Sharpening the focus on mathematics: Designing, implementing and evaluating MathTASK activities for mathematics teacher education

Irene Biza[a], Lina Kayali[a], Bruna Moustapha-Corrêa[b], Elena Nardi[a] and Athina Thoma[a]

i.biza@uea.ac.uk, L.Kayali@uea.ac.uk, bruna.correa@uniriotec.br, e.nardi@uea.ac.uk, A.Thoma@uea.ac.uk

[a]University of East Anglia, Norwich, UK

[b]Universidade Federal do Estado do Rio de Janeiro, Rio de Janeiro, Brasil

Abstract

MathTASK is a research and development programme that engages mathematics teachers with challenging and highly contextualised classroom situations in the form of tasks (mathtasks). Teacher responses to these tasks reveal their mathematical and pedagogical discourses and provide opportunities to articulate, reflect and reform said discourses. These tasks have been used as instruments for research as well as teacher education and professional development in the UK, Greece and Brazil. In this chapter, we first introduce the MathTASK programme and a mathtask example. We then present a summary of theoretical constructs that have emerged in the course of analysis of MathTASK data. We then present the general principles in using mathtasks into research and teacher education and we exemplify these principles through four examples, each addressing different issues of mathematics teaching and learning, and each developed with different educational levels and contexts in mind. We conclude with observations on the benefits of using mathtasks as a means to trigger and facilitate mathematics teachers' reflection on their practice.

Introduction

Mathematics teachers have high aspirations when they enter the classroom. They want their students to understand, appreciate and enjoy mathematics. Often though, what they face in the classroom is nowhere near these aspirations: students' responses may not make sense, addressing individual needs is difficult, the class does not cooperate, technology is confusing and the resources not exactly what is needed[1]. MathTASK[2], a research and development programme that brings together researchers, mathematics teacher educators[3] (thereafter teacher educators) and teachers from the UK, Greece and Brazil, aims to help teachers deal with the challenging situations they often face in the classroom - and, ultimately, to help mathematics teachers transform their aspirations into effective classroom strategies. To this aim, we design situation-specific tasks for mathematics teachers and then invite teachers to engage with these tasks. We call these tasks *mathtasks*. Tasks are presented to teachers as short narratives that comprise a classroom situation where a teacher and students deal with a mathematical problem and a conundrum that may arise from the different responses to the problem put forward by different students. The mathematical problem, the student responses and

[1] See a brief animation that describes MathTASK at: https://youtu.be/gt0HZBfBBGI.

[2] We use MathTASK (https://www.uea.ac.uk/groups-and-centres/a-z/mathtask) when we refer to the overall programme and its principles, whereas we use mathtask to refer to specific tasks designed with the principles of the MathTASK.

[3] Mathematics teacher educators are those who engage with the education of pre- or in- service teachers.

the teacher reactions are all inspired by the vast array of issues that typically emerge in the complexity of the mathematics classroom and that prior research highlights as seminal. MathTASK so far has focused on four sets of these issues: different or potentially flawed approaches to the mathematical problem taken by different class members; classroom management issues triggered by the exchanges during the lesson and interfering with students' mathematical learning; creative, or not, tensions emerging from the use of digital resources in mathematical problem solving; and, inclusion in mathematical activity of typically under-included learners, such as learners with some disability. Teachers are invited to engage with these tasks through reflecting, responding in writing and discussing. At the heart of MathTASK is the claim that, through setting out from – and sharpening the focus on – particular elements of mathematics embedded in classroom situations that are likely to occur in actual practice, consistent, specific and research-informed mathematics pedagogies can emerge. Our chapter aims to offer evidence in support of this claim.

Specifically, in this chapter we first introduce mathtasks and present the general principles in designing and using mathtasks for research and *teacher education*[4] purposes. We illustrate these in one example of a mathtask. We then present a summary of theoretical constructs that have emerged in the course of analysis of MathTASK data. We continue with four examples, each from a study conducted by at least one of the authors. We conclude with a brief discussion of the benefits of using mathtasks into research and teacher education.

Studying the discourses of mathematics teachers

The focus of our work is the exploration of teachers' pedagogical and mathematical discourses in their preparation for teaching and in the reflection on their own teaching practices, especially in relation to their interaction with their educators (e.g. in undergraduate or postgraduate course for teachers) or with their colleagues (e.g. when they discuss their teaching during their daily routine or during an in-service professional development course). Teacher education courses expect teachers to transform the theoretical input of these courses into what they do in their everyday work in the classroom. This transformation has been described before by constructs such as Chevallard's (1985) *transposition didactique*, Lampert's teachers' *dilemmas and commitments* (e.g. 1985), Shulman's (1986, 1987) *pedagogical content knowledge*, Hill and Ball's (2004) *mathematical knowledge for teaching* and Rowland and colleagues' (Turner & Rowland 2011) *Knowledge Quartet*. Over the years, these concepts have evolved. For example, the attention of initial works was on the knowledge that teachers need to *possess* to become *effective* in their teaching (Shulman 1986, 1987). Shulman's typology is seminal, and it has been the starting point for several studies, which pay attention, for example, to actual events in the mathematics classroom (e.g. mathematical knowledge for teaching of Hill and Ball. 2004).

Recently, partly in the spirit of rapidly emerging discursive approaches in mathematics education research (e.g. Kieran, Forman & Sfard, 2002), attention has shifted towards *mathematical discourse for teaching* (Cooper, 2014). This shift is in recognition of the different discourses involved in teaching practice, pedagogical and mathematical, and pays attention to how these discourses present in different professions involved in these practices, e.g. teachers, policy makers, teacher educators and mathematicians who educate teachers. Our work is embedded in these developments: we endorse this recent perspective about teaching practice as engagement with certain professional or academic

[4]With teacher education, we mean any course that aims towards teachers' learning. This can be either an *initial teacher education* course for undergraduate or postgraduate students who aspire to become teachers (pre-service teachers) or a *professional development* course for those who already have a teaching profession (in-service teachers) and they would like to enhance their knowledge and professional practice. In this chapter, we use mostly *teacher education* and we specify if this is *initial teacher education* or *professional development* only if we want to refer to a specific course for pre- or in- service teachers, respectively.

discourses. And, we explore how we can access, and help develop, teacher discourses for research, teacher education and professional development purposes.

Additionally, research has reported the overt discrepancy between theoretically and out of context expressed teacher views about mathematics and pedagogy and actual practice (e.g. Speer, 2005; Thompson, 1992). Speer (2005) claims, for example, that, instead of discussing about teaching practices in the abstract, a discussion of these in a concrete context can provide shared understanding between researchers and participating teachers of the beliefs that are attributed by researchers to teachers. With this observation in mind, in our work we start from specific classroom situations that can provide a trigger for exchanges and build shared insights between researchers and teachers. Specifically, we invite pre- and in- service teachers to reflect on fictional but realistic and research grounded classroom situations (mathtasks) that include a mathematical problem and a reaction by one or more students (and a teacher) to this problem (Biza Nardi, 2019; Biza, Nardi & Joel, 2015; Biza, Nardi & Zachariades 2007, 2009, 2014, 2018; Nardi, Biza & Zachariades 2012). We discuss the MathTASK design principles in the next section.

Design, implementation and evaluation principles in MathTASK

Using tasks for research about mathematics teaching and teachers and for teacher education

In the literature, the word task is used in different ways (Leont'ev, 1975; Christiansen & Walter, 1986; Mason & Johnston-Wilder 2006) and often conveys that tasks are mediating tools for teaching and learning mathematics. In the case of teacher education, a task can be used to trigger teachers' reflection and to explore their mathematical knowledge for teaching as well as their pedagogical and epistemological perceptions and beliefs. An appropriately designed task, which addresses complex purposes, affords opportunity to engage with aspects of mathematics, didactical strategies, pedagogical theory and epistemological beliefs. We see all these aspects crucial in teachers' diagnostic proficiency when they deal with unexpected situations in the classroom that demand immediate reaction.

In the field of mathematics teacher education, significant attention has been paid to the nature, role and use of tasks. For example, parts of the Handbook of Mathematics Teacher Education (Tirosh & Wood 2009) have focused on works that integrate tasks into teacher education. Also, a special issue of Journal of Mathematics Teacher Education (2007) edited by Zaslavsky, Watson and Mason, as well as the book edited by Zaslavsky and Sullivan (2011), signal this interest.

Additionally, a substantial body of research explores the use of cases, that is, "any description of an episode or incident that can be connected to the knowledge base for teaching" (Carter, 1999, p. 174), in mathematics teacher education and research (see, for example, a review in Markovits and Smith (2008)). Shulman (1992) envisioned the case method

> … as a strategy for overcoming many of the most serious deficiencies in the education of teachers. Because they are contextual, local, and situated – as are all narratives – cases integrate what otherwise remains separated. (p.28)

Over the years, this key idea has gained substantial momentum in mathematics teacher education, whether in the shape of brief classroom situations used as prompts (e.g. Erens & Eichler, 2013; Dreher, Nowinska & Kuntze, 2013) or, in the shape of more extended 'imagined' classroom dialogues, such as Zazkis, Sinclair and Liljedahl's (2013) 'lesson plays'. As Zazkis et al. (2013) write "[w]ith this imagination, attention and awareness are developed in "slow motion", having a complete control of the situation and ability to replay or redress it, rather than "thinking on one's feet" and making in the moment decisions" (p. 29). The task design we put forward in our study resonates well with these works by identifying classroom fictional but realistic critical incidents and transforming them into tasks for teachers. Before describing the mathtask design principles, we discuss first about critical incidents and their role in research and teacher education.

Critical incidents

Critical incidents have been used extensively in teacher education programmes in the form of brief reflective accounts, written by teachers, on classroom situations they have observed or experienced as a part of their training (e.g. Goodell, 2006; Potari & Psycharis, 2018). According to Skott (2001), "critical incidents of practice" are instances of when a teacher makes classroom decisions taking into account several motives some of which can be conflicting, vital to the teacher's school mathematics priorities, and crucial for the development of classroom interactions and students' learning. Tripp (2012) describes a critical incident as an ordinary event or routine that tells the trends, purposes, and routines of a teacher's practice; it becomes critical when someone chooses to see it as such. In his view this can be "problematic" as it is dependent on one's interpretation (p.28). Goodell (2006) argues that "a critical incident can be thought of as an everyday event encountered by a teacher in his or her practice that makes the teacher question the decisions that were made, and provides an entry to improving teaching" (p.224). It is believed that reflections on critical incidents can play an important role for teachers' learning (Goodell, 2006; Hole & McEntee, 1999; Potari & Psycharis, 2018; Skott, 2001; Tripp, 2012). Skott (2001) argues that critical incidents of practice (CIPs) are useful in two aspects:

> First, they provide a window on the role of teachers' school mathematical priorities when these are challenged as informants of teaching practice by the emergence of multiple motives of their activities. Second, CIPs may prove significant for the long-term development of a teacher's school mathematical priorities. (p. 19)

Thus, identifying critical incidents and having teachers reflecting on them "may turn the classroom into a learning environment for teachers as well as for students" (Skott, 2001, p. 4), and consequently for researchers. Deep reflection on critical incidents inspires teachers to think of what happened, why it happened, what it could mean and what its implications are (Hole & McEntee, 1999). Additionally, Goodell (2006) claims that asking teachers to identify critical incidents and produce reflective accounts followed up by group discussions addresses the concern that previous research on teacher education has expressed on the lack of structure on teacher reflection and their challenges with looking objectively at school-based experiences and benefit from them (e.g. Pultorak, 1993). In our work, we expand this claim further: we argue that by familiarising teachers with pre-prepared critical incidents (mathtasks) has the potential to introduce them to a practice of identification and communication of what might be critical for them and to structure their reflections on it. We return to what types of incident might be considered in our work as critical in the next section which presented the design principles of MathTASK.

Design principles

In MathTASK, a critical incident is a classroom event or an instance of when teachers have to take a decision about how they would react. The choice of the incident is grounded on issues that research and experience have identified as seminal; it is focused enough to promote teachers' structured reflections; and, it is broad enough to open a meta-discussion on more general issues related to the teaching of mathematics. For example, at the heart of the teaching situations in our tasks are pivotal moments in the growth of learners' mathematical thinking. These moments are akin to what Leatham, Peterson, Stockero and Van Zoest (2015) call Mathematically Significant Pedagogical Opportunities to build on Student Thinking (MOSTs), which are "instances of student thinking that have considerable potential at a given moment to become the object of rich discussion about important mathematical ideas" (p. 90). Specifically, we see identifying and facilitating the ways in which teachers recognise MOSTs and optimise these opportunities as they diagnose the issues in a classroom situation and address these issues in their practice (mathematical and pedagogical) as a core aim of our work. In this sense, the situations in mathtasks satisfy the three characteristics of MOST: "student mathematical thinking, mathematically significant, and pedagogical opportunity" (p. 91).

We propose the use of mathtasks in teacher education to explore, assess and develop teachers' Mathematical Discourse for Teaching (Cooper, 2014). Additionally, with these tasks, we aim to address the complex set of considerations that teachers take into account when they determine their actions. To this aim, we draw on what Herbst and colleagues (e.g. Herbst and Chazan 2003) describe as the practical rationality of teaching (PRT). We delve into these considerations and findings from our previous research on the spectrum of warrants (SW) secondary mathematics teachers put forward in order to justify the decisions they intend to make in their classroom: empirical–personal, empirical–professional, institutional–curricular, institutional–epistemological, a priori–epistemological, a priori–pedagogical and evaluative (Nardi, et al. 2012, see a more elaborate presentation of these characterisation in the next section).

Additionally, we are interested in teachers' competences in identifying mathematical and pedagogical issues and the mathematical and pedagogical discourse they endorse in such identification. To this aim, we draw on Cooper's (2014) Mathematical Discourse for Teaching (MDT). Additionally, we draw on what Rowland and colleagues (Turner & Rowland 2011) describe as Foundation – one of the four features of the Knowledge Quartet (KQ), with the other three being Connection, Transformation and Contingency – namely, amongst others, the 'overt subject knowledge, theoretical underpinning of pedagogy, use of terminology' (p. 200). Additionally, we see Ball and colleagues' (Ball, Thames & Phelps, 2008), Horizon Content Knowledge (HCK) – "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" and "the vision useful in seeing connections to much later mathematical ideas" (p. 403) – as a useful component of mathematical knowledge for teaching that brings together mathematical and curricular content.

In this respect, in designing these tasks we bear in mind the following principles:

- The mathematical content of the task concerns a topic or an issue that is known for its subtlety or for causing difficulty to students, this information is drawn from the literature and/or teaching experience (MOSTs: student mathematical thinking, mathematically significant).
- The student's response reflects this subtlety (or lack of) or difficulty and provides an opportunity for the teacher to reflect on and demonstrate the ways in which s/he would help the student achieve subtlety or overcome difficulty (MOSTs: pedagogical opportunity).
- The teacher's pedagogical approach concerns mathematical, pedagogical and epistemological issues that are known for their subtlety or for being challenging to teachers (PRT, SW)
- Mathematical content and student/teacher responses provide a context in which teachers' discourses are evidenced (MDT), also in relation to teachers' knowledge, beliefs and intended practices (mathematical, pedagogical and epistemological) that are allowed to surface (MKT, HCK, KQ).
- Mathematical content and students'/teachers' actions and interactions are contextualised to the curriculum and the educational context teachers are familiar (e.g. contextual information about the class and students level allows the teachers to situate themselves as teachers of that class)

Learning objectives in using mathtasks

Use of mathtasks with pre- or in-service teachers has the following learning objectives:

A. General

- Identifying student mathematical errors
- Noticing and valuing student contributions in a lesson
- Preparing and reflecting on a reaction in a teaching situation
- Evaluating pedagogical approach followed by another teacher (when a reaction from a teacher is offered)
- Evaluating and juxtaposing solutions offered by students (when more than one solution are included in the incident)
- Appreciating the value and drawbacks of different solutions
- Appreciating the value or the drawbacks of technological tools

B. Specific to the content of the teaching situation under discussion

- Learning about specific mathematical topics and the teaching of these topics
- Appreciating different facets of a mathematical activity (e.g. reasoning, proving, visualising, etc.)
- The potentialities and challenges of using technology in the teaching of specific mathematical topics or activities.

Structure and Format of a mathtask

We started working on the development of these tasks in 2005 (initial format can be found in Biza et al. 2007). Each task is based on a teaching situation, which is fictional, yet derived from findings in prior research. Over the years, we have deployed various versions of the situation-specific task design. So far, the structure of a mathtask is:

- A classroom situation *context* is described (e.g. the level of the class, the setting of the class, etc.)
- A *mathematical problem* is given by the teacher to the students
- A *classroom situation* follows in the form of:
  - one student response;
  - more than one student responses;
  - student(s) response(s) and reaction from (or dialogue with) a teacher; or
  - student(s) response(s) and reaction from (or dialogue with) a teacher that is followed up with a dialogue between teachers.
- A list of *questions* that invite participants to engage with and reflect upon the situation, such as:
  - solve the mathematical problem;
  - reflect on the aims of using this mathematical problem in a class;
  - identify the issues in the classroom situation; or,
  - propose how you would react in a similar classroom situation if you were the teacher of the class.

The format of a mathtask always starts with a written introduction in which the context and the mathematical problem are given and closes with the list of questions. The format of the classroom situation varies and would be:

- written in a script, very often in the form of a dialogue, where the work of the students on the problem is provided or
- a video either from a real student-teacher interaction (cartoonised for anonymity purposes) or a screen capture of student work with few pauses, in significant moments, where the participant is invited to respond and discuss.

Using a pre-designed mathtask

Mathtasks can be used for research and teacher education purposes, with pre- or in- service teachers, who work individually and or in groups. In workshops organised by researchers or teacher educators, mathtasks are given to teachers who read, respond in writing and then discuss their responses in groups or in a plenary discussion. There is an opportunity for teachers to revisit and amend their initial responses to the task after the end of the discussion by using a different coloured pen. Differences in the responses before and after the discussion can indicate potential shifts in teachers' discourses about mathematics and pedagogy. Especially for research purposes, interviews with teachers (individually or in focused groups) on their responses to the task can give more insight on the views they expressed in the written responses. Recently, mathtasks are used in teacher education programmes as an introduction of teachers to the idea of what a critical incident is and as an intermediate step before starting to prepare their own critical incidents. Beyond events organised by researchers or teacher educators, teachers can use the mathtasks in their discussion with colleagues, in their regular departmental meetings or in their informal discussion between teaching. Additionally, mathtasks have been used in the formative and summative assessment of mathematics education courses.

When mathtasks are used for assessment purposes, tasks are chosen according to the learning objectives of the course and responses are assessed by following these objectives. For example, when mathtasks are used for the introduction of mathematics students to mathematics education, we aim to see how the students (e.g. teacher students) use the mathematical as well as the mathematical education content. In this case, we can assess the responses according to the four characteristics of *Consistency, Specificity, Reification of pedagogical discourse* and *Reification of mathematical discourse.* We elaborate these terms later in this chapter after exemplifying the structure and the design principles of MathTASK with one example in the next section.

Exemplification of the MathTASK design principles: The "Simplification Task"

The principles we discussed earlier are demonstrated in the "Simplification Task" (Biza et al., 2015). In Figure 1, the mathtask is with comments on the side that explain its design.

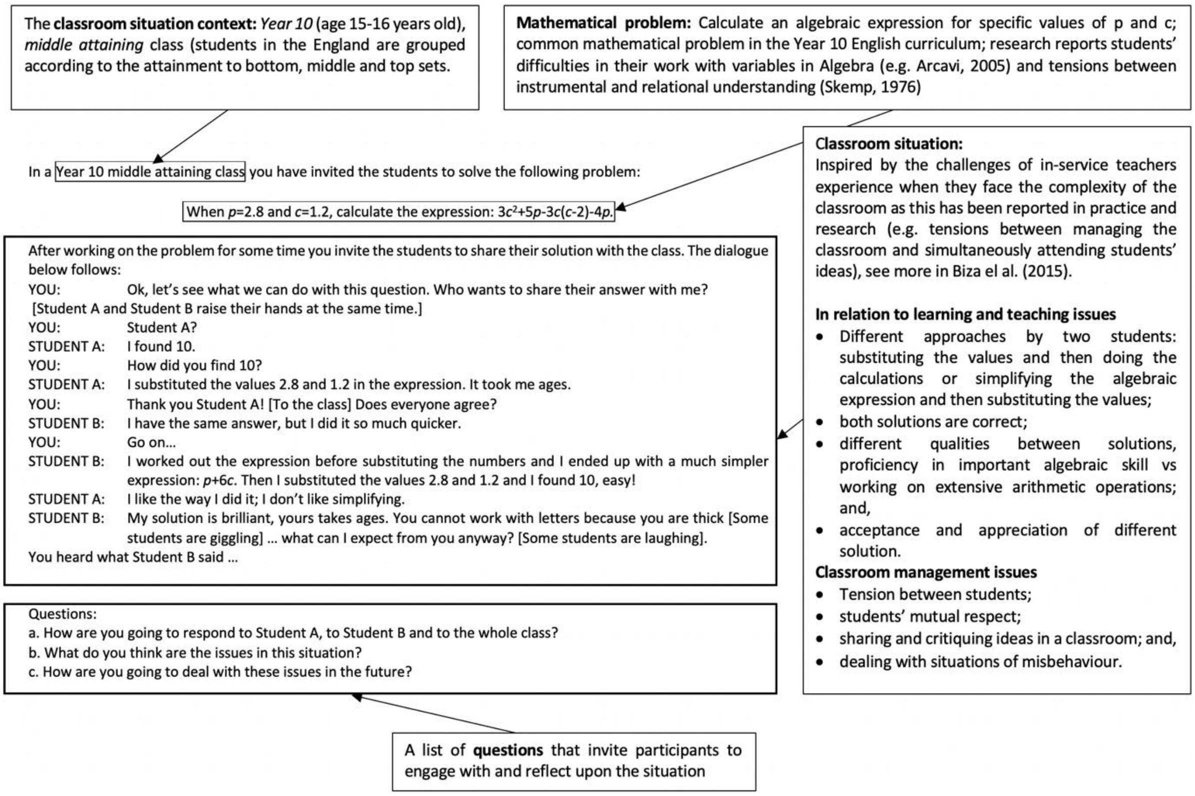


Figure 1: The Simplification Task (Biza et al., 2015, p. 188) annotated

We return to exemplifying four more mathtasks later in the chapter. First though, we present a summary of theoretical constructs that have emerged in the course of analysis of MathTASK data.

Theoretical constructs proposed by the use of mathtasks

Findings from the use of mathtasks in research have revealed the complex set of considerations mathematics teachers take into account when they make decisions or reflect upon their teaching. To give an example, when we asked a mathematics teacher if they would "accept a graph-based argument as proof", he replied:

> Mathematically, in the classroom, I would welcome it at lesson-level and I would analyse it and praise it, but not in a test". Asked to elaborate, he said: "Through [the graph-based argument] I would try to lead the discussion towards a normal proof...with the definition, the slope, the derivative, etc.". Asked to justify he said: "This is what we, mathematicians, have learnt so far. To ask for precision. ... we have this axiomatic principle in our minds. ... And this is what is required in the exams. And we are supposed to prepare the students for the exams. (Biza et al. 2009, p. 34)

The teacher above seems to approach visual argumentation from three different and interconnected perspectives: the restrictions of the current educational setting, in this case the university entrance examinations; the epistemological constraints with regard to what makes an argument a proof within the mathematical community; and, finally, the pedagogical role of visual argumentation as a means towards the construction of formal mathematical knowledge. These three perspectives reflect three roles that a mathematics teacher needs to balance: educator (responsible for facilitating students' mathematical learning), mathematician (accountable for introducing the normal practices of the mathematical community) and professional (responsible for preparing candidates for one of the most important examinations of their student career).

This observation led us to the analysis of the arguments put forward by secondary mathematics teachers in their written responses to a classroom situation described in one mathtask and the follow up interviews. Our analysis aimed to discern, differentiate and discuss the range of influences (epistemological, pedagogical, curricular, professional and personal) on the arguments teachers put forward in their scripts and interviews. We focused particularly on the warrants of these arguments, in the light of Toulmin's (1958) model of informal arguments and Freeman's (2005) classification of warrants, and we proposed the following classification:

- an a priori warrant is, for example, resorting to a mathematical theorem or definition (a priori–epistemological) or resorting to a pedagogical principle (a priori–pedagogical);
- an institutional warrant is, for example, a justification of a pedagogical choice on the grounds of it being recommended or required in a textbook (institutional–curricular) or on the grounds that it reflects the standard practices of the mathematics community (institutional–epistemological);
- an empirical warrant is, for example, the citation of a frequent occurrence in the classroom (according to the arguer's teaching experiences, empirical–professional) or resorting to personal learning experiences in mathematics (empirical–personal);
- an evaluative warrant is a justification of a pedagogical choice on the grounds of a personally held view, value or belief. (Nardi et al. 2012, pp. 160-161).

In a different study, we analysed teachers' responses to mathtasks in relation to their competencies in diagnosing issues in students' responses and to respond to these issues. The analysis suggested a typology of four interrelated characteristics of teachers' responses:

- *Consistency:* how consistent a response is in the way it conveys the link between the respondent's stated beliefs and their intended practice,
- *Specificity:* how contextualised and specific a response is to the teaching situation in the task,
- *Reification of pedagogical discourse:* how reified the pedagogical discourse of the response is in order to describe the pedagogical and didactical issues of the classroom situations and the intended practice presented in the script, and
- *Reification of mathematical discourse:* how reified the mathematical discourse of the response is in relation to the identification of the underpinning mathematical content of the classroom situations and the transformation of this mathematical content into the intended practice presented in the script. (Biza et al. 2018, p.64)

The use of the term *reification* above draws on discursive perspectives such as Sfard's (2008), where reification is defined as the gradual turning of processes into objects. Discourses, Sfard writes, change in a "chain of intermittent expansion and compression" (p. 118). Reification is the key element of compression which can be endogenous – resulting from saming within one particular discourse - and exogenous which "conflates several discourses into one" (p. 122). Reification is a response to what discursive researchers see as our innate "need for closure" (p. 184) in our use of signifiers and brings at least two potent gains: increasing the communicative effectiveness of discourse and increasing the practical effectiveness of discourse. For example, in an educational system that follows grouping of students according to their ability, a dominant approach in the UK, a *top set student* characterisation has reified a certain learning ability, expectations in performance, a set of appropriate tasks and, very often, certain behaviours in the classroom. We believe that a potent use of these characteristics can serve as an instrument for the analysis of teacher reflections on their own practice. We now illustrate how the design, implementation and evaluation principles of the MathTASK programme have materialised in four different examples.

Example 1. Problematizing the use of letters in algebra

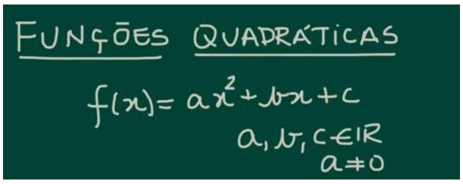
Algebra plays a significant role in the Brazilian school curriculum. However, students find the content difficult, especially because of the *letters* and the different roles these letters may have: unknown, variables, coefficients, parameters, abstract symbols, etc.. Teachers who introduce their students to algebra, very often deal with questions such as: How do students think about the letters in their mathematics class? What about if the same letter is used for different purposes? or: How can we make students aware of the different uses of letters?

The mathtask (Figure 2) we present in this example aims to address these questions by triggering problematization of how letters are used in algebra. It was designed and used by the third author in a professional master for school mathematics teachers in a Brazilian institution. In what follows, first we present the mathtask and the evidence from research and practice that motivated its design. We then discuss its implementation in a lesson for in-service teachers.

The mathtask and its design

In the mathtask (Figure 2), the teacher gives the students a set of mathematical problems with quadratic functions (e.g. finding the coefficients or calculating specific values). One of the students, Bruna, asks: "how are we going to solve this function?" which makes teacher to wonder: "why the students keep saying that they need to "solve the function"?". This discussion about "solving the function" is central in the design of this mathtask. We use it to problematize the algebraic representation of equations and functions, and, also their teaching. Bruna in this classroom situation probably does not realise that the x of a function is not the same as the x of an equation. In Brazil, teachers spend almost the whole 9o ano do Ensino Fundamental studying how to solve quadratic equations and, then in the next year, the quadratic functions are introduced to the students. However, solving the equation $ax^2 + bx + c = 0$ and dealing with the function $f(x) = ax^2 + bx + c$ use very similar notation and are very different at the same time. Students who are used to solve quadratic equations and see x deployed in the context of functions, do not realise the difference and can easily apply well known routines of solving equation although those are not relevant.

In a classroom of the 1o ano do Ensino Médio (1st year of the High School), teacher Victor begins the quadratic functions' study. He starts writing on the blackboard as indicate and continues the lesson highlighting the coefficients and how is it possible to determine images of specific values.



Then the class starts solving some exercises selected by the teacher.

Exercícios

1. Indique os coeficientes das funções a seguir.
    a. $f(x) = 5x^2 + x - 7$
    b. $f(x) = 4 - x^2$
    c. $f(x) = x^2 - x$
    d. $f(x) = -4 + 2x^2 - 9x$

2. Considere $f(x) = x^2 - 3x + 4$. Determine.
    a. $f(1)$
    b. $f(2)$
    c. $f(0)$

d. $f(-1)$
e. x, tal que $f(x) = 4$
f. x, tal que $f(x) = 2$
g. x, tal que $f(x) = 0$

Aline and Bruna sit together to solve the list.

Bruna: How are we going to solve this function?

Aline: This question here is just to indicate the coefficients.

Bruna: Ok. But what about the other one? Don't we might solve?

Teacher Victor was intrigued with Bruna's question about "solving functions". After the class he meets a colleague and tells him.

Victor: I teach for years and until now, I cannot understand why the students keep saying that they need to "solve the function".

Questions:

a. What does lie behind Bruna's request to "solve the function" and Victor's observation?

b. As a teacher, how would you approach to a student that wants to "solve the function"?

Figure 2. A mathtask used in a professional master for school mathematics teachers

Our aim with this task is, therefore, to catch up the attention for the similarities and differences that equations and functions might have, especially in relation to the different roles of the letters in algebra as *unknowns, variables* and *coefficients*. The *unknown* is a quantity that it is not known, usually temporarily, and satisfies a given equation (e.g. in 3x+7=10). A *variable* is a quantity that varies according to certain conditions (e.g. x in $f(x) = x$, where x is any real number). *Coefficients* are considered as known quantities represented in a general way with a letter. In a generic equation, determining coefficients is something arbitrary, while determining the unknown is through the (not arbitrary) solution of the equation (see a summary in Table 1). The salient similarity of the quadratic equation ($ax^2 + bx + c = 0$) and the function ($f(x) = ax^2 + bx + c$) representations might veil the difference between unknown (x in the equation) and variable (x in the function) and coefficient (*a* in both equation and function). Especially, the use of the letter x to represent both unknown in equations and variables in functions might, therefore, confuse students.

Table 1: Different uses of letters in algebraic expressions and equations (the table was created by the ideas presented in Roque (2012))

Letters as...	**Examples**	**Characterisation**
unknowns (x)	$3x + 7 = 10$ $x^2 + 7x = 49$	Quantities that we do not know and satisfy a given equation
variables (x e y)	$f(x) = x$ $x^2 + y^2 = 1$	Quantities we do not know and can take arbitrary values determined by certain conditions
coefficients (a e b e c)	$y = ax + b$ $ax^2 + bx + c = 0$	Quantities we consider as known in certain algebraic expressions to describe a general case

Even though the teacher might not be the one who teaches letters as variables (e.g. at 1a série do Ensino Médio), it is important to be aware of these potential issues while teaching letters as unknowns. Similarly, while teaching variables (e.g. at 9o ano do Ensino Fundamental), the teacher should consider that the students might already have established routines with letters as unknowns. This teacher awareness is important for students' preparation of what is coming and for the anticipation of potential conflicts between what they know and what is new to them. We believe that such awareness can be established with appropriate teacher preparation that broadens teachers' mathematical and pedagogical discourse by strengthening their confidence with the mathematical content; identifying connections between mathematical ideas; demonstrating how mathematical objects, in our case letters in algebra, may have different uses and meanings; and, learning how these connections can be integrated in mathematics teaching (see the discussion about Horizon Content Knowledge, Ball, Thames & Phelps, 2008) and the Foundation and Connection dimensions of the Knowledge Quartet (Turner & Rowland, 2011) and Mathematical Discourse for Teaching (Cooper, 2014). The mathtask in Figure 2 aims at such preparation and concerns mathematical content that, although will not be used necessarily in the class, it will enrich teachers' discourses and will influence their decisions in their actual teaching.

Discussion around question (a) of the mathtask, offers the possibility to talk about the epistemological difference between variables and unknowns, as summarized in Table 1. Since coefficients also appear in the generic expression of equations and functions, they also can be discussed; examples are welcome to explain, for instance, the arbitrariness or not of the letters. Whilst in question (b), there are possibilities to address classroom issues and potential approaches to these issues when they emerge in class. In this case, the teachers' teaching experience might surprise us with reports of common students' mistakes or strategies developed by the teachers to deal with similar situations.

Using mathtask in a professional master for school mathematics teachers

The mathtask in Figure 2 earlier was used in the context of the PhD study of the third author (Moustapha-Corrêa, 2020; Moustapha-Corrêa et al., 2019; Moustapha-Corrêa et al., 2021). In-service teachers who attended this lesson, recognized their own practices in a situation similar to the one in the mathtask. Some of them did not realize that, in the transition from the study of the equations to the study of functions, attention needs to be paid to the differences between unknowns and variables. On the other hand, the same group of teachers argued that in some situations, such as 2(a), 2(b) and 2(c) (Figure 2), they "solve a function", to find the solution of the problem. The mathtask, as presented here, served, therefore, its purposes.

Specifically, it seems that some of the teachers who participated in this study were not used to highlighting the difference between equations and functions to their students. They were shifting between unknowns and variables without considering these as not the *same* – what Giraldo and Roque (2014) called *naturalizar*. The different uses of the letters in algebra made them different objects in the mathematical discourse of the classroom. Not highlighting these differences may create a commognitive conflict (Sfard, 2008) between what the teacher and the textbook say and what the students respond to the tasks. A commognitive conflict may occur when the same word is being used in different ways by the discussants, especially when they are not aware of these differences. In the case of "solving functions", this happens because students have not shifted their discourse about letters and teachers need to be aware and ready to address this issue in their teaching.

In future designs of mathtasks, the issue of "solving functions" also could be addressed through a classroom situation, where the students who are asked to study the properties of a quadratic function, they set it equal to zero and solve the resulting equation, regardless of what is asked in the exercise or by the teacher.

Example 2. Technology as a visual mediator: "what do you see?"

The data that inspired this mathtask example are from the PhD study of the second author that looks at teachers' work with resources (Kayali, 2019; Kayali & Biza, 2017, 2021). A resource here is defined as "anything that can possibly intervene in [a teacher's] activity", it can be an artefact (e.g. a pen), a teaching material (e.g. worksheet), or even a social interaction (e.g. a conversation with a colleague)

(Gueudet, Buteau, Mesa, & Misfeldt, 2014, p. 142). Our interest in teachers' work with resources arose from the two-way influence between resources and teachers (i.e. resources influence and are influenced by teachers); therefore, exploring the interactions between the two may help identify opportunities to develop teaching (Kayali & Biza, 2021; Gueudet et al., 2014). Setting out from this interest, a series of lessons were observed with one mathematics teacher, Adam, who taught in a British secondary school.

Observations that led to the mathtask design

Lesson context

This lesson was the first one we observed for Adam. At the time of the observation, Adam had four years of teaching experience, during which he taught students aged 12-18 years. He holds a degree in economics and a postgraduate certificate in education[5] for teaching mathematics in secondary schools, and was about to finish his master's degree in education[6]. This is because in the UK it is not necessary for a teacher to hold a mathematics degree to teach mathematics, instead a degree that has a mathematical component is enough along with a postgraduate certificate in education which is a initial teacher education course. The first lesson observed for Adam was taught to Year 12 class (17-18 years old) and was audio-recorded. The lesson was taught in a classroom that had an interactive whiteboard and one computer for teacher's use. The teacher's computer had two mathematics-education software Autograph (www.autograph-maths.com) and Geogebra (www.geogebra.org). The focus of the observation was on Adam's use of resources, especially his use of mathematics-education software, in this case Autograph or Geogebra, which he mentioned as frequently used in his teaching.

Lesson overview

The observation was on a revision lesson about solving simultaneous linear and modulus equations (i.e. equations that include absolute value). Adam started by moving a stick in the air in order to draw a specific graph, and asking the students to recognise the graph. One of these graphs was the *sine* graph, but the students seemed to be confused about what graphs were being drawn. After the stick activity, Adam asked his students to solve some problems that were displayed on the board. All the problems apart from one (which was designed by Adam) were chosen from the textbook. During the lesson, Adam used Autograph to check the answers given by the students, he entered the functions and the graphs were projected on the board. Then a discussion and demonstration of the algebraic solution was led by him on the whiteboard. When two of the students finished with the problems on the board earlier than the rest of the class, Adam gave them an extension question which he might have suggested spontaneously in response to the need of extra work. The extension question was in two parts: the first asked for two different modulus functions that do not intersect, the second asked for two that intersect once. "Is that possible? Can you give me two that intersect once?", Adam asked the class, and the dialogue below followed:

Student A: $y = |x|$ and $y = 2|x|$, shift across

[Adam plotted the graphs in Autograph (Figure 3)]

[5] The postgraduate certificate in education (PGCE) is a one- or two-year postgraduate course for teacher training ("Teaching- What is a PGCE?," 2019). It is one of the routes to qualify as a teacher in England, and requires the applicant to hold an undergraduate degree in mathematics or a closely related subject ("Teaching- Eligibility for teacher training", 2019). If the applicant's degree is not in mathematics, s/he can enrol in a subject knowledge course ("Teaching- Subject knowledge enhancement (SKE) courses", 2019).

[6] The master's in Educational Practice and Research is a part-time postgraduate course for education professionals (mostly teachers) with an interest in extending their professional development by studying for a Master's level degree ("MA Educational Practice and Research, UEA", 2019).

Adam: Oh, ya it is.

Student A: Ya, you've translated it.

Student B: *y* = | *x* – 4 | and *y* = 2 | *x* |.

[Adam plotted the graphs in Autograph (Figure 4), looked at the graphs on Autograph and nodded in what seemed like a hesitant agreement.]

Student C: Change the slope.

Adam amended the equations as student C suggested and wrote y = 2 | x – 4 | and y = 2 | x | without commenting on student B's answer

[Adam did not follow up student B's response or student C's correction but moved straight to a completely different activity with which he concluded the lesson.]

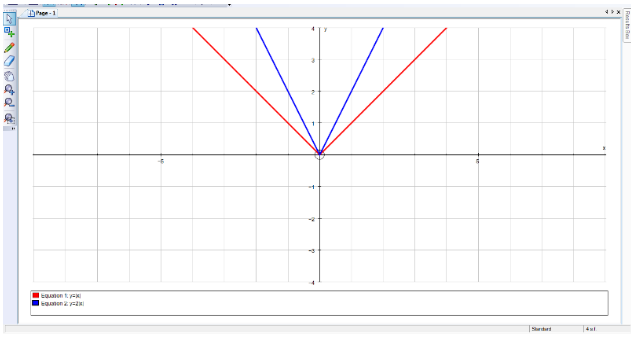


Figure 3: Student A's answer on Autograph (*y* = | *x* | and *y* = 2 | *x* |)

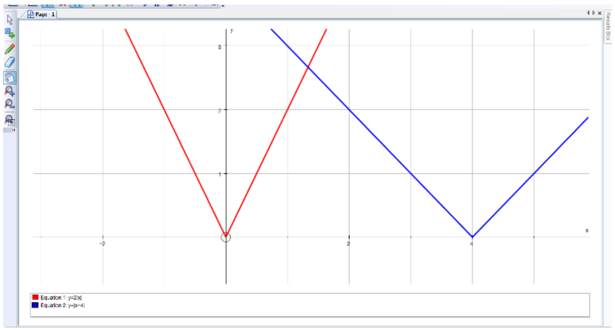


Figure 4: Student B's answer on Autograph (*y* = | *x* – 4 | and *y* = 2 | *x* |)

What is interesting in this lesson story and why?

Mathematics education software influence teachers' actions as they are adapted to provide access to mathematical knowledge; it is noted that they make teaching more complex (e.g. if students know more about technology than their teacher) and tasks more challenging for teachers to design (Clark-Wilson & Noss, 2015). This complexity can lead to "hiccups" which are unexpected moments or events in the classroom that occur due to the use of technology (Clark-Wilson & Noss, 2015). Unexpected classroom moments and events were also addressed by Rowland, Thwaites, and Jared (2015) in a more general context (i.e. not only in relation to technology) under the "Contingency" dimension of the Knowledge Quartet. For example, a "contingent" moment can be due to unexpected students' contributions. Rowland et al.'s (2015) contingency dimension looks at teachers' responses to such contributions, their responses to the (un)availability of resources, their use of opportunities that arise in the classroom and whether they deviate from their planned lesson agendas.

The lesson observation addressed in this example sheds light on this complexity and on the unexpected or unplanned. On one hand, it shows Adam's appreciation of Autograph ease of use as a tool for visual representation, so he used the software to check students' work and present graphical solutions before going for algebraic ones. On the other hand, Adam seemed confused by Autograph when it came to student B's answer with which he seemed to hesitantly agree. This might be because only one intersection point was visible within the displayed part of the graph (see Figure 4). In this case, Adam missed the opportunity to use the full affordances of Autograph (the zooming in/out feature in this case) in order to improve student's B answer and to explain the correct answer to the rest of the class. There was no evidence that the rest of the class, apart from student C, realised where the problem was and how it was amended. Using the language of the Knowledge Quartet (Rowland et al., 2015), this was a contingent moment that occurred due to unplanned students' answers and contributions, and it seemed that the teacher here missed the opportunity to reflect on it.

The mathtask and its design

Based on the observation above, we created a task that reflected the above classroom situation particularly in relation to the use of technology. A team of mathematics education researchers and practising teachers looked at the classroom scenario and recognised that the extension question that Adam used in his lesson (asking for two different modulus functions that do not intersect and two that intersect once) could be the basis for a task to share with teachers. The team suggested replacing Autograph with Geogebra, because Geogebra is a free software and hence it was more accessible to teachers from different schools. Thus, the task in Figure 5 was produced.

In a Year 12 lesson about simultaneous modulus equations, students are asked the following question:

"Give two modulus functions which have graphs that intersect only once"

The teacher and the students have access to the Geogebra software.

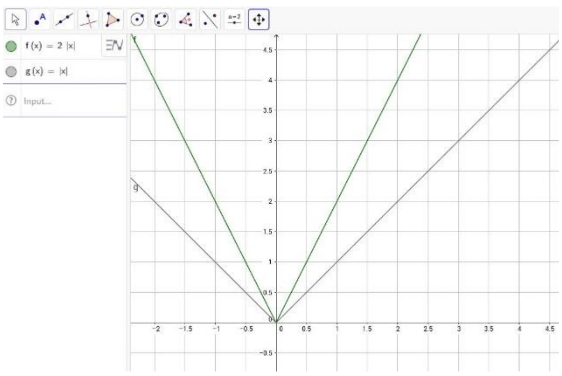
After a while, the following conversation occurs:

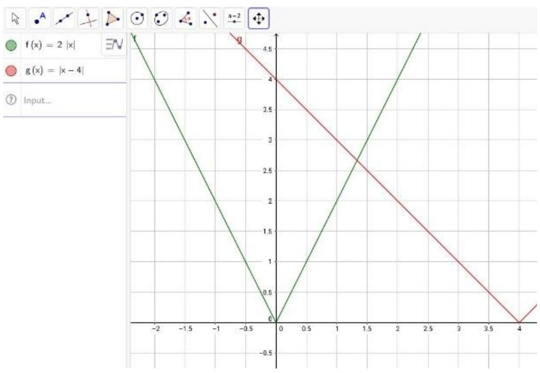
Student A: The two modulus functions that I found are $y = |x|$ and $y = 2|x|$.

Student B: I found a different pair: $y = |x - 4|$ and $y = 2|x|$.

Teacher: Let's check these solutions in GeoGebra.

The teacher produces the graphs of student A's and B's suggestions in GeoGebra.





Student A Student B

Questions:

a. What do you think are the issues emerging from the solutions proposed by students A and B?

b. What are the aims of doing this activity in class?
c. If you were the teacher, what would you do next, in relation to responding to each of these students and to the whole class?
d. How would you use the GeoGebra software, or another software, to support your responses to the above?

Figure 5: A mathtask for Simultaneous Modulus Equations

Using mathtask for reflection and professional development of school teachers

In service teachers who attended a MathTASK teacher education (in this case professional development) event in the UK were given this *mathtask* and were invited to reflect on and discuss issues around the use of technology. The discussion was around the use of technology, as well as the use of more than one approach to solve the question. Issues related to the software were identified in 13 teachers' responses and included: the value of the visual representation of the graphs and solutions and the zooming in/out feature of the software; and, whether the software should be a primary or supplementary tool of teaching in such situation. Besides, some teachers suggested that classroom discussion should be encouraged in this case, and that it could be started using open questions like "what do you see?". Others suggested graphing one function and creating another one that can be manipulated using sliders for the gradient and/or y-intercept to allow students to explore what would happen when these values changed. In this conversation, we valued teachers' engagement with the mathematical content related to the *mathtask* (e.g. solving modulus equations, translation of graphs, the role of the gradients, etc.); the mathematical meaning and the pedagogical role of representations; the role of technology in the visualisation of mathematical ideas and its pedagogical affordances and drawbacks (see about the focus on mathematical and pedagogical discourses in Biza et al., 2018). Also, we credited the use of the *mathtask* in keeping the discussion structured and contextualised on specific issues (see about the focus on the specificity in teacher responses in Biza & Nardi, 2019 and Biza et al., 2018). Overall, the task facilitated bringing to the fore issues about technology use and mathematics teaching/learning. We therefore see the main aim of the *mathtask* under discussion as achieved.

Example 3: What does $a|b$ mean in university mathematics?

The mathtask we discuss in this section regards mathematics teaching at university level and intends to support university mathematics lecturers'[7] professional development. Recently, higher education institutions in the UK invest more resources in the preparation of lecturers for teaching. Until now, this preparation was mostly on general pedagogy (e.g., management of learning, use of resources or assessment) and less on the pedagogy of a specific discipline, in our case mathematics. The proposed mathtask focuses on the teaching of mathematics at university level with attention to both mathematics and pedagogy and draws on findings from the PhD study of the fifth author on students' expected and actual engagement with university mathematical discourses in the context of final-year examination questions (Thoma, 2018; Thoma & Nardi, 2017; 2018). We start from a sample of these findings that led to the design of this mathtask.

Observations that led to the mathtask design

The sample of data we present here is from the analysis of twenty-two students' examination scripts from the end of year examination of a first year module on Sets, Numbers, Proofs and Probability in a

[7] In the UK context, teachers at university are usually called lecturers and a session that is led by the teacher is called lecture. In the usual structure of the lectures, the lecturer presents and the students attend by keeping notes with occasional contributions. There are also other types of teaching, such as seminars in which students work on problem sheets and the lecturer supports them with answering questions they may have.

UK mathematics department (Thoma & Nardi, 2018). One of the questions that students were given in the exams is presented in Figure 6.

(a) Suppose a, b, d, m, n are integers. Give the definition of what is meant by saying that d is a divisor of a. Using this, prove that if d is a divisor of a and d is a divisor of b, then d is a divisor of $ma + nb$.

(b) Use the Euclidean algorithm to find the greatest common divisor d of 123 and 45. Hence (or otherwise) find integers m, n with $123m + 45n = d$.

(c) Do there exist integers s, t such that $123s + 45t = 7$? Explain your answer carefully.

Figure 6: Exam question in the Sets, Numbers, Proofs and Probability module (Thoma & Nardi, 2018, p. 168)

One student wrote the response we can see in Figure 7. In this response, the student writes the d/a and d/b where d is the divisor of a and b. The results of these fractions *m* and *n* respectively are considered as integers in student's response, although they are not. This is conflicting with the introduction of the variables *m* and *n* in the task as integers. Thoma and Nardi (2018) suggest that the student was asked to engage with the discourses of different number sets, the integer numbers and the real numbers and, while working with integers, the student regarded them as real numbers and vice versa. This error "occurred because the students did not constrain the narrative that they produced within a specific numerical context". Thoma and Nardi (ibid) call this a manifestation of an underlying commognitive conflict that relates to not "working within the appropriate numerical context" (p. 168). This observation led to the mathtask we propose in what follows.

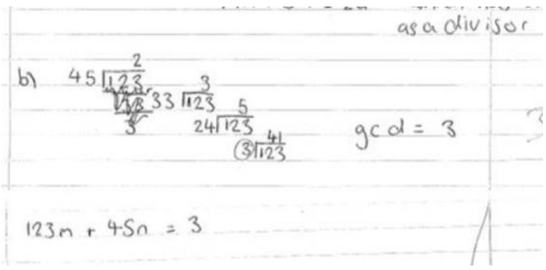


Figure 7: A student response to the exam question in Figure 6 (Thoma & Nardi, 2018, p. 169)

The mathtask and its design

The mathtask in Figure 8 regards first-year undergraduate mathematics classes and captures a scene from a seminar class. In the UK context, after the lectures, the students have seminar classes where they go through problems related to the content presented in the lectures. The main aim of these seminars is for the students to go through the problems either individually, or with their peers. During the seminars, in the university where this study was conducted, there are usually about 20 students and one or two tutors (lecturers or doctoral students). The tutors go around the class, and they answer questions the students might have. The following situation occurs in one of the seminar classes in a first-year undergraduate mathematics module. The task starts by giving a brief description of the context. The reader who engages with this task is asked to take the position of the lecturer. The lecturer is presented with a solution from two students and is asked to reflect on how they would respond to these students.

First-year undergraduate students have been introduced to the concept of the divisor in last week's lecture. Now they are in a seminar class, and they are working on a problem sheet with questions on the concept of divisors. The students work on the problem sheet either in pairs or individuallu. You are going around and checking what they are doing. You see two students discussing their work on the following problem.

"Prove that if *a*|*b* and *a*|*c* then *a*| *b+c,* where *a*, *b* and *c* are integers."

They seem to have reached a solution, and they are discussing. You look at what they wrote, and you see the following:

$$\frac{a}{b} = k \text{ and } \frac{a}{c} = l,$$

$$\text{So } b = \frac{a}{k} \text{ and } c = \frac{a}{l}.$$

$$\frac{a}{b+c} = \frac{a}{\frac{a}{k}+\frac{a}{l}} = \frac{1}{\frac{k+l}{kl}} = \frac{kl}{k+l}$$

You hear them commenting:

Student A: I think that's it. We have that a divides *b* and that *a* divides *c* and we showed that a divides *b*+*c*.

Student B: Yes, I think we are done. That was easy.

Questions:

a. How would you solve this mathematical problem?

b. What could be the aim of using this problem in class?

c. What issues would you raise in your response to these students?

Figure 8: The Number Theory mathtask "a divides b"

This mathtask follows the methodology of the MathTASK programme, also inspired by the methodology used by Iannone and Nardi (2005), and Nardi (2008) who asked undergraduate mathematics lecturers to discuss selected students' written responses. Its main aim is to trigger the reflection on aspects such as: use of symbolism; use of terminology; and, the transition between various numerical domains. We discuss these aspects also by referring to relevant research literature.

The appropriateness of the convention of using the symbol "|" to illustrate the divisibility property is raised by Kontorovich (2019). The participants of his study are mathematicians who discuss mathematical conventions and the suitability of the symbols used. They raised the issue regarding the symmetrical property of the symbol "|" and the discrepancy with the non-symmetrical relation of *a*|*b*. Similarly, Zazkis (1998) discusses meanings of the term divisor. The word divisor is being used to mean the number which is used to divide by in the context of division and in the context of number theory the word divisor is signalling an integer. Specifically, in number theory, *a*|*b* where *a* and *b* are integers means that *b* is an integer multiple of *a*. In other words, $b = ka$, where *a*, *b*, and *k* are integers.

In the mathtask in Figure 8, the students have translated $a|b$ which means *"a* divides *b"* as the quotient between *a* and *b* ($a \div b$). The students are introducing the symbol *k* to indicate the result of the quotient, and similarly they introduce the symbol *l* for $a \div c$. They manipulate the expression to create the sum between *b+c* and their manipulation results in a fraction. However, they do not comment on the numerical domain of the variables *k*, *l* and the resulting fraction $\frac{kl}{k+l}$. The result of a quotient between two integer numbers is not necessarily an integer. This issue with using symbols without providing explicit information regarding the numerical domain of the variable is also documented in research (Biehler & Kempen's, 2013; Epp, 2011; Thoma, 2018; Thoma & Nardi, 2017; 2018). The definition of *a* divides *b* means that *b* is an integer multiple of *a*. Consequently, the result of the division should be in the integers. However, these students deal with the variables without specifying the numerical set that they belong and without considering the constraints that division has in the set of integers, since the set of integers is not closed under division. Furthermore, the symbol of divisibility ("|") and the symbol of divide ("/") are very similar. But they convey very different relationships between numbers. For instance, when we have a and b integers and we write "$a \mid b$", there is an integer k such that $b=ka$. On the other hand, when we write "a/b" we say "divide a by b". In the latter situation, there are no restrictions regarding the numerical domains of the variables a and b. The quotient of the division is not necessarily an integer number and neither are the variables a and b. In this case, the relationship can be represented as $a=bk$, where a, b and k are real numbers.

We posit that the discussion of this mathtask can provide opportunities for university teachers to reflect on: the use of variables and notations for mathematical operations; the introduction of variables by specifying the numerical set where these variables belong; and, whether the result of a division belongs in the same numerical set as the numbers being divided. Similarly, the aim of this task is to raise discussion around the object of the divisor and the different uses of this term in various mathematical questions, which the students might be familiar with from their secondary school years but also their undergraduate studies. In addition to discussing these potential commognitive conflicts, another issue that the task raises are the constraints of integers regarding the operation of division. This could also lead to a discussion regarding the various numerical domains and the examination of the closure of various operations in those domains. Finally, engagement with this mathtask may give the opportunity to discuss the different uses and meanings of mathematical symbols and how the transition of students between mathematical areas or/and educational levels (e.g. from secondary school to university) may influence or challenge their learning of mathematics.

Example 4: Can a blind learner's unconventional description of a square-based pyramid challenge ableist perspectives on mathematics teaching?

While social justice has been a concern for many researchers interested in building more equitable mathematics classrooms, until recently, attention to disabled learners has been scarce. In particular, it is only recently that this research is starting to gain momentum in mathematics teacher education research and development. Furthermore, where discourses about disabled students exist, they tend to underestimate their potential for learning mathematics (Gervasoni & Lindenskov, 2011). These discourses have been described as "ableist", where ableism is "a network of beliefs, processes and practices that produces a particular kind of self and body (the corporeal standard) that is projected as the perfect, species-typical and therefore essential and fully human. Disability then, is cast as a diminished state of being human." (Campbell, 2001, p.44). There are signs of change though as shown by a small, yet growing, body of research that explores how ableist assumptions contribute to the creation of disabling learning environments in which learners with cognitive, emotional, physical and or sensory configurations that differ from what is currently defined as socially desirable and normal are disadvantaged (Healy & Powell 2013). It was with this desire to challenge ableism, particularly in the context of teacher education and professional development, that the project, CAPTeaM (**C**hallenging **A**bleist **P**erspectives on the **Tea**ching of **M**athematics), was conceived.

Opting for mathtask research design elements in the CAPTeaM project

To explore and challenge ableism, particularly in the context of teacher education and professional development, we develop and trial mathtasks that encourage teachers to reflect upon the challenges of teaching mathematics to disabled students. In what follows, we illustrate how the MathTASK design principles were implemented in the design of a CAPTeaM mathtask and we also offer evidence of the extent to which pre- and in-service teachers' engagement with the mathtasks contribute to reflections on the inclusion of disabled learners in mathematics lessons. We then present a theoretical construct that emerged from grounded analyses of data collected through the use of CAPTeaM mathtasks and now informs our analyses of data across the different parts of the project. One output of these analyses is that teacher education and professional development programmes need to question more explicitly (often ableist) teacher perspectives on what constitutes a normal mathematics classroom.

CAPTeaM is a collaborative project involving researchers and pre- and in-service teachers in Brazil and the UK. Two British Academy International Partnership and Mobility Scheme grants have enabled us to combine the different research foci of two research teams (in the UK, this is the team behind MathTASK; in Brazil, this is *Rumo à Educação Matemática Inclusiva* team, *Towards an Inclusive Mathematics Education*) in a reciprocal manner. The team designs mathtasks which aim at providing opportunities for pre- and in-service teachers to reflect upon issues related to the inclusion of disabled mathematics learners in their classes. The tasks emphasise different issues related to inclusion and challenge what we identify as ableist assumptions in different ways.

The CAPTeaM mathtasks and their design: principles and one example

The design of tasks involve the selection by members of the Brazilian team of episodes of mathematical interactions between students and teachers from the database of video evidence collected in the different studies of their research programme. As Nardi, Healy, Biza and Fernandes (2016) write:

> "…the design principle behind the selection process [is] the idea of highlighting the mathematical agency of disabled students: instead of attempting to determine "normal" or "ideal" achievement and positioning those who deviate from supposed norms as problematic and in need of remediation, attention should be directed to how students' mathematical ideas may develop differently and what pedagogical strategies are appropriate for supporting these developmental trajectories. The aim [is] hence to locate episodes representative of the successful mathematical practices associated with particular forms of interacting with the world – practices of learners who see with their hands and ears, who speak with their hands, whose visual memory is more efficient than their verbal memory, or, have other interesting ways of interacting with the world. We [opt] for episodes involving the use of interesting and valid mathematical strategies, but in which the properties and relations were expressed in unconventional or surprising forms." (p.349).

Using the MathTASK approach, each episode is inserted as a video clip into a brief narrative about a fictional mathematics classroom. We then invite the participants to assume the role of the teacher of this class and evaluate the interactions of the disabled students presented in the video clips – first individually and in written responses to a set of questions and then in a group discussion (which we take observation notes from and also video/audio-record).

We now present an example of a CAPTeaM mathtask, *André and the pyramid* (Figures 9, 10, 11). The video clip used in this task shows a short episode from an activity in which a blind student proposes a description of a square-based pyramid (Figures 10, 11). More details on the research context in which this activity was used are in Healy and Fernandes (2011).

Imagine you are teaching a class about three-dimensional geometric figures. As the students work on exploring how they would describe what a square-based pyramid is to someone who doesn't know, you move around the class to observe their strategies. You notice many are counting faces, edges and vertices. André, who is blind, has been working with materials, such as 3D solids. He offers this description. [Video clip follows]

Questions:

a. What is André proposing as a description of a square-based pyramid?

b. What do you do next?

c. What do you think are the issues in this situation?

d. What prior experience do you have in dealing with these issues?

e. What prior experience do you have in supporting the mathematical learning of blind students in your classroom?

f. How confident do you feel about including blind students in your classroom?

Figure 9. Example of a CAPTeaM mathtask: *André and the pyramid*

The 27sec video shows a blind student, André, describing his view of a square-based pyramid. As he spoke, André moved his fingers along the edges that join the vertices at the base of the pyramid (Figure 10) to the vertex at its apex (Figure 11) (stills from this video are presented in Nardi, Healy & Biza 2015, p. 55):

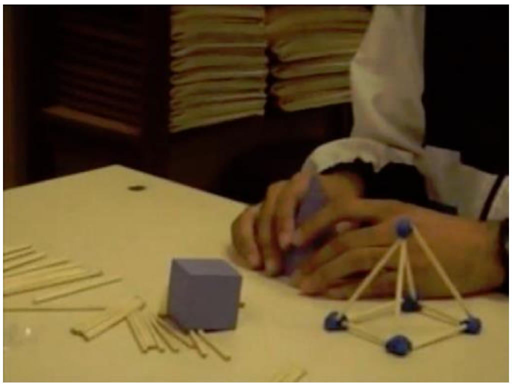


Figure 10: Feeling the vertices of the base.

Figure 11: Indicating the vertex at the apex.

Andre says: "I would say that the part underneath is square… the base… is square… And as you go up, they get, the sides of the square get smaller… Until they form a point here on top (moves his fingers along the edges to the vertex at the apex of the pyramid)."

Using a CAPTeaM mathtask to explore teacher perspectives on the inclusion of disabled learners in mathematics lessons

Grounded analyses of data collected in written protocols of responses to this mathtask as well as video-recordings of its plenary discussions (Nardi et al, 2016; Nardi, Healy & Biza, 2018) have led to five themes:

1. *Value and Attuning:* to what extent a respondent attunes to and values the disabled learner's contribution(s), and how, if at all, s/he attends to the particularities of their mathematical agency or adapts to the restriction imposed on the communication;
2. *Classroom Management:* how the respondent manages the classroom after the contribution by the disabled learner has been made;
3. *Experience and Confidence*: how experienced and confident the respondent claims to be in teaching students with the disability exemplified in the mathtask;
4. *Institutional Possibilities and Constraints:* what institutional possibilities and constraints the respondent identifies as crucial to the teaching of students featured in the mathtask;
5. *Resignification:* evidence of respondent's reconsideration of their views and intended practices in the light of engaging with the mathtask.

Our evidence suggests that those who engage with CAPTeaM mathtasks are encouraged to think about how the mathematical agency of disabled students might be supported or restricted by aspects of the learning environments in which they experience mathematics and to recognise that they are not *a priori* mathematically deficient. We posit that our tasks are successful in motivating the pre- and in-service teachers to rethink the notion of the "normal" student. We believe that this is an important step towards preparing teachers to work with learners with disabilities and influencing how they choose to organise the learning activities they offer to all their students. We are aware of a caveat though: our choice to embed the mathtasks in classroom settings that the teachers are likely to experience (or have experienced), may have contributed to the edifying of a different norm, the normal classroom. Building an inclusive school mathematics requires the deconstruction of this notion too and it is often up to what follows the written response to the mathtask (e.g. a plenary discussion and/or an opportunity to engage not with one but a suite of mathtasks) that amplifies the opportunity to start imagining what a truly inclusive mathematics classroom might look like.

Conclusions

This book chapter presents our work on the MathTASK programme that designs and engages mathematics teachers with classroom situations (mathtasks) for research and teacher education purposes. This work draws on previous studies that use specific *cases* or classroom incidents (real of fictional) in teacher education (e.g., Shulman, 1992; Zazkis et al., 2013) and proposes a design and use of classroom situations that brings the mathematical content upfront together with the pedagogy. In the sections of this chapter, we presented the theoretical underpinning that influence this work; the design, implementation and evaluation principles which we demonstrated through one example of a mathtask; and, the theoretical constructs proposed by the use of these tasks. We then presented four examples of mathtasks that have been designed for different purposes: in a professional master for school mathematics teachers; for reflection and professional development of school mathematics teachers; for reflection and professional development of university mathematics teachers; and, to explore teacher perspectives on the inclusion of disabled learners in mathematics lessons. The first three examples are associated with the PhD studies of the second, third and fifth author of this chapter. Specifically, in example 1, mathtasks were designed and applied with a twofold purpose: to educate in-service teachers and to conduct research on those teachers' discursive shifts on what is mathematics and what mathematical truth is (Moustapha-Corrêa, 2020; Moustapha-Corrêa et al., 2019; Moustapha-Corrêa et al., 2021). In example 2 and example 3, mathtask design was influenced by research observation from a secondary mathematics classroom ((Kayali, 2019; Kayali & Biza, 2017, 2021), in the former, and first year university mathematics assessment practices (Thoma, 2018; Thoma & Nardi, 2018) in the latter. Example 1 and example 2 are targeting secondary mathematics teaching in Brazil and UK, respectively and have been used in in-service teacher professional development. Example 3 is a new direction of our work and aims towards university mathematics teachers' professional development, this is an area that is developing quickly in the UK and USA. Example 4 is part of the project CAPTeaM that engages teachers across educational levels and in different national and institutional contexts with reflection on the inclusion of disabled learners in mathematics.

In all examples, the mathematical content is central and always intertwined with the pedagogy of mathematics teaching. Teachers very often act at the boundaries of the teaching discourses (grounded on their experiences as students or as teachers), the mathematical discourses (grounded on the mathematical component of their education) and the pedagogical discourses (grounded on the pedagogical component of their education. MathTASK programme aims to bring these discourses together.

Additionally, research has indicated that discussion on specific classroom situations offers structure in teachers' arguments and helps them to express their views about teaching (Goodell, 2006; Speer, 2005). In line with these observations, we credit the MathTASK design for the contextualisation of teachers' reflections.

Overall, we see the situation-specific task design we propose and the theoretical findings from the use of mathtasks in research – classification of warrants (Nardi et al., 2012) and typology of four characteristics (Biza et al., 2012) – as potent research tools and components of formative and summative assessment in teacher education programmes. By accentuating the specificity of the classroom situation, we invite teachers to reflect upon students' (and another teacher's) approaches and imagine their own intended practice. We thus gain insight into teachers' views and, crucially, challenging aspects of these views.

Teachers who participated in MathTASK workshops said that: "[t]hese activities made me reflect on my teaching practice" or "[m]y engagement with these tasks helped me deepen my own mathematical knowledge" or "[m]y engagement with these tasks helped me anticipate students' answers and their mistakes as well as their different ways of solving or approaching mathematical concepts". This balance between mathematics and pedagogy in teachers' reflections is exactly at the heart of MathTASK.

Acknowledgement

The collaborative work between our institutions on which we draw in this chapter is facilitated through Higher Education Impact funds (HEIF) received for the MathTASK programme at UEA and a British Academy International Partnership and Mobility grant [PM160190]. We also acknowledge CAPES (Programa de Doutorado Sanduíche no Exterior – 88881.133350/2016-01) for support to the doctoral study of the third author through a scholarship for a doctoral internship at Rutgers University (New Jersey, USA).

References

Arcavi, A. (2005). Developing and using symbol sense in mathematics. For the Learning of Mathematics, 14(3), 42–47.

Ball, D., Thames, H. M., & Phelps, G. (2008). Content knowledge for teaching. *Journal of Teacher Education*, *59*(5), 389–407.

Biehler, R., & Kempen, L. (2013). Students' use of variables and examples in their transition from generic proof to formal proof. In B. Ubuz, C. Haser & M.A. Mariotti (Eds.), *Proceedings of the Eigth Congress of the European Mathematical Society for Research in Mathematics Education* (pp. 86-95). Ankara, Turkey: Middle East Technical University and ERME.

Biza, I. & Nardi, E. (2019). Scripting the experience of mathematics teaching: The value of student teacher participation in identifying and reflecting on critical classroom incidents *International Journal for Lesson and Learning Studies, 9*(1), 43-56.

Biza, I., Nardi, E., & Joel, G. (2015). Balancing classroom management with mathematical learning: Using practice-based task design in mathematics teacher education. *Mathematics Teacher Education and Development, 17(2), 182-198.*

Biza, I., Nardi, E., & Zachariades, T. (2007). Using tasks to explore teacher knowledge in situation-specific contexts. *Journal of Mathematics Teacher Education*, *10*, 301-309.

Biza, I., Nardi, E., & Zachariades, T. (2009). Teacher beliefs and the didactic contract on visualization. *For the Learning of Mathematics, 29*(3), 31-36.

Biza, I., Nardi, E., & Zachariades, T. (2018). Competences of mathematics teachers in diagnosing teaching situations and offering feedback to students: Specificity, consistency and reification of pedagogical and mathematical discourses. In T. Leuders, J. Leuders, & K. Philipp (Eds.), *Diagnostic*

Competence of Mathematics Teachers. Unpacking a complex construct in teacher education and teacher practice, (pp. 55-78). New York: Springer.

Biza, I., Nardi, E., & Zachariades, T. (2014). Utilizando tarefas de situação específica para explorar o conhecimento matemático e as crenças pedagógicas: Exemplos de Álgebra e Análise. In T. Rogue & V. Giraldo (Eds.), *O saber do professor de matemática: ultrapassando a dicotomia entre didática e conteúdo* (pp. 329-378). Rio de Janeiro, Brazil: Editora Ciência Moderna Ltda.

Carter, K. (1999). What is a case? What is not a case? In M. A. Lundeberg, B. B. Levin, & H. L. Harrington (Eds.), *Who learns what from cases and how? The research base for teaching and learning with cases.* Mahwah NJ: Lawrence Erlbaum Associates.

Chevallard, Y. (1985). *La transposition didactique [The didactic transposition].* Grenoble: La Pensée

Sauvage Éditions.

Christiansen, B., & Walter, G. (1986). Task and activity. In B. Christiansen, A.-G. Howson & M. Otte (Eds.), *Perspectives on mathematics education: Papers submitted by members of the Bacomet Group* (pp. 243-307). Dordrecht: D. Reide.

Clark-Wilson, A., & Noss, R. (2015). Hiccups within technology mediated lessons: A catalyst for mathematics teachers' epistemological development. *Research in Mathematics Education, 17*(2), 92-109.

Cooper, J. (2014). Mathematical discourse for teaching: A discursive framework for analyzing professional development. In *Proceedings of the Joint Meeting of PME* (Vol. 38, pp. 337–344).

Dreher, A., Nowinska, E., & Kuntze, S. (2013). Awareness of dealing with multiple representations in the mathematics classroom - A study with teachers in Poland and Germany. In Lindmeier, A. M. & Heinze, A. (Eds.). *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 249-256. Kiel, Germany: PME.

Epp, S. S. (2011). Variables in mathematics education. In *International Congress on Tools for Teaching Logic* (pp. 54-61). Springer, Berlin, Heidelberg.

Erens, R., & Eichler, A. (2013). Belief systems' change - from preservice to trainee high school teachers on calculus. In Lindmeier, A. M. & Heinze, A. (Eds.) *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 281-288. Kiel, Germany: PME.

Freeman, J. B. (2005). Systematizing Toulmin's warrants: An epistemic approach. *Argumentation, 19*(3), 331–346.

Gervasoni, A., & Lindenskov, L. (2011). Students with "special rights" for mathematics education. In B. Atweh, M. Graven & P. Valero (Eds.), *Mapping equity and quality in mathematics education* (pp. 307–324). New York, NY: Springer.

GIRALDO, V. A.; ROQUE, T. M. História e Tecnologia na Construção de um Ambiente Problemático para o Ensino de Matemática. In GIRALDO, V. A. & ROQUE, T. M. (Org.), O Saber do Professor de Matemática: Ultrapassando a Dicotomia entre Didática e Conteúdo. Rio de Janeiro: Editora Ciência Moderna Ltda, pp. 9-37, 2014.

Goodell, J. E. (2006). Using Critical Incident Reflections: A Self-Study as a Mathematics Teacher Educator. *Journal of Mathematics Teacher Education, 9*(3), 221-248.

Gueudet, G., Buteau, C., Mesa, V., & Misfeldt, M. (2014). Instrumental and Documentational Approaches: From Technology Use to Documentation Systems in University Mathematics Education. *Research in Mathematics Education, 16*(2), 139-155.

Healy, L., & Fernandes, S.H.A.A. (2011). The role of gestures in the mathematical practices of those who do not see with their eyes. *Educational Studies in Mathematics, 77*, 157-174.

Healy, L. & Powell, A.B. (2013). Understanding and overcoming "disadvantage" in learning mathematics. In M.A. Clements, A. Bishop, C. Keitel, J. Kilpatrick, & F. Leung (Eds.), *Third international handbook of mathematics education*. (pp. 69–100). NL: Springer.

Herbst, P., & Chazan, D. (2003). Exploring the practical rationality of mathematics teaching through conversations about videotaped episodes: The case of engaging students in proving. *For the Learning of Mathematics*, 23(1), 2–14.

Hole, S., & McEntee, G. H. (1999). Reflection Is at the Heart of Practice. *Educational Leadership, 56*(8), 34-37.

Hill, H.C., & Ball, D.L. (2004). Learning mathematics for teaching: Results from California's mathematics professional development institutes. *Journal for Research in Mathematics Education*, *35*(5), 330–351.

Iannone, P., & Nardi, E. (2005). On the pedagogical insight of mathematicians:'Interaction'and 'transition from the concrete to the abstract'. *The Journal of Mathematical Behavior*, *24*(2), 191-215.

Kayali, L. (2021 Mathematics teachers work with resources: Four cases of secondary teachers using technology. (Unpublished doctoral dissertation). University of East Anglia, Norwich, UK.

Kayali, L., & Biza, I. (2017). "One of the beauties of Autograph is … that you don't really have to think": Integration of resources in mathematics teaching. In T. Dooley & G. Gueudet (Eds.), *Proceedings of the 10th Conference of European Research in Mathematics Education* (pp. 2406-2413). Dublin: Dublin City University.

Kayali, L., & Biza, I. (2021). Scheming and Re-scheming: Secondary Mathematics Teachers' Use and Re-use of Resources. *Digital Experiences in Mathematics Education.* Online First *https://doi.org/10.1007/s40751-021-00091-x*.

Kieran, C., Forman, E. A., & Sfard, A. (2002). *Learning discourse: Discursive approaches to research in mathematics education.* Kluwer Academic Publishers.

Kontorovich, I. (2019) What can be 'annoying' about mathematical conventions? Analysing exchanges of mathematically competent discursants. In U. T. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education*, (pp. xxxx-yyyy). Utrecht, the Netherlands: Freudenthal Group & Freudenthal Institute, Utrecht University and ERME.

Lampert, M. (1985). How do teachers manage to teach? Perspectives on problems in practice. *Harvard*

Educational Review, 55(2), 178–194.

Leatham, K.R., Peterson, B.E., Stockero, S.L., & Van Zoest, L.R. (2015). Conceptualizing mathematically significant pedagogical opportunities to build on student thinking. *Journal for Research in Mathematics Education*, *46*(1), 88–124.

Leont'ev, A. (1975). *Dieyatelinocti, soznaine, i lichynosti [Activity, consciousness, and personality]*. Moskva: Politizdat.

MA Educational Practice and Research (Part Time). (2019). Retrieved from https://www2.uea.ac.uk/study/postgraduate/taught-degree/detail/ma-educational-practice-and-research-3years-part-time

Markovits, Z., & Smith, M.S. (2008). Cases as tools in mathematics teacher education. In Tirosh, D. & Wood, T. (Eds.), *The international handbook of mathematics teacher education: Volume 2, Tools and Processes in Mathematics Teacher Education* (pp. 39-65). Rotterdam: Sense Publishers.

Mason, J., & Johnston-Wilder, S. (2006). *Designing and using mathematical tasks*. York, UK: QED Press.

Moustapha-Corrêa, B. (2020). Rumo a uma Postura Problematizadora na Formação de Professores de Matemática: Articulando Práticas Históricas e Práticas de Sala de Aula. (Unpublished doctoral dissertation). Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brazil.

Moustapha-Corrêa, B., Bernardes, A., & Giraldo, V. (2019). Historical Tasks to Foster Problematization. U. T. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education*, (pp. 2149-2157). Utrecht, the Netherlands: Freudenthal Group & Freudenthal Institute, Utrecht University and ERME.

Moustapha-Corrêa, B., Bernardes, A., Giraldo, V., Biza, I., & Nardi, E. (2021). Fostering shifts of mathematics teachers' pedagogical and epistemological discourses through history-focussed and situation-specific tasks. *Journal of Mathematical Behavior (Special Issue: Advances in Commognitive Research), 61,* 100840.

Nardi, E. (2008). *Amongst mathematicians: Teaching and learning mathematics at university level*. Springer Science & Business Media.

Nardi, E., Biza, I., & Zachariades, T. (2012). Warrant' revisited : Integrating mathematics teachers' pedagogical and epistemological considerations into Toulmin's model of argumentation. *Educational Studies in Mathematics, 79*(2), 157-173.

Nardi, E., Healy, L., & Biza, I. (2015). The CAPTeaM Project (Challenging Ableist Perspectives on the Teaching of Mathematics): A preliminary report. In Adams, G. (Ed.) *Proceedings of the British Society for Research into the Learning of Mathematics Day Conference 35*(2), 51-57.

Nardi, E., Healy, L., & Biza, I. (2017). 'Feeling' the mathematics of disabled learners: Supporting teachers towards valuing, attuning, integrating and resignifying in an inclusive mathematics classroom**.** In R. Hunter, M. Civil, B. Herbel-Eisenmann, N. Planas & D. Wagner (Eds.) *Mathematical discourse that breaks barriers and creates space for marginalized learners* (pp. 147-170). The Netherlands: SENSE Publications.

Nardi, E., Healy, L., Biza, I. & Fernandes, S.H.A.A (2016). Challenging ableist perspectives on the teaching of mathematics through situation-specific tasks. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education* (PME) (Vol. 3, pp. 347–354). Szeged, Hungary: PME.

Potari, D., & Psycharis, G. (2018). Prospective Mathematics Teacher Argumentation While Interpreting Classroom Incidents. In M. E. Strutchens, R. Huang, D. Potari, & L. Losano (Eds.), *Educating Prospective Secondary Mathematics Teachers* (pp. 169-187). ICME-13 Monographs: Springer.

Pultorak, E. G. (1993). Facilitating reflective thought in novice teachers. *Journal of Teacher Education, 44*, 288–295.

ROQUE, T. História da Matemática: Uma visão crítica, desfazendo mitos e lendas. Rio de Janeiro: Zahar. 2012.

Rowland, T., Thwaites, A., & Jared, L. (2015). Triggers of contingency in mathematics teaching. *Research in Mathematics Education, 17*(2), 74-91.

Sfard, A. (2008). *Thinking as communicating. Human development, the growth of discourse, and mathematizing*. New York, NY: Cambridge University Press.

Shulman, L.S. (1992). Toward a pedagogy of cases. In J. H. Shulman (Ed.), *Case methods in teacher education* (pp. 1–29). New York, Teachers College Press.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4–14.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review, 57*(1), 1–22.

Skemp, R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 77*, 20– 26.

Skott, J. (2001). The emerging practices of a novice teacher: The roles of his school mathematics images. *Journal of Mathematics Teacher Education, 4*(1), 3- 28.

Speer, M.N. (2005). Issues of methods and theory in the study of mathematics teachers' professed and attributed beliefs. *Educational Studies in Mathematics*, *58*(3), 361–391.

Teaching- Subject knowledge enhancement (SKE) courses. (2019). Retrieved from https://getintoteaching.education.gov.uk/explore-my-options/teacher-training-routes/subject-knowledge-enhancement-ske-courses

Teaching- What is a PGCE? (2019). Retrieved from https://getintoteaching.education.gov.uk/explore-my-options/teacher-training-routes/pgce

Teaching- Eligibility for teacher training. (2019). Retrieved from https://getintoteaching.education.gov.uk/eligibility-for-teacher-training

Thoma, A. (*2018*). *Transition to university mathematical discourses: A commognitive analysis of first year examination tasks, lecturers' perspectives on assessment and students' examination scripts.* (Unpublished doctoral dissertation). University of East Anglia, Norwich, UK.

Thoma, A., & Nardi, E. (2017). Discursive shifts from school to university mathematics and lecturer assessment practices: Commognitive conflicts regarding variables. In T. Dooley & G. Gueudet (Eds.), *Proceedings of the Tenth Congress of the European Mathematical Society for Research in Mathematics Education* (pp. 2266-2273). Dublin, Ireland: DCU Institute of Education and ERME.

Thoma, A., & Nardi, E. (2018). Transition from School to University Mathematics: Manifestations of Unresolved Commognitive Conflict in First Year Students' Examination Scripts. *International Journal of Research in Undergraduate Mathematics Education*, *4*(1), 161-180.

Thompson, A. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 122–127). New York: Macmillan.

Tirosh, D., & Wood, T. (Eds.). (2009). *The International Handbook of Mathematics Teacher Education* (Vol. 2). Rotterdam: Sense publishers.

Toulmin, S. (1958). *The uses of argument.* Cambridge, UK: Cambridge University Press.

Tripp, D. (2012). *Critical incidents in teaching: Developing professional judgement* (Classic ed.). London; New York: Routledge.

Turner, F., & Rowland, T. (2011) The Knowledge Quartet as an Organizing Framework for Developing and Deepening Teachers' Mathematics Knowledge (pp 195-212). In Rowland, T. and Ruthven, K. (2011) (Eds.) *Mathematical Knowledge in Teaching*. London and New York: Springer.

Zaslavsky, O., & Sullivan, P. (Eds.). (2011). *Constructing knowledge for teaching: Secondary mathematics tasks to enhance prospective and practicing teacher learning*. New-York: Springer.

Zaslavsky, O., Watson, A., & Mason, J. (2007). Special issue: The nature and role of tasks in mathematics teachers' education. *Journal of Mathematics Teacher Education*, *10*(4-6).

Zazkis, R., Sinclair, N., & Liljedahl, P. (2013). *Lesson play in mathematics education: A tool for research and professional development*. New-York: Springer.

Zazkis, R. (1998). Divisors and quotients: Acknowledging polysemy. *For the learning of mathematics*, *18*(3), 27-30.