



3-Preprojective Algebras of Type D

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Abstract

We present a family of selfinjective algebras of type D, which arise from the 3-preprojective algebras of type A by taking a \mathbb{Z}_3 -quotient. We show that a subset of these are themselves 3-preprojective algebras, and that the associated 2-representation-finite algebras are fractional Calabi-Yau. In addition, we show our work is connected to modular invariants for $SU(3)$.

Keywords 3-Preprojective · 2-Representation-finite · Fractional Calabi-Yau · Jacobian algebras · $SU(3)$ modular invariants

Mathematics Subject Classification (2010) 16D50; 16E35 · 16S35

1 Introduction

Overview

Many mathematical objects admit a classification in terms of Dynkin diagrams, perhaps the most famous example being complex semisimple Lie algebras. A theorem of Gabriel says that the path algebra of a quiver has finite representation type if and only if its underlying graph is an ADE Dynkin diagram. The group \mathbb{Z}_2 acts on the type A diagrams by rotating them through π . One obtains the type D diagrams through quotienting by this action, duplicating the fixed vertex whenever one exists (see Table 1).

A related classification appears in [7, 8], where Evans and Pugh study Jacobian algebras of so-called ADE graphs, introduced by Di Francesco and Zuber in work on $SU(3)$ modular invariants [4]. The group \mathbb{Z}_3 acts on the type A graphs by rotating them through $2\pi/3$. The type D graphs arise from type A by taking \mathbb{Z}_3 -orbifolds, which amounts to quotienting by the action and triplicating the fixed vertex, whenever one exists. The type A algebras are well-studied. Indeed, in §4 we show they are isomorphic to the 3-preprojective algebras of type A. This article can be seen as an exploration of the type D algebras from the perspective of higher homological algebra.

In [20], Iyama introduced d -representation-finite algebras: algebras which have global dimension at most d , and whose module category has a d -cluster tilting subcategory. One can define higher analogues of the Auslander-Reiten translates which restrict to this subcategory, and hence gain some understanding of the representation theory of the algebra, even if it has wild representation type.

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Table 1 Type D Dynkin diagrams (and tadpole diagrams) arising as \mathbb{Z}_2 -quotients of type A diagrams

Type A Dynkin diagram	\mathbb{Z}_2 -quotient
A_2 : $\bullet \text{ --- } \bullet$	T_1 : $\bullet \text{ --- } \circlearrowright$
A_3 : $\bullet \text{ --- } \circ \text{ --- } \bullet$	D_3 : $\bullet \text{ --- } \begin{matrix} \circ \\ \diagdown \\ \bullet \\ \diagup \\ \circ \end{matrix}$
A_4 : $\bullet \text{ --- } \bullet \text{ --- } \bullet \text{ --- } \bullet$	T_2 : $\bullet \text{ --- } \bullet \text{ --- } \circlearrowright$
A_5 : $\bullet \text{ --- } \bullet \text{ --- } \circ \text{ --- } \bullet \text{ --- } \bullet$	D_4 : $\bullet \text{ --- } \bullet \text{ --- } \begin{matrix} \circ \\ \diagdown \\ \bullet \\ \diagup \\ \circ \end{matrix}$
\vdots	\vdots

Herschend and Iyama showed that d -representation-finiteness is closely linked to the fractional Calabi-Yau property [16]. If an algebra Λ has finite global dimension, the bounded derived category of its module category has a Serre functor, an autoequivalence satisfying a certain duality. If a power of the Serre functor is given by a shift, Λ is said to be fractional Calabi-Yau. This property was introduced by Kontsevich to generalise properties of Calabi-Yau manifolds, which are important in theoretical physics.


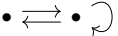
Given a d -representation-finite algebra Λ , one can construct its $(d + 1)$ -preprojective algebra Π . Note that Π is always selfinjective, and if Λ is basic then Π is Frobenius, meaning it is isomorphic to its dual as a Π - Π -bimodule, provided one twists by some automorphism σ . We call σ the Nakayama automorphism of Π . Grant showed that a d -representation-finite algebra is fractional Calabi-Yau if and only if the Nakayama automorphism of its $(d + 1)$ -preprojective algebra has finite order [13].

In this article we present a family of selfinjective algebras we call type D, which are Morita equivalent to skew group algebras of the 3-preprojective algebras of type A. Our definition is standalone, in the sense that it makes no reference to type A, and we prove the Morita equivalence by showing that our definition agrees with a construction of Giovannini and Pasquali [10]. In §4 we construct isomorphisms between our algebras and the type D algebras of Evans and Pugh.

We show that one in three of the selfinjective algebras of type D are 3-preprojective. By considering their Nakayama automorphisms, we show that the corresponding 2-representation-finite algebras are fractional Calabi-Yau. Finally, in §7 we give recipes to construct 2-Auslander-Reiten quivers for these algebras, on which one can see the fractional Calabi-Yau property quite explicitly.

Notation 1.1 Following the convention in [7, 8], we denote by \mathcal{D}^s the quiver which arises from \mathcal{A}^s (and label the corresponding algebras accordingly). However, only every third quiver is the quiver of a 3-preprojective algebra. We could have chosen to only label those with a D. However, it is not obvious how one should index in this case (see Table 2).

Table 2 Two options for notation. See Figure 1 for the missing quivers

Quiver	Our notation	3-preprojective algebra?	Alternative notation
	\mathcal{D}^2	\times	T^2
	\mathcal{D}^3	\times	T^2
-	\mathcal{D}^4	✓	\mathcal{D}^2
-	\mathcal{D}^5	\times	T^2
-	\mathcal{D}^6	\times	T^2
-	\mathcal{D}^7	✓	\mathcal{D}^2
⋮	⋮	⋮	⋮

Main results

By *algebra* we mean associative, unital, finite-dimensional algebra over $k = \mathbb{C}$. Modules are taken to be right modules unless otherwise stated. If p, q are paths in some quiver, pq means “first p then q ”. We take \mathbb{N} to contain 0.

Denote by $>$ the strict lexicographic order on \mathbb{N}^3 . Let $\omega: \mathbb{N}^3 \rightarrow \mathbb{N}^3, \omega(x_0, x_1, x_2) = (x_1, x_2, x_0)$.

Definition 1.2 Let $s \in \mathbb{Z}, s \geq 2$. Define

$$Q_0 = \{x \in \mathbb{N}^3 \mid x_0 + x_1 + x_2 = s - 1, x > \omega(x), x > \omega^2(x)\},$$

$$Q_1 = \bigcup_{i,j=0}^2 \{\alpha_{i,j}: x \rightarrow \omega^j(x) + f_i \mid x, \omega^j(x) + f_i \in Q_0\},$$

where $f_0 = (-1, 1, 0), f_1 = (0, -1, 1)$ and $f_2 = (1, 0, -1)$.

If $s \not\equiv 1 \pmod{3}$, let \mathcal{D}^s be the quiver with vertices $\mathcal{D}_0^s = Q_0$ and arrows $\mathcal{D}_1^s = Q_1$.

If $s = 3t + 1$ for some $t \in \mathbb{Z}^+$, write $X = (t, t, t)$, and let \mathcal{D}^s be the quiver with vertices

$$\mathcal{D}_0^s = Q_0 \cup \{X_0, X_1, X_2\}$$

(i.e. take three copies of X indexed by $\{0, 1, 2\}$) and arrows

$$\mathcal{D}_1^s = Q_1 \cup \{\beta_k: X - f_0 \rightarrow X_k, \gamma_k: X_k \rightarrow X + f_2 \mid k = 0, 1, 2\}.$$

Notation 1.3 An arrow $\alpha_{i,j}: x \rightarrow \omega^j(x) + f_i$ in \mathcal{D}^s is uniquely determined by its source and indices, so we often denote it $e_x \alpha_{i,j}$. If $j = 0$ we simplify notation further and just write $e_x \alpha_i$. Our convention when drawing \mathcal{D}^s is to label the “ α ” arrows using their indices alone.

Some examples of \mathcal{D}^s are presented in Fig. 1.

Let (Q, W) be a quiver with potential (QP) - a quiver together with a linear combination of cycles. By formally differentiating W with respect to the arrows of Q , one obtains the *Jacobian algebra* $\mathcal{J}(Q, W) = kQ / \langle \partial_\alpha W \mid \alpha \in Q_1 \rangle$. See e.g. [3, §2.1] for a full exposition.

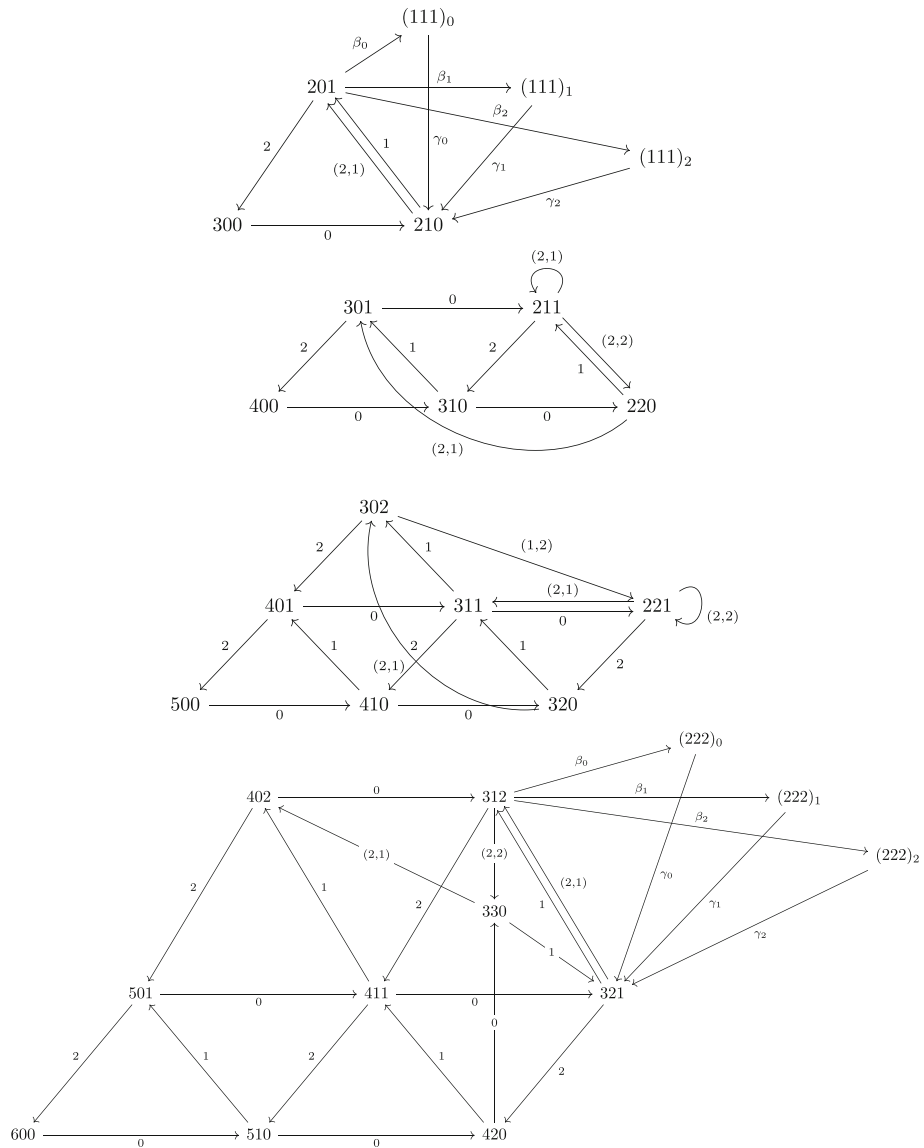


Fig. 1 Top to bottom: $\mathcal{D}^4, \mathcal{D}^5, \mathcal{D}^6, \mathcal{D}^7$

Definition 1.4 For each $s \geq 2$, we define a Jacobian algebra $\Pi_{\mathcal{D}}^s = \mathcal{J}(\mathcal{D}^s, W_{\mathcal{D}}^s)$ over \mathbb{C} . The potential is

$$W_{\mathcal{D}}^s = \sum_c \lambda_{\mathcal{D}}(c)c,$$

where the sum is taken over all 3-cycles in \mathcal{D}^s , and $\lambda_{\mathcal{D}}$ is defined as follows.

For $x, y \in \mathbb{N}^3$, write $x \sim y$ if $x = y, \omega(x) = y$ or $\omega^2(x) = y$. Take expressions involving indices of arrows mod 3.

1. Suppose $c = e_x \alpha_{i_0, j_0} \alpha_{i_1, j_1} \alpha_{i_2, j_2}$ is a product of three distinct arrows. If $j_0 + j_1 + j_2 = 0$ then

$$\lambda_{\mathcal{D}}(c) = \begin{cases} 1 & \text{if } (i_0 + j_0, i_1 - j_2, i_2) \sim (0, 1, 2), \\ -1 & \text{if } (i_0 + j_0, i_1 - j_2, i_2) \sim (0, 2, 1). \end{cases}$$

Otherwise, $\lambda_{\mathcal{D}}(c) = 0$.

2. If $c = e_x \alpha_{i,j}^3$ for a loop $e_x \alpha_{i,j}$ then

$$\lambda_{\mathcal{D}}(c) = \begin{cases} \frac{1}{3} & \text{if } (i + j, i - j, i) \sim (0, 1, 2), \\ -\frac{1}{3} & \text{if } (i + j, i - j, i) \sim (0, 2, 1). \end{cases}$$

3. If $s \equiv 1 \pmod{3}$ then for each $k \in \{0, 1, 2\}$,

$$\lambda_{\mathcal{D}}(\alpha_1 \beta_k \gamma_k) = -1, \quad \lambda_{\mathcal{D}}(\alpha_{2,1} \beta_k \gamma_k) = \zeta^k,$$

where $\zeta = e^{2\pi i/3}$.

Example 1.5 The potential on \mathcal{D}^4 is

$$W_{\mathcal{D}}^4 = \alpha_0 \alpha_1 \alpha_2 + \cancel{0 \alpha_0 \alpha_{2,1} \alpha_2} + \sum_{k=0}^2 (\zeta^k \alpha_{2,1} \beta_k \gamma_k - \alpha_1 \beta_k \gamma_k).$$

Hence $\Pi_{\mathcal{D}}^4$ is the algebra $\mathbb{C}\mathcal{D}^4$ modulo the relations

$$\begin{aligned} & \alpha_1 \alpha_2, \quad \alpha_0 \alpha_1, \\ & \sum_{k=0}^2 \zeta^k \beta_k \gamma_k, \quad \alpha_2 \alpha_0 - \sum_{k=0}^2 \beta_k \gamma_k, \\ & \{\zeta^k \gamma_k \alpha_{2,1} - \gamma_k \alpha_1 \mid k = 0, 1, 2\}, \quad \{\zeta^k \alpha_{2,1} \beta_k - \alpha_1 \beta_k \mid k = 0, 1, 2\}. \end{aligned}$$

The potential on \mathcal{D}^5 is given by

$$W_{\mathcal{D}}^5 = e_{400} \alpha_0 \alpha_1 \alpha_2 + e_{310} \alpha_0 \alpha_1 \alpha_2 - e_{301} \alpha_0 \alpha_2 \alpha_1 + e_{301} \alpha_0 \alpha_{2,2} \alpha_{2,1} - e_{211} \alpha_{2,1} \alpha_{2,2} \alpha_1 + \frac{1}{3} e_{211} \alpha_{2,1}^3.$$

In §2 we recall the 3-preprojective algebras of type A, which we denote $\Pi_{\mathcal{A}}^s$. The group \mathbb{Z}_3 acts on these by automorphisms, so we can consider the skew group algebras $\Pi_{\mathcal{A}}^s \# \mathbb{Z}_3$. In §3 we apply a construction of [10] to prove the following.

Theorem 1 (3.8) For each $s \geq 2$, $\Pi_{\mathcal{D}}^s$ is Morita equivalent to $\Pi_{\mathcal{A}}^s \# \mathbb{Z}_3$.

In type A, one can always take cuts of $\Pi_{\mathcal{A}}^s$ to obtain 2-representation-finite algebras $\Pi_{\mathcal{A}}^s / \langle C \rangle$, which are fractional Calabi-Yau. In §5 we show that the situation is different in type D: it is possible to take cuts of $\Pi_{\mathcal{D}}^s$ to obtain 2-representation-finite algebras if and only if $s \equiv 1 \pmod{3}$.

Theorem 2 (5.5) If $s \equiv 1 \pmod{3}$ and $C \subset \mathcal{D}_1^s$ is a cut, then $\Pi_{\mathcal{D}}^s / \langle C \rangle$ is 2-representation-finite, and its 3-preprojective algebra is $\Pi_{\mathcal{D}}^s$.

In §6 we prove that the resulting 2-representation-finite algebras are fractional Calabi-Yau.

Theorem 3 (6.6) Let $s = 3t + 1$, where $t \in \mathbb{Z}^+$. For any cut $C \subset \mathcal{D}_1^s$, $\Pi_{\mathcal{D}}^s / \langle C \rangle$ is fractional Calabi-Yau of dimension $2t/(t + 1)$.

2 3-preprojective algebras of type A

We recall some background on d -representation-finite algebras and their $(d + 1)$ -preprojective algebras, before presenting an important example.

Definition 2.1 [18, Def 2.1-2] Let Λ be an algebra, $d \in \mathbb{Z}^+$. We call $M \in \text{mod } \Lambda$ a d -cluster tilting object if

$$\begin{aligned} \text{add } M &= \{N \in \text{mod } \Lambda \mid \text{Ext}_{\Lambda}^i(M, N) = 0 \ \forall i \in \{1, \dots, d - 1\}\} \text{ and} \\ \text{add } M &= \{N \in \text{mod } \Lambda \mid \text{Ext}_{\Lambda}^i(N, M) = 0 \ \forall i \in \{1, \dots, d - 1\}\}. \end{aligned}$$

We call Λ d -representation-finite if $\text{gl. dim } \Lambda \leq d$ and $\text{mod } \Lambda$ contains a d -cluster tilting object.

Let Λ be a d -representation-finite algebra. The d -Auslander-Reiten translates are [19, Thm 1.4.1]

$$\begin{aligned} \tau_d &:= \tau \Omega^{d-1} : \underline{\text{mod}} \Lambda \rightarrow \overline{\text{mod}} \Lambda, \\ \tau_d^- &:= \tau^- \Omega^{1-d} : \overline{\text{mod}} \Lambda \rightarrow \underline{\text{mod}} \Lambda, \end{aligned}$$

where τ, τ^- are the classical Auslander-Reiten translates and Ω, Ω^- are the syzygy and cosyzygy functors, respectively (see e.g. [1, §IV.2]). These translates allow us to define a generalisation of the classical preprojective algebra of a quiver.

Definition 2.2 [16, §2] Let Λ be a d -representation-finite algebra. The $(d + 1)$ -preprojective algebra of Λ is

$$\Pi(\Lambda) = \bigoplus_{i \geq 0} \text{Hom}_{\Lambda}(\Lambda, \tau_d^{-i} \Lambda).$$

If $f : \Lambda \rightarrow \tau_d^{-i} \Lambda$ and $g : \Lambda \rightarrow \tau_d^{-j} \Lambda$, their product is

$$gf = \tau_d^{-i}(g) \circ f : \Lambda \rightarrow \tau_d^{-(i+j)} \Lambda.$$

There is a natural \mathbb{Z} -grading on $\Pi(\Lambda)$, called the *tensor grading*, where the degree i part is $\text{Hom}_{\Lambda}(\Lambda, \tau_d^{-i} \Lambda)$.

2-representation-finite algebras are particularly well-understood, thanks to a result of Herschend and Iyama. We need the following notion.

Definition 2.3 [17, Def 3.1] If (Q, W) is a QP, then to each subset $C \subseteq Q_1$ we associate a grading g_C on kQ , given on arrows by

$$g_C(\alpha) = \begin{cases} 1 & \text{if } \alpha \in C, \\ 0 & \text{else.} \end{cases}$$

A subset $C \subseteq Q_1$ is called a *cut* if W is homogeneous of degree 1 with respect to g_C . If C is a cut then g_C induces a grading on $\mathcal{J}(Q, W)$, and we call the degree 0 part $\mathcal{J}(Q, W)_C$ a *truncated Jacobian algebra*.

Note that $\mathcal{J}(Q, W)_C \cong \mathcal{J}(Q, W)/\langle C \rangle$.

As in [17, Def 3.6], we call a QP (Q, W) *selfinjective* if $\mathcal{J}(Q, W)$ is a selfinjective algebra. For general background on selfinjective, Frobenius and symmetric algebras see e.g. [9].

Theorem 2.4 [17, Thm 3.11]

1. If (Q, W) is a selfinjective QP and $C \subseteq Q_1$ is a cut, then $\mathcal{J}(Q, W)_C$ is 2-representation-finite, and its 3-preprojective algebra is $\mathcal{J}(Q, W)$.
2. Every basic 2-representation-finite algebra arises this way.

If $\mathcal{J}(Q, W)_C$ is basic 2-representation-finite, then the grading g_C on $\mathcal{J}(Q, W)$ corresponds to the tensor grading.

We now recall the 3-preprojective algebras of type A.

Definition 2.5 [18, Def 5.1] Let $s \in \mathbb{Z}, s \geq 2$. Let \mathcal{A}^s be the quiver with vertices

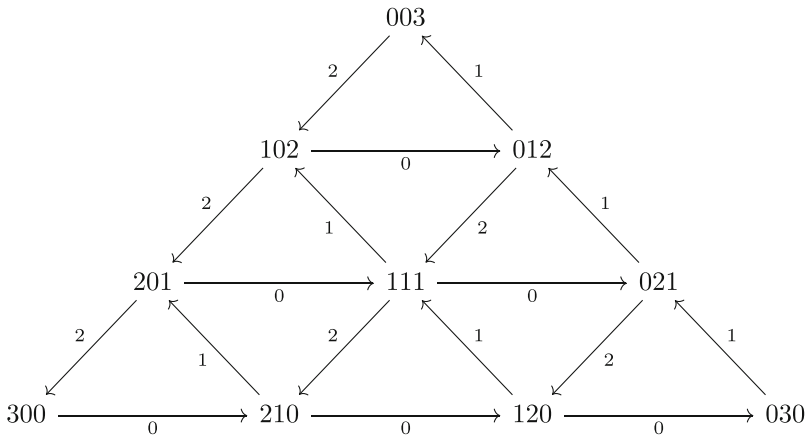
$$\mathcal{A}_0^s = \{x \in \mathbb{N}^3 \mid x_0 + x_1 + x_2 = s - 1\}$$

and arrows

$$\mathcal{A}_1^s = \bigcup_{i=0}^2 \{\alpha_i : x \longrightarrow x + f_i \mid x, x + f_i \in \mathcal{A}_0^s\},$$

where $f_0 = (-1, 1, 0), f_1 = (0, -1, 1)$ and $f_2 = (1, 0, -1)$.

For example, \mathcal{A}^4 is the following quiver.



Definition 2.6 For each $s \geq 2$, we define a Jacobian algebra $\Pi_{\mathcal{A}}^s = \mathcal{J}(\mathcal{A}^s, W_{\mathcal{A}}^s)$ over \mathbb{C} . The potential is

$$W_{\mathcal{A}}^s = \sum_c \lambda_{\mathcal{A}}(c)c,$$

where the sum is taken over all 3-cycles $c = e_x \alpha_{i_0} \alpha_{i_1} \alpha_{i_2}$ in \mathcal{A}^s , and

$$\lambda_{\mathcal{A}}(c) = \begin{cases} 1 & \text{if } (i_0, i_1, i_2) \sim (0, 1, 2), \\ -1 & \text{if } (i_0, i_1, i_2) \sim (0, 2, 1). \end{cases}$$

Informally, this is the sum of all anti-clockwise 3-cycles minus the sum of all clockwise 3-cycles.

Remark 2.7 If $\alpha_i e_x$ lies on an edge of \mathcal{A}^s then $\partial_{\alpha_i e_x} W_{\mathcal{A}}^s = e_x \alpha_{i+1} \alpha_{i-1}$, while if $\alpha_i e_x$ is an internal arrow then $\partial_{\alpha_i e_x} W_{\mathcal{A}}^s = e_x (\alpha_{i+1} \alpha_{i-1} - \alpha_{i-1} \alpha_{i+1})$. Hence $\Pi_{\mathcal{A}}^s$ is the path algebra of \mathcal{A}^s modulo the relations

1. each length two path which starts and ends on the same edge of \mathcal{A}^s and whose midpoint is not on that edge is zero,
2. each rhombus in \mathcal{A}^s commutes.

Thus $\Pi_{\mathcal{A}}^s$ is precisely the algebra called $\widehat{\Lambda}^{(2,s)}$ in [18, Def 5.1].

Notation 2.8 Let $s \geq 2$. Recall $\omega: \mathbb{N}^3 \rightarrow \mathbb{N}^3, \omega(x_0, x_1, x_2) = (x_1, x_2, x_0)$. It is clear that ω permutes \mathcal{A}_0^s , inducing an automorphism of \mathcal{A}^s such that

$$(x \xrightarrow{i} x + f_i) \mapsto (\omega(x) \xrightarrow{i-1} \omega(x) + f_{i-1}).$$

This is well-defined since $\omega(x + f_i) = \omega(x) + f_{i-1}$. We abuse notation and write ω for the induced automorphism of $\mathbb{C}\mathcal{A}^s$.

Proposition 2.9 [16, Thm 3.5]. *For any $s \geq 2$, the algebra $\Pi_{\mathcal{A}}^s$ is Frobenius, and its Nakayama automorphism is ω .*

Theorem 2.4 then implies that, for any $s \geq 2$ and cut $C \subset \mathcal{A}_1^s, \Pi_{\mathcal{A}}^s/\langle C \rangle$ is 2-representation-finite, and its 3-preprojective algebra is $\Pi_{\mathcal{A}}^s$ [18, Prop 5.48].

Closely related to d -representation-finiteness is the fractional Calabi-Yau property. For an algebra Λ with $\text{gl. dim } \Lambda < \infty$, let $\mathcal{D} = \mathbf{D}^b(\text{mod } \Lambda)$ be its bounded derived category. Denote by $\Sigma: \mathcal{D} \rightarrow \mathcal{D}$ the shift functor, and by

$$\nu = - \otimes_{\Lambda}^{\mathbf{L}} D\Lambda: \mathcal{D} \rightarrow \mathcal{D}$$

the derived Nakayama functor, where $D = \text{Hom}_{\mathbb{C}}(-, \mathbb{C})$. Note that ν is a Serre functor, meaning $\text{Hom}_{\mathcal{D}}(X, Y) \cong D \text{Hom}_{\mathcal{D}}(Y, \nu X)$ naturally in X and Y [14, §4.6]. By [2, Prop 3.3], there exists a natural isomorphism $n: \nu\Sigma \rightarrow \Sigma\nu$ making (ν, n) a *triangle functor* - for a definition see [21, §2.5].

Definition 2.10 [21, §2.6] Let Λ be an algebra with $\text{gl. dim } \Lambda < \infty$. Then Λ is *fractional Calabi-Yau* of dimension N/m if there exists an isomorphism of triangle functors

$$(\nu, n)^m \cong (\Sigma, -\text{id}_{\Sigma^2})^N$$

in \mathcal{D} for some $N, m \in \mathbb{Z}, m \neq 0$.

Remark 2.11 One should treat N/m as a pair of integers, not as a rational number.

Example 2.12 [5] For any cut $C \subset \mathcal{A}_1^s, \Pi_{\mathcal{A}}^s/\langle C \rangle$ is fractional Calabi-Yau of dimension $2(s-1)/(s+2)$.

3 Morita equivalence of $\Pi_{\mathcal{D}}^s$ and $\Pi_{\mathcal{A}}^s \# \mathbb{Z}_3$

Let Λ be an algebra and G be a finite group acting on Λ . Recall the *skew group algebra* $\Lambda \# G$ is the algebra with underlying vector space $\Lambda \otimes_k kG$, and multiplication defined by

$$(a \otimes g)(b \otimes h) = ag(b) \otimes gh$$

for $a, b \in \Lambda$ and $g, h \in G$.

In [10], the authors consider the case $\Lambda = \mathcal{J}(Q, W)$ is a Jacobian algebra and G is a finite cyclic group. If certain assumptions are satisfied, they construct a QP (\tilde{Q}, \tilde{W}) such that $\mathcal{J}(\tilde{Q}, \tilde{W})$ is Morita equivalent to $\Lambda \# G$. These assumptions are satisfied when (Q, W) is *strongly planar* and G acts by rotations [10, Lem 6.5].

Definition 3.1 [17, Def 8.1] Let (Q, W) be a QP. Its *canvas* $X_{(Q,W)}$ is the 2-dimensional CW complex defined as follows.

1. $X_{(Q,W)}^0 = Q_0$.
2. The 1-cells are indexed by Q_1 , and the attaching maps $\phi_\alpha^1: [0, 1] \rightarrow Q^0$ satisfy $\phi_\alpha^1(0) = s(\alpha)$ and $\phi_\alpha^1(1) = t(\alpha)$. Let $\varepsilon_\alpha^1: [0, 1] \rightarrow X_{(Q,W)}$ be a characteristic map extending ϕ_α^1 .
3. The 2-cells are indexed by cycles $c = \alpha_0 \cdots \alpha_{l-1}$ appearing in W , and the attaching maps $\phi_c^2: S^1 \rightarrow X_{(Q,W)}^1$ satisfy

$$\phi_c^2 \left(\cos \left(\frac{2\pi}{l} (i + t) \right), \sin \left(\frac{2\pi}{l} (i + t) \right) \right) = \varepsilon_{\alpha_i}^1 (t)$$

for integers $0 \leq i < s$ and real numbers $0 \leq t < 1$.

Informally, the 1-skeleton is the underlying graph of Q , and to obtain the canvas we glue a 2-cell to each cycle appearing in W . For general background on CW complexes see e.g. [15, §0].

Definition 3.2 [10, Def 6.3] We call a QP (Q, W) *strongly planar* if there is an embedding $\varepsilon: X_{(Q,W)} \rightarrow \mathbb{R}^2$ such that $\text{Im } \varepsilon$ is homeomorphic to a disk.

Lemma 3.3 For all $s \geq 2$, $(\mathcal{A}^s, W_{\mathcal{A}}^s)$ is strongly planar.

Proof Identify \mathbb{R}^2 with the plane $P = \{x \in \mathbb{R}^3 \mid x_0 + x_1 + x_2 = s - 1\}$. Embed the 0-cells (vertices of \mathcal{A}^s) in the obvious way, and embed each 1-cell $D_{e_x \alpha_i}^1$ as the line segment in P joining x and $x + f_i$. The induced embedding $\varepsilon: X_{(\mathcal{A}^s, W_{\mathcal{A}}^s)} \rightarrow P$ satisfies $\text{Im } \varepsilon = \{x \in P \mid x_0, x_1, x_2 \geq 0\}$, which is a closed triangle lying in the plane and therefore homeomorphic to a disk. □

Definition 3.4 [10, Def 6.4] Let (Q, W) be a strongly planar QP, and let G be a finite cyclic group acting on kQ . Then G is said to act on (Q, W) by *rotations* if

1. there is an embedding $X_{(Q,W)} \rightarrow \mathbb{R}^2$ such that the action of a generator of G is induced by a rotation of the plane;
2. the action of G is faithful;
3. every cycle c appearing in W is one of the following types:
 - (i) c goes through no vertices fixed by G ;
 - (ii) c goes through precisely one vertex fixed by G (counted with multiplicity);
 - (iii) c goes through precisely one vertex not fixed by G (counted with multiplicity);
 - (iv) c goes through only vertices fixed by G .

Lemma 3.5 For all $s \geq 2$, \mathbb{Z}_3 acts on $(\mathcal{A}^s, W_{\mathcal{A}}^s)$ by rotations.

Proof Recall the automorphism ω of $C\mathcal{A}^s$ from Notation 2.8. Clearly $\omega^3 = \text{id}$, so there is a group action

$$\begin{aligned} \mathbb{Z}_3 &\rightarrow \text{Aut}(C\mathcal{A}^s), \\ j &\mapsto \omega^j. \end{aligned}$$

1. Let $\varepsilon: X_{(\mathcal{A}^s, W_{\mathcal{A}}^s)} \rightarrow P$ be the embedding of Lemma 3.3. Then ω is induced by rotating the plane P clockwise through $\frac{2\pi}{3}$, about the point $(\frac{s-1}{3}, \frac{s-1}{3}, \frac{s-1}{3})$.
2. Since $s \geq 2$, $(s - 1, 0, 0)$ is a vertex of \mathcal{A}^s not fixed by ω or ω^2 , so the action of \mathbb{Z}_3 is faithful.

3. A vertex $x = (x_0, x_1, x_2) \in \mathcal{A}_0^s$ is fixed by \mathbb{Z}_3 if and only if $x_0 = x_1 = x_2$. Since $x_0 + x_1 + x_2 = s - 1$, this occurs if and only if $s = 3t + 1$ for some $t \in \mathbb{Z}^+$, in which case there is a unique fixed vertex (t, t, t) . Let c be a cycle appearing in $W_{\mathcal{A}}^s$. Then c passes through any given vertex of \mathcal{A}^s at most once. Since there is at most one vertex fixed by \mathbb{Z}_3 , c is either type (i) or type (ii). \square

Theorem 3.6 [10, Thm 3.20] *Let (Q, W) be a strongly planar QP and G be a finite cyclic group acting by rotations. Then G acts on $\mathcal{J}(Q, W)$, and there is an explicit construction of a QP (\tilde{Q}, \tilde{W}) and an idempotent $\eta \in \mathcal{J}(Q, W)\#G$ such that $\mathcal{J}(\tilde{Q}, \tilde{W}) \cong \eta(\mathcal{J}(Q, W)\#G)\eta$.*

In particular, there exists a QP $(\tilde{\mathcal{A}}^s, \tilde{W}_{\mathcal{A}}^s)$ such that $\mathcal{J}(\tilde{\mathcal{A}}^s, \tilde{W}_{\mathcal{A}}^s)$ is Morita equivalent to $\Pi_{\mathcal{A}}^s\#\mathbb{Z}_3$. We will describe the construction in this case, under the headings ‘Vertices’, ‘Arrows’ and ‘Potential’, and conclude that in fact $(\tilde{\mathcal{A}}^s, \tilde{W}_{\mathcal{A}}^s) = (D^s, W_D^s)$.

Vertices

Let $V_1 = \{x \in \mathcal{A}_0^s \mid x \succ \omega(x), x \succ \omega^2(x)\}$. Writing $X = (\frac{s-1}{3}, \frac{s-1}{3}, \frac{s-1}{3})$, let $V_2 = \{X\}$ if $s \equiv 1 \pmod{3}$, and $V_2 = \emptyset$ otherwise. Now $V_1 \sqcup V_2$ is a complete set of representatives of the \mathbb{Z}_3 -orbits of vertices of \mathcal{A}^s , and V_2 contains precisely the vertices fixed by \mathbb{Z}_3 . By [10, Notation 3.9-11], $\tilde{\mathcal{A}}_0^s = V_1$ if $s \not\equiv 1 \pmod{3}$, and $\tilde{\mathcal{A}}_0^s = V_1 \cup \{X_k \mid k = 0, 1, 2\}$ if $s \equiv 1 \pmod{3}$.

Arrows

For each arrow in \mathcal{A}_1^s , we fix a representative of its \mathbb{Z}_3 -orbit, and define arrows(s) in $\tilde{\mathcal{A}}_1^s$ corresponding to the representatives [10, Notation 3.13]. There are three cases.

1. Consider an arrow in \mathcal{A}_1^s between two vertices not fixed by \mathbb{Z}_3 . There is a unique arrow in its orbit whose target is in $\tilde{\mathcal{A}}_0^s$, and it must be of the form $\alpha_i : \omega^j(x) \rightarrow \omega^j(x) + f_i$ for some $x \in \tilde{\mathcal{A}}_0^s, i, j \in \{0, 1, 2\}$. We define a corresponding arrow $\alpha_{i,j} : x \rightarrow \omega^j(x) + f_i$ in $\tilde{\mathcal{A}}_1^s$.
2. Suppose $s \equiv 1 \pmod{3}$. There are three arrows in \mathcal{A}_1^s whose target is X , all in the same \mathbb{Z}_3 -orbit. Only one of them has its source in $\tilde{\mathcal{A}}_0^s$, namely $\alpha_0 : X - f_0 \rightarrow X$. We define three corresponding arrows $\{\beta_k : X - f_0 \rightarrow X_k \mid k = 0, 1, 2\}$ in $\tilde{\mathcal{A}}_1^s$.
3. Suppose $s \equiv 1 \pmod{3}$. There are three arrows in \mathcal{A}_1^s whose source is X , all in the same \mathbb{Z}_3 -orbit. Only one of them has its target in $\tilde{\mathcal{A}}_0^s$, namely $\alpha_2 : X \rightarrow X + f_2$. We define three corresponding arrows $\{\gamma_k : X_k \rightarrow X + f_2 \mid k = 0, 1, 2\}$ in $\tilde{\mathcal{A}}_1^s$.

Potential

We fix a complete set $C_1 \sqcup C_2$ of representatives of the \mathbb{Z}_3 -orbits of 3-cycles in \mathcal{A}^s , where C_1 contains cycles that do not pass through the fixed vertex X , and C_2 contains cycles that do. To each $c \in C_1$ we associate a cycle \tilde{c} in $\tilde{\mathcal{A}}^s$, and to each $c \in C_2$ we associate three cycles $\tilde{c}_0, \tilde{c}_1, \tilde{c}_2$ in $\tilde{\mathcal{A}}^s$ [10, Notation 3.17].

1. Consider a 3-cycle in \mathcal{A}^s that does not pass through X . Choose a representative c of its \mathbb{Z}_3 -orbit that passes through at least one vertex in $\tilde{\mathcal{A}}_0^s$. Then c is of the form

$$x \xrightarrow{l_0} y' \xrightarrow{l_1} z' \xrightarrow{l_2} x$$

where $x \in \widetilde{\mathcal{A}}_0^s$, $\{l_0, l_1, l_2\} = \{0, 1, 2\}$, $y' = x + f_{l_0}$ and $z' = x - f_{l_2}$. There exist unique $j_0, j_2 \in \{0, 1, 2\}$ such that $y := \omega^{j_0}(y') \in \widetilde{\mathcal{A}}_0^s$ and $z := \omega^{-j_2}(z') \in \widetilde{\mathcal{A}}_0^s$.

To simplify notation moving forward, write $i_0 = l_0 - j_0$, $i_1 = l_1 + j_2$, $i_2 = l_2$ and $j_1 = -(j_0 + j_2)$. Define \widetilde{c} to be the cycle

$$x \xrightarrow{(i_0, j_0)} y \xrightarrow{(i_1, j_1)} z \xrightarrow{(i_2, j_2)} x$$

in $\widetilde{\mathcal{A}}^s$. For a proof \widetilde{c} that exists as claimed, see Remark 3.7.

- Suppose $s \equiv 1 \pmod{3}$. There are six cycles in $W_{\mathcal{A}}^s$ passing through X , in two disjoint \mathbb{Z}_3 -orbits. As representatives of these orbits we choose

$$\begin{aligned} c^- : X + f_2 &\xrightarrow{1} X - f_0 \xrightarrow{0} X \xrightarrow{2} X + f_2, \\ c^+ : X + f_2 &\xrightarrow{0} X - f_1 \xrightarrow{1} X \xrightarrow{2} X + f_2. \end{aligned}$$

For each $k \in \{0, 1, 2\}$, we define the cycles

$$\begin{aligned} \widetilde{c}_k^- : X + f_2 &\xrightarrow{1} X - f_0 \xrightarrow{\beta_k} X_k \xrightarrow{\gamma_k} X + f_2, \\ \widetilde{c}_k^+ : X + f_2 &\xrightarrow{(2,1)} X - f_0 \xrightarrow{\beta_k} X_k \xrightarrow{\gamma_k} X + f_2 \end{aligned}$$

in $\widetilde{\mathcal{A}}^s$. We also define $p(c^-) = 0$, $p(c^+) = -1$. Informally, this will adjust for the fact $X - f_1 \notin \mathcal{A}_0^s$ but $\omega(X - f_1) \in \mathcal{A}_0^s$.

By [10, Notation 3.18], the potential on $\widetilde{\mathcal{A}}^s$ is given by

$$\widetilde{W}_{\mathcal{A}}^s = \sum_{c \in C_1} \lambda_{\mathcal{A}}(c) \frac{|\mathbb{Z}_3 \cdot c|}{3} \widetilde{c} + \sum_{c \in C_2} \lambda_{\mathcal{A}}(c) \sum_{k=0}^2 \zeta^{-p(c)k} \widetilde{c}_k.$$

Remark 3.7 We show the cycle \widetilde{c} defined above exists. Recall c is the cycle

$$x \xrightarrow{l_0} \omega^{-j_0}(y) \xrightarrow{l_1} \omega^{j_2}(z) \xrightarrow{l_2} x$$

in \mathcal{A}^s , where $x, y, z \in \widetilde{\mathcal{A}}_0^s$, $\{l_0, l_1, l_2\} = \{0, 1, 2\}$ and $j_0, j_2 \in \{0, 1, 2\}$.

We find the arrows in $\widetilde{\mathcal{A}}^s$ induced by each arrow in c . Note that $\omega^j(e_x \alpha_i) = e_{\omega^j(x)} \alpha_{i-j}$ for all $i, j \in \{0, 1, 2\}$ and $x \in \mathcal{A}_0^s$. Recall we write $i_0 = l_0 - j_0$, $i_1 = l_1 + j_2$, $i_2 = l_2$ and $j_1 = -(j_0 + j_2)$.

- Consider $x \xrightarrow{l_0} \omega^{-j_0}(y)$. Applying ω^{j_0} , we see that the arrow in its \mathbb{Z}_3 -orbit with target in $\widetilde{\mathcal{A}}_0^s$ is $(\omega^{j_0}(x) \xrightarrow{l_0-j_0} y) = (\omega^{j_0}(x) \xrightarrow{i_0} y)$. Hence, there is an arrow $x \xrightarrow{(i_0, j_0)} y$ in $\widetilde{\mathcal{A}}^s$.
- Consider $\omega^{-j_0}(y) \xrightarrow{l_1} \omega^{j_2}(z)$. Applying ω^{-j_2} , we see that the arrow in its \mathbb{Z}_3 -orbit with target in $\widetilde{\mathcal{A}}_0^s$ is $(\omega^{-(j_0+j_2)}(y) \xrightarrow{l_1+j_2} z) = (\omega^{j_1}(y) \xrightarrow{i_1} z)$. Hence, there is an arrow $y \xrightarrow{(i_1, j_1)} z$ in $\widetilde{\mathcal{A}}^s$.
- Note that $\omega^{j_2}(z) \xrightarrow{l_2} x$ already has target in $\widetilde{\mathcal{A}}_0^s$ so, using $i_2 = l_2$, there is an arrow $z \xrightarrow{(i_2, j_2)} x$ in $\widetilde{\mathcal{A}}_0^s$.

Hence the cycle \widetilde{c} given by $x \xrightarrow{(i_0, j_0)} y \xrightarrow{(i_1, j_1)} z \xrightarrow{(i_2, j_2)} x$ exists in $\widetilde{\mathcal{A}}^s$ as claimed.

We now prove Theorem 1. The reader may wish to recall Definitions 1.2 and 1.4.

Theorem 3.8 For all $s \geq 2$, $(\widetilde{\mathcal{A}}^s, \widetilde{W}_{\mathcal{A}}^s) = (\mathcal{D}^s, W_{\mathcal{D}}^s)$. Hence, $\Pi_{\mathcal{D}}^s \cong \eta(\Pi_{\mathcal{A}}^s \# \mathbb{Z}_3)\eta$ for some idempotent $\eta \in \Pi_{\mathcal{A}}^s \# \mathbb{Z}_3$.

Proof By inspection, we have that $\widetilde{\mathcal{A}}_0^s = \mathcal{D}_0^s$ and $\widetilde{\mathcal{A}}_1^s \subseteq \mathcal{D}_1^s$. To see that $\mathcal{D}_1^s \subseteq \widetilde{\mathcal{A}}_1^s$, let $e_x \alpha_{i,j} \in \mathcal{D}_1^s$. Then $x, \omega^j(x) + f_i \in \widetilde{\mathcal{A}}_0^s$, so certainly $\omega^j(x) \in \mathcal{A}_0^s$. Thus $e_{\omega^j(x)} \alpha_i$ is an arrow in \mathcal{A}_1^s between two vertices not fixed by \mathbb{Z}_3 , whose target is in $\widetilde{\mathcal{A}}_0^s$. Hence $e_x \alpha_{i,j} \in \widetilde{\mathcal{A}}_1^s$ by construction, and $\widetilde{\mathcal{A}}^s = \mathcal{D}^s$.

Note that both $\widetilde{W}_{\mathcal{A}}^s$ and $W_{\mathcal{D}}^s$ are linear combinations of 3-cycles in \mathcal{D}^s . So to show $\widetilde{W}_{\mathcal{A}}^s = W_{\mathcal{D}}^s$ it is enough to check, for every 3-cycle c in \mathcal{D}^s , that its coefficient $\widetilde{\lambda}_{\mathcal{A}}(c)$ in $\widetilde{W}_{\mathcal{A}}^s$ is equal to $\lambda_{\mathcal{D}}(c)$.

1. Suppose $c = e_x \alpha_{i_0, j_0} \alpha_{i_1, j_1} \alpha_{i_2, j_2}$ is the product of three distinct arrows. If $j_0 + j_1 + j_2 \neq 0$, then c is not in $\widetilde{W}_{\mathcal{A}}^s$ by construction, so $\widetilde{\lambda}_{\mathcal{A}}(c) = 0 = \lambda_{\mathcal{D}}(c)$. Otherwise, $c = \widetilde{d}$, where d is the cycle $e_x \alpha_{i_0 + j_0} \alpha_{i_1 - j_2} \alpha_{i_2}$ in \mathcal{A}^s . Hence

$$\begin{aligned} \widetilde{\lambda}_{\mathcal{A}}(c) &= \lambda_{\mathcal{A}}(d) \frac{|\mathbb{Z}_3 \cdot d|}{3} \\ &= \lambda_{\mathcal{A}}(d) \\ &= \begin{cases} 1 & \text{if } (i_0 + j_0, i_1 - j_2, i_2) \sim (0, 1, 2), \\ -1 & \text{if } (i_0 + j_0, i_1 - j_2, i_2) \sim (0, 2, 1) \end{cases} \\ &= \lambda_{\mathcal{D}}(c). \end{aligned}$$

2. Suppose $c = e_x \alpha_{i,j}^3$ for a loop $e_x \alpha_{i,j}$. If $3j \neq 0$ then c is not in $\widetilde{W}_{\mathcal{A}}^s$ by construction, so $\widetilde{\lambda}_{\mathcal{A}}(c) = 0 = \lambda_{\mathcal{D}}(c)$. Otherwise, $c = \widetilde{d}$, where d is the cycle $e_x \alpha_{i+j} \alpha_{i-j} \alpha_i$ in \mathcal{A}^s . Hence

$$\begin{aligned} \widetilde{\lambda}_{\mathcal{A}}(c) &= \lambda_{\mathcal{A}}(d) \frac{|\mathbb{Z}_3 \cdot d|}{3} \\ &= \frac{\lambda_{\mathcal{A}}(d)}{3} \\ &= \begin{cases} \frac{1}{3} & \text{if } (i + j, i - j, i) \sim (0, 1, 2), \\ -\frac{1}{3} & \text{if } (i + j, i - j, i) \sim (0, 1, 2) \end{cases} \\ &= \lambda_{\mathcal{D}}(c). \end{aligned}$$

3. Suppose $s \equiv 1 \pmod{3}$ and let c be a 3-cycle in \mathcal{D}^s passing through X_k for some $k \in \{0, 1, 2\}$. Then either $c = \alpha_1 \beta_k \gamma_k = \widetilde{c}_k^-$ or $c = \alpha_{2,1} \beta_k \gamma_k = \widetilde{c}_k^+$. Now

$$\begin{aligned} \widetilde{\lambda}_{\mathcal{A}}(\alpha_1 \beta_k \gamma_k) &= \lambda_{\mathcal{A}}(c^-) \zeta^{-p(c^-)k} = -1 = \lambda_{\mathcal{D}}(\alpha_1 \beta_k \gamma_k), \\ \widetilde{\lambda}_{\mathcal{A}}(\alpha_{2,1} \beta_k \gamma_k) &= \lambda_{\mathcal{A}}(c^+) \zeta^{-p(c^+)k} = \zeta^k = \lambda_{\mathcal{D}}(\alpha_{2,1} \beta_k \gamma_k) \end{aligned}$$

and we are done. □

4 Connection with operator algebras

In [7, 8], Evans and Pugh study Jacobian algebras of the quivers \mathcal{A}^s and \mathcal{D}^s (considered as the $SU(3)$ \mathcal{ADE} graphs of Di Francesco and Zuber [4]) with respect to different potentials. We show that their algebras are isomorphic to those considered in this article. Note that the quiver we call \mathcal{A}^s they call $\mathcal{A}^{(s+2)}$, and likewise \mathcal{D}^s corresponds to $\mathcal{D}^{(s+2)}$.

Type A

In the following we write $q = e^{\pi i/(s+2)}$, where $s \geq 2$ is the index of the quiver \mathcal{A}^s . For each $n \in \mathbb{Z}^+$, we define the *quantum number* $[n] = (q^n - q^{-n})/(q - q^{-1})$.

Definition 4.1 [7, Thm 5.1] For each $s \geq 2$, define a potential on \mathcal{A}^s by

$$V_{\mathcal{A}}^s = \sum_c \mu_{\mathcal{A}}(c)c,$$

where the sum runs over all 3-cycles in \mathcal{A}^s and, for $x = (x_0, x_1, x_2) \in \mathcal{A}_0^s$,

$$\begin{aligned} \mu_{\mathcal{A}}(e_x \alpha_0 \alpha_1 \alpha_2) &= \sqrt{[x_1 + 1][x_1 + 2][x_2 + 1][x_2 + 2][x_1 + x_2 + 2][x_1 + x_2 + 3]}/[2], \\ \mu_{\mathcal{A}}(e_x \alpha_0 \alpha_2 \alpha_1) &= \sqrt{[x_1 + 1][x_1 + 2][x_2][x_2 + 1][x_1 + x_2 + 2][x_1 + x_2 + 3]}/[2]. \end{aligned}$$

Remark 4.2 We believe there is a small typo in [7, Thm 5.1]. Namely, the formula (14) should have $[k + m + 3][k + m + 4]$ in place of $[k + m + 2][k + m + 3]$. Translating into our notation gives the above definition.

The following lemma will be the key tool in proving $\Pi_{\mathcal{A}}^s \cong \mathcal{J}(\mathcal{A}^s, V_{\mathcal{A}}^s)$.

Lemma 4.3 [6, Prop 3.7] *Let (Q, W) be a QP and let Q' be a quiver. Any algebra isomorphism $f: kQ \rightarrow kQ'$ induces an algebra isomorphism $\mathcal{J}(Q, W) \cong \mathcal{J}(Q', f(W))$.*

We first show that one can replace the commutativity relations in $\Pi_{\mathcal{A}}^s$ with anti-commutativity relations, inspired by work in [11, §3.3].

Lemma 4.4 *Let $s \geq 2$. Denote by $|W_{\mathcal{A}}^s|$ the potential on \mathcal{A}^s given by the sum of all 3-cycles, each with coefficient 1. Then $\Pi_{\mathcal{A}}^s \cong \mathcal{J}(\mathcal{A}^s, |W_{\mathcal{A}}^s|)$.*

Proof Let $\text{par}_i(x) = (-1)^{s-x_i+1}$ for all $x = (x_0, x_1, x_2) \in \mathcal{A}_0^s$ and $i \in \{0, 1, 2\}$. Let $\varphi: \mathbb{C}\mathcal{A}^s \rightarrow \mathbb{C}\mathcal{A}^s$ be the algebra automorphism induced by $e_x \alpha_i \mapsto \text{par}_i(x)e_x \alpha_i$. Consider a cycle c in $W_{\mathcal{A}}^s$. If $c = e_x \alpha_0 \alpha_1 \alpha_2$ then $\lambda_{\mathcal{A}}(c) = 1$ so

$$\begin{aligned} \varphi(\lambda_{\mathcal{A}}(c)c) &= \text{par}_0(x) \text{par}_1(x + f_0) \text{par}_2(x - f_2)c \\ &= (-1)^{3s+1-(x_0+x_1+x_2)}c \\ &= (-1)^{2s+2}c \\ &= c, \end{aligned}$$

where we used that $x_0 + x_1 + x_2 = s - 1$. Similarly if $c = e_x \alpha_0 \alpha_2 \alpha_1$ then one can check $\varphi(\lambda_{\mathcal{A}}(c)c) = c$. Hence $\varphi(W_{\mathcal{A}}^s) = |W_{\mathcal{A}}^s|$, so $\Pi_{\mathcal{A}}^s \cong \mathcal{J}(\mathcal{A}^s, |W_{\mathcal{A}}^s|)$ by Lemma 4.3. \square

We can now make our conclusion.

Proposition 4.5 *For all $s \geq 2$, $\Pi_{\mathcal{A}}^s \cong \mathcal{J}(\mathcal{A}^s, V_{\mathcal{A}}^s)$.*

Proof We show $\mathcal{J}(\mathcal{A}^s, |W_{\mathcal{A}}^s|) \cong \mathcal{J}(\mathcal{A}^s, V_{\mathcal{A}}^s)$, at which point the statement follows by Lemma 4.4. Let

$$\begin{aligned} \text{coef}_0(x) &= \sqrt[4]{[x_1 + 1][x_1 + 2][x_1 + x_2 + 2][x_1 + x_2 + 3]}/\sqrt[3]{[2]}, \\ \text{coef}_1(x) &= \sqrt[4]{[x_1][x_1 + 1][x_2 + 1][x_2 + 2]}/\sqrt[3]{[2]}, \\ \text{coef}_2(x) &= \sqrt[4]{[x_2][x_2 + 1][x_1 + x_2 + 1][x_1 + x_2 + 2]}/\sqrt[3]{[2]} \end{aligned}$$

for all $x = (x_0, x_1, x_2) \in \mathcal{A}_0^s$. Let $\psi : \mathbb{C}\mathcal{A}^s \rightarrow \mathbb{C}\mathcal{A}^s$ be the algebra automorphism induced by $e_x \alpha_i \mapsto \text{coef}_i(x) e_x \alpha_i$. In view of Lemma 4.3, it is enough to show that $\psi(|W_{\mathcal{A}}^s|) = V_{\mathcal{A}}^s$. Indeed, it is straightforward to calculate

$$\begin{aligned} \psi(e_x \alpha_0 \alpha_1 \alpha_2) &= \text{coef}_0(x) \text{coef}_1(x + f_0) \text{coef}_2(x - f_2) e_x \alpha_0 \alpha_1 \alpha_2 = \mu_{\mathcal{A}}(e_x \alpha_0 \alpha_1 \alpha_2) e_x \alpha_0 \alpha_1 \alpha_2, \\ \psi(e_x \alpha_0 \alpha_2 \alpha_1) &= \text{coef}_0(x) \text{coef}_2(x + f_0) \text{coef}_1(x - f_1) e_x \alpha_0 \alpha_2 \alpha_1 = \mu_{\mathcal{A}}(e_x \alpha_0 \alpha_2 \alpha_1) e_x \alpha_0 \alpha_2 \alpha_1, \end{aligned}$$

so we are done. □

Type D

To get from $(\mathcal{A}^s, W_{\mathcal{A}}^s)$ to $(\mathcal{A}^s, V_{\mathcal{A}}^s)$, the only change we make is to multiply the coefficient of each cycle in the potential by a non-zero constant. The canvas does not change, i.e. $X_{(\mathcal{A}^s, W_{\mathcal{A}}^s)} = X_{(\mathcal{A}^s, V_{\mathcal{A}}^s)}$. In particular, $(\mathcal{A}^s, V_{\mathcal{A}}^s)$ is strongly planar and \mathbb{Z}_3 acts by rotations. Thus we can apply the construction of [10, §3.2-3] to $(\mathcal{A}^s, V_{\mathcal{A}}^s)$. The specific coefficients in the potential do not play a role in the construction of the quiver, nor of the idempotent. Hence we obtain a potential $\widetilde{V}_{\mathcal{A}}^s$ such that $\mathcal{J}(\mathcal{D}^s, \widetilde{V}_{\mathcal{A}}^s) \cong \eta(\mathcal{J}(\mathcal{A}^s, V_{\mathcal{A}}^s) \# \mathbb{Z}_3) \eta$, where η is as in Theorem 3.8. Furthermore, a cycle appears in $W_{\mathcal{A}}^s$ if and only if it appears in $V_{\mathcal{A}}^s$, so the same cycles appear in $\widetilde{W}_{\mathcal{A}}^s$ and $\widetilde{V}_{\mathcal{A}}^s$. That is to say,

$$\widetilde{V}_{\mathcal{A}}^s = \sum_{c \in C_1} \mu_{\mathcal{A}}(c) \frac{|\mathbb{Z}_3 \cdot c|}{3} \widetilde{c} + \sum_{c \in C_2} \mu_{\mathcal{A}}(c) \sum_{k=0}^2 \zeta^{-p(c)k} \widetilde{c}_k.$$

Definition 4.6 [7, Thm 6.1, 6.2] For each $s \geq 2$, define a potential on \mathcal{D}^s by

$$V_{\mathcal{D}}^s = \sum_c \mu_{\mathcal{D}}(c) c,$$

where the sum runs over all 3-cycles in \mathcal{D}^s , and $\mu_{\mathcal{D}}$ is defined as follows.

If $s = 3t + 1$ for some $t \in \mathbb{Z}^+$, then

$$\begin{aligned} \mu_{\mathcal{D}}(\alpha_1 \beta_k \gamma_k) &= \zeta^k \frac{[t] \sqrt{[t+1]^3 [t+2]}}{\sqrt{3}[2]}, \\ \mu_{\mathcal{D}}(\alpha_{2,1} \beta_k \gamma_k) &= \zeta^{-k} \frac{[t+2] \sqrt{[t][t+1]^3}}{\sqrt{3}[2]}. \end{aligned}$$

For all other cycles c , including when $s \not\equiv 1 \pmod{3}$, $\mu_{\mathcal{D}}(c) = \widetilde{\mu}_{\mathcal{A}}(c)$.

Remark 4.7 In [7, Thm 6.2], the authors explicitly give the coefficients of six cycles, and say the rest are given by the ‘‘corresponding’’ cycles in \mathcal{A}^s . Hence, we set $\mu_{\mathcal{D}}(c) = \widetilde{\mu}_{\mathcal{A}}(c)$ for all other cycles c . Once we make this assumption, we already get the correct coefficients for four of the distinguished cycles, so we omit them in the above definition.

Proposition 4.8 *Let $s \geq 2$. Then $\Pi_{\mathcal{D}}^s \cong \mathcal{J}(\mathcal{D}^s, V_{\mathcal{D}}^s)$.*

Proof First, we show that $\mathcal{J}(\mathcal{D}^s, \widetilde{V}_{\mathcal{A}}^s) \cong \mathcal{J}(\mathcal{D}^s, V_{\mathcal{D}}^s)$. To this end, let $\chi : \mathbb{C}\mathcal{D}^s \rightarrow \mathbb{C}\mathcal{D}^s$ be the algebra automorphism such that $\chi(\beta_k) = \frac{\zeta^k}{\sqrt{3}} \beta_k$ and χ is the identity on all other arrows.

We check that $\chi(\widetilde{\mu}_{\mathcal{A}}(c)c) = \mu_{\mathcal{D}}(c)c$ for all 3-cycles c in \mathcal{D}^s , whence $\chi(\widetilde{V}_{\mathcal{A}}^s) = V_{\mathcal{D}}^s$ and the claim follows by Lemma 4.3.

If c is not one of the distinguished cycles in Definition 4.6, $\chi(\widetilde{\mu}_{\mathcal{A}}(c)c) = \widetilde{\mu}_{\mathcal{A}}(c)c = \mu_{\mathcal{D}}(c)c$ by definition.

If $c = \alpha_1\beta_k\gamma_k$ then $c = \widetilde{c}_k^-$, where $c^- = e_{(t+1,t-1,t)}\alpha_0\alpha_2\alpha_1$ (see the construction of the potential in §3). So

$$\begin{aligned} \chi(\widetilde{\mu}_{\mathcal{A}}(c)c) &= \frac{\zeta^k}{\sqrt{3}}\widetilde{\mu}_{\mathcal{A}}(c)c \\ &= \frac{\zeta^k}{\sqrt{3}}\mu_{\mathcal{A}}(c^-)\zeta^{-p(c^-)k}c \\ &= \zeta^k \frac{[t+1]\sqrt{[t][t+2][2t+1][2t+2]}}{\sqrt{3}[2]}c \\ &= \zeta^k \frac{[t]\sqrt{[t+1]^3[t+2]}}{\sqrt{3}[2]}c \\ &= \mu_{\mathcal{D}}(c)c, \end{aligned}$$

where we used that $p(c^-) = 0$ and that $[n] = [3t + 3 - n]$ for all $1 \leq n \leq 2t + 2$ [7, Lem 4.3].

Similarly, if $c = \alpha_{2,1}\beta_k\gamma_k$ then $c = \widetilde{c}_k^+$, where $c^+ = e_{(t+1,t,t-1)}\alpha_0\alpha_1\alpha_2$. Recalling $p(c^+) = -1$, one calculates as above to show $\chi(\widetilde{\mu}_{\mathcal{A}}(c)c) = \mu_{\mathcal{D}}(c)c$. Therefore $\mathcal{J}(\mathcal{D}^s, \widetilde{V}_{\mathcal{A}}^s) \cong \mathcal{J}(\mathcal{D}^s, V_{\mathcal{D}}^s)$.

By Proposition 4.5, we have an isomorphism $\psi: \Pi_{\mathcal{A}}^s \rightarrow \mathcal{J}(A^s, V_{\mathcal{A}}^s)$, which induces an isomorphism $\psi \otimes \text{id}: \Pi_{\mathcal{A}}^s \# \mathbb{Z}_3 \rightarrow \mathcal{J}(A^s, V_{\mathcal{A}}^s) \# \mathbb{Z}_3$. Note that $\eta = \sum d_i \otimes e_j$ for some idempotents $d_i \in \Pi_{\mathcal{A}}^s$ (which are length zero) and $e_j \in \mathbb{C}\mathbb{Z}_3$ [10, Notation 3.11]. Since ψ acts trivially on the d_i , $(\psi \otimes \text{id})(\eta) = \eta$, and we have a chain of isomorphisms

$$\Pi_{\mathcal{D}}^s \cong \eta(\Pi_{\mathcal{A}}^s \# \mathbb{Z}_3)\eta \xrightarrow{\psi \otimes \text{id}} \eta(\mathcal{J}(A^s, V_{\mathcal{A}}^s) \# \mathbb{Z}_3)\eta \cong \mathcal{J}(\mathcal{D}^s, \widetilde{V}_{\mathcal{A}}^s) \xrightarrow{\chi} \mathcal{J}(\mathcal{D}^s, V_{\mathcal{D}}^s).$$

□

5 Cuts of $\Pi_{\mathcal{D}}^s$

We would like to emulate type A, and take cuts of $\Pi_{\mathcal{D}}^s$ to obtain some 2-representation-finite algebras of type D. This turns out to be impossible if $s \not\equiv 1 \pmod{3}$.

Lemma 5.1 *Let $s \geq 2$, $s \not\equiv 1 \pmod{3}$. There are zero cuts of $(\mathcal{D}^s, W_{\mathcal{D}}^s)$.*

Proof If $s = 3t$ for some $t \in \mathbb{Z}^+$, there is a loop $\alpha_{2,2}: x \rightarrow x$ in \mathcal{D}_1^s , where $x = (t, t, t - 1)$. By Definition 1.4, the 3-cycle $e_x\alpha_{2,2}^3$ appears in $W_{\mathcal{D}}^s$ with non-zero coefficient. For any subset $C \subseteq \mathcal{D}_1^s$, either $g_C(e_x\alpha_{2,2}^3) = 0$ or $g_C(e_x\alpha_{2,2}^3) = 3$, so C is not a cut of $(\mathcal{D}^s, W_{\mathcal{D}}^s)$.

If $s = 3t + 2$ for some $t \in \mathbb{N}$, there is a loop $\alpha_{2,1}: y \rightarrow y$ in \mathcal{D}_1^s , where $y = (t + 1, t, t)$. As before, $e_y\alpha_{2,1}^3$ appears in $W_{\mathcal{D}}^s$ with non-zero coefficient, and the same argument shows there are no cuts of $(\mathcal{D}^s, W_{\mathcal{D}}^s)$. □

Thus, it is impossible for $\Pi_{\mathcal{D}}^s$ to be a 3-preprojective algebra if $s \not\equiv 1 \pmod{3}$. We will see that if $s \equiv 1 \pmod{3}$, then $\Pi_{\mathcal{D}}^s$ is indeed a 3-preprojective algebra.

Definition 5.2 [10, Def 7.5] Let (Q, W) be a QP , and let G be a group acting on $\mathcal{J}(Q, W)$. We call a cut $C \subseteq Q_1$ G -invariant if $g \cdot \alpha \in C$ for all $\alpha \in C$. We say a $QP(Q, W)$ has enough $(G$ -invariant) cuts if, for every $\alpha \in Q_1$, there exists a $(G$ -invariant) cut C with $\alpha \in C$.

Lemma 5.3 Let $s = 3t + 1$ for some $t \in \mathbb{Z}^+$. Then $(\mathcal{D}^s, W_{\mathcal{D}}^s)$ has enough cuts.

Proof Consider an arrow $\delta \in \mathcal{D}_1^s$. There are three possibilities.

1. $\delta = e_x \alpha_{i,j}$ for some $x \in \mathcal{D}_0^s, i, j \in \{0, 1, 2\}$. In this case let $\delta' = e_{\omega^j(x)} \alpha_i \in \mathcal{A}_1^s$.
2. $\delta = \beta_k$ for some $k \in \{0, 1, 2\}$. In this case let $\delta' = \alpha_0 e_X \in \mathcal{A}_1^s$, where $X = (t, t, t)$.
3. $\delta = \gamma_k$ for some $k \in \{0, 1, 2\}$. In this case let $\delta' = e_X \alpha_2 \in \mathcal{A}_1^s$.

By [10, Prop 8.2], $(\mathcal{A}^s, W_{\mathcal{A}}^s)$ has enough \mathbb{Z}_3 -invariant cuts, so there exists a \mathbb{Z}_3 -invariant cut $C' \subset \mathcal{A}_1^s$ containing δ' . By the recipe in [10, Prop 7.3],

$$C = \{e_x \alpha_{i,j} \mid x \in Q_0, e_{\omega^j(x)} \alpha_i \in C'\} \cup \{\beta_0, \beta_1, \beta_2 \mid \alpha_0 e_X \in C'\} \cup \{\gamma_0, \gamma_1, \gamma_2 \mid e_X \alpha_2 \in C'\} \tag{1}$$

is a cut of $(\mathcal{D}^s, W_{\mathcal{D}}^s)$. Since $\delta' \in C'$, we have $\delta \in C$, so $(\mathcal{D}^s, W_{\mathcal{D}}^s)$ has enough cuts. \square

Lemma 5.4 [10, Cor 2.6] Let Λ be a Frobenius algebra, whose Nakayama automorphism σ generates $\text{Im}(G) \subseteq \text{Aut}(\Lambda)$. Then $\Lambda \# G$ is symmetric.

We now prove Theorem 2.

Theorem 5.5 For all $s \geq 2$, $\Pi_{\mathcal{D}}^s$ is symmetric. If $s \equiv 1 \pmod{3}$ and $C \subset \mathcal{D}_1^s$ is a cut, then $\Pi_{\mathcal{D}}^s / \langle C \rangle$ is 2-representation-finite, and its 3-preprojective algebra is $\Pi_{\mathcal{D}}^s$.

Proof Proposition 2.9 says that $\Pi_{\mathcal{A}}^s$ is Frobenius. Its Nakayama automorphism ω generates $\text{Im}(\mathbb{Z}_3) \subseteq \text{Aut}(\Pi_{\mathcal{A}}^s)$, so we can apply Lemma 5.4 to conclude that $\Pi_{\mathcal{A}}^s \# \mathbb{Z}_3$ is symmetric. Being symmetric is Morita invariant, so $\Pi_{\mathcal{D}}^s$ is also symmetric. In particular, $(\mathcal{D}^s, W_{\mathcal{D}}^s)$ is a selfinjective QP , so if $s \equiv 1 \pmod{3}$ and $C \subset \mathcal{D}_1^s$ is a cut, then $\Pi_{\mathcal{D}}^s / \langle C \rangle$ is 2-representation-finite by Theorem 2.4.

We can say more.

Proposition 5.6 [10, Thm 7.9] Let (Q, W) be a strongly planar selfinjective QP , with a group G acting by rotations and enough G -invariant cuts. Then all truncated Jacobian algebras $\mathcal{J}(\tilde{Q}, \tilde{W})_C$ are derived equivalent.

Corollary 5.7 Let $s \equiv 1 \pmod{3}$. All truncations $\Pi_{\mathcal{D}}^s / \langle C \rangle$ are derived equivalent.

6 $\Pi_{\mathcal{D}}^s / \langle C \rangle$ is fractional Calabi-Yau

We use a theorem of Grant, which relates the Nakayama automorphism of a $(d + 1)$ -preprojective algebra to the fractional Calabi-Yau property of the associated d -representation-finite algebras.

Throughout, let $\Lambda = \mathcal{J}(Q, W)_C$ be a basic 2-representation-finite algebra, and let σ be the Nakayama automorphism of $\Pi = \mathcal{J}(Q, W)$.

Lemma 6.1 [20, Prop 1.3], [12, Prop 3.2].

- (a) There exists a function $l: Q_0 \rightarrow \mathbb{Z}$ such that $D(\Lambda e_i) \cong \tau_2^{-l(i)}(\sigma(e_i)\Lambda)$ for all $i \in Q_0$.

(b) For all $i \in Q_0$, $e_i \Pi \cong D(\Pi\sigma(e_i))\{-l(i)\}$ as graded Π -modules.

Here $\{-\}$ denotes a grading shift, i.e. $(M\{n\})_r = M_{r+n}$.

Our main tool will be the following theorem.

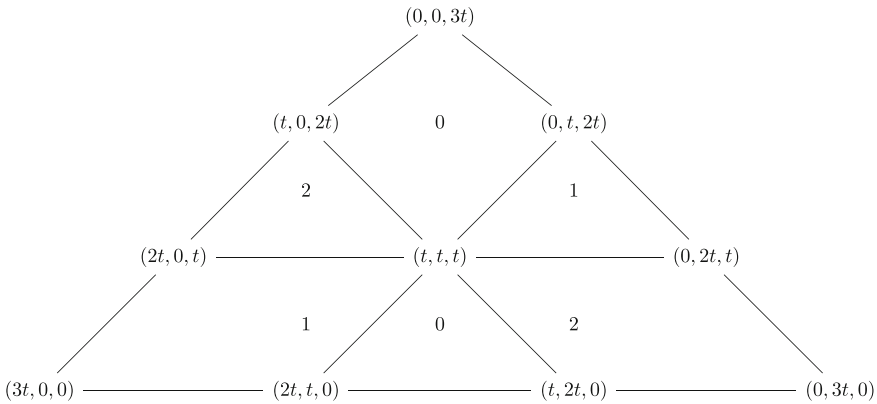
Theorem 6.2 *If there exists $k \in \mathbb{Z}^+$ such that $\sigma^k = \text{id}$, then*

- (a) *there exists $N \in \mathbb{Z}$ such that $\sum_{j=1}^k l(\sigma^j(i)) = N$ for all $i \in Q_0$;*
- (b) *Λ is fractional Calabi-Yau of dimension $2N/(k + N)$.*

Proof In [12, §3.4], it is demonstrated that if $f \in \text{Hom}_\Lambda(e_i \Lambda, \tau_2^{-r}(e_j \Lambda))$ then $\sigma(f) \in \text{Hom}_\Lambda(\sigma(e_i) \Lambda, \tau_2^{l(i)-l(j)-r}(\sigma(e_j) \Lambda))$. In other words, if $f \in \Pi_r$ (the degree r part of Π) and $i, j \in Q_0$, then $e_j f e_i \in \Pi_{r+l(i)-l(j)}$. This means (σ, l) is a *degree adjusted automorphism* of Π , so [13, Lem 6.15] gives us (a). By Lemma 6.1(b), (σ, l) is a *tr-graded Nakayama automorphism* of Π , so [13, Thm 6.14] yields (b). For definitions of the italicised terms see [13]. □

To apply this to $\Pi_{\mathcal{D}}^s$, we need a technical lemma. In view of Corollary 5.7, we can restrict our attention to a particular cut.

Definition 6.3 Let $s = 3t + 1$. Define a \mathbb{Z}_3 -invariant cut $K' \subset \mathcal{A}_1^s$ using the following diagram.



Here, the label i indicates that all arrows α_i in that region (including those on the edge) should be cut. We denote by K the induced cut of $(\mathcal{D}^s, W_{\mathcal{D}}^s)$, and by g_K the induced grading on $\Pi_{\mathcal{D}}^s$.

See Figure 2 for K and K' in the case $t = 2$.

Remark 6.4 Note that all arrows in K' appear in the leftmost set of (1) - see the proof of Lemma 5.3. Hence, we have that

$$e_x \alpha_{i,j} \in K \iff e_{\omega_j(x)} \alpha_i \in K'.$$

In the following, $\text{Soc}(M)$ denotes the *socle* (maximal semisimple submodule) of a module M .

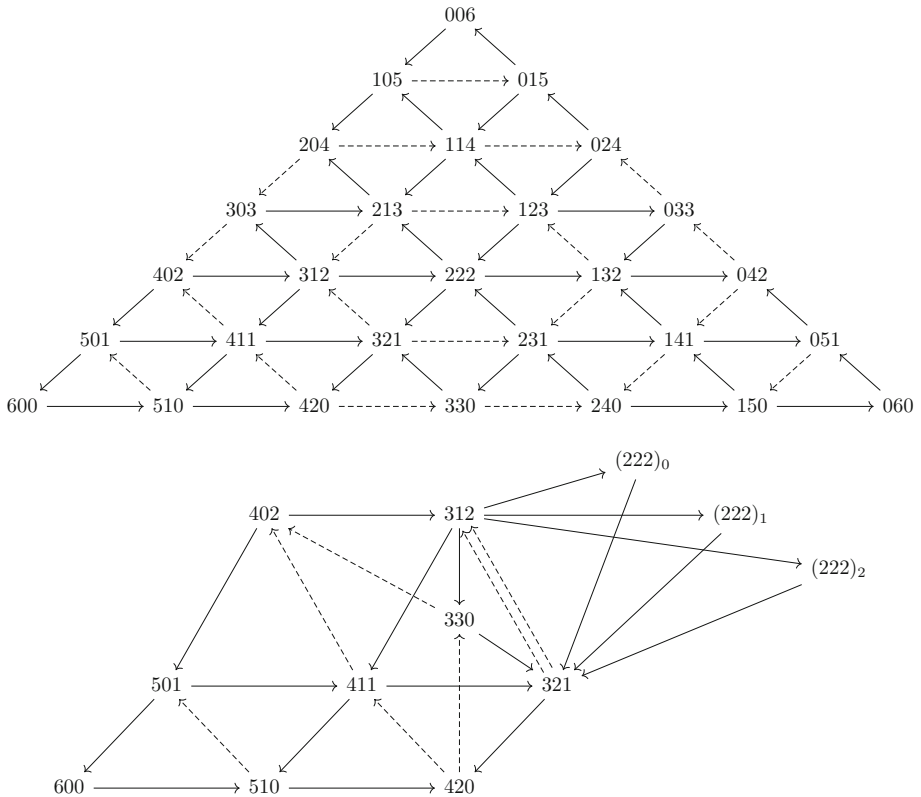


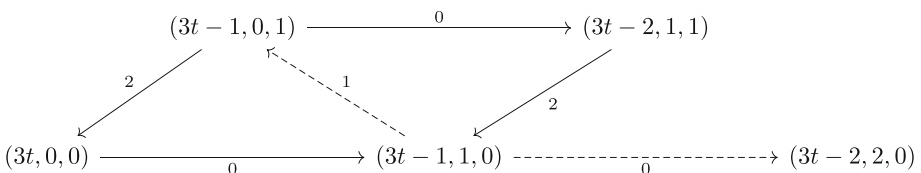
Fig. 2 The \mathbb{Z}_3 -invariant cut K' of \mathcal{A}^7 and the induced cut K of \mathcal{D}^7 , indicated by the dashed arrows

Lemma 6.5 *Let $s = 3t + 1$, $\Pi = \Pi_{\mathcal{D}}^s$ and $x = (3t, 0, 0) \in \mathcal{D}_0^s$. Then $\text{Soc}(e_x \Pi) = \langle p \rangle$, where*

$$p = \begin{cases} e_x \alpha_0^{\frac{3t-1}{2}} \alpha_{2,1} \alpha_2^{\frac{3t-1}{2}} & \text{if } t \text{ is odd,} \\ e_x \alpha_0^{\frac{3t}{2}} \alpha_{2,1} \alpha_2^{\frac{3t-2}{2}} & \text{if } t \text{ is even.} \end{cases}$$

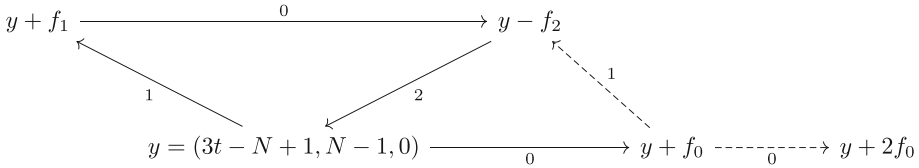
In particular, $g_K(p) = t$.

Proof Suppose $t \geq 3$ is odd (the even case is analogous, and $t = 1$ can be seen from Example 8.1). The proof proceeds by recursively computing $e_x \Pi[n]$, the space of paths starting at x with length n . Clearly $e_x \Pi[0] = \langle e_x \rangle$ and $e_x \Pi[1] = \langle e_x \alpha_0 \rangle$. There are two ways to extend $e_x \alpha_0$ to a path of length 2:



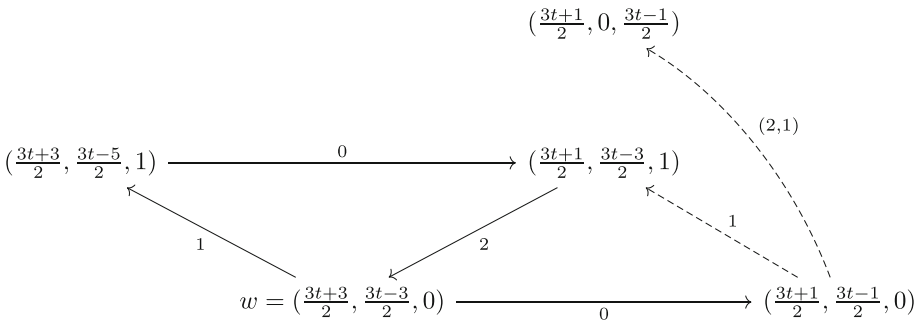
We have $\partial_{\alpha_2 e_x} W_D^S = e_x \alpha_0 \alpha_1$, so $e_x \alpha_0 \alpha_1 = 0$. There is no relation reducing $e_x \alpha_0^2$, since the two arrows lie in 3-cycles which do not share a common edge. So certainly $e_x \alpha_0^2 \neq 0$ and $e_x \Pi[2] = \langle e_x \alpha_0^2 \rangle$.

Now let $2 \leq N \leq \frac{3t-3}{2}$ and assume $e_x \Pi[N] = \langle e_x \alpha_0^N \rangle$. There are two ways to extend $e_x \alpha_0^N$ to a path of length $N + 1$:



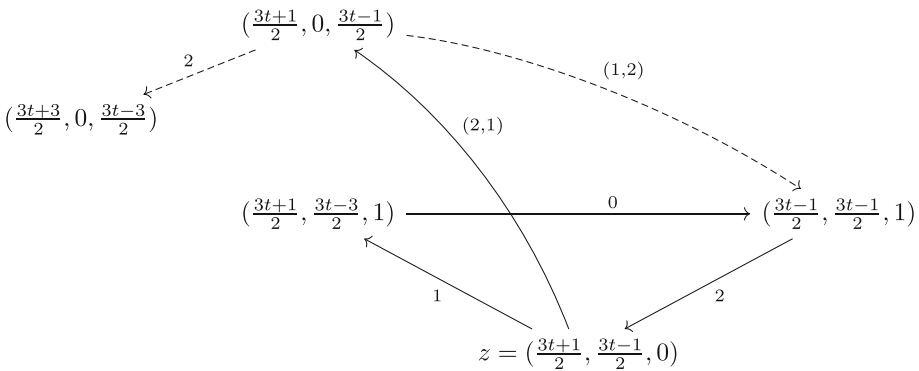
We have $\partial_{\alpha_2 e_y} W_D^S = e_y (\alpha_0 \alpha_1 - \alpha_1 \alpha_0)$. Now $e_x \Pi[N] = \langle e_x \alpha_0^N \rangle$, so $e_x \alpha_0^{N-1} \alpha_1 = 0$. Hence $e_x \alpha_0^N \alpha_1 = e_x \alpha_0^{N-1} \alpha_1 \alpha_0 = 0$. As before there is no relation reducing $e_x \alpha_0^{N+1}$, so this path generates $e_x \Pi[N + 1]$. Therefore $e_x \Pi[n] = \langle e_x \alpha_0^n \rangle$ for all $n \leq \frac{3t-1}{2}$.

There are two ways to extend $e_x \alpha_0^{\frac{3t-1}{2}}$ to a path of length $\frac{3t+1}{2}$:



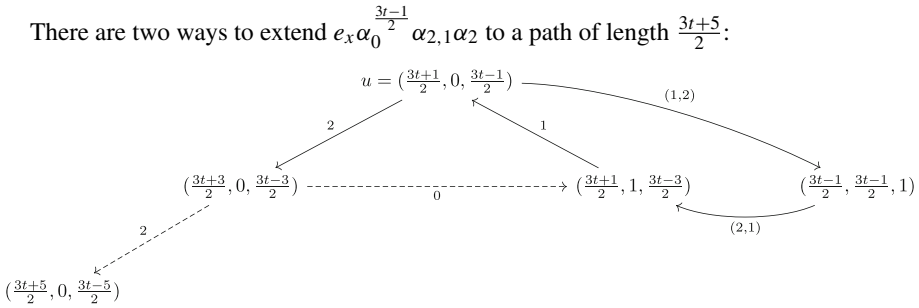
We have $\partial_{\alpha_2 e_w} W_D^S = e_w (\alpha_0 \alpha_1 - \alpha_1 \alpha_0)$, so $e_x \alpha_0^{\frac{3t-1}{2}} \alpha_1 = e_x \alpha_0^{\frac{3t-3}{2}} \alpha_1 \alpha_0 = 0$. There is no relation reducing $e_x \alpha_0^{\frac{3t-1}{2}} \alpha_{2,1}$ so this path generates $e_x \Pi[\frac{3t+1}{2}]$.

There are two ways to extend $e_x \alpha_0^{\frac{3t-1}{2}} \alpha_{2,1}$ to a path of length $\frac{3t+3}{2}$:



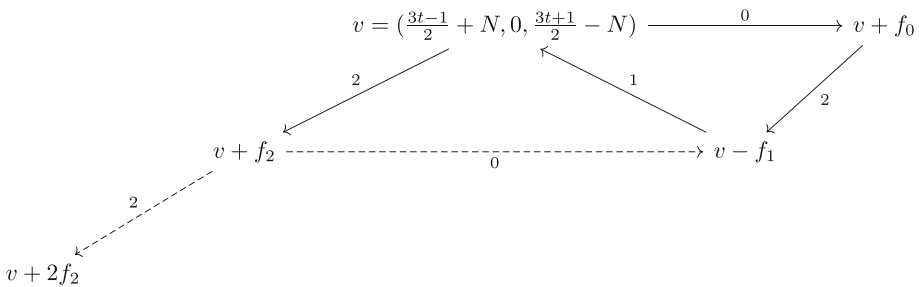
We have $\partial_{\alpha_2 e_z} W_D^S = e_z (\alpha_{2,1} \alpha_{1,2} - \alpha_1 \alpha_0)$, so $e_x \alpha_0^{\frac{3t-1}{2}} \alpha_{2,1} \alpha_{1,2} = e_x \alpha_0^{\frac{3t-1}{2}} \alpha_1 \alpha_0 = 0$. There is no relation reducing $e_x \alpha_0^{\frac{3t-1}{2}} \alpha_{2,1} \alpha_2$ so this path generates $e_x \Pi[\frac{3t+3}{2}]$.

There are two ways to extend $e_x \alpha_0^{\frac{3t-1}{2}} \alpha_{2,1} \alpha_2$ to a path of length $\frac{3t+5}{2}$:



We have $\partial_{\alpha_1 e_u} W_D^S = e_u (\alpha_2 \alpha_0 - \alpha_{1,2} \alpha_{2,1})$, so $e_x \alpha_0^{\frac{3t-1}{2}} \alpha_{2,1} \alpha_2 \alpha_0 = e_x \alpha_0^{\frac{3t-1}{2}} \alpha_{2,1} \alpha_{1,2} \alpha_{2,1} = 0$. There is no relation reducing $e_x \alpha_0^{\frac{3t-1}{2}} \alpha_{2,1} \alpha_2^2$ so this path generates $e_x \Pi[\frac{3t+5}{2}]$.

Now let $2 \leq N \leq \frac{3t-3}{2}$ and assume $e_x \Pi[\frac{3t+1}{2} + N] = \langle e_x \alpha_0^{\frac{3t-1}{2}} \alpha_{2,1} \alpha_2^N \rangle$. There are two ways to extend $e_x \alpha_0^{\frac{3t-1}{2}} \alpha_{2,1} \alpha_2^N$ to a path of length $\frac{3t+3}{2} + N$:



We have $\partial_{\alpha_1 e_v} W_D^S = e_v (\alpha_2 \alpha_0 - \alpha_0 \alpha_2)$, so $e_x \alpha_0^{\frac{3t-1}{2}} \alpha_{2,1} \alpha_2^N \alpha_0 = e_x \alpha_0^{\frac{3t-1}{2}} \alpha_{2,1} \alpha_2^{N-1} \alpha_0 \alpha_2 = 0$ by assumption. There is no relation reducing $e_x \alpha_0^{\frac{3t-1}{2}} \alpha_{2,1} \alpha_2^{N+1}$ so this path generates $e_x \Pi[\frac{3t+3}{2} + N]$. Therefore $e_x \Pi[\frac{3t+1}{2} + n] = \langle e_x \alpha_0^{\frac{3t-1}{2}} \alpha_{2,1} \alpha_2^n \rangle$ for all $n \leq \frac{3t-1}{2}$.

Notice that $t(e_x \alpha_0^{\frac{3t-1}{2}} \alpha_{2,1} \alpha_2^{\frac{3t-1}{2}}) = x$. There is only one arrow with source x , namely $e_x \alpha_0$ (see the first figure in the proof). But since $\partial_{\alpha_1 e_{(3t-1,0,1)}} W_D^S = e_{(3t-1,0,1)} (\alpha_2 \alpha_0 - \alpha_0 \alpha_2)$, we have $e_x \alpha_0^{\frac{3t-1}{2}} \alpha_{2,1} \alpha_2^{\frac{3t-1}{2}} \alpha_0 = e_x \alpha_0^{\frac{3t-1}{2}} \alpha_{2,1} \alpha_2^{\frac{3t-3}{2}} \alpha_0 \alpha_2 = 0$, so $e_x \Pi[n] = 0$ for all $n > 3t$.

Using Remark 6.4,

$$g_K(p) = g_{K'} \left(e_x \alpha_0^{\frac{3t-1}{2}} \right) + g_{K'} \left(\alpha_2^{\frac{3t+1}{2}} e_x \right).$$

By definition of K' ,

$$g_{K'} \left(e_x \alpha_0^{\frac{3t-1}{2}} \right) = \frac{t-1}{2}, \quad g_{K'} \left(\alpha_2^{\frac{3t+1}{2}} e_x \right) = \frac{t+1}{2},$$

so $g_K(p) = t$. □

We now prove Theorem 3.

Theorem 6.6 *Let $s = 3t + 1$. For any cut $C \subset \mathcal{D}_1^S$, $\Pi_D^S / \langle C \rangle$ is fractional Calabi-Yau of dimension $2t/(t+1)$.*

Proof We prove the statement for $C = K$. This is enough, since all truncations of $\Pi_{\mathcal{D}}^s$ are derived equivalent by Corollary 5.7. Theorem 6.2(a) with $\sigma = \text{id}$ and $k = 1$ implies the function $l: \mathcal{D}_0^s \rightarrow \mathbb{Z}$ from Lemma 6.1 takes a constant value N . Let $x = (3t, 0, 0) \in \mathcal{D}_0^s$. By Lemma 6.1(b), $e_x \Pi \cong D(\Pi e_x)\{-N\}$ as graded Π -modules. In Lemma 6.5 we demonstrated that $\text{Soc}(e_x \Pi)$ is generated by a path p of degree t , so $e_x \Pi$ exists in degrees 0 through t . Since we also have $p \in \Pi e_x$, we deduce that $D(\Pi e_x)$ exists in degrees $-t$ through 0, and that $N = t$. Therefore $\Pi_{\mathcal{D}}^s/\langle C \rangle$ is fractional Calabi-Yau of dimension $2t/(t + 1)$ by Theorem 6.2(b). \square

Definition 6.7 [16, Def 1.2] If the function $l: Q_0 \rightarrow \mathbb{Z}$ from Lemma 6.1 takes a constant value $N \in \mathbb{Z}$, we say Λ is $(N + 1)$ -homogeneous.

The following is immediate from the proof of Theorem 6.6.

Corollary 6.8 Let $s = 3t + 1$. For any cut $C \subset \mathcal{D}_1^s$, $\Pi_{\mathcal{D}}^s/\langle C \rangle$ is $(t + 1)$ -homogeneous.

7 2-Auslander-Reiten quivers

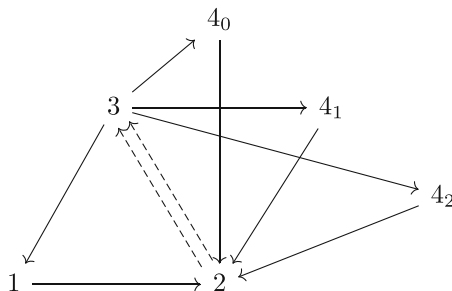
The general theory in this section is from [20, §1].

Let Λ be a d -representation-finite algebra with a d -cluster tilting object $M \in \text{mod } \Lambda$. Then $\text{add } M$ is equal to the subcategory $\mathcal{M} := \text{add}\{\tau_d^{-i} \Lambda \mid i \geq 0\} \subset \text{mod } \Lambda$ [20, Thm 1.6]. We call \mathcal{M} the d -cluster tilting subcategory of $\text{mod } \Lambda$.

The functors τ_d, τ_d^- play the role of Auslander-Reiten translates on this subcategory: they induce mutually quasi-inverse equivalences $\tau_d: \underline{\mathcal{M}} \rightarrow \overline{\mathcal{M}}$ and $\tau_d^-: \overline{\mathcal{M}} \rightarrow \underline{\mathcal{M}}$ between the stable and costable categories.

Thus, it makes sense to consider a d -Auslander-Reiten quiver of $\text{mod } \Lambda$, whose vertices are indecomposable objects of \mathcal{M} , and the number of arrows from X to Y is equal to the dimension of $\text{rad}_{\mathcal{M}}(X, Y)/\text{rad}_{\mathcal{M}}^2(X, Y)$, where rad denotes the Jacobson radical [20, Def 6.1].

Example 7.1 Let $\Pi = \Pi_{\mathcal{D}}^4$, $C = \{\alpha_1, \alpha_{2,1}\}$ and $\Lambda = \Pi/\langle C \rangle$. To simplify notation relabel as follows.

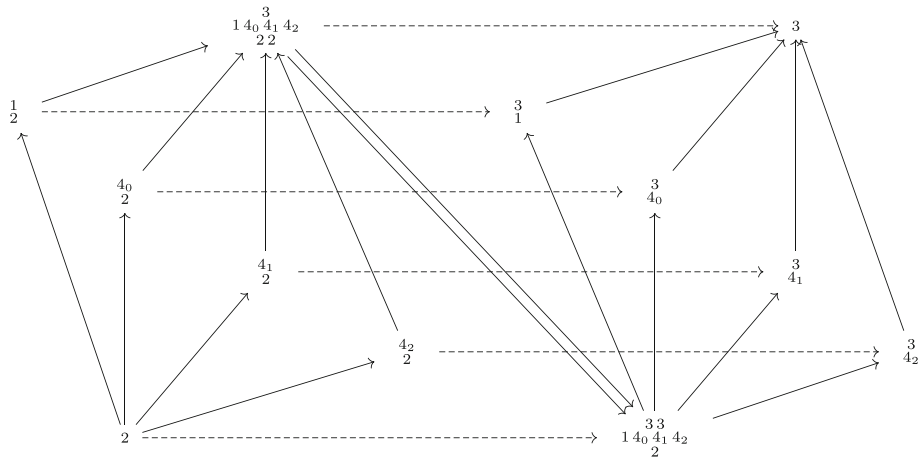


Using the definition $\tau_2^- = \tau^- \Omega^-$, one computes the radical series

$$e_1 \Lambda = \frac{1}{2}, \quad e_2 \Lambda = 2, \quad e_3 \Lambda = 1 \begin{matrix} 4_0 \\ 4_1 \\ 4_2 \end{matrix} \begin{matrix} 4_2 \\ 4_1 \\ 4_0 \end{matrix}, \quad e_{4_k} \Lambda = \frac{4^k}{2},$$

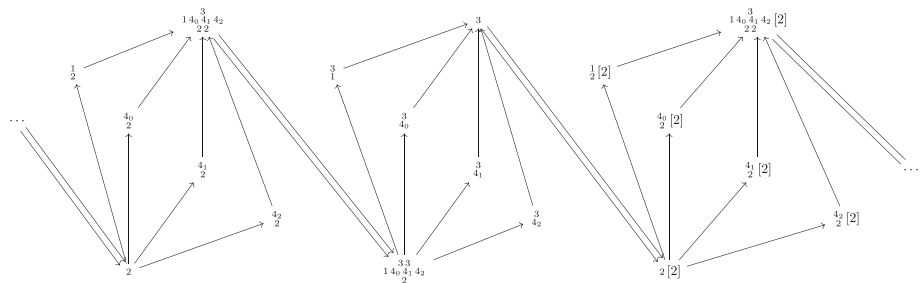
$$\tau_2^-(e_1 \Lambda) = \frac{3}{1}, \quad \tau_2^-(e_2 \Lambda) = 1 \begin{matrix} 3 \\ 4_0 \\ 4_1 \\ 4_2 \end{matrix} \begin{matrix} 4_2 \\ 4_1 \\ 4_0 \end{matrix}, \quad \tau_2^-(e_3 \Lambda) = 3, \quad \tau_2^-(e_{4_k}) = \frac{3}{4_k}.$$

The 2-Auslander-Reiten quiver of $\mathcal{M} \subset \text{mod } \Lambda$ is as follows. The dashed lines indicate the action of τ_2^- .



Given a d -representation-finite algebra Λ , there is also a notion of higher Auslander-Reiten theory on its bounded derived category $\mathcal{D} = \mathbf{D}^b(\text{mod } \Lambda)$. Here the role of Auslander-Reiten translate is played by the autoequivalence $\nu_d := \nu \circ \Sigma^{-d} : \mathcal{D} \rightarrow \mathcal{D}$, and the d -cluster tilting subcategory is $\mathcal{U} := \text{add}\{\nu_d^{-i} \mid i \in \mathbb{Z}\} \subset \mathcal{D}$. This definition is justified by the fact that for any $X \in \mathcal{M}$ with no injective summands, $\tau_d^- X \cong \nu_d^- X$.

Example 7.2 The 2-Auslander-Reiten quiver of $\mathcal{U} \subset \mathcal{D}$ is as follows.



On \mathcal{U} , the Serre functor ν has orbits

$$\begin{aligned}
 2 &\mapsto 1\ 4_0\ 4_1\ 4_2 / 2 \mapsto 2[2], \\
 1/2 &\mapsto 3/1 \mapsto 1/2[2], \\
 4_k/2 &\mapsto 3/4_k \mapsto 4_k/2[2], \\
 1\ 4_0\ 4_1\ 4_2 / 2\ 2\ 2 &\mapsto 3 \mapsto 1\ 4_0\ 4_1\ 4_2 / 2\ 2\ 2[2],
 \end{aligned}$$

aligning with the fact Λ is 2/2-fractional Calabi-Yau.

By homogeneity (Corollary 6.8), if $\Lambda = \Pi_D^{3t+1}/\langle C \rangle$, the Serre functor on \mathcal{U} has disjoint orbits

$$e_x \Lambda \mapsto \tau_2^-(e_x \Lambda) \mapsto \dots \mapsto \tau_2^{1-t}(e_x \Lambda) \mapsto D(\Lambda e_x) \mapsto e_x \Lambda[2t]$$

for each $x \in \mathcal{D}_0^{3t+1}$. The orbits are disjoint since $\tau_2^{-i}(e_x \Lambda) \cong \tau_2^{-j}(e_y \Lambda)$ implies $\tau_2^{j-i}(e_x \Lambda) \cong e_y \Lambda$. Since $e_y \Lambda$ is projective, we must have $i = j$, whence $x = y$ since Λ is basic.

We give a recipe to construct the 2-Auslander-Reiten quiver of a 2-representation-finite algebra of type D.

Proposition 7.3 *Let $s = 3t + 1$, C be a cut of $\Pi = \Pi_{\mathcal{D}}^s$, and $\Lambda = \Pi / \langle C \rangle$. The 2-Auslander-Reiten quiver of $\mathcal{U} \subset \mathbf{D}^b(\text{mod } \Lambda)$ is isomorphic to the quiver Γ with vertices*

$$\Gamma_0 = \{(x, i) \mid x \in \mathcal{D}_0^s, i \in \mathbb{Z}\},$$

where there is an arrow $(x, i) \rightarrow (y, j)$ in Γ_1 if and only if

1. $i = j$ and there is an arrow $y \rightarrow x$ in $\mathcal{D}_1^s \setminus C$, or
2. $j = i + 1$ and there is an arrow $y \rightarrow x$ in C .

The 2-Auslander-Reiten quiver of $\mathcal{M} \subset \text{mod } \Lambda$ is isomorphic to the full subquiver Γ' of Γ on the vertices

$$\Gamma'_0 = \{(x, i) \mid x \in \mathcal{D}_0^s, 0 \leq i \leq t\}.$$

Proof The vertices of the 2AR quiver of \mathcal{U} are the isoclasses of its indecomposable objects. Now

$$\begin{aligned} \mathcal{U} &= \text{add}\{v_2^{-i} \Lambda \mid i \in \mathbb{Z}\} \\ &= \text{add} \left\{ v_2^{-i} \left(\bigoplus_{x \in \mathcal{D}_0^s} e_x \Lambda \right) \mid i \in \mathbb{Z} \right\} \\ &= \text{add} \left\{ \bigoplus_{x \in \mathcal{D}_0^s} v_2^{-i}(e_x \Lambda) \mid i \in \mathbb{Z} \right\} \\ &= \text{add}\{v_2^{-i}(e_x \Lambda) \mid x \in \mathcal{D}_0^s, i \in \mathbb{Z}\}. \end{aligned}$$

Since Λ is basic, $\{v_2^{-i}(e_x \Lambda) \mid x \in \mathcal{D}_0^s, i \in \mathbb{Z}\}$ is the set of isoclasses of indecomposable objects, which is clearly in bijection with the set Γ_0 defined above.

Note [16, §2] that $\Pi \cong \bigoplus_{r \in \mathbb{Z}} \text{Hom}_{\mathcal{U}}(\Lambda, v_2^{-r} \Lambda)$, so for all $x, y \in \mathcal{D}_0^s$ and $i, j \in \mathbb{Z}$,

$$\text{Hom}_{\mathcal{U}}(v_2^{-i}(e_x \Lambda), v_2^{-j}(e_y \Lambda)) \cong \text{Hom}_{\mathcal{U}}(e_x \Lambda, v_2^{i-j}(e_y \Lambda)) \cong e_y \Pi_{j-i} e_x$$

as vector spaces, where Π_{j-i} is the degree $j - i$ part of Π .

We now show statements 1 and 2 of the proposition.

1. Suppose $i = j$. Then $\text{Hom}_{\mathcal{U}}(v_2^{-i}(e_x \Lambda), v_2^{-i}(e_y \Lambda))$ has a basis consisting of paths from y to x in Λ (since the degree zero part of Π is Λ). In particular, the irreducible morphisms from $v_2^{-i}(e_x \Lambda)$ to $v_2^{-i}(e_y \Lambda)$ in \mathcal{U} (i.e. arrows $(x, i) \rightarrow (y, i)$ in Γ_1) are in bijection with arrows $y \rightarrow x$ in $\mathcal{D}_1^s \setminus C$.
2. Suppose $j = i + 1$. Then $\text{Hom}_{\mathcal{U}}(v_2^{-i}(e_x \Lambda), v_2^{-(i+1)}(e_y \Lambda))$ has a basis consisting of degree one paths from y to x in Π . In particular, the irreducible morphisms from $v_2^{-i}(e_x \Lambda)$ to $v_2^{-(i+1)}(e_y \Lambda)$ in \mathcal{U} (i.e. arrows $(x, i) \rightarrow (y, i + 1)$ in Γ_1) are in bijection with arrows $y \rightarrow x$ in C .

These are all the arrows in Γ_1 . Indeed, if $j < i$ then $\text{Hom}_{\mathcal{U}}(v_2^{-i}(e_x\Lambda), v_2^{-j}(e_y\Lambda)) = 0$ since Π is non-negatively graded. Conversely if $j > i + 1$, then no morphism in $\text{Hom}_{\mathcal{U}}(v_2^{-i}e_x\Lambda, v_2^{-j}(e_y\Lambda))$ is irreducible, since it corresponds to a path in Π from y to x of degree $j - i > 1$, which can be factorised into two paths of lower degree since Π is generated in degree 1.

To see Γ' is as claimed, note that its set of vertices is $\{v_2^{-i}(e_x\Lambda) \mid x \in \mathcal{D}_0^s, 0 \leq i \leq t\}$, which is clearly in bijection with the set Γ'_0 above. This is because Λ is $(t + 1)$ -homogeneous, so $\tau_2^{-i}(e_x\Lambda) \cong v_2^{-i}(e_x\Lambda)$ for $0 \leq i \leq t$, and $\tau_2^{-(t+1)}(e_x\Lambda) = 0$. □

8 Further questions

We present some possible directions for further research in this area.

- 1. Basis and dimension:** Finding a basis for $\Pi_{\mathcal{D}}^s$ can be difficult, even for small s .

Example 8.1 Let $\Pi = \Pi_{\mathcal{D}}^4$ (see Figure 1 and Example 1.5). Then

$$\begin{aligned} e_{300}\Pi &= \langle e_{300}, \alpha_0, \alpha_0\alpha_{2,1}, \alpha_0\alpha_{2,1}\alpha_2 \rangle, \\ e_{210}\Pi &= \langle e_{210}, \alpha_1, \alpha_{2,1}, \alpha_1\beta_0, \alpha_1\beta_1, \alpha_1\beta_2, \alpha_{2,1}\alpha_2, \alpha_{2,1}\alpha_2\alpha_0 \rangle, \\ e_{201}\Pi &= \langle e_{201}, \alpha_2, \beta_0, \beta_1, \beta_2, \alpha_2\alpha_0, \beta_0\gamma_0, \alpha_2\alpha_0\alpha_{2,1} \rangle, \\ e_{(111)_k}\Pi &= \langle e_{(111)_k}, \gamma_k, \gamma_k\alpha_1, \gamma_k\alpha_1\beta_k \rangle \text{ for all } k \in \{0, 1, 2\}, \end{aligned}$$

so we obtain a basis for Π . In particular, $\dim_{\mathbb{C}} \Pi = 32$.

One calculates as in Lemma 6.5, recursively computing the space of paths starting at each vertex with a given length. To make these calculations we had to derive some additional relations:

- $\alpha_1\beta_0\gamma_0 = \zeta^2\alpha_1\beta_1\gamma_1 = \zeta\alpha_1\beta_2\gamma_2$ and $\alpha_{2,1}\beta_0\gamma_0 = \alpha_{2,1}\beta_1\gamma_1 = \alpha_{2,1}\beta_2\gamma_2$;
- $\beta_0\gamma_0\alpha_1 = \zeta^2\beta_1\gamma_1\alpha_1 = \zeta\beta_2\gamma_2\alpha_1$ and $\beta_0\gamma_0\alpha_{2,1} = \beta_1\gamma_1\alpha_{2,1} = \beta_2\gamma_2\alpha_{2,1}$;
- if $i \neq j$ then $\gamma_i\alpha_1\beta_j = \gamma_i\alpha_{2,1}\beta_j = 0$.

To see (a), note that

$$\begin{aligned} \alpha_{2,1}\beta_2\gamma_2 &= -\zeta\alpha_{2,1}\beta_0\gamma_0 - \zeta^2\alpha_{2,1}\beta_1\gamma_1 = -\zeta\alpha_1\beta_0\gamma_0 - \zeta\alpha_1\beta_1\gamma_1, \text{ but also} \\ \alpha_{2,1}\beta_2\gamma_2 &= \zeta\alpha_1\beta_2\gamma_2 = -\zeta^2\alpha_1\beta_0\gamma_0 - \alpha_1\beta_1\gamma_1, \end{aligned}$$

so $\zeta\alpha_1\beta_0\gamma_0 + \zeta\alpha_1\beta_1\gamma_1 = \zeta^2\alpha_1\beta_0\gamma_0 + \alpha_1\beta_1\gamma_1$, which rearranges to give $\alpha_1\beta_0\gamma_0 = \zeta^2\alpha_1\beta_1\gamma_1$. One similarly shows $\alpha_1\beta_0\gamma_0 = \zeta\alpha_1\beta_2\gamma_2$, and the fact $\alpha_1\beta_k\gamma_k = \zeta^k\alpha_{2,1}\beta_k\gamma_k$ for all $k \in \{0, 1, 2\}$ implies the second statement.

Part (b) is demonstrated analogously. To see (c), assume $i \neq j$. Then $\gamma_i\alpha_1\beta_j = \zeta^i\gamma_i\alpha_{2,1}\beta^j$, but also $\gamma_i\alpha_1\beta_j = \zeta^j\gamma_i\alpha_{2,1}\beta_j$. This is a contradiction unless $\gamma_i\alpha_1\beta_j = \gamma_i\alpha_{2,1}\beta_j = 0$.

- d > 2:** The $(d + 1)$ -preprojective algebras of type A are defined for all $d \in \mathbb{Z}^+$ ($d = 1$ recovers the classical preprojective algebra of a type A quiver). Furthermore, quotienting by the ideal generated by a cut gives a d -representation-finite algebra [18, §5]. Let $\Pi_{\mathcal{A}}$ be a $(d + 1)$ -preprojective algebra of type A, for $d > 2$. Some questions are:

- Can we find a group G - maybe \mathbb{Z}_{d+1} - acting on $\Pi_{\mathcal{A}}$ by automorphisms?
- Can we generalise Definitions 1.2 and 1.4 to give an algebra $\Pi_{\mathcal{D}}$ that is Morita equivalent to $\Pi_{\mathcal{A}}\#G$?

- (c) Is $\Pi_{\mathcal{D}}$ the $(d + 1)$ -preprojective algebra of some d -representation-finite algebra Λ - maybe only when its index is $1 \pmod{d + 1}$?
- (d) Is Λ fractional Calabi-Yau?

The methods used would likely have to be different to those used in this article. Perhaps most significantly, when $d > 2$ one no longer has the Herschend-Iyama classification (Theorem 2.4).

3. **Auslander algebras:** In [20], Iyama studied iterated Auslander algebras of Dynkin quivers. If one takes the Auslander algebra of a linearly oriented type A quiver, one obtains a 2-representation-finite algebra of type A. For $s = 3t + 1$ and a cut C of $\Pi_{\mathcal{D}}^s$, is there a type D quiver whose Auslander algebra is $\Lambda = \Pi_{\mathcal{D}}^s / \langle C \rangle$?

If C contains one element of $\{e_{(t+1,t,t-1)}\alpha_1, e_{(t+1,t,t-1)}\alpha_{2,1}\}$, then it also contains the other. This is because there must be precisely one cut arrow in each of the cycles $\alpha_1\beta_0\gamma_0$ and $\alpha_{2,1}\beta_0\gamma_0$. Hence, either Λ or C contains two parallel arrows between a pair of vertices. As such, the 2-Auslander-Reiten quiver of $\mathcal{M} \subset \text{mod } \Lambda$ contains parallel arrows (see Proposition 7.3).

In contrast, if Q is a type D quiver and Γ is its Auslander algebra, the 2-Auslander-Reiten quiver of $\mathcal{M} \subset \text{mod } \Gamma$ contains no parallel arrows - use the recipe of [20, Def 6.11]. This means Γ and Λ are not even Morita equivalent, since the module category of a 2-representation-finite algebra has a *unique* 2-cluster-tilting subcategory [20, Thm 1.6]. We cannot argue similarly with $\mathcal{U} \subset \mathbf{D}^b(\text{mod } \Lambda)$. Indeed, 2-representation-finiteness is not preserved under derived equivalence (see [16, Rem 1.6] for a nice example). So, the question becomes: is there a type D quiver whose Auslander algebra is derived equivalent to Λ ?

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