Digital Currencies and the Macroeconomy

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Abstract

The present research explores the interaction between macro-finance fundamentals and the private currency ecosystem, examining their economic effects through both theoretical modeling and empirical analysis. First, we investigate the relationship between crypto-currencies, proxied by Bitcoin and Ether, and key macroeconomic variables such as the United States term spread, Volatility Index (VIX), and breakeven inflation. Our findings reveal no significant link between crypto-currency returns and the term spread, suggesting investors do not consider economic cycles when trading crypto-currencies. However, extreme low VIX values correlate with high crypto volatility, with upper tail dependence reaching 3.7% to 7.6%. Second, we develop a one-period theoretical model where government-backed currencies and crypto-currencies serve as media of exchange for differentiated goods, showing that while fiat money is neutral, crypto-currencies are non-neutral due to mining costs and labor reallocation. In other words, cash being costless and crypto-currencies being costly lead to different equilibrium implications for policy decisions. In a dynamic model, cash-in-advance constraints lead to non-neutrality of money, with crypto-currencies introducing additional distortions through transaction fees and labor shifts. We recommend further research on how extreme events affect the relationship between crypto-currencies and macroeconomic variables, and propose exploring the coexistence of central bank digital currencies alongside fiat and crypto-currencies to better understand their long-term macroeconomic implications.

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Introduction

The idea of private currencies has been discussed in the literature for at least five decades (Hayek, 1976). However, the debate, both from academic and policy-making perspectives, experienced a pause for a relatively long period. On the academic side, research has primarily focused on currency substitution between competing economies through the lens of search-theoretic and cash-in-advance frameworks (Matsuyama et al., 1993; Lucas Jr, 1982). These approaches have provided robust predictions on the conditions under which one currency becomes more valued than another in trade settings.

Interests in privately-issued currencies gained prominence with the introduction of Bitcoin, the first widely successful private currency. Since then, various initiatives have emerged with different technological designs. In some models, currency growth follows a deterministic process with an upper bound (Nakamoto, 2008), while in others, such constraints are less defined. The optimal design of private currencies is a dynamic research field closely linked to the literature on monetary policy (Chiu and Koeppl, 2017).

Private currencies are often referred to as electronic or digital currencies in the literature. This designation can be a source of confusion, as the vast majority of broad money balances in the modern era are also electronic (Barrdear and Kumhof, 2016). One reason cited in the latter for this terminology is the settlement protocol in the crypto-currency environment, which requires solving algorithmic puzzles. This distinction is included for clarification but has limited significance for understanding the rest of the papers. For instance, Chapter 1 focuses on crypto-currencies as speculative assets and their connection to macroeconomic fundamentals. In Chapters 2 and 3, we provide a stylized and tractable framework for analyzing crypto-currencies as a means of payment. Moving forward, we will clarify the use of digital currencies when there is potential ambiguity between broad money supply and crypto-currencies.

In this research, we address several key questions regarding the fundamental nature of private currencies:

- What are the macro-financial fundamentals driving returns and volatility on the

Crypto-currency Market (CM)?

- Can crypto-currencies be valued as a medium of exchange in a competitive economy?
- What are the policy implications arising from the coexistence of fiat money (cash) and crypto-currency?

We address the first question in Chapter 1. Our analysis uses Bitcoin (BTC) and Ethereum (ETH) as representatives of the CM. As of 30 June 2024, these two crypto-currencies account for approximately 65% of the CM (Cryptocompare, 2024). We begin by characterizing the nature of the CM as an unregulated and volatile market. Next, we employ the copula framework to study the co-movement between CM returns, the U.S. term spread, the Volatility Index (VIX), and breakeven inflation. The copula method enables us to break down and illustrate the relationships between variables, particularly during extreme events, by focusing on upper and lower tail dependence. Furthermore, we perform multiple robustness checks to verify the reliability of our results. The main insights from Chapter 1 are:

- There is no clear pattern emerges indicating that returns on the CM tend to cluster around any specific quantile of the U.S. term spread (see Figure 1.4 and Table 1.4);
- We find evidence that extreme low VIX values (5% quantile) are correlated with high predicted volatility (90% quantile) in the CM (see Figure 1.5);
- Our analysis reveals that crypto-currencies respond differently to economic indicators than traditional assets, with significant effects at extreme return levels, especially post-COVID, warranting further research on extreme events.

Our static model in Chapter 2 addresses question 2. The theoretical framework begins with the assumption that cash is required to purchase consumption goods, while a portion of the population opts for an alternative payment method, in this case crypto-currency. Consumers use crypto-currency to avoid value-added tax on goods (pecuniary benefit) and to maintain anonymity (non-pecuniary benefit). An innovative aspect of this chapter is the inclusion of both pecuniary and non-pecuniary factors in consumer preferences, allowing us to endogenize the stock of goods purchased with crypto-currencies. We derive several conclusions, outlining the conditions under which both currencies are valued as media of exchange:

- Our results show three potential outcomes: all goods purchased with money, all with crypto-currency, or a mix of both, depending on the relative transaction costs and fees;

- While fiat money remains neutral, the costly mining of crypto-currencies introduces non-neutrality, affecting labor allocation and leading to shifts in real wages and unemployment in the goods production sector.

Our dynamic model in Chapter 3 addresses question 3. We analyze the coexistence of fiat money and privately-issued currencies in a dynamic model where all factors of production are compensated in fiat money. This introduces a cash-in-advance constraint that impacts both consumption and investment, marking a significant departure from the static model, which lacked capital investment. Using both neoclassical and endogenous growth frameworks, we derive several key insights from the model. The key takeaways are as follows:

- Our main result shows that money is neither neutral nor super-neutral, as a money supply shock raises inflation, reducing consumption growth and affecting capital and labor allocation, with the impact magnified by the substitutability between money and crypto-purchased goods.

Our Chapter 1 analysis shows weak links between crypto-currency returns and traditional economic indicators, suggesting that crypto markets may function independently and require tailored regulatory and investment approaches. We suggest using higher-frequency data to better capture CM responses to policy changes and investigating bubble formation. For the theoretical part, incorporating both pecuniary and non-pecuniary features into the dynamic framework and extending the model to include Central Bank Digital Currencies (CBDCs) would deepen the analysis of interactions between fiat money, crypto-currencies, and CBDCs, enhancing the current study.

Overall, this thesis consists of three chapters. Chapter 1 provides an empirical analysis of the CM. Chapter 2 introduces a static model of private currencies, focusing on the conditions for the coexistence of fiat money and crypto-currencies. Chapter 3 presents a dynamic model to address certain inconsistencies in the static model, specifically the non-neutrality of cash. Each chapter presents clear research questions and offers plausible conclusions with policy and academic implications.

Chapter 1

Volatility on the Crypto-currency Market: A Copula-GARCH Approach

Abstract

This study analyzes the relationship between crypto-currencies, proxied by a Bitcoin (BTC) and Ether (ETH) index, and key macroeconomic variables from April 2013 to May 2024. We focus on US term spread, Volatility Index (VIX), and 5-year breakeven inflation as predictors. Our findings reveal no significant dependence between returns and the term spread, suggesting investors do not consider policy paths or economic cycles when trading crypto-currencies. In contrast, extreme low VIX values are linked to high Crypto-currency Market (CM) volatility, with upper tail dependence estimated at 3.7% and 7.6% using Gumbel-Hougaard and Joe copulas, respectively. Our copula modeling exercise also shows a weak correlation of crypto-currency returns with breakeven inflation. Robustness checks, including a sub-sample analysis and variable transformation, confirm these results. We find that while crypto-currencies exhibit weak links to certain financial fundamentals, they respond differently to economic indicators compared to traditional assets, showing increased returns during restrictive monetary policies. The study highlights a need for further research integrating extreme events with dynamic time series analysis to better understand these relationships.

1.1 Introduction

The crypto-currency's secondary market rise is a singular case study in the financial literature. The market has gone from 0 in valuation in January 2010 to more than 2.4 trillion United States Dollar (USD) as of 31 July 2024. The unconventionally high returns on crypto-currencies is a possible reason for this expansion. Between 1 April 2014 and 31 May 2024, the price of BTC has multiplied by a factor of 574. In contrast, the highest performing stock in the S&P500, NVIDIA, has seen an increase of its equity price by a factor of 22. The exceedingly high returns coupled with low entry barriers have turned crypto-currencies into an attractive class of assets for retail investors. More recently, the CM activity has also been amplified with an influx of institutional investments. As a consequence, this nascent market has been subjected to important scrutiny work from regulators and academics alike.

A major impediment with crypto-currencies is the unstable fluctuation around the mean returns. The average annual volatility of the BTC, proxied by the standard deviation of the returns distribution, oscillated around 82% between April 2014 and May 2024. We evaluated the average annual volatility of the S&P500 at 20% over the same time period. So, the BTC price is approximately four times more volatile than the S&P500 index. By traditional standards, the crypto-currency trading is an extremely high-risk financial activity.

Are there financial and economic drivers to explain price fluctuations on the CM? A strand of the emerging literature identifies interest rates on government bonds as a major candidate to explain crypto-currency prices (Karau, 2021; Aboura, 2022). The same argument is also prevalent in economically inclined newspapers (The Economist, 2022; Financial Times, 2022). In fact, the reasoning supposes that higher interest rates on government bonds crowd capital out of the CM. Similarly, low interest rates on government securities increase both investors' risk-taking attitude and the attractiveness of the CM. The latter reasoning is similar to the risk-shifting mechanism studied in Rajan (2006) and Borio and Zhu (2012). Put differently, the explanation posits a trade-off between holding crypto-currencies and government bonds, which is a variation of the classical trade-off in portfolio construction with risky and risk-free assets.

Our research addresses two levels of inconsistencies in the current literature. On the one hand, it is likely the interest rate channel identified in the literature arises from an inadequate interpretation. For instance, Aboura (2022) argues that the March 2020 interest rate cut in the US was instrumental to the subsequent crypto-currency bullish run. However, the same period witnessed the inception of multiple fiscal transfer packages directed to households and small enterprises across the globe. Higher household savings, driven in part by the pandemic-related restrictions, may have instead dictated retail investors' preference for crypto-currencies[1]. On the other hand, a crypto-currency price response to interest rate change does not lead to clear-cut conclusions. Publications in this area often report incoherent crypto-currency price reactions, which depends on various policy set-ups and the country considered for the object of the analysis (Karau, 2021; Aboura, 2022). Consequently, these contrasts lessen the relevance of these studies in practical decision-making related to the CM.

Our study re-examines the relationship between price variation in the CM and the term spread. We focus on the direction and steepness of the yield curve, specifically the sign and magnitude of the curve's slope. The slope reflects both intertemporal changes in interest rates and expectations about market conditions, which form the basis of the expectations theory. Implicitly, we test whether the direction of the yield curve is informative for crypto-currency returns. Our analysis provides mild evidence supporting this hypothesis, suggesting the potential for a further dependence study that combines the CM with other risky markets. For the extension to risky markets, we use the VIX as a measure of expected volatility. Similar to the yield curve, the VIX incorporates uncertainty information about 500 leading firms in the US economy. Our final bivariate dependence analysis examines the relationship between price variation in the CM and the US 5-year breakeven inflation rate. As a measure of expected inflation, the breakeven rate is crucial for understanding how investors align their decisions with future market conditions. Identifying synchronization or asymmetry in fluctuations between CM price movements and the variables discussed in this section could provide a starting point for rationalizing investment decisions in the CM.

We use the copula framework to model the dependence between returns in the CM and three other variables: the term spread, the VIX, and breakeven inflation. The copula is a widely recommended approach for studying the dependence between continuous random variables when the Pearson correlation and related techniques are insufficient to capture the true dependence structure. Patton (2006) illustrates the relevance of the copula technique in analyzing cases of asymmetric relationships, such as between exchange rates. Other econometric applications of the copula method in finance and economics are also described in Patton (2006). In this paper, our main objective is to measure the co-movement between price fluctuations in the CM and the aforementioned variables. The chosen technique also allows for the decomposition and visualization of dependence between the variables in terms of extreme events, known as upper and lower tail dependence in the copula literature. For instance, a tendency

[1] See Dossche et al. (2021) for an overview on household savings increase in the euro area during the pandemic and the allocation of a sizable part of them to financial investments.

for extreme low returns in the CM to cluster with extreme low term spread values would indicate lower tail dependence between the two variables. This scenario could arise from a flattened yield curve, signaling recessionary periods or an easing monetary policy stance.

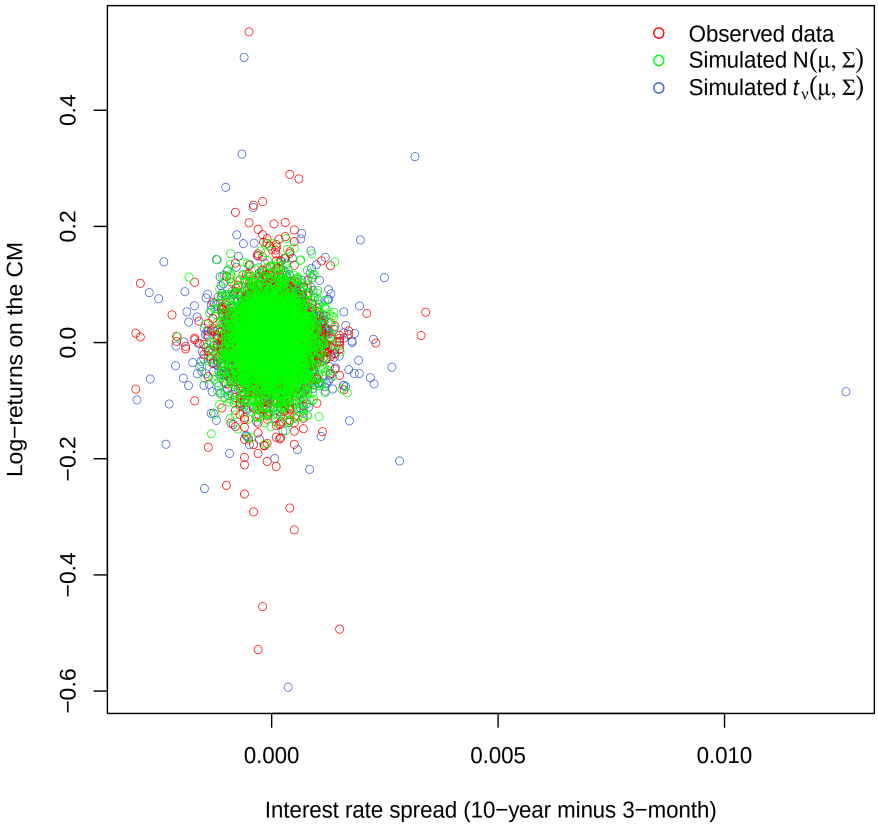
We illustrate the existence of extreme events between log-returns on the CM and the term spread in a simulation analysis in Figure 1.1. Initially, the two observed series are plotted against each other. We then extract the mean vector and the covariance matrix of the series to simulate bivariate normal and Student's t distributions based on these estimates. Figure 1.1 shows a clear departure from normality and highlights extreme events best captured by the bivariate Student's t distribution. Using the copula technique, we provide estimates of the magnitude and statistical significance of these extreme events.

The dataset used in this research covers the period 01 April 2013 to 31 May 2024. The returns on the CM are proxied by an index aggregating BTC and Ether (ETH). The market capitalization of these two crypto-currencies represents more than 65% of the CM for the first quarter of 2024 (Cryptocompare, 2024). Regarding the term spread, it is computed for the US government bond market. Not only the latter is the largest of such markets, but also US treasuries are held by investors across the globe. The reasoning underlying the choice of the VIX and the 5-year breakeven inflation is also driven by the preeminence of the US financial system. Therefore, the term spread, the VIX and the breakeven inflation are the best candidates to gauge investors' possible trade-off with the CM.

Our first set of results shows no evidence of dependence between the CM and the government bond market. In particular, there is no revealed pattern for returns on the CM to cluster with a particular quantile of the interest rate spread. We illustrate this result through the plot of the marginal distributions of both variables in Figure 1.4. The low estimates of both the dependence and the tail parameters in Table 1.4 support the graphical representation of the two variables. So, investors on the CM seem to give no weigh to expected policy paths and uncertainty about economic cycles in their decisions related to the CM as measured by the yield curve slope.

The dependence between volatility on the CM and the VIX exhibits a more complex picture. Conversely, we find evidence that extreme low VIX values (5% quantile) are correlated with high predicted volatility (90% quantile) on the CM (see Figure 1.5). We transform the VIX variable (100 minus the VIX) and express this observation in terms of an upper tail dependence structure, which has an established mathematical formula in the copula literature. The likelihood of observing the cluster of these extreme events is estimated at approximately 3.7% and 7.6% for the Gumbel-Hougaard and Joe copulas, respectively. Given that the VIX is a 1-month ahead estimate, we generate

Figure 1.1: Daily log-returns on the CM and the yield slope variation



Notes: This figure assesses the departure from the normality assumption in the log-returns on the CM and the first difference of the term spread series. Since bivariate normality implies normality of the margins, we use univariate tests to establish the underlying statistical distribution of the two series. Formal univariate tests, including Shapiro-Wilk (p-value=0.000), D'Agostino's (p-value=0.000), and Jarque-Bera (p-value=0.000), reject the normality assumption at the 1% significance level. Similar conclusions apply for comparisons between the log-returns on the CM and the other two variables (available upon request).

a similar CM volatility measure and re-evaluate the tail dependence probability. The upper tail dependence estimates sit at around 3.6% and 8%. Based on these results, market participants should expect overlapping between low VIX and high CM volatility values for 9 or 19 days in a trading calendar year if the Gumbel-Hougaard and Joe copulas are considered.

We also find non-statistically significant estimates in the copula modeling exercise between log-returns on the CM and breakeven inflation. Overall, the copula technique's results remain consistent in the presence of possible nominal and real economic drivers of CM prices, specifically the term spread and breakeven inflation in this analysis.

To ensure the robustness of our findings, we conduct several additional checks detailed in subsection 1.4.6. First, we perform a sub-sample analysis covering the period from 1 January 2020 to 31 May 2024, coinciding with increased institutional investor participation and greater regulatory recognition of CM activities. Surprisingly, even within this focused period, we find no evidence of significant co-movement between returns on the CM, the interest rate spread, and breakeven inflation. This consistent result suggests that the entry of institutional investors has not strengthened the linkage between crypto-currencies and broader macroeconomic variables. Furthermore, similar conclusions emerge when we transform the variables into weekly frequency to mitigate the potential issue of non-synchronous closing times, identified as a significant challenge in crypto-currency analyses (Alexander and Dakos, 2020). Lastly, we investigate the possibility of a stronger linkage between the CM and the technology sector using the NASDAQ index. However, our copula-based figure, presented in Figure 1.7, rejects this hypothesis.

Recent literature highlights an increased interconnectedness between crypto-currencies and macroeconomic fundamentals in the post-COVID era. To explore this, we use the sub-sample analysis discussed in the previous paragraph. Then, we fit a linear rolling quantile regression to examine how predictors affect log-returns at the 0.05 and 0.95 percentiles. Our findings indicate that a 1% increase in predictors significantly decreases log-returns at the 0.05 quantile, while the effect is not significant at the median (0.5). At the 0.95 quantile, the effect becomes positive and significant. Controlling for the two other variables reduces individual effects, with only the VIX significantly impacting median returns, particularly until late 2023. This suggests that crypto-currencies, a form of risky assets, respond differently to economic indicators compared to traditional assets, potentially showing increased returns during restrictive monetary policies. The unclear post-COVID conclusion in this paper calls for further research that integrates extreme events with dynamic time series analysis.

In terms of economic knowledge, we provide evidence that crypto-currencies are weakly linked to some financial fundamentals, in particular uncertainty information contained in the interest rate spread and the VIX. Unlike the copula technique, our simple quantile model uncovers some significant relationship between the dependent variables and the log-returns on the CM. The high proportion of ties in the interest rate spread (84%) and the breakeven inflation (94%) is a potential source of problems for the copula estimates. This issue may undermine the continuity assumption made by the Sklar's theorem on the marginal distribution of the mentioned series and affect the reliability of the copula estimates (see Hofert et al. (2019)).

As explained above, our study finds that crypto-currency returns exhibit limited dependence with traditional economic indicators such as the term spread, VIX, and breakeven inflation. This suggests that CM may operate independently of the broader financial system, posing unique challenges for risk assessment. Our additional analysis further reveals that crypto-currency returns are primarily driven by momentum effects and internal market dynamics, including responses to major crypto-specific events such as Bitcoin halvings and Ethereum upgrades. Policymakers and investors should therefore exercise caution, as traditional economic signals may not reliably predict CM behavior. These findings underscore the importance of tailored regulatory frameworks and investment strategies informed by internal market dynamics.

The remainder of this work is organized as follows. Section 2 presents the current state of the financial econometric literature on crypto-currencies as an asset class. Section 3 gives an overview of the data used in this paper. Section 4 discusses the estimation of the copula and various robustness check strategies. Finally, section 5 concludes the paper.

1.2 Related Literature

A starting point of this investigation is the literature on rational expectations and the term structure of interest rates. In particular, our contribution extends the long standing debate of the linkages between interest rates and risky assets into the CM literature. This work's methodology follows the approach in Estrella and Mishkin (1996) for the spread choice. In terms of early findings, numerous publications argue the existence of a significant link between the term structure of interest rates and economic activity (see e.g., Mishkin, 1990; Estrella and Hardouvelis, 1991; Ang et al., 2006). Work by Zhou (1996) and Boudoukh et al. (1997) also depict an important relation between interest rates on US government securities and equity returns. However, other work in the field cast doubt on the predictive power and the use of the yield slope in predicting future economic trends (see e.g., Shiller et al., 1983; Campbell and Shiller, 1991). Our analysis applies the copula framework to both the entire sample and sub-samples of the dataset to detect possible interlinkages between the spread (the VIX and the breakeven inflation also) and returns on the CM. As a statistical tool, the copula technique lays out a straightforward approach to test this relationship while keeping the core theoretical underpinnings of the expectations theory intact.

This research adds to the empirical literature researching the linkages between crypto assets, economic policies and the traditional class of financial securities. Earlier studies have found significant impacts of monetary policy decisions on crypto-currency price valuations. Focusing on BTC alone, Karau (2021) uncovers a strong connection between monetary policy stances and the BTC price. Corbet et al.

(2020) also observe similar links between monetary decisions and crypto-currencies. On a different approach, but closely related to our analysis, Akyildirim et al. (2020) pinpoint the existence of a correlation between crypto-currencies and uncertainty on the stock markets, proxied by implied volatility measures. Our analysis offers an integrated investigation of the dependence between the CM, the risky and the risk-free market. Compared to the findings reported in this paragraph, our copula analysis finds no substantial relationship between the CM, the interest rate spread, the VIX and the breakeven inflation. The ADL model presents a nuanced picture with strong and statistically significant effects of the breakeven inflation on the log-returns on the CM. In a nutshell, these results make a case for the use of dynamic models (models with lags) in the analysis of crypto prices.

From a broader perspective, our research offers practical insights into price movements on the CM. A pioneered thinking on this question is from Böhme et al. (2015), who see the bitcoin money growth model as an inherent cause for the shallow market issue. Given the widespread use of crypto-currencies for financial trading purposes, publications on price fluctuations on the CM have expanded largely over the recent years. Regarding the stylized facts, Zhang et al. (2018) analyze the returns of 8 leading crypto-currencies and detect the existence of heavy tails, a pattern towards long memory, and a powerful feature of volatility clustering. Similarly, Hu et al. (2019) find a significant dissimilarity in the returns distribution for a sample of over 200 virtual currencies, which would probably indicate some restraints in generalizing findings for a class of crypto-currencies to the entire CM. These well-established statistical facts are supported in numerous volatility modeling publications (see also Bariviera, 2017; Jiang et al., 2018). Our research finds the existence of persistent conditional volatility on the CM and shared properties with traditional financial time series. Unlike Böhme et al. (2015), we have not identified the monetary structure of BTC and ETH to be a driving factor in their price variations.

There is indeed considerable research examining the relationship between the CM and traditional market drivers, offering mixed perspectives on their linkage. For instance, Bouri et al. (2017) suggest BTC may hedge global uncertainty primarily at shorter investment horizons and under specific market conditions. Similar to our conclusions, Trabelsi (2018) and Corbet et al. (2018) emphasize limited volatility spillovers between crypto-currencies and traditional asset classes, proposing crypto-assets as diversification instruments due to their relative market isolation. Conversely, Antonakakis et al. (2019) indicate increasing spillover dynamics, particularly during periods of heightened financial turbulence such as the COVID-19 pandemic, highlighting potential systemic risks. Similarly, Mroua et al. (2024) find that Bitcoin's causal influence on traditional assets varies across volatility regimes, exerting stronger impacts during turbulent market periods. Meanwhile, Benigno and Rosa (2023) underscore a striking macroeconomic disconnect, showing that Bitcoin prices appear largely insensitive to traditional economic news and monetary policy announcements, complicating its classification as either a macroeconomic hedge or speculative asset. On a different note, Foley et al. (2019) view BTC primarily as a vehicle for criminal activities with minimal fundamental economic value. These findings reflect divergent viewpoints regarding the CM and its primary connections to broader macroeconomic factors.

1.3 Data and Summary Statistics

1.3.1 Data construction methodology

We obtain statistics on crypto-currencies, the term spread on US government instruments, and the VIX from FirstRate Data, a trusted service provider, and the Federal Reserve Bank of Saint Louis respectively. BTC and ETH prices are observed daily at 5:00 PM, Eastern Time. The Exchange Rate Index (ERI) for the CM is made up BTC and ETH. As of 30 June 2024, BTC and ETH account for 65% of the overall market capitalization of the CM (see Cryptocompare, 2024). So, these two crypto-currencies are representative of the crypto exchange activities[2].

Observations for BTC/USD are available from 01 April 2013 to 31 May 2024, while data for ETH/USD are accessible from 11 March 2016 to 31 May 2024. To compute an index of the two rates at a given time t, the Dow Jones methodology is implemented as follows: $\frac{P_t^{BTC}+P_t^{ETH}}{n}$, where P_t^{BTC}, P_t^{ETH}, and n represent the exchange rate of BTC, the exchange rate of ETH, and the number of price series, respectively. From 01 April 2013 to 10 March 2016, the ERI is simply equal to the price of BTC in USD. We later modify the formula to accommodate the introduction of ETH and ensure no sudden jumps that can affect the copula analysis in the next section. Although ETH trading activities started in 2015, FirstRate's series begin in March 2016. Noise affecting ETH price in the early trading days might explain this choice[3]. We introduce a different divisor in the formula to compute the ERI from 11 March 2016 until the end of the series. The divisor is calculated as the summation of the prices divided by the previous day's index $\left(\frac{P_t^{BTC}+P_t^{ETH}}{ERI_{t-1}}\right)$. Instead of dividing by the number of crypto-currencies, the sum of the two crypto-currency prices is divided by the divisor in the

[2] Attempting to include the top 10 and 20 crypto-currencies to the index basket has not changed the results reported later in this analysis.

[3] Larger service providers, such as Bloomberg, have even shorter time series for ETH. Their dataset dates back to 2018.

modified formula $ERI_t = \frac{P_t^{BTC}+P_t^{ETH}}{\text{Divisor}}$[4]. We subsequently refer to the variation of the index as the crypto-currency returns or simply X_t^{ERI} in the next section.

The study relies primarily on the term spread to offer a comprehensive analysis of the CM. The spread used in this work is the difference between the interest rates on the 10-year treasury note and the 3-month treasury bill. Estrella and Mishkin (1996) provide a thorough empirical analysis of the strength of the interest rate spread considered here in predicting macroeconomic cycles. More specifically, the conclusion of their investigation shows a relatively significant long-term prediction capability of the interest rate spread between the 10-year note and the 3-month bill. We use a similar motivation to study how price movements on the CM can be approximated by information contained in the interest rate spread. Mathematically, the yield on a zero-coupon government bond with a 1 US dollar face value is defined as $y_t = \left[\frac{1}{P_0(0,t)}\right]^{\frac{1}{t}} - 1$, with $P_0(0,t)$ describing the price of the bond quoted (and purchased) at time 0 and expiring in t periods. Formally, our spread variable is defined as $Spread_t = y_{10} - y_{0.25}$.

The empirical section also encompasses two uncertainty and forward measures. First, the VIX measures the expected volatility regarding the S&P500. Second, the 5-year breakeven inflation gives the market expectation of the average inflation for a 5-year horizon. The co-movement analysis of these variable with returns on the CM would give us valuable information regarding market participants on the CM.

Overall, the dataset at hand contains 2915 daily observations. The latter are sampled on business days only. Regarding missing values (0.57% of the total number of observations), the identified cases are filled up according to the Kalman filter approach. As opposed to linear interpolation or related techniques, the latter is preferred due to its ability to replace missing values while preserving the existing trend or seasonal pattern observed in the series.

In the following sections, we refer to returns on the CM, the first difference of the term spread or interest rate spread, the VIX, the first difference of the 5-year breakeven inflation as X_t^{ERI}, X_t^{Spread}, X_t^{VIX} and $X_t^{Breakeven}$, respectively.

1.3.2 Summary Statistics

Table 1.1 reports descriptive statistics of the main variables used in this research. We compute daily returns on the CM as $X_t^{ERI} = log\left(\frac{ERI_t}{ERI_{t-1}}\right)$. Returns on the CM are left-skewed and characterized by an excess kurtosis. Average returns for the sample period oscillate around 0.2% for the CM. Note that the daily log-returns distribution

[4] Plugging the divisor expression into the ERI formula reveals that the index remains constant on 10 March 2016 and 11 March 2016. However, as we continue to iterate forward to compute the index, the values begin to change. The divisor effectively prevents the massive jump that would have occurred if we had used the basic ERI formula.

range from -53% to 54%. X_t^{ERI} range conveys evidence of extreme fluctuations in crypto-currency trading activities. In contrast, the first difference of the interest rate spread, denoted $X_t^{Spread} = Spread_t - Spread_{t-1}$, has shown less variation. The average value is close to 0 with a skewness of -0.059. Overall, the spread distribution reflects the monetary conditions following the Great Financial Crisis, where interest rates on government securities were relatively low and stable until the recent rise in price inflation.

Table 1.1: Summary statistics of the main variables

Variable	N	Mean	Std. Dev.	Min	Max	Kurtosis	Skewness
X_t^{ERI}	2914	0.002	0.051	-0.529	0.535	19.805	-0.670
X_t^{Spread}	2914	-0.000	0.001	-0.003	0.003	6.511	-0.059
X_t^{VIX}	2914	17.712	7.004	9.140	82.690	16.974	2.728
$X_t^{Breakeven}$	2914	0.000	0.0004	-0.003	0.002	8.561	-0.299

Notes: This table reports basic statistics for log-returns on the CM, change in the spread series, and the VIX.

The logarithmic of ERI in Figure 1.2 shows an increasing trend throughout the entire sample. Compared to 01 April 2013, the index was multiplied by more than 500 on 31 May 2024. As evidenced by the top-right panel of Figure 1.2, volatility clustering has been persistent for the period covered in this research. The two most sizable volatility bursts in the log-returns of ERI occurred around April and December 2013. The first one emerged from news of liquidity issues faced by two pioneered crypto exchanges (BitInstant and Mt. Gox) to meet their obligations towards their investors. As a result, the price of bitcoin nosedived 60% on 11 April 2013 before realizing a rebound of 32% seven days later. The second corresponded to the warning issued by the People's Bank of China on 05 December 2013 against the use of BTC in financial transactions. BTC price declined by 58% following the announcement and regained much of its value on 09 December 2013 (52%). It is worth mentioning that the introduction of ETH in the sample (purple vertical line) does not lead to any abnormal change in the fluctuations observed on the CM. The observation also goes for the vertical red line, which indicates the true inception date of ETH on 30 July 2015.

In the volatility modelling process, the emphasis is on the change in the term spread series, meaning X_t^{Spread}. In fact, the spread series, the second row of Figure 1.2, is non-stationary in level [5]. So, the change in the spread, which is stationary, is important for the conditional copula modelling process. The latter uses the Generalized Autoreregressive Conditional Heteroskedasticity (GARCH)

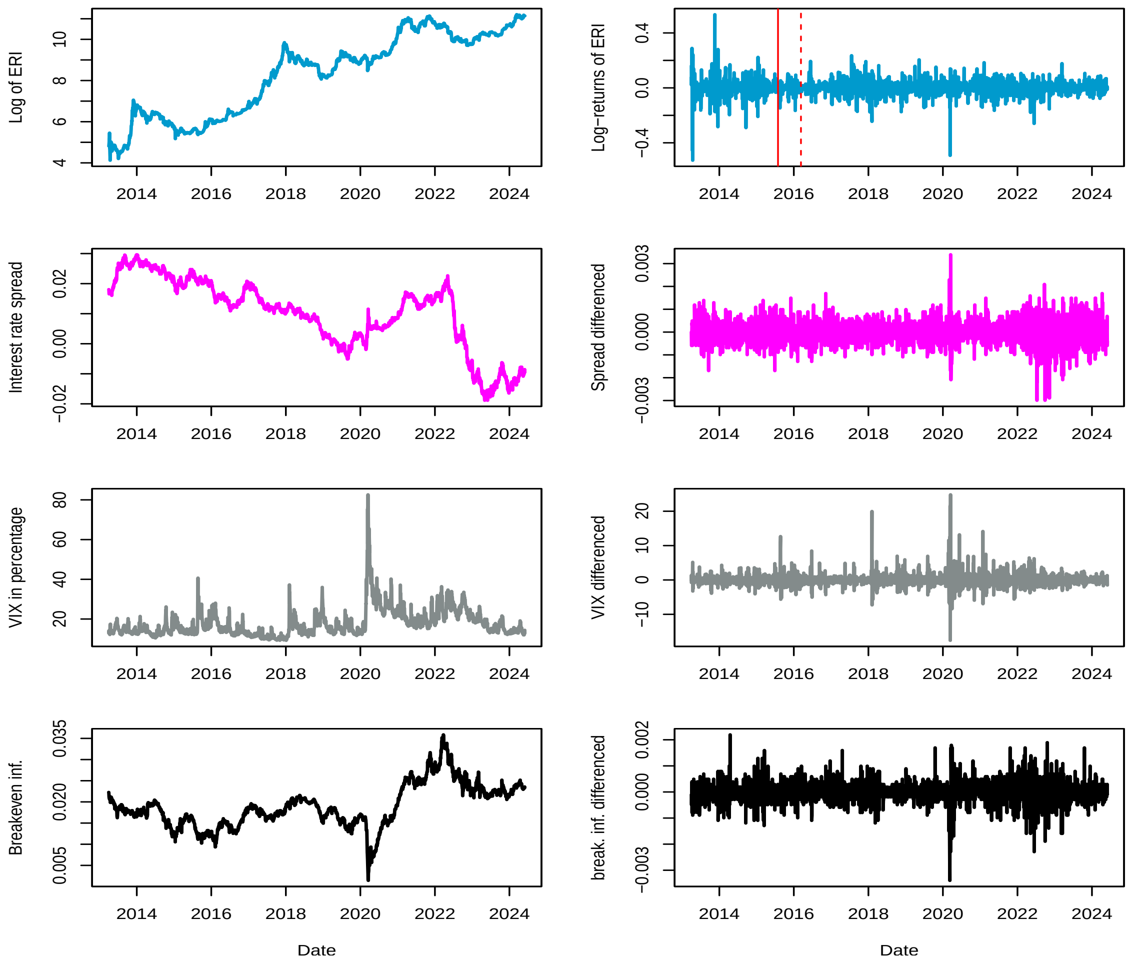
[5] See the Autocorrelation Function (ACF) in Figure 1.10 of the appendix for an analysis of each series' departure from the stationarity assumption

modelling framework as an input to compute dependence parameters, making the stationarity of the margins a critical element. Furthermore, the stationarity of the margins are important to ensure the application of the copula diagnostic tests (Hofert et al., 2019). In terms of economic interpretation, X_t^{Spread} captures similar information as the level series, meaning investors attitude towards future change in the economy. However, evidence of volatility clustering seems to be less dominant in X_t^{Spread} in comparison to X_t^{ERI} as reported in Figure 1.2. Aside a few episodes of peaks and drops, fluctuations in X_t^{Spread} are constrained within -0.003% and 0.003%.

Inspecting the logarithmic of ERI and the spread in level shows a divergent trend between 2014 and 2020. As the spread between long-term and short-term interest rates was shrinking, crypto-currencies prices increased significantly in value. Yet, this makes a compelling case for the use of the first difference of the spread series. Successive reductions of the spread between the two interest rates turn into negative values in the first difference transformation. The negative signs in X_t^{Spread} will be useful when studying co-movement with X_t^{ERI}. So, the existence of negative values for both X_t^{ERI} and X_t^{Spread} at comparable time periods would be critical for the tail dependence in the bivariate copula analysis.

The last two variables of Table 1.1 and Figure 1.2 are the VIX and the breakeven inflation. The most important observation is the synchronization of volatility movements with returns on the CM around early 2021.

Figure 1.2: Plot of the variables in level and their first difference transformations



Notes: This figure presents the four main variables of the study. The red line illustrates the effective issuance date of ETH (30 July 2015), whereas the purple one corresponds to the introduction period of ETH in the present sample (11 March 2016).

1.4 Model Specification and Results

1.4.1 Theoretical Motivation

As explained in the introduction, this study is built on the assumption of a trade-off between crypto-currencies and traditional financial instruments. We start with the intuition of a possible negative correlation between crypto-currencies and the latter class of assets. The reasoning underlying the relationship hinges on the expectations theory. For instance, a positive spread or an upward sloping curve is interpreted as a signal of future short-term interest rate hikes or economic expansion. We would expect investors on the CM to capture these signals of possible higher rewards from the wider financial market and opt for safer assets (bonds). The resulting outflow of capital from the CM would induce a negative relationship between the two markets.

Our second focus is on the VIX and breakeven inflation. These two variables allow us to broaden our analysis of the CM beyond the traditional trade-off between risky and risk-free assets. Specifically, we examine the co-movement between the CM, investors' fear, and expected inflation.

1.4.2 Overview of the Copula Theory

A bivariate copula modelling implies finding parameter estimates for each variable and the dependence between them. The methodology starts with classical probability descriptions for continuous random variables. For instance, a 2-dimension random vector $\mathbf{X} = (X_t^1, X_t^2)$ can be defined by its joint Cumulative Distribution Function (CDF) noted $H(x^1, x^2) = P(X_t^1 \leq x^1, X_t^2 \leq x^2)$ or in terms of the respective margins $F_1(x^1) = P(X_t^1 \leq x^1)$ and $F_2(x^2) = P(X_t^2 \leq x^2)$.[6] The Sklar's theorem states that $H(x^1, x^2)$ can be transformed into a function C, denoted copula, giving information on both margins and the dependence between the two variables. Formally, Sklar's theorem stipulates that:

$$H(x^1, x^2) = C(F_1(x^1), F_2(x^2)), (x^1, x^2) \in R^2. \tag{1.1}$$

A useful transformation involves applying the integral transform theorem on the component of each margin to make the arguments of $H(.)$ uniformly distributed over the interval $[0, 1]$. So, the copula function can now be written in terms of uniform components as:

$$C(u^1, u^2) = P(F_1(X_t^1) \leq u^1, F_2(X_t^2) \leq u^2). \tag{1.2}$$

Statistical properties underlying the relevance of copula are subject to a relatively dense literature. One key element of the copula pertains to its role in detecting complex dependence structure between random variables. For instance, contrary to a parametric dependence measure such as the Pearson correlation that is restricted to two variables (which should be linearly linked), copula can be generalized for any k-dimension vector of random variables. Hence, a k-dimension copula will simple be written as

$$C(x^1, x^2 ..., x^k) = (F_1(x^1), F_2(x^2), ..., F_k(x^k)), \tag{1.3}$$

which is a mapping of $[0, 1]^k \to [0, 1]$.

So far, the account presented in this segment touches upon a basic definition of the copula framework. However, numerous variants of copulas have been developed

[6]The mathematical notation in this section and the following ones are based on Nelsen (2003), Mikosch (2006) and Hofert et al. (2019).

in the recent literature. Table 1.2 presents the main copula families encounter in empirical work in finance.[7] The level of the θ parameter controls the dependence between the random variables at hand. In fact, a crucial difference among families of copulas lie in the notion of tail dependence. The latter is a conditional probability that measures the likelihood of both X^1 and X^2 facing an extreme event (or lie above/below a certain quantile denote q). After setting up a given quantile, the lower (τ^L) and upper tail (τ^U) are given by

$$\begin{aligned}\tau^L &= \lim_{q\to 0^+} P[(X_t^1 < F_1^{-1}(q) | X_t^2 < F_2^{-1}(q)] = \lim_{q\to 0^+} \tfrac{C(q,q)}{q}\\ \tau^U &= \lim_{q\to 1^-} P[(X_t^1 > F_1^{-1}(q) | X_t^2 > F_2^{-1}(q)] = \lim_{q\to 0^+} \tfrac{1-2q+C(q,q)}{q}.\end{aligned}$$

Table 1.2: Summary of some of the widely used copula families

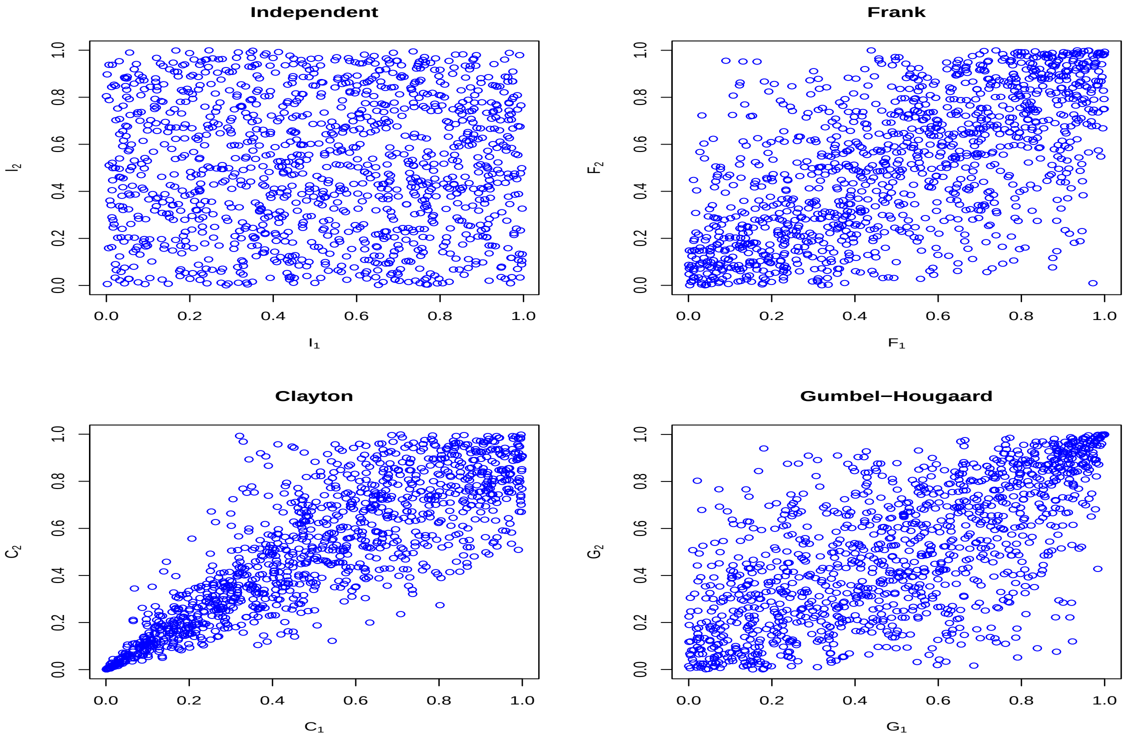
Type	**Parameter (θ)**	$C(u_1,u_2)$	τ^L	τ^U
Normal	[-1,1]	$N_\theta(\phi^{-1}(u_1), \phi^{-1}(u_2))$	0	0
Student's t	[-1,1]	$t_{\theta,v}\left(t_v^{-1}(u_1), t_v^{-1}(u_2)\right)$	$2t_{v+1(w)}$	$2t_{v+1(w)}$
Clayton	$(0,\infty)$	$(u_1^{-\theta} + u_2^{-\theta} - 1)^{\frac{-1}{\theta}}$	$2^{\frac{-1}{\theta}}$	0
Frank	$(0,\infty)$	$\frac{1}{\theta}\log\left[1 + \frac{(e^{-\theta u_1}-1)(e^{-\theta u_2}-1)}{e^{-\theta}-1}\right]$	0	0
Gumbel-Hougaard	$[1,\infty)$	$\exp\left[-\left((\log u_1)^\theta + (\log u_2)^\theta\right)^{\frac{1}{\theta}}\right]$	0	$2-2^{\frac{1}{\theta}}$
Joe	$[1,\infty)$	$1-\left[(1-u_1)^\theta + (1-u_2)^\theta - (1-u_1)^\theta(1-u_2)^\theta\right]^{\frac{1}{\theta}}$	0	$2-2^{\frac{1}{\theta}}$

Notes: In-depth explanation is provided in McNeil et al. (2015) and the references in the theoretical section. The tail dependence of the Student's t is obtained as the CDF of a univariate distribution with $v+1$ degrees of freedom, with $w = \frac{-\sqrt{v+1}\sqrt{1-\rho}}{\sqrt{1+\rho}}$. ϕ and N are denoted CDF of a univariate standard normal distribution and CDF of a bivariate normal distribution, respectively.

Unconditional and conditional estimations of copula parameters are conducted in section 1.4. The choice of the suitable copula technique is supported by a mix of graphical and goodness-of-fit tests. For illustration purposes, four copula representations are generated in Figure 1.3. The Frank, the Clayton and the Gumbel-Hougaard replicate the graphical pattern in table Table 1.2 in terms of tail dependence (concentration of observations either near the point $(0,0)$ or $(1,1)$, which are situations of lower and upper tail dependence respectively). Thus, a graphical analysis contains a paramount role in copula evaluation.

[7]Elliptical copulas (Normal and Student's t) and Archimedian copulas (Clayton, Gumbel, Joe and Frank) are among the most used in financial studies. McNeil et al. (2015) and Hofert et al. (2019) explore in great detail other techniques, such as copulas related to the extreme value theorem.

Figure 1.3: Graphical representation of four simulated bivariate copula families



Notes: Each figure is generated with a sample size of 1200. The θ parameter is arbitrarily set to 5 for Frank, 4 for Clayton and 2 for Gumbel-Hougaard.

1.4.3 Crypto-currency Market and Investors' Expectation

Similar to any classical estimation framework, working with copulas involves estimating the parameter θ reported in Table 1.2 and identifying the best copula class to model the joint distribution of the variables in question. In the context of financial time series, an important step is to remove characteristics inherent to each variable to obtain an unbiased dependence measure. The most evident characteristic is the serial dependence that often exists in financial time series. The preferred approach in the literature is to assume an ARMA-GARCH process for each variable and extract the standardized residuals for the analysis of the dependence structure. This approach aligns with a fundamental aspect of copula theory, which is that the dependence between random variables is unaffected by their marginal distributions (invariance property of copula).

Estimates for the bivariate copula in this paper are computed in two steps. The first step entails formulating a stochastic process driving the conditional returns and variance of the log-returns on the CM as well as the interest rate spread. In the second step, the standardized residuals from step 1 are used to estimate the dependence between the two variables. The conditional copula specification for returns on the

CM and the change in the spread involves a simple mathematical twist with regard to the case in the previous section known as unconditional copula. Again, using Hofert et al. (2019) notations, the conditional joint distribution, $H_{\mathcal{G}_{t-1}}(.)$, between X_t^{ERI} and X_t^{Spread} is given by:

$$H_{\mathcal{G}_{t-1}}(x^{ERI}, x^{Spread}) = P(X_t^{ERI} \leq x^{ERI}, X_t^{Spread} \leq x^{Spread} | \mathcal{G}_{t-1}), (x^{ERI}, x^{Spread}) \in R^2. \tag{1.4}$$

In this set up, $\mathcal{G}_{t-1}$ is the information available at $t-1$ regarding X_t^{ERI} and X_t^{Spread} and their underlying dependence structure. x^{ERI} and x^{Spread} are an observation of the log-returns on the CM and the interest rate spread for a given date. Again, Sklar's theorem allows to write the joint conditional distribution to be formulated in terms of conditional copula as (see Equation 1.1)

$$H_{\mathcal{G}_{t-1}}(x^{ERI}, x^{Spread}) = C_{\mathcal{G}_{t-1}}(F_{\mathcal{G}_{t-1,ERI}}, (x^{ERI}), F_{\mathcal{G}_{t-1,Spread}}, (x^{Spread})). \tag{1.5}$$

We obtain the standardized residuals for each variable from an ARMA(0,0)-GARCH(1,1) process with no mean. subsection 1.A.2 explains the autoregressive order followed in the specification below for the two random variables:

$$\begin{aligned} X_t^{ERI} &= \epsilon_t^{ERI} \\ (\delta_t^{ERI})^2 &= \omega + \alpha_1(\epsilon_{t-1}^{ERI})^2 + \beta_1(\delta_{t-1}^{ERI})^2 \\ \epsilon_t^1 &= \delta_t^{ERI} e_t^{ERI} \\ e_t^{ERI} &\overset{iid}{\sim} t_v \end{aligned} \tag{1.6}$$

$$\begin{aligned} X_t^{Spread} &= \epsilon_t^{Spread} \\ (\delta_t^{Spread})^2 &= \omega + \alpha_1(\epsilon_{t-1}^{Spread})^2 + \beta_1(\delta_{t-1}^{Spread})^2 \\ \epsilon_t^{Spread} &= \delta_t^{Spread} e_t^{Spread} \\ e_t^{Spread} &\overset{iid}{\sim} \text{GED}(\kappa), \end{aligned} \tag{1.7}$$

Where ϵ_t^{ERI} is the innovation or shock driving X_t^{ERI}. $(\delta_t^{ERI})^2$ defines the conditional variance at time t. Finally, e_t^{ERI} is the standardized residuals that is student's t distributed in the case of the first expression. The same description applies for Equation 1.7, with the exception that e_t^{Spread} follows a Generalized Error Distribution (GED). subsection 1.A.2 offers an overview on the fitness of the two equations with alternative modelling choice of e_t^{ERI} and e_t^{Spread}.

Parameters of equations 6 and 7 are estimated from the joint density function of $H(.)$ using the MLE. The likelihood notation takes the form of a joint density product of the marginal densities and the copula density as follows:

$$f(\mathbf{X};\Omega,\psi) = f_1(X_t^{ERI};\Omega_1)f_2(X_t^{Spread};\Omega_2)c(u^{ERI},u^{Spread};\psi). \tag{1.8}$$

We apply the two-stage approach by first estimating all margin parameters in Ω_1 and Ω_2. The vector of copula parameters (ψ) are again computed via the MLE in the second stage. These two steps are visible from the log likelihood of the joint density, where the sum of the marginal log likelihoods and the copula log likelihood form the first and the second stage respectively. The log-likelihood expression is written as

$$\mathcal{L}f(\Omega,\psi;X) = \ln f_1(X_t^{ERI};\Omega_1) + \ln f_2(X_t^{Spread};\Omega_2) + \ln c(u^{ERI},u^{Spread};\psi). \tag{1.9}$$

Note the expressions 2 and 9 write the copula function in terms of the uniform margins u. This transformation is crucial to the estimation of the copula parameters. As explained above, the process requires extracting the standardized residuals of the ARMA-GARCH processes and apply the integral transform theorem in order to obtain the uniform margins from the empirical distribution of the residuals. In this paper, we follow the recommendation in Hofert et al. (2019), where the uniform margins in the copula density are estimated by:

$$U_t^i = \frac{1}{n+1}(R_t^i), \tag{1.10}$$

with R_t^i, the rank of a residual observation in the dataset and i a given variable. The position or the rank is determined by the time index t. The U_t^i sample is known as pseudo-observations in the literature.

Estimates of the marginal series show significant volatility persistence with the sum of α_1 and β_1 being close to 1 as reported in Table 1.3. The persistence is, however, much higher in the case of the CM, which unequivocally subscribes to the description of volatility clustering. According to the half-life calculation, it takes 693 days for the conditional variance to revert to 50% of its long term level following a shock on the CM. The half-life estimate is around 22 days for the interest rate spread. From an investment perspective, crypto-currencies as an asset class would require active risk management strategies to hedge against risks spanning over many years. The estimates for the degrees of freedom (v) and the shape parameter (κ) fall within the range expected. Both cases highlight that the standard residuals obtained from the two GARCH equations have heavier tails than the standard normal distribution.

Table 1.4 reports the dependence parameter estimates for a set of copula families widely used in empirical finance. None of the estimates is statistically significant at the conventional significance levels. The relatively large degree of freedom (32) is evidence that a normal copula would be preferred to the student's t family in the context of this study. The low value of the dependence parameter for the different copula families indicates weak dependence between fluctuations on the CM and the term structure. The computed tail probabilities are approximately zero for the different copula types. So, observed extreme events on the two markets are likely to be unrelated.

Table 1.3: ARMA-GARCH estimates of returns on the CM and the treasury yield spread

	X_t^{ERI}	X_t^{Spread}
	GARCH(1,1)	GARCH(1,1)
ω	$6.5{\times}10^{-6***}$	0.000
	(0.000)	0.000
α_1	0.142***	0.067***
	(0.000)	(0.014)
β_1	0.857***	0.903***
	(0.020)	(0.015)
v	3.056***	
	(0.159)	
κ		1.509***
		(0.000)
Log. Likelihood	5294.387	18171.69
BIC	-3.6228	-12.461
N. obs.	2914	2914

Note: In this table, v and κ are estimates of the degrees of freedom and the shape parameter in a GARCH model with student's t and GED innovations respectively. In parenthesis are the standard errors of the estimates.
***Significant at the 1 percent level.
**Significant at the 5 percent level.
*Significant at the 10 percent level

We plot the uniform transformed components of the standardized residual series in Figure 1.4. In accordance with the copula estimates, no clear dependence pattern is observable between the CM and the US government bond market. The graph rather illustrates the case of an independence copula. In theory, a clayton parameter estimate oscillating around zero or a gumbel parameter around 1 is a sign of independence copula. We formally test the null hypothesis that the relation between the two markets is no different from an independence copula structure [8]. Unsurprisingly, we find no evidence against the null hypothesis (p-value=0.932). To ensure the result of this

[8] This test evaluates whether the Kendall's rank correlation is statistically different from zero or not. The correlation level is 0.001, which is not different from zero according to the test. As a side note, this test is possible because rank-based correlation parameter can be written as a function of the underlying copula between the two variables.

Table 1.4: Estimates of different copula family parameters

	Normal	Student's t	Clayton	Frank	Gumbel-Hougaard	Joe
θ	0.004	0.004	0.031	0.038	1.001***	1.003***
	(0.018)	(0.019)	(0.019)	(0.11)	(0.011)	(0.016)
τ^L		2.267×10^{-6}	1.472×10^{-10}***			
		(5.281×10^{-6})	(2.87×10^{-12})			
τ^U		2.267×10^{-6}			2.065×10^{-8}***	2.065×10^{-8}
		(5.281×10^{-6})			(2.251×10^{-10})	(9.313×10^{-7})
Deg. of freedom		32***				
		(0.000)				
Log. Likelihood	0.01847	1.321	1.33	0.058	-1.626×10^{-6}	-1.631×10^{-6}
AIC	-1.963	-5.359	-1.34	-1.884	-4	-4
N. obs.	2914	2914	2914	2914	2914	2914

Notes: In the spirit of Table 1.2, the statistical significance of the Joe and Gumbel-Hungaard is tested as $H_0 : \theta = 1$ and $H_1 : \theta > 1$. The standard errors for the tail probabilities, in parenthesis, are computed with the delta method, since the latter is a transformation of θ.
***Significant at the 1 percent level.
**Significant at the 5 percent level.
*Significant at the 10 percent level .

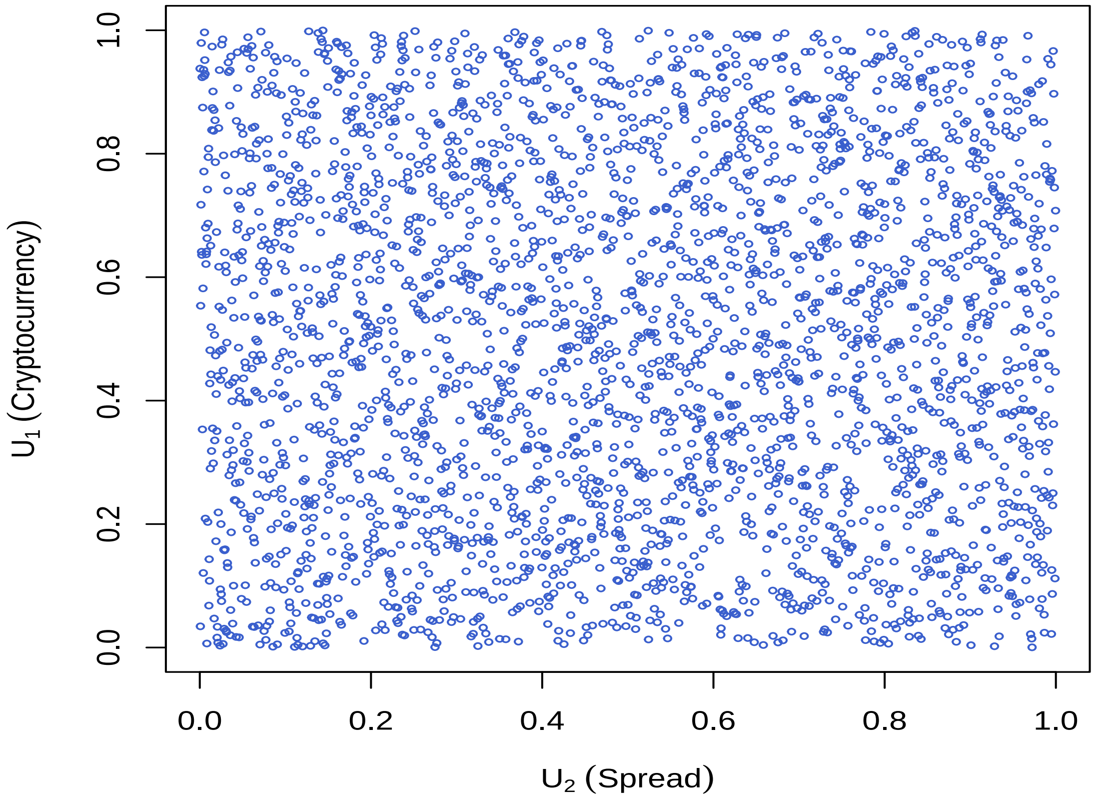
analysis is not a mere consequence of the copula specification, we test for the existence of a time-varying dependence structure using Bücher et al. (2014)'s method. The latter accounts for deviations in the dependence parameter due to the changing nature of the dependence between the two margins or abrupt structural breaks in the two series. We find strong evidence against a time-varying copula (p-value = 0.847). This first set of results is indeed a robust illustration of the independence nature of the two markets modelled through the bivariate copula, which is the antipode of the rational expectation theory. In the sense that the slope of the yield curve for the selected maturities is not relevant in explaining fluctuations on the CM. We will later go back to the dynamic nature of the analysis and robustness considerations.

1.4.4 Crypto-currency Market and Volatility Anticipation

We shift our attention to study the link between expected volatility on traditional risky markets and price fluctuations on the CM in this section. The GARCH-based estimate of the crypto price volatility is used for the CM. As explained in the introduction, the S&P 500 is our proxy for the overall stock market. As such, we use the VIX to measure expected volatility regarding the global stock market. The VIX is often seen in the literature as a proxy for investors' sentiment and uncertainty about the future state of the equity market (see Bekaert et al., 2013). So, the existence of a significant dependence between the VIX and the volatility on the CM would help pinpoint possible fundamentals driving crypto-currency price movements.

Unlike Bekaert et al. (2013) that breaks the VIX down into an uncertainty and a

Figure 1.4: Dependence representation between the spread and the CM



Notes: This figure presents the residuals extracted from equations 6 and 7. The residuals are transformed to be distributed between 0 and 1 (Pseudo-observations). To operate the transformation, we use the rescaled empirical distribution function approach. Hofert et al. (2019) gives a detailed explanation of this technique.

risk-aversion component, the estimations below use raw values of the VIX as downloaded from the Federal reserve Bank of Saint-Louis. Our interest is simply the VIX component in level. In other terms, we want to appraise how volatility on both markets relates to each other. A primary investigation shows that high volatility on the CM tends to synchronize with low volatility expectations on the S&P 500 (See top-left panel of Figure 1.5). To better capture the latter observation and compute a tail dependence probability, we reverse the distribution of the VIX by subtracting the index from 100. The synchronization is now translated into an upper tail dependence representation, which is computed in the last column of Table 1.5 and visible in the top-right panel of Figure 1.5.

The observed overlap between high volatility in the CM and low levels of the VIX reflects a pattern well-documented in the systemic risk literature. For instance, Borio and Drehmann (2009) argue that periods of low volatility, especially in conventional risk measures such as the VIX, can be misleading. These phases of subdued volatility often coincide with excessive risk-taking behaviours, including increased leverage. In their analysis, they refer to this phenomenon as the "*instability paradox*." In this context, low volatility does not truly reflect a low-risk environment, but rather a state of complacency that precedes financial instability (Borio and Drehmann, 2009).

In our analysis, the presence of high volatility on the CM during times of relatively low VIX levels can be interpreted as a reflection of disintermediation, where investors, in pursuit of speculative gains, reallocate funds from crypto-currencies into more traditional equity markets. The outflow of capital to equities can accelerate the sell-off in crypto-currencies and amplify daily realized volatility in the CM. In turn, the transferred funds may be used to support leveraged positions or other risk-enhancing strategies in the equity market. Within the framework developed by Borio and Drehmann (2009), the VIX fails to capture such dynamics, as it functions as a contemporaneous indicator of financial distress and lacks forward-looking capacity. This narrative aligns with the broader argument that financial instability often builds during periods of apparent calm, when standard risk measures fail to capture the accumulation of leverage, liquidity mismatches, and tail risks across the system.

This interpretation supports the view of the CM not as an isolated phenomenon, but as increasingly interconnected with the broader financial cycle, albeit with distinct dynamics such as a greater impact of microstructure factors in exchange rate determination.

We present semiparametric copula estimates for the equity market and the CM in Table 1.5. In the previous section, we conduct the dependence estimation work using the residuals of the GARCH processes. We used the residuals to account for the volatility clustering feature of the returns series in the copula parameter estimation. In this section, we directly generate the pseudo-observations (with Equation 1.10) using the values of the VIX in level and the GARCH-based predicted volatility from equation 6. Then, the copula dependence parameters are estimated between the two variables via the MLE [9]. In line with the previous results, the dependence parameter estimates are relatively low for the different copula families. Nonetheless, unlike results presented in the previous section, the tail dependence probability is non-negligible (3.7%) for the

[9]A semiparametric estimate avoids the steps of specifying a GARCH structure for the VIX. In the present analysis, the semiparametric choice does not alter the conclusion if we were to estimates the dependence coefficients parametrically (see Figure 1.15). In addition to the simplicity of the semiparametric approach, the GARCH estimates for a VIX series would be hard to make sense of as opposed to price series where the mean and variance equations have meaningful financial implications. In Figure 1.15, we provide proof that a fully parametric set up, similar to the previous section, would not change the conclusion of mild dependence structure found between the two random variables. The GARCH process is estimated using the first difference of the VIX variable.

Gumbel-Hougaard and the Joe copulae (7.6 %). So, if a trader were to combine the ERI and the S&P 500 index, she should expect the volatility on each market to go in different directions every 27 days with the Gumbel-Hougaard and every 13 days with the Joe. However, judging from the AIC, the Gumbel-Hougaard offers a better fit than the Joe Copula and would therefore be a stronger statistical framework to study the relationship between the variables (with the student's t copula being the best model).

Table 1.5: CM and VIX copula estimates

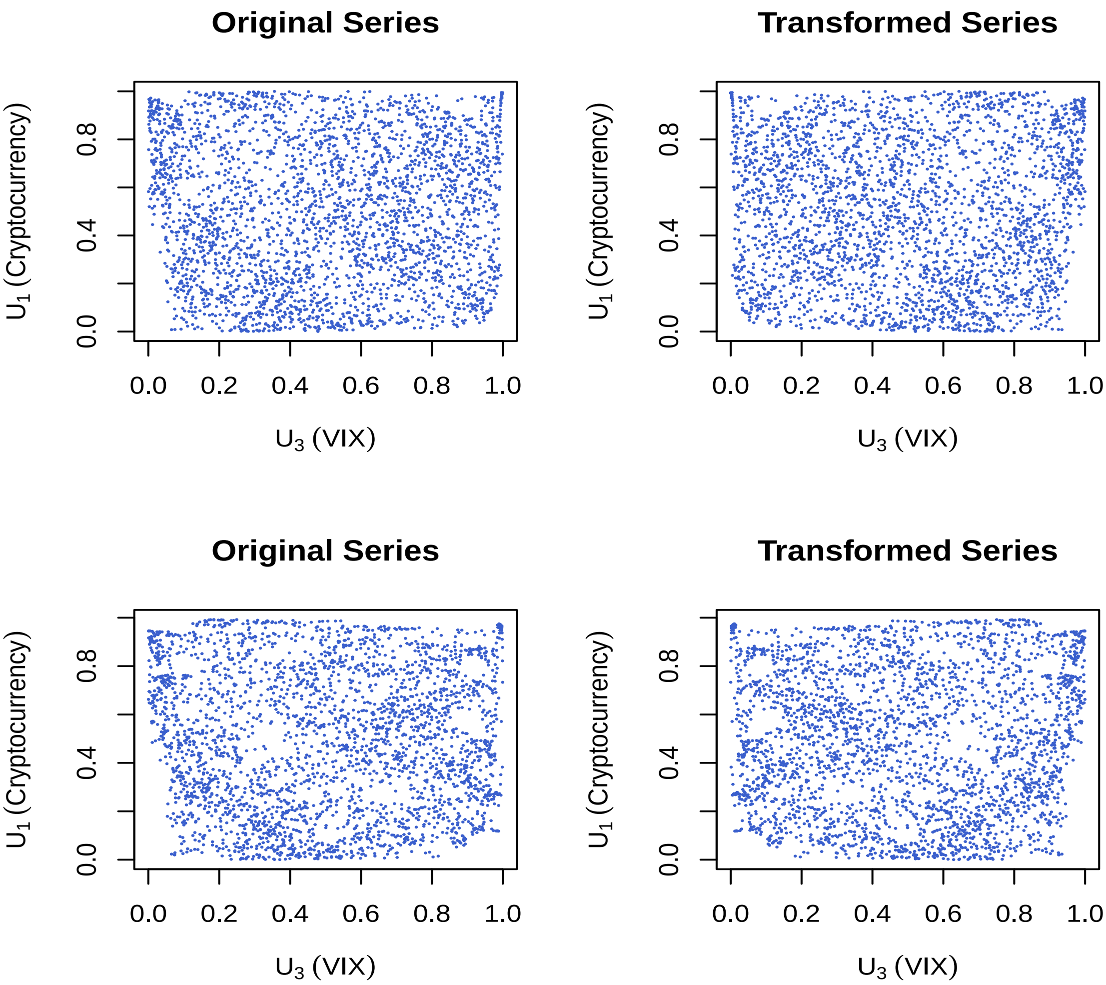
	Normal	Student's t	Clayton	Frank	Gumbel-Hougaard	Joe
θ	0.024	0.062***	0.016	0.084	1.028***	1.059 ***
	(0.018)	0.019	(0.025)	(0.113)	(0.013)	(0.019)
τ^L		5.325×10^{-15}	8.723×10^{-19}***			
		(1.23×10^{-9})	(1.887×10^{-20})			
τ^U		5.325×10^{-15}			0.037***	0.076***
		(1.23×10^{-9})			(0.009)	(0.001)
Deg. of freedom		91				
		(0.000)				
Log. Likelihood	0.827	1.052	-3.886	0.2913	3.708	7.572
AIC	-0.345	-5.8951	-11.771	-1.417	3.417	8.013
N. obs.	2914	2914	2914	2914	2914	2914

Notes: This table presents estimates of some of the widely used copulas. In parentheses are standard errors of the estimates. In the spirit of Table 1.2, the statistical significance of the Joe and Gumbel-Hungaard is tested as $H_0 : \theta = 1$ and $H_1 : \theta > 1$. The standard errors for the tail probabilities are computed with the delta method. The MLE failed to estimate the dependence parameter in the Clayton case. We instead estimate the parameter via the method-of-moment (Spearman's rho). Hofert et al. (2019) gives a detailed explanation of this technique.
***Significant at the 1 percent level.
**Significant at the 5 percent level.
*Significant at the 10 percent level

The pseudo-observations of the two variables are plotted in Figure 1.5. As explained above, the first column gives a representation of the VIX along with the volatility on the CM. The second column, labelled as transformed series, flips the cluster of observations near the point $(0, 1)$ to the point (1,1). So, the Gumbel-Hougaard and the Joe upper tail dependence probabilities output the odds of having this cluster of points.

We also display a representation of the VIX along with a GARCH-based forward volatility estimate for the CM in the bottom panels of Figure 1.5. Given the VIX is a forward-looking variable, investors' forward volatility expectation for the CM may show a stronger response to change in the VIX than the instantaneous volatility analysis conducted in the previous paragraphs. The forward volatility, computed as $\frac{1}{22}\sum_{k=1}^{22}\sigma_{t+k}$, is a rolling ahead moving average over 22 trading days (or one calendar month), with k being the one period ahead index. In terms of level, the new dependence parameter estimate is close to the results of Table 1.5. We find no major difference in the likelihood of observing joint extreme movements on both markets. The tail dependence probabilities are now 3.6% and 8% for the Gumbel-Hougaard and the Joe techniques respectively (see Table 1.9 in the appendix).

Figure 1.5: Dependence representation between the VIX and CM volatility



Notes: This figure presents the VIX and the volatility on the CM. The first row gives the VIX and the predicted volatility at time t. The second row is gives the VIX and the forward predicted volatility over 22 trading days. The values are transformed to be distributed over the interval 0 and 1. The original series plot the pseudo-observations with no transformation. However, the plots with the transformed series come from subtracting 100 from the VIX.

The dependence observed in this section, though modest, highlights that the CM and the global stock market may value expected uncertainty differently, leading to distinct price reactions. Drawing on the leverage effect, which suggests a negative correlation between volatility and returns, one explanation could be that anticipated low volatility in the S&P 500 boosts expected future returns, making the market more attractive and driving capital inflows. In this scenario, risk-averse investors might favor the S&P 500 over the riskier CM, reducing transaction activity on the CM and increasing its price fluctuations. This outcome exemplifies the risk-shifting mechanism and its impact on returns distribution in the CM.

1.4.5 Crypto-currencies and inflation expectation

The conclusion reported in subsection 1.4.3 would be similar if interest rates on Treasury Inflation-Protected Securities (TIPS) were to be used rather than nominal ones. In fact, it is theoretically sound to assume investors care about real earnings and this fact should reflect in the dependence between the returns on the CM with interest rates on TIPS depending on the state of the inflation expectation (high or low). However, previous studies found no significant evidence for investors to hold more inflation-protected financial instruments when inflation expectations are high (Shiller, 2015; Fleckenstein et al., 2014). We observe similar patterns in the dependence structure between crypto-currency price movements and the 5-year US breakeven inflation.

In the conditional copula framework, we model the breakeven inflation (first difference) as an ARMA(0,0)-GARCH(1,1) with GED residuals. Model checking and justifications for this GARCH order can be found in Figure 1.16. The copula estimates are reported in Table 1.6 and the scatter plot of the marginal distribution in Figure 1.17 of the appendix. Estimates of the copula and the tail dependence parameters are low and statistically not significant. The graph of the uniform-transformed margins shows no sign of tail dependence or relationship between the two variables. Bücher et al. (2014)'s test for time-varying copula provides weak evidence against a constant conditional copula model (p-value=0.0504). Therefore, the results obtained in this section might be a consequence of time-varying changes in the underlying dependence structure between the two variables. Given the weak evidence against the suitability of the bivariate constant copula, we keep the results and further explore the changing relationship between the two variables in the context of a simple linear quantile regression in the next section.

1.4.6 Further Interpretation

Contrary to insights popularized in business and financial magazines, the results of this research fall short of establishing a significant correlation between the crypto-currency market and the spread on the US government bonds [10]. We also find mild evidence linking the volatility on the CM with the expected volatility on the S&P500 index,

[10] This analysis by The Economist (2022) is one of these news articles explaining the April 2022 price drop by the rising interest rate on US government debt instruments.

Table 1.6: CM and inflation expectation copula estimates

	Normal	Student's t	Clayton	Frank	Gumbel-Hougaard	Joe
θ	0.044	0.043	0.072	0.226	1.008	1.001
	(0.018)	(0.019)	(0.02)	(0.111)	(0.011)	(0.013)
τ^L		4.349×10^{-7}	6.831×10^{-5}			
		(0.017)	(1.384×10^{-6})			
τ^U		4.349×10^{-7}			0.011	2.746×10^{-10}
		(0.017)			(0.000)	(2.709×10^{-10})
Deg. of freedom		39*				
		(16.62)				
Log. Likelihood	2.829	3.756	7.244	2.05	0.321	-1.669×10^{-7}
AIC	3.657	1.512	10.488	2.099	-3.358	-4
N. obs.	2914	2914	2914	2914	2914	2914

Notes: This table presents estimates of some of the widely used copulas. In parentheses are standard errors of the estimates. In the spirit of Table 1.2, the statistical significance of the Joe and Gumbel-Hungaard is tested as $H_0 : \theta = 1$ and $H_1 : \theta > 1$. The standard errors for the tail probabilities are computed with the delta method. The MLE failed to estimate the dependence parameter in the Frank case. We instead estimate the parameter via the method-of-moment. Hofert et al. (2019) gives a detailed explanation on the use of this technique.
***Significant at the 1 percent level.
**Significant at the 5 percent level.
*Significant at the 10 percent level

which is in a stark contrast with a similar study conducted by Akyildirim et al. (2020). The same weak evidence is also reported in the context of the breakeven inflation. This lack of connection would suggest that investors on the CM give little weigh to the slope of the yield curve, the VIX indicator and the expected inflation in their investment decisions. However, the first two variables are traditionally seen as strong predictors of change in financial conditions and business cycles (see e.g., Estrella and Mishkin, 1996). Therefore, movements in the slope of the yield curve and the VIX contain valuable economic information for portfolio construction and investment decisions related to the mainstream markets. So, it is crucial to pinpoint elements that can explain the results found in the context of the CM analysis.

Sticky updating about future uncertainty may play a role in the weak dependence between the CM and the stock market (see e.g., Lochstoer and Muir, 2022). This would be a result of investors taking time to incorporate new information regarding future uncertainty from the S&P 500 into their crypto-currency investment decisions. In this setting, models that can account for the lag in the response of the CM to the volatility forecast of the S&P500 would be more insightful than the copula framework. Moreover, investors may value idiosyncratic risks more than a broad indicator like the VIX. This would, for instance, express in CM participants having a stronger reaction to change in crypto-currency regulations than broader uncertainty information hidden in the VIX, such as changes in the monetary policy stance or gloomy economic forecasts.

Significant inter-temporal variation of the interconnectedness between the CM and the other markets may also explain the weak variation reported in section 1.4. In a recent paper, Iyer (2022) reports a deeper connexion between crypto-currencies and the US stock market following the COVID-19 shock. The author finds a pearson correlation of 0.01 and 0.36 for the sub-periods 2017-2019 and 2020-2021 respectively. This implies that long time series can hide or offset recent correlational developments between the CM and other markets. In our analysis, we indirectly control for this issue by using the test for point detection. The point detection (stationarity) test has the advantage of identifying structural breaks that affect the margins and the copula parameter. As reported in section 1.4, we found no clear evidence of non-stationarity for the four core variables. So, it is unlikely that the weak dependence estimates computed for the log-returns on the CM and the other variables are sample-dependent.

1.4.7 Robustness Checks

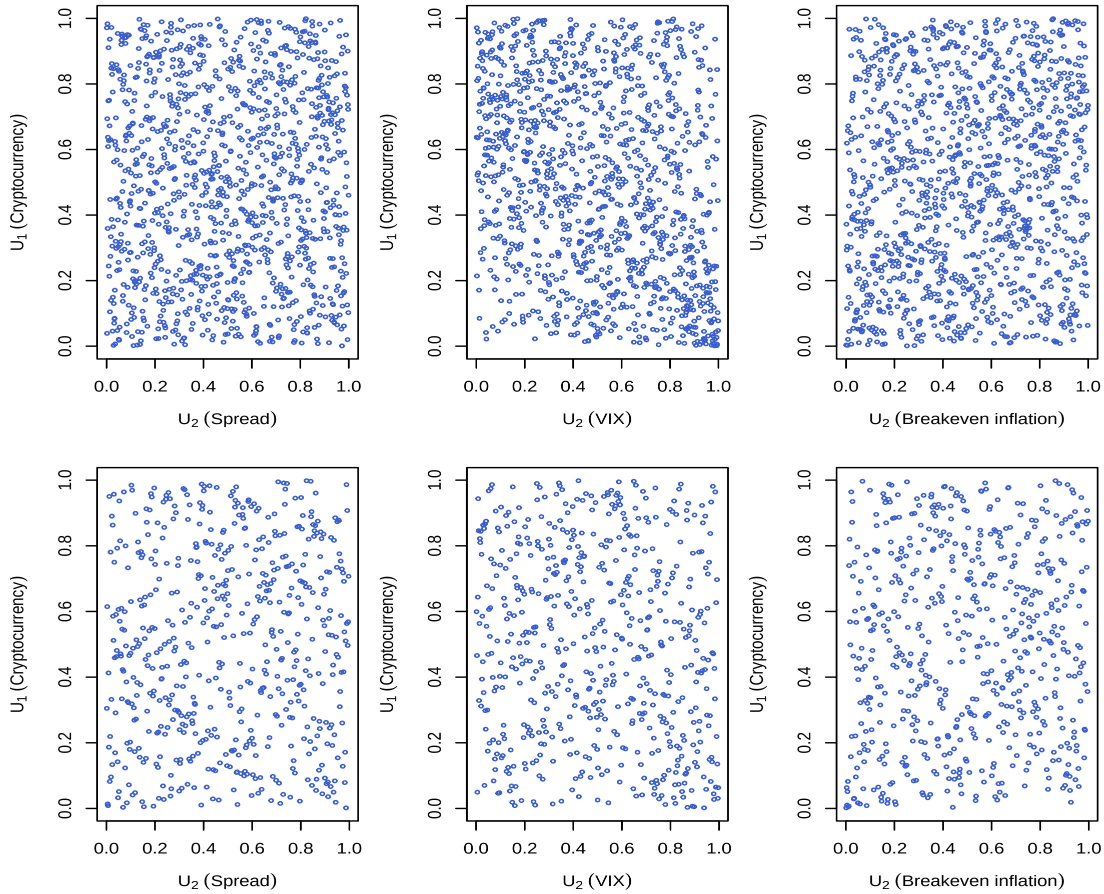
In this subsection, we conduct robustness checks using a sub-sample analysis (2020–2024). The latter confirms a weak relationship between CM and macroeconomic indicators despite increased institutional interests. We further address statistical issues by using weekly observations. The findings with the lower frequency data go in the same direction. Additionally, we examine the NASDAQ to test potential CM connections with the technology sector, but find no significant link. Lastly, a quantile regression analysis shows that CM returns respond uniquely to macroeconomic variables, highlighting the need for future research into momentum and internal market dynamics.

1.4.7.1 Sub-sample Analysis

We extend the analysis to address statistical issues that could impact the observed interconnectedness between the CM and other markets. Challenges with crypto-currency data, such as non-synchronous closing times and early trading noise, are considered (Alexander and Dakos, 2020) . To mitigate these issues, we focus on a sub-sample from 1 January 2020 to 31 May 2024, aligning with the period analyzed in Iyer (2022) to facilitate comparison. Additionally, we use weekly returns to reduce the potential impact of non-synchronous closing times, enabling a more accurate assessment of price variations in the CM.

The pseudo-observations plotted below are generated from the standardized residuals of a GARCH process. For simplicity, we estimate a GARCH of the same order as the previous sections and maintain the same distributional assumption on the standardized residuals. Figure 1.6 shows no deviations from our previous conclusions, both for the daily and weekly observations. Summing up, recent periods, meaning post-covid era, characterized by a growing public oversight over crypto trading activities does not rule out the weak relationship found in the whole sample between crypto-currency returns and the slope of the yield curve. It is important to note that the period in question also corresponds to the advent of smart money into the crypto-currency investment sphere [11]. It would be natural to expect economic indicators such as the slope of the yield curve to be correlated with crypto-currency prices over this period. This deduction stems form the fact that institutional investments ought to follow some technical rules and be aligned with market conditions. Otherwise, rationalizing crypto-currency investments decisions would be a difficult task. So, there is an imperative obligation to shed light on why crypto-currency prices seem to be detached from market indicators derived from the state of the world's economy.

Figure 1.6: Dependence representation between log-returns on the CM, interest rate spread, the VIX and the 5-year breakeven inflation



Notes:

[11]See Fidelity (2021) for an overview on institutional investors on the CM.

The non-dependence relationship for the period 2020-2024 is in stark contrast with a growing literature linking the CM with macro-financial variables over this time span. Iyer (2022), cited above, is among the researchers reporting a significant economic nexus. However, her exercise gives little context on the choice of this specific window to conduct the analysis. There is also a weak discussion on the macroeconomic fundamentals driving this uptick in the correlation between the two variables. In the subsequent paragraphs, we estimate a simple rolling quantile regression to shed light on a possible time-varying interconnectedness between the CM and the other variables retained in this analysis.

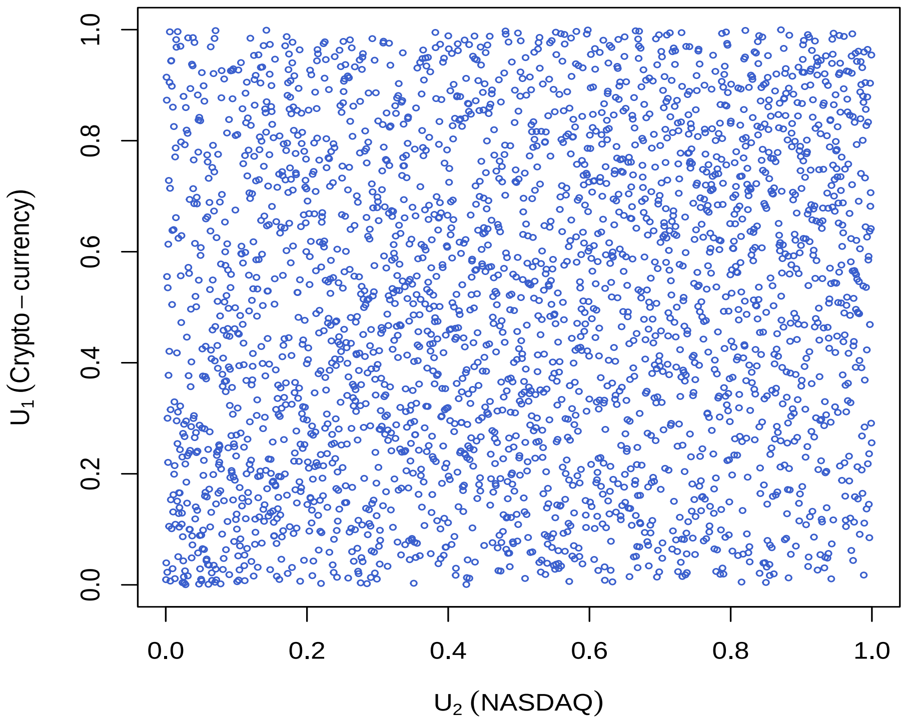
1.4.7.2 The CM and the NASDAQ

We now turn our attention to the NASDAQ, given its proximity to CM activities through the technology sector. The empirical strategy mirrors that of the preceding analyses in this chapter. The rationale for including the NASDAQ in the robustness check lies in the direct exposure of several NASDAQ-listed companies – such as Coinbase Global Inc., Marathon Digital Holdings, Riot Platforms, Cipher Mining, BTCS Inc., and MicroStrategy Incorporated – to the CM. Identifying whether this specific market segment serves as the primary channel connecting the CM to traditional asset classes is crucial. For clarity, we plot the marginal distributions of the standardized residuals from a GARCH(1,1) model with student's t-distributed errors in Figure 1.7. The lack of strong correlation evident in both tails of the distribution suggests that the NASDAQ does not constitute a distinct pathway through which the CM interacts significantly with the broader macroeconomic environment.

1.4.7.3 Quantile Regression Analysis

Our final robustness check involves estimating a linear rolling quantile regression model. The motivation in this exercise is twofold. First, it gives us the possibility to test the existence of changing relationship between the variables over time in the sense that there are significant economic and statistical effects of policy variables on the CM log-returns. Second, it allows a final test regarding the suitability of the Archimedean and Elliptical copula structures to explain the problem at hand. Since the copula theory revolves around the idea of extreme events, we take the same approach to the quantile regression to report results for the 0.05 and 0.95 quantiles. We also plot the slope coefficient of the predictor variables at the 0.5 quantile (median) for comparison. For a better appraisal, we report the slope coefficients as a set of plots.

Figure 1.7: Dependence between log-returns on the CM and the NASDAQ index returns.



Notes: The figure is a scatterplot of pseudo-observations derived from the standardized residuals of GARCH(1,1) models with Student's t-distributed errors, fitted to the return series of a crypto-currency index (ERI) and the NASDAQ index, using a sample of 2,798 daily observations spanning the period from April 2, 2013 to May 10, 2024.

The rolling regression takes the form

$$Q_{X_t^{ERI}}(\tau \mid X_t) = X_t \beta_t(\tau)$$

where $Q_{X_t^{ERI}}(\tau \mid X_t)$ is the predicted value of the log-returns of ERI at a specified quantile conditional on the predictors, t is a window of size 252 observations, τ is a given quantile and $\beta_t(\tau)$ a vector of the slope coefficients of the predictor variables for each τ. The vector X_t can be thought as having the three predictors, meaning the interest rate spread, the VIX and the breakeven inflation. In the regression process, we regress X_t^{ERI} on the variable one at a time and plot the slope coefficients in Figure 1.8. Then, we regress X_t^{ERI} on all three covariates at the same time and reports the coefficients in Figure 1.9.

Figure 1.8 reports evidence of time-varying effect of the predictors on the log-returns of ERI. The first key observation is an apparent uniform response of the CM to the three predictors at a given quantile. The first column of Figure 1.8, meaning at $\tau = 0.05$, shows that a 1% rise in the macro-financial variables (their first difference) leads to a decrease in the log-returns on the CM. The reported effect is statistically significant across the whole sample period. The response of the 0.5 quantile of log-returns to a 1% increase in the predictors is, however, not statistically significant. The sign of the response becomes positive and significant at the 0.95 quantile of X_t^{ERI}.

When controlling for two of the covariates, the effect of the studied macro-financial variable vanishes (Figure 1.9). 0 is in the 95% confidence bands for the two extreme quantiles, meaning $\tau = 0.05$ and $\tau = 0.95$. The response of the median values of X_t^{ERI} to a 1% increase in the VIX would be the only result that stands out. An increase in uncertainty around 2021 has the effect of reducing the median returns on the CM until late 2023. The effect prolongs until the end of the sample period with the magnitude reduces significantly.

One possible economic interpretation is that the policy variables (interest rates) and the VIX carry information regarding the state of the economy. In this regard, working with one variable at a time would be enough to render a plausible effect of policy variables or market sentiments on the lower and upper tail distributions of the log-returns on the CM. Karau (2021) follows a similar approach, where he includes the variables one at a time and proceeds to see the effect with all the variables being considered at once. Now, the question remains to explain the opposite effect of the covariates on the log-returns of ERI at the lower and upper quantiles.

The first column of Figure 1.8 measures how negative economic outcomes, meaning a 1% increase in a particular covariate, affect the lower tail of a risky asset. These outcomes have the effect of tightening monetary and financial conditions. In such a context, investors opt for a risk-off sentiment, which can exacerbate losses in the lower tail of the CM. In addition, financial episodes of this nature are often accompanied by more liquidity constraints. Given the unregulated nature of the CM, increased liquidity constraints have the power to lead to significant price drops with downward pressures on returns in the 0.05 quantile of X_t^{ERI}. This suggests that crypto-currencies respond to anticipated inflation, increased interest rates, and negative economic sentiments similarly to other traditional financial asset classes.

The results for the upper tail are counter-intuitive regarding established views in the asset pricing literature. Risky assets are expected to move in the opposite direction to policy rates. However, one can argue that higher spreads, anticipated inflation, and increased fear towards the S&P500 may increase the appeal of the CM, as shown in the third column of Figure 1.9. As explained above, a 1% increase in all the covariates signals possible economic downturn cycles, which can negatively affect the performance of traditional risky assets. An increase in returns on the CM during restrictive monetary policy would position this market as a competing asset class to fixed-income securities. This view requires micro-level data on investors' transactions to understand the dynamic between the bond market and the CM.

Regarding the post-COVID era, no particular trends emerge, as noted in Iyer (2022) and Karau (2021). The latter adopts a high-frequency identification strategy and is better suited to isolate the response of the CM to economic shocks. In any case, a natural extension for this study would be to combine the idea of extreme events with a dynamic time series framework to study the response of the lower and upper tail distribution to specified economic shocks. Our approach to include all the 3 variables is to make our benchmark exercise comparable to Karau (2021). We hold the view that X_t^{Spread}, X_t^{VIX} and $X_t^{Breakeven}$ are redundant variables. Hence, the results in Figure 1.8 are indeed more reliable.

From a practical standpoint, this study helps clear up some misconceptions about potential factors driving crypto-currency prices. However, the crypto-currency network remains a fast evolving environment with various aspects to understand. The relative absence of entry restrictions makes crypto-currencies accessible to investors around the world. As such, the CM would probably be one the most diverse investors pool in existence. As a potential downside, this diversity might entail a degree of asymmetry in the financial literacy and the risk attitude of CM market participants. The two factors are likely to affect price variations on the CM. A strand of the behavioral financial literature has for long studied how these distortions feed in trading behaviors (biases) observed from market participants [12]. This facet of the analysis is not covered in this research and would be a promising line of investigation for the CM.

1.4.7.4 Momentum and Microstructure

We analyze momentum and event significance using a two-step approach. First, an ARMA model is estimated using regularized regression (Least Absolute Shrinkage and Selection Operator, LASSO) to identify relevant AR and MA lags. This method ensures a parsimonious specification by highlighting essential temporal dependencies and reducing the risk of overfitting

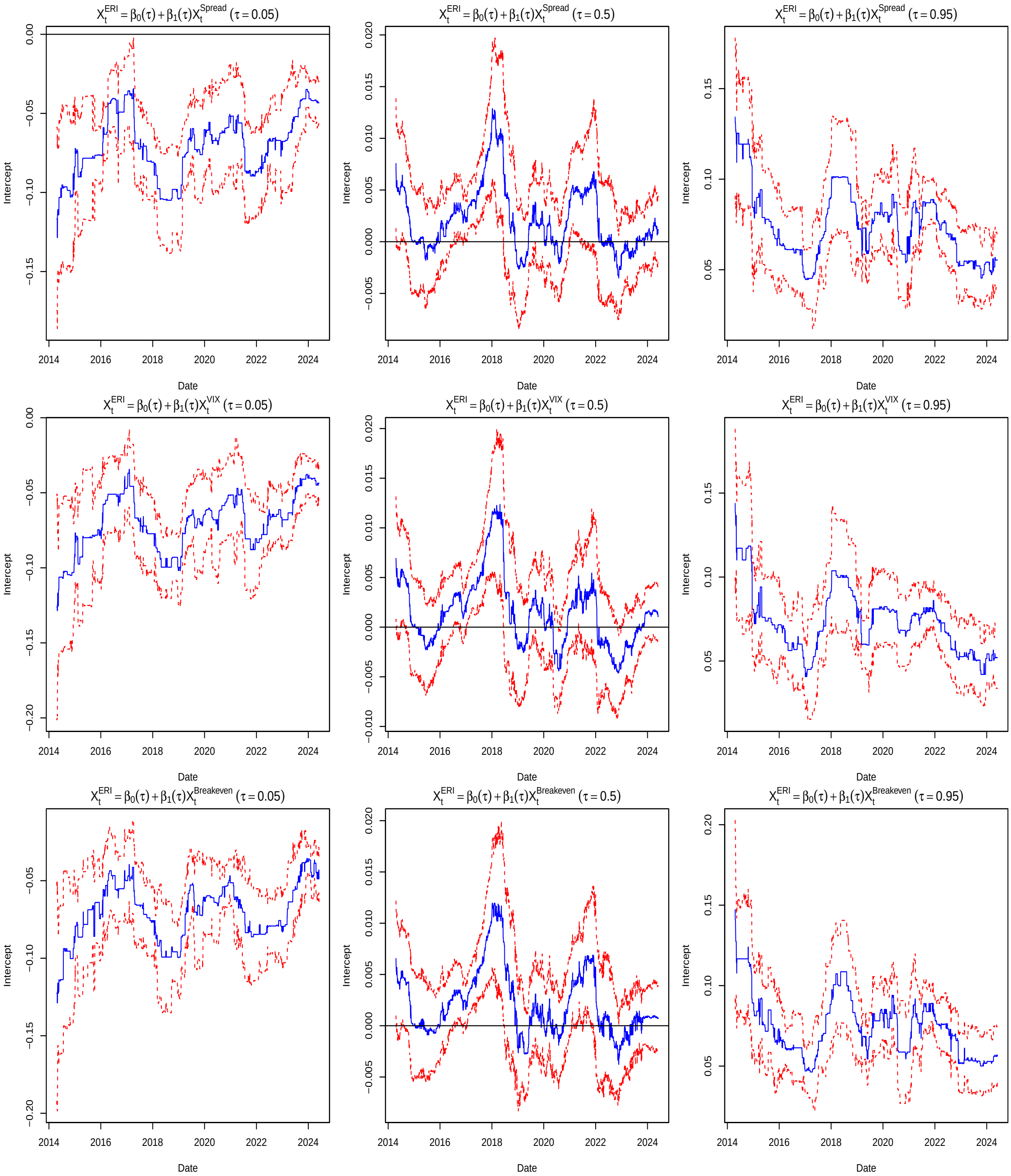
In the second step, the AR and MA lags selected via LASSO inform the specification of an ARMAX model, incorporating an exogenous dummy variable to evaluate the impact of significant crypto-currency-specific events or internal CM dynamics on price movements. The ARMAX model is given by:

$$ERI_t = \alpha + \sum_{i \in \mathcal{P}} \phi_i ERI_{t-i} + \sum_{j \in \mathcal{Q}} \theta_j \varepsilon_{t-j} + \delta D_t + \varepsilon_t,$$

where ERI_t denotes the CM return at time t, $\mathcal{P}$ and $\mathcal{Q}$ represent the sets of selected AR and MA lags, D_t is a dummy variable equal to 1 on event dates (Bitcoin halvings: July 9, 2016; May 11, 2020; Ethereum upgrades: London Hard Fork on August 5,

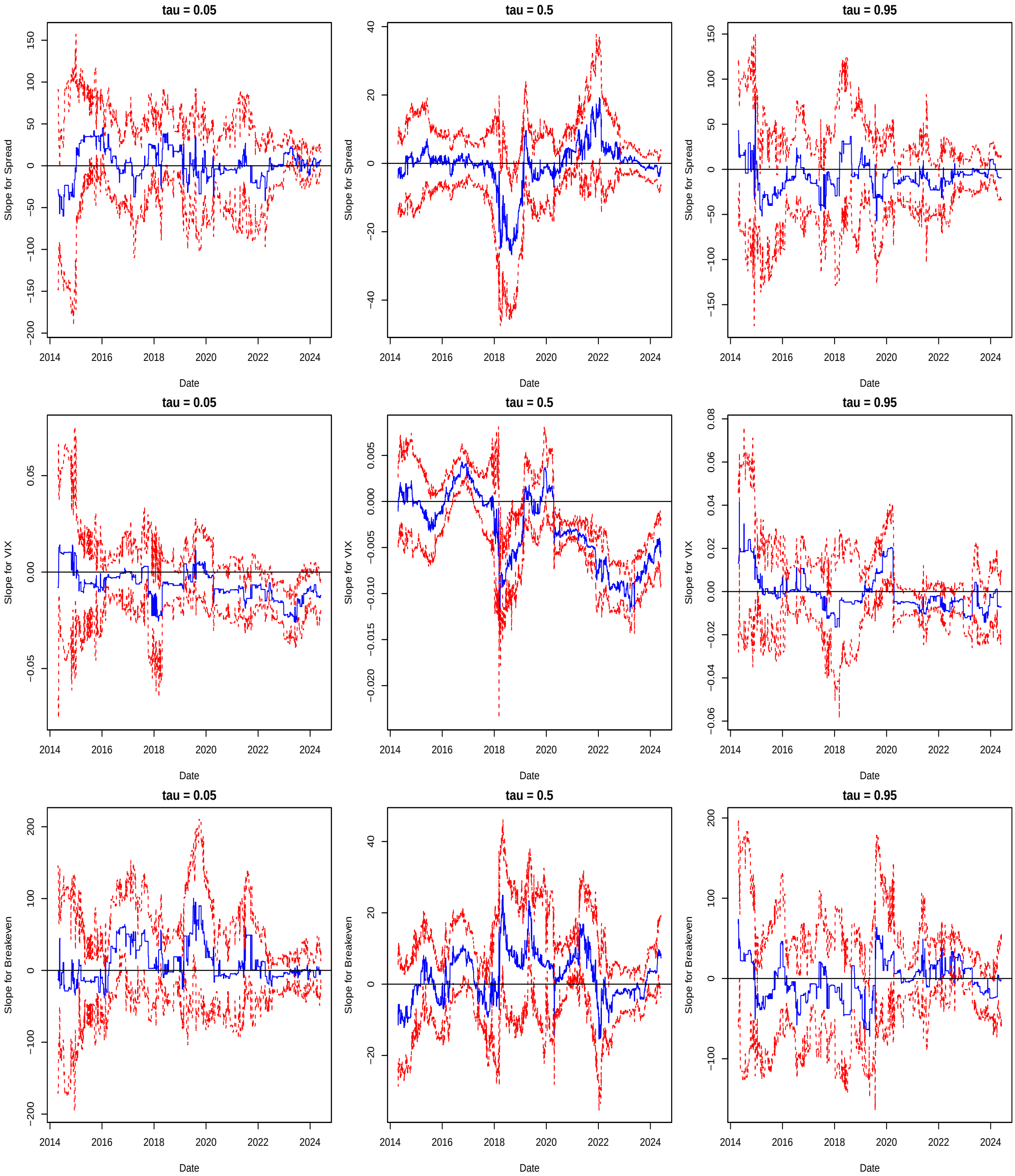
[12] See Liu et al. (2022) for a brief summary of this strand of the behavioral finance literature.

Figure 1.8: Effect of Macro-financial Variables on log-returns on the CM (single regression)



Notes: This graph reports the slope of a rolling quantile regression of returns on the CM against the term spread, the VIX, and breakeven inflation. We regress the returns on the CM on each covariate one at a time. The blue line in this plot is the $\beta_t(\tau)$ computed over a window of 252 observation. The 95% confidence intervals were formed by scaling the full sample covariance.

Figure 1.9: Effect of Macro-financial Variables on log-returns on the CM (Multivariate regression)



Notes: This graph reports the slope of a rolling quantile regression of returns on the CM against the term spread, the VIX, and breakeven inflation. We regress the returns on the CM on all three covariates at the same time. The blue line in this plot is the $\beta_t(\tau)$ computed over a window of 252 observations. The 95% confidence intervals were formed by scaling the full sample covariance.

2021; the Merge on September 15, 2022) and 0 otherwise, and ε_t represents a white noise error term.[13] These events are selected due to their substantial structural or narrative impacts on the CM, generating significant media attention and influencing investor behavior.

The empirical results summarized in Table 1.7 suggest crypto-currency returns are influenced predominantly by internal market dynamics. The inclusion of the event dummy variable leads some AR and MA coefficients to lose statistical significance due to absorbed variation. Significant autoregressive terms at lags 4 and 10 indicate momentum effects consistent with trend-following or herd behavior in speculative markets. Conversely, the negative AR(1) coefficient implies short-term mean reversion, potentially capturing swift market corrections following abrupt price changes. Additionally, statistically significant MA terms at lags 2 and 3 highlight the lasting impact of recent market shocks. Critically, the negative and significant coefficient of the event dummy variable supports the hypothesis that prices typically decline following major crypto-specific announcements, aligning with the "buy the rumor, sell the news" trading strategy commonly practiced in speculative periods. Overall, these findings emphasize that crypto-currency price dynamics are driven by some endogenous factors or internal dynamics.

Table 1.7: ARMAX Model of BTC Returns with Event Dummy

Row	α	ϕ_1	ϕ_4	ϕ_{10}	ϕ_{12}	ϕ_{13}	ϕ_{18}	ϕ_{21}	ϕ_{22}	ϕ_{28}	ϕ_{30}	θ_2	θ_3	θ_7	θ_{11}	θ_{29}	δ
Estimate	0.002^{**}	-0.069^{***}	0.057^{***}	0.060^{***}	0.032	-0.018	0.025	-0.027	0.028	0.032^{*}	0.030^{*}	0.033^{*}	0.052^{***}	0.023	0.029	0.026	-0.050^{*}
Std. Error	(0.001)	(0.019)	(0.019)	(0.019)	(0.019)	(0.019)	(0.019)	(0.019)	(0.019)	(0.018)	(0.018)	(0.020)	(0.020)	(0.020)	(0.019)	(0.018)	(0.028)
Observations		2,649															
R^2		0.026															

Note: $^{*}p < 0.1$; $^{**}p < 0.05$; $^{***}p < 0.01$

1.5 Conclusion

Our analysis finds no strong dependence between the CM and the term spread in the copula modelling framework. This result dismisses the importance of the yield curve in investment decision-making related to the CM. Such an outcome corresponds to numerically low copula dependence and tail estimates. Our interpretation is supported for the entire sample and a sub-sample analysis. The latter sample covers the post-2020 period, which has seen an expansion of the crypto activities to institutional investors. Our results display the same negligible dependence between the two markets for the sub-sample consideration. Therefore, information about future changes in monetary policy and economic cycles contained in the slope of the yield curve does not matter

[13] The permitted lags for each set $\mathcal{P}$ and $\mathcal{Q}$ range from 1 to 30 to adequately capture momentum dynamics.

for returns on the CM. The conclusion is similar for an extension of the analysis for the breakeven inflation.

We extend the same methodology to the analysis of the dependence between price fluctuation on the CM and the VIX. The VIX is known in the literature to be a measure of investors' fear and uncertainty about future market conditions. Our results provide weak evidence that low VIX estimates tend to correspond with high volatility on the CM. This tail dependence relationship becomes stronger when looking at the VIX with a forward-looking volatility estimate for the CM. We interpret this "low-high" volatility result as a shift in resources allocation between the two markets. In times of low volatility, investors, namely the risk-averse ones, would substitute crypto-currencies for stocks. In the end, the outflow of money would nourish uncertainty and cause the volatility on the CM to spike up.

A robustness check with a linear rolling quantile regression analysis reveals that macroeconomic predictors have a differential impact on crypto-currency returns across quantiles. Specifically, a 1% increase in predictors leads to a significant decrease in log-returns at the 0.05 quantile, while the effect at the median quantile (0.5) is not significant. At the 0.95 quantile, the effect becomes positive and significant. Additionally, controlling for other variables (the two other covariates) highlights that only the VIX significantly influences median returns, particularly up to late 2023. These findings suggest that crypto-currencies respond uniquely to economic indicators compared to traditional assets, with potential for increased returns during restrictive monetary policies.

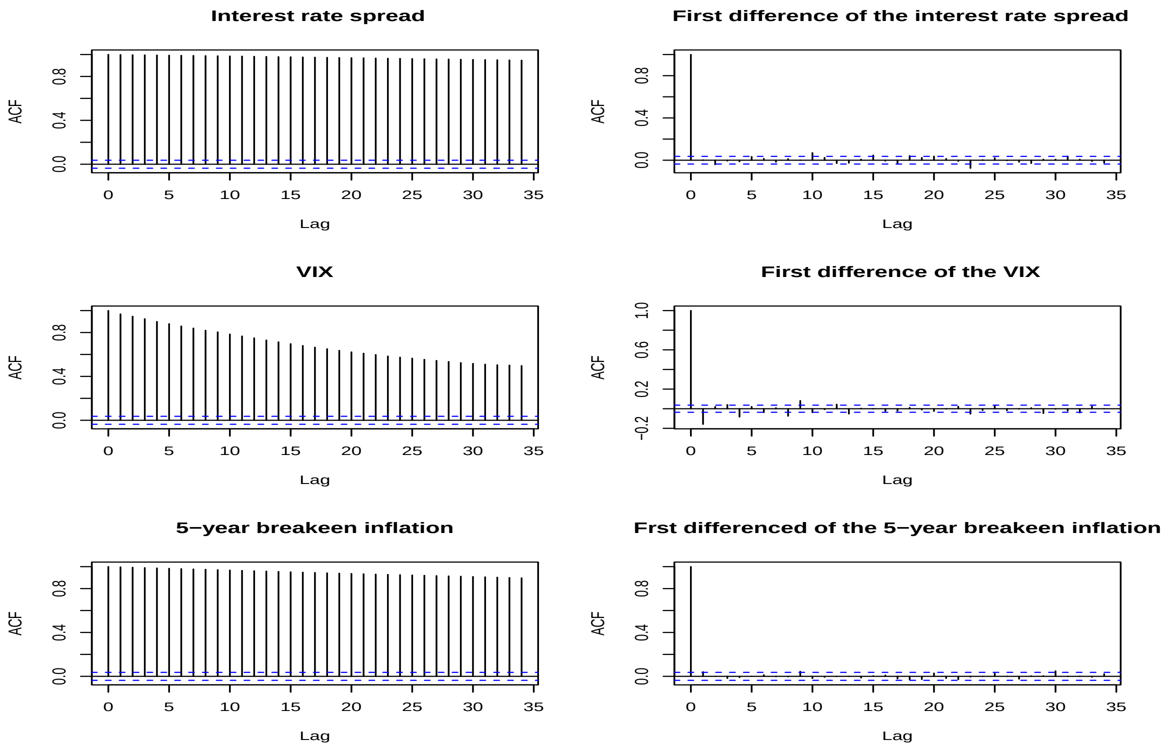
Moreover, our findings emphasize the critical importance of momentum effects and internal market dynamics in shaping crypto-currency prices. Major crypto-specific events, such as Bitcoin halvings and Ethereum protocol upgrades can influence market dynamics, independent of macroeconomic factors. Further research should deepen exploration into these internal market dynamics and momentum factors. Possible areas include the use of granular data to enhance the predictive understanding of crypto-currency price movements.

In conclusion, our findings challenge the literature that suggests a significant direct link between the CM, monetary policy, and the government bond market (see e.g., Karau, 2021). This is evidenced by the weak correlation between the CM log-returns, the term spread, the VIX and the breakeven inflation. We propose two avenues for further research: using higher-frequency data to better capture the CM's response to policy decisions and utilizing granular data to explore how "new era thinking" or bubble formation, as discussed by Shiller (2015), might influence CM price dynamics.

1.A Appendix

1.A.1 Supplementary information

Figure 1.10: ACF of the interest rate spread, the VIX and the breakeven inflation



Notes: This figure presents the ACF and the PACF of the interest rate spread, the VIX and the breakeven inflation. It is clear from the ACF that the original series stem from a non-stationary process. Note that the horizontal dashed lines (in red) represent the 95% confidence interval.

1.A.2 Supplementary information to the modelling section

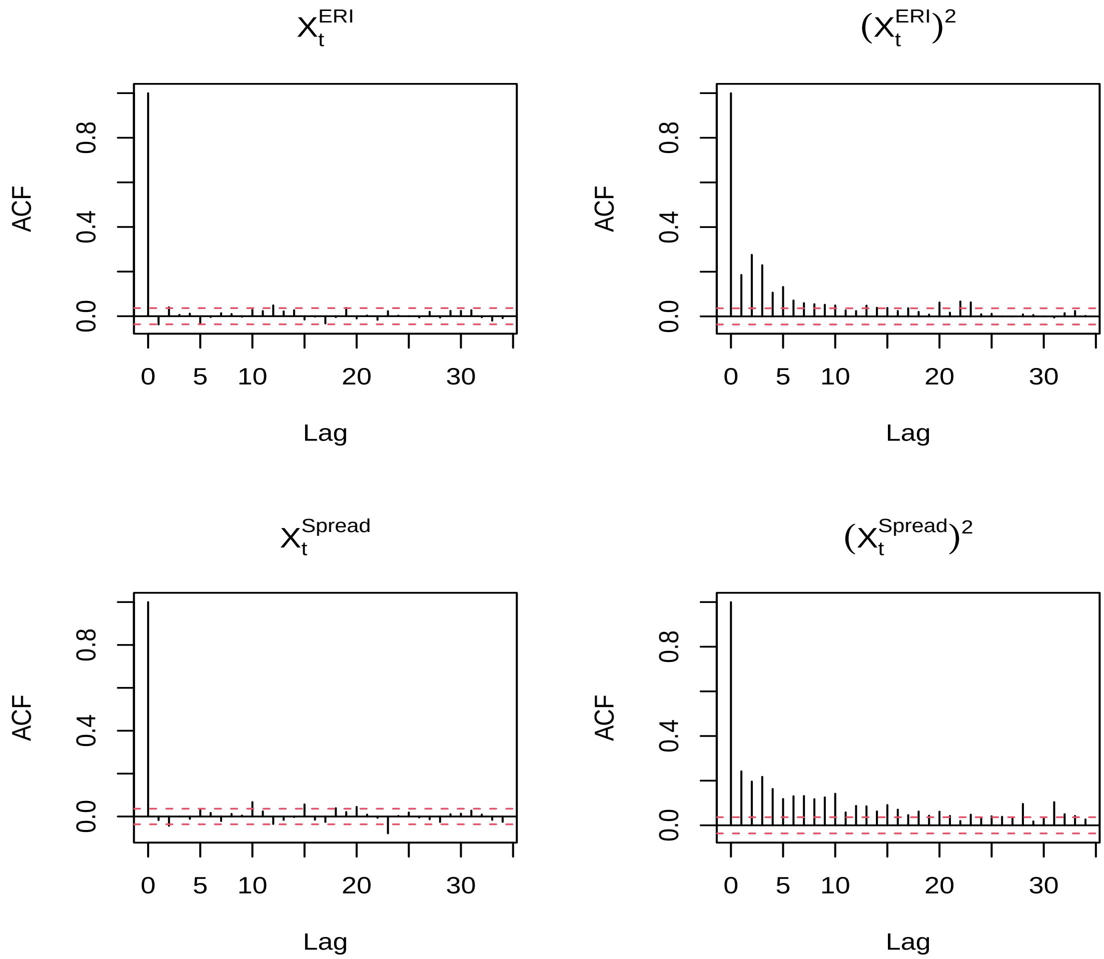
In choosing the model choice, we start with plots of the correlation between different lags of both series. Figure 1.11 reveals weak evidence of correlation between successive lag values of X_t^{ERI} and X_t^{Spread}. On the contrary, the squared of both series exhibit significant correlation between contemporaneous and past values. We apply the Ljung-Box test to confirm the existence of serial dependence in the series. There is evidence against serial dependence for X_t^{ERI} and X_t^{Spread} at low lag components. However, the squared transformation of both variables show existence of strong serial dependence of up to lag 12 or more.

We use stepwise regression to select the number of lags to enter the conditional equations 6 and 7. The ”auto.arima” algorithm in R and similar techniques in Python are straightforward approaches to implement the lag selection process. The returns series is fed into the algorithm and the best autoregressive model is selected based on the Bayesian Information Criteria (BIC). We construct the variance equation with the squared component of the series, meaning $(X_t^{ERI})^2$ and $(X_t^{Spread})^2$. Obviously, the latter has its own shortcomings but it is a more structured process than deducting lag order form the AFC/PACF reading. The BIC selects ARMA(1,0)-GARCH(1,4) and ARMA(0,0)-GARCH(1,1) for returns on the CM and change in the spread, respectively. In both cases, no conditional mean is suggested by the algorithm.

We use the MLE to compute parameters of the model order suggested by the stepwise regression framework. Estimates are reported in Table 1.8. In passing, we provide estimates of alternative models for comparison purposes. In the case of X_t^{ERI}, we run a GARCH(1,1) and a TARCH(1,1,1). A GARCH(1,1) sits between the suggested GARCH(1,4) and the TARCH(1,1,1). The GARCH(1,1) represents a simpler framework (less parameters to be estimated), whereas the TARCH (1,1,1) stands as a more complex formulation. In the case of X_t^{Spread}, we accept the GARCH(1,1) since it is a standard model used in the literature for this kind of work. According to the information criterion reported in Table 1.8, the smaller GARCH(1,1) would be a good starting point to model the conditional variance of X_t^{ERI}.

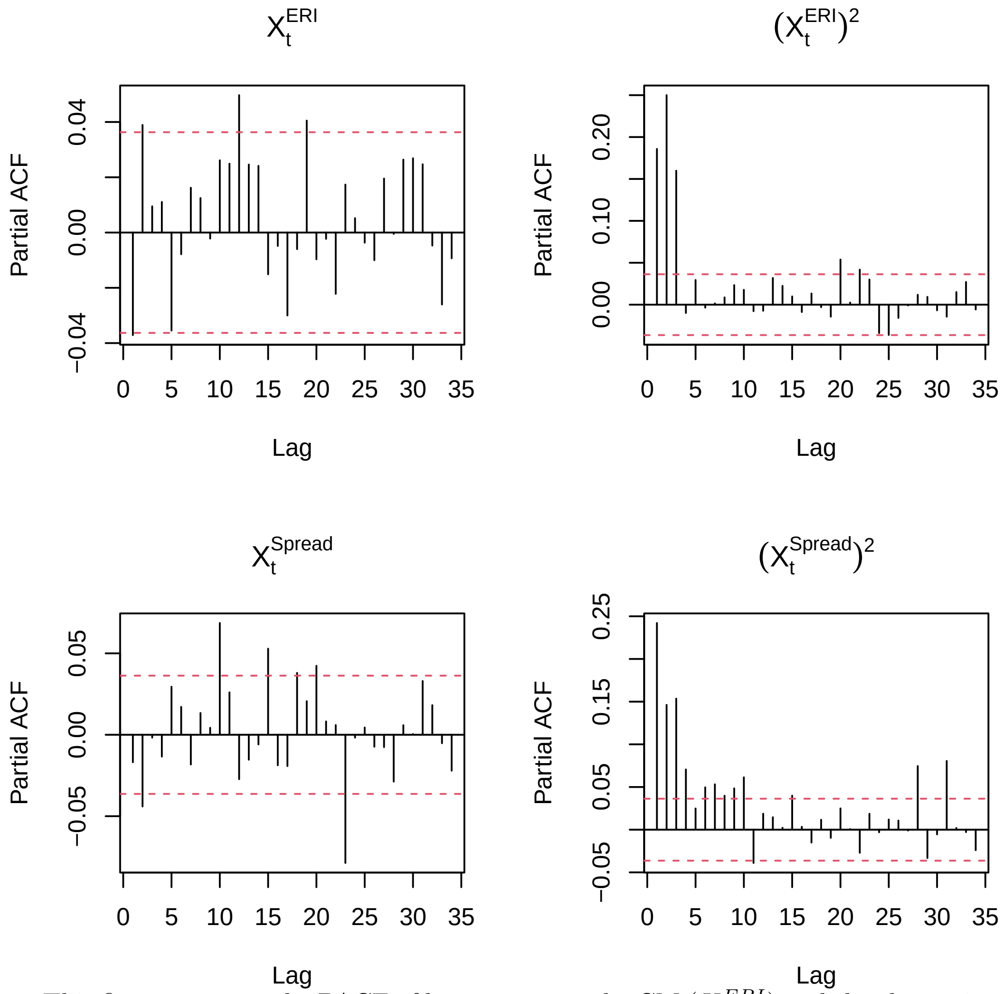
To conclude this section, we inspect the distribution of the standardized residuals against some theoretical processes in order to rationalize the choice of e_t in Equation 1.6 and Equation 1.7.

Figure 1.11: Autocorrelation function of log-returns and change in the interest rate spread



Notes: This figure presents the ACF of log-returns on the CM (X_t^{ERI}) and the change in the spread (X_t^{Spread}) . The horizontal dashed lines (in red) represent the 95% confidence interval of a white noise series.

Figure 1.12: Partial autocorrelation of log-returns and change in the interest rate spread



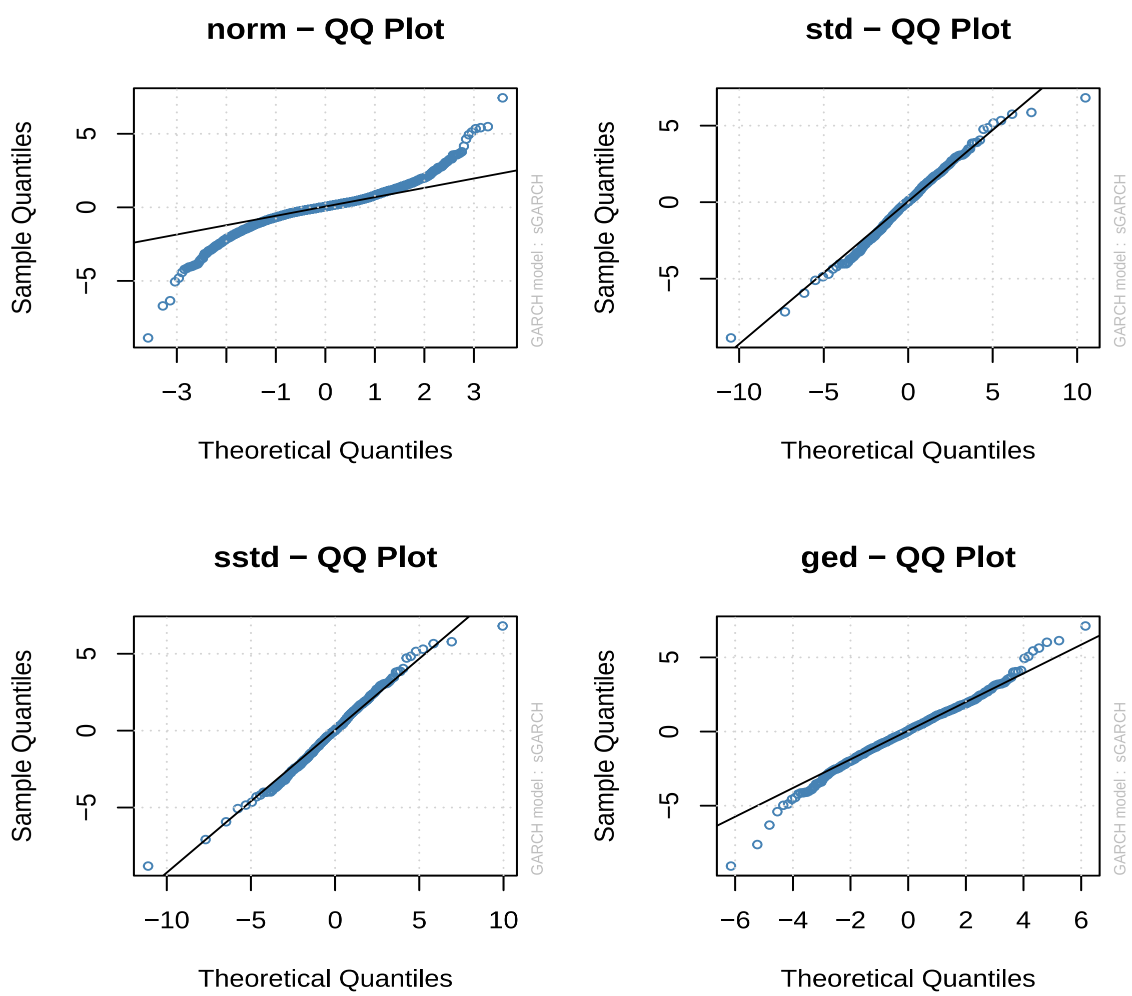
Notes: This figure presents the PACF of log-returns on the CM (X_t^{ERI}) and the change in the spread (X_t^{Spread}) . The horizontal dashed lines (in red) represent the 95% confidence interval of a white noise series.

Table 1.8: Model comparison for log-returns on the CM and change in interest rate spread.

	X_t^{ERI}		
	GARCH(1,1)	GARCH(1,4)	TARCH(1,1,1)
ω	0.00013***	0.0001***	0.004***
	(0.000)	(0.000)	(0.000)
α_1	0.161***	0.161***	0.182***
	(0.017)	(0.019)	(0.015)
α_2			
α_3			
β_1	0.798***	0.796***	0.803***
	(0.019)	(0.158)	(0.017)
β_2		0.000	
		(0.190)	
β_3		0.000	
		(0.072)	
β_4		0.001	
		(0.092)	
γ_1			0.094**
			(0.040)
AIC	-3.379	-3.372	-3.375
BIC	-3.371	-3.358	-3.365

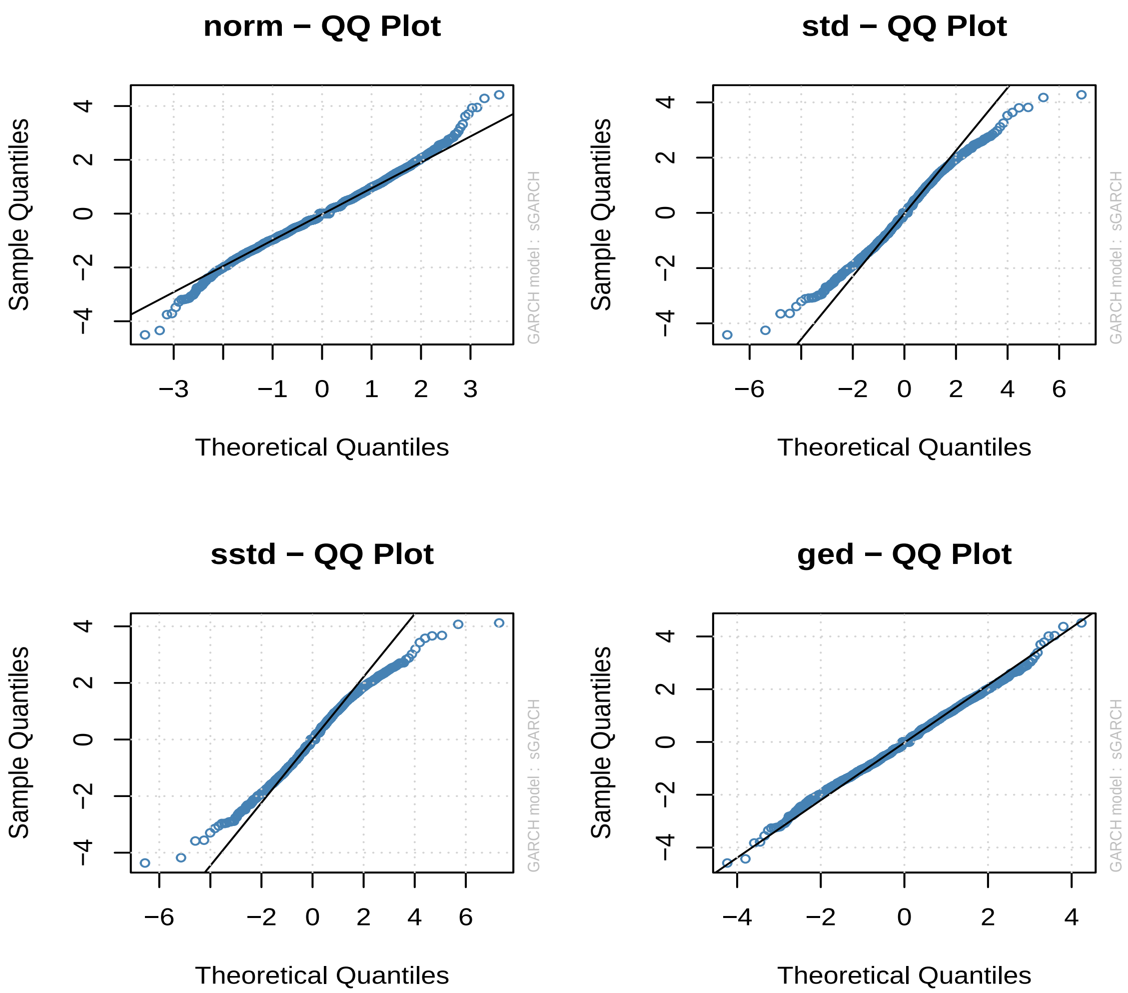
Notes: This table presents different model candidates for the conditional variance of X_t^{ERI}.
***Significant at the 1 percent level.
**Significant at the 5 percent level.
*Significant at the 10 percent level.

Figure 1.13: Comparison of different distribution assumption for X_t^{ERI} innovations



Notes: This figure plots the residuals of the ARMA(0,0,0)-GARCH(1,1) process against the normal, the student's t, the skewed student's t, and the Generalized Error Distribution (GED). The normal assumption offers the worst fit out of all the assumptions considered to model the innovation process in Equation 1.6. The Student's t and the Skewed student's t distributions offer similar results.

Figure 1.14: Comparison of different distribution assumption for X_t^{Spread} innovations



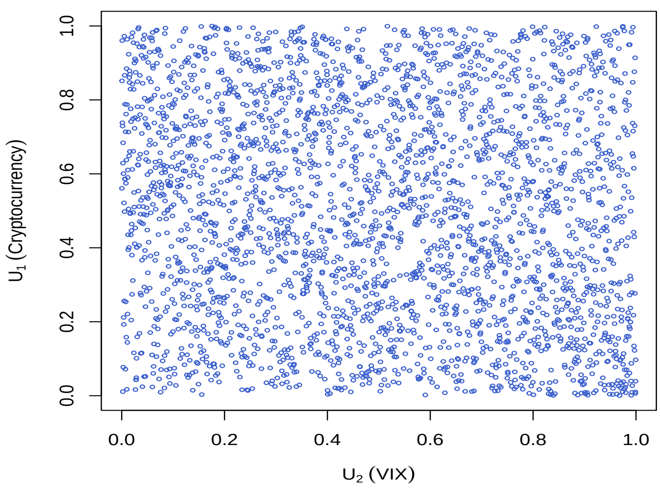
Notes: This figure plots the residuals of the ARMA(0,0,0)-GARCH(1,1) process against the normal, the student's t, the skewed student's t, and the GED. The GED offers the best result out of all the assumptions considered to model the innovation process in Equation 1.7.

Table 1.9: CM and VIX copula estimates

	Normal	Student's t	Clayton	Frank	Gumbel-Hougaard	Joe
θ	0.035**	0.034***	0.039**	0.158	1.028 ***	1.063***
	(0.02)	(0.020)	(0.023)	(0.118)	(0.014)	(0.022)
τ^L		0.000	0.000			
		(1×10^{-8})	(0.000)			
τ^U		0.000			0.036***	0.08***
		(1×10^{-8})			(0.001)	(0.001)
Deg. of freedom		4.8×10^{4}***				
		(0.000)				
Log. Likelihood	1.674	1.672	-9.623	1.025	3.025	6.558
AIC	1.347	-4.656	-23.247	0.05	2.05	9.116

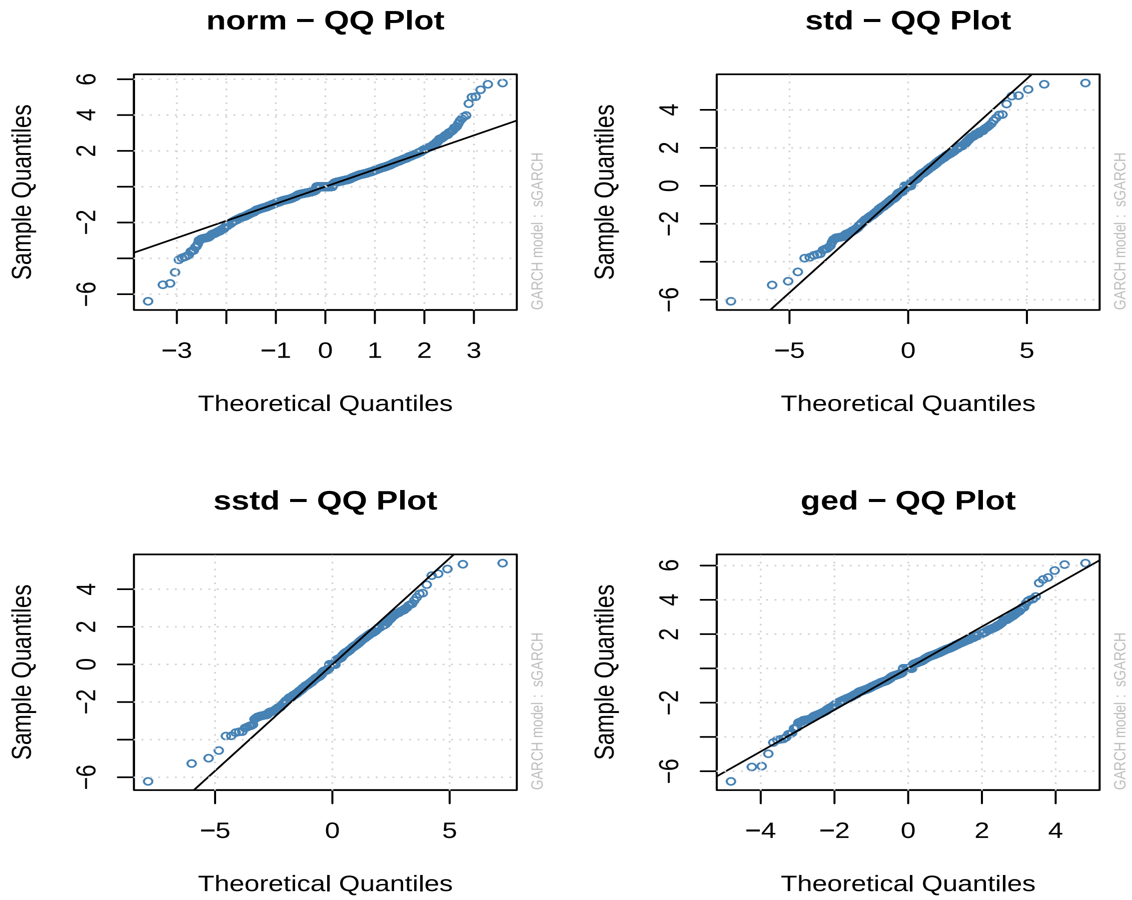
Notes: This table presents estimates of some of the widely used copulas. In parentheses are standard errors of the estimates. In the spirit of Table 1.2, the statistical significance of the Joe and Gumbel-Hungaard is tested as $H_0 : \theta = 1$ and $H_1 : \theta > 1$. The standard errors for the tail probabilities are computed with the delta method. The MLE failed to estimate the dependence parameter in the Clayton case. We instead estimate the parameter via the method-of-moment (Spearman's rho). Hofert et al. (2019) gives a detailed explanation of this technique.
***Significant at the 1 percent level.
***Significant at the 1 percent level.
**Significant at the 5 percent level.
*Significant at the 10 percent level

Figure 1.15: Dependence representation between the VIX and the CM



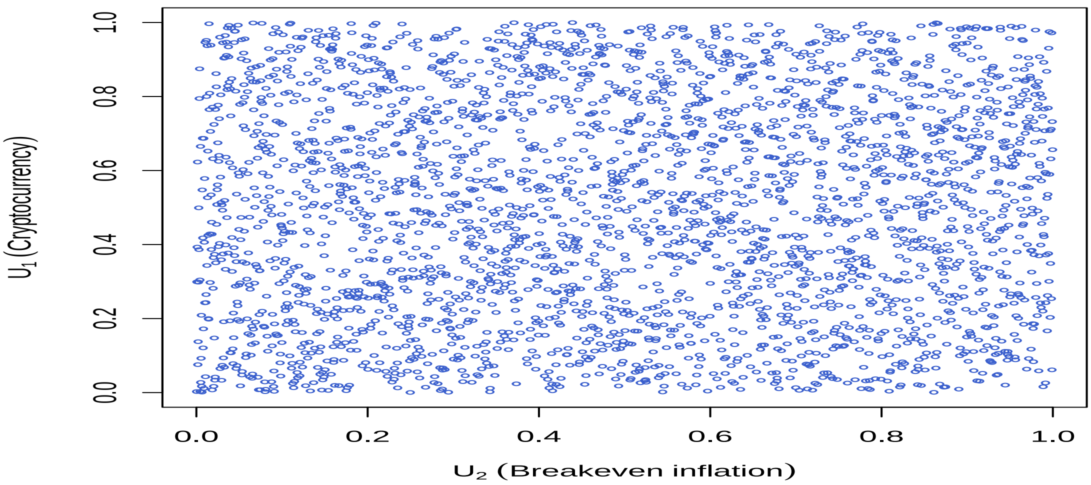
Notes: This figure presents the residuals extracted from GARCH process. The residuals are transformed to be distributed between 0 and 1 (Pseudo-observations).

Figure 1.16: Residuals of the breakeven GARCH(1,1) model



Notes: This set of plots presents the 4 assumptions considered to model the volatility of the breakeven inflation. The GED shows a better fit with respect to the other distributions. The intercept of the conditional variance equation is approximately zero (same for the mean equation). The conditional variance equation can be written as $\delta^2_{4t} = 0.058\epsilon^2_{4t-1} + 0.901\delta^2_{4t-1}$. The GED shape parameter is estimated to be 0.9.

Figure 1.17: Pseudo-observation of log-returns on the CM and the first difference of the inflation expectation



Notes: This figure presents the pseudo-observations of the residuals (uniform transformation) from the GARCH equations for the log-returns on the CM and the 5-year breakeven inflation (the differenced series). The spread of the points shows no particular dependence structure.

Chapter 2

The Transactive Role of Crypto-currencies: A Theoretical Perspective

Abstract

We build a theoretical model where both fiat money and crypto-currencies are used as media of exchange for differentiated goods. Crypto-currencies offer pecuniary benefits, such as avoiding consumption taxes, and non-pecuniary benefits like transaction privacy, while non-users face utility losses that grow with available goods. We identify an endogenous threshold good where consumers are indifferent between government-backed money and privately-issued currency, leading to three equilibrium scenarios: all goods purchased with fiat money, all with crypto-currency, or a mix of both. Our model predicts that, while fiat money is neutral, crypto-currencies are non-neutral due to mining costs, which affect labor allocation.

2.1 Introduction

Crypto-currencies emerged with the promise of eliminating frictions that lead to high transaction costs in traditional money-based economies (Nakamoto, 2008). This alternative medium of payment presents consumers with a decision-making problem regarding which currency to use for day-to-day transactions—whether to choose fiat money or crypto-currency. In a two-country model without capital controls, the choice of currency would depend on the real exchange rate between the two. However, crypto-currencies offer more than just liquidity services. For instance, anonymity in transactions is a non-pecuniary benefit linked to crypto-currencies, adding complexity to this decision. Therefore, the challenge lies in how crypto-currencies can coexist with fiat money. This raises critical questions: what factors determine the stock of goods purchased with either form of currency? And what are the welfare-enhancing attributes associated with higher adoption of crypto-currencies?

Current theories remain insufficient in explaining the determinants of demand for *crypto-purchased goods* and *money-purchased goods*. For example, Marchiori (2021) tackles this issue in a cash-in-advance model where both crypto-currencies and fiat money are used for consumption payments. However, the model imposes a strict exogenous constraint, assigning one fixed set of goods for crypto-currency purchases and another for cash purchases. This leaves unresolved the question of what drives consumers to use crypto-currency for transactions. Similarly, Benigno et al. (2022) develops a related theoretical model but focuses on the monetary policy implications of having both currencies in circulation. Consequently, both models and other related frameworks provide limited insights into the key questions raised earlier.

We propose a one-period cash-in-advance model in which both fiat money and privately issued currency are accepted as media of exchange for a continuum of differentiated goods. However, an exogenous rule requires that all production factors be remunerated in money terms. We assume heterogeneity in access to crypto-currencies, reflecting differences in information and technology (IT) know-how between crypto users and non-users. In subsection 2.5.4, we show that an increase in the crypto-currency accessibility parameter has real economic consequences, particularly in terms of consumption and labor reallocation across sectors of the economy

The originality of our analysis lies in modeling crypto-currencies as a tool that provides both *pecuniary* and *non-pecuniary* benefits to consumers. On the pecuniary side, crypto-currencies allow consumers to bypass value-added taxes on consumption goods. On the non-pecuniary side, they impose a utility loss on non-users by offering privacy in transactions. This approach endogenizes the demand for crypto-currencies in purchasing goods. As opposed to Marchiori's analysis, consumers have the discretion to buy any good with either fiat money or crypto-currencies in our model. Based on this framework, we argue in subsection 2.3.2 for the existence of a *threshold good*, where consumers are indifferent between using fiat money or crypto-currency, giving the latter a defined role as a medium of exchange in the market.

Our first set of results lead to three possible outcomes based on the relative transaction costs of purchased goods. In the first scenario, all goods are purchased with money due to high crypto-fees or low transaction costs for money-purchased goods. In the second scenario, low crypto-fees or high taxes on money-related payments lead to the exclusive use of crypto-currencies for transactions. The third scenario introduces a threshold, where both currencies are used, with the set of goods purchased with money expanding as crypto-fees rise or consumption taxes on money payments fall.

In our model, fiat money is costlessly produced and distributed to final goods consumers. We exclude savings in the form of money holdings or capital investments. This leads to an equilibrium solution where money is neutral in the economy, which is a standard result in the real business cycle literature. On the other hand, mining crypto-currencies is costly in the model. As a result, crypto-currencies exhibit non-neutrality in the system because the presence of crypto-currency mining activity affects labor allocation across sectors. This effect is visible in the numerical simulation below, where we show that a positive investment shock in the mining sector influences real wages and drives up unemployment in the goods production sector.

To develop a tractable model with interpretable solutions, this chapter and the next one omit certain aspects of CM dynamics. For instance, we do not assign a speculative role to Bitcoin or other crypto-currencies in our model. Makarov and Schoar (2021) argues that until June 2021, 90% of observed BTC transactions were economically non-meaningful[1]. Moreover, their analysis highlights a market bias toward the speculative use of crypto-currencies. The authors also note a steep concentration of BTC holdings among a few account holders, raising questions about the price formation mechanism in the CM environment. These elements could serve as the foundation for an extension incorporating a fully developed general equilibrium model with a financial sector and various frictions affecting the exchange rate between crypto-currencies and fiat currencies.

The remainder of the paper progresses as follows. We first present the connection between our results and the existing literature in section 2.2. We proceed to model the different components of our general equilibrium framework in section 2.3. Then,

[1]See Foley et al. (2019) for a discussion on the use of BTC in organized crime activities.

we explore the equilibrium conditions emerging from the optimization problem in section 2.4. Finally, we provide a numerical analysis to study the response of the endogenous variables following a shock to the exogenous constants in the final section.

2.2 Related Literature

The analysis in this paper advances the growing literature on the co-existence of government-backed and privately-issued currencies. Work on currency competition and the concept of private currencies is well established in the monetary literature (see Hayek (1976) and Kareken and Wallace (1981)). Benigno et al. (2022) provide an extensive discussion of currency competition and its various ramifications with the crypto-currency framework. For the sake of clarity, we focus on contributions related to crypto-currencies and highlight the place of our analysis within this body of work.

A closely related analytical setup to ours is the analysis by Schilling and Uhlig (2019b) on the medium of exchange role of fiat money and crypto-currencies. Their model predicts the existence of an endogenous good for which consumers are indifferent between fiat money and crypto-currencies. Other elements, such as the difference in transaction costs for crypto-using and money-using consumers, are also present in our paper. A key point of departure between the two frameworks lies in our treatment of crypto-currencies as an instrument that facilitates privacy in goods transactions, which is explicitly modeled in our framework. Other contributions in this line of research abstract from the privacy aspect and present further modeling divergences from our analysis. For instance, Marchiori (2021) restricts the transactive role of crypto-currencies to a specific set of goods. In Fernández-Villaverde and Sanches (2019), the existence of crypto-currencies can lead to an undesirable equilibrium where the stock of money in circulation fails to meet transaction needs. Additionally, Yu (2023) and Schilling and Uhlig (2019) derive a role for crypto-currencies when their rate of return matches that of fiat money in the market. Another currency competition model is proposed by Zhu and Hendry (2019), which focuses on price stability. Our paper, however, does not incorporate most of the features discussed in these works. Instead, we focus on deriving the conditions that make currency substitution possible in an economy with both private currency and fiat money.

2.3 Static Model with Purely Transactive Currencies

2.3.1 Basics

Consider a static economy where consumption transactions can be performed using two different currencies. One currency, called *money*, acts as the legal tender and is issued by the government. The other currency is represented by privately-issued virtual coins and is called *crypto-currency*. Households supply labor and physical capital to firms and receive factor incomes that, as a result of exogenous rules, must be paid with *money*. The hypothesis that all production factors must be remunerated using the legal tender of the economy plays the role of a cash-in-advance (CIA) constraint in a one-period economy. However, the goods market is open to alternative means of payments, and the crypto-currency can be used to purchase consumption goods. Households can use part of their money stock to purchase the crypto-currency on *exchange platforms* that charge fees for their services. At the same time, firms receiving payments in crypto-currency will use the exchange platform to convert crypto-payments into money that will be used to remunerate labor and capital owners. Like consumers, firms will be charged a *crypto-transaction fee* since exchange platforms bear the costs of validating every transaction that involves crypto-currency.

The static environment has specific characteristics. Since households spend all their income in one period and the crypto-currency is only used for transaction purposes, the stock of crypto-currency returns in full to the exchange platform when all exchanges are completed. The final state of the crypto-market would be radically different in a dynamic model including multiple periods and saving-investment opportunities. Nonetheless, this static model offers important insights on how the introduction of a crypto-currency may affect the equilibrium in CIA-constrained economies, as we will see in the dynamic model of the next chapter.

2.3.2 Consumers

The economy is populated by L consumers, each consuming (a continuous finite mass of) N different goods indexed by $n \in [0, N]$. Each good n may in principle be purchased using money or crypto-currency. However, only a fraction ϵ of the L consumers has access to crypto-currencies and can freely decide which goods to buy with either means of payment. The remaining $(1-\epsilon)\,L$ consumers purchase all the N goods using money. We treat $\epsilon \in (0, 1)$ as an exogenous parameter that we can manipulate to investigate important properties of the model. Letting $\epsilon \to 0$ we can study the equilibrium of a benchmark economy without crypto-currency and compare its predictions to those obtained in the general case, $0 < \epsilon < 1$, as well as in the opposite polar case where every household has access to crypto-currency, $\epsilon \to 1$. It is safe to argue that $0 < \epsilon < 1$ is a realistic hypothesis – e.g., because crypto-currencies are not used by households with insufficient IT literacy or equipment. However, the key rationale for our hypothesis $\epsilon < 1$ is that it provides us with a free parameter whereby we can assess the impact of market-size shocks – that is, exogenous changes in the potential demand for crypto-currencies – and more generally the welfare effects of crypto-currencies.

In order to distinguish individual variables that refer to either type of households, we will use 'tildas': for any variable x associated with consumers having access to the crypto-currency, the same variable for 'crypto-less consumers' will be denoted by $\tilde{x}$. The next two sub-sections specify the expenditure problem for each type of consumer in turn.

2.3.2.1 The crypto-less consumer

Consider an individual within the set of $(1-\epsilon)\,L$ consumers having *no access* to the crypto-currency. Total utility from consumption is an integral of well-behaved sub-utility functions,

$$\tilde{U} \equiv \int_0^N \tilde{u}_n\left(\tilde{c}\left(n\right)\right) dn \tag{2.1}$$

where $\tilde{c}\left(n\right)$ is the consumed quantity of the n-th good.

The crypto-less consumer purchases all goods using money, which entails different types of transaction costs. The literature suggests a long list of private costs associated with money payments that could be circumvented using alternatives like crypto-currencies. Some costs are *non-pecuniary* – e.g., lack of anonymity, legal constraints, personal time costs generated by bureaucracy, red-tape, transaction-recording and similar administrative duties – and are typically connected to the nature of the good being purchased regardless of its market value: the transaction per se creates disutility and can thus be modeled as a non-distortionary 'tax' in terms of utility. Other costs are *pecuniary* – e.g., credit-card and money-transfer fees, intermediation costs, consumption taxes imposed by governments on traceable money transactions – and can be either lump-sum or distortionary. To cover all bases, our model includes both pecuniary and non-pecuniary costs.

We capture non-pecuniary costs by introducing a simple disutility term: if good n is purchased with money, the associated net satisfaction is

$$\tilde{u}_n\left(\tilde{c}\left(n\right)\right) = \ln\left[\tilde{c}\left(n\right)\cdot\left(1-\delta\left(n\right)\right)\right] \text{ with } 0 \leqslant \delta\left(n\right) < 1. \tag{2.2}$$

The disutility parameter $\delta(n)$ is good-specific because non-pecuniary transaction costs may vary substantially across types of goods. The logarithmic form (2.2) rules out distortions in the sense that, under utility-maximizing conditions, goods characterized by different disutility parameters $\delta(\cdot)$ will capture identical expenditure shares. In other words, non-pecuniary transaction costs are a non-distortionary tax in terms of utility.

The pecuniary costs of money-purchased goods, instead, are represented by a proportional fee: consumers buying good n using *money* will spend

$$p(n) \cdot \tilde{c}(n) \cdot (1+\tau),$$

where $p(n)$ is the market price of the good in terms of money, and $\tau > 0$ is the fee rate. The money-transaction fee τ may be interpreted in several ways – e.g., as a credit-card fee charged by private intermediaries, a consumption tax set by the government, a sunk monetary cost not collected by other agents. For our purposes, the key characteristic is that the money-transaction fee will not apply if the same good is purchased using the crypto-currency. In this model, τ is a consumption tax rate that the government is able to impose exclusively on recorded money payments for consumption. This will imply a tax-avoidance benefit from using crypto-currency[2]. Alternative models in which τ is a transaction fee paid to banks will likely yield similar results as long as the same fee does not apply to crypto-payments. The expenditure problem of the crypto-less consumer is

$$\max_{\{\tilde{c}(n)\}} \tilde{U} = \int_0^N \ln\left[\tilde{c}(n) \cdot (1-\delta(n))\right] dn$$

subject to

$$\tilde{x} = \int_0^N p(n)\,\tilde{c}(n) \cdot (1+\tau)\, dn \tag{2.3}$$

where $\tilde{x}$ is individual spending on consumption goods. The first order conditions imply identical expenditure shares for each good,

$$p(n)\,\tilde{c}(n) \cdot (1+\tau) = \tilde{x}/N \quad \text{for each } n \in [0, N]. \tag{2.4}$$

[2]This property of the model is empirically plausible: there is widespread consensus that the use of crypto-currencies can facilitate tax-avoidance as well as similar shadow-economy activities. The latter point is explained and documented in a recent publication of the Bank for International Settlements (BIS) – see the BIS (2023) report for details. Other narratives on the use of crypto-currencies include a greater penetration in places with tight capital control (see Makarov and Schoar (2019)).

As previously noted, non-pecuniary costs do not distort expenditure shares: crypto-less consumers simply suffer a deadweight utility loss by purchasing goods using money since they do not have access to the crypto-currency. The individual income constraint of crypto-less consumers reads

$$\tilde{x} = w + rk_i + \tilde{g} \tag{2.5}$$

were w is the prevailing wage rate, r is the rental rate of individually-owned capital k_i, and $\tilde{g}$ represents lump-sum transfers from the government. The implicit hypotheses of (2.5) are that each individual supplies one unit of homogeneous labor that is remunerated at the same rate w by all firms and sectors, and rents k_i units of capital to goods-producing firms obtaining the same rental rate r. The goverment uses transfers $\tilde{g}$ to rebate the proceeds from the consumption tax to households.

2.3.2.2 The crypto-user: preferences

Consider an individual within the set of ϵL consumers having access to the crypto-currency. In the present environment, the justification for the existence of the crypto-currency is that using money to purchase at least some types of goods entails private transaction costs that exceed those implied by using the crypto-currency. In general, the utility of crypto-using consumers can be written as

$$U \equiv \int_0^{\bar{n}} u_j\left(c\left(j\right)\right) dj + \int_{\bar{n}}^{N} u_i\left(c\left(i\right)\right) di \tag{2.6}$$

where $c\left(n\right)$ is the physical quantity of the n-th good consumed, for any $n \in [0, N]$. The right hand side of (2.6) distinguishes between subsets of goods purchased using different means of payment: the subset indexed by $j \in [0, \bar{n})$ is purchased using *money* and the associated utility u_j includes non-pecuniary costs just like expression (2.2) above,

$$u_j\left(c\left(j\right)\right) = \ln\left[c\left(j\right) \cdot \left(1 - \delta\left(j\right)\right)\right] \text{ for } j \in [0, \bar{n}), \tag{2.7}$$

whereas the subset of goods indexed by $i \in [\bar{n}, N]$ is purchased using the *crypto-currency* and the associated utility does not include the disutility term:

$$u_i\left(c\left(i\right)\right) = \ln c\left(i\right) \text{ for } i \in [\bar{n}, N]. \tag{2.8}$$

Expressions (2.6)-(2.8) implicitly define a *threshold good*, indexed by $n = \bar{n}$, which splits the set $[0, N]$ by payment characteristics. In related literature, different means of payments are associated to different goods in a pre-determined way. For example, in the Marchiori (2021) model there exists one 'cash good' that can only be purchased with money and one 'virtual good' that can only be purchased with crypto-currency by assumption. In our analysis, instead, all the N goods can in principle be purchased using either type of currency, but consumers choose which goods to pay with either method according to utility maximization, so that the *threshold good* $\bar{n}$ is endogenously determined by preferences and market conditions. Therefore, our model justifies the existence of the crypto-currency for transactive purposes, and will predict that changing market conditions, or exogenous shocks on relevant parameters, will affect the transactive demand for crypto-currency even along the extensive margin via changes in the endogenous threshold $\bar{n}$.

The existence of a threshold good depends on the distribution of disutility terms across goods. We model such distribution in the simplest way by specifying $\delta(n)$ as a function that, under very mild assumptions, determines an interior cut-off point $\bar{n} \in (0, N)$ whereby both the resulting subsets, $[0, \bar{n})$ and $[\bar{n}, N]$, are non-empty. This result (i.e., the existence of subsets of goods purchased with different means of payments) hinges on the existence of good-specific costs associated with money payments.[3] In the next two subsections, we solve the expenditure problem of the crypto-user and then determine the threshold good by specifying a suitable function $\delta(n)$.

2.3.2.3 The crypto-user: expenditure problem

We solve the consumer problem in two steps. We firstly derive the utility-maximizing conditions taking $\bar{n}$ as given. Secondly, we derive the no-arbitrage condition for the threshold good $\bar{n}$ taking expenditure shares as given. The key elements are the opportunity costs of alternative means of payments. Purchasing $c(n)$ units of good n using money requires, as we know, spending

$$p(n) \cdot c(n) \cdot (1 + \tau). \tag{2.9}$$

Purchasing the same good using crypto-currency requires the consumer to convert the necessary amount of money holdings into crypto-currency and purchase the good from the producer by transferring the crypto-currency to the latter. The exchange platform will charge a fee for the currency exchange service. Crypto-transaction fees are proportional to the amount of crypto-currency involved: exchange platforms apply the rate φ^C to consumers selling money against crypto. The total cost to the consumer,

[3] Besides all the possible interpretations of δ and τ, what matters for our analysis is that at least one type of money transaction costs – pecuniary or non-pecuniary – is good-specific. Our results are essentially the same if we exclude pecuniary transaction costs ($\tau = 0$) while keeping $\delta(n)$ good-specific, or vice versa, if we exclude disutility costs ($\delta = 0$) and assume, instead, good-specific pecuniary costs, $\tau(n)$. Modelling the general case where both $\delta(n)$ and $\tau(n)$ are good-specific creates unnecessary algebraic complications without yielding further economic insight.

expressed in terms of money, is

$$Q \cdot p^{*}(n) \cdot c(n) \cdot \left(1+\varphi^{C}\right), \tag{2.10}$$

where $p^{*}(n)$ is the price of the n-th good expressed in crypto-currency units, Q is the *nominal exchange rate* – i.e., the units of money needed to purchase one unit of crypto-currency besides the fee rate φ^{C} that the exchange platform charges on consumers selling money against crypto [4].

Expenditure problem for given $\bar{n}$. Given the existence of a unique interior cut-off point $\bar{n} \in (0, N)$, the expenditure problem solved by the consumer is

$$\max_{\{c(j),c(i)\}} \int_{0}^{\bar{n}} \ln\left[c(j)(1-\delta(j))\right] dj + \int_{\bar{n}}^{N} \ln c(i)\, di$$

subject to

$$x = \int_{0}^{\bar{n}} p(j)\, c(j)(1+\tau)\, dj + \int_{\bar{n}}^{N} Qp^{*}(i)\, c(i)\left(1+\varphi^{C}\right) di, \tag{2.11}$$

where x is consumption expenditure per capita in monetary terms. Denoting by λ the multiplier for constraint (2.11), the first order conditions read

$$1 = \lambda p(j)\, c(j)(1+\tau) \quad \text{for each } j \in [0, \bar{n}), \tag{2.12}$$

$$1 = \lambda Q p^{*}(i)\, c(i)\left(1+\varphi^{C}\right) \quad \text{for each } i \in [\bar{n}, N]. \tag{2.13}$$

Combining these expressions to eliminate λ, we obtain identical expenditure levels for each good,

$$Qp^{*}(i)\, c(i)\left(1+\varphi^{C}\right) = \frac{x}{N} \text{ and } p(j)\, c(j)(1+\tau) = \frac{x}{N}. \tag{2.14}$$

Aggregation of goods by type of payment yields

$$\int_{0}^{\bar{n}} p(j)\, c(j)(1+\tau) .dj = \bar{n} \cdot p(j)\, c(j)(1+\tau) = \frac{\bar{n}}{N} \cdot x, \tag{2.15}$$

$$\int_{\bar{n}}^{N} Qp^{*}(i)\, c(i)\left(1+\varphi^{C}\right) di = (N-\bar{n}) \cdot Qp^{*}(i)\, c(i)\left(1+\varphi^{C}\right) = \frac{N-\bar{n}}{N} \cdot x. \tag{2.16}$$

[4] For simplicity, we assume a single exchange rate at which conversions take place. As documented in Makarov and Schoar (2021), crypto-currency activities can lead to persistent price differences across exchanges. The modeling approach in this chapter and the next aligns more closely with the subset of crypto-currencies known as stablecoins. Except in special cases, stablecoins have evolved with a fixed exchange rate relative to the fiat currency on which their value is based.

The individual income constraint of crypto-using consumers reads

$$x = w + rk_i + g \tag{2.17}$$

and has the same interpretation as (2.5). For future reference, note that result (2.14) implies

$$\frac{c(j)}{c(i)} = \frac{Qp^*(i)}{p(j)} \cdot \frac{1+\varphi^C}{1+\tau}. \tag{2.18}$$

Expression (2.18) is the relative demand of crypto-purchased versus money-purchased goods.

Conditions for using money versus crypto-currency. In order to choose the best payment option for a given good $n \in [0, N]$, the consumer compares opportunity costs in terms of utility. We can think of this choice as a sub-problem in which the consumer compares the utility levels enjoyed by spending a fixed amount of income $\hat{x}$ on good n using alternative payment methods, and then chooses the method yielding the highest utility. From (2.9) and (2.10), the hypothetical consumption levels attained under money- and crypto-payments are

$$\tilde{c}' = \frac{\hat{x}}{p(n) \cdot (1+\tau)} \quad \text{and} \quad \tilde{c}'' = \frac{\hat{x}}{Qp^*(n)\, c(n) \left(1+\varphi^C\right)}, \tag{2.19}$$

where $\tilde{c}'$ is purchased using money and $\tilde{c}''$ is purchased using the crypto-currency. Calculating the associated utility levels $u_n(\tilde{c}')$ and $u_n(\tilde{c}'')$ from (2.7)-(2.7), the welfare gap reads

$$u_n(\tilde{c}') - u_n(\tilde{c}'') = \ln\left[\frac{Qp^*(n)}{p(n)} \cdot \frac{\left(1+\varphi^C\right)(1-\delta(n))}{(1+\tau)}\right]. \tag{2.20}$$

Therefore, the condition for using the crypto-currency, $u_n(\tilde{c}') \leqslant u_n(\tilde{c}'')$, is

$$Qp^*(n) \cdot \left(1+\varphi^C\right) \leqslant p(n) \cdot \frac{1+\tau}{1-\delta(n)}. \tag{2.21}$$

Inequality (2.21) describes the situation in which *crypto-payments are superior to money-payments*: the utility cost of purchasing good n with money exceeds the utility cost of using the crypto-currency for the same purpose. When this inequality holds, consumers will use crypto-currency to purchase the n-th good.[5] When (2.21) is violated, they will use money.

[5] We are arbitrarily assuming that, in case of strict equality in (2.21), the indifferent consumer will opt for crypto-currency payment.

2.3.2.4 The threshold good

Under mild assumptions, there exists a unique good $n = \bar{n}$ acting as a threshold good – that is, $\bar{n}$ is interior and splits the mass of N goods in two non-empty subsets of goods purchased via different payment methods. On the demand side, the threshold condition is set by the consumers' indifference between money and crypto-currency payments: the threshold good $n = \bar{n}$ is characterized by (2.21) holding as a strict equality,

$$Qp^*(\bar{n}) \cdot \left(1 + \varphi^C\right) = p(\bar{n}) \cdot \frac{1+\tau}{1-\delta(\bar{n})}. \tag{2.22}$$

2.3.2.5 Goods producers: no-arbitrage pricing

On the supply side, the prices $p(n)$ and $p^*(n)$ must obey a no-arbitrage condition that makes firms producing goods indifferent between receiving money or crypto-currency payments. Assume perfect competition among producers in each sector $n \in [0, N]$ and full access of firms to both means of payment. Price-taking firms will adopt a combination of prices, $p(n)$ and $p^*(n)$, that yields zero profits irrespective of which currency is used in the transaction. Each unit of good sold yields either $p(n)$ units of money or $p^*(n)$ units of crypto-currency that the firm needs to re-convert into money in order to remunerate the factors of production. Hence, selling the good versus crypto-currency yields a net marginal revenue of $Qp^*\left(1 - \varphi^F\right)$ units of money – where the last term includes the re-conversion fee rate φ^F that the firm pays to the exchange platform for seeling crypto versus money. The *no-arbitrage condition for firms* thus reads

$$Qp^*(n) \cdot \left(1 - \varphi^F\right) = p(n) \quad \text{for each } n \in [0, N]. \tag{2.23}$$

2.3.2.6 Critical condition for the threshold good

Combining the supply-side condition (2.23) with the demand-side condition (2.22), we obtain the equilibrium condition for overall no-arbitrage between money-payments and crypto-payments for goods,

$$\frac{1+\varphi^C}{1-\varphi^F} = \frac{1+\tau}{1-\delta(\bar{n})}. \tag{2.24}$$

Expression (2.24) defines a unique fixed point $\bar{n}$ under a number of circumstances. In our model we posit the following linear relationship

$$\delta(n) = \beta \cdot (n/N) \qquad \text{with } 0 < \beta < 1. \tag{2.25}$$

Assumption (2.25) introduces a ranking within the mass of goods: $n \in [0, N]$ becomes an index that sorts consumption goods by increasing levels of private disutility from money use. Purchasing good $n = 0$ with money does not generate any direct disutility (besides the utility loss induced by pecuniary transaction costs, τ). Purchasing with money other goods bears increasing disutility as n increases. The last good in the list, $n = N$, carries the highest direct disutility from money payments, $\delta(N) = \beta$. The restriction $\beta < 1$ ensures $1 - \delta(N) > 0$, so that the associated utility level $u_N = \ln[c(N) \cdot (1 - \delta(N))]$ is well defined. From (2.24) and (2.25), we obtain the results summarized in the following

Proposition. The optimal payment method for any good $n \in [0, N]$ is determined by

$$\delta(n) < 1 - \frac{(1+\tau)\left(1-\varphi^F\right)}{1+\varphi^C} \Longrightarrow \text{Money} \tag{2.26}$$

$$\delta(n) \geqslant 1 - \frac{(1+\tau)\left(1-\varphi^F\right)}{1+\varphi^C} \Longrightarrow \text{Crypto-currency} \tag{2.27}$$

Given $\delta(n) = \beta \cdot (n/N)$, there are three possible scenarios. Scenario I: if $0 < \beta < 1 - \frac{(1+\tau)\left(1-\varphi^F\right)}{1+\varphi^C}$, all the N goods are purchased using money. Scenario II: if $1 - \frac{(1+\tau)\left(1-\varphi^F\right)}{1+\varphi^C} < 0 < \beta$, all the N goods are purchased using the crypto-currency. Scenario III: if

$$0 < 1 - \frac{(1+\tau)\left(1-\varphi^F\right)}{1+\varphi^C} < \beta \tag{2.28}$$

there exists a unique threshold good $\bar{n} \in (0, N)$ such that all goods $n \in [0, \bar{n})$ are purchased using money, whereas all goods $n \in [\bar{n}, N]$ are purchased using the crypto-currency. The threshold is determined by

$$\bar{n} = \frac{1+\varphi^C - (1+\tau)\left(1-\varphi^F\right)}{1+\varphi^C} \cdot \frac{N}{\beta} \tag{2.29}$$

Proof. Inequalities (2.26)-(2.27) follow directly by substituting (2.23) into (2.21) and solving for $\delta(n)$. Using (2.25) to substitute $\delta(n)$ in (2.24) yields result (2.29). Scenarios I-III follow from the parameter restrictions that would respectively imply $\bar{n} > N$, $\bar{n} < 0$ and $0 < \bar{n} < N$ in (2.29).

Scenario I arises when even good N, the one with the highest disutility from money payments, yields higher utility when purchased with money due to relatively high crypto-fees and/or relatively low transaction costs for money. Scenario II is the opposite case in which φ^C and φ^F are relatively low and/or τ is relatively high: it can only arise if pecuniary costs for money are strictly positive, $\tau > 0$, and strong enough to more than compensate for the effects of crypto-fees. Scenario III refers to equilibria with an *interior threshold good*, where both currencies (money and crypto) are used to purchase different subsets of goods. Expression (2.29) confirms the most intuitive properties of the threshold index, namely,

$$\frac{\partial \bar{n}}{\partial \varphi^C} > 0, \ \frac{\partial \bar{n}}{\partial \varphi^F} > 0, \ \text{and} \ \frac{\partial \bar{n}}{\partial \tau} < 0,$$

that is, the mass of goods exclusively purchased using money $\bar{n}$ is higher the higher the crypto-fee rates and the lower the consumption tax on money payments, τ.

2.3.3 Production of goods

Each good $n \in [0, N]$ is produced by an indefinitely large set of competitive firms – henceforth called 'sector n' – that take prices on input and output markets as given. Despite diminishing marginal returns at the firm level, learning-by-doing spillovers at the sectoral level induce constant marginal returns to capital – that is, a constant real interest rate – in the spirit of Romer (1986) and Romer (1989). Assuming identical technologies across producers of each good $n \in [0, N]$ guarantees a symmetric equilibrium where the economy's overall bundle of consumption goods is produced according to an AK technology.

Each firm exploits the production function $y(\cdot) = k(\cdot)^{\alpha} (\bar{a}\ell(\cdot))^{1-\alpha}$ where y is output, k is physical capital, ℓ is labor, $\bar{a}$ is workers' productivity, $\alpha \in (0, 1)$ is an elasticity parameter, and $(\cdot)$ stands for firms and/or sectoral indices to simplify the notation.[6] At the firm level, labor productivity is $\bar{a}$ is taken as given and profit maximization yields the usual first-order conditions

$$r = \alpha \frac{p(n) y(n)}{k(n)}, \tag{2.30}$$

$$w = (1-\alpha) \frac{p(n) y(n)}{\ell(n)}, \tag{2.31}$$

where $y(n)$ is total output of the n-th good, $k(n)$ and $\ell(n)$ are capital and labor used in the n-th sector, r is the market rental rate of capital and w is the prevailing wage rate.[7] At the sectoral level – i.e., across all producers of the n-th good – there are learning-by-doing spillovers whereby the use of capital increases workers' productivity.

[6] A complete notation would require to specify the number of firms producing good n and indexing inputs at the firm and at the sectoral levels accordingly. We avoid using the complete notation by discussing exclusively the functional forms that arise from Romer's (1986) model at the sectoral level – see Romer (1986) and Romer (1989) for details.

[7] We are assuming competitive input markets and fully mobile homogeneous inputs so that r and w are equalized across sectors producing different goods and are taken as given by each firm.

We postulate the spillover function $\bar{a} = A^{\frac{1}{1-\alpha}} \left(k\left(n\right) / \ell\left(n\right) \right)$ whereby the productivity of each worker increases with the capital-labor ratio in the relevant sector. The intuition is that capital use induces complementary efficiency gains: each worker uses machines and a more intense use of machines in the sector makes each unit of labor more efficient. The spillover function implies that sectoral output becomes linear in sectoral capital,

$$y\left(n\right) = Ak\left(n\right) \text{ for each } n \in \left[0, N\right], \tag{2.32}$$

like in standard growth models *á la* Romer (1989). Consequently, the equilibrium interest and wage rates are

$$r = p\left(n\right) \cdot \alpha A, \tag{2.33}$$

$$w = p\left(n\right) \cdot \left(1-\alpha\right) A \cdot k\left(n\right) / \ell\left(n\right) \tag{2.34}$$

Expressions (2.33)-(2.34) imply the standard functional distribution of income whereby capital rents capture a fraction α of output value while labor incomes capture the residual fraction,

$$p\left(n\right) y\left(n\right) = \underbrace{rk\left(n\right)}_{\alpha p(n) y(n)} + \underbrace{w\ell\left(n\right)}_{(1-\alpha) p(n) y(n)} \quad \text{for each } n \in \left[0, N\right]. \tag{2.35}$$

Importantly, the symmetric equilibrium produces price equalization and input-ratio equalization across sectors. As each firm satisfies (2.33) in each sector n, each good will be sold at the same price,

$$p\left(n\right) = p \text{ for each } n \in \left[0, N\right]. \tag{2.36}$$

Since the exchange rate Q is not good-specific, result (2.36) implies $p^*\left(n\right) = p^*$ for each $n \in [0, N]$ as well, which by firms' no-arbitrage pricing (2.23) implies

$$p^*\left(n\right) = p^*, \qquad p^* = \frac{p}{Q \cdot \left(1 - \varphi^F\right)} \quad \text{for each } n \in \left[0, N\right]. \tag{2.37}$$

Similarly, wage equalization across firms implies the same capital-labor ratio $k\left(n\right) / \ell\left(n\right)$ in each sector in view of (2.34). For future reference, we define

$$L^Y \equiv \int_0^{\bar{n}} \ell(n) dn + \int_{\bar{n}}^{N} \ell(n) dn, \tag{2.38}$$

where L^Y is total employment in the production of consumption goods.

2.3.4 Aggregate expenditures

Since p and p^* are identical across goods – see (2.36) and (2.37) – we can write real consumption indices by consumer type and payment type as follows:

$$\begin{cases} \tilde{c}(j) = \frac{1}{p(1+\tau)}\frac{\tilde{x}}{N} & \text{and} \quad c(j) = \frac{1}{p(1+\tau)}\frac{x}{N} \quad \text{for each } j \in [0, \bar{n}), \\ \tilde{c}(i) = \frac{1}{p(1+\tau)}\frac{\tilde{x}}{N} & \text{and} \quad c(i) = \frac{1}{Qp^*(1+\varphi^C)}\frac{x}{N} \quad \text{for each } i \in [\bar{n}, N], \end{cases} \tag{2.39}$$

where the expressions in the top row refer to non-binary goods (purchased by crypto-less and crypto-using consumers, respectively) and those in the bottom row refers to binary goods: the only crypto-purchased quantity is $c(i)$ with $i \in [\bar{n}, N]$.

Expressions (2.39) imply that all crypto-less consumers purchase the same amount of each good. Crypto-using consumers, instead, purchase different quantities depending on the means of payment. Under the assumed preferences, in particular, crypto-using individuals purchase *less* units of crypto-paid goods relative to the units of goods they purchase with money. The reason is a substitution effect: crypto-payments allow the consumer to avoid the non-pecuniary costs of money-purchases and thus yield more utility for each unit of good purchased. Under the assumed preferences, the ability to extract higher utility via non-pecuniary benefits prompts agents to *reduce* the purchased quantity $c(i)$ holding the goods' expenditure share unchanged, x/N, unchanged. In fact, the relative demand (2.18) and the indifference condition (2.22) with $p^*(\bar{n}) = p^*$ and $p(\bar{n}) = p$ imply

$$\frac{c(i)}{c(j)} = \frac{p(1+\tau)}{Qp^*(1+\varphi^C)} = (1-\delta(\bar{n})) < 1 \tag{2.40}$$

for each $j \in [0, \bar{n})$ and each $i \in [\bar{n}, N]$. By aggregating real indices over consumption goods, we obtain individually-purchased units of non-binary and binary goods,

$$\begin{cases} \int_0^{\bar{n}} \tilde{c}(j)\,dj = \frac{\bar{n}}{p(1+\tau)}\frac{\tilde{x}}{N} & \text{and} \quad \int_0^{\bar{n}} c(j)\,dj = \frac{\bar{n}}{p(1+\tau)}\frac{x}{N} \\ \int_{\bar{n}}^{N} \tilde{c}(i)\,di = \frac{N-\bar{n}}{p(1+\tau)}\frac{\tilde{x}}{N} & \text{and} \quad \int_{\bar{n}}^{N} c(i)\,di = \frac{N-\bar{n}}{Qp^*(1+\varphi^C)}\frac{x}{N} \end{cases}. \tag{2.41}$$

Individual expenditures for each type of agent read

$$\tilde{x} = p(1+\tau)\int_0^{\bar{n}} \tilde{c}(j)\,dj + p(1+\tau)\int_{\bar{n}}^{N} \tilde{c}(i)\,di, \tag{2.42}$$

$$x = p(1+\tau)\int_0^{\bar{n}} c(j)\,dj + Qp^*\left(1+\varphi^C\right)\int_{\bar{n}}^{N} c(i)\,di. \tag{2.43}$$

Multiplying by the relevant population size of each consumer category, $(1-\epsilon)L$ and

ϵL, we have that total spending in the economy is

$$X = (1-\epsilon) L\tilde{x} + \epsilon L x = \underbrace{Lp(1+\tau)\left[(1-\epsilon)\int_0^N \tilde{c}(n)\,dn + \epsilon\int_0^{\bar{n}} c(j)\,dj\right]}_{\text{gross spending on money-purchased goods } = X^m} + \underbrace{\epsilon LQp^*\left(1+\varphi^C\right)\int_{\bar{n}}^N c(i)\,di}_{\text{gross spending on crypto-purchased goods } = X^b} = X^m + X^b \tag{2.44}$$

which distinguishes between money-paid and crypto-paid goods and specifies that these are *gross* expenditures, that is, they include consumption taxes and fees paid to the exchange platform. For future reference, we cab rewrite aggregate gross spending on crypto-paid goods as

$$X^b = \epsilon LQp^*\left(1+\varphi^C\right)\int_{\bar{n}}^N c(i)\,di = Lp(1+\tau)\frac{\epsilon}{1-\delta(\bar{n})}\int_{\bar{n}}^N c(i)\,di \tag{2.45}$$

where the last term follows by substituting $Qp^*\left(1+\varphi^C\right) = p(1+\tau)/(1-\delta(\bar{n}))$ from (2.22). Hence, we can alternatively rewrite (2.44) as

$$X = Lp(1+\tau)\left[(1-\epsilon)\int_0^N \tilde{c}(n)\,dn + \epsilon\int_0^{\bar{n}} c(j)\,dj + \frac{\epsilon}{1-\delta(\bar{n})}\int_{\bar{n}}^N c(i)\,di\right], \tag{2.46}$$

which is, again, total spending in terms of money.

2.3.5 The exchange platform

We model the exchange platform as a competitive sector where an indefinite number of 'crypto-exchange firms' provide services to consumers and firms and bear the cost of validating these currency transactions. In this model, validation is the activity that crypto-exchange firms must perform in every exchange operation between crypto-currency and money – which includes both selling the crypto-currency to consumers and repurchasing it from final producers. The exchange platform as a whole purchases B^H units of the crypto-currency from crypto-extractors – which represents another sector employing labor: see next subsection – and employs L^H workers to perform validation activities. Since labor is homogeneous and fully mobile between the final goods' production sector and the exchange platform, the wage rate w will be equalized between these sectors.

At the aggregate level, the *monetary inflows* of the exchange platform are represented by fees charged on consumers selling money against crypto – that is, money inflows for the platform – and by fees charged on producing firms that sell crypto against money – that is, money retained from the outflows reaching final goods' producers:

$$\begin{aligned}\text{Platform money inflows} &= \epsilon L \cdot \left[Qp^{*}\left(1+\varphi^{C}\right) - Qp^{*}\left(1-\varphi^{F}\right)\right] \cdot \int_{\bar{n}}^{N} c\left(i\right) di = \\ &= \left(\varphi^{C}+\varphi^{F}\right) \cdot Q \cdot \epsilon L p^{*} \cdot \int_{\bar{n}}^{N} c\left(i\right) di = \\ &= \left(\varphi^{C}+\varphi^{F}\right) \cdot Q \cdot B^{H},\end{aligned} \tag{2.47}$$

where the last term comes from the fact that the stock of crypto-currency in circulation must match the consumers' total demand, $B^{H} = \epsilon L p^{*} \cdot \int_{\bar{n}}^{N} c\left(i\right) di$. The monetary outflows of the exchange platform comprise the sectoral wage bill, wL^{H}, and the monetary expenses QB^{H} associated with the purchases of crypto-currency from crypto-extractors at the wholesale exchange rate Q. This structure has two implicit assumptions. First, consumers cannot purchase the crypto-currency directly from the crypto-extractors, they need to go through crypto-exchange firms that invest the necessary amount of labor in validation activities. Second, the crypto-exchange firms' commitment to repurchase the crypto-currency in circulation from manufacturing firms versus money is honoured without any uncertainty – e.g., because such commitment is perfectly enforceable by rule of law. Zero profits in the crypto-sector thus require

$$\left(\varphi^{C}+\varphi^{F}\right) QB^{H} = wL^{H} + QB^{H}.$$

There are many ways to model the behavior of crypto-exchange firms consistently with the above zero-profit condition. The simplest structure hinges on linear returns to labor in validation activities. Suppose that the exchange platform comprises H competitive firms indexed by $h \in [0, H]$. Crypto-firm h purchases b_h units of crypto-currency from crypto-extractors at the wholesale rate Q, and hires $\ell_h^C + \ell_h^F$ workers to perform validation activities, where ℓ_h^C is the number of workers validating crypto-purchases by consumers and ℓ_h^F is the number of workers validating crypto-sales by manufacturing firms. The profits of the crypto-exchange firm thus read

$$\pi_h = Qb_h \cdot \left(\varphi^{C}+\varphi^{F}-1\right) - w\ell_h^C - w\ell_h^F.$$

The validation of crypto-transactions requires an amount of work time that depends on the number of crypto-currency units to be verified. Since both types of exchange transactions involve the same number of crypto-currency units, we can set without loss of generality $\ell_h^C = \ell_h^F = \ell_h^{\cdot}$ and, accordingly, $\varphi^C = \varphi^F = \varphi^{\cdot}$. Formally, suppose that the transfer of one unit of crypto-currency in either direction requires $\xi > 0$ units of labor, so that $\ell^{\cdot}_h = \xi b_h$. We can rewrite the profits of the crypto-exchange firm as

$$\pi_h = Qb_h \cdot (2\varphi^{\cdot} - 1) - w2\ell^{\cdot}_h = Qb_h \cdot (2\varphi^{\cdot} - 1) - w \cdot 2\xi b_h. \tag{2.48}$$

All firms take prices Q and w as given, and compete a la Bertrand in setting the fees $\varphi^{\cdot}$ which will result in the equality between marginal revenues and marginal costs. Given linear returns, each firm will charge the equilibrium fee rate associated with the zero profit condition $Q\,(2\varphi^{\cdot} - 1) = w2\xi$, that is,

$$\varphi^{\cdot} = \frac{1}{2} + \xi \cdot \frac{w}{Q}. \tag{2.49}$$

Note that in order to satisfy the restriction $\varphi^F < 1$, parameter ξ needs to satisfy ex-post the restriction $\xi < (1/2) \cdot (Q/w)$ in equilibrium. At the aggregate level, the zero profit condition reads

$$(2\varphi^{\cdot} - 1) \cdot QB^H = wL^H. \tag{2.50}$$

Substituting $\varphi^{\cdot}$ from (2.49) into (2.50) yields total labor employed in the exchange platform as a function of the total crypto-currency in circulation

$$L^H = 2\xi \cdot B^H, \tag{2.51}$$

where B^H is determined by crypto-extractors as discussed below.

2.3.6 Crypto-extractors and potential supply

The model incorporates an important distinction between crypto-currency in circulation, B^H, and potential crypto-currency supply. On the one hand, the total crypto-currency in circulation is the relevant notion of supply in the market-clearing condition for goods' transactions: B^H matches the total amount of crypto-currency units used by consumers in crypto-payments for goods,

$$\underbrace{B^H}_{\text{Crypto-currency in circulation}} = \underbrace{\epsilon L p^* \cdot \int_{\bar{n}}^{N} c\,(i)\, di.}_{\text{Transactive demand}} \tag{2.52}$$

On the other hand, the *potential supply of crypto-currency*, denoted by B^S, is a fixed number of crypto-currency units representing the stock from which the B^H units in circulation are extracted. In our model, B^S is an exogenous constant: the potential supply of crypto-currency can be thought of as a mass of virtual coins with no inherent value, costlessy created by an external entity – which bears similarities to money supply, M^S. However, differently from money supply, the potential supply of crypto-currency is not put into circulation for free and its the end-users of crypto-currency do not have direct access to B^S. Access is restricted to firms – henceforth called 'crypto-extracting firms' – that pay a fixed startup cost as well as variable "mining costs" to extract units that can be sold to crypto-exchange firms against money at the wholesale exchange rate Q. The number of crypto-extracting firms, S, is endogenously determined by free entry leading to a symmetric equilibrium with zero profits – i.e., a situation in which operative profits cover exactly the fixed startup cost.

Consider a single crypto-extracting firm indexed by $s \in [0, S]$. Setting up the firm incurs a fixed labor cost, ℓ_s^f, which can be thought of as real resources to be invested in obtaining access to extraction activities. The firm then hires ℓ_s^m workers to extract $b_s(\ell_s^m)$ units of crypto-currency from the stock B^S according to the technology

$$b_s(\ell_s^m) = \left(\bar{b} \cdot \ell_s^m\right)^{\varsigma}, \qquad 0 < \varsigma < 1, \tag{2.53}$$

where $\bar{b}$ is a labor efficiency parameter that firm s takes as given. The firm's profits read

$$\pi_s = Q \cdot b_s(\ell_s^m) - w\ell_s^m - w\ell_s^f = Q \cdot \left(\bar{b} \cdot \ell_s^m\right)^{\varsigma} - w\ell_s^m - w\ell_s^f \tag{2.54}$$

and the first order condition with respect to ℓ_s^m implies

$$\varsigma \cdot Q \cdot \left(\bar{b} \cdot \ell_s^m\right)^{\varsigma} = w\ell_s^m. \tag{2.55}$$

Substituting (2.55) back into the profit equation yields

$$\pi_s = (1 - \varsigma) \cdot Q \cdot \left(\bar{b} \cdot \ell_s^m\right)^{\varsigma} - w\ell_s^f. \tag{2.56}$$

The combination of price-taking behavior and decreasing marginal returns to mining, $\varsigma < 1$, generates potentially positive profits that more than compensate for the fixed cost $w\ell_s^f$. However, in this case, the entry of more firms in the extraction sector can squeeze profits by increasing the difficulty each firm faces in extraction until each firm makes zero profits. A simple way to model this outcome is to assume that the efficiency of each worker employed in extraction, $\bar{b}$, increases with the stock of potential supply B^S and decreases with the total number of workers competing for it,

$$\bar{b} \equiv \vartheta \frac{B^S}{\int_0^S \ell_s^m ds}, \qquad \vartheta > 0. \tag{2.57}$$

In a symmetric equilibrium where ℓ_s^m is the same for each firm, substitution of (2.57) into (2.53) yields the extraction level

$$b_s\left(\ell_s^m\right) = \left(\vartheta \frac{B^S}{S\ell_s^m} \cdot \ell_s^m\right)^{\varsigma} = \left(\vartheta \frac{B^S}{S}\right)^{\varsigma}. \tag{2.58}$$

Since per-firm extraction declines with the total number of firms, profits per firm decline with S. A symmetric zero-profit equilibrium will hold under free entry when the number of firm reaches the critcal level $S = \bar{S}$ given by

$$(1-\varsigma)\, Q \left(\vartheta \frac{B^S}{\bar{S}}\right)^{\varsigma} = w\ell_s^f \quad \rightarrow \quad \bar{S} = \vartheta B^S \cdot \left(\frac{1-\varsigma}{\ell_s^f} \cdot \frac{Q}{w}\right)^{\frac{1}{\varsigma}}. \tag{2.59}$$

Importantly, the amount of labour hired in extraction activities at the firm level is determined by the fixed cost: solving the first order condition (2.55) for ℓ_s^m yields

$$\ell_s^m = \varsigma \cdot \frac{Q}{w} \cdot \left(\vartheta \frac{B^S}{\bar{S}}\right)^{\varsigma} = \frac{\varsigma}{1-\varsigma} \cdot \ell_s^f. \tag{2.60}$$

Aggregating across firms, total employment is

$$L^S = \bar{S} \cdot \left(\ell_s^m + \ell_s^f\right) = \bar{S} \cdot \frac{\ell_s^f}{1-\varsigma} = \vartheta \left(\frac{1-\varsigma}{\ell_s^f}\right)^{\frac{1-\varsigma}{\varsigma}} B^S \cdot \left(\frac{Q}{w}\right)^{\frac{1}{\varsigma}}, \tag{2.61}$$

and the zero profit condition reads

$$w = \frac{B^H Q}{L^S}. \tag{2.62}$$

Total production can be written as

$$B^H = \bar{S} \cdot \left(\vartheta \frac{B^S}{\bar{S}}\right)^{\varsigma} = \left(\vartheta B^S\right)^{\varsigma} \left(\bar{S}\right)^{1-\varsigma} = \vartheta B^S \cdot \left(\frac{1-\varsigma}{\ell_s^f} \cdot \frac{Q}{w}\right)^{\frac{1-\varsigma}{\varsigma}}$$

which can be useful for future reference.

2.3.7 Aggregate income

Since there are no savings in this static economy, total expenditures must match total incomes. Distinguishing among sources of income by sector, we can rewrite aggregate incomes Y^i as

$$Y^i = w\left[L^H + L^s\right] + \left[wL^Y + rK\right] + Lg. \tag{2.63}$$

where the right hand side specifies the incomes received by workers employed in exchange and extracting activities, $w\left[L^H + L^s\right]$, and by owners of the inputs in goods' production, $wL^Y + rK$. The three components of total expenditures satisfy the following equations. First, the wage bill of the crypto-currency sector is determined by (3.12) together with the market-clearing and the zero profit conditions, respectively (2.52) and (2.62)

$$w\left[L^H + L^s\right] = \left(\varphi^C + \varphi^F\right) \cdot Q \cdot \epsilon L p^* \cdot \int_{\bar{n}}^{N} c(i)\, di. \tag{2.64}$$

Second, total factor payments to the inputs producing consumption goods equal the market value of the resulting output sold by firms,

$$wL^Y + rK = p(1-\epsilon) L \int_0^N \tilde{c}(n)\, dn + p\epsilon L \int_0^{\bar{n}} c(j)\, dj + \left(1 - \varphi^F\right) Q \epsilon L p^* \int_{\bar{n}}^{N} c(i)\, di. \tag{2.65}$$

Third, total net transfers to households consist of lump-sum tax rebates, i.e., the government revenues from the consumption tax applied to all money-purchased goods,

$$Lg = L\tau p(1-\epsilon) \int_0^N \tilde{c}(n)\, dn + L\tau p\epsilon \int_0^{\bar{n}} c(j)\, dj. \tag{2.66}$$

It can be easily verified that Y^i coincides with X in expression (2.46), that is, the aggregate constraint requiring total expenditures to match total incomes is satisfied. By substituting (2.64), (2.65) and (2.66) in (2.63), we have

$$\begin{aligned} Y^i &= Qp^*\left(1+\varphi^C\right) \epsilon L \int_{\bar{n}}^{N} c(i)\, di + \\ &\quad + p(1-\epsilon) L \int_0^N \tilde{c}(n)\, dn + p\epsilon L \int_0^{\bar{n}} c(j)\, dj + \\ &\quad + \tau \cdot p(1-\epsilon) L \int_0^N \tilde{c}(n)\, dn + \tau \cdot p\epsilon L \int_0^{\bar{n}} c(j)\, dj, \end{aligned}$$

where we can substitute $Qp^*\left(1+\varphi^C\right) = p(1+\tau) / (1-\delta(\bar{n}))$ from (2.22) to obtain

$$\begin{aligned} Y^i &= p(1+\tau) \cdot \frac{\epsilon}{1-\delta(\bar{n})} L \int_{\bar{n}}^{N} c(i)\, di + \\ &\quad + p(1+\tau)(1-\epsilon) L \int_0^N \tilde{c}(n)\, dn + p(1+\tau)\epsilon L \int_0^{\bar{n}} c(j)\, dj, \end{aligned}$$

that is,

$$Y^i = p(1+\tau)\left[(1-\epsilon) L \int_0^N \tilde{c}(n)\, dn + \epsilon L \int_0^{\bar{n}} c(j)\, dj + \frac{\epsilon}{1-\delta(\bar{n})} L \int_{\bar{n}}^{N} c(i)\, di\right], \tag{2.67}$$

where the right hand side of (2.67) coincides with the right hand side of (2.46), implying $Y^i = X$.

Henceforth, we assume that the government rebates the revenues of the consumption tax to all consumers without distinction, i.e., in transfers of equal amounts for both crypto-less and crypto-using consumers. Equal transfers per capita, $g = \tilde{g}$, imply equal income and expenditure levels across the two categories of consumers: from (2.5) and (2.17), we obtain $x = \tilde{x}$. This implies that all individuals consume the same units of money-purchased goods regardless of the consumer type: going back to (2.41), we have

$$\tilde{c}(i) = \tilde{c}(j) = c(j) = c^m \text{ for each } j \in [0, \bar{n}), \tag{2.68}$$

where $\tilde{c}(j)$ and $\tilde{c}(i)$ are purchased by crypto-less individuals, while $c(j)$ are purchased by individuals that have access to the crypto-currency but prefer to use money for such goods. Considering $c(i)$ with $i \in [\bar{n}, N]$, instead, the units of crypto-purchased goods are strictly less than c^m in view of (2.40), which then implies

$$c(i) = (1 - \delta(\bar{n})) \cdot c^m = c^b \text{ for each } i \in [\bar{n}, N]. \tag{2.69}$$

Results (2.68)-(2.69) imply that total income/expenditure can be written in more compact form as

$$X = Y^i = Lp(1 + \tau) N \cdot c^m, \tag{2.70}$$

which will be useful later.

2.4 Equilibrium

2.4.1 Money and Crypto-currency

Total money in circulation results from an exogenous rule whereby all factor incomes and net taxes must be paid in terms of *money* issued by the government. The rule says that the nominal money stock equals the value of total incomes, $M^s = Y^i$. As a consequence, real money supply (in terms of consumption goods) equals real income,

$$\frac{M^s}{p} = \frac{Y^i}{p} = L(1 + \tau) N \cdot c^m \tag{2.71}$$

where the last term comes from (2.70). As for the crypto-currency, the market clearing condition is (2.52) and can be rewritten using (2.68)-(2.69) in real terms as

$$\frac{B^H}{p^*} = \epsilon L \left(N - \bar{n}\right)\left(1 - \delta\left(\bar{n}\right)\right) \cdot c^m \tag{2.72}$$

which says that the supply of crypto-currency must match the (transactive) demand for crypto-currency.

The relevant equations for the real exchange rate between money and crypto-currency are the no-arbitrage equation for producers (2.23) and for consumers (2.22), which we report here for the sake of exposition,

$$\frac{Qp^*}{p} = \frac{1}{1 - \varphi^F} \tag{2.73}$$

$$\frac{Qp^*}{p} = \frac{1 + \tau}{\left(1 - \delta\left(\bar{n}\right)\right) \cdot \left(1 + \varphi^C\right)}. \tag{2.74}$$

2.4.2 Input markets

In equilibrium, the labour market clears so as to ensure that total labor supply matches total employment,

$$L = L^Y + L^H + L^S. \tag{2.75}$$

Similarly, the capital market requires

$$K = \int_0^{\bar{n}} k(j)dj + \int_{\bar{n}}^N k(i)di \tag{2.76}$$

From the profit-maximizing conditions of goods' producers, result (2.36) shows that goods' prices are symmetric, $p\left(n\right) = p$ for each n, so that every firm producing goods employs the same capital-labor ratio: from (2.34), we have

$$k\left(n\right) = \frac{1}{\left(1 - \alpha\right) A} \cdot \frac{w}{p} \cdot \ell\left(n\right) \text{ for each } n \in \left[0, N\right]. \tag{2.77}$$

Integrating both sides over the N goods, and substituting the aggregate labor and capital constraints (2.75) and (2.76), yields $K\left(1 - \alpha\right) A = \left(w/p\right) \cdot L^Y$, which we can solve for the equilibrium real wage as

$$\frac{w}{p} = \left(1 - \alpha\right) AK \cdot \frac{1}{L^Y}. \tag{2.78}$$

The relationship between the real wage and employment in the other sector where labor is employed, the exchange platform, follow from the zero profit condition for crypto-exchange firms in aggregate terms,

$$\frac{w}{p} = (2\varphi^{\cdot} - 1) \cdot \frac{QB^H}{p} \cdot \frac{1}{L^H}. \tag{2.79}$$

It is useful to remember that (2.79) combined with the equilbrium fee (2.49) yields $L^H = 2\xi \cdot B^H$ independently of labor demand of crypto-extracting firms. The latter, in the present variant of the model, reads

$$\frac{w}{p} = \frac{QB^H}{p} \cdot \frac{1}{L^S}. \tag{2.80}$$

2.4.3 Goods market equilibrium

The goods' market equilibrium is characterized by the aggregate budget constraint of goods-producing firms,

$$\int_0^N py\,(n)\,dn = wL^Y + rK, \tag{2.81}$$

and by the expenditure-income equality (2.70). Since $y\,(n) = Ak\,(n)$ for each $n \in [0, N]$ from result (2.32), total production of goods equals $\int_0^N py\,(n)\,dn = pAK$. Using this result to substitute the left hand side of (2.81), and using (2.70) to substitute the left hand side of (2.81), we obtain

$$pAK = Lp \cdot [N - \epsilon\,(N - \bar{n})\,\delta\,(\bar{n})] \cdot c^m,$$

from which the real consumption index c^m can be expressed as

$$c^m = \frac{AK}{L \cdot [N - \epsilon\,(N - \bar{n})\,\delta\,(\bar{n})]}. \tag{2.82}$$

We show now have all the necessary elements to solve for the key equilbrium variables: see below.

2.5 Solution procedure and numerical results

2.5.1 Reduced system

In order to solve for the equilibrium values of the endogenous variables, we can follow a two-step procedure. First, we build a reduced system that collects a subset of the equilibrium relationships to deliver solutions for a subset of endogenous variables. Second, we use the remaining equilibrium relationships – which are implicit in the reduced system – to determine all other endogenous variables. The reduced system reads

$$\bar{n} = \frac{1+\varphi-(1+\tau)(1-\varphi)}{1+\varphi}\cdot\frac{N}{\beta}, \tag{2.83}$$

$$\delta(\bar{n}) = \frac{\beta}{N}\cdot\bar{n} \tag{2.84}$$

$$c^m = \frac{AK}{L\cdot[N-\epsilon(N-\bar{n})\delta(\bar{n})]} \tag{2.85}$$

$$p = \frac{M^s}{L(1+\tau)N\cdot c^m} \tag{2.86}$$

$$\frac{QB^H}{p} = \frac{\epsilon L(N-\bar{n})(1-\delta(\bar{n}))}{1-\varphi}\cdot c^m \tag{2.87}$$

$$\frac{w}{p} = (1-\alpha)AK\cdot\frac{1}{L^Y} \tag{2.88}$$

$$\frac{w}{p} = (2\varphi-1)\cdot\frac{QB^H}{p}\cdot\frac{1}{L^H} \tag{2.89}$$

$$\frac{w}{p} = \frac{QB^H}{p}\cdot\frac{1}{L^S} \tag{2.90}$$

$$L = L^Y+L^H+L^S \tag{2.91}$$

$$L^S = \vartheta B^S\cdot\left(\frac{1-\varsigma}{\ell_s^f}\cdot\frac{Q}{w}\right)^{\frac{1}{\varsigma}}\cdot\frac{\ell_s^f}{1-\varsigma} \tag{2.92}$$

$$\varphi = \frac{1}{2}+\xi\cdot\frac{w}{Q} \tag{2.93}$$

Equations (2.83)-(2.84) determine the equilibrium threshold good and the associated critical level of $\delta(n)=\delta(\bar{n})$ according to Proposition 1. Equation (2.85) is the final goods' market clearing condition determining consumption of money-purchased goods. Equation (2.86) is the cash-in-advance constraint determining the money price of final goods for a given supply of money. Equation (2.87) is the market clearing condition in the crypto-currency market (2.72) – requiring that the crypto-currency in circulation matches transactive demand from consumers – rewritten in terms of money-purchased goods. Equations (2.88)-(2.90) are the labor demand schedules of, respectively, the final goods' producers, crypto-exchange firms, and crypto-extracting firms. Equation (2.91) is the labor market clearing condition. Equation (2.92) is the zero-profit condition for crypto-extracting firms. Equation (2.93) is the equilibrium fee charged by crypto-exchange firms, implying zero profits in the exchange platform. This system of 11 equations determines 11 endogenous variables, namely,

$$\bar{n}, \delta(\bar{n}), c^m, p, \frac{QB^H}{p}, \frac{w}{p}, L^Y, L^H, L^S, \frac{Q}{w}, \varphi. \tag{2.94}$$

The equilibrium values of $(p, \frac{w}{p}, \frac{Q}{w})$ imply solutions for the levels of the key nominal variables (p, w, Q). Combining these results with the equilibrium fee φ, we obtain the real exchange rate $\frac{1}{1-\varphi} = \frac{Qp^*}{p}$ and the price of crypto-purchased goods $p^* = \frac{1}{1-\varphi}\frac{p}{Q}$. From (2.69), the consumption of crypto-purchased goods is $c^b = (1 - \delta(\bar{n})) \cdot c^m$, and the amount crypto-currency in circulation is $B^H = \epsilon L (N - \bar{n})(1 - \delta(\bar{n})) \cdot p^* \cdot c^m$. Aggregate real expenditure is $X/p = L(1+\tau) N \cdot c^m$. Finally, we can calculate welfare – and more specifically, the utility levels of crypto-using and crypto-less consumers – as

$$\begin{aligned} U &\equiv \int_0^{\bar{n}} \ln[c(j)(1-\delta(j))]\, dj + \int_{\bar{n}}^N \ln c(i)\, di = \\ &= \bar{n} \cdot \ln c^m + \int_0^{\bar{n}} \ln\left(1 - \frac{\beta}{N} \cdot n\right) dn + (N - \bar{n}) \ln c^b \end{aligned} \tag{2.95}$$

and

$$\begin{aligned} \tilde{U} &\equiv \int_0^N \ln[\tilde{c}(n) \cdot (1 - \delta(n))]\, dn \\ &= N \cdot \ln c^m + \int_0^N \ln\left(1 - \frac{\beta}{N} \cdot n\right) dn, \end{aligned} \tag{2.96}$$

respectively. A numerical illustration is reported below.

2.5.2 Benchmark equilibrium: a numerical illustration

We use the reduced system (2.83)-(2.93) to numerically evaluate the equilibrium levels of the endogenous variables. The calibration of the exogenous constants in Table 1 adheres to the mathematical constraints underlying the construction of our one-period model. The baseline results are then reported in Table 1 and the subsequent ones. Figure 1 illustrates the variation in utility levels over the goods space for crypto-using and crypto-less consumers, as formulated in expressions 2.95 and 2.96. Overall, our baseline results align with the expected signs and magnitudes of the model's key variables. For instance, the threshold good $\bar{n}$ satisfies the restriction implied by the proposition in subsection 2.3.2.6. The distribution of the stock of goods purchased with either means of payment is consistent with real-world observations, where crypto-currencies are marginally used in goods transactions. In terms of utility gains, crypto-using consumers strictly dominate crypto-less consumers. The latter finding gives us a strong basis to study how various policy and non-policy decisions affect the welfare level of the two consumer types in our economy. Below, we assess the resulting effect of a partial change in some of the key exogenous constants around the benchmark equilibrium solutions.

Table 2.1: Exogenous constants and their values

Variable/Parameter	τ	N	β	M	L	ϵ	A	K	α	ς	l_s^f	B^S	ξ	ϑ
Value	0.05	50	0.75	1000	200	0.5	0.10	10000	0.4	0.25	10	1000	0.02	1

2.5.3 Policy shocks: money growth, crypto-currency supply and tax hike

Effects of money growth. Given the chosen exogenous constants, a 10% increase in money growth pushes up the market price of money-purchased goods and the nominal wage by a comparable proportion. This results in an appreciation of the nominal exchange rate between money and crypto-currency to maintain identical expenditure levels between crypto-less and crypto-using consumers. Real expenditures, demand for consumption goods, and overall consumer welfare remain unchanged under the studied monetary shock. The model's prediction aligns with the real business cycle literature, where an aggregate money supply shock affects only prices in the economy, in this case p and w.

Effects of crypto-currency growth. In equilibrium, a positive crypto-currency supply shock affects both the nominal and real variables of the model. Initially, the increase in crypto-currency circulation drives up the demand for exchange rate conversion service, leading to a rise in φ. The shock also triggers a rise in the market price of crypto-purchased goods. In response, consumers substitute crypto-purchased goods with money-purchased goods. This price inflation reduces real aggregate consumption in the economy and decreases real expenditures. In the labour sector, employment rises in the goods production and the crypto-exchange sectors whereas the extraction sector sees a halt in employment activity. Given that labour compensation is homogeneous across sectors, the lower employment activity in the extraction sector forces the labour market to clear at a lower nominal wage compared to the benchmark simulation. The crypto shock deteriorates welfare for both crypto-using and crypto-less consumers. The welfare loss for money-using consumers derives from consuming goods with an increased disutillity penalty. As explained above, the crypto price inflation enlarges the subset of goods purchased with money. The downside comes with the fact that each additional good added to the subset of money-purchased good bears higher disutility for the consumer and reduces the associated utility. In the case of crypto-using consumers, the welfare loss reflects the impact of inflation on the real consumption. Overall, the crypto-currency supply shock has real effects and impacts consumer welfare, as reflected in the changes in the utility levels in Table 2.

Effects of a consumption tax hike. A 10% consumption tax negatively impacts the quantity of money-purchased goods, putting downward pressure on the market price of supplied goods. The resulting positive income effect boosts the real consumption in the economy. However, the demand for crypto-purchased goods grow faster than money-purchased ones under the 10% tax increase. On the labour side, we observe a contraction in employment in goods production, leading to a decline in the nominal wage rate. Meanwhile, the increased demand for crypto-purchased goods combined with a higher demand for crypto-currencies raise employment in exchange and extraction activities. In aggregate terms, real expenditures rise due to the boost in real consumption. It is worth noting that the specified shock pushes down crypto fees. This explains a scale effect in the crypto sector where higher activities attract new entrants and drives down the market price for service. In conclusion, both consumer types enjoy higher utility as a result of stronger purchasing power as reported in Table 2.

Table 2.2: Benchmark results and policy shock analysis

Variable	**Baseline**	ΔM^S	ΔB^S	$\Delta\tau$
φ	0.5084	0.5084	0.5086	0.5083
$\bar{n}$	43.8506	43.8506	43.8633	43.7391
$\delta(\bar{n})$	0.6578	0.6578	0.6579	0.6561
c^m	0.1042	0.1042	0.1042	0.1043
c^b	0.0357	0.0357	0.0356	0.0359
p	0.9139	1.0052	0.9139	0.9089
p^*	0.2635	0.2635	0.2701	0.2616
Q	7.0554	7.7610	6.8852	7.0673
w	2.9488	3.2437	2.9487	2.9378
L^Y	185.9434	185.9434	185.9678	185.6357
L^H	0.2311	0.2311	0.2363	0.2349
L^S	13.8255	13.8255	13.7959	14.1294
B^H	5.7784	5.7784	5.9082	5.8734
$\bar{S}$	1.0369	1.0369	1.0347	1.0597
$\frac{w}{Q}$	0.4180	0.4180	0.4283	0.4157
$\frac{Qp^*}{p}$	2.0340	2.0340	2.0349	2.0338
$\frac{w}{p}$	3.2268	3.2268	3.2264	3.2321
$\frac{QB^H}{p}$	44.6120	44.6120	44.5105	45.6680
$\frac{B^H}{B^S}$	0.0058	0.0058	0.0054	0.0059
$\frac{X}{P}$	1094.2611	1094.2611	1094.1793	1100.1924
U	-139.0447	-139.0447	-139.0519	-138.9817
$\tilde{U}$	-139.9599	-139.9599	-139.9636	-139.9271

2.5.4 Population shocks versus crypto-currency access

Effects of an increase in L. An increase in population size exerts pressure on the labour market in the form of excess supply as reported in Table 3. The real wage shifts downward to reflect the combined effect of a lower nominal wage on the labour market and inflationary pressures stemming from crypto-purchased goods. The population shock increases the demand for the mass of goods purchased with crypto-currencies and lowers the crypto fees, through the same scale effect mechanism explained in the section above. Following the shock, real expenditures go up, reflecting the volume effect generated by a higher population size. On the crypto-currency side, the population growth leads to higher demand of currency for transactions. In a nutshell, a population shocks affects all segments of crypto-backed activities. The shock is welfare-deteriorating for both crypto-using and crypto-less consumers, which comes from the contraction in real consumption induced by the reduction in real purchasing power.

Effects of an increase in ϵ. A higher penetration of crypto-currencies as a means of payment increases the consumption of crypto-purchased goods. The market reacts by putting downward pressures on the price of money-purchased goods. A direct consequence of a positive shock to ϵ is a rise in employment in crypto-related activities, as shown in Table 3. In other words, the shock causes a shift in labour force from goods production to crypto-related occupations. As in the above analysis, higher activities in crypto-backed activities lower transaction fees charged by exchange platforms. The deflationary impact on both types of goods increases the purchasing power of consumers in the economy. A higher real wage drives up expenditures and positively affects the utility derived by different types of consumers.

Table 2.3: Benchmark results and population shock analysis

Variable	Baseline	ΔL	$\Delta\epsilon$
φ	0.5084	0.5082	0.5082
$\bar{n}$	43.8506	43.8382	43.8386
$\delta(\bar{n})$	0.6578	0.6576	0.6576
c^m	0.1042	0.0947	0.1047
c^b	0.0357	0.0324	0.0358
p	0.9139	0.9138	0.9099
p^*	0.2635	0.2827	0.2553
Q	7.0554	6.5725	7.2465
w	2.9488	2.6809	2.9582
L^Y	185.9434	204.5115	184.5604
L^H	0.2311	0.2487	0.2481
L^S	13.8255	15.2398	15.1915
B^H	5.7784	6.2163	6.2015
$\bar{S}$	1.0369	1.1430	1.1394
$\frac{w}{q}$	0.4180	0.4079	0.4082
$\frac{Qp^*}{p}$	2.0340	2.0332	2.0332
$\frac{w}{p}$	3.2268	2.9338	3.2510
$\frac{QB^H}{p}$	44.6120	44.7109	49.3871
$\frac{B^H}{B^S}$	0.0058	0.0062	0.0062
$\frac{X}{p}$	1094.2611	1094.3409	1098.9790
U	-139.0447	-143.8032	-138.8264
$\tilde{U}$	-139.9599	-144.7218	-139.7448

2.5.5 Technology shocks

Effects of an increase in A. The 10% productivity shock does not affect the distribution of goods across the different types of consumers as reported in table 4. The effects are similar to those observed in related neoclassical models, where technology-driven shocks improve production processes and increase aggregate output in the economy. The higher supply of goods exerts downward pressure on prices and strengthens real wages. This, in turn, increases real expenditures by a comparable percentage, leading to a positive impact on the utility enjoyed by different types of consumers.

Effects of an increase in ℓ_s^f*.* Higher fixed capital investment in the extraction activity reduces transaction fees on crypto-purchased goods. This can be interpreted as a network effect, where exchange platforms respond by improving their efficiency in validating crypto-backed transactions, thereby lowering costs for other economic agents in the economy. The increased investment triggers a reallocation of the labour force, leading to higher employment in the extraction industry. Nominal wages readjust to clear the labour market at a higher rate. As a result of the higher wage bill, some firms exit the extraction industry, negatively impacting the stock of crypto-currency in circulation. The combined effect of higher nominal wages and lower prices boosts aggregate consumption and expenditures. Overall, the investment shock improves the welfare of both crypto-using and crypto-less consumers.

Table 2.4: Benchmark results and technology shock analysis

Variable	**Baseline**	ΔA	Δl_s^f
φ	0.5084	0.5084	0.5078
$\bar{n}$	43.8506	43.8506	43.8143
$\delta(\bar{n})$	0.6578	0.6578	0.6572
c^m	0.1042	0.1146	0.1042
c^b	0.0357	0.0392	0.0357
p	0.9139	0.8308	0.9137
p^*	0.2635	0.2395	0.2445
Q	7.0554	7.0554	7.5910
w	2.9488	2.9488	2.9493
L^Y	185.9434	185.9434	185.8737
L^H	0.2311	0.2311	0.2162
L^S	13.8255	13.8255	13.9101
B^H	5.7784	5.7784	5.4044
$\bar{S}$	1.0369	1.0369	0.9484
$\frac{w}{q}$	0.4180	0.4180	0.3885
$\frac{Qp^*}{p}$	2.0340	2.0340	2.0316
$\frac{w}{p}$	3.2268	3.5495	3.2280
$\frac{QB^H}{p}$	44.6120	49.0731	44.9019
$\frac{B^H}{B^S}$	0.0058	0.0058	0.0054
$\frac{X}{p}$	1,094.2611	1,203.6872	1,094.4946
U	-139.0447	-134.2792	-139.0243
$\tilde{U}$	-139.9599	-135.1944	-139.9492

2.6 Conclusion

Our model sheds light on the dynamics of currency competition between fiat money and crypto-currencies, highlighting the conditions under which each form of currency is used for transactions. We demonstrate that the coexistence of both currencies hinges on the relative transaction costs, tax policies, and privacy benefits associated with each. The introduction of a threshold good, where consumers are indifferent between fiat and crypto-currency, reveals the intricate balance between the two currencies in the market. While fiat money remains neutral in our framework, crypto-currencies introduce real economic effects due to the costs associated with mining and its impact on labor distribution. Our findings suggest that the adoption of crypto-currencies has broader macroeconomic implications, particularly through changes in consumption patterns, labor reallocation, and sectoral employment. Future research should explore how these dynamics evolve over time and under different monetary policy regimes, considering the growing role of digital currencies in the global economy.

Chapter 3

A Dynamic Model of Crypto-currencies

Abstract

We analyze the coexistence of cash (fiat money) and privately-issued currencies (crypto-currencies) in a dynamic model where all factors of production are paid in fiat money. This introduces a cash-in-advance constraint that affects both consumption and investment, leading to non-neutrality of money. Crypto-currencies add distortions through labor reallocation and transaction fees. Using flexible utility specifications, we explore the impact of substitutability between money and crypto-purchased goods. Our main result is that an increase in the money supply raises inflation and shifts labor allocation, affecting growth dynamics. While broader economic variables remain stable, real wages are highly sensitive to changes in consumer preferences and crypto-fees, underscoring the impact of private digital currencies on the economy's long-term trajectory.

3.1 Introduction

We study the coexistence of fiat money and crypto-currencies in a dynamic model where all production factors are legally required to be compensated in fiat money, the legal tender. This legal requirement for all transactions to be conducted in a government-backed currency is equivalent to a comprehensive cash-in-advance (CIA) constraint, which mandates that money must cover both consumption expenditures and investment in physical capital. The introduction of physical capital investment and dynamic considerations allows us to capture the non-neutrality of both fiat money and crypto-currencies, which does not occur in models where both media of exchange serve purely transactional functions.

The CIA constraint also extends to exchanges of fiat money for crypto-currencies, which are then used to purchase crypto-paid consumption goods. We model the representative consumer's preference for either form of payment through different mathematical specifications of the instantaneous utility function. Initially, the consumer derives utility by combining money-purchased and crypto-purchased goods through a Cobb-Douglas function. Later, we generalize our analysis by considering a Constant Elasticity of Substitution (CES) framework, which offers more flexibility to study the limiting cases of complementarity and substitutability between fiat money and crypto-currencies.

Additionally, we abstract from explicitly modeling the pecuniary and non-pecuniary benefits of crypto-purchased goods, as was done in our static framework. Instead, we assume that consumers perceive money-purchased and crypto-purchased goods as yielding different utilities, even though firms view these goods as identical from a production standpoint.

Our first key result is that money is neither neutral nor super-neutral in our system. We provide both analytical and numerical derivations of this non-neutrality in section 3.8, within both the neoclassical and endogenous growth frameworks. A shock in the money supply growth rate propagates through the economy by raising inflation, which in turn reduces real consumption growth due to the increased cost of holding money. This affects capital accumulation and labor allocation as the CIA constraint forces adjustments in consumption and investment. The impact of this shock is amplified by the degree of substitutability between money-purchased and crypto-purchased goods, leading to shifts in real wages and sectoral labor allocation.

The introduction of crypto-currencies alters the dynamics of monetary non-neutrality in our model. While money is non-neutral due to the cash-in-advance (CIA) constraint on capital purchases, where inflation raises the cost of holding money, crypto-currencies add another layer of distortion. Labor is diverted from goods production to the operation of crypto-exchange platforms, reducing the capital-labor ratio and leading to under-accumulation of capital and lower consumption levels. Furthermore, crypto-currency fees distort the balance between money-purchased and crypto-purchased goods, creating additional inefficiencies in the steady state economy.

In subsection 3.8.3, we illustrate how key economic variables respond to changes in the elasticity of substitution between money-purchased and crypto-purchased goods over time. Despite steady growth in capital, output, and consumption, the elasticity of substitution has little effect on these broader growth trends, indicating that technological progress is the main driver of long-term economic growth. However, real wages are much more sensitive, rising faster when crypto-purchased goods are more easily substituted for money-purchased goods, as labor is allocated more efficiently. In contrast, with stronger complementarity between goods, wage growth is slower. Price levels remain stable, reflecting the relatively small impact of substitution on overall cost structures. These results highlight that while broader growth is driven by technology, labor markets and wages are more responsive to changes in consumer preferences.

This framework enables us to examine the interaction between fiat money and crypto-currencies within an economy and how legal and economic constraints shape the dynamic allocation of resources across sectors. Our analysis offers valuable insights into the effects of monetary policy and exchange platform fees on an economy's long-term growth trajectory, providing new perspectives on the role of private digital currencies in modern monetary systems.

As stated in the introduction, we use digital currencies to refer specifically to privately issued currencies or crypto-currencies, as opposed to fiat or state-backed currencies. This terminology aligns with the relevant literature, where crypto-currencies and digital currencies are often used interchangeably (Barrdear and Kumhof, 2016). Our primary focus is to assign a transactional role to crypto-currencies and analyze their impact on key economic outcomes in equilibrium.

The remainder of the paper progresses as follows. We first present the connection between our results and the existing literature in section 3.2. We then present our dynamic model in section 3.3. Then, we explore the solutions emerging from the optimization problem in section 3.4. Finally, we provide a numerical analysis to study the response of the endogenous variables following a shock to the exogenous constants in the final section.

3.2 Related Literature

This paper builds on the literature of private currencies, monetary economics, and currency competition. We start with the monetary model of Marchiori (2021), where a cash-in-advance constraint requires consumers to exchange part of their money holdings for crypto-currencies to purchase specific goods. While Marchiori (2021) focuses on Bitcoin supply growth, our analysis examines how exchange platform fees and consumer preferences for crypto-purchased goods affect the economy. In this sense, we offer a partial equilibrium analysis, abstracting from the mining sector, similar to other models in the literature (see Schilling and Uhlig (2019b), Lotz and Vasselin (2019), Benigno et al. (2022)).

Another important contribution to the literature is highlighting key policy implications regarding the crypto-money linkage. For instance, Schilling and Uhlig (2019) argue that welfare remains unaffected in a monetary model with crypto-currency price dynamics. However, we provide evidence in Table 3.4 and Table 3.9 that a shock to exchange platform transaction fees can distort welfare. The welfare level varies depending on the functional forms of the utility function. Fernández-Villaverde and Sanches (2019) characterizes the equilibrium welfare level in the presence of a private currency as wasteful, where the authority fails to provide the necessary amount of money for transactions. Our model does not include variables to make such a statement.

3.3 Model Setup

3.3.1 Cash-in-advance and dynamic budget constraints

Time is continuous and indexed by $t \in [0, \infty)$. The economy is populated by a constant number of L identical households purchasing consumption goods with a combination of fiat money issued by the government (henceforth, money) and crypto-currency purchased on exchange platforms. Each consumer is infinitely-lived and maximizes intertemporal lifetime utility

$$U \equiv \int_0^\infty e^{-\rho t} \ln \left[u \left(c_t^m, c_t^x \right) \right] dt = \int_0^\infty e^{-\rho t} \ln \left[\left(c_t^m \right)^\theta \left(c_t^x \right)^{1-\theta} \right] dt, \tag{3.1}$$

where $\rho > 0$ is the utility discount rate, and c_t^m and c_t^x indicate units of the consumption good purchased at time t by means of money and crypto-currency, respectively.

Money is printed costlessly by the government and transferred to households via lump-sum transfers. The single consumer uses money to purchase new capital, to directly purchase c_t^m units of output or to purchase units of crypto-currency that are then used to purchase c_t^x units of output. Denoting aggregate real investment in physical capital by $\dot{K}_t$, aggregate nominal money by M_t and the aggregate units of purchased crypto-currency by S_t, the CIA constraint that applies to money holdings at the individual level reads

$$\frac{M_t}{L} = P_t \frac{\dot{K}_t}{L} + P_t c_t^m + Q_t \left(1 + \delta_t\right) \frac{S_t}{L}, \tag{3.2}$$

where Q_t is the nominal exchange rate between money and crypto-currency and δ_t are fees paid by the household to acquire the crypto-currency from exchange platforms – i.e., the units of money needed by households to purchase one unit of crypto on the market are $Q_t \left(1 + \delta_t\right)$[1]. The budget for crypto-paid consumption goods is subject to the parallel *crypto-CIA constraint*

$$\frac{S_t}{L} = P_t^* c_t^x. \tag{3.3}$$

From (3.2) and (3.3), the *combined CIA constraint* reads

$$\frac{M_t}{L} = P_t \frac{\dot{K}_t}{L} + P_t c_t^m + Q_t \left(1 + \delta_t\right) P_t^* c_t^x. \tag{3.4}$$

Each household supplies labor (inelastically) to firms and owns a fraction $1/L$ of the existing capital stock K_t that firms use as an input in goods' production. The household dynamic budget constraint in money terms reads

$$P_t \frac{\dot{K}_t}{L} + \frac{\dot{M}_t}{L} = w_t + r_t \frac{K_t}{L} + \frac{V_t}{L} - P_t c_t^m - Q_t \left(1 + \varphi_t\right) P_t^* c_t^x \tag{3.5}$$

where w_t is the monetary wage rate, r_t is the rate of return to capital in terms of money, V_t equals aggregate lump-sum transfers from the government to all households. We henceforth normalize total population (workforce) to unity, $L = 1$, and transform (3.5) in real terms by defining real money as $m_t = M_t/P_t$, real money transfers as $v_t = V_t/P_t$, and the real exchange rate $q_t = \frac{Q_t P_t^*}{P_t}$, from which we obtain

$$\dot{K}_t + \dot{m}_t = \frac{1}{P_t}\left(w_t + r_t K_t\right) + v_t - c_t^m - q_t \left(1 + \delta_t\right) c_t^x - m_t \pi_t. \tag{3.6}$$

Similarly, the combined CIA constraint (3.4) can be rewritten as

$$m_t = \dot{K}_t + c_t^m + q_t \left(1 + \delta_t\right) c_t^x. \tag{3.7}$$

[1] Our model assumes a uniform exchange rate across crypto-currency exchanges, thereby excluding potential frictions that affect price discovery in the CM. This assumption aligns more closely with a special class of crypto-currencies known as stablecoins.

The household problem consists of maximizing present-value utility (3.1) subject to the constraints (3.6) and (3.7). As usual in the literature, we postulate that CIA constraints hold as strict equalities – i.e., both the CIA constraint on money (3.2) and the CIA constraint on the crypto-currency (3.3) are binding because both currencies are strictly dominated by physical capital in terms of rate of returns. We are thus focusing on environments where the rate of money deflation ($-\pi_t = -\dot{P}_t/P_t$) is smaller than the market rental rate of capital – which is the case in any economy with positive inflation – and where agents do not accumulate crypto-currency as an asset – which is guaranteed by a similar return-dominance condition that we will formulate and impose ex post via parameter restrictions.

3.3.2 Production

All consumption goods are produced with the same constant returns to technology by a competitive sector: total final output equals $Y_t = F\left(K_t, a_t L_t^y\right)$, where L^y is labor employed in goods production and a_t is labor productivity. Real output is sold to households either as consumption or as new physical capital:

$$F\left(K_t, a_t L_t^y\right) = c_t^m + c_t^x + \dot{K}_t = C_t + \dot{K}_t \tag{3.8}$$

where we have defined $C_t \equiv Lc_t^m + Lc_t^x$ as aggregate real consumption and L is normalized to unity. From the producers point of view, there is perfect substitutability among the three uses of the final good, which implies price equalization: by no-arbitrage logic, each unit of output must yield P_t units of money. New capital and money-paid consumption goods indeed have the same money price P_t. For crypto-paid consumption goods, the production sector will charge a crypto-price P_t^* that generates the same unit revenue after conversion.

After selling c_t^x units against crypto-currency, producers will convert the associated crypto-payments into money so as to compensate production factors. Converting the $P_t^* c_t^x$ units of crypto received from customers into money involves paying a proportional fee to exchange platforms. We set the fee rate for firms equal to δ_t, the same fee rate paid by households acquiring the crypto-currency. The net money revenue from selling crypto-paid goods thus equals $Q_t\left(1-\delta_t\right) P_t^* c_t^x$ in terms of money. By no-arbitrage with the revenue that firms would obtain by selling the same units against money, $P_t c_t^x$, it follows that the crypto-price of crypto-purchased goods equals

$$P_t^* = \frac{P_t}{Q_t\left(1-\delta_t\right)}. \tag{3.9}$$

The total profits of the final sector in terms of money can thus be written as

$P_tF(K_t, a_tL_t^y) - r_tK_t - w_tL_t^y$, and constant returns to scale imply the zero-profit condition

$$P_tF(K_t, a_tL_t^y) = r_tK_t + w_tL_t^y = P_tC_t + P_t\dot{K}_t. \tag{3.10}$$

Note that the no-arbitrage condition (3.9) implies that the real exchange rate equals

$$q_t = \frac{Q_tP_t^*}{P_t} = \frac{1}{1-\delta_t}, \tag{3.11}$$

so that positive growth in exchange fees implies a real appreciation of the crypto-currency.

3.3.3 Exchange platform

We model the exchange platform as a competitive sector where an indefinite number of 'crypto-exchange firms' provide services to consumers and firms and bear the cost of validating these currency transactions. In this model, validation is the activity that crypto-exchange firms must perform in every exchange operation between crypto-currency and money – which includes both selling the crypto-currency to consumers and repurchasing it from final producers. The exchange platform as a whole trades $S_t = P_t^*c_t^x$ units of the crypto-currency and employs $1 - L_t^y$ workers to perform validation activities. Since labor is homogeneous and fully mobile between the final goods' production sector and the exchange platform, the wage rate w_t will be equalized between these sectors.

At the aggregate level, the *monetary inflows* of the exchange platform are represented by fees charged on consumers selling money against crypto – that is, money inflows for the platform – and by fees charged on producing firms that sell crypto against money – that is, money retained from the outflows reaching final goods' producers:

$$\begin{aligned}\text{Platform money inflows} &= QP_t^*(1+\delta_t)c_t^x - QP_t^*(1+\delta_t)c_t^x = \\ &= 2\delta_t \cdot Q_t \cdot \underbrace{P_t^*c_t^x}_{S_t}.\end{aligned} \tag{3.12}$$

The monetary outflows of the exchange platform equal the wage bill, $w_t(1 - L_t^y)$. Zero profits in the crypto-sector thus require

$$2\delta_tQ_t \cdot \underbrace{P_t^*c_t^x}_{S_t} = w_t(1 - L_t^y). \tag{3.13}$$

There are many ways to model the behavior of crypto-exchange firms consistently with the above zero-profit condition. We leave this part of the model unspecified for the sake of generality: as we show in sections 3.5 and 3.6, many relevant results can be established, for different variants of the model, by simply imposing the zero-profit condition (3.13) without assuming a specific technology for the exchange platform. Further results for the main variants of the model – the 'neoclassical case' and the 'AK case' – will be obtained later under specific technology assumptions for both goods production and exchange platform.

3.3.4 Aggregate constraints and equivalence

This subsection briefly (i) derives the aggregate constraint of the economy in money terms and (ii) verifies the equivalence between the CIA constraint imposed on expenditures (3.7) and the legal requirement that all factor incomes must be paid using the legal tender.

(i) Aggregate constraint of the economy in money terms. By combining the zero profit condition of the production sector (3.10) with the household budget constraint (3.6), we obtain

$$\dot{K}_t = F\left(K_t, a_t L_t^y\right) + \frac{w_t\left(1 - L_t^y\right)}{P_t} - c_t^m - q_t\left(1 + \delta_t\right) c_t^x + \left(v_t - m_t \pi_t - \dot{m}_t\right) \qquad (3.14)$$

Using the definition of real exchange rate $q_t = \frac{Q_t P_t^*}{P_t}$, we can rewrite the zero profit condition for the exchange platform (3.13) as

$$2\delta_t q_t P_t c_t^x = w_t\left(1 - L_t^y\right). \qquad (3.15)$$

Substituting (3.15) in (3.14) and rearranging terms yields

$$\dot{K}_t = F\left(K_t, a_t L_t^y\right) - c_t^m - q_t\left(1 - \delta_t\right) c_t^x + \left(v_t - m_t \pi_t - \dot{m}_t\right)$$

which, after substituting $q_t\left(1 - \delta_t\right) = 1$ from (3.11), becomes

$$P_t \dot{K}_t + P_t c_t^m + P_t c_t^x = P_t F\left(K_t, a_t L_t^y\right) + P_t\left[v_t - m_t \pi_t - \dot{m}_t\right]. \qquad (3.16)$$

Expression (3.16) is the aggregate resource constraint of the economy in money terms. In real terms, it reduces to the goods market clearing condition because, under a binding CIA constraint, real money transfers v_t represent the increase in real money holdings,

$$v_t = \frac{\dot{M}_t}{P_t} = \frac{\dot{M}_t}{M_t} \cdot m_t = \left(\frac{\dot{m}_t}{m_t} + \pi_t \right) \cdot m_t = \dot{m}_t + m_t \pi_t, \tag{3.17}$$

so that the last term in square brackets in (3.16) cancels out, and dividing boths sides by P_t yields the market clearing condition (3.8).

(ii) Equivalence between CIA constraint and legal requirement on factor payments. Recalling the CIA constraint in real terms, use (3.11) to rewrite (3.7) as

$$m_t = \dot{K}_t + c_t^m + \frac{1+\delta_t}{1-\delta_t} c_t^x, \tag{3.18}$$

which, after some manipulation, yields

$$m_t = \dot{K}_t + c_t^m + c_t^x + \frac{2\delta_t}{1-\delta_t} \cdot c_t^x. \tag{3.19}$$

Multiplying both sides of (3.19) by P_t and using again (3.11) yields

$$M_t = \underbrace{P_t \dot{K}_t + P_t c_t^m + P_t c_t^x}_{r_t K_t + w_t L_t^y} + \underbrace{2\delta_t \cdot q_t P_t c_t^x}_{w_t \left(1 - L_t^y\right)} \tag{3.20}$$

where the last term coincides with the exchange plarform's total wage bill by the zero profit condition (3.15). Result (3.20) confirms that the CIA constraint imposed on expenditures (3.7) is equivalent to the assumed legal requirement that all factor incomes must be paid using money, the legal tender.

3.4 Intertemporal choices and equilibrium notions

3.4.1 Utility maximizing conditions

The household problem consists of maximizing present-value utility (3.1) subject to the constraints (3.6) and (3.7). The current-value Hamiltonian for the household problem can be written as

$$\begin{aligned} &= \ln\left[\left(c_t^m\right)^\theta \left(c_t^x\right)^{1-\theta} \right] + \lambda_t^K \dot{K}_t + \lambda_t^M \dot{m}_t + \lambda_t^S \left[m_t - \dot{K}_t - c_t^m - q_t \left(1+\delta_t\right) c_t^x \right] = \\ &= \ln\left[\left(c_t^m\right)^\theta \left(c_t^x\right)^{1-\theta} \right] + \lambda_t^K I_t + \lambda_t^M \dot{m}_t + \lambda_t^S \left[m_t - I_t - c_t^m - q_t \left(1+\delta_t\right) c_t^x \right], \end{aligned} \tag{3.21}$$

where we have defined capital investment as $\dot{K} = I$. This allows us to treat real money m and capital K as state variables while c_t^m, c_t^x and I_t act as control variables; all prices are taken as given under perfect foresight, λ_t^K is the shadow price of capital accumulation, λ_t^M is the shadow price of money accumulation, and λ_t^S is the Khun-Tucker multiplier attached to the CIA constraint. Replacing $\dot{m}_t$ by means of expression (3.6), and collecting terms for $\dot{K} = I$, the Hamiltonian (3.21) can be rewritten as

$$\begin{aligned} = \; & \ln\left[\left(c_t^m\right)^\theta \left(c_t^x\right)^{1-\theta}\right] + \left(\lambda_t^K - \lambda_t^S - \lambda_t^M\right) \cdot I_t + \\ & +\lambda_t^M \left[\frac{1}{P_t}\left(w_t + r_t K_t\right) + v_t - c_t^m - q_t\left(1+\delta_t\right) c_t^x - m_t \pi_t\right] + \\ & +\lambda_t^S \cdot \left[m_t - c_t^m - q_t\left(1+\delta_t\right) c_t^x\right]. \end{aligned} \tag{3.22}$$

The necessary conditions for utility maximization are therefore

$$\begin{aligned} c_t^m &= 0 & &\rightarrow \frac{\theta}{c_t^m} - \lambda_t^M - \lambda_t^S = 0 \\ c_t^x &= 0 & &\rightarrow \frac{1-\theta}{c_t^x} - \left(\lambda_t^M - \lambda_t^S\right) q_t\left(1+\delta_t\right) = 0 \\ I_t &= 0 & &\rightarrow \lambda_t^K - \lambda_t^S = \lambda_t^M \\ K_t &= \rho\lambda_t^K - \dot{\lambda}_t^K & &\rightarrow \rho\lambda_t^K - \dot{\lambda}_t^K = \lambda_t^M \frac{r_t}{P_t} \\ M_t &= \rho\lambda_t^M - \dot{\lambda}_t^M & &\rightarrow \rho\lambda_t^M - \dot{\lambda}_t^M = \lambda_t^S - \lambda_t^M \pi_t \end{aligned}$$

along with the transversality conditions

$$\lim_{t\to\infty} \lambda_t^K K_t e^{-\rho t} = 0, \tag{3.23}$$

$$\lim_{t\to\infty} \lambda_t^M m_t e^{-\rho t} = 0. \tag{3.24}$$

For future reference, we can rewrite the utility-maximizing conditions as

$$\frac{\theta}{c_t^m} = \lambda_t^K \tag{3.25}$$

$$\frac{1-\theta}{c_t^x} = \lambda_t^K \cdot q_t\left(1+\delta_t\right) \tag{3.26}$$

$$\lambda_t^K = \lambda_t^M + \lambda_t^S \tag{3.27}$$

$$\frac{\dot{\lambda}_t^K}{\lambda_t^K} = \rho - \frac{\lambda_t^M}{\lambda_t^K} \cdot \frac{r_t}{P_t} \tag{3.28}$$

$$\frac{\dot{\lambda}_t^M}{\lambda_t^M} = \rho + \pi_t - \frac{\lambda_t^K - \lambda_t^M}{\lambda_t^M} \tag{3.29}$$

Equations (3.25) and (3.26) imply that the ratio between money-paid and crypto-paid consumption goods is determined by tastes and by the gross real exchange rate between money and crypto-currency:

$$\frac{c_t^m}{c_t^x} = \frac{\theta}{1-\theta} \cdot q_t \left(1+\delta_t\right) = \frac{\theta}{1-\theta} \cdot \frac{1+\delta_t}{1-\delta_t} \tag{3.30}$$

where the last term follows from (3.11). As intuitive, the share of consumption in money-purchased goods increases with higher crypto-fees.

The co-state equations (3.28)-(3.29) can be reduced to a single differential equation by defining the composite multiplier $\lambda_t^R \equiv \lambda_t^M / \lambda_t^K$, which evolves over time according to

$$\frac{\dot{\lambda}_t^R}{\lambda_t^R} = \frac{\dot{\lambda}_t^M}{\lambda_t^M} - \frac{\dot{\lambda}_t^K}{\lambda_t^K} = \pi_t - \frac{\lambda_t^K - \lambda_t^M}{\lambda_t^M} + \frac{\lambda_t^M}{\lambda_t^K} \cdot \frac{r_t}{P_t}$$

that is,

$$\frac{\dot{\lambda}_t^R}{\lambda_t^R} = \pi_t - \frac{1-\lambda_t^R}{\lambda_t^R} + \lambda_t^R \cdot \frac{r_t}{P_t}. \tag{3.31}$$

Equation (3.31) determines the joint dynamics of the shadow values of money and capital and will be used later to determine the properties of long-run equilibria.

3.4.2 Steady-state and BGP equilibria

The work-horses of dynamic macroeconomics suggest considering two reference notions of long-run equilibria. The first characterizes models of exogenous growth, i.e., Ramsey-like economies where diminishing returns to capital drive down the interest rate over time and imply that, in the absence of productivity growth, consumption and output per capita are stationary in the long run. In the present context, if we assume that the production function of the final sector exhibits diminishing returns to capital (at the firm and at the aggregate level) alongside a constant exogenous level of labor productivity, we obtain a Ramsey-like economy that should, at least in principle, admit a *steady-state equilibrium* in the long run characterized by constant consumption levels. We investigate this point in the first variant of our model, which we label as the 'neoclassical case'.

The second variant of the model is suggested by the endogenous growth literature. In this class of models, the economy's rate of return is sustained in the long run by endogenous forces that eliminate strictly diminishing returns to accumulable factors, implying persistent consumption growth in the long run. In the present context, if we assume that the production function of the final sector incorporates learning-by-doing spillovers through labor productivity – whereby capital exhibits diminishing returns at the firm level but non-diminishing returns at the aggregate level – we obtain a Romer-like economy (Romer (1989)) that should, at least in principle, admit a *balanced growth path equilibrium* delivering sustained growth in consumption and output in the long run. We will refer to this variant of the model as to the 'AK case'.

3.5 The Neoclassical case

This section describes the general properties of the steady state equilibrium. These properties hold regardless of the specific technology used by the exchange platform and are generally valid for any static CRS production function in goods production (i.e., a linearly homogeneous technology with constant labor productivity: $a_t = a > 0$). In section 3.8 we will specify technologies for both the exchange platform and the goods sector to derive further results on the impact of technology shocks.

3.5.1 Consumption and money non-neutrality

Consider an equilibrium with constant consumption. From (3.25) and (3.26), stationarity in c_t^m and c_t^x requires a constant multiplier λ_t^K as well as constant crypto-fees,

$$\frac{\mathrm{d}}{\mathrm{d}t} q_t \left(1 + \delta_t\right) = \frac{\mathrm{d}}{\mathrm{d}t} \frac{1 + \delta_t}{1 - \delta_t} = 0,$$

which will be the case for suitable specifications of the technology of the exchange platform. From (3.28), the steady state $\dot{\lambda}_t^K = 0$ requires that the real rental rate for capital equals the utility dicount rate weighted by the composite multiplier $\lambda_t^R \equiv \lambda_t^M / \lambda_t^K$ previously defined,

$$\frac{r_t}{P_t} = \frac{1}{\lambda_t^R} \cdot \rho. \tag{3.32}$$

Since ρ is constant and r_t/P_t equals the physical marginal product of capital, a constant real interest rate requires $\dot{\lambda}_t^R = 0$ in (3.31), which yields

$$\lambda_t^R = \frac{1}{1 + \rho + \pi_t}. \tag{3.33}$$

By combining (3.32) with (3.33), a steady-state equilibrium in the neoclassical case is characterized by the real rate of return

$$\frac{r_t}{P_t} = \rho \cdot \left(1 + \rho + \pi_t\right). \tag{3.34}$$

Expression (3.34) shows three important results. First, money is not neutral: a nominal variable – the money inflation rate, π_t – affects real variables in equilibrium – the physical marginal product of capital, r_t/P_t. Second, a neoclassical steady state with constant real interest requires the inflation rate to be constant over time, which in turn imposes a restriction on monetary growth (i.e., a constant money growth rate set by the authority). Third, inflation tends to reduce capital accumulation. In the standard Ramsey model without a CIA constraint on capital purchases, the steady-state condition $r_t/P_t = \rho$ implies a lower interest rate and a higher capital-labor ratio than condition (3.34) – provided that the money inflation rate is $\pi_t > -\rho$. In other words, unless we observe substantial deflation, the cash-in-advance constraint implies under-accumulation of capital and inefficiently low consumption.

The economic intuition for non-neutrality of money is that the CIA constraint on new capital purchases forces agents to keep money to make real investment but positive inflation increases the real cost of holding money, which affects the real return to investment from the household point of view. This source of non-neutrality does not apply to the crypto-currency – in fact, we have not postulated that cyrpto-currency is necessary to purchase real investment. The crypto-currency is non-neutral for other reasons, namely, the fact that its circulation absorbs real resources (in the form of labor employed in exchange platforms). This point is clarified below.

3.5.2 Non-neutralities: money versus crypto

In order to assess the role of the crypto-currency, impose the conditions for a neoclassical steady state in the CIA constraint: setting $\dot{K}_t = 0$ in (3.7), we obtain

$$m_t = c_t^m + \frac{1+\delta_t}{1-\delta_t} c_t^x = \frac{1}{\theta} c_t^m \tag{3.35}$$

where the last term follows from substituting the utility-maximizing consumption ratio (3.30). From (3.35), a steady state in consumption implies a steady state in real money supply, $\dot{m}_t = 0$, which means that the money inflation rate equals the growth rate of money supply. Assuming that the monetary authority lets nominal grow at the constant rate g^M, the inflation rate is constant as well,

$$\pi_t = \dot{M}_t/M_t \equiv g^M. \tag{3.36}$$

Since π_t only depends on money growth, the dynamics of the supply of crypto-currency do not affect the steady-state condition (3.34) through this channel: money inflation is independent of crypto inflation. However, the existence of the crypto-market does affect the real interest rate in (3.34) through a *labor reallocation effect*. In a neoclassical world, the physical marginal product of capital depends on the capital-labor ratio in goods' production, K_t/L_t^y, and L_t^y is in turn affected by the fact that part of the workforce, $L - L_t^y$, is at the same time employed in exchange platforms. Since the crypto-market subtracts resources – in this case, labor inputs – that would have been otherwise used in goods production, the existence of the crypto-currency exherts an additional pressure towards under-accumulation and inefficiently low consumption levels in the steady state. In fact, if all the workforce L could be employed in goods production, condition (3.34) would be met with an identical capital-labor ratio – say, $K_t'/L = K_t/L_t^y$ – but such ratio would be associated to higher levels of capital and output ($L > L_t^y$ would imply $K_t' > K_t$).

Another effect of the crypto-market is that the fee rate δ_t distorts the relative expenditure shares of money-purchased and crypto-purchased goods, which is immediately evident from (3.30). In this respect, the extent of the distortion depends on the technology of the exchange platform and on the resulting level of fees. We will present a complete analytical derivation of the balanced growth equilibrium under a specific technology for the exchange platform in section 3.8.

3.6 The AK case

Assume that the final good sector comprises an indefinite number of firms exploiting the same technology displaying constant returns to scale at the firm level. Despite diminishing marginal returns to both labor and capital at the firm level, learning-by-doing spillovers at the sectoral level induce constant marginal returns to capital – that is, a constant real interest rate – in the spirit of Romer (1986) and Romer (1989). Assuming identical technologies across firms guarantees a symmetric equilibrium where the economy's final consumption good is produced according to an AK technology.

3.6.1 Goods production with spillovers

Assume that the final good sector comprises an indefinite number of firms indexed by n. Each firm exploits the production function $y_{n,t} = {k_{n,t}}^{\alpha} \left(\bar{a}_t \ell_{n,t}^y\right)^{1-\alpha}$ where $y_{n,t}$ is output, $k_{n,t}$ is physical capital, $\ell_{n,t}^y$ is labor, $\bar{a}_t$ is workers' productivity, $\alpha \in (0,1)$ is an elasticity parameter. At the firm level, labor productivity $\bar{a}_t$ is taken as given, and profit maximization yields the usual first-order conditions

$$\frac{r_t}{P_t} = \alpha \frac{y_{n,t}}{k_{n,t}}, \quad (3.37)$$

$$\frac{w_t}{P_t} = (1-\alpha) \frac{y_{n,t}}{\ell_{n,t}^y}, \quad (3.38)$$

Since firms use identical technologies, the capital-labor ratio is the same in each firm and coincides with the capital-labor ratio at the sectoral level, $k_{n,t}/\ell_{n,t}^y = K_t/L_t^y$. Assume learning-by-doing spillovers at the sectoral level whereby the use of capital increases workers' productivity. We postulate the spillover function $\bar{a} = A^{\frac{1}{1-\alpha}} \left(K_t/L_t^y\right)$, which implies that the productivity of each worker increases with the capital intensity of the sector. The intuition is that capital use induces complementary efficiency gains: each worker uses machines and a more intense use of machines in the sector makes each unit of labor more efficient. Substituting the spillover function in firms' technologies, sectoral output becomes linear in sectoral capital,

$$Y_t = AK_t, \tag{3.39}$$

like in standard growth models *á la* Romer (1989). Consequently, the equilibrium interest and wage rates are

$$\frac{r_t}{P_t} = \alpha A, \tag{3.40}$$

$$\frac{w_t}{P_t} = (1-\alpha) \cdot A \cdot \left(K_t / L_t^y\right). \tag{3.41}$$

The fact that the real return to capital αA is constant creates the possibility of balanced growth paths (BGPs), that is, scenarios in which the economy exhibits sustained endogenous growth in the long run. Given the non-neutrality of money and crypto-currency, the natural question is whether nominal variables will affect not only income levels but also income growth in the long run. The next subsection tackles this issue in general terms without assuming a specific technology for the exchange platform.

3.6.2 Balanced growth equilibrium: general properties

Consider a balanced growth equilibrium where consumption levels of both goods grow at the constant rate

$$g^C = \frac{\dot{c}_t^m}{c_t^m} = \frac{\dot{c}_t^x}{c_t^x}$$

and there is a constant rate of crypto-fees, $\dot{\delta}_t = 0$, which will be the case for suitable specifications of the technology of the exchange platform. From (3.28), a constant growth rate $-\dot{\lambda}_t^K / \lambda_t^K = g^C > 0$ requires

$$g^C = \frac{\lambda_t^M}{\lambda_t^K} \cdot \alpha A - \rho > 0 \tag{3.42}$$

where we have substituted the interest rate $r_t/P_t = \alpha A$ from (3.40). From (3.42), balanced growth requires $\lambda_t^R \equiv \lambda_t^M / \lambda_t^K$ to be constant as well: imposing $\dot{\lambda}_t^R = 0$ in (3.31) yields the second-order equation

$$\alpha A \cdot \left(\lambda_t^R\right)^2 + \lambda_t^R \left(1 + \pi_t\right) - 1 = 0$$

with positive root given by

$$\lambda_t^R = \frac{\sqrt{(1+\pi_t)^2 + 4\alpha A} - (1+\pi_t)}{2\alpha A}. \tag{3.43}$$

Substituting this result into (3.42), the balanced growth rate equals

$$g^C \equiv \frac{\dot{c}_t^m}{c_t^m} = \frac{\dot{c}_t^x}{c_t^x} = \frac{\sqrt{(1+\pi_t)^2 + 4\alpha A} - (1+\pi_t)}{2} - \rho \tag{3.44}$$

Result (3.44) shows that money is neither neutral nor superneutral: the inflation rate affects real growth in a BGP equilibrium. In particular, the derivative of the balanced growth rate with respect to π_t equals

$$\frac{\partial g^C}{\partial \pi_t} = -\frac{4\alpha A + 2\pi_t + \pi_t^2}{2(1+\pi_t)^2 + 8\alpha A} \tag{3.45}$$

and is strictly negative for any positive (or even negative, but relatively small) rate of money inflation. That is, positive inflation slows down real growth in this model. The reason is, conceptually, the same as that in the neoclassical case: the CIA constraint on new capital purchases forces agents to keep money to make real investment but positive inflation increases the real cost of holding money, which affects the real return to investment from the household point of view. Differently from the neoclassical case, where the interest rate determines the *stationary level* of consumption in the steady state, in the AK variant of the model the interest rate determines the *growth rate* of consumption along the balanced growth path. Therefore, in the AK case, the negative effect money inflation on the real return to investment translates into a negative effect on the economy's growth rate.

The transmission from monetary policy to inflation can be addressed by imposing the conditions for a BGP equilibrium in the CIA constraint. Setting $\dot{K}_t = g^C K_t$ in (3.7), we obtain

$$m_t = g^C K_t + c_t^m + \frac{1+\delta_t}{1-\delta_t} c_t^x = g^C K_t + \frac{1}{\theta} c_t^m \tag{3.46}$$

where the last term follows from substituting the utility-maximizing consumption ratio (3.30). Since a BGP requires capital and consumption to grow at rate g^C, the ratio c_t^m / K_t must be constant: denoting this (endogenous) variable as $\chi_{CD}^m \equiv c_t^m / K_t$ we can rewrite (3.46) as[2]

$$m_t = \left(g^C + \frac{\chi_{CD}^m}{\theta} \right) \cdot K_t. \tag{3.47}$$

[2]Subsection 3.8.4 includes a complete derivation of the equilibrium value of χ_{CD}^m.

Equation (3.47) implies that a constant growth rate g^C requires that real money supply grows over time at the same constant rate, $\dot{m}_t/m_t = \dot{K}_t/K_t = g^C$. This in turn means that a constant growth rule for nominal money supply, $\dot{M}_t/M_t = g^M$, will imply a constant inflation rate π and a constant real growth rate for the economy g^C that satisfies the BGP relation

$$g^M = \pi + g^C. \tag{3.48}$$

Using (3.44) to substitute g^C in (3.48) and rearranging terms yields

$$\sqrt{(1+\pi_t)^2 + 4\alpha A} - (1-\pi_t) = 2\left(g^M + \rho\right) \tag{3.49}$$

The above results obey a precise causality: given the exogenous monetary rule set by authorities, the growth rate of money supply g^M determines inflation π according to (3.49). The inflation rate π then determines the economy's real growth rate g^C according to equation (3.44).

Since money inflation only depends on nominal money growth, the dynamics of the supply of crypto-currency do not affect real growth through this channel: money inflation, $\pi_t = \dot{P}_t/P_t$, is independent of crypto inflation, $\pi_t^* = \dot{P}_t^*/P_t^*$. The main consequence of the crypto-currency is a permanent change in the level of the real wage induced by a *labor reallocation effect*. Expression (3.41) implies that in a BGP equilibrium – where capital grows at rate $\bar{g}_t$ while employment levels L_t^y and $L - L_t^y$ are stationary – the real wage will grow at the balanced rate $\bar{g}_t$ while sectoral employment determines a permanent level effect: the higher the employment in the exchange platform $L - L_t^y$, the lower the *levels* of the equilibrium real wage $w_t/P_t = (1/L_t^y)\cdot(1-\alpha)\,AK_t$ along the BGP. We will present a complete analytical derivation of the balanced growth equilibrium under a specific technology for the exchange platform in section 3.8.

3.7 Substitutability and money-crypto interactions

In this section, we extend the model to replace Cobb-Douglas perferences with a CES utility function. The next section shows how the relevant dynamic system changes when money-purchased and crypto-purchased goods are allowed to be strict complements or strict substitutes. The subsequent sections derive general results for neoclassical steady-state equilibria and for BGP equilibria with endogenous growth in the same vein as the previous sections.

3.7.1 Intertemporal choices under CES preferences

Suppose that the instantaneous utility function $u\left(c_t^m, c_t^x\right)$ in (3.1) is replaced by the CES form

$$u\left(c_t^m, c_t^x\right) = \left[\theta \cdot \left(c_t^m\right)^{\frac{\sigma-1}{\sigma}} + (1-\theta) \cdot \left(c_t^x\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}, \tag{3.50}$$

where $\sigma > 0$ is the elasticity of substitution between money-purchased and crypto-purchased goods. When $\sigma < 1$, the consumer perceives the two types of consumption as strict complements. When $\sigma > 1$, the consumer perceives the two types of consumption as strict substitutes. Letting $\sigma \to 1$, the utility function reduces to the Cobb-Douglas form $u\left(c_t^m, c_t^x\right) = \left(c_t^m\right)^\theta \left(c_t^x\right)^{1-\theta}$ assumed before. In this modified model, the household maximizes intertemporal utility

$$U \equiv \int_0^\infty e^{-\rho t} \ln\left[u\left(c_t^m, c_t^x\right)\right] dt$$

subject to the same dynamic constraints considered before. Proceeding with the same steps shown in section 3.4, we obtain a system of utility-maximizing conditions in which the marginal utilities are not separable: both $\partial u/\partial c_t^m$ and $\partial u/\partial c_t^x$ depend on money-purchased *and* on crypto-purchased quantities, c_t^m and c_t^x. More precisely, system (3.25)-(3.29) is replaced by

$$\frac{1}{u\left(c_t^m, c_t^x\right)} \cdot \frac{\partial u}{\partial c_t^m} = \lambda_t^K \tag{3.51}$$

$$\frac{1}{u\left(c_t^m, c_t^x\right)} \cdot \frac{\partial u}{\partial c_t^x} = \lambda_t^K \cdot q_t\left(1+\delta_t\right) \tag{3.52}$$

$$\lambda_t^K = \lambda_t^M + \lambda_t^S \tag{3.53}$$

$$\frac{\dot{\lambda}_t^K}{\lambda_t^K} = \rho - \frac{\lambda_t^M}{\lambda_t^K} \cdot \frac{r_t}{P_t} \tag{3.54}$$

$$\frac{\dot{\lambda}_t^M}{\lambda_t^M} = \rho + \pi_t - \frac{\lambda_t^K - \lambda_t^M}{\lambda_t^M} \tag{3.55}$$

The key difference with respect to the model with Cobb-Douglas utility is that relative expenditure shares now depend on relative prices. By combining (3.51) with (3.52), we obtain the utility-maximizing condition

$$\frac{\partial u}{\partial c_t^m} \cdot q_t\left(1+\delta_t\right) = \frac{\partial u}{\partial c_t^x},$$

where we can substitute the marginal utilities calculated from (3.50),

$$\frac{\partial u}{\partial c_t^m} = \theta \cdot \left(\frac{u\left(c_t^m, c_t^x\right)}{c_t^m}\right)^{\frac{1}{\sigma}} \quad \text{and} \quad \frac{\partial u}{\partial c_t^x} = (1-\theta) \cdot \left(\frac{u\left(c_t^m, c_t^x\right)}{c_t^x}\right)^{\frac{1}{\sigma}}, \tag{3.56}$$

obtaining the consumption ratio

$$\frac{c_t^m}{c_t^x} = \left[\frac{\theta}{1-\theta} \cdot q_t \left(1+\delta_t\right)\right]^{\sigma}. \tag{3.57}$$

Dividing both sides by $q_t\left(1+\delta_t\right)$ yields the real expenditure ratio, i.e., the expenditure on money-purchased goods relative to that on crypto-purchased goods,

$$\frac{c_t^m}{q_t\left(1+\delta_t\right)c_t^x} = \left(\frac{\theta}{1-\theta}\right)^{\sigma} \cdot \left(\frac{1-\delta_t}{1+\delta_t}\right)^{1-\sigma}, \tag{3.58}$$

where we have used (3.11) to eliminate the real exchange rate on the right hand side. Expression (3.58) shows that a change in the crypto-fee has generally ambiguous effects on relative expenditures. If the consumer perceives money-purchased and crypto-purchased goods as complements, $\sigma < 1$, an increase in the fee rate δ_t prompts them to reduce the left hand side of (3.58) – that is, to reduce relative spending on money-purchased goods to spend a higher fraction of consumption expenditure on crypto-purchased goods. Viceversa, if the household perceives money-purchased and crypto-purchased goods as substitutes, $\sigma > 1$, an increase in the fee rate δ_t prompts them to reduce the expenditure share on crypto-purchased goods. Armed with this result, we can now investigate the general properties of neoclassical steady-state equilibria and of BGP equilibria under CES preferences.

3.7.2 Neoclassical steady state with CES preferences

In this subsection, we consider a neoclassical steady state equilibrium with constant consumption. The first part of the analysis is very similar to that in section 3.5, with small differences that we emphasize below. The distortions induced by the crypto-market become more evident in the complete analytical solution, which clarifies reallocation effects and their consequences for the steady-state capital stock.

From (3.51) and (3.52), stationarity in c_t^m and c_t^x requires a constant multiplier λ_t^K as well as constant crypto-fees,

$$\frac{\mathrm{d}}{\mathrm{d}t} q_t\left(1+\delta_t\right) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1+\delta_t}{1-\delta_t} = 0,$$

which will be the case for suitable specifications of the technology of the exchange platform. From (3.54), the steady state $\dot{\lambda}_t^K = 0$ requires that the real rental rate for capital equals the utility dicount rate weighted by the composite multiplier $\lambda_t^R \equiv \lambda_t^M/\lambda_t^K$ previously defined: $r_t/P_t = \rho/\lambda_t^R$. Since ρ is constant and r_t/P_t equals the physical marginal product of capital, a constant real interest rate requires $\dot{\lambda}_t^R = 0$ in

(3.31), which yields $\lambda_t^R = 1/\left(1+\rho+\pi_t\right)$ and, hence, a steady-state real rate of return

$$\frac{r_t}{P_t} = \rho \cdot \left(1+\rho+\pi_t\right).$$

As noted before, (i) money is not neutral and (ii) inflation tends to reduce capital accumulation. In order to assess the role of the crypto-currency, impose the conditions for a neoclassical steady state in the CIA constraint: setting $\dot{K}_t = 0$ in (3.18), we obtain

$$m_t = c_t^m \cdot \left[1 + \left(\frac{1-\theta}{\theta}\right)^{\sigma} \cdot \left(\frac{1+\delta_t}{1-\delta_t}\right)^{1-\sigma}\right] \tag{3.59}$$

where the last term follows from substituting the consumption expenditures ratio (3.58). Since $\dot{\delta}_t = 0$ by construction of the steady state, result (3.59) implies that stationary consumption is associated with $\dot{m}_t = 0$, that is, the money inflation rate equals the growth rate of money supply set by the authority, $\pi_t = \dot{M}_t/M_t$. Since π_t only depends on money growth, the dynamics of the supply of crypto-currency do not affect the steady-state condition (3.34) through this channel: money inflation equals $\pi_t = \dot{M}_t/M_t$ and is therefore independent of crypto-currency supply. This conclusion also holds with Cobb-Douglas preferences, as shown in subsection 3.5. However, differently from the model with Cobb-Douglas preferences, the degree of substitutability between money-purchased and crypto-purchased goods affects the price level. The right hand side of (3.59) shows that when consumers perceive money-purchased and crypto-purchased goods as strict complements (substitutes), a higher fee tends to increase (reduce) the equilibrium real money supply at given consumption levels. The reason is that under complementarity (substitutability), higher fees prompt consumers to spend relatively more on crypto-purchased goods, exherting a downward pressure on the relative price of money-purchased goods and, hence, an upward pressure on the equilibrium real money supply at given consumption levels.

Besides the effect on price levels, it should be remembered that the crypto-currency is not neutral because, as shown in subsection 3.5, it affects the real interest rate in (3.34) through a *labor reallocation effect*: the physical marginal product of capital depends on the capital-labor ratio in goods' production, K_t/L_t^y, and L_t^y is in turn affected by employment in exchange platforms via the labor market.

3.7.3 Balanced growth equilibrium with CES preferences

In this subsection, we consider a BGP equilibrium with sustained growth generated by the technology described in subsection 3.6.1: sectoral spillovers induce linear returns to capital at the aggregate level, $Y_t = AK_t$, with real factor rewards given by $r_t/P_t = \alpha A$ and $w_t/P_t = (1-\alpha) \cdot A \cdot (K_t/L_t^y)$.

The general properties of the BGP are as follows. Consider a balanced growth equilibrium where consumption levels of both goods grow at the constant rate $g^C = \dot{c}_t^m/c_t^m = \dot{c}_t^x/c_t^x$, and there is a constant rate of crypto-fees, $\dot{\delta}_t = 0$, which will be the case for suitable specifications of the technology of the exchange platform. Time-differentiating (3.51) and (3.52) with $\dot{\delta}_t = 0$ we obtain

$$\frac{\dot{\lambda}_t^K}{\lambda_t^K} = \frac{1}{\frac{\partial u}{\partial c_t^m}} \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial u}{\partial c_t^m} - \frac{\dot{u}\left(c_t^m, c_t^x\right)}{u\left(c_t^m, c_t^x\right)}, \tag{3.60}$$

$$\frac{\dot{\lambda}_t^K}{\lambda_t^K} = \frac{1}{\frac{\partial u}{\partial c_t^x}} \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial u}{\partial c_t^x} - \frac{\dot{u}\left(c_t^m, c_t^x\right)}{u\left(c_t^m, c_t^x\right)}. \tag{3.61}$$

From (3.50), the growth rate of utility equals

$$\frac{\dot{u}\left(c_t^m, c_t^x\right)}{u\left(c_t^m, c_t^x\right)} = \frac{\theta \cdot \left(c_t^m\right)^{\frac{\sigma-1}{\sigma}} g^C + (1-\theta) \cdot \left(c_t^x\right)^{\frac{\sigma-1}{\sigma}} g^C}{\theta \cdot \left(c_t^m\right)^{\frac{\sigma-1}{\sigma}} + (1-\theta) \cdot \left(c_t^x\right)^{\frac{\sigma-1}{\sigma}}} = g^C. \tag{3.62}$$

From (3.56), the growth rate of marginal utility for either good reads

$$\frac{1}{\frac{\partial u}{\partial c_t^i}} \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial u}{\partial c_t^i} = \frac{1}{\sigma} \left[\frac{\dot{u}\left(c_t^m, c_t^x\right)}{u\left(c_t^m, c_t^x\right)} - g^C \right] = 0 \tag{3.63}$$

Substituting results (3.62)-(3.63) in either (3.60) or (3.61) yields $\dot{\lambda}_t^K/\lambda_t^K = -g^C$. Hence, from (3.54) and the constant interest rate $r_t/P_t = \alpha A$, we have

$$g^C = \frac{\lambda_t^M}{\lambda_t^K} \cdot \alpha A - \rho \tag{3.64}$$

which implies a constant composite multiplier $\lambda_t^R = \lambda_t^M/\lambda_t^K$. Setting $\dot{\lambda}_t^R = 0$ in (3.31) yields again result (3.43) and thereby the balanced growth rate

$$g^C \equiv \frac{\dot{c}_t^m}{c_t^m} = \frac{\dot{c}_t^x}{c_t^x} = \frac{\sqrt{\left(1+\pi_t\right)^2 + 4\alpha A} - \left(1+\pi_t\right)}{2} - \rho. \tag{3.65}$$

As noted before, any positive (or even negative, but relatively small) rate of money inflation reduces g^C because money inflation increases the cost of holding money – and holding money is necessary to have the liquidity needed to make real investment. Result (3.65) shows that the balanced growth rate under CES preferences is the same as in the Cobb-Douglas case with $\sigma = 1$. However, the current hypothesis that goods can be perceived as complements modifies the impact of monetary policy on the general price level. To see this formally, use result (3.58) to write real consumption expenditures as

$$c_t^m + q_t (1+\delta_t) c_t^x = c_t^m \cdot \left[1 + \left(\frac{1-\theta}{\theta}\right)^\sigma \cdot \left(\frac{1+\delta_t}{1-\delta_t}\right)^{1-\sigma}\right] \tag{3.66}$$

and impose the conditions for a BGP equilibrium in the CIA constraint: setting $\dot{K}_t = g^C K_t$ in (3.7), we obtain

$$m_t = \bar{g}_t K_t + c_t^m + q_t (1+\delta_t) c_t^x = \bar{g}_t K_t + c_t^m \cdot \left[1 + \left(\frac{1-\theta}{\theta}\right)^\sigma \cdot \left(\frac{1+\delta_t}{1-\delta_t}\right)^{1-\sigma}\right] \tag{3.67}$$

where the last term follows from (3.66). Since a BGP requires capital and consumption to grow at rate g^C, the ratio c_t^m/K_t must be constant: denoting this (endogenous) variable as $\chi^m \equiv c_t^m/K_t$ we can rewrite (3.67) as[3]

$$m_t = \left\{ g^C + \chi^m \left[1 + \left(\frac{1-\theta}{\theta}\right)^\sigma \cdot \left(\frac{1+\delta_t}{1-\delta_t}\right)^{1-\sigma}\right]\right\} \cdot K_t. \tag{3.68}$$

Equation (3.68) implies that a constant growth rate g^C requires that real money supply grows over time at the same constant rate, $\dot{m}_t/m_t = \dot{K}_t/K_t = g^C$. This in turn means that a constant growth rule for nominal money supply, $\dot{M}_t/M_t = g^M$, will imply a constant inflation rate π and a constant real growth rate for the economy g^C that satisfies the BGP relation $g^M = \pi + g^C$. As shown before (cf. equation (3.49) in the previous section), the growth rate of money supply g^M determines inflation π according to

$$\sqrt{(1+\pi)^2 + 4\alpha A} - (1-\pi) = 2\left(g^M + \rho\right),$$

and the inflation rate π then determines the economy's real growth rate g^C according to equation (3.65). The novel result contained in (3.68) is that the degree of substitutability between money-purchased and crypto-purchased goods directly affects the whole time path of the price level. Substituing $m_t = M_t/P_t$ in (3.68) and rearranging terms, we obtain

$$P_t = \frac{1}{g^C + \chi^m \left[1 + \left(\frac{1-\theta}{\theta}\right)^\sigma \cdot \left(\frac{1+\delta_t}{1-\delta_t}\right)^{1-\sigma}\right]} \cdot \frac{M_t}{K_t}. \tag{3.69}$$

Result (3.69) implies that crypto-fees permanently reduce (increase) the money price level when consumers perceive money-purchased and crypto-purchased goods as

[3] Subsection 3.8.4 includes a complete derivation of the equilibrium value of χ^m.

complements (substitutes). In particular, if the monetary authority sets a constant money growth rule from time zero onwards and the crypto-fee rate is constant from time zero onwards, the whole time path of the price level is given by

$$P_t = \frac{1}{g^C + \chi^m \left[1 + \left(\frac{1-\theta}{\theta}\right)^{\sigma} \cdot \left(\frac{1+\delta}{1-\delta}\right)^{1-\sigma}\right]} \cdot \frac{M_0}{K_0} \cdot e^{\pi t}. \tag{3.70}$$

Expression (3.70) shows that, for a given chosen monetary policy rule g^M, which determines real growth g^C and the inflation rate π, the elasticity of substitution σ and the crypto-fee rate δ determine how high or low the initial price level P_0, and thereby all subsequent price levels, will be. Under complementarity, $\sigma < 1$, a higher δ yields a lower price level because higher crypto-fees prompt consumers to reduce their relative demand for money-purchased goods. Under substitutability, $\sigma > 1$, a higher δ yields a higher price level because higher crypto-fees prompt consumers to increase their relative demand for money-purchased goods. Since δ is positively related to the real exchange rate – see equation (3.11) – it follows that a real appreciation of the crypto-currency induced by higher crypto-fees affects the price level of money-purchased goods permanently and in opposite directions depending on the value of the elasticity of substitution σ.

As we have shown in section 3.6.2, the crypto-market permanently affects real wage levels via a *labor reallocation effect*: the higher the employment in the exchange platform $L - L_t^y$, the lower the *levels* of the equilibrium real wage $w_t/P_t = (1/L_t^y) \cdot (1-\alpha) AK_t$ along the BGP. This result is obviously confirmed in the model with CES preferences.

3.8 Complete derivations and shocks

This section presents full analytical derivations of (i) the Neoclassical steady state and (ii) the BGP equilibrium for the extended model with CES preferences, which allows us to derive more general results (the predictions for the model with Cobb-Douglas preferences can be obtained as a special case by setting $\sigma = 1$). For each variant of the model, we obtain a reduced system of equilibrium relationships that determines all endogenous variables and allows us to investigate the effect of exogenous shocks.

3.8.1 Exchange platform: specifics

Assume that the exchange platform is a competitive sector with free entry of 'exchange firms'. Each firm n hires $\ell_{n,t}^x$ workers to perform currency exchange operations according to a linear technology: each worker's cost to the firm is proportional to the monetary value of the transaction, with proportionality factor $\xi > 0$,

$$w_t = \xi \cdot \eta_{j,t}, \tag{3.71}$$

where w_t is the wage rate prevailing in the labor market and $\eta_{j,t}$ is the money value of the transaction performed by agent j. Total employment in the currency exchange sector, $L_t^x = \sum_n \ell_{n,t}^x = 1 - L_t^y$, satisfies the demand for currency conversion. Therefore, the sectoral wage bill reads

$$w_t \cdot (1 - L_t^y) = \xi \cdot \int_0^{L_t^x} \eta_{j,t} dj = \xi \cdot (P_t c_t^x + P_t c_t^x), \tag{3.72}$$

where the last term on the right hand side is the market clearing condition for exchange services whereby the money value of total transactions includes those (i) requested by consumers purchasing c_t^x and those (ii) requested by firms selling c_t^x. Exchange firms take the exchange rate as given and set the crypto-fee rate in Bertrand competition. The resulting zero-profit condition, as shown in subsection 3.3.3, is $2\delta_t Q_t P_t^* c_t^x = w_t (1 - L_t^y)$ and can be rewritten in real terms as

$$2\delta_t q_t c_t^x = \frac{w_t}{P_t} \cdot (1 - L_t^y). \tag{3.73}$$

From (3.72) and (3.73), it follows that $\delta_t q_t = \xi$. Combining this result with the real exchange rate in (3.11), the crypto-fee rate associated with zero profits in the exchange platform reads

$$\delta_t = \frac{\xi}{1 + \xi} \equiv \delta \tag{3.74}$$

which is constant over time. We now have all the elements to derive anlytically the neoclassical steady state equilibrium and the BGP equilibrium in the AK model.

3.8.2 Neoclassical steady state: full derivation

In the neoclassical case, we normalize labor productivity $a_t = 1$ and assume a Cobb-Douglas production function $Y_t = (K_t)^\alpha (L_t^y)^{1-\alpha}$ for the final setor. The profit-maximizing conditions yield the demand schedules for capital and labor,

$$\frac{r_t}{P_t} = \alpha \cdot \left(\frac{L_t^y}{K_t}\right)^{1-\alpha}, \tag{3.75}$$

$$\frac{w_t}{P_t} = (1 - \alpha) \cdot \left(\frac{K_t}{L_t^y}\right)^{\alpha}. \tag{3.76}$$

Combining (3.75) with the steady-state condition for the interest rate (3.34) and steady-state inflation rate $\pi_t = g^M$ from (3.36) yields the capital-labor ratio for the final sector in the neoclassical steady state,

$$\frac{K_t}{L_t^y} = \left[\frac{\alpha}{\rho \cdot (1 + \rho + g^M)}\right]^{\frac{1}{1-\alpha}}. \tag{3.77}$$

From the zero-profit condition in the exchange platform (3.73) and the equilibrium fee rate (3.74), labor demand by currency-exchange firms is

$$\frac{w_t}{P_t} = 2\frac{\delta_t q_t}{1 - L_t^y} c_t^x = \frac{2\xi}{1 - L_t^y} c_t^x. \tag{3.78}$$

The equilibrium in the labor market is characterized by real wage equalization which, from (3.76) and (3.78), implies

$$\frac{K_t}{L_t^y} = \left[\frac{2\xi}{1-\alpha} \cdot \frac{c_t^x}{1 - L_t^y}\right]^{\frac{1}{\alpha}}. \tag{3.79}$$

Using (3.11) and (3.74), the ratio between money-purchased and crypto-purchased goods (3.57) equals

$$\frac{c_t^m}{c_t^x} = \left[\frac{\theta\,(1 + 2\xi)}{1 - \theta}\right]^{\sigma}. \tag{3.80}$$

Using (3.74), the steady-state level of real money supply (3.35) equals

$$m_t = c_t^m \cdot \left[1 + \left(\frac{1-\theta}{\theta}\right)^{\sigma} \cdot (1 + 2\xi)^{1-\sigma}\right]. \tag{3.81}$$

The goods' market clearing condition (3.8) in the steady state implies

$$c_t^x = (K_t)^{\alpha}\,(L_t^y)^{1-\alpha} - c_t^m. \tag{3.82}$$

Reduced system (neoclassical steady state). Equations (3.77), (3.79), (3.80), (3.81) and (3.82) form a *reduced equilibrium system* that allows us to determine the steady state values of inputs and consumption levels – and thereby all the related endogenous variables of interest – in the neoclassical steady state:

$$\frac{K_{ss}}{L_{ss}^{y}} = \left[\frac{\alpha}{\rho\cdot(1+\rho+g^{M})}\right]^{\frac{1}{1-\alpha}} \tag{3.83}$$

$$\frac{K_{ss}}{L_{ss}^{y}} = \left[\frac{2\xi}{1-\alpha}\cdot\frac{c_{ss}^{x}}{1-L_{ss}^{y}}\right]^{\frac{1}{\alpha}} \tag{3.84}$$

$$\frac{c_{ss}^{m}}{c_{ss}^{x}} = \left[\frac{\theta\,(1+2\xi)}{1-\theta}\right]^{\sigma} \tag{3.85}$$

$$m_{ss} = \left[1+\left(\frac{1-\theta}{\theta}\right)^{\sigma}\cdot(1+2\xi)^{1-\sigma}\right]\cdot c_{ss}^{m} \tag{3.86}$$

$$c_{ss}^{x} = (K_{ss})^{\alpha}\,(L_{ss}^{y})^{1-\alpha} - c_{ss}^{m} \tag{3.87}$$

The reduced system (3.83)-(3.87) comprises five equations determining five unknowns: capital K_{ss}, labor employed in the final sector L_{ss}^{y}, consumption of money-purchased goods c_{ss}^{m}, consumption of crypto-purchased goods c_{ss}^{x}, and real money holdings m_{ss}. The exogenous parameters reflect technologies (α,ξ), preferences (θ,σ,ρ) and the monetary policy rule set by the authority, $\dot{M}_t/M_t = g^M$. The equilibrium values $(K_{ss}, L_{ss}^{y}, c_{ss}^{m}, c_{ss}^{x}, m_{ss})$ allow us to calculate real factor prices r_t/P_t and w_t/P_t from (3.75)-(3.76), the crypto-fee rate from (3.74), the real exchange rate from (3.11), and steady-state utility $u\,(c_t^m, c_t^x)$ from (3.50). The next subsection presents some numerical results describing the effects of exogenous shocks.

3.8.3 Neoclassical steady state: numerical analysis

In this subsection, we introduce a numerical illustration of the neoclassical steady state and study cases of strict complementarity, strict substitutability, and Cobb-Douglas preferences. We then proceed to assess the effects of exogenous changes in the growth rate of nominal money, in the crypto-fee rate (due to an exogenous rise in ξ), and in the taste parameter θ on the endogenous variables in the reduced system above. Parameter values are reported in Table 3.1 along with the equilibrium level of the endogenous variables in Table 3.2.

Table 3.1: Parameter values.

Preferences	Technology	Monetary policy rule
$\theta = 0.3$	$\alpha = 0.3$	$g^M = 0.045$
$\rho = 0.02$	$\xi = 0.05$	

Table 3.2: Benchmark results.

	K_{ss}	L^y_{ss}	c^m_{ss}	c^x_{ss}	$\frac{r}{p}$	$\frac{w}{p}$	ϕ	q	$u\left(c^m_t, c^x_t\right)$
$\sigma = 0.5$	40.3413	0.9219	1.1660	1.6982	0.0213	2.1748	0.0476	1.050	1.4937
$\sigma = 1$	39.8858	0.9115	0.9073	1.9246	0.0213	2.1748	0.0476	1.050	-
$\sigma = 1.5$	39.4957	0.9026	0.6857	2.1185	0.0213	2.1748	0.0476	1.050	1.5754

3.8.3.1 Neoclassical shock analysis

An increase in g^M (faster monetary growth). A 10% increase in the money supply leads to monetary non-neutrality, reflected in a decline in capital stock (K_{ss}), overall consumption (c^m_{ss} and c^x_{ss}), and real wages ($\frac{w}{p}$) across all substitution levels (σ) in Table 3.3. The reduction in capital investment is driven by inflation eroding real savings, while consumption decreases due to reduced purchasing power. Utility declines more sharply when money and crypto-currencies are substitutes ($\sigma = 1.5$) because consumers shift more heavily toward crypto-currencies, amplifying the negative impact of rising transaction costs. The rental rate of capital ($\frac{r}{p}$) increases due to reduced capital availability, while crypto-fees (ϕ) and the real exchange rate (q) remain unchanged. This monetary non-neutrality arises from inflationary pressure, negatively impacting the economy's key variables and altering the allocation of resources between sectors.

Table 3.3: Shock analysis (10% increase in the money supply).

	K_{ss}	L^y_{ss}	c^m_{ss}	c^x_{ss}	$\frac{r}{p}$	$\frac{w}{p}$	ϕ	q	$u\left(c^m_t, c^x_t\right)$
$\sigma = 0.5$	40.0990↓	0.9219	1.1639↓	1.6952↓	0.0214↑	2.1709↓	0.0476	1.050	1.4910↓
$\sigma = 1$	39.6463↓	0.9115	0.9057↓	1.9211↓	0.0214↑	2.1709↓	0.0476	1.050	-
$\sigma = 1.5$	39.2585↓	0.9026	0.6845↓	2.1146↓	0.0214↑	2.1709↓	0.0476	1.050	1.5725↓

Note: The upward (downward) arrow indicates an increase (decrease) relative to the benchmark values reported in Table 3.2. The absence of an arrow signifies no change compared to the benchmark.

An increase in ξ (which raises fees, δ). As shown in Table 3.4, a 10% increase in ξ raises crypto-currency transaction costs, leading to a decline in capital stock (K_{ss}), labor in the goods sector (L^y_{ss}), and consumption of both money (c^m_{ss}) and crypto-purchased goods (c^x_{ss}) across all σ levels, except for a small increase in c^m_{ss} when $\sigma = 1.5$ as consumers shift away from crypto-purchased goods. Real wages remain unchanged, but the crypto-fee (ϕ) and real exchange rate (q) rise, reflecting higher transaction costs. Utility falls due to reduced consumption, with the largest impact seen when fiat money and crypto-currency are substitutes ($\sigma = 1.5$).

A reduction in θ (higher taste for crypto-purchased goods). A 10% decrease in θ leads to an increase in consumption of crypto goods (c^x_{ss}) across all σ levels in Table 3.5. This shift reduces the consumption of money-purchased goods (c^m_{ss}) and decreases both

Table 3.4: Shock analysis (10% increase in the fee structure).

	K_{ss}	L^y_{ss}	c^m_{ss}	c^x_{ss}	$\frac{r}{p}$	$\frac{w}{p}$	ϕ	q	$u\left(c^m_t, c^x_t\right)$
$\sigma = 0.5$	40.0350↓	0.9149↓	1.1603↓	1.6822↓	0.0213	2.1748	0.0521↑	1.055↑	1.4822↓
$\sigma = 1$	39.5470↓	0.9038↓	0.9051↓	1.9027↓	0.0213	2.1748	0.0521↑	1.055↑	-
$\sigma = 1.5$	39.1285↓	0.8942↓	0.6863↑	2.0918↓	0.0213	2.1748	0.0521↑	1.055↑	1.5603↓

Note: The upward (downward) arrow indicates an increase (decrease) relative to the benchmark values reported in Table 3.2. The absence of an arrow signifies no change compared to the benchmark.

capital stock (K_{ss}) and labor allocated to the goods sector (L^y_{ss}). Utility ($u(c^m_t, c^x_t)$) increases due to the higher consumption of crypto goods, with the most pronounced increase seen when money and crypto are substitutes ($\sigma = 1.5$). Real wages ($\frac{w}{p}$) and the crypto fee (ϕ) remain unchanged, while the rate of return on capital ($\frac{r}{p}$) experiences a slight decrease when $\sigma = 1.5$. This suggests that a stronger preference for crypto-purchased goods and a reallocation of resources towards crypto-based consumption, affecting production and investment patterns in the economy.

Table 3.5: Shock analysis: 10% decrease in θ.

	K_{ss}	L^y_{ss}	c^m_{ss}	c^x_{ss}	$\frac{r}{p}$	$\frac{w}{p}$	ϕ	q	$u\left(c^m_t, c^x_t\right)$
$\sigma = 0.5$	40.2477↓	0.9198↓	1.1129↓	1.7447↑	0.0213	2.1748	0.0476	1.050	1.5128↑
$\sigma = 1$	39.7244↓	0.9078↓	0.8156↓	2.0048↑	0.0213	2.1748	0.0476	1.050	-
$\sigma = 1.5$	39.3006↓	0.8981↓	0.5749↓	2.2154↑	0.0213	2.1748	0.0476	1.050	1.6270↑

Note: The upward (downward) arrow indicates an increase (decrease) relative to the benchmark values reported in Table 3.2. The absence of an arrow signifies no change compared to the benchmark.

3.8.4 BGP equilbrium: full derivation

As shown in subsection 3.6.1, final output in the AK model equals $Y_t = AK_t$ and the real rental rate for capital is $r_t/P_t = \alpha A$. From (3.41), labor demand in the final sector implies a real wage $w_t/P_t = (1-\alpha) \cdot A \cdot (K_t/L^y_t)$, whereas, irrespective of the final sector's technology, labor demand in the exchange platform is given by (3.78). Therefore, wage equalization in the labor market implies

$$\frac{L^y_t}{1 - L^y_t} = \frac{A\,(1-\alpha)}{2\xi} \cdot \frac{K_t}{c^x_t}. \tag{3.88}$$

Since the crypto-fee rate $\delta_t = \xi/\left(1+\xi\right)$ is constant over time and the monetary authority is assumed to follow a constant money growth rule $\dot{M}_t/M_t = g^M$, the AK model admits a permanent BGP equilibrium such that the economy exhibits a constant growth rate from time zero onwards. This implies that, differently from the neoclassical case where we focus on steady-state results – the AK model allows us to build a reduced equilibrium system determining the entire time path of the economy. The key relationship to derive is the equilibrium ratio of consumption to capital which, in this class of models, is a jump variable that settles in its only permanent feasible steady state from time zero onwards. From $Y_t = AK_t$ and (3.8), the growth rate of capital obeys

$$\frac{\dot{K}_t}{K_t} = A - \frac{c_t^m + c_t^x}{K_t} = A - \frac{C_t}{K_t}. \tag{3.89}$$

From (3.65), the growth rate of consumption equals

$$\frac{\dot{C}_t}{C_t} = \frac{\sqrt{(1+\pi_t)^2 + 4\alpha A} - (1+\pi_t)}{2} - \rho \equiv g^C. \tag{3.90}$$

The above expressions imply that, defining $\chi_t \equiv C_t/K_t$, the growth rate of the consumption-capital ratio obeys

$$\frac{\dot{\chi}_t}{\chi_t} = g^C - A + \chi_t, \tag{3.91}$$

which is a dynamically unstable equation whose uniques steady state is

$$\chi_* = A - g^C = A + \rho - \frac{\sqrt{(1+\pi_t)^2 + 4\alpha A} - (1+\pi_t)}{2}. \tag{3.92}$$

It can be shown by standard arguments that setting $\chi_t = \chi_*$ in each $t \in [0, \infty)$ is the only solution that is compatible with (i) the conditions for intertemporal utility maximization and with (ii) satisfying the capital accumulation constraint along the entire time path.[4] Therefore, the BGP equilibrium is characterized by a constant consumption-capital ratio from time zero onwards, $\chi_t = \chi_*$ in each $t \in [0, \infty)$.

Using (3.11) and (3.74), the ratio between money-purchased and crypto-purchased goods (3.57) equals

$$\frac{c_t^m}{c_t^x} = \left[\frac{\theta\,(1+2\xi)}{1-\theta}\right]^{\sigma}. \tag{3.93}$$

Equation (3.92) allows us to the determine the ratio between money-purchased consumption and capital. Since aggregate consumption equals

$$C_t = c_t^m + c_t^x = c_t^m \cdot \left\{1 + \left[\frac{1-\theta}{\theta\,(1+2\xi)}\right]^{\sigma}\right\}, \tag{3.94}$$

[4]The intuition is that choosing a different consumption-capital ratio at time zero, $\chi_0 \gtrless \chi_*$, would generate – from equation (3.91) – explosive dynamics in χ_t which would violate either the consumers' transversality conditions in the long run (due to overaccumulation of capital) or the aggregate resource constraint (3.89) in finite time (due to overconsumption).

the ratio $\chi_t^m \equiv c_t^m / K_t$ will be constant over time and equal to

$$\chi_t^m = \frac{c_t^m}{K_t} = \frac{1}{1 + \left[\frac{1-\theta}{\theta(1+2\xi)}\right]^\sigma} \cdot \frac{C_t}{K_t} = \frac{1}{1 + \left[\frac{1-\theta}{\theta(1+2\xi)}\right]^\sigma} \cdot \chi_*,$$

that is,

$$\chi^m = \frac{A - g^C}{1 + \left[\frac{1-\theta}{\theta(1+2\xi)}\right]^\sigma}. \tag{3.95}$$

Expression (3.95) determines the variable χ^m that we have previously introduced in equation (3.68) and confirms that it is constant over time. Similarly, letting $\sigma = 1$, expression (3.95) determines the variable χ_{CD}^m that we have previously introduced in equation (3.47). We have now all the elements to build a reduced system for the BGP equilibrium in the AK model.

Reduced system (BGP equilibrium). The following *reduced equilibrium system* allows us to determine four key endogenous variables – namely, the inflation rate, the balanced growth rate (of real consumption, output and capital), the consumption-capital ratio, and employment in the final sector (and, residually, in the exchange platform) – along the balanced growth path of the AK model:

$$g^M = \frac{\sqrt{(1+\pi)^2 + 4\alpha A} - (1-\pi)}{2} - \rho \tag{3.96}$$

$$g^C = g^M - \pi \tag{3.97}$$

$$\chi^m = \frac{A - g^C}{1 + \left[\frac{1-\theta}{\theta(1+2\xi)}\right]^\sigma} \tag{3.98}$$

$$\frac{L^y}{1-L^y} = \frac{A(1-\alpha)}{2\xi} \cdot \frac{1}{\chi^m} \cdot \left[\frac{\theta(1+2\xi)}{1-\theta}\right]^\sigma \tag{3.99}$$

Equation (3.96) follows immediately from (3.49) and determines the inflation rate π given the monetary growth rate g^M set by the authority. Equation (3.97) follows immediately from (3.48) and determines the BGP growth rate g^C. Equation (3.98) follows from the above analysis – eq.(3.95) – and determines the ratio of consumption in money-purchased goods to physical capital. Equation (3.99) follows from substituting (3.92) and (3.95) into the condition for wage equalization in the labor market (3.88), and determines employment in the final sector, L^y, as well as employment in the exchange platform, $1 - L^y$.

Since the economy exhibits a BGP equilibrium from time zero onwards, the determination of (π, g^C, χ^m, L^y) in the reduced system allows us to calculate the whole time paths of the main variables of interest according to the following equations: capital, output and consumption are given by

$$K_t = K_0 \cdot e^{g_C \cdot t}, \tag{3.100}$$

$$Y_t = AK_t = K_0 \cdot e^{g_C \cdot t}, \tag{3.101}$$

$$C_t = (C_t/K_t) \cdot K_t = \chi_* \cdot K_t = \left(A - g^C\right) \cdot K_0 \cdot e^{g_C \cdot t}, \tag{3.102}$$

whereas the real wage and price level are given by[5]

$$\frac{w_t}{P_t} = (1-\alpha) \cdot A \cdot (1/L^y) \cdot K_0 \cdot e^{g_C \cdot t} \tag{3.103}$$

$$P_t = \frac{1}{g^C + \chi^m \left[1 + \left(\frac{1-\theta}{\theta}\right)^{\sigma} \cdot (1+2\xi)^{1-\sigma}\right]} \cdot \frac{M_0}{K_0} \cdot e^{\pi t} \tag{3.104}$$

where K_0 is exogenously given and M_0 is exogenously set by the authority.

3.8.5 BGP equilibrium: numerical analysis

The following subsection presents a numerical illustration of the balanced growth path for different values of the elasticity of substitution – covering the cases of strict complementarity, strict substitutability, and Cobb-Douglas preferences – and evaluates, for each of these three basline scenarios, the effects of exogenous changes in the growth rate of nominal money, in the crypto-fee rate (due to an exogenous rise in ξ), and in the taste parameter θ. First, we report the fixed parameter values and the results for the baseline scenrio in Table 3.6 and Table 3.7.

Table 3.6: Parameter values.

Preferences	**Technology**	**Monetary policy rule**
$\theta = 0.3$	$\alpha = 0.3$	$g^M = 0.045$
$\rho = 0.02$	$\xi = 0.05$	
	$A = 0.16$	

Table 3.7: Benchmark results.

	π	g^C	χ^m	L^y	$\frac{c_t^m}{c_t^x}$
$\sigma = 0.5$	0.0199	0.0251	0.0549	0.9333	0.6866
$\sigma = 1$	0.0199	0.0251	0.0432	0.9243	0.4714
$\sigma = 1.5$	0.0199	0.0251	0.0330	0.9166	0.3237

[5]The time path of the real wage in (3.103) follows straightforwardly from equation (3.41). The time path of the money price in (3.104) follows from equation (3.70) after substituting the equilibrium fee rate (3.74).

3.8.6 BGP shock analysis

An increase in g^M (faster monetary growth). A 10% increase in the money supply (g^M), as reported in Table 3.8, leads to a rise in inflation (π) across all cases, regardless of the elasticity of substitution (σ). As expected, the real consumption growth rate (g^C) decreases, indicating the negative effect of higher inflation on real consumption. The ratio of money-purchased goods to capital (χ^m) increases, suggesting a shift towards more money-purchased goods as inflation rises. However, this increase is more pronounced when money and crypto are complements ($\sigma = 0.5$) and less so when they are substitutes ($\sigma = 1.5$). Labor allocation to the goods production sector (L^y) declines slightly as money becomes more abundant, reflecting a reallocation of labor resources. Lastly, the ratio of money-purchased to crypto-purchased goods (c_t^m/c_t^x) falls as σ increases, implying that when money and crypto-currency are substitutes, consumers favor crypto-purchased goods more heavily after the shock.

Table 3.8: Shock Analysis: 10% Increase in the Money Supply

	π	g^C	χ^m	L^y	$\frac{c_t^m}{c_t^x}$
$\sigma = 0.5$	0.0246↑	0.0249↓	0.0550↑	0.9332↓	0.6866
$\sigma = 1$	0.0246↑	0.0249↓	0.0433↑	0.9242↓	0.4714
$\sigma = 1.5$	0.0246↑	0.0249↓	0.0330↑	0.9165↓	0.3237

Note: The upward (downward) arrow indicates an increase (decrease) relative to the benchmark values reported in Table 3.7. The absence of an arrow signifies no change compared to the benchmark.

An increase in ξ (which raises fees, δ). In Table 3.9, a 10% increase in the fee structure (ξ) leads to no change in inflation (π) and the consumption growth rate (g^C) across all cases, regardless of the elasticity of substitution (σ). However, the ratio of money-purchased goods to capital (χ^m) increases, indicating a shift towards money-purchased goods as the cost of crypto-related transactions rises. This increase in χ^m is larger when money and crypto are complements ($\sigma = 0.5$) and less so when they are substitutes ($\sigma = 1.5$). Labor allocation to the goods production sector (L^y) declines, reflecting a reduction in the productive sector as crypto becomes more costly to use. The ratio of money-purchased to crypto-purchased goods (c_t^m/c_t^x) rises, suggesting that higher fees for crypto transactions push consumers to favor money-purchased goods, with this effect being strongest when the two goods are more substitutable. This analysis highlights the role of transaction costs in shifting consumer preferences between money and crypto, and its impact on real variables in the BGP framework.

Table 3.9: Shock Analysis: 10% increase in the fee structure

	π	g^C	χ^m	L^y	$\frac{c_t^m}{c_t^x}$
$\sigma = 0.5$	0.0199	0.0251	0.0551↑	0.9273↓	0.6897↑
$\sigma = 1$	0.0199	0.0251	0.0435↑	0.9176↓	0.4757↑
$\sigma = 1.5$	0.0199	0.0251	0.0333↑	0.9093↓	0.3281↑

Note: The upward (downward) arrow indicates an increase (decrease) relative to the benchmark values reported in Table 3.7. The absence of an arrow signifies no change compared to the benchmark.

A reduction in θ (higher taste for crypto-purchased goods). As reported in Table 3.10, a 10% reduction in θ results in no change in inflation (π) and consumption growth (g^C) across all values of the elasticity of substitution (σ). However, the ratio of money-purchased goods to capital (χ^m) decreases, indicating a shift toward crypto-purchased goods. This reduction in χ^m is more pronounced when the two goods are complements ($\sigma = 0.5$) and less significant when they are substitutes ($\sigma = 1.5$). Labor allocation to the goods production sector (L^y) also decreases, reflecting a reduced need for money-purchased goods as the economy adapts to the higher preference for crypto-currency transactions. The ratio of money-purchased to crypto-purchased goods (c_t^m/c_t^x) decreases sharply, showing that consumers are opting more for crypto-purchased goods, with the largest decline occurring when the goods are more substitutable ($\sigma = 1.5$). This shift highlights the influence of consumer preferences on the allocation of resources in the economy.

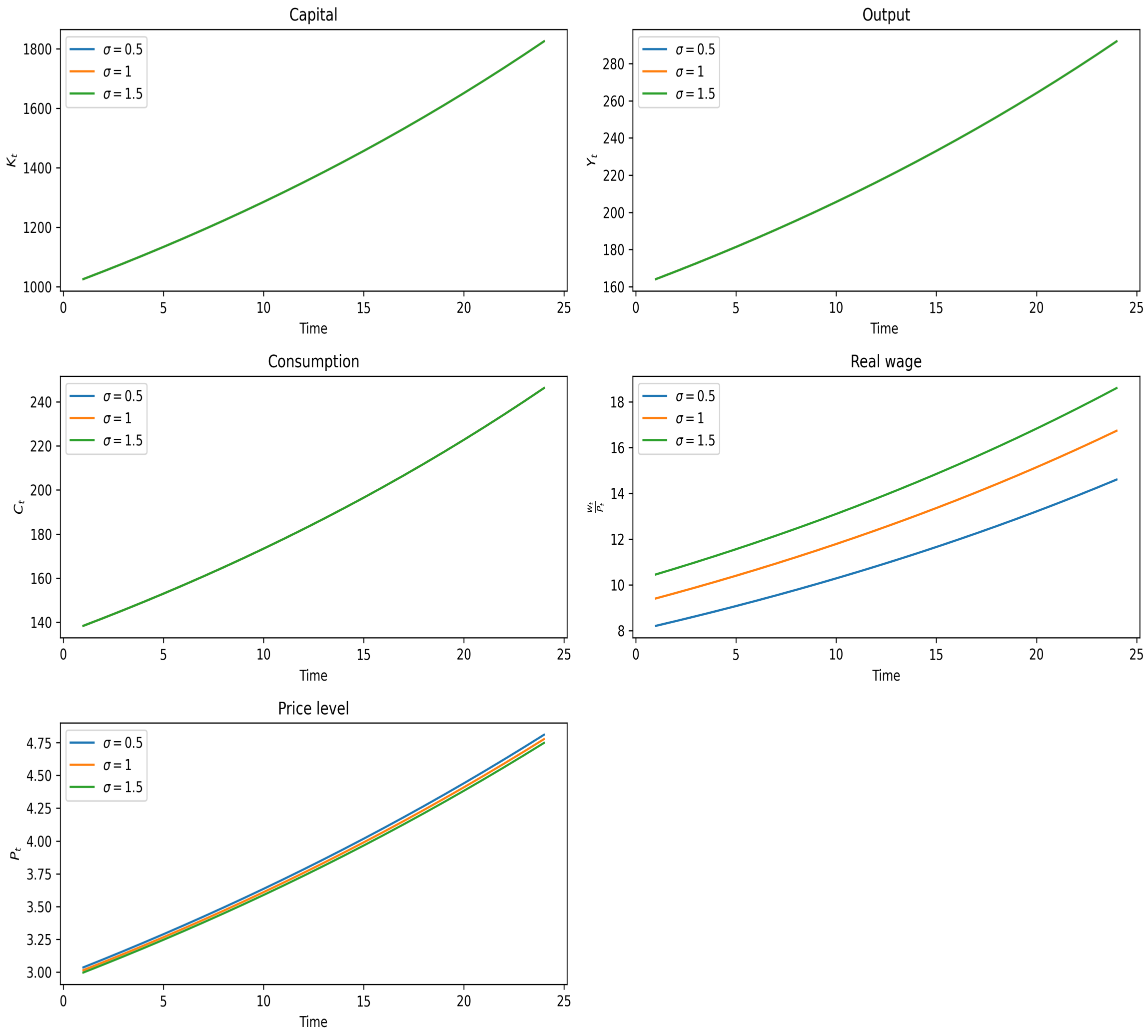
Table 3.10: Shock analysis: 10% decrease in θ.

	π	g^C	χ^m	L^y	$\frac{c_t^m}{c_t^x}$
$\sigma = 0.5$	0.0199	0.0251	0.0525↓	0.9315↓	0.6378↓
$\sigma = 1$	0.0199	0.0251	0.0390↓	0.9211↓	0.4068↓
$\sigma = 1.5$	0.0199	0.0251	0.0278↓	0.9127↓	0.2595↓

Note: The upward (downward) arrow indicates an increase (decrease) relative to the benchmark values reported in Table 3.7. The absence of an arrow signifies no change compared to the benchmark.

Figure 3.1 demonstrates that capital, output, and consumption grow steadily over time but remain largely unaffected by variations in the elasticity of substitution ($\sigma = 0.5$, $\sigma = 1$, and $\sigma = 1.5$). This suggests that the broader growth trajectory of the economy is driven by technology rather than consumer preferences between money-purchased and crypto-purchased goods. However, real wages are highly sensitive to changes in σ, with greater substitutability ($\sigma = 1.5$) leading to faster wage growth due to more efficient labor allocation. In contrast, when goods are more complementary ($\sigma = 0.5$), wage growth is slower. The price level shows only slight variation, rising more slowly with greater substitutability, reflecting the lower cost pressures from crypto-purchased goods. Overall, the impact of elasticity is most visible in real wages, while price levels and aggregate economic variables remain relatively stable.

Figure 3.1: Evolution of the key model variables along the BGP



3.9 Conclusion

This paper explores how the coexistence of fiat money and crypto-currencies shapes economic outcomes in a dynamic setting. We highlight that crypto-currencies disrupt resource allocation, particularly by diverting labor from traditional sectors and adding transaction costs, amplifying the non-neutrality of money. While key growth indicators like capital and output remain relatively stable, shifts in labor allocation and real wages are more responsive to changes in crypto fees and consumer preferences. A promising direction to improve on this work would be to incorporate the idea of pecuniary and non-pecuniary features in the dynamic framework. Up until now, we have assumed that crypto goods are needed. Although the consumption ratio is determined endogenously, future research could take a similar approach to the static model to determine a threshold good where consumers are indifferent between payment methods. Moreover, extending the model to include Central Bank Digital Currencies (CBDCs) would provide a valuable avenue for studying interactions among fiat money, crypto-currencies, and CBDCs, along with their influence on consumer preferences. This extension would greatly enhance the current analysis.

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