mathrm{ext}}) \to K^b(\overline{\mathbb{S}}_2^{\mathrm{ext}})$ induces a morphism of $\widehat{\mathbb{S}}_2^{\mathrm{ext}}$-birepresentations $E_\Omega\colon \widehat{\mathbf{\Omega}} \to \overline{\mathbf{\Omega}}$, which restricts and corestricts to a morphism of $\widehat{\mathbb{S}}_2^{\mathrm{ext}}$-birepresentations $E_{\Omega'}\colon \widehat{\mathbf{\Omega}'} \to \overline{\mathbf{\Omega}'}$.*

Proof. On objects, the functor E_Ω is defined by the identity map, of course. Given a morphism $f\colon \mathrm{A} \to \mathrm{B}$ in $\widehat{\mathbf{\Omega}}$ with only one non-zero component $f^{r,s}\colon \mathrm{A} \to \mathrm{B}\mathrm{Y}^{r,s}$ in $K^b(\widehat{\mathbb{S}}_2^{\mathrm{ext}})$, for some $r,s \in \mathbb{Z}$, define $E_\Omega(f)$ as the morphism in $\overline{\mathbf{\Omega}}$ whose only one non-zero component $E_\Omega(f)^{0,s-r}$ is given by postcomposing $E(f)$ with the isomorphism $\mathrm{Y}^{r,s} \cong \mathrm{Y}^{0,s-r}$ in $K^b(\overline{\mathbb{S}}_2^{\mathrm{ext}})$. This definition can be extended to all morphisms in $\hom_{\widehat{\mathbf{\Omega}}}(\mathrm{A},\mathrm{B})$ by additivity.

The definition implies immediately that E_Ω is $\mathbb{R}$-linear and sends identity morphisms to identity morphisms, and the hexagon identities and binaturality of ζ guarantee that it also preserves composition. Finally, it is a morphism of left $\widehat{\mathbb{S}}_2^{\mathrm{ext}}$ birepresentations because the isomorphisms $\mathrm{Y}^{r,s} \cong \mathrm{Y}^{0,s-r}$ in the definition of E_Ω are applied to the rightmost factor of the objects.

The last claim of the lemma follows immediately. □

The following result is an immediate consequence of Corollary 5.25 and Lemma 5.27.

Corollary 5.28. *The functor $E_{\Omega'}$ induces a morphism of $\widehat{\mathbb{S}}_2^{\mathrm{ext}}$ birepresentations $E_{\Omega'}\colon \mathbf{U} \to \overline{\mathbf{\Omega}'}$ (for which we use the same notation).*

We do not know whether $E_{\Omega'}$ is an equivalence. It is certainly essentially surjective, because it is the identity on objects, but it is unclear whether it is fully faithful.

The above fits exactly the conditions of [FKQ, Theorem/Definition 1.1]: the triangulated category $\mathcal{T} := K^b(\overline{\mathbb{S}}_2^{\mathrm{ext}})$ is naturally endowed with the dg-enhancement $\mathcal{A} := C^b_{dg}(\overline{\mathbb{S}}_2^{\mathrm{ext}})$ (the dg-category of bounded complexes in $\overline{\mathbb{S}}_2^{\mathrm{ext}}$), and the complex $F := \mathrm{Y}^{0,1} = \mathrm{T}_1\mathrm{B}_\rho$ naturally yields a dg-bimodule such that the induced endofunctor $H^0(F)$ on $H^0(\mathcal{A}) \cong K^b(\widehat{\mathbb{S}}_2^{\mathrm{ext}})$ is an equivalence (with inverse induced by $\mathrm{Y}^{0,-1}$). This means that there is a canonical triangulated orbit category $\mathbf{W}$ of $K^b(\overline{\mathbb{S}}_2^{\mathrm{ext}})$ under the action of $\mathrm{Y}^{0,1}$, which in the notation of [FKQ, Theorem/Definition 1.1] is defined by

$$\mathbf{W} := H^0\left(\mathrm{pretr}\left(C^b_{dg}(\overline{\mathbb{S}}_2^{\mathrm{ext}})/\mathrm{Y}^{0,\mathbb{Z}}\right)\right).$$

The left $\widehat{\mathbb{S}}_2^{\mathrm{ext}}$-action clearly commutes with the right action by $\mathrm{Y}^{0,1}$, hence $\mathbf{W}$ is a triangulated left $\widehat{\mathbb{S}}_2^{\mathrm{ext}}$-birepresentation.

By [FKQ, Theorem 3.17], there is a natural dg-functor

$$Q\colon C^b_{dg}\left(\overline{\mathbb{S}}_2^{\mathrm{ext}}\right) \to \mathrm{pretr}\left(C^b_{dg}\left(\overline{\mathbb{S}}_2^{\mathrm{ext}}\right)/\mathrm{Y}^{0,\mathbb{Z}}\right), \tag{155}$$

which induces a triangle functor $Q^\triangle := H^0(Q)$ between the homotopy categories

$$Q^\triangle\colon K^b\left(\overline{\mathbb{S}}_2^{\mathrm{ext}}\right) \to \mathbf{W}. \tag{156}$$

As $Q^\triangle$ is a morphism of triangulated $\widehat{\mathbb{S}}_2^{\mathrm{ext}}$-birepresentations, it induces a morphism of $[\widehat{\mathbb{S}}_2^{\mathrm{ext}}]_\oplus$-modules between the triangulated Grothendieck groups

$$\left[Q^\triangle\right]_\triangle\colon \left[K^b\left(\overline{\mathbb{S}}_2^{\mathrm{ext}}\right)\right]_\triangle \to [\mathbf{W}]_\triangle. \tag{157}$$

A well-known general result on Grothendieck groups, see e.g. [Ro], implies that $[K^b(\overline{\mathbb{S}}_2^{\mathrm{ext}})]_\triangle \cong [\overline{\mathbb{S}}^{\mathrm{ext}}]_\oplus \cong \overline{H}_2^{\mathrm{ext}}$. Thus (157) yields a morphism of $\widehat{H}_2^{\mathrm{ext}}$-modules

$$q\colon \overline{H}_2^{\mathrm{ext}} \to [\mathbf{W}]_\triangle.$$

We claim that this morphism in an epimorphism. Indeed, consider the dg functor Q in (155). The objects of $C^b_{dg}(\overline{\mathbb{S}}_2^{\mathrm{ext}})/Y^{0,\mathbb{Z}}$ are the same as those of $C^b_{dg}(\overline{\mathbb{S}}_2^{\mathrm{ext}})$. In taking the pretriangulated closure, we add cones, but those do not generate new elements in the Grothendieck group, proving the claim.

By construction, horizontal composition with $\mathrm{Y}^{0,1}$ is naturally isomorphic to the identity functor on $\mathbf{W}$, so q factors through W by (150), resulting in a commutative triangle

$$\begin{array}{ccc} \overline{H}_2^{\mathrm{ext}} & \longrightarrow & W \\ & \searrow^{q} & \downarrow{\scriptstyle \gamma_W} \\ & & [\mathbf{W}]_\triangle \end{array}$$

Surjectivity of q implies surjectivity of γ_W. Recall also that $W^{\mathbb{C}(q)}$ is simple, see Proposition 5.2. However, we do not know that $[\mathbf{W}]_{\triangle}^{\mathbb{C}(q)}$ is non-zero, which is why we have to settle for a conjecture.

Conjecture 5.29. *The morphism*

$$\gamma_W^{\mathbb{C}(q)} \colon W^{\mathbb{C}(q)} \to [\mathbf{W}]_{\triangle}^{\mathbb{C}(q)}$$

is an isomorphism.

Finally, recall the morphism of $\widehat{\mathbb{S}}_2^{\mathrm{ext}}$ birepresentations $E_{\Omega'} \colon \mathbf{U} \to \overline{\mathbf{\Omega}'}$ from Corollary 5.28. By [FKQ, Remark 3.12], there is an equivalence of wide finitary $\widehat{\mathbb{S}}_2^{\mathrm{ext}}$ birepresentations

$$\overline{\mathbf{\Omega}} \cong H^0(C_{dg}^b(\overline{\mathbb{S}}_2^{\mathrm{ext}})/Y^{0,\mathbb{Z}}).$$

Precomposition with $E_{\Omega'}$ and postcomposition with the canonical functor $H^0(C_{dg}^b(\overline{\mathbb{S}}_2^{\mathrm{ext}})/Y^{0,\mathbb{Z}}) \to H^0(\mathrm{pretr}(C_{dg}^b(\overline{\mathbb{S}}_2^{\mathrm{ext}})/Y^{0,\mathbb{Z}}))$ yields a morphism of $\widehat{\mathbb{S}}_2^{\mathrm{ext}}$ birepresentations

$$\Pi_{\mathbf{W}}^{\mathbf{U}} \colon \mathbf{U} \to \mathbf{W}. \tag{158}$$

We conjecture that $\Pi_{\mathbf{W}}^{\mathbf{U}}$ categorifies the projection π_W^U in Proposition 5.5, meaning that the following diagram commutes

$$\begin{array}{ccccc} U & & \xrightarrow{\quad\pi_W^U\quad} & & W \\ \Big\downarrow{\scriptstyle \gamma_U} & & & & \Big\downarrow{\scriptstyle \gamma_W} \\ [\mathbf{U}]_{\oplus} & \xrightarrow{[\Pi_{\mathbf{W}}^{\mathbf{U}}]} & [\mathbf{W}]_{\oplus} & \longrightarrow & [\mathbf{W}]_{\triangle} \end{array}$$

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