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Introducing a temperature adjustment to make Wien's law a more accurate approximation of Planckian blackbody radiation in the visible range

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Abstract: Wien's law is well known to approximate the Planckian blackbody radiator equation, in the visible range, for temperatures less than 4000 Kelvin. As temperature increases, however, the approximation degrades and ultimately for high temperatures the spectra generated by Wien's law have significantly different spectral shapes compared to the actual Planckian lights. Accordingly, as temperature T increases, the chromaticity of a Wien spectrum at T diverges from that of a Planck spectrum of the same T. That is in spectral space. However, in (u', v') chromaticity space, the Planck and Wien loci are closely parallel, so it is plausible that, for any target Planck spectrum at T^P, there is a Wien temperature T^W that produces nearly the same chromaticity (hence nearly the same spectrum). In this paper, we derive a temperature adjustment function $f()$ such that the Wien spectrum calculated with the temperature T^W is close to the Planck spectrum calculated at any given T^P: $T^W = f(T^P)$. We investigate the utility of this result in the context of locus filter theory. A locus filter has the property that when it is applied to any Wien illuminant the resulting filtered light also is well described by Wien's Law. Our temperature adjustment formula, in effect, extends locus filter theory so that it also applies to Planckian lights.

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1. Introduction

Planck's law defines blackbody radiation using only one parameter: the color temperature typically measured in Kelvin [1]. Color temperature elucidates how the color appearance of a heated blackbody radiator shifts within a specific temperature range. When the temperature increases from 1000 to 100,000 Kelvin (equivalently, we write from 1000K to 100,000K), the color shifts from red to orange to yellowish-white, to bluish-white and then to blue. Warm colors are associated with low color temperatures, while cool colors are associated with high color temperatures.

Wien's displacement law [2,3] is an alternative formula that approximates Planck's law. Although the two formulae are not the same, they generate very similar spectra for low color temperatures (i.e., less than 4000 K). However, the generated spectra are different the higher the temperatures become. Although Wien's law can generate spectra different (than the actual desired Planckian) its construction compared to Planck's formula is advantageous. Indeed, it is easy to show that the logarithm of the set of all Wien spectra spans a 2-dimensional linear subspace [4]. This result simplifies mathematical derivations and is employed in a variety of research works in the field of computer vision [5–13].

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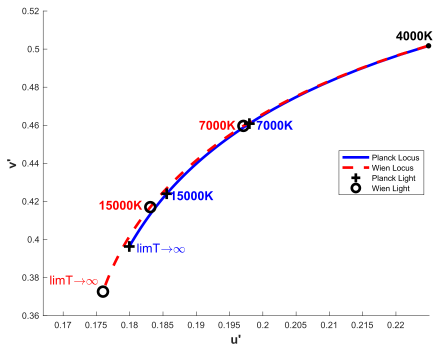
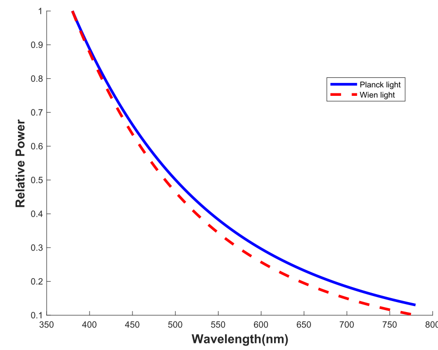
Figure 1(a) plots the Planckian and Wien loci in the CIE $u'v'$ chromaticity diagram. The $u'v'$ diagram is often used as it is approximately perceptually uniform in the sense that color stimuli separated by a small Euclidean distance (denoted $D_{u'v'}$) are perceived to be similar in color [3].



(a)



(b)

Fig. 1. (a) Planckian locus (solid blue line) compared to the Wien locus (dashed red line) in $u'v'$ diagram. Positions of 7000K and 15000K Planck (blue text, below the loci) and Wien (red text above the loci) are shown. For a 4000K temperature, the Wien and Planck $u'v'$ coordinates are almost coincident (see black text). (b) the spectral power distribution of Planck light compared to the Wien light for a color temperature of 15000K (where maximum power is normalised to 1).

Moreover, the same $D_{u'v'}$ calculated at different locations in the $u'v'$ diagram corresponds to approximately the same magnitude of *perceived* color difference. When the $D_{u'v'}$ for a pair of stimuli is greater than 0.004, it is deemed to be just noticeably different [14].

On the $u'v'$ chromaticity diagram color temperatures vary from warmer (e.g., 4000 K) to cooler (e.g., 20000 K) from right to left. It is clear that for low color temperatures, the two loci are quite similar, but they diverge a little as the temperature increases.

The $u'v'$ coordinates for 15000K Planckian and Wien lights are calculated and plotted - as a cross (Planckian) and circle (Wien) - in Fig. 1(a). The cross and circle fall on respectively the Planckian and Wien loci. The temperatures written in blue (below the loci) are for points on the Planckian locus (with those temperatures). And, those written in red are for points on the Wien locus (above the loci). The $u'v'$ point for 4000K is almost coincident for Wien and Planck and so is plotted as a single black point. Notice that the position of a given temperature on the Wien locus is below the corresponding temperature on the Planckian locus.

In Fig. 1(b) we plot the Wien spectrum for a 15000K temperature and the corresponding 15000K Planckian spectrum. In Fig. 1(b) both spectra are normalized so their maxima equal 1. Notice that the Wien spectrum is steeper in the shorter wavelengths indicating a cooler color appearance. Further, we calculate the Euclidean distance in the $u'v'$ space. The $D_{u'v'} = 0.007$ (between cross and circle) indicating that the corresponding spectra (see Fig. 1(b)) would be perceived to have different colors [14].

Our paper starts with the question: Does there exist a Wien illuminant - with temperature T^W - that is closer in spectral shape to a Planckian with a temperature T^P where $T^W \neq T^P$. Intuitively, the answer seems to be yes. It is worth noting that the Planck and Wien loci are almost coincident, but the Wien locus extends further. For any T^P on the Planckian locus, there must be a T^W whose Wien spectrum lies very close to the Planck spectrum of the given T^P.

In this paper, we derive such a temperature adjustment function $f()$ so that the Wien spectrum calculated with the temperature $T^W = f(T^P)$ (Planckian temperature) is a very close approximation to the desired Planckian spectrum. We find that the function $f()$ can be well modeled by a

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$\mathcal{L}$ook-up-table, as a $\mathcal{P}$olynomial (in Mired units) or as a parameterized $\mathcal{A}$rctangent (in Kelvin) of temperature. Respectively, these 3 functions are denoted $f^{\mathcal{L}}()$, $f^{\mathcal{P}}()$ and $f^{\mathcal{A}}()$. Numerical experiments demonstrate that all three functions deliver similar performance (are equally effective at converting a target Planck temperature to a Wien counterpart). However, the arctangent and polynomial formulations have the advantage that they are one-to-one functions and can be inverted.

This invertibility is a useful property when we approximate a Planckian with a Wien counterpart in Locus Filter theory [4]. In [4], Deeb and Finlayson defined a transmissive filter - used to change the color of a light - to be a Locus Filter if and only if the filter always maps a Wien illuminant to some other counterpart which is also defined by Wien's equation. Effectively, the adjustment functions $f()$ - derived in this paper - extend the Locus Filter theory so that it also applies to Planckian blackbody radiators.

2. Preliminary equations

The Wien displacement law describes the blackbody radiation E^W as a function of color temperature T and wavelength λ [3]:

$$E^W(\lambda, T) = kc_1\lambda^{-5}e^{-\frac{c_2}{T\lambda}} \tag{1}$$

where c_1 and c_2 are constants equal to $3.74183 \times 10^{-16} Wm^2$ and $1.4388 \times 10^{-2} mK$, respectively. Additionally, the scalar k modulates the intensity of the Wien light. The Wien and Planck functions both describe similar spectra for low (say <4000 K) color temperatures. For a blackbody radiator, the spectral radiant emittance E^P is calculated using Planck's formula [3]:

$$E^P(\lambda, T) = kc_1\lambda^{-5}(e^{\frac{c_2}{T\lambda}} - 1)^{-1} \tag{2}$$

where c_1 and c_2 are as defined for Eq. (1), and, again, we admit a scalar k to modulate the power of the light.

Let us quantify how similar Wien lights are to actual Planckians in terms of the shape of spectra in the range of the visible spectrum. To do this, we will adopt a discrete approximation of the lights. Each light spectrum will be represented as an 81-component vector (corresponding to the spectral power distribution of the lights in the visible range from 380 to 780 Nanometres (nm) at 5 nm sampling). Now, given Planckian and Wien-approximation vectors, denoted as $\underline{E}^P$ and $\underline{E}^W$, we can calculate the angle between the vectors as a measure of similarity that is independent of intensity. We define the angular error as:

$$\text{AngularError}(\underline{E}^P, \underline{E}^W) = acos\left(\frac{\underline{E}^P.\underline{E}^W}{||\underline{E}^P||.||\underline{E}^W||}\right) \tag{3}$$

where '.' denotes the vector dot-product, ||.|| is the vector 2-norm and *acos* is the inverse cosine. The angular error, by construction, is independent of the magnitude of the vectors of the underlying spectra. In considering the angular error between spectra and – later in the Subsection 3.3 – the $D_{u'v'}$ measure, we acknowledge that this is just one of many possible measures. For a different spectral recovery test, Agarla et al. [15] considered 13 measures. They demonstrated a strong correlation between many error measures (such as a normalized RMSE) and the angular measure we adopt. In this paper, we choose angular error simply as an exemplar of the available error metrics.

As a complement to the angular error metric, we can calculate the chromaticity difference $D_{u'v'}$ between Planck and Wien lights as the Euclidean distance on the CIE 1976 $u'v'$ chromaticity diagram. Let us denote the $u'v'$ coordinates of a Planckian and Wien light, respectively as

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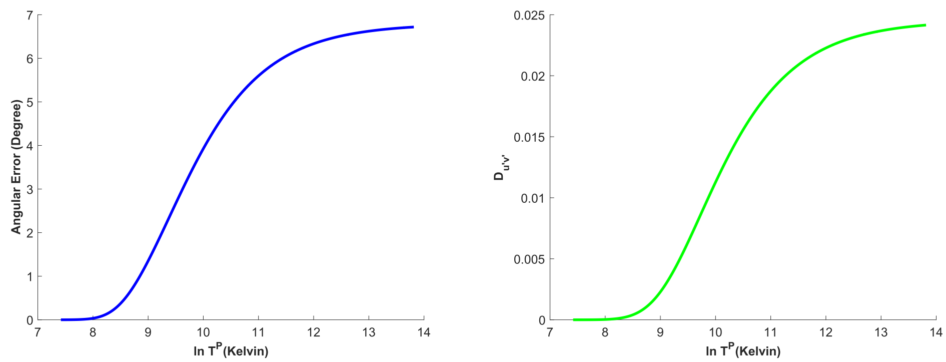
(u'^P, v'^P) and (u'^W, v'^W). The Euclidean distance, $D_{u'v'}$, between these points is calculated as:

$$D_{u'v'} = \sqrt{(u'^P - u'^W)^2 + (v'^P - v'^W)^2} \tag{4}$$

Figure 2 plots the errors for Planckians and Wien lights where color temperature is increasing (x-axis is natural logarithm of color temperature and the y-axis is the error - angular and $D_{u'v'}$ - between Planckian and Wien spectrum of that temperature). Here, the temperature range is 1667 up to 1,000,000 K (we will use this domain for all numerical experiments discussed in this paper). As temperature increases, both angular error and $D_{u'v'}$ also increase. Significantly, we see that as the temperature increases the $D_{u'v'}$ error is more than 0.004 and this indicates that the perceived color of the Planck and Wien Pair would be visually noticeably different to a human observer [14]. From Fig. 2(b), the $D_{u'v'}$ threshold is approximately at $lnT^P = 9.2$ (i.e., $T^P = 10300K$). According to Fig. 2(a), this corresponds to an angular error of about 2 degrees.



(a) (b)

Fig. 2. Error between the spectra of Planckian and Wien lights with the same color temperature in terms of (a) Angular Error, and (b) $D_{u'v'}$.

3. Estimating a Planckian light using the Wien formula

We would like to determine a correction function $f()$ such that for Planckian light with a color temperature of T^P, denoted as $E^P(\lambda, T^P)$, the Wien approximation for the corrected temperature, $E^W(\lambda, f(T^P))$, is closer to the Planckian light than when using T directly in Wien's formula. We seek a function $f()$ such that

$$E^P(\lambda, T) \approx kE^W(\lambda, f(T)) \tag{5}$$

where k is a scalar. Throughout this paper, we denote the *best* corrected Wien temperature (which produces a light with a spectral shape similar to a Planckian with temperature T^P) as

$$T^W = f(T^P) \tag{6}$$

Substituting Eq. (6) into Eq. (5):

$$E^P(\lambda, T) \approx kE^W(\lambda, T^W) \tag{7}$$

3.1. Correcting Wien color temperature using a look-up-table

Here we find the best – integer Plackian temperature mapping to integer Wien – look-up-table that implements the correction function, which we denote $f^{\mathcal{L}}()$. In order to solve for $f^{\mathcal{L}}()$ we will again represent spectra as vectors in the discrete domain. Respectively, $\underline{E}^P$ and $\underline{E}^W$ denote Planckian and Wien lights. Now, suppose we have a set of Planckian color temperatures

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$\tau^P = [T_1^P, T_2^P, \ldots, T_N^P]$ the corresponding Planckian spectra are in the set $\psi^P = [\underline{E}_1^P, \underline{E}_2^P, \ldots, \underline{E}_N^P]$. For the ith Planckian temperature T_i^P we find the *integer* Wien temperature T_i^W which minimizes:

$$\min_{T_i^W} AngularError(\underline{E}_i^P, \underline{E}_i^W)\ ,\ T_i^W \in \Omega \tag{8}$$

or

$$\min_{T_i^W} D_{u'v'}(\underline{E}_i^P, \underline{E}_i^W),\ T_i^W \in \Omega \tag{9}$$

$T_i^W, T_i^P \in \Omega$ where $\Omega = [1667, 10^6]$ Kelvin.

For conciseness of exposition, in the text that follows we exclusively present derivations where the Angular error between spectra is minimised (we will return to $D_{u'v'}$ in Section 3.3). Respectively, the sets of Wien temperatures and spectra that minimize Angular error are denoted as τ^W and ψ^W.

The minimization in Eq. (8) is solved simply by searching for the best answer. The best correction is found for each of our Planckian temperatures and these pairs, taken together, define the best look-up-table (LUT) function denote $f^{\mathcal{L}}$:

$$\tau_i^W = f^{\mathcal{L}}(\tau_i^P)\ ,\ i = 1, 2, \ldots, N \tag{10}$$

Intuitively from Fig. 1, we expect that the best Wien spectrum that approximates a Planckian will have a smaller corrected color temperature, $T^W < T^P$. Moreover, we also expect relationship between T^P and T^W to be increasing. In Fig. 3, we plot the Wien corrected temperature in Kelvin against the (natural) logarithm of the Planckian temperature (where we are minimizing the spectral angular error). The function $f^{\mathcal{L}}()$ delivers corrected temperature that is always less than the actual Planckian temperature. $f^{\mathcal{L}}()$ is an increasing function. Notice that the range of $f^{\mathcal{L}}()$, [1667, 30179] is much less than its domain $[1667, 10^6]$ (approx. 7.4 to 13.8 in the log units shown in the plot).

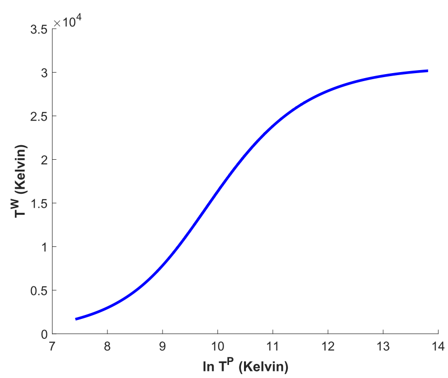

Fig. 3. Corrected Wien color temperatures using look-up-table through angular error minimization.

One issue with $f^{\mathcal{L}}()$ is that a pair of consecutive Planckian temperature T^P, $T^P + 1$ can map to the same Wien corrected temperature: $f^{\mathcal{L}}(T^P) = f^{\mathcal{L}}(T^P + 1)$. As an example, $f^{\mathcal{L}}(2519) = f^{\mathcal{L}}(2520) = 2519$. This happens because T^W is always equal/less than T^P, and since both these temperatures are integers, with the integer range of the former smaller than the domain of the latter, the function cannot be one-to-one. There must be duplicates. The existence of duplicates implies that $f^{\mathcal{L}}$ is not an invertible function.

As a final comment, we acknowledge that in building this look-up-table we are choosing to map integer input temperatures to integer outputs. We could, of course, have adopted a finer (or indeed coarser) sampling of the temperature scale. Moreover, we could have adopted some interpolation scheme so that the LUT has continuity. However, rather than focusing our efforts on finding the best LUT we instead, in the next subsection, derive 2 concise analytic formulae which have the advantage of being much more compact transfer functions than a LUT. Moreover, in our results section, we show that these functions deliver very good temperature correction.

3.2. Analytic functions to correct Wien color temperature

Now, we seek analytic expressions such that:

$$\tau_i^W \approx f(\tau_i^P) \tag{11}$$

where $f()$ is a continuous and invertible function.

In the minimizations that follow we will find the optimal parameters for a *training set* and then evaluate how well the parameterised function works in general. We will assume that the look-up-table function $f^{\mathcal{L}}()$, derived in the last section, can be used to generate ground truth Wien temperatures.

Judd [16] suggested that reciprocal color temperature (say Mired) would be a more convenient parameter for general use than color temperature itself, because differences in reciprocal color temperature are proportional to the corresponding chromaticity differences. The use of reciprocal temperature instead of color temperature is already fairly prevalent due to the form of the Wien radiation law. Therefore, we will choose 600 Mired temperatures as our training set.

The Mired color temperature (micro-reciprocal-degree) [16] of a temperature T is defined as

$$T^M = Mired(T) = \frac{10^6}{T} \tag{12}$$

Relative to Eq. (12), the interval Ω maps to $\Omega^M = [1, 600]$ in Mired Units. While we sample the temperatures of interest in Mired units, we need to convert them back to corresponding temperatures in Kelvin, so the set of Planckian temperatures we wish to approximate becomes:

$$\tau_i^P = Mired^{-1}(i)\ ,\ i \in \{1, 2, \dots, 600\} \tag{13}$$

Because each τ_i^P is not an integer value we calculate the corresponding τ_i^W by linear interpolation. For $i \in \{1, 2, 3, \dots, 600\}$

$$\begin{aligned} \alpha_i &= \tfrac{\lceil \tau_i^P \rceil - \tau_i^P}{\lceil \tau_i^P \rceil - \lfloor \tau_i^P \rfloor} \\ \tau_i^W &= \alpha_i f^{\mathcal{L}}(\lfloor \tau_i^P \rfloor) + (1-\alpha_i) f^{\mathcal{L}}(\lceil \tau_i^P \rceil) \end{aligned} \tag{14}$$

where $\lfloor . \rfloor$ and $\lceil . \rceil$ are respectively the floor and ceiling operators (mapping real numbers to their closest integer counterpart below and above).

We now present two analytic derivations of the functions $f()$. In both cases we will find the parameters of $f()$ by trying to predict τ_i^W from τ_i^P. In the arctangent derivation the optimisation is carried out directly with respect to these temperatures. In the second Polynomial derivation we will carry out the optimisation in Mired units.

3.2.1. Arctangent correction function

Our first analytic correction function is based on the arctangent function. Actually, we choose the arctangent function because one distinct characteristic of the arctangent is its slow approach to the asymptotes ($\pm\infty$), making it a good fit for our values (temperatures). Furthermore, for Planckian lights (T^P), we do not use negative temperatures, so the positive section of the arctangent function is adequate. Now, we define $f^{\mathcal{A}}()$ as:

$$f^{\mathcal{A}}(T^P) = d_1 atan(\frac{T^P}{d_2}) \tag{15}$$

where d_1 and d_2 are found by minimizing:

$$\min_{d_1,d_2} \sum_i ||\tau_i^W - d_1 atan(\frac{\tau_i^P}{d_2})||^2 \tag{16}$$

Equation (16) was minimized using Nelder-Mead simplex searching method [17]. To 2 decimal places, $d_1 = 18973.32$ and $d_2 = 18726.82$.

We wish to check whether $f^{\mathcal{A}}$ is a strictly increasing function. We calculate its derivative:

$$\frac{\delta f^{\mathcal{A}}}{\delta T}(T^P) = \frac{d_1}{1 + (\frac{T^P}{d_2})^2} \tag{17}$$

The *atan*() function has a useful property: it is a strictly increasing, one-to-one function. Thus it follows that every unique Planckian temperature has a unique Wien counterpart. Further the minimum temperature we consider is $T^P = 1667$ Kelvin and this maps to $T^W = 1684$ Kelvin.

Later, it will be useful to map a Wien temperature to its Planckian counterpart using Eq. (18).

$$T^P = d_2\ tan(\frac{T^W}{d_1}) \tag{18}$$

3.2.2. Polynomial correction function

Let $f^{\mathcal{P}}()$ denote the polynomial that *best* maps Planckian temperatures to corresponding Wien counterparts where $f^{\mathcal{P}}()$ is found by regression. Significantly, the regression is not carried out directly - in Kelvin units - but rather we first convert temperatures to Mired units. Remember in Mired units the domain of interest is $\Omega^M = [1, 600]$ and we will be calculating a polynomial expansion of values in this range.

Clearly, for an *nth-order* polynomial, the term 600^n will start to dominate the optimization (compared to the lower-order terms). Concomitantly, high-order polynomial regressions can suffer from numerical accuracy problems (due to these high powers). Thus, we will formulate our polynomial regression in normalised mired units, defined:

$$T^N = NMired(T) = \frac{10^6}{600T} = \frac{10^4}{6T} = \frac{1666.7}{T} \tag{19}$$

Clearly, in normalised mired units the interval Ω^M maps to $\Omega^{MN} = [\frac{1}{600}, 1]$. When Eq. (19) is applied to vectors of temperatures we write $\underline{w} = NMired(\underline{v})$. Component-wise, $w_i = NMired(v_i)$.

Now, we define our polynomial mapping function as $f^{\mathcal{P}}()$:

$$f^{\mathcal{P}}(T^P) = NMired^{-1}(poly(NMired(T^P); \underline{q})) \tag{20}$$

In Eq. (20), $poly(x; \underline{q})$ denotes the polynomial expansion of a scalar x where the coefficients of the polynomial are defined by the vector $\underline{q}$. As an example, $poly(x; [a_0\ a_1\ a_2]^t) = a_0 + a_1 x + a_2 x^2$. According to this notation, a coefficient vector with $n + 1$ component vectors defines an order n polynomial expansion with an offset term. Additionally, we define the function $polyvec(x, n)$ which returns an $n + 1$ component row vector of x raised to the powers 1 through n with a '1' to denote the offset, $polyvec(x, n) = [1\ x\ x^2\ \cdots\ x^n]$.

Clearly, $poly(x;\underline{q}) = polyvec(x,n).\underline{q}$ (i.e. an $1 \times n+1$ vector multiplying a $n+1 \times 1$ vector, we calculate the dot-product of the two vectors).

To find a good instantiation of $f^{\mathcal{P}}()$, we need to find a $\underline{q}$ that makes the error, err, small.

$$err = \sum_i ||NMired(\tau_i^W) - poly(NMired(\tau_i^P);\underline{q})||^2 \tag{21}$$

A reasonable way to determine $\underline{q}$ is to carryout a regression in NMired units. Let us define an $N \times (n+1)$ matrix A where (in the ith row of A) $A_i = polyvec(NMired(\tau_i^P), n)$. Then, we find the $\underline{q}$ that minimizes:

$$\min_{\underline{q}} ||A\underline{q} - NMired(\underline{\tau}^W)|| \tag{22}$$

Equation 22 is optimally solved using the Moore-Penrose inverse: $\underline{q} = [A^tA]^{-1}A^tNMired(\underline{\tau}^W)$ [18] (^{t} denotes matrix transpose).

How well we fit the data depends on the order of the polynomial used. The larger the number of terms the better we fit the data. But, if we use too many terms the polynomial is likely overly fit to the data at hand and will not generalise well to unseen data [19]. We found that there was some benefit of using a 5th-order polynomial but no benefit of using any higher-order expansion. The 'solved for' best 5th-order coefficients are shown in Table 1.

Table 1. The coefficients that define the correction function $f^{\mathcal{P}}()$.

5th-Polynomial coefficient	Value
q_0	0.05345
q_1	0.5415
q_2	1.6057
q_3	−2.7681
q_4	2.3133
q_5	−0.7467

Table 2. The coefficients that define the correction function $f^{\mathcal{P}}()$ using the Wien corrected temperatures by $D_{u'v'}$ minimization.

5th-degree-Polynomial coefficient	Value
q_0	0.057397
q_1	0.50396
q_2	1.7085
q_3	−2.8779
q_4	2.3518
q_5	−0.7444

Let us now consider whether our polynomial function is strictly increasing. First, we note that the polynomial is actually applied in normalised mired units. Suppose for two temperatures T_1^P and T_2^P, where $T_2^P > T_1^P$, we have the corresponding corrected Wien temperatures T_1^W and T_2^W. The increasing property - that we would like $f^{\mathcal{P}}()$ to exhibit - would imply that $T_2^W > T_1^W$. In normalised mired units, it is evident that $NMired(T_1^P) > NMired(T_2^P)$ and $NMired(T_1^W) > NMired(T_2^W)$ (the ordinal relations flip). However, they both always flip together and this implies that a function in Kelvin that is increasing is an increasing function in normalised mired units (and vice versa).

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Thus, to show that $f^{\mathcal{P}}()$ is increasing we need only show that $poly(x; \underline{q})$ is increasing in the normalised mired interval $x \in [1/600, 1]$ (where $x = NMired(T^P)$). Using the coefficients from Table 1, we differentiate:

$$\frac{\delta poly(x; \underline{q})}{\delta x} = 0.5415 + 3.2115x - 8.3042x^2 + 9.2534x^3 - 3.7334x^4 \tag{23}$$

The derivative in Eq. (23) has one real root at $x = 1.369$ which is outside of our normalised mired interval. As $\frac{\delta poly}{\delta x}(\frac{1}{600}; \underline{q}) = 0.5468$ and $f^{\mathcal{P}}(1667) = 1668.40$ is also positive the polynomial is positive and strictly increasing in the domain of interest.

3.3. Correction functions through $D_{u'v'}$ minimization

Here we show the best arctangent in Eq. (24) and polynomial fitting (see Table 2) when we used the Wien corrected temperatures obtained through $D_{u'v'}$ minimization.

$$f^{\mathcal{A}}(T^P) = 17932.75\ atan(\frac{T^P}{17545.53}) \tag{24}$$

3.4. Locus filters and Planckian lights

In its simplest guise, the locus filter $F_{Locus}(\lambda, T_{lf})$ is a transmissive filter that maps a Wien illuminant with temperature T_1 to another Wien light with temperature T_2:

$$F_{Locus}(\lambda, T_{lf})E^W(\lambda, T_1) = E^W(\lambda, T_2) \tag{25}$$

Of course the locus filter is the ratio of the second light divided by the first. In [4], this filter can be written as:

$$F_{Locus}(\lambda, T_{lf}) = e^{-\frac{c_2}{T_{lf}\lambda}} \tag{26}$$

where T_{lf} denotes the Locus Filter Temperature (LFT), which is equal to:

$$T_{lf} = \frac{1}{\frac{1}{T_2} - \frac{1}{T_1}} \tag{27}$$

Locus filters have the interesting and unique property that a filter defined by a given pair of Wien-type lights maps any light on the Wien locus to another Wien-type light [4]. If $F_{Locus}(\lambda, T_{lf})$ is applied to $E^W(\lambda, T_3)$ then

$$\begin{aligned} &F_{Locus}(\lambda, T_{lf})E^W(\lambda, T_3) = E^W(\lambda, T_4) \\ &T_4 = \frac{1}{\frac{1}{T_{lf}} + \frac{1}{T_3}} \end{aligned} \tag{28}$$

Following from Eq. (27) is straightforward to show that high LFTs induce *weak* filters while low LFTs induce *strong* filters, regardless of the sign (negative or positive) of LFT. Here, weak and strong, respectively *mean* close to or far from a filter that has a uniform transmission across all wavelengths in the visible spectrum.

In the last section we found the T^W which when inserted into Wien's equation results in a spectrum that was closest in shape to a Planckian with temperature T^P, where $T^W = f(T^P)$. In the discussion that follows we assume the correction function $f()$ is invertible. This means $f()$ could be the arctangent, $f^{\mathcal{A}}()$, or Polynomial, $f^{\mathcal{P}}()$, variants (see Eqs. (15) and (20)) as we previously demonstrated that both are one-to-one functions. But, the look-up-table $f^{\mathcal{L}}()$ is not one-to-one so not uniquely invertible and cannot be used. Without loss of generality, we can write

$$E^W(\lambda, T_1) \approx kE^P(\lambda, f^{-1}(T_1)) \tag{29}$$

since $T_1 = f(f^{-1}(T_1))$, and for T_2

$$E^W(\lambda, T_2) \approx kE^P(\lambda, f^{-1}(T_2)) \tag{30}$$

Now, we substitute Eqs. (29) and (30) into Eq. (25):

$$E^P(\lambda, f^{-1}(T_2)) \approx F_{Locus}(\lambda, T_{lf})E^P(\lambda, f^{-1}(T_1)) \tag{31}$$

Equation (31) teaches that a locus filter calculated - according to Locus Filter theory - for two Wien lights has the property that it is also a Locus filter for two Planckian lights. In Eq. (31) a Locus filter maps one Planckian light to another. Given a Locus Filter defined by T_{lf} and a Planckian temperature T_1^P, what is the Planckian temperature of the filtered light? By substitution Eq. (32) follows. Assuming the correction function $f()$ is accurate, a Planckian light with temperature T_1^P filtered by a locus filter with T_{lf} results in a second Planckian with temperature T_2^P:

$$T_2^P = f^{-1}\left(\frac{1}{\frac{1}{T_{lf}} + \frac{1}{f(T_1^P)}}\right) \tag{32}$$

Locus Filter theory was developed in a very general sense [4]. The domain of Temperatures for Wien lights and for Locus Filter Temperatures was $(-\infty, +\infty)$. A locus filter (for any LFT) applied to any Wien light resulted in another Wien light. The *trick* to achieving such a general theory was to admit negative temperature lights [20].

In Ref. [20], we have shown that the Planckian locus stops in the middle of the chromaticity diagram and cannot extend further (see Fig. 4). However, Wien's law provides the opportunity to generate a longer locus. As seen in Fig. 4, unlike the normal Wien locus (blue solid line) which sweeps from monochromatic red ($\lim_{T\to 0^+}$) to infinite temperature which is a somewhat desaturated blue ($\lim_{T\to\infty^+}$) an additional negative temperature locus was admitted. This negative locus (blue dotted line) sweeps from monochromatic blue ($\lim_{T\to 0^-}$) to the same point of *+ve* infinity ($\lim_{T\to\infty^+} = \lim_{T\to\infty^-}$). Indeed, negative temperatures do not extend the Planckian locus via Planck's law, as Wien's law does. Thus, without recourse to an extended Wien locus, we

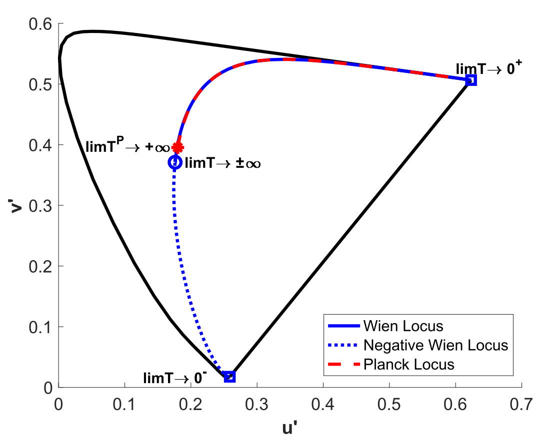


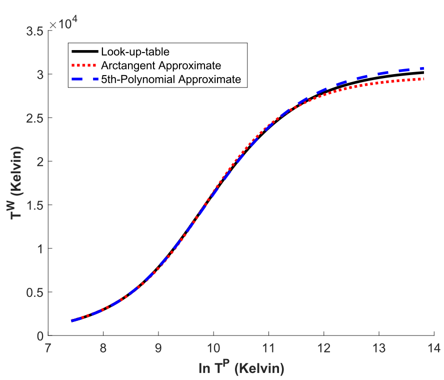
Fig. 4. Planckian locus compared to the negative Wien locus [20].

might have a blue Locus Filter filtering a bluish Planckian light. The result can be a bluer light than the infinite-temperature Planckian (red filled circle in Fig. 4) and, therefore, cannot be interpreted as Planckian light. Thus, our correction function $f()$ extends Locus Filter Theory to Planckian lights with a caveat: not all filtered Planckian lights are themselves Planckian (though, empirically, many are).

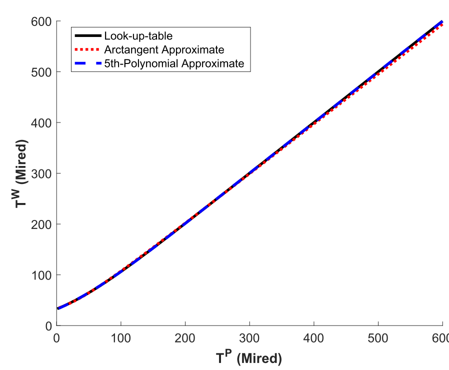
4. Numerical experiments and results

4.1. Applying correction functions

To find the best analytical functions that model the relationship between Planckian color temperatures and the Wien-corrected counterparts, we used a *training set* ranging from 1 to 600 Mired with 1 Mired step, corresponding to color temperatures from 1667 to 10^6 Kelvin with non-uniform steps. Figure 5(a) shows the corrected Wien temperatures versus Planckian temperatures, in Kelvin units. Here, the x-axis is the natural logarithm of actual desired Planckian temperature, and the y-axis is the temperature that drives Wien's equation such that the resulting spectrum is close (in terms of angular error) to the Planckian light. The solid black line shows the temperature conversion using the look-up-table. In dashed line, we show the conversion using *5th-order* polynomial. The average distance between LUT and polynomial fitting is just 15 degrees Kelvin. The dotted line records the conversion using the arctangent function. Here the average distance between LUT and arctangent curves is just 43 Kelvin. In Fig. 5(b), we replot the relationship between Planck and corrected Wien temperatures merely in Mired units. Both analytic fits using the 5th-order polynomial and arctangent functions almost overlay the LUT mapping, indicating that the analytic functions work similarly to the look-up-table.



(a)



(b)

Fig. 5. The relationship between Planckian color temperature and the corrected Wien temperature using the look-up-table mapping, 5th order polynomial, and arctan function in (a) Kelvin, and (b) Mired units.

To illustrate what the corrected Wien temperature means, let us return to our example in Fig. 1(b). In Fig. 6, we show in a solid line the 15000K Planckian spectrum. The dotted line is the spectrum of light generated by corrected Wien temperature, i.e., $T^W = 12866K$. In fact, this example is corrected by the LUT method in terms of angular error minimization. The dashed line is for the 15000K Wien formula. It is evident that by correcting the color temperature that drives Wien's function we arrive at a much more similar spectrum. Indeed, the angular error reduces from 3 to 0.6 degrees.

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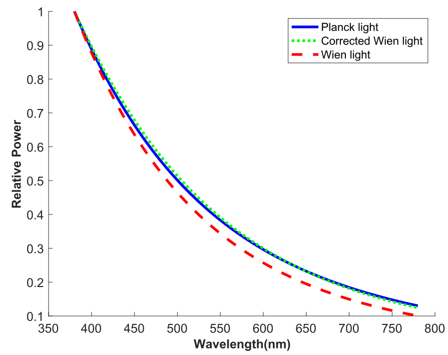


Fig. 6. A 15000K Planckian light and the corrected temperature of 12866K Wien light compared to the 15000K Wien light, all spectra are normalized to have a maximum power equal to one.

4.2. Accuracy of analytical correction functions

To assess how well the analytical correction functions (i.e., $f^{\mathcal{A}}$ and $f^{\mathcal{P}}$) work, we generate the actual Planckian light spectrum for temperature T^P and its Wien approximation for temperature $T^W = f^{\mathcal{A}}(T^P)$ and $T^W = f^{\mathcal{P}}(T^P)$. This calculation is carried out in the discrete domain so both spectra are represented as 81-vectors (a sampling from 380 to 780 nm with a 5 nm step). Then, to determine the closeness of the spectrum-pair we use Eq. (3) to calculate the angular error.

For every Planckian spectrum in the domain Ω (approximately a million spectra), we generate corresponding Wien spectra according to the methods set forth in this paper. The error in the approximation is calculated as the angle between the two spectra. The errors of different correction algorithms are reported in Table 3.

Table 3. Statistics of the angular error between Planckian lights and their equivalent Wien lights when the color temperature is uncorrected and when corrected with different functions.

	Uncorrected	$f^{\mathcal{L}}$corrected	$f^{\mathcal{A}}$corrected	$f^{\mathcal{P}}$corrected
Mean (Ω)	6.4	0.66	0.68	0.67
Mean (Ω^S)	0.94	0.31	0.44	0.31
95th quant. (Ω)	6.7	0.68	0.69	0.68
Max (Ω)	6.7	0.69	0.71	0.69

The first column of Table 3 records the "uncorrected" error: the Planckian color temperature is used directly in the Wien equation. Then, we correct the Planckian temperature using the functions $f^{\mathcal{L}}$, $f^{\mathcal{A}}$, and $f^{\mathcal{P}}$, with the results reported in the second, third, and fourth columns, respectively.

The rows of the table record 4 error statistics. In the first row we record the mean angular error for the domain Ω. In the second row we record the mean angular error for a smaller ^{S} subset of the temperature domain: $\Omega^S = [3500, 10000]$, which contains 6501 color temperatures that are broadly indicative of the lights we encounter in the natural world. In the 3rd and 4th rows we record the 95% quantile and maximum errors (for Ω).

We see that the maximum error - the angle between the Planckian and Wien temperature spectra is 6.72 degrees when no correction is carried out, and , 0.69, 0.69, and 0.71 when, respectively, LUT, 5th-order polynomial, and arctangent are used. To 2 decimal places, a 5th-order polynomial has the same max error as the look-up-table. For the arctangent, the max error is about 0.71 which is almost as good.

Since the chromaticity of lights is an important factor, Table 4 presents the results of the color chromaticity difference ($D_{u'v'}$) between Planckian lights and their corrected counterparts. Here, the efficiency of our fit becomes even more evident. The results are rounded to three decimal places, and it is clear that the means of Ω and Ω^S both decrease significantly through our correction functions. For example, even the max of Ω for all correction functions is less than the threshold [14]. In comparison the uncorrected maximum error is 6 times the threshold.

Table 4. Statistics of the $D_{u'v'}$ between Planckian lights' chromaticity and their equivalent Wien lights when the color temperature is uncorrected and when corrected with different functions.

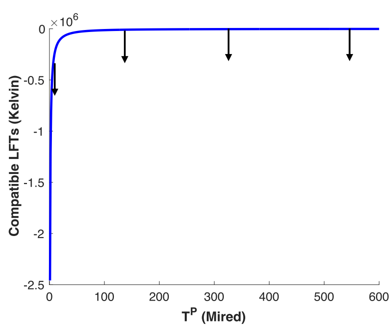
	Uncorrected	$f^{\mathcal{L}}$corrected	$f^{\mathcal{A}}$corrected	$f^{\mathcal{P}}$corrected
Mean (Ω)	0.023	0.002	0.001	0.002
Mean (Ω^S)	0.002	0.000	0.000	0.000
95th quant. (Ω)	0.024	0.002	0.002	0.002
Max (Ω)	0.024	0.002	0.002	0.002

4.3. Locus filtering of Planckian lights

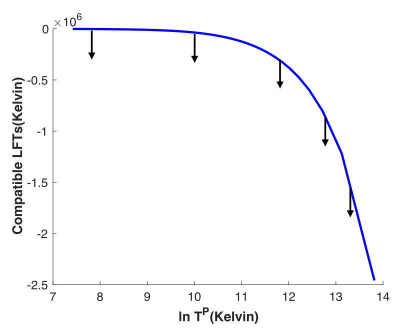
A locus filter maps a Wien illuminant with one temperature so it has the same shape as a second Wien light with another temperature. A useful property of locus filters is that any locus filter applied to any Wien light results in a second Wien illuminant [4]. In Subsection 3.4, we extended the Locus Filter theory to apply to Planckian lights. However, we remarked that not all filtered Planckian lights are themselves Planckian. Let us consider this point in more detail.

We say that a given locus filter is *compatible* with the set of Planckian lights which, when filtered, are also describable by Planck's Equation. Filters that make lights more warm have a positive LFT and can always be applied to all Planckian lights. However, a filter that is bluish, and has a negative LFT, is only compatible with a subset of the Planckian lights. A bluish Planckian light filtered by a bluish filter can result in an even bluer light (that is beyond the end of the Planckian locus, i.e., the infinity point).

In Fig. 7, for negative LFTs only, we delimit the set of possible LFTs when filtering a Planckian light of a given temperature, where the conversion to and from Wien lights is implemented in Eqs. (15) and (18), using the arctangent conversion function and its inverse. The area below the blue line delimits all the LFTs that are possible for lights of a given temperature (Mired units, left graph and Kelvin, right graph). For this test we assume Planckian temperatures are in the range (0, 1000000].



(a)



(b)

Fig. 7. Compatible Locus Filter Temperatures (Negative LFTs) for Planckian temperatures in terms of (a) Mired, and (b) Kelvin

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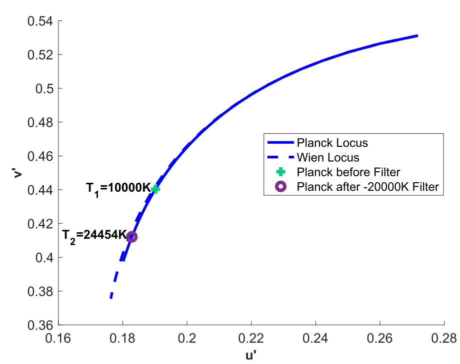
In Table 5, we apply some Locus Filters with compatible LFTs (with positive and negative values) to several Planckian lights and calculate the color temperature of the Filtered Planckian. The first three columns of the table show the change in temperature when applying filters with a locus filter with positive LFT values of 5000, 7500, and 10000 to different Planckian lights. The last two columns are for some compatible negative LFTs, −20000 and −15000. As an example, a 4000K Planckian light filtered by a locus filter with LFT = 5000 is found to shift to a Planckian with color temperature 2200K. However, the same light shifts to 5040K when filtered with a locus filter with $LFT = -20000$.

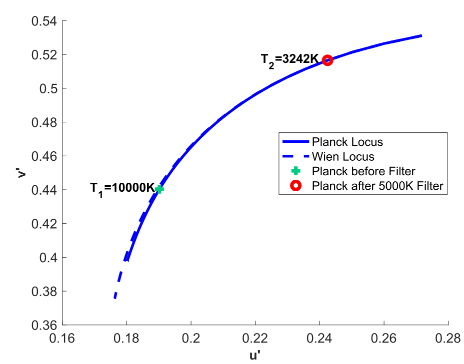
Table 5. Color temperature shifts after applying locus filters on Planckian lights. The first column records the color temperature in Kelvin for 4 input lights, ranging from 4000K to 10000K. Other columns show the LFT ranging from −20000 to 10000. In position (i, j) we see the output color temperature for the ith input light filtered by a locus filter with the jth LFT.

input Planck temperature	output Planck temperature for locus filter temperature (LFT) of				
	(5000)	**(7500)**	**(10000)**	**(−20000)**	**(−15000)**
4000	2200	2588	2838	5040	5522
6000	2686	3287	3703	8800	10478
8000	3011	3791	4357	14365	20274
10000	3242	4166	4863	24454	65430

In Fig. 8, we illustrate how a Planckian radiator behaves before and after applying a locus filter in the CIE (u', v') color chromaticity diagram. A 10000K Planckian light is filtered with a locus filter with a negative locus filter temperature ($LFT = -20000$) makes it cooler (see Fig. 8(a)). Also, in Fig. 8(b), the same light is filtered with a locus filter with a positive temperature (LFT = 5000) making it warmer. In both cases, the new filtered lights are, as they must be, on the Planckian locus.



(a)



(b)

Fig. 8. CIE(u', v') chromaticity of a 10000K Planckian radiator before and after applying a locus filter (a) with −20000K LFT, and (b) 5000K LFT

4.4. Discussion

For many applications, there is no inherent *complexity* in using Planck's formula. This includes rendering images using Planckian lights with the same correlated color temperature as an actual illuminant, which is useful in the formulation of Color Rendering Indices [21], among other applications.

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However, in computer vision, the derivation of photometric invariants - combinations of camera responses which are invariant to a change in light color temperature - the use of Wien's formula was crucial. Indeed, in the foundational paper [5] the authors point out that for high color temperatures, Wien's approximation only produces spectra being "broadly" similar to Planckian lights of the same temperature. Consequently, the derivation and empirical evaluation of their photometric invariant is only for lights with temperatures up to 10000K. Through the temperature correction function derived here, we can use Wien's formula for all lights. Concomitantly, the theory of photometric invariants [5] is extended to account for illuminants described by the Planckian locus.

Another compelling reason to be interested in using Wien's - with our temperature correction - instead of Planck's formula is in regard to Locus Filters (and color filters in general). A locus filter [4] has the property that it will always map a Wien illuminant of a given temperature T to a new Wien light with a derived temperature T'. Where T' is computed given knowledge of the starting temperature T and the locus filter at hand. The Planckian locus stops at the infinity temperature which corresponds to a bluish light with low saturation. Filtering blue lights with a blue color filter (including blue locus filters) must result in an even bluer light, which cannot be described using Planck's formula and so Planck's equation is not sufficient for describing all typical lights. Wien's formula does not have this *stopping point*. And, as we briefly discussed in this paper, as we move towards colors even bluer than that generated by an infinite Planck temperature we get new blue lights that are described by negative temperatures (see Fig. 4). In [22], it is shown how by re-expressing Planckian lights using Wien's formula, Locus filter theory can be applied - in its full generality - to Planckian lights.

5. Conclusion

Planck's famous equation defining a black-body radiator is tolerably well approximated by a simpler formula called Wien's approximation. However, for higher color temperatures (say T> 4000 Kelvin), the Wien approximation is not as accurate, and the difference between the actual Planckian and the Wien approximation becomes visually noticeable for temperatures exceeding 10000 K. In this paper, we have shown that we can map a Planckian color temperature T^P to a newly converted temperature $T^W = f(T^P)$ in such a way that the spectra generated by Wien's approximation become much closer to the desired Planckian. We defined two analytical correction functions, polynomial, and arctangent functions that can map any Planckian color temperature to the Wien domain. The arctangent function is an elegant formula with an invertible property. Through the application of our correction functions, the generated spectra are very similar to the desired Planckian spectra. Using the analytic correction function, we found that the theory of locus filter is extended to the Planckian radiators. Then, applying locus filters with different temperatures to the Planckian lights generated new lights over the Planckian locus.

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Data availability. No data were generated or analyzed in the presented research.

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