Towards the Jacquet Conjecture on the Local Converse Problem for p-adic GL_n

Dihua Jiang Chufeng Nien Shaun Stevens

Abstract

The Local Converse Problem is to determine how the family of the local gamma factors $\gamma(s, \pi \times \tau, \psi)$ characterizes the isomorphism class of an irreducible admissible generic representation π of $\mathrm{GL}_n(F)$, with F a non-archimedean local field, where τ runs through all irreducible supercuspidal representations of $\mathrm{GL}_r(F)$ and r runs through positive integers. The Jacquet conjecture asserts that it is enough to take $r = 1, 2, \ldots, \left[\frac{n}{2}\right]$. Based on arguments in the work of Henniart and of Chen giving preliminary steps towards the Jacquet conjecture, we formulate a general approach to prove the Jacquet conjecture. With this approach, the Jacquet conjecture is proved under an assumption which is then verified in several cases, including the case of level zero representations.

Keywords. Irreducible admissible representation, Whittaker model, Local gamma factor, Local converse theorem

1 Introduction

Let π be an irreducible (admissible) generic representation of $\mathrm{G}_n := \mathrm{GL}_n(F)$, where F is a locally compact non-archimedean local field. We may assume that $n \geqslant 2$, since the discussion in this paper for $n = 1$ is trivial. Attached to π is the family of local gamma factors $\gamma(s, \pi \times \tau, \psi)$, with τ any irreducible generic admissible representations of any G_r, in the sense of Jacquet, Piatetski-Shapiro and Shalika ([13]), which can also be defined through the Langlands-Shahidi method ([19]). Here ψ is a nontrivial additive character

D. Jiang: School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA; e-mail: dhjiang@math.umn.edu

C. Nien: Department of Mathematics, National Cheng Kung University and National Center for Theoretical Sciences (South), Tainan 701, Taiwan; e-mail: nienpig@mail.ncku.edu.tw

S. Stevens: School of Mathematics, University of East Anglia, Norwich Research Park, Norwich, NR4 7TJ, UK; e-mail: Shaun.Stevens@uea.ac.uk

Mathematics Subject Classification (2010): Primary 11F70, 22E50; Secondary 11F85, 22E55

of F; the definition of this family of local gamma factors is recalled in Section 2. It is natural to ask how this family of invariants yields information about the representation π.

In this paper, we consider the *Local Converse Problem* for G_n, which is to find the least integer n_0 such that the family of local gamma factors $\gamma(s, \pi \times \tau, \psi)$, with τ running through all irreducible generic representations of G_r for $r = 1, \ldots, n_0$, determines the irreducible generic representation π of G_n up to isomorphism. It is an easy consequence of the work of Jacquet, Piatetski-Shapiro and Shalika ([13]) that $n_0 \leqslant n$. The work of Henniart ([11]) shows that $n_0 \leqslant n - 1$, and the works of J.-P. Chen ([7] and [8]) and Cogdell and Piatetski-Shapiro ([9]) show that $n_0 \leqslant n-2$, for $n \geqslant 3$. A stronger statement (when $n > 4$) is the following conjecture, which is usually attributed to H. Jacquet.

Conjecture 1.1 (The Jacquet Conjecture on the Local Converse Problem). *Let π_1 and π_2 be irreducible generic smooth representations of G_n. If their local gamma factors $\gamma(s, \pi_1 \times \tau, \psi)$ and $\gamma(s, \pi_2 \times \tau, \psi)$ are equal, as functions in the complex variable s, for all irreducible generic representations τ of G_r, with $r = 1, \ldots, \left[\frac{n}{2}\right]$, then π_1 and π_2 are equivalent as representations of G_n.*

It is clear that the work ([11, 7, 9, 8]) confirms that for $2 \leqslant n \leqslant 4$, Conjecture 1.1 is a theorem. Indeed, for $n = 2$, Conjecture 1.1 was proved in 1970 by Jacquet and Langlands in their well-known book ([12]), and for $n = 3$ it was proved in 1979 by Jacquet, Piatetski-Shapiro and Shalika ([14]). Following a standard argument, which was already known to the experts in the 1980s, we deduce in Section 2.4 that Conjecture 1.1 is equivalent to the following conjecture.

Conjecture 1.2. *Assume that π_1 and π_2 are irreducible unitarizable supercuspidal representations of G_n. If their local gamma factors $\gamma(s, \pi_1 \times \tau, \psi)$ and $\gamma(s, \pi_2 \times \tau, \psi)$ are equal as functions in the complex variable s, for all irreducible supercuspidal representations τ of G_r with $r = 1, \ldots, \left[\frac{n}{2}\right]$, then π_1 and π_2 are equivalent as representations of G_n.*

Since any irreducible supercuspidal representation of G_n has a nontrivial Whittaker model, it is natural to use this property, combined with the local functional equation of the local Rankin–Selberg convolution for $\mathrm{G}_n \times \mathrm{G}_r$, to figure out a possible approach to prove Conjecture 1.2. This is in fact the idea behind the previous attacks on the Local Converse Problem ([14, 11, 7, 8]). In this paper, we add a new idea to the argument in order to attempt to reduce the twists down to $r = 1, \ldots, \left[\frac{n}{2}\right]$, i.e. Conjecture 1.2. The idea is to find Whittaker functions satisfying some special properties.

Let U_n be the unipotent radical of the standard Borel subgroup B_n of G_n, which consists of all upper-triangular matrices. Denote by P_n the mirabolic subgroup of G_n, consisting of matrices with last row equal to $(0, \ldots, 0, 1)$. We also fix a standard non-degenerate character ψ_n of U_n (see Section 2.1) so that all Whittaker functions are implicitly ψ_n-Whittaker functions.

Definition 1.3. Let π be an irreducible unitarizable supercuspidal representations of G_n and let $\mathbf{K}$ be a compact-mod-centre open subgroup of G_n. A (non-zero) Whittaker function W_π for π is called $\mathbf{K}$-*special* if it satisfies:

$$W_{\pi_i}(g^{-1}) = \overline{W_{\pi_i}(g)} \text{ for all } g \in \mathbf{K},$$

and $\mathrm{Supp} W_\pi \subset \mathrm{U}_n\mathbf{K}$, where $^-$ denotes complex conjugation.

Definition 1.4. Let π_1 and π_2 be irreducible unitarizable supercuspidal representations of G_n with the same central character. Let W_{π_1} and W_{π_2} be (non-zero) Whittaker functions for π_1 and π_2, respectively. We call (W_{π_1}, W_{π_2}) a *special pair* (of Whittaker functions) for (π_1, π_2) if there exists a compact-mod-centre open subgroup $\mathbf{K}$ of G_n such that W_{π_1} and W_{π_2} are both $\mathbf{K}$-special and

$$W_{\pi_1}(p) = W_{\pi_2}(p), \text{ for all } p \in \mathrm{P}_n.$$

If a special pair of Whittaker functions as in Definition 1.4 exists for (π_1, π_2), we can prove that the representations π_1 and π_2 are distinguished by their families of local gamma factors $\gamma(s, \pi_i \times \tau, \psi)$, for τ irreducible supercuspidal representations of G_r, with $r = 1, \ldots, \left[\frac{n}{2}\right]$, by using a refinement of the argument in [7] and [8]. This approach was successfully carried out by the second-named author in [17] for general linear groups over finite fields. The key point is to find a refined decomposition for G_n which reflects the symmetry carried in Definition 1.3. We recall in Section 3.1 this refined decomposition.

Theorem 1.5. *Let π_1 and π_2 be irreducible unitarizable supercuspidal representations of G_n. Assume that a special pair (W_{π_1}, W_{π_2}) exists for (π_1, π_2). If the local gamma factors $\gamma(s, \pi_1 \times \tau, \psi)$ and $\gamma(s, \pi_2 \times \tau, \psi)$ are equal as functions in the complex variable s, for all irreducible supercuspidal representations τ of G_r with $r = 1, \ldots, \left[\frac{n}{2}\right]$, then $W_{\pi_1} = W_{\pi_2}$ and π_1 and π_2 are equivalent as representations of G_n.*

In certain cases, one can prove the existence of special pairs for irreducible unitarizable supercuspidal representations of G_n by using the construction of supercuspidal representations in terms of maximal simple types of Bushnell and Kutzko ([6]) and the explicit construction of Bessel functions of supercuspidal representations due to Paškūnas and the third-named author ([18]). Given an irreducible supercuspidal representation π of G_n, one of the invariants associated to it, by Bushnell and Henniart in [2], is its *endo-class* $\Theta(\pi)$. We prove:

Proposition 1.6. *Let π_1, π_2 be irreducible unitarizable supercuspidal representations of G_n with the same endo-class. Then there is a special pair (W_{π_1}, W_{π_2}) for (π_1, π_2).*

Theorem 1.5 with Proposition 1.6 implies, for example, that two level zero irreducible unitarizable supercuspidal representations π_1, π_2 of G_n can be distinguished by the set of local gamma factors $\gamma(s, \pi_i \times \tau, \psi)$, for all irreducible supercuspidal representations τ of G_r with $r = 1, 2, \ldots, \left[\frac{n}{2}\right]$. In fact, this is a special case of a more general result, as follows.

Attached to an irreducible supercuspidal representation π of G_n, via its endo-class $\Theta(\pi)$, is an invariant which we call its *degree* $\deg(\pi)$. The degree is an integer dividing n: for example, $\deg(\pi) = 1$ if and only if π is a twist of a level zero representation; and if $\deg(\pi) < n$ then π is invariant under a non-trivial unramified character twist, though the converse is not true. By using formulae on the conductors of pairs of supercuspidal representations from [5, 4], we immediately obtain the following corollary.

Corollary 1.7. *Let π_1 and π_2 be irreducible unitarizable supercuspidal representations of G_n and suppose that $\deg(\pi_1) < n$. If the local gamma factors $\gamma(s, \pi_1 \times \tau, \psi)$ and $\gamma(s, \pi_2 \times \tau, \psi)$ are equal as functions in the complex variable s, for all irreducible supercuspidal representations τ of G_r with $r = 1, \ldots, \left[\frac{n}{2}\right]$, then π_1 and π_2 are equivalent as representations of G_n.*

We end the paper with some discussion of the scope of the methods used here, in particular of the obstacles to extending to the case $\deg(\pi_1) = n$ (see Remark 5.3).

2 Basics in the local Rankin–Selberg convolution

We start by recalling the basic facts about Whittaker models of irreducible generic representations of G_n and local gamma factors of Rankin–Selberg convolution type over the locally compact non-archimedean local field F. We denote by $\mathfrak{o}_F$ the ring of integers in F, by $\mathfrak{p}_F$ the prime ideal in $\mathfrak{o}_F$, and by $\mathfrak{k}_F$ the residue field of F, of cardinality q and characteristic p; we also write $|\cdot|$ for the absolute value on F, normalized to have image $q^{\mathbb{Z}}$. We use analogous notation for extensions of F. We also fix, once and for all, an additive character ψ of F which is trivial on $\mathfrak{p}_F$ but nontrivial on $\mathfrak{o}_F$.

2.1 Whittaker models

Let Q_n be the standard parabolic subgroup of G_n corresponding to the partition $(n-1, 1)$. Then

$$\mathrm{Q}_n = \mathrm{Z}_n \mathrm{P}_n,$$

where Z_n is the center of G_n, and P_n is the mirabolic subgroup.

Definition 2.1. A character ψ_{U_n} of U_n is called *non-degenerate* if its normalizer in B_n is $\mathrm{Z}_n\mathrm{U}_n$. We denote by ψ_n the *standard* non-degenerate character given by

$$\psi_n(u) = \psi\left(\sum_{i=1}^{n-1} u_{i,i+1}\right),$$

for $u = (u_{i,j}) \in \mathrm{U}_n$.

We call an irreducible smooth representation (π, V_π) of G_n *generic* if there is a non-degenerate character ψ_{U_n} of U_n such that the Hom-space

$$\mathrm{Hom}_{\mathrm{G}_n}(V_\pi, \mathrm{Ind}_{\mathrm{U}_n}^{\mathrm{G}_n}(\psi_{\mathrm{U}_n}))$$

is nonzero. By the uniqueness of local Whittaker models, this Hom-space is at most one-dimensional. Since the non-degenerate characters of U_n are all conjugate under B_n, we see that π is generic if and only if

$$\mathrm{Hom}_{\mathrm{G}_n}(V_\pi, \mathrm{Ind}_{\mathrm{U}_n}^{\mathrm{G}_n}(\psi_n)) \cong \mathrm{Hom}_{\mathrm{U}_n}(V_\pi|_{\mathrm{U}_n}, \psi_n)$$

is non-zero, where the isomorphism comes from Frobenius reciprocity.

Assume that π is generic. We fix a nonzero functional ℓ in $\mathrm{Hom}_{\mathrm{U}_n}(V_\pi|_{\mathrm{U}_n}, \psi_n)$ (which is unique up to scalar). The *Whittaker function* attached to a vector $v \in V_\pi$ is defined by

$$W_v(g) := \ell(\pi(g)(v)), \text{ for all } g \in \mathrm{G}_n.$$

It is easy to see that W_v belongs to $\mathrm{Ind}_{\mathrm{U}_n}^{\mathrm{G}_n}(\psi_n)$ and

$$\mathcal{W}(\pi, \psi_n) := \{W_v \mid v \in V_\pi\}$$

is called the ψ_n-*Whittaker model* of π, or simply the Whittaker model of π. It is clear that the Whittaker model of π is independent of the choice of the nonzero functional ℓ.

For any $W_v \in \mathcal{W}(\pi, \psi_n)$, define

$$\widetilde{W_v}(g) := W_v(w_n \cdot {}^t g^{-1}),$$

for $g \in \mathrm{G}_n$, where w_n is the longest Weyl group element of G_n, with 1's on the second diagonal and zeros elsewhere, and ${}^t g$ denotes the transpose of g. Then one can check that the function $\widetilde{W_v}$ belongs to the ψ_n^{-1}-Whittaker model of the contragredient $\widetilde{\pi}$ of π, that is,

$$\widetilde{W_v} \in \mathcal{W}(\tilde{\pi}, \psi_n^{-1}) \subset \mathrm{Ind}_{\mathrm{U}_n}^{\mathrm{G}_n}(\psi_n^{-1}).$$

It is a basic fact that any irreducible supercuspidal representation of G_n is generic ([10, Theorem B]). We recall the following properties of the restriction of an irreducible generic representation (π, V_π) of G_n to the subgroup P_n, which can be viewed as the starting point of our approach to prove the Jacquet conjecture for G_n.

Theorem 2.2 ([1, §5])**.** *With the notation fixed as above, the following hold.*

(i) $\mathrm{Ind}_{\mathrm{U}_n}^{\mathrm{P}_n}(\psi_n)$ *is irreducible as a representation of* P_n.

(ii) *If* π *is a generic representation of* G_n*, then* $\mathrm{Ind}_{\mathrm{U}_n}^{\mathrm{P}_n}(\psi_n)$ *is a* P_n*-subrepresentation of* $\mathcal{W}(\pi, \psi_n)|_{\mathrm{P}_n}$.

(iii) *If* π *is an irreducible supercuspidal representation of* G_n*, then* $\pi|_{\mathrm{P}_n}$ *is equivalent to* $\mathrm{Ind}_{\mathrm{U}_n}^{\mathrm{P}_n}(\psi_n)$ *as representations of* P_n.

2.2 Local gamma factors

Next we review the basic setting of local gamma factors attached to a pair of irreducible generic representations, for details of which we refer to [13].

Let $n, r \geqslant 1$ be integers and let π and τ be irreducible generic representations of G_n and G_r, respectively, with central characters ω_π and ω_τ respectively. Let $W_\pi \in \mathcal{W}(\pi, \psi_n)$ be a Whittaker function of π and $W_\tau \in \mathcal{W}(\tau, \psi_r^{-1})$ be a Whittaker function of τ. Since it is the only case of interest to us here, we suppose that $n > r$.

If j is an integer for which $n - r - 1 \geqslant j \geqslant 0$, a local zeta integral for the pair (π, τ) is defined by

$$\mathcal{Z}(W_\pi, W_\tau, s; j) := \int_g \int_x W_\pi \begin{pmatrix} g & 0 & 0 \\ x & \mathrm{I}_j & 0 \\ 0 & 0 & \mathrm{I}_{n-r-j} \end{pmatrix} W_\tau(g) |\det g|^{s - \frac{n-r}{2}} dx dg,$$

where the integration in the variable g is over $\mathrm{U}_r \backslash \mathrm{G}_r$ and the integration in the variable x is over $\mathrm{Mat}_{j \times r}(F)$. Jacquet, Piatetski-Shapiro, and Shalika proved in [13] the following theorem.

For $g \in \mathrm{G}_n$, we denote by R_g the right translation action of g on functions from G_n to $\mathbb{C}$, and we put $w_{n,r} = \begin{pmatrix} \mathrm{I}_r & 0 \\ 0 & w_{n-r} \end{pmatrix}$.

Theorem 2.3 ([13, Section 2.7]). *With notation as above, the following hold.*

(i) *Each integral $\mathcal{Z}(W_\pi, W_\tau, \Phi, s; j)$ is absolutely convergent for* $\mathrm{Re}(s)$ *sufficiently large and is a rational function of q^{-s}. More precisely, for fixed j, the integrals $\mathcal{Z}(W_\pi, W_\tau, s; j)$ span a fractional ideal (independent of j)*

$$\mathbb{C}[q^s, q^{-s}] L(s, \pi \times \tau)$$

of the ring $\mathbb{C}[q^s, q^{-s}]$, where the local L-factor $L(s, \pi \times \tau)$ has the form $P(q^s)^{-1}$, with $P \in \mathbb{C}[x]$ and $P(0) = 1$.

(ii) *For $n - r - 1 \geqslant j \geqslant 0$, there is a factor $\epsilon(s, \pi \times \tau, \psi)$ independent of j, such that*

$$\frac{\mathcal{Z}(R_{w_{n,r}} \widetilde{W}_\pi, \widetilde{W}_\tau, 1 - s; n - r - j - 1)}{L(1 - s, \widetilde{\pi} \times \widetilde{\tau})} = \omega_\tau(-1)^{n-1} \epsilon(s, \pi \times \tau, \psi) \frac{\mathcal{Z}(W_\pi, W_\tau, s; j)}{L(s, \pi \times \tau)}.$$

(iii) *There are $c \in \mathbb{C}^\times$ and $f = f(\pi \times \tau, \psi) \in \mathbb{Z}$ such that*

$$\epsilon(s, \pi \times \tau, \psi) = c q^{-fs}.$$

The local gamma factor attached to a pair of representations π and τ is defined in [13] by

$$\gamma(s,\pi\times\tau,\psi)=\epsilon(s,\pi\times\tau,\psi)\frac{L(1-s,\widetilde{\pi}\times\widetilde{\tau})}{L(s,\pi\times\tau)}. \tag{2.4}$$

Then the functional equation in Part (ii) of Theorem 2.3 can be rewritten

$$\mathcal{Z}(R_{w_{n,r}}\widetilde{W_\pi},\widetilde{W_\tau},1-s;n-r-j-1)=\omega_\tau(-1)^{n-1}\gamma(s,\pi\times\tau,\psi)\mathcal{Z}(W_\pi,W_\tau,s,j). \tag{2.5}$$

We also remark that the local gamma factor $\gamma(s,\pi\times\tau,\psi)$ determines the conductor $f(\pi\times\tau,\psi)$, since it is the leading power of q^{-s} in a power series expansion for $\gamma(s,\pi\times\tau,\psi)$.

2.3 Central characters

In this section, we show that the well-known result that local gamma factors determine the central character. We begin by recalling the following result on the stability of local gamma factors, which follows from [15, Proposition 2.7].

Proposition 2.6. *Let π be an irreducible generic representation of G_n with $n\geq 2$. Then there exits m_π such that, for any character χ of $F^\times$ of conductor $m\geqslant m_\pi$ and any $c\in\mathfrak{p}^{-m}$ satisfying $\chi(1+x)=\psi(cx)$, for $x\in\mathfrak{p}_F^{[\frac{m}{2}]+1}$, we have*

$$L(s,\pi\times\chi)=1 \text{ and } \epsilon(s,\pi\times\chi,\psi)=\omega_\pi(c)^{-1}\epsilon(s,1\times\chi,\psi)^n.$$

Proof. Although this is not quite the statement of [15, Proposition 2.7], this statement is included in the proof (see page 323 of *op. cit.*). □

Corollary 2.7. *Let π_1, π_2 be irreducible generic representations of G_n. If their local gamma factors $\gamma(s,\pi_1\times\chi,\psi)$ and $\gamma(s,\pi_2\times\chi,\psi)$ are equal as functions in the complex variable s, for any character χ of $F^\times$, then $\omega_{\pi_1}=\omega_{\pi_2}$.*

Proof. For $i=1,2$, let $m_{\pi_i},m_{\widetilde{\pi}_i}$ be the numbers given by Proposition 2.6 and put $m_0=\max\{m_{\pi_i},m_{\widetilde{\pi}_i}\mid i=1,2\}$. For χ a character of $F^\times$ of conductor $m\geqslant m_0$, we have $\epsilon(s,\pi_i\times\chi,\psi)=\gamma(s,\pi_i\times\chi,\psi)$, by (2.4) and Proposition 2.6.

For any $c\in\mathfrak{p}^{-m}\setminus\mathfrak{p}^{1-m}$, with $m\geqslant m_0$, there exists a character χ_c character of conductor m such that $\chi_c(1+x)=\psi(cx)$, for $x\in\mathfrak{p}_F^{[\frac{m}{2}]+1}$; thus Proposition 2.6 implies

$$\omega_{\pi_1}(c)=\omega_{\pi_2}(c).$$

Since any element of $F^\times$ can be expressed as the quotient of two elements of valuation at most $-m$, we deduce that $\omega_{\pi_1}=\omega_{\pi_2}$. □

2.4 Reduction from generic to supercuspidal

This section is devoted to reducing Conjecture 1.1 to Conjecture 1.2. In other words, if the Local Converse Theorem for twisting by generic representations of rank up to $\left[\frac{n}{2}\right]$ holds for unitarizable supercuspidal representations, then it also holds for general generic smooth representations.

Let π be an irreducible generic smooth representation of G_n. From the classification of irreducible smooth representations of G_n [20, Theorem 9.7], π is the unique irreducible generic subquotient of a standard parabolically induced representation

$$\tau_1|\cdot|^{z_1} \times \cdots \times \tau_t|\cdot|^{z_t},$$

where each τ_i is an irreducible unitarizable supercuspidal representation of G_{n_i}, with $n = \sum_{i=1}^t n_i$, and

$$z_1 \geqslant \cdots \geqslant z_t$$

are real numbers. Moreover $(\tau_1, \ldots, \tau_t)$ and $(z_1, \ldots, z_t)$ are uniquely determined up to a permutation σ such that $z_{\sigma(i)} = z_i$, and any such tuples give rise to an irreducible generic representation of G_n in this way. By the multiplicativity of the local gamma factors ([13, Theorem 3.1]),we have

$$\gamma(s, \pi \times \tau, \psi) = \prod_{i=1}^{t} \gamma(s + z_i, \tau_i \times \tau, \psi), \tag{2.8}$$

for all irreducible generic representations τ of G_r. We also observe that there is at most one index i such that $n_i > \left[\frac{n}{2}\right]$.

Proposition 2.9 ([16, Section 3.2])**.** *With notation as above, assume that τ is irreducible, unitarizable and supercuspidal.*

(i) *If $\prod_{i=1}^t \gamma(s+z_i, \tau_i \times \tau, \psi)$ has a real pole (respectively, zero) at $s = s_0$, then $\tau \simeq \widetilde{\tau}_i$ and $s_0 = 1 - z_i$ (respectively, $s_0 = -z_i$), for some $i \in \{1, \ldots, t\}$.*

(ii) *For each $j = 1, \ldots, t$, the product $\prod_{i=1}^t \gamma(s + z_i, \tau_i \times \widetilde{\tau}_j, \psi)$ has a real pole and zero. Moreover, if $j = 1$ then there is a zero at $s = -z_1$, and if $j = t$ then there is a pole at $s = 1 - z_t$.*

Note that the assumption in [16], that F is of characteristic zero, is not used in the proof of this since it requires only the multiplicativity of local gamma factors.

Corollary 2.10. *With notation as above, suppose also that τ_i' are irreducible unitarizable supercuspidal representation of $\mathrm{G}_{n_i'}$, for $1 \leqslant i \leqslant t'$, with $n = \sum_{i=1}^{t'} n_i'$, and that $z_1' \geqslant \cdots \geqslant z_{t'}'$ are real numbers. Suppose $m \geqslant \left[\frac{n}{2}\right]$ and*

$$\prod_{i=1}^{t} \gamma(s + z_i, \tau_i \times \tau, \psi) = \prod_{i=1}^{t'} \gamma(s + z_i', \tau_i' \times \tau, \psi),$$

for all irreducible unitarizable supercuspidal representations τ *of* G_r*, with* $r = 1, 2, \ldots, m$*. Then* $t = t'$ *and there is a permutation* σ *of* $\{1, \ldots t\}$ *such that:*

(i) $n_i = n'_{\sigma(i)}$*, for all* $i = 1, \ldots, t$*;*

(ii) $\gamma(s + z_i, \tau_i \times \tau, \psi) = \gamma(s + z'_{\sigma(i)}, \tau'_{\sigma(i)} \times \tau, \psi)$*, for all irreducible unitarizable supercuspidal representations* τ *of* G_r*, with* $r = 1, 2, \ldots, m$ *and* $i = 1, \ldots, t$*;*

(iii) $\tau_i \simeq \tau'_{\sigma(i)}$ *and* $z_i = z'_{\sigma(i)}$*, for all* i *such that* $n_i \leqslant \left[\frac{n}{2}\right]$*.*

Proof. The proof is by induction on t. If $t = 1$ but $t' > 1$ then $n'_j \leqslant \left[\frac{n}{2}\right]$, for some j, and, by Proposition 2.9(i), $\prod_{i=1}^{t'} \gamma(s + z'_i, \tau'_i \times \widetilde{\tau}'_j, \psi)$ has a real pole while $\gamma(s + z_1, \tau_1 \times \widetilde{\tau}'_j, \psi)$ does not, which is absurd. Thus $t' = 1$ and there is nothing more to prove.

Now assume $t \geqslant 2$ and note that either n_1 or n_t is at most $\left[\frac{n}{2}\right]$. Suppose first that $n_t \leqslant \left[\frac{n}{2}\right]$. Then $\prod_{i=1}^{t} \gamma(s + z_i, \tau_i \times \widetilde{\tau}_t, \psi)$ has a pole at $1 - z_t$ so, by Proposition 2.9(i), there is an integer $1 \leqslant j \leqslant t'$ such that $\tau'_j \simeq \tau_t$ and $z'_j = z_t$. Hence $\tau'_j|\cdot|^{z'_j} \simeq \tau_t|\cdot|^{z_t}$ and $\gamma(s + z_t, \tau_t \times \tau, \psi) = \gamma(s + z'_j, \tau'_j \times \tau, \psi)$, for all irreducible generic representations τ of G_r, for all r. In particular, we deduce

$$\prod_{i=1}^{t-1} \gamma(s + z_i, \tau_i \times \tau, \psi) = \prod_{i=1, i \neq j}^{t'} \gamma(s + z'_i, \tau'_i \times \tau, \psi),$$

for all irreducible unitarizable supercuspidal representations τ of G_r, with $r = 1, 2, \ldots, m$. The result now follows from the inductive hypothesis.

Finally, if $n_1 \leqslant \left[\frac{n}{2}\right]$ then $\prod_{i=1}^{t} \gamma(s + z_i, \tau_i \times \widetilde{\tau}_1, \psi)$ has a zero at $-z_1$ so, by Proposition 2.9(i), there is an integer $1 \leqslant j \leqslant t'$ such that $\tau'_j \simeq \tau_1$ and $z'_j = z_1$. The result then follows as in the first case. □

Putting Corollary 2.10 with $m = \left[\frac{n}{2}\right]$ together with the multiplicativity of local gamma factors (2.8) and the classification of irreducible generic representations [20, Theorem 9.7], we see that Conjecture 1.2 implies Conjecture 1.1.

3 Special pairs and the local converse theorem

3.1 Preliminary results

We begin by recalling some useful lemmas, which form the technical steps of the proof.

Lemma 3.1 ([15, Section 3.2])**.** *Let* t *be a positive integer and let* H *be a complex smooth function on* G_t *with compact support modulo* U_t *satisfying*

$$H(ug) = \psi_t(u)H(g),$$

for all $u \in \mathrm{U}_t, g \in \mathrm{G}_t$*. If*

$$\int_{\mathrm{U}_t \backslash \mathrm{G}_t} H(g) W_\tau(g) dg = 0,$$

for all $W_\tau \in \mathcal{W}(\tau, \psi_t^{-1})$*, with* τ *running through all irreducible generic representations of* G_t*, then* $H \equiv 0$*.*

From [8, Section 3.1], we have the generalized Bruhat decomposition:

$$\mathrm{G}_n = \bigsqcup_{i=0}^{n-1} \mathrm{U}_n \alpha^i \mathrm{Q}_n,$$

where $\alpha = \begin{pmatrix} 0 & \mathrm{I}_{n-1} \\ 1 & 0 \end{pmatrix}$.

Definition 3.2. Given two functions H_1 and H_2 on G_n, if

$$H_1(x) = H_2(x), \text{ for all } x \in \mathrm{U}_n \alpha^i \mathrm{Q}_n,$$

then we say that H_1 and H_2 *agree on height* i*.*

Assume that (W_{π_1}, W_{π_2}) is a special pair for (π_1, π_2), as in Definition 1.4, so that W_{π_1} and W_{π_2} agree on height $i = 0$. The condition on local gamma factors in the statement Conjecture 1.2, via Corollary 2.7 and the following proposition, implies the agreement of W_{π_1} and W_{π_2} on height i, for $i = 0, \ldots, \left[\frac{n}{2}\right]$.

Proposition 3.3 ([8, Proposition 3.1])**.** *Fix an integer* $1 \leqslant r < n$*. Let* π_1 *and* π_2 *be irreducible supercuspidal representations of* G_n *with the same central character, and let* W_{π_1}, W_{π_2} *be Whittaker functions, for* π_1, π_2 *respectively, which coincide on* P_n*. If the local gamma factors* $\gamma(s, \pi_1 \times \tau, \psi)$ *and* $\gamma(s, \pi_2 \times \tau, \psi)$ *are equal as functions in the complex variable* $s \in \mathbb{C}$*, for all irreducible generic representations* τ *of* G_r*, then the two Whittaker functions* W_{π_1}*,* W_{π_2} *agree on height* r*.*

We are going to use the functional equations together with the properties of special pairs of Whittaker functions in order to show that if a special pair (W_{π_1}, W_{π_2}) agree on height i, for $i = 0, \ldots, \left[\frac{n}{2}\right]$, then they are in fact equal. To do so, we apply a refined decomposition of G_n, whose finite field version was a key ingredient in the proof of the Jacquet Conjecture on the Local Converse Problem for G_n over finite fields in [17].

Proposition 3.4 ([17, Proposition 3.8])**.** *The following (non-disjoint) decomposition holds:*

$$\mathrm{G}_n = \bigcup_{0 \leqslant r \leqslant \left[\frac{n}{2}\right], n - \left[\frac{n}{2}\right] \leqslant k \leqslant n} \mathrm{U}_n \alpha^r \mathrm{Q}_n \alpha^k \mathrm{U}_n.$$

3.2 Proof of Theorem 1.5

Let π_1, π_2 be irreducible supercuspidal representations of G_n and let (W_{π_1}, W_{π_2}) be a special pair for (π_1, π_2). Let $\mathbf{K}$ be the compact-mod-centre open subgroup of G_n such that W_{π_i} are $\mathbf{K}$-special. By hypothesis, the local gamma factors $\gamma(s, \pi_1 \times \tau, \psi)$ and $\gamma(s, \pi_2 \times \tau, \psi)$ are equal as functions in the complex variable s, for all irreducible supercuspidal representations τ of G_r with $r = 1, \ldots, \left[\frac{n}{2}\right]$. This condition can be extended for all irreducible generic smooth representations τ of G_n by the multiplicativity of local gamma factors. Moreover, by Corollary 2.7, π_1, π_2 have the same central character. The proof goes in three steps.

Step (1). By Proposition 3.3, $W_{\pi_1}(g) = W_{\pi_2}(g)$, for

$$g \in \bigcup_{0 \leqslant r \leqslant \left[\frac{n}{2}\right]} \mathrm{U}_n \alpha^r \mathrm{Q}_n = \bigcup_{0 \leqslant r \leqslant \left[\frac{n}{2}\right]} \mathrm{U}_n \alpha^r \mathrm{Q}_n \alpha^n \mathrm{U}_n.$$

Step (2). For $g = q\alpha^k u \in \mathrm{Q}_n \alpha^k \mathrm{U}_n \cap \mathbf{K}$, with $n - \left[\frac{n}{2}\right] \leqslant k \leqslant n$, and $i = 1, 2$, we have

$$W_{\pi_i}(q\alpha^k u) = \overline{W_{\pi_i}((q\alpha^k u)^{-1})} = \overline{W_{\pi_i}(u^{-1}\alpha^{n-k} q^{-1})},$$

since W_{π_i} is $\mathbf{K}$-special. Since $u^{-1}\alpha^{n-k}q^{-1} \in \mathrm{U}_n \alpha^{n-k} \mathrm{Q}_n$, from **Step (1)** it follows that

$$W_{\pi_1}(q\alpha^k u) = W_{\pi_2}(q\alpha^k u).$$

Thus W_{π_1}, W_{π_2} agree on $\mathrm{Q}_n \alpha^k \mathrm{U}_n \cap \mathbf{K}$ and hence on $\mathrm{Q}_n \alpha^k \mathrm{U}_n \cap \mathrm{U}_n \mathbf{K}$, since they are both ψ_n-Whittaker functions. Since $\mathrm{Supp} W_{\pi_i} \subset \mathrm{U}_n \mathbf{K}$, we deduce that $W_{\pi_1}(g) = W_{\pi_2}(g)$, for all

$$g \in \bigcup_{n - \left[\frac{n}{2}\right] \leqslant k \leqslant n} \mathrm{Q}_n \alpha^k \mathrm{U}_n = \bigcup_{n - \left[\frac{n}{2}\right] \leqslant k \leqslant n} \mathrm{U}_n \alpha^0 \mathrm{Q}_n \alpha^k \mathrm{U}_n.$$

Step (3). It remains to consider the case of $g \in \mathrm{U}_n \alpha^r \mathrm{Q}_n \alpha^k \mathrm{U}_n$, with $1 \leqslant r \leqslant \left[\frac{n}{2}\right]$ and $n - \left[\frac{n}{2}\right] \leqslant k \leqslant n-1$. For any fixed $u \in \mathrm{U}_n$ and $p \in \mathrm{P}_n$, **Step (2)** implies that

$$R_{p\alpha^k u} W_{\pi_1}(q) = R_{p\alpha^k u} W_{\pi_2}(q),$$

for all $q \in \mathrm{P}_n$, where we recall that R_g denotes the right translation action by g on the Whittaker functions. We apply the functional equation (2.5) for $j = n-r-1$ to the Whittaker functions $R_{p\alpha^k u} W_{\pi_i}$ for $i = 1, 2$ and any Whittaker function W_τ in $\mathcal{W}(\tau, \psi_r^{-1})$.

The local zeta function $\mathcal{Z}(R_{p\alpha^k u} W_{\pi_i}, W_\tau, s; n-r-1)$ is given by the following integral

$$\int_h \int_x R_{p\alpha^k u} W_{\pi_i} \begin{pmatrix} h & 0 & 0 \\ x & \mathrm{I}_{n-r-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} W_\tau(h) |\det h|^{s - \frac{n-r}{2}} dx dh$$

where the integration in the variable h is over $\mathrm{U}_r\backslash\mathrm{G}_r$ and the integration in the variable x is over $\mathrm{Mat}_{(n-r-1)\times r}(F)$. Hence we obtain

$$\mathcal{Z}(R_{p\alpha^k u}W_{\pi_1}, W_\tau, s; n-r-1) = \mathcal{Z}(R_{p\alpha^k u}W_{\pi_2}, W_\tau, s; n-r-1).$$

Since $\gamma(s,\pi_1\times\tau,\psi)=\gamma(s,\pi_2\times\tau,\psi)$, by the functional equation (2.5) for $j=n-r-1$, we obtain

$$\mathcal{Z}(R_{w_{n,r}}\widetilde{R_{p\alpha^k u}W_{\pi_1}}, \widetilde{W}_\tau, 1-s; 0) = \mathcal{Z}(R_{w_{n,r}}\widetilde{R_{p\alpha^k u}W_{\pi_2}}, \widetilde{W}_\tau, 1-s; 0).$$

Thus, from the definition of these zeta integrals,

$$\int_g \left(R_{w_{n,r}}\widetilde{R_{p\alpha^k u}W_{\pi_1}} - R_{w_{n,r}}\widetilde{R_{p\alpha^k u}W_{\pi_2}}\right)\begin{pmatrix} g & 0 \\ 0 & \mathrm{I}_{n-r}\end{pmatrix} |\det(g)|^{s-\frac{n-r}{2}}\widetilde{W}_\tau(g)dg = 0,$$

for all generic representations τ of G_r, where the integration in the variable h is over $\mathrm{U}_r\backslash\mathrm{G}_r$. From Lemma 3.1, we deduce that

$$R_{w_{n,r}}\widetilde{R_{p\alpha^k u}W_{\pi_1}}\begin{pmatrix} g & 0 \\ 0 & \mathrm{I}_{n-r}\end{pmatrix} = R_{w_{n,r}}\widetilde{R_{p\alpha^k u}W_{\pi_2}}\begin{pmatrix} g & 0 \\ 0 & \mathrm{I}_{n-r}\end{pmatrix}$$

for all $p\in\mathrm{P}_n$, $u\in\mathrm{U}_n$ and $g\in\mathrm{G}_r$. Now by definition, for $i=1,2$,

$$\begin{aligned} R_{w_{n,r}}\widetilde{R_{p\alpha^k u}W_{\pi_i}}\begin{pmatrix} g & 0 \\ 0 & \mathrm{I}_{n-r}\end{pmatrix} &= R_{p\alpha^k u}W_{\pi_i}\left(w_n\begin{pmatrix} {}^t g^{-1} & 0 \\ 0 & I_{n-r}\end{pmatrix}{}^t w_{n,r}^{-1}\right) \\ &= W_{\pi_i}\left(\begin{pmatrix} 0 & \mathrm{I}_{n-r} \\ w_r{}^t g^{-1} & 0\end{pmatrix} p\alpha^k u\right). \end{aligned}$$

Hence we obtain the identity

$$W_{\pi_1}\left(\begin{pmatrix} 0 & \mathrm{I}_{n-r} \\ w_r{}^t g^{-1} & 0\end{pmatrix} p\alpha^k u\right) = W_{\pi_2}\left(\begin{pmatrix} 0 & \mathrm{I}_{n-r} \\ w_r{}^t g^{-1} & 0\end{pmatrix} p\alpha^k u\right),$$

for all $p\in\mathrm{P}_n$, $u\in\mathrm{U}_n$ and $g\in\mathrm{G}_r$. In particular, taking $g=w_r$ we obtain

$$W_{\pi_1}(\alpha^r p\alpha^k u) = W_{\pi_2}(\alpha^r p\alpha^k u),$$

for all $p\in\mathrm{P}_n$ and $u\in\mathrm{U}_n$. This proves that $W_{\pi_1}(g)=W_{\pi_2}(g)$, for

$$g\in\mathrm{U}_n\alpha^r\mathrm{Q}_n\alpha^k\mathrm{U}_n$$

with $1\leqslant r\leqslant\left[\frac{n}{2}\right]$ and $n-\left[\frac{n}{2}\right]\leqslant k\leqslant n-1$. This completes **Step (3)**.

By combining the results from all three **Steps** above, we obtain that

$$W_{\pi_1}(g) = W_{\pi_2}(g), \text{ for all } g\in\mathrm{G}_n.$$

By the uniqueness of local Whittaker models for irreducible smooth representations of G_n, the two Whittaker models $\mathcal{W}(\pi_1,\psi_n)$ and $\mathcal{W}(\pi_2,\psi_n)$ have trivial intersection unless π_1 and π_2 are equivalent as representations of G_n, which completes the proof of Theorem 1.5.

4 Supercuspidals with the same endo-class

$\mathbf{K}$-special Whittaker functions are Whittaker functions of G_n with certain symmetry when restricted to $\mathbf{K}$. The Bessel functions of irreducible supercuspidal representations of G_n constructed by Paškūnas and the third-named author in [18] are such examples. We recall from [18] the basics of these Bessel functions, which rely on the construction theory of supercuspidal representations of G_n in terms of maximal simple types of Bushnell and Kutzko [6]. We will use the standard notation from [6] and [18].

4.1 Bessel functions

We begin by recalling from [18, Section 5] the general formulation of Bessel functions. Let $\mathcal{K}$ be an open compact-modulo-center subgroup of G_n and let $\mathcal{U} \subset \mathcal{M} \subset \mathcal{K}$ be compact open subgroups of $\mathcal{K}$. Let τ be an irreducible smooth representation of $\mathcal{K}$ and let Ψ be a linear character of $\mathcal{U}$. Take an open normal subgroup $\mathcal{N}$ of $\mathcal{K}$, which is contained in $\mathrm{Ker}(\tau) \cap \mathcal{U}$. Let χ_τ be the (trace) character of τ. The associated *Bessel function* $\mathcal{J} : \mathcal{K} \to \mathbb{C}$ of τ is defined by

$$\mathcal{J}(g) := [\mathcal{U} : \mathcal{N}]^{-1} \sum_{u \in \mathcal{U}/\mathcal{N}} \Psi(h^{-1})\chi_\tau(gu).$$

This is independent of the choice of $\mathcal{N}$. The basic properties of this Bessel function which we will need are given below.

Proposition 4.1 ([18, Proposition 5.3])**.** *Assume that the data introduced above satisfy the following:*

- $\tau|_{\mathcal{M}}$ *is an irreducible representation of* $\mathcal{M}$*; and*
- $\tau|_{\mathcal{M}} \cong \mathrm{Ind}_{\mathcal{U}}^{\mathcal{M}}(\Psi)$.

Then the Bessel function $\mathcal{J}$ *of* τ *enjoys the following properties:*

(i) $\mathcal{J}(1) = 1$*;*

(ii) $\mathcal{J}(hg) = \mathcal{J}(gh) = \Psi(h)\mathcal{J}(g)$ *for all* $h \in \mathcal{U}$ *and* $g \in \mathcal{K}$*;*

(iii) *if* $\mathcal{J}(g) \neq 0$*, then* g *intertwines* Ψ*; in particular, if* $m \in \mathcal{M}$*, then* $\mathcal{J}(m) \neq 0$ *if and only if* $m \in \mathcal{U}$*;*

When the representation τ is also unitarizable, the Bessel function enjoys another symmetry property, as in the finite field case in [17].

Lemma 4.2. *In the situation of Proposition 4.1, assume further that τ is unitarizable. Then*

$$\mathcal{J}(g) = \overline{\mathcal{J}(g^{-1})}, \quad \textit{for } g \in \mathcal{K}.$$

Proof. Note that Ψ is unitary, since it is a character of the compact group $\mathcal{U}$. That is $\overline{\Psi(g^{-1})} = \Psi(g)$. Since χ_τ is also unitary and $\chi_\tau(gh) = \chi_\tau(hg)$, for $g, h \in \mathcal{K}$, we get

$$\begin{aligned}
\overline{\mathcal{J}(g^{-1})} &:= [\mathcal{U}:\mathcal{N}]^{-1} \overline{\sum_{u \in \mathcal{U}/\mathcal{N}} \Psi(u^{-1})\chi_\tau(g^{-1}u)} \\
&= [\mathcal{U}:\mathcal{N}]^{-1} \sum_{u \in \mathcal{U}/\mathcal{N}} \Psi(u)\chi_\tau(u^{-1}g) \\
&= [\mathcal{U}:\mathcal{N}]^{-1} \sum_{u \in \mathcal{U}/\mathcal{N}} \Psi(u)\chi_\tau(gu^{-1}) \\
&= [\mathcal{U}:\mathcal{N}]^{-1} \sum_{u \in \mathcal{U}/\mathcal{N}} \Psi(u^{-1})\chi_\tau(gu) \\
&= \mathcal{J}(g), \text{ for } g \in \mathcal{K}.
\end{aligned}$$

The penultimate equality follows from the substitution $u \mapsto u^{-1}$ and the normality of $\mathcal{N}$ in $\mathcal{U}$. ☐

4.2 Maximal simple types

Following [6, Section 6], the irreducible supercuspidal representations of G_n are classified by means of maximal simple types (J, λ), where J is a compact open subgroup of G_n and λ is an irreducible representation of J. More precisely, (J, λ) is introduced as follows. We refer to [6] for precise definitions of the objects introduced here.

Let $V = F^n$, an n-dimensional vector space over F with standard basis. Thus we identify $\mathrm{Aut}_F(V)$ with G_n and $A = \mathrm{End}_F(V)$ with $\mathrm{Mat}_{n\times n}(F)$. Let $\mathfrak{A}$ be a principal hereditary $\mathfrak{o}_F$-order in A with Jacobson radical $\mathfrak{P}$. Define $\mathbf{U}^0(\mathfrak{A}) = \mathbf{U}(\mathfrak{A}) = \mathfrak{A}^\times$ and for $m \geqslant 1$, define $\mathbf{U}^m(\mathfrak{A}) = 1 + \mathfrak{P}^m$. For $m \geqslant 0$, choose $\beta \in A$ such that $\beta \in \mathfrak{P}^{-m} \setminus \mathfrak{P}^{1-m}$, $E = F[\beta]$ is a field extension of F, and $E^\times$ normalizes $\mathfrak{A}$. Provided an additional technical condition is satisfied (namely $k_F(\beta) < 0$), these data give a principal simple stratum $[\mathfrak{A}, m, 0, \beta]$ of A. Take $J = J(\beta, \mathfrak{A})$, $J^1 = J^1(\beta, \mathfrak{A})$, and $H^1 = H^1(\beta, \mathfrak{A})$ as defined in [6, Section 3]. Denote by $\mathcal{C}(\mathfrak{A}, \beta, \psi)$ the set of simple (linear) characters of H^1 as defined in [6, Section 3].

Recall from [6, Section 6] the following definition of maximal simple types.

Definition 4.3. The pair (J, λ) is called a *maximal simple type* if one of the following holds:

(a) $J = J(\beta, \mathfrak{A})$ is an open compact subgroup associated to a simple stratum $[\mathfrak{A}, m, 0, \beta]$ of A as above, such that, if $E = F[\beta]$ and $B = \mathrm{End}_E(V)$, then $\mathfrak{B} = \mathfrak{A} \cap B$ is a maximal $\mathfrak{o}_E$-order in B. Moreover, there exists a simple character $\theta \in \mathcal{C}(\mathfrak{A}, \beta, \psi)$ such that

$$\lambda \cong \kappa \otimes \sigma,$$

where κ is a β-extension of the unique irreducible representation η of $J^1 = J^1(\beta, \mathfrak{A})$, which contains θ, and σ is the inflation to J of an irreducible cuspidal representation of

$$J/J^1 \cong \mathbf{U}(\mathfrak{B})/\mathbf{U}^1(\mathfrak{B}) \cong \mathrm{GL}_r(\mathfrak{k}_E),$$

where $r = n/[E : F]$.

(b) $(J, \lambda) = (\mathbf{U}(\mathfrak{A}), \sigma)$, where $\mathfrak{A}$ is a maximal hereditary $\mathfrak{o}_F$-order in A and σ is the inflation to $\mathbf{U}(\mathfrak{A})$ of an irreducible cuspidal representation of

$$\mathbf{U}(\mathfrak{A})/\mathbf{U}^1(\mathfrak{A}) \cong \mathrm{GL}_n(\mathfrak{k}_F).$$

We will regard case (b) formally as a special case of case (a) by setting $\beta = 0$ and $E = F$, and θ, η, κ trivial. In either case, we put $\mathbf{J} = E^\times J$. With these data, any irreducible supercuspidal representation π of G_n is of the form

$$\pi \cong \text{c-Ind}_{\mathbf{J}}^{\mathrm{G}_n}(\Lambda),$$

for some choice of $(\mathbf{J}, \Lambda)$, where $\Lambda|_J = \lambda$. We call such a pair $(\mathbf{J}, \Lambda)$ and *extended maximal simple type*.

For π an irreducible supercuspidal representation of G_n, any two extended maximal simple types in π are conjugate in G_n. This fact allows one to associate some invariants to π. The simple character θ in the construction of an extended maximal simple type for π determines an *endo-class* $\Theta = \Theta(\pi)$ as defined in [2]. We do not recall precisely the definition of endo-class: it is a class for a certain equivalence relation on functions which take values in simple characters. For $i = 1, 2$, let $\theta_i \in \mathcal{C}(\mathfrak{A}_i, \beta_i, \psi)$ be simple characters for G_n. If θ_1, θ_2 have the same endo-class then they intertwine in G_n; if, moreover, the hereditary orders $\mathfrak{A}_1, \mathfrak{A}_2$ are isomorphic then θ_1, θ_2 are conjugate in G_n.

Although the field extension E/F involved in the construction of a maximal simple type in π is not uniquely determined, its residue degree and ramification index are in fact invariants of the endo-class $\Theta = \Theta(\pi)$ and we write

$$f(\Theta) = f(E/F),\ e(\Theta) = e(E/F),\ \deg(\Theta) = [E : F].$$

These are then also invariants of π so we write $\deg(\pi) = \deg(\Theta)$ and call it the *degree* of π. We also remark that the $\mathfrak{o}_F$-period of the hereditary order $\mathfrak{A}$ in the construction of any maximal simple type in π is $e(\Theta)$.

4.3 Explicit Whittaker functions

Let π be an irreducible *unitarizable* supercuspidal representation of G_n. By [3, Proposition 1.6], there is an extended maximal simple type $(\mathbf{J},\Lambda)$ in π such that

$$\mathrm{Hom}_{\mathrm{U}_n\cap\mathbf{J}}(\psi_n,\Lambda)\neq 0.$$

Since Λ restricts to a multiple of some simple character $\theta\in\mathcal{C}(\mathfrak{A},\beta,\psi)$, one obtains that $\theta(u)=\psi_n(u)$ for all $u\in\mathrm{U}_n\cap H^1$. As in [18, Definition 4.2], one defines a character $\Psi_n:(J\cap\mathrm{U}_n)H^1\to\mathbb{C}^\times$ by

$$\Psi_n(uh):=\psi_n(u)\theta(h), \tag{4.4}$$

for all $u\in J\cap\mathrm{U}_n$ and $h\in H^1$. By [18, Theorem 4.4], the data

$$\mathcal{K}=\mathbf{J},\ \tau=\Lambda,\ \mathcal{M}=(J\cap\mathrm{P}_n)J^1,\ \mathcal{U}=(J\cap\mathrm{U}_n)H^1,\ \text{and}\ \Psi=\Psi_n$$

satisfy the conditions in Proposition 4.1 and hence define a Bessel function $\mathcal{J}$.

Now we define a function $W_\pi:\mathrm{G}_n\to\mathbb{C}$ by

$$W_\pi(g):=\begin{cases}\psi_n(u)\mathcal{J}(j) & \text{if } g=uj \text{ with } u\in\mathrm{U}_n,\ j\in\mathbf{J},\\ 0 & \text{otherwise,}\end{cases} \tag{4.5}$$

which is well-defined by Proposition 4.1(ii). Then, by [18, Theorem 5.8], W_π is a Whittaker function for π. Moreover, since π is unitarizable, the same is true of Λ, so W_π is a $\mathbf{J}$-special Whittaker function for π, by Lemma 4.2. By Proposition 4.1, the restriction of W_π to P_n has a particularly simple description: for $g\in\mathrm{P}_n$,

$$W_\pi(g)=\begin{cases}\Psi_n(g) & \text{if } g\in(J\cap\mathrm{U}_n)H^1;\\ 0 & \text{otherwise.}\end{cases} \tag{4.6}$$

4.4 Proof of Proposition 1.6

Let π_1, π_2 be irreducible unitarizable supercuspidal representations of G_n with the same endo-class Θ. We will use all the notation of Definition 4.3 but with subscripts ${}_1, {}_2$.

Let $(\mathbf{J}_1,\Lambda_1)$ be an extended maximal simple type in π_1 such that

$$\mathrm{Hom}_{\mathrm{U}_n\cap\mathbf{J}_1}(\psi_n,\Lambda_1)\neq 0.$$

By [18, Remark 4.15], we may assume that the pair (U_n,ψ_n) arises from the construction of [18, Theorem 3.3]. This construction, which produces a particular maximal unipotent subgroup and non-degenerate character, depends only on the simple character θ_1. Thus, by [18, Corollary 4.13], the space $\mathrm{Hom}_{\mathrm{U}_n\cap\mathbf{J}_1}(\psi_n,\Lambda)$ is non-zero for *any* extended maximal simple type $(\mathbf{J}_1,\Lambda)$ containing θ_1.

Now let $(\mathbf{J}_2, \Lambda_2)$ be any extended maximal simple type in π_2. The hereditary orders $\mathfrak{A}_i$ have the same period $e(\Theta)$ so are conjugate in G_n; replacing $\mathfrak{A}_2$ by a conjugate if necessary, we assume they are equal. Then the simple characters θ_1, θ_2 are conjugate in G_n, by definition of endo-equivalence; again, replacing θ_2 by a conjugate if necessary, we assume they are equal. Now $\mathbf{J}_i$ is the G_n-normalizer of θ_i so we have $\mathbf{J}_1 = \mathbf{J}_2$. Hence, by the remarks above,

$$\mathrm{Hom}_{\mathrm{U}_n \cap \mathbf{J}_1}(\psi_n, \Lambda_2) \neq 0.$$

Thus the characters Ψ_n^1, Ψ_n^2 as defined in (4.4) are equal. Finally, by (4.6), the $\mathbf{J}_1$-special Whittaker functions W_{π_1}, W_{π_2} defined by (4.5) agree on P_n. Thus (W_{π_1}, W_{π_2}) is a special pair for (π_1, π_2), which completes the proof of Proposition 1.6.

Remark 4.7. In the proof of the existence of a special pair, we do not in fact use that the endo-classes for π_1, π_2 coincide, but only that $\mathbf{J}_1, \mathbf{J}_2$ are contained in a common compact-modulo-center open subgroup of G, that $H_1^1 \cap \mathrm{P}_n = H_2^1 \cap \mathrm{P}_n$ and that θ_1, θ_2 coincide on $H_1^1 \cap \mathrm{P}_n$. This is significantly weaker: for example, if $\deg(\pi_1) = n$ and β_1 is a *minimal* element (see, for example, [6, Section 1.4] for the definition) then $H_1^1 \cap \mathrm{P}_n = U^{[\frac{m}{2}]+1}(\mathfrak{A}_1) \cap \mathrm{P}_n$.

5 Conductors of pairs

In this section we will prove Corollary 1.7. The techniques here are entirely different, relying on the explicit computation of conductors of pairs of supercuspidal representations from [5] and their application in [4].

5.1 Endo-classes

In [4], Bushnell and Henniart define a function $\mathfrak{F}$ of pairs of endo-classes with the property that, for π, τ irreducible supercuspidal representations of $\mathrm{G}_n, \mathrm{G}_r$ respectively, with $n > r \geqslant 1$, the conductor satisfies

$$f(\pi \times \widetilde{\tau}, \psi) = nr(\mathfrak{F}(\Theta(\pi), \Theta(\tau)) + 1). \tag{5.1}$$

Moreover, [4, Theorem C] says that this function characterizes endo-classes in the following way: for endo-classes Θ_1, Θ_2,

$$\mathfrak{F}(\Theta_1, \Theta_2) \geqslant \mathfrak{F}(\Theta_1, \Theta_1), \tag{5.2}$$

with equality if and only if $\Theta_1 = \Theta_2$.

The final ingredient we need is that, given an endo-class Θ, there is an irreducible supercuspidal representation τ of G_r with endo-class Θ whenever r is a multiple of the degree $\deg(\Theta)$. This is immediate from the definitions of endo-class in [3] and of maximal simple types.

5.2 Proof of Corollary 1.7

Let $\pi_1,\ \pi_2$ be unitarizable irreducible supercuspidal representations of G_n, with endo-classes $\Theta_1,\ \Theta_2$ respectively, and suppose that $\deg(\pi_1) < n$. Suppose the local gamma factors $\gamma(s, \pi_1 \times \tau, \psi)$ and $\gamma(s, \pi_2 \times \tau, \psi)$ are equal as functions in the complex variable s, for all irreducible supercuspidal representations τ of G_r with $r = 1, \dots, \left[\frac{n}{2}\right]$.

Put $r := \deg(\pi_1)$, which is a proper divisor of n; in particular, $r \leqslant \left[\frac{n}{2}\right]$. Let τ be an irreducible supercuspidal representation τ of G_r with endo-class $\Theta(\tau) = \Theta_1$. Then, by (5.1) and hypothesis, we have

$$\mathfrak{F}(\Theta_1, \Theta_1) = \frac{f(\pi_1 \times \widetilde{\tau}, \psi)}{nr} - 1 = \frac{f(\pi_2 \times \widetilde{\tau}, \psi)}{nr} - 1 = \mathfrak{F}(\Theta_2, \Theta_1).$$

We deduce, from (5.2), that $\Theta_1 = \Theta_2$. Then Proposition 1.6 and Theorem 1.5 combine to imply that π_1 is equivalent to π_2.

Remark 5.3. The restriction of a simple character $\theta \in \mathcal{C}(\mathfrak{A}, \beta, \psi)$ to the groups $H^{t+1} = H^1 \cap \mathbf{U}^{t+1}(\mathfrak{A})$, $t \geqslant 0$, determines a family of endo-classes. By considering these endo-classes, rather than just those coming from the simple character θ, it seems likely that one could prove a more general version of Corollary 1.7 by generalizing the function $\mathfrak{F}$.

However, even in the most optimistic scenario, this will leave the case where, for $i = 1, 2$, we have $\deg(\pi_i) = n$ and any simple character $\theta_i \in \mathcal{C}(\mathfrak{A}, \beta_i, \psi)$ in π_i has β_i a minimal element. One would need to prove that the equality of local gamma factors implies that one can assume $\beta_1 \equiv \beta_2 \pmod{\mathfrak{P}^{-\left[\frac{m}{2}\right]}}$ to enable us to construct a special pair of Whittaker functions (see Remark 4.7). However, even the case $n = 3$ seems to be very difficult to analyze directly via the explicit construction of supercuspidal representations, even in the tame case.

Acknowledgments. The work of the first-named author is supported in part by NSF grants DMS–1001672 and DMS–1301567. Both the first and the second-named authors would like to thank the Chinese Academy of Sciences for invitation and support over the years when their research work related to this paper was in process. The second-named author is supported by NSC101-2918-I-006-003 and would like to thank the University of East Anglia for providing a stimulating environment during the execution period of the project. The third-named author is supported by EPSRC grant EP/H00534X/1. We thank the referee for some helpful comments, in particular for encouraging us to remove an unnecessary hypothesis on the characteristic.

References

[1] Bernšteĭn, I. N., Zelevinskiĭ, A. V.: Representations of the group $GL(n, F)$, where F is a local non-Archimedean field, Uspehi Mat. Nauk **31**, no. 3 (189), 5–70 (1976) (in Russian); English transl.: Russian Math. Surveys **31**, no. 3, 1–68 (1976)

[2] Bushnell, C. J., Henniart, G.: Local tame lifting for $\mathrm{GL}(N)$. I. Simple characters, Inst. Hautes Études Sci. Publ. Math. no. 83, 105–233 (1996)

[3] Bushnell, C. J., Henniart, G.: Supercuspidal representations of GL_n: explicit Whittaker functions, J. Algebra **209**, 270–287 (1998)

[4] Bushnell, C. J., Henniart, G.: Local tame lifting for $\mathrm{GL}(n)$. IV. Simple characters and base change, Proc. London Math. Soc. (3) **87**, 337–362 (2003)

[5] Bushnell, C. J., Henniart, G. M., Kutzko, P. C.: Local Rankin-Selberg convolutions for GL_n: explicit conductor formula, J. Amer. Math. Soc. **11**, 703–730 (1998)

[6] Bushnell, C. J., Kutzko, P. C.: The admissible dual of $\mathrm{GL}(N)$ via compact open subgroups, Annals of Mathematics Studies, vol. 129, Princeton University Press, Princeton, NJ (1993)

[7] Chen, J.-P. J.: Local Factors, Central Characters, and Representations of the General Linear Group over Non-Archimedean Local Fields, Thesis, Yale University (1996)

[8] Chen, J.-P. J.: The $n \times (n-2)$ local converse theorem for $\mathrm{GL}(n)$ over a p-adic field, J. Number Theory **120**, 193–205 (2006)

[9] Cogdell, J. W., Piatetski-Shapiro, I. I.: Converse theorems for GL_n. II, J. Reine Angew. Math. **507**, 165–188 (1999)

[10] Gel′fand, I. M., Kazhdan, D. A.: Representations of the group $\mathrm{GL}(n, K)$ where K is a local field, In: Lie groups and their representations (Proc. Summer School, Bolyai János Math. Soc., Budapest, 1971), Halsted, New York, 95–118 (1975)

[11] Henniart, G.: Caractérisation de la correspondance de Langlands locale par les facteurs ϵ de paires, Invent. Math. **113**, 339–350 (1993)

[12] Jacquet, H., Langlands, R. P.: Automorphic forms on $\mathrm{GL}(2)$, Lecture Notes in Mathematics, Vol. 114, Springer-Verlag, Berlin (1970)

[13] Jacquet, H., Piatetski-Shapiro, I. I., Shalika, J. A.: Rankin-Selberg convolutions, Amer. J. Math. **105**, 367–464 (1983)

[14] Jacquet, H., Piatetski-Shapiro, I. I., Shalika, J. A.: Automorphic forms on $\mathrm{GL}(3)$. I, Ann. of Math. (2) **109**, 169–212 (1979)

[15] Jacquet, H., Shalika, J.: A lemma on highly ramified ϵ-factors, Math. Ann. **271**, 319–332 (1985)

[16] Jiang, D., Soudry, D.: The local converse theorem for $\mathrm{SO}(2n+1)$ and applications, Ann. of Math. (2) **157**, 743–806 (2003)

[17] Nien, C.: A proof of finite field analogue of Jacquet's conjecture, Amer. J. Math. **136**, 653–674 (2014)

[18] Paškūnas, V., Stevens, S.: On the realization of maximal simple types and epsilon factors of pairs, Amer. J. Math. **130**, 1211–1261 (2008)

[19] Shahidi, F.: Fourier transforms of intertwining operators and Plancherel measures for $\mathrm{GL}(n)$, Amer. J. Math. **106**, 67–111 (1984)

[20] Zelevinsky, A. V.: Induced representations of reductive $\mathfrak{p}$-adic groups. II. On irreducible representations of $\mathrm{GL}(n)$, Ann. Sci. École Norm. Sup. (4) **13**, 165–210 (1980)