Additive Cellular Automata and Volume Growth

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Abstract: A class of dynamical systems associated to rings of S–integers in rational function fields is described. General results about these systems give a rather complete description of the well–known dynamics in one–dimensional additive cellular automata with prime alphabet, including simple formulæ for the topological entropy and the number of periodic configurations. For these systems the periodic points are uniformly distributed along some subsequence with respect to the maximal measure, and in particular are dense. Periodic points may be constructed arbitrarily close to a given configuration, and rationality of the dynamical zeta function is characterized. Throughout the emphasis is to place this particular family of cellular automata into the wider context of S–integer dynamical systems, and to show how the arithmetic of rational function fields determines their behaviour. Using a covering space the dynamics of additive cellular automata are related to a form of hyperbolicity in completions of rational function fields. This expresses the topological entropy of the automata directly in terms of volume growth in the covering space.
Keywords: Cellular automata, Entropy, Rational function field, Adele ring, Hyperbolic dynamics

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1 Introduction

Cellular automata are a particular class of dynamical system studied by von Neumann [24] as a primitive model for self–reproduction. Since then they have been widely studied in a variety of contexts in physics, biology and computer science. A detailed discussion with extensive references may be found in Wolfram's paper [31]. The state space of a cellular automaton is particularly simple: it consists in one dimension of a one–dimensional array with values taken from a fixed finite alphabet, and their evolution in time is determined by a finite or local rule. Nonetheless, the global dynamical behaviour of time evolutes of a cellular automata may exhibit extremely intricate behaviour and – in complete generality – understanding global dynamical invariants may be genuinely intractable ([7, 14]).

In this paper two restrictions are placed on the cellular automata: first, that the alphabet have cardinality a prime (though the methods apply equally well to prime–power alphabets once 'additivity' is interpreted in a way that reflects a finite field structure on the alphabet). Second, that the local rule determining the time evolution be 'additive'. This latter restriction is very strong, and forces the cellular automata to be an endomorphism of a compact abelian group. The measurable structure of these systems has been completely determined, [19].

Recently, an arithmetically natural class of algebraic dynamical systems, the so–called S–integer systems, has been studied ([5, 27, 28]). These systems arise as extensions of simple algebraic dynamical systems, and they have two features of particular interest. Firstly, their structure may be studied using tools from number theory (in particular, the use of an adelic covering space to relate the entropy of the complicated dynamics of the automata to the simple volume-growth dynamics of the automata lifted to the covering space). Secondly, the collection of all such systems extending a given initial system is parametrized in a natural way by a probability space, giving some meaning to the idea of 'typical' behaviour for algebraic dynamical systems. Special cases of S–integer systems include the additive cellular automata on prime–power alphabets, and results from [5] apply to give alternate proofs of the results of Favati *et al.* and Margara [10]. The arithmetic structure at work also gives additional information: for example, the periodic points are not only dense but uniformly distributed with respect to the maximal measure along time sequences where the number of periodic points grows. The algebraic structure of finite characteristic fields gives a method for constructing periodic points arbitrarily close to any given point.

Many of the results presented here are well–known; in particular Corollary 1 and 3, and Theorems 4 and 5 may be found for example in the work of

Margara *et al.*, [4], [8], [10], and [22].

The paper is organized as follows. In Section 2 standard notation is fixed and the elementary properties of linear cellular automata are recalled. Section 3 introduces the S-integer dynamical systems and shows how they contain some simple additive cellular automata. In Section 4 the main dynamical properties of these automata are studied using methods from S-integers: topological entropy, numbers and distribution of periodic points. In Section 5 a quantitative denseness of periodic points of periodic points is exhibited, and Section 6 contains a summary and some remarks. Finally, Section 7 gives a short review of the number theory used in the paper.

2 Notation for cellular automata

Let A be a finite set or *alphabet*, and let Σ_A denote the two–sided sequence space

$$\Sigma_A = A^{\mathbb{Z}} = \{x = (x_i)_{i\in\mathbb{Z}} \mid x_i \in A \ \forall\ i \in \mathbb{Z}\}.$$

The set Σ_A will also be written $\Sigma_{|A|}$ since only the cardinality of the alphabet matters. The metric on $\Sigma_{|A|}$ defined by

$$\rho(x, y) = \sum_{i=-\infty}^{\infty} 2^{-|i|} d(x_i, y_i), \tag{1}$$

where d is any metric on the finite set A, makes $\Sigma_{|A|}$ into a compact metric space. The left shift $\sigma : \Sigma_{|A|} \longrightarrow \Sigma_{|A|}$ defined by

$$(\sigma(x))_i = x_{i+1} \tag{2}$$

is a homeomorphism of this compact metric space.

A *cellular automaton* is a continuous map $\alpha : \Sigma_{|A|} \longrightarrow \Sigma_{|A|}$ that commutes with σ. The evolution of a configuration $x \in \Sigma_{|A|}$ under α is called *temporal*, and under σ *spatial*. An easy consequence of the compactness of $\Sigma_{|A|}$ is that any such map α must be given by a *local rule*: there is a neighbourhood size k and a map

$$f : A^{2k+1} \longrightarrow A$$

with the property that

$$(\alpha(x))_i = f(x_{i-k}, \ldots, x_i, \ldots, x_{i+k}),$$

(this is an observation due to Curtis, Lyndon and Hedlund, [12]).

A similar definition may be made for automata on one–sided shift spaces: define $\Sigma^+_{|A|}$ to be the one–sided shift space $A^{\mathbb{N}}$, sum from 0 to ∞ only in

(1), and define σ^+ to be the continuous $|A|$–to–one map defined by (2) for $i \geq 0$ only. Any continuous σ^+–commuting map $\alpha : \Sigma_{|A|}^+ \to \Sigma_{|A|}^+$ is given by a one–sided local rule of the form

$$f : A^{k+1} \to A$$

with the property that

$$(\alpha(x))_i = f(x_i, \dots, x_{i+k})$$

for all $i \geq 0$.

If the alphabet is written $A = \{0, 1, \dots, a-1\}$ for some a, identified with the integers mod a under addition, then a cellular automaton is called *additive* if it is an endomorphism of the group structure on Σ_a or Σ_a^+. It is clear that this holds if and only if the local rule is of the form

$$f(x_{-k}, \dots, x_0, \dots, x_k) = a_{-k}x_{-k} + \dots + a_0x_0 + \dots a_kx_k \bmod \mathrm{a} \tag{3}$$

for some coefficients $a_{-k}, \dots, a_k \in A$: if $\alpha : \Sigma_a \to \Sigma_a$ is an additive cellular automaton, then $\hat{\alpha}$ is a homomorphism of the dual group $\widehat{\Sigma_a} = (\mathbb{Z}/a\mathbb{Z})[u^{\pm 1}]$ that commutes with multiplication by u (the dual of the spatial shift map). It follows that $\widehat{\alpha}$ – and hence α – is determined by the polynomial

$$\widehat{\alpha}(1) = a_{-k}u^{-k} + \dots + a_0x_0 + \dots a_ku^k \bmod \mathrm{a}$$

from which (3) follows. For a one–sided state space, all the a_i with $i < 0$ are required to be zero. The Tychonoff topology on the compact group coincides with the topology defined by the metric (1).

Surjective cellular automata preserve the Haar measure on the compact group Σ_n or Σ_n^+, and this measure coincides with the independent identically distributed $(\frac{1}{n}, \dots, \frac{1}{n})$ measure. With the exception of the proof of Theorem 4 we shall not be interested in measure–theoretic aspects of cellular automata – [20] has some precise results on statistical phenomena in the evolution of cellular automata.

3 S–integer dynamical systems

In this section we introduce a family of dynamical systems defined using the arithmetic of rational function fields: the examples below show how they relate to additive cellular automata. In order to make this paper self–contained, we include proofs in simple cases: in particular, we give proofs only for the case of finite sets S.

Let k denote an $\mathbb{A}$–field of positive characteristic: that is, a rational function field of the form $\mathbb{F}_p(t)$ where $\mathbb{F}_p$ is a field with p elements, or a finite algebraic extension of such a field. Associated to k is a set of places $P(k)$: each element of $P(k)$ is an equivalence class of valuations. We abuse notation slightly by identifying a *prime element* for each place with a corresponding valuation (see Chapter III, §1 of [30] for the precise formulation).

Example 1 The simplest case is the field $k = \mathbb{F}_p(t)$ itself. For each monic irreducible polynomial $\nu \in \mathbb{F}_p[t]$ there is a distinct place $\nu \in P(k)$ with corresponding valuation given by

$$|f|_\nu = p^{-\mathrm{ord}_\nu(f)\cdot\deg(\nu)},$$

where $\mathrm{ord}_\nu(f)$ is the signed multiplicity with which ν divides the rational function f. There is in addition one exceptional place given by $\nu(t) = t^{-1}$, with corresponding valuation defined by

$$|f(t)|_\nu = |f(t)|_{t^{-1}} = |f(t^{-1})|_t.$$

It is conventional to regard this exceptional place as the 'infinite' one, and to write $P_\infty(k) = \{t^{-1}\}$.

The next examples show how the valuations work in practice. The first is a polynomial and the second is a rational function.

Example 2 [1] Let $p = 7$ and consider the polynomial

$$f(t) = t^6 + 2t^5 + 3t^4 + 5t^3 + 6t^2 + t + 4.$$

This may be factorized using standard methods (from Chapter 4 of [18], for example) into

$$f(t) = (t + 3)(t^2 + t + 3)(t^3 + 5t^2 + 5t + 2).$$

Each of the three factors is irreducible over $\mathbb{F}_7$ (see Table C in the Appendix of [18]). This allows us to calculate all the valuations of f. The three finite valuations corresponding to irreducible polynomials that divide f,

$$|f|_{t+3} = 7^{-(1)(1)} = \tfrac{1}{7}; \quad |f|_{t^2+t+3} = 7^{-(1)(2)} = \tfrac{1}{49}; \quad |f|_{t^3+5t^2+5t+2} = 7^{-(1)(3)} = \tfrac{1}{343}.$$

Then the infinite valuation

$$|f(t)|_{t^{-1}} = |f(t^{-1})|_t = \left|\frac{1 + 2t + 3t^2 + 5t^3 + 6t^4 + t^5 + 4t^6}{t^6}\right|_t = 7^{-(-6)(1)} = 117649.$$

Finally, for $\nu(t)$ any irreducible polynomial other than those appearing as factors of f,

$$|f|_\nu = 7^{-(0)(\deg(\nu))} = 1.$$

[2] As an illustration of how valuations work for rational functions, let $p = 2$ and consider the rational function $f(t) = \frac{1+t^2}{t}$. Then

$$\left|\frac{1+t^2}{t}\right|_t = 2^{-(1)(-1)} = 2, \quad \left|\frac{1+t^2}{t}\right|_{1+t} = 2^{-(1)(2)} = \frac{1}{4},$$

(since, over $\mathbb{F}_2$, $1+t^2 = (1+t)^2$), and

$$\left|\frac{1+t^2}{t}\right|_{t^{-1}} = \left|\frac{1+t^{-2}}{t^{-1}}\right|_t = \left|\frac{t^2+1}{t}\right|_t = 2^{-(1)(-1)} = 2.$$

For all $\nu \notin \{t, 1+t, t^{-1}\}$ we have $|f|_\nu = 1$ since ν does not divide f.

For the general case – in which k is a finite extension field of $\mathbb{F}_p(t)$ for some prime p, there are finitely many valuations of k with the property that they restrict to a given $\nu \in P(\mathbb{F}_p(t))$ for each ν: the details are in [30].

Definition 1 Let $k = \mathbb{F}_p(t)$. Given an element $\xi \in k\backslash\{0\}$, and any set $S \subset P(k)\backslash P_\infty$ with the property that $|\xi|_w \leq 1$ for all $w \notin S \cup P_\infty$, define a dynamical system $(X, \alpha) = (X^S, \alpha^{(S,\xi)})$ as follows. The compact abelian group X is the dual group to the discrete countable group of S–integers R_S in k, defined by

$$R_S = \{x \in k \colon |x|_w \leq 1 \text{ for all } w \notin S \cup P_\infty\}.$$

The continuous group endomorphism $\alpha : X \to X$ is dual to the monomorphism $\widehat{\alpha} : R_S \to R_S$ defined by $\widehat{\alpha}(x) = \xi x$.

To explain this definition and to show how it relates to cellular automata, consider the following examples.

Example 3 [1] Let $k = \mathbb{F}_p(t)$, $S = \emptyset$, and $\xi = t$. Then $R_S = \mathbb{F}_p[t]$, and so $X = \widehat{R_S} = \prod_{i=0}^{\infty}\{0, 1, \ldots, p-1\} = \Sigma_p^+$. The map α is therefore the full one–sided shift on p symbols. Equivalently, the map α is the cellular automaton with one–sided state space and with local rule $f(x_0, x_1) = x_1$.

[2] Let $k = \mathbb{F}_p(t)$, $S = \{t\}$, and $\xi = t$. Recall that the valuation corresponding to t is $|g|_t = p^{-\mathrm{ord}_t(g)}$, so $|t|_t = p^{-1}$. The ring of S–integers is

$$R_S = \{g \in \mathbb{F}_p(t) \colon |g|_w \leq 1 \text{ for all } w \neq t, t^{-1}\} = \mathbb{F}_p[t^{\pm 1}].$$

The dual of R_S is then $\prod_{-\infty}^{\infty}\{0,1,\dots,p-1\} = \Sigma_p$, and in this case α is the full two–sided shift on p symbols. Equivalently, the map α is the cellular automaton with local rule $f(x_{-1}, x_0, x_1) = x_1$.

[3] Let $k = \mathbb{F}_p(t)$, $S = \{t\}$, and $\xi = 1 + t$. Then $X = \Sigma_p$ is the two–sided shift space on p symbols, and α is the cellular automaton with local rule $f(x_{-1}, x_0, x_1) = x_0 + x_1$.

[4] Let $k = \mathbb{F}_p(t)$, $S = \{t, 1 + t\}$, and $\xi = 1 + t$. Then α is the invertible extension of the cellular automaton in [3]. The $\mathbb{Z}^2$ dynamics under both the temporal and spatial maps for this example is a version of Ledrappier's example [17].

[5] Fix the characteristic to be $p = 2$ and $S = \{t\}$. Then $X_S = \Sigma_2$, the full 2-shift. Following Favati *et al.* in [10], additive local rules for cellular automata with $k = 1$ have a natural parametrization: associate the local rule

$$f(x_{-1}, x_0, x_1) = ax_{-1} + bx_0 + cx_1$$

to the natural number

$$n_f = f(0,0,0)\cdot 2^0 + f(0,0,1)\cdot 2^1 + \cdots + f(1,1,0)\cdot 2^6 + f(1,1,1)\cdot 2^7.$$

By suitable choice of the Laurent polynomial ξ in Definition 1 we produce the following examples.

Polynomial f	Rule number n_f
0	0
1	204
t	170
t^{-1}	240
$t^{-1}+1$	60
$1+t$	102
$t^{-1}+t$	90
$t^{-1}+1+t$	150

Other examples - in which the set S includes some finite valuations - give certain isometric extensions of additive cellular automata (see [5, 28] for the details).

4 Dynamical properties

Let α now be any uniformly continuous map of a metric space (X, ρ). A set $E \subset X$ is said to be (n, ϵ)*–separated under* α if for every pair $x \neq y$ in E

there is an $m \in \{0, 1, \ldots, n-1\}$ with the property that $\rho(\alpha^m(x), \alpha^m(y)) > \epsilon$. For each compact set $K \subset X$, let

$$s_K(n, \epsilon) = \max\{|E| : E \subset K \text{ is } (n, \epsilon) - \text{separated under } \alpha\},$$

$$h_K(\alpha, \epsilon) = \limsup_{n \to \infty} \frac{1}{n} \log s_K(n, \epsilon), \text{ and}$$

$$h_K(\alpha) = \lim_{\epsilon \searrow 0} h_K(\alpha, \epsilon),$$

(the expression under the limit means the limit is taken as ϵ decreases to zero). Finally, define the *topological entropy* of α to be

$$h(\alpha) = \sup_K h_K(\alpha). \tag{4}$$

Notice that if X is compact, then $h_X(\alpha) = h(\alpha)$.

The topological entropy of a map is a crude global measure of the exponential complexity of the structure of the orbits of the map.

Theorem 1 *The topological entropy of the S–integer system $(X^S, \alpha^{(S,\xi)})$ is given by*

$$h(\alpha^{(S,\xi)}) = \sum_{w \in S \cup P_\infty} \log^+ |\xi|_w \tag{5}$$

The proof of this result motivates the viewpoint adopted here. Roughly speaking, the number theory (the adele ring) provides a covering space for the cellular automata, and the complicated dynamics of the automata lifts to a 'linearised' dynamics on the covering space. General results about covering spaces show that the topological entropy of the automata coincides with the rate of volume growth of the lifted map – expressed in equation (6) below.
Proof. This is shown in [5], Theorem 4.1 using the adelic method of [21]. A very simple proof is outlined here for S finite. This is easier than the general case because there are no Archimedean places to deal with, the arithmetic 'dimension' is one, and the topology on the covering space is simply the product topology.

According to the Appendix, the group R_S embeds as a discrete subgroup of $\prod_{\nu \in S \cup P_\infty} k_\nu$ with compact quotient, and there is a map $p : k_S \to k_s/\Delta(R_S)$; Theorem 6 means that there is a commutative diagram expressing the adelic covering space k_S as follows:

$$\begin{tikzcd}
k_S \arrow[r, "\tilde{\alpha}"] \arrow[d, "p"'] & k_S \arrow[d, "p"] \\
k_S/\Delta(R_S) \cong X^S \arrow[r, "\alpha^{(S,\xi)}"] & X^S \cong k_S/\Delta(R_S)
\end{tikzcd}$$

Figure 1: The adelic covering space

in which the map p is a local isometry and $\tilde{\alpha}$ denotes multiplication by ξ in each coordinate..

It follows by Theorems 9 and 20 in [3] that

$$h(\alpha) = h(\tilde{\alpha}) = \lim_{\epsilon \searrow 0} \limsup_{n \to \infty} -\frac{1}{n} \log \mu \left(\bigcap_{j=0}^{n-1} \tilde{\alpha}^{-j}(B_\epsilon) \right) \tag{6}$$

where B_ϵ is the metric open ball of radius ϵ around the identity, μ is Haar measure on the locally compact group $\prod_{\nu\in S\cup P_\infty} k_\nu$, and $\tilde{\alpha}$ is the lifted map $(x_\nu)_{\nu\in S\cup P_\infty} \mapsto (\xi x_\nu)_{\nu\in S\cup P_\infty}$ on the covering space $\prod_{\nu\in S\cup P_\infty} k_\nu$.

Since S is finite, we may use the max metric on $\prod_{\nu\in S\cup P_\infty} k_\nu$. It follows that

$$B_\epsilon = \{(x_\nu) : |x|_\nu < \epsilon \;\forall\; \nu \in S \cup P_\infty\}.$$

Now the covering map from $\prod_{\nu\in S\cup P_\infty} k_\nu$ onto X^S gives a local portrait of the hyperbolicity.

For example, if $S \cup P_\infty = \{\nu_1, \nu_2, \nu_3\}$ say, and $|\xi|_{\nu_1} > 1$, $|\xi|_{\nu_2} > 1$, $|\xi|_{\nu_3} < 1$ then the local dynamics in a neighbourhood of the identity in X^S is illustrated in Figure 2. The box B_ϵ is transformed under $\tilde{\alpha}^{-1}$ (multiplication by ξ^{-1}) into a squashed box with sides of length $2\epsilon|\xi|_{\nu_1}^{-1}$, $2\epsilon|\xi|_{\nu_2}^{-1}$, $2\epsilon|\xi|_{\nu_3}^{-1}$ in the directions corresponding to ν_1, ν_2, ν_3 respectively.

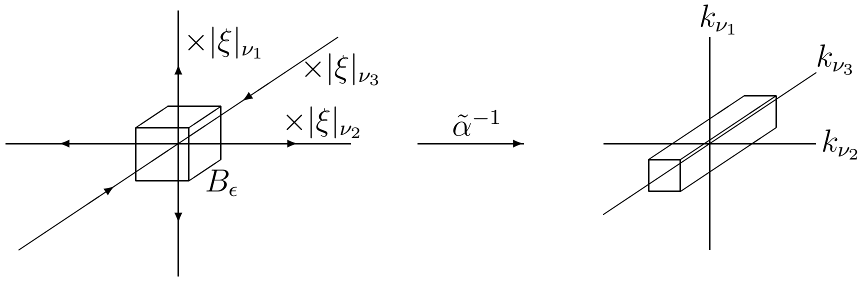

Figure 2: Multiplying B_ϵ by ξ^{-1} for $S \cup P_\infty = \{\nu_1, \nu_2, \nu_3\}$

In the covering space the effect of multiplying the box B_ϵ by ξ^{-1} gives

$$\tilde{\alpha}^{-j}(B_\epsilon) = \{(x_\nu) : |\xi^j x|_\nu < \epsilon \;\forall\; \nu \in S\cup P_\infty\} = \{(x_\nu) : |x|_\nu < \epsilon/|\xi|_\nu^j \;\forall\; \nu \in S\cup P_\infty\}.$$

Thus the set

$$D(n,\epsilon) = \bigcap_{j=0}^{n-1} \tilde{\alpha}^{-j}(B_\epsilon)$$

is a 'box' with one side for each term $\nu \in S \cup P_\infty$, and the 'length' of each side is

$$\min\{\epsilon, \epsilon/|\xi|_\nu, \epsilon/|\xi|_\nu^2, \dots, \epsilon/|\xi|_\nu^{n-1}\} = \begin{cases} \epsilon & \text{if } |\xi|_\nu \le 1, \\ \epsilon/|\xi|_\nu^{n-1} & \text{if } |\xi|_\nu > 1. \end{cases} \tag{7}$$

It follows that

$$\mu\left(D(n,\epsilon)\right) = \epsilon^{|S\cup P_\infty|} \cdot \left(\prod_{\nu:|\xi|_\nu>1} |\xi|_\nu^{n-1} \right)^{-1},$$

which when substituted into (6) gives the formula (5). □

The Haar measure μ is *maximal* in the sense that the measure–theoretic entropy of α with respect to μ coincides with the topological entropy $h(\alpha)$ by [2].

Example 4 [1] The simplest application of equation (5) is to give the entropy of the full shift on p symbols: let α be the S–integer dynamical system corresponding to $k = \mathbb{F}_p(t)$, $S = \{t\}$ and $\xi = t$. Then $h(\alpha) = \log p$ arising from the one term $|t|_{t^{-1}} = |\frac{1}{t}|_t = p^{-(1)(1)} = p$ in (5). Here the local hyperbolicity portrait in the covering space is shown in Figure 3, showing that the system is hyperbolic.

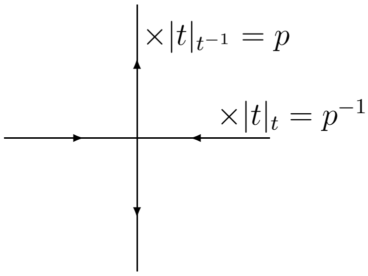

Figure 3: Multiplication by t is hyperbolic for $S = \{t\}$

[2] A less trivial example is the following. Consider the additive cellular automata on $\Sigma_7 = \{0, 1, \dots, 6\}^{\mathbb{Z}}$ defined by the local rule

$$f(x_0, x_1, \dots, x_6) = 4x_0 + x_1 + 6x_2 + 5x_3 + 3x_4 + 2x_5 + x_6.$$

This map is given by the S–integer dynamical system $\alpha = \alpha^{(k,S,\xi)}$ with $k = \mathbb{F}_7(t)$, $S = \{t\}$ and $\xi = t^6 + 2t^5 + 3t^4 + 5t^3 + 6t^2 + t + 4 \in \mathbb{F}_7[t]$. Using the factorization in Example 2[1] and equation (5) we see that $h(\alpha) = 6 \cdot \log 7$.
[3] Consider the 'rule 90' cellular automata in Example 3[5]. This corresponds to the S–integer dynamical system with $k = \mathbb{F}_2(t)$, $S = \{t\}$ and $\xi = t^{-1} + t$. Over $\mathbb{F}_2$ we have

$$t^{-1} + t = \frac{1 + t^2}{t} = \frac{(1 + t)^2}{t}$$

as a factorization into irreducibles. Using Example 2[2] and formula (5) we see that

$$h(\alpha) = \log \left| \frac{(1 + t)^2}{t} \right|_{t^{-1}} + \log \left| \frac{(1 + t)^2}{t} \right|_t = 2 \cdot \log 2.$$

Of course the expressions arising in Example 4 for S–integer systems which have the special structure of additive cellular automata can be simplified.

Corollary 1 *An additive cellular automaton $\alpha : \Sigma_p \to \Sigma_p$ with local rule*

$$f(x_{-\ell}, \ldots, x_0, \ldots, x_r) = a_{-\ell}x_{-\ell} + \cdots + a_0x_0 + \cdots + a_rx_r,$$

$(a_{-\ell}, a_r \neq 0)$ has topological entropy

$$h(\alpha) = \begin{cases} r \cdot \log p & \text{if } r \geq -\ell \geq 0, \\ (\ell + r) \cdot \log p & \text{if } \ell, r \geq 0, \\ \ell \cdot \log p & \text{if } -\ell \leq r \leq 0. \end{cases} \tag{8}$$

Proof. The cellular automaton is given by the S–integer dynamical system with $k = \mathbb{F}_p(t)$, $S = \{t\}$ and $\xi = a_{-\ell}t^{-\ell} + \cdots + a_rt^r$. Simply evaluate (5) for the valuations t and t^{-1}. Assume first that $\ell, r \geq 0$. Then

$$|\xi|_t = \left| \frac{a_\ell + \cdots + a_rt^{r+\ell}}{t^\ell} \right|_t = p^{-(-\ell)(1)} = p^\ell;$$

$$|\xi|_{t^{-1}} = |\xi(t^{-1})|_t = \left| \frac{a_\ell t^{\ell+r} + \cdots + a_r}{t^r} \right|_t = p^{-(-r)(1)} = p^r.$$

Summing gives $h(\alpha) = r \cdot \log p + \ell \cdot \log p$.

For $r \geq -\ell \geq 0$, ξ is a polynomial in t so $|\xi|_t \leq 1$ and $|\xi|_{t^{-1}} = p^r$. The case $-\ell \leq r \leq 0$ is similar. □

Corollary 1 is a simple instance of a more general principle concerning directional entropies in zero–dimensional algebraic dynamical systems of dimension at least 2: in that setting the directional entropies are determined by 'widths' of the support of certain polynomials by [15, 16]. It also gives the entropy of mixed dynamics (involving spatial and temporal motion): the map $\sigma^n\alpha^m$ is given by the S–integer dynamical system with $S = \{t\}$ and $\xi = t^n \cdot \left(a_{-\ell}t^{-\ell} + \cdots + a_rt^r\right)$ so the corresponding entropy is given by a similar formula.

Recall that a map α preserving a probability measure μ is *ergodic* if any measurable set A with $\mu(A\Delta\alpha^{-1}(A)) = 0$ has $\mu(A) = 0$ or 1. Before turning to periodic points, notice that a simple application of the Halmos criterion for ergodicity of compact group endomorphisms in [11] shows that an additive cellular automata is ergodic for the preserved Haar measure if and only if the polynomial corresponding to the local rule is non–constant. It follows from the formula below that for ergodic additive cellular automata there are only finitely many periodic points of each period.

Theorem 2 *Let $\alpha : \Sigma_p \to \Sigma_p$ be the S–integer dynamical system with $k = \mathbb{F}_p(t)$, S finite and ξ non–constant. The number of points with period n under α is*

$$\mathrm{Fix}_n(\alpha) = \prod_{\nu \in S \cup P_\infty} |\xi^n - 1|_\nu \, .$$

The upper growth rate exists and coincides with the topological entropy:

$$\limsup_{n\to\infty} \frac{1}{n}\log \mathrm{Fix}_n(\alpha) = h(\alpha).$$

Moreover, a sequence $n_j \to \infty$ with the property that $\frac{1}{n_j}\log \mathrm{Fix}_{n_j}(\alpha) \to h(\alpha)$ can be identified explicitly.

A sequence along which the growth rate of periodic points gives the entropy comes from the last line of the proof.

Proof. Use the covering space construction from the Appendix again. Recall that the group $X_S = \hat{R_S}$ sits as a quotient of the product $Y = \prod_{\nu\in S\cup P_\infty} k_\nu$ by the discrete subgroup R_S. Let F be a fundamental domain for this quotient which has finite Haar measure (see Appendix). Standard harmonic analysis (for example, [13] Volume 1) shows that

$$\mathrm{Fix}_n(\alpha) = |\ker(\alpha^n - 1)| = \mu\left((\tilde{\alpha}^n - 1)F\right) = \mathrm{mod}_Y(\tilde{\alpha}^n - 1) = \prod_{\nu\in S\cup P_\infty} |f(t)^n - 1|\,,$$

where mod_Y is the 'module' (scaling of Haar measure) in the locally compact group Y.

Turning now to the upper growth rate, note that the erratic behaviour of periodic configurations in cellular automata still arises in the additive setting (cf. Example 5[2] below), so there are some difficulties. The proof here comes from [5], included for completeness. Write

$$\frac{1}{n}\log \mathrm{Fix}_n(\alpha) = \frac{1}{n}\sum_{\nu\in S\cup P_\infty:\xi\notin r_\nu^*} \log|\xi^n - 1|_\nu + \frac{1}{n}\sum_{\nu\in S'} \log|\xi^n - 1|_\nu,$$

where S' is the subset of S defined by

$$S' = \{\nu \in S : \xi \in r_\nu^*\},$$

and $r_\nu^* = \{x \in k : |x|_\nu = 1\}$. Split S' into two sets A and B, where

$$A = \{\nu \in S' : |\xi - 1|_\nu = 1\}$$

and

$$B = \{\nu \in S' : |\xi - 1|_\nu < 1\}.$$

For each ν_j $(j = 1, \dots, m) \in A$ we can associate integers $d_1, \dots, d_m \geq 2$ such that $|\xi^n - 1|_{\nu_j} = 1$ if and only if $d_j \not| n$.

Consider next the valuations $\nu_1, \dots, \nu_l \in B$. If $\nu \in B$ we may write $\xi = 1 + \sum_{i=1}^{\infty} a_i\pi^i$, where a_i and π are as above, and $|\xi - 1|_\nu = p^{-s}$ where

$s = \frac{1}{e}\min\{i : a_i \neq 0\} > 0$ and $\mathrm{ord}_\nu(\pi) = \frac{1}{e}$. For each $\nu_j \in B$ label such s by s_j, the coefficients a_i by $a_i(j)$ and π by π_j. Then

$$\begin{aligned}\frac{1}{n}\sum_{\nu\in B}\log|\xi^n-1|_\nu &= \frac{1}{n}\sum_{\nu\in B}\log|\xi-1|_\nu + \frac{1}{n}\sum_{\nu\in B}\log|\xi^{n-1}+\cdots+\xi+1|_\nu \\ &= \frac{1}{n}\sum_{j=1}^{l}\log|\pi_j|_{\nu_j}^{s_j} + \frac{1}{n}\sum_{j=1}^{l}\log\left|n+\sum_{i=1}^{\infty}b_i(j){\pi_j}^i\right|_{\nu_j},\end{aligned}$$

for computable coefficients $b_i(j) \in r_{\nu_j}$ and $j = 1,\dots,l$. This expression tends to zero if $p\not| n$. Hence

$$\frac{1}{n}\sum_{\nu\in S'}\log|\xi^n-1|_\nu \to 0 \text{ as } n\to\infty$$

through the set $\{n \geq 1 : p \not| n, d_j \not| n \text{ for } j = 1,\dots,m\}$. It follows that $p^+(\alpha) = h(\alpha)$. □

Corollary 2 *Let $\alpha : \Sigma_p \to \Sigma_p$ be an ergodic additive cellular automata corresponding to the S–integer dynamical system with $k = \mathbb{F}_p(t)$, $S = \{t\}$ and $\xi \in \mathbb{F}_p[t^{\pm1}]$. Then there are*

$$\mathrm{Fix}_n(\alpha) = |\xi^n-1|_t \cdot |\xi^n-1|_{t^{-1}}$$

points of period n. If $q_1, q_2, \dots$ is an enumeration of the primes, then

$$\lim_{m\to\infty}\frac{1}{q_m}\mathrm{Fix}_{q_m}(\alpha) = h(\alpha) > 0.$$

A consequence of Theorem 2 is that the periodic points are dense – indeed, along any sequence with the number of periodic points going to infinity they are uniformly distributed with respect to Haar measure.

Lemma 1 *Let α be an ergodic S–integer dynamical system as in Theorem 2. If $n_j \to \infty$ is any sequence of times for which $\mathrm{Fix}_{n_j}(\alpha) \to \infty$, then the uniform periodic point measures at times n_j converge weakly to Haar measure as $j \to \infty$.*

That is, under the hypotheses of Lemma 1, for any continuous complex–valued function ϕ on Σ_p,

$$\frac{1}{\mathrm{Fix}_{n_j}(\alpha)}\sum_{x:\alpha^n(x)=x}\phi(x) \longrightarrow \int_{\Sigma_p}\phi\, d\mu. \tag{9}$$

Proof. Let the corresponding S–integer dynamical system be given by $k = \mathbb{F}_p(t)$, S and ξ as usual. Since finite combinations of characters are dense in the space of continuous functions on the compact group X_S, if (9) fails to be true it must fail with $\phi = r$ for some non–trivial character $r \in R_S\backslash\{0\}$. This requires there to be a subsequence $n_{j(m)} \to \infty$ for which $r \in (\xi^{n_{j(m)}} - 1)\, R_S$ for all m. By the formula used in the proof of Theorem 2 this requires

$$\infty > \prod_{\nu \in S \cup P_\infty} |r|_\nu = \left| \frac{R_S}{r \cdot R_S} \right| \geq \left| \frac{R_S}{(\xi^{n_{j(m)}} - 1) \cdot R_S} \right| = \mathrm{Fix}_{n_{j(m)}}(\alpha) \to \infty,$$

which is impossible. □

Corollary 3 *If $\alpha : \Sigma_p \to \Sigma_p$ is an ergodic additive cellular automata, then the set of periodic points is dense, and there are sequences of times along which the periodic points are uniformly distributed with respect to the preserved Haar measure.*

The precise behaviour of the periodic points in any non–trivial cellular automaton is erratic. In the examples below we give some information for certain cases. Even for additive cellular automata, there may be positive logarithmic growth rates other than the entropy. These examples are part of a wider investigation into periodic point behaviour for S–integer dynamical systems in [5]. One surprising result is that there are examples for which S may be infinite but

$$\limsup_{n \to \infty} \frac{1}{n} \log \mathrm{Fix}_n(\alpha) = h(\alpha)$$

still holds. In fact Corollary 3 in [28] shows that additive linear cellular automata with prime alphabet must have this property for almost every set S in the sense of probability, for all values of the prime p excepting at most two.

A delicate measure of the complexity of the periodic point structure of any continuous map is given by the dynamical zeta function

$$\zeta_\alpha(z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{Fix}_n(\alpha).$$

In particular, if this function is rational, then the number of periodic points of period n grows in a simple recurrent fashion in n.

Example 5 [1] The simplest case is, as usual, the full shift on p symbols. This is given by $k = \mathbb{F}_p(t)$, $S = \{t\}$ and $\xi = t$. Using Theorem 2 we have that

$$\mathrm{Fix}_n(\alpha) = |t^n - 1|_{t^{-1}} |t^n - 1|_t = p^n,$$

so in this case $\frac{1}{n} \log \mathrm{Fix}_n(\alpha)$ converges to $h(\alpha) = \log p$. The dynamical zeta function is rational, given by

$$\zeta_\alpha(z) = \frac{1}{1 - pz}.$$

[2] Let $k = \mathbb{F}_p(t)$, $S = \{t\}$ and $\xi = 1 + t$. This is the additive cellular automata with local rule given by

$$f(x_0, x_1) = x_0 + x_1.$$

If $p = 2$ this is 'rule 102' in the sense of Example 3[5]. The local hyperbolicity portrait is shown in Figure 4, which indicates why this system is non–hyperbolic.

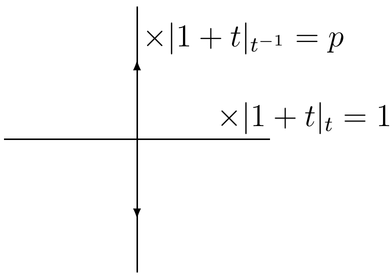

Figure 4: Local effect of multiplication by $1 + t$

Using Theorem 2 we have

$$\mathrm{Fix}_n(\alpha) = |(1+t)^n - 1|_{t^{-1}} |(1+t)^n - 1|_t = p^n \cdot \left| t^n + \binom{n}{1} t^{n-1} + \cdots + \binom{n}{n-1} t \right|_t. \tag{10}$$

It follows that the exact number of points of period n depends on the vanishing properties of binomial coefficients modulo the prime p. The following simple argument (for details, see Section 9 of [5]) gives some insight into how complicated the periodic points really are – and shows that the dynamical zeta function must be irrational. Write $n = q \cdot p^{\mathrm{ord}_p(n)}$ (that is, factor the prime p out of n as many times as possible). Then using (10) we see that $\mathrm{Fix}_n(\alpha) = p^n \cdot p^{-p^{\mathrm{ord}_p(n)}} = p^{n(1-1/q)}$ since by construction q does not divide p.

It follows that for any sequence $n_j \to \infty$ with $n_j p^{-\mathrm{ord}_p(n_j)} = q$ for some fixed q, we have

$$\lim_{n_j \to \infty} \frac{1}{n_j} \log \mathrm{Fix}_{n_j}(\alpha) = \left(1 - \frac{1}{q}\right) \log p.$$

That is, for this example the set

$$\left\{ \frac{1}{n} \log \mathrm{Fix}_n(\alpha) \right\}$$

has infinitely many limit points. The complex behaviour seen here seems to be prevalent for most S–integer dynamical systems – see [29].

The non-hyperbolicity is manifested in the extremely complex dynamics. This is illustrated in Figure 5, where the time evolution of a random initial configuration is shown for $p = 3$ (the elements '0','1', and '2' in $\mathbf{F}_3$ are coded white, grey, black respectively in Figure 5).

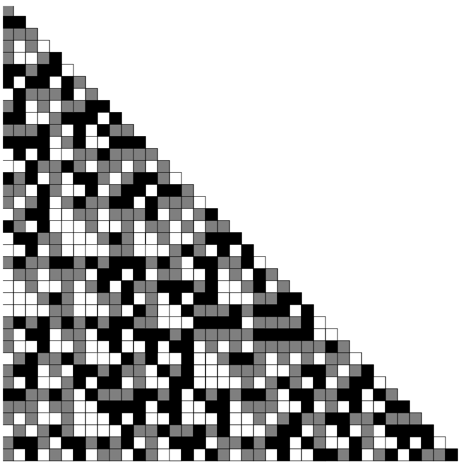

Figure 5: Time evolution in a non-hyperbolic example

The last result in this section is a generalization of Example 5[2] that covers all additive cellular automata on prime alphabet.

Theorem 3 *If α is an additive cellular automaton on Σ_p with p prime, and with local rule corresponding to the polynomial*

$$\xi(t) = a_{-\ell}t^{-\ell} + \cdots + a_r t^r$$

with $a_{-\ell}, a_r \neq 0$, then the dynamical zeta function of α is rational if and only if $\ell = r$ or ℓ and r are both positive.

Proof. If $\ell = r$ then $\xi(t) = a_r t^r$, so Theorem 2 gives

$$\mathrm{Fix}_n(\alpha) = |a_r^n t^{nr} - 1|_t \times |a_r^n t^{nr} - 1|_{t^{-1}} = p^{nr},$$

so

$$\zeta_\alpha(z) = \frac{1}{1 - p^r z}.$$

If both ℓ and r are positive, then

$$|\xi(t)^n - 1|_{t^{-1}} = \left|a_{-\ell}^n t^{n\ell} + \cdots + a_r^n t^{-nr} - 1\right|_t = p^{nr}.$$

On the other hand,

$$|\xi(t)^n - 1|_t = \left|a_{-\ell}^n t^{-n\ell} + \cdots + a_r^n t^{nr} - 1\right|_t = p^{n\ell},$$

so by Theorem 2 there are $p^{n(r+\ell)}$ points of period n and

$$\zeta_\alpha(z) = \frac{1}{1 - p^{r+\ell} z}.$$

For the remaining case, we may write

$$\xi(t) = a_0 + a_\ell t^\ell + \cdots + a_r t^r,$$

with $a_\ell, a_r \neq 0$ (the case in which only negative powers of t are involved is similar). Then

$$|\xi(t)^n - 1|_{t^{-1}} = \left|a_0^n + \cdots + a_r^n t^{-nr}\right|_t = \left|\frac{a_0^n t^{rn} + \cdots + a_r^n}{t^{rn}}\right|_t = p^{rn}.$$

To compute the other part of the periodic point formula, write $n = qp^{\mathrm{ord}_p(n)}$. Then, since q does not divide p,

$$|\xi(t)^n - 1|_t = \left|\left(a_0^q + qa_0^{q-1}a_\ell t^\ell + \cdots + a_r^q t^{rq}\right)^{n/q} - 1\right|_t = \left|(a_0^n - 1) + Dt^{\ell n/q} + \ldots\right|_t$$

where D is not divisible by p. It follows that

$$\mathrm{Fix}_n(\alpha) = \begin{cases} p^{rn} & \text{if } a_0^n \not\equiv 1, \\ p^{n(r-\ell/q)} & \text{if } a_0^n \equiv 1. \end{cases} \tag{11}$$

From this we may exhibit infinitely many limit points for the set $\{\frac{1}{n}\log \mathrm{Fix}_n(\alpha)\}$, showing that ζ_α cannot be rational.

□

5 Constructing periodic points and 'chaotic' behaviour

Finally, we use methods from ergodic theory and the arithmetic viewpoint above to give an alternative proof of the result in [10] that additive ergodic cellular automata on prime alphabets are 'chaotic' in the sense of Devaney. Recall that a continuous map α on a compact metric space (X, d) is *regionally transitive* if for every pair of open sets U, V in X there is an $n \in \mathbb{N}$ with $\alpha^n(U) \cap V \neq \emptyset$, has *dense periodic points* if the set $\bigcup_{n\in\mathbb{N}} \mathrm{Fix}_n(\alpha)$ is dense in X, and has *sensitive dependence on initial conditions* if there is a constant $\delta > 0$ such that for all $x \in X$ and any open set $U \ni x$ there is a $y \in U$ such that $\sup_{n\in\mathbb{N}} d(\alpha^n(x), \alpha^n(y)) > \delta$. Following [9], a map satisfying all three properties is called 'chaotic'.

Theorem 4 *An ergodic additive cellular automata on Σ_p for p prime is regionally transitive, has dense periodic points, and has sensitive dependence on initial conditions.*

Proof. Since the invariant Haar measure is a Borel measure whose support is all of the space Σ_p, regional transitivity follows at once from ergodicity (see [25], page 151-152 for the details). The set of periodic points is certainly dense. Finally, it is well–known that sensitive dependence to initial conditions follows from regional transitivity and dense periodic points by [1]. □

Theorem 5 *Let α be an ergodic additive cellular automata on Σ_p for p prime, and let $x \in \Sigma_p$ be any initial configuration. Then there is a simple procedure for constructing a point of finite period n within distance ϵ of x. Moreover, the period n may be chosen smaller than a constant times* $\log(1/\epsilon)$.

Proof. If the support of the local rule of the automaton is a singleton (that is, the corresponding polynomial is a monomial) then the automaton is a power of the shift and the result is obvious.

Assume next that the cellular automaton has a local rule that looks back as well as forward: then the corresponding polynomial in $\mathbb{F}_p[t^{\pm 1}]$ is

$$\xi(t) = c_{-\ell}t^{-\ell} + \cdots + c_0 + \cdots + c_\ell t^\ell,$$

with both $c_{-\ell}$ and c_ℓ non–zero, and $\ell > 0, r > 0$. Then by the 'freshman's dream' in characteristic p we have

$$\xi^{p^k}(t) = c_{-\ell}^{p^k} t^{-\ell\cdot p^k} + \cdots + c_0^{p^k} + \cdots + c_\ell^{p^k} t^{\ell\cdot p^k}. \tag{12}$$

This means that the map α^{p^k} is the additive cellular automaton with local rule corresponding to the polynomial (12). Since the support of this polynomial lies only on points whose coordinates are multiples of p^k, it is clear that a point y which is fixed under α^{p^k} may be constructed by defining y_j to be x_j for all j with $|j| < p^k$, and then simply using the local rule (12) and the requirement that $\alpha^{p^k}(y) = y$ to write down the remaining coordinates.

Now assume that the local rule only depends on strictly positive co–ordinates: that is, the corresponding polynomial is of the form

$$\xi(t) = c_\ell t^\ell + \cdots + c_r t^r$$

with $c_\ell, c_r \neq 0$ and $r > \ell > 0$. Then the same construction works: the automaton α^{p^k} has local rule corresponding to the polynomial

$$c_\ell^{p^k} t^{\ell \cdot p^k} + \cdots + c_r^{p^k} t^{r \cdot p^k},$$

and it is straightforward to extend a finite configuration $(x_j)_{|j|<p^k}$. The case of strictly negative co–ordinates is similar.

The remaining case is where the polynomial corresponding to the local rule has the form

$$\xi(t) = c_0 + c_\ell t^\ell + \cdots + c_r t^r$$

for some $c_\ell, c_r, c_0 \neq 0$ and $r \geq \ell > 0$. In this case there may be only one point of period p^k (for example, the 'rule 102') cellular automaton has this property) so we need to use a different argument. By the argument used above,

$$\xi(t)^{p^k} = c_0^{p^k} + c_\ell^{p^k} t^{\ell \cdot p^k} + \cdots + c_r^{p^k} t^{r \cdot p^k}.$$

It follows that if D is the coefficient of t^ℓ in $\xi(t)^{p^k - 1}$, then

$$D \cdot c_0 + c_\ell c_0^{p^k - 1} \equiv 0$$

in $\mathbb{F}_p$. In particular, $D \equiv 0 \bmod p$. On the other hand, since $c_0 \neq 0$ in $\mathbb{F}_p$, $c_0^{p^k-1} = c_0^{(p-1)(1+p+\cdots+p^{k-1})} \equiv 1$, so the polynomial corresponding to the local rule of the automaton α^{p^k-1} is

$$\xi(t)^{p^k-1} = 1 + D \cdot t^\ell + \cdots + c_r^{p^k-1} t^{r \cdot (p^k-1)}.$$

Now construct a point y with period $p^k - 1$ under α as follows. The point y must lie in the kernel of the map corresponding to multiplication by

$$\xi(t)^{p^k-1} - 1 = D \cdot t^\ell + \cdots + c_r^{p^k-1} t^{r \cdot (p^k-1)}.$$

It follows that y must solve the equations

$$\sum_{j=\ell}^{r\cdot(p^k-1)} y_{j+N} \equiv 0$$

in $\mathbb{F}_p$ for all $N \in \mathbb{Z}$. It is clear that a solution can be found for which $y_j = x_j$ for any $\big(r(p^k-1)-\ell\big)-1$ specified j, and choosing these to be the central co–ordinates of x gives the result.

The estimate on the size of period needed follows from a simple calculation: two points in Σ_p that agree on all coordinates j with $|j| < s$ are distance on the order of 2^{-s} apart under ρ. $\square$

This result gives a very simple and general construction for this class of cellular automata.

Example 6 [1] To illustrate the simple case of Theorem 5, we find a periodic point that agrees with the point

$$x = \ldots \underbrace{011\hat{0}101}\ldots$$

(the hat indicates the zero position) on the indicated positions for the 'rule 90' cellular automata. The polynomial defining the local rule here is $\xi(t) = t^{-1}+t \in \mathbb{F}_2[t^{\pm 1}]$, so following the procedure in Theorem 5 we write down

$$(t^{-1}+t)^4 = t^{-4}+4t^{-2}+6+4t^2+t^4 = t^{-4}+t^4$$

mod 2. Then it is clear that we may write down a point y that is fixed by α^4 and that agrees with the displayed positions in x. In fact there are two such points:

$$y = \ldots 100\underbrace{011\hat{0}101}011\ldots$$

and

$$y' = \ldots 101\underbrace{011\hat{0}101}111\ldots$$

[2] Now consider the same point

$$x = \ldots \underbrace{011\hat{0}101}\ldots$$

for the 'rule 102' cellular automaton. Following the procedure, we will need $k = 4$, so the point y will be a point in the kernel of the automaton with local rule corresponding to

$$(1+t)^15 - 1 = t + t^2 + \cdots + t^15.$$

Thus a point may be constructed by appending 8 arbitrary symbols to either side and then using the rule that any fifteen adjacent symbols in y must sum to zero to build the rest of the point:

$$y = \ldots 00\underbrace{011\hat{0}101}\underbrace{00000000}01\ldots$$

6 Conclusion and remarks

[1] The machinery developed for S–integer dynamical systems may be used to give insights into the dynamical behaviour of the special class of additive cellular automata with prime alphabets. Very exact information about how the periodic configurations lie in the space is shown, and a simple formula for the topological entropy and for counting periodic points is arrived at.

[2] The method used involves a 'linear' covering space k_S that comprises a direct product of fields. The lifted map $\tilde{\alpha}$ may be thought of as having generalized eigenvalues on this space, and the modulus of these eigenvalues is given by the set $\{|\xi|_\nu\}_{\nu \in S \cup P_\infty}$. It is reasonable to view an additive cellular automaton as hyperbolic if this set does not contain 1. For example, the eigenvalues for the 'rule 90' cellular automaton are found in Example 2[2] to have sizes 2 (since here $S = \{t\}$ and $|t^{-1} + t|_{t^{-1}} = |t^{-1} + t|_t = 2$). The eigenvalues for the 'rule 102' cellular automaton has one eigenvalue of size 2 (since $|1 + t|_{t^{-1}} = 2$) and one of size 1 (since $|1 + t|_t = 1$). In this setting, Theorem 3 shows that additive cellular automata have rational zeta function only when they are hyperbolic, and in general the hyperbolic systems have much more straightforward dynamics.

Another manifestation of the simple dynamical consequences of hyperbolicity is the following. If α is hyperbolic, then it is easy to check that it either has local rule corresponding to a monomial or a polynomial involving both negative and positive powers of t. Since additive cellular automata on prime alphabets are automatically bipermutative, a result of Shereshevsky and Afraimovich [26] applies to show that any such cellular automaton is topologically conjugate to some power of the one–sided full shift on p symbols.

[3] Additive cellular automata in general (with arbitrary alphabet) are not directly amenable to the S–integer formalism.

[4] Some results on higher–dimensional cellular automata are available: in [23] it is proved that an ergodic additive cellular automaton in two dimensions has infinite topological entropy. The method of proof is to exhibit subsystems that are periodic in one spatial dimension that have dense periodic points. An easy consequence of this work is therefore that additive cellular automata in two dimensions are chaotic in the sense of Devaney. Other dynamical properties of higher–dimensional cellular automata are in [22].

[5] The volume-growth approach to computing entropy used here does not extend to non-linear automata, because the maximal measure is usually not homogeneous in the sense of [3]. Exact calculations for certain individual non-linear automata have been carried out – an example is [6] – but in

general the problem is completely intractable by [14].

7 Appendix

For completeness, we give a short introduction to the relevant parts of number theory used above. The full story – the main theorems for adele rings of rational function fields – is in [30] and in [5].

Let G be a locally compact abelian group. A *character* on G is a continuous homomorphism $\chi : G \to \mathbb{S}^1$. The set of characters forms a group $\widehat{G}$ under multiplication, and when endowed with the topology of uniform convergence on compact sets, $\hat{G}$ is again a locally compact abelian group. The results on harmonic analysis used below are all standard and may be found in [13] for example.

Let $k = \mathbb{F}_p(t)$, and let $|\cdot|_\nu$ be a valuation on k defined as in Section 3. The valuation $|\cdot|_\nu$ defines a metric

$$d_\nu(x, y) = |x - y|_\nu$$

on the field k. Notice that this is an *ultrametric* in that a stronger form of the triangle inequality is true:

$$d_\nu(x, y) \leq \max\{d_\nu(x, z), d_\nu(z, y)\}$$

for all $x, y, z \in k_\nu$. The completion (in the sense of metric spaces) of k with respect to d_ν is a *local field* k_ν. Each local field k_ν has a maximal compact subring,

$$r_\nu = \{x \in k_\nu : |x|_\nu \leq 1\},$$

(closed under addition since the metric d_ν is an ultrametric). The invertible elements in the ring r_ν form the multiplicative group

$$r_\nu^* = \{x \in k_\nu : |x|_\nu = 1\}.$$

The field k_ν is then a locally compact non–discrete topological field, and so $\widehat{k_\nu}$ is isomorphic to k_ν (Chapter II,§5 in [30]). The explicit form of this isomorphism is important. Define a non–trivial character on k_ν by writing the elements of k_ν in the form

$$x = \sum_{i=m}^{\infty} a_i \pi^i \tag{13}$$

for some coefficients $a_i \in \mathbb{F}_p$, $m \in \mathbb{Z}$ and π a chosen element of k (Chapter I, §4 of [30]) and then setting $\chi(x) = \psi(a_{-1})$ for an arbitrary non–trivial character ψ on $\mathbb{F}_p$. Notice that the elements of r_ν in the notation (13) are exactly those with $m \in \mathbb{N}$. Then the map

$$\theta : k_\nu \to \widehat{k_\nu} \tag{14}$$

defined by $\theta(a)(x) = \chi(ax)$ is an isomorphism of topological groups between k_ν and $\widehat{k_\nu}$.

Now let S denote any finite set of finite valuations on k, and write $T = S \cup P_\infty$. Let

$$k_S = \prod_{\nu \in T} k_\nu;$$

elements of k_S are called S-adeles. Since k_S is a finite product of locally compact non–discrete fields, we have

$$\widehat{k_S} \cong k_S, \tag{15}$$

an isomorphism of topological groups. However, a specially constructed isomorphism will be needed later.

Recall that the ring of S–integers in the global field k is defined in Definition 1,

$$R_S = \{x \in k \colon |x|_w \leq 1 \text{ for all } w \notin T\}.$$

The map

$$\Delta(x) = (x, x, \ldots, x) \in \prod_{\nu \in T} k_\nu$$

is an injective homomorphism $\Delta : R_S \mapsto k_S$. The image of Δ is a copy of R_S sitting inside k_S, and the main observation is the following.

Theorem 6 *The subgroup $\Delta(R_S)$ is a discrete subgroup of k_S with compact quotient. Moreover, there is an isomorphism between the quotient $k_S/\Delta(R_S)$ and the dual group $\widehat{R_S}$.*

The map $k_S \to k_S/\Delta(R_S)$ is the covering map used in the proof of Theorem 1

The isomorphism used to prove Theorem 6 has to be constructed with some care: starting with the character χ in equation (14) for $\nu = t^{-1}$ extend it to the character $\chi'(x, y) = \chi(x)$ on $\{(x, y) : x \in k_{t^{-1}}, y \in \prod_{\nu \in S} k_\nu\}$. Then χ' can be extended uniquely to a character $\bar{\chi}$ on k_S that is trivial on $\Delta(k)$. Then any character on k_S may be written in the form $(x, y) \mapsto \bar{\chi}(ax, by)$ for $x \in k_{t^{-1}}$ and $y \in \prod_{\nu \in S} k_\nu$. One may then check that the map from k_S to $\widehat{k_S}$ that sends (a, b) to that character has the desired properties. For the full details, see Chapter IV, §2 of [30].

For S infinite (in fact, for S comprising all the places of k) and for general k (that is, allowing k to be a number field as well) this is one of the 'main theorems' in adelic number theory: see Chapter IV, §2 in [30].

Proof. Let d_S denote the maximum metric on k_S:

$$d_S\left((x_\nu),(y_\nu)\right)=\max_{\nu\in T}\{d_\nu(x_\nu,y_\nu)\}. \tag{16}$$

Let $x\in R_S$ be a non–zero element. Then x is of the form $\frac{h}{g}$ where $h,g\in\mathbb{F}_p[t]$ have no factors in common and g can only be divisible by polynomials corresponding to valuations in S. If $x=h$ is actually a polynomial, then $|h|_{t^{-1}}=p^{\deg(h)}\geq 1$ by a calculation similar to that in Example 2[2], so

$$d_S\left((x_\nu),0\right)\geq 1$$

by (16). If g is non–constant, then it must be divisible by some irreducible polynomial corresponding to one of the finite valuations $\nu\in S$, so

$$|\tfrac{h}{g}|_\nu\geq p,$$

showing again that

$$d_S\left((x_\nu),0\right)\geq 1$$

by (16). It follows that every element of the subgroup $\Delta(R_S)\backslash\{0\}$ is distance at least 1 from zero, so the subgroup is discrete. This implies that $\Delta(R_S)$ is a closed subgroup of k_S, and general results on duality show that

$$\widehat{R_S}\cong\widehat{\Delta(R_S)}\cong k_S/\Delta(R_S)^\perp,$$

where

$$\Delta(R_S)^\perp=\{\chi\in\widehat{k_S}:\chi(x)=1\ \forall\ x\in\Delta(R_S)\}$$

is the annihilator of $\Delta(R_S)$ in the dual group $\widehat{k_S}$. Now a careful examination of the exact form of the isomorphism constructed as described in the discussion after Theorem 6 shows that the subgroup $\Delta(R_S)^\perp$ is the image of $\Delta(R_S)$. Thus

$$\widehat{R_S}\cong\widehat{\Delta(R_S)}\cong k_S/\Delta(R_S).$$

This also shows that the quotient $k_S/\Delta(R_S)$ is compact since it is the dual group of the discrete group R_S. □

It remains only to exhibit a fundamental domain with finite volume for the quotient map (this is needed in the proof of Theorem 2). Of course Theorem 6 shows that such a domain must exist – the argument below gives a simple description.

Theorem 7 *A fundamental domain for the quotient map*

$$k_S \longrightarrow \widehat{R_S}$$

may be chosen with finite measure.

Proof. Let

$$R_S^{(\nu)} = \{x \in R_S : |x|_w \leq 1 \ \forall \ w \in T \backslash \{\nu\}\}$$

for each $\nu \in T$. Assume first that $\nu \in S$. Then it is clear that

$$R_S^{(\nu)} \cap r_\nu = \mathbb{F}_p \tag{17}$$

since $R_S^{(\nu)}$ comprises those rational functions $\frac{h}{g} \in R_S$ with the property that only powers of the polynomial corresponding to ν appear in g, so intersecting with r_ν means the denominator must be constant. On the other hand, h must be constant since $|\frac{h}{g}|_{t^{-1}} \leq 1$, so if $\frac{h}{g} \in k^{(\nu)} \cap r_\nu$ both h and g are constants. If ν is the infinite place the same proof works with t replaced by t^{-1}, showing that (17) holds for all $\nu \in T$.

Now any element of k_ν may be written as a sum of an element of r_ν and an element of $R_S^{(\nu)}$ (this is easy to see using the notation (13) for elements of k_ν), so

$$k_\nu = R_S^{(\nu)} + r_\nu. \tag{18}$$

Now let

$$F = \prod_{\nu \in T} r_\nu.$$

Then $\Delta(R_S) \cap F = \Delta(\mathbb{F}_p)$, which is finite, and $k_S = \Delta(R_S) + F$. Since F is an open compact subgroup of k_S, this shows that the quotient map has a compact, hence finite measure, fundamental domain. $\square$

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