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Community for Undergraduate Learning in the Mathematical Sciences

The CULMS Newsletter

CULMS is the Community for Undergraduate Learning in the Mathematical Sciences.

This newsletter is for mathematical science providers at universities with a focus on teaching and learning.

Each issue will share local and international knowledge and research as well as provide information about resources and conferences.

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Editorial: The Birth of a New Journal

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Background

When a new baby is born there are usually a number of people who have assisted along the way in making the birth a happy event. Apart from grandparents and other family and friends, there are the doctors, nurses, midwives, obstetricians and others, who have all played a crucial part. One thing is certain though; when the baby is safely born all share in the happiness of the parents. It is very gratifying to know that the same has been true with regard to the period of gestation and the impending birth of the new journal, *International Journal of Research in Undergraduate Mathematics Education* (IJRUME), which will make its first appearance early in 2015. Over a number of years leading educators have made a strong case for such a journal, especially senior members of the Special Interest Group of the Mathematical Association of America on Research in Undergraduate Mathematics Education (SIGMAA on RUME).

Why is such a journal needed? In the last ten years or more the field of research in university level mathematics education has become a very active research domain, but the publications have been dispersed among a number of journals rather than collected together in one place, in order to make them more accessible. The vision for IJRUME is that it will become the premier international journal dedicated to university mathematics education research. Unlike other journals, there will be no restriction on the upper level of mathematical content. This new journal will immediately provide a common outlet for at least three leading research groups. The first is SIGMAA on RUME, which holds an annual conference with over 200 attendees. Second is the growing group of around 150 undergraduate mathematics education researchers associated with the biennial DELTA conference, which is committed to improving undergraduate mathematics and statistics education. The third research community is that comprising a working group specifically devoted to University Mathematics Education at the biennial Congress of European Research in Mathematics Education (CERME).

IJRUME is dedicated to the interests of post-secondary mathematics education. One of its aims is to help in the further development of a strong community of mathematicians and mathematics educators interested in undergraduate mathematics education. The journal will welcome original research, including empirical, theoretical, and methodological reports of learning and teaching of undergraduate and graduate students. It will present research at the undergraduate level that reflects on theoretical perspectives and results of empirical studies in mathematics education, including those studies that seek to describe best practice. It will aim to be inclusive, covering university mathematics education for students in the mathematical sciences and related STEM and other disciplines, such as engineering and economics, as well as the training of future mathematics teachers.

Once a baby is born the extended family, medical professionals and others don't immediately lose interest in it. On the contrary they continue to take an interest and monitor its progress. The Editors-in-Chief of the new journal, Karen Marrongelle, Chris Rasmussen and myself, have been very gratified that not only have so many leading international researchers in university mathematics education expressed their happiness at

the impending birth of the journal but have also promised their continued support by agreeing to be a member of the Editorial Board. It is important to the editors that the journal be truly international in its scope and not be closely linked to any particular country, region or group, so having board members who represent many different geographical areas was an important issue. I list the members of the Editorial Board here, with their affiliations, so that this coverage can be explicitly seen and thank these people for providing such strong support:

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We are grateful that Springer has agreed to support the journal, and hope that you, as a reader of the CULMS Newsletter, will agree that this new journal is long overdue and that you will also want to support it. You can do this in a number of ways. First, by letting your colleagues know of its existence and pointing them to the website at Springer.com/40753. Second, by logging in to the journal's editorial website at http://www.editorialmanager.com/rund and volunteering to referee manuscripts. Third, and extremely importantly, is by submitting your research manuscripts to the journal, using the same editorial website.

Of course, this new journal is not the only instrument promoting undergraduate learning and teaching and nothing mentioned above takes away from the support that we all want give to the CULMS Newsletter. We hope you will still send your shorter opinion pieces and other relevant articles to the CULMS Newsletter, which will continue to play an important role in raising issues and disseminating ideas about undergraduate mathematical sciences, as we see in this current issue.

Editorial

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Do Bold Shakeups of the *Learning-Teaching Agreement* Work? A *Commognitive* Perspective on a LUMOS Low Lecture Innovation

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Mathematics undergraduates, and their lecturers, often describe the transition into university mathematics as a process of enculturation into new mathematical practices and new ways of constructing and conveying mathematical meaning (Nardi, 1996). What characterises the breadth and intensity of this enculturation varies according to factors such as (Artigue, Kent & Batanero, 2007): student background and preparedness for university level studies of mathematics; the aims and scope of each of the courses that the students take in the early days of their arrival at university; how distant the pedagogical approaches taken in these courses are from those taken in the secondary schools that the students come from; the students' affective dispositions towards the subject and their expectations for what role mathematics is expected to play in their professional life. On their part, lecturers' views on their pedagogical role may also vary according to factors such as (Nardi, 2008): length of teaching experience; type of courses (pure, applied, optional, compulsory etc.) they teach; perceptions of the goals of university mathematics teaching (such as to facilitate access to the widest possible population of participants in mathematics or select those likely to push the frontiers of the discipline); and, crucially, institutional access to innovative practices, e.g. through funded, encouraged and acknowledged research into such practices.

In this paper I draw on my experiences as a member of the International Advisory Board of the LUMOS project (Barton & Paterson, 2013) to comment on aspects of aforementioned student enculturation. Here I see this enculturation as the adaptation of different ways to act and communicate mathematically. I take a perspective on these ways to act and communicate as *discourses* and I treat the changes to the mathematical and pedagogical perspectives of those who act as *discursive shifts*. To this purpose, I deploy the approach introduced by Anna Sfard (2008) and known as the *commognitive approach*.

Shakeup of the *Learning-Teaching Agreement*, Episode 1: First Impressions

The five MATHS108 students arrived in the small, cosy meeting room where their first experience of the LUMOS project innovation known as *Engagement Sessions* was about to kick off. Their preparation for the session consisted of engaging with the open task of exploring functions from R to RxR. The students expected to be invited to share their explorations with the lecturer and the group. What were these *newcomers* to the practices of university mathematics to make of this open task? I wondered. What were their expectations of the *old-timer* who led the session? And, what kind of bearing, if any, was the slightly unexpected nature of the task – these students are so far accustomed to working with functions from R to R – bound to have on the session?

In this paper I take the position of the *commognitive* perspective (Sfard, 2008) to offer an account of what unfolded in the remainder of the session. First I introduce briefly this theoretical perspective – adapting an abridged version of the presentation in Nardi, Ryve,

Stadler & Viirman (2014). Then I outline the pedagogical innovations of the LUMOS project (Barton & Paterson, 2013) that the observed session is a component of. I then offer a few *commognitive* snapshots from the session. Finally, I conclude with a consideration of the events that followed the session and with a few remarks on the overall experience.

The *Commognitive* Perspective

The *commognitive* perspective is one of the discursive approaches to research in mathematics education, often described as *participationist*. These are typically juxtaposed to those labelled *acquisitionist* and espouse the socio-cultural tenet that learning occurs in, and is co-constituted by, situational, cultural and historical milieu. Much like other *participationist* approaches, the *commognitive* perspective places firm emphasis on the view of human thinking as a type of communication and aims to examine what Sfard (2008) describes, in true Vygotskian form, as the social nature of the individual.

The main tenet of the *commognitive* approach is that communication in mathematical learning is not merely an aid to, or a component of, thinking. Sfard's position on communication is that it is almost tantamount to the thinking itself; and, that learning is change in one's participation in well-defined forms of activity. Thinking, in this framework, is conceptualized as communicating with oneself, where 'communication may be diachronic or synchronic, with others or with oneself, predominantly verbal or with the help of any other symbolic system' (Sfard, 2008, p. 28). Speech then is no longer a window to thought but its determining element; therefore thought and speech are inseparable. The interlocutor is constrained by the situation in which the communication takes place and influences it in return.

In this sense, learning mathematics is initiation to a discourse, where discourse is meant as a type of communication that characterizes a particular community, here that of mathematics. Sfard's perspective has become known as the *commognitive* framework, with the hybrid term *commognition* emphasising the interrelatedness, almost inseparability, of cognition and communication and refers 'to those phenomena that are traditionally included in the term cognition, as well as to those usually associated with interpersonal exchanges' (p. 83). Mathematical learning – the initiation into the discourses of mathematics – generally involves substantial discursive shifts for learners; and, the teaching of mathematics involves the facilitating of these shifts.

Communication through written or spoken language, and manipulation of physical objects and artefacts, are the main means to the discursive ends of teaching and learning. Specifically, a discourse is made distinct (pp. 133-135) by a community's *word use* (mathematical and colloquial terms), *visual mediators* (graphs, diagrams, symbols, physical props), *endorsed narratives* (written or spoken texts, such as definitions, theorems and proofs, which describe objects, processes and relationships among those, and are subject to endorsement according to community rules) and *routines* (regularly employed and well-defined practices of the community, such as defining, conjecturing, proving, estimating, generalising and abstracting).

Mathematical learning can be *object-level* (namely resulting in the 'growth in the number and complexity of *endorsed narratives* and routines' (p. 300)) and *meta-level* (namely 'express[ing] itself in the change in the *metarules* of the discourse' (p. 300)). Mathematical communication involves incessant transitions from signifiers to other entities that Sfard calls *realizations* of the signifiers (words or symbols that 'function as nouns', p. 154). Realizations are 'perceptually accessible entities' (p. 155) and can be vocal (spoken words) and visual (written words/symbols, iconic, concrete and gestural).

Of tremendous importance in mathematics are symbolic realizations which bring about considerable 'generative power' (p. 159) of the discourse. *Substantiation of a narrative* is the process through which we 'become convinced that a narrative can be endorsed' (p. 231) – such as proof.

Unlike colloquial discourses on material things, *discursive objects* ('the objects of mathematical exchange', p. 135) are 'featured as something that can perhaps be 'represented' with visual means, but never really shown' (p. 135). Distinguished from *discursive objects* are what Sfard defines as *primary objects*, 'perceptually accessible entities existing independently of human discourses' (p. 169). Mathematical discourse can be *colloquial* (namely, 'used in everyday life and developing spontaneously, mediated visually mainly by *primary objects* pre-existing the discourse' (p. 296)) and *literate* (namely, 'mediated mainly by symbolic artefacts created specifically for the sake of communication' (p. 299)).

In the light of above, mathematical communication depends enormously on the interlocutors' handling of signifiers, such as their use of words. Differentiated *word use* by different interlocutors emerges then as a great challenge of mathematical communication, and is one type of the challenge that Sfard calls *commognitive conflict*. This conflict is not always acknowledged by interlocutors and is often resolved in an 'imperceptible manner, by gradual mutual adjusting of [the interlocutors'] discursive ways.' (p. 145). This adjusting involves a power-related conceding to one of the present discourses, ultimately accepted by the interlocutors as privileged and paradigmatic (e.g. acknowledging the lecturer as the 'ultimate substantiator', p. 234). Successful exposure to, and resolution of, a *commognitive conflict* requires 'a voluntary alignment of the discursants' (p. 283) with the conditions of the *learning-teaching agreement* among the discursants (a notion close to Brousseau's notion of *didactic contract*). This requirement for agreement covers: the leading discourse; the *rules* that the discursants ('newcomers' and 'old-timers', p. 284) agree to work by; and, the *nature of the expected change* in the discourse (p. 283-5). Finally, Sfard emphasises the strong ethical dimension ('entails tolerance and solidarity') of the *learning-teaching agreement* construct.

The *commognitive* perspective has been deployed so far in about a dozen university mathematics education studies – grouped and outlined in (Nardi et al., 2014) as studies of student-student and lecturer-student interactions as well as lecturer practices. Nardi et al. posit that this perspective offers a potent set of lenses through which to consider mathematical interaction at university level. Here the brief *commognitive* account of observing / participating in a *low lecture* session of the LUMOS project hopefully also demonstrates this potency. In doing do, I also aim that the key *commognitive* terms presented in this section will come into more meaningful being through their use in this account. First though I briefly introduce the LUMOS project and its component innovations. This introduction is based on information shared by one of the projects leaders, Bill Barton, and also on his CULMS paper with Judy Paterson (2013).

LUMOS and the *Low Lecture* Innovation

LUMOS (*Learning in Undergraduate Mathematics: Output Spectrum*) is a two-year project funded by Ako Aotearoa and the Teaching & Learning Research Initiative (TLRI) and led by Bill Barton and Judy Paterson at the University of Auckland Department of Mathematics. Its team of 31, mostly from this department, is working towards the aim of understanding how course delivery at class level can achieve desired learning outcomes for undergraduate mathematics. It is expected that the project will generate evidence that

different types of courses contribute to student learning in different ways, and therefore recommend that the department encourages a variety of pedagogical practices. Three *course innovations* are currently under trial: *team-based learning*, *intensive technology* and *low lecture*. The third of these, *low lecture*, is the focus of this paper.

There are three key ideas behind *low lecture*. First, lectures are far from the best means of imparting information or developing skills. They are however useful for material overviews, demonstrating model ways of communicating mathematical ideas and enthusing *newcomers* with the skill and fluency that can often be found in the communicational practices of *old-timers*. Second, optimal learning takes place when students are engaged individually or in groups. Third, responsibility for learning content and acquiring skills is handed back to students using print and online resources, but with regular self / lecturer monitoring of progress.

Low lecture was trialled for the first time in 2013, with 14 MATHS108 students. Faculty members, as members of the LUMOS team, run the trial on an extra to load basis. The trial consists of one lecture per week for the duration of the semester and three 2-hour *Engagement Sessions* for which students need to prepare for in advance, as well as write up a report for afterwards. These reports substitute assignments. The remaining parts of MATHS108 (tutorials, tests and final written examination) stay the same.

This paper was inspired by my experience of observing / participating in the first *Engagement Session* of a group of five students (thereafter Students B, N, J, D and A). The session was run by a member of the LUMOS team (thereafter Lecturer L). The students were seated counter-clockwise in this order, from the right side of L (I was seated on the left of L), around a rotund table, arranged in the middle of a small meeting room.

The students arrived with evidence of their preparatory work towards the session in hand. One – N, the only female in the group – also had her laptop with online access, and often used, during the session. Throughout, the ambience was convivial and highly respectful of all. The students granted me permission to join the session and seemed comfortable with – or at least did not seem to mind – my presence. L and I reassured them that they can ask me to leave in any given moment. Gratefully they did not; hence, the account that follows, based on notes jotted down right after the session and generated following my reflection upon the session from a *commognitive* standpoint. The account aims broadly at addressing the questions listed in *Episode 1* at the start of this paper.

Shakeup of the *Learning-Teaching Agreement*, *Episode 2:* Commognition

At the very start of the session L reminds the students that its overall aim is to set out from the request in the preparation sheet and spend some time prior to the session on this request: you are already familiar with functions from R to R, such as $f(x)=x^2$, and perhaps with functions from RxR to R such as $f(x,y)=3x-y^2$. In preparation for this session, *explore* the idea of functions from R to RxR. L had set two tasks for this exploration: first, propose a notation for this type of function; second, devise a relationship of this type and explore how you would secure that it is a function, what its range of values would be, how its graph would look like and what its behaviour be for very small or very large values of x. The preparation sheet ends with a request to devise a second function of this type and repeat the exploration with a view to comparing with the first doing the same process. The students are reminded that they will be expected to communicate the outcomes of their exploration and that means to do so will be available in the room for them to do so. At the session L also reminds them that a 4-page report, an account of their pre-session efforts (p.1), their take on the exchanges during the session (p.2-3) and their further explorations

soon after (p.4).

L's final wording, 'happy *mathematising*', at the end of the preparation sheet, encapsulates succinctly and explicitly the *discursive object* of the activity that the students are invited to participate in. His overall demeanour and utterances throughout the session also convey exactly that this session is about engaging with the *routines* of a mathematician (he lists several of these in at least two occasions, including hypothesising, justifying, proving, visualizing, extrapolating etc.). The students' responses to these *metadiscursive* utterances – particularly when L asks them to cease activity for a moment to heed what they are doing, and how – is rather mute: they seem keen and confident to act but perhaps less so to take up this invitation for reflective distancing from the action. It took no more than a few seconds for them to return to the vicarious discussion of their exploratory work though and – on the grounds of this discussion – I was left with no doubt that their take on the purpose of the session was essentially congruent to that of L! Sfard (2008) speaks of mathematical *routines* in terms of *deeds, rituals* and *explorations* (p. 223 onwards) and it would be hard to perceive what was happening here as anything other than *explorations.*

I now wish to dedicate the remaining part of this account to the other two questions posed in the beginning of this paper. The first concerns some features of the students' *exploratory* work, particularly in relation to the slightly unexpected nature of the task (a reversal of the familiar perspective on functions from '*RxR* to *R*' to '*R* to *RxR*'). The second concerns evidence of the students' – and L's – perceptions of the *learning-teaching agreement* at issue.

With regard to the first I was alerted - and on a couple of occasions slightly alarmed – by the likelihood of *commognitive conflict* that the students' *word use* and *visual mediation* may result in. The students' standard approach to *substantiation* was to endorse or reject a narrative about the *objects* at stake through indications in favour, or against, a claim on screen, or roughly produced *visual mediators* on paper. Combined with their generally non-standard use of *symbolic realizations* (notation, graphs and related terms), the ingredients seemed to be there for *commognitive conflict*. In line with the task given by L, which included a request to consider how a graph of a function from *R* to *RxR* would look like, I was on various occasions under the impression that the narratives that most of the students generated concerned functions that looked more like $f+g$, fg, fog, rather than *f: R→RxR*; and that the essential question 'how does a function from *R* to *RxR* look like?' was not pursued as directly as I might have expected. *Word use* was sufficiently variable. – Sfard distinguishes between *passive, routine-driven, phrase-driven* and *object-driven use* (p. 181-2) – even though with the understandable plenty of deictic language, aimed at screen or paper, I found it difficult to pin down whether the *focus* and *object* of the exchanges was always well-coordinated amongst the six interlocutors (L and the five students). A similar observation applies to the students' loose, non-standard deployments of notation. L seems also alerted to this and on several occasions he draws on his *ultimate substantiator* status to alert the students to this too one occasion is the use of the expression 'x^2 over *cosx*' – see *Episode 3*). To me Student A's proposition to introduce the notation $t→(f(t), g(t))$ was the closest the group came to a standard notational *realization* of the *object* at hand. Finally, while I was awed at the confidence with which the students deployed online software to generate complex and attractive visual *realizations* of their suggestions that were gazed at from all angles, I also noted that these were hardly interpreted or explicitly connected to the task set by L.

This brings me to the second set of thoughts I promised above, on the evidence of the students' – and L's – perceptions of the *learning-teaching agreement* at issue. I was

simply amazed at these *newcomers'* very open expectations of the *old-timer*, L, who led the session. It is this openness which brought about my use of 'shakeup' in the title of this paper. Certainly the ethics requirement for 'tolerance and solidarity' was amply met. More impressively the power-related conceding to one of the present discourses ultimately accepted by the interlocutors as privileged and paradigmatic resulted in the conceding to the discursive path proposed not by L (!) but by one of the students (D, who proposed a complex endorsement of the idea of a function from R to RxR in which the first rule for x becomes the x axis for the second rule). In all this, L coordinated the intense exchanges with admirable distancing, in fact with minimal use of his *ultimate substantiator* status. And this is where the grandest element of aforementioned shakeup lies, L's conceding of much of his status. Is this liberating, perplexing to the students, both? How does it sit alongside the rest of these students' experiences at this university? They seem comfortable with L's serious shakeup of the *learning-teaching agreement.* Will they stay so throughout? When, if at all, will they demand a reinstatement of this status in the form of a demand for specific assessment (for example) of their proposed narratives (on functions from R to RxR, and more broadly)?

Throughout I admired how L was uniformly open to these narratives and how he held back from encouraging their *endorsement* or rejection. I also admired how he sustained a mental list of proposed narratives that there had been no time to pursue (a little concerning in my view, Student A's). To me this demonstrated pedagogical sensitivity in the pragmatic context of limited time (and Student D's more vocal presence attracting perhaps more attention than Student A's). Student N appeared to experience the most obvious discovery moments but Student J's gestural language and body positioning also suggested so, particularly when 3D images started appearing on screen. Only Student B appeared minimally participant, and quietly perplexed.

Shakeup of the *Learning-Teaching Agreement*, Episode 3: The Aftermath

I left the session impressed by its buzz and warmth, and a little anxious about not having worked on Student A's proposed narrative. The events that followed that evening largely appeased that anxiety: Student A wrote to L with an imaginative account of Student D's idea (omitted here due to limitations of space). He had nobly conceded to the temporary dominance of another student's proposed narrative and made the most of it. To me this implies that regardless of any clear-cut answers to the questions I posed at the start – after all I was there for a tiny part of the *low lecture* trial only – for at least the two hours of the *Engagement Session* I witnessed, these *newcomers* slipped comfortably into the shoes of the *old-timers*, with all the fallibility and excitement that walking in these shoes entails. For that alone, surely this is an innovative path worth treading.

Acknowledgement

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Using the 'Three Point Framework' to Focus Teachers' Attention During Lesson Preparation

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Lesson planning can help to navigate the complexity of teaching mathematics. It is a core skill that is assumed of every teacher, and yet, planning a good lesson or lecture can be challenging. Particularly for the beginning teachers and lecturers, it might not be easy for them to think of appropriate questions, or select models and representations to present concepts accurately. In this article, I will share how a 'Three-Point Framework' can be used to analyse teachers' lesson planning, and propose how the framework can be used to direct teachers' attention on relevant issues during lesson preparation.

Teaching mathematics is a complex process because teachers have to focus their attention on the content, students and their teaching at the same time. One way to navigate this complexity is through effective lesson planning or preparation. Through effective lesson planning, teachers have opportunities to understand mathematics content deeply, think about and design more effective instructional strategies (Li, Chen, & Kulm, 2009). Moreover, lesson planning can be conceptualised as a manifestation of teachers' general pedagogical knowledge (Blömeke et al., 2008). This is also true in the context of teaching mathematics at the undergraduate level, where lectures and seminars are the dominant modes of teaching. For example, the quality of questions and how lecturers pose them to students can influence students' notion of mathematics and doing mathematics (Mason, 2000). Similarly, the use of appropriate models, examples and representations can help promote mathematical thinking amongst students (Thomas, 2008) and facilitate students' transition from school to university mathematics. Hence, the ability to plan lessons can be viewed as an important component of teaching expertise.

Despite being a core competency assumed of every teacher (and lecturer), effective lesson planning could be a challenging endeavour for teachers. Without careful attention to relevant details, teachers might miss out on important aspects of mathematical content and design of tasks. It is possible, even for mathematically competent teachers, to fail to notice the necessary conditions for applying standard procedures (Klymchuk & Thomas, 2011). To illustrate my point, I recall an incident in which a group of teachers came together to plan a lesson on linear graphs as part of a Lesson Study conducted in their school.

Vignette 1.

The teachers planned to use graphing calculator as a means for students to explore linear functions of the form $y = mx + c$ by varying the values of m and c through a series of tasks. In this particular segment of the lesson, teachers designed an exploratory task for students to direct their attention to how the values of c might affect the graph of the linear function. More specifically, they wanted students to see that "as the value of c increases with a fixed m, the lines are parallel and moves vertically upward". In their lesson plan, the teachers anticipated a possible misconception that some students might think that the line "moved diagonally across" instead of "vertically" when the value of c changed (See

Figure 1). In the lesson plan, the teachers did not indicate any specific points to take note of and assumed that an explanation that the "movement of the line is vertical" would suffice.
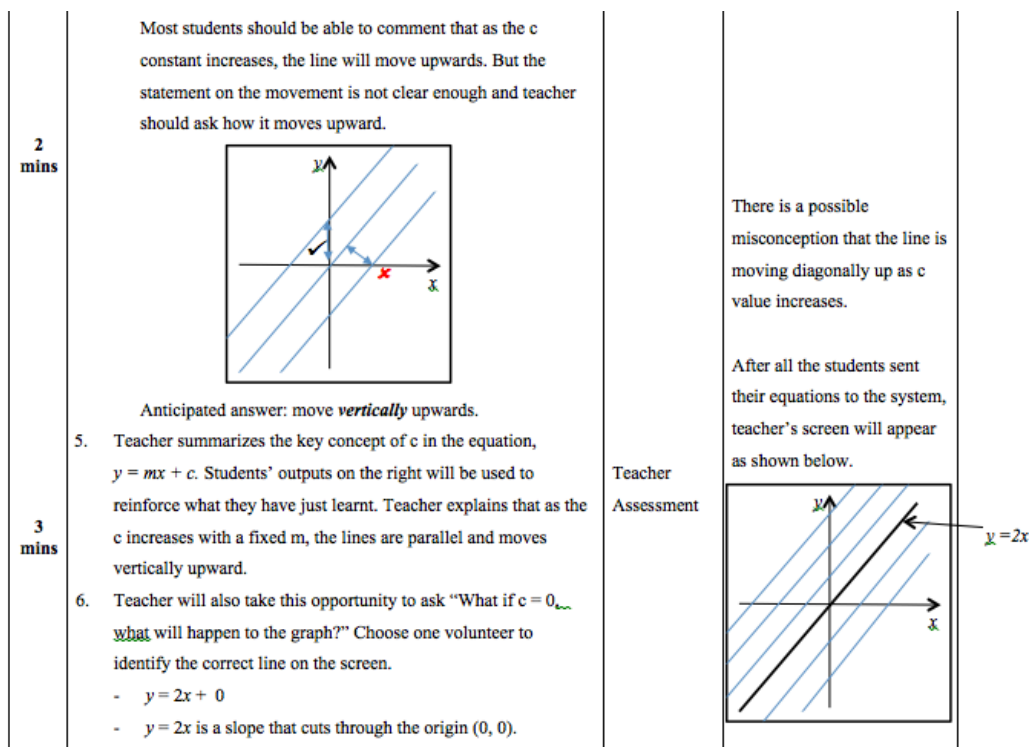

Figure 1. Excerpt from the lesson plan.

During the lesson, the students, as expected, seemed to have this notion that the lines "moved diagonally" instead of "vertically". Even though the teacher highlighted that the movement is supposed to be vertical, many of the students could not understand why it has to be so. The teacher was not able to offer a convincing explanation on the spot and stated it as a rule instead.

This brief episode highlights the need for teachers to be able to attend to students' thinking, make sense of students' responses, and decide how to respond in a manner that enhances students' mathematical thinking. Teacher noticing—the ability to see and interpret mathematical or instructional details in order to make decisions during teaching—is a construct that researchers have used to unpack decision-making processes during teaching (Jacobs, Lamb, & Philipp, 2010; van Es, 2011). In this article, I will describe the notion of noticing, and introduce how the 'Three-Point Framework' (Yang & Ricks, 2012) can be used to direct teachers' attention. By focusing on relevant details, teachers' noticing can be made to be more productive during lesson planning. I will also illustrate how the construct can be used to analyse teachers' lesson plan and demonstrate how this construct can be used to facilitate teachers' attention on relevant details during lesson preparation.

Productive Noticing Using the Three-Point Framework

According to Mason (2002), noticing is a set of practices that work together to sensitise teachers' awareness in order to respond freshly in classroom situations. These practices include "reflecting systematically; recognising choices and alternatives;

preparing and noticing possibilities; and validating with others" (Mason, 2002, p. 95). Many researchers view noticing as consisting of two main processes: "attending to particular events and making sense of events in an instructional setting" (Sherin, Jacobs, & Philipp, 2011, p. 5), but Jacobs et al. (2010) also include how teachers decide to respond to instructional events in order to link the intended responses to the two main processes of noticing. This triad view of noticing—attending to; making sense of; and deciding to respond—ties in with Mason's (2002) idea that noticing should bring to the mind of teachers a different way to respond.

Most studies on teacher noticing focuses on the use of video technology, or teachers' reflection after the lessons are conducted (Sherin et al., 2011). One issue with these approaches is the lack of focus on preparation to notice. As Mason (2002) put it, "noticing is an act of attention, and as such is not something you can decide to do all of a sudden. It has to happen to you, through the exercise of some internal or external impulse or trigger" (p. 61). More specifically, Mason (2002) highlights advance preparation to notice, and the use of prior experience to enhance noticing in order to have a different act in mind. So, in the context of lesson preparation, teachers have opportunities to reflect systematically based on what they observe, recognise choices and prepare to teach differently and responsively to students.

Studies on teacher noticing tend to focus on noticing a wide range of details, and it is still not clear whether an explicit focus can help in improving teacher noticing (Star, Lynch, & Perova, 2011). To investigate whether an explicit focus can lead to more productive noticing, Choy (2013) has proposed the use of Yang and Ricks' (2012) Three-point framework—key point; difficult point; and critical point—to direct teachers' attention on the more relevant issues in teaching. According to Yang and Ricks (2012), the *key point* refers to key mathematical concept to be taught in the lesson; the *difficult point* refers to cognitive obstacle faced by students when they attempt to learn the key point; while the *critical point* refers to the approach taken by teachers to help students overcome the difficult point. By incorporating the 'Three-Point Framework' into the practices of noticing, I characterise teachers' noticing as productive when they are able to:

- attend to specific details related to the key point, difficult point or critical point that could potentially lead to new responses;
- relate these details to prior knowledge and experiences to gain new understanding for instruction (key point and difficult point);
- combine this new understanding to decide how to respond (critical point) to instructional events.

This characterisation of productive mathematical noticing uses the 'three points' to direct teachers' attention to specific details of what they notice, and provides a way to examine whether the lesson plan has the potential to address the students' learning difficulties. Such an analysis might raise teachers' awareness of possible issues related to the design of the tasks so that more targeted modifications could be made to the lesson plan. In the next section, the lesson idea shown in Vignette 1 is analysed and some modifications are suggested using the 'Three Point Framework' (See Table 1).

Analysing a Lesson Plan Using the 'Three Point Framework'

Notwithstanding the limitation that teachers' intentions are inferred from their detailed lesson plan, the framework can still provide a useful way for us to think about the design of the lesson. For instance, teachers in Vignette 1 were clear about the *key point* and the *difficult point*. They were able to provide rather detailed description of the objective of this

segment and pointed out a possible difficulty that students might face. See Table 1 for an analysis of the lesson idea shown in Vignette 1.

Table 1: Analysis of lesson idea in Vignette 1

What was noticed	Processes of noticing		
	Attending to	*Making Sense of*	*Deciding to respond*
Key Point	Teachers wanted to bring across the idea that "as the value of *c* increases with a fixed *m*, the lines are parallel and moves vertically upward".		Use graphing calculator as a means for students to explore linear functions of the form $y = mx + c$ by varying the values of *c* while keeping *m* constant.
Difficult Point	Students might think that the line "moved diagonally across" instead of "vertically" when the value of *c* changed.		Highlight to students that the line move "vertically" and not "diagonally".
Critical Point			

However, the link between the response and what was attended to could have been made clearer using the idea of multiple representations to develop representational versatility (Thomas, 2008). Although the teachers planned for a slide that reveals their awareness of multiple representations (See *Figure 2*), they did not make explicit links between the tasks and the idea of multiple representations.

In the tasks that followed (See *Figure 1*), there was no indication that the connections between numerical, graphical and symbolic representations were made explicit. For example, while students had opportunities to use the graphing calculator to explore the linear function by varying values of *m* and of *c*, they were not tasked to look at the relationships between the numerical values and the graphs of the function, nor make links between the parameter *c*, the numerical values and the graphs. It seemed that the main purpose of the lesson is to explore the connection between symbolic and graphical representation (See *Figure 2*) *without using the numerical representation*. However, the teachers also stated, "it is important for students to see the connection between the 3 representations", and this would have contradicted what they were trying to do. In a way, the *critical point*—that students need to see the connection between the three representations—was missing when the lesson plan was analysed using the 'Three-Point Framework'. Without the *critical point*, the decisions made, such as "highlight to students that the line moved vertically" seemed to be disconnected from the key understanding teachers were trying to highlight.

2. It's important for students to see the connection between the 3 representations.

Missing connection

3 mins

Given an equation of a straight line, they should be able to create a table of x and y values and draw a straight line graph. But if they are to skip the numeric portion, most students will probably have difficulties relating the graphs to the given equations.
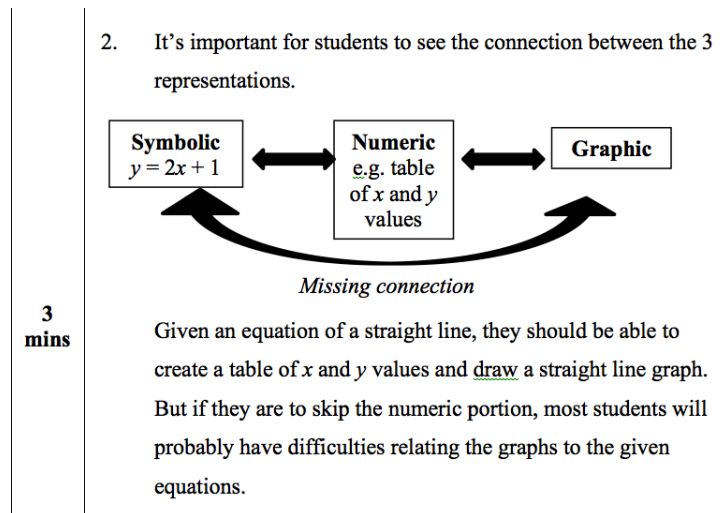
Figure 2. Teachers' awareness of multiple representations

Moreover, the *critical point* would have provided teachers an approach that could help students to overcome the *difficult point* in order to see the *key point*. That would have ensured the coherence between the design and intent of the tasks. If teachers have taken note of the *critical point*, which is to make the connections between the three representations more explicit, they might have designed the tasks or responded to students differently. For example, bearing in mind the critical point, teachers could have directed students' attention on the numerical values of the function to see that the y values have increased by the same value (depending on c) for different values of x. That would have highlighted a vertical movement instead of a diagonal one. The same increase in y values across different values of x for the same m could also have hinted the parallel movement. Then in order to do this, teachers need to make sure that students have opportunities to see the connections between the three representations when they work through the tasks. Likewise, focusing on why students might have the difficulty that was identified might provide the means to overcome it. By maintaining a focus on the 'Three Points', the teachers could have come up with tasks that could potentially direct students' attention on the connection between the representations. Some suggested modifications to the tasks are shown in Table 2.

The main point about the analysis is not to determine *the* "best" way to teach a key point, but rather, to generate different possibilities based on what teachers notice about the three points. The suggested changes in Table 2 are not the only possible set of changes. The idea is to attend to noteworthy details of a lesson plan and think about them to generate possible teaching strategies. This is analogous to an experimenter who tries to observe carefully before designing an experiment to test his hypothesis. Hiebert, Morris, and Glass (2003) propose that viewing lesson as a form of experiment can provide a rigorous way to improve the teaching of Mathematics.

Table 2: Suggested modifications to lesson idea based on Vignette 1

What was noticed	Processes of noticing		
	Attending to	*Making Sense of*	*Deciding to respond*
Key Point	Teachers wanted to bring across the idea that "as the value of c increases with a fixed m, the lines are parallel and moves vertically upward".	To see vertical and parallel movement, students can compare the y values for the same x across the different linear functions, say, $y = 2x$ and $y = 2x + 1$.	Use graphing calculator as a means for students to explore linear functions of the form $y = mx + c$ by varying the values of c while keeping m constant. Ask students to record values of x, y for the different linear functions in a table.
Difficult Point	Students might think that the line "moved diagonally across" instead of "vertically" when the value of c changed.	The misconception could be due to several reasons: a. Students did not see the connection between the different representations. b. Students might be distracted by the x-*intercepts* of the parallel lines and concluded the diagonal movement.	Ask students to compare the values of y for the same values of x across the different linear functions. Ask them to see if there is any connection with the value of c. It might be clearer to restrict the domain of the linear function so that the vertical movement is more apparent (See Figure 3).
Critical Point	Highlight the connections between the numerical, graphical and symbolic representations of the linear functions. Guide students to see that x and y coordinates of a point on the graph is related to the algebraic expression.		Provide opportunities for students to gain access to and work with all three representations during the tasks.

The language of the 'Three Points' can be used to clarify the learning goal(s), specify the learning tasks used to help students achieve them, provide justification for every aspect of the lesson before it is carried out in class. This is done through the consideration of various facets of mathematical content, students' learning difficulties and possible teaching approaches. Hence, the 'Three-Point Framework' provides a way for teachers to articulate their thinking when planning a lesson.
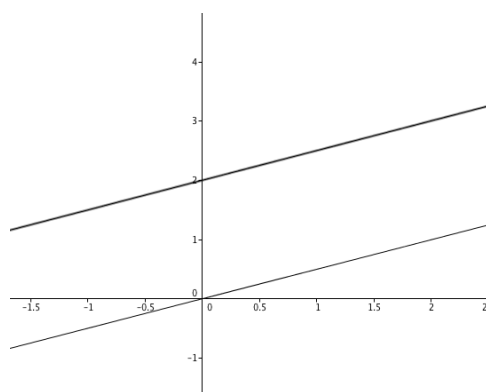
Figure 3. Restriction of domain.

Concluding Remarks

The purpose of noticing is to bring to the minds of teachers' different ways to respond when planning to teach. For noticing to be productive during lesson planning, teachers should be specific about the learning goal and provide rationale for the decisions made in the plan. More importantly, the design of the tasks should address the key learning difficulty or misconception directly. This can help to shift the emphasis from making "appropriate spontaneous decisions" when implementing a lesson to making "appropriate predictions and decisions" during lesson planning (Hiebert et al., 2003, p. 209). By directing teachers' attention explicitly to the three points, and making sense of the details of these three points, they are more likely to decide on a response that could enhance students' understanding of Mathematics.

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Pitfalls Along the Problem Solving Path: Ignored Constraints, Over-Generalisations, and Unchallenged Assumptions.

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We examine the mathematical behaviour of a group of five students during one problem solving session in a third year discrete mathematics course. They failed to solve the problem because they ignored explicit constraints, over-generalised earlier examples and left a number of erroneous assumptions unchallenged. Their behaviour was typical of the novice behaviour described by Schoenfeld (1985; 1992; 2009). We ask what we, and they, might do to encourage them to behave more like experts.

Introduction

In this paper we examine the mathematical behaviour of a group of five students during one problem solving session in a third year discrete mathematics course. The students were engaged but spectacularly unsuccessful. Over-generalising, ignoring constraints and leaving assumptions unchallenged all contributed to this lack of success. We suggest reasons for their behaviour, and then consider what they, and we as teachers, might do to develop more critical behaviour.

The Study and Data Collection

Our data comes from a study that examined the conversations between a team of five students as they worked on problems during the course. The twelve week course was divided in to six modules. Each of these culminated in a team task requiring the students to apply the ideas learnt in the module. By the time the task discussed in this paper took place the team had worked together on both short questions and longer application tasks for nine weeks. They had got to know one another and appeared relaxed about asking each other, and the lecturer, questions. Evidence suggested that a leadership hierarchy had been established.

The study is qualitative involving "noting pattern, themes, categories and regularities" (Cohen, Manion & Morrisson, 2007, p. 461) within the data. Recognising that "social research should be conducted in natural uncontrived, real world settings with as little intrusiveness as possible by the researcher" (Cohen &Manion, 2007 p. 168) we chose to take a purely observational approach to the data collection. Every effort was made to keep the research as unobtrusive as possible both for the benefit and comfort of the team and to keep the discourse data as natural and representative of the team's interactions as was possible. Consequently, the data collected consisted entirely of audio recordings of the team's discussions as they worked on the task. A recording device was placed on the centre of the desk at the beginning of the task and collected at the end of the class.

At weekly research meetings we examined transcripts of the conversational data for patterns of behaviour and sequences of actions. As observers we were aware that inferences linking knowledge and thought to behaviour may vary from person to person. This is, as Cohen and Manion (2007) superbly state, the "glory and the headache" (p. 461) of qualitative research. This approach has a measure of subjectivity and we recognise that different researchers may have made different interpretations.

The Task

The culminating task in this module required the construction of a Balanced Incomplete Block Design (BIBD). Wikipedia gives the following definition of a BIBD:

> Given a finite set X (of elements called points) and integers k, r, λ 1, we define a 2 - design (or BIBD, standing for balanced incomplete block design) B to be a family of k-element subsets of X, called blocks, such that the number r of blocks containing x in X is not dependent on which x is chosen, and the number λ of blocks containing given distinct points x and y in X is also independent of the choices. A BIBD can be represented by its parameters (b, v, r, k, λ) where v is the number of elements of x, b the number of blocks, r the number of blocks containing a given point, k the number of points in a block and λ the number of blocks containing 2 (or more generally t) points.

The following equations link the parameters:

1. bk = vr
2. λ = r(k-1)/(v-1)

The team had 50 minutes to solve the task. They needed to come to a consensus on their solution since the team was allowed to hand in only one solution. They should have been prepared to solve the problem below: they were familiar with the necessary equations and underlying theory and had done similar questions in lectures.

Task 3 — BIBD construction

Each team will be given a stack of 80 cards (10 of each of 8 colours).

Throughout this task, do not consider BIBDs with $k \leq 2$ or $k \geq v - 2$, as these are in a sense trivial.

Answers with $v > 8$ or $r > 10$ are not acceptable; you must be able to construct each BIBD from the cards provided.

You may write on the cards if you want.

1. Find parameters for the largest* BIBD which *might* be constructed using the cards.
   The parameters must be consistent with the known relationships between the parameters.

2. Construct the largest* BIBD you can with the cards (potentially with the parameters above).

Either: Ask for a photograph to be taken of your BIBD (more than one photo can be taken).
   Or: Accurately describe a construction for your answer BIBD.
   Or: Write out the blocks in any other format.

*The size of a BIBD, for the purpose of marking this task, will be *bk*, the number of cards used. It may not be possible to use all of the cards.

Figure 1. BIBD Task

Solving the Problem with Expert Help

When we worked through the task with the course lecturer, he explained that what we were looking for was an integer value of λ that satisfied the constraints and maximised *bk = vr*. He emphasised that the students knew that "in combinatorics, if your answer is not an integer, you have done something wrong". He added: "Simply put, combinatorics is about counting the number of ways an event or group can occur or be arranged or finding

the optimal case for such a group or event."

We followed his lead as illustrated in Figure 2. We mainly worked in symbolic mode, (Tall, 2008), not even touching the cards, experimenting with different values of *v* and *r* since the constraints on these were explicitly given. We started with the largest possible pair, *v* =10 and *r* = 8, giving a product of 80. The value of λ for this was not an integer for any choices of *b* and *k*; so we tried the pair that gave the next highest product, as depicted. Working through the largest products (*vr*) finds a solution of *v* = 7, *r* = 9, *b* = 21 and *k* = 3, giving $\lambda = 3$.

Figure 2. Working with lecturer

We tried all possible integer products so that *bk* = *vr*, keeping in mind that *k* > 2 and *k* < *v*-2. When we worked together, we skipped over building the BIBD and assumed that if the parameters fit, a BIBD would exist (this is not always true). We should have tried trial and error to construct the BIBD – this is not too hard, especially with a team keeping track of what to try next

Analysis of the Students' Behaviour Based on Conversational Data

At the outset we imagined that the students would attack the problem in a similar manner. This did not prove to be the case. From the very beginning the students ignored the constraints given for *v* and *r* which set them off on what proved to be an unproductive path. Once on this path, their problems were compounded by their over-generalising from earlier examples and from their acceptance of unfounded assumptions.

Ignored Constraints:

There were a total of 612 lines of transcript in the 50 minutes.

> Student C (line 3): We are trying to choose different values of **b and k** to find a different block design that works.

A little later:

> Student C (line 73): So we tested out the largest block design to see if that was possible but our λ was not an integer value so we saw it cannot be done for that b and k.

No one challenged this approach or mentioned the statement about constraints explicitly given in the problem. After 40 minutes their focus is still on maximising *bk.*

Student G (line 397): "We are still maximising bk but now there are 5 varieties so get rid of one colour."

Why did they focus on b and k rather than v and r? And having done so why did they stick with this choice and never consider approaching the problem in a different manner? We suggest that they chose to look at *bk* before *vr* as that approach appeared to fit an enactive or conceptual-embodied mode of understanding best (Bruner, 1966; Tall, 2008). They were given the set of 80 cards and then instructed to create the BIBD using them. It was this that they could choose to have photographed as evidence of their success. The physical construction of a BIBD is more closely related to the product of *bk* than *vr*. The parameters *b* and *k* represent the physical height and width of the block design since *b* = the number of blocks (height) and *k*= size of a block (width). Both *bk* and *vr* describe the magnitude of a BIBD, yet only *bk* defines its physical dimensions.

Their behaviour suggests that they were working in an embodied mode, seeing the variables *b* (number of blocks) and *k* (size of each block) and the cards in a concrete manner. This appears to have inhibited their ability to work symbolically, to grasp the significance of, or even notice, the constraints on *v* and *r*. Was it the presence of the cards that inhibited the move to a symbolic way of working?

Why, when problems arose with using values of *b* and *k* explicitly, did the team not go back and begin using the (constrained) values of *v* and *r* to make their task simpler? Schoenfeld (1985, 1992, 2009) argues that novice problem solvers frequently stick stubbornly to one approach regardless of its success. He describes how they "read, make a decision quickly, and pursue that direction come hell or high water" (Schoenfeld, 2009, p 61). By contrast, an expert problem solver when confronted with a similarly novel task will constantly reassess their method and move back in their problem solving process to find a more apt technique.

Already in trouble, the students then stumbled over a further obstacle in the problem solving path. They over-generalised the relationships between the parameters in earlier examples of BIBDs they had encountered (Olivier, 1989; Mason, 1988; Watson & Mason, 2002): in their 'folding back' they collected inappropriate prior knowledge (Pirie & Kieren, 1994).

Over-Generalisation of Examples:

They were convinced that it is not simply the products *bk* and *vr* that have to be equal, but that the factors also have to be the same. This further constrained the possibilities they could consider and since it is not correct, prevented them finding an optimal solution.

Student G (line 25): "We want to talk about the biggest, so the maximum amount of b, the maximum amount of v."

Student G (line 50): "Yep so 8k equals 8r, so there is k must equal r."

Why did they believe b=v, and k = r? We argue that they over-generalized on the basis of two previous examples (Watson & Mason, 2002). Throughout the course examples, both practical and theoretical, have been used to illustrate key ideas: double counting, recognising symmetries etc. The first example they met was one in which thirteen part time ice cream salesmen tested combinations of two milk shake flavours from thirteen for palatability. The question asked was how many flavours each worker should taste-test in all combinations so that every combination of flavours would be taste-tested exactly once and that every worker would be allocated the same number of flavours to test.

The lasting impact of this example is evidenced in the fact that the students frequently

refer to variables in the current task, which was about coloured cards not about milkshakes, in terms of flavours.

   Student C (line 40): How many flavours are there?

A theoretical network with which students were very familiar is the Fano plane. Both of these examples were 'symmetric'. The BIBD for trialling combinations of milkshake flavours was (*13,13,7,7,4*) and the one for the Fano plane (*7,7,3,3,1*). In both of these *b* was equal to *v* and *k* was equal to *r*. This is not a feature of BIBDs in general and was never stated as a rule by the lecturer. However, it appears to have been picked up by at least the dominant member of the group and then they all assumed it to be true as they tried to work out an arrangement for the cards.

On a number of occasions it is clear the students assume $b = v$ and $k = r$, without the relationship being stated explicitly. Given that the best solution was:

   $(b, v, r, k, \lambda) = (21,7,9,3,3)$

the effect of this misconception was to exclude the best answer and other reasonable answers from the domain of feasible solutions.

The team's initial approach would have suggested that the team's understanding was situated in the physical representation that would lead to the observation that the maximum *b* is 40. Interestingly, however, they later lose sight of this and take the maximum to be 10, the maximum value of *v*. They are using the constraints given for *v* and *r* for *b* and *k*. The team never attempted to use a value of $b > 10$.

   Student B(line 70) : Yeah, I feel like this does but that's just a guess because we started
   from b being the maximum and then we knew that 10 wouldn't work, 9 wouldn't work,
   8 wouldn't work, so we went to seven.

The lecturer expressed the opinion that to him, this was the most surprising (and disappointing) aspect of the group's work on the problem.

Not all previous examples had the same impact on their thinking. They failed to remember that a very similar one they had done in class clearly showed that the magnitude of *bk* and *vr* should be at least 56, it did not show up in their example space (Watson & Mason, 2002). In that case $(b,v,r,k,\lambda) = (14,8,7,4,3)$. The fact that its construction is more complicated than the 'milk shakes' or the very familiar Fano plane may explain why students do not have it readily accessible in their useful example space.

   Towards the end of the session we hear a student say:

   35 was largest in terms of parameters and then 28 was largest we could construct. So 35
   so we have (7, 7, 5, 5 7).

This was the final reference to the biggest BIBD their group could theoretically construct. They in fact settled on a value of 28 when they could not physically construct a 7x5 BIBD. They do acknowledge that they are not really happy with their solution.

   Student B (line 115): It's really unlikely that would be the answer, but this is the largest
   we thought we could construct.

Why were they so sure $b=v$? We contend that they over-generalised on the basis of two previous examples (Watson & Mason, 2002). This is also evidence of their having example spaces that were not sufficiently developed and differentiated to deal with the problem. The team also shows evidence of 'folding back' and collecting inappropriate prior knowledge (Pirie & Kieren, 1994). Folding back and collecting inappropriate prior knowledge offers an explanation for the generation of over-generalisations. Another

explanation could be that they had difficulty in identifying a suitable simpler example – as Polya (1945) would have us do. (Schoenfeld, personal communication)

What clouds our judgement at times like this? It appears that having chosen a path they can see no way of getting off it. It is interesting that two examples had such an impact while the third was forgotten. What is it that makes some examples memorable and readily accessed and indeed over generalised during problem solving? How might we signal the world of difference between the generalising necessary for generating mathematical ideas and connections (Mason, 1988) and potentially dangerous over-generalising that leads to misconceptions(Olivier, 1989)?

In addition to ignoring constraints and over-generalising, the students showed little capacity for challenging assumptions.

Unchallenged Assumptions:

The team based their arguments on a number of ungrounded assumptions. One of them was that $k-\lambda$ must itself be the square of an integer, a notion that the team refer to as "the perfect square." This is a false constraint as it only applies to specific symmetric BIBDs.

> Student G (line 27): I'd have λ equal to 6, then I'd have $(k-\lambda)$ equals to 1 which is equal to 1 squared, so it does satisfy that.

It was at this point in the discussion this constraint first appeared. It was invoked by Student G without any challenge from the rest of the team. The team members take this constraint on board and proceed to work with it in solving the task. This false assumption causes them to not even consider a number of potentially correct solutions. Not until over two thirds of the way through the task does Student G realise the fallacy of this constraint and that $k-\lambda$ is not required to be the square of some other integer.

> Student G (line 213): Now, we'd want everything to be a four. Oh no wait a second, b is not even. b is not even, so $k-\lambda$ does not have to be equal to a square.

Whether the entire group agreed upon this extra constraint or whether the rest of the group were following the lead of a member of high standing within the group is unclear. It is our view that the team was too willing to accept the assertions of the team member who had become established as the leader, or had not developed a mechanism for challenging them.

The team incorporated a second false assumption into their thinking after an interchange with the lecture. What appears to have occurred was that they did not understand that the question the lecturer answered was not the one they were trying to ask. This raises the question of what is attended to by various participants in a conversation. During the discussion of the role of k and λ the students and the lecturer were attending to different parts of what was being said. It is also possible that they take from the fact that the lecturer ignores the statement about $k-\lambda$ being a square that it is correct. Student G had become very unsure about the relationship between k and λ.

> Student G (line 37): Can I ask a question? If this condition here is a perfect square, if you have k and λ, are there any more restrictions on λ in terms of its size relative to these other parameters. Could you have $k-\lambda = 0$?

> Lecturer: if you have $k-\lambda = 0$ what does that mean?

When the lecturer asked the team to consider what it means for the k and λ to be equal, they gave a correct explanation:

Student G (line 41): It means that the amount of pairs, the amount each time a pair appears is the same as the size of the blocks.

However the students take away from this discussion with the lecturer, the rule that k cannot equal λ. The team inferred, from the lecturer posing a question back to them, that he was pointing out an error when he was not. This condition, that he had no intention of imposing, ensures that they will not reach an optimal solution since for this BIBD, $k = \lambda = 3$.

Student G (line 194): But also, k - λ , I don't think that can be smaller than or equal to 0. Because that would mean there is more λ than there is the size of the block.

Student C: No we can't do that.

The group goes on checking that λ is not equal to k for the rest of the task.

Student C (line 403): You can't have λ equal to k.

Student G (line 404): Oh right 5 minus... Yeah ok. Do you think then? ... So maybe we need to go to 7 times 4. Do we have anything other than 7 times 4?

Why did the students accept Student G's assertion about 'the perfect square'? Why did they hear the lecturer's question as a statement? In the final section we consider questions the study raises and ways in which the students might be empowered to avoid the pitfalls in the path and how we, as lecturers, might enable students to behave in a more expert manner: to stand back and look at the path they have chosen and see that it is potentially a dead end.

Questions and Suggestions

Why did none of the students think to go back and re-read the task carefully? Why did they over-generalise the two familiar examples and largely ignore very relevant information from a more directly analogous example? Are there ways of becoming more aware of what aspects of a dialogue different participants attend to? Did the presence of manipulatives somehow inhibit students' 'movement' to thinking symbolically?

To all of these one could argue that when we are confused we do silly or counter-productive things, and the more confused we are the worse it gets. But experts do fewer silly things, self regulate and monitor themselves more effectively and consequently waste less time on fruitless searches than novices do (Schoenfeld, 1992). So the question for us, as lecturers, and for students who want to behave in more expert ways, is how might we encourage and develop expert-like behaviour.

The first concrete suggestion coming out of this study is the nomination of a Devil's Advocate to the team: a person who for the duration of a task is charged with challenging all assumptive statements and monitoring the conversation for mismatches between questions asked and answers given. We suggest that they be charged with asking the following questions: How do you know that …? Do you mean ….? What do we know about ….? What were we told …. ? What are we trying to find? For each new task a new advocate is nominated until all members of the team have played this role.

In successful teams there are almost certainly members, or at least one member, who perform this role. However students all need to be empowered to challenge one another and being a 'designated challenger' would mean that this role does not always fall to the same person. Probably more importantly, less assertive, quieter students need to be mandated to take on this responsibility on a regular basis. It would be interesting to see what impact this might have on students' behaviour when doing tasks alone.

The model of delivery in this course is Team Based Learning (TBL). In this model roles in the team are explicitly not designated, as compared for example with De Bono's hats model (1985). However, the analysis of the team's interactions in this task suggests that in some teams such a person does not emerge spontaneously and that they would play a very useful role. In this study less assertive students in particular might have 'found their voice' through this mechanism and Student G's hold on the decision making process could have been loosened. The peril of ill-founded assumptions may well be more obvious in Mathematics than in some other subjects in which TBL has been used.

This study exposes the role power plays in a group problem solving process. It is clear in this study that more credence was given to both the lecturer's question and Student G's assertions. The lecturer's question was heard as a statement and Student G's statements were not challenged. How might we encourage a climate in which students examine statements critically regardless of who makes them?

In this case the students set off on a path suggested without taking time to process the question. There was no discussion about which side of the equation to exploit and explore, to decide which one related more closely to the given constraints. In 1945 Polya suggested that when struggling to understand the question, one should attempt to rephrase the question in one's own words. We suggest requiring students, before they proceed, to restate the question in their own words. To avoid collusion they write down their interpretation and then share it with the group. At this stage discrepancies can be to be addressed. Provided all students do not make a very similar misinterpretation, this method should lead to clearer conception of what is required for everyone in the team.

We have two suggestions for lecturers. Firstly that we call the students' attention to aspects of examples that are specific to the situation to support the development of a more carefully articulated example space. The students in this study behaved like school students who assume that right angle triangles need to be oriented with their base parallel to the bottom of the page.

Secondly, an open, and for us unanswered question, concerns why using concrete examples appears to have interfered with the students' ability to solve a new problem. In subsequent discussions that lecturer opined that "most year 3 students are mostly at the 'image' levels of the Pirie-Kieren model of understanding (1994) for much of combinatorics – abstraction is hard. Perhaps if the activity had started with the cards, but ended with a question that sits beyond the cards it might have nudged their understanding more effectively."

Concluding Remarks

For novices there are many possible pitfalls along the problem solving path. As we wrote about the three that were evidenced in this particular episode we were constantly aware of our role as lecturers. We wondered what warning signals we could incorporate into our modelling of the problem solving processes in class and how these might alter the students' behaviour. We are interested in exploring the effect a Devil's Advocate might have on the team's ability to develop the monitoring and self-regulatory behaviour so essential to expert problem solving (Schoenfeld, 1992, 2009, 2013). In conclusion imagine if the students had said:

> How do you know it is a perfect square? What do you mean when you say "What do we know about k?" and "Let's all check what we were told carefully." A very different pathway might have emerged.

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Graduate Training and Research: Millennium Tools for Regional Development: An East African Perspective

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The issue of graduate research and training in any country is very important and needs to be critically examined. The material progress of a country is said to be directly proportional to the number of research papers published by the researchers (both Masters and PhDs) in the country. There are many challenges facing graduate schools in East Africa. Some of them are due to internal factors and others due to external factors. Some of the common problems found in public graduate schools in East Africa include weak organization of research, insufficient funding of postgraduate training and research, poor remuneration of research personnel among others. The lack of government and private sector support for universities, especially graduate training and research, complicates matters more. Future sustainability of graduate training and research will require close collaboration between the government, Non-Governmental Organizations (NGOs) and private sector industries with the universities conducting research. This paper briefly explores more of the above issues raised concerning universities, with an East African perspective.

Introduction

There are a number of challenges for graduate schools in East Africa and Africa as a whole. East African countries of Kenya, Uganda and Tanzania have very high poverty levels that inevitably affect higher education funding by government and private sector. This is because those institutions having a lot of constraints would prefer to fund other crucial sectors of the economy. This has led to a number of problems that hinder promotion of postgraduate training and research in public universities. Some of the common problems in public graduate schools include weak organization of research, insufficient capacity building, perceived gaps between academics and policy makers, insufficient funding of postgraduate research and training among others.

To improve the relevance of postgraduate research and training, there is a need for reforms, which will allow higher education to provide greater support to national economic and social development. Regional and institutional collaboration in research will in future constitute key elements of the solution to higher educational challenges.

Discussion

It is a fact that translation of research into development is weak in East African countries. In Kenya, for example, the budget for research is a mere 0.01% of the total budget. With this view of the research by the government there has developed a lack of coherent planning to link research, education and development.

Science and technology are of no use if not put into practice. We are not living in the age of mathematician Gauss who gave a remarkable method of drawing a regular 17-sided figure, which was never put to use anywhere. But Gauss was proud of this and not of the massive contribution he had made to Mathematics and Sciences. Gauss was of the belief that Mathematics should be learnt for its aesthetic sense. Sadly, most of the researches

done in Africa, and especially East African countries do not see the light of the day. They are shelved somewhere to gather dust. Chances of translation of this research into development are very low indeed. As already mentioned public research is not an end itself and this research is intended to generate knowledge that can be used to improve service delivery systems, policies and practices.

For use of the research, potential end users especially the government must appreciate the relevance of particular findings and commit themselves to use them. One way of achieving this is by involving potential users of the research in the research process itself. However this noble idea may be problematic to implement in countries like Kenya and in the East African countries in general, because the countries mentioned do not have a clear research plan at a national level based on their national needs. Subsequently this makes it difficult for training institutions to develop properly their research plans tailored to national needs.

The graduate schools are not entirely blameless in this aspect of lack of properly guided research plans. Most of the universities in East Africa tailor their research to attract donor funding. As such their ownership of the research inevitably declines. This brings about a number of problems.

Firstly, it continues to perpetuate the myth in the government and private circles that universities are just elitist and completely autonomous from the rest of the society. As such, nothing that is done there is of possible use to the general public. After all the universities are said to be doing research for the consumption of a completely different audience, people resident in donor countries

Secondly, the donor agencies tend to fund only research in particular pet areas, meaning the universities just tailor their research to meet those demands and the vicious circle goes on. That's why the universities may over-concentrate in one area of research. As a result they can get personnel only trained in that restricted area (leaving certain areas of research grossly neglected) with anticipated problems when donors leave, as they eventually do.

Thirdly, it is a fact that most institutions lack adequate skills to manage research systems. The coordination among the departments in a single university is one area that needs a mention. Efforts to establish a multidisciplinary research team effort are for the most part lacking. Each department will do its own thing without involving others.

How can the gap between academics who do research and policy makers in the government and the private sectors be bridged? Key persons who will support the implementation of the research findings need to be involved in the research. This will partially find the solution for perennial problems like lack of research funds and facilities as those can be easily provided if there is goodwill from the policy makers.

Dissemination of research findings needs to be improved in order to benefit the intended beneficiary. The research institutions need to open themselves up by "taking" their research to the various decision makers in the government and private sectors. The decision makers in the government and private sector will then be aware of what kind of research is being undertaken in their areas of jurisdiction. As already stated most graduate schools undertake their research in isolation such that collaborations between local universities (and even departments in a single university) are very rare. Talk of one hand not knowing what the other is doing and you have your point.

The process of cross-country partnerships especially between East African universities is very important. Some of the reasons for this are stated below.

Firstly, this is cost effective. Collaboration usually takes care of limited resources. For example universities in Nordic countries have perfected this system. Since those

countries have small populations per each country, a Nordic Academy for Advanced studies was established in 1990 in order to bring closer collaborations there.

Secondly, the 'donor' funding in this era is geared towards collaborative research. As such internationalization of research training is the key. Funding diversification should be sought. Reliance on donor funding from limited sources particularly Western countries is not sustainable in the long run. The research undertaken should be applicable in a wide range of settings. Management of graduate schools in East Africa should be streamlined. The main thrust of the research should be geared towards adequate response to individual, institutional and societal needs and aspirations. Graduate training is the engine behind research and development of any country. In addition postgraduate research students are mainly the vehicles of development of any country. The material progress of any country is directly proportional to the number of research papers published by researchers (Masters and Ph.D.s in that country).

Supervision of research and postgraduate training is part of any university's core business. Universities ought to recognize that the work of postgraduate research students forms a vital part of any institution's overall research. Students contribute to a university research profile. Hence excellence in supervisory practice should be stressed as it helps the students fulfill their potential and also contributes to the institution's research profile.

As already stated the success of postgraduate training and research depends on the individual supervisors of the students, the institution pooling with respect to postgraduate research and the extent to which administrative structures and procedures are designed to assist research students. Examples of the above are the libraries and support for direct costs of student research. In many universities in Europe and America, where success of postgraduate research has been seen, it is assumed that acceptance of student into a Doctoral programme or into a Masters programme implies that the direct costs necessary for research are available.

Postgraduate students should be represented in relevant postgraduate committees whose mandate should include acting as a formal mechanism to develop policy, monitor students' progress and make recommendations about infrastructure resources and their allocation.

Postgraduate students should be encouraged to attend interest group seminars that should be made mandatory. This would result in a pool of ideas, which can benefit everybody.

In addition, high quality education and training of research students require appropriate infrastructure resources provided by the institution. This is because completion rates, submission time and satisfaction with the graduate research programmes are closely related to infrastructure facilities provided by the university, for example, provision of computer facilities for data analysis, writing and information access. Post graduate students, consistent with supervision policy should be encouraged to present their work at conferences.

The universities can sponsor around 50% of travel costs and conference registration. To ensure sustainability of graduate training there is a need to increase scholarships and stipends, and encourage close collaboration between the government, NGOs and private sector industries with the universities conducting research. When scholarships are awarded, the university can save by giving scholarships to post graduate students who can be required to teach the undergraduates some units. The money that could have been used to pay part-timers can be directed to support research activities instead.

The university management should be more proactive and solicit for research funds from various sources. The Vice Chancellors as the main fundraisers should spearhead this

effort. In order to facilitate this more, university management can delegate and therefore provide the Deans of Faculties and Directors of Institutes with facilities like vehicles and allowances to enable them to travel, attend seminars and visit other places where they can solicit for research funds.

In the same vein more money should be allocated to provide publicity of the programmes offered by the universities and especially the research being conducted. In this sense marketing is very important. The current situation where residents of Kenya, for instance, cannot tell what postgraduate programmes are offered at Makerere University, as well research being conducted there, is very sad indeed.

Marketing should be done through advertisements in newspapers, magazines, and use of brochures, posters and educational exhibitions/seminars. The use of current technology like Internet is also very important. Facilitation for senior university staff to travel throughout the country and to foreign countries as well to sell the postgraduate programmes is imperative. As well as doing all the above the remuneration of university researchers needs to be looked at afresh. This is to remove the current brain drain to better paying research jobs in developed countries, which, if let to continue, will eventually make the research work in East African countries unsustainable.

Recommendations

Firstly, regular checks on progress of postgraduate students should be done. This is through conducting of seminars for Masters and PhD students where they can exchange their views. Universities should support those seminars.

Secondly, there should be a decrease in the duration taken to complete Masters and PhD studies. This is partly caused by the delay by external examiners in submitting their reports. For this to be successful board members of defence committees should be duly rewarded.

Thirdly, personnel manning Boards of Postgraduate Studies in East African countries should meet periodically to discuss and sort out the common problems facing the Boards. Related to this latter point is the need for universities to ensure there are adequate facilities in personnel and finances and in general infrastructure such as communication and accommodations. There should be internationalization of research training to ensure international standards for national research.

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