The discursive footprint of learning across mathematical domains: The case of the tangent line

Irene Biza

School of Education and Lifelong Learning, University of East Anglia, Norwich, UK

Abstract

This paper employs the commognitive frame (Sfard, 2008) to investigate how experiences with tangents across mathematical domains leave their marks on students' subsequent work with tangents. To this aim, I introduce the notion of the *discursive footprint* of tangents and its characteristics by reviewing how tangents are used across mathematical domains in school textbooks. Manifestations of this footprint were sought in 182 undergraduate mathematics students' responses to a questionnaire about tangents by labelling their responses and by identifying patterns in the endorsed narratives. Manifestations include the identification of characteristics of sole (and combination of) discourses (geometry, algebra, calculus, mathematical analysis) in student responses. Five themes emerged from the analysis: *apparent replication of word use in different narratives; geometry-local hybrid discourse; endorsement of conflicting narratives; enrichment of familiar narratives with new words;* and, *mathematical analysis as a subsuming discourse*. Finally, I discuss the potency of the discursive footprint in research and teaching.

Keywords: Tangent line, discursive footprint, mathematical discourse, commognition, precedents, subsuming discourses.

1. Introduction

There are mathematical topics that appear in different domains of mathematics potentially with different uses in each of these domains (*cross-curricular topics* in Kontorovich, 2018). The tangent line to a curve for which we have seen a wide range of definitions (Artigue, 1990) is one of these topics. For example, the tangent line appears in geometry (e.g., tangent to a circle), in algebra (e.g., tangent to parabola and other Cartesian curves), in calculus (e.g., tangent to a function graph at a point where the function is differentiable) and in mathematical analysis (e.g., tangent to a function graph as a line with the limit of the difference quotient as its slope). Research has attributed students' challenges with tangent lines (thereafter tangents) to their different uses within different mathematical domains (e.g., Biza, Christou & Zachariades, 2008; Castela, 1995; Vinner, 1991). The work presented in this paper contributes to this body of research by drawing on the commognitive frame (Sfard, 2008) – that sees mathematics as a discourse and the learning of mathematics as a communication act within this discourse – to gain more insight into learning about tangents across mathematical domains. I see learning about tangents as a longitudinal process through education and everyday experiences (*precedent events*, Lavie, Steiner & Sfard, 2019) that leave their mark on how students' discourse about tangents is shaped. The discursive characteristics of how tangents are dealt with in different mathematical domains in the course of students' studies are what I call the *discursive footprint* of tangents. In this paper, first, I conceptualise the notion of discursive footprint of tangents and, then, I identify manifestations of this footprint in student responses to a questionnaire related to tangents.

2. Students' learning about tangents in prior research

Prior research has reported students' difficulties with tangents, especially in the case of function graphs. Specifically, students often do not accept as a tangent a line that has many points in common with the graph (e.g., $f(x)=\sin x$ at $x_0=\pi/4$)

or coincides with part, or the whole, of the graph (e.g., when the curve is a straight line). Also, students have difficulty to accept tangents at inflection points (e.g., $f(x)=x^3$ at $x_0=0$), in which the tangent splits the graph into two semi-planes. Similarly, it can be not easy to decide whether there is a tangent at points in which the limit of the difference quotient is not defined (e.g., $f(x)=|x|$ at $x_0=0$) or when it is infinite (e.g., $f(x)=\sqrt{|x|}$ at $x_0=0$) (Biza et al., 2008; Castela, 1995; Vinner, 1991).

These difficulties have been attributed *inter alia* to how tangents are defined in different mathematical domains and what properties tangents have in these domains (Biza et al., 2008; Castela, 1995; Winicki & Leikin, 2000; Vinner, 1991). In geometry, for example, the tangent to a circle is a line that has one point in common with the circle while, in calculus, the tangent at a point of a function graph is a line that has this point in common with the graph and its slope is the value of the derivative of the function at this point. Differences in definitions affect perspectives on tangency. For example, tangency may describe a *global* relationship between a curve and a line (e.g., the whole circle has one point in common with the line) or a *local* relationship between a curve and a line (e.g., tangency at a specific point of a function graph regards how the graph is at this specific point and not as a whole) (Biza et al., 2008; Maschietto, 2008). In addition, different mathematical terms – such as secants, slope, limit, derivative, etc. (Sierpinska, 1985) – are involved in different definitions and uses of tangents. Students are asked to navigate across definitions, properties, perspectives and terms without necessarily being aware of their underpinning differences.

In addition, some studies highlight the influence of curricular, pedagogical and institutional practices on how tangents are used. For example, Castela (1995) argues that not considering differences between global and local perspectives in teaching confuses students. In addition, the use of tangents across educational contexts is not uniform: for example, tangents are used differently in vocational education mathematics courses and pure mathematics courses (Pinto & Moreira, 2007). Also, a tangent line perspective on derivative is common for students affiliated with mathematics departments but not for those affiliated with mechanical engineering departments who encounter tangency largely from a rate of change perspective (Bingolbali & Monaghan, 2008).

The influence of students' previous experiences and the educational context in which they learn about tangents is important to my work about tangents. In earlier stages of this work (Biza et al., 2008; Biza & Zachariades, 2010), three clusters of upper secondary and university student perspectives on tangency were identified: *analytical local*, *geometrical global* and *intermediate*. The analytical local perspective is closer to how tangents are used in calculus and mathematical analysis, where a tangent is defined at a point (locally, see also, Maschietto, 2008) through derivatives or limiting positions of secants. Biza et al. (2008) and Biza and Zachariades (2010) showed that students with this perspective typically accept a line as a tangent even if it has many points in common with the graph. The geometrical global perspective is closer to how tangents are used in geometry. Students with this perspective typically do not accept a tangent line that has more than one point in common with the graph (see also, Winicki & Leikin, 2000; Vinner, 1991; Castela 1995). Intermediate perspectives are between the other two: for example, students employ properties from geometry in a local part of the graph (such as that the tangent line should have one point in common with the curve at a neighbourhood of the tangency point, see also Vinner, 1991).

These three perspectives were identified through analysis of students' written responses to a questionnaire (Biza, et al., 2008; Biza & Zachariades, 2010). This analysis had focussed on whether the responses are correct or not.

Recently (Biza 2019), after noticing the broad spectrum in the ways students use and argue about tangents, the subtlety of which is not reflected in the three aforementioned perspectives, I returned to a more systematic analysis of these responses. For example, for a student who responded that "a line ε is not a tangent to a function graph because this line intersects the function graph at another point", the response had been characterised as evidencing geometrical global perspective. However, the word "intersect" in the response might have been used with different meanings: "intersect" may mean that a line and a graph have another common point in which the line cuts across the graph; or, it may mean that the line and the graph have another common point regardless of whether the line cuts across the graph or not (e.g., the line might be a tangent to the graph at this point). Different meanings of "intersecting" could affect the responses in other parts of the questionnaire. Such differences could not be captured by looking only at whether a student response, or justification, is correct or not. Furthermore, in the three aforementioned perspectives, I could see elements of what students had learned in previous years intertwined together. However, my previous analysis did not account for the subtlety of the effect that these previous experiences had on students' responses about tangents. I now draw on the commognitive frame (Sfard, 2008) to formulate the research question of this paper.

3. Learning about tangents as a change of discourse

According to the commognitive frame (Sfard, 2008), mathematics is a discourse established within a certain community and defined by four characteristics: *word use*, *visual mediators*, *narratives* and *routines* (Sfard, 2008). Word use includes the use of mathematical terms, such as tangent, derivative or slope, as well as everyday words with a specific meaning within mathematics (such as *intersect* or *point*). Visual mediators are "visible objects that are operated upon as part of the process of communication" (ibid, p. 133) such as function graphs, geometrical figures or symbols, as well as physical objects. Narratives "denote any sequence of utterances, spoken or written, framed as a description of objects, of relations between objects, or of activities with or by objects" (ibid, p. 223). Narratives are "subject to endorsement, modification or rejection according to rules defined by the community (such as definitions, theorems and proofs)" (Nardi, Ryve, Stadler & Viirman, 2014, p. 184). For example, 'a tangent line is a line that has one point in common with a curve' might be an endorsed narrative for tangents in the case of a circle in geometry but not in mathematical analysis. Routines "include regularly employed and well-defined practices that are used in distinct, characteristic ways by a community (such as defining, conjecturing, proving, estimating, generalising and abstracting)" (ibid, p. 184). For example, in geometry, finding the tangent to a circle at a point A means drawing a line which is perpendicular to the radius at this point, whereas, in algebra, finding the tangent involves using the formula of the circle and calculating the tangent line equation. Of particular interest to this paper are routines related to *substantiation* of an endorsed – or to be endorsed – narrative. This is a process in which the person who engages with a mathematical activity is "convinced that the narrative can be endorsed" (Sfard, 2008, p. 231).

I now outline briefly the use of tangency in the discourse of four mathematical domains: geometry, algebra, calculus and mathematical analysis. In geometry discourse, geometrical figures are key objects (e.g., points, lines, circles etc.). Thus, whether a line is a tangent to a circle is substantiated through narratives about these objects, such as the one common point, and mediated by drawings that visualise these objects, such the drawing of the one common point. In algebra discourse, symbolic formulae in algebraic expressions are the key objects and curves are realised through their formulae (e.g., a circle of radius one, with centre at (0,0) as $x^2+y^2=1$). Thus, whether a line is a tangent to a curve is substantiated through narratives about these formulae, such as the solutions of simultaneous equations, and mediated by formulae and drawings in the Cartesian plane. In calculus discourse, functions are the key objects and curves that meet certain conditions are realised through function graphs (e.g., a semi-circle of radius one, with centre at (0,0) as $y = \sqrt{1-x^2}$). Thus, whether a line is a tangent to a curve is substantiated through narratives about functions, such as the calculation of the derivative at a point, and mediated by the notations and drawings related to the object of function. In mathematical analysis discourse, similarly to calculus, functions are the key objects. However, whether a line is a tangent to a curve is substantiated with narratives that involve limits and convergence.

There is a thread that connects utterances across these discourses, what Sfard (2008) calls an "isomorphic" mapping of the utterances of one discourse to another (p. 122). Curves that meet certain conditions (e.g., the semi-circle) might be seen as geometrical figures, as Cartesian curves with algebraic formulae or as function graphs. The mathematical analysis discourse conflates the isomorphic parts of geometry, algebra and calculus discourses in relation to the tangent line. For example, the tangent line as the limiting position of secant lines in mathematical analysis works well for the tangent line to a circle, to a parabola and to any function graph. Sfard (2008) sees the discourse that conflates the isomorphic parts of other discourses into one discourse as a *subsuming discourse*, and in this sense the mathematical analysis discourse is the subsuming discourse of the isomorphic parts of geometry, algebra and calculus discourses. However, such subsuming - and the thread that connects the different discourses - is not explicitly traced in the teaching where communication often occurs within one of the aforementioned discourses while paying limited attention to similarities or differences across discourses.

I see previous experiences with tangents as precedent events that leave their marks on how a student reacts to a certain *task situation*, namely a situation "in which a person considers herself bound to act—to do something" (Lavie et al., 2019, p. 159). A person considers a current task situation similar to precedents she has dealt with before (or has seen another person dealing with) and may repeat what was done in the past. Lavie et al. (2019, p. 160) claim that the search for precedents is "restricted to a precedent-search-space (PSS, for short) arising as if of itself in a situation in which we find ourselves". School experiences of tangents have an essential role in the generation of the PSS, and, in this paper, I aim to shed light on how students' prior experiences with tangents shape their discursive activity in task situations involving tangents as the students enter university mathematics studies.

Having described the different uses of tangents in different mathematical domains and setting out from the assumption that students' prior experiences of tangents in school constitute aforementioned precedents, I propose the notion of discursive footprint. The discursive footprint captures students' curricular experiences of tangents and comprises word use around tangents, narratives of what a tangent line is, visual mediators related to tangents and routines employed to determine and substantiate whether a line is tangent to a certain curve. Such experiences were discerned from the curricular materials (in the case of this study school textbooks) students had been taught with in the past and before participating in the study. These experiences span different mathematical domains. I conjecture that the discursive footprint of tangents impacts students' subsequent work with tangents. Narratives about tangents that the students have seen in the past may be echoed in their subsequent responses but with changes. Sometimes, a *hybrid discourse* – namely, a discourse "that is not yet the way it should be, but not the way it was before" (Sfard, 2008, p. 269) – might be developed by learners. So, I expect to see such hybridisation in student responses. Also, I expect to see student responses which may endorse conflicting narratives – what Kontorovich (2019) calls evidence of *intra-commognitive conflicts*. Figuring out the discursive characteristics of students' previous curricular experiences (curriculum-generated PSS) will help to pin down *which* of those characteristics come across in student responses to task situations related to tangents and *how*. In a nutshell, in this paper, I investigate the following research question: *What are the manifestations of the discursive footprint of tangents in students' responses to task situations about tangents?*

4. Participants and Methods

4.1 Participants and data collection

The data analysed in this paper are responses to a questionnaire administered to 182 mathematics undergraduate students attending a first-year calculus course in two universities in Greece. Similarly to other recent studies (e.g., Kontorovich 2019; Thoma & Nardi, 2018), students' discursive activity is seen in individual written responses to questionnaire tasks. Participation in the study was voluntary and not part of any course assessment. Participants had completed compulsory education up to Year 9 (age 15) and post-compulsory in Years 10-12 (age 15-18) with specialisation in mathematics in Years 11 and 12. Admission to the undergraduate mathematics programme was based on a national examination at the end of Year 12. The participants had been taught about tangents at school level (section 4.3) but not yet at university, as the questionnaire was administered at the beginning of their first year.

4.2 The questionnaire

The part of the questionnaire I use in the analysis presented in this paper[1] consists of five tasks[2] in which the students were asked to: explain the meaning of a tangent line in their own words (Q1); describe its properties (Q2); determine if a line drawn on a given graph is a tangent line (Q3, fourteen graphs); define the tangent line (Q6); and, apply the tangent line formula on specific functions (Q8) – see Figure 1.

The proposed graphs in Q3 and functions in Q8 were chosen to reflect students' common challenges with tangents that previous research has highlighted (e.g., Biza et al., 2008; Castela, 1995; Vinner, 1991). For example, situations where the line in question has more than one point in common with the graph (e.g., Q3.2, Q3.3, Q3.11, Q3.13 or Q8a) or where the line passes through an inflection point (e.g., Q3.6, Q3.10, Q3.12 or Q8b) or an edge point (e.g., Q3.5, Q3.7 or Q3.9). The questionnaire was initially designed with the intention to investigate students' perspectives on tangents (Biza et al., 2008; Biza & Zachariades, 2010). In this previous research, the analysis was based mostly on the characterisation of responses as correct/incorrect. In this paper, I repurpose the analysis of the data from the same group of first year university mathematics students (Biza & Zachariades, 2010) in order to identify manifestations of the discursive footprint of tangents. While the questionnaire was not designed with the commognitive theory (Sfard, 2008) in mind, I see the questionnaire as a collection of task situations in which students were invited to engage. Specifically,

---

[1] The questionnaire, textbook excerpts and student responses have been translated from Greek.

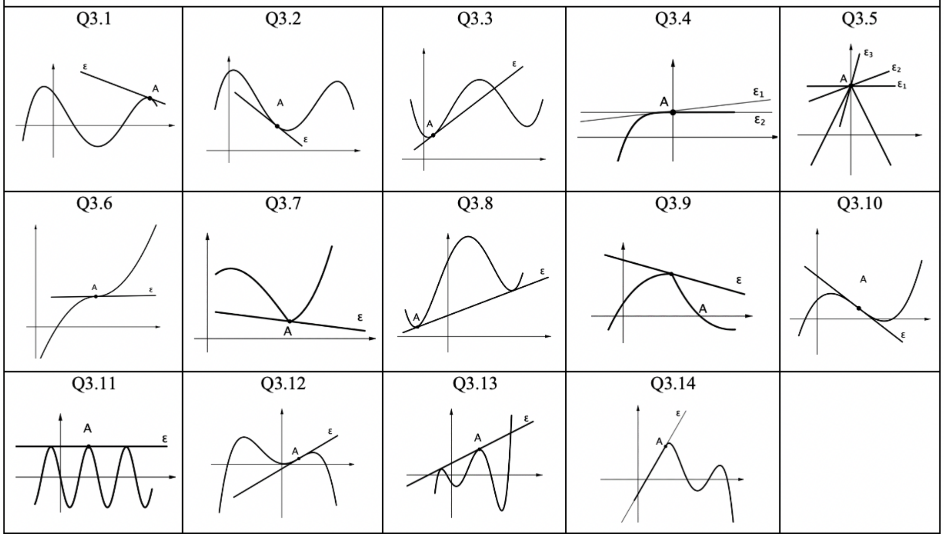
[2] I have kept the original numbering of the tasks. Only responses to Q1-Q3, Q6 and Q8 have been considered in the analysis presented in this paper. Q4, Q5 and Q7 invited students to sketch tangent lines or write tangent formulae. In those tasks, many of the respondents did not provide justifications for their sketches or formulae. As justifications are essential in the analysis presented in this paper, in the absence of such justifications in the students' responses to Q4, Q5 and Q7, I have not considered responses to these tasks here.

Q1, Q2 and Q6 invite students to generate narratives about tangent line, its properties and its definition. In Q3, students are asked to determine whether a drawn line is a tangent of a given graph and to substantiate their claims. In this task , the function formulae were not provided. There was only the visual prompt, the diagrams of the function graphs. As a result, the task situation did not guide participants explicitly towards a familiar course of action, such as using the formula to calculate the derivative: the participants had to make this judgement from the graph alone. The mediating role of these diagrams can be discerned only if participants mention this role explicitly in their justifications (e.g., if they say "the point *A* of the graph is …" or if they draw on the provided diagrams). In contrast, Q8 invites a calculation of the formula of the tangent, a familiar task situation that is commonly used in the textbooks that the students had been taught with.

Q1. Explain, in simple words, what you are thinking when you hear the term "tangent line".

Q2. Write as many properties as you can think of about the relationship between a curve and its tangent at a point *A*.

Q3. Which of the lines that are drawn in the following figures are tangent lines of the corresponding graph at point *A*? Justify your answers.



[…]

Q6. What is the definition of the tangent line of a function graph at its point *A*?

[…]

Q8. Calculate the equation of the tangent line of the graph of the function *f* at the point $A(x_0, f(x_0))$ in the following cases:

a. $f(x) = \begin{cases} x^2, & x < 0 \\ 0, & x \geq 0 \end{cases}$ for $x_0 = 0$ b. $f(x) = (x-2)^3 + 3$ for $x_0 = 2$

Figure 1. Tasks Q1, Q2, Q3, Q6 and Q8 from the questionnaire

4.3 The tangent line in the curriculum

In order to trace the discursive footprint of tangents in the curriculum, I reviewed how tangents are presented in the textbooks, participants had been taught with in secondary school[3]. The first appearance of tangents is in the Year 7 Geometry textbook (Alibinisis, Antypas, Efstathopoulos, Klaoudatos & Papapstavridis, 1994), and, there, a tangent line to a circle is defined as the line that has only one point in common with the circle. Also, the distance of this line from the centre of the circle is equal to the radius. The tangent line to the circle is introduced again in the Year 10 Euclidean Geometry textbook (Argyropoulos et al., 2003) through similar properties. In the same Year's Algebra textbook

[3] In the Greek educational context, teachers and students follow the syllabus and textbooks approved by the Greek Ministry of Education for each educational level. Teachers and students may use additional materials and resources at their discretion. For the purpose of this paper, only the Ministry approved textbooks have been analysed as they form the core body of resources used in teaching.

(Andreadakis et al., 1998), the tangent line is seen also in tasks that require finding the common points between a line and a circle or between a line and a parabola with known formulae. Here, the line is a tangent to the circle (or the parabola) if the simultaneous equations lead to a quadratic equation with a double solution (e.g., $a(x\text{-}b)^2=0$). Then, in the Year 11 Algebra textbook (Adamopoulos et al., 1998), aimed at those who have chosen pathways with specialisation in mathematics, the tangent line to conic sections is introduced. For a parabola, an ellipse and a hyperbola, the tangent line is defined as the limiting position of secants. This definition is used in the proof of the formula of the tangent line, but not in the exercises that follow, which focus mostly on applications of formulae. Finally, in the Year 12 Calculus textbook (Andreadakis et al., 1998), again, for the specialisation in mathematics pathways, the tangent line to a circle is redefined with the use of the limiting position of secants and then this new definition is used to introduce the tangent to the function graph that has slope equal to the limits of the slopes of the secants (derivative).

Table 1 distinguishes between tangents in different mathematical discourses as found in the textbooks by accounting for the characteristic word use, narratives, visual mediators, and routines to determine and substantiate whether a line is tangent to a certain curve. As Table 1 demonstrates, a rich set of narratives were found in the exposition of the textbooks. However, it is often the case that – in the exercise part of the textbooks and in the examination questions on which students are assessed – engagement with only a small part of these narratives is required (Biza & Zachariades, 2010).

4.4 Seeking manifestations of the discursive footprint of tangents in the students' responses

To identify manifestations of the discursive footprint of tangents in students' responses, first, each student response was labelled according to the discursive characteristics identified in the textbooks – discursive footprint (Table 1). For example, a response that uses words such as "one point in common" or "circle" or "vertical to the radius", is labelled as part of the geometry discourse (labelled as G, as I explain in 5.1). When characteristics of more than one discourses are manifested in the same response, the response was labelled accordingly (e.g., geometry and calculus, G+C, as I explain also in 5.1).

Then, I looked for patterns in the endorsed narratives about tangents (e.g., what a tangent line is and what properties it has) in the responses, particularly in relation to substantiation routines (e.g., why a drawn line in Q3 is or is not acceptable as a tangent) and in relation to word use (e.g., what is the meaning of 'intersect' in phrases such as 'the line intersects the curve'). Use of notation and drawings, namely visual mediators, had an auxiliary support to this analysis. I analysed each participant's responses across the questionnaire tasks. More attention was put on scripts that had more than one labels. For example, if a response to Q2 endorses the narrative that the line should have only one point in common with the curve (G), is this narrative used in the substantiation of why a line is accepted (or not) as a tangent in the graphs of Q3? Is it combined with calculus (C) or mathematical analysis (AN) narratives and how? Furthermore, I sought differences and similarities between responses to the same or similar questionnaire tasks across participants. For example, how do different participants use the word 'intersect' in Q3.3 (another common point) and Q3.8 (another tangency point)? How do they justify their choices in Q3.6, Q3.10 or Q3.12 in which point *A* is an inflection point? Also, I looked for responses in which a certain characterisation dominated across participants (see Table 2). For example, how do different participants use the narrative that the line should have only one point in common with the curve together with the routine of finding the derivative (C+G)? Findings of this analysis are presented in 5.2.

Discourse	Tangent to…	Key words and phrases	Narratives	Routines	Visual mediators
Geometry (G)	Circle Semicircle (or part of a circle circumference)	• One point in common/(common) point(s) • Distance • Radius • Vertical • Plane/Line- Semi-plane/line • Circle • (Right) Angle(s) • Intersect • Split • External, Internal, side (words indicating the relative position of the line and the circle)	• The line has only one point in common with the circle. • Every point of the line, except the tangency point, is external to the circle. • The line keeps the circle at the same semi-plane. • The line has a geometrical property that describes the relationship between the line and the circle, e.g., the distance of the centre of the circle from the line is equal to the radius.	• Sketching tangents to a given circle • Recognising and characterising sketched lines in relation to the circle (secant, external or tangent) • Calculating radii and distances • Using geometrical properties to solve geometrical problems	• Drawings of geometrical objects on the Euclidean plane (no axis or grid) • Labels of objects on the Euclidean plane (e.g., points, angles and lines)
Algebra (ALG)	Conic Sections (circle, ellipse, parabola or hyperbola)	• One point in common/(Common) point(s) • Formula(e) • Equation(s)/simultaneous equations • Slope/Direction coefficient[4] • Double solution/root(s) • Vector(s) • Tangent of angle or tanθ	• The line is defined by a given algebraic formula[5]. • The system of simultaneous equations of the line and the curve has a solution of multiplicity more than 1 (multiplicity 2 is called *double solution*). • Curve and line have the same slope.	• Finding the formula of a tangent line • Solving simultaneous equations • Using the formula to identify properties of the curves and the tangent line	• Drawings on the Cartesian plane • Mathematical formulae • Labels of objects on the Cartesian plane (e.g., points, angles and lines)
Calculus (C)	Function graphs	• (Common) point(s) • Function graph • Derivative/differentiability • Formula: $y-f(x_0)=f'(x_0)(x-x_0)$ • Slope • Direction coefficient • Concavity/concave (up/down) • Inflection/corner/edge point(s) • Increasing/decreasing function(s)	• The line has a point in common with the curve and its slope is equal to the value of the derivative at this point.	• Calculating the derivative to find the slope of the tangent line (or function graph) • Finding the formula of a tangent line • Examining if a given line is a tangent • Identifying conditions under which a given line is a tangent • Finding a tangent passing through a given point outside the graph • Calculating the area between the function graph and the tangent line (in integrals)	• Function graphs on the Cartesian plane • Formulae in symbols • Labels of objects on the Cartesian plane (e.g., points, angles and lines)
Mathematical Analysis (AN)	Function graphs	Key words of the Calculus (C) discourse used together with: • Limit(s) • Limit(s) of difference quotient/slopes • Limiting position of secants	Narratives of the Calculus (C) discourse substantiated by: • If *A* and *B* are two points on the curve, the line is the limiting position of the secants *AB* when *B* approaching *A*. • The line has slope equal to the limit of the slopes of the secant lines. • The line has slope equal to the limit of the difference quotient.	Routines of the Calculus (C) discourse with: • Calculating the limit of difference quotient • Calculating the limit of slopes • Substantiating formulae and properties of tangents with limits	Visual mediators used in the Calculus (C) discourse with: • Limit notation • Limiting position of secants

Table 1. Discursive footprint of tangents in the textbooks the students were taught with

[4] In the Greek curriculum, the "direction coefficient" is the coefficient *m* in *y*=*mx*+*b*, that indicates the slope of a line.

[5] An algebraic formula that does not require the calculation of derivatives or limits, such as the formula of the tangent of a hyperbola at a given point.

5. Manifestations of the discursive footprint of tangents in students' responses

5.1 Overview

Table 2 presents an overview of the labels I used to analyse the data with reference to the mathematical discourses and examples from students' responses to the questionnaire. In the examples provided in the table, underlined text indicates discursive characteristics mapped in Table 1 while the corresponding discourse is indicated in square brackets. As I expected, there were responses with evidence of engagement with more than one discourses. For example, ALG+G indicates engagement with both algebra and geometry discourses. Also, I found narratives, such as [tangent is] "the line that has one common point with the curve in a region of the tangency point", that were not found in the textbooks (Table 1). Such responses evidence characteristics of the geometry discourse (G), e.g., one common point, used in a local neighbourhood of the tangency point. I labelled these responses as *geometry-local* (GL), instead of geometry (G), although key characteristics of the geometry discourse were used. I elaborate this characterisation further in section 5.2.

Discourse	Example of student responses
Geometry (G)	"No [it is not tangent], the line has 2 points in common [G] with the function graph"
Algebra (ALG)	"Yes, line ε is tangent at A, the slope [ALG] is equal to the direction coefficient [ALG] of the line"
Calculus (C)	"A function which is differentiable [C] at a point $A(x_0,f(x_0))$ has a tangent at this point […] its formula is $y-f(x_0)=f'(x_0)(x-x_0)$ [C]"
Mathematical Analysis (AN)	"The line satisfies this formula [sic] and $\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=\lambda$" [use of limits characterise the AN discourse]
Geometry-Local (GL)	"[It is tangent, b]ecause if we consider a small region (κ, γ) around the point A [GL] where [the line] ε is tangent we can see that [line] ε does not touch any other point [GL]"
Algebra and Geometry (ALG+G)	"The [coordinates of the] point *A* verifies the [formulae of the] line and the figure [ALG]. If the figure is circle, the line from the centre of the circle cuts the tangent vertically [G]."
Algebra and Geometry-Local (ALG+GL)	"The function graph and the tangent (ε) have only one point in common in a 'neighbourhood' around this point (the point *A*) [GL]. It [the tangent] has as its slope [ALG] the value of the tangent of the angle [ALG] that is formed between the *x* axis and the function graph [sic]" [her emphasis]
Algebra and Calculus (ALG+C)	"$\tan_\varepsilon=f'(x_0)$ [ALG, C] If we consider the point $A(x_0, y_0)$ then the equation of the function [sic] is $y-f(x_0)=f'(x_0)(x-x_0)$ [C]"
Algebra and Mathematical Analysis (ALG+AN)	"The slope [ALG] of the tangent is the derivative of the curve […] $\tan\omega=\lambda=\frac{dy}{dx}=\frac{y_2-y_1}{x_2-x_1}\Rightarrow\frac{dy}{dx}=\frac{f(x+\Delta x)-f(x)}{\Delta x}\ \lim\frac{\Delta y}{\Delta x}=f'(x)$" [use of limits characterise the AN discourse]
Calculus and Geometry (C+G)	"One common point [G] only and $f'(A)=\lambda_\varepsilon$ [C]"
Mathematical Analysis and Geometry (AN+G)	"The tangent line of a function graph C_f at a point *A* of a function graph is the limiting position of a line passing through the points *A, M*, while *M* approaches towards *A* infinitely [AN]. The tangent splits the plane into two semi-planes, of which only one contains the shape [G]"
Calculus and Geometry-Local (C+GL)	"$A(x_0,f(x_0))$ $y-y_0=\lambda(x-x_0)$ $f'(x_0)=\lambda$ direction coefficient [C]. It is a unique line with only one common point near to *A* [GL]"
Calculus, Algebra and Geometry (C+ALG+G)	"The tangent is a line, between which [the line] and the figure there is 1 common point, the *A* [G]. The slope of the tangent is the rate of change and the derivative of the point *A* [C] which is on a line of the figure and is characterised by a function [sic] $\tan\vartheta=f'(x)$ [ALG]"
Mathematical Analysis, Algebra and Geometry (AN+ALG+G)	"$f'(x_A)=\lambda$ the direction coefficient. $f'(x_A)=\lim_{x\to x_0}\frac{f(x)-f(x_A)}{x-x_A}$ [AN]. At this point it [the line] has one 'double' [his emphasis] common point with C_f [ALG]. It can have other common points [G] with C_f, $x\neq x_A$"
Calculus, Algebra and Geometry-Local (C+ALG+GL)	"Line ε is [tangent] because *f* is differentiable at x_A [C] and ε has one (double) [ALG] common point with C_f in an interval $(x_A-\kappa, x_A+\kappa)$, $\kappa>0$ and very small [GL]"
Inconclusive	"Yes, because it is tangent[6]"

Table 2. Overview of the labels used to characterise student responses

[6] Such responses did not provide enough evidence to attribute them to a certain discourse according to the characteristics in Table 1.

The distinction between calculus and mathematical analysis was not always a clear-cut in student responses. In this characterisation, I followed the convention to label a response as: calculus (C) when derivative and properties of the function (and function graph) related to the derivative (e.g., concavity, inflection point etc.) were mentioned without any reference to convergence or limits; and, as mathematical analysis (AN) when limits or/and some type of convergence were mentioned. Also, I note that some of the responses are not mathematically complete or accurate – mathematical correctness of the responses is not the focus of this paper and I see the inclusion of a variety of responses in Table 2 as a poignant reminder of this. In what follows, I elaborate the manifestations of the discursive footprint of tangents in the students' responses by discussing five themes that emerged from further scrutiny of the labelled scripts: *apparent replication of word use in different narratives; geometry-local hybrid discourse; endorsement of conflicting narratives; enrichment of familiar narratives with new words* and, *mathematical analysis as a subsuming discourse.*

5.2 Manifestations of the discursive footprint in student responses: Five themes

Apparent replication of word use in different narratives

Here, I discuss how word use from mathematical discourses found in the textbooks is replicated in the task situations that the questionnaire presented. The following examples are taken from responses that evidence characteristics attributed to the geometry discourse (G in Table 2).

Students[7] S[6] and S[92] write in Q3.3:

"Line ε is not a tangent to the function graph because it [the line] intersects it [function graph]" (S[6])

"Line ε is not a tangent [to the function graph] because its extension intersects it [function graph]" (S[92])

Although, both S[6] and S[92] use the verb "to intersect" ("τέμνει" in Greek), the use of this verb is not the same as we see in their responses to Q3.8 (and similarly to Q3.11):

S[6] rejects the line[8] because "it has 2 common points with the function graph"

S[92] accepts the line because "it does not intersect the curve. Simply, [line] ε is tangent [to the curve], apart from point *A,* at another point [of the curve]"

Then, they both reject the line in Q3.13. Both students engage with the geometry discourse[9]. However, it seems that they use the same verb, "to intersect", differently: for S[6], intersection is used to describe the number of common points ('only one point in common' in Table 1) whereas, for S[92], intersection means also cutting across or splitting the graph (e.g., 'the line keeps the circle at the same semi-plane' in Table 1). This difference is more evident in their responses

---

[7]Student script numbering was done in no particular order. "He" and "She" reflect gender choice made by the participants on the cover page of the questionnaire.

[8] I use 'reject the line' (or 'accept the line') to indicate whether the respondent has rejected (or accepted) that the sketched line ε is tangent to a function graph in Q3.

[9] For simplicity, when I say "student S engages with discourse X", I mean that "there is evidence in the student response of characteristics attributed to discourse X according to the mapping presented in Table 1". Similarly, when I say "student S endorses narrative Y", I mean that "there is evidence in the student's response of endorsing narrative Y".

to Q2:

“The figure and the tangent have one common point” (S[6])

“[…] All the points of the curve, apart from point *A* are located on the left or the right side of the [line]” (S[92])

In consistency with his response to Q2, S[92] rejects the line when *A* is an inflection point (Q3.6, Q3.10 and Q3.12).

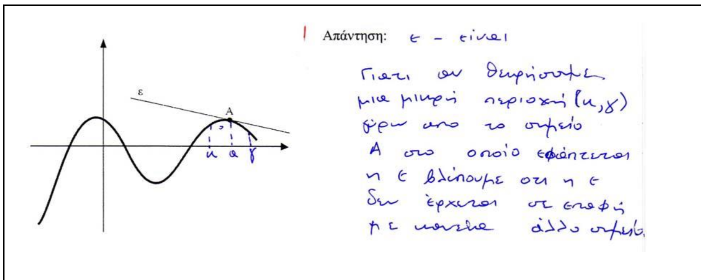
In another example, S[119] also rejects the line in Q3.2 because “the extension of the line ε intersects the curve in a second point. Thus, [the line] splits the curve in two semi-planes”. Again, I see the verb “to intersect” as used for the identification of common points but also to describe the division of the curve in different semi planes. It seems that, for S[119], the tangent line should satisfy both conditions. He writes in Q1: “Tangent line is that line which intersects the curve in only one point and the same [line] or its extension does not separate the curve into two semi-planes”. Consistently with this description, he rejects the line in both Q3.8, because “it has more than one common points”, and Q3.6, because “although it [the line] has only one common point with the curve it [the line] separates it [the curve] into two semi-planes”.

These responses use words such as “intersect”, “common point” and “semi-planes” that the students have seen before in geometry. However, such use varies across responses indicating different narrative about tangent line: the line has ‘only one point in common with the curve’ or ‘the line keeps the curve at the same semi-plane’ or both. This is what I see as *apparent replication of word use in different narratives* about tangents.

Geometry-local hybrid discourse

Here, I discuss narratives that I found in the students’ responses but not in the textbooks. The following examples are from responses containing narratives attributed to what I label as geometry-local discourse in Table 2 (GL).

S[123] writes in response to Q1: “The tangent line is a line which is tangent to a point *A* of a function graph and in a small interval around it [the point *A*] it [the line] does not intersect the function graph”. A similar narrative appears in the substantiation of why the line should be endorsed as a tangent in Q3.1 and Q3.2 (Figure 2). It seems that, despite endorsing ‘the one point in common’ narrative (Table 1) to justify why S[123] sees this as a tangent line, this narrative is substantiated locally, around point *A*.



Answer: **ε** – it is [tangent]

Because if we consider a small interval (**κ**, **γ**) around the point A where [the line] **ε** is tangent we can see that [the line] **ε** does not touch any other point

Figure 2. S[123]’s response to Q3.1

Similarly, in Q3.2, S[124] claims that line *ε* is a tangent because “point *A* is common point of the line *ε* and the function graph and the part of the function graph which is close to *A* is at the same side of the line *ε*”. It seems that, for S[124], ‘the line keeps the circle at the same semi-plane’ narrative (Table 1) is also substantiated locally: “close to *A*”.

Other students accept the line as a tangent in an interval but not in general. For example, S[21] writes in response to Q3.8: "In (-∞,0] it [the line] is tangent at *A* and in [0, +∞) it [the line] is tang. at point *B* [and points as *B* the other common point on the graph]". Similarly, S[88] writes in response to Q3.2 "No [it is not a tangent] in general because it cuts the function graph. However, in a small interval (δ>0) (x-δ, x+δ) it is" and sketches an interval (x_1, x_2) in the graph that contains point *A* but not the other point.

There is evidence in these responses of endorsing narratives that are similar to those of the geometry discourse (e.g., one point in common). However, these narratives are substantiated in a neighbourhood of the tangency point. The difference between the responses above is that, for S[21] and S[88], the line's satisfaction of geometrical properties substantiates its tangency locally but not in general. In contrast, for S[123] and S[124], satisfaction of the same properties locally substantiates tangency in general. Thus, tangency is substantiated through endorsing a narrative not found in the textbooks that applies geometrical properties in a local area of the tangency point. New key words signify a neighbourhood around the point *A*, such as "interval", "close to point *A*", etc. and visual mediators include indications of such neighbourhood in the graph (Figure 2). Also, there is evidence in the responses of engaging with routines that concern the application of geometrical properties around the tangency point. I consider this as evidence of an emerging hybrid discourse (Sfard, 2008), what I label as geometry-local discourse (GL in Table 2), in which geometrical properties are applied locally. In calculus and mathematical analysis discourses, the attention is local as well, although the properties are not geometrical. I see GL as a *generation of a hybrid discourse* towards C or AN but not there yet.

Endorsement of conflicting narratives

Here, I discuss students' engagement with different discourses in the same response (e.g., C+G in Table 2). The following examples are from one script in which I see evidence of *endorsing conflicting narratives*.

S[125] writes in response to Q1: "A function which is differentiable at a point $A(x_0,f(x_0))$ has a tangent at this point. The tangent is a line which has one point in common with the graph of the function and its formula is $y-f(x_0)=f'(x_0)(x-x_0)$". The same student writes the definition for the tangent line to a function graph in Q6: "If $A(x_0,f(x_0))$ is the tangency point with the function we define [as tangent line] $y-f(x_0)=f'(x_0)(x-x_0)$". It seems that S[125] engages with the calculus discourse. However, in her response to Q1, there is a reference to "one point in common", the use of which becomes clearer in her responses to other tasks. First, she rejects the line in Q3.2 because "[a]lthough the function is differentiable at *A* thus it has a tangent, the ε that passes through *A,* if we extend it, it will have another common point with the function and thus it is not a tangent". Later, in Q3.10, she writes: "The [line] ε is tangent because the function is differentiable at *A* and they [the line and the graph] have only one common point".

It seems that, for S[125], tangency is substantiated through the endorsement of different narratives: the 'derivative of the function at a point' (calculus, Table 1) and the 'only one point in common with the circle' (geometry, Table 1). In Q3.2, in which the one point in common property is not satisfied while the function looks differentiable, S[125] expresses this mismatch between the two narratives ("although") before concluding that ε "is not a tangent". In Q3.10, both narratives are satisfied, and, thus, she accepts the line. However, in Q3.5, Q3.7 and Q3.9, she rejects the line because the "function is not differentiable" while in Q3.14 she rejects the line because "although the function has a tangent to *A*, it [the graph] has infinite common points with the [line] ε".

In these responses, I see narratives endorsed in the calculus discourse together with *conflicting narratives* endorsed in the geometry discourse, without explicit evidence that the student is aware of such conflict (*intra-commognitive conflicts*, Kontorovich (2019))

Enrichment of familiar narratives with new words

Here, I discuss the use of words that are introduced in calculus and were not used previously in geometry or algebra, e.g., “inflection point”. I sought such examples in responses endorsing what I label as calculus (C) or mathematical analysis (AN) narratives in Table 2 and then I looked at responses at other task situations across the script from the same students. The following examples are from responses to task situations where point *A* is an inflection point of the graph (e.g., Q3.6, Q3.10 and Q3.12).

Students S[93] and S[249] write in Q3.6:

“My impression is that it is not [tangent] … because the concavity [of the curve] changes …” (S[93])

“It is tangent because the concavity changes at *A*” (S[249])

I sought further what the students mean by “concavity” in their responses to other questionnaire tasks. S[93], first, writes in Q1 that a tangent “is a line that has only one common point with the function. Also, all the function points are in the left or the right side of the tangent (at least the points nearby)”. So, the narrative ‘line keeps the curve at the same semi-plane’ (geometry, G, Table 1), at least around point *A* (geometry-local, GL, Table 2), is endorsed by S[93] and he uses the change of “concavity” (a word used in calculus, C, Table 1) to describe how the curve passes from one side of the line to other.

On the other hand, S[249] lists the following properties in Q2: “1) [the tangent] should intersect the curve at no point 2) the first derivative should exist 3) when the concavity changes”. So, although she rejects the line in Q3.2 and Q3.3 because the line “intersects the curve”, it seems that, to her, the graph in Q3.6 is not a case of intersection. In Q3.6 “the concavity changes” and this is one of the properties she has listed in Q2. Thus, she accepts that line ε is a tangent in Q3.6 and Q3.10 but not in Q3.12 “because if we extend it, it will intersect the curve”. It seems that S[249] uses the “concavity” (a word used in calculus, C, Table 1) as a reason to accept tangency at inflection points as exceptional cases in which the “no point” intersection (geometry, G, Table 1) is not satisfied.

In these responses, *familiar narratives* (here from geometry) have been *enriched with new words* (here from calculus). The new words are used either to justify previously endorsed narratives in new cases of curves in calculus or to explain contradictions between previously endorsed narratives and such new cases of curves.

Mathematical analysis as a subsuming discourse

Here, I discuss responses endorsing narratives from several discourses (e.g., AN+ALG+G in Table 2). The following examples are from one script – first presented briefly in (Biza, 2019).

In his response to Q1, S[149] writes:

> [The tangent] is a line that is tangent to C_f at the tangency point. That is [the line] has ‘two’ [his emphasis] common points with C_f the distance of which is infinitely small and thus we consider that it [C_f] has a double point.

and then in Q2:

> $f'(x_A) = \lambda$ the direction coefficient. $f'(x_A) = \lim_{x \to x_0} \frac{f(x)-f(x_A)}{x-x_A}$. At this point it [the line] has one 'double' [his emphasis] common point with C_f. It can have other common points with C_f, $x \neq x_A$ [he sketches the graph in Figure 3].

In Q3.2 and Q3.3, S[149] accepts the line as a tangent because "it satisfies all the conditions" and, in Q3.6, he writes: "[line] ε is [tangent] because f is differentiable at x_A and ε has one (double) common point with C_f in the region (x_A-κ, x_A+κ), κ>0 and very small"

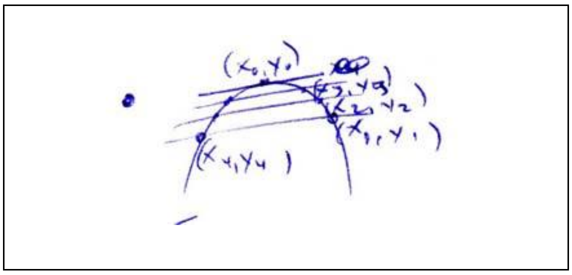


Figure 3. S[149]'s response to Q2

In Q3.7, the student rejects the line because: *"C_f* has two tangent semi-lines at *A* which, however, do not have the same slope" while he accepts the line in Q3.14 because "it consists of two semi-lines. The one is tangent to C_f at *A* for $x \leq x_A$ and the other to C_f at *A* for $x \geq x_A$. They have the same direction, so they form a tangent line".

In these responses, the student still uses words from geometry ('common points') and algebra ('direction coefficient'), but the meaning of these words appears different from the meanings that I discerned in the Geometry and Algebra textbooks (Table 1). That a line is a tangent is substantiated through recourse to the 'convergence of secant lines' narrative, as the line with 'slope the limit of the difference quotient' (Table 1). Further, words hitherto used in algebra and geometry are now used in connection to derivative, convergence and limits. Although the endorsed narratives evidenced in the student's responses are not an exact match to those that I found in the textbooks, I see evidence in these responses where the mathematical analysis discourse (convergence) has *subsumed* the isomorphic parts of geometry, algebra and calculus discourses.

6. Discursive footprint as a tool to analyse students' learning about tangents across mathematical domains

The study presented in this paper focuses on learning about tangents across mathematical domains. I see learning as a longitudinal process where previous experiences (precedent events, Lavie et al., 2019) leave their mark on how students' discourse about tangents takes shape. In earlier work (Biza et al., 2008; Biza & Zachariades, 2010), colleagues and I proposed three clusters of upper secondary and university student perspectives on tangency (*analytical local*, *geometrical global* and *intermediate*). However, these clusters did not showcase the variation of ways in which students engage with tangents and did not demonstrate how students' previous experiences leave their marks on such engagement. Motivated by this observation, in this paper, I revisit the responses of 182 undergraduate mathematics students to a questionnaire about tangents to investigate and explain the variation in students' work with tangents.

To this purpose, I draw on the commognitive frame (Sfard, 2008) and I propose the notion of the discursive footprint of tangents. This footprint captures students' curricular experiences of tangents and comprises word use around tangents, narratives of what a tangent line is, visual mediators related to tangents and routines employed to determine and substantiate whether a line is tangent to a certain curve. I reviewed how tangents are presented in the textbooks that participants had been taught with in secondary school and I outlined the discursive footprint and its characteristics in Table 1. Then, I looked for manifestations of the discursive footprint in student responses, firstly, by labelling responses according to the discursive characteristics of the footprint (5.1) and, secondly, by looking for patterns in the endorsed narratives about tangents in the same script and in the same or similar questionnaire tasks across scripts (5.2).

The discursive footprint was manifested in student responses either with characteristics of sole discourses (geometry, algebra, calculus, mathematical analysis) as well as with a combination of characteristics of different discourses (e.g., geometry and algebra). I summarised the manifestations of the discursive footprint in student responses in five themes. The first theme – *apparent replication of word use in different narratives* – highlights that the same word might be used with different meanings and stresses the importance of examining the narratives in which this word is used. The second theme – *geometry-local hybrid discourse* – highlights that new discourses may emerge, in which the students' discursive activity is no longer what it used to be, but is not yet what it is conventionally expected to be. The third theme – *endorsement of conflicting narratives* – highlights situations where contradictory narratives from different discourses may appear together in the same response. The fourth theme – *enrichment of familiar narratives with new words* – highlights the persistence of familiar narratives about what a tangent line is with the addition of new words. Finally, the fifth theme – *mathematical analysis as a subsuming discourse* – highlights the connection across discourses, namely when isomorphic parts of geometry, algebra and calculus are subsumed by mathematical analysis. Subsuming discourses is what I would expect from students when they see tangents in different mathematical domains. However, in the data I analysed, this took place explicitly only in a small number of responses.

Previous research has spotted cases of tangency that confuse students (Biza et al., 2008; Castela, 1995; Vinner, 1991). Those cases have been connected with the variation of ways in which tangents are defined across mathematical domains (Winicki & Leikin, 2000; Vinner, 1991), especially in relation to the shift from a global (geometry) to a local perspective (calculus or mathematical analysis) (Biza et al., 2008; Maschietto, 2008). Such studies, including my earlier work, have argued that students' previous experiences conflict with subsequent learning about tangents. With the commognitive analysis proposed in this paper, attention is drawn on students' learning about tangents as a discursive activity. Thus, even if students do not provide a mathematically accurate response, the analysis provides evidence that some learning did take place. Such learning can be seen through the lens of students' previous experiences – or PSS as per Lavie et al. (2019). With the introduction of the notion of discursive footprint of tangents and its use in the analysis of student responses, I elaborate further how the PSS, specifically the curriculum-generated PSS, can be taken into consideration more systematically. Firstly, I propose a way to conceptualise precedents in the form of the discursive characteristics of the footprint by identifying these characteristics in school textbooks. Secondly, I demonstrate how the mapping of the discursive footprint can be applied in the analysis of student responses. I conjecture that there is potency of this type of analysis not only for the tangent line, but, also for other mathematical topics that are present across mathematical domains. Finally, I discern the pedagogical potential of the discursive footprint notion for teaching, as a descriptor of curriculum-generated PSS as well as a predictor and evaluation tool of students' work on task situations.

Acknowledgements

I would like to thank the Research in Mathematics Education group at the University of East Anglia, for their comments on earlier versions of this paper, and especially, Elena Nardi, Athina Thoma, Evi Papadaki and Bruna Moustapha-Corrêa for the inspiring conversations about the potentialities of commognitive analysis.

References

Adamopoulos, L., Viskadourakis, B., Gavalas, D., Polyzos, G., & Sverkos, A. (1998). *Μαθηματικά Θετικής και Τεχνολογικής Κατεύθυνσης, Β΄ Τάξη Ενιαίου Λυκείου* [Mathematics for specialised pathways, Lyseum B΄]. Athens, Greece: OEDB.

Alibinisis, A., Antypas, Z., Efstathopoulos, E., Klaoudatos, N., & Papapstavridis, S. (1994). *Μαθηματικά Α΄ Γυμνασίου* [Mathematics, Gymnasium A']. Athens, Greece: OEDB.

Andreadakis, S., Katsargyris, V., Metis, S., Brouchoutas, K., Papastavrisdis, S., & Polyzos, G. (1998). *Μαθηματικά, Γ΄ τάξη Ενιαίου Λυκείου, Θετική και Τεχνολογική Κατεύθυνση*. [Mathematics for specialised pathways, Lyseum C΄]. Athens, Greece: OEDB.

Andreadakis, S., Katsargyris, V., Papastavridis, S., Polyzos, G., & Sverkos, A. (1998). *Άλγεβρα, Α΄ τάξη Ενιαίου Λυκείου* [Algebra Lyseum A΄]. Athens, Greece: OEDB.

Argyropoulos, H., Vlamos, P., Katsoulis, G., Markatis, S., & Sideris, P. (2003). *Ευκλείδεια Γεωμετρία, Α΄ και Β΄ Λυκείου* [Euclidean Geometry Lyceum A΄ and B΄]. OEDB. Athens, Greece.

Artigue, M. (1990). Épistémologie et Didactique. *Recherches en Didactiques des Mathématiques, 10*(2-3), 241-286.

Bingolbali, E., & Monaghan, J. (2008). Concept image revisited. *Educational Studies in Mathematics, 68*, 19-35.

Biza, I. (2019). Calculus as a discursive bridge for Algebra, Geometry and Analysis: The case of tangent line. In J. Monaghan, E. Nardi and T. Dreyfus (Eds.), *Calculus in upper secondary and beginning university mathematics – Conference proceedings.* Kristiansand, Norway: MatRIC. Retrieved on 20/12/2019 from https://matric-calculus.sciencesconf.org/.

Biza, I., Christou, C., & Zachariades, T. (2008). Student perspectives on the relationship between a curve and its tangent in the transition from Euclidean Geometry to Analysis. *Research in Mathematics Education, 10*(1), 53-70.

Biza, I., & Zachariades, T. (2010). First year mathematics undergraduates' settled images of tangent line. *The Journal of Mathematical Behavior, 29*(4), 218-229.

Castela, C. (1995). Apprendre avec et contre ses connaissances antérieures: Un example concret, celui de la tangente. *Recherches en Didactiques des Mathématiques, 15*(1), 7-47.

Kontorovich, I. (2018). Why Johnny struggles when familiar concepts are taken to a new mathematical domain: Towards a polysemous approach. *Educational Studies in Mathematics*, *97*(1), 5-20.

Kontorovich, I. (2019). What can be more challenging than square-rooting from squared things? In G. Hine, S. Blackley, & A. Cooke (Eds.), *Proceedings of the 42nd annual conference of the Mathematics Education Research Group of Australasia)* pp. 412-419. Perth: MERGA.

Lavie, I., Steiner, A., & Sfard, A. (2019). Routines we live by: From ritual to exploration. *Educational Studies in Mathematics*, *101*(2), 153-176.

Maschietto, M. (2008). Graphic calculators and micro-straightness: Analysis of a didactic engineering. *International Journal of Computers for Mathematical Learning, 13*(3), 207–230.

Nardi, E., Ryve, A., Stadler, E., & Viirman, O. (2014). Commognitive analyses of the learning and teaching of mathematics at university level: the case of discursive shifts in the study of Calculus. *Research in Mathematics Education 16*(2), 182-198.

Pinto, M., & Moreira, V. (2007). School practices with the mathematical notion of tangent line. In A. Watson & P. Winbourne (Eds.), *New directions for situated cognition in mathematics education* (pp. 261-285). USA: Springer.

Sierpinska, A. (1985). Obstacles épistémologiques relatifs à la notion de limite. *Recherches en Didactiques des Mathématiques, 6*(1), 5-67.

Sfard, A. (2008). *Thinking as communicating. Human development, the growth of discourse, and mathematizing.* New York, NY: Cambridge University Press.

Thoma, A., & Nardi, E. (2018). Transition from school to university mathematics: Manifestations of unresolved commognitive conflict in first year students' examination scripts. *International Journal of Research in Undergraduate Mathematics Education, 4*(1), 161-180.

Vinner, S. (1991). The role of definitions in the teaching and learning of Mathematics. In D. O. Tall (Ed.), *Advanced mathematical thinking* (pp. 65-81). Dordrecht, The Netherlands: Kluwer.

Winicki-Landman, G., & Leikin, R. (2000). On equivalent and non-equivalent definition: Part 1. *For the learning of Mathematics, 20*(1), 17-21.