“Points”, “slopes” and “derivatives”: Substantiations of narratives about tangent line in university mathematics students’ discourses

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This paper reports from a study on first year university mathematics students’ meaning making of tangent line, especially in their transition between mathematical contexts: algebraic, geometrical and analytical. The analysis draws on the commognitive approach (Sfard, 2008) in order to identify characteristics of responses to a questionnaire in which 182 students were asked to explain in simple words the tangent line, describe its properties, provide its definition and identify if a drawn line is a tangent of a given curve. Findings suggest that students engage with analytical, geometrical and algebraic discourses in their substantiations about tangents, sometimes by engaging with more than one discourse in the same response or/and across different responses in the same script.

Keywords: Tangent line, mathematical discourse, derivative, narratives, routines.

Introduction

Research reports students’ difficulties with their meaning making of the tangent line to a function graph. These difficulties have been attributed to students’ encounter with the tangent line in different mathematical contexts (e.g. Euclidean geometry, analytic geometry or analysis), disconnection between algebraic or analytical approaches (e.g. rate of change, slope, derivative or tangent line formula) and graphical approaches (e.g. limiting position of secants, visualisation of tangents) and differences between a global perspective (relation of the line and curve as a whole) and the local perspective (relation of the line to the curve at a specific point). Most challenging cases are: when the tangent has more than one common point with the graph (e.g. $f(x)=\sin x$ at $\pi/4$) or coincides with the graph or a part of it (e.g. when the curve is a straight line); tangency at inflection points (e.g. $f(x)=x^3$ at 0); and, points in which the limit of the difference quotient from the left and the right are different real numbers (e.g. $f(x)=|x|$ at 0) or infinity (e.g. $f(x)=\sqrt{|x|}$ at 0) (Biza, Christou & Zachariades, 2008; Castela, 1995; Park, 2015; Vinner, 1991).

In this paper, I draw on my previous research on students’ perspectives about tangent line (Biza et al., 2008; Biza & Zachariades, 2010) by analysing not only what lines students recognise as tangent or not, but also by considering how they justify their choices. The conjecture I examine here is that students use a range of arguments from geometry, algebra and analysis to justify their choices that go beyond the correctness or not of these choices. With this analysis, my objective is to gain insight into how students make meaning of mathematical objects – in the case of this paper tangents – through their communication about them. To this aim, I analyse first year university mathematics students’ responses to a questionnaire about tangent line by drawing on the commognitive approach (Sfard, 2008). In what follows, I introduce the main tenets of the commognitive approach that I employed in the analysis, and the methodology of the study. Then, I present preliminary findings from the analysis and I discuss them also in relation to potential implications for teaching.

Theoretical underpinnings of the study

According to the commognitive approach (Sfard, 2008) communication about mathematics in written or verbal responses is not a *window* to thinking but an inseparable part of this thinking that makes sense only in the context in which this communication takes place. A mathematical discourse is defined by four characteristics: *word use*, *visual mediators*, *narratives* and *routines*. *Word use* includes the use of mathematical terms (e.g., in the context of this study, 'tangent', 'derivative' or 'direction coefficient') as well as everyday words with a specific meaning within mathematics (such as 'touch', 'region' or 'point'). *Visual mediators* include mediators of mathematical meaning (e.g., function graphs, diagrams, geometrical figures or symbols) as well as physical objects. *Narratives* include texts, written or spoken, which describe objects and processes as well as relationships among those (e.g., definitions, theorems or proofs), and are subject to endorsement, modification or rejection according to rules defined by the community (e.g., 'a tangent line is a line that has one common point with a curve' is an endorsed narrative for tangents in Euclidean geometry but not in analysis). *Routines* include regularly employed and well-defined practices that are used in distinct, characteristic ways by the community (such as defining, conjecturing, proving, estimating, generalising and abstracting). Sfard elaborates three kinds of routines: *deeds, explorations* and *rituals* where explorations are categorised into *substantiations, recall* or *constructions* (ibid, pp. 223–245). Recently, there has been increasing interest in discursive approaches and, especially, in university mathematics teaching and learning research, discursive approaches are gaining more momentum (Nardi, Ryve, Stadler, & Viirman, 2014) in the investigation of university teachers' discursive practices (e.g. Park 2015; Viirman, 2015) and student learning (e.g. Güçler, 2016).

In this study, students' responses to a questionnaire are seen as acts of communication and thus part of their meaning making about tangent lines. Mathematical routines such as investigating if a line is a tangent (see question Q3 in Figure 1) can be explorations that include recall of previously endorsed narratives, substantiations of narratives about *why* a line is (or is not) tangent or constructions of new objects such as formulae and graphs. However, there are differences in the mathematical discourses about tangency in analysis, geometry and algebra. For example, in Euclidean geometry, whether a line is tangent or not depends on the number of common points and the relative position between the line and the curve (geometrical routines) because a tangent line to a circle has one common point and keeps the circle to one side (geometrical narratives). In analysis, tangency is checked locally (analytical routine) and is defined by the derivative at a point (analytical endorsed narratives) which is the slope of the line (algebraic narrative). In algebra, the tangent line will be justified through calculations (algebraic routine) of the slope and defined through its equation or the vector that gives its direction (algebraic narratives). Identifying how students' responses to a questionnaire engage with these discourses is the focus of this paper.

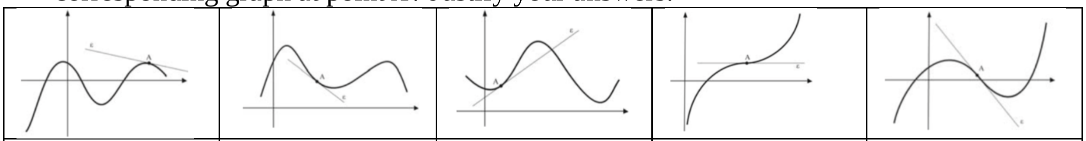
Methodology

Data reported in this paper were collected from a questionnaire administered to 182 first year university students (97 female) from mathematics departments in Greek universities. All participants had been taught about the tangent line in Euclidean and analytic geometry, and in elementary analysis courses in Years 10, 11 and 12, but not yet at university as the questionnaire was administered at the beginning of their first year. The questionnaire included tasks (see a sample of questions from the questionnaire in Figure 1) in which the students were asked to explain in their own words the tangent line (Q1); to describe properties of it (Q2); to identify if a drawn line is a tangent line of a given curve (Q3); to construct the tangent line, if it exists, of a given curve through a specific point on the curve or outside the graph (Q4 and Q5); to provide definitions (Q6), to write the formula, and to apply the formula in specific cases (Q7 and Q8). In questions Q3, Q4 and Q5 only the graph was provided and no formula of the corresponding curve was given; students were asked to identify or construct the tangents based on the graphs and justify their choices. The proposed curves were chosen to reflect students' common difficulties with tangent lines identified by previous research (Biza et al., 2008; Castela, 1995; Vinner, 1991). For example, the corresponding line: had more than one common point with the curve (e.g., in Figure 1, Q3.b and Q3.c in comparison to Q3.a challenge the geometrical routine of checking the number of common points and the relative position between the line and the curve) or passed through an inflection point (e.g., in Figure 1, Q3.d and Q3.e challenge the geometrical routine of the relative position between the line and the curve) – for more about the questionnaire design see Biza et al. (2008).

Q1: Explain, in simple words, what you are thinking when you hear the term "tangent line".

Q2: Write as many properties as you can think of about the relationship between a curve and its tangent line at a point *A*.

Q3: Which of the lines that are drawn in the following figures are tangent lines of the corresponding graph at point *A*? Justify your answers.



Q3.a	Q3.b	Q3.c	Q3.d	Q3.e

Q6: What is the definition of the tangent line of a function graph at its point *A*?

Figure 1: Questionnaire sample

In earlier analysis (Biza & Zachariades, 2010), student choices in questions Q3, Q4, Q5, Q7 and Q8 were characterised according to their correctness and analysed quantitatively. This analysis suggested a classification of students regarding their perspectives on tangent line and its relation with the corresponding curve into three groups with *analytical local* perspectives (closer to the tangent line in the context of analysis – 25.8%); *geometrical global* perspectives (more relevant to the tangent line in the context of geometry – 17.6%); *intermediate* perspectives between the analytical local and the geometrical global perspectives (56.6%). Although this classification indicated a spectrum of students' perspectives about tangency, it does not grasp the subtlety of these perspectives as they were evident in students' choices and justifications of these choices. To this aim, student responses to questions Q1, Q2 and Q6 and their justifications in questions Q3, Q4 and Q5 were analysed qualitatively. Part of this analysis focuses on the mathematical discourses students engaged in in their responses (analytical, geometrical and algebraic) with specific emphasis on the words used, routines and narratives, substantiation of these narratives, how this discourse is related to their choices (correct or not) in the questionnaire and the consistency of student responses across the questionnaire. This paper discusses preliminary findings from the 182 student responses to the items: Q1, Q2, Q3a-e and Q6 presented in Figure 1.

Student justifications on why the sketched line is a tangent or not

Justifications students offered in order to accept or reject a tangent line when it does not have any other common point and keeps the graph at the same semi-plane (Q3.a) or when it has other common points sketched (Q3.c) or not (Q3.b) are summarised in Table 1.

Justification	Script Example
Rejection of the line as a tangent	
Common points between line and curve, global view	"No [it is not a tangent], the line has 2 points in common with the function graph"
Relation of the line and the curve, global view	"No it is not a tangent, although it touches[1] the function graph at the point *A*, it cuts [the graph] at another point"
Relative position of the line and the curve, global view	"[The line] splits the curve in two semi-planes"
Only local acceptance of the tangency	"Not [a tangent] in general […] in a small interval (δ>0 (x-δ, x+δ) it is [tangent]"
Derivative / differentiability	"Although the function is differentiable at *A* and thus it has a tangent, the extension of the [line] ε that goes through *A* has another common point with the function and as a result it is not a tangent"
Acceptance of the line as a tangent	
Common points between line and curve, local view	"[It is a tangent, b]ecause if we consider a small region (κ, γ) around the point *A* where [the line] ε is tangent we can see that [the line] ε does not touch any other point"
Relation of the line and the curve, local view	"Yes [it is tangent] because it touches exactly at [the point] *A* and it does not cut it [the graph]"
Relative position of the line and the curve, local view	"The part of the function graph which is close to the point *A* is located at the same side of the line ε"
Common points between line and curve, global view	"It [the line] has one common point with the curve"
Relative position of the line and the curve, global view	"$f(x)>(\varepsilon)$"
Slope of the line	"Yes, the line ε is tangent at *A*, the slope equals to the direction coefficient of the line"[2]
Derivative / differentiability	"It is [tangent] because it has slope [equals to] the derivative of the function at this point"
Opposite rays	"The rays ε_1, ε_2 which are tangents at *A* are opposite"
Other	"There is only one tangent at the point *A*" or "There is a limit which is the same from the left and the right side or "ε: it is tangent, the point *A* is defined and belongs to the domain of the graph"

Table 1: Student choice justifications to questions Q3.a, Q3.b and Q3.c

[1] Data have been translated from Greek to English. In Greek, the noun *tangent [line]* (*εφαπτομένη [ευθεία]*) and verbs such as *being tangent, abut, touch* (*εφάπτεται*) have the same origin. In Greek, the excerpt: "No, it is not a tangent because although it touches the graph …" sounds contradictory ("Όχι, δεν είναι εφαπτομένη γιατί αν και εφάπτεται στη γρ. παράσταση …"), one explanation is that the noun "tangent" draws on the mathematical discourse, whereas the verb "touches" draws on the everyday discourse.

[2] In the Greek curriculum, the "direction coefficient" is the coefficient m in $y=mx+b$, that indicates the slope of a line.

Justifications students offered in order to accept or reject a tangent line when the tangency point is an inflection point (Q3.d and Q3.e) are summarised in Table 2.

Justification	Script Example
Rejection of the line as a tangent	
Common points between line and curve, global view	"No [it is not a tangent], because the curve and the line cut each other in several points"
Relation of the line and the curve, global view	"It is not [a tangent] because [the line] penetrates the curve"
Relative position of the line and the curve, global or local view	"It [the line] intersects the function graph by going to its both sides"
Inflection point / concavity change	"It [the line] is not tangent because *A* is inflection point"
Change of function formula	"Because the formula of the graph changes"
Derivative / differentiability	"The graph does not have tangent at [the point] *A* because the graph is an image of a function *f* and $A(x_0,f(x_0))$, $f'(x)=\kappa$ for $x<x_0$ and $f'(x)=\lambda$ for $x>x_0$ $\kappa\neq\lambda$ close to x_0"
Solution of the corresponding system of simultaneous equations (line and curve)	"[The line] ε is not tangent because the system line – curve has one solution and not a double solution"
Other	"more than 1 lines can be sketched through point *A* with at least one common point with the graph" or "if we consider the figure as two figures with *A* as the unique common point the line ε is a tangent of the two figures. If we consider the figure as a whole the [line] is not a tangent of this figure»
Acceptance of the line as a tangent	
Common points between line and curve, local view	"The ε has only point in common with C_f in the region $(x_A-\kappa, x_A+\kappa)$, $\kappa>0$ and very small"
Common points between line and curve, global view	"The line ε is tangent because it has one common point with the curve and the concavity of the graph changes at this point" (these participants rejected the line when it had more than one points in common)
Opposite rays	"It is [tangent] but for the right part of the function after *A*, tangent is the right part of the tangent and respectively for the left [part]"
Slope of the line	"Yes, the line ε is tangent at *A*, the slope equals to the direction coefficient of the line"
Derivative / differentiability	"derivative equals to the slope of the tangent"
Inflection point / concavity change	"It is [tangent] and the [point] *A* is inflection point"
Other	"The ε is tangent at the point *A* especially internal"

Table 2: Student choice justifications to questions Q3.d and Q3.e

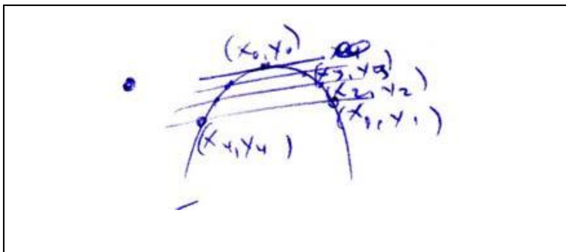
In both set of questions students engage with analytical, geometrical or algebraic discourses. They use narratives such as derivative, differentiability, intervals, regions close to the tangency point, inflection point or concavity (analytical discourse); common points, relative position of curves, same-plane or ray (geometrical discourse); and, slope or system of simultaneous equations (algebraic discourse). Routines include checking for common points or for the relative position between line and curve (geometrical discourse) or for derivatives (analytical discourse) or slopes (algebraic discourse). Routines are applied locally around the point *A* (analytical discourse) or globally for the whole figure (geometrical discourse). Indicatively, of the justifications offered in Q3.c, 83.3% were geometrical (either global or local); 12.1% analytical; and, 4.6% a mixture of analytical and geometrical/algebraic. Whereas, in Q3.d, 70% were geometrical (either global or local); 2% algebraic; 22% analytical; and, 6% a mixture of geometrical and analytic/algebraic.

Similarly, the word use includes verbal descriptions as well as terms and symbols from geometry, analysis and algebra. The relation of the line and the curve especially at the point *A* are described in a range of ways, not necessarily with consistent (in terms of the different discourses) meaning. For example, in questions Q3.d and Q3.e where point *A* is an inflection point the line *intersects* (*τέμνει*); *pierces* (*τρυπάει*); *cuts* (*κόβει*); *crosses* (*διαπερνά*) or *bisects* (*διχοτομεί*). The same word can be used with a range of meanings in different scripts or across the same script. For example, "intersection" may mean the common point regardless of the position of the curve in relation to the line, whereas in other cases (in the same or different scripts) it means the split of the curve into parts. The analysis aimed to identify evidence not only regarding the common points but also regarding the overall relation of the line and the curve and their relative position. These subtle differences are not always evident in student responses.

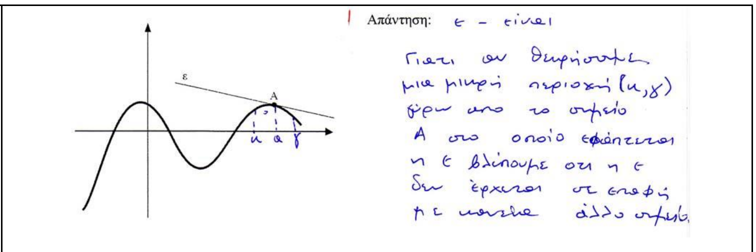
Furthermore, the analysis indicated the use of endorsed narratives from more than one discourse in the same response or/and across responses within the same script. For example, student S[149] who performed well in all questionnaire items, writes in Q2:

> $f'(x_A)=\lambda$ the direction coefficient. $f'(x_A)=\lim_{x\to x_A}\frac{f(x)-f(x_A)}{x-x_A}$ [analytical narrative]. At this point it [the line] has one "double" [his emphasis] common point with C_f [algebraic narrative]. It can have other common points with C_f, $x\neq x_A$ [geometrical narrative]

and he sketches the graph in Figure 2a. In Q3.b and Q3.c he accepts the line because "it satisfies all the conditions" and in Q3.d he writes: "The [line] ε is [tangent] because f is differentiable at x_A and ε has one (double) common point with C_f in the region $(x_A-\kappa,\ x_A+\kappa)$, $\kappa>0$ and very small" [a mixture of analytical and algebraic endorsed narratives applied locally].



2a: S[149]'s response to Q2



2b: S[123]'s response to Q3.a

Figure 2: Student responses

Student S[123], on the other hand, who had difficulties with accepting tangency at an inflection point writes in question Q1: "The tangent line is a line that touches a graph at a point *A* and in a small area around it [the point] it does not intersect the graph" [geometrical narratives applied locally]. Then, in question Q2 he writes: "The slope of the tangent at the point $A=(x_0, f(x_0))$ is equal to $f'(x_0)$" [analytical narrative]. In Q3.a and Q3.b he accepts the line and justifies the choice: "Because if we consider a small interval (κ, γ) around the point *A* where [the line] ε is tangent we can see that [the line] ε does not touch any other point" (Figure 2b) [geometrical narratives applied locally]. In Q3.d he responds "ε – it is not [tangent] because it bisects *f"* [geometrical narrative].

Another student (S[261]), who also had problems with tangency at inflection points, writes in Q1:

“Gradient (λ), the tangency point, the formula of the line, $f(x)=\lambda=(y_2-y_1)/(x_2-x_1)$, $M(x,y)$, $y_2-y_1=f(x)(x_2-x_1)$” [algebraic narratives]. Then in Q2 she responds: “The [point] *A* is a tangency point and belongs to the figure. It satisfies the equation of the tangent as well as of the figure. It is $\lambda=\tan\omega$ (the angle between the figure and x'x)” [algebraic narratives]. In Q3.a and Q3.b she accepts the line and justifies the choice: “(ε) is tangent at *A* [the line is] at the same side of the graph” [geometrical narratives applied globally]. In Q3.c she writes “(ε) is tangent at *A* only. <u>There</u> [the line is] at the same side of the graph” [geometrical narratives applied locally]. I have highlighted “there” in her response as an indication of the focus at the area around point *A*. In question Q3.d she responds: “(ε) not tangent. It intersects with the graph going through both its sides” [geometrical narratives applied locally]. Finally, in Q6 she responds by using mainly geometrical narratives with symbolisation from analysis:

> The tangent line of a function graph at a tangency point $A=(x_0, f(x_0))$, that belongs to the function and the tangent, is a line that intersects the function without going from its one side to the other but remains at the same side of the function with $\boxed{\text{only}}$ one common point the [point] *A*. [her emphasis]

Discussion

This paper reports on my first attempt to draw on the commognitive approach to analyse 182 first year university mathematics students’ justifications about tangent line. My initial conjecture was that students engage with different discourses (geometrical, algebraic or analytical) even for the substantiation of similar choices regarding the tangent line. Although the findings presented in this paper cover only a small slice of the data, in terms of questionnaire items, I would say that there is evidence supporting my conjecture. Students engage with analytical, geometrical or algebraic discourses in terms of the endorsed narratives and routines I identified in their responses. Also, the word use includes verbal descriptions as well as terms and symbols from geometry, analysis and algebra. Additionally, the same justification may engage with more than one discourse or/and different discourses across the script. Also, it seems that in several responses there are arguments that use analytical endorsed narratives (derivative etc.) applied through geometrical routines (check the tangency globally). I note here that my analysis considers students’ responses in relation to the discourses of the mathematical community in different mathematical areas and not in relation to their correctness. Use of analytical narratives, for example, do not necessarily ‘secure’ the correctness of the response and, the other way around, a correct choice does not necessarily draw on a coherent and consistent justification. Furthermore, the type of the task may also affect the type of discourse students engage in. For example, a graphical question (e.g. sketch the tangent line) may trigger geometrical discourses whereas an algebraic question (e.g. calculate the formula) may trigger analytical or algebraic ones. The reported findings are indicative and further investigation is in process also in relation to inconsistencies within a response with potential commognitive conflicts between geometric, algebraic and analytical discourses of tangent lines as well as to the extent student responses are mediated by the task formation.

This work contributes to our insight into what students bring with them when they join post-secondary mathematics courses and I credit to the commognitive approach the deepening of this insight in the case of students' meaning making of tangency. I envisage teaching implications of the outcomes of this analysis in calculus or analysis introductory courses. For example, the observed mismatch between lecturers' and students' discourses (Park, 2015) would be dealt by explicitly addressing commognitive conflicts with the use of appropriately selected examples – see (Biza, 2011) about the role of examples in student meaning making of tangency.

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