BIMODULES OVER UNIFORMLY ORIENTED A_n QUIVERS WITH RADICAL SQUARE ZERO

VOLODYMYR MAZORCHUK AND XIAOTING ZHANG

Abstract. We start with observing that the only connected finite dimensional algebras with finitely many isomorphism classes of indecomposable bimodules are the quotients of the path algebras of uniformly oriented A_n-quivers modulo the radical square zero relations. For such algebras we study the (finitary) tensor category of bimodules. We describe the cell structure of this tensor category, determine existing adjunctions between its 1-morphisms and find a minimal generating set with respect to the tensor structure. We also prove that, for the algebras mentioned above, every simple transitive 2-representation of the 2-category of projective bimodules is equivalent to a cell 2-representation.

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Keywords: bimodule; quiver; left cell; fiat 2-category; simple transitive 2-representation; cell 2-representation

1. Introduction and description of the results

Finitary 2-categories were introduced in [MM1] as "finite dimensional" counterparts of, in general, "infinite dimensional" 2-categories which were studied in the categorification literature on the borderline between algebra and topology during the last twenty years, see [BFK, CR, Kh, KL, Ro, St]. The series [MM1, MM2, MM3, MM4, MM5, MM6] of papers develops basics of abstract representation theory for finitary 2-categories. Classical examples of finitary 2-categories are: the 2-category of Soergel bimodules over the coinvariant algebra of a finite Weyl group, see [MM1, Subsection 7.1], and the 2-category of projective functors over finite dimensional associative algebra, see [MM1, Subsection 7.3]. Further examples of finitary 2-categories were constructed and studied in [GM1, GM2, Xa, Zh1, Zh2], see also the above mentioned series [MM1]–[MM6].

In the present paper we consider a new natural class of examples of finitary 2-categories. The paper started from the following question:

For which finite-dimensional algebras, the corresponding tensor category of bimodules is finitary?

Although we suspect that the answer to this question could be known to specialists in representation type, we did not manage to find any explicit answer in the literature (the closest relevant reference we found is [Le]). In Theorem 1 we show that the only finite dimensional algebras over an algebraically closed field, for which the number of isomorphism classes of bimodules is finite, are those algebras whose connected components are radical square zero quotients of uniformly oriented path algebras of type A Dynkin quivers. This result motivates the rest of the paper where we take a closer look at the tensor category of bimodules over such algebras.

We start with an attempt to understand the combinatorics of left-, right- and two-sided cells of this tensor category. These cells are natural generalizations of Green's relations for semigroups from [Gr] to the setup of tensor categories, see [MM2, Section 3]. Apart from $\Bbbk$-split bimodules, that is bimodules of the form $X \otimes_{\Bbbk} Y$, where X is a left module and Y is a right module, see [MMZ], the remaining bimodules can be split into four families, which we call W, M, N and S, motivated by the shape of the diagram of a bimodule. All $\Bbbk$-split bimodules always form the maximum two-sided cell which is easy to understand.

To describe the remaining structure, we introduce several combinatorial invariants of bimodules, called *left support* and *right support* and also the number of *valleys* in the diagram of a bimodule. We show that these invariants, in combination with bimodule types, classify left, right and two-sided cells. For example, two-sided cells are classified, in the case of non $\Bbbk$-split bimodules, by the number of valleys in bimodule diagrams. This result is a first step in understanding combinatorial structure for bimodule categories over arbitrary finite dimensional algebras, the latter question forming the core of our motivation. We also give an explicit description for all adjoint pairs of functors formed by our bimodules. This description covers only some bimodules as, in general, the right adjoint of tensoring with a bimodule is not exact and hence is not isomorphic to tensoring with some (possibly different) bimodule.

Furthermore, we find a minimal generating set for our tensor category, with respect to the tensor structure. It consists of the identity bimodule and three additional bimodules in the two-sided cell closest to the one formed by the identity bimodule, with respect to the two-sided order. To prove the statement, we give explicit formulae for tensor products of each of these three bimodules with all other indecomposable bimodules.

Finally, we study simple transitive of the 2-category of projective bimodules over our algebras. Classification of such 2-representations is a natural problem which was considered, for various classes of 2-categories in [MM5, MM6, Zh1, Zi, MZ, MaMa, KMMZ, MT, MMMT, MMZ]. It also has interesting applications, see [KiM1]. A natural class of simple transitive 2-representations is given by the so-called *cell* 2-*representations* constructed in [MM1, MM2]. For the 2-category of projective bimodules over a finite dimensional algebra A, it is known that cell 2-representations exhaust all simple transitive 2-representations if A is self-injective (see [MM5, MM6]), if $A = \Bbbk[x, y]/(x^2, y^2, xy)$ (see [MMZ]) and if A is the radical square zero quotient of the path algebra of a uniformly oriented quiver of type A_2 or A_3 (see [MZ]). In the present paper we extend this result to all directed algebras admitting a non-zero projective-injective module, see Theorem 19. We recover, with a much shorter and much more elegant proof, the main results of [MZ], however, our approach is strongly inspired by [MZ, Section 3]. Our approach to the proof of Theorem 19 contains some new general ideas which could help to attack similar problems for other finitary 2-categories. We note that there are natural examples of 2-categories which have simple transitive 2-representations that are not equivalent to cell 2-representations, see [MaMa, KMMZ, MT, MMMT].

The paper is organized as follows: Section 2 contains the material related to the formulation and proof of Theorem 1. Section 3 studies combinatorics of bimodules. The main results of this section which provide a combinatorial description of the cell structure are collected in Subsection 3.5. In Theorem 15 of Section 4, we describe a minimal generating set of our tensor category with respect to the tensor structure.

Finally, Section 5 contains the material related to classification of simple transitive 2-representations.

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2. Characterization via representation type

2.1. **Main object of study.** Throughout the paper we fix an algebraically closed field $\Bbbk$. For $n \in \{1, 2, 3, \dots\}$, we denote by A_n the quotient of the path algebra of the quiver

$$\mathtt{1} \xrightarrow{\alpha_1} \mathtt{2} \xrightarrow{\alpha_2} \mathtt{3} \xrightarrow{\alpha_3} \dots \xrightarrow{\alpha_{n-1}} \mathtt{n} \tag{1}$$

modulo the relations that the product of any two arrows is zero. In particular, we have $A_1 \cong \Bbbk$, A_2 is isomorphic to the algebra of upper triangular 2×2 matrices with coefficients in $\Bbbk$ and $\mathrm{Rad}(A_n)^2 = 0$, for any n. If n is fixed or clear from the context, we will simply write A for A_n.

We denote by

- A-mod the category of finite dimensional left A-modules;
- mod-A the category of finite dimensional right A-modules;
- A-mod-A the category of finite dimensional A-A-bimodules.

For $i = 1, 2, \dots, n$, we denote by e_i the trivial path at the vertex $\mathtt{i}$. Thus we have a primitive decomposition $1 = e_1 + e_2 + \dots + e_n$ of the identity $1 \in A$. Then $P_i = Ae_i$ is an indecomposable projective in A-mod and we denote by L_i the simple top of P_i. Further, we denote by I_i the indecomposable injective envelope of L_i. Note that P_i has dimension 2, for $i = 1, 2, \dots, n-1$, and $P_n = L_n$. Similarly, I_i has dimension 2, for $i = 2, 3, \dots, n$, and $I_1 = L_1$. Moreover, $P_i \cong I_{i+1}$, for $i = 1, 2, \dots, n-1$.

2.2. **Bimodule representation type.** The main result of this subsection is the following statement. We suspect that this claim should be known to experts, but we failed to find any explicit reference in the literature.

Theorem 1. *Let B be a finite dimensional associative $\Bbbk$-algebra. Then the following conditions are equivalent:*

(*a*) *The category B-mod-B has finitely many isomorphism classes of indecomposable objects.*

(*b*) *Each connected component of B is Morita equivalent to some A_n.*

As B-mod-B is equivalent to $B \otimes_{\Bbbk} B^{\mathrm{op}}$-mod, condition (a) is equivalent to saying that $B \otimes_{\Bbbk} B^{\mathrm{op}}$ is of finite representation type.

2.3. **Proof of the implication** (a)⇒(b)**.** Let B be a basic finite dimensional associative $\Bbbk$-algebra such that the category B-mod-B has finitely many isomorphism classes of indecomposable objects. Note that this, in particular, implies that B has finite representation type. Consider the Gabriel quiver $Q = (Q_0, Q_1)$ of B, where Q_0 is the set of vertices and Q_1 the set of arrows. Then, for any $i, j \in Q_0$, we have at most one arrow from i to j for otherwise B would surject onto the Kronecker algebra and thus have infinite representation type.

Next we claim that Q has no loops. Indeed, if Q would have a loop, B would have a quotient isomorphic to the algebra $D := \Bbbk[x]/(x^2)$ of dual numbers. However, the algebra $D \otimes_{\Bbbk} D^{\mathrm{op}}$ has a quotient isomorphic to $\Bbbk[x, y]/(x^2, y^2, xy)$ and the latter has infinite representation type since we have an infinite family of pairwise non-isomorphic indecomposable 2-dimensional modules of this algebra on which x and y act via

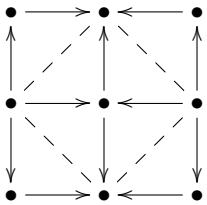
$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}, \quad \text{where } \lambda \in \Bbbk,$$

respectively (note that $\Bbbk$, being algebraically closed, is infinite). This is a contradiction.

Next we claim that each vertex of Q has indegree at most 1 and outdegree at most 1. We prove the claim for indegrees and, for outdegrees, the proof is similar. If Q has a vertex with indegree at least two, then, taking the above into account, Q has a subgraph of the form:

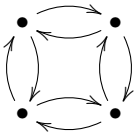
$$i \longrightarrow j \longleftarrow k \tag{2}$$

Then B has a quotient isomorphic to the path algebra C of (2). We claim that $C \otimes_{\Bbbk} C^{\mathrm{op}}$ has infinite representation type thus giving us a contradiction. Indeed, $C \otimes_{\Bbbk} C^{\mathrm{op}}$ is isomorphic to the quotient of the path algebra of the solid part of the following quiver:

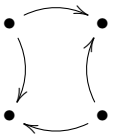


modulo the commutativity relations indicated by dotted lines. In the middle of this picture we see an orientation of the affine Dynkin diagram of type $\tilde{D}_4$. The corresponding centralizer subalgebra thus has infinite representation type. It follows that $C \otimes_{\Bbbk} C^{\mathrm{op}}$ has infinite representation type.

Next we claim that Q does not have any components of the form $\bullet \rightleftarrows \bullet$. If the latter were the case, then $B \otimes_{\Bbbk} B^{\mathrm{op}}$ would have a quotient isomorphic to the path algebra of

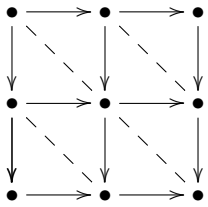


modulo the relations that all squares commute. The latter has a quotient isomorphic to the path algebra corresponding to the following orientation



of an affine Dynkin quiver of type $\tilde{A}_4$ and thus has infinite representation type, a contradiction.

The above shows that Q is a disjoint union of graphs of the form (1). We now only need to show that $\mathrm{Rad}(B)^2 = 0$. If $\mathrm{Rad}(B)^2 \neq 0$, then B has a quotient which is isomorphic to the path algebra F of (1), for $n = 3$. Then $F \otimes_{\Bbbk} F^{\mathrm{op}}$ is isomorphic to the quotient of the path algebra of the solid part of the following quiver:

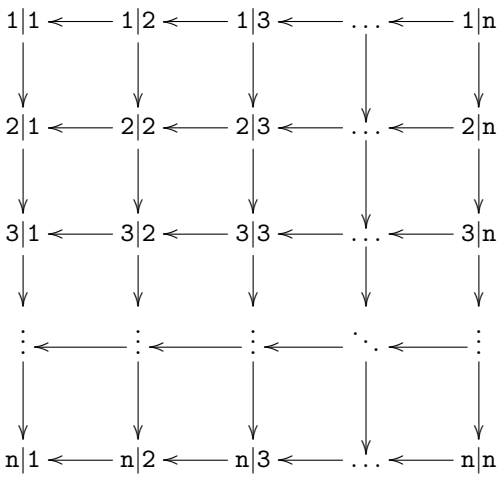


modulo the commutativity relations indicated by dotted lines. Similarly to the previous paragraph, this algebra has a centralizer subalgebra which is the path algebra of a type $\tilde{D}_4$ quiver. Therefore it has infinite representation type.

2.4. **Proof of the implication** (b)⇒(a)**.** Note that $A_n/(e_n) \cong A_{n-1}$, for any n. Therefore, for all m, n, there is a full embedding of the category of A_n-A_m-bimodules into the category of A_k-A_k-bimodules, where $k = \max(m, n)$. Using additivity and the fact that Morita equivalence, by definition, does not affect representation type, we obtain that it is sufficient to prove that the category A_n-mod-A_n has finitely many isomorphism classes of indecomposable objects, for every n.

The algebra $A_n \otimes_{\Bbbk} A_n^{\mathrm{op}}$ is the quotient of the path algebra of the following quiver:

(3)



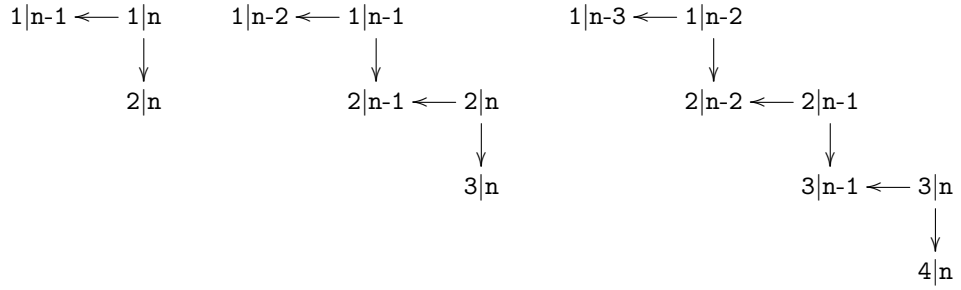
modulo the relations that all squares commute and the product of any two horizontal or any two vertical arrows is zero.

The algebra $A_n \otimes_{\Bbbk} A_n^{\mathrm{op}}$ is thus a special biserial algebra in the sense of [BR, WW]. According to these two references, each indecomposable module over a special biserial algebra is either a string modules or a band module or a non-uniserial projective-injective module. In our case, there are certainly finitely many indecomposable non-uniserial projective-injective modules (they correspond to commutative squares in our quiver).

We claim that, in the case of $A_n \otimes_{\Bbbk} A_n^{\mathrm{op}}$, there are only finitely many string modules (up to isomorphism) and that there are no band modules. This follows from the

form of the relations. Indeed, the maximal (with respect to inclusion) strings avoiding zero relations are:

(4)



and so on (in total, there are $2n-2$ such maximal strings). We see that edges of these strings never intersect and that the strings never close into bands (corresponding to primitive cyclic words in some references). Consequently, there are no band modules and only finitely many string modules. The claim follows.

An exact enumeration of isomorphism classes of indecomposable objects in the category A_n-mod-A_n is given in the next subsection.

2.5. Enumeration of indecomposable A_n-A_n-bimodules.

Proposition 2. *For $n = 1, 2, \dots$, the category A_n-mod-A_n contains exactly*

$$\frac{4n^3 + 3n^2 - 7n + 3}{3}$$

isomorphism classes of indecomposable objects.

Proof. For $n = 1$, the claim is clear as $A_1 \cong \mathbb{C}$ and thus $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \cong \mathbb{C}$ is a simple algebra and thus has exactly one isomorphism class of indecomposable modules. For $n > 1$, we have $(n-1)^2$ indecomposable projective-injective objects in A_n-mod-A_n. From Subsection 2.4, we know that the remaining indecomposable objects correspond to string $A_n \otimes_{\Bbbk} A_n^{\mathrm{op}}$-module. A string module is uniquely identified by the string the module is supported at, that is by a substring of one of the maximal strings as shown in (4).

For each $k = 3, 5, 7, \dots, 2n-1$, there are exactly two maximal strings with k vertices. A string with k vertices supports $k(k+1)/2$ string modules (which correspond to connected substrings). Note that simple modules are supported just by vertices and there are $n^2 - 2$ vertices, that is all but the left upper and right lower corners, which belong to two different maximal substrings and hence are counted twice above. Putting all this together and simplifying gives the necessary formula. □

3. Combinatorics of A-A-bimodules

3.1. **Cells.** The main aim of the section is to describe the cell combinatorics of A-A-bimodules in the sense of [MM2, Section 3]. Denote by $\mathcal{S}$ the set of isomorphism classes of indecomposable A-A-bimodules. Recall that the *left preorder* $\leq_L$ is defined as follows: for $X, Y \in \mathcal{S}$ we have $X \leq_L Y$ provided that Y is isomorphic to a direct summand of $Z \otimes_A X$, for some A-A-bimodule Z. An equivalence class with respect to $\leq_L$ is called a *left cell* and the corresponding equivalence relation is denoted by $\sim_L$. The *right preorder* $\leq_R$ and the corresponding *right cells* and $\sim_R$ are defined similarly using tensoring over A from the right. The *two-sided preorder* $\leq_J$

and the corresponding *two-sided cells* and $\sim_J$ are defined similarly using tensoring over A from both sides.

The preorder $\leq_L$ induces naturally a partial order on the set of all left cells. Abusing notation, we denote this partial order also by $\leq_L$. Similarly for $\leq_R$ and the set of all right cells, and for $\leq_J$ and the set of all two-sided cells.

3.2. **$\Bbbk$-split A-A-bimodules.** For simplicity, in this section we will denote by $\mathbb{N}_n$ the set $\{0, 1, 2, \dots, n-1\}$ and $\mathbb{N}_n^*$ the set $\{1, 2, \dots, n-1\}$.

An A-A-bimodule X is called $\Bbbk$-*split*, cf. [MMZ], provided that X is isomorphic to a direct sum of A-A-bimodules of the form $M \otimes_\Bbbk N$, where $M \in A$-mod and $N \in$ mod-A.

We will often argue using bimodule action graphs. We will depict the left action using vertical arrows and the right action using horizontal arrows. Following the proof of Proposition 2, we can now list all non-zero indecomposable $\Bbbk$-split A-A-bimodules (up to isomorphism) and describe their action graphs as follows:

- the projective-injective bimodules $Ae_i \otimes_\Bbbk e_{j+1}A$, where $i, j \in \mathbb{N}_n^*$:

$$\begin{array}{ccc} \mathtt{i} \mid \mathtt{j} & \longleftarrow & \mathtt{i} \mid \mathtt{j{+}1} \\ \downarrow & & \downarrow \\ \mathtt{i{+}1} \mid \mathtt{j} & \longleftarrow & \mathtt{i{+}1} \mid \mathtt{j{+}1}; \end{array}$$

- the simple bimodules $\mathtt{i} \mid \mathtt{j}$, where $i, j \in \mathbb{N}_{n+1}^*$, that is, string $A \otimes_\Bbbk A^{\mathrm{op}}$-modules of dimension 1 (we will denote the bimodule $\mathtt{i} \mid \mathtt{j}$ by L_{ij});

- string $A \otimes_\Bbbk A^{\mathrm{op}}$-modules of dimension 2:

$$\begin{array}{l} \mathtt{i} \mid \mathtt{j}, \\ \quad\downarrow \\ \mathtt{i{+}1} \mid \mathtt{j} \end{array} \quad i \in \mathbb{N}_n^*, j \in \mathbb{N}_{n+1}^*; \qquad \mathtt{i} \mid \mathtt{j} \longleftarrow \mathtt{i} \mid \mathtt{j{+}1}, \quad i \in \mathbb{N}_{n+1}^*, j \in \mathbb{N}_n^*.$$

Note that the projective A-A-bimodules $Ae_n \otimes_\Bbbk e_{j+1}A$, where $j \in \mathbb{N}_n^*$, and also $Ae_i \otimes_\Bbbk e_1 A$, where $i \in \mathbb{N}_n^*$, belong to the last type of indecomposables (and they are not injective).

We denote by $\mathcal{J}_\Bbbk$ the set of all $\Bbbk$-split elements in $\mathcal{S}$.

Proposition 3.

(*i*) *The set $\mathcal{J}_\Bbbk$ is the unique maximal two-sided cell in $\mathcal{S}$.*

(*ii*) *For each indecomposable right A-module N, the set of elements in $\mathcal{J}_\Bbbk$ of the form $X \otimes_\Bbbk N$, for some $X \in A$-mod, forms a left cell. Moreover, each left cell in $\mathcal{J}_\Bbbk$ is of such form.*

(*iii*) *For each indecomposable left A-module K, the set of elements in $\mathcal{J}_\Bbbk$ of the form $K \otimes_\Bbbk Y$, for some $Y \in$ mod-A, forms a right cell. Moreover, each right cell in $\mathcal{J}_\Bbbk$ is of such form.*

Proof. For $X \in \mathcal{J}_\Bbbk$ and Y any A-A-bimodule, both $X \otimes_A Y$ and $Y \otimes_A X$ are obviously $\Bbbk$-split, so $\Bbbk$-split bimodules form a tensor ideal in A-mod-A. For any indecomposable K_1 and K_2 in A-mod and any indecomposable $X \in$ mod-A, we have

$$(K_1 \otimes_\Bbbk A) \otimes_A (K_2 \otimes_\Bbbk X) \cong (K_1 \otimes_\Bbbk X)^{\oplus \dim K_2}.$$

This implies claim (ii) and claim (iii) is proved similarly. This, combined with the fact that $\Bbbk$-split bimodules form a tensor ideal in A-mod-A, also implies claim (i), completing the proof. □

3.3. **A-A-bimodules which are not $\Bbbk$-split.** Consider the action graph Γ_M for a string $A \otimes_{\Bbbk} A^{\mathrm{op}}$-module M. Then, directly from the construction of string modules, we can make the following easy observations:

- the indegree of each vertex in Γ_M is at most two;
- the outdegree of each vertex in Γ_M is at most two;
- each vertex of Γ_M is either a source or a sink (or both);
- M is simple if and only if both the indegree and the outdegree of each vertex in Γ_M is zero.

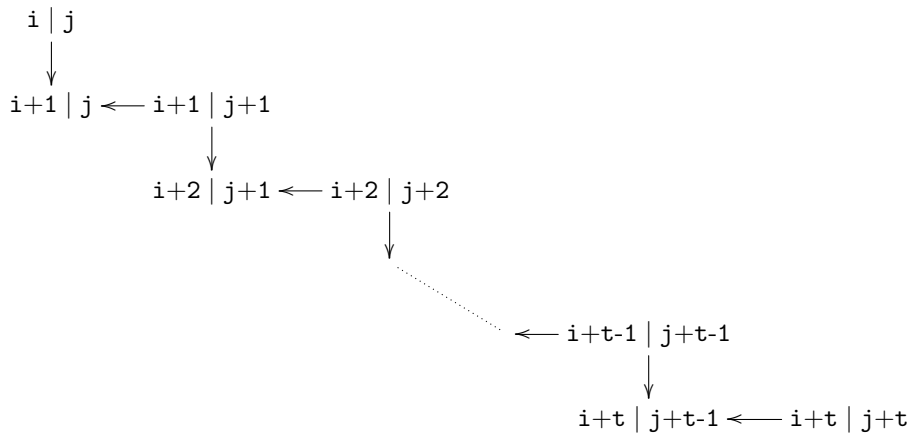
A vertex of Γ_M of indegree exactly two will be called a *valley*. A typical example of a valley in an action graph is the white vertex of the following graph:

We denote by $\mathbf{v}(M)$ the number of valleys in Γ_M. Clearly, $0 \leq \mathbf{v}(M) \leq n-1$, moreover, the only M for which $\mathbf{v}(M) = n-1$ is the regular bimodule $M \cong {}_AA_A$. For any $A \otimes_{\Bbbk} A^{\mathrm{op}}$-module N which is not projective-injective, define

$$\mathbf{v}(N) := \max\{\mathbf{v}(M) : M \text{ is a string module and is as a direct summand of } N\}.$$

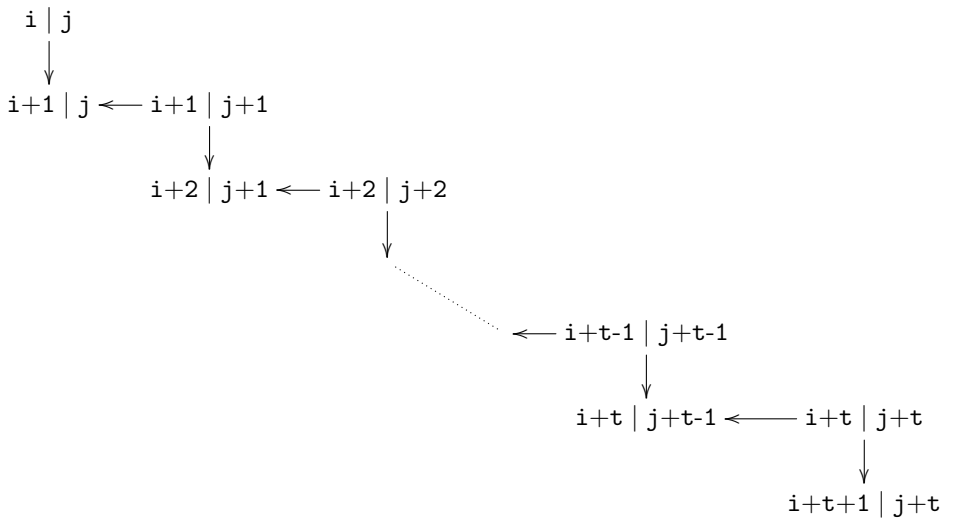
Indecomposable A-A-bimodules that are not $\Bbbk$-split correspond to string $A \otimes_{\Bbbk} A^{\mathrm{op}}$-modules whose action graphs have k vertices, where $3 \leq k \leq 2n-1$. Below we list all such bimodules, fix notation for them and describe the corresponding action graphs. We use the number of valleys in the action graph as a parameter, denoted by t. Our choice for notation is motivated by the shape of the action graph.

Bimodules W_{ij}^t. For any $t \in \mathbb{N}_n^*$ and $i, j \in \mathbb{N}_{n-t+1}^*$, we denote by W_{ij}^t the following A-A-bimodule:

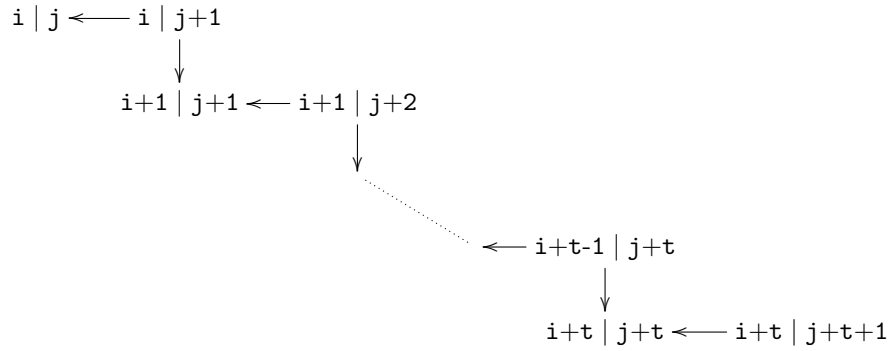


In particular, we have ${}_AA_A \cong W_{11}^{n-1}$.

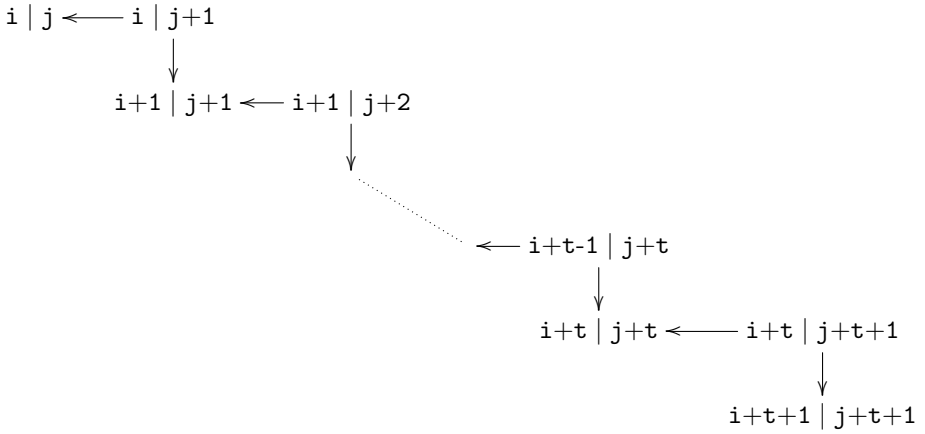
Bimodules S_{ij}^t. For any $t \in \mathbb{N}_{n-1}^*$, $i \in \mathbb{N}_{n-t}^*$, and $j \in \mathbb{N}_{n-t+1}^*$, we denote by S_{ij}^t the following A-A-bimodule:



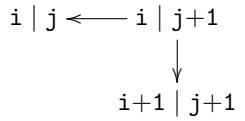
Bimodules N_{ij}^t. For any $t \in \mathbb{N}_{n-1}^*$, $i \in \mathbb{N}_{n-t+1}^*$ and $j \in \mathbb{N}_{n-t}^*$, we denote by N_{ij}^t the following A-A-bimodule:



Bimodules M_{ij}^t. For any $t \in \mathbb{N}_{n-1}$ and $i, j \in \mathbb{N}_{n-t}^*$, we denote by M_{ij}^t the following A-A-bimodule:



In particular, $\mathrm{Hom}_{\Bbbk}({}_AA_A, \Bbbk) \cong M_{11}^{n-2}$. As a concrete example, the action graph of the bimodule M_{ij}^0, where $i, j \in \mathbb{N}_n^*$, is:



Let M be an indecomposable A-A-bimodule listed in Subsection 3.2-3.3. Then vertices in Γ_M correspond to the *standard basis* of M denoted $\mathbf{B}(M)$. We will often identify vertices in Γ_M with this standard basis. Note the following:

- every vertex of non-zero indegree generates a simple sub-bimodule;
- every vertex of outdegree two generates a subbimodule which is isomorphic to some M_{ij}^0.

3.4. **Supports.** For an A-A-bimodule X, we define the *left support* of X as the set of all $i \in \{1, 2, \dots, n\}$ such that $e_i X \neq 0$. The left support of X will be denoted $\mathrm{Lsupp}(X)$. Similarly, the *right support* of X, denoted $\mathrm{Rsupp}(X)$, is the set of all $i \in \{1, 2, \dots, n\}$ such that $Xe_i \neq 0$. Note that, for any indecomposable A-A-bimodule X, both $\mathrm{Lsupp}(X)$ and $\mathrm{Rsupp}(X)$ are convex subset of $\{1, 2, \dots, n\}$. Here $X \subset \{1, 2, \dots, n\}$ is *convex* if, for any $x, y, z \in \{1, 2, \dots, n\}$, the properties $x, z \in X$ and $x < y < z$ imply $y \in X$.

We define the *width* $\mathbf{w}(X)$ and the *height* $\mathbf{h}(X)$ of X to be, respectively,

$$\mathbf{w}(X) := 1 + \max\{i \in \mathrm{Rsupp}(X)\} - \min\{i \in \mathrm{Rsupp}(X)\},$$
$$\mathbf{h}(X) := 1 + \max\{i \in \mathrm{Lsupp}(X)\} - \min\{i \in \mathrm{Lsupp}(X)\}.$$

Note that, for an indecomposable X, we have $|\mathbf{w}(X) - \mathbf{h}(X)| \leq 1$.

It is worth noting that indecomposable bimodules in the families N and S are uniquely determined (inside the set of isomorphism classes of indecomposable A-A-bimodules) by their left and right supports and that for bimodules in these families we always have $|\mathbf{w}(X) - \mathbf{h}(X)| = 1$. Further, each indecomposable bimodule in the families M and W is uniquely determined by its left and right support inside its family. Moreover, for each bimodule X in the families M and W, there is a unique bimodule in the other family (i.e. in W if X is in M and vice versa) which shares the left and right supports with X. Note that, if X of the M-family and Y of the W-family share the left and right supports, then $\mathbf{v}(Y) > \mathbf{v}(X)$.

The following lemma contains a crucial observation for our combinatorial description. The claim follows directly from the definitions.

Lemma 4. *For any A-A-bimodules X and Y, we have*

$$\mathrm{Lsupp}(X \otimes_A Y) \subset \mathrm{Lsupp}(X) \quad \text{and} \quad \mathrm{Rsupp}(X \otimes_A Y) \subset \mathrm{Rsupp}(Y).$$

3.5. **Description of cells.** For $v \in \mathbb{N}_n$, denote by $\mathcal{J}_v$ the set of all non $\Bbbk$-split elements in $\mathcal{S}$ having exactly v valleys. Note that $\mathcal{J}_{n-1}$ consists just of the identity A-A-bimodule ${}_AA_A$. Recall from [CM, Subsection 2.3] that a two-sided cell is called *idempotent* provided that it contains three elements F, G and H (not necessarily different), such that F is isomorphic to a direct summand of $\mathrm{G} \otimes_A \mathrm{H}$.

In this subsection we present our main results. We start by describing two-sided cells in $\mathcal{S}$.

Theorem 5.

(*i*) *Each $\mathcal{J}_v$ is a two-sided cell in $\mathcal{S}$, moreover, each two-sided cell in $\mathcal{S}$ coincides with either $\mathcal{J}_{\Bbbk}$ or with one of the $\mathcal{J}_v$'s.*

(*ii*) *The two-sided cells of $\mathcal{S}$ are linearly ordered as follows:*

$$\mathcal{J}_{\Bbbk} \geq_J \mathcal{J}_0 \geq_J \mathcal{J}_1 \geq_J \cdots \geq_J \mathcal{J}_{n-1}.$$

(*iii*) *All two-sided cells but $\mathcal{J}_0$ are idempotent.*

As left and right cells in $\mathcal{J}_{\Bbbk}$ are already described in Proposition 3, it is left to consider left and right cells formed by non $\Bbbk$-split bimodules.

Theorem 6. *Let $v \in \mathbb{N}_n$.*

(*i*) *If $v > 0$, then, for each $j \in \mathbb{N}^*_{n-v+1}$, the set of all bimodules of type W and S in $\mathcal{J}_v$ with right support $\{j, j+1, \dots, j+v\}$ forms a left cell in $\mathcal{J}_v$.*

(*ii*) *For each $j \in \mathbb{N}^*_{n-v}$, the set of all bimodules of type M and N in $\mathcal{J}_v$ with right support $\{j, j+1, \dots, j+v+1\}$ forms a left cell in $\mathcal{J}_v$.*

(*iii*) *Each left cell in $\mathcal{J}_v$ is of the form given by* (i) *or* (ii).

Theorem 7. *Let $v \in \mathbb{N}_n$.*

(*i*) *If $v > 0$, then, for each $j \in \mathbb{N}^*_{n-v+1}$, the set of all bimodules of type W and N in $\mathcal{J}_v$ with left support $\{j, j+1, \dots, j+v\}$ forms a right cell in $\mathcal{J}_v$.*

(*ii*) *For each $j \in \mathbb{N}^*_{n-v}$, the set of all bimodules of type M and S in $\mathcal{J}_v$ with left support $\{j, j+1, \dots, j+v+1\}$ forms a right cell in $\mathcal{J}_v$.*

(*iii*) *Each right cell in $\mathcal{J}_v$ is of the form given by* (i) *or* (ii).

As an immediate consequence from Theorems 5, 6 and 7, we have:

Corollary 8. *All two-sided cells in $\mathcal{J}$ are* strongly regular *in the sense that we have $|\mathcal{L} \cap \mathcal{R}| = 1$, for any left cell $\mathcal{L}$ in $\mathcal{J}$ and any right cell $\mathcal{R}$ in $\mathcal{J}$.*

Regular two-sided cells play an important role in the theory developed in [MM1]–[MM6], see also [KiM2].

The algebra A has an involutive anti-automorphism $\imath$ which swaps e_1 with e_n, e_2 with e_{n-1}, α_1 with α_{n-1} and so on. Using this anti-automorphism, we can define an anti-involution on the tensor category A-mod-A which we, abusing notation, also denote by $\imath$. This anti-involution swaps the sides of bimodules and twists both action by $\imath$. Consequently, the multisemigroup $\mathcal{S}$ has an involutive anti-automorphism. This anti-automorphism makes the statements of Theorem 6 and Theorem 7 equivalent. Remainder of the section is devoted to the proof of Theorems 5 and 6.

3.6. **Proof of Theorem 6.** We start by introducing the following notation: for a subset $U \subset \mathbb{Z}$ and for $r \in \mathbb{Z}$, we will denote by $U[r]$ the set $\{i + r \,:\, i \in U\}$.

Lemma 9. *Let us fixed a type, W, M, S or N, of non-$\Bbbk$-split A-A-bimodules. Then all bimodules of this type and with fixed right support belong to the same left cell.*

Proof. Denote by φ the endomorphism of A which sends

$$e_i \mapsto \begin{cases} e_{i+1}, & i+1 \leq n; \\ 0, & \text{else}; \end{cases} \qquad \alpha_i \mapsto \begin{cases} \alpha_{i+1}, & i+1 < n; \\ 0, & \text{else}. \end{cases}$$

Set $\tilde{e} = e_2 + e_3 + \cdots + e_n = \varphi(1)$. Then $\tilde{e}A$ has the natural structure of an A-A-bimodule where the right action of a is given by multiplication with a and the left action of a is given by multiplication with $\varphi(a)$. We will denote this bimodule by ${}^{\varphi}\tilde{e}A$. Similarly, $A\tilde{e}$ has the natural structure of an A-A-bimodule where the right action of a is given by multiplication with $\varphi(a)$ and the left action of a is given by multiplication with a. We will denote this bimodule by $A\tilde{e}^{\varphi}$.

We have $A\tilde{e} = \tilde{e}A\tilde{e}$. Consequently, the multiplication map

$${}^{\varphi}\tilde{e}A \otimes_A A\tilde{e}^{\varphi} \to {}^{\varphi}\tilde{e}A\tilde{e}^{\varphi}$$

is an isomorphism of A-A-bimodules. Set $I := Ae_nA$. Then, mapping $1 + I \mapsto \tilde{e}$, gives rise to an isomorphism of A-A-bimodules between A/I and ${}^{\varphi}\tilde{e}A\tilde{e}^{\varphi}$. In particular, it follows that ${}^{\varphi}\tilde{e}A \otimes_A A\tilde{e}^{\varphi} \cong {}_A(A/I)_A$.

This means that, if $X \in A$-mod is annihilated by I, then, tensoring X (from the left) first with $A\tilde{e}^{\varphi}$ and then with ${}^{\varphi}\tilde{e}A$, gives X back. If X has, additionally, the structure of an indecomposable A-A-bimodule, then tensoring with $A\tilde{e}^{\varphi}$ just twists the left action of A on X by φ and hence does not change the type (W, M, S or N) of X. In particular, we have

$$\mathrm{Lsupp}(A\tilde{e}^{\varphi} \otimes_A X) = (\mathrm{Lsupp}(X))[1] \quad \text{and} \quad \mathrm{Rsupp}(A\tilde{e}^{\varphi} \otimes_A X) = \mathrm{Rsupp}(X).$$

Starting now with X such that $1 \in \mathrm{Lsupp}(X)$ and applying this procedure inductively, we will obtain that all bimodules of the same type as X and with the same right support as X belong to the left cell of X. This proves the claim. □

Lemma 10. *Bimodules of types W and S with the same right support belong to the same left cell.*

Proof. After Lemma 9, it is enough to prove this claim for any two particular bimodules of types W and S with the same right support. Let X be the unique indecomposable subquotient of ${}_AA_A$ of type W with right support $\{j, j+1, \dots, k\}$. Assume that $k < n$ and let Y be the unique indecomposable subquotient of ${}_AA_A$ of type S with right support $\{j, j+1, \dots, k\}$. Then Y surjects onto X with one-dimensional kernel.

Let $Q = A(e_j + e_{j+1} + \cdots + e_k)A$ be a subbimodule of the regular bimodule ${}_AA_A$. Consider the short exact sequence

$$0 \to Q \to A \to \mathrm{Coker} \to 0$$

of bimodules. By construction, $\mathrm{Coker} \otimes_A X = 0$ and hence $Q \otimes_A X$ surjects onto X. Consequently, we either have $Q \otimes_A X \cong X$ or $Q \otimes_A X \cong Y$. To determine which of these cases takes place, we use adjunction:

$$\mathrm{Hom}_{A\text{-}A}(Q \otimes_A X, Y) \cong \mathrm{Hom}_{A\text{-}A}(X, \mathrm{Hom}_{A\text{-}}(Q, Y)).$$

It is now easy to check that $\mathrm{Hom}_{A\text{-}}(Q, Y) \cong X$ and thus the right hand side of the above isomorphism is non-zero. As $\mathrm{Hom}_{A\text{-}A}(X, Y) = 0$, it follows that $Q \otimes_A X = Y$.

If we denote by Q' the quotient of the above bimodule Q by the subbimodule $e_{k+1}Q$, then we again have either $Q' \otimes_A Y \cong X$ or $Q' \otimes_A Y \cong Y$. By adjunction,

$$\mathrm{Hom}_{A\text{-}A}(Q' \otimes_A Y, Y) \cong \mathrm{Hom}_{A\text{-}A}(Y, \mathrm{Hom}_{A\text{-}}(Q', Y)).$$

However, now Q' is a proper quotient of Q and one sees that the dimension of $\mathrm{Hom}_{A\text{-}}(Q', Y)$ is strictly smaller than that of $\mathrm{Hom}_{A\text{-}}(Q, Y)$. Therefore $\mathrm{Hom}_{A\text{-}}(Q', Y)$ is a proper subbimodule of X which yields

$$\mathrm{Hom}_{A\text{-}A}(Q' \otimes_A Y, Y) = 0.$$

This implies $Q' \otimes_A Y \cong X$ and hence X and Y do belong to the same left cell.

In the case $k = n$ the arguments are similar with the only difference that X has to be changed to ${}^{\varphi}\tilde{e}A \otimes_A X$. This works only if $j > 1$. If $j = 1$ and $k = n$, then there are no A-A-bimodules of type S with such right support. □

Lemma 11. *Bimodules of types M and N with the same right support belong to the same left cell.*

Proof. This is similar to the proof of Lemma 10 and is left to the reader. □

Lemma 12. *Bimodules of types W and M cannot belong to the same left cell.*

Proof. Let X be an A-A-bimodule of type W and Y an A-A-bimodule of type M. Taking Lemma 4 into account, suppose that $\mathrm{Rsupp}(X) = \mathrm{Rsupp}(Y)$ and consider the right A-modules X_A and Y_A. Then from our explicit description of bimodules we can see that X_A is not a quotient of any module in $\mathrm{add}(Y_A)$. Therefore no $Z \otimes_A Y_A$ can have X_A as a quotient, let alone direct summand. The claim follows. □

Claims (i) and (ii) of Theorem 6 follow from Lemmata 9–12. Claim (iii) follows from claims (i) and (ii) and classification of indecomposable A-A-bimodules.

3.7. **Proof of Theorem 5.** Claim (i) follows from Theorems 6 and Theorem 7. That fact that $\mathcal{J}_v$, for $v > 1$, are idempotent follows from the proof of Lemma 10.

To prove that two-sided cells are linearly ordered, for $j = 1, 2, \dots, n-1$, consider the quotient A-A-bimodule $Q_j := A/A(e_{j+2}+\cdots+e_n)A$ of A (in particular, $Q_{n-1} = A$). We have $Q_j \in \mathcal{J}_j$, for all j, in fact, $Q_j \cong W_{11}^j$. Consider also the A-A-bimodule $\mathrm{Hom}_{\Bbbk}(Q_j, \Bbbk) \cong M_{11}^{j-1} \in \mathcal{J}_{j-1}$. We can, in fact, interpret Q_j as the identity bimodule for the algebra $B := A/A(e_{j+2} + \cdots + e_n)A$. After this interpretation it is clear that

$$W_{11}^j \otimes_B M_{11}^{j-1} \cong W_{11}^j \otimes_A M_{11}^{j-1} \cong M_{11}^{j-1}.$$

Claim (ii) follows.

To complete the proof of Theorem 5, it remains to show that $\mathcal{J}_0$ is not idempotent. Note that $\mathcal{J}_0$ only consists of bimodules of type M. Let $X \otimes_A Y$ be the tensor product of two bimodules from $\mathcal{J}_0$. We need to check that $X \otimes_A Y$ is $\Bbbk$-split. This can be done by a direct computation or, alternatively, argued theoretically as follows.

If $\mathrm{Rsupp}(X) \cap \mathrm{Lsupp}(Y) = \varnothing$, then $X \otimes_A Y = 0$. If $\mathrm{Rsupp}(X) = \{i, i+1\}$ and $\mathrm{Lsupp}(Y) = \{i+1, i+2\}$, then, tensoring Y with the short exact sequence

$$0 \to X' \to X \to \mathrm{Coker} \to 0,$$

where $X??'$ is simple with $\mathrm{Rsupp}(X') = \{i\}$ and using $X' \otimes_A Y = 0$, implies $X \otimes_A Y \cong \mathrm{Coker} \otimes_A Y$ is $\Bbbk$-split as Coker is $\Bbbk$-split.

If $\mathrm{Rsupp}(X) = \{i+1, i+2\}$ and $\mathrm{Lsupp}(Y) = \{i, i+1\}$, then, tensoring Y with the short exact sequence

$$0 \to X' \to X \to \mathrm{Coker} \to 0,$$

where X' is simple with $\mathrm{Rsupp}(X') = \{i+1\}$, we obtain $\mathrm{Coker} \otimes_A Y = 0$ and $X' \otimes Y = 0$ implying $X \otimes_A Y = 0$ which is certainly $\Bbbk$-split.

If $\mathrm{Rsupp}(X) = \mathrm{Lsupp}(Y) = \{i, i+1\}$, then we tensor Y with the short exact sequence

$$0 \to X' \to X \to \mathrm{Coker} \to 0,$$

where X' is simple with $\mathrm{Rsupp}(X') = \{i\}$. Then $\mathrm{Coker} \otimes_A Y = 0$ and thus the two-dimensional bimodule $X' \otimes_A Y$ surjects onto $X \otimes_A Y$. Therefore $X \otimes_A Y$ has dimension at most two and thus is $\Bbbk$-split. The proof of Theorem 5 is complete.

3.8. **Adjunctions between indecomposable A-A-bimodules.** In this subsection we will describe all pairs of bimodules which correspond to adjoint pairs of functors.

Lemma 13. *For any $X, Y \in A$-mod-A, the pair $(X \otimes_A -, Y \otimes_A -)$ is an adjoint pair of endofunctors of A-mod if and only if X is projective as a left A-module and $\mathrm{Hom}_A(X, A) \cong Y$ as A-A-bimodules.*

Proof. As $(X \otimes_A -, \mathrm{Hom}_A(X, -))$ is an adjoint pair of functors, we have

$$Y \otimes_A - \cong \mathrm{Hom}_A(X, -),$$

in particular, the latter functor must be exact and hence X must be left A-projective. As exact functors are uniquely determined by their action on the category of projective A-modules, we get an isomorphism $\mathrm{Hom}_A(X, A) \cong Y$ of a A-A-bimodule. The converse implication is straightforward. □

An indecomposable A-A-bimodule X which is not $\Bbbk$-split is left projective if and only if it belongs to the following set:

$$\{W_{11}^{n-1}\} \cup \{W_{n-t|j}^{t}, S_{ij}^{t}, t \in \mathbb{N}_{n-1}^{*}, i \in \mathbb{N}_{n-t}^{*}, j \in \mathbb{N}_{n-t+1}^{*}\}. \tag{5}$$

For an A-A-bimodule X, we will denote by X the endofunctor $X \otimes_A -$ of A-mod. As $W_{11}^{n-1} \cong {}_AA_A$, the functor W_{11}^{n-1} is self adjoint. Considering the right adjoint of the functors given by tensoring with bimodules in (5), we obtain the following proposition.

Proposition 14. *For any $t \in \mathbb{N}_{n-1}^{*}$, we have the following:*

(*a*) *for any $j \in \mathbb{N}_{n-t+1}^{*}$, the pair $(\mathrm{W}_{n-t|j}^{t}, \mathrm{N}_{j|n-t-1}^{t})$ is an adjoint pair of functors;*

(*b*) *for any $j \in \mathbb{N}_{n-t+1}^{*}$, the pair $(\mathrm{S}_{1j}^{t}, \mathrm{W}_{j1}^{t})$ is an adjoint pair of functors;*

(*c*) *for any $i \in \mathbb{N}_{n-t}^{*} \setminus \{1\}$ and $j \in \mathbb{N}_{n-t+1}^{*}$, the pair $(\mathrm{S}_{ij}^{t}, \mathrm{N}_{j|i-1}^{t})$ is an adjoint pair of functors.*

Proof. We start by proving claim (a) any claim (c) is proved similarly. As $W_{n-t|j}^{t}$ is left projective, the right adjoint $\mathrm{W}_{n-t|j}^{t}$ is automatically exact. We need to prove that $\mathrm{Hom}_A(W_{n-t|j}^{t}, A) \cong N_{j|n-t-1}^{t}$.

As $W_{n-t|j}^{t}$ is indecomposable, as a bimodule, so is $\mathrm{Hom}_A(W_{n-t|j}^{t}, A)$. Now, recall that indecomposable A-A-bimodules of type N are uniquely determined by their left and right supports. Therefore it is enough to check that the bimodules $\mathrm{Hom}_A(W_{n-t|j}^{t}, A)$ and $N_{j|n-t-1}^{t}$ have the same left support and the same right support.

The left A-action on $X := \mathrm{Hom}_A(W^t_{n-t|j}, A)$ comes from the right A-action on $W^t_{n-t|j}$. Because of our notation, the minimum s for which e_i does not annihilate $W^t_{n-t|j}$ on the right is $s = j$. Furthermore, as a left A-module, $W^t_{n-t|j}$ has $t+1$ direct summands. This implies that

$$\mathrm{Lsupp}(X) = \{j, j+1, \dots, j+t\} = \mathrm{Lsupp}(N^t_{j|n-t-1}).$$

The direct summand Ae_{n-t+s}, for $s = 0, 1, \dots, t$, of $W^t_{n-t|j}$ maps only to the direct summands Ae_{n-t+s} and $Ae_{n-t+s-1}$ of A. This implies that

$$\mathrm{Rsupp}(X) = \{n-t-1, n-t, \dots, n\} = \mathrm{Rsupp}(N^t_{j|n-t-1}).$$

The claim follows.

Claim (b) is also proved analogously to claim (a) with one additional remark: we determine the right adjoint using its support. For claim (b), the support argument implies that the right adjoint might be either of type W or of type M. However, exactness of this right adjoint determines its type uniquely as W. The rest is completely analogous to claim (a). □

4. A minimal generating set

4.1. **The main result of the section.** The main result of this subsection is the following:

Theorem 15. *Assume $n \geq 3$.*

(*i*) *The category A-mod-A coincides with the minimal subcategory of A-mod-A containing*

$$\{W^{n-1}_{11} \cong {}_AA_A,\ W^{n-2}_{21},\ N^{n-2}_{11},\ S^{n-2}_{12}\} \tag{6}$$

and closed under isomorphisms, tensor products and taking direct sums and direct summands.

(*ii*) *The set* (6) *is minimal in the sense that no proper subset of* (6) *has the property described in* (i).

In the case $n = 1$, we have just one indecomposable bimodule. In the case $n = 2$, we have $W^1_{11} \cong {}_AA_A$, its dual M^1_{11}, and nine indecomposable $\Bbbk$-split bimodules. In particular, three of the bimodules in the list (6) do not exist. A direct adjustment of (6), however, gives, for $n = 2$, the following minimal generating system:

$$\{W^1_{11}, M^0_{11}, L_{21}, Ae_1 \otimes_k e_2A\}$$

which is easy to check by a straightforward computation (which uses the isomorphism $M^0_{11} \otimes_A M^0_{11} \cong L_{12}$). Another, more obvious, minimal generating system, for $n = 2$, is the following:

$$\{W^1_{11}, M^0_{11}, L_{11}, L_{22}, Ae_1 \otimes_k e_2A\}.$$

4.2. **Comments on the proof of Lemma 9.** Note that some of the bimodules listed in (6) appeared already in Lemma 9. Namely, we have $A\tilde{e}^{\varphi} \cong W^{n-2}_{21}$ and ${}^{\varphi}\tilde{e}A \cong N^{n-2}_{11}$ in A-mod-A. We can similarly describe the A-A-bimodule S^{n-2}_{12} as being isomorphic to $A\tilde{e'}^{\psi}$, where $\tilde{e'} = e_1 + e_2 + \cdots + e_{n-1}$ and $\psi := \imath \circ \varphi \circ \imath$ (see Subsection 3.5 for the definition of $\imath$). Note that $\tilde{e'} = \psi(1)$ and that ψ sends

$$e_i \mapsto \begin{cases} e_{i-1}, & i-1 \geq 1; \\ 0, & \text{else}; \end{cases} \qquad \alpha_i \mapsto \begin{cases} \alpha_{i-1}, & i-1 \geq 1; \\ 0, & \text{else}. \end{cases}$$

Making a parallel with the proof of Lemma 9, we also have the A-A-bimodule ${}^{\psi}\tilde{e'}A$. This bimodule is isomorphic to W_{21}^{n-2}. In fact, the anti-involution $\imath$ on A-mod-A fixes both W_{11}^{n-1} and W_{21}^{n-2} but swaps N_{11}^{n-2} with S_{12}^{n-2}. Similarly to the proof of Lemma 9, we have

$$ {}^{\psi}\tilde{e'}A \otimes_A A\tilde{e'}^{\psi} \cong {}^{\psi}\tilde{e'}A\tilde{e'}^{\psi} \cong {}_A(A/I')_A,$$

where $I' = Ae_1A = Ae_1$. This means that, if $X \in$ mod-A is annihilated by I', then, tensoring X first with ${}^{\psi}\tilde{e'}A$ and then with $A\tilde{e'}^{\psi}$ (both from the right), gives X back. If X has the structure of an indecomposable A-A-bimodule, then tensoring with ${}^{\psi}\tilde{e'}A$ just twists the right action of A on X by ψ and hence does not change the type (W, M, S or N) of X. Moreover, we have

$$\mathrm{Lsupp}(X \otimes_A {}^{\psi}\tilde{e'}A) = \mathrm{Lsupp}(X) \quad \text{and} \quad \mathrm{Rsupp}(X \otimes_A {}^{\psi}\tilde{e'}A) = (\mathrm{Rsupp}(X))[-1]$$

(here we use the notation $U[-1]$ from Subsection 3.6). This fact can be used to prove the "other side" version of Lemma 9, which consequently contributes to the proof of Theorem 7.

4.3. **Auxiliary lemmata.** By construction, the action graph Γ_X of an A-A-bimodule X is a subgraph of the graph (3). In what follows, for a fixed A-A-bimodule X, we will describe the outcome of tensoring of X with W_{21}^{n-2}, N_{11}^{n-2}, and S_{12}^{n-2}, both from the left and from the right, using combinatorial manipulations with the graph Γ_X, considering the latter as a subgraph of (3).

Lemma 16. *Let X be an indecomposable A-A-bimodule.*

(*i*) *The action graph of the A-A-bimodule $W_{21}^{n-2} \otimes_A X$ is obtained from Γ_X by moving the latter vertically one step down and then cutting off all vertices and arrows which fall outside the graph in figure* (3).

(*ii*) *The action graph of the A-A-bimodule $X \otimes_A W_{21}^{n-2}$ is obtained from Γ_X by moving the latter horizontally one step to the left and then cutting off all vertices and arrows which fall outside the graph in figure* (3).

Proof. From the proof of Lemma 9, it follows that claim (i) is true as soon as $IX = 0$ (recall that $I = Ae_nA = e_nA$). Note that in this case, no "cutting off" is necessary.

In general, we have $W_{21}^{n-2} \cong A\tilde{e}^{\varphi}$ and $IX = e_nX$ is an A-A-subbimodule of X. If $e_nX \neq 0$, then we have $A\tilde{e}^{\varphi} \otimes_A e_nX = 0$ and hence $A\tilde{e}^{\varphi} \otimes_A X \cong A\tilde{e}^{\varphi} \otimes_A (X/e_nX)$. As $I(X/e_nX) = 0$, we can apply the argument from the previous paragraph. Factoring e_nX out corresponds precisely to "cutting off" those vertices and edges which fall outside the graph in figure (3) after the move. This completes the proof of claim (i).

From Subsection 4.2 it follows that claim (ii) is true as soon as $XI' = 0$. Note that in this case, no "cutting off" is necessary.

In general, we have $W_{21}^{n-2} \cong {}^{\psi}\tilde{e'}A$ and $XI' = Xe_1$ is an A-A-subbimodule of X. If $Xe_1 \neq 0$, then we have $Xe_1 \otimes_A {}^{\psi}\tilde{e'}A = 0$ and hence

$$X \otimes_A {}^{\psi}\tilde{e'}A \cong (X/Xe_1) \otimes_A {}^{\psi}\tilde{e'}A.$$

As $(X/Xe_1)I' = 0$, we can apply the argument from the previous paragraph. Factoring Xe_1 out corresponds precisely to "cutting off" those vertices and edges which fall outside the graph in figure (3) after the move. This completes the proof of claim (ii). □

Let X be an indecomposable A-A-bimodule. A full subgraph Γ of Γ_X will be called *thick* provided that, for any arrow $x \to y$ in Γ_X, the condition $x \in \Gamma$ implies $y \in \Gamma$.

Lemma 17. *Let X be an indecomposable and not $\Bbbk$-split A-A-bimodule.*

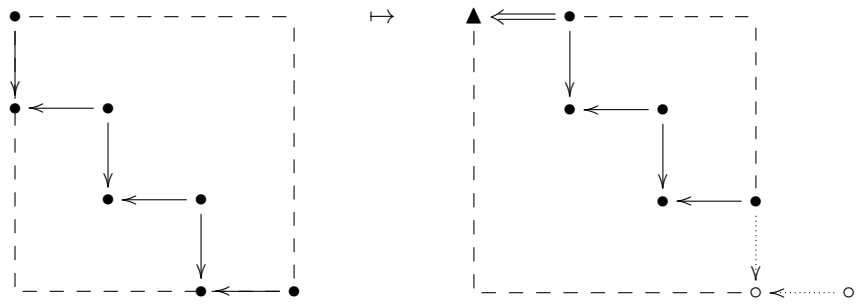
(*i*) *The action graph of the A-A-bimodule $N_{11}^{n-2} \otimes_A X$ is obtained from Γ_X by moving the latter vertically one step up and then cutting off all vertices and edges which fall outside the graph in figure* (3).

(*ii*) *If $\dim_{\Bbbk} Xe_1 = 2$, then the action graph of the A-A-bimodule $X \otimes_A N_{11}^{n-2}$ is obtained from Γ_X in the following three steps:*

- *first we move Γ_X one step to the right;*
- *then we cut off the thick subgraph generated by all vertices which fall outside the graph in figure* (3)*, we denote the resulting graph Γ;*
- *finally, we add to Γ one new vertex and one new arrow as follows: let v be the north-west corner of Γ, then we add to Γ the immediate west neighbor w of v and the arrow connecting v to w.*

(*iii*) *If $\dim_{\Bbbk} Xe_1 \neq 2$, then the action graph of the A-A-bimodule $X \otimes_A N_{11}^{n-2}$ is obtained from Γ_X by moving the latter horizontally one step to the right and then cutting off the thick subgraph generated by all vertices which fall outside the graph in figure* (3).

A good example to illustrate the procedure described in Lemma 17(ii) is the obvious isomorphism $W_{11}^{n-1} \otimes_A N_{11}^{n-2} \cong N_{11}^{n-2}$ based on the fact that $W_{11}^{n-1} \cong {}_AA_A$. In particular, this example shows that the second step of the procedure described in Lemma 17(ii) can lead to elimination of some vertices which do not fall outside the graph in figure (3). For $n = 4$, the transformation in this example can be depicted explicitly as follows:

(7) 

Here dashed lines indicate the boundaries of the graph in figure (3), dotted arrows and white vertices are the ones which are deleted during the second step of the procedure described in Lemma 17 (ii). Finally, the triangle vertex and the double arrow are the ones which are added during the last step of the procedure described in Lemma 17 (ii).

Proof. Assume that $e_1X = 0$ and let Y be an A-A-bimodule such that Γ_Y is obtained by moving Γ_X one step up (note that this is well-defined as $e_1X = 0$). Then Y has the same type as X and

$$\mathrm{Lsupp}(Y) = \mathrm{Lsupp}(X)[-1] \quad \text{and} \quad \mathrm{Rsupp}(Y) = \mathrm{Rsupp}(X).$$

From Lemma 16 (i) it follows that $X \cong W_{21}^{n-2} \otimes_A Y$. This, together with the discussion in Subsection 4.2, implies $N_{11}^{n-2} \otimes_A X \cong Y$ in A-mod-A. Claim (i) in the case $e_1X = 0$ follows.

Assume now that $e_1X \neq 0$. Consider the following short exact sequence of A-A-bimodules:

$$0 \to \tilde{e}X \to X \to \mathrm{Coker} \to 0, \tag{8}$$

where $\tilde{e}X \to X$ is the natural inclusion. In this sequence we have that ${}_A\mathrm{Coker}$ is semisimple, moreover, $e_i\mathrm{Coker} = 0$, for all $i > 1$. Recall that $N_{11}^{n-2} \cong {}^{\varphi}\tilde{e}A$. Then ${}^{\varphi}\tilde{e}A \otimes_A \mathrm{Coker} = 0$. The functor $N_{11}^{n-2} \otimes_A \_$ is exact as N_{11}^{n-2} is right projective (or due to Proposition 14 (a)). Therefore, applying $N_{11}^{n-2} \otimes_A \_$ to (8), gives $N_{11}^{n-2} \otimes_A \tilde{e}X \cong N_{11}^{n-2} \otimes_A X$. We can now apply the previous paragraph to $N_{11}^{n-2} \otimes_A \tilde{e}X$ and thus complete the proof of claim (i).

We have $\mathrm{Lsupp}(X \otimes_A {}^{\varphi}\tilde{e}A) \subset \mathrm{Lsupp}(X)$ by Lemma 4. Similarly to the proof of Lemma 9, if $XI = 0$, then, tensoring X (from the right) with ${}^{\varphi}\tilde{e}A$ and then with $A\tilde{e}^{\varphi}$, gives X back. Consequently, in the case $XI = 0$ we have

$$\mathrm{Lsupp}(X \otimes_A {}^{\varphi}\tilde{e}A) = \mathrm{Lsupp}(X).$$

Consider the short exact sequence

$$0 \to {}^{\varphi}\tilde{e}Ae_1 \to {}^{\varphi}\tilde{e}A \to \mathrm{Coker}' \to 0 \tag{9}$$

of A-A-bimodules where ${}^{\varphi}\tilde{e}Ae_1 \to {}^{\varphi}\tilde{e}A$ is the natural inclusion. Then we have $\mathrm{Coker}' \cong W_{12}^{n-2}$. Using Lemma 16 (ii) or the fact that $e_1A\tilde{e}^{\varphi} = 0$, we have

$$\mathrm{Coker}' \otimes_A A\tilde{e}^{\varphi} \cong {}^{\varphi}\tilde{\mathrm{e}}\mathrm{A} \otimes_A A\tilde{e}^{\varphi}.$$

Therefore, tensoring X (from the right) with Coker' and then with $A\tilde{e}^{\varphi}$ also gives X back. Applying the functor $X \otimes_A \_$ to (9), we thus obtain

$$\mathrm{Rsupp}(X \otimes_A {}^{\varphi}\tilde{e}A) \subset \mathrm{Rsupp}(X \otimes_A {}^{\varphi}\tilde{e}Ae_1) \cup \mathrm{Rsupp}(X \otimes_A \mathrm{Coker}').$$

As tensor functors are right exact, we also have

$$\mathrm{Rsupp}(X \otimes_A \mathrm{Coker}') \subset \mathrm{Rsupp}(X \otimes_A {}^{\varphi}\tilde{e}A).$$

Just like in the proof of Lemma 9, tensoring with W_{12}^{n-2} twists the right A-action on X by φ and hence does not change the type (W, M, S, or N) of X. In particular, we have

$$\mathrm{Lsupp}(X \otimes_A \mathrm{Coker}') = \mathrm{Lsupp}(X) \quad \text{and} \quad \mathrm{Rsupp}(X \otimes_A \mathrm{Coker}') = \mathrm{Rsupp}(X)[1].$$

Therefore, considering the action graph of $X \otimes_A \mathrm{Coker}'$ corresponds to the first step of claim (ii).

Note that $\mathrm{Rsupp}(X \otimes_A {}^{\varphi}\tilde{e}Ae_1) \subset \{1\}$. Hence, the above arguments imply

$$(\mathrm{Rsupp}(X))[1] \subset \mathrm{Rsupp}(X \otimes_A {}^{\varphi}\tilde{e}A) \subset \{1\} \cup (\mathrm{Rsupp}(X))[1].$$

Now, if $\dim_{\Bbbk} Xe_1 = 2$, then $\mathbf{B}(X)$ contains the vertices $\mathtt{j} \mid \mathtt{1}$ and $\mathtt{j{+}1} \mid \mathtt{1}$, for some j. Moreover, we obtain that the A-A-bimodule $X \otimes_A {}^{\varphi}\tilde{e}Ae_1$ is of dimension one with basis $\mathtt{j} \mid \mathtt{1} \otimes \alpha_1$ (here it is important that X is not $\Bbbk$-split). The image of this element in $X \otimes_A {}^{\varphi}\tilde{e}A$ is also non-zero. This implies that

$$\mathrm{Rsupp}(X \otimes_A {}^{\varphi}\tilde{e}A) = \{1\} \cup (\mathrm{Rsupp}(X))[1].$$

Therefore the sequence

$$0 \to X \otimes_A {}^{\varphi}\tilde{e}Ae_1 \to X \otimes_A {}^{\varphi}\tilde{e}A \to X \otimes_A \mathrm{Coker}' \to 0$$

is exact, which completes the proof of claim (ii) in the case $XI = 0$.

If $\dim_{\Bbbk} Xe_1 \neq 2$, then we have $X \otimes_A {}^{\varphi}\tilde{e}Ae_1 = 0$ as ${}_A({}^{\varphi}\tilde{e}Ae_1)$ is simple and X is not $\Bbbk$-split. This implies that $X \otimes_A {}^{\varphi}\tilde{e}A \cong X \otimes_A \mathrm{Coker}'$, which establishes claim (iii) in the case $XI = 0$.

If $XI \neq 0$, then we have $XI \otimes_A {}^{\varphi}\tilde{e}A = 0$ as the left action of e_n annihilates ${}^{\varphi}\tilde{e}A$. Hence we obtain $X \otimes_A {}^{\varphi}\tilde{e}A \cong (X/XI) \otimes_A {}^{\varphi}\tilde{e}A$ so that we can reduce our consideration to the previous case $XI = 0$. This proves claims (ii) and (iii) in full generality and we are done. □

Lemma 18. *Let X be an indecomposable and not $\Bbbk$-split A-A-bimodule.*

(*i*) *If $\dim e_n X = 2$, then the action graph of the A-A-bimodule $S_{12}^{n-2} \otimes_A X$ is obtained from Γ_X in the following three steps:*

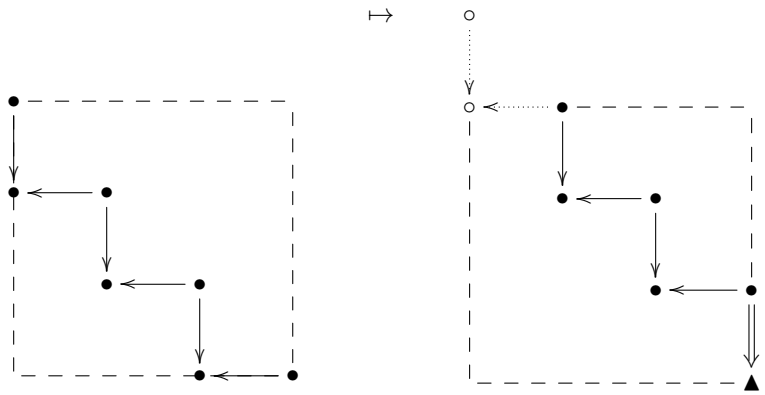
- *first we move Γ_X vertically one step up;*
- *then we cut off the thick subgraph generated by all vertices which fall outside the graph in figure* (3)*, we denote the resulting graph Γ;*
- *finally, we add to Γ one new vertex and one new arrow as follows: let v be the south-east corner of Γ, then we add to Γ the immediate south neighbor w of v and the arrow connecting v to w.*

(*ii*) *If $\dim e_n X \neq 2$, then the action graph of the A-A-bimodule $S_{12}^{n-2} \otimes_A X$ is obtained from Γ_X by moving the latter vertically one step up and then cutting off the thick subgraph generated by all vertices which fall outside the graph in figure* (3)*.*

(*iii*) *The action graph of the A-A-bimodule $X \otimes_A S_{12}^{n-2}$ is obtained from Γ_X by moving the latter one step to the right and then cutting off all vertices and edges which fall outside the graph in figure* (3)*.*

Proof. Observing that we have an isomorphism $S_{12}^{n-2} \cong A\tilde{e'}^{\psi}$ of A-A-bimodules, the proof is similar to that of Lemma 17 and is left to the reader. □

Again, a good example to illustrate the procedure described in Lemma 18 (i) is the obvious isomorphism $S_{12}^{n-2} \otimes_A W_{11}^{n-1} \cong S_{12}^{n-2}$ based on the fact that $W_{11}^{n-1} \cong {}_A A_A$. In particular, this example shows that the second step of the procedure described in Lemma 18 (i) can lead to elimination of some vertices which do not fall outside the graph in figure (3). For $n = 4$, the transformation in this example can be depicted (using the same conventions as in (7)) explicitly as follows:



4.4. **Proof of Theorem 15.** Let $\mathcal{X}$ denote the minimal subcategory of A-mod-A containing (6) and closed under isomorphisms, tensor products and taking direct sums and direct summands. To prove Theorem 15 (i), we need to show that all indecomposable A-A-bimodules belong to $\mathcal{X}$.

Clearly, $W_{11}^{n-1} \in \mathcal{X}$ as it appears in (6). Further, $M_{11}^{n-2} \in \mathcal{X}$ as, for example,

$$S_{12}^{n-2} \otimes_A (W_{21}^{n-2} \otimes_A N_{11}^{n-2}) \cong S_{12}^{n-2} \otimes_A N_{21}^{n-2} \cong M_{11}^{n-2},$$

where we used Lemma 17 (ii) for the first isomorphism and Lemma 18 (i) for the second one.

Starting from W_{11}^{n-1}, M_{11}^{n-2}, S_{12}^{n-2} and N_{11}^{n-2}, and applying the manipulations described in Lemma 16 (i)-(ii), Lemma 17 (i) and Lemma 18 (iii), it is clear that we can obtain action graphs of all indecomposable A-A-bimodules which are not isomorphic to $Ae_i \otimes_{\Bbbk} e_{j+1}A$, where $i, j \in \mathbb{N}_n^*$. Hence all such bimodules must belong to $\mathcal{X}$, in particular, all indecomposable string A-A-bimodules of dimension two are in $\mathcal{X}$.

Finally, tensoring indecomposable string A-A-bimodules of dimension two with each other, we can get all bimodules of the form $Ae_i \otimes_{\Bbbk} e_{j+1}A$, where $i, j \in \mathbb{N}_n^*$. Claim (i) of Theorem 15 follows.

Now, let us prove claim (ii). Let T be a proper subset of (6) and $\mathcal{X}_T$ the minimal subcategory of A-mod-A containing T and closed under isomorphisms, tensor products and taking direct sums and direct summands. We have to show that $\mathcal{X}_T$ is a proper subcategory of A-mod-A. If $W_{11}^{n-1} \notin T$, then the latter claim follows directly from Theorem 5 (ii).

If $W_{21}^{n-2} \notin T$, then, from Lemmata 17 and 18 it follows that the only possible manipulations with actions graphs are to move them up or to the right. As we can only start with action graphs of bimodules from T, it follows that, in particular, $W_{21}^{n-2} \notin \mathcal{X}_T$. A similar argument also shows that, necessarily, either $S_{12}^{n-2} \in T$ or $N_{11}^{n-2} \in T$.

Assume that T is (6) without S_{12}^{n-2}. Then, from Lemmata 16 and 17, it follows that all bimodules in $\mathcal{X}_T$ are either $\Bbbk$-split or of type W or N. Assume that T is (6) without N_{11}^{n-2}. Then, from Lemmata 16 and 18, it follows that all bimodules in $\mathcal{X}_T$ are either $\Bbbk$-split or of type W or S. Claim (ii) follows and the proof of Theorem 15 is complete.

5. Simple transitive 2-representations of projective A-A-bimodules

5.1. **Finitary 2-categories and their 2-representations.** In this section we switch from the concrete algebras A_n considered above to general finite dimensional algebras A.

For a finite dimensional $\Bbbk$-algebra A, consider the 2-category $\mathscr{C}_A$ of projective endofunctors of A-mod, see [MM1, Subsection 7.3] (we note that this 2-category depends on the choice of a small category equivalent to A-mod). We assume that A is basic and connected and let $\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_k = 1$ be a primitive decomposition of the identity in A. The 2-category $\mathscr{C}_A$ has one object $\mathtt{i}$. A complete list of representatives of isomorphism classes of indecomposable A-A-bimodules which contribute to 1-morphisms in $\mathscr{C}_A$ consists of the regular bimodule ${}_AA_A$, which corresponds to the identity 1-morphism $\mathbb{1}_{\mathtt{i}}$, and indecomposable projective bimodules $A\varepsilon_i \otimes \varepsilon_j A$,

where $i, j = 1, 2, \dots, k$, which correspond to indecomposable 1-morphisms respectively denoted by F_{ij}. Finally, 2-morphisms in $\mathscr{C}_A$ are given by homomorphisms of A-A-bimodules.

A *finitary 2-representation* of $\mathscr{C}_A$ is a functorial action, denoted $\mathbf{M}$, on a category $\mathbf{M}(\mathtt{i})$ equivalent to B-proj of projective modules over some finite dimensional $\Bbbk$-algebra B. All such 2-representations form a 2-category, denoted $\mathscr{C}_A$-afmod, where 1-morphisms are 2-natural transformations and 2-morphisms are modifications, see [MM3] for details.

A finitary 2-representation $\mathbf{M}$ is called *transitive* if $\mathbf{M}(\mathtt{i})$ has no proper $\mathscr{C}_A$-invariant, idempotent split and isomorphism closed additive subcategories. A transitive 2-representation $\mathbf{M}$ is called *simple* if $\mathbf{M}(\mathtt{i})$ has no proper $\mathscr{C}_A$-invariant ideals, see [MM5, MM6] for details.

Classical examples of simple transitive 2-representations are so-called *cell 2-representations* as defined in [MM1, MM2]. The 2-category $\mathscr{C}_A$ has, up to equivalence, two cell 2-representations:

- The cell 2-representation $\mathbf{C}_{\{\mathbb{1}_\mathtt{i}\}}$ which is given as the quotient of the left regular action of $\mathscr{C}_A$ on $\mathscr{C}_A(\mathtt{i}, \mathtt{i})$ by the unique maximal $\mathscr{C}_A$-invariant left ideal.
- The cell 2-representation $\mathbf{C}_{\{\mathrm{F}_{i1}\}}$ which is given (up to equivalence) by the defining action of $\mathscr{C}_A$ on A-proj.

5.2. **The main result of the section.** The main result of this section is the following:

Theorem 19. *Assume that A has a non-zero projective injective module and is directed in the sense that $\varepsilon_i A \varepsilon_j = 0$ whenever $i < j$ and $\varepsilon_i A \varepsilon_i = \Bbbk \varepsilon_i$, for all i. Then every simple transitive 2-representation of $\mathscr{C}_A$ is equivalent to a cell 2-representation.*

The algebra A_n from Subsection 2.1 obviously satisfies both assumption of Theorem 19. Therefore Theorem 19 provides a classification of simple transitive 2-representation of $\mathscr{C}_{A_n}$. In fact, any quotient of the path algebra of the quiver (1) satisfies both assumption of Theorem 19. There are of course many other algebras which satisfy these assumptions, for example incidence algebras of finite posets having the minimum and the maximum element (for example, the Boolean of a finite set) and many others. In the cases $A = A_2$ and $A = A_3$, Theorem 19 is proved in [MZ].

The rest of this section is devoted to the proof of Theorem 19.

5.3. **Notational preparation.** We let $\mathbf{M}$ be a simple transitive 2-representation of $\mathscr{C}_A$ and denote by B a basic $\Bbbk$-algebra such that $\mathbf{M}(\mathtt{i})$ is equivalent to B-proj. Let $\epsilon_1 + \epsilon_2 + \dots + \epsilon_r = 1$ be a primitive decomposition of the identity in B. For $i, j = 1, 2, \dots, r$, we denote by G_{ij} the endofunctor of B-mod given by tensoring with the indecomposable projective B-B-bimodule $B\epsilon_i \otimes \epsilon_j B$. We note that $r \neq k$, in general.

Without loss of generality we may assume that $\mathbf{M}$ is faithful. Indeed, the 2-category $\mathscr{C}_A$ is simple, see [MMZ, Subsection 3.2]. Therefore, if $\mathbf{M}$ is not faithful, then $\mathbf{M}(\mathrm{F}_{ij}) = 0$, for all i, j. The quotient of $\mathscr{C}_A$ by the 2-ideal generated by all F_{ij} satisfies the assumptions of [MM5, Theorem 18] and hence $\mathbf{M}$ is equivalent to a cell 2-representation by [MM5, Theorem 18].

So, from now on, $\mathbf{M}$ is faithful, in particular, each $\mathbf{M}(\mathrm{F}_{ij})$ is non-zero. As A has a non-zero projective-injective module, by [MZ, Section 3], each $\mathbf{M}(\mathrm{F}_{ij})$ is a projective endofunctor of B-mod, that is, is isomorphic to a direct sum of some G_{st}, $s,t\in\{1,2,\dots,r\}$, possibly even with multiplicities. For $i,j=1,2,\dots,k$, we denote by

- X_{ij} the set of all $s\in\{1,2,\dots,r\}$ such that G_{st} is isomorphic to a direct summand of $\mathbf{M}(\mathrm{F}_{ij})$, for some $t\in\{1,2,\dots,r\}$;
- Y_{ij} the set of all $t\in\{1,2,\dots,r\}$ such that G_{st} is isomorphic to a direct summand of $\mathbf{M}(\mathrm{F}_{ij})$, for some $s\in\{1,2,\dots,r\}$.

As each $\mathbf{M}(\mathrm{F}_{ij})$ is non-zero, each X_{ij} and each Y_{ij} is not empty.

Recall that A is assumed to have a non-zero projective-injective module. Then there exist $i_0,j_0\in\{1,2,\dots,k\}$ such that $A\varepsilon_{i_0}\cong\mathrm{Hom}_{\Bbbk}(\varepsilon_{j_0}A,\Bbbk)$. In this case, for every $q\in\{1,2,\dots,k\}$, the pair

$$(\mathrm{F}_{qj_0},\mathrm{F}_{i_0q}) \tag{10}$$

is an adjoint pair of 1-morphisms, see [MZ, Lemma 5].

For $s=1,2,\dots,r$, we denote by P_s the indecomposable projective B-module $B\epsilon_s$ and by L_s the simple top of P_s. Whenever it does not lead to any confusion, we will use *action notation* and simply write e.g. $\mathrm{F}_{ij}\,M$, for $M\in\mathbf{M}(\mathtt{i})$ and $M\in B$-mod, instead of $\mathbf{M}(\mathrm{F}_{ij})(M)$.

5.4. Analysis of the sets X_{ij} and Y_{ij}.

Lemma 20. *For $i=1,2,\dots,k$, we have $X_{ij_1}=X_{ij_2}$, for all $j_1,j_2\in\{1,2,\dots,k\}$. Similarly, for $j=1,2,\dots,k$, we have $Y_{i_1j}=Y_{i_2j}$, for all $i_1,i_2\in\{1,2,\dots,k\}$.*

Proof. We prove the first claim, the proof of the second one is similar. We have

$$\left(A\varepsilon_i\otimes\varepsilon_{j_1}A\right)\otimes_A\left(A\varepsilon_{j_1}\otimes\varepsilon_{j_2}A\right)\cong A\varepsilon_i\otimes\varepsilon_{j_2}A^{\oplus\dim(\varepsilon_{j_1}A\varepsilon_{j_1})}.$$

Note that $\dim(\varepsilon_{j_1}A\varepsilon_{j_1})>0$. Therefore F_{ij_2} is isomorphic to a direct summand of $\mathrm{F}_{ij_1}\circ\mathrm{F}_{j_1j_2}$. This implies $X_{ij_2}\subset X_{ij_1}$ (as multiplication on the right cannot create new indexing idempotents on the left). By symmetry, we also have $X_{ij_1}\subset X_{ij_2}$. The claim follows. □

After Lemma 20, for $i=1,2,\dots,k$, we may denote by X_i the common value of all X_{ij}, where $j\in\{1,2,\dots,k\}$. Similarly, for $j=1,2,\dots,k$, we may denote by Y_j the common value of all Y_{ij}, where $i\in\{1,2,\dots,k\}$.

Lemma 21. *We have $X_1\cup X_2\cup\dots\cup X_k=\{1,2,\dots,r\}$.*

Proof. The set $X_1\cup X_2\cup\dots\cup X_k$ indexes those projectives that can be obtained using the action of $\mathscr{C}_A$. Therefore the claim follows immediately from transitivity of $\mathbf{M}$. □

Lemma 22. *For every $q=1,2,\dots,k$, we have $X_q=Y_q$.*

Proof. Consider the pair $(\mathrm{F}_{qj_0},\mathrm{F}_{i_0q})$ of adjoint 1-morphisms given by (10). By adjunction, for $s\in\{1,2,\dots,r\}$, we have

$$\mathrm{Hom}_B(\mathrm{F}_{qj_0}\,B,L_s)\cong\mathrm{Hom}_B(B,\mathrm{F}_{i_0q}\,L_s),$$

in particular, the left hand side is non-zero if and only if the right hand side is non-zero. At the same time, the left hand side is non-zero if and only if P_s appears

in $\mathrm{F}_{qj_0} B$, that is, $s \in X_q$; while the right hand side is non-zero if and only if $\mathrm{F}_{i_0 q} L_s \neq 0$, that is, $s \in Y_q$. The claim follows. □

As an immediate consequence of Lemmata 21 and 22, we have:

Corollary 23. *We have* $Y_1 \cup Y_2 \cup \cdots \cup Y_k = \{1, 2, \dots, r\}$.

Note that, until this point, we never used that A is directed. It will, however, be crucial for the following two lemmata.

Lemma 24. *For* $i_1 \neq i_2 \in \{1, 2, \dots, k\}$, *we have* $X_{i_1} \cap X_{i_2} = \varnothing$.

Proof. Let $s \in X_{i_1} \cap X_{i_2}$. Then $s \in Y_{i_1} \cap Y_{i_2}$ by Lemma 22. Without loss of generality we may assume $i_1 < i_2$. Then we have that $\mathbf{M}(\mathrm{F}_{i_2 i_1}) \circ \mathbf{M}(\mathrm{F}_{i_2 i_1})$ contains a direct summand isomorphic to $\mathrm{G}_{xs} \circ \mathrm{G}_{sy}$, for some $x, y \in \{1, 2, \dots, r\}$. Note that $\mathrm{G}_{xs} \circ \mathrm{G}_{sy} \neq 0$ as $\epsilon_s B \epsilon_s \neq 0$. Consequently, we obtain

$$\mathbf{M}(\mathrm{F}_{i_2 i_1} \circ \mathrm{F}_{i_2 i_1}) \cong \mathbf{M}(\mathrm{F}_{i_2 i_1}) \circ \mathbf{M}(\mathrm{F}_{i_2 i_1}) \neq 0.$$

At the same time, we have $\mathrm{F}_{i_2 i_1} \circ \mathrm{F}_{i_2 i_1} = 0$ since $\varepsilon_{i_1} A \varepsilon_{i_2} = 0$ by our assumption that A is directed, a contradiction. The claim follows. □

Lemma 25. *For* $i = 1, 2, \dots, k$, *we have* $|X_i| = 1$.

Proof. Let $X_i = \{s_1, s_2, \dots, s_m\}$. For a fixed s_q, the additive closure of all $\mathrm{F}_{xi} L_{s_q}$, where $x \in \{1, 2, \dots, n\}$, is a non-zero $\mathscr{C}_A$-invariant subcategory of B-proj and hence must coincide with B-proj by transitivity of $\mathbf{M}$. This implies that all $\mathrm{G}_{s_y s_q}$, where $y \in \{1, 2, \dots, m\}$, do appear as direct summands of $\mathbf{M}(\mathrm{F}_{ii})$.

As A is directed, we have $\varepsilon_i A \varepsilon_i = \Bbbk \varepsilon_i$ and hence $\mathrm{F}_{ii} \circ \mathrm{F}_{ii} \cong \mathrm{F}_{ii}$. Therefore

$$\mathbf{M}(\mathrm{F}_{ii}) \circ \mathbf{M}(\mathrm{F}_{ii}) \cong \mathbf{M}(\mathrm{F}_{ii}) \tag{11}$$

as well. Let h denote the multiplicity of $\mathrm{G}_{s_1 s_1}$ in $\mathbf{M}(\mathrm{F}_{ii})$. From the previous paragraph we know that $h > 0$. Clearly, $\mathrm{G}_{s_1 s_1}$ appears as a direct summand of $\mathrm{G}_{s_1 s_1} \circ \mathrm{G}_{s_1 s_1}$. Therefore (11) implies $h^2 \leq h$, that is $h = 1$.

If $m > 1$, then $\mathrm{G}_{s_1 s_1}$ appears as a direct summand of $\mathrm{G}_{s_1 s_2} \circ \mathrm{G}_{s_2 s_1}$. Therefore in this case (11) fails and we are done. □

As a consequence of Lemmata 21 and 25, we have $k = r$. Without loss of generality we may now choose ϵ_i's such that each F_{ij} acts via G_{ij}.

Corollary 26. *For* $i, j = 1, 2, \dots, k$, *we have* $\dim(\varepsilon_i A \varepsilon_j) = \dim(\epsilon_i B \epsilon_j)$.

Proof. As each F_{st} acts via G_{st}, the claim follows by comparing

$$\mathrm{F}_{si} \circ \mathrm{F}_{jt} \cong \mathrm{F}_{st}^{\oplus \dim(\varepsilon_i A \varepsilon_j)} \quad \text{and} \quad \mathrm{G}_{si} \circ \mathrm{G}_{jt} \cong \mathrm{G}_{st}^{\oplus \dim(\epsilon_i B \epsilon_j)}.$$

□

5.5. **Completing the proof of Theorem 19.** After the preparation in Subsection 5.4, the proof of Theorem 19 is similar to the arguments in [MZ, Section 6] or [MaMa, Subsection 4.9]. Consider the principal 2-representation $\mathbf{P}_{\mathtt{i}} := \mathscr{C}_A(\mathtt{i}, \_)$ of $\mathscr{C}_A$, that is the left regular action of $\mathscr{C}_A$ on $\mathscr{C}_A(\mathtt{i}, \mathtt{i})$. The additive closure of all F_{i1}, where $i = 1, 2, \dots, k$, in $\mathscr{C}_A(\mathtt{i}, \mathtt{i})$ is $\mathscr{C}_A$-invariant and gives a 2-representation which we denote by $\mathbf{N}$. The latter has a unique $\mathscr{C}_A$-invariant left ideal $\mathbf{I}$ and the corresponding quotient is exactly the cell 2-representation, see [MM2, Subsection 6.5].

Mapping $\mathbb{1}_{\mathtt{i}}$ to $L_1 \in B$-mod, gives rise to a 2-natural transformation Φ from $\mathbf{N}$ to $\mathbf{M}$ which, because of the results in Subsection 5.4, sends indecomposable objects to indecomposable objects. Since, by Corollary 26, the Cartan matrices of A and B coincide, it follows that Φ must annihilate $\mathbf{I}$ and hence induce an equivalence between the cell 2-representation $\mathbf{N}/\mathbf{I}$ and $\mathbf{M}$. This completes the proof.

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V. M.: Department of Mathematics, Uppsala University, Box. 480, SE-75106, Uppsala, SWEDEN, email: mazor@math.uu.se

X. Z.: Department of Mathematics, Uppsala University, Box. 480, SE-75106, Uppsala, SWEDEN, email: xiaoting.zhang@math.uu.se