FINITARY BIREPRESENTATIONS OF FINITARY BICATEGORIES

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ABSTRACT. In this paper, we discuss the generalization of finitary 2-representation theory of finitary 2-categories to finitary birepresentation theory of finitary bicategories. In previous papers on the subject, the classification of simple transitive 2-representations of a given 2-category was reduced to that for certain subquotients. These reduction results were all formulated as bijections between equivalence classes of 2-representations. In this paper, we generalize them to biequivalences between certain 2-categories of birepresentations. Furthermore, we prove an analog of the double centralizer theorem in finitary birepresentation theory.

CONTENTS

1. INTRODUCTION

Finitary 2-representation theory of finitary 2-categories, which is the categorical analog of finite dimensional representation theory of finite dimensional algebras, has evolved considerably in the last decade, see e.g. [MM1, MM2, MM3, MM5, MM6, ChMa, ChMi, KMMZ, MMMT1, MMMZ] and references therein.

The main reason for restricting the framework to 2-representations of 2-categories, was to avoid technical difficulties which naturally arise when one considers birepresentations of bicategories in general: by weakening the axioms, the proofs of most results require more and bigger diagrams, whose commutativity is not always easy to show. However, in our work on the 2-representation theory of Soergel bimodules of finite Coxeter type, it became clear that the 2-categorical setup is really too restrictive. Dealing with concrete examples of a certain weak categorical structure as if they were strict and justifying the oversimplification by referring to well-known general and abstract strictification theorems, becomes untenable at some point. For example, the (conjectural) classification of the so-called simple transitive 2-representations of Soergel bimodules involves certain subquotients of Soergel bimodules which are naturally bicategories but not 2-categories. Moreover, the well-known classification of the simple transitive birepresentations of these bicategories depends on the associator in an essential way (cf. [EGNO, Example 7.4.10 and Corollary 7.12.20]).

The main purpose of this paper is therefore to discuss the generalization of some important foundational results on finitary 2-representation theory to finitary birepresentation theory. By discussing, we mean formulating those results carefully in the greatest possible generality (or at least as generally as we currently can) and proving them in detail whenever the proof is not straightforward and cannot be found in the literature. A lot of the results in this paper will not surprise the experts, but we think that it is important to have the statements and their proofs, which sometimes involve quite complicated diagrams (e.g. the proof of Theorem 4.16), in written form somewhere in the literature. However, the paper also contains new results, as we will discuss in the next paragraphs, which are also intended as a brief and incomplete overview of birepresentation theory.

In the series of papers mentioned above, the first key tool for studying the structure of finitary 2-representations, is the weak Jordan–Hölder theorem [MM5, Subsection 3.5]. Just like the usual Jordan–Hölder theorem in the representation theory of finite dimensional algebras, the weak Jordan–Hölder theorem shows that any finitary 2-representation admits a filtration by 2-subrepresentations and an associated sequence of so-called *simple transitive* subquotients, which play the role of the simples in 2-representation theory. This sequence is an essential invariant of the 2-representation and for that reason the main focus in 2-representation theory has been on the problem of classifying simple transitive 2-representations so far. For the rest of this introduction, we will refer to this problem as the *Classification Problem*. Fortunately, the generalization of the weak Jordan–Hölder theorem to finitary birepresentations is straightforward. The Classification Problem for a given finitary 2-category $\mathscr{C}$ can be subdivided into several smaller classification problems by taking advantage of the so-called *cell structure* of $\mathscr{C}$, which was introduced in [MM1, Section 4]. The set of isomorphism classes of indecomposable 1-morphisms of $\mathscr{C}$ is naturally endowed with three preorders, called the left, the right and the two-sided preorders, generalizing Green's relations for semigroups [Gre] the well-known Kazhdan–Lusztig preorders on the Hecke algebras of Coxeter groups [KL]. Just as in Kazhdan–Lusztig theory, the associated equivalence classes are called left, right and two-sided cells and are partially ordered. By [ChMa, Subsection 3.2], for each simple transitive 2-representation of $\mathscr{C}$, there is a unique maximal two-sided cell of $\mathscr{C}$ that is not annihilated by the 2-representation, called its *apex*. This shows that one can address the Classification Problem for $\mathscr{C}$ "one apex at a time". The generalization of these results to finitary birepresentations of finitary bicategories is also straightforward.

The next trick is to reduce the Classification Problem for $\mathscr{C}$ even further. For that, one has to assume that the $\mathscr{C}$ is *fiat*, meaning that it is endowed with a weak categorical involution satisfying certain additional conditions (for some results it is not strictly necessary for the auto-equivalence to be involutory, but that is a technicality we do

not want to discuss here). In the context of tensor categories, these notions relate to rigidity/pivotal structures. The involution maps each left cell to a right cell, called its *dual*, and vice-versa, and the intersection of a left cell and its dual is called a *diagonal $\mathcal{H}$-cell*. For any diagonal $\mathcal{H}$-cell $\mathcal{H}$ in any two-sided cell $\mathcal{J}$, one can naturally define a subquotient $\mathscr{C}_{\mathcal{H}}$ of $\mathscr{C}$ which is also fiat and contains at most two cells: the trivial one (containing the identity 1-morphism) and $\mathcal{H}$ (i.e. $\mathscr{C}_{\mathcal{H}}$ has two cells if $\mathcal{H}$ does not contain the identity and one cell otherwise), both of which are left, right and two-sided. In [MMMZ, Subsection 4.2], it was shown that there is a bijection between the set of equivalence classes of simple transitive 2-representations of $\mathscr{C}$ with apex $\mathcal{J}$ and the set of equivalence classes of simple transitive 2-representations of $\mathscr{C}_{\mathcal{H}}$ with apex $\mathcal{H}$. The generalization in this paper, which we call *strong $\mathcal{H}$-cell reduction*, is two-fold: not only do we prove it in the context of finitary birepresentations of *fiab* bicategories (the bicategorical analog of fiat 2-categories), but we also formulate it as a biequivalence between two 2-categories of simple transitive birepresentations, rather than a mere bijection between sets of equivalence classes of simple transitive birepresentations. Both the formulation of this generalization and its proof are much more involved than the original counterparts in [MMMZ]. They are the content of Theorems 4.30 and 4.31, but require a lot of technical preparation, which is also new and starts in Subsection 2.6.

A key ingredient in strong $\mathcal{H}$-reduction is the relation between the birepresentations of a given finitary bicategory $\mathscr{C}$ and the coalgebras in $\mathscr{C}$, which are 1-morphisms together with comultiplication and counit 2-morphisms satisfying weak versions of coassociativity and counitality. This relation, which generalizes Ostrik's results in the context of tensor categories [Os, Theorem 3.1] (see also [EGNO] and references therein), was first studied in [MMMT1] and [MMMZ] in the context of finitary 2-categories (a major difference with Ostrik's results being that tensor categories are abelian, whereas finitary 2-categories are only additive), but is vastly generalized here, in Sections 3 and 4. In this case, the generalization is three-fold: as before, everything is now done in the context of birepresentations and bicategories and the key results are now formulated in terms of biequivalences between certain bicategories rather than mere bijections between sets of equivalence classes, but in Subsection 4.5 we additionally show how to avoid the (injective) abelianization of $\mathscr{C}$ (under an additional assumption of $\mathcal{J}$-*simplicity*), which was used in [MMMT1] and [MMMZ] to generalize Ostrik's results. This is done by using an operation on coalgebras which we call *framing* and can be seen as a categorical generalization of conjugation. The concept of framing as such is not new, see e.g. the proof of [EGNO, Theorem 7.12.11], but our application of it to avoid abelianization is new to the best of our knowledge.

Finally, in Section 5 we state and prove the *double centralizer theorem* for simple transitive birepresentations of fiab bicategories (Theorem 5.2). This is the analog of [EGNO, Theorem 7.12.11] in our context. If $\mathscr{C}$ is semisimple, the two theorems and their proofs coincide, but in general there is a subtle but important difference, which we will explain in Section 5.

2. Finitary bicategories and birepresentations

In this section we briefly discuss finitary and (quasi) fiab bicategories and finitary birepresentations. The notions and results in bicategory theory and birepresentation theory that are logically intertwined, are presented together.

2.1. **Notation.** Throughout the paper, a *category* is always assumed to be essentially small, meaning that it is equivalent to a small category (i.e. one whose classes of objects and morphisms are sets), and is denoted by a letter such as $\mathcal{C}$. A *bicategory* is always assumed to be essentially small as well, meaning that it is biequivalent to a small bicategory (i.e. one whose classes of objects, 1-morphisms and 2-morphisms are sets), and is denoted by a letter such as $\mathscr{C}$. These assumptions, which are satisfied by all examples of our interest, are necessary to avoid set-theoretic problems (for some more comments, see [EGNO, Section 1.1]). Precise definitions and standard terminology for bicategories, pseudofunctors etc. can be found in many sources e.g. [Be], [Lei, Section 1.0], [ML] or [Gr, Chapter I,3], and the reader is referred to these texts for details. Their strict versions, i.e. when all coherers are identities, are called 2-*categories*, 2-*functors* etc.

Let us summarize some further notation:

- Objects in categories (which are not morphism categories in bicategories) are denoted by letters such as $X \in \mathcal{C}$, and morphisms by $f \in \mathcal{C}$.
- Objects in bicategories are denoted by letters such as $\mathtt{i} \in \mathscr{C}$, 1-morphisms by those such as $F \in \mathscr{C}$, 2-morphisms by Greek letters such as $\alpha \in \mathscr{C}$, and the corresponding category of morphisms from $\mathtt{i}$ to $\mathtt{j}$ is denoted by $\mathscr{C}(\mathtt{i}, \mathtt{j})$.
- Identity 1-morphisms are denoted by $\mathbb{1}_{\mathtt{i}}$ and identity 2-morphisms by id_F, where the subscripts are sometimes omitted.
- We write $FG = F \circ_{\mathsf{h}} G$ for the composition of 1-morphisms (which is always horizontal), and $\circ_{\mathsf{v}}$ and $\circ_{\mathsf{h}}$ for the vertical and horizontal compositions of 2-morphisms, respectively. For both compositions we use the operator convention, e.g. the source and the target of FG are equal to the source of G and the target of F, respectively;
- a bicategory consists of a quadruple $\mathscr{C} = (\mathscr{C}, \alpha, \upsilon^l, \upsilon^r)$, where α is the associator and υ^l and υ^r are the left and right unitors.

We simplify the notation of $\alpha, \upsilon^l, \upsilon^r$ by only indicating the 1-morphisms in their subscripts, but not the objects, e.g. for any 1-morphisms $F \in \mathscr{C}(\mathtt{i}, \mathtt{j})$, $G \in \mathscr{C}(\mathtt{j}, \mathtt{k})$, $H \in \mathscr{C}(\mathtt{k}, \mathtt{l})$ with $\mathtt{i}, \mathtt{j}, \mathtt{k}, \mathtt{l} \in \mathscr{C}$, the associator and unitors give isomorphisms

$$\alpha_{H,G,F} := \alpha_{H,G,F}^{\mathtt{lkji}} \colon (HG)F \xrightarrow{\cong} H(GF),$$

$$\upsilon_F^l := (\upsilon_F^l)^{\mathtt{ji}} \colon \mathbb{1}_{\mathtt{j}}F \xrightarrow{\cong} F, \quad \upsilon_F^r := (\upsilon_F^r)^{\mathtt{ji}} \colon F\mathbb{1}_{\mathtt{i}} \xrightarrow{\cong} F.$$

Recall that $\upsilon_{\mathbb{1}_{\mathtt{i}}}^l = \upsilon_{\mathbb{1}_{\mathtt{i}}}^r$. In several proofs we will use the following commutative diagrams (cf. [Ke, Equations (6) and (7)]):

(2.1)

$$
\begin{array}{ccc}
 & GF & \\
\upsilon_{GF}^l \Big\uparrow & \xleftarrow{\upsilon_G^l \circ_{\mathsf{h}} \mathrm{id}_F} & \\
\mathbb{1}_{\mathtt{k}}(GF) & \xrightarrow[\alpha_{\mathbb{1}_{\mathtt{k}},G,F}^{-1}]{} & (\mathbb{1}_{\mathtt{k}}G)F
\end{array}
\quad , \quad
\begin{array}{ccc}
 & GF & \\
\upsilon_{GF}^r \Big\uparrow & \xleftarrow{\mathrm{id}_G \circ_{\mathsf{h}} \upsilon_F^r} & \\
(GF)\mathbb{1}_{\mathtt{i}} & \xrightarrow[\alpha_{G,F,\mathbb{1}_{\mathtt{i}}}]{} & G(F\mathbb{1}_{\mathtt{i}})
\end{array}
\quad ,
$$

for any $F \in \mathscr{C}(\mathtt{i}, \mathtt{j}), G \in \mathscr{C}(\mathtt{j}, \mathtt{k})$ and any $\mathtt{i}, \mathtt{j}, \mathtt{k} \in \mathscr{C}$.

Example 2.1. A monoidal category can be identified with the endomorphism category of a one-object bicategory, where the monoidal product is defined by the horizontal

composition. This monoidal category is strict if and only if the bicategory is a 2-category.

Example 2.2. Given two bicategories $\mathscr{C}$ and $\mathscr{D}$, the pseudofunctors between them together with the strong transformations of pseudofunctors and modifications form a bicategory $[\mathscr{C}, \mathscr{D}]$. If $\mathscr{D}$ is a 2-category, then one can show that $[\mathscr{C}, \mathscr{D}]$ is also a 2-category, see e.g. [Gr, Chapter I,3.3].

A *biideal* $\mathcal{I}$ in a bicategory $\mathscr{C}$ consists of an ideal $\mathcal{I}(\mathtt{i}, \mathtt{j})$ inside each $\mathscr{C}(\mathtt{i}, \mathtt{j})$, such that for any 2-morphisms $\beta \in \mathscr{C}(\mathtt{k}, \mathtt{l}), \gamma \in \mathcal{I}(\mathtt{j}, \mathtt{k}), \zeta \in \mathscr{C}(\mathtt{i}, \mathtt{j})$, the horizontal composition satisfies $\beta \circ_\mathsf{h} \gamma \circ_\mathsf{h} \zeta \in \mathcal{I}(\mathtt{i}, \mathtt{l})$. Finally, let $\mathscr{C}^{\mathrm{op}}$, $\mathscr{C}^{\mathrm{co}}$ and $\mathscr{C}^{\mathrm{co,op}}$ denote the bicategories obtained from $\mathscr{C}$ by reversing only the horizontal composition, only the vertical composition and both compositions, respectively.

2.2. **Finitary and fiab bicategories.** Let $\Bbbk$ be any field.

Definition 2.3. A *finitary* category $\mathcal{C}$ (over $\Bbbk$) is a $\Bbbk$-linear additive idempotent split category with finitely many isomorphism classes of indecomposable objects and finite dimensional morphism spaces.

Definition 2.4. We say that a bicategory $\mathscr{C}$ is *multifinitary* if $\mathscr{C}$ has finitely many objects, for all $\mathtt{i}, \mathtt{j} \in \mathscr{C}$ the categories $\mathscr{C}(\mathtt{i}, \mathtt{j})$ are finitary, and horizontal composition $\circ_\mathsf{h}$ of 2-morphisms is $\Bbbk$-bilinear. If additionally the identity 1-morphism on each object is indecomposable, then $\mathscr{C}$ is called *finitary*.

Definition 2.5. A *quasi (multi)fiab bicategory* is a (multi)finitary bicategory $\mathscr{C}$ together with an object-preserving $\Bbbk$-linear biequivalence $\star \colon \mathscr{C} \to \mathscr{C}^{\mathrm{co,op}}$ with the property that, for every pair $\mathtt{i}, \mathtt{j} \in \mathscr{C}$ and every $\mathrm{F} \in \mathscr{C}(\mathtt{i}, \mathtt{j})$, there exist adjunction 2-morphisms $\mathrm{ev}_\mathrm{F} \colon \mathrm{F}\mathrm{F}^\star \to \mathbb{1}_\mathtt{j}$ and $\mathrm{coev}_\mathrm{F} \colon \mathbb{1}_\mathtt{i} \to \mathrm{F}^\star\mathrm{F}$ such that the diagrams

$$
\begin{array}{ccccc}
\mathrm{F} & = & \mathrm{F} & \qquad & \mathrm{F}^\star & = & \mathrm{F}^\star \\
{\scriptstyle (v_\mathrm{F}^r)^{-1}}\downarrow & & \uparrow{\scriptstyle v_\mathrm{F}^l} & & {\scriptstyle (v_{\mathrm{F}^\star}^l)^{-1}}\downarrow & & \uparrow{\scriptstyle v_{\mathrm{F}^\star}^r} \\
\mathrm{F}\mathbb{1}_\mathtt{i} & & \mathbb{1}_\mathtt{j}\mathrm{F} & , & \mathbb{1}_\mathtt{i}\mathrm{F}^\star & & \mathrm{F}^\star\mathbb{1}_\mathtt{j} \\
{\scriptstyle \mathrm{id}_\mathrm{F}\circ_\mathsf{h}\mathrm{coev}_\mathrm{F}}\downarrow & & \uparrow{\scriptstyle \mathrm{ev}_\mathrm{F}\circ_\mathsf{h}\mathrm{id}_\mathrm{F}} & & {\scriptstyle \mathrm{coev}_\mathrm{F}\circ_\mathsf{h}\mathrm{id}_{\mathrm{F}^\star}}\downarrow & & \uparrow{\scriptstyle \mathrm{id}_{\mathrm{F}^\star}\circ_\mathsf{h}\mathrm{ev}_\mathrm{F}} \\
\mathrm{F}(\mathrm{F}^\star\mathrm{F}) & \xrightarrow{\;\alpha_{\mathrm{F},\mathrm{F}^\star,\mathrm{F}}^{-1}\;} & (\mathrm{F}\mathrm{F}^\star)\mathrm{F} & & (\mathrm{F}^\star\mathrm{F})\mathrm{F}^\star & \xrightarrow{\;\alpha_{\mathrm{F}^\star,\mathrm{F},\mathrm{F}^\star}\;} & \mathrm{F}^\star(\mathrm{F}\mathrm{F}^\star)
\end{array}
$$

commute.

If $\mathrm{Id}_\mathscr{C}$ and $(\star)^2$ are equivalent in $[\mathscr{C}, \mathscr{C}]$, then $\mathscr{C}$ is called *(multi)fiab*.

Following [MM2], the strict version of a (quasi) (multi)fiab bicategory is called a *(quasi) (multi)fiat* 2-*category*.

Given a quasi (multi)fiab bicategory, there is also a quasi-inverse of $\star \colon \mathscr{C} \to \mathscr{C}^{\mathrm{co,op}}$, which is usually denoted by the same symbol but applied to the left side of 1- and 2-morphisms, e.g. $^\star\mathrm{F}$ instead of $\mathrm{F}^\star$. There are then additional 2-morphisms $\mathrm{ev}_\mathrm{F}' \colon {}^\star\mathrm{F}\mathrm{F} \to \mathbb{1}_\mathtt{i}$ and $\mathrm{coev}_\mathrm{F}' \colon \mathbb{1}_\mathtt{j} \to \mathrm{F}(^\star\mathrm{F})$. In this paper we will only use this inverse in the multifiab case, where we can identify $\mathrm{F}^\star$ and $^\star\mathrm{F}$ and, in particular, obtain

$$
\mathrm{ev}_\mathrm{F}' = \left[\mathrm{F}^\star\mathrm{F} \xrightarrow{\cong} \mathrm{F}^\star\mathrm{F}^{\star\star} \xrightarrow{\mathrm{ev}_{\mathrm{F}^\star}} \mathbb{1}_\mathtt{i} \right], \quad \mathrm{coev}_\mathrm{F}' = \left[\mathbb{1}_\mathtt{j} \xrightarrow{\mathrm{coev}_{\mathrm{F}^\star}} \mathrm{F}^{\star\star}\mathrm{F}^\star \xrightarrow{\cong} \mathrm{F}\mathrm{F}^\star \right].
$$

These satisfy the conditions expressed by the commutative diagrams

$$
\begin{array}{ccc}
\mathrm{F}^{\star} =\!=\!=\!=\!= \mathrm{F}^{\star} & \quad & \mathrm{F} =\!=\!=\!=\!= \mathrm{F} \\
(v_{\mathrm{F}^{\star}}^{r})^{-1} \downarrow \qquad \uparrow v_{\mathrm{F}^{\star}}^{l} & & (v_{\mathrm{F}}^{l})^{-1} \downarrow \qquad \uparrow v_{\mathrm{F}}^{r} \\
\mathrm{F}^{\star}\mathbb{1}_{\mathrm{j}} \qquad \mathbb{1}_{\mathrm{i}}\mathrm{F}^{\star} & , & \mathbb{1}_{\mathrm{j}}\mathrm{F} \qquad \mathrm{F}\mathbb{1}_{\mathrm{i}} \\
\mathrm{id}_{\mathrm{F}^{\star}}\circ_{\mathrm{h}}\mathrm{coev}_{\mathrm{F}}' \downarrow \qquad \uparrow \mathrm{ev}_{\mathrm{F}}'\circ_{\mathrm{h}}\mathrm{id}_{\mathrm{F}^{\star}} & & \mathrm{coev}_{\mathrm{F}}'\circ_{\mathrm{h}}\mathrm{id}_{\mathrm{F}} \downarrow \qquad \uparrow \mathrm{id}_{\mathrm{F}}\circ_{\mathrm{h}}\mathrm{ev}_{\mathrm{F}}' \\
\mathrm{F}^{\star}(\mathrm{FF}^{\star}) \xrightarrow[\alpha_{\mathrm{F}^{\star},\mathrm{F},\mathrm{F}^{\star}}^{-1}]{} (\mathrm{F}^{\star}\mathrm{F})\mathrm{F}^{\star} & & (\mathrm{FF}^{\star})\mathrm{F} \xrightarrow[\alpha_{\mathrm{F},\mathrm{F}^{\star},\mathrm{F}}]{} \mathrm{F}(\mathrm{F}^{\star}\mathrm{F})
\end{array}
$$

For readers familiar with string diagrams, we note that the diagrams for $\mathrm{ev}_{\mathrm{F}}'$ and $\mathrm{coev}_{\mathrm{F}}'$ are obtained from the ones for ev_{F} and $\mathrm{coev}_{\mathrm{F}}$ by inverting the orientation.

Remark 2.6. The term *weakly fiat 2-category* was introduced in [MM5, Subsection 2.5]. The corresponding notion for bicategories would be *weakly fiab*, but we decided to change *weakly* to *quasi*, and use that terminology from now on, to avoid confusion with the notion of weakness in bicategories.

Remark 2.7. Quasi fiab and fiab one-object bicategories correspond to rigid and pivotal monoidal categories in the terminology of e.g. [EGNO], respectively. Multifinitary bicategories are the additive analog of multitensor categories, cf. [EGNO, Definition 4.1.1].

Example 2.8. A particular class of quasi fiab one-object bicategories is that of fusion categories, which are semisimple rigid monoidal categories, e.g. $\mathscr{V}\mathrm{ect}^{\omega}(G)$, the category of G-graded vector spaces whose monoidal product is twisted by a 3-cocycle on G, and $U_q(\mathfrak{g})\text{-}\mathrm{mod}_{\mathrm{ss}}$, the semisimplified module category of the quantum group associated to a complex finite dimensional semisimple Lie algebra $\mathfrak{g}$ for q a root of unity.

Example 2.9. Let $\Bbbk = \mathbb{C}$ and let $W = (W,\mathtt{S})$ be a Coxeter group with its reflection representation. To these data one can associate the one-object bicategory of *Soergel bimodules* $\mathscr{S} = \mathscr{S}_{\mathbb{C}}(W,\mathtt{S})$, which categorifies the Hecke algebra of W such that the indecomposable 1-morphisms C_w correspond to the Kazhdan–Lusztig basis elements c_w, for $w \in W$. The one-object bicategory $\mathscr{S}$ can be defined over the polynomial algebra, as in e.g. [EW3], or over the coinvariant algebra, as in e.g. [So1]. For finite W, the bicategory $\mathscr{S}$ is finitary when defined over the coinvariant algebra.
Based on results in [EW3], Lusztig [Lu, §18.5] associated with each two-sided cell $\mathcal{J}$ of W a semisimple one-object bicategory $\mathscr{A}_{\mathcal{J}}$, called the *asymptotic limit* or the *asymptotic bicategory*, which categorifies the direct summand of the asymptotic Hecke algebra corresponding to $\mathcal{J}$ (or, in Lusztig's terminology, the J-ring associated with $\mathcal{J}$). By [EW1, Section 5], the monoidal category $\mathscr{S}$ is pivotal for any W, and so is $\mathscr{A}_{\mathcal{J}}$ for any $\mathcal{J}$ of W.
Thus, for any finite Coxeter group, Soergel bimodules over the coinvariant algebra form a one-object fiab bicategory in our terminology.

Example 2.10. There are also fiab bicategories with more than one object which play an important role in birepresentation theory. For example, the bicategory of *singular Soergel bimodules* [Wi] associated with a Coxeter group $W = (W,\mathtt{S})$, whose objects are indexed by the parabolic subsets of $\mathtt{S}$. This is why we do not restrict our setup to rigid or pivotal monoidal categories, but always consider (quasi) fiab bicategories in general.

The *abelianizations* discussed in [MMMT1, Section 3] carry over verbatim to the bicategorical setting. Indeed, a multifinitary bicategory $\mathscr{C}$ admits an injective and a projective abelianization, denoted by $\underline{\mathscr{C}}$ and $\overline{\mathscr{C}}$, respectively. These are such that their morphism categories are abelian and $\mathscr{C}$ is biequivalent to the 2-full subbicategory of injective, respectively projective, 1-morphisms. Note that all the 2-morphisms and coheres from

$\mathscr{C}$ extend to $\underline{\mathscr{C}}$ and $\overline{\mathscr{C}}$, and we will usually not distinguish between the ones for $\mathscr{C}$, or $\underline{\mathscr{C}}$ and $\overline{\mathscr{C}}$ to ease notation.

Remark 2.11. Even if $\mathscr{C}$ is multifiab, its abelianizations need not be, but * gives rise to an antiequivalence between $\underline{\mathscr{C}}$ and $\overline{\mathscr{C}}$.

Similarly, finitary birepresentations, which will be discussed in Subsection 2.3, admit abelianizations.

2.3. **Finitary birepresentations.**

Definition 2.12. We let $\mathfrak{A}_{\Bbbk}^f$ denote the 2-category of finitary categories, $\Bbbk$-linear functors (recall that any $\Bbbk$-linear functor between $\Bbbk$-linear, additive categories is automatically additive) and natural transformations.

Let $\mathscr{C}$ be a finitary bicategory, defined over $\Bbbk$.

Definition 2.13. A *finitary (left) birepresentation of $\mathscr{C}$* is a (covariant) $\Bbbk$-linear pseudofunctor $\mathbf{M}\colon \mathscr{C} \to \mathfrak{A}_{\Bbbk}^f$.

Concretely, a finitary birepresentation $\mathbf{M}$ of $\mathscr{C}$ associates
- a finitary category $\mathbf{M}(\mathtt{i})$, defined over $\Bbbk$, to every $\mathtt{i} \in \mathscr{C}$;
- a $\Bbbk$-linear functor $\mathbf{M}_{\mathtt{ji}}\colon \mathscr{C}(\mathtt{i},\mathtt{j}) \to \mathfrak{A}_{\Bbbk}^f\big(\mathbf{M}(\mathtt{i}),\mathbf{M}(\mathtt{j})\big)$ to every pair $\mathtt{i},\mathtt{j} \in \mathscr{C}$;
- a natural isomorphism

$$\mathbf{M}(\mathtt{i}) \underset{\mathrm{Id}_{\mathbf{M}(\mathtt{i})}}{\overset{\mathbf{M}_{\mathtt{ii}}(\mathbb{1}_{\mathtt{i}})}{\rightrightarrows}}{}^{\iota_{\mathtt{i}}} \mathbf{M}(\mathtt{i})$$

   to every $\mathtt{i} \in \mathscr{C}$;
- a natural isomorphism

$$
\begin{array}{ccc}
\mathscr{C}(\mathtt{j},\mathtt{k}) \boxtimes \mathscr{C}(\mathtt{i},\mathtt{j}) & \xrightarrow{\quad \circ_h \quad} & \mathscr{C}(\mathtt{i},\mathtt{k}) \\
{\scriptstyle \mathbf{M}_{\mathtt{kj}}\boxtimes \mathbf{M}_{\mathtt{ji}}}\downarrow & \overset{\mu_{\mathtt{kji}}}{\Longrightarrow} & \downarrow{\scriptstyle \mathbf{M}_{\mathtt{ki}}} \\
\mathfrak{A}_{\Bbbk}^f\big(\mathbf{M}(\mathtt{j}),\mathbf{M}(\mathtt{k})\big) \boxtimes \mathfrak{A}_{\Bbbk}^f\big(\mathbf{M}(\mathtt{i}),\mathbf{M}(\mathtt{j})\big) & \xrightarrow{\quad \circ_h \quad} & \mathfrak{A}_{\Bbbk}^f\big(\mathbf{M}(\mathtt{i}),\mathbf{M}(\mathtt{k})\big)
\end{array}
$$

   to every triple $\mathtt{i},\mathtt{j},\mathtt{k} \in \mathscr{C}$.

These data are such that the diagrams

(2.2)

$$
\begin{array}{ccc}
\mathbf{M}_{\mathtt{ji}}(\mathrm{F})\mathbf{M}_{\mathtt{ii}}(\mathbb{1}_{\mathtt{i}}) & \xrightarrow{\mu_{\mathtt{jii}}^{\mathrm{F},\mathbb{1}_{\mathtt{i}}}} & \mathbf{M}_{\mathtt{ji}}(\mathrm{F}\mathbb{1}_{\mathtt{i}}) \\
{\scriptstyle \mathrm{id}_{\mathbf{M}_{\mathtt{ji}}(\mathrm{F})}\circ_h \iota_{\mathtt{i}}}\downarrow & & \downarrow{\scriptstyle \mathbf{M}_{\mathtt{ji}}(\upsilon_{\mathrm{F}}^r)} \\
\mathbf{M}_{\mathtt{ji}}(\mathrm{F})\mathrm{Id}_{\mathbf{M}(\mathtt{i})} & = & \mathbf{M}_{\mathtt{ji}}(\mathrm{F})
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{M}_{\mathtt{jj}}(\mathbb{1}_{\mathtt{j}})\mathbf{M}_{\mathtt{ji}}(\mathrm{F}) & \xrightarrow{\mu_{\mathtt{jji}}^{\mathbb{1}_{\mathtt{j}},\mathrm{F}}} & \mathbf{M}_{\mathtt{ji}}(\mathbb{1}_{\mathtt{j}}\mathrm{F}) \\
{\scriptstyle \iota_{\mathtt{j}}\circ_h \mathrm{id}_{\mathbf{M}_{\mathtt{ji}}(\mathrm{F})}}\downarrow & & \downarrow{\scriptstyle \mathbf{M}_{\mathtt{ji}}(\upsilon_{\mathrm{F}}^l)} \\
\mathrm{Id}_{\mathbf{M}(\mathtt{j})}\mathbf{M}_{\mathtt{ji}}(\mathrm{F}) & = & \mathbf{M}_{\mathtt{ji}}(\mathrm{F})
\end{array}
,
$$

$$
\begin{array}{ccc}
\big(\mathbf{M}_{\mathtt{lk}}(\mathrm{H})\mathbf{M}_{\mathtt{kj}}(\mathrm{G})\big)\mathbf{M}_{\mathtt{ji}}(\mathrm{F}) & = & \mathbf{M}_{\mathtt{lk}}(\mathrm{H})\big(\mathbf{M}_{\mathtt{kj}}(\mathrm{G})\mathbf{M}_{\mathtt{ji}}(\mathrm{F})\big) \\
{\scriptstyle \mu_{\mathtt{lkj}}^{\mathrm{H},\mathrm{G}}\circ_h \mathrm{id}_{\mathbf{M}_{\mathtt{ji}}(\mathrm{F})}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\mathbf{M}_{\mathtt{lk}}(\mathrm{H})}\circ_h \mu_{\mathtt{kji}}^{\mathrm{G},\mathrm{F}}} \\
\mathbf{M}_{\mathtt{lj}}(\mathrm{HG})\mathbf{M}_{\mathtt{ji}}(\mathrm{F}) & & \mathbf{M}_{\mathtt{lk}}(\mathrm{H})\mathbf{M}_{\mathtt{ki}}(\mathrm{GF}) \\
{\scriptstyle \mu_{\mathtt{lji}}^{\mathrm{HG},\mathrm{F}}}\downarrow & & \downarrow{\scriptstyle \mu_{\mathtt{lki}}^{\mathrm{H},\mathrm{GF}}} \\
\mathbf{M}_{\mathtt{li}}\big((\mathrm{HG})\mathrm{F}\big) & \xrightarrow{\mathbf{M}_{\mathtt{li}}(\alpha_{\mathrm{H},\mathrm{G},\mathrm{F}})} & \mathbf{M}_{\mathtt{li}}\big(\mathrm{H}(\mathrm{GF})\big)
\end{array}
$$

(2.3) at left.

commute for all $\mathtt{i},\mathtt{j},\mathtt{k},\mathtt{l} \in \mathscr{C}$ and all $\mathrm{F} \in \mathscr{C}(\mathtt{i},\mathtt{j}), \mathrm{G} \in \mathscr{C}(\mathtt{j},\mathtt{k}), \mathrm{H} \in \mathscr{C}(\mathtt{k},\mathtt{l})$.

Example 2.14. If $\mathscr{C}$ is a finitary 2-category, then any finitary 2-representation of $\mathscr{C}$ is a finitary birepresentation (namely, one whose coherers are trivial).

Example 2.15. For any $\mathtt{i} \in \mathscr{C}$, the *principal birepresentation* (also called *Yoneda birepresentation*) $\mathbf{P}_{\mathtt{i}} := \mathscr{C}(\mathtt{i}, \_)$ is a finitary birepresentation of $\mathscr{C}$. If $\mathscr{C}$ is a finitary 2-category, then the principal birepresentations are all finitary 2-representations.

Let $\mathbf{M}$ and $\mathbf{N}$ be two finitary birepresentations of $\mathscr{C}$.

Definition 2.16. A *morphism of finitary birepresentations* $\Phi\colon \mathbf{M} \to \mathbf{N}$ is a $\Bbbk$-linear strong transformation of pseudofunctors.

Concretely, a morphism of birepresentations $\Phi\colon \mathbf{M} \to \mathbf{N}$ associates

- a $\Bbbk$-linear functor $\Phi_{\mathtt{i}}\colon \mathbf{M}(\mathtt{i}) \to \mathbf{N}(\mathtt{i})$ to each $\mathtt{i} \in \mathscr{C}$;
- a natural isomorphism

$$
\begin{array}{ccc}
\mathscr{C}(\mathtt{i},\mathtt{j}) & \xrightarrow{\;\mathbf{M}_{\mathtt{ji}}\;} & \mathfrak{A}_{\Bbbk}^{f}\big(\mathbf{M}(\mathtt{i}),\mathbf{M}(\mathtt{j})\big) \\
{\scriptstyle \mathbf{N}_{\mathtt{ji}}}\downarrow & \overset{\phi_{\mathtt{ji}}}{\Longrightarrow}\nearrow & \downarrow{\scriptstyle \Phi_{\mathtt{j}}\circ_{\mathtt{h}}-} \\
\mathfrak{A}_{\Bbbk}^{f}\big(\mathbf{N}(\mathtt{i}),\mathbf{N}(\mathtt{j})\big) & \xrightarrow[-\circ_{\mathtt{h}}\Phi_{\mathtt{i}}]{} & \mathfrak{A}_{\Bbbk}^{f}\big(\mathbf{M}(\mathtt{i}),\mathbf{N}(\mathtt{j})\big)
\end{array}
$$

to every pair $\mathtt{i},\mathtt{j} \in \mathscr{C}$.

These data are such that the diagrams

$$
(2.4) \quad
\begin{array}{ccccc}
\mathbf{N}_{\mathtt{ii}}(\mathbb{1}_{\mathtt{i}})\Phi_{\mathtt{i}} & \xrightarrow{\;\phi_{\mathtt{ii}}^{\mathbb{1}_{\mathtt{i}}}\;} & & & \Phi_{\mathtt{i}}\mathbf{M}_{\mathtt{ii}}(\mathbb{1}_{\mathtt{i}}) \\
{\scriptstyle \iota_{\mathtt{i}}^{\mathbf{N}}\mathrm{id}_{\Phi_{\mathtt{i}}}}\downarrow & & & & \downarrow{\scriptstyle \mathrm{id}_{\Phi_{\mathtt{i}}}\iota_{\mathtt{i}}^{\mathbf{M}}} \\
\mathrm{Id}_{\mathbf{N}(\mathtt{i})}\Phi_{\mathtt{i}} & = & \Phi_{\mathtt{i}} & = & \Phi_{\mathtt{i}}\mathrm{Id}_{\mathbf{M}(\mathtt{i})}
\end{array} ,
$$

(2.5)

$$
\begin{array}{ccccc}
\mathbf{N}_{\mathtt{kj}}(G)\mathbf{N}_{\mathtt{ji}}(F)\Phi_{\mathtt{i}} & \xrightarrow{\mathrm{id}_{\mathbf{N}_{\mathtt{kj}}(G)}\phi_{\mathtt{ji}}^{F}} & \mathbf{N}_{\mathtt{kj}}(G)\Phi_{\mathtt{j}}\mathbf{M}_{\mathtt{ji}}(F) & \xrightarrow{\phi_{\mathtt{kj}}^{G}\mathrm{id}_{\mathbf{M}_{\mathtt{ji}}(F)}} & \Phi_{\mathtt{k}}\mathbf{M}_{\mathtt{kj}}(G)\mathbf{M}_{\mathtt{ji}}(F) \\
{\scriptstyle \nu_{\mathtt{kji}}^{G,F}\mathrm{id}_{\Phi_{\mathtt{i}}}}\downarrow & & & & \downarrow{\scriptstyle \mathrm{id}_{\Phi_{\mathtt{k}}}\mu_{\mathtt{kji}}^{G,F}} \\
\mathbf{N}_{\mathtt{ki}}(GF)\Phi_{\mathtt{i}} & & \xrightarrow{\phi_{\mathtt{ki}}^{GF}} & & \Phi_{\mathtt{k}}\mathbf{M}_{\mathtt{ki}}(GF)
\end{array}
$$

commute for all $\mathtt{i},\mathtt{j},\mathtt{k} \in \mathscr{C}$ and all $F \in \mathscr{C}(\mathtt{i},\mathtt{j}), G \in \mathscr{C}(\mathtt{j},\mathtt{k})$.
Note that we will omit the symbol $\circ_{\mathtt{h}}$ for the horizontal composition of 2-morphisms if it causes no confusion, e.g. (2.4), (2.5) and so on.
Let $\Phi, \Psi\colon \mathbf{M} \to \mathbf{N}$ be two morphisms of birepresentations.

Definition 2.17. A *modification* $\sigma\colon \Phi \to \Psi$ is a modification between the strong $\Bbbk$-linear transformations.

Concretely, a modification $\sigma\colon \Phi \to \Psi$ associates a natural transformation $\sigma_{\mathtt{i}}\colon \Phi_{\mathtt{i}} \to \Psi_{\mathtt{i}}$ to every $\mathtt{i} \in \mathscr{C}$. These data are such that the diagram

$$
\begin{array}{ccc}
\mathbf{N}_{\mathtt{ji}}(F)\Phi_{\mathtt{i}} & \xrightarrow{\mathrm{id}_{\mathbf{N}_{\mathtt{ji}}(F)}\sigma_{\mathtt{i}}} & \mathbf{N}_{\mathtt{ji}}(F)\Psi_{\mathtt{i}} \\
{\scriptstyle \phi_{\mathtt{ji}}^{F}}\downarrow & & \downarrow{\scriptstyle \psi_{\mathtt{ji}}^{F}} \\
\Phi_{\mathtt{j}}\mathbf{M}_{\mathtt{ji}}(F) & \xrightarrow{\sigma_{\mathtt{j}}\mathrm{id}_{\mathbf{M}_{\mathtt{ji}}(F)}} & \Psi_{\mathtt{j}}\mathbf{M}_{\mathtt{ji}}(F)
\end{array}
$$

commutes for all $\mathtt{i},\mathtt{j} \in \mathscr{C}$ and all $F \in \mathscr{C}(\mathtt{i},\mathtt{j})$.
We say that two finitary birepresentations $\mathbf{M},\mathbf{N}$ of $\mathscr{C}$ are *equivalent* if there are morphisms of birepresentations $\Phi\colon \mathbf{M} \to \mathbf{N}$ and $\Psi\colon \mathbf{N} \to \mathbf{M}$ and invertible modifications $\Psi\Phi \xrightarrow{\cong} \mathrm{Id}_{\mathbf{M}}$ and $\Phi\Psi \xrightarrow{\cong} \mathrm{Id}_{\mathbf{N}}$.

Definition 2.18. As in Example 2.2, the fact that $\mathfrak{A}_{\Bbbk}^{f}$ is a $\Bbbk$-linear additive 2-category implies that

$$\mathscr{C}\text{-afmod} := [\mathscr{C}, \mathfrak{A}_{\Bbbk}^{f}]$$

is a $\Bbbk$-linear additive 2-category, called the 2-*category of finitary birepresentations of* $\mathscr{C}$.

Recall the following terminology, where add denotes the *additive closure*, meaning the closure under taking finite direct sums and summands.

Definition 2.19. Let $\mathscr{C}$ be a multifinitary bicategory and $\mathbf{M}$ a finitary birepresentation of $\mathscr{C}$. Then

(i) the birepresentation $\mathbf{M}$ is *generated by* $X \in \mathbf{M}(\mathtt{i})$, for some $\mathtt{i} \in \mathscr{C}$, if the embedding

$$\mathrm{add}\big\{\mathbf{M}_{\mathtt{ji}}(\mathrm{F})X \mid \mathrm{F} \in \mathscr{C}(\mathtt{i}, \mathtt{j})\big\} \hookrightarrow \mathbf{M}(\mathtt{j})$$

is an equivalence for all $\mathtt{j} \in \mathscr{C}$;
(ii) the birepresentation $\mathbf{M}$ is *cyclic* if it is generated by some $X \in \mathbf{M}(\mathtt{i})$ for some $\mathtt{i} \in \mathscr{C}$;
(iii) the birepresentation $\mathbf{M}$ is *transitive* if it is non-zero and is generated by any non-zero $X \in \mathbf{M}(\mathtt{i})$ for any $\mathtt{i} \in \mathscr{C}$.

By definition, a $\mathscr{C}$-*stable ideal* $\mathbf{I}$ of a finitary birepresentation $\mathbf{M}$ of $\mathscr{C}$ is the assignment of an ideal $\mathbf{I}(\mathtt{i}) \subseteq \mathbf{M}(\mathtt{i})$ to each $\mathtt{i} \in \mathscr{C}$, such that $\mathbf{M}_{\mathtt{ji}}(\mathrm{F})\big(\mathbf{I}(\mathtt{i})\big) \subseteq \mathbf{I}(\mathtt{j})$, for all $\mathrm{F} \in \mathscr{C}(\mathtt{i}, \mathtt{j})$. We say that $\mathbf{I}$ is *proper* if $\{0\} \subsetneq \mathbf{I}(\mathtt{i}) \subsetneq \mathbf{M}(\mathtt{i})$ for some $\mathtt{i} \in \mathscr{C}$.

Definition 2.20. A finitary birepresentation $\mathbf{M}$ is said to be *simple transitive* if it has no proper $\mathscr{C}$-stable ideals.

It follows immediately from Definition 2.20 that any simple transitive birepresentation is transitive. The converse is false in general, but every transitive birepresentation $\mathbf{M}$ of $\mathscr{C}$ has a unique simple transitive quotient, by the straightforward generalization of [MM5, Lemma 4] to bicategories.

Definition 2.21. We use the following $1, 2$-full 2-subcategories of $\mathscr{C}$-afmod (where $1, 2$-full means 1-full and 2-full, by definition):

(i) $\mathscr{C}$-cfmod denotes the one consisting of all cyclic (finitary) birepresentations;
(ii) $\mathscr{C}$-tfmod denotes the one consisting of all transitive (finitary) birepresentations;
(iii) $\mathscr{C}$-stmod denotes the one consisting of all simple transitive (finitary) birepresentations.

Note that we have $\mathscr{C}$-stmod $\subseteq \mathscr{C}$-tfmod $\subseteq \mathscr{C}$-cfmod $\subseteq \mathscr{C}$-afmod.
Recall that we want to generalize some of the results in e.g. [MMMT1] and [MMMZ] to the weak setup. Fortunately, most previous results carry over to this more general framework, due to two strictification theorems:

- every multifinitary bicategory $\mathscr{C}$ is biequivalent to a multifinitary 2-category, by the classical strictification results in this setting (see e.g. [GPS, Section 1.4] or [Lei, Section 2.3]);
- if $\mathscr{C}$ is a multifinitary 2-category, then its 2-category of finitary birepresentations is biequivalent to its 2-category of finitary 2-representations, by [Pow, Section 4.2].

A particular example of results that carry over verbatim, and that we will need later, is the following *weak Jordan–Hölder theorem*, cf. [MM5, Section 4].

Theorem 2.22. *Let $\mathscr{C}$ be a multifinitary bicategory. For any finitary birepresentation $\mathbf{M}$ of $\mathscr{C}$, there is a finite filtration by subbirepresentations of $\mathscr{C}$*

$$0 = \mathbf{M}_0 \subsetneq \mathbf{M}_1 \subsetneq \cdots \subsetneq \mathbf{M}_m = \mathbf{M},$$

where every $\mathbf{M}_k$ *generates a* $\mathscr{C}$*-stable ideal* $\mathbf{I}_k$ *in* $\mathbf{M}_{k+1}$*, such that* $\mathbf{M}_{k+1}/\mathbf{I}_k$ *is transitive and has a unique associated simple transitive quotient* $\mathbf{L}_{k+1}$*. Up to equivalence and ordering, the set* $\{\mathbf{L}_k \mid 1 \leq k \leq m\}$ *is an invariant of* $\mathbf{M}$*.*

For any finitary birepresentation $\mathbf{M}$, we call those simple transitive birepresentations $\mathbf{L}_k$, where $1 \leq k \leq m$, defined as in Theorem 2.22 the *weak Jordan–Hölder constituents* of $\mathbf{M}$.
We also briefly need the following.

Definition 2.23. We let $\mathfrak{R}_\Bbbk$ denote the 2-category of abelian finitary categories, i.e. objects are $\Bbbk$-linear additive categories which are equivalent to categories of finitely generated modules over finite dimensional associative $\Bbbk$-algebras, 1-morphisms are $\Bbbk$-linear functors and 2-morphisms are natural transformations.
An *abelian finitary (left) birepresentation* of $\mathscr{C}$ is a $\Bbbk$-linear pseudofunctor $\mathbf{M} \colon \mathscr{C} \to \mathfrak{R}_\Bbbk$. A finitary birepresentation $\mathbf{M}$ can be extended to an abelian birepresentation $\underline{\mathbf{M}}$, so that $\underline{\mathbf{M}}(\mathtt{i}) := \underline{\mathbf{M}(\mathtt{i})}$ for any $\mathtt{i} \in \mathscr{C}$. We can similarly abelianize $\mathbf{M}$ projectively to obtain $\overline{\mathbf{M}}$ with $\overline{\mathbf{M}}(\mathtt{i}) := \overline{\mathbf{M}(\mathtt{i})}$ for any $\mathtt{i} \in \mathscr{C}$.

The notions we have seen above for additive finitary birepresentations carry over verbatim to abelian finitary birepresentations and we leave it to the reader to write out the details.

Remark 2.24. We only need abelian birepresentations very rarely in this paper and refer to [MM2] for a comparison of finitary and abelian birepresentations.

2.4. **The additive closure of a bicategory.** Let $\mathcal{C}_j$ be additive categories for $j = 1, \ldots, n$. We define $\bigoplus_{j=1}^n \mathcal{C}_j$ to be the additive category whose objects are formal direct sums $X_1 \oplus \cdots \oplus X_s$, where $X_q \in \mathcal{C}_j$ for some $j = \jmath(X_q) \in \{1, \ldots, n\}$. Morphisms in $\mathrm{Hom}_{\bigoplus_{j=1}^n \mathcal{C}_j}(X_1 \oplus \cdots \oplus X_s, Y_1 \oplus \cdots \oplus Y_t)$ are matrices of morphisms

$$(f_{pq})_{p=1,\ldots,t;q=1,\ldots,s}, \text{ where } f_{pq} \in \begin{cases} \mathrm{Hom}_{\mathcal{C}_j}(X_q, Y_p) & \text{if } \jmath(X_q) = \jmath(Y_p) = j, \\ \{0\} & \text{otherwise.} \end{cases}$$

Composition is given by matrix multiplication. The additive structure is given by concatenation, i.e. $(X_1 \oplus \cdots \oplus X_s) \oplus (Y_1 \oplus \cdots \oplus Y_t) := X_1 \oplus \cdots \oplus X_s \oplus Y_1 \oplus \cdots \oplus Y_t$. Let $\mathscr{C}$ now be a finitary bicategory. Define $\mathscr{C}^\oplus$ as follows. It has one object $\bullet$ and $\mathscr{C}^\oplus(\bullet, \bullet) = \bigoplus_{\mathtt{j},\mathtt{k} \in \mathscr{C}} \mathscr{C}(\mathtt{j}, \mathtt{k})$, as defined above. If $\mathrm{F}_q \in \mathscr{C}(\mathtt{j}, \mathtt{k})$ for some $\mathtt{j}, \mathtt{k} \in \mathscr{C}$, we set $\mathtt{i}_{s(\mathrm{F}_q)} = \mathtt{j}$ and $\mathtt{i}_{t(\mathrm{F}_q)} = \mathtt{k}$ for source and target, respectively.
Composition of 1-morphisms is given by

$$(\mathrm{F}_1 \oplus \cdots \oplus \mathrm{F}_s)(\mathrm{G}_1 \oplus \cdots \oplus \mathrm{G}_t)$$
$$:= \mathrm{F}_1\mathrm{G}_1 \oplus \cdots \oplus \mathrm{F}_1\mathrm{G}_t \oplus \mathrm{F}_2\mathrm{G}_1 \oplus \cdots \oplus \mathrm{F}_s\mathrm{G}_1 \oplus \ldots \mathrm{F}_s\mathrm{G}_t,$$

where we omit components that are not defined, which we interpret as being zero. Vertical composition of 2-morphisms is defined componentwise.
Given a matrix f of morphisms in $\mathrm{Hom}_{\mathscr{C}^\oplus(\bullet,\bullet)}(\mathrm{F}_1 \oplus \cdots \oplus \mathrm{F}_s, \mathrm{F}_1' \oplus \cdots \oplus \mathrm{F}_{s'}')$ and g in $\mathrm{Hom}_{\mathscr{C}^\oplus(\bullet,\bullet)}(\mathrm{G}_1 \oplus \cdots \oplus \mathrm{G}_t, \mathrm{G}_1' \oplus \cdots \oplus \mathrm{G}_{t'}')$, their horizontal composition $f \circ_\mathsf{h} g$ is a matrix whose $(p'q', pq)$-component is given by $f_{p'p} \circ_\mathsf{h} g_{q'q}$, whenever this makes sense, and 0 otherwise.
Taking into account that

$$\big((\mathrm{F}_1 \oplus \cdots \oplus \mathrm{F}_s)(\mathrm{G}_1 \oplus \cdots \oplus \mathrm{G}_t)\big)(\mathrm{H}_1 \oplus \cdots \oplus \mathrm{H}_u) = \bigoplus_{p,q,r} (\mathrm{F}_p\mathrm{G}_q)\mathrm{H}_r,$$

$$(\mathrm{F}_1 \oplus \cdots \oplus \mathrm{F}_s)\big((\mathrm{G}_1 \oplus \cdots \oplus \mathrm{G}_t)(\mathrm{H}_1 \oplus \cdots \oplus \mathrm{H}_u)\big) = \bigoplus_{p,q,r} \mathrm{F}_p(\mathrm{G}_q\mathrm{H}_r),$$

both with the ordering on the summands given by the reverse lexicographic ordering on the indices, the associator is given by the diagonal matrix of the respective associators. To define the identity 1-morphism in $\mathscr{C}^{\oplus}$, fix an ordering $\mathtt{i}_1 < \cdots < \mathtt{i}_m$ on the objects of $\mathscr{C}$. The identity 1-morphism is then given by $\mathbb{1}_{\mathtt{i}_1} \oplus \cdots \oplus \mathbb{1}_{\mathtt{i}_m}$. Note that reordering produces an isomorphic 1-morphism. The right unitor is given by the component unitors in $\mathscr{C}$. Indeed, $(F_1 \oplus \cdots \oplus F_s)(\mathbb{1}_{\mathtt{i}_1} \oplus \cdots \oplus \mathbb{1}_{\mathtt{i}_m})$ will only have s direct summands of the form $F_p \mathbb{1}_{\mathtt{i}_{s(F_p)}}$, so the right unitor will be the diagonal $s \times s$-matrix of the corresponding unitors in $\mathscr{C}$. Similarly, the left unitor will be a permutation matrix with the unitors from $\mathscr{C}$ as entries, since $(\mathbb{1}_{\mathtt{i}_1} \oplus \cdots \oplus \mathbb{1}_{\mathtt{i}_m})(F_1 \oplus \cdots \oplus F_s)$ has summands $\mathbb{1}_{\mathtt{i}_{t(F_p)}} F_p$, but ordered according to the ordering on the $\mathtt{i}_{t(F_p)}$.

Lemma 2.25. *If $\mathscr{C}$ is a finitary bicategory, then $\mathscr{C}^{\oplus}$ is a multifinitary bicategory.*

Proof. The pentagon axiom and the compatibility of left and right unitors follow immediately from the same axioms for $\mathscr{C}$. The stated properties of $\mathscr{C}^{\oplus}(\bullet, \bullet)$ are inherited from the same properties for $\mathscr{C}$. Observe that $\mathscr{C}^{\oplus}$ is not finitary since the identity 1-morphism is not indecomposable, but $\mathscr{C}^{\oplus}$ is multifinitary. $\square$

The following lemma is immediate.

Lemma 2.26. *If $\mathscr{C}$ is (quasi) fiab, then $\mathscr{C}^{\oplus}$ is (quasi) multifiab.*

We now explain how to go back and forth between birepresentations of $\mathscr{C}$ and $\mathscr{C}^{\oplus}$. Given a finitary birepresentation $\mathbf{M}$ of $\mathscr{C}$, we can define a birepresentation $\mathbf{M}^{\oplus}$ by

- $\mathbf{M}^{\oplus}(\bullet) = \bigoplus_{\mathtt{i} \in \mathscr{C}} \mathbf{M}(\mathtt{i})$;
- $\mathbf{M}^{\oplus}(F_1 \oplus \cdots \oplus F_s) = \mathbf{M}(F_1) \oplus \cdots \oplus \mathbf{M}(F_s)$ for a 1-morphism $F_1 \oplus \cdots \oplus F_s$;
- $\mathbf{M}^{\oplus}\big((\beta_{pq})_{p=1,\ldots,t;q=1,\ldots,s}\big) = \big(\mathbf{M}(\beta)_{pq}\big)_{p=1,\ldots,t;q=1,\ldots,s}$ for a 2-morphism

$$\beta = (\beta_{pq})_{p=1,\ldots,t;q=1,\ldots,s} \colon F_1 \oplus \cdots \oplus F_s \to G_1 \oplus \cdots \oplus G_t.$$

Here we interpret the actions of direct sums of functors and their natural transformations on our chosen biproduct of additive categories in the evident way.

Conversely, given a birepresentation $\mathbf{N}$ of $\mathscr{C}^{\oplus}$, we can associate a birepresentation $\mathbf{N}'$ of $\mathscr{C}$ by noting that projection onto $\mathbb{1}_{\mathtt{i}}$ as a direct summand of the identity 1-morphism in $\mathscr{C}^{\oplus}$ defines an endomorphism of the identity functor on $\mathbf{N}(\bullet)$, and we thus have a decomposition $\mathbf{N}(\bullet) = \bigoplus_{\mathtt{i} \in \mathscr{C}} \mathbf{N}(\bullet)_{\mathtt{i}}$. We can then define:

- $\mathbf{N}'(\mathtt{i}) = \mathbf{N}(\bullet)_{\mathtt{i}}$ for any object $\mathtt{i} \in \mathscr{C}$;
- $\mathbf{N}'(F) = \mathbf{N}(F)$ for any 1-morphism F in $\mathscr{C}(\mathtt{i}, \mathtt{j})$, where $\mathtt{i}, \mathtt{j} \in \mathscr{C}$;
- $\mathbf{N}'(\beta) = \mathbf{N}(\beta)$ for any 2-morphism $\beta \colon F \to G$, where $F, G \in \mathscr{C}(\mathtt{i}, \mathtt{j})$ and $\mathtt{i}, \mathtt{j} \in \mathscr{C}$.

It is immediate that $(\mathbf{M}^{\oplus})'$ is equivalent to $\mathbf{M}$ and $(\mathbf{N}')^{\oplus}$ is equivalent to $\mathbf{N}$, which proves the following proposition.

Proposition 2.27. *There is a biequivalence of 2-categories $\mathscr{C}^{\oplus}$-afmod $\simeq \mathscr{C}$-afmod.*

2.5. **Cell theory.** The theory of cells carries over verbatim from finitary 2-categories to multifinitary bicategories. Let us briefly recall its main features; details and references can be found in [MM1, Subsection 4.5], [ChMa, Subsection 3.2], [MM5, Section 3] and [MMMZ, Subsection 4.2].

For each multifinitary bicategory $\mathscr{C}$, one defines the *left partial preorder* $\geq_L$ on indecomposable 1-morphisms by

$F \geq_L G \Leftrightarrow$ there exists H such that F is isomorphic to a direct summand of HG.

One then defines *left cells*, denoted by $\mathcal{L}$, to be the equivalence classes with respect to $\geq_L$, on which $\geq_L$ naturally induces a partial order denoted by the same symbol. Similarly, one defines the *right* and *two-sided partial preorders* $\geq_R$ and $\geq_J$ and their corresponding *right cells* and *two-sided cells*, denoted by $\mathcal{R}$ and $\mathcal{J}$ respectively. Note

that the source map $\mathtt{i}_{s(-)}$ is constant on each left cell and and the target map $\mathtt{i}_{t(-)}$ is constant on each right cell.

Example 2.28.

  (i) A fusion category $\mathscr{C}$ has only one (left, right and two-sided) cell, because, for any 1-morphism $\mathrm{F} \in \mathscr{C}$, the decomposition of both $\mathrm{FF}^\star$ and $\mathrm{F}^\star\mathrm{F}$ contains the identity on the unique object.
 (ii) Recall that for any finite Coxeter group, the one-object bicategory of Soergel bimodules $\mathscr{S}$ is finitary when it is defined over the coinvariant algebra. The (left, right, or two-sided, respectively) cells and cell orders in $\mathscr{S}$ correspond to the Kazhdan–Lusztig [KL] (left, right, or two-sided, respectively) cells and orders of W by the Soergel–Elias–Williamson categorification theorem [EW3, Theorem 1.1]. This remains true when $\mathscr{S}$ is defined over the polynomial algebra and/or the Coxeter group is non-finite.

For any left cell $\mathcal{L}$, one can define the so-called *cell birepresentation* $\mathbf{C}_{\mathcal{L}}$ as follows: Let $\mathtt{i}$ be the source of $\mathcal{L}$. Define a subbirepresentation $\mathbf{M}^{\geq \mathcal{L}}$ of the principal birepresentation $\mathbf{P}_{\mathtt{i}}$, using the induced action of $\mathscr{C}$ on

$$\mathrm{add}\big(\{\mathrm{F} \mid \mathrm{F} \geq_L \mathcal{L}\}\big).$$

Then $\mathbf{M}^{\geq \mathcal{L}}$ has a unique maximal ideal $\mathbf{I}$ and we define

$$\mathbf{C}_{\mathcal{L}} := \mathbf{M}^{\geq \mathcal{L}}/\mathbf{I},$$

which is always a simple transitive birepresentation.

Example 2.29.

  (i) The (unique) cell birepresentation of a fusion category coincides with its regular birepresentation, for which the action is defined by the monoidal product.
 (ii) The cell birepresentations of $\mathscr{S}$, for any Coxeter group W, categorify the Kazhdan–Lusztig [KL] cell representations of the Hecke algebra of W, by the Soergel–Elias–Williamson categorification theorem [EW3, Theorem 1.1].

Let $\mathscr{C}$ be a multifinitary bicategory. By the bicategorical analog of [ChMa, Subsection 3.2], any transitive birepresentation $\mathbf{M}$ of $\mathscr{C}$ has an associated invariant called *apex*, which is the unique two-sided cell $\mathcal{J}$ of $\mathscr{C}$ not annihilated by $\mathbf{M}$ that is maximal with respect to the two-sided order $\geq_J$.

Example 2.30. Suppose that $\mathscr{C}$ is quasi multifiab. Let $\mathcal{L}$ be a left cell inside a two-sided cell $\mathcal{J}$ of $\mathscr{C}$. Then the apex of the cell birepresentation $\mathbf{C}_{\mathcal{L}}$ is equal to $\mathcal{J}$.

Definition 2.31. Let $\mathscr{C}$ be a multifinitary bicategory. Denote by $\mathscr{C}\text{-afmod}_{\mathcal{J}}$ the $1, 2$-full 2-subcategory of $\mathscr{C}\text{-afmod}$ consisting of the finitary birepresentations whose weak Jordan–Hölder constituents all have apex $\mathcal{J}$. With respect to those $1, 2$-full 2-subcategories of $\mathscr{C}\text{-afmod}$ in Definition 2.21, we denote by

  (i) $\mathscr{C}\text{-cfmod}_{\mathcal{J}}$ the one consisting of all cyclic (finitary) birepresentations whose weak Jordan–Hölder constituents all have apex $\mathcal{J}$;
 (ii) $\mathscr{C}\text{-tfmod}_{\mathcal{J}}$ the one consisting of all transitive (finitary) birepresentations with apex $\mathcal{J}$;
(iii) $\mathscr{C}\text{-stmod}_{\mathcal{J}}$ the one consisting of all simple transitive (finitary) birepresentations with apex $\mathcal{J}$.

Again, we have $\mathscr{C}\text{-stmod}_{\mathcal{J}} \subseteq \mathscr{C}\text{-tfmod}_{\mathcal{J}} \subseteq \mathscr{C}\text{-cfmod}_{\mathcal{J}} \subseteq \mathscr{C}\text{-afmod}_{\mathcal{J}}$.

Inside each two-sided cell, we define $\mathcal{H}$-*cells* as the intersection of left and right cells. Note that $\mathtt{i}_{s(\mathcal{H})}$ need not be equal to $\mathtt{i}_{t(\mathcal{H})}$ in general. Any two-sided cell $\mathcal{J}$ is the disjoint union of the $\mathcal{H}$-cells it contains. If $\mathscr{C}$ is quasi multifiab, then * exchanges the

left and right cells inside each two-sided cell. For any left cell $\mathcal{L}$ inside a two-sided cell $\mathcal{J}$, the intersection

$$\mathcal{H}(\mathcal{L}) := \mathcal{L} \cap \mathcal{L}^\star \subseteq \mathcal{J}$$

is called the $\mathcal{H}$-cell *associated* to $\mathcal{L}$. Note that all 1-morphisms in an $\mathcal{H}$-cell $\mathcal{H}(\mathcal{L})$ are 1-endomorphisms of one fixed $\mathtt{i} \in \mathscr{C}$, which we call the *source of $\mathcal{H}$*. By the generalization of [MM1, Proposition 17] to quasi multifiab bicategories, each left cell $\mathcal{L}$ contains a unique distinguished 1-morphism $\mathrm{D} = \mathrm{D}(\mathcal{L})$, called *Duflo involution*. If $\mathscr{C}$ is multifiab, then every associated $\mathcal{H}$-cell $\mathcal{H}(\mathcal{L})$ is stable under $\star$ and is called a *diagonal $\mathcal{H}$-cell*. Since both $\mathrm{D} = \mathrm{D}(\mathcal{L})$ and $\mathrm{D}^\star$ belong to $\mathcal{L}$ (c.f. [MM1, Proposition 17]), we have $\mathrm{D} \in \mathcal{H}(\mathcal{L})$ in this case.

Lemma 2.32. *Let $\mathscr{C}$ be a multifinitary bicategory and $\mathbf{M} \in \mathscr{C}\text{-tfmod}_{\mathcal{J}}$ and $\mathcal{H}$ be any $\mathcal{H}$-cell inside $\mathcal{J}$. Then there exists some non-zero object $X \in \mathbf{M}(\mathtt{i}_{s(\mathcal{H})})$ which is not annihilated by $\mathcal{H}$.*

Proof. Let $\mathbf{M}$ be a transitive birepresentation of $\mathscr{C}$ with apex $\mathcal{J}$. Assume that the category $\mathbf{M}(\mathtt{i}_{s(\mathcal{H})})$ is annihilated by $\mathcal{H}$ and note that each $\mathbf{M}(\mathtt{j})$, where $\mathtt{j} \neq \mathtt{i}_{s(\mathcal{H})}$, is also annihilated by $\mathcal{H}$ by definition. We deduce that $\mathbf{M}$ annihilates $\mathcal{H}$ and hence annihilates $\mathcal{J}$ as $\mathrm{add}(\mathcal{J}) \subset \mathscr{C}\mathrm{add}(\mathcal{H})\mathscr{C}$, a contradiction. $\square$

Lemma 2.33. *Let $\mathscr{C}$ be a multifinitary bicategory. Any $\mathbf{M}$ in $\mathscr{C}\text{-afmod}_{\mathcal{J}}$ is cyclic, that is*

$$\mathscr{C}\text{-afmod}_{\mathcal{J}} = \mathscr{C}\text{-cfmod}_{\mathcal{J}}.$$

Moreover, for any $\mathcal{H}$-cell $\mathcal{H}$ inside $\mathcal{J}$, there exists a generator $X \in \mathbf{M}(\mathtt{i})$ of $\mathbf{M}$ such that, for any $\mathrm{F} \in \mathcal{H}$, $\mathbf{M}_{\mathtt{ji}}(\mathrm{F})X$ also generates $\mathbf{M}$, where $\mathtt{i} := \mathtt{i}_{s(\mathcal{H})}$ and $\mathtt{j} := \mathtt{i}_{t(\mathcal{H})}$.

Proof. Let $\mathbf{M} \in \mathscr{C}\text{-afmod}_{\mathcal{J}}$ and recall the existence of weak Jordan–Hölder series from Theorem 2.22. Let $0 \subset \mathbf{M}_1 \subsetneq \cdots \subsetneq \mathbf{M}_m = \mathbf{M}$ be a filtration of $\mathbf{M}$ by subbirepresentations such that each subquotient is transitive with apex $\mathcal{J}$. For each $q \in \{1, \ldots, m\}$, by Lemma 2.32, we can choose $X_q \in \mathbf{M}_q(\mathtt{i})$ such that $X_q \notin \mathbf{M}_{q-1}(\mathtt{i})$ and $\mathbf{M}_{\mathtt{ji}}(\mathrm{F}) X_q \notin \mathbf{M}_{q-1}(\mathtt{j})$ for any $\mathrm{F} \in \mathcal{H}$. Then, setting $X = X_1 \oplus \cdots \oplus X_q$, both X and $\mathbf{M}_{\mathtt{ji}}(\mathrm{F}) X$ generate $\mathbf{M}$. The statements follow. $\square$

2.6. **Quotients of bicategories via cell theory.** The main reason for introducing the various subbicategories of $\mathscr{C}\text{-afmod}$ associated with a two-sided cell $\mathcal{J}$ in Definition 2.31, is the reduction of the classification problem of all simple transitive birepresentations of $\mathscr{C}$ to that of the simple transitive birepresentations with apex $\mathcal{J}$ (where $\mathcal{J}$ is arbitrary but fixed). As explained in the introduction, we will show how to reduce the Classification Problem even further by strong $\mathcal{H}$-reduction in Theorems 4.30 and 4.31, when $\mathscr{C}$ is multifiab. But before we can do that, we first have to prepare the ground. In this subsection we therefore show how the aforementioned subbicategories of $\mathscr{C}\text{-afmod}$, and certain generalizations of them, are related to certain (sub)quotients of $\mathscr{C}$. At the end of this subsection, we will indicate more precisely the relation with strong $\mathcal{H}$-reduction.

For now, let $\mathscr{C}$ just be a multifinitary bicategory (i.e. not necessarily (quasi) multifiab) and $\mathcal{J}$ a two-sided cell of $\mathscr{C}$. We denote by $\mathcal{I}_{\not\leq_J \mathcal{J}}$ the biideal in $\mathscr{C}$ generated by id_F for all $\mathrm{F} \not\leq_J \mathcal{J}$. The quotient $\mathscr{C}/\mathcal{I}_{\not\leq_J \mathcal{J}}$ is a multifinitary bicategory whose two-sided cells correspond exactly to the two-sided cells $\mathcal{J}'$ of $\mathscr{C}$ satisfying $\mathcal{J}' \leq_J \mathcal{J}$. In particular, it has a unique maximal two-sided cell, corresponding to $\mathcal{J}$. If $\mathscr{C}$ is (quasi) multifiab, then $\mathscr{C}/\mathcal{I}_{\not\leq_J \mathcal{J}}$ is (quasi) multifiab. By Theorem 2.22, it is easy to understand the relation between the finitary birepresentations of $\mathscr{C}$ and those of $\mathscr{C}/\mathcal{I}_{\not\leq_J \mathcal{J}}$. Let $\mathscr{C}\text{-afmod}_{\leq_J \mathcal{J}}$ be the $1, 2$-full 2-subcategory of $\mathscr{C}\text{-afmod}$ consisting of birepresentations whose weak Jordan–Hölder constituents all have apex $\leq_J \mathcal{J}$. Similarly, we have the $1, 2$-full 2-subcategory $\mathscr{C}\text{-cfmod}_{\leq_J \mathcal{J}}$. In the following, we will define various 2-functors, some of

which will be local equivalences. Here, we use the terminology that a pseudofunctor is a *local equivalence* if it induces equivalences on the morphism categories (but is not necessarily essentially surjective on objects).

Theorem 2.34. *Let $\mathscr{C}$ be a multifinitary bicategory and $\mathcal{J}$ a two-sided cell in $\mathscr{C}$. The pullback via the 2-full projection $\mathscr{C} \to \mathscr{C}/\mathcal{I}_{\nleq \mathcal{J}}$ defines a 2-functor*

$$(2.6) \qquad\qquad \mathscr{C}/\mathcal{I}_{\nleq \mathcal{J}}\text{-afmod} \to \mathscr{C}\text{-afmod}_{\leq \mathcal{J}},$$

which is a local equivalence. It can be restricted to a local equivalence

$$(2.7) \qquad\qquad \mathscr{C}/\mathcal{I}_{\nleq \mathcal{J}}\text{-cfmod} \to \mathscr{C}\text{-cfmod}_{\leq \mathcal{J}},$$

and, for any two-sided cell $\mathcal{J}' \leq_J \mathcal{J}$, to biequivalences

$$(2.8) \qquad\qquad \mathscr{C}/\mathcal{I}_{\nleq \mathcal{J}}\text{-tfmod}_{\mathcal{J}'} \xrightarrow{\simeq} \mathscr{C}\text{-tfmod}_{\mathcal{J}'},$$

$$(2.9) \qquad\qquad \mathscr{C}/\mathcal{I}_{\nleq \mathcal{J}}\text{-stmod}_{\mathcal{J}'} \xrightarrow{\simeq} \mathscr{C}\text{-stmod}_{\mathcal{J}'}.$$

The local equivalences (2.6) *and* (2.7) *preserve weak Jordan–Hölder series and, for any two-sided cell $\mathcal{J}' \leq_J \mathcal{J}$, they descend to a local equivalence*

$$(2.10) \qquad\qquad \mathscr{C}/\mathcal{I}_{\nleq \mathcal{J}}\text{-cfmod}_{\mathcal{J}'} \to \mathscr{C}\text{-cfmod}_{\mathcal{J}'}.$$

If $\mathscr{C}$ is quasi multifiab, then (2.6) *is a biequivalence and hence so are* (2.7) *and* (2.10).

Proof. Note that for any $\mathbf{M} \in \mathscr{C}/\mathcal{I}_{\nleq \mathcal{J}}\text{-afmod}$ we have $\mathcal{I}_{\nleq \mathcal{J}} \subseteq \mathrm{ann}(\mathbf{M})$. Then $\mathcal{I}_{\nleq \mathcal{J}}$ is annihilated by all weak Jordan–Hölder constituents of $\mathbf{M}$ which implies that the latter all have apex $\leq_J \mathcal{J}$. Thus the pullback (2.6) is well-defined and obviously a local equivalence which can be restricted to local equivalences (2.7)-(2.9). For any birepresentation $\mathbf{M}$ in $\mathscr{C}\text{-tfmod}_{\mathcal{J}'}$, respectively $\mathscr{C}\text{-stmod}_{\mathcal{J}'}$, we also have $\mathcal{I}_{\nleq \mathcal{J}} \subseteq \mathrm{ann}(\mathbf{M})$ since $\mathrm{apex}(\mathbf{M}) = \mathcal{J}' \leq_J \mathcal{J}$. Thus $\mathbf{M}$ belongs to $\mathscr{C}/\mathcal{I}_{\nleq \mathcal{J}}\text{-tfmod}_{\mathcal{J}'}$, respectively $\mathscr{C}/\mathcal{I}_{\nleq \mathcal{J}}\text{-stmod}_{\mathcal{J}'}$. Therefore both (2.8) and (2.9) are biequivalences. It follows from the definition and biequivalences (2.8)-(2.9) that the local equivalence (2.6), respectively (2.7), preserves weak Jordan–Hölder series and descends to a local equivalences (2.10) for any $\mathcal{J}' \leq_J \mathcal{J}$.

Now assume that $\mathscr{C}$ is quasi multifiab. It suffices to prove that (2.6) is essentially surjective since essential surjectivity of its restrictions (2.7) and (2.10) is straightforward. For any $\mathbf{M} \in \mathscr{C}\text{-afmod}_{\leq \mathcal{J}}$, let $0 = \mathbf{M}_0 \subsetneq \mathbf{M}_1 \subsetneq \cdots \subsetneq \mathbf{M}_m = \mathbf{M}$ be a filtration by subbirepresentations as in Theorem 2.22. Without loss of generality, we assume that $m = 2$. Since $\mathcal{I}_{\nleq \mathcal{J}} \subseteq \mathrm{ann}(\mathbf{M}_1) \cap \mathrm{ann}(\mathbf{M}_2/\mathbf{I}_1)$, where $\mathbf{I}_1$ is the $\mathscr{C}$-stable ideal in $\mathbf{M}_2$ generated by $\mathbf{M}_1$, we obtain $\mathcal{I}_{\nleq \mathcal{J}} \circ_{\mathsf{h}} \mathcal{I}_{\nleq \mathcal{J}} \subseteq \mathrm{ann}(\mathbf{M}_2) = \mathrm{ann}(\mathbf{M})$. Recall that each left cell $\mathcal{L}$ in a quasi multifiab bicategory $\mathscr{C}$ contains the Duflo involution $\mathrm{D} := \mathrm{D}(\mathcal{L})$ and, in fact, by the generalization of [MM1, Proposition 17] to quasi multifiab bicategories, each 1-morphism FD contains F as a direct summand for any $\mathrm{F} \in \mathcal{L}$. Hence, $\mathcal{I}_{\nleq \mathcal{J}} \circ_{\mathsf{h}} \mathcal{I}_{\nleq \mathcal{J}}$ contains id_{F} for all $\mathrm{F} \nleq_J \mathcal{J}$, that is to say, $\mathrm{id}_{\mathrm{F}} \in \mathrm{ann}(\mathbf{M})$ for all $\mathrm{F} \nleq_J \mathcal{J}$. Finally, we have $\mathcal{I}_{\nleq \mathcal{J}} \subseteq \mathrm{ann}(\mathbf{M})$, which completes the proof. $\qquad\square$

By Lemma 2.33, for each $\mathcal{J}' \leq_J \mathcal{J}$, the local equivalence (2.10) can be written as

$$(2.11) \qquad\qquad \mathscr{C}/\mathcal{I}_{\nleq \mathcal{J}}\text{-afmod}_{\mathcal{J}'} \to \mathscr{C}\text{-afmod}_{\mathcal{J}'},$$

which is a biequivalence provided that $\mathscr{C}$ is quasi multifiab.

Definition 2.35. Let $\mathcal{J}$ be a two-sided cell in a multifinitary bicategory $\mathscr{C}$. Then $\mathscr{C}$ is called *$\mathcal{J}$-simple* if any non-zero biideal of $\mathscr{C}$ contains the identity 2-morphisms of all 1-morphisms in $\mathcal{J}$.

By the analog of [MM2, Theorem 15] for multifinitary bicategories, examples of $\mathcal{J}$-simple multifinitary bicategories are not hard to find: for any two-sided cell $\mathcal{J}$ of a multifinitary bicategory $\mathscr{C}$, there is a unique quotient bicategory $\mathscr{C}_{\leq \mathcal{J}}$ that is $\mathcal{J}$-simple and whose two-sided cells correspond exactly to those of $\mathscr{C}/\mathcal{I}_{\nleq \mathcal{J}}$. The bicategory $\mathscr{C}_{\leq \mathcal{J}}$

is called the $\mathcal{J}$-*simple quotient of* $\mathscr{C}$ and is unique up to biequivalence. If $\mathscr{C}$ is (quasi) multifiab, then $\mathscr{C}_{\leq\mathcal{J}}$ is also (quasi) multifiab.

Remark 2.36. Note that $\mathscr{C}/\mathcal{I}_{\nleq\mathcal{J}}$ is, in general, not $\mathcal{J}$-simple. However, the $\mathcal{J}$-simple quotients of $\mathscr{C}$ and $\mathscr{C}/\mathcal{I}_{\nleq\mathcal{J}}$ are biequivalent. By definition, the two-sided cells of $\mathscr{C}_{\leq\mathcal{J}}$ are the same as those of $\mathscr{C}/\mathcal{I}_{\nleq\mathcal{J}}$, but the 2-morphism spaces of the $\mathcal{J}$-simple quotient are smaller in general.

Example 2.37. If $\mathscr{C}$ is semisimple, $\mathscr{C}/\mathcal{I}_{\nleq\mathcal{J}}$ and $\mathscr{C}_{\leq\mathcal{J}}$ coincide.

The above example is special, because in general $\mathscr{C}/\mathcal{I}_{\nleq\mathcal{J}}$ and $\mathscr{C}_{\leq\mathcal{J}}$ do not coincide. To show why, let us give one simple example:

Example 2.38. Let $D \cong \Bbbk[x]/(x^2)$ be the algebra of dual numbers and D-proj the category of complex finite dimensional projective D-modules. Then take $\mathscr{C}$ to be the one-object finitary 2-category of $\Bbbk$-linear endofunctors of D-proj that are isomorphic to direct sums of copies of the identity functor Id. By definition, the 2-morphisms of $\mathscr{C}$ are the natural transformations between those endofunctors. Note that $\mathscr{C}$ has only one two-sided cell $\mathcal{J}$: the one containing only the isomorphism class of Id. Therefore, $\mathscr{C}/\mathcal{I}_{\nleq\mathcal{J}} \cong \mathscr{C}$ and

$$\mathrm{End}_{\mathscr{C}/\mathcal{I}_{\nleq\mathcal{J}}}(\mathrm{Id}) \cong \mathrm{End}_{\mathscr{C}}(\mathrm{Id}) \cong D.$$

However,

$$\mathrm{End}_{\mathscr{C}_{\leq\mathcal{J}}}(\mathrm{Id}) \cong \Bbbk,$$

because (x) is the unique maximal ideal of D and $D/(x) \cong \Bbbk$. Thus, $\mathscr{C}_{\leq\mathcal{J}}$ is a proper quotient of $\mathscr{C}/\mathcal{I}_{\nleq\mathcal{J}}$. Note that $\mathscr{C}_{\leq\mathcal{J}}$ is semisimple in this case, but that need not be true in general.

The pullback

(2.12) $$\mathscr{C}_{\leq\mathcal{J}}\text{-afmod} \to \mathscr{C}/\mathcal{I}_{\nleq\mathcal{J}}\text{-afmod}$$

via the 2-full projection $\mathscr{C}/\mathcal{I}_{\nleq\mathcal{J}} \to \mathscr{C}_{\leq\mathcal{J}}$ is a local equivalence. It is not a biequivalence in general, because not every finitary birepresentation of $\mathscr{C}/\mathcal{I}_{\nleq\mathcal{J}}$ is equivalent to the pullback of a birepresentation of $\mathscr{C}_{\leq\mathcal{J}}$, e.g. the birepresentation defined by the natural action of $\mathscr{C}/\mathcal{I}_{\nleq\mathcal{J}}$ on the additive closure of $\mathcal{J}$ inside $\mathscr{C}/\mathcal{I}_{\nleq\mathcal{J}}$. Restricting (2.12) gives the pullback

(2.13) $$\mathscr{C}_{\leq\mathcal{J}}\text{-cfmod} \to \mathscr{C}/\mathcal{I}_{\nleq\mathcal{J}}\text{-cfmod},$$

which is also a local equivalence and descends to local equivalences

(2.14) $$\mathscr{C}_{\leq\mathcal{J}}\text{-tfmod}_{\mathcal{J}'} \to \mathscr{C}/\mathcal{I}_{\nleq\mathcal{J}}\text{-tfmod}_{\mathcal{J}'},$$

(2.15) $$\mathscr{C}_{\leq\mathcal{J}}\text{-stmod}_{\mathcal{J}'} \to \mathscr{C}/\mathcal{I}_{\nleq\mathcal{J}}\text{-stmod}_{\mathcal{J}'}.$$

for any two-sided cell $\mathcal{J}' \leq_J \mathcal{J}$. The pullbacks (2.12) and (2.13) both preserve weak Jordan–Hölder series and can be restricted to a local equivalence

$$\mathscr{C}_{\leq\mathcal{J}}\text{-afmod}_{\mathcal{J}'} = \mathscr{C}_{\leq\mathcal{J}}\text{-cfmod}_{\mathcal{J}'} \to \mathscr{C}/\mathcal{I}_{\nleq\mathcal{J}}\text{-cfmod}_{\mathcal{J}'} = \mathscr{C}/\mathcal{I}_{\nleq\mathcal{J}}\text{-afmod}_{\mathcal{J}'},$$

for any two-sided cell $\mathcal{J}' \leq_J \mathcal{J}$, where the two equalities hold by Lemma 2.33. Moreover, if $\mathscr{C}$ is quasi multifiab, the local equivalence (2.15) for $\mathcal{J}' = \mathcal{J}$ is a biequivalence, see the proof of Proposition 4.22.

By precomposing (2.6) with (2.12), we obtain the pullback

(2.16) $$\mathscr{C}_{\leq\mathcal{J}}\text{-afmod} \to \mathscr{C}\text{-afmod}_{\leq\mathcal{J}},$$

which is a local equivalence. Similarly, we have the local equivalence

(2.17) $$\mathscr{C}_{\leq\mathcal{J}}\text{-cfmod} \to \mathscr{C}\text{-cfmod}_{\leq\mathcal{J}}.$$

Both (2.16) and (2.17) can be restricted to a series of local equivalences

$$(2.18) \qquad \mathscr{C}_{\leq \mathcal{J}}\text{-tfmod}_{\mathcal{J}'} \to \mathscr{C}\text{-tfmod}_{\mathcal{J}'},$$

$$(2.19) \qquad \mathscr{C}_{\leq \mathcal{J}}\text{-stmod}_{\mathcal{J}'} \to \mathscr{C}\text{-stmod}_{\mathcal{J}'},$$

$$(2.20) \qquad \mathscr{C}_{\leq \mathcal{J}}\text{-afmod}_{\mathcal{J}'} = \mathscr{C}_{\leq \mathcal{J}}\text{-cfmod}_{\mathcal{J}'} \to \mathscr{C}\text{-cfmod}_{\mathcal{J}'} = \mathscr{C}\text{-afmod}_{\mathcal{J}'},$$

for any two-sided cell $\mathcal{J}' \leq_J \mathcal{J}$, where the two equalities appearing in the last local equivalence hold by Lemma 2.33. Moreover, if $\mathscr{C}$ is quasi multifiab, the local equivalence (2.19) for $\mathcal{J}' = \mathcal{J}$ is a biequivalence, cf. Proposition 4.22.

Denote by $\mathscr{C}_{(\mathcal{J})}$ be the 2-full subcategory of $\mathscr{C}/\mathcal{I}_{\nleq \mathcal{J}}$ whose objects are all $\mathtt{i}_{s(F)}, \mathtt{i}_{t(F)}$ for $F \in \mathcal{J}$, and whose morphism categories are given by

$$\bigoplus_{\mathtt{i},\mathtt{j} \in \mathscr{C}_{(\mathcal{J})}} \mathscr{C}_{(\mathcal{J})}(\mathtt{i},\mathtt{j}) := \mathrm{add}\{F, \mathbb{1}_{\mathtt{i}} \mid F \in \mathcal{J}, \mathtt{i} \in \mathscr{C}_{(\mathcal{J})}\}.$$

Define the 2-full subcategory $\mathscr{C}_{\mathcal{J}}$ of $\mathscr{C}_{\leq \mathcal{J}}$ similarly. By definition, $\mathcal{J}$ is the only two-sided cell of $\mathscr{C}_{(\mathcal{J})}$ and $\mathscr{C}_{\mathcal{J}}$ not necessarily consisting only of identity 1-morphisms. If $\mathscr{C}$ is (quasi) multifiab, then $\mathscr{C}_{(\mathcal{J})}$ and $\mathscr{C}_{\mathcal{J}}$ are also (quasi) multifiab.

Lemma 2.39. *Suppose that $\mathscr{C}$ is multifinitary. Then $\mathscr{C}_{\mathcal{J}}$ is $\mathcal{J}$-simple. Moreover, $\mathscr{C}_{\mathcal{J}}$ is the $\mathcal{J}$-simple quotient of $\mathscr{C}_{(\mathcal{J})}$.*

Proof. This follows from $\mathcal{J}$-simplicity of $\mathscr{C}_{\leq \mathcal{J}}$ and the fact that the unique maximal biideal of $\mathscr{C}_{(\mathcal{J})}$ not containing identities on 1-morphisms in $\mathcal{J}$ is the restriction of the analogous biideal of $\mathscr{C}/\mathcal{I}_{\nleq \mathcal{J}}$. □

Pulling back via the 2-fully faithful embedding $\mathscr{C}_{\mathcal{J}} \to \mathscr{C}_{\leq \mathcal{J}}$ yields a 2-functor

$$(2.21) \qquad \mathscr{C}_{\leq \mathcal{J}}\text{-afmod} \to \mathscr{C}_{\mathcal{J}}\text{-afmod},$$

which can be restricted to 2-functors

$$(2.22) \qquad \mathscr{C}_{\leq \mathcal{J}}\text{-tfmod}_{\mathcal{J}} \to \mathscr{C}_{\mathcal{J}}\text{-tfmod}_{\mathcal{J}},$$

$$(2.23) \qquad \mathscr{C}_{\leq \mathcal{J}}\text{-stmod}_{\mathcal{J}} \to \mathscr{C}_{\mathcal{J}}\text{-stmod}_{\mathcal{J}}.$$

Indeed, for a birepresentation $\mathbf{M} \in \mathscr{C}_{\leq \mathcal{J}}\text{-tfmod}_{\mathcal{J}}$, its underlying category is equal to $\mathrm{add}\{\mathbf{M}(F)X \mid F \in \mathcal{J}\}$ for any non-zero object X in $\mathbf{M}(\mathtt{i})$ for some $\mathtt{i}$. Thus the 2-functor (2.22) is well-defined. Since $\mathcal{J}$ is the unique maximal two-sided cell in both $\mathscr{C}_{\leq \mathcal{J}}$ and $\mathscr{C}_{\mathcal{J}}$, any proper $\mathscr{C}_{\mathcal{J}}$-stable ideal of $\mathbf{M} \in \mathscr{C}_{\leq \mathcal{J}}\text{-tmod}_{\mathcal{J}}$ is $\mathscr{C}_{\leq \mathcal{J}}$-stable as well, which implies that (2.23) is also well-defined. Since the 2-functor (2.21) preserves weak Jordan–Hölder series, it restricts to a 2-functor

$$(2.24) \qquad \mathscr{C}_{\leq \mathcal{J}}\text{-afmod}_{\mathcal{J}} \to \mathscr{C}_{\mathcal{J}}\text{-afmod}_{\mathcal{J}}.$$

In Theorem 4.28 (cf. also Remark 4.34), provided that $\mathscr{C}$ is quasi multifiab, we show that (2.23) is a biequivalence, which can be viewed as the restriction of (2.24). The latter is a local equivalence by Theorem 4.33. If $\mathscr{C}$ is quasi multifiab, composing the biequivalences in (2.19) for $\mathcal{J}' = \mathcal{J}$ and (2.23) yields a biequivalence

$$\mathscr{C}_{\mathcal{J}}\text{-stmod}_{\mathcal{J}} \to \mathscr{C}\text{-stmod}_{\mathcal{J}},$$

see Theorem 4.28 for details. Recall that a diagonal $\mathcal{H}$-cell of a multifiab bicategory is the intersection of a left cell and its dual. Of crucial importance for the birepresentation theory of $\mathscr{C}$, cf. Theorems 4.30 and 4.31, is the following.

Definition 2.40. Suppose that $\mathscr{C}$ is multifiab and let $\mathcal{J}$ be a two-sided cell of $\mathscr{C}$ and $\mathcal{H} \subseteq \mathcal{J}$ a diagonal $\mathcal{H}$-cell with source $\mathtt{i} \in \mathscr{C}$. Define the 2-full subcategory $\mathscr{C}_{(\mathcal{H})}$ of $\mathscr{C}_{(\mathcal{J})}$ with single object $\mathtt{i}$ and

$$\mathscr{C}_{(\mathcal{H})}(\mathtt{i},\mathtt{i}) := \mathrm{add}\{F, \mathbb{1}_{\mathtt{i}} \mid F \in \mathcal{H}\}.$$

Define the 2-full subcategory $\mathscr{C}_{\mathcal{H}}$ of $\mathscr{C}_{\mathcal{J}}$ similarly.

If $\mathbb{1}_i$ does not belong to $\mathcal{H}$, then $\mathscr{C}_{(\mathcal{H})}$ and $\mathscr{C}_{\mathcal{H}}$ have precisely two cells, which are both left, right and two-sided: the trivial cell $\{\mathbb{1}_i\}$ and the non-trivial cell $\mathcal{H}$. Note that both $\mathscr{C}_{(\mathcal{H})}$ and $\mathscr{C}_{\mathcal{H}}$ are multifiab, because $\mathcal{H}$ is preserved by $\star$ when $\mathscr{C}$ is multifiab.

Lemma 2.41. *The bicategory $\mathscr{C}_{\mathcal{H}}$ is $\mathcal{H}$-simple. Moreover, it is the $\mathcal{H}$-simple quotient of $\mathscr{C}_{(\mathcal{H})}$.*

Proof. Consider the cell birepresentation $\mathbf{C}_{\mathcal{L}}$ of $\mathscr{C}_{\mathcal{J}}$, where $\mathcal{H} = \mathcal{H}(\mathcal{L})$, and note that it is 2-faithful by $\mathcal{J}$-simplicity of $\mathscr{C}_{\mathcal{J}}$. By the generalization of [KMMZ, Theorem 2] to bicategories, the action of each $\mathrm{F} \in \mathcal{H}$ is represented via $\mathbf{C}_{\mathcal{L}}$ by a projective bimodule over the underlying algebra of $\mathbf{C}_{\mathcal{L}}$. Let $\mathrm{D} = \mathrm{D}(\mathcal{L})$ be the Duflo involution in $\mathcal{L}$, which also belongs to $\mathcal{H}$ (see Subsection 2.5). Then, by the generalization of [MM1, Lemma 12] to bicategories, D does not annihilate any simples indexed by elements of $\mathcal{H}$ in the (projective) abelianization of $\mathbf{C}_{\mathcal{L}}$. Therefore, given $\mathrm{F}, \mathrm{G} \in \mathcal{H}$ and a non-zero $\alpha\colon \mathrm{F} \to \mathrm{G}$ in $\mathscr{C}_{\mathcal{H}}$, the 2-morphism

$$(\mathrm{id}_\mathrm{D} \circ_\mathsf{h} \alpha) \circ_\mathsf{h} \mathrm{id}_\mathrm{D}\colon (\mathrm{DF})\mathrm{D} \to (\mathrm{DG})\mathrm{D}$$

is non-zero. As $\mathbf{C}_{\mathcal{L}}(\mathrm{D})$ is a projective bimodule, the morphism $\mathbf{C}_{\mathcal{L}}\big(\mathrm{id}_\mathrm{D} \circ_\mathsf{h} \alpha \circ_\mathsf{h} \mathrm{id}_\mathrm{D}\big)$ is not a radical morphism in the category of bimodules. Thus, $(\mathrm{id}_\mathrm{D} \circ_\mathsf{h} \alpha) \circ_\mathsf{h} \mathrm{id}_\mathrm{D}$ contains, as a direct summand, an isomorphism from some non-zero summand of $(\mathrm{DF})\mathrm{D}$ to some summand of $(\mathrm{DG})\mathrm{D}$. Therefore, $\mathscr{C}_{\mathcal{H}}$ is $\mathcal{H}$-simple and the second claim follows by definition. $\square$

In Theorem 4.32, we will show that there is a biequivalence between $\mathscr{C}\text{-stmod}_{\mathcal{J}}$ and $\mathscr{C}_{\mathcal{H}}\text{-stmod}_{\mathcal{H}}$.

3. Coalgebras and comodules in bicategories

In this section, let $\mathscr{C}$ be a multifinitary bicategory.

3.1. **Coalgebras and comodules.** The following definitions are analogs of those in [EGNO, Section 7.8].

Definition 3.1. A *coalgebra* $\mathrm{C} = (\mathrm{C}, \delta_\mathrm{C}, \epsilon_\mathrm{C})$ in $\mathscr{C}$ consists of a 1-morphism $\mathrm{C} \in \mathscr{C}(\mathtt{i}, \mathtt{i})$, for some object $\mathtt{i} \in \mathscr{C}$, a comultiplication 2-morphism $\delta_\mathrm{C}\colon \mathrm{C} \to \mathrm{CC}$ and a counit 2-morphism $\epsilon_\mathrm{C}\colon \mathrm{C} \to \mathbb{1}_\mathtt{i}$. These should satisfy the usual coassociativity and counitality axioms of a coalgebra, i.e. the diagrams

$$
\begin{array}{ccccc}
\mathrm{CC} & \xleftarrow{\;\delta_\mathrm{C}\;} & \mathrm{C} & \xrightarrow{\;\delta_\mathrm{C}\;} & \mathrm{CC} \\
{\scriptstyle \delta_\mathrm{C}\circ_\mathsf{h}\mathrm{id}_\mathrm{C}}\downarrow & & & & \downarrow{\scriptstyle \mathrm{id}_\mathrm{C}\circ_\mathsf{h}\delta_\mathrm{C}} \\
(\mathrm{CC})\mathrm{C} & & \xrightarrow{\;\alpha_{\mathrm{C},\mathrm{C},\mathrm{C}}\;} & & \mathrm{C}(\mathrm{CC})
\end{array}
,
$$

$$
\begin{array}{ccccccc}
\mathrm{C} & =\!\!=\!\!= & \mathrm{C} & & \mathrm{C} & =\!\!=\!\!= & \mathrm{C} \\
{\scriptstyle \delta_\mathrm{C}}\downarrow & & \uparrow{\scriptstyle v_\mathrm{C}^l} & & {\scriptstyle \delta_\mathrm{C}}\downarrow & & \uparrow{\scriptstyle v_\mathrm{C}^r} \\
\mathrm{CC} & \xrightarrow[{\scriptstyle \epsilon_\mathrm{C}\circ_\mathsf{h}\mathrm{id}_\mathrm{C}}]{} & \mathbb{1}_\mathtt{i}\mathrm{C} & & \mathrm{CC} & \xrightarrow[{\scriptstyle \mathrm{id}_\mathrm{C}\circ_\mathsf{h}\epsilon_\mathrm{C}}]{} & \mathrm{C}\mathbb{1}_\mathtt{i}
\end{array}
,
$$

should commute.

Example 3.2.
  (i) The identity 1-morphism $\mathbb{1}_\mathtt{i}$, for any $\mathtt{i} \in \mathscr{C}$, is naturally a coalgebra.
  (ii) In finite dihedral type, there is an explicit construction of coalgebras in $\mathscr{S}$ corresponding to ADE Dynkin diagrams [MMMT1, Section 7].

Definition 3.3. Let $C, D \in \mathscr{C}(\mathtt{i}, \mathtt{i})$ be coalgebras in $\mathscr{C}$. A *homomorphism of coalgebras* in $\mathscr{C}$ is a 2-morphism $\phi \colon C \to D$ such that the diagrams

$$
\begin{array}{ccc}
C & \xrightarrow{\ \phi\ } & D \\
{\scriptstyle \delta_C}\downarrow & & \downarrow{\scriptstyle \delta_D} \\
CC & \xrightarrow[\phi \circ_h \phi]{} & DD
\end{array}
\quad , \qquad
\begin{array}{ccc}
C & \xrightarrow{\quad \phi \quad} & D \\
& {\scriptstyle \epsilon_C}\searrow \quad \swarrow{\scriptstyle \epsilon_D} & \\
& \mathbb{1}_{\mathtt{i}} &
\end{array}
$$

commute.

The coalgebras C and D are *isomorphic* if there exists an invertible homomorphism between them.

Next, let us recall the definitions of left, right and bicomodules in $\mathscr{C}$.

Definition 3.4. Let $C \in \mathscr{C}(\mathtt{i}, \mathtt{i})$ be a coalgebra in $\mathscr{C}$. A *left C-comodule* $M = (M, \delta_{C,M})$ in $\mathscr{C}$ consists of a 1-morphism $M \in \mathscr{C}(\mathtt{j}, \mathtt{i})$, for some object $\mathtt{j} \in \mathscr{C}$, and a 2-morphism $\delta_{C,M} \colon M \to CM$, called the *left coaction*, such that the diagrams

$$
\begin{array}{ccc}
CM \xleftarrow{\ \delta_{C,M}\ } M \xrightarrow{\ \delta_{C,M}\ } CM \\
{\scriptstyle \delta_C \circ_h \mathrm{id}_M}\downarrow \qquad\qquad\qquad \downarrow{\scriptstyle \mathrm{id}_C \circ_h \delta_{C,M}} \\
(CC)M \xrightarrow[\ \alpha_{C,C,M}\ ]{} C(CM)
\end{array}
\quad , \qquad
\begin{array}{ccc}
M & =\!=\!=\!=\!= & M \\
{\scriptstyle \delta_{C,M}}\downarrow & & \uparrow{\scriptstyle v_M^l} \\
CM & \xrightarrow[\epsilon_C \circ_h \mathrm{id}_M]{} & \mathbb{1}_{\mathtt{i}} M
\end{array}
$$

commute. The definition of a *right C-comodule* $M = (M, \delta_{M,C})$ in $\mathscr{C}$ is similar.

Definition 3.5. For coalgebras $C \in \mathscr{C}(\mathtt{i}, \mathtt{i})$ and $D \in \mathscr{C}(\mathtt{j}, \mathtt{j})$ in $\mathscr{C}$, a *C-D-bicomodule* $M = (M, \delta_{C,M}, \delta_{M,D})$ in $\mathscr{C}$ is a left C- and a right D-comodule in $\mathscr{C}(\mathtt{j}, \mathtt{i})$ such that the diagram

$$
\begin{array}{ccc}
MD \xleftarrow{\ \delta_{M,D}\ } M \xrightarrow{\ \delta_{C,M}\ } CM \\
{\scriptstyle \delta_{C,M} \circ_h \mathrm{id}_D}\downarrow \qquad\qquad\qquad \downarrow{\scriptstyle \mathrm{id}_C \circ_h \delta_{M,D}} \\
(CM)D \xrightarrow[\ \alpha_{C,M,D}\ ]{} C(MD)
\end{array}
$$

commutes.

The definitions of *homomorphisms* of left, right and bicomodules should now be clear and are omitted for brevity.

Remark 3.6. There are, of course, also the dual notions of *algebras* and *modules* in $\mathscr{C}$. Their definition can be obtained from the above by inverting all arrows.

Coalgebras, comodules, bicomodules and the respective homomorphisms in the injective abelianization $\underline{\mathscr{C}}$ or in the projective abelianization $\overline{\mathscr{C}}$ are defined just as in $\mathscr{C}$.

We say that a coalgebra D is a *subcoalgebra* of another coalgebra C if there is a monic 2-morphism $\phi \colon D \to C$ that is a homomorphism of coalgebras. A subcoalgebra of C is called *proper* if it is neither zero nor isomorphic to C. A coalgebra C is *cosimple* if it has no proper subcoalgebras.

Example 3.7. A cosimple coalgebra in $\mathscr{V}\mathrm{ect}$ is a cosimple coalgebra in the usual sense.

Example 3.8. Let $\mathbb{1}$ and s denote the two simple 1-morphisms in $\mathscr{V}\mathrm{ect}(\mathbb{Z}/2\mathbb{Z})$. Then $C = \mathbb{1} \oplus s$ has an essentially unique structure of a cosimple coalgebra in $\mathscr{V}\mathrm{ect}(\mathbb{Z}/2\mathbb{Z})$. The forgetful functor $\mathscr{V}\mathrm{ect}(\mathbb{Z}/2\mathbb{Z}) \to \mathscr{V}\mathrm{ect}$ is monoidal, so C is also a coalgebra in $\mathscr{V}\mathrm{ect}$. However, it is not cosimple in $\mathscr{V}\mathrm{ect}$, as $\mathbb{1}$ and s are mapped to isomorphic 1-morphisms by the forgetful functor.

3.2. **Cotensor product of bicomodules.** Let us briefly review the cotensor product of bicomodules over a coalgebra in $\mathscr{C}$ (or in any bicategory having equalizers), see also e.g. [MMMZ, Subsection 3.3].

Let M be a right C-comodule and N a left C-comodule in $\mathscr{C}$. The *cotensor product* of M and N over C, denoted by $M\square_C N$, is by definition the equalizer of the diagram

$$MN \xrightarrow[\;(\alpha_{M,C,N})\circ_v(\delta_{M,C}\mathrm{id}_N)\;]{\mathrm{id}_M\delta_{C,N}} M(CN) \quad\rightsquigarrow\quad \begin{array}{c} MN \\ {\scriptstyle\delta_{M,C}\mathrm{id}_N}\swarrow \qquad \searrow{\scriptstyle\mathrm{id}_M\delta_{C,N}} \\ (MC)N \xrightarrow[\;\alpha_{M,C,N}\;]{} M(CN) \end{array} \quad.$$

Due to coassociativity of the right coaction, $\delta_{M,C}$ induces a left comodule isomorphism $\delta_{M,C}\colon M \xrightarrow{\cong} M\square_C C$, see [MMMZ, Lemma 5] for this statement in the strict setting. Similarly, $\delta_{C,N}$ induces a right comodule isomorphism $\delta_{C,N}\colon N \xrightarrow{\cong} C\square_C N$. Furthermore, the associator in $\mathscr{C}$ induces an associator for the cotensor product.

Lemma 3.9. *Suppose that* K *is a right* C-*comodule,* M *a* C-D-*bicomodule and* N *a left* D-*comodule, all in* $\mathscr{C}$. *Then* $\alpha_{K,M,N}$ *induces a natural 2-isomorphism, for which we use the same notation,*

$$(3.1) \qquad \alpha_{K,M,N}\colon (K\square_C M)\square_D N \xrightarrow{\cong} K\square_C(M\square_D N).$$

Proof. First we claim that $\alpha_{K,M,N}$ induces two intermediate natural 2-isomorphisms, for which we also use the same notation,

$$(3.2)\quad \alpha_{K,M,N}\colon (K\square_C M)N \xrightarrow{\cong} K\square_C(MN), \quad \alpha_{K,M,N}\colon (KM)\square_D N \xrightarrow{\cong} K(M\square_D N).$$

We only prove the existence of the first one, which is the one we need below. The existence of the second one can be proved analogously. Consider the following diagram.

The vertical faces commute: the faces labeled 1 by naturality of the associator, the one labeled 2 by the pentagon coherence condition for the associator, and the triangle labeled 3 by definition of $\delta_{C,MN}$. Since all the vertical maps are isomorphisms, this implies that $\alpha_{K,M,N}$ induces a 2-isomorphism between the equalizer of the top triangle, which is $(K\square_C M)N$ since right composition with N is left exact, and the equalizer of the bottom triangle, which is $K\square_C(MN)$. This proves the existence of the first natural

2-isomorphism in (3.2). Next, consider

$$
\begin{array}{ccc}
(K\square_C M)N & \xrightarrow{\ \delta_{K\square_C M,D}\mathrm{id}_N\ } & ((K\square_C M)D)N
\end{array}
$$

Again, all vertical faces commute: the left and back quadrilaterals labeled 1 by naturality of the induced associator, the right pentagon labeled 2 by the pentagon coherence condition for the induced associator, and the triangle labeled 3 by definition of $\delta_{K\square_C M,D}$. Recall that equalizers are unique up to isomorphisms, which implies that the induced associators in (3.2) are also natural and satisfy the pentagon coherence condition. Again, the vertical maps are 2-isomorphisms, so $\alpha_{K,M,N}$ induces a 2-isomorphism between the equalizer of the top triangle, which is $(K\square_C M)\square_D N$, and the equalizer of the bottom triangle, which is $K\square_C(M\square_D N)$. $\qquad\square$

Corollary 3.10. *Coalgebras, bicomodules and bicomodule homomorphisms in $\underline{\mathscr{C}}$ form a bicategory, in which horizontal composition is given by the cotensor product.*

We will denote the bicategory of coalgebras, bicomodules and bicomodule homomorphisms in $\underline{\mathscr{C}}$ by $\mathscr{B}\mathrm{icom}_{\underline{\mathscr{C}}}$.

3.3. **Coalgebras and bicomodules under pseudofunctors.** Let $\Phi\colon \mathscr{C}\to\mathscr{D}$ be a $\Bbbk$-linear pseudofunctor between two multifinitary bicategories with structural 2-isomorphisms

$$
\phi_{F,G}\colon \Phi(FG)\xrightarrow{\cong}\Phi(F)\Phi(G),\quad \phi_{\mathtt{i}}\colon \Phi(\mathbb{1}_{\mathtt{i}})\xrightarrow{\cong}\mathbb{1}_{\Phi(\mathtt{i})}.
$$

Denote by $\underline{\Phi}\colon \underline{\mathscr{C}}\to\underline{\mathscr{D}}$ its extension to the abelianizations, which is left exact by definition.

Lemma 3.11. *The $\Bbbk$-linear pseudofunctor $\underline{\Phi}$ induces a $\Bbbk$-linear pseudofunctor, for which we use the same notation, $\underline{\Phi}\colon \mathscr{B}\mathrm{icom}_{\underline{\mathscr{C}}}\to\mathscr{B}\mathrm{icom}_{\underline{\mathscr{D}}}$.*

Proof. The proof consists of five parts, the first four of which are straightforward:

(i) If $C=(C,\delta_C,\epsilon_C)$ is a coalgebra in $\underline{\mathscr{C}}(\mathtt{i},\mathtt{i})$, then the 1-morphism $\underline{\Phi}(C)$ is a coalgebra in $\underline{\mathscr{D}}(\underline{\Phi}(\mathtt{i}),\underline{\Phi}(\mathtt{i}))$, with comultiplication and counit

$$
\delta_{\underline{\Phi}(C)}:=\big[\underline{\Phi}(C)\xrightarrow{\underline{\Phi}(\delta_C)}\underline{\Phi}(CC)\xrightarrow{\phi_{C,C}}\underline{\Phi}(C)\underline{\Phi}(C)\big],
$$

$$
\epsilon_{\underline{\Phi}(C)}:=\big[\underline{\Phi}(C)\xrightarrow{\underline{\Phi}(\epsilon_C)}\underline{\Phi}(\mathbb{1}_{\mathtt{i}})\xrightarrow{\phi_{\mathtt{i}}}\mathbb{1}_{\underline{\Phi}(\mathtt{i})}\big].
$$

(ii) If $M=(M,\delta_{C,M})$ is a left C-comodule in $\underline{\mathscr{C}}$, then $\underline{\Phi}(M)$ is a left $\underline{\Phi}(C)$-comodule in $\underline{\mathscr{D}}$ with left coaction

$$
\delta_{\underline{\Phi}(C),\underline{\Phi}(M)}:=[\underline{\Phi}(M)\xrightarrow{\underline{\Phi}(\delta_{C,M})}\underline{\Phi}(CM)\xrightarrow{\phi_{C,M}}\underline{\Phi}(C)\underline{\Phi}(M)].
$$

(iii) If $M = (M, \delta_{M,C})$ is a right C-comodule in $\underline{\mathscr{C}}$, then $\underline{\Phi}(M)$ is a right $\underline{\Phi}(C)$-comodule in $\underline{\mathscr{D}}$ with right coaction

$$\delta_{\underline{\Phi}(M),\underline{\Phi}(C)} := \left[\underline{\Phi}(M) \xrightarrow{\underline{\Phi}(\delta_{M,C})} \underline{\Phi}(MC) \xrightarrow{\phi_{M,C}} \underline{\Phi}(M)\underline{\Phi}(C) \right].$$

(iv) If C and D are two coalgebras and M is a C-D-bicomodule in $\underline{\mathscr{C}}$, then $\underline{\Phi}(M)$ is a $\underline{\Phi}(C)$-$\underline{\Phi}(D)$ bicomodule in $\underline{\mathscr{D}}$ with bicoactions defined by the previous two points.

(v) Let C be a coalgebra in $\underline{\mathscr{C}}$. If M is a right C-comodule and N is a left C-comodule in $\underline{\mathscr{C}}$, then there is a 2-isomorphism

$$\underline{\Phi}(M\square_C N) \xrightarrow{\cong} \underline{\Phi}(M)\square_{\underline{\Phi}(C)}\underline{\Phi}(N)$$

in $\underline{\mathscr{D}}$. To prove this claim, consider the following diagram, where we distinguish the associators in the abelianizations of $\mathscr{C}$ and $\mathscr{D}$ by superscripts:

As before, all lateral faces commute due to naturality and the coherence condition of $\phi_{\_,\_}$, as well as the definition of $\delta_{\underline{\Phi}(M),\underline{\Phi}(C)}$ and $\delta_{\underline{\Phi}(C),\underline{\Phi}(N)}$. Since all vertical arrows are 2-isomorphisms, this implies that the equalizer of the top triangle, which is $\underline{\Phi}(M\square_C N)$ by left exactness of $\underline{\Phi}$, is isomorphic to the equalizer of the bottom triangle, which is $\underline{\Phi}(M)\square_{\underline{\Phi}(C)}\underline{\Phi}(N)$. $\qquad\square$

4. Coalgebras and birepresentations

As before, throughout this section $\mathscr{C}$ is assumed to be a multifinitary bicategory. We will recall how finitary birepresentations of $\mathscr{C}$ give rise to coalgebras in $\underline{\mathscr{C}}$ and vice versa. This is a bicategorical version of [MMMT1], which in turn was inspired by [Os].

4.1. **The finitary birepresentation associated to a coalgebra.** Let $C \in \underline{\mathscr{C}}(i,i)$ be a coalgebra. For every $j \in \mathscr{C}$, take $\mathrm{comod}_{\underline{\mathscr{C}}}(C)_j$ to be the category of right C-comodules and comodule homomorphisms in $\underline{\mathscr{C}}(i,j)$. Then there is a birepresentation of $\underline{\mathscr{C}}$ which assigns

- the category $\mathrm{comod}_{\underline{\mathscr{C}}}(C)_j$ to an object $j \in \underline{\mathscr{C}}$;
- the functor

$$X_{kj} \circ_h - : \mathrm{comod}_{\underline{\mathscr{C}}}(C)_j \to \mathrm{comod}_{\underline{\mathscr{C}}}(C)_k$$

to a 1-morphism X_{kj} in $\underline{\mathscr{C}}(j,k)$, for $j,k \in \underline{\mathscr{C}}$;

- the natural transformation

$$\beta_{\mathtt{kj}} \circ_{\mathtt{h}} - : X_{\mathtt{kj}} \circ_{\mathtt{h}} - \to Y_{\mathtt{kj}} \circ_{\mathtt{h}} -$$

to a 2-morphism $\beta_{\mathtt{kj}} \colon X_{\mathtt{kj}} \to Y_{\mathtt{kj}}$ in $\underline{\mathscr{C}}(\mathtt{j},\mathtt{k})$, for $\mathtt{j},\mathtt{k} \in \underline{\mathscr{C}}$;
- the natural isomorphism

$$
\begin{array}{ccc}
 & \overset{\mathbb{1}_{\mathtt{j}} \circ_{\mathtt{h}} -}{\longrightarrow} & \\
\mathrm{comod}_{\underline{\mathscr{C}}}(\mathrm{C})_{\mathtt{j}} & \Downarrow \iota_{\mathtt{j}} & \mathrm{comod}_{\underline{\mathscr{C}}}(\mathrm{C})_{\mathtt{j}} \\
 & \underset{\mathrm{Id}_{\mathrm{comod}_{\underline{\mathscr{C}}}(\mathrm{C})_{\mathtt{j}}}}{\longrightarrow} &
\end{array}
$$

to each object $\mathtt{j} \in \underline{\mathscr{C}}$, where $\iota_{\mathtt{j}}$ is the natural transformation induced by the left unitor υ^l_- in $\mathscr{C}$;
- the natural isomorphism

$$
\begin{array}{ccc}
\underline{\mathscr{C}}(\mathtt{k},\mathtt{l}) \boxtimes \underline{\mathscr{C}}(\mathtt{j},\mathtt{k}) \boxtimes \mathrm{comod}_{\underline{\mathscr{C}}}(\mathrm{C})_{\mathtt{j}} & \xrightarrow{\circ_{\mathtt{h}} \boxtimes \mathrm{Id}} & \underline{\mathscr{C}}(\mathtt{j},\mathtt{l}) \boxtimes \mathrm{comod}_{\underline{\mathscr{C}}}(\mathrm{C})_{\mathtt{j}} \\
{\scriptstyle \mathrm{Id} \boxtimes \circ_{\mathtt{h}}} \Big\downarrow & {\scriptstyle \mu_{\mathtt{lkj}} := \alpha^{-1}_{\mathtt{lkji}}} \nearrow & \Big\downarrow {\scriptstyle \circ_{\mathtt{h}}} \\
\underline{\mathscr{C}}(\mathtt{k},\mathtt{l}) \boxtimes \mathrm{comod}_{\underline{\mathscr{C}}}(\mathrm{C})_{\mathtt{k}} & \xrightarrow{\circ_{\mathtt{h}}} & \mathrm{comod}_{\underline{\mathscr{C}}}(\mathrm{C})_{\mathtt{l}}
\end{array}
$$

to each triple of objects $\mathtt{j},\mathtt{k},\mathtt{l} \in \underline{\mathscr{C}}$.

The *underlying category* of this birepresentation is defined as

$$\mathrm{comod}_{\underline{\mathscr{C}}}(\mathrm{C}) := \bigoplus_{\mathtt{j} \in \mathscr{C}} \mathrm{comod}_{\underline{\mathscr{C}}}(\mathrm{C})_{\mathtt{j}}.$$

This birepresentation of $\underline{\mathscr{C}}$ restricts to a finitary birepresentation of $\mathscr{C}$ sending each $\mathtt{j} \in \mathscr{C}$ to $\mathrm{inj}_{\underline{\mathscr{C}}}(\mathrm{C})_{\mathtt{j}}$, the full subcategory of injective objects in $\mathrm{comod}_{\underline{\mathscr{C}}}(\mathrm{C})_{\mathtt{j}}$. The underlying category of this restriction is defined as

$$\mathrm{inj}_{\underline{\mathscr{C}}}(\mathrm{C}) := \bigoplus_{\mathtt{j} \in \mathscr{C}} \mathrm{inj}_{\underline{\mathscr{C}}}(\mathrm{C})_{\mathtt{j}}.$$

We will use the notation $\mathbf{comod}_{\underline{\mathscr{C}}}(\mathrm{C})$ and $\mathbf{inj}_{\underline{\mathscr{C}}}(\mathrm{C})$ for these two birepresentations, respectively.

We record a useful fact describing objects in finitary birepresentations:

Lemma 4.1. *If $\mathscr{C}$ is quasi multifiab, then, for a coalgebra C in $\underline{\mathscr{C}}(\mathtt{i},\mathtt{i})$, the category $\mathrm{inj}_{\underline{\mathscr{C}}}(\mathrm{C})_{\mathtt{j}}$ is the additive closure of $\{\mathrm{GC} \mid \mathrm{G} \in \mathscr{C}(\mathtt{i},\mathtt{j})\}$ inside $\mathrm{comod}_{\underline{\mathscr{C}}}(\mathrm{C})_{\mathtt{j}}$.*

Proof. First note that, since C is an injective C-comodule and $\mathscr{C}$ is multifiab, GC is an injective C-comodule for any 1-morphism G in $\mathscr{C}$.

Any $\mathrm{X} \in \mathrm{comod}_{\underline{\mathscr{C}}}(\mathrm{C})_{\mathtt{j}}$ embeds into XC, due to counitality and the coaction being a comodule morphism (note that we are not claiming that this embedding is split in $\mathrm{comod}_{\underline{\mathscr{C}}}(\mathrm{C})_{\mathtt{j}}$). Suppose that $\mathrm{X} \in \mathrm{inj}_{\underline{\mathscr{C}}}(\mathrm{C})_{\mathtt{j}}$ and that $\mathrm{X}_0 \to \mathrm{X}_1$ is an injective presentation of X in $\underline{\mathscr{C}}$, where $\mathrm{X}_0, \mathrm{X}_1$ are 1-morphisms in $\mathscr{C}$. Then XC embeds further into $\mathrm{X}_0\mathrm{C}$, which is injective. The claim follows. $\qquad\square$

We call a morphism of finitary birepresentations *exact* if it extends to an exact morphism of the corresponding abelianized birepresentations, i.e. its component functors extend to exact functors between the (injective) abelianizations of the component categories. We say that a functor between additive categories is an *injective functor* if it is injective in the category of functors between the injective abelianizations. We call a morphism of finitary birepresentations *injective* if its extension to the corresponding abelianized birepresentations is given by injective functors.

Lemma 4.2. *Assume that $\mathscr{C}$ is quasi multifiab, let C and D be two coalgebras and M a biinjective C-D-bicomodule in $\underline{\mathscr{C}}$. Cotensoring defines an exact morphism of birepresentations of $\mathscr{C}$*

$$\_\square_{\mathrm{C}}\mathrm{M}\colon \mathbf{comod}_{\mathscr{C}}(\mathrm{C}) \to \mathbf{comod}_{\mathscr{C}}(\mathrm{D})$$

which sends injective objects in the underlying categories to injective objects. In particular, it restricts to a morphism of birepresentations of $\mathscr{C}$

$$\_\square_{\mathrm{C}}\mathrm{M}\colon \mathbf{inj}_{\mathscr{C}}(\mathrm{C}) \to \mathbf{inj}_{\mathscr{C}}(\mathrm{D}).$$

Proof. Since M is injective as a left C-comodule, it is a direct summand of CF for some 1-morphism F in $\mathscr{C}$, in view of Lemma 4.1. The cotensor functor is therefore a direct summand of right multiplication by F, which is exact, so exactness of $\_\square_{\mathrm{C}}\mathrm{M}$ follows.

Similarly, if N is an injective right C-comodule, it is a direct summand of GC for some 1-morphism G in $\mathscr{C}$. Moreover, M being injective as a right D-comodule, it is a direct summand of HD for some 1-morphism H in $\mathscr{C}$. Thus $\mathrm{N}\square_{\mathrm{C}}\mathrm{M}$ is a direct summand of $\mathrm{GC}\square_{\mathrm{C}}\mathrm{HD} \cong \mathrm{G}(\mathrm{HD}) \cong (\mathrm{GH})\mathrm{D}$ which is an injective right D-comodule, so $\mathrm{N}\square_{\mathrm{C}}\mathrm{M}$ is itself injective as a right D-comodule. This completes the proof. □

Finally, note that if $f\colon \mathrm{M} \to \mathrm{N}$ is a homomorphism between two biinjective C-D-bicomodules M, N in $\underline{\mathscr{C}}$, then

$$\_\square_{\mathrm{C}}f\colon \_\square_{\mathrm{C}}\mathrm{M} \to \_\square_{\mathrm{C}}\mathrm{N}$$

defines a modification.

4.2. **Morita–Takeuchi theory in bicategories.**
We start by discussing the notion of Morita–Takeuchi equivalence in finitary bicategories (MT equivalence for short).

Definition 4.3. We say that two coalgebras C and D in $\mathscr{C}$ are *MT equivalent* if

$$\mathbf{inj}_{\underline{\mathscr{C}}}(\mathrm{C}) \simeq \mathbf{inj}_{\underline{\mathscr{C}}}(\mathrm{D})$$

as birepresentations of $\mathscr{C}$.

The following theorem is a straightforward generalization of [MMMT1, Theorem 5.1] in the context of bicategories, so we omit the proof. It resembles the classical Morita–Takeuchi equivalence for coalgebras over a field.

Theorem 4.4. *Two coalgebras C and D in $\underline{\mathscr{C}}$ are MT equivalent if and only if there exist a C-D-bicomodule M and a D-C-bicomodule N, and bicomodule isomorphisms*

$$f\colon \mathrm{C} \xrightarrow{\cong} \mathrm{M}\square_{\mathrm{D}}\mathrm{N}, \quad g\colon \mathrm{D} \xrightarrow{\cong} \mathrm{N}\square_{\mathrm{C}}\mathrm{M}$$

in $\underline{\mathscr{C}}$ such that we have commuting diagrams

$$
\begin{array}{ccccc}
\mathrm{C}\square_{\mathrm{C}}\mathrm{M} & \xleftarrow{\delta_{\mathrm{C,M}}} & \mathrm{M} & \xrightarrow{\delta_{\mathrm{M,D}}} & \mathrm{M}\square_{\mathrm{D}}\mathrm{D} \\
{\scriptstyle f\square_{\mathrm{C}}\mathrm{id}_{\mathrm{M}}}\downarrow & & & & \downarrow{\scriptstyle \mathrm{id}_{\mathrm{M}}\square_{\mathrm{D}}g} \\
(\mathrm{M}\square_{\mathrm{D}}\mathrm{N})\square_{\mathrm{C}}\mathrm{M} & \xrightarrow{\quad\alpha_{\mathrm{M,N,M}}\quad} & & & \mathrm{M}\square_{\mathrm{D}}(\mathrm{N}\square_{\mathrm{C}}\mathrm{M})
\end{array}\;,
$$

$$
\begin{array}{ccccc}
\mathrm{D}\square_{\mathrm{D}}\mathrm{N} & \xleftarrow{\delta_{\mathrm{D,N}}} & \mathrm{N} & \xrightarrow{\delta_{\mathrm{N,C}}} & \mathrm{N}\square_{\mathrm{C}}\mathrm{C} \\
{\scriptstyle g\square_{\mathrm{D}}\mathrm{id}_{\mathrm{N}}}\downarrow & & & & \downarrow{\scriptstyle \mathrm{id}_{\mathrm{N}}\square_{\mathrm{C}}f} \\
(\mathrm{N}\square_{\mathrm{C}}\mathrm{M})\square_{\mathrm{D}}\mathrm{N} & \xrightarrow{\quad\alpha_{\mathrm{N,M,N}}\quad} & & & \mathrm{N}\square_{\mathrm{C}}(\mathrm{M}\square_{\mathrm{D}}\mathrm{N})
\end{array}\;.
$$

Remark 4.5. Note that M and N are automatically biinjective if they satisfy the conditions in Theorem 4.4.

Suppose that $\Phi\colon \mathscr{C} \to \mathscr{D}$ is a $\Bbbk$-linear pseudofunctor between two multifinitary bicategories. The following corollary follows immediately from Lemma 3.11 and Theorem 4.4:

Corollary 4.6. *The extension* $\underline{\Phi}\colon \underline{\mathscr{C}} \to \underline{\mathscr{D}}$ *sends MT equivalent coalgebras in* $\underline{\mathscr{C}}$ *to MT equivalent coalgebras in* $\underline{\mathscr{D}}$.

4.3. **The internal cohom construction.** Let $\mathscr{C}$ be a multifinitary bicategory, $\mathbf{M}$ a finitary birepresentation of $\mathscr{C}$, and let $\underline{\mathbf{M}}$ denote the corresponding abelian birepresentation of $\underline{\mathscr{C}}$ (see Definition 2.23).
For all $X \in \mathbf{M}(\mathtt{j})$, $Y \in \mathbf{M}(\mathtt{i})$ there is a representable left exact functor

$$\Gamma_{X,Y}\colon \underline{\mathscr{C}}(\mathtt{i},\mathtt{j}) \to \mathscr{V}\mathrm{ect}, \quad \mathrm{F} \mapsto \mathrm{Hom}_{\underline{\mathbf{M}}(\mathtt{j})}\big(X, \underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})\, Y\big).$$

This means that there exists a 1-morphism $[Y, X] \in \underline{\mathscr{C}}(\mathtt{i},\mathtt{j})$, called the *internal cohom* from Y to X, and a natural isomorphism

$$\gamma_{Y,X}\colon \mathrm{Hom}_{\underline{\mathscr{C}}(\mathtt{i},\mathtt{j})}\big([Y,X],\mathrm{F}\big) \xrightarrow{\cong} \mathrm{Hom}_{\underline{\mathbf{M}}(\mathtt{j})}\big(X, \underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})\, Y\big), \text{ for all } \mathrm{F} \in \underline{\mathscr{C}}(\mathtt{i},\mathtt{j}).$$

By Yoneda's lemma, the pair $\big([Y,X], \gamma_{Y,X}\big)$ is unique up to a unique natural isomorphism in the following sense. If $\big([Y,X], \gamma_{Y,X}\big)$ and $\big([Y,X]', \gamma'_{Y,X}\big)$ are both internal cohoms from Y to X, then there exists a unique 2-isomorphism $\phi\colon [Y,X] \to [Y,X]'$ such that

$$\gamma_{Y,X}^{\text{-}1}\big(\gamma'_{Y,X}(\_)\big) = \_ \circ_{\mathsf{v}} \phi$$

as natural isomorphisms

$$\mathrm{Hom}_{\underline{\mathscr{C}}(\mathtt{i},\mathtt{j})}([Y,X]',\mathrm{F}) \xrightarrow{\cong} \mathrm{Hom}_{\underline{\mathscr{C}}(\mathtt{i},\mathtt{j})}([Y,X],\mathrm{F}), \text{ for all } \mathrm{F} \in \underline{\mathscr{C}}(\mathtt{i},\mathtt{j}).$$

The coevaluation $\mathrm{coev}_{Y,X}\colon X \to \underline{\mathbf{M}}_{\mathtt{ji}}\big([Y,X]\big)Y$ in $\underline{\mathbf{M}}(\mathtt{j})$ is defined as the image of $\mathrm{id}_{[Y,X]}$ under the isomorphism

$$(4.1) \qquad \gamma_{Y,X}\colon \mathrm{Hom}_{\underline{\mathscr{C}}(\mathtt{i},\mathtt{j})}\big([Y,X],[Y,X]\big) \xrightarrow{\cong} \mathrm{Hom}_{\underline{\mathbf{M}}(\mathtt{j})}\big(X, \underline{\mathbf{M}}_{\mathtt{ji}}([Y,X])Y\big).$$

Using the coevaluation morphisms, we can define a canonical 2-morphism

$$\delta_{Z,Y,X}\colon [Z,X] \to [Y,X][Z,Y],$$

for all $X \in \mathbf{M}(\mathtt{i})$, $Y \in \mathbf{M}(\mathtt{j})$, $Z \in \mathbf{M}(\mathtt{k})$ as follows. Consider the morphism τ defined by

$$
\begin{array}{c}
X \xrightarrow{\quad \mathrm{coev}_{Y,X} \quad} \underline{\mathbf{M}}_{\mathtt{ij}}\big([Y,X]\big)Y \\
\underline{\mathbf{M}}_{\mathtt{ij}}([Y,X])\mathrm{coev}_{Z,Y} \\
\underline{\mathbf{M}}_{\mathtt{ij}}\big([Y,X]\big)\underline{\mathbf{M}}_{\mathtt{jk}}\big([Z,Y]\big)Z \xrightarrow{(\mu_{\mathtt{ijk}}^{[Y,X],[Z,Y]})_Z} \underline{\mathbf{M}}_{\mathtt{ik}}\big([Y,X][Z,Y]\big)Z
\end{array}.
$$

The 2-morphism $\delta_{Z,Y,X}$ is defined as the image $\gamma_{Z,X}^{\text{-}1}(\tau)$ under the isomorphism

$$\gamma_{Z,X}\colon \mathrm{Hom}_{\underline{\mathscr{C}}(\mathtt{k},\mathtt{i})}\big([Z,X],[Y,X][Z,Y]\big) \xrightarrow{\cong} \mathrm{Hom}_{\underline{\mathbf{M}}(\mathtt{i})}\big(X, \underline{\mathbf{M}}_{\mathtt{ik}}([Y,X][Z,Y])Z\big).$$

Lemma 4.7. *For all* $X \in \mathbf{M}(\mathtt{l})$, $Y \in \mathbf{M}(\mathtt{k})$, $Z \in \mathbf{M}(\mathtt{j})$, $W \in \mathbf{M}(\mathtt{i})$, *there is a commutative diagram*

$$(4.2)
\begin{array}{ccc}
[Z,X][W,Z] & \xleftarrow{\delta_{W,Z,X}} [W,X] \xrightarrow{\delta_{W,Y,X}} & [Y,X][W,Y] \\
\delta_{Z,Y,X}\,\mathrm{id}_{[W,Z]} \downarrow & & \downarrow \mathrm{id}_{[Y,X]}\delta_{W,Z,Y} \\
\big([Y,X][Z,Y]\big)[W,Z] & \xrightarrow{\alpha_{[Y,X],[Z,Y],[W,Z]}} & [Y,X]\big([Z,Y][W,Z]\big)
\end{array}.
$$

Proof. By the isomorphisms in (4.1), the commutativity of the diagram in (4.2) is equivalent to the commutativity of the boundary of the diagram

$$X$$

the diagram with nodes and morphisms:

- $\underline{\mathbf{M}}_{1i}([W,X])W$ ① $\underline{\mathbf{M}}_{1j}([Z,X])Z$ ② $\underline{\mathbf{M}}_{1k}([Y,X])Y$ ③ $\underline{\mathbf{M}}_{1i}([W,X])W$
- with arrows $\mathrm{coev}_{W,X}$, $\mathrm{coev}_{Z,X}$, $\mathrm{coev}_{Y,X}$, $\mathrm{coev}_{W,X}$ from X
- $\underline{\mathbf{M}}_{1j}([Z,X])\mathrm{coev}_{W,Z}$, $\underline{\mathbf{M}}_{1j}(\delta_{Z,Y,X})_Z$, $\underline{\mathbf{M}}_{1k}([Y,X])\mathrm{coev}_{Z,Y}$, $\underline{\mathbf{M}}_{1k}([Y,X])\mathrm{coev}_{W,Y}$
- $\underline{\mathbf{M}}_{1i}(\delta_{W,Z,X})_W$, $\underline{\mathbf{M}}_{1i}(\delta_{W,Y,X})_W$
- $\underline{\mathbf{M}}_{1k}([Y,X])\underline{\mathbf{M}}_{kj}([Z,Y])Z$
- $(\mu_{1kj}^{[Y,X],[Z,Y]})_Z$, $\underline{\mathbf{M}}_{1k}([Y,X])\underline{\mathbf{M}}_{kj}([Z,Y])\mathrm{coev}_{W,Z}$
- ④ ⑤ ⑥
- $\underline{\mathbf{M}}_{1j}([Z,X])\underline{\mathbf{M}}_{ji}([W,Z])W$ $\underline{\mathbf{M}}_{1j}([Y,X][Z,Y])Z$ $\underline{\mathbf{M}}_{1k}([Y,X])\underline{\mathbf{M}}_{kj}([Z,Y])\underline{\mathbf{M}}_{ji}([W,Z])W$ $\underline{\mathbf{M}}_{1k}([Y,X])\underline{\mathbf{M}}_{ki}([W,Y])W$
- $\underline{\mathbf{M}}_{1j}(\delta_{Z,Y,X})\underline{\mathbf{M}}_{ji}([W,Z])W$, $(\mu_{1kj}^{[Y,X],[Z,Y]})\underline{\mathbf{M}}_{ji}([W,Z])W$, $\underline{\mathbf{M}}_{1k}([Y,X])\underline{\mathbf{M}}_{ki}(\delta_{W,Z,Y})W$
- $\underline{\mathbf{M}}_{1j}([Y,X][Z,Y])\mathrm{coev}_{W,Z}$, $\underline{\mathbf{M}}_{1k}([Y,X])(\mu_{kji}^{[Z,Y],[W,Z]})_W$, $(\mu_{1ki}^{[Y,X],[W,Y]})_W$
- $(\mu_{1ji}^{[Z,X],[W,Z]})_W$
- $\underline{\mathbf{M}}_{1i}([Z,X][W,Z])W$ ⑦ $\underline{\mathbf{M}}_{1j}([Y,X][Z,Y])\underline{\mathbf{M}}_{ji}([W,Z])W$ ⑧ $\underline{\mathbf{M}}_{1k}([Y,X])\underline{\mathbf{M}}_{ki}([Z,Y][W,Z])W$ ⑨ $\underline{\mathbf{M}}_{1i}([Y,X][W,Y])W$
- $\underline{\mathbf{M}}_{1i}(\delta_{Z,Y,X}\circ_h\mathrm{id})_W$, $(\mu_{1ji}^{[Y,X][Z,Y],[W,Z]})_W$, $(\mu_{1ki}^{[Y,X],[Z,Y][W,Z]})_W$, $\underline{\mathbf{M}}_{1i}(\mathrm{id}\circ_h\delta_{W,Z,Y})_W$
- $\underline{\mathbf{M}}_{1i}(([Y,X][Z,Y])[W,Z])W \xrightarrow{\underline{\mathbf{M}}_{1i}(\alpha_{[Y,X],[Z,Y],[W,Z]})_W} \underline{\mathbf{M}}_{1i}([Y,X]([Z,Y][W,Z]))W$

We note that

- the facets labeled 1, 2, 3 and 6 commute by definition of $\delta_{W,Z,X}$, $\delta_{Z,Y,X}$, $\delta_{W,Y,X}$ and $\delta_{W,Z,Y}$, respectively;
- the facets labeled 4, 5, 7 and 9 commute by naturality of $\underline{\mathbf{M}}_{1j}(\delta_{Z,Y,X})$, μ_{1kj}, μ_{1ji} and μ_{1ki}, respectively;
- the facet labeled 8 commutes due to the coherence condition for μ in (2.3) and the fact that $\underline{\mathbf{M}}_{1k}([Y,X])(\mu_{kji}^{[Z,Y],[W,Z]})_W = \big(\mathrm{id}_{\underline{\mathbf{M}}_{1k}([Y,X])} \circ_h \mu_{kji}^{[Z,Y],[W,Z]}\big)_W$.

Commutativity of these facets implies that all paths in the above diagram from X at the top to $\underline{\mathbf{M}}_{1i}\big([Y,X]([Z,Y][W,Z])\big)W$ at the bottom are equal, in particular, the two paths around the boundary, which is exactly what we had to show. $\square$

For every $X \in \mathbf{M}(i)$ there is also a canonical 2-morphism

$$\epsilon_X \colon [X,X] \to \mathbb{1}_i,$$

defined as the image of $\big(\iota_i^{-1}\big)_X$ under the isomorphism

$$\mathrm{Hom}_{\underline{\mathbf{M}}(i)}\big(X, \underline{\mathbf{M}}_{ii}(\mathbb{1}_i)X\big) \xrightarrow{\cong} \mathrm{Hom}_{\underline{\mathscr{C}}(i,i)}\big([X,X], \mathbb{1}_i\big),$$

where ι_j was defined in Definition 2.13.

Lemma 4.8. *For every $X \in \mathbf{M}(i)$, $Y \in \mathbf{M}(j)$, we have commutative diagrams*

(4.3)

$$[X,Y] \xrightarrow{\delta_{X,X,Y}} [X,Y][X,X]$$
$$(v_{[X,Y]}^r)^{-1} \searrow \quad \downarrow \mathrm{id}_{[X,Y]}\epsilon_X ,$$
$$[X,Y]\mathbb{1}_i$$

$$[X,Y] \xrightarrow{\delta_{X,Y,Y}} [Y,Y][X,Y]$$
$$(v_{[X,Y]}^l)^{-1} \searrow \quad \downarrow \epsilon_Y\,\mathrm{id}_{[X,Y]} .$$
$$\mathbb{1}_j[X,Y]$$

Proof. We only prove commutativity of the left diagram in (4.3). Commutativity of the right diagram can be proved by similar arguments.

First, consider the commutative diagram

$$(4.4) \quad \begin{array}{ccc} \operatorname{Hom}_{\mathscr{C}(\mathtt{i},\mathtt{i})}\big([X,X],[X,X]\big) & \xrightarrow{\cong} & \operatorname{Hom}_{\underline{\mathbf{M}}(\mathtt{i})}\big(X,\underline{\mathbf{M}}_{\mathtt{i}\mathtt{i}}([X,X])X\big) \\ {\scriptstyle \epsilon_X \circ_{\mathsf{v}} -}\Big\downarrow & & \Big\downarrow{\scriptstyle \underline{\mathbf{M}}_{\mathtt{i}\mathtt{i}}(\epsilon_X)_X \circ_{\mathsf{v}} -} \\ \operatorname{Hom}_{\mathscr{C}(\mathtt{i},\mathtt{i})}\big([X,X],\mathbb{1}_{\mathtt{i}}\big) & \xrightarrow{\cong} & \operatorname{Hom}_{\underline{\mathbf{M}}(\mathtt{i})}\big(X,\underline{\mathbf{M}}_{\mathtt{i}\mathtt{i}}(\mathbb{1}_{\mathtt{i}})X\big) \end{array} \quad .$$

By comparing the image of $\operatorname{id}_{[X,X]} \in \operatorname{Hom}_{\mathscr{C}(\mathtt{i},\mathtt{i})}\big([X,X],[X,X]\big)$ under the two maps to $\operatorname{Hom}_{\underline{\mathbf{M}}(\mathtt{i})}\big(X,\underline{\mathbf{M}}_{\mathtt{i}\mathtt{i}}(\mathbb{1}_{\mathtt{i}})X\big)$ corresponding to the two paths in (4.4), we see that

$$(4.5) \quad \underline{\mathbf{M}}_{\mathtt{i}\mathtt{i}}(\epsilon_X)_X \circ_{\mathsf{v}} \operatorname{coev}_{X,X} = (\iota_{\mathtt{i}})_X^{-1}.$$

The next observation is that commutativity of the left diagram in (4.3) is equivalent to commutativity of the boundary of

Commutativity of the boundary of this diagram follows from commutativity of the internal facets. The latter commute due to

- the definition of $\delta_{X,X,Y}$ for the facet labeled 1;
- (4.5) for the facets labeled 2;
- naturality of $\mu_{\mathtt{j}\mathtt{i}\mathtt{i}}$ for the facet labeled 3;
- the left coherence condition in (2.2) for the facet labeled 4.

This completes the proof. $\qquad\square$

The following proposition is an immediate consequence of Lemmas 4.7 and 4.8.

Proposition 4.9. *Let $\mathbf{M}$ be a finitary birepresentation of $\mathscr{C}$. For any $X \in \underline{\mathbf{M}}(\mathtt{i})$ and any $Y \in \underline{\mathbf{M}}(\mathtt{j})$,*

(i) the triple $\big([X,X],\delta_{X,X,X},\epsilon_X\big)$ is a coalgebra in $\underline{\mathscr{C}}$;

(ii) the triple $\big([X,Y],\delta_{X,Y,Y},\delta_{X,X,Y}\big)$ is a $[Y,Y]$-$[X,X]$-bimodule in $\underline{\mathscr{C}}$.

As in [MMMT1], we will often use the notation C^X for the coalgebra $[X,X]$. The following theorem is the analog of [MMMT1, Theorem 4.7] for quasi fiab bicategories and finitary birepresentations. The proof is entirely analogous and therefore omitted.

Theorem 4.10. *Assume that $\mathscr{C}$ is quasi multifiab and $\mathbf{M}$ is a finitary birepresentation of $\mathscr{C}$. Let $X \in \mathbf{M}(\mathtt{i})$ be a generator of $\mathbf{M}$. Then there is an equivalence of finitary birepresentations*

$$\mathbf{M} \simeq \mathbf{inj}_{\underline{\mathscr{C}}}(\mathrm{C}^X)$$

such that $Y \mapsto [X,Y]$ for all $Y \in \mathbf{M}(\mathtt{j})$.

Remark 4.11. The existence of a single generator can be an obstacle to applying Theorem 4.10 in the setup of quasi multifiab bicategories, since such a generator might not exist in a single $\mathbf{M}(\mathtt{i})$. However, we can always pass to the birepresentation $\mathbf{M}^{\oplus}$ of $\mathscr{C}^{\oplus}$, which will have a generator. We can thus always associate a coalgebra in $\mathscr{C}^{\oplus}$ to $\mathbf{M}$.

Corollary 4.12. *If, in the setup of Theorem 4.10, X, Y are two generators of $\mathbf{M}$, then C^X and C^Y are MT equivalent coalgebras in $\underline{\mathscr{C}}$.*

4.4. **Framing coalgebras.** Let $\mathscr{C}$ be a quasi multifiab bicategory. Recall that, for all 1-morphisms $\mathrm{F} \in \mathscr{C}$, the tuple $(\mathrm{F}, \mathrm{F}^{\star})$ forms an adjoint pair in $\mathscr{C}$.

Lemma 4.13. *If $\mathrm{C} \in \underline{\mathscr{C}}(\mathtt{i}, \mathtt{i})$ is a coalgebra in $\underline{\mathscr{C}}$ such that $0 \neq (\mathrm{FC})\mathrm{F}^{\star}$ for some 1-morphism $\mathrm{F} \in \mathscr{C}(\mathtt{i}, \mathtt{j})$, then the 1-morphism $(\mathrm{FC})\mathrm{F}^{\star} \in \underline{\mathscr{C}}(\mathtt{j}, \mathtt{j})$ has a coalgebra structure in $\underline{\mathscr{C}}$ with comultiplication*

$$\delta_{(\mathrm{FC})\mathrm{F}^{\star}} := \big(\mathrm{id}_{(\mathrm{FC})\mathrm{F}^{\star}} \circ_{\mathsf{h}} \alpha_{\mathrm{F},\mathrm{C},\mathrm{F}^{\star}}^{-1}\big) \circ_{\mathsf{v}} \alpha_{(\mathrm{FC})\mathrm{F}^{\star},\mathrm{F},\mathrm{CF}^{\star}} \circ_{\mathsf{v}} \big(\alpha_{\mathrm{FC},\mathrm{F}^{\star},\mathrm{F}}^{-1} \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{CF}^{\star}}\big)$$
$$\circ_{\mathsf{v}} \big((\mathrm{id}_{\mathrm{FC}} \circ_{\mathsf{h}} \mathrm{coev}_{\mathrm{F}}) \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{CF}^{\star}}\big) \circ_{\mathsf{v}} \big((v_{\mathrm{FC}}^{r})^{-1} \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{CF}^{\star}}\big) \circ_{\mathsf{v}} \alpha_{\mathrm{FC},\mathrm{C},\mathrm{F}^{\star}}$$
$$\circ_{\mathsf{v}} \big(\alpha_{\mathrm{F},\mathrm{C},\mathrm{C}}^{-1} \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{F}^{\star}}\big) \circ_{\mathsf{v}} \big((\mathrm{id}_{\mathrm{F}} \circ_{\mathsf{h}} \delta_{\mathrm{C}}) \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{F}^{\star}}\big)$$

and counit

$$\epsilon_{(\mathrm{FC})\mathrm{F}^{\star}} := \mathrm{ev}_{\mathrm{F}} \circ_{\mathsf{v}} (v_{\mathrm{F}}^{r} \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{F}^{\star}}) \circ_{\mathsf{v}} \big((\mathrm{id}_{\mathrm{F}} \circ_{\mathsf{h}} \epsilon_{\mathrm{C}}) \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{F}^{\star}}\big).$$

Proof. Associativity and counitality for $(\mathrm{FC})\mathrm{F}^{\star}$ follow from those for C, the coherence conditions for the associator and the unitors of $\underline{\mathscr{C}}$ and the adjunction conditions for F. $\qquad\square$

Remark 4.14. The 1-morphism $\mathrm{F}(\mathrm{CF}^{\star})$, if nonzero, acquires a coalgebra structure via the isomorphism $\alpha_{\mathrm{F},\mathrm{C},\mathrm{F}^{\star}} \colon (\mathrm{FC})\mathrm{F}^{\star} \xrightarrow{\cong} \mathrm{F}(\mathrm{CF}^{\star})$.

Remark 4.15. If $\mathscr{C}$ is a fiat 2-category, we can picture the coalgebra structure of $\mathrm{FCF}^{\star} = (\mathrm{FC})\mathrm{F}^{\star} = \mathrm{F}(\mathrm{CF}^{\star})$ from Lemma 4.13 in the form of string diagrams. Using solid black strands for C and dotted blue strands for F and $\mathrm{F}^{\star}$, we denote the 2-morphisms $\delta_{\mathrm{C}}, \epsilon_{\mathrm{C}}, \mathrm{ev}_{\mathrm{F}}, \mathrm{coev}_{\mathrm{F}}$ by

In this diagrammatic notation, the comultiplication and counit of $\mathrm{FCF}^{\star}$ become

This explains our choice of the term *framing*. Using these diagrams, the proof of Lemma 4.13 becomes an easy exercise in planar topology and many of the statements below also have natural topological interpretations.

The idea to use duals for the construction of (co)algebras is not new, see e.g. [Mu, Section 3] for framings of the identity object in a strict tensor category, although we do not know of any reference for the general content of Lemma 4.13 (either in the framework of 2-categories or bicategories).

Note that for any $\mathrm{F} \in \mathscr{C}(\mathtt{i}, \mathtt{j})$ the adjoint pair $(\mathrm{F}, \mathrm{F}^{\star})$ gives rise to the natural isomorphism

$$\mathrm{Hom}_{\underline{\mathscr{C}}(\mathtt{k},\mathtt{j})}(\mathrm{FH}, \mathrm{G}) \xrightarrow{\cong} \mathrm{Hom}_{\underline{\mathscr{C}}(\mathtt{k},\mathtt{i})}(\mathrm{H}, \mathrm{F}^{\star}\mathrm{G}),$$

where $G \in \underline{\mathscr{C}}(k, j), H \in \underline{\mathscr{C}}(k, i)$, given by sending $\beta \in \mathrm{Hom}_{\underline{\mathscr{C}}(k,j)}(FH, G)$ to the element

$$(\mathrm{id}_{F^\star} \circ_h \beta) \circ_v \alpha_{F^\star, F, H} \circ_v (\mathrm{coev}_F \circ_h \mathrm{id}_H) \circ_v (v_H^l)^{-1} \in \mathrm{Hom}_{\underline{\mathscr{C}}(k,i)}(H, F^\star G)$$

with inverse given by sending $\gamma \in \mathrm{Hom}_{\underline{\mathscr{C}}(k,i)}(H, F^\star G)$ to

$$v_G^l \circ_v (\mathrm{ev}_F \circ_h \mathrm{id}_G) \circ_v \alpha_{F, F^\star, G}^{-1} \circ_v (\mathrm{id}_F \circ_h \gamma) \in \mathrm{Hom}_{\underline{\mathscr{C}}(k,j)}(FH, G).$$

We also have the natural isomorphism

$$\mathrm{Hom}_{\underline{\mathscr{C}}(j,k)}(HF^\star, G) \xrightarrow{\cong} \mathrm{Hom}_{\underline{\mathscr{C}}(i,k)}(H, GF),$$

where $G \in \underline{\mathscr{C}}(j, k), H \in \underline{\mathscr{C}}(i, k)$, given by sending $\beta \in \mathrm{Hom}_{\underline{\mathscr{C}}(j,k)}(HF^\star, G)$ to the element

$$(\beta \circ_h \mathrm{id}_F) \circ_v \alpha_{H, F^\star, F}^{-1} \circ_v (\mathrm{id}_H \circ_h \mathrm{coev}_F) \circ_v (v_H^r)^{-1} \in \mathrm{Hom}_{\underline{\mathscr{C}}(i,k)}(H, GF)$$

with inverse given by sending $\gamma \in \mathrm{Hom}_{\underline{\mathscr{C}}(i,k)}(H, GF)$ to

$$v_G^r \circ_v (\mathrm{id}_G \circ_h \mathrm{ev}_F) \circ_v \alpha_{G, F, F^\star} \circ_v (\gamma \circ_h \mathrm{id}_{F^\star}) \in \mathrm{Hom}_{\underline{\mathscr{C}}(j,k)}(HF^\star, G).$$

Theorem 4.16. *Suppose that, additionally to the hypotheses of Theorem 4.10,* F *is a* 1-*morphism in* $\mathscr{C}(i, j)$ *such that* $\mathbf{M}_{ji}(F) X$ *generates* $\mathbf{M}$. *Then the* 1-*morphism* $(FC^X)F^\star \in \underline{\mathscr{C}}$ *with coalgebra structure defined in Lemma 4.13 is, up to isomorphism, the coalgebra associated with the object* $\mathbf{M}_{ji}(F) X$.

Proof. By adjunction and the natural isomorphism $\gamma_{X,X}$, we have natural isomorphisms

(4.6)
$$\mathrm{Hom}_{\underline{\mathbf{M}}(j)}\big(\underline{\mathbf{M}}_{ji}(F) X, \underline{\mathbf{M}}_{jj}(G)\underline{\mathbf{M}}_{ji}(F) X\big) \cong \mathrm{Hom}_{\underline{\mathbf{M}}(j)}\big(\underline{\mathbf{M}}_{ji}(F) X, \underline{\mathbf{M}}_{ji}(GF) X\big)$$
$$\cong \mathrm{Hom}_{\underline{\mathbf{M}}(i)}\big(X, \underline{\mathbf{M}}_{ij}(F^\star)\underline{\mathbf{M}}_{ji}(GF) X\big)$$
$$\cong \mathrm{Hom}_{\underline{\mathbf{M}}(i)}\big(X, \underline{\mathbf{M}}_{ii}(F^\star(GF)) X\big)$$
$$\cong \mathrm{Hom}_{\underline{\mathscr{C}}(i,i)}\big(C^X, F^\star(GF)\big)$$
$$\cong \mathrm{Hom}_{\underline{\mathscr{C}}(i,j)}(FC^X, GF)$$
$$\cong \mathrm{Hom}_{\underline{\mathscr{C}}(j,j)}\big((FC^X)F^\star, G\big),$$

for all 1-morphisms $G \in \underline{\mathscr{C}}(j, j)$. Below we will give and use these isomorphisms explicitly, e.g. the first and third isomorphisms are given by $(\mu_{jji}^{G,F})_X \circ_v -$ and $(\mu_{iji}^{F^\star, GF})_X \circ_v -$, respectively.

Considering $G = \mathbb{1}_j$, we now prove that $\epsilon_{(FC^X)F^\star}$ (recall the notation $\epsilon_X := \epsilon_{C^X}$) is the image of $(\iota_j^{-1})_{\underline{\mathbf{M}}_{ji}(F)X}$ under the isomorphisms in (4.6). It suffices to show that the image of $\epsilon_{(FC^X)F^\star}$ and the image of $(\iota_j^{-1})_{\underline{\mathbf{M}}_{ji}(F) X}$ under those isomorphisms coincide in any of the morphism spaces appearing in (4.6). On one hand, the first isomorphism

$$\mathrm{Hom}_{\underline{\mathbf{M}}(j)}\big(\underline{\mathbf{M}}_{ji}(F) X, \underline{\mathbf{M}}_{jj}(\mathbb{1}_j)\underline{\mathbf{M}}_{ji}(F) X\big) \xrightarrow{\cong} \mathrm{Hom}_{\underline{\mathbf{M}}(j)}\big(\underline{\mathbf{M}}_{ji}(F) X, \underline{\mathbf{M}}_{ji}(\mathbb{1}_jF) X\big)$$

sends $(\iota_j^{-1})_{\underline{\mathbf{M}}_{ji}(F) X}$ to $\underline{\mathbf{M}}_{ji}\big((v_F^l)^{-1}\big)_X$, by (2.2). The second isomorphism

$$\mathrm{Hom}_{\underline{\mathbf{M}}(j)}\big(\underline{\mathbf{M}}_{ji}(F) X, \underline{\mathbf{M}}_{ji}(\mathbb{1}_jF) X\big) \xrightarrow{\cong} \mathrm{Hom}_{\underline{\mathbf{M}}(i)}\big(X, \underline{\mathbf{M}}_{ij}(F^\star)\underline{\mathbf{M}}_{ji}(\mathbb{1}_jF) X\big)$$

sends $\underline{\mathbf{M}}_{ji}\big((v_F^l)^{-1}\big)_X$ to

$$\Big(\mathrm{id}_{\underline{\mathbf{M}}_{ij}(F^\star)} \circ_h \underline{\mathbf{M}}_{ji}\big((v_F^l)^{-1}\big)_X\Big) \circ_v (\mu_{iji}^{F^\star, F})_X^{-1} \circ_v \underline{\mathbf{M}}_{ii}(\mathrm{coev}_F)_X \circ_v (\iota_i^{-1})_X$$
$$= (\mu_{iji}^{F^\star, \mathbb{1}_jF})_X^{-1} \circ_v \underline{\mathbf{M}}_{ii}\big(\mathrm{id}_{F^\star} \circ_h (v_F^l)^{-1}\big)_X \circ_v \underline{\mathbf{M}}_{ii}(\mathrm{coev}_F)_X \circ_v (\iota_i^{-1})_X,$$

where the equality holds by the naturality of $\mu_{\mathtt{iji}}$. Further, the latter is sent to $\underline{\mathbf{M}}_{\mathtt{ii}}\big(\mathrm{id}_{\mathrm{F}^\star} \circ_\mathsf{h} (v_\mathrm{F}^l)^{-1}\big)_X \circ_\mathsf{v} \underline{\mathbf{M}}_{\mathtt{ii}}(\mathrm{coev}_\mathrm{F})_X \circ_\mathsf{v} (\iota_\mathtt{i}^{-1})_X$ under the third isomorphism in (4.6) for $\mathrm{G} = \mathbb{1}_\mathtt{j}$. On the other hand, consider the commutative diagram

(4.7)

Its commutativity follows from

- naturality of $(v^r)^{-1}$ for the facet labeled 1 (noting that $(v^r)_\mathrm{H}^{-1} = (v_\mathrm{H}^r)^{-1}$ for any 1-morphism $\mathrm{H} \in \mathscr{C}$);
- the interchange law for the facets labeled 2 and 6;
- naturality of α^{-1} for the facets labeled 3 and 7;
- the right diagram in (2.1) for the facet labeled 4;
- the triangle coherence condition of the unitors for the facet labeled 5.

The inverse $\mathrm{Hom}_{\underline{\mathscr{C}}}\big((\mathrm{FC}^X)\mathrm{F}^\star, \mathbb{1}_\mathtt{j}\big) \xrightarrow{\cong} \mathrm{Hom}_{\underline{\mathscr{C}}}\big(\mathrm{FC}^X, \mathbb{1}_\mathtt{j}\mathrm{F}\big)$ of the last isomorphism in (4.6) sends $\epsilon_{(\mathrm{FC}^X)\mathrm{F}^\star}$ to the composite of the paths going right and then down to the bottom along the boundary of the diagram in (4.7). This composite, due to the commutativity of the diagram, is the same as the composite of the path going first down and then right, and down as the final step. Since $v_{\mathbb{1}_\mathtt{i}}^l = v_{\mathbb{1}_\mathtt{i}}^r$, the latter equals

(4.8) $(\mathrm{ev}_\mathrm{F} \circ_\mathsf{h} \mathrm{id}_\mathrm{F}) \circ_\mathsf{v} \alpha_{\mathrm{F},\mathrm{F}^\star,\mathrm{F}}^{-1} \circ_\mathsf{v} (\mathrm{id}_\mathrm{F} \circ_\mathsf{h} \mathrm{coev}_\mathrm{F}) \circ_\mathsf{v} (\mathrm{id}_\mathrm{F} \circ_\mathsf{h} \epsilon_X).$

Consider now the diagram

(4.9)

which commutes due to

- naturality of $(v^l)^{-1}$ for the facets labeled 1 and 4;
- the interchange law for the facets labeled 2 and 5;
- naturality of α and α^{-1} for the facets labeled 3, 6, 8 and 11, respectively;
- the left diagram in (2.1) for the facet labeled 7;
- the pentagon coherence condition of the associator for the facet labeled 9;
- the adjunction condition of the adjoint pair $(F, F^\star)$ for the facet labeled 10;
- the triangle coherence condition of the unitors for the facet labeled 12.

Then the inverse $\mathrm{Hom}_{\mathscr{C}}(FC^X, \mathbb{1}_j F) \xrightarrow{\cong} \mathrm{Hom}_{\mathscr{C}}\big(C^X, F^\star(\mathbb{1}_j F)\big)$ of the fifth isomorphism in (4.6) sends (4.8) to the composite of the paths going right and then down to the bottom along the boundary of the diagram in (4.9). As before, this is the same as the composite of the paths going down and then right, i.e.,

(4.10) $$\big(\mathrm{id}_{F^\star} \circ_h (v_F^l)^{-1}\big) \circ_v \mathrm{coev}_F \circ_v \epsilon_X.$$

The inverse $\mathrm{Hom}_{\mathscr{C}}(C^X, F^\star(\mathbb{1}_j F)) \xrightarrow{\cong} \mathrm{Hom}_{\underline{\mathbf{M}}(\mathtt{i})}(X, \underline{\mathbf{M}}_{\mathtt{ii}}\big(F^\star(\mathbb{1}_j F)\big) X)$ of the fourth isomorphism in (4.6) sends (4.10) to

$$\underline{\mathbf{M}}_{\mathtt{ii}}(\mathrm{id}_{F^\star} \circ_h (v_F^l)^{-1})_X \circ_v \underline{\mathbf{M}}_{\mathtt{ii}}(\mathrm{coev}_F)_X \circ_v \underline{\mathbf{M}}_{\mathtt{ii}}(\epsilon_X)_X \circ_v \mathrm{coev}_{X,X},$$

which equals $\underline{\mathbf{M}}_{\mathtt{ii}}\big(\mathrm{id}_{\mathrm{F}^\star} \circ_{\mathsf{h}} (v_{\mathrm{F}}^l)^{-1}\big)_X \circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ii}}(\mathrm{coev}_{\mathrm{F}})_X \circ_{\mathsf{v}} (\iota_{\mathtt{i}}^{-1})_X$, by (4.5). Hence, we obtain that $\epsilon_{(\mathrm{FC}^X)\mathrm{F}^\star} \in \mathrm{Hom}_{\underline{\mathscr{C}}}\big((\mathrm{FC}^X)\mathrm{F}^\star, \mathbb{1}_{\mathtt{j}}\big)$ is equal to the image of $(\iota_{\mathtt{j}}^{-1})_{\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})X}$ under the isomorphisms in (4.6).

Next, taking $\mathrm{G} = (\mathrm{FC}^X)\mathrm{F}^\star$ in (4.6), we have to determine the image of $\mathrm{id}_{(\mathrm{FC}^X)\mathrm{F}^\star}$ in $\mathrm{Hom}_{\underline{\mathbf{M}}(\mathtt{j})}\big(\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})\,X, \underline{\mathbf{M}}_{\mathtt{jj}}\big((\mathrm{FC}^X)\mathrm{F}^\star\big)\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})X\big)$ under the (inverses of the) isomorphisms in (4.6), which is equal to $\mathrm{coev}_{\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})X, \underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})X}$. In detail, the inverse of the last isomorphism sends $\mathrm{id}_{(\mathrm{FC}^X)\mathrm{F}^\star}$ to

$$(4.11) \qquad \alpha_{\mathrm{FC}^X,\mathrm{F}^\star,\mathrm{F}}^{-1} \circ_{\mathsf{v}} (\mathrm{id}_{\mathrm{FC}^X} \circ_{\mathsf{h}} \mathrm{coev}_{\mathrm{F}}) \circ_{\mathsf{v}} (v_{\mathrm{FC}^X}^r)^{-1}$$

in $\mathrm{Hom}_{\underline{\mathscr{C}}}\big(\mathrm{FC}^X, ((\mathrm{FC}^X)\mathrm{F}^\star)\mathrm{F}\big)$.

Further, the inverse $\mathrm{Hom}_{\underline{\mathscr{C}}}\big(\mathrm{FC}^X, ((\mathrm{FC}^X)\mathrm{F}^\star)\mathrm{F}\big) \xrightarrow{\cong} \mathrm{Hom}_{\underline{\mathscr{C}}}\big(\mathrm{C}^X, \mathrm{F}^\star(((\mathrm{FC}^X)\mathrm{F}^\star)\mathrm{F})\big)$ of the fifth isomorphism sends (4.11) to

$$(\mathrm{id}_{\mathrm{F}^\star} \circ_{\mathsf{h}} \alpha_{\mathrm{FC}^X,\mathrm{F}^\star,\mathrm{F}}^{-1}) \circ_{\mathsf{v}} \big(\mathrm{id}_{\mathrm{F}^\star} \circ_{\mathsf{h}} (\mathrm{id}_{\mathrm{FC}^X} \circ_{\mathsf{h}} \mathrm{coev}_{\mathrm{F}})\big) \circ_{\mathsf{v}} \big(\mathrm{id}_{\mathrm{F}^\star} \circ_{\mathsf{h}} (v_{\mathrm{FC}^X}^r)^{-1}\big)$$
$$\circ_{\mathsf{v}} \alpha_{\mathrm{F}^\star,\mathrm{F},\mathrm{C}^X} \circ_{\mathsf{v}} (\mathrm{coev}_{\mathrm{F}} \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{C}^X}) \circ_{\mathsf{v}} (v_{\mathrm{C}^X}^l)^{-1}.$$

This 2-morphism equals

$$(4.12) \qquad (\mathrm{id}_{\mathrm{F}^\star} \circ_{\mathsf{h}} \alpha_{\mathrm{FC}^X,\mathrm{F}^\star,\mathrm{F}}^{-1}) \circ_{\mathsf{v}} \big(\mathrm{id}_{\mathrm{F}^\star} \circ_{\mathsf{h}} (\mathrm{id}_{\mathrm{FC}^X} \circ_{\mathsf{h}} \mathrm{coev}_{\mathrm{F}})\big) \circ_{\mathsf{v}} (\mathrm{id}_{\mathrm{F}^\star} \circ_{\mathsf{h}} \alpha_{\mathrm{F},\mathrm{C}^X,\mathbb{1}_{\mathtt{i}}}^{-1})$$
$$\circ_{\mathsf{v}} \alpha_{\mathrm{F}^\star,\mathrm{F},\mathrm{C}^X \mathbb{1}_{\mathtt{i}}} \circ_{\mathsf{v}} (\mathrm{coev}_{\mathrm{F}} \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{C}^X \mathbb{1}_{\mathtt{i}}}) \circ_{\mathsf{v}} (v_{\mathrm{C}^X \mathbb{1}_{\mathtt{i}}}^l)^{-1} \circ_{\mathsf{v}} (v_{\mathrm{C}^X}^r)^{-1}$$

due to commutativity of

To see that this diagram commutes we note that

- the facets labeled 1 and 3 commute by naturality of $(v^l)^{-1}$ and α, respectively;
- the facet labeled 2 commutes by the interchange law;
- the facet labeled 4 commutes by the right diagram in (2.1).

The map $\mathrm{Hom}_{\underline{\mathscr{C}}}\big(\mathrm{C}^X, \mathrm{F}^\star(((\mathrm{FC}^X)\mathrm{F}^\star)\mathrm{F})\big) \xrightarrow{\cong} \mathrm{Hom}_{\underline{\mathbf{M}}(\mathtt{i})}\big(X, \underline{\mathbf{M}}_{\mathtt{ii}}(\mathrm{F}^\star(((\mathrm{FC}^X)\mathrm{F}^\star)\mathrm{F}))\,X\big)$ sends (4.12) to

$$\underline{\mathbf{M}}_{\mathtt{ii}}\big(\mathrm{id}_{\mathrm{F}^\star} \circ_{\mathsf{h}} \alpha_{\mathrm{FC}^X,\mathrm{F}^\star,\mathrm{F}}^{-1}\big)_X \circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ii}}\big(\mathrm{id}_{\mathrm{F}^\star} \circ_{\mathsf{h}} (\mathrm{id}_{\mathrm{FC}^X} \circ_{\mathsf{h}} \mathrm{coev}_{\mathrm{F}})\big)_X \circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ii}}\big(\mathrm{id}_{\mathrm{F}^\star} \circ_{\mathsf{h}} \alpha_{\mathrm{F},\mathrm{C}^X,\mathbb{1}_{\mathtt{i}}}^{-1}\big)_X$$
$$\circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ii}}\big(\alpha_{\mathrm{F}^\star,\mathrm{F},\mathrm{C}^X \mathbb{1}_{\mathtt{i}}}\big)_X \circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ii}}\big(\mathrm{coev}_{\mathrm{F}} \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{C}^X \mathbb{1}_{\mathtt{i}}}\big)_X \circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ii}}\big((v_{\mathrm{C}^X \mathbb{1}_{\mathtt{i}}}^l)^{-1}\big)_X$$
$$\circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ii}}\big((v_{\mathrm{C}^X}^r)^{-1}\big)_X \circ_{\mathsf{v}} \mathrm{coev}_{X,X}.$$

The latter, under the composite of the inverses of the third and second isomorphisms, is sent to

$$
\left(\iota_{\mathrm{j}}\right)_{\underline{\mathbf{M}}_{\mathrm{ji}}\left(\left((\mathrm{F}C^X)\mathrm{F}^\star\right)\mathrm{F}\right)X} \circ_{\mathrm{v}} \underline{\mathbf{M}}_{\mathrm{jj}}\left(\mathrm{ev}_{\mathrm{F}}\right)_{\underline{\mathbf{M}}_{\mathrm{ji}}\left(\left((\mathrm{F}C^X)\mathrm{F}^\star\right)\mathrm{F}\right)X} \circ_{\mathrm{v}} \left(\mu_{\mathrm{jij}}^{\mathrm{F},\mathrm{F}^\star}\right)_{\underline{\mathbf{M}}_{\mathrm{ji}}\left(\left((\mathrm{F}C^X)\mathrm{F}^\star\right)\mathrm{F}\right)X}
$$

$$
\circ_{\mathrm{v}} \left(\underline{\mathbf{M}}_{\mathrm{ji}}(\mathrm{F})\left(\mu_{\mathrm{iji}}^{\mathrm{F}^\star,\left((\mathrm{F}C^X)\mathrm{F}^\star\right)\mathrm{F}}\right)_X^{-1}\right) \circ_{\mathrm{v}} \left(\underline{\mathbf{M}}_{\mathrm{ji}}(\mathrm{F})\underline{\mathbf{M}}_{\mathrm{ii}}\left(\mathrm{id}_{\mathrm{F}^\star} \circ_{\mathrm{h}} \alpha_{\mathrm{F}C^X,\mathrm{F}^\star,\mathrm{F}}^{-1}\right)_X\right)
$$

(4.13)
$$
\circ_{\mathrm{v}} \left(\underline{\mathbf{M}}_{\mathrm{ji}}(\mathrm{F})\underline{\mathbf{M}}_{\mathrm{ii}}\left(\mathrm{id}_{\mathrm{F}^\star} \circ_{\mathrm{h}} \left(\mathrm{id}_{\mathrm{F}C^X} \circ_{\mathrm{h}} \mathrm{coev}_{\mathrm{F}}\right)\right)_X\right) \circ_{\mathrm{v}} \left(\underline{\mathbf{M}}_{\mathrm{ji}}(\mathrm{F})\underline{\mathbf{M}}_{\mathrm{ii}}\left(\mathrm{id}_{\mathrm{F}^\star} \circ_{\mathrm{h}} \alpha_{\mathrm{F},C^X,\mathbb{1}_{\mathrm{i}}}^{-1}\right)_X\right)
$$

$$
\circ_{\mathrm{v}} \left(\underline{\mathbf{M}}_{\mathrm{ji}}(\mathrm{F})\underline{\mathbf{M}}_{\mathrm{ii}}\left(\alpha_{\mathrm{F}^\star,\mathrm{F},C^X\mathbb{1}_{\mathrm{i}}}\right)_X\right) \circ_{\mathrm{v}} \left(\underline{\mathbf{M}}_{\mathrm{ji}}(\mathrm{F})\underline{\mathbf{M}}_{\mathrm{ii}}\left(\mathrm{coev}_{\mathrm{F}} \circ_{\mathrm{h}} \mathrm{id}_{C^X\mathbb{1}_{\mathrm{i}}}\right)_X\right)
$$

$$
\circ_{\mathrm{v}} \left(\underline{\mathbf{M}}_{\mathrm{ji}}(\mathrm{F})\underline{\mathbf{M}}_{\mathrm{ii}}\left((\upsilon_{C^X\mathbb{1}_{\mathrm{i}}}^l)^{-1}\right)_X\right) \circ_{\mathrm{v}} \left(\underline{\mathbf{M}}_{\mathrm{ji}}(\mathrm{F})\underline{\mathbf{M}}_{\mathrm{ii}}\left((\upsilon_{C^X}^r)^{-1}\right)_X\right)
$$

$$
\circ_{\mathrm{v}} \left(\underline{\mathbf{M}}_{\mathrm{ji}}(\mathrm{F})\mathrm{coev}_{X,X}\right),
$$

which is an element in $\mathrm{Hom}_{\underline{\mathbf{M}}(\mathrm{j})}\left(\underline{\mathbf{M}}_{\mathrm{ji}}(\mathrm{F})\,X, \underline{\mathbf{M}}_{\mathrm{ji}}\left(\left((\mathrm{F}C^X)\mathrm{F}^\star\right)\mathrm{F}\right)X\right)$.

Consider now the diagram

in which the path going down the left side and then right along the bottom includes all but the first two morphisms in (4.13). The diagram commutes by

- naturality of μ_{jii} for the facets labeled 1, 3, 5, 7, 9 and 11;
- the triangle coherence condition of the unitors for the facet labeled 2;
- naturality of α for the facets labeled 4, 8, 10 and 12;
- the pentagon coherence condition of the associator for the facet labeled 6;
- the diagram in (2.3) for the facet labeled 13;

- naturality of $\mu_{\mathtt{jji}}$ for the facet labeled 14;
- the right diagram in (2.2) for the facet labeled 15.

Let us also consider the diagram

which commutes by

- the adjunction condition of the adjoint pair $(\mathrm{F}, \mathrm{F}^\star)$ for the facet labeled 1;
- naturality of α for the facet labeled 2;
- the first condition in (2.1) for the facet labeled 3;
- the interchange law for the facets labeled 4, 6 and 8;
- naturality of υ^l for the facets labeled 5, 7 and 9.

In this diagram, going from the top right corner to the bottom right corner by first going all the way left, then down and then right again corresponds to going right and then down in the previous diagram, starting from the second entry in the first row. Hence, we obtain that (4.13) equals

$$
(4.14)
\begin{aligned}
&\underline{\mathbf{M}}_{\mathtt{ji}}(\alpha^{-1}_{\mathrm{FC}^X,\mathrm{F}^\star,\mathrm{F}})X \circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{id}_{\mathrm{FC}^X} \circ_{\mathsf{h}} \mathrm{coev}_\mathrm{F})X \circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ji}}(\alpha^{-1}_{\mathrm{F},\mathrm{C}^X,\mathbb{1}_\mathtt{i}})X \\
&\circ_{\mathsf{v}} \left(\mu^{\mathrm{F},\mathrm{C}^X\mathbb{1}_\mathtt{i}}_{\mathtt{jii}}\right)_X \circ_{\mathsf{v}} \left(\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})\underline{\mathbf{M}}_{\mathtt{ii}}((\upsilon^r_{\mathrm{C}^X})^{-1})_X\right) \circ_{\mathsf{v}} (\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})\mathrm{coev}_{X,X}) \\
&= \underline{\mathbf{M}}_{\mathtt{ji}}(\alpha^{-1}_{\mathrm{FC}^X,\mathrm{F}^\star,\mathrm{F}})X \circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{id}_{\mathrm{FC}^X} \circ_{\mathsf{h}} \mathrm{coev}_\mathrm{F})X \circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ji}}((\upsilon^r_{\mathrm{FC}^X})^{-1})_X \\
&\circ_{\mathsf{v}} \left(\mu^{\mathrm{F},\mathrm{C}^X}_{\mathtt{jii}}\right)_X \circ_{\mathsf{v}} (\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})\mathrm{coev}_{X,X}),
\end{aligned}
$$

due to commutativity of the diagram

where the left square and the right triangle commute by naturality of $\mu_{\mathtt{jii}}$ and the right diagram in (2.1), respectively. This shows that $\mathrm{coev}_{\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})X,\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})\,X}$, i.e. the image of (4.14) under the inverse of the first isomorphism in (4.6), is equal to

$$(\mu_{\mathtt{jji}}^{(\mathrm{FC}^X)\mathrm{F}^\star,\mathrm{F}})_X^{-1} \circ_\mathrm{v} \underline{\mathbf{M}}_{\mathtt{ji}}(\alpha_{\mathrm{FC}^X,\mathrm{F}^\star,\mathrm{F}}^{-1})_X \circ_\mathrm{v} \underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{id}_{\mathrm{FC}^X} \circ_\mathrm{h} \mathrm{coev}_\mathrm{F})_X \circ_\mathrm{v} \underline{\mathbf{M}}_{\mathtt{ii}}((v_{\mathrm{FC}^X}^r)^{-1})_X$$
$$\circ_\mathrm{v} (\mu_{\mathtt{jii}}^{\mathrm{F},\mathrm{C}^X})_X \circ_\mathrm{v} (\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})\mathrm{coev}_{X,X}).$$

By (4.6), we have $\mathrm{C}^{\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})X} \cong (\mathrm{FC}^X)\mathrm{F}^\star$ as 1-morphisms and the comultiplication $\delta_{\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})X}$ corresponds to the element

$$f := \left(\mu_{\mathtt{jjj}}^{(\mathrm{FC}^X)\mathrm{F}^\star,(\mathrm{FC}^X)\mathrm{F}^\star}\right)_{\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})X} \circ_\mathrm{v} \left(\underline{\mathbf{M}}_{\mathtt{jj}}\big((\mathrm{FC}^X)\mathrm{F}^\star\big)\mathrm{coev}_{\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})X,\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})X}\right)$$
$$\circ_\mathrm{v} \mathrm{coev}_{\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})X,\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})X}$$

in $\mathrm{Hom}_{\underline{\mathbf{M}}(\mathtt{j})}\left(\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})X, \underline{\mathbf{M}}_{\mathtt{jj}}\big(((\mathrm{FC}^X)\mathrm{F}^\star)((\mathrm{FC}^X)\mathrm{F}^\star)\big)\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})X\right)$. It remains to show that f corresponds to the comultiplication of $(\mathrm{FC}^X)\mathrm{F}^\star$ via the isomorphisms in (4.6), for $\mathrm{G} = \big((\mathrm{FC}^X)\mathrm{F}^\star\big)\big((\mathrm{FC}^X)\mathrm{F}^\star\big)$. First consider the diagram

which commutes by

- naturality of $\mu_{\mathtt{jii}}^{\mathrm{F},\mathrm{C}^X}$ for the facet labeled 1;
- naturality of $\underline{\mathbf{M}}_{\mathtt{ii}}\big((v_{\mathrm{FC}^X}^r)^{-1}\big)$ for the facets labeled 2 and 3;
- naturality of $\underline{\mathbf{M}}_{\mathtt{ji}}\big(\mathrm{id}_{\mathrm{FC}^X} \circ_\mathrm{h} \mathrm{coev}_\mathrm{F}\big)$ for the facets labeled 4 and 5;
- naturality of $\underline{\mathbf{M}}_{\mathtt{ji}}\big(\alpha_{\mathrm{FC}^X,\mathrm{F}^\star,\mathrm{F}}^{-1}\big)$ for the facets labeled 6 and 7;
- naturality of $\mu_{\mathtt{jji}}^{(\mathrm{FC}^X)\mathrm{F}^\star,\mathrm{F}}$ for the facets labeled 8 and 9;
- the diagram below (4.14) for the facets labeled 10 and 11.

To simplify notation, we set $\mathrm{H}_1 := \mathrm{F}^\star\mathrm{F}$ and $\mathrm{H}_2 := (\mathrm{FC}^X)\mathrm{F}^\star$. Then coev_F is a 2-morphism from $\mathbb{1}_\mathtt{i}$ to H_1 and $\alpha_{\mathrm{FC}^X,\mathrm{F}^\star,\mathrm{F}}^{-1}$ is a 2-morphism from $(\mathrm{FC}^X)\mathrm{H}_1$ to $\mathrm{H}_2\mathrm{F}$.

Consider the diagram

$$
\begin{array}{c}
\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{FC}^X)\underline{\mathbf{M}}_{\mathtt{ii}}(C^X\mathbb{1}_{\mathtt{i}})X \xrightarrow{\ \underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{FC}^X)\underline{\mathbf{M}}_{\mathtt{ii}}\left(\mathrm{id}_{C^X}\circ_{\mathtt{h}}\mathrm{coev}_{\mathrm{F}}\right)_X\ } \underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{FC}^X)\underline{\mathbf{M}}_{\mathtt{ii}}(C^X\mathrm{H}_1)X \xrightarrow{\ \underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{FC}^X)\underline{\mathbf{M}}_{\mathtt{ii}}\left(\alpha^{-1}_{C^X,\mathrm{F}^\star,\mathrm{F}}\right)_X\ } \underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{FC}^X)\underline{\mathbf{M}}_{\mathtt{ii}}((C^X\mathrm{F}^\star)\mathrm{F})X
\end{array}
$$

This diagram commutes by

- naturality of $\underline{\mathbf{M}}_{\mathtt{ii}}\big((\upsilon^r_{\mathrm{FC}^X})^{-1}\big)$ for the facets labeled 1 and 2;
- naturality of $\underline{\mathbf{M}}_{\mathtt{ji}}\big(\mathrm{id}_{\mathrm{FC}^X}\circ_{\mathtt{h}}\mathrm{coev}_{\mathrm{F}}\big)$ for the facets labeled 3 and 4;
- naturality of $\underline{\mathbf{M}}_{\mathtt{ji}}\big(\alpha^{-1}_{\mathrm{FC}^X,\mathrm{F}^\star,\mathrm{F}}\big)$ for the facets labeled 5 and 6;
- naturality of $\big(\mu^{\mathrm{H}_2,\mathrm{F}}_{\mathtt{jji}}\big)^{-1}$ for the facets labeled 7 and 8;
- naturality of $\mu_{\mathtt{jii}}$ for the facets labeled 9 and 10;
- naturality of α^{-1} for the facet labeled 11;
- the pentagon coherence condition of the associator for the facet labeled 12.

Further, we consider

This diagram is commutative by

- naturality of $\underline{\mathbf{M}}_{\mathtt{ii}}\big((\upsilon^r_{\mathrm{FC}^X})^{-1}\big)$ for the facet labeled 1;
- naturality of $\mu_{\mathtt{jij}}$ for the facets labeled 2, 4 and 6;
- naturality of $\underline{\mathbf{M}}_{\mathtt{ji}}\big(\mathrm{id}_{\mathrm{FC}^X} \circ_{\mathtt{h}} \mathrm{coev}_{\mathrm{F}}\big)$ for the facet labeled 3;
- naturality of $\underline{\mathbf{M}}_{\mathtt{ji}}\big(\alpha^{-1}_{\mathrm{FC}^X,\mathrm{F}^\star,\mathrm{F}}\big)$ for the facet labeled 5;
- naturality of $\big(\mu^{\mathrm{H}_2,\mathrm{F}}_{\mathtt{jji}}\big)^{-1}$ for the facet labeled 7;
- the diagram in (2.3) for the facets labeled 8 and 9;
- naturality of $\mu_{\mathtt{jji}}$ and $\mu_{\mathtt{jjj}}$ for the facets labeled 10 and 11 respectively.

In the last three diagrams, the last column of the previous diagram coincides with the first column of the next, so we can glue these three diagrams from left to right into one big diagram. By commutativity of this big diagram, the two paths along its boundary from northwest to southeast represent the same 2-morphism. The path which first goes down and then to the right, after precomposing the composite 2-morphism with $\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})\mathrm{coev}_{X,X}$, corresponds to f. By considering the other path along the boundary of the big diagram, we obtain

$$
\begin{aligned}
f =& \underline{\mathbf{M}}_{\mathtt{ji}}\big(\mathrm{id}_{\mathrm{H}_2} \circ_{\mathtt{h}} \alpha^{-1}_{\mathrm{F},\mathrm{C}^X,\mathrm{F}^\star}\big)_{\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})X} \circ_{\mathtt{v}} \underline{\mathbf{M}}_{\mathtt{jj}}\big(\alpha_{\mathrm{H}_2,\mathrm{F},\mathrm{C}^X\mathrm{F}^\star}\big)_{\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})X} \\
& \circ_{\mathtt{v}} \underline{\mathbf{M}}_{\mathtt{jj}}\big(\alpha^{-1}_{\mathrm{FC}^X,\mathrm{F}^\star,\mathrm{F}} \circ_{\mathtt{h}} \mathrm{id}_{\mathrm{C}^X\mathrm{F}^\star}\big)_{\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})X} \circ_{\mathtt{v}} \underline{\mathbf{M}}_{\mathtt{jj}}\big((\mathrm{id}_{\mathrm{FC}^X} \circ_{\mathtt{h}} \mathrm{coev}_{\mathrm{F}}) \circ_{\mathtt{h}} \mathrm{id}_{\mathrm{C}^X\mathrm{F}^\star}\big)_{\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})X} \\
& \circ_{\mathtt{v}} \underline{\mathbf{M}}_{\mathtt{jj}}\big((\upsilon^r_{\mathrm{FC}^X})^{-1} \circ_{\mathtt{h}} \mathrm{id}_{\mathrm{C}^X\mathrm{F}^\star}\big)_{\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})X} \circ_{\mathtt{v}} \big(\mu^{\mathrm{FC}^X,\mathrm{C}^X\mathrm{F}^\star}_{\mathtt{jij}}\big)_{\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})X} \\
& \circ_{\mathtt{v}} \big(\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{FC}^X)\big(\mu^{\mathrm{C}^X\mathrm{F}^\star,\mathrm{F}}_{\mathtt{jji}}\big)^{-1}_X\big) \circ_{\mathtt{v}} \big(\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{FC}^X)\underline{\mathbf{M}}_{\mathtt{ii}}(\alpha^{-1}_{\mathrm{C}^X,\mathrm{F}^\star,\mathrm{F}})X\big) \\
& \circ_{\mathtt{v}} \big(\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{FC}^X)\underline{\mathbf{M}}_{\mathtt{ii}}\big(\mathrm{id}_{\mathrm{C}^X} \circ_{\mathtt{h}} \mathrm{coev}_{\mathrm{F}}\big)_X\big) \circ_{\mathtt{v}} \big(\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{FC}^X)\underline{\mathbf{M}}_{\mathtt{ii}}\big((\upsilon^r_{\mathrm{C}^X})^{-1}\big)_X\big) \\
& \circ_{\mathtt{v}} \big(\mu^{\mathrm{F},\mathrm{C}^X}_{\mathtt{jii}}\big)_{\underline{\mathbf{M}}_{\mathtt{ii}}(\mathrm{C}^X)X} \circ_{\mathtt{v}} \big(\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})\underline{\mathbf{M}}_{\mathtt{ii}}(\mathrm{C}^X)\mathrm{coev}_{X,X}\big) \\
& \circ_{\mathtt{v}} \big(\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})\mathrm{coev}_{X,X}\big)
\end{aligned}
$$

Similarly, applying naturality of μ and using the diagram in (2.3), we obtain that the composite $\left(\mu_{\mathtt{jji}}^{\mathrm{H_2H_2,F}}\right)_X \circ_{\mathsf{v}} f$ equals

$$\underline{\mathbf{M}}_{\mathtt{ji}}\big((\mathrm{id}_{\mathrm{H_2}} \circ_{\mathsf{h}} \alpha_{\mathrm{F,C}^X,\mathrm{F}^\star}^{-1}) \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{F}}\big)_X \circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ji}}(\alpha_{\mathrm{H_2,F,C}^X\mathrm{F}^\star} \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{F}})_X$$
$$\circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ji}}\big((\alpha_{\mathrm{FC}^X,\mathrm{F}^\star,\mathrm{F}}^{-1} \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{C}^X\mathrm{F}^\star}) \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{F}}\big)_X$$
$$\circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ji}}\Big(\big((\mathrm{id}_{\mathrm{FC}^X} \circ_{\mathsf{h}} \mathrm{coev}_{\mathrm{F}}) \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{C}^X\mathrm{F}^\star}\big) \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{F}}\Big)_X$$
$$\circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ji}}\Big(\big((\upsilon_{\mathrm{FC}^X}^r)^{\text{-}1} \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{C}^X\mathrm{F}^\star}\big) \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{F}}\Big)_X \circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ji}}(\alpha_{\mathrm{FC}^X,\mathrm{C}^X\mathrm{F}^\star,\mathrm{F}}^{-1})_X$$
$$\circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ji}}\big(\mathrm{id}_{\mathrm{FC}^X} \circ_{\mathsf{h}} \alpha_{\mathrm{C}^X,\mathrm{F}^\star,\mathrm{F}}^{-1}\big)_X \circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ji}}\big(\mathrm{id}_{\mathrm{FC}^X} \circ_{\mathsf{h}} (\mathrm{id}_{\mathrm{C}^X} \circ_{\mathsf{h}} \mathrm{coev}_{\mathrm{F}})\big)_X$$
$$\circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ji}}\big(\mathrm{id}_{\mathrm{FC}^X} \circ_{\mathsf{h}} (\upsilon_{\mathrm{C}^X}^r)^{\text{-}1}\big)_X \circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ji}}(\alpha_{\mathrm{F,C}^X,\mathrm{C}^X}^{-1})_X \circ_{\mathsf{v}} \big(\mu_{\mathtt{jii}}^{\mathrm{F,C}^X\mathrm{C}^X}\big)_X$$
$$\circ_{\mathsf{v}} \Big(\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})\big(\mu_{\mathtt{iii}}^{\mathrm{C}^X,\mathrm{C}^X}\big)_X\Big) \circ_{\mathsf{v}} \Big(\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})\underline{\mathbf{M}}_{\mathtt{ii}}(\mathrm{C}^X)\mathrm{coev}_{X,X}\Big)$$
$$\circ_{\mathsf{v}} \big(\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})\mathrm{coev}_{X,X}\big)$$
$$= \underline{\mathbf{M}}_{\mathtt{ji}}\big((\mathrm{id}_{\mathrm{H_2}} \circ_{\mathsf{h}} \alpha_{\mathrm{F,C}^X,\mathrm{F}^\star}^{-1}) \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{F}}\big)_X \circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ji}}(\alpha_{\mathrm{H_2,F,C}^X\mathrm{F}^\star} \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{F}})_X$$
$$\circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ji}}\big((\alpha_{\mathrm{FC}^X,\mathrm{F}^\star,\mathrm{F}}^{-1} \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{C}^X\mathrm{F}^\star}) \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{F}}\big)_X$$
$$\circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ji}}\Big(\big((\mathrm{id}_{\mathrm{FC}^X} \circ_{\mathsf{h}} \mathrm{coev}_{\mathrm{F}}) \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{C}^X\mathrm{F}^\star}\big) \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{F}}\Big)_X$$
$$\circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ji}}\Big(\big((\upsilon_{\mathrm{FC}^X}^r)^{\text{-}1} \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{C}^X\mathrm{F}^\star}\big) \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{F}}\Big)_X \circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ji}}(\alpha_{\mathrm{FC}^X,\mathrm{C}^X\mathrm{F}^\star,\mathrm{F}}^{-1})_X$$
$$\circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ji}}\big(\mathrm{id}_{\mathrm{FC}^X} \circ_{\mathsf{h}} \alpha_{\mathrm{C}^X,\mathrm{F}^\star,\mathrm{F}}^{-1}\big)_X \circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ji}}\big(\mathrm{id}_{\mathrm{FC}^X} \circ_{\mathsf{h}} (\mathrm{id}_{\mathrm{C}^X} \circ_{\mathsf{h}} \mathrm{coev}_{\mathrm{F}})\big)_X$$
$$\circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ji}}\big(\mathrm{id}_{\mathrm{FC}^X} \circ_{\mathsf{h}} (\upsilon_{\mathrm{C}^X}^r)^{\text{-}1}\big)_X \circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ji}}(\alpha_{\mathrm{F,C}^X,\mathrm{C}^X}^{-1})_X \circ_{\mathsf{v}} \big(\mu_{\mathtt{jii}}^{\mathrm{F,C}^X\mathrm{C}^X}\big)_X$$
$$\circ_{\mathsf{v}} \Big(\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})\underline{\mathbf{M}}_{\mathtt{ii}}\big(\delta_{X,X,X}\big)_X\Big) \circ_{\mathsf{v}} \big(\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})\mathrm{coev}_{X,X}\big),$$

where the equality follows from the definition of $\delta_{X,X,X} = \delta_{\mathrm{C}^X}$.
Consider the diagram

which commutes due to

- naturality of $\mu_{\mathtt{jii}}$ for the facet labeled 1;
- naturality of $(v^r)^{-1}$ for the facet labeled 2;
- the interchange law for the facets labeled 3 and 8;
- the right diagram in (2.1) for the facets labeled 4 and 5;
- naturality of α^{-1} (respectively α) for the facets labeled 6, 9 and 12 (respectively 10);
- the pentagon coherence condition of the associator for the facets labeled 7 and 11.

By commutativity, the 2-morphisms corresponding to the paths along the boundary of this diagram from northwest to southwest are equal. The path going straight down corresponds to the composite of seven consecutive factors of the expression for $\left(\mu_{\mathtt{jji}}^{\mathrm{H_2 H_2, F}}\right)_X \circ_{\mathsf{v}} f$ above (reading from left to right, these are the factors six until twelve). Replacing those factors by the ones corresponding to the other path along the boundary in the diagram above, yields the equation

(4.15)
$$
\begin{aligned}
\left(\mu_{\mathtt{jji}}^{\mathrm{H_2 H_2, F}}\right)_X \circ_{\mathsf{v}} f = {}& \underline{\mathbf{M}}_{\mathtt{ji}}\big((\mathrm{id}_{\mathrm{H_2}} \circ_{\mathsf{h}} \alpha_{\mathrm{F},\mathrm{C}^X,\mathrm{F}^\star}^{-1}) \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{F}}\big)_X \circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ji}}(\alpha_{\mathrm{H_2},\mathrm{F},\mathrm{C}^X \mathrm{F}^\star} \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{F}})_X \\
& \circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ji}}\big((\alpha_{\mathrm{FC}^X,\mathrm{F}^\star,\mathrm{F}}^{-1} \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{C}^X \mathrm{F}^\star}) \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{F}}\big)_X \\
& \circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ji}}\Big(\big((\mathrm{id}_{\mathrm{FC}^X} \circ_{\mathsf{h}} \mathrm{coev}_{\mathrm{F}}) \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{C}^X \mathrm{F}^\star}\big) \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{F}}\Big)_X \\
& \circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ji}}\Big(\big((v_{\mathrm{FC}^X}^r)^{-1} \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{C}^X \mathrm{F}^\star}\big) \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{F}}\Big)_X \circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ji}}(\alpha_{\mathrm{FC}^X,\mathrm{C}^X,\mathrm{F}^\star} \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{F}})_X \\
& \circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ji}}\big((\alpha_{\mathrm{F},\mathrm{C}^X,\mathrm{C}^X}^{-1} \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{F}^\star}) \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{F}}\big)_X \circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ji}}\Big(\big((\mathrm{id}_{\mathrm{F}} \circ_{\mathsf{h}} \delta_{X,X,X}) \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{F}^\star}\big) \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{F}}\Big)_X \\
& \circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ji}}(\alpha_{\mathrm{FC}^X,\mathrm{F}^\star,\mathrm{F}}^{-1})_X \circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{id}_{\mathrm{FC}^X} \circ_{\mathsf{h}} \mathrm{coev}_{\mathrm{F}})_X \circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ji}}\Big((v_{\mathrm{FC}^X}^r)^{-1}\Big)_X \\
& \circ_{\mathsf{v}} \big(\mu_{\mathtt{jii}}^{\mathrm{F},\mathrm{C}^X}\big)_X \circ_{\mathsf{v}} \big(\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})\mathrm{coev}_{X,X}\big).
\end{aligned}
$$

Now we prove that $\delta_{\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})X}$ is equal to $\delta_{(\mathrm{FC}^X)\mathrm{F}^\star}$, as defined in Lemma 4.13. On one hand, the composite of the first three isomorphisms in (4.6) sends f to

(4.16)
$$
\begin{aligned}
& \big(\mu_{\mathtt{iji}}^{\mathrm{F}^\star,(\mathrm{H_2 H_2})\mathrm{F}}\big)_X \circ_{\mathsf{v}} \Big(\underline{\mathbf{M}}_{\mathtt{ij}}(\mathrm{F}^\star)\big(\mu_{\mathtt{jji}}^{\mathrm{H_2 H_2, F}}\big)_X\Big) \circ_{\mathsf{v}} \big(\underline{\mathbf{M}}_{\mathtt{ij}}(\mathrm{F}^\star)\big)f \\
& \circ_{\mathsf{v}} \big(\mu_{\mathtt{iji}}^{\mathrm{F}^\star,\mathrm{F}}\big)_X^{-1} \circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ii}}(\mathrm{coev}_{\mathrm{F}}) \circ_{\mathsf{v}} (\iota_{\mathtt{i}}^{-1})_X
\end{aligned}
$$

in $\mathrm{Hom}_{\underline{\mathbf{M}}(\mathtt{i})}\Big(X, \underline{\mathbf{M}}_{\mathtt{ii}}\big(\mathrm{F}^\star((\mathrm{H_2 H_2})\mathrm{F})\big)X\Big)$.

On the other hand, chasing the image of $\delta_{\mathrm{H_2}}$, where $\mathrm{H_2} = (\mathrm{FC}^X)\mathrm{F}^\star$, the last isomorphism

$$
\mathrm{Hom}_{\mathscr{C}}(\mathrm{H_2}, \mathrm{H_2 H_2}) \xrightarrow{\cong} \mathrm{Hom}_{\mathscr{C}}\big(\mathrm{FC}^X, (\mathrm{H_2 H_2})\mathrm{F}\big)
$$

sends $\delta_{\mathrm{H_2}}$ to

(4.17) $(\delta_{\mathrm{H_2}} \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{F}}) \circ_{\mathsf{v}} \alpha_{\mathrm{FC}^X,\mathrm{F}^\star,\mathrm{F}}^{-1} \circ_{\mathsf{v}} (\mathrm{id}_{\mathrm{FC}^X} \circ_{\mathsf{h}} \mathrm{coev}_{\mathrm{F}}) \circ_{\mathsf{v}} (v_{\mathrm{FC}^X}^r)^{-1}.$

The fifth isomorphism

$$
\mathrm{Hom}_{\mathscr{C}}\big(\mathrm{FC}^X, (\mathrm{H_2 H_2})\mathrm{F}\big) \cong \mathrm{Hom}_{\mathscr{C}}\Big(\mathrm{C}^X, \mathrm{F}^\star\big((\mathrm{H_2 H_2})\mathrm{F}\big)\Big)
$$

sends (4.17) to

(4.18)
$$
\begin{aligned}
& \Big(\mathrm{id}_{\mathrm{F}^\star} \circ_{\mathsf{h}} \big(\delta_{\mathrm{H_2}} \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{F}}\big)\Big) \circ_{\mathsf{v}} \big(\mathrm{id}_{\mathrm{F}^\star} \circ_{\mathsf{h}} \alpha_{\mathrm{FC}^X,\mathrm{F}^\star,\mathrm{F}}^{-1}\big) \circ_{\mathsf{v}} \Big(\mathrm{id}_{\mathrm{F}^\star} \circ_{\mathsf{h}} \big(\mathrm{id}_{\mathrm{FC}^X} \circ_{\mathsf{h}} \mathrm{coev}_{\mathrm{F}}\big)\Big) \\
& \circ_{\mathsf{v}} \big(\mathrm{id}_{\mathrm{F}^\star} \circ_{\mathsf{h}} (v_{\mathrm{FC}^X}^r)^{-1}\big) \circ_{\mathsf{v}} \alpha_{\mathrm{F}^\star,\mathrm{F},\mathrm{C}^X}^{-1} \circ_{\mathsf{v}} \big(\mathrm{coev}_{\mathrm{F}} \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{C}^X}\big) \circ_{\mathsf{v}} (v_{\mathrm{C}^X}^l)^{-1}.
\end{aligned}
$$

The fourth isomorphism

$$\operatorname{Hom}_{\mathscr{C}}\Big(\mathrm{C}^X, \mathrm{F}^\star\big((\mathrm{H}_2\mathrm{H}_2)\mathrm{F}\big)\Big) \xrightarrow{\cong} \operatorname{Hom}_{\underline{\mathbf{M}}(\mathbf{i})}\Big(X, \underline{\mathbf{M}}_{\mathtt{ii}}\big(\mathrm{F}^\star\big((\mathrm{H}_2\mathrm{H}_2)\mathrm{F}\big)\big)X\Big)$$

sends (4.18) to

$$\underline{\mathbf{M}}_{\mathtt{ii}}\Big(\mathrm{id}_{\mathrm{F}^\star} \circ_{\mathsf{h}} \big(\delta_{\mathrm{H}_2} \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{F}}\big)\Big)_X \circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ii}}\big(\mathrm{id}_{\mathrm{F}^\star} \circ_{\mathsf{h}} \alpha^{-1}_{\mathrm{FC}^X, \mathrm{F}^\star, \mathrm{F}}\big)_X$$

$$(4.19) \qquad \circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ii}}\Big(\mathrm{id}_{\mathrm{F}^\star} \circ_{\mathsf{h}} \big(\mathrm{id}_{\mathrm{FC}^X} \circ_{\mathsf{h}} \mathrm{coev}_{\mathrm{F}}\big)\Big)_X \circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ii}}\big(\mathrm{id}_{\mathrm{F}^\star} \circ_{\mathsf{h}} (\upsilon^r_{\mathrm{FC}^X})^{-1}\big)_X$$

$$\circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ii}}\big(\alpha^{-1}_{\mathrm{F}^\star, \mathrm{F}, \mathrm{C}^X}\big)_X \circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ii}}\big(\mathrm{coev}_{\mathrm{F}} \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{C}^X}\big)_X \circ_{\mathsf{v}} \underline{\mathbf{M}}_{\mathtt{ii}}\big((\upsilon^l_{\mathrm{C}^X})^{-1}\big)_X$$

$$\circ_{\mathsf{v}} \mathrm{coev}_{X,X}.$$

Finally, consider the diagram

which commutes due to

- naturality of $\iota_{\mathtt{i}}^{-1}$ for the facet labeled 1;
- the right diagram in (2.2) for the facet labeled 2;
- naturality of $\mu_{\mathtt{iii}}$ for the facet labeled 3;
- naturality of $\underline{\mathbf{M}}_{\mathtt{ii}}(\mathrm{coev}_{\mathrm{F}})$ for the facet labeled 4;
- the diagram in (2.3) for the facet labeled 5;

- naturality of $\left(\mu_{\mathtt{iji}}^{\mathrm{F}^\star,\mathrm{F}}\right)^{-1}$ for the facet labeled 6;
- naturality of $\mu_{\mathtt{iji}}$ for the facets labeled 7, 8, 9 and 10.

As above, commutativity guarantees that the 2-morphisms corresponding to the paths along the boundary from northwest to southwest coincide. The path which goes straight down corresponds to the 2-morphisms in (4.19), which is therefore equal to

$$\left(\mu_{\mathtt{iji}}^{\mathrm{F}^\star,(\mathrm{H}_2\mathrm{H}_2)\mathrm{F}}\right)_X \circ_\mathsf{v} \left(\underline{\mathbf{M}}_{\mathtt{ij}}(\mathrm{F}^\star)\underline{\mathbf{M}}_{\mathtt{ji}}(\delta_{\mathrm{H}_2} \circ_\mathsf{h} \mathrm{id}_{\mathrm{F}})_X\right) \circ_\mathsf{v} \left(\underline{\mathbf{M}}_{\mathtt{ij}}(\mathrm{F}^\star)\underline{\mathbf{M}}_{\mathtt{ji}}(\alpha_{\mathrm{FC}^X,\mathrm{F}^\star,\mathrm{F}}^{-1})_X\right)$$

$$\circ_\mathsf{v} \left(\underline{\mathbf{M}}_{\mathtt{ij}}(\mathrm{F}^\star)\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{id}_{\mathrm{FC}^X} \circ_\mathsf{h} \mathrm{coev}_{\mathrm{F}})_X\right) \circ_\mathsf{v} \left(\underline{\mathbf{M}}_{\mathtt{ij}}(\mathrm{F}^\star)\underline{\mathbf{M}}_{\mathtt{ji}}((v_{\mathrm{FC}^X}^r)^{-1})_X\right)$$

$$\circ_\mathsf{v} \left(\underline{\mathbf{M}}_{\mathtt{ij}}(\mathrm{F}^\star)(\mu_{\mathtt{jii}}^{\mathrm{F},\mathrm{C}^X})_X\right) \circ_\mathsf{v} \left(\underline{\mathbf{M}}_{\mathtt{ij}}(\mathrm{F}^\star)\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})\mathrm{coev}_{X,X}\right)$$

$$\circ_\mathsf{v} (\mu_{\mathtt{iji}}^{\mathrm{F}^\star,\mathrm{F}})_X^{-1} \circ_\mathsf{v} \underline{\mathbf{M}}_{\mathtt{ii}}(\mathrm{coev}_{\mathrm{F}})_X \circ_\mathsf{v} (\iota_{\mathtt{i}})_X^{-1}.$$

This expression coincides precisely with that in (4.16), considering (4.15) and the definition of $\delta_{\mathrm{H}_2} = \delta_{(\mathrm{FC}^X)\mathrm{F}^\star}$ as in Lemma 4.13. This completes the proof. $\qquad\square$

As in [MMMT1, Corollary 5.2] (see also [EGNO, Lemma 7.9.4]), the internal co-homs $[X, \underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})\,X]$ and $[\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})\,X, X]$ are the biinjective $(\mathrm{FC}^X)\mathrm{F}^\star$-$\mathrm{C}^X$- respectively C^X-$(\mathrm{FC}^X)\mathrm{F}^\star$-bicomodules inducing this MT equivalence. Firstly, noting that $\mathrm{C}^X = [X, X]$ and $(\mathrm{FC}^X)\mathrm{F}^\star \cong [\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})\,X, \underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})\,X]$, we have

$$\begin{aligned}
\mathrm{Hom}_{\underline{\mathscr{C}}(\mathtt{i},\mathtt{j})}\big([X, \underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})\,X], \mathrm{G}\big) &\cong \mathrm{Hom}_{\underline{\mathbf{M}}(\mathtt{j})}\big(\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})\,X, \underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{G})\,X\big) \\
&\cong \mathrm{Hom}_{\underline{\mathbf{M}}(\mathtt{i})}\big(X, \underline{\mathbf{M}}_{\mathtt{ij}}(\mathrm{F}^\star)\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{G})\,X\big) \\
&\cong \mathrm{Hom}_{\underline{\mathbf{M}}(\mathtt{i})}\big(X, \underline{\mathbf{M}}_{\mathtt{ii}}(\mathrm{F}^\star\mathrm{G})\,X\big) \\
&\cong \mathrm{Hom}_{\underline{\mathscr{C}}(\mathtt{i},\mathtt{i})}(\mathrm{C}^X, \mathrm{F}^\star\mathrm{G}) \\
&\cong \mathrm{Hom}_{\underline{\mathscr{C}}(\mathtt{i},\mathtt{j})}(\mathrm{FC}^X, \mathrm{G})
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Hom}_{\underline{\mathscr{C}}(\mathtt{j},\mathtt{i})}\big([\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})\,X, X], \mathrm{H}\big) &\cong \mathrm{Hom}_{\underline{\mathbf{M}}(\mathtt{i})}\big(X, \underline{\mathbf{M}}_{\mathtt{ij}}(\mathrm{H})\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})\,X\big) \\
&\cong \mathrm{Hom}_{\underline{\mathbf{M}}(\mathtt{i})}\big(X, \underline{\mathbf{M}}_{\mathtt{ii}}(\mathrm{HF})\,X\big) \\
&\cong \mathrm{Hom}_{\underline{\mathscr{C}}(\mathtt{i},\mathtt{i})}(\mathrm{C}^X, \mathrm{HF}) \\
&\cong \mathrm{Hom}_{\underline{\mathscr{C}}(\mathtt{j},\mathtt{i})}(\mathrm{C}^X\mathrm{F}^\star, \mathrm{H})
\end{aligned}$$

for all 1-morphisms $\mathrm{G} \in \underline{\mathscr{C}}(\mathtt{i},\mathtt{j})$ and $\mathrm{H} \in \underline{\mathscr{C}}(\mathtt{j},\mathtt{i})$. Therefore, there are isomorphisms of 1-morphisms

$$(4.20) \qquad [X, \underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})\,X] \cong \mathrm{FC}^X \quad \text{and} \quad [\underline{\mathbf{M}}_{\mathtt{ji}}(\mathrm{F})\,X, X] \cong \mathrm{C}^X\mathrm{F}^\star.$$

Corollary 4.17. *Under the same assumptions as in Theorem 4.16, the coalgebras C^X and $(\mathrm{FC}^X)\mathrm{F}^\star \cong \mathrm{F}(\mathrm{C}^X\mathrm{F}^\star)$ are MT equivalent. Moreover, the MT equivalence is realized by the bicomodules FC^X and $\mathrm{C}^X\mathrm{F}^\star$, whose right and left C^X-comodule structures, respectively, are the canonical ones and whose left and right $(\mathrm{FC}^X)\mathrm{F}^\star$-comodule structures, respectively, are given by*

$$\alpha_{\mathrm{FC}^X,\mathrm{F}^\star,\mathrm{FC}^X}^{-1} \circ_\mathsf{v} \left(\mathrm{id}_{\mathrm{FC}^X} \circ_\mathsf{h} \alpha_{\mathrm{F}^\star,\mathrm{F},\mathrm{C}^X}\right) \circ_\mathsf{v} \left(\mathrm{id}_{\mathrm{FC}^X} \circ_\mathsf{h} (\mathrm{coev}_{\mathrm{F}} \circ_\mathsf{h} \mathrm{id}_{\mathrm{C}^X})\right)$$

$$\circ_\mathsf{v} \left(\mathrm{id}_{\mathrm{FC}^X} \circ_\mathsf{h} (v_{\mathrm{C}^X}^l)^{-1}\right) \circ_\mathsf{v} \alpha_{\mathrm{F},\mathrm{C}^X,\mathrm{C}^X}^{-1} \circ_\mathsf{v} \left(\mathrm{id}_{\mathrm{F}} \circ_\mathsf{h} \delta_{\mathrm{C}^X}\right)$$

and

$$\left(\mathrm{id}_{\mathrm{C}^X\mathrm{F}^\star} \circ_\mathsf{h} \alpha_{\mathrm{F},\mathrm{C}^X,\mathrm{F}^\star}^{-1}\right) \circ_\mathsf{v} \alpha_{\mathrm{C}^X\mathrm{F}^\star,\mathrm{F},\mathrm{C}^X\mathrm{F}^\star} \circ_\mathsf{v} \left(\alpha_{\mathrm{C}^X,\mathrm{F}^\star,\mathrm{F}}^{-1} \circ_\mathsf{h} \mathrm{id}_{\mathrm{C}^X\mathrm{F}^\star}\right)$$

$$\circ_\mathsf{v} \left((\mathrm{id}_{\mathrm{C}^X} \circ_\mathsf{h} \mathrm{coev}_{\mathrm{F}}) \circ_\mathsf{h} \mathrm{id}_{\mathrm{C}^X\mathrm{F}^\star}\right) \circ_\mathsf{v} \left((v_{\mathrm{C}^X}^r)^{-1} \circ_\mathsf{h} \mathrm{id}_{\mathrm{C}^X\mathrm{F}^\star}\right)$$

$$\circ_\mathsf{v} \alpha_{\mathrm{C}^X,\mathrm{C}^X,\mathrm{F}^\star} \circ_\mathsf{v} \left(\delta_{\mathrm{C}^X} \circ_\mathsf{h} \mathrm{id}_{\mathrm{F}^\star}\right).$$

Proof. The first statement follows immediately from Corollary 4.12 and Theorem 4.16. The proof of the second statement is more involved. On the one hand, by the first isomorphism in (3.2), we have

$$(\mathrm{F}\mathrm{C}^X)\mathrm{F}^\star \cong (\mathrm{F}\mathrm{C}^X \square_{\mathrm{C}^X} \mathrm{C}^X)\mathrm{F}^\star \cong (\mathrm{F}\mathrm{C}^X)\square_{\mathrm{C}^X}(\mathrm{C}^X\mathrm{F}^\star).$$

On the other hand, to prove $(\mathrm{C}^X\mathrm{F}^\star)\square_{(\mathrm{F}\mathrm{C}^X)\mathrm{F}^\star}(\mathrm{F}\mathrm{C}^X) \cong \mathrm{C}^X$, we consider the following diagram

(4.21)

$$
\begin{array}{c}
(\mathrm{C}^X\mathrm{F}^\star)\square_{(\mathrm{F}\mathrm{C}^X)\mathrm{F}^\star}(\mathrm{F}\mathrm{C}^X) \xleftarrow{\ \beta\ } (\mathrm{C}^X\mathrm{F}^\star)(\mathrm{F}\mathrm{C}^X) \xrightarrow{\delta_{\mathrm{C}^X\mathrm{F}^\star,(\mathrm{F}\mathrm{C}^X)\mathrm{F}^\star}\circ_\mathrm{h}\mathrm{id}_{\mathrm{F}\mathrm{C}^X}} \big((\mathrm{C}^X\mathrm{F}^\star)((\mathrm{F}\mathrm{C}^X)\mathrm{F}^\star)\big)\big(\mathrm{F}\mathrm{C}^X\big) \\[2mm]
\theta \nwarrow \quad \gamma \nearrow \qquad \mathrm{id}_{\mathrm{C}^X\mathrm{F}^\star}\circ_\mathrm{h}\delta_{(\mathrm{F}\mathrm{C}^X)\mathrm{F}^\star,\mathrm{F}\mathrm{C}^X} \searrow \qquad \swarrow \alpha_{\mathrm{C}^X\mathrm{F}^\star,(\mathrm{F}\mathrm{C}^X)\mathrm{F}^\star,\mathrm{F}\mathrm{C}^X} \\[2mm]
\mathrm{C}^X \qquad\qquad (\mathrm{C}^X\mathrm{F}^\star)\big(((\mathrm{F}\mathrm{C}^X)\mathrm{F}^\star)(\mathrm{F}\mathrm{C}^X)\big)
\end{array}
$$

where $\delta_{(\mathrm{F}\mathrm{C}^X)\mathrm{F}^\star,\mathrm{F}\mathrm{C}^X}$ and $\delta_{\mathrm{C}^X\mathrm{F}^\star,(\mathrm{F}\mathrm{C}^X)\mathrm{F}^\star}$ are the left and right $(\mathrm{F}\mathrm{C}^X)\mathrm{F}^\star$-coaction 2-morphisms, respectively, and γ is given by

$$\alpha_{\mathrm{C}^X\mathrm{F}^\star,\mathrm{F},\mathrm{C}^X} \circ_\mathrm{v} \big(\alpha^{-1}_{\mathrm{C}^X,\mathrm{F}^\star,\mathrm{F}} \circ_\mathrm{h} \mathrm{id}_{\mathrm{C}^X}\big) \circ_\mathrm{v} \big((\mathrm{id}_{\mathrm{C}^X} \circ_\mathrm{h} \mathrm{coev}_\mathrm{F}) \circ_\mathrm{h} \mathrm{id}_{\mathrm{C}^X}\big)$$
$$\circ_\mathrm{v} \big((v^r_{\mathrm{C}^X})^{-1} \circ_\mathrm{h} \mathrm{id}_{\mathrm{C}^X}\big) \circ_\mathrm{v} \delta_{\mathrm{C}^X}.$$

Now we claim that γ equalizes the right triangle in the diagram (4.21), which, by the universal property of the equalizer, implies that there exists a unique 2-morphism θ such that the left triangle in the diagram (4.21) commutes. Consider the diagram

which commutes due to

- coassociativity of δ_{C^X} for the facet labeled 1;
- naturality of $(v^r)^{-1}$ for the facet labeled 2;
- the interchange law for the facets labeled 3, 4, 8, 10, 13, 14, 15, 16, 17 and 18;
- naturality of α and α^{-1} for the facets labeled 5, 7, 9, 11, 12, 19, 20, 21, 22 and 23;
- the triangle coherence condition of the unitors for the facet labeled 6,

and the diagram
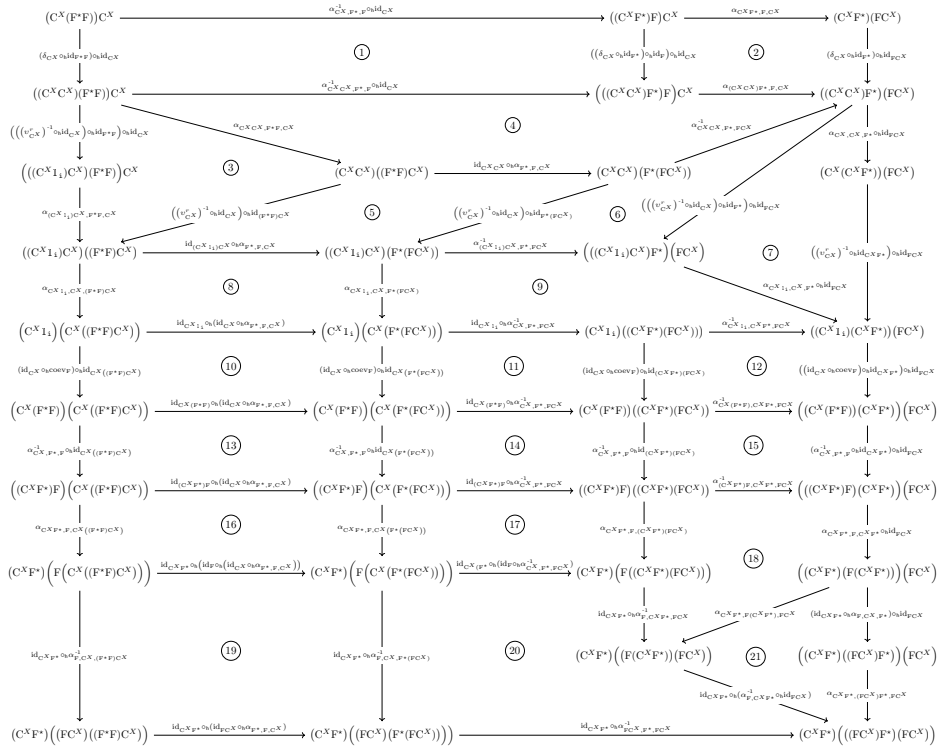
which commutes due to

- naturality of α and α^{-1} for the facets labeled 1, 2, 3, 6, 7, 8, 12, 15, 16, 17, 19 and 21;
- the pentagon coherence condition of the associator for the facets labeled 4, 9, 18 and 20;
- the interchange law for the facets labeled 5, 10, 11, 13 and 14.

The last column of the former diagram coincides with the first column of the latter one, so we can glue the above two diagrams from left to right. Commutativity of the resulting big diagram proves the claim, as the two paths along its boundary from northwest to southeast correspond precisely to the two paths along the boundary of the right triangle in (4.21) precomposed with γ.

By applying F to the diagram in (4.21) from the left, we obtain the upper part of the diagram

Note that $F\big((C^X F^\star)\square_{(FC^X)F^\star}(FC^X)\big)$ is the equalizer of the right upper triangle, since the functor F is left exact when acting on $\mathrm{comod}_{\mathscr{C}}(C^X)$. As in the proof of Lemma 3.9, the associator $\alpha^{-1}_{F,C^X F^\star,FC^X}$ induces the 2-isomorphism φ (see the second isomorphism in (3.2)), such that the pentagon labeled 1 commutes. All other vertical facets also commute: the facet labeled 2 by naturality of the associator, the one labeled 3 by the pentagon coherence condition for the associator, and the triangle labeled 4 by definition of $\delta_{F(C^X F^\star),(FC^X)F^\star}$. Further, $\alpha^{-1}_{F,C^X F^\star,FC^X} \circ_{\mathsf{v}} (\mathrm{id}_F \circ_{\mathsf{h}} \gamma)$ equalizes the right bottom triangle, whence, by the universal property of kernels, the 2-morphism $\varphi \circ_{\mathsf{v}} (\mathrm{id}_F \circ_{\mathsf{h}} \theta)$ provides a 2-isomorphism

$$\big(F(C^X F^\star)\big)\square_{(FC^X)F^\star}\big(FC^X\big) \cong (FC^X)F^\star\square_{(FC^X)F^\star}\big(FC^X\big) \cong FC^X.$$

In particular, $\mathrm{id}_F \circ_{\mathsf{h}} \theta$ provides a 2-isomorphism

$$F\big((C^X F^\star)\square_{(FC^X)F^\star}(FC^X)\big) \cong FC^X.$$

Analogously, for any $H \in \mathscr{C}$, we have the 2-isomorphisms

$$\big(HF\big)\big((C^X F^\star)\square_{(FC^X)F^\star}(FC^X)\big) \cong H\Big(F\big((C^X F^\star)\square_{(FC^X)F^\star}(FC^X)\big)\Big)$$
$$\cong H(FC^X)$$
$$\cong (HF)C^X$$

where the first and third 2-isomorphisms are given by the associator and the second 2-isomorphism is induced by $\mathrm{id}_H(\mathrm{id}_F \circ_{\mathsf{h}} \theta)$. By naturality of the associator, the composite of the above three 2-isomorphisms equals $\mathrm{id}_{HF} \circ \theta$. Therefore we have

$$K\big((C^X F^\star)\square_{(FC^X)F^\star}(FC^X)\big) \cong KC^X.$$

for any direct summand K of HF with some $H \in \mathscr{C}$, and this is functorial. Note that

$$(KC^X)\square_{C^X}\big((C^X F^\star)\square_{(FC^X)F^\star}(FC^X)\big) \cong \big((KC^X)\square_{C^X}(C^X F^\star)\big)\square_{(FC^X)F^\star}(FC^X)$$
$$\cong \big(K(C^X F^\star)\big)\square_{(FC^X)F^\star}(FC^X)$$
$$\cong K\big((C^X F^\star)\square_{(FC^X)F^\star}(FC^X)\big)$$
$$\cong KC^X$$

where the first 2-isomorphism is given by the associator, the second and third ones are due to the second 2-isomorphism in (3.2), and the fourth one is induced by θ. Combining this with the fact that FC^X generates $\mathrm{inj}_{\mathscr{C}}(C^X)$, we see that the natural

transformation $\_\Box_{\mathrm{C}^X}\theta\colon \_\Box_{\mathrm{C}^X}\mathrm{C}^X \to \_\Box_{\mathrm{C}^X}\big((\mathrm{C}^X\mathrm{F}^\star)\Box_{(\mathrm{FC}^X)\mathrm{F}^\star}(\mathrm{FC}^X)\big)$ is an isomorphism. This implies that θ is a 2-isomorphism. $\square$

4.5. Avoiding abelianizations.

Proposition 4.18. *Let $\mathscr{C}$ be a $\mathcal{J}$-simple quasi multifiab bicategory and $\mathrm{F} \in \mathcal{J}$. The pseudofunctor*

$$(\mathrm{F}\_)\mathrm{F}^\star\colon \underline{\mathscr{C}} \to \underline{\mathscr{C}}, \quad \mathrm{G} \mapsto (\mathrm{FG})\mathrm{F}^\star$$

takes values in $\mathrm{inj}(\underline{\mathscr{C}}) \cong \mathscr{C}$.

Proof. Let us consider the $\mathcal{J} \boxtimes \mathcal{J}^{\mathrm{op}}$-simple fiab bicategory

$$\mathscr{C}^{\mathrm{e}} = \mathscr{C} \boxtimes \mathscr{C}^{\mathrm{op}},$$

cf. [MM6, Section 6 and Proposition 21]. Note that $\mathscr{C}$ is a birepresentation of $\mathscr{C}^{\mathrm{e}}$, and thus, by $\mathcal{J}$-simplicity, $\mathrm{add}(\mathcal{J})$ is a simple transitive birepresentation of $\mathscr{C}^{\mathrm{e}}$. By construction, $\mathrm{add}(\mathcal{J})$ has apex $\mathcal{J}\boxtimes\mathcal{J}^{\mathrm{op}}$ in $\mathscr{C}^{\mathrm{e}}$. From the straightforward generalization of [KMMZ, Theorem 2] to bicategories, we know that $(\mathrm{FX})\mathrm{F}^\star$ is injective in $\underline{\mathrm{add}(\mathcal{J})}$ for any $\mathrm{X} \in \underline{\mathrm{add}(\mathcal{J})}$. Finally, since for any simple 1-morphism L in $\underline{\mathscr{C}}$ we have

$$(\mathrm{FL})\mathrm{F}^\star = 0 \quad \Leftrightarrow \quad \mathrm{L}\text{ is not supported in }\mathcal{J},$$

cf. [MM6, Proposition 26], the claim follows. $\square$

Theorem 4.19. *Let $\mathscr{C}$ be a $\mathcal{J}$-simple quasi multifiab bicategory and $\mathbf{M}$ a transitive birepresentation of $\mathscr{C}$ with apex $\mathcal{J}$. Then, for any $X \in \mathbf{M}(\mathtt{i}), Y \in \mathbf{M}(\mathtt{j})$, the 1-morphism $[X,Y]$ belongs to $\mathscr{C}(\mathtt{i},\mathtt{j})$ (not only to $\underline{\mathscr{C}}(\mathtt{i},\mathtt{j})$).*

Proof. Let $X \in \mathbf{M}(\mathtt{i}), Y \in \mathbf{M}(\mathtt{j})$, fix an arbitrary $\mathcal{H}$-cell $\mathcal{H}$ inside $\mathcal{J}$ and denote $\mathtt{k} := \mathtt{i}_{s(\mathcal{H})}$ and $\mathtt{t} := \mathtt{i}_{t(\mathcal{H})}$. By Lemma 2.33, we can choose a generator $Z \in \mathbf{M}(\mathtt{k})$ such that, for any $\mathrm{F} \in \mathcal{H}$, $\mathbf{M}_{\mathtt{tk}}(\mathrm{F})Z$ also generates $\mathbf{M}$. Therefore, there exist 1-morphisms $\mathrm{G} \in \mathscr{C}(\mathtt{t},\mathtt{i}), \mathrm{H} \in \mathscr{C}(\mathtt{t},\mathtt{j})$ such that

$$\mathbf{M}_{\mathtt{ik}}(\mathrm{GF})\, Z \cong \mathbf{M}_{\mathtt{it}}(\mathrm{G})\mathbf{M}_{\mathtt{tk}}(\mathrm{F})\, Z \cong X \oplus X',$$
$$\mathbf{M}_{\mathtt{jk}}(\mathrm{HF})\, Z \cong \mathbf{M}_{\mathtt{jt}}(\mathrm{H})\mathbf{M}_{\mathtt{tk}}(\mathrm{F})\, Z \cong Y \oplus Y'.$$

for some $X' \in \mathbf{M}(\mathtt{i})$ and $Y' \in \mathbf{M}(\mathtt{j})$.
Next, recall from (4.20) that

$$[\mathbf{M}_{\mathtt{ik}}(\mathrm{GF})\, Z, \mathbf{M}_{\mathtt{jk}}(\mathrm{HF})\, Z] \cong \big((\mathrm{HF})[Z,Z]\big)(\mathrm{GF})^\star \cong \Big(\mathrm{H}\big((\mathrm{F}[Z,Z])\mathrm{F}^\star\big)\Big)\mathrm{G}^\star,$$

where the last isomorphism is obtained by using the associator several times and $(\mathrm{GF})^\star = \mathrm{F}^\star\mathrm{G}^\star$. By Proposition 4.18, we know that $(\mathrm{F}[Z,Z])\mathrm{F}^\star$ belongs to $\mathscr{C}$ for all 1-morphisms F and thus $[\mathbf{M}_{\mathtt{ik}}(\mathrm{GF})\, Z, \mathbf{M}_{\mathtt{jk}}(\mathrm{HF})\, Z]$ also belongs to $\mathscr{C}$. Since the internal cohom is additive in both entries, we see that $[X,Y]$ is a direct summand of $[\mathbf{M}_{\mathtt{ik}}(\mathrm{GF})\, Z, \mathbf{M}_{\mathtt{jk}}(\mathrm{HF})\, Z]$ and therefore, it belongs to $\mathscr{C}$ as well. $\square$

Example 4.20. For any coalgebra 1-morphism C in $\underline{\mathscr{C}}$ we have

$$\mathrm{C} \cong [\mathrm{C},\mathrm{C}],$$

as follows e.g. from [ChMi, Lemma 3]. However, this does not contradict Theorem 4.19, since a coalgebra C which is strictly in $\underline{\mathscr{C}}$ will correspond to a birepresentation $\mathbf{M}$ that is either not transitive or has smaller apex.

4.6. **Simple transitive birepresentations and coalgebras.** Simple transitive birepresentations correspond to particularly nice coalgebras (compare [MMMT1, Corollary 4.9] and [MMMZ, Corollary 12]).

Proposition 4.21. *Let $\mathscr{C}$ be a quasi multifiab bicategory and $\mathcal{J}$ a two-sided cell. If $\mathbf{M} \in \mathscr{C}\text{-afmod}_{\mathcal{J}}$, then, for any fixed left cell $\mathcal{L}$ inside $\mathcal{J}$, there is a 1-morphism $\mathrm{C} \in \underline{\mathrm{add}\big(\mathcal{H}(\mathcal{L})\big)} \subseteq \underline{\mathrm{add}(\mathcal{J})}$ which has a coalgebra structure in $\underline{\mathscr{C}}$ such that*

(4.22)
$$\mathbf{M} \simeq \mathbf{inj}_{\underline{\mathscr{C}}}(\mathrm{C}).$$

If, moreover, $\mathscr{C}$ is $\mathcal{J}$-simple, then we can choose $\mathrm{C} \in \mathrm{add}\big(\mathcal{H}(\mathcal{L})\big) \subseteq \mathrm{add}(\mathcal{J})$. If $\mathbf{M} \in \mathscr{C}\text{-stmod}_{\mathcal{J}}$, then such a coalgebra C is cosimple. Conversely, if $\mathrm{C} \in \underline{\mathrm{add}(\mathcal{J})}$ is a cosimple coalgebra in $\underline{\mathscr{C}}$, then $\mathbf{inj}_{\underline{\mathscr{C}}}(\mathrm{C})$ is a simple transitive birepresentation of $\mathscr{C}$ with apex $\mathcal{J}$.

Proof. Due to the biequivalence (2.11) for $\mathcal{J}' = \mathcal{J}$, without loss of generality, we may assume that $\mathcal{J}$ is the unique maximal two-sided cell of $\mathscr{C}$.
Set $\mathcal{H} := \mathcal{H}(\mathcal{L})$ and let $\mathtt{i}$ be the source of $\mathcal{H}$. By Lemma 2.33, for any $\mathbf{M} \in \mathscr{C}\text{-afmod}_{\mathcal{J}}$, there is a generator $X \in \mathbf{M}(\mathtt{i})$ of $\mathbf{M}$ such that, for any $\mathrm{F} \in \mathcal{H}$, $\mathbf{M}_{\mathtt{ii}}(\mathrm{F})X$ also generates $\mathbf{M}$. By Theorem 4.10, there is a biequivalence

$$\mathbf{M} \simeq \mathbf{inj}_{\underline{\mathscr{C}}}(\mathrm{C}^X),$$

where $\mathrm{C}^X \in \underline{\mathscr{C}}(\mathtt{i}, \mathtt{i})$. By Corollary 4.12 and Theorem 4.16, for any $\mathrm{F} \in \mathcal{H}$, the coalgebra C^X is MT equivalent to $\mathrm{C}^{\mathbf{M}_{\mathtt{ii}}(\mathrm{F})X} \cong (\mathrm{F}\mathrm{C}^X)\mathrm{F}^{\star} \in \underline{\mathscr{C}}(\mathtt{i}, \mathtt{i})$. Suppose that C^X is given by $\mathrm{C}_1^X \xrightarrow{\beta} \mathrm{C}_2^X$ in $\underline{\mathscr{C}}(\mathtt{i}, \mathtt{i})$. Then $(\mathrm{F}\mathrm{C}^X)\mathrm{F}^{\star}$ is given by

$$(\mathrm{F}\mathrm{C}_1^X)\mathrm{F}^{\star} \xrightarrow{(\mathrm{id}_{\mathrm{F}} \circ_{\mathrm{h}} \beta) \circ_{\mathrm{h}} \mathrm{id}_{\mathrm{F}^{\star}}} (\mathrm{F}\mathrm{C}_2^X)\mathrm{F}^{\star}.$$

Since $\mathcal{J}$ is the unique maximal two-sided cell of $\mathscr{C}$, the 1-morphisms $(\mathrm{F}\mathrm{C}_1^X)\mathrm{F}^{\star}$ and $(\mathrm{F}\mathrm{C}_2^X)\mathrm{F}^{\star}$ belong to $\mathrm{add}(\mathcal{H})$, whence $\mathrm{C} := (\mathrm{F}\mathrm{C}^X)\mathrm{F}^{\star}$ belongs to $\underline{\mathrm{add}(\mathcal{H})}$. This proves the first claim of the proposition.
If $\mathscr{C}$ is $\mathcal{J}$-simple, then C already belongs to $\mathrm{add}(\mathcal{H})$ as a result of Theorem 4.19.
If $\mathbf{M} \in \mathscr{C}\text{-stmod}_{\mathcal{J}}$, then the coalgebra C satisfying (4.22) is cosimple by the generalization of [MMMZ, Corollary 12] to bicategories.
For the converse statement, first observe that for cosimple C, the birepresentation $\mathbf{inj}_{\underline{\mathscr{C}}}(\mathrm{C})$ is transitive by the generalization of [ChMi, Theorem 20 (ii)] to bicategories. The generalization of [MMMZ, Corollary 12] to bicategories then implies that it is simple transitive. If $\mathbf{inj}_{\underline{\mathscr{C}}}(\mathrm{C})$ annihilates $\mathcal{J}$, then we obtain $\mathrm{C}\mathrm{C} = 0$ since $\mathrm{C} \in \underline{\mathrm{add}(\mathcal{J})}$, which is a contradiction. Therefore the apex of $\mathbf{inj}_{\underline{\mathscr{C}}}(\mathrm{C})$ being $\mathcal{J}$ follows from the maximality of $\mathcal{J}$. $\qquad\square$

Proposition 4.22. *If $\mathscr{C}$ is quasi multifiab, then, for any two-sided cell $\mathcal{J}$ in $\mathscr{C}$, there is a biequivalence*

$$\mathscr{C}_{\leq \mathcal{J}}\text{-stmod}_{\mathcal{J}} \xrightarrow{\simeq} \mathscr{C}\text{-stmod}_{\mathcal{J}}.$$

Proof. By (2.19) for $\mathcal{J}' = \mathcal{J}$, we already know that there is a local equivalence

$$\mathscr{C}_{\leq \mathcal{J}}\text{-stmod}_{\mathcal{J}} \to \mathscr{C}\text{-stmod}_{\mathcal{J}}.$$

It remains to prove that any simple transitive birepresentation of $\mathscr{C}$ with apex $\mathcal{J}$ descends to $\mathscr{C}_{\leq \mathcal{J}}$. Due to the biequivalence (2.9) for $\mathcal{J}' = \mathcal{J}$, without loss of generality, we can assume that $\mathcal{J}$ is the unique maximal two-sided cell of $\mathscr{C}$, i.e. that $\mathscr{C} \simeq \mathscr{C}/\mathcal{I}_{\not\leq \mathcal{J}}$. Let $\mathcal{I}$ be the biideal of $\mathscr{C}$ such that $\mathscr{C}_{\leq \mathcal{J}} \simeq \mathscr{C}/\mathcal{I}$, i.e. $\mathcal{I}$ is the maximal biideal of $\mathscr{C}$ not containing id_X for any $X \in \mathcal{J}$.
Now, suppose that $\mathbf{M} \in \mathscr{C}\text{-stmod}_{\mathcal{J}}$. Since $\mathrm{apex}(\mathbf{M}) = \mathcal{J}$, the annihilator of $\mathbf{M}$ is contained in $\mathcal{I}$. We need to show that this inclusion is an equality, so suppose that

$\alpha\colon \mathrm{X} \to \mathrm{Y}$ is a 2-morphism in $\mathscr{C}$ not belonging to the annihilator of $\mathbf{M}$. By Proposition 4.21, there is a coalgebra $\mathrm{C} \in \underline{\mathrm{add}(\mathcal{J})} \subseteq \underline{\mathscr{C}}$ such that

$$\mathbf{M} \simeq \mathbf{inj}_{\underline{\mathscr{C}}}(\mathrm{C}).$$

By this equivalence of birepresentations, there exists a 1-morphism $\mathrm{F} \in \mathcal{J}$ such that $\alpha \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{FC}} \colon \mathrm{X}(\mathrm{FC}) \to \mathrm{Y}(\mathrm{FC})$ is non-zero in $\mathrm{inj}_{\underline{\mathscr{C}}}(\mathrm{C})$, whence the left $\mathscr{C}$-stable ideal in $\mathrm{inj}_{\underline{\mathscr{C}}}(\mathrm{C})$ generated by $\alpha \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{FC}}$ is equal to $\mathrm{inj}_{\underline{\mathscr{C}}}(\mathrm{C})$ by simple transitivity. In particular, this left $\mathscr{C}$-stable ideal contains some id_{G} with $\mathrm{G} \in \underline{\mathrm{add}(\mathcal{J})}$. We claim that therefore $\alpha \notin \mathcal{I}$. To prove this claim, assume that C is given by $\mathrm{C}_1 \xrightarrow{\beta} \mathrm{C}_2 \in \underline{\mathscr{C}}$. Then $\alpha \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{FC}}$ is given by the commutative square

$$\begin{array}{ccc}
\mathrm{X}(\mathrm{FC}_1) & \xrightarrow{\mathrm{id}_{\mathrm{X}} \circ_{\mathsf{h}}(\mathrm{id}_{\mathrm{F}} \circ_{\mathsf{h}} \beta)} & \mathrm{X}(\mathrm{FC}_2) \\
{\scriptstyle \alpha \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{FC}_1}} \downarrow & & \downarrow {\scriptstyle \alpha \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{FC}_2}} \\
\mathrm{Y}(\mathrm{FC}_1) & \xrightarrow[\mathrm{id}_{\mathrm{Y}} \circ_{\mathsf{h}}(\mathrm{id}_{\mathrm{F}} \circ_{\mathsf{h}} \beta)]{} & \mathrm{Y}(\mathrm{FC}_2)
\end{array} \;.$$

Since the left $\mathscr{C}$-stable ideal in $\underline{\mathscr{C}}$ generated by $\alpha \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{FC}}$ contains id_{G} with $\mathrm{G} \in \underline{\mathrm{add}(\mathcal{J})}$, the left $\mathscr{C}$-stable ideal in $\mathscr{C}$ generated by $\alpha \circ_{\mathsf{h}} \mathrm{id}_{\mathrm{FC}_1}$ contains some id_{K} with $\mathrm{K} \in \overline{\mathrm{add}(\mathcal{J})}$. The latter left $\mathscr{C}$-stable ideal is contained in the biideal of $\mathscr{C}$ generated by α, whence $\alpha \notin \mathcal{I}$. We conclude that $\mathrm{ann}(\mathbf{M}) = \mathcal{I}$, which is what we had to prove. $\square$

4.7. Bicomodules and birepresentations. Let $\mathscr{C}$ be a multifinitary bicategory.

Definition 4.23. We define $\mathscr{BB}\mathrm{icom}_{\underline{\mathscr{C}}}$ to be the *bicategory of biinjective bicomodules over coalgebras in $\underline{\mathscr{C}}$*, whose objects, 1-morphisms and 2-morphisms are coalgebras, biinjective bicomodules and bicomodule homomorphisms in $\underline{\mathscr{C}}$, respectively. Horizontal composition is defined by the cotensor product over coalgebras and vertical composition is defined by the composition of bicomodule homomorphisms. For each object C in $\mathscr{BB}\mathrm{icom}_{\underline{\mathscr{C}}}$, the identity 1-morphism $\mathbb{1}_{\mathrm{C}}$ is given by C, seen as a C-C-bicomodule. For each 1-morphism M in $\mathscr{BB}\mathrm{icom}_{\underline{\mathscr{C}}}$, the identity 2-morphism is simply the identity bicomodule endomorphism of M.

By (3.1) and the explanations above it, as well as the fact that the cotensor product over a coalgebra of two biinjective comodules is again biinjective, $\mathscr{BB}\mathrm{icom}_{\underline{\mathscr{C}}}$ is indeed a bicategory.

Definition 4.24. For various $1, 2$-full 2-subcategories $\mathscr{D}$ of $\mathscr{C}$-afmod appearing in Definitions 2.21 and 2.31, we define the associated 2-subcategories $\mathscr{D}^{\mathrm{ex}}$ with the objects being the same as those of $\mathscr{D}$, the 1-morphisms being the exact morphisms of finitary birepresentations and 2-morphisms being all modifications.

We have

$$\mathscr{C}\text{-stmod}^{\mathrm{ex}} \subset \mathscr{C}\text{-tfmod}^{\mathrm{ex}} \subset \mathscr{C}\text{-cfmod}^{\mathrm{ex}} \subset \mathscr{C}\text{-afmod}^{\mathrm{ex}}$$

and, due to Lemma 2.33,

$$\mathscr{C}\text{-stmod}^{\mathrm{ex}}_{\mathcal{J}} \subset \mathscr{C}\text{-tfmod}^{\mathrm{ex}}_{\mathcal{J}} \subset \mathscr{C}\text{-cfmod}^{\mathrm{ex}}_{\mathcal{J}} = \mathscr{C}\text{-afmod}^{\mathrm{ex}}_{\mathcal{J}}.$$

Note that all finitary birepresentations of $\mathscr{C}^{\oplus}$ are cyclic. Furthermore, all morphisms between simple transitive birepresentations with the same apex of a given fiab bicategory are exact, as the following proposition shows. This is the analog of [EGNO, Proposition 7.6.9] in our context and its proof follows the same reasoning, except that we have to invoke [KMMZ, Theorem 2] at some point.

Proposition 4.25. *Suppose that $\mathscr{C}$ is quasi (multi)fiab. For any two-sided cell $\mathcal{J}$ of $\mathscr{C}$, the bicategories $\mathscr{C}\text{-stmod}^{ex}_{\mathcal{J}}$ and $\mathscr{C}\text{-stmod}_{\mathcal{J}}$ are equal.*

Proof. Let $\mathbf{M}$, $\mathbf{N}$ be two simple transitive birepresentations of $\mathscr{C}$ with apex $\mathcal{J}$ and let $\Phi\colon \mathbf{M} \to \mathbf{N}$ be a $\Bbbk$-linear homomorphism of birepresentations. We have to show that its extension $\underline{\Phi}\colon \underline{\mathbf{M}} \to \underline{\mathbf{N}}$ is exact.

Before we do that, we first prove an auxiliary result. For $\mathtt{i}, \mathtt{j} \in \mathscr{C}$, let

$$\mathrm{C}_{\mathtt{j},\mathtt{i}} := \bigoplus_{\mathrm{X} \in \mathscr{C}(\mathtt{i},\mathtt{j}) \cap \mathcal{J}} \mathrm{X} \ \in \mathrm{add}(\mathcal{J})$$

and notice that, by adjunction, $\mathrm{C}_{\mathtt{j},\mathtt{i}}^{\star} \cong \mathrm{C}_{\mathtt{i},\mathtt{j}}$.

Claim. The endofunctors $\underline{\mathbf{M}}(\mathrm{C}_{\mathtt{j},\mathtt{i}})$ and $\underline{\mathbf{N}}(\mathrm{C}_{\mathtt{j},\mathtt{i}})$ are both projective and injective in the category of left exact endofunctors and they do not annihilate any objects in $\underline{\mathbf{M}}(\mathtt{i})$ and $\underline{\mathbf{N}}(\mathtt{i})$, respectively.

The first part of the claim follows from simple transitivity of $\mathbf{M}$, $\mathbf{N}$ and [KMMZ, Theorem 2]. Let us show the second part of the claim for $\underline{\mathbf{M}}(\mathrm{C}_{\mathtt{j},\mathtt{i}})$, the argument for $\underline{\mathbf{N}}(\mathrm{C}_{\mathtt{j},\mathtt{i}})$ being analogous. Suppose to the contrary that $L \in \underline{\mathbf{M}}(\mathtt{i})$ is a simple object such that $\underline{\mathbf{M}}(\mathrm{C}_{\mathtt{j},\mathtt{i}})L = 0$. Let Q be the direct sum of all indecomposable injectives in $\underline{\mathbf{M}}(\mathtt{j})$, i.e. the direct sum of all indecomposables in $\mathbf{M}(\mathtt{j})$. By adjunction, we have

$$\mathrm{Hom}_{\underline{\mathbf{M}}}\big(L, \underline{\mathbf{M}}(\mathrm{C}_{\mathtt{i},\mathtt{j}})Q\big) \cong \mathrm{Hom}_{\mathbf{M}}\big(\underline{\mathbf{M}}(\mathrm{C}_{\mathtt{j},\mathtt{i}})L, Q\big) = 0.$$

However, this means that the injective hull of L has multiplicity zero in the decomposition of $\underline{\mathbf{M}}(\mathrm{C}_{\mathtt{i},\mathtt{j}})Q = \mathbf{M}(\mathrm{C}_{\mathtt{i},\mathtt{j}})Q$, which contradicts transitivity of $\mathbf{M}$. This completes the proof of the claim.

Now, suppose that the above homomorphism $\underline{\Phi}$ is not exact. Then there exists an object $\mathtt{i}$ and a short exact sequence of objects in $\underline{\mathbf{M}}(\mathtt{i})$

$$0 \to X \to Y \to Z \to 0$$

such that its image under $\underline{\Phi}$

$$0 \to \underline{\Phi}(X) \to \underline{\Phi}(Y) \to \underline{\Phi}(Z) \to 0$$

is not exact in $\underline{\mathbf{N}}(\mathtt{i})$. The claim implies that

$$0 \to \underline{\mathbf{M}}(\mathrm{C}_{\mathtt{j},\mathtt{i}})X \to \underline{\mathbf{M}}(\mathrm{C}_{\mathtt{j},\mathtt{i}})Y \to \underline{\mathbf{M}}(\mathrm{C}_{\mathtt{j},\mathtt{i}})Z \to 0$$

is split exact, while

$$0 \to \underline{\mathbf{N}}(\mathrm{C}_{\mathtt{j},\mathtt{i}})\underline{\Phi}(X) \to \underline{\mathbf{N}}(\mathrm{C}_{\mathtt{j},\mathtt{i}})\underline{\Phi}(Y) \to \underline{\mathbf{N}}(\mathrm{C}_{\mathtt{j},\mathtt{i}})\underline{\Phi}(Z) \to 0$$

is not exact. But this is a contradiction, since the latter sequence is isomorphic to

$$0 \to \underline{\Phi}\big(\underline{\mathbf{M}}(\mathrm{C}_{\mathtt{j},\mathtt{i}})X\big) \to \underline{\Phi}\big(\underline{\mathbf{M}}(\mathrm{C}_{\mathtt{j},\mathtt{i}})Y\big) \to \underline{\Phi}\big(\underline{\mathbf{M}}(\mathrm{C}_{\mathtt{j},\mathtt{i}})Z\big) \to 0$$

and $\underline{\Phi}$ preserves split exactness.

This shows that $\underline{\Phi}$ is exact and completes the proof of the proposition. $\square$

Theorem 4.26. *Let $\mathscr{C}$ be a quasi multifiab bicategory. The assignment*

$$\mathrm{C} \mapsto \mathbf{inj}_{\mathscr{C}}(\mathrm{C}),$$

$$\mathrm{M} \mapsto \mathbf{inj}_{\mathscr{C}}(\mathrm{C}) \xrightarrow{\_ \Box_{\mathrm{C}} \mathrm{M}} \mathbf{inj}_{\mathscr{C}}(\mathrm{D}),$$

$$(\mathrm{M} \xrightarrow{f} \mathrm{N}) \mapsto (\_ \Box_{\mathrm{C}} \mathrm{M} \xrightarrow{\_} \_ \Box_{\mathrm{C}} \mathrm{N}),$$

defines a biequivalence

$$(4.23) \qquad \mathscr{B}\mathscr{B}\mathrm{icom}_{\mathscr{C}^{\oplus}} \simeq \mathscr{C}^{\oplus}\text{-afmod}^{\mathrm{ex}},$$

which restricts to a biequivalence

$$(4.24) \qquad \mathscr{B}\mathscr{B}\mathrm{icom}_{\mathscr{C}} \simeq \mathscr{C}\text{-cfmod}^{\mathrm{ex}}.$$

Proof. The pseudofunctor $\mathscr{BB}\mathrm{icom}_{\underline{\mathscr{C}}\oplus} \to \mathscr{C}^{\oplus}\text{-afmod}^{\mathrm{ex}}$ is well-defined by Lemmas 3.9 and 4.2. It is a biequivalence due to Theorems 4.4 and 4.10. When we restrict to coalgebras and biinjective bicomodules in $\underline{\mathscr{C}}$ on one side of the biequivalence, we have to restrict to cyclic birepresentations of $\mathscr{C}$ on the other side, because we need a generator $X \in \mathbf{M}(\mathtt{i})$, for some $\mathtt{i} \in \mathscr{C}$, in order to define C^X via the internal cohom construction. $\qquad\qquad\square$

The following corollary follows immediately from Proposition 2.27 and (4.23).

Corollary 4.27. *Let $\mathscr{C}$ be a quasi multifiab bicategory. Then there is a biequivalence*

$$\mathscr{BB}\mathrm{icom}_{\underline{\mathscr{C}}\oplus} \simeq \mathscr{C}\text{-afmod}^{\mathrm{ex}}.$$

Let $\mathrm{add}_{\mathscr{C}_{\leq \mathcal{J}}}(\mathcal{J})$ be the additive closure of $\mathcal{J}$ inside $\mathscr{C}_{\leq \mathcal{J}}$ and let $\mathscr{BB}\mathrm{icom}^{\mathrm{cos}}_{\mathrm{add}_{\mathscr{C}_{\leq \mathcal{J}}}(\mathcal{J})}$ be the 1-dense and 2-full subbicategory of $\mathscr{BB}\mathrm{icom}_{\mathscr{C}_{\leq \mathcal{J}}}$ of biinjective bicomodules over cosimple coalgebras in $\mathrm{add}_{\mathscr{C}_{\leq \mathcal{J}}}(\mathcal{J})$. For the multifinitary bicategory $\mathscr{C}_{\mathcal{J}}$, one can also define $\mathrm{add}_{\mathscr{C}_{\mathcal{J}}}(\mathcal{J})$ and $\mathscr{BB}\mathrm{icom}^{\mathrm{cos}}_{\mathrm{add}_{\mathscr{C}_{\mathcal{J}}}(\mathcal{J})}$. Since $\mathscr{C}_{\mathcal{J}}$ is a 2-full subbicategory of $\mathscr{C}_{\leq \mathcal{J}}$, we have

$$(4.25) \quad \mathrm{add}_{\mathscr{C}_{\mathcal{J}}}(\mathcal{J}) = \mathrm{add}_{\mathscr{C}_{\leq \mathcal{J}}}(\mathcal{J}) \quad \text{and} \quad \mathscr{BB}\mathrm{icom}^{\mathrm{cos}}_{\mathrm{add}_{\mathscr{C}_{\mathcal{J}}}(\mathcal{J})} = \mathscr{BB}\mathrm{icom}^{\mathrm{cos}}_{\mathrm{add}_{\mathscr{C}_{\leq \mathcal{J}}}(\mathcal{J})}.$$

Theorem 4.28. *If $\mathscr{C}$ is quasi multifiab, then there are the following biequivalences:*

$$\mathscr{C}\text{-stmod}_{\mathcal{J}} \simeq \mathscr{C}_{\leq \mathcal{J}}\text{-stmod}_{\mathcal{J}} \simeq \mathscr{C}_{\mathcal{J}}\text{-stmod}_{\mathcal{J}} \simeq \mathscr{BB}\mathrm{icom}^{\mathrm{cos}}_{\mathrm{add}_{\mathscr{C}_{\leq \mathcal{J}}}(\mathcal{J})}.$$

Proof. By Proposition 4.25, all instances of stmod in this theorem are equal to $\mathrm{stmod}^{\mathrm{ex}}$. Bearing this in mind, the first biequivalence is due to the restriction of the biequivalence in Proposition 4.22. By Proposition 4.21, therefore, the biequivalence in Theorem 4.26 restricts to a biequivalence

$$(4.26) \qquad\qquad \mathscr{BB}\mathrm{icom}^{\mathrm{cos}}_{\mathrm{add}_{\mathscr{C}_{\leq \mathcal{J}}}(\mathcal{J})} \simeq \mathscr{C}_{\leq \mathcal{J}}\text{-stmod}_{\mathcal{J}},$$

By (4.25) and $\mathcal{J}$-simplicity of $\mathscr{C}_{\mathcal{J}}$, we also have a biequivalence

$$\mathscr{BB}\mathrm{icom}^{\mathrm{cos}}_{\mathrm{add}_{\mathscr{C}_{\leq \mathcal{J}}}(\mathcal{J})} = \mathscr{BB}\mathrm{icom}^{\mathrm{cos}}_{\mathrm{add}_{\mathscr{C}_{\mathcal{J}}}(\mathcal{J})} \simeq \mathscr{C}_{\mathcal{J}}\text{-stmod}_{\mathcal{J}},$$

which is indeed a restriction of (4.26). $\qquad\qquad\square$

Remark 4.29. Note that Theorems 4.26 and 4.28 also prove that

$$\mathscr{BB}\mathrm{icom}^{\mathrm{cos}}_{\underline{\mathrm{add}(\mathcal{J})}} \simeq \mathscr{BB}\mathrm{icom}^{\mathrm{cos}}_{\underline{\mathrm{add}_{\mathscr{C}_{\leq \mathcal{J}}}(\mathcal{J})}} \simeq \mathscr{BB}\mathrm{icom}^{\mathrm{cos}}_{\mathrm{add}_{\mathscr{C}_{\leq \mathcal{J}}}(\mathcal{J})}.$$

4.8. Strong $\mathcal{H}$-reduction. Let $\mathscr{C}$ be a multifiab bicategory, $\mathcal{J}$ a two-sided cell in $\mathscr{C}$ and $\mathcal{H}$ a diagonal $\mathcal{H}$-cell inside $\mathcal{J}$. Assume that $\mathtt{i}$ is the source of $\mathcal{H}$. Recall from Lemma 2.33 that

$$\mathscr{C}_{(\mathcal{J})}\text{-afmod}_{\mathcal{J}} = \mathscr{C}_{(\mathcal{J})}\text{-cfmod}_{\mathcal{J}} \quad \text{and} \quad \mathscr{C}_{(\mathcal{H})}\text{-afmod}_{\mathcal{H}} = \mathscr{C}_{(\mathcal{H})}\text{-cfmod}_{\mathcal{H}},$$

which implies that

$$\mathscr{C}_{(\mathcal{J})}\text{-afmod}^{\mathrm{ex}}_{\mathcal{J}} = \mathscr{C}_{(\mathcal{J})}\text{-cfmod}^{\mathrm{ex}}_{\mathcal{J}} \quad \text{and} \quad \mathscr{C}_{(\mathcal{H})}\text{-afmod}^{\mathrm{ex}}_{\mathcal{H}} = \mathscr{C}_{(\mathcal{H})}\text{-cfmod}^{\mathrm{ex}}_{\mathcal{H}}.$$

We denote by $\mathscr{BB}\mathrm{icom}_{\mathscr{C}_{(\mathcal{J})}, \mathrm{add}(\mathcal{H})}$, respectively $\mathscr{BB}\mathrm{icom}_{\mathscr{C}_{(\mathcal{H})}, \mathrm{add}(\mathcal{H})}$, the bicategory of coalgebras, biinjective bicomodules and comodule homomorphisms in $\underline{\mathrm{add}(\mathcal{H})}$ inside $\mathscr{C}_{(\mathcal{J})}$, respectively $\mathscr{C}_{(\mathcal{H})}$.

Theorem 4.30. *There are biequivalences*

$$\mathscr{C}_{(\mathcal{J})}\text{-afmod}^{\mathrm{ex}}_{\mathcal{J}} = \mathscr{C}_{(\mathcal{J})}\text{-cfmod}^{\mathrm{ex}}_{\mathcal{J}} \simeq \mathscr{BB}\mathrm{icom}_{\underline{\mathscr{C}_{(\mathcal{J})}}, \underline{\mathrm{add}(\mathcal{H})}}$$

$$\simeq \mathscr{BB}\mathrm{icom}_{\underline{\mathscr{C}_{(\mathcal{H})}}, \underline{\mathrm{add}(\mathcal{H})}} \simeq \mathscr{C}_{(\mathcal{H})}\text{-cfmod}^{\mathrm{ex}}_{\mathcal{H}} = \mathscr{C}_{(\mathcal{H})}\text{-afmod}^{\mathrm{ex}}_{\mathcal{H}}.$$

Proof. We claim that the last biequivalence in the first row and the middle biequivalence in the second row are obtained by restricting the biequivalence in (4.24). Indeed, (4.24) provides biequivalences

$$\mathscr{BB}\mathrm{icom}_{\underline{\mathscr{C}_{(\mathcal{J})}}} \simeq \mathscr{C}_{(\mathcal{J})}\text{-cfmod}^{\mathrm{ex}} \quad \text{and} \quad \mathscr{BB}\mathrm{icom}_{\underline{\mathscr{C}_{(\mathcal{H})}}} \simeq \mathscr{C}_{(\mathcal{H})}\text{-cfmod}^{\mathrm{ex}}$$

and Proposition 4.21 guarantees that the restriction of the pseudofunctor in Theorem 4.26 to coalgebras in $\overline{\mathrm{add}(\mathcal{H})}$ is still essentially surjective on objects when corestricting to $\mathscr{C}_{(\mathcal{J})}\text{-cfmod}^{\mathrm{ex}}_{\mathcal{J}}$ respectively $\mathscr{C}_{(\mathcal{H})}\text{-cfmod}^{\mathrm{ex}}_{\mathcal{H}}$.

Finally, $\mathscr{BB}\mathrm{icom}_{\underline{\mathscr{C}_{(\mathcal{H})}},\overline{\mathrm{add}(\mathcal{H})}}$ is naturally isomorphic to $\mathscr{BB}\mathrm{icom}_{\underline{\mathscr{C}_{(\mathcal{J})}},\overline{\mathrm{add}(\mathcal{H})}}$, consisting of the same objects, 1-morphisms and 2-morphisms, just considered in different ambient bicategories. $\square$

Passing to the $\mathcal{J}$-simple and $\mathcal{H}$-simple quotients $\mathscr{C}_{\mathcal{J}}$ and $\mathscr{C}_{\mathcal{H}}$, respectively, and defining $\mathscr{BB}\mathrm{icom}_{\mathscr{C}_{\mathcal{J}},\mathrm{add}(\mathcal{H})}$ and $\mathscr{BB}\mathrm{icom}_{\mathscr{C}_{\mathcal{H}},\mathrm{add}(\mathcal{H})}$ as the bicategories of coalgebras, bicomodules and comodule homomorphisms in $\mathrm{add}(\mathcal{H})$ inside $\underline{\mathscr{C}_{\mathcal{J}}}$ and $\underline{\mathscr{C}_{\mathcal{H}}}$, respectively, we obtain the following.

Theorem 4.31. *There are biequivalences*

$$\mathscr{C}_{\mathcal{J}}\text{-afmod}^{\mathrm{ex}}_{\mathcal{J}} = \mathscr{C}_{\mathcal{J}}\text{-cfmod}^{\mathrm{ex}}_{\mathcal{J}} \simeq \mathscr{BB}\mathrm{icom}_{\mathscr{C}_{\mathcal{J}},\mathrm{add}(\mathcal{H})}$$

$$\simeq \mathscr{BB}\mathrm{icom}_{\mathscr{C}_{\mathcal{H}},\mathrm{add}(\mathcal{H})} \simeq \mathscr{C}_{\mathcal{H}}\text{-cfmod}^{\mathrm{ex}}_{\mathcal{H}} = \mathscr{C}_{\mathcal{H}}\text{-afmod}^{\mathrm{ex}}_{\mathcal{H}}.$$

Proof. The only thing to note is that, under the assumption of $\mathcal{J}$-simplicity, the coalgebra $\mathrm{C}^{\mathbf{M}_{\mathrm{ii}}(\mathrm{F})X} \cong (\mathrm{FC}^{X})\mathrm{F}^{\star}$ in Proposition 4.21 belongs indeed to $\mathrm{add}(\mathcal{H})$ by Proposition 4.18. $\square$

We deduce the following consequence, which we (also) call *strong $\mathcal{H}$-reduction*.

Theorem 4.32. *Let $\mathscr{C}$ be a fiab bicategory, and fix a two-sided cell $\mathcal{J}$ of $\mathscr{C}$ as well as diagonal $\mathcal{H}$-cell $\mathcal{H} \subset \mathcal{J}$. Then there is a biequivalence*

$$\mathscr{C}\text{-stmod}_{\mathcal{J}} \simeq \mathscr{C}_{\mathcal{H}}\text{-stmod}_{\mathcal{H}}.$$

Proof. Bearing Proposition 4.25 in mind, the statement follows by Theorems 4.28 and 4.31. $\square$

4.9. **An extra biequivalence.** The goal of this subsection is to prove that the 2-functor in (2.24) is a local equivalence.

Theorem 4.33. *Let $\mathscr{C}$ be a quasi multifiab bicategory and $\mathcal{J}$ a two-sided cell in $\mathscr{C}$. Then the 2-functor*

$$\mathscr{C}_{\leq \mathcal{J}}\text{-afmod}_{\mathcal{J}} \to \mathscr{C}_{\mathcal{J}}\text{-afmod}_{\mathcal{J}},$$

defined in (2.24), is a local equivalence.

Proof. Recall from Subsection 2.6 that (2.24) is well-defined. Let $\mathbf{M}$ and $\mathbf{N}$ be two arbitrary birepresentations in $\mathscr{C}_{\leq \mathcal{J}}\text{-afmod}_{\mathcal{J}}$. Since 2-faithfulness of (2.24) is obvious, it suffices to prove 1- and 2-fullness, or in other words essential surjectivity and fullness of the induced functor

$$(4.27) \qquad \mathrm{Hom}_{\mathscr{C}_{\leq \mathcal{J}}\text{-afmod}_{\mathcal{J}}}(\mathbf{M}, \mathbf{N}) \to \mathrm{Hom}_{\mathscr{C}_{\mathcal{J}}\text{-afmod}_{\mathcal{J}}}(\mathbf{M}, \mathbf{N}).$$

By the abelianized version of Proposition 4.21 and the fact that $\mathscr{C}_{\leq \mathcal{J}}$ is $\mathcal{J}$-simple, there exist two 1-morphisms C and D in $\mathrm{add}(\mathcal{J})$ which have coalgebra structures in $\mathscr{C}_{\leq \mathcal{J}}$ such that

$$\underline{\mathbf{M}} \simeq \mathbf{comod}_{\underline{\mathscr{C}_{\leq \mathcal{J}}}}(\mathrm{C}) \quad \text{and} \quad \underline{\mathbf{N}} \simeq \mathbf{comod}_{\underline{\mathscr{C}_{\leq \mathcal{J}}}}(\mathrm{D}),$$

as birepresentations of $\mathscr{C}_{\leq \mathcal{J}}$. Both equivalences can be restricted to equivalences

$$\mathbf{M} \simeq \mathbf{inj}_{\underline{\mathscr{C}_{\leq \mathcal{J}}}}(\mathrm{C}) \quad \text{and} \quad \mathbf{N} \simeq \mathbf{inj}_{\underline{\leq \mathscr{C}_{\mathcal{J}}}}(\mathrm{D}),$$

as birepresentations of $\mathscr{C}_{\leq \mathcal{J}}$. Since $\mathscr{C}_{\mathcal{J}}$ is a 2-full subbicategory of $\mathscr{C}_{\leq \mathcal{J}}$, the coalgebra structures of C and D in $\mathrm{add}(\mathcal{J})$ both restrict to $\mathscr{C}_{\mathcal{J}}$. Via the 2-functor in (2.24), the above equivalences descend to

$$\underline{\mathbf{M}} \simeq \mathbf{comod}_{\underline{\mathscr{C}_{\mathcal{J}}}}(\mathrm{C}) \quad \text{and} \quad \underline{\mathbf{N}} \simeq \mathbf{comod}_{\underline{\mathscr{C}_{\mathcal{J}}}}(\mathrm{D}),$$

as birepresentations of $\mathscr{C}_{\mathcal{J}}$, respectively, and

$$\mathbf{M} \simeq \mathbf{inj}_{\underline{\mathscr{C}_{\mathcal{J}}}}(\mathrm{C}) \quad \text{and} \quad \mathbf{N} \simeq \mathbf{inj}_{\underline{\mathscr{C}_{\mathcal{J}}}}(\mathrm{D}),$$

as birepresentations of $\mathscr{C}_{\mathcal{J}}$.

Let Φ be any morphism of birepresentations in $\mathrm{Hom}_{\mathscr{C}_{\mathcal{J}}\text{-afmod}_{\mathcal{J}}}(\mathbf{M}, \mathbf{N})$. Then the induced morphism $\underline{\Phi}$ from $\underline{\mathbf{M}}$ to $\underline{\mathbf{N}}$ is left exact by definition. Hence the functor underlying $\underline{\Phi}$ can be represented by cotensoring with some C-D-bicomodule $\mathrm{X} \in \underline{\mathscr{C}_{\mathcal{J}}} \subseteq \underline{\mathscr{C}_{\leq \mathcal{J}}}$, which is injective as a right D-comodule since it sends injective right C-comodules to injective right D-comodules. It is clear that the functor $\_\square_{\mathrm{C}}\mathrm{X}$ is an element of $\mathrm{Hom}_{\mathscr{C}_{\leq \mathcal{J}}\text{-afmod}_{\mathcal{J}}}\big(\mathbf{inj}_{\underline{\mathscr{C}_{\mathcal{J}}}}(\mathrm{C}), \mathbf{inj}_{\underline{\mathscr{C}_{\mathcal{J}}}}(\mathrm{D})\big)$ by Lemma 3.9. This implies that $\Phi \in \mathrm{Hom}_{\mathscr{C}_{\leq \mathcal{J}}\text{-afmod}_{\mathcal{J}}}(\mathbf{M}, \mathbf{N})$ and our functor (4.27) are essentially surjective. Since modifications correspond to homomorphisms of C-D-bicomodules, fullness of (4.27) is also clear and the statement is proved. $\qquad\square$

Remark 4.34. In fact, the local equivalence in Theorem 4.33 restricts to a biequivalence $\mathscr{C}_{\leq \mathcal{J}}\text{-stmod}_{\mathcal{J}} \simeq \mathscr{C}_{\mathcal{J}}\text{-stmod}_{\mathcal{J}}$, cf Theorem 4.28.

5. The double centralizer theorem

Throughout this section, let $\mathscr{C}$ be a fiab bicategory, $\mathcal{H}$ a diagonal $\mathcal{H}$-cell and $\mathbf{M}$ a simple transitive birepresentation of $\mathscr{C}_{\mathcal{H}}$ with apex $\mathcal{H}$. By Proposition 4.21 there is a cosimple coalgebra $\mathrm{C} \in \mathrm{add}(\mathcal{H})$ such that

$$\mathbf{M} \simeq \mathbf{inj}_{\underline{\mathscr{C}_{\mathcal{H}}}}(\mathbf{C}),$$

and by Lemma 4.1 we have

(5.1) $$\mathrm{inj}_{\underline{\mathscr{C}_{\mathcal{H}}}}(\mathbf{C}) = \mathrm{add}\{\mathrm{GC} \mid \mathrm{G} \in \mathscr{C}_{\mathcal{H}}\},$$

where the additive closure is taken inside $\mathrm{comod}_{\underline{\mathscr{C}_{\mathcal{H}}}}(\mathrm{C})$.

Let $\mathscr{E}\mathrm{nd}_{\mathscr{C}_{\mathcal{H}}}(\mathbf{M})$ denote the one-object 2-category of endomorphisms (of finitary birepresentations) of $\mathbf{M}$ and recall that $\mathscr{E}\mathrm{nd}_{\mathscr{C}_{\mathcal{H}}}(\mathbf{M}) = \mathscr{E}\mathrm{nd}^{\mathrm{ex}}_{\mathscr{C}_{\mathcal{H}}}(\mathbf{M})$ by Proposition 4.25. Further, let $\mathscr{B}_{\mathbf{M}} := (\mathrm{C})\mathscr{B}\mathrm{iinj}_{\underline{\mathscr{C}_{\mathcal{H}}}}(\mathrm{C})$ denote the one-object finitary bicategory of biinjective C-bicomodules in $\underline{\mathscr{C}_{\mathcal{H}}}$, with the horizontal composition being given by $\_\square_{\mathrm{C}}\_$. By Proposition 4.21, there is a biequivalence

(5.2) $$\mathscr{E}\mathrm{nd}_{\mathscr{C}_{\mathcal{H}}}(\mathbf{M}) \simeq \mathscr{B}^{\mathrm{op}}_{\mathbf{M}},$$

where the right biaction of $\mathscr{B}_{\mathbf{M}}$ on $\mathbf{inj}_{\underline{\mathscr{C}_{\mathcal{H}}}}(\mathbf{C})$ is given by $\_\square_{\mathrm{C}}\_$. By (5.2), $\mathbf{M}$ can be viewed as a left birepresentation of $\mathscr{B}^{\mathrm{op}}_{\mathbf{M}}$ or, equivalently, a right birepresentation of $\mathscr{B}_{\mathbf{M}}$.

Lemma 5.1. $\mathbf{M}$ *is transitive as a birepresentation of* $\mathscr{B}^{\mathrm{op}}_{\mathbf{M}}$.

Proof. For every $\mathrm{X} \in \mathrm{inj}_{\underline{\mathscr{C}_{\mathcal{H}}}}(\mathbf{C})$ and $\mathrm{Y} \in \mathrm{add}(\mathcal{H})$, we have $\mathrm{C}(\mathrm{YC}) \in \mathscr{B}_{\mathbf{M}}$ and

$$\mathrm{X}\square_{\mathrm{C}}\big(\mathrm{C}(\mathrm{YC})\big) \cong (\mathrm{X}\square_{\mathrm{C}}\mathrm{C})(\mathrm{YC}) \cong \mathrm{X}(\mathrm{YC}) \cong (\mathrm{XY})\mathrm{C}.$$

The result now follows from (5.1) and the fact that $\mathcal{H}$ is a right cell of $\mathscr{C}_{\mathcal{H}}$. $\qquad\square$

Since the biaction of $\mathscr{C}_{\mathcal{H}}$ and $\mathscr{E}\mathrm{nd}_{\mathscr{C}_{\mathcal{H}}}(\mathbf{M})$ on $\mathbf{M}$ weakly commute by definition, there is a *canonical pseudofunctor*

$$\mathrm{can} \colon \mathscr{C}_{\mathcal{H}} \to \mathscr{E}\mathrm{nd}^{\mathrm{ex}}_{\mathscr{E}\mathrm{nd}_{\mathscr{C}_{\mathcal{H}}}(\mathbf{M})}(\mathbf{M}).$$

For every $X \in \operatorname{add}(\mathcal{H})$, the endofunctor $\mathbf{M}(X)$ of $\mathbf{M}(\mathtt{i})$ is injective by the dual version of [KMMZ, Theorem 2], so in particular it is exact. (Recall that we call an endofunctor of an additive category injective if it is injective in the category of endofunctors of the injective abelianization.) The identity $\mathbb{1}_{\mathtt{i}}$ in $\mathscr{C}_{\mathcal{H}}$, where $\mathtt{i}$ is the source of $\mathcal{H}$, acts by the identity functor, which is not injective but is, of course, exact. The following theorem, which we call the *double centralizer theorem*, is the analog of [EGNO, Theorem 7.12.11] for fiab bicategories and simple transitive birepresentations. In its formulation, the superscript inj denotes the injective morphisms, i.e. those realized by injective endofunctors.

Theorem 5.2. *The canonical pseudofunctor is fully faithful on 2-morphisms and essentially surjective on 1-morphisms when restricted to $\operatorname{add}(\mathcal{H})$ and corestricted to $\mathscr{E}\mathrm{nd}^{\operatorname{inj}}_{\mathscr{E}\mathrm{nd}_{\mathscr{C}_{\mathcal{H}}}(\mathbf{M})}(\mathbf{M})$.*

The proof follows similar reasoning as the proof of [EGNO, Theorem 7.12.11], but we have to adapt some of the arguments to our setting, because $\mathscr{C}_{\mathcal{H}}$ is not abelian and $\mathbb{1}_{\mathtt{i}}$ does not act on $\mathbf{M}(\mathtt{i})$ by an injective endofunctor, as already remarked.

Before we give the proof of Theorem 5.2, let us recall some general facts about duality and coactions and point out some consequences. Since these facts are well-known and not difficult to check, we omit their proofs, see also Remark 5.3. Suppose that C is a coalgebra in $\mathscr{C}_{\mathcal{H}}$ and let $Y \in \operatorname{inj}_{\underline{\mathscr{C}_{\mathcal{H}}}}(C)$. Then $Y^{\star} \in (C)\operatorname{inj}_{\underline{\mathscr{C}_{\mathcal{H}}}}$, with the left C-coaction $\delta_{C,Y^{\star}}$ being defined as the composite of (recall ev' and $\overline{\mathrm{coev}}'$ from below Definition 2.5)

$$Y^{\star} \xrightarrow{(v^r_{Y^{\star}})^{-1}} Y^{\star}\mathbb{1}_{\mathtt{i}} \xrightarrow{\mathrm{id}_{Y^{\star}}\circ_{\mathrm{h}}\mathrm{coev}'_Y} Y^{\star}(YY^{\star}) \xrightarrow{\mathrm{id}_{Y^{\star}}\circ_{\mathrm{h}}(\delta_{Y,C}\circ_{\mathrm{h}}\mathrm{id}_{Y^{\star}})} Y^{\star}\big((YC)Y^{\star}\big) \rightharpoondown$$
$$\xrightarrow{\alpha^{-1}_{Y^{\star},Y,CY^{\star}}\circ_{\mathrm{v}}(\mathrm{id}_{Y^{\star}}\circ_{\mathrm{h}}\alpha_{Y,C,Y^{\star}})}$$
$$\rightharpoondown (Y^{\star}Y)(CY^{\star}) \xrightarrow{\mathrm{ev}'_Y\circ_{\mathrm{h}}\mathrm{id}_{CY^{\star}}} \mathbb{1}_{\mathtt{i}}(CY^{\star}) \xrightarrow{\quad v^l_{CY^{\star}} \quad} CY^{\star}.$$

This implies that $Y^{\star}Y \in \mathscr{B}_{\mathbf{M}}$ and that the following diagrams commute:

$$(5.3) \quad \begin{array}{ccccccccc} Y^{\star}Y & \xrightarrow{\delta_{C,Y^{\star}}\circ_{\mathrm{h}}\mathrm{id}_Y} & (CY^{\star})Y & \xrightarrow{\alpha_{C,Y^{\star},Y}} & C(Y^{\star}Y) & \xrightarrow{\mathrm{id}_C\circ_{\mathrm{h}}\mathrm{ev}'_Y} & C\mathbb{1}_{\mathtt{i}} & \xrightarrow{v^r_C} & C \\ \| & & & & & & & & \| \\ Y^{\star}Y & \xrightarrow[\mathrm{id}_Y\circ_{\mathrm{h}}\delta_{Y,C}]{} & Y^{\star}(YC) & \xrightarrow[\alpha^{-1}_{Y^{\star},Y,C}]{} & (Y^{\star}Y)C & \xrightarrow[\mathrm{ev}'_Y\circ_{\mathrm{h}}\mathrm{id}_C]{} & \mathbb{1}_{\mathtt{i}}C & \xrightarrow[v^l_C]{} & C \end{array} \quad ,$$

$$(5.4) \quad \begin{array}{ccc} \mathbb{1}_{\mathtt{i}} \xrightarrow{\mathrm{coev}'_Y} YY^{\star} & \xrightarrow{\delta_{Y,C}\circ_{\mathrm{h}}\mathrm{id}_{Y^{\star}}} & (YC)Y^{\star} \\ & \searrow^{\mathrm{id}_Y\circ_{\mathrm{h}}\delta_{C,Y^{\star}}} & \downarrow^{\alpha_{Y,C,Y^{\star}}} \\ & & Y(CY^{\star}) \end{array} \quad .$$

Now, let $X \in \mathscr{C}_{\mathcal{H}}$, $Y \in \operatorname{inj}_{\underline{\mathscr{C}_{\mathcal{H}}}}(C)$ and $Z \in \mathscr{B}_{\mathbf{M}}$. For any $f \in \operatorname{Hom}_{\underline{\mathscr{C}_{\mathcal{H}}}}(X, Y\square_C Z)$, define $\tilde{f} := \iota_{Y,Z}\circ_{\mathrm{h}} f \in \operatorname{Hom}_{\underline{\mathscr{C}_{\mathcal{H}}}}(X, YZ)$, where $\iota_{Y,Z}\colon Y\square_C Z \hookrightarrow YZ$ is the canonical embedding. Then $g \in \operatorname{Hom}_{\underline{\mathscr{C}_{\mathcal{H}}}}(X, YZ)$ satisfies $g = \tilde{f}$, for some $f \in \operatorname{Hom}_{\underline{\mathscr{C}_{\mathcal{H}}}}(X, Y\square_C Z)$, if and only if

$$(5.5) \qquad \alpha_{Y,C,Z} \circ_{\mathrm{v}} (\delta_{Y,C} \circ_{\mathrm{h}} \mathrm{id}_Z) \circ_{\mathrm{v}} g = (\mathrm{id}_Y \circ_{\mathrm{h}} \delta_{C,Z}) \circ_{\mathrm{v}} g.$$

Taking $X = \mathbb{1}_{\mathtt{i}}$ and $Z = Y^{\star}$, we see that commutativity of the diagram in (5.4) means that coev'_Y factors through $Y\square_C Y^{\star}$, i.e.

$$\mathrm{coev}'_Y = \widetilde{\mathrm{coev}^{C,\prime}_Y},$$

where $\operatorname{coev}_Y^{C,\prime} \in \operatorname{Hom}_{\mathscr{C}_{\mathcal{H}}}(\mathbb{1}_i, Y\square_C Y^\star)$. This, in turn, implies that $Y^\star Y$ is a coalgebra in $\mathscr{B}_M$, with comultiplication $\delta_{Y^\star Y}^C$ being the composite of

(5.6)

$$Y^\star Y \xrightarrow{(v_{Y^\star}^r)^{-1}\circ_h \mathrm{id}_Y} (Y^\star \mathbb{1}_i)Y \xrightarrow{\mathrm{id}_{Y^\star}\circ_h \operatorname{coev}_Y^{C,\prime}\circ_h \mathrm{id}_Y} \big(Y^\star(Y\square_C Y^\star)\big)Y \xrightarrow{\cong} (Y^\star Y)\square_C(Y^\star Y)$$

and counit $\epsilon_{Y^\star Y}^C$ being the composite of either one of the rows in (5.3). Checking coassociativity and counitality is an easy but tedious exercise in diagram-chasing, which we leave to the reader. We only note that to check counitality, one has to use commutativity of (5.3).
Finally, let

(5.7)
$$\delta_{Y^\star Y} := \widetilde{\delta_{Y^\star Y}^C} \quad \epsilon_{Y^\star Y} := \epsilon_C \circ_v \epsilon_{Y^\star Y}^C.$$

Then $(Y^\star Y, \delta_{Y^\star Y}, \epsilon_{Y^\star Y})$ is a coalgebra in $\mathscr{C}_{\mathcal{H}}$. As a matter of fact, it is exactly the coalgebra structure on $Y^\star Y$ which we defined in Lemma 4.13, if we consider $Y^\star Y$ as the framing of the coalgebra $\mathbb{1}_i$ by $Y^\star$ in $\mathscr{C}_{\mathcal{H}}$, namely,

$$\delta_{Y^\star Y} = \alpha_{Y^\star Y, Y^\star, Y} \circ_v (\alpha_{Y^\star, Y, Y^\star}^{-1} \circ_h \mathrm{id}_Y)\circ_v$$
$$\big((\mathrm{id}_{Y^\star}\circ_h \operatorname{coev}_Y')\circ_h \mathrm{id}_Y\big)\circ_v \big((v_{Y^\star}^r)^{-1}\circ_h \mathrm{id}_Y\big)'$$
$$\epsilon_{Y^\star Y} = \operatorname{ev}_Y'.$$

Remark 5.3. The facts above are easy to see in the strict setting using string diagrams. For example, (5.3) reads as

while $\delta_{Y^\star Y}$ and $\epsilon_{Y^\star Y}$ can be depicted as

for which coassociativity and counitality are easy exercises in planar topology. Similarly, the various constructions which we will use below also have string diagrammatic interpretations. For example,

is the picture of the map in (5.10).

Proof of Theorem 5.2. By (5.2) we can interpret the canonical pseudofunctor as a pseudofunctor (with the same name) $\operatorname{can}\colon \mathscr{C}_{\mathcal{H}} \to \mathscr{E}\mathrm{nd}_{\mathscr{B}_M^{\mathrm{op}}}^{\mathrm{ex}}(\mathbf{M})$.
By Lemma 5.1 and the internal cohom construction, there is an equivalence of right birepresentations of $\mathscr{B}_M$

$$\mathbf{inj}_{\mathscr{C}_{\mathcal{H}}}(C) \simeq \big([C,C]\big)\mathbf{inj}_{\mathscr{B}_M}.$$

A crucial ingredient is the following claim.

Claim 1. We have

(5.8) $$[C, C] \cong C^\star C \text{ in } \mathscr{B}_{\mathbf{M}}$$

and the implied equivalence

(5.9) $$\mathbf{inj}_{\underline{\mathscr{C}_{\mathcal{H}}}}(C) \simeq (C^\star C)\mathbf{inj}_{\underline{\mathscr{B}_{\mathbf{M}}}}$$

is given explicitly by $X \mapsto C^\star X$.

Claim 1 then implies that

$$\mathscr{E}\mathrm{nd}^{\mathrm{ex}}_{\mathscr{E}\mathrm{nd}_{\mathscr{C}_{\mathcal{H}}}(\mathbf{M})}(\mathbf{M}) \simeq (C^\star C)\mathscr{B}\mathrm{iinj}_{\underline{\mathscr{B}_{\mathbf{M}}}}(C^\star C)$$

and 1-morphisms in $\mathscr{E}\mathrm{nd}^{\mathrm{inj}}_{\mathscr{E}\mathrm{nd}_{\mathscr{C}_{\mathcal{H}}}(\mathbf{M})}(\mathbf{M})$, under this equivalence, correspond to injective $(C^\star C)$-bicomodules in $\underline{\mathscr{B}_{\mathbf{M}}}$, which all live in $\mathscr{B}_{\mathbf{M}}$. We can then trace the canonical pseudofunctor through the equivalences to a pseudofunctor

$$\mathrm{can} \colon \mathscr{C}_{\mathcal{H}} \to (C^\star C)\mathscr{B}\mathrm{iinj}_{\underline{\mathscr{B}_{\mathbf{M}}}}(C^\star C).$$

It is on that level that we prove fully faithfulness on 2-morphisms and essential surjectivity on 1-morphisms when restricting to $\mathrm{add}(\mathcal{H})$ and corestricting to injective $(C^\star C)$-bicomodules.

We now proceed to prove Claim 1, i.e. the isomorphism (5.8) and the explicit description of the equivalence it implies. Both follow from the natural isomorphism

$$\mathrm{Hom}_C(X, Y\square_C Z) \cong \mathrm{Hom}_{\underline{\mathscr{B}_{\mathbf{M}}}}(Y^\star X, Z)$$

for $X, Y \in \mathbf{inj}_{\underline{\mathscr{C}_{\mathcal{H}}}}(C)$ and $Z \in \underline{\mathscr{B}_{\mathbf{M}}}$, which we claim is given by

(5.10) $$f \mapsto \upsilon^l_Z \circ_\mathrm{v} (\mathrm{ev}'_Y \circ_\mathrm{v} \circ_\mathrm{h} \mathrm{id}_Z) \circ_\mathrm{v} \alpha^{-1}_{Y^\star, Y, Z} \circ_\mathrm{v} (\mathrm{id}_{Y^\star} \circ_\mathrm{h} \tilde{f}).$$

To verify this, we have to show that the natural transformation is well-defined and that it is an isomorphism.

Since f is assumed to be a right C-comodule homomorphism, it is clear from the definition of the natural transformation that the image of f is also a right C-comodule homomorphism. The fact that the image of f is also a left C-comodule homomorphism follows from the assumption that the target of f is $Y\square_C Z$, as the following commutative diagram shows:

Commutativity follows from

- the interchange law, naturality of the associator and the pentagon coherence condition of the associator for the facet labeled 1;
- commutativity of (5.3) and naturality of the associator for the facet labeled 2;
- the triangle coherence condition of the unitors for the facet labeled 3;
- equation (5.5) for the facet labeled 4;
- naturality of the associator for the facets labeled 5 and 7;
- naturality of the associator and the pentagon coherence condition of the associator for the facet labeled 6;
- the interchange law for the facet labeled 8;

- naturality of the unitor and (2.1) for the facet labeled 9.

This finishes the proof that the image of f belongs to $\mathrm{Hom}_{\underline{\mathscr{B}_{\mathbf{M}}}}(Y^\star X, Z)$, in other words, that the natural transformation above is well-defined.

Its inverse is given by sending any $h \in \mathrm{Hom}_{\underline{\mathscr{B}_{\mathbf{M}}}}(Y^\star X, Z)$ to the unique 2-morphism $h' \in \mathrm{Hom}_C(X, Y\square_C Z)$ such that

$$\widetilde{h'} = (\mathrm{id}_Y \circ_h h) \circ_v \alpha_{Y,Y^\star,X} \circ_v (\mathrm{coev}'_Y \circ_h \mathrm{id}_X) \circ_v (v_X^l)^{-1}.$$

The fact that the latter composite satisfies (5.5) implies well-definition. The statement that the map $h \mapsto h'$ is the inverse of (5.10) follows from naturality of the unitors and associator, the interchange law, the pentagon coherence condition for the associator, the triangle coherence condition of the unitors and the zigzag relation for coevaluations and evaluations.

This completes the proof of Claim 1, since it shows that $[Y, X] \cong Y^\star X$ which, setting $X = Y = C$, yields (5.8). Moreover, the equivalence in (5.9) is then given by $X \mapsto [C, X] \cong C^\star X$.

It is not hard to see that the coalgebra structure of $C^\star C \cong [C, C]$ in $\mathscr{B}_{\mathbf{M}}$ given by the internal cohom construction coincides with the one given in (5.6) and the text above and below it. As we explained in (5.7) and the text above and below it, this coalgebra structure in $\mathscr{B}_{\mathbf{M}}$ induces one in $\mathscr{C}_{\mathcal{H}}$.

It follows that $C^\star$ is naturally a left $C^\star C$-comodule in $(C)\mathrm{inj}_{\underline{\mathscr{C}_{\mathcal{H}}}}$, with coaction defined by

$$\delta_{C^\star C, C^\star} := \alpha_{C^\star, C, C^\star}^{-1} \circ_v (\mathrm{id}_{C^\star} \circ_h \mathrm{coev}'_C) \circ_v (v_{C^\star}^r)^{-1},$$

and C is naturally a right $C^\star C$-comodule, with coaction defined by

$$\delta_{C, C^\star C} := \alpha_{C, C^\star, C} \circ_v (\mathrm{coev}'_C \circ_h \mathrm{id}_C) \circ_v (v_C^l)^{-1}.$$

Coassociativity follows from the interchange law, naturality of the associator and unitors, and the pentagon coherence condition of the associator. Counitality follows from the zigzag relations.

To show that the canonical pseudofunctor $\mathrm{can} \colon \mathscr{C}_{\mathcal{H}} \to (C^\star C)\mathscr{B}\mathrm{iinj}_{\underline{\mathscr{B}_{\mathbf{M}}}}(C^\star C)$ is fully faithful on 2-morphisms and essentially surjective on 1-morphisms, when restricted to $\mathrm{add}(\mathcal{H})$, we need the following.

Claim 2. The canonical pseudofunctor $\mathrm{can} \colon \mathscr{C}_{\mathcal{H}} \to (C^\star C)\mathscr{B}\mathrm{iinj}_{\underline{\mathscr{B}_{\mathbf{M}}}}(C^\star C)$ is explicitly given by

$$F \mapsto C^\star(FC).$$

We already know that the equivalence in (5.9) is given by $X \to C^\star X$. Since F, viewed as an endomorphism of $\mathbf{inj}_{\mathscr{C}_{\mathcal{H}}}(C)$, sends X to FX, it follows that under the equivalence in (5.9), it sends $C^\star X$ to $C^\star(FX)$. We therefore have to show that there is a natural isomorphism

$$X \cong (C\square_{C^\star C}C^\star)X,$$

for $X \in \mathbf{inj}_{\mathscr{C}_{\mathcal{H}}}(C)$. By Lemma 5.1, it suffices to prove that there is a natural isomorphism in $\mathscr{C}_{\mathcal{H}}$

$$(5.11) \hspace{3cm} G \cong (C\square_{C^\star C}C^\star)G,$$

for $G \in \mathrm{add}(\mathcal{H})$. Finally, since $\mathcal{H}$ is a right cell of $\mathscr{C}_{\mathcal{H}}$, it suffices to prove (5.11) when $G = CH$, for some $H \in \mathrm{add}(\mathcal{H})$. In this case, the isomorphism in (5.11) is immediate, since $C \cong C\square_{C^\star C}(C^\star C) \cong (C\square_{C^\star C}C^\star)C$. Note that the isomorphism

$$CH \cong (C\square_{C^\star C}C^\star)(CH),$$

is natural in CH, as it is given by

$$(\mathrm{coev}_C^{C^\star C,\prime} \circ_h \mathrm{id}_{CH}) \circ_v (v_{CH}^l)^{-1},$$

where $\mathrm{coev}_{\mathrm{C}}^{\mathrm{C}^\star \mathrm{C},\prime}$ is defined just as $\mathrm{coev}_{\mathrm{Y}}^{\mathrm{C},\prime}$ below (5.5). This completes the proof of Claim 2.

We are now ready to complete the proof of Theorem 5.2. The assignment

$$\mathrm{can}^{-1}\colon \mathrm{Z} \mapsto \mathrm{C}\square_{\mathrm{C}^\star\mathrm{C}}(\mathrm{Z}\square_{\mathrm{C}^\star\mathrm{C}}\mathrm{C}^\star)$$

defines a pseudofunctor in the opposite direction. Clearly, $\mathrm{can} \circ \mathrm{can}^{-1}$ is naturally isomorphic to the identity on $(\mathrm{C}^\star\mathrm{C})\mathscr{B}\mathrm{iinj}_{\underline{\mathscr{B}}_{\mathbf{M}}}(\mathrm{C}^\star\mathrm{C})$, as follows from

$$(\mathrm{C}^\star\mathrm{C})\square_{\mathrm{C}^\star\mathrm{C}}(\mathrm{Z}\square_{\mathrm{C}^\star\mathrm{C}}(\mathrm{C}^\star\mathrm{C})) \cong \mathrm{Z}$$

for any $\mathrm{Z} \in (\mathrm{C}^\star\mathrm{C})\mathscr{B}\mathrm{iinj}_{\underline{\mathscr{B}}_{\mathbf{M}}}(\mathrm{C}^\star\mathrm{C})$. On the other hand, $\mathrm{can}^{-1} \circ \mathrm{can}$ is naturally isomorphic to the identity on $\mathrm{add}(\mathcal{H})$, due to (5.11) and its analog for tensoring with $\mathrm{C}\square_{\mathrm{C}^\star\mathrm{C}}\mathrm{C}^\star$ on the right in $\mathrm{add}(\mathcal{H})$. Since we already know that can takes values in injective $(\mathrm{C}^\star\mathrm{C})$-bicomodules when restricted to $\mathrm{add}(\mathcal{H})$, these two natural isomorphisms imply that can^{-1} takes values in $\mathrm{add}(\mathcal{H})$ when restricted to injective $(\mathrm{C}^\star\mathrm{C})$-bicomodules and, moreover, that can is fully faithful on 2-morphisms and essentially surjective on 1-morphisms when restricted to $\mathrm{add}(\mathcal{H})$ and corestricted to injective $(\mathrm{C}^\star\mathrm{C})$-bicomodules. $\square$

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