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Few new reals

David Asperó
School of Mathematics, University of East Anglia
Norwich NR4 7TJ, UK
d.aspero@uea.ac.uk

Miguel Angel Mota*
Departamento de Matemáticas
Instituto Tecnológico Autónomo de México (ITAM)
01080 Mexico City, Mexico
motagaytan@gmail.com

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We introduce a new method for building models of CH, together with Π_2 statements over $H(\omega_2)$, by forcing. Unlike other forcing constructions in the literature, our construction adds new reals, although only $\aleph_1$-many of them. Using this approach, we build a model in which a very strong form of the negation of Club Guessing at ω_1 known as Measuring holds together with CH, thereby answering a well-known question of Moore. This construction can be described as a finite-support weak forcing iteration with side conditions consisting of suitable graphs of sets of models with markers. The CH-preservation is accomplished through the imposition of copying constraints on the information carried by the condition, as dictated by the edges in the graph.

Keywords: CH-preservation; measuring; side conditions; adding reals.

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1. Introduction

The problem of building models of consequences, at the level of $H(\omega_2)$, of classical forcing axioms in the presence of the Continuum Hypothesis (CH) has a long history, starting with Jensen's landmark result that Suslin's Hypothesis is compatible with CH [10]. Much of the work in this area is due to Shelah (see [22]), with contributions also by other people (see e.g. [2, 6, 12, 13, 19] or [20]). Most of the work in the area done so far proceeds by showing that some suitable countable support iteration

*Corresponding author.

whose iterands are proper forcing notions not adding new reals fails to add new reals also at limit stages.

There are (nontrivial) limitations to what can be achieved in this area. One conclusive example is the main result from [6], which highlights a strong global limitation: There is no model of CH satisfying a certain mild large cardinal assumption and realizing all Π_2 statements over the structure $H(\omega_2)$ that can be forced, using proper forcing, to hold together with CH. In fact there are two Π_2 statements over $H(\omega_2)$, each of which can be forced, using proper forcing, to hold together with CH — for one of them we need an inaccessible limit of measurable cardinals — and whose conjunction implies $2^{\aleph_0} = 2^{\aleph_1}$.

The above example is closely tied to the following well-known obstacle to not adding reals, which appears in [11] (see also [12]) and which is more to the point in the context of this paper[a]: Given a ladder system $\vec{C} = (C_\delta : \delta \in \mathrm{Lim}(\omega_1))$ (i.e. each C_δ is a cofinal subset of δ of order type ω), let $\mathsf{Unif}(\vec{C})$ denote the statement that for every coloring $F : \mathrm{Lim}(\omega_1) \longrightarrow \{0, 1\}$ there is a function $G : \omega_1 \longrightarrow \{0, 1\}$ with the property that for every $\delta \in \mathrm{Lim}(\omega_1)$ there is some $\alpha < \delta$ such that $G(\xi) = F(\delta)$ for all $\xi \in C_\delta \backslash \alpha$ (where, given an ordinal α, $\mathrm{Lim}(\alpha)$ is the set of limit ordinals below α). We say that G uniformizes F on $\vec{C}$. Given $\vec{C}$ and F as above there is a natural forcing notion, let us call it $\mathcal{Q}_{\vec{C},F}$, for adding a uniformizing function for F on $\vec{C}$ by initial segments. It takes a standard exercise to show that $\mathcal{Q}_{\vec{C},F}$ is proper, adds the intended uniformizing function, and does not add reals. However, any long enough iteration of forcings of the form $\mathcal{Q}_{\vec{C},F}$, even with a fixed $\vec{C}$, will necessarily add new reals. As a matter of fact, the existence of a ladder system $\vec{C}$ for which $\mathsf{Unif}(\vec{C})$ holds cannot be forced together with CH in any way whatsoever, as this statement actually implies $2^{\aleph_0} = 2^{\aleph_1}$. The argument is well known and may be found for example in [11] and in [12].

In this paper, we distance ourselves from the tradition of iterating forcing without adding reals and tackle the problem of building interesting models of CH with an entirely different approach: starting with a model of CH, we build a forcing which adds new reals,[b] albeit only $\aleph_1$-many of them.

In [7], a framework for building finite-support forcing iterations incorporating systems of countable models as side conditions was developed (see also [3, 8, 9] for further elaborations). These iterations arise naturally in, for example, situations in which one is interested in building a forcing iteration of length κ (where κ is relatively long) which is proper and which, in addition, does not collapse cardinals.[c] Much of what we will say in the next few paragraphs will probably make sense

[a] We will revisit this obstacle in Sec. 2.2 with the purpose of addressing the following question: Why do our methods work with the present application (forcing $\mathsf{Measuring}$) and not with the problem of forcing $\mathsf{Unif}(\vec{C})$ (for any given $\vec{C}$)?

[b] As it turns out, the construction resembles a classical finite-support iteration, and in fact it adds Cohen reals.

[c] For example if, as in [7], we want to force certain instances of the Proper Forcing Axiom (PFA) together with $2^{\aleph_0} = \kappa > \aleph_2$.

only to readers with at least some familiarity with the framework as presented, for example, in [7].

In the situations we are referring to here, one typically aims at a construction which in fact has the $\aleph_2$-chain condition, and in order to achieve this goal it is natural to build the iteration in such a way that conditions be of the form (F, Δ), for F a (finitely supported) κ-sequence of working parts, and with Δ being a set of models with markers, i.e. a set of ordered pairs (N, ρ), where N is a countable elementary submodel of $H(\kappa)$, possibly enhanced with some predicate $T \subseteq H(\kappa)$, and where $\rho \in N \cap \kappa$. N is one of the models for which we will try to "force" each working part $F(\alpha)$, for every stage $\alpha \in N \cap \rho$, to be generic for the generic extension of N up to that stage; thus, ρ is to be seen as a "marker" that tells us that N is to be seen as "active" as a side condition at least up to stage ρ.

In order for the construction to have the $\aleph_2$-chain condition and be proper, it is often necessary to start from a model of CH and require that the domain of Δ be a set of models with suitable symmetry properties. We call (finite) sets of models having these properties T-symmetric systems (for a fixed $T \subseteq H(\kappa)$). One of these properties, and the one on which we will focus our attention in a moment, is the following: In a T-symmetric system $\mathcal{N}$, if N and N' are both in $\mathcal{N}$ and $N \cap \omega_1 = N' \cap \omega_1$, then there is a (unique) isomorphism $\Psi_{N,N'}$ between the structures $(N; \in, T, \mathcal{N} \cap N)$ and $(N'; \in, T, \mathcal{N} \cap N')$ which, moreover, is the identity on $N \cap N'$.

At this point one could take a step back and analyze the pure side condition forcing $\mathcal{P}_0$ by itself. This forcing $\mathcal{P}_0$, which we can naturally see as the first stage of our construction, consists of all finite T-symmetric systems of submodels, ordered by reverse inclusion. $\mathcal{P}_0$ first appeared in the literature in [24]. It is a relatively well-known fact, and was noted in [9],[d] that forcing with $\mathcal{P}_0$ adds Cohen reals, although not too many; in fact it adds exactly $\aleph_1$-many of them. This may be somewhat surprising given that $\mathcal{P}_0$ adds, by finite approximations, a new rather large object (a symmetric system covering all of $H(\kappa)^V$).[e] The argument for this is contained in the proof of Lemma 3.16 from this paper, but it will nonetheless be convenient at this point to sketch it here.

Let us assume, towards a contradiction, that CH holds and there is a sequence $(\dot{r}_\nu)_{\nu<\omega_2}$ of $\mathcal{P}_0$-names which some condition $\mathcal{N}$ forces to be distinct subsets of ω. Without loss of generality we may take each $\dot{r}_\nu$ to be a member of $H(\kappa)$. For each ν we can pick N_ν to be a sufficiently correct countable model — meaning that $(N_\nu; \in, T^*) \preccurlyeq (H(\kappa); \in, T^*)$ for a suitably expressive predicate $T^* \subseteq H(\kappa)$ — containing all relevant objects, which in this case includes $\mathcal{N}$ and $\dot{r}_\nu$. As CH holds, we may find distinct indices ν and ν' such that there is a unique isomorphism $\Psi_{N_\nu,N_{\nu'}}$ between the structures $(N_\nu; \in, T^*, \mathcal{N}, \dot{r}_\nu)$ and $(N_{\nu'}; \in, T^*, \mathcal{N}, \dot{r}_{\nu'})$ fixing $N_\nu \cap N_{\nu'}$. But then $\mathcal{N}^* = \mathcal{N} \cup \{N_\nu, N_{\nu'}\}$ is a condition in $\mathcal{P}_0$ forcing that $\dot{r}_\nu = \dot{r}_{\nu'}$. The point is

[d]See also [18].
[e]Incidentally, $\mathcal{P}_0$ is in fact strongly proper, and so each new real it adds is in fact contained in an extension of V by some Cohen real. The preservation of CH by $\mathcal{P}_0$ was exploited in [16].

that if $n \in \omega$ and $\mathcal{N}'$ is any condition extending $\mathcal{N}^*$ and forcing $n \in \dot{r}_\nu$, then $\mathcal{N}'$ is in fact compatible with a condition $\mathcal{M} \in N_\nu$ forcing the same thing. This is true since $\mathcal{N}^*$ is an $(N_\nu, \mathcal{P}_0)$-generic condition. But then $\Psi_{N_\nu, N_{\nu'}}(\mathcal{M})$ is a condition forcing $n \in \Psi_{N_\nu, N_{\nu'}}(\dot{r}_\nu) = \dot{r}_{\nu'}$ (since, by taking T^* expressive enough, we may assume the forcing relation for $\mathcal{P}_0$ to be definable in $(H(\kappa); \in, T^*)$ without parameters). Finally, if $\mathcal{N}''$ is any common extension of $\mathcal{N}'$ and $\mathcal{M}$, then $\mathcal{N}''$ forces also that $n \in \dot{r}_{\nu'}$, since it extends $\Psi_{N_\nu, N_{\nu'}}(\mathcal{M})$ as $\Psi_{N_\nu, N_{\nu'}}(\mathcal{M}) \subseteq \mathcal{N}''$ by the symmetry requirement.[f]

$\mathcal{P}_0$ has received some attention in the literature. For example, Todorčević proved that $\mathcal{P}_0$ adds a Kurepa tree (see [18]). Also, [18] presents a mild variant of $\mathcal{P}_0$ which not only preserves CH but actually forces $\diamondsuit$.

The iterations with symmetric systems of models as side conditions that we were referring to before do not preserve CH, and in fact they force $2^{\aleph_0} = \kappa > \aleph_1$. The reason is of course that there are no symmetry requirements on the working parts. Actually, even if the first stage of the iterations — which is, essentially, $\mathcal{P}_0$ — preserves CH, the iterations are in fact designed to add new reals at all later (successor) stages.

Something one may naturally envision at this point is the possibility to build a suitable forcing with systems of models (with markers) as side conditions while strengthening the symmetry constraints, so as to make them apply not only to the side condition part of the forcing but also to the working parts; one would hope to exploit the above idea in order to show that the forcing thus constructed preserves CH, and would of course like to be able to do that while at the same time forcing some interesting statement. In this paper, we implement this idea by proving that a very strong form of the failure of Club Guessing at ω_1 known as Measuring (see [12]) that follows from PFA can be forced adding new reals while, nevertheless, preserving CH.

Definition 1.1. Measuring holds if and only if for every sequence $\vec{C} = (C_\delta : \delta \in \omega_1)$, if each C_δ is a closed subset of δ in the order topology, then there is a club $C \subseteq \omega_1$ such that for every $\delta \in C$ there is some $\alpha < \delta$ such that either

- $(C \cap \delta) \backslash \alpha \subseteq C_\delta$ or
- $(C \backslash \alpha) \cap C_\delta = \emptyset$.

In the above definition, we say that C *measures* $\vec{C}$. Measuring is of course equivalent to its restriction to club-sequences $\vec{C}$ on ω_1, i.e. to sequences of the form $\vec{C} = (C_\delta : \delta \in \mathrm{Lim}(\omega_1))$, where each C_δ is a club of δ. It is also not difficult to see that Measuring can be rephrased as the assertion that the algebra $\mathcal{P}(\omega_1)/\,\mathsf{NS}_{\omega_1}$ — where NS_{ω_1} denotes the nonstationary ideal on ω_1 — forces that $\mathcal{C}^V_{\omega_1}$ is a base for an ultrafilter on the Boolean subalgebra of $\mathcal{P}(\omega_1^V)$ generated by the closed sets as

[f]It is worth noticing the resemblance of this argument with Shelah's argument for showing that CH gets preserved by the limit of any countable support iteration of length less than ω_2 of proper forcings of size at most $\aleph_1$ (see e.g. the proof of [1, Theorem 2.10]).

computed in the generic ultrapower $M = V/\dot{G}$, where $\mathcal{C}^V_{\omega_1}$ denotes the club filter on ω_1 in V.

A partial order $\mathbb{P}$ is $\aleph_2$-Knaster if for every sequence $(q_\xi : \xi < \omega_2)$ of $\mathbb{P}$-conditions there is a set $I \subseteq \omega_2$ of cardinality $\aleph_2$ such that q_ξ and $q_{\xi'}$ are compatible for all ξ, $\xi' \in I$. Of course, every $\aleph_2$-Knaster partial order has the $\aleph_2$-chain condition.

Our main theorem is the following.

Theorem 1.2 (CH). *Let $\kappa \geq \omega_2$ be a regular cardinal such that $2^{<\kappa} = \kappa$. Then there is a partial order $\mathcal{P} \subseteq H(\kappa)$ with the following properties*:

(1) *$\mathcal{P}$ is proper.*
(2) *$\mathcal{P}$ is $\aleph_2$-Knaster.*
(3) *$\mathcal{P}$ forces the following statements*:
    (a) *Measuring*;
    (b) *CH*;
    (c) $2^{\aleph_1} = \kappa$.

Theorem 1.2 answers a question of Moore, who asked if Measuring is compatible with CH (see [12] or [21]). The relative consistency of Measuring with CH has also been obtained recently by Golshani and Shelah in [14], where they have actually shown that every countable support iteration of the natural proper posets for adding a club of ω_1 measuring a given club-sequence by countable approximations fails to add new reals.[g] Prior to [14], the strongest failures of Club Guessing at ω_1 known to be within reach of the forcing iteration methods for producing models of CH without adding new reals (see [23]) were only in the region of the negation of weak Club Guessing at ω_1, $\neg$WCG, which is the statement that for every ladder system $(C_\delta : \delta \in \mathrm{Lim}(\omega_1))$ there is a club $C \subseteq \omega_1$ having finite intersection with each C_δ.[h] Moore upon learning about an earlier version of Theorem 1.2, asked whether Measuring implies that there are non-constructible reals. This question was aimed at addressing the issue whether or not adding new reals is a necessary feature of any successful approach to forcing Measuring + CH, and it obtains a negative answer by the Golshani–Shelah result.

Our construction is a sequence $\langle \mathcal{P}_\beta : \beta \leq \kappa \rangle$ which is not a forcing iteration, in the usual sense of $\mathcal{P}_\alpha$ being a complete suborder of $\mathcal{P}_\beta$ for all $\alpha < \beta \leq \kappa$, but which nevertheless has a sufficiently nice property; it is what we will refer to as a *weak forcing iteration.* This means that for all $\alpha < \beta$, every $\mathcal{P}_\alpha$-condition is a $\mathcal{P}_\beta$-condition, for all p_0, $p_1 \in \mathcal{P}_\alpha$, if $p_1 \leq_{\mathcal{P}_\alpha} p_0$, then $p_1 \leq_{\mathcal{P}_\beta} p_0$,[i] and, moreover, every

[g]It is straightforward to see that these natural forcings for adding a given instance of Measuring do not add reals; however, before [14] it was not known whether their countable support iterations also (consistently) have this property.
[h]Measuring implies $\neg$WCG. To see this, suppose $(C_\delta : \delta \in \mathrm{Lim}(\omega_1))$ is a ladder system and $D \subseteq \omega_1$ is a club measuring it. Then every limit point $\delta \in D$ of limit points of D is such that $D \cap C_\delta$ is bounded in δ since no tail of $D \cap \delta$ can possibly be contained in C_δ as C_δ has order type only ω.
[i]Although it not be the case that if $p_1 \leq_{\mathcal{P}_\beta} p_0$, then $p_1 \leq_{\mathcal{P}_\alpha} p_0$. In other words, $\mathcal{P}_\alpha$ need not be a suborder of $\mathcal{P}_\beta$.

predense subset of $\mathcal{P}_\alpha$ is also predense in $\mathcal{P}_\beta$. Using this piece of terminology, our construction can be roughly described as a finitely supported weak forcing iteration $\langle \mathcal{P}_\beta : \beta \leq \kappa \rangle$ in which conditions come together with a side condition consisting of a graph of edges $\{(N_0, \rho_0), (N_1, \rho_1)\}$, where each (N_i, ρ_i) is a model with marker, with suitable structural properties. Given any such edge $\{(N_0, \rho_0), (N_1, \rho_1)\}$, $N_0 \cong N_1$. Furthermore, all the information carried by the condition — including both its working part and its side condition — contained in N_0 and attached to any $\alpha \in N_0 \cap \rho_0$ such that $\Psi_{N_0,N_1}(\alpha) < \rho_1$ (where Ψ_{N_0,N_1} is the unique isomorphism between $(N_0; \in)$ and $(N_1; \in)$) is to be copied over into N_1 by Ψ_{N_0,N_1}. This copying will be crucially used in the proof of CH-preservation[j] and also in other parts of the proof of Theorem 1.2 (most notably in the proof of the $\aleph_2$-chain condition). The working part consists of conditions for natural forcing notions adding instances of Measuring.

Rather than delving into more details here, we direct the reader to the actual construction in Sec. 2.

1.1. *Some observations on extensions of* Measuring

We conclude this introduction by briefly considering some extensions of Measuring.

It is immediate to see that Measuring is equivalent to the statement that if $(\mathcal{C}_\delta : \delta \in \mathrm{Lim}(\omega_1))$ is such that each $\mathcal{C}_\delta$ is a countable collection of closed subsets of δ, then there is a club of ω_1 measuring all members of $\mathcal{C}_\delta$ for each δ. We may thus consider the following family of strengthenings of Measuring.

Definition 1.3. Given a cardinal κ, Meas_κ holds if and only if for every family $\mathcal{C}$ consisting of closed subsets of ω_1 and such that $|\mathcal{C}| \leq \kappa$ there is a club $C \subseteq \omega_1$ with the property that for every $D \in \mathcal{C}$ and every $\delta \in C$ there is some $\alpha < \delta$ such that either

- $(C \cap \delta) \backslash \alpha \subseteq D$ or
- $((C \cap \delta) \backslash \alpha) \cap D = \emptyset$.

$\mathsf{Meas}_{\aleph_0}$ is trivially true in ZFC. Also, it is clear that Meas_κ implies Meas_λ whenever $\lambda < \kappa$, and that $\mathsf{Meas}_{\aleph_1}$ implies Measuring.

Recall that the splitting number, $\mathfrak{s}$, is the minimal cardinality of a splitting family, i.e. of a collection $\mathcal{X} \subseteq [\omega]^{\aleph_0}$ such that for every $Y \in [\omega]^{\aleph_0}$ there is some $X \in \mathcal{X}$ such that $X \cap Y$ and $Y \backslash X$ are both infinite.

In the proof of Fact 1.4, if $(C_\delta : \delta \in \mathrm{Lim}(\omega_1))$ is a ladder system on ω_1, we write $(C_\delta(n))_{n<\omega}$ to denote the strictly increasing enumeration of C_δ. Also, $[\alpha, \beta) = \{\xi \in \mathsf{Ord} : \alpha \leq \xi < \beta\}$ for all ordinals $\alpha \leq \beta$.

Fact 1.4. $\mathsf{Meas}_\mathfrak{s}$ is false.

Proof. Let $\mathcal{X} \subseteq [\omega]^{\aleph_0}$ be a splitting family. Let $(C_\delta)_{\delta \in \mathrm{Lim}(\omega)}$ be a ladder system on ω_1 such that $C_\delta(n)$ is a successor ordinal for each $\delta \in \mathrm{Lim}(\omega_1)$ and $n < \omega$, and

[j]See also [4] for another forcing construction using edges in order to preserve GCH.

let $\mathcal{C}$ be the collection of all sets of the form

$$Z_\delta^X = \bigcup\{[C_\delta(n), C_\delta(n+1)) : n \in X\} \cup \{\delta\}$$

for some $\delta \in \mathrm{Lim}(\omega_1)$ and $X \in \mathcal{X}$. Let D be a club of ω_1, let $\delta < \omega_1$ be a limit point of D, and let

$$Y = \{n < \omega : [C_\delta(n), C_\delta(n+1)) \cap D \neq \emptyset\}.$$

Let $X \in \mathcal{X}$ be such that $X \cap Y$ and $Y \backslash X$ are infinite. Then $Z_\delta^X \cap D$ and $D \backslash Z_\delta^X$ are both cofinal in δ. Hence, D does not measure $\mathcal{C}$. □

The following is proved in joint work of the first author with John Krueger.

Theorem 1.5 ([5]). *$\mathsf{Meas}_{\aleph_1}$ can be forced over any model of* ZFC *and follows from* *BPFA.*

Another natural way to strengthen $\mathsf{Measuring}$ is to allow, in the sequence to be measured, not just closed sets, but also sets of higher complexity (from a descriptive set-theoretic point of view). The version of $\mathsf{Measuring}$ where one considers sequences $\vec{X} = (X_\delta : \delta \in \mathrm{Lim}(\omega_1))$, with each X_δ an open subset of δ in the order topology, is of course equivalent to $\mathsf{Measuring}$. A natural next step would therefore be to consider sequences in which each X_δ is a countable union of closed sets. This is of course the same as allowing each X_δ to be an arbitrary subset of δ. Let us call the corresponding statement $\mathsf{Measuring}^*$.

Definition 1.6. $\mathsf{Measuring}^*$ holds if and only if for every sequence $\vec{X} = (X_\delta : \delta \in \mathrm{Lim}(\omega_1))$, if $X_\delta \subseteq \delta$ for all δ, then there is some club $C \subseteq \omega_1$ such that for every $\delta \in C$, a tail of $C \cap \delta$ is either contained in or disjoint from X_δ.

It is easy to see that $\mathsf{Measuring}^*$ is false in ZFC. As a matter of fact, given a stationary and co-stationary $S \subseteq \omega_1$, there is no club of ω_1 measuring $\vec{X} = (S \cap \delta : \delta \in \mathrm{Lim}(\omega_1))$. In fact, if C is any club of ω_1, then both $C \cap S \cap \delta$ and $(C \cap \delta) \backslash S$ are cofinal subsets of δ for each δ in the club of limit points in ω_1 of both $C \cap S$ and $C \backslash S$.

The status of $\mathsf{Measuring}^*$ is more interesting in the absence of the Axiom of Choice. Let $\mathcal{C}_{\omega_1} = \{X \subseteq \omega_1 : C \subseteq X \text{ for some club } C \text{ of } \omega_1\}$.

Observation 1.1 (ZF $+\mathcal{C}_{\omega_1}$ is a normal filter on ω_1). Suppose $\vec{X} = (X_\delta : \delta \in \mathrm{Lim}(\omega_1))$ is such that

(1) $X_\delta \subseteq \delta$ for each δ.
(2) For each club $C \subseteq \omega_1$,
   (a) there is some $\delta \in C$ such that $C \cap X_\delta \neq \emptyset$ and
   (b) there is some $\delta \in C$ such that $(C \cap \delta) \backslash X_\delta \neq \emptyset$.

Then there is a stationary and co-stationary subset of ω_1 definable from $\vec{X}$.

Proof. We have two possible cases. The first case is when for all $\alpha < \omega_1$, either

- $W^0_\alpha = \{\delta < \omega_1 : \alpha \notin X_\delta\}$ is in $\mathcal{C}_{\omega_1}$ or
- $W^1_\alpha = \{\delta < \omega_1 : \alpha \in X_\delta\}$ is in $\mathcal{C}_{\omega_1}$.

For each $\alpha < \omega_1$ let W_α be W^ϵ_α for the unique $\epsilon \in \{0, 1\}$ such that $W^\epsilon_\alpha \in \mathcal{C}_{\omega_1}$, and let $W^* = \Delta_{\alpha<\omega_1} W_\alpha \in \mathcal{C}_{\omega_1}$. Then $X_{\delta_0} = X_{\delta_1} \cap \delta_0$ for all $\delta_0 < \delta_1$ in W^*. It then follows, by (2), that $S = \bigcup_{\delta \in W^*} X_\delta$, which of course is definable from $\vec{X}$, is a stationary and co-stationary subset of ω_1. Indeed, suppose $C \subseteq \omega_1$ is a club, and let us fix a club $D \subseteq W^*$. There is then some $\delta \in C \cap D$ and some $\alpha \in C \cap D \cap X_\delta$. But then $\alpha \in S$ since $\delta \in W^*$ and $\alpha \in W^* \cap X_\delta$. There is also some $\delta \in C \cap D$ and some $\alpha \in C \cap D$ such that $\alpha \notin X_\delta$, which implies that $\alpha \notin S$ by a symmetrical argument, using the fact that $X_{\delta_0} = X_{\delta_1} \cap \delta_0$ for all $\delta_0 < \delta_1$ in W^*.

The second possible case is that in which there is some $\alpha < \omega_1$ with the property that both W^0_α and W^1_α are stationary subsets of ω_1. But now we can let S be W^0_α, where α is first such that W^0_α is stationary and co-stationary. □

It is worth comparing the above observation with Solovay's classic result that an ω_1-sequence of pairwise disjoint stationary subsets of ω_1 is definable from any given ladder system on ω_1 (working in the same theory).

Corollary 1.7 (ZF + $\mathcal{C}_{\omega_1}$ is a normal filter on ω_1). *The following are equivalent*:

(1) *$\mathcal{C}_{\omega_1}$ is an ultrafilter on ω_1.*
(2) *Measuring*.*
(3) *For every sequence $(X_\delta : \delta \in \mathrm{Lim}(\omega_1))$, if $X_\delta \subseteq \delta$ for each δ, then there is a club $C \subseteq \omega_1$ such that either*
   - *$C \cap \delta \subseteq X_\delta$ for every $\delta \in C$ or*
   - *$C \cap X_\delta = \emptyset$ for every $\delta \in C$.*

Proof. (3) trivially implies (2), and by the observation (1) implies (3). Finally, to see that (2) implies (1), note that the argument right after the definition of Measuring* uses only ZF together with the regularity of ω_1 and the negation of (1). □

In particular, the strong form of Measuring* given by (3) in the above observation follows from ZF together with the Axiom of Determinacy.

Much of the notation used in this paper follows the standards set forth in [15, 17]. Other, less standard, pieces of notation will be introduced as needed. The rest of this paper is structured as follows. In Sec. 2, we construct a sequence $(\mathcal{P}_\beta : \beta \leq \kappa)$ of forcing notions. In Sec. 3, we prove the relevant facts about this construction which will show $\mathcal{P}_\kappa$ to witness the conclusion of Theorem 1.2. Section 3.4 contains some remarks on why our construction in Sec. 2 cannot possibly be adapted to force Unif($\vec{C}$) for any ladder system $\vec{C}$ (which, as we already mentioned, is well known to be incompatible with CH), and on the (closely related) obstacles towards building models of reasonable forcing axioms together with CH using the present approach.

2. The Main Construction

The theorem we will prove in this and the following section, we recall, is the following.

Theorem 2.1 (CH). *Let $\kappa \geq \omega_2$ be a regular cardinal such that $2^{<\kappa} = \kappa$. Then there is a partial order $\mathcal{P} \subseteq H(\omega_2)$ with the following properties*:

(1) *$\mathcal{P}$ is proper.*
(2) *$\mathcal{P}$ is $\aleph_2$-Knaster.*
(3) *$\mathcal{P}$ forces the following statements*:

  (a) *Measuring*;
  (b) *CH*;
  (c) $2^{\aleph_1} = \kappa$.

In this section, we present the construction of a certain sequence $(\mathcal{P}_\beta : \beta \leq \kappa)$ of forcing notions. In Sec. 3, we will prove that $\mathcal{P}_\kappa$ is a forcing $\mathcal{P}$ witnessing the conclusion of Theorem 2.1.

We start out by fixing some pieces of notation that will be used in both this and the following section. If N is a set such that $N \cap \omega_1 \in \omega_1$, δ_N denotes this intersection. δ_N is also called *the height of N*.

Given $P \subseteq H(\kappa)$ and $N \subseteq H(\kappa)$, we will tend to write (N, P) as short hand for $(N, P \cap N)$. Also, if N_0 and N_1 are $\in$-isomorphic elementary submodels of $H(\kappa)$, we refer to the unique $\in$-isomorphism $\Psi : (N_0; \in) \rightarrow (N_1; \in)$ as Ψ_{N_0,N_1}.

We will make use of the following notion of symmetric system from [7].

Definition 2.2. Let $T \subseteq H(\kappa)$ and let $\mathcal{N}$ be a finite collection of countable subsets of $H(\kappa)$. We say that $\mathcal{N}$ *is a T-symmetric system* if and only if the following holds:

(1) For every $N \in \mathcal{N}$, $(N; \in, T)$ is an elementary substructure of $(H(\kappa); \in, T)$.
(2) Given N_0 and N_1 in $\mathcal{N}$, if $\delta_{N_0} = \delta_{N_1}$, then there is a unique isomorphism

$$\Psi_{N_0,N_1} : (N_0; \in, T) \rightarrow (N_1; \in, T).$$

  Furthermore, Ψ_{N_0,N_1} is the identity on $N_0 \cap N_1$.
(3) For all N_0, N_1, $M \in \mathcal{N}$, if $M \in N_0$ and $\delta_{N_0} = \delta_{N_1}$, then $\Psi_{N_0,N_1}(M) \in \mathcal{N}$.
(4) For all N and M in $\mathcal{N}$, if $\delta_M < \delta_N$, then there is $N' \in \mathcal{N}$ such that $\delta_{N'} = \delta_N$ and $M \in N'$.

Taking up a suggestion of Inamdar, we call condition (4) *the shoulder axiom*.

Strictly speaking, the phrase "T-symmetric system" is ambiguous in general since $H(\kappa)$ may not be determined by T. However, in all practical cases $(\bigcup T) \cap \mathsf{Ord} = \kappa$, so T does determine $H(\kappa)$ in these cases.

We will talk about *symmetric systems* in some contexts in which T is clear or irrelevant.

The following two amalgamation lemmas are proved in [7].

Lemma 2.3. *Let $T \subseteq H(\kappa)$ and let $\mathcal{N}$ be a T-symmetric system. Let $N \in \mathcal{N}$ and let $\mathcal{M} \in N$ be a T-symmetric system such that $\mathcal{N} \cap N \subseteq \mathcal{M}$. Let*

$$\mathcal{W}(\mathcal{N}, \mathcal{M}, N) := \mathcal{N} \cup \{\Psi_{N,N'}(M) : M \in \mathcal{M}, N' \in \mathcal{N}, \delta_{N'} = \delta_N\}.$$

Then $\mathcal{W}(\mathcal{N}, \mathcal{M}, N)$ is the $\subseteq$-minimal T-symmetric system $\mathcal{W}$ such that $\mathcal{N} \cup \mathcal{M} \subseteq \mathcal{W}$.

Given $T \subseteq H(\kappa)$ and $\mathcal{N}_0$ and $\mathcal{N}_1$, T-symmetric systems, let us write $\mathcal{N}_0 \cong_T \mathcal{N}_1$ if $|\mathcal{N}_0| = |\mathcal{N}_1| = n$, for some $n < \omega$, and there are enumerations $(N_i^0 : i < n)$ and $(N_i^1 : i < n)$ of $\mathcal{N}_0$ and $\mathcal{N}_1$, respectively, for which there is an isomorphism

$$\Psi : \left(\bigcup \mathcal{N}_0; \in, N_i^0, T\right)_{i<n} \to \left(\bigcup \mathcal{N}_1; \in, N_i^1, T\right)_{i<n}$$

which is the identity on $(\bigcup \mathcal{N}_0) \cap (\bigcup \mathcal{N}_1)$.

Lemma 2.4. *Let $T \subseteq H(\kappa)$ and let $\mathcal{N}_0$ and $\mathcal{N}_1$ be T-symmetric systems such that $\mathcal{N}_0 \cong_T \mathcal{N}_1$. Then $\mathcal{N}_0 \cup \mathcal{N}_1$ is the $\subseteq$-minimal T-symmetric system $\mathcal{W}$ such that $\mathcal{N}_0 \cup \mathcal{N}_1 \subseteq \mathcal{W}$.*

We will recursively build a sequence $(\mathcal{P}_\beta : \beta \leq \kappa)$ of forcing notions, together with a sequence of predicates $(\Phi_\alpha : \alpha < \kappa)$. Theorem 2.1 will be witnessed by $\mathcal{P}_\kappa$. Given $\beta < \kappa$ we let

$$\mathcal{T}_\beta = \{N \in [H(\kappa)]^{\aleph_0} : (N; \in, \Phi_\beta) \preccurlyeq (H(\kappa); \in \Phi_\beta)\}.$$

Let $\mathrm{Succ}(\kappa)$ denote the set of successor ordinals below κ. To start with, let us fix a function $\Phi : \mathrm{Succ}(\kappa) \to H(\kappa)$ with the property that $\{\alpha \in \mathrm{Succ}(\kappa) : \Phi(\alpha) = x\}$ is unbounded in κ for each $x \in H(\kappa)$ (which exists by $2^{<\kappa} = \kappa$), and let Φ_0 be the satisfaction predicate for the structure $(H(\kappa); \in, \Phi)$. Also, given any $\beta > 0$, Φ_β will uniformly encode, among other things, the sequences $(\Phi_\alpha : \alpha < \beta)$ and $(\mathrm{Sat}(\Phi_\alpha) : \alpha < \beta)$, where $\mathrm{Sat}(\Phi_\alpha)$ denotes the satisfaction predicate for the structure $(H(\kappa); \in, \Phi_\alpha)$.

We will call an ordered pair (N, ρ), where

- N is a countable elementary submodel of $(H(\kappa); \in, \Phi_0)$;
- $\rho \in N \cap \kappa$ and
- $N \in \mathcal{T}_{\alpha+1}$ for every $\alpha \in N \cap \rho$,

a *model with marker*.[k]

[k]In the definition of $\mathcal{P}_\beta$, we will assume $\Phi_{\alpha+1}$ has been defined for all $\alpha < \beta$. While defining $\mathcal{P}_\beta$, we will refer to the notion of model with marker. In that case, the marker ρ will be at most β, and hence $\Phi_{\alpha+1}$ — and therefore $\mathcal{T}_{\alpha+1}$ — will be defined for all $\alpha \in N \cap \rho$.

If (N, ρ) is a model with marker, we will sometimes say that ρ *is the marker of* (N, ρ).

In our forcing construction, we will use models with markers (N, ρ) in a crucial way. The presence of the marker ρ will tell us that N is to be seen as "active" for all stages in $N \cap \rho$.

Given an unordered pair

$$e = \{(N_0, \rho_0), (N_1, \rho_1)\}$$

of models with markers, we will call e an *edge* in case

(1) $N_0 \cong N_1$;
(2) for every $\alpha \in N_0 \cap \rho_0$, if $\bar{\alpha} = \Psi_{N_0,N_1}(\alpha) < \rho_1$, then Ψ_{N_0,N_1} is an isomorphism between

$$(N_0; \in, \Phi_{\alpha+1})$$

and

$$(N_1; \in, \Phi_{\bar{\alpha}+1}).$$

We note that, in the above definition, (N_0, ρ_0) and (N_1, ρ_1) may or may not be distinct. Hence, an edge may contain two models with markers or may just be the singleton $\{(N, \rho)\}$ of a model with marker (N, ρ).

Also, we call an ordered pair $\langle (N_0, \rho_0), (N_1, \rho_1)\rangle$ a *directed edge* if $\{(N_0, \rho_0), (N_1, \rho_1)\}$ is an edge. If $\mathcal{G}$ is a set of edges, we say that a directed edge $\langle (N_0, \rho_0), (N_1, \rho_1)\rangle$ *comes from* $\mathcal{G}$ if $\{(N_0, \rho_0), (N_1, \rho_1)\} \in \mathcal{G}$.

If $e = \langle (N_0, \rho_0), (N_1, \rho_1)\rangle$ is a directed edge, we write Ψ_e for Ψ_{N_0,N_1}.

If $\beta < \kappa$, we say that an edge $\{(N_0, \rho_0), (N_1, \rho_1)\}$ is *below* β if $\rho_0 \leq \beta$ and $\rho_1 \leq \beta$.

Given a set $\mathcal{G}$ of edges,[1] we denote $\bigcup \mathcal{G}$ by $\Delta(\mathcal{G})$; i.e. $\Delta(\mathcal{G})$ is the set of models with markers (N, ρ) for which there is some (N', ρ') such that $\{(N, \rho), (N', \rho')\} \in \mathcal{G}$.

Given a directed edge $e = \langle (N_0, \rho_0), (N_1, \rho_1)\rangle$ and an edge $e' = \{(N'_0, \rho'_0), (N'_1, \rho'_1)\}$ such that

- $e' \in N_0$;
- $\max\{\rho'_0, \rho'_1\} \leq \rho_0$ and
- $\Psi_{N_0,N_1}(\max\{\rho'_0, \rho'_1\}) \leq \rho_1$,

we denote

$$\{(\Psi_{N_0,N_1}(N'_0), \Psi_{N_0,N_1}(\rho'_0)), (\Psi_{N_0,N_1}(N'_1), \Psi_{N_0,N_1}(\rho'_1))\}$$

by $\Psi_e(e')$.

Fact 2.5. Suppose $e = \langle (N_0, \rho_0), (N_1, \rho_1)\rangle$ is a directed edge and $e' = \{(N'_0, \rho'_0), (N'_1, \rho'_1)\}$ is an edge such that

- $e' \in N_0$;

[1]We think of sets of edges as graphs, hence the choice of the letter $\mathcal{G}$ in this context.

- $\max\{\rho'_0, \rho'_1\} \leq \rho_0$ and
- $\Psi_{N_0,N_1}(\max\{\rho'_0, \rho'_1\}) \leq \rho_1$.

Then $\Psi_e(e')$ is an edge.

Proof. For $i \in \{0, 1\}$, let $N''_i = \Psi_{N_0,N_1}(N'_i)$. Then, for each i, the elementarity of Ψ_{N_0,N_1}, together with the fact that $N'_0 \cong N'_1$ and $\rho'_i \in N'_i$, implies that $N''_0 \cong N''_1$ and $\Psi_{N_0,N_1}(\rho'_i) \in N''_i$. Furthermore, for each $\alpha \in N'_i \cap \rho'_i$, the fact that Ψ_{N_0,N_1} is also an isomorphism between the structures $(N_0; \in, \Phi_{\alpha+1})$ and $(N_1; \in, \Phi_{\bar{\alpha}+1})$, for $\bar{\alpha} = \Psi_{N_0,N_1}(\alpha)$, together with $(N'_i; \in, \Phi_{\alpha+1}) \preccurlyeq (N_0; \in, \Phi_{\alpha+1})$, implies that

$$(N''_i; \in, \Phi_{\bar{\alpha}+1}) \preccurlyeq (N_1; \in, \Phi_{\bar{\alpha}+1}) \preccurlyeq (H(\kappa); \in, \Phi_{\bar{\alpha}+1}).$$

Hence, $(N''_i, \Psi_{N_0,N_1}(\rho'_i))$ is a model with marker. Finally, if α and $\bar{\alpha}$ are as above, with $i = 0$, $\beta = \Psi_{N'_0,N'_1}(\alpha)$ and $\alpha^\dagger := \Psi_{N''_0,N''_1}(\bar{\alpha}) = \Psi_{N_0,N_1}(\beta) < \Psi_{N_0,N_1}(\rho'_1)$, then letting $\alpha^* = \max\{\alpha, \beta\}$ and $\alpha^{**} = \Psi_{N_0,N_1}(\alpha^*)$ and using the fact that $(N'_0; \in, \Phi_{\alpha+1}) \cong (N'_1; \in, \Phi_{\Psi_{N'_0,N'_1}(\alpha)+1})$ and that Ψ_{N_0,N_1} is also an isomorphism between $(N_0; \in, \Phi_{\alpha^*+1})$ and $(N_1; \in, \Phi_{\alpha^{**}+1})$, we get that $(N''_0; \in, \Phi_{\bar{\alpha}+1}) \cong (N''_1; \in, \Phi_{\alpha^\dagger+1})$. To see this, simply use that $(N'_0; \in, \Phi_{\alpha+1}) \preccurlyeq (N_0; \in, \Phi_{\alpha+1})$, $(N'_1; \in, \Phi_{\beta+1}) \preccurlyeq (N_0; \in, \Phi_{\beta+1})$ and, if $\alpha^* > \min\{\alpha, \beta\}$, also that Φ_{α^*+1} codes the satisfaction relation of $(H(\kappa); \in, \Phi_{\min\{\alpha,\beta\}+1})$. □

Given a set $\mathcal{G}$ of edges, we say that $\mathcal{G}$ is *closed under restrictions* if $\{(N_0, \alpha_0), (N_1, \alpha_1)\} \in \mathcal{G}$ whenever $\{(N_0, \rho_0), (N_1, \rho_1)\} \in \mathcal{G}$, $\alpha_0 \in N_0 \cap (\rho_0 + 1)$ and $\alpha_1 \in N_1 \cap (\rho_1 + 1)$. Also, we say that $\mathcal{G}$ is *closed under copying* in case for every directed edge $e = \langle (N_0, \rho_0), (N_1, \rho_1) \rangle$ coming from $\mathcal{G}$ and every edge $e' = \{(N'_0, \rho'_0), (N'_1, \rho'_1)\} \in \mathcal{G}$, if $e' \in N_0$, $\max\{\rho'_0, \rho'_1\} \leq \rho_0$, and $\Psi_{N_0,N_1}(\max\{\rho'_0, \rho'_1\}) \leq \rho_1$, then $\Psi_e(e') \in \mathcal{G}$.

If Δ is a set of models with markers and $\beta < \kappa$, we let

$$\mathcal{N}^\Delta_\beta = \{N : (N, \beta) \in \Delta\}.$$[m]

We say that a set $\mathcal{G}$ of edges is *sticky* in case for every ordinal α and for all N_0, $N_1 \in \mathcal{N}^{\Delta(\mathcal{G})}_{\alpha+1}$, if $\delta_{N_0} = \delta_{N_1}$, then $\{(N_0, \alpha + 1), (N_1, \alpha + 1)\} \in \mathcal{G}$.[n]

Given sets $\mathcal{G}_0$ and $\mathcal{G}_1$ of edges, we say that $\mathcal{G}_0$ *and* $\mathcal{G}_1$ *are compatible* in case for all $\alpha < \kappa$ and N_0, $N_1 \in \mathcal{N}^{\Delta(\mathcal{G}_0)}_{\alpha+1} \cup \mathcal{N}^{\Delta(\mathcal{G}_1)}_{\alpha+1}$ such that $\delta_{N_0} = \delta_{N_1}$ we have that $(N_0; \in, \Phi_{\alpha+1}) \cong (N_1; \in, \Phi_{\alpha+1})$. If this is the case, then there is a $\subseteq$-minimum sticky set $\mathcal{G}$ of edges including both $\mathcal{G}_0$ and $\mathcal{G}_1$ and which is closed under restrictions and closed under copying. We denote this set $\mathcal{G}$ by $\mathcal{G}_0 \oplus \mathcal{G}_1$.

If $\mathcal{G}$ is a set of edges, we denote by $\mathbb{M}(\mathcal{G})$ some canonically chosen structure with universe $\bigcup \mathrm{dom}(\Delta(\mathcal{G}))$ coding $\mathcal{G}$ and

$$\left\langle \left(\alpha, \Phi_{\alpha+1} \cap \bigcup \mathrm{dom}(\Delta(\mathcal{G}))\right) : \alpha \in \bigcup \{N \cap \rho : (N, \rho) \in \Delta(\mathcal{G})\} \right\rangle.$$

[m]Note that if $\mathcal{G}$ is a set of edges closed under restrictions and $\Delta = \Delta(\mathcal{G})$, then $\mathcal{N}^\Delta_0$ is the same thing as $\mathrm{dom}(\Delta)$.

[n]In particular, if $\mathcal{G}$ is sticky, then $\{(N, \alpha + 1)\} \in \mathcal{G}$ for every ordinal α and every $N \in \mathcal{N}^{\Delta(\mathcal{G})}_{\alpha+1}$.

Also, we consider the following form of the isomorphism relation $\cong_T$ for T-symmetric systems, for sets of edges: If $\mathcal{G}_0$ and $\mathcal{G}_1$ are sets of edges, we write $\mathcal{G}_0 \cong \mathcal{G}_1$ in case there is an isomorphism $\Psi : \mathbb{M}(\mathcal{G}_0) \to \mathbb{M}(\mathcal{G}_1)$ which is the identity on $(\bigcup \mathrm{dom}(\Delta(\mathcal{G}_0))) \cap (\bigcup \mathrm{dom}(\Delta(\mathcal{G}_1)))$.

We will use the following easy extension of Lemma 2.4.

Lemma 2.6. *Let $\mathcal{G}_0$ and $\mathcal{G}_1$ be sticky sets of edges closed under restrictions and under copying. Suppose $\mathcal{G}_0 \cong \mathcal{G}_1$. Then $\mathcal{G}_0 \oplus \mathcal{G}_1$ is the union of $\mathcal{G}_0 \cup \mathcal{G}_1$ and the set of unordered pairs $\{(N_0, \alpha_0 + 1), (N_1, \alpha_1 + 1)\}$ such that $\delta_{N_0} = \delta_{N_1}$, $\alpha_0 \in N_0$, $\alpha_1 \in N_1$, and for which there is some $\alpha \geq \alpha_0$, α_1 such that $N_0 \in \mathcal{N}_{\alpha+1}^{\Delta(\mathcal{G}_0)}$ and $N_1 \in \mathcal{N}_{\alpha+1}^{\Delta(\mathcal{G}_1)}$.*[o] *Hence, if, in addition, $\mathcal{N}_0^{\Delta(\mathcal{G}_0)}$ and $\mathcal{N}_0^{\Delta(\mathcal{G}_1)}$ are Φ_0-symmetric systems and $\mathcal{N}_{\alpha+1}^{\Delta(\mathcal{G}_0)}$ and $\mathcal{N}_{\alpha+1}^{\Delta(\mathcal{G}_1)}$ are $\Phi_{\alpha+1}$-symmetric systems for each $\alpha < \kappa$, then $\mathcal{N}_0^{\Delta(\mathcal{G}_0 \oplus \mathcal{G}_1)}$ is a Φ_0-symmetric system and $\mathcal{N}_{\alpha+1}^{\Delta(\mathcal{G}_0 \oplus \mathcal{G}_1)}$ is a $\Phi_{\alpha+1}$-symmetric system for each $\alpha < \kappa$.*

If $\mathcal{G}$ is a set of edges and $\alpha < \kappa$, we let

$$\mathcal{G}|_\alpha = \{\{(N_0, \rho_0), (N_1, \rho_1)\} \in \mathcal{G} : \rho_0, \rho_1 \leq \alpha\}.$$

We will need the following easy lemma.

Lemma 2.7. *Suppose $\mathcal{G}$ is a sticky set of edges closed under restrictions and under copying. Suppose $\mathcal{N}_0^{\Delta(\mathcal{G})}$ is a Φ_0-symmetric system and $\mathcal{N}_{\alpha+1}^{\Delta(\mathcal{G})}$ is a $\Phi_{\alpha+1}$-symmetric system for each $\alpha < \kappa$. Let $\alpha_0 < \kappa$. Then the following holds*:

(1) *$\mathcal{G}|_{\alpha_0}$ is a sticky set of edges closed under restrictions and under copying.*
(2) *$\mathcal{N}_\alpha^{\Delta(\mathcal{G}|_{\alpha_0})} = \mathcal{N}_\alpha^{\Delta(\mathcal{G})}$ for every $\alpha \leq \alpha_0$. In particular, $\mathcal{N}_0^{\Delta(\mathcal{G}|_{\alpha_0})}$ is a Φ_0-symmetric system and for each $\alpha < \kappa$, $\mathcal{N}_{\alpha+1}^{\Delta(\mathcal{G}|_{\alpha_0})}$ is a $\Phi_{\alpha+1}$-symmetric system.*

Given functions $f_0, \dots, f_n$, for some $n < \omega$, we let

$$f_n \circ \cdots \circ f_0$$

be f_0 if $n = 0$; if $n > 0$, we let this expression denote the function f with domain the set of x such that for every $i < n$, $x \in \mathrm{dom}(f_i \circ \cdots \circ f_0)$ and $(f_i \circ \cdots \circ f_0)(x) \in \mathrm{dom}(f_{i+1})$, and such that for every $x \in \mathrm{dom}(f)$, $f(x) = f_n((f_{n-1} \circ \cdots \circ f_0)(x))$.

If $\vec{\mathcal{E}} = (\langle (N_0^i, \rho_0^i), (N_1^i, \rho_1^i)\rangle : i < n)$, for some $n < \omega$, is a sequence of pairs of models with markers such that $N_0^i \cong N_1^i$ for all $i < n$, we denote $\Psi_{N_0^{n-1}, N_1^{n-1}} \circ \cdots \circ \Psi_{N_0^0, N_1^0}$ by $\Psi_{\vec{\mathcal{E}}}$. We also let $\delta_{\vec{\mathcal{E}}} = \{\delta_{N_0^i} : i < n\}$.

If $\mathcal{G}$ is a set of edges and $a \in H(\kappa)$, we call $\langle a, \vec{\mathcal{E}}\rangle$ a *$\mathcal{G}$-thread* if $\vec{\mathcal{E}}$ is a finite sequence of directed edges coming from $\mathcal{G}$ and $a \in \mathrm{dom}(\Psi_{\vec{\mathcal{E}}})$.

Given a set $\mathcal{G}$ of edges and an ordinal $\alpha < \kappa$, we say that

$$\langle \alpha, (\langle (N_0^i, \rho_0^i), (N_1^i, \rho_1^i)\rangle : i \leq n)\rangle$$

[o] We note that, in particular, $\mathcal{G}_0$ and $\mathcal{G}_1$ are compatible, and so $\mathcal{G}_0 \oplus \mathcal{G}_1$ exists.

is a *connected $\mathcal{G}$-thread* in case the following holds:

(1) $\langle\alpha, (\langle(N_0^i, \rho_0^i), (N_1^i, \rho_1^i)\rangle : i \leq n)\rangle$ is a $\mathcal{G}$-thread.
(2) $\alpha \in N_0^0 \cap (\rho_0^0 + 1)$ and $\Psi_{N_0^0, N_1^0}(\alpha) < \rho_1^0 + 1$.
(3) If $n > 0$, then $\langle(\Psi_{N_0^0, N_1^0}(\alpha), (\langle(N_0^i, \rho_0^i), (N_1^i, \rho_1^i)\rangle : 0 < i \leq n)\rangle$ is a connected $\mathcal{G}$-thread.

If $\mathcal{G}$ is a set of edges and (δ, α), $(\delta, \bar{\alpha}) \in \omega_1 \times \kappa$, we say that $(\delta, \bar{\alpha})$ *is $\mathcal{G}$-accessible from* (δ, α) if

- $\bar{\alpha} = \alpha$ or
- there is a connected $\mathcal{G}$-thread $\langle\alpha, \vec{\mathcal{E}}\rangle$ such that $\bar{\alpha} = \Psi_{\vec{\mathcal{E}}}(\alpha)$ and $\delta \leq \min(\delta_{\vec{\mathcal{E}}})$.

In the proof of Lemma 2.8, if

$$\vec{\mathcal{E}} = (\langle(N_0^i, \rho_0^i), (N_1^i, \rho_1^i)\rangle : i < n)$$

is a sequence of ordered edges, we will denote the sequence

$$(\langle(N_1^{n-1-i}, \rho_1^{n-1-i}), (N_0^{n-1-i}, \rho_0^{n-1-i})\rangle : i < n)$$

by $(\vec{\mathcal{E}})^{-1}$.

We will need the following counterpart of Lemma 2.3 for sets of edges.

Lemma 2.8. *Let $\beta < \kappa$. Let $\mathcal{G}_0$ be a sticky set of edges below β closed under restrictions and under copying and such that $\mathcal{N}_0^{\Delta(\mathcal{G}_0)}$ is a Φ_0-symmetric system and $\mathcal{N}_{\alpha+1}^{\Delta(\mathcal{G}_0)}$ is a $\Phi_{\alpha+1}$-symmetric system for each $\alpha < \kappa$. Let $N \in \mathcal{N}_\beta^{\Delta(G_0)}$. Suppose $\mathcal{G}_1 \in N$ is a sticky set of edges below β closed under restrictions and under copying and such that $\mathcal{N}_0^{\Delta(\mathcal{G}_1)}$ is a Φ_0-symmetric system and $\mathcal{N}_{\alpha+1}^{\Delta(\mathcal{G}_1)}$ is a $\Phi_{\alpha+1}$-symmetric system for each $\alpha < \kappa$. Suppose $\mathcal{G}_0 \cap N \subseteq \mathcal{G}_1$. Finally, suppose that for every $Q \in \mathrm{dom}(\Delta(\mathcal{G}_0)) \cap N$, $\mathcal{G}_1 \cap Q = \mathcal{G}_0 \cap Q$. Let $\mathcal{G}^*$ be the union of the following sets:*

(1) $\mathcal{G}_0$.
(2) *The set $\mathcal{G}_2$ consisting of unordered pairs of the form*

$$\{(\Psi_{\vec{\mathcal{E}}}(N_0), \Psi_{\vec{\mathcal{E}}}(\rho_0)), (\Psi_{\vec{\mathcal{E}}}(N_1), \Psi_{\vec{\mathcal{E}}}(\rho_1))\},$$

where $\{(N_0, \rho_0), (N_1, \rho_1)\} \in \mathcal{G}_1$, $\langle\{N_0, N_1\}, \vec{\mathcal{E}}\rangle$ is a $\mathcal{G}_0$-thread with $\min(\delta_{\vec{\mathcal{E}}}) = \delta_N$, and $\langle\rho_0, \vec{\mathcal{E}}\rangle$ and $\langle\rho_1, \vec{\mathcal{E}}\rangle$ are connected $\mathcal{G}_0$-threads.
(3) *The set $\mathcal{G}_3$ consisting of unordered pairs of the form*

$$\{(M_0, \alpha_0), (M_1, \alpha_1)\}$$

such that $\delta_{M_0} = \delta_{M_1}$ and for which there is some $\alpha < \beta$ such that $\{(M_0, \alpha + 1)\} \in \mathcal{G}_2$, $\{(M_1, \alpha + 1)\} \in \mathcal{G}_2$, $\alpha_0 \in M_0 \cap (\alpha + 2)$ and $\alpha_1 \in M_1 \cap (\alpha + 2)$.

Then $\mathcal{G}^*$ is a sticky set of edges closed under restrictions and under copying, $\mathcal{N}_0^{\Delta(\mathcal{G}^*)}$ is a Φ_0-symmetric system, and $\mathcal{N}_{\alpha+1}^{\Delta(\mathcal{G}^*)}$ is a $\Phi_{\alpha+1}$-symmetric system for each $\alpha < \kappa$.

Proof. It is immediate to check that, by our construction, $\mathcal{G}^*$ is closed under restrictions. Also, it is clear that $\mathcal{N}_0^{\Delta(\mathcal{G}^*)} = \mathcal{N}_0^{\Delta(\mathcal{H})}$, where

$$\mathcal{H} = \mathcal{G}_0 \cup \{\{(\Psi_{N,N'}(M), 0)\} : M \in \mathcal{N}_0^{\Delta(\mathcal{G}_1)}, N' \in \mathcal{N}_0^{\Delta(\mathcal{G}_0)}, \delta_{N'} = \delta_N\}.$$

Hence, by Lemma 2.3, $\mathcal{N}_0^{\mathcal{G}^*}$ is a Φ_0-symmetric system. We will now prove, for every $\alpha < \beta$, that $\mathcal{N}_{\alpha+1}^{\Delta(\mathcal{G}^*)}$ is a $\Phi_{\alpha+1}$-symmetric system. The point that needs the most work is the verification of the shoulder axiom for $\mathcal{N}_{\alpha+1}^{\Delta(\mathcal{G}^*)}$, which we will go through next.

For this, given M_0^*, $M_1^* \in \mathcal{N}_{\alpha+1}^{\mathcal{G}^*}$ such that $\delta_{M_0^*} < \delta_{M_1^*}$, it is enough to show that there is some $M_1^{**} \in \mathcal{N}_{\alpha+1}^{\Delta(\mathcal{G}^*)}$ such that $\delta_{M_1^{**}} = \delta_{M_1^*}$ and $M_0^* \in M_1^{**}$. If $\delta_{M_0^*} \geq \delta_N$, then M_0^* and M_1^* are both in $\mathrm{dom}(\Delta(\mathcal{G}_0))$ and so we are done by the shoulder axiom for $\mathcal{N}_{\alpha+1}^{\Delta(\mathcal{G}_0)}$. Hence, we will assume in what follows that $\delta_{M_0^*} < \delta_N$. If $M_0^* \in \mathcal{N}_{\alpha+1}^{\Delta(\mathcal{G}_0)}$, then we may of course assume that $M_1^* \notin \mathcal{N}_{\alpha+1}^{\Delta(\mathcal{G}_0)}$. It then follows, by the definition of $\mathcal{G}_2$, together with the stickiness of $\mathcal{G}_0$ and the shoulder axiom for $\mathcal{N}_{\alpha+1}^{\Delta(\mathcal{G}_0)}$, that there is a sequence $\vec{\mathcal{E}}$ such that $\langle M_0^*, \vec{\mathcal{E}}\rangle$ is a $\mathcal{G}_0$-thread with $\min(\delta_{\vec{\mathcal{E}}}) = \delta_N$, $\langle \alpha+1, \vec{\mathcal{E}}\rangle$ is a connected $\mathcal{G}_0$-thread, and $\Psi_{\vec{\mathcal{E}}}(M_0^*) \in N$. Then $M_0 := \Psi_{\vec{\mathcal{E}}}(M_0^*) \in \mathrm{dom}(\Delta(\mathcal{G}_0)) \cap N$, and therefore $M_0 \in \mathrm{dom}(\Delta(\mathcal{G}_1))$.

For $i = 0$, 1, let us fix $\alpha_i < \beta$, $M_i \in \mathcal{N}_{\alpha_i+1}^{\Delta(\mathcal{G}_1)}$ and $\vec{\mathcal{E}}_i$ such that $\langle (M_i, \alpha_i+1), \vec{\mathcal{E}}_i\rangle$ is a $\mathcal{G}_0$-thread, $\min(\delta_{\vec{\mathcal{E}}_i}) = \delta_N$ and $\langle \alpha_i+1, \vec{\mathcal{E}}_i\rangle$ is a connected $\mathcal{G}_0$-thread. Suppose $\alpha = \Psi_{\vec{\mathcal{E}}_0}(\alpha_0) = \Psi_{\vec{\mathcal{E}}_1}(\alpha_1)$ and $\delta_{M_0} < \delta_{M_1}$. By the analysis in the previous paragraph, in order to show the shoulder axiom for $\mathcal{N}_{\alpha+1}^{\Delta(\mathcal{G}^*)}$ it will suffice to prove that there is some $M_1' \in \mathcal{N}_{\alpha+1}^{\Delta(\mathcal{G}^*)}$ such that $\delta_{M_1'} = \delta_{M_1}$ and $\Psi_{\vec{\mathcal{E}}_0}(M_0) \in M_1'$. By, if necessary, appending suitable ordered edges from $\mathcal{G}_0$ at the right places using stickiness of $\mathcal{G}_0$ and the shoulder axiom for $\mathcal{N}_{\gamma+1}^{\Delta(\mathcal{G}_0)}$ for appropriate γ — these places could be the beginning or the end of $\vec{\mathcal{E}}_0$, the beginning or the end of $\vec{\mathcal{E}}_1$, or somewhere inside $\vec{\mathcal{E}}_0$ or $\vec{\mathcal{E}}_1$ — we obtain $\vec{\mathcal{E}}_0'$ and $\vec{\mathcal{E}}_1'$ such that

$$\Psi_{\vec{\mathcal{E}}_1'}^{-1} \circ \Psi_{\vec{\mathcal{E}}_0'} : (N; \in) \to (N; \in)$$

is an isomorphism. But then $\Psi_{\vec{\mathcal{E}}_1'}^{-1} \circ \Psi_{\vec{\mathcal{E}}_0'} \restriction N$ is of course the identity on N, which implies that $\alpha_0 = \alpha_1$ since $\Psi_{\vec{\mathcal{E}}_1'}^{-1} \circ \Psi_{\vec{\mathcal{E}}_0'}(\alpha_0) = \alpha_1$ from the way we have constructed $\vec{\mathcal{E}}_0'$ and $\vec{\mathcal{E}}_1'$ from $\vec{\mathcal{E}}_0$ and $\vec{\mathcal{E}}_1$, respectively. Now, by the shoulder axiom for $\mathcal{N}_{\alpha_0+1}^{\Delta(\mathcal{G}_1)}$, we can find $M_1^\dagger \in \mathcal{N}_{\alpha_0+1}^{\Delta(\mathcal{G}_1)}$ such that $\delta_{M_1^\dagger} = \delta_{M_1}$ and $M_0 \in M_1^\dagger$, and $M_1^{**} := \Psi_{\vec{\mathcal{E}}_0}(M_1^\dagger)$ is then a model in $\mathcal{N}_{\alpha+1}^{\mathcal{G}^*}$ as desired.

Similarly, by an argument as in the above proof of the shoulder axiom, we can see that if M_0, $M_1 \in \mathcal{N}_{\alpha+1}^{\Delta(\mathcal{G}^*)}$ are such that $\delta_{M_0} = \delta_{M_1}$, then $(M_0; \in, \Phi_{\alpha+1}) \cong (M_1; \in, \Phi_{\alpha+1})$. More specifically, and as in the proof of the shoulder axiom, we may assume that we are in the case in which for each $i \in \{0,1\}$ there are $\alpha_i < \beta$, $M_i^- \in \mathcal{N}_{\alpha_i+1}^{\Delta(\mathcal{G}_1)}$ and $\vec{\mathcal{E}}_i$ such that $\langle (M_i^-, \alpha_i+1), \vec{\mathcal{E}}_i\rangle$ is a $\mathcal{G}_0$-thread, $\min(\delta_{\vec{\mathcal{E}}_i}) = \delta_N$,

$\langle \alpha_i + 1, \vec{\mathcal{E}}_i \rangle$ is a connected $\mathcal{G}_0$-thread and $\Psi_{\vec{\mathcal{E}}_i}(M_i^-) = M_i$. To see that $(M_0; \in, \Phi_{\alpha+1}) \cong (M_1; \in, \Phi_{\alpha+1})$, we notice that $\alpha_0 = \alpha_1$ as in the previous argument and therefore $(M_0^-; \in, \Phi_{\alpha_0+1}) \cong (M_1^-; \in, \Phi_{\alpha_1+1})$. Also, by the same construction as in the argument in the proof of the shoulder axiom, we may obtain $\vec{\mathcal{E}}_0' = (\langle (N_0^{i,0}, \rho_0^{i,0}), (N_1^{i,0}, \rho_1^{i,0}) \rangle : i \leq n_0)$ and $\vec{\mathcal{E}}_1' = (\langle (N_0^{i,1}, \rho_0^{i,1}), (N_1^{i,1}, \rho_1^{i,1}) \rangle : i \leq n_1)$ from $\vec{\mathcal{E}}_0$ and $\vec{\mathcal{E}}_1$, so that $\mathrm{dom}(\vec{\mathcal{E}}_0') = \mathrm{dom}(\vec{\mathcal{E}}_1') = N$, $\Psi_{\vec{\mathcal{E}}_0'}(M_0^-) = M_0$ and $\Psi_{\vec{\mathcal{E}}_1'}(M_0^-) = M_1$. But then the desired conclusion holds since

$$\Psi_{\vec{\mathcal{E}}_0'} : (N; \in, \Phi_{\alpha_0+1}) \to (N_1^{n_0,0}; \in, \Phi_{\alpha+1})$$

and

$$\Psi_{\vec{\mathcal{E}}_1'} : (N; \in, \Phi_{\alpha_0+1}) \to (N_1^{n_1,1}; \in, \Phi_{\alpha+1})$$

are isomorphisms. The proof that $(\Psi_{M_0,M_1}(M), \alpha+1) \in \Delta(\mathcal{G}^*)$ whenever M_0, M_1 are as above and $M \in \mathcal{N}_{\alpha+1}^{\Delta(\mathcal{G}^*)} \cap M_0$, which concludes the proof that $\mathcal{N}_{\alpha+1}^{\Delta(\mathcal{G}^*)}$ is a $\Phi_{\alpha+1}$-symmetric system, is contained in the argument in the next paragraph.

We now show that $\mathcal{G}^*$ is closed under copying. For this, suppose $e = \{(M_0, \rho_0), (M_1, \rho_1)\} \in \mathcal{G}^*$ and $e' = \{(M_0', \rho_0'), (M_1', \rho_1')\} \in \mathcal{G}^* \cap M_0$ are such that $\max\{\rho_0', \rho_1'\} \leq \rho_0$ and $\Psi_{N_0,N_1}(\max\{\rho_0', \rho_1'\}) \leq \rho_1$, and let us prove that $\Psi_{M_0,M_1}(e') \in \mathcal{G}^*$. The case when $\delta_{M_0} \geq \delta_N$ follows from the construction of $\mathcal{G}_2$ — in this case of course M_0, $M_1 \in \mathcal{N}_{\alpha+1}^{\Delta(\mathcal{G}_0)}$. Now, suppose $\delta_{M_0} < \delta_N$. If $e \in \mathcal{G}_2$, then the conclusion follows from the construction of $\mathcal{G}_2$ and the hypothesis that $Q \cap \mathcal{G}_1 = Q \cap \mathcal{G}_0$ for every $Q \in \mathrm{dom}(\Delta(\mathcal{G}_0)) \cap N$. In order to finish this proof it thus remains to consider the case in which $e \in \mathcal{G}_3$. We then have that there is $\alpha + 1 \geq \rho_0$, ρ_1 such that the edges $\{(M_0, \alpha+1)\}$ and $\{(M_1, \alpha+1)\}$ are both in $\mathcal{G}_2$. Hence there are $\alpha^* < \beta$ and $\{(M_0^*, \alpha^*+1)\}$, $\{(M_1^*, \alpha^*+1)\} \in \mathcal{G}_1$ such that $M_0 = \Psi_{\vec{\mathcal{E}}_0}(M_0^*)$ and $M_1 = \Psi_{\vec{\mathcal{E}}_1}(M_1^*)$ for suitable $\vec{\mathcal{E}}_0$ and $\vec{\mathcal{E}}_1$ as in the definition of $\mathcal{G}_2$ such that $\Psi_{\vec{\mathcal{E}}_0}(\alpha^*) = \Psi_{\vec{\mathcal{E}}_1}(\alpha^*) = \alpha$. Since then $\{(M_0^*, \alpha^*+1), (M_1^*, \alpha^*+1)\} \in \mathcal{G}_1$ by stickiness of $\mathcal{G}_1$ and $\Psi_{\vec{\mathcal{E}}_0}^{-1}(e') \in \mathcal{G}_1 \cap M_0^*$, $e^* := \Psi_{M_0^*,M_1^*}(\Psi_{\vec{\mathcal{E}}_0}^{-1}(e')) \in \mathcal{G}_1$. This finishes the proof in this case since then $\Psi_{M_0,M_1}(e') = \Psi_{\vec{\mathcal{E}}_1}(e^*) \in \mathcal{G}_2 \subseteq \mathcal{G}^*$.

Finally, we note that stickiness of $\mathcal{G}^*$ holds at $\alpha + 1$ (i.e. the unordered pair $\{(M_0, \alpha+1), (M_1, \alpha+1)\} \in \mathcal{G}^*$ for all M_0, $M_1 \in \mathcal{N}_{\alpha+1}^{\Delta(\mathcal{G}^*)}$ such that $\delta_{M_0} = \delta_{M_1}$) since, by the definition of $\mathcal{G}_2$, we can assume that $\{(M_0, \alpha+1), (M_1, \alpha+1)\} \notin \mathcal{G}_0$, $\delta_{M_0} = \delta_{M_1} < \delta_N$, and hence

$$\{(M_0, \alpha+1), (M_1, \alpha+1)\} \in \mathcal{G}_3. \qquad \square$$

Remark 2.9. The set $\mathcal{G}^*$ in the proof of Lemma 2.8 is precisely $\mathcal{G}_0 \oplus \mathcal{G}_1$.

Remark 2.10. The main reason for requiring our sets of edges $\mathcal{G}$ to be sticky, rather than simply asking that $\mathcal{N}_{\alpha+1}^{\Delta(\mathcal{G})}$ be a $\Phi_{\alpha+1}$-symmetric system for each α, it to secure the above amalgamation lemma. As observed by Inamdar, this lemma does not hold if we do not require stickiness.

We will call a function F *pertinent* if $\mathrm{dom}(F) \in [\mathrm{Succ}(\kappa)]^{<\omega}$ and for every $\alpha \in \mathrm{dom}(F)$, $F(\alpha) = (b_\alpha, d_\alpha)$, where

- $b_\alpha \in [\mathrm{Lim}(\omega_1) \times \omega_1]^{<\omega}$ is a regressive function (i.e. $b_\alpha(\delta) < \delta$ for each $\delta \in \mathrm{dom}(b_\alpha)$);
- $d_\alpha \in [\omega_1 \times H(\kappa)]^{<\omega}$.

In the above situation, we will often refer to b_α and d_α as, respectively, b^F_α and d^F_α. Also, if $\alpha \notin \mathrm{dom}(F)$, b^F_α and d^F_α are both defined to be the empty set.

Given an ordered pair $q = (F, \mathcal{G})$, where F is a function and $\mathcal{G}$ is a set of edges, we will denote F and $\mathcal{G}$ by, respectively, F_q and $\mathcal{G}_q$. Given $\alpha \in \mathrm{dom}(F_q)$, we will denote $b^{F_q}_\alpha$ and $d^{F_q}_\alpha$ by, respectively, b^q_α and d^q_α.

If $q = (F_q, \mathcal{G}_q)$, where F_q and $\mathcal{G}_q$ are as above, and $\beta < \kappa$, we let $\mathcal{N}^q_\beta$ stand for $\mathcal{N}^{\Delta(\mathcal{G}_q)}_\beta$. If G is a set of ordered pairs as above, we denote by $\mathcal{N}^G_\beta$ the set $\bigcup\{\mathcal{N}^q_\beta : q \in G\}$.

Given $q = (F_q, \mathcal{G}_q)$, where F_q and $\mathcal{G}_q$ are as above, and given $N \subseteq H(\kappa)$, we denote by $q \restriction N$ the ordered pair $(F_q \restriction\restriction N, \mathcal{G}_q \cap N)$, where $F_q \restriction\restriction N$ is the function with domain $\mathrm{dom}(F_q) \cap N$ such that

$$(F_q \restriction\restriction N)(\alpha) = (b^q_\alpha \cap N, d^q_\alpha \cap N)$$

for each $\alpha \in \mathrm{dom}(F) \cap N$.

Also, given $q = (F_q, \mathcal{G}_q)$ as above, $\delta < \omega_1$ and $\alpha < \kappa$, we denote by $\Xi^{q,\alpha}_\delta$ the set of ordinals $\bar\alpha$ such that $(\delta, \bar\alpha)$ is $\mathcal{G}_q$-accessible from (δ, α), $\bar\alpha \in \mathrm{dom}(F_q)$ and $\delta \in \mathrm{dom}(b^q_{\bar\alpha})$.

We will now define our sequence $(\mathcal{P}_\beta : \beta \leq \kappa)$ and $(\Phi_\beta : \beta < \kappa)$. As we said before, Theorem 2.1 will be witnessed by $\mathcal{P}_\kappa$. We already defined Φ_0.

Given $\alpha \leq \kappa$, $\dot G_\alpha$ will be the canonical $\mathcal{P}_\alpha$-name for the generic filter added by $\mathcal{P}_\alpha$. We will denote the forcing relation for $\mathcal{P}_\alpha$ by $\Vdash_\alpha$, and the extension relation for $\mathcal{P}_\alpha$ by $\leq_\alpha$.

Given any $\alpha < \kappa$, and assuming $\mathcal{P}_\alpha$ has been defined, we let $\dot C^\alpha$ be some canonically chosen (using Φ) $\mathcal{P}_\alpha$-name for a club-sequence on ω_1^V for which the following holds:

- If $\Phi(\alpha)$ is a $\mathcal{P}_\alpha$-name for a club-sequence on ω_1, then $\dot C^\alpha = \Phi(\alpha)$.
- If $\Phi(\alpha)$ is not a $\mathcal{P}_\alpha$-name for a club-sequence on ω_1, then $\dot C^\alpha$ is a $\mathcal{P}_\alpha$-name for $\vec C$, where $\vec C \in V$ is some fixed club-sequence on ω_1.

Given $\delta \in \mathrm{Lim}(\omega_1)$, we let $\dot C^\alpha_\delta$ be a $\mathcal{P}_\alpha$-name for $\dot C^\alpha(\delta)$ (where $\dot C^\alpha(\delta)$ of course refers to the δth member of $\dot C^\alpha$).

We are finally in a position to define our construction. Let $\beta < \kappa$, and suppose $\mathcal{P}_\alpha$, Φ_α and $\Phi_{\alpha+1}$ have been defined for each $\alpha < \beta$. Suppose, in addition, that for all $\bar\alpha < \alpha < \beta$, every $\mathcal{P}_{\bar\alpha}$-name is also a $\mathcal{P}_\alpha$-name. We aim to define $\mathcal{P}_\beta$ and $\Phi_{\beta+1}$, and also Φ_β if $\beta < \kappa$ is a nonzero limit ordinal.

An ordered pair $q = (F_q, \mathcal{G}_q)$ is a $\mathcal{P}_\beta$-condition if and only if it has the following properties:

(1) $\mathcal{G}_q$ is a sticky set of edges below β closed under restrictions and under copying, and such that

   (a) $\mathcal{N}_0^{\Delta(\mathcal{G}_q)}$ is a Φ_0-symmetric system;
   (b) for every $\alpha < \beta$, $\mathcal{N}_{\alpha+1}^{\Delta(\mathcal{G}_q)}$ is a $\Phi_{\alpha+1}$-symmetric system.

(2) F_q is a pertinent function with $\mathrm{dom}(F_q) \subseteq \beta$.

(3) For every $\alpha < \beta$, *the restriction of* q *to* α, $q|_\alpha$, is a condition in $\mathcal{P}_\alpha$, where

$$q|_\alpha := (F_q \restriction \alpha, \mathcal{G}_q|_\alpha).$$

(4) If $\alpha \in \mathrm{dom}(F_q)$, then $F_q(\alpha) = (b^q_\alpha, d^q_\alpha)$ has the following properties:

   (a) For every $\delta \in \mathrm{dom}(b^q_\alpha)$ there is some $N \in \mathcal{N}^q_{\alpha+1}$ such that $\delta = \delta_N$.
   (b) For every $N \in \mathcal{N}^q_{\alpha+1}$ and $\delta \in \mathrm{dom}(b^q_\alpha)$, if $b^q_\alpha(\delta) < \delta_N < \delta$ and $\beta = \alpha + 1$, then $q|_\alpha \Vdash_\alpha \delta_N \notin \dot{C}^\alpha_\delta$.
   (c) For every $N \in \mathcal{N}^q_{\alpha+1}$, $(\delta, a) \in d^q_\alpha \cap N$ and $N' \in \mathcal{N}^q_{\alpha+1}$, if $\delta_{N'} = \delta_N$, then $(\delta, \Psi_{N,N'}(a)) \in d^q_\alpha$.
   (d) For every $(\delta, a) \in d^q_\alpha$ and $N \in \mathcal{N}^q_{\alpha+1}$, if $\delta < \delta_N$, then there is some $N' \in \mathcal{N}^q_{\alpha+1}$ such that $\delta_{N'} = \delta_N$ and $a \in N'$.

(5) Suppose $\beta = \alpha + 1$. For every $N \in \mathcal{N}^q_{\alpha+1}$, if $\Xi^{q,\alpha}_{\delta_N} \neq \emptyset$, then $q|_\alpha$ forces that for every $a \in N$ there is some $M \in \mathcal{N}^{\dot{G}_\alpha}_\alpha \cap \mathcal{T}_{\alpha+1} \cap N$ such that

   (a) $a \in M$ and
   (b) $\delta_M \notin \bigcup\{\dot{C}^{\bar{\alpha}}_{\delta_N} : \bar{\alpha} \in \Xi^{q,\alpha}_{\delta_N}\}$.[p]

(6) Suppose $\{(N_0, \rho_0), (N_1, \rho_1)\} \in \mathcal{G}_q$, $\alpha \in \mathrm{dom}(F_q) \cap N_0 \cap \rho_0$ and $\bar{\alpha} = \Psi_{N_0,N_1}(\alpha) < \rho_1$. Then

   (a) $\bar{\alpha} \in \mathrm{dom}(F_q)$;
   (b) $b^q_\alpha \cap N_0 = b^q_{\bar{\alpha}} \cap N_1$;
   (c) Ψ_{N_0,N_1} "$d^q_\alpha = d^q_{\bar{\alpha}} \cap N_1$.

(7) The following holds for every $\alpha < \beta$ and every $N \in \mathcal{N}^q_{\alpha+1}$:

   (a) For all $Q \in \mathcal{N}^q_{\alpha+1} \cap N$ and $(\delta_0, \delta_1) \in b^q_\alpha$, if $\delta_1 < \delta_Q < \delta_0$ and $\delta_0 < \delta_N$, then there is some $p \in \mathcal{P}_\alpha \cap N$ such that $q|_\alpha \leq_\alpha p$ and $p \Vdash_\alpha \delta_Q \notin \dot{C}^\alpha_{\delta_0}$.
   (b) For every $Q \in \mathcal{N}^q_{\alpha+1} \cap N$, if $\Xi^{(q \restriction N)|_{\alpha+1},\alpha}_{\delta_Q} \neq \emptyset$, then there is some $p \in \mathcal{P}_\alpha \cap N$ such that $q|_\alpha \leq_\alpha p$ and such that p forces that for every $a \in Q$ there is some $M \in \mathcal{N}^{\dot{G}_\alpha}_\alpha \cap \mathcal{T}_{\alpha+1} \cap Q$ with $a \in M$ and $\delta_M \notin \bigcup\{\dot{C}^{\bar{\alpha}}_{\delta_Q} : \bar{\alpha} \in \Xi^{(q \restriction N)|_{\alpha+1},\alpha}_{\delta_Q}\}$.[q]

[p] It is worth noting that clauses (4)(b) and (5) only apply when $\beta = \alpha + 1$. Also, notice that item (b) in (5) makes sense since, in the situation of this clause, every $\mathcal{P}_{\bar{\alpha}}$-name is itself a $\mathcal{P}_\alpha$-name by our working hypothesis.

[q] Just to be clear, $\Xi^{(q \restriction N)|_{\alpha+1},\alpha}_{\delta_Q}$ is of course the set of ordinals $\bar{\alpha}$ such that $(\delta_Q, \bar{\alpha})$ is $(\mathcal{G}_q)|_{\alpha+1} \cap N$-accessible from (δ_Q, α), $\bar{\alpha} \in \mathrm{dom}(F_q) \cap N$ and $\delta_Q \in \mathrm{dom}(b^q_{\bar{\alpha}})$.

Given $\mathcal{P}_\beta$-conditions q_i, for $i = 0, 1$, $q_1 \leq_\beta q_0$ if and only if the following holds:

(1) $\mathrm{dom}(F_{q_0}) \subseteq \mathrm{dom}(F_{q_1})$ and for every $\alpha \in \mathrm{dom}(F_{q_0})$,

  (a) $b^{q_0}_\alpha \subseteq b^{q_1}_\alpha$ and
  (b) $d^{q_0}_\alpha \subseteq d^{q_1}_\alpha$.

(2) $\mathcal{G}_{q_0} \subseteq \mathcal{G}_{q_1}$
(3) For every $\{(N_0, \rho_0), (N_1, \rho_1)\} \in \mathcal{G}_{q_0}$ and $\alpha \in N_0 \cap (\rho_0 + 1)$, the following holds:

  (a) If $\Psi_{N_0,N_1}(\alpha) > \beta$, then $\mathcal{N}^{q_1}_\alpha \cap N_0 = \mathcal{N}^{q_0}_\alpha \cap N_0$.
  (b) If $\alpha \in \mathrm{dom}(F_{q_1}) \cap \rho_0$ and $\Psi_{N_0,N_1}(\alpha) \geq \beta$, then

    (i) if $b^{q_1}_\alpha \cap N_0 \neq \emptyset$, then $\alpha \in \mathrm{dom}(F_{q_0})$ and $b^{q_1}_\alpha \cap N_0 = b^{q_0}_\alpha \cap N_0$;
    (ii) if $d^{q_1}_\alpha \cap N_0 \neq \emptyset$, then $\alpha \in \mathrm{dom}(F_{q_0})$ and $d^{q_1}_\alpha \cap N_0 = d^{q_0}_\alpha \cap N_0$.

We will refer to clause (7) of the definition of $\mathcal{P}_\beta$ holding for q by saying that q is *N-saturated below β*.

Fact 2.11. $\leq_\beta$ is a transitive relation.

Proof. Let q_0, q_1, $q_2 \in \mathcal{P}_\beta$ and suppose $q_1 \leq_\beta q_0$ and $q_2 \leq_\beta q_1$. In order to show that $q_2 \leq_\beta q_0$, it suffices to verify (3) as all other clauses are trivial. For this, let $\{(N_0, \rho_0), (N_1, \rho_1)\} \in \mathcal{G}_{q_0}$, $\alpha \in N_0 \cap (\rho_0 + 1)$ and $\bar{\alpha} = \Psi_{N_0,N_1}(\alpha)$, and let us assume that $\bar{\alpha} > \beta$. We will prove that $\mathcal{N}^{q_2}_\alpha \cap N_0 = \mathcal{N}^{q_0}_\alpha \cap N_0$. (The argument taking care of (3)(b) is the same.)

Since $\mathcal{G}_{q_0} \subseteq \mathcal{G}_{q_1} \subseteq \mathcal{G}_{q_2}$, by (3)(a) in the definition of $q_2 \leq_\beta q_1$ we have that $\mathcal{N}^{q_2}_\alpha \cap N_0 = \mathcal{N}^{q_1}_\alpha \cap N_0$. Since $\mathcal{N}^{q_1}_\alpha \cap N_0 = \mathcal{N}^{q_0}_\alpha \cap N_0$ by (3)(a) in the definition of $q_1 \leq_\beta q_0$, we have that $\mathcal{N}^{q_1}_\alpha \cap N_0 = \mathcal{N}^{q_0}_\alpha \cap N_0$. Putting these two equalities together it follows that $\mathcal{N}^{q_2}_\alpha \cap N_0 = \mathcal{N}^{q_0}_\alpha \cap N_0$. □

We still need to define $\Phi_{\beta+1}$, and Φ_β if $\beta < \kappa$ is a nonzero limit ordinal.

Let $\Vdash^*_\beta$ denote the restriction of the forcing relation $\Vdash_\beta$ for $\mathcal{P}_\beta$ to formulas involving only names in $H(\kappa)$. Then we let $\Phi_{\beta+1} \subseteq H(\kappa)$ canonically code the satisfaction relation for the structure

$$(H(\kappa); \Phi_\beta, \mathcal{P}_\beta, \Vdash^*_\beta).$$

Finally, if $\beta < \kappa$ is a nonzero limit ordinal, we let Φ_β be a subset of $H(\kappa)$ canonically coding $(\Phi_\alpha : \alpha < \beta)$.

We will assume that the definition of $(\Phi_\beta : \beta < \kappa)$ is uniform in β.

Finally, we define $\mathcal{P}_\kappa = \bigcup_{\beta<\kappa} \mathcal{P}_\beta$.

3. Proving Theorem 2.1

We will now prove the relevant lemmas that, together, will show $\mathcal{P}_\kappa$ to witness Theorem 2.1.

Given partial orders $\mathbb{P}$ and $\mathbb{Q}$, we will say that $\mathbb{P}$ is a *weak suborder of* $\mathbb{Q}$ in case $\mathrm{dom}(\mathbb{P}) \subseteq \mathrm{dom}(\mathbb{Q})$ and for all p_0, $p_1 \in \mathrm{dom}(\mathbb{P})$, if $p_1 \leq_{\mathbb{P}} p_0$, then $p_1 \leq_{\mathbb{Q}} p_0$. Thus,

$\mathbb{P}$ is a suborder of $\mathbb{Q}$ in case it is a weak suborder of $\mathbb{Q}$ and for all $p_0, p_1 \in \mathrm{dom}(\mathbb{P})$ we have that if $p_1 \leq_{\mathbb{Q}} p_0$, then $p_1 \leq_{\mathbb{P}} p_0$.

It is clear that if $\mathbb{P}$ is a weak suborder of $\mathbb{Q}$, then every $\mathbb{P}$-name is itself also a $\mathbb{Q}$-name.

Our first two lemmas are obvious.

Lemma 3.1. *For all $\alpha < \beta \leq \kappa$, $\mathcal{P}_\alpha$ is a weak suborder of $\mathcal{P}_\beta$.*[r]

On the other hand, it is not true in general that for all $\alpha < \beta$, $\mathcal{P}_\alpha$ is a suborder of $\mathcal{P}_\beta$.[s]

Lemma 3.2. *For every $\beta < \kappa$, $\mathcal{P}_\beta$ and $\Vdash^*_\beta$ are uniformly $($in $\beta)$ definable over the structure $(H(\kappa); \in, \Phi_{\beta+1})$ without parameters.*

Given partial orders $\mathbb{P}$ and $\mathbb{Q}$, we will say that $\mathbb{P}$ *is a weak complete suborder of* $\mathbb{Q}$ in case $\mathbb{P}$ is a weak suborder of $\mathbb{Q}$ and every predense subset of $\mathbb{P}$ is also predense in $\mathbb{Q}$ (i.e. if $D \subseteq \mathbb{P}$ is predense in $\mathbb{P}$, then for every $q \in \mathbb{Q}$ there are $p \in D$ and $r \in \mathbb{Q}$ such that $r \leq_{\mathbb{Q}} p$ and $r \leq_{\mathbb{Q}} q$). Also, we will call a sequence $\langle \mathbb{P}_\alpha : \alpha \leq \lambda \rangle$ of forcing notions a *weak forcing iteration* if for all $\alpha < \beta$, $\mathbb{P}_\alpha$ is a weak complete suborder of $\mathbb{P}_\beta$.

Given partial orders $\mathbb{P}$ and $\mathbb{Q}$ such that $\mathbb{P}$ is a weak suborder of $\mathbb{Q}$, we call a function $\pi : \mathbb{Q} \to \mathbb{P}$ a *weak projection of* $\mathbb{Q}$ *onto* $\mathbb{P}$ in case for every $q \in \mathbb{Q}$ and every condition $p \in \mathbb{P}$ such that $p \leq_{\mathbb{P}} \pi(q)$ there is some $r \in \mathbb{Q}$ such that $r \leq_{\mathbb{Q}} p$ and $r \leq_{\mathbb{Q}} q$. In this situation $\mathbb{P}$ is clearly a weak complete suborder of $\mathbb{Q}$.

Our sequence $(\mathcal{P}_\beta : \beta \leq \kappa)$ is a weak forcing iteration. In fact, given $\alpha < \beta \leq \kappa$, the function sending $q \in \mathcal{P}_\beta$ to $q|_\alpha$ is a weak projection of $\mathcal{P}_\beta$ onto $\mathcal{P}_\alpha$. This is an immediate consequence of the following lemma, the proof of which is straightforward thanks to clause (3) in the definition of the extension relation $\leq_\alpha$.

Lemma 3.3. *Let $\alpha < \beta \leq \kappa$, let $q \in \mathcal{P}_\beta$ and $r \in \mathcal{P}_\alpha$, and suppose $r \leq_\alpha q|_\alpha$. Then*

$$(F_q \cup F_r, \mathcal{G}_q \cup \mathcal{G}_r)$$

is a condition in $\mathcal{P}_\beta$ extending both q and r in $\mathcal{P}_\beta$.

Given $\alpha < \beta \leq \kappa$, $q \in \mathcal{P}_\beta$ and $r \in \mathcal{P}_\alpha$ extending $q|_\alpha$, we write $q \oplus r$ to denote the common extension

$$(F_q \cup F_r, \mathcal{G}_q \cup \mathcal{G}_r)$$

of q and r defined in the statement of Lemma 3.3.

[r]This lemma shows, in particular, that for all $\alpha < \beta$, every $\mathcal{P}_\alpha$-name is also a $\mathcal{P}_\beta$-name, and hence that our construction $(\mathcal{P}_\beta : \beta \leq \kappa)$ is well defined.
[s]See Remark 3.4.

Given an edge $\{(M_0, \gamma_0), (M_1, \gamma_1)\}$, we will write

$$\langle\{(M_0, \gamma_0), (M_1, \gamma_1)\}\rangle$$

to denote the $\subseteq$-least set of edges containing $\{(M_0, \gamma_0), (M_1, \gamma_1)\}$ and closed under restrictions, i.e. the set

$$\{\{(M_0, \alpha_0), (M_1, \alpha_1)\} : \alpha_0 \in M_0 \cap (\gamma_0 + 1), \alpha_1 \in M_1 \cap (\gamma_1 + 1)\}.$$

Remark 3.4. As we have just seen, our construction is a weak forcing iteration, and in fact, given any $\alpha < \beta \leq \kappa$, the function sending $q \in \mathcal{P}_\beta$ to $q|_\alpha$ is a weak projection of $\mathcal{P}_\beta$ onto $\mathcal{P}_\alpha$. However, it is not an iteration in the usual sense. Actually, it is easy to find ordinals $\alpha < \beta$ and conditions q_0, $q_1 \in \mathcal{P}_\alpha$ such that $q_1 \leq_\beta q_0$ and yet q_0 and q_1 are actually incompatible in $\mathcal{P}_\alpha$. For example, for some high enough β, we can consider $\mathcal{P}_\beta$-conditions $q_0 = (\emptyset, \mathcal{G}_0)$ and $q_1 = (\emptyset, \mathcal{G}_1)$, where

- $\mathcal{G}_0 = \langle\{(N_0, \rho_0), (N_1, \rho_1)\}\rangle$;
- $\mathcal{G}_1$ is the union of
  - — $\mathcal{G}_0$;
  - — $\langle\{(M, \rho_0)\}\rangle$ and
  - — $\{\{(\Psi_{N_0,N_1}(M), \gamma)\} : \gamma \in \Psi_{N_0,N_1}(M) \cap \rho_1\}$,

and where $\rho_0 < \rho_1$, $M \in N_0$, (M, ρ_0) is a model with marker, and $\Psi_{N_0,N_1}(\rho_0) > \rho_1$. Let $\alpha = \rho_1$. Then $q_1 \leq_\beta q_0$ but q_0 and q_1 are incompatible in $\mathcal{P}_\alpha$ since every $r \in \mathcal{P}_\alpha$ such that $r \leq_\alpha q_0$, q_1 would have to be such that $M \in \mathcal{N}^r_{\rho_0}$ (since it would extend q_1) and $M \notin \mathcal{N}^r_{\rho_0}$ (since it would extend q_0 and since $\Psi_{N_0,N_1}(\rho_0) > \rho_1$).

The following lemma will be used in the proofs of Lemmas 3.11 and 3.16.

Lemma 3.5. *Let $\beta < \kappa$ and $q \in \mathcal{P}_\beta$. Suppose $\{(N_0, \rho_0), (N_1, \rho_1)\} \in \mathcal{G}_q$, $\alpha \in N_0 \cap \rho_0$, $\dot{a} \in N_0$ is a $\mathcal{P}_\alpha$-name, $\varphi(x)$ is a formula in the language of set theory, $(q \restriction N_0)|_\alpha \in \mathcal{P}_\alpha$ and $(q \restriction N_0)|_\alpha \Vdash_\alpha \varphi(\dot{a})$. Suppose $\alpha^* := \Psi_{N_0,N_1}(\alpha) < \rho_1$. Then $\Psi_{N_0,N_1}((q \restriction N_0)|_\alpha) = (q \restriction N_1)|_{\alpha^*} \in \mathcal{P}_{\alpha^*}$, $\Psi_{N_0,N_1}(\dot{a})$ is a $\mathcal{P}_{\alpha^*}$-name and $(q \restriction N_1)|_{\alpha^*} \Vdash_{\alpha^*} \varphi(\Psi_{N_0,N_1}(\dot{a}))$.*

Proof. By Lemma 3.2 and since

$$\Psi_{N_0,N_1} : (N_0; \in, \Phi_{\alpha+1}) \to (N_1; \in, \Phi_{\alpha^*+1})$$

is an isomorphism, we have that $\Psi_{N_0,N_1}((q \restriction N_0)|_\alpha)$ is a $\mathcal{P}_{\alpha^*}$-condition and $\Psi_{N_0,N_1}(\dot{a})$ is a $\mathcal{P}_{\alpha^*}$-name. And since $(q \restriction N_0)|_\alpha \Vdash_\alpha \varphi(\dot{a})$, we also have that

$$\Psi_{N_0,N_1}((q \restriction N_0)|_\alpha) \in \mathcal{P}_{\alpha^*}$$

and

$$\Psi_{N_0,N_1}((q \restriction N_0)|_\alpha) \Vdash_{\alpha^*} \varphi(\Psi_{N_0,N_1}(\dot{a}))$$

again by Lemma 3.2 and the fact that

$$\Psi_{N_0,N_1} : (N_0; \in, \Phi_{\alpha+1}) \to (N_1; \in, \Phi_{\alpha^*+1})$$

is an isomorphism. Finally, clause (6) in the definition of condition, and the closure of $\mathcal{G}_q$ under copying, together entail that

$$\Psi_{N_0,N_1}((q \restriction N_0)|_\alpha) = (q \restriction N_1)|_{\alpha^*}. \qquad \square$$

3.1. *Properness and $\aleph_2$-c.c.*

The goal of this section is to show both the properness and the $\aleph_2$-chain condition of all members $\mathcal{P}_\beta$ of our construction. Our first lemma shows, given a $\mathcal{P}_\beta$-condition q and an edge $\{(N_0, \rho_0), (N_1, \rho_1)\}$ below β such that $q \in N_0 \cap N_1$, how to add $\{(N_0, \rho_0), (N_1, \rho_1)\}$ to q.

Lemma 3.6. *Let $\beta < \kappa$, $q \in \mathcal{P}_\beta$, and let $\{(N_0, \rho_0), (N_1, \rho_1)\}$ be an edge below β such that $q \in N_0 \cap N_1$. Let $\mathcal{G}^*$ be the union of $\mathcal{G}_q$ and $\langle\{(N_0, \rho_0), (N_1, \rho_1)\}\rangle$. Then $q^* = (F_q, \mathcal{G}^*)$ is a condition in $\mathcal{P}_\beta$ extending q.*

Proof. This is immediate since $\mathcal{G}^*$ is the $\subseteq$-minimal sticky set of edges closed under restrictions and such that $\mathcal{G}_q \cup \{(N_0, \rho_0), (N_1, \rho_1)\} \subseteq \mathcal{G}^*$. $\square$

The proof of the following lemma is the same as that of the previous lemma.

Lemma 3.7. *Let $\beta^* \leq \kappa$, $q \in \mathcal{P}_\beta$ and $N \preccurlyeq H(\kappa)$ such that $N \in \mathcal{T}_{\beta+1}$ for every $\beta \in N \cap \beta^*$. Suppose $q \in N$. Then there is an extension $q^* \in \mathcal{P}_{\beta^*}$ of q such that $\{(N, \beta)\} \in \mathcal{G}_{q^*}$ for every $\beta \in N \cap \beta^*$.*

It will be convenient to prove the $\aleph_2$-chain condition and our main properness result in the same lemma, by a simultaneous induction. This will be the content of Lemma 3.11. Before getting there, it will be useful to introduce some pieces of notation and some technical lemmas.

The following lemma, which is immediate, asserts a useful interpolation property of the extension relation.

Lemma 3.8. *Let $\beta < \kappa$, $q \in \mathcal{P}_\beta$ and $N \in \mathcal{N}_0^q$. Suppose $q \restriction N \in \mathcal{P}_\beta$, and let $p \in \mathcal{P}_\beta \cap N$ be a condition such that $q \leq_\beta p$. Then $q \leq_\beta q \restriction N$ and $q \restriction N \leq_\beta p$.*

Lemma 3.9. *Let $\beta < \kappa$, $q \in \mathcal{P}_\beta$ and $N \in \mathcal{N}_\beta^q$. Then $q \restriction N \in \mathcal{P}_\beta$.*

Proof. We prove, by induction on $\alpha \leq \beta$, that

$$(q \restriction N)|_\alpha := ((F_q \restriction\restriction N) \restriction \alpha, (\mathcal{G}_q \cap N)|_\alpha)$$

is a condition in $\mathcal{P}_\alpha$.

Clause (1) in the definition of condition holds for $(q \restriction N)|_\alpha$ due to the fact that if $\mathcal{N}$ is a symmetric system and $M \in \mathcal{N}$, then $\mathcal{N} \cap M$ is also a symmetric system. Clauses (2), (6) and (7) are trivial, and clause (3) follows from the induction hypothesis. All subclauses in (4) except for (4)(b) are trivial. Finally, (4)(b) holds by clause (a) in the definition of N-saturatedness below β together with Lemma 3.8, and (5) holds by clause (b) in the definition of N-saturatedness below β together with, again, Lemma 3.8. $\square$

We will also need the following technical lemma, which is an immediate consequence of Lemma 2.8.

Lemma 3.10. *Let $\alpha < \beta < \kappa$, $q \in \mathcal{P}_\beta$, $N \in \mathcal{N}_0^q$, $t \in \mathcal{P}_\beta \cap N$, and suppose $q \restriction N \in \mathcal{P}_\beta$ and $t \leq_\beta q \restriction N$.*[t] *Suppose for every $Q \in \mathcal{N}^{\Delta(\mathcal{G}_q)} \cap N$, $Q \cap \mathcal{G}_t = Q \cap \mathcal{G}_q$. Let $p \in \mathcal{P}_\alpha$, and suppose $p \leq_\alpha q|_\alpha$ and $p \leq_\alpha t|_\alpha$. Let $q' = q \oplus p$ and let $\mathcal{G} = \mathcal{G}_{q'} \oplus \mathcal{G}_t$. Then $\mathcal{G}$ is a sticky set of edges closed under restrictions and under copying and such that $\mathcal{N}_0^{\Delta(\mathcal{G})}$ is a Φ_0-symmetric system and $\mathcal{N}_{\alpha+1}^{\Delta(\mathcal{G})}$ is a $\Phi_{\alpha+1}$-symmetric system for every $\alpha < \beta$.*

Proof. This is by an application of Lemma 2.8 with $\mathcal{G}_{q'}$ and $\mathcal{G}_{t'}$, where $t' = t \oplus (p \restriction N)$. $\square$

Given a set $\mathcal{G}$ of edges and a pertinent function F such that $\mathrm{dom}(F) \subseteq \bigcup \mathrm{dom}(\Delta(\mathcal{G}))$, we define *the closure of F via edges coming from $\mathcal{G}$* to be the function F^* with domain the set X of ordinals of the form $\Psi_{\vec{\mathcal{E}}}(\alpha)$, for some $\alpha \in \mathrm{dom}(F)$ and some connected $\mathcal{G}$-thread $\langle \alpha, \vec{\mathcal{E}} \rangle$, defined by letting $F^*(\bar{\alpha})$ be, for every $\bar{\alpha} \in X$, the ordered pair $(b_{\bar{\alpha}}^{F^*}, d_{\bar{\alpha}}^{F^*})$, where

- $b_{\bar{\alpha}}^{F^*} = b_{\bar{\alpha}}^{F} \cup b_{\bar{\alpha}}^{F'}$,[u] where $b_{\bar{\alpha}}^{F'}$ is the union of the collection of sets of the form $\Psi_{\vec{\mathcal{E}}}``b_\alpha^F$, for some $\alpha \in \mathrm{dom}(F)$ and some connected $\mathcal{G}$-thread $\langle \alpha, \vec{\mathcal{E}} \rangle$ with $\bar{\alpha} = \Psi_{\vec{\mathcal{E}}}(\alpha)$;[v]
- $d_{\bar{\alpha}}^{F^*} = d_{\bar{\alpha}}^{F} \cup d_{\bar{\alpha}}^{F'}$, where $d_{\bar{\alpha}}^{F'}$ is the union of the collection of sets of the form $\Psi_{\vec{\mathcal{E}}}``d_\alpha^F$, for some $\alpha \in \mathrm{dom}(F)$ and some connected $\mathcal{G}$-thread $\langle \alpha, \vec{\mathcal{E}} \rangle$ with $\bar{\alpha} = \Psi_{\vec{\mathcal{E}}}(\alpha)$.

We will denote this function F^* by $\mathrm{cl}_{\mathcal{G}}(F)$.

Also, given pertinent functions F_0 and F_1 and given $\alpha \in \mathrm{dom}(F_0) \cap \mathrm{dom}(F_1)$, let $F_0(\alpha) + F_1(\alpha)$ denote

$$(b_\alpha^{F_0} \cup b_\alpha^{F_1}, d_\alpha^{F_0} \cup d_\alpha^{F_1}).$$

We will then denote by $F_0 + F_1$ the function F with domain $\mathrm{dom}(F_0) \cup \mathrm{dom}(F_1)$ defined by letting

- $F(\alpha) = F_\epsilon(\alpha)$ for all $\epsilon \in \{0, 1\}$ and $\alpha \in \mathrm{dom}(F_\epsilon) \setminus \mathrm{dom}(F_{1-\epsilon})$ and
- $F(\alpha) = F_0(\alpha) + F_1(\alpha)$ for all $\alpha \in \mathrm{dom}(F_0) \cap \mathrm{dom}(F_1)$.

[t] The hypothesis that $q \restriction N \in \mathcal{P}_\beta$ is actually not needed; if we drop it, then $t \leq_\beta q \restriction N$ needs to be replaced by a hypothesis to the effect that the relevant forms of clauses (1) and (2) in the definition of $\leq_\beta$ hold between t and $q \restriction N$.

[u] Recall that $b_{\bar{\alpha}}^F$ is defined to be $\emptyset$ if $\bar{\alpha} \notin \mathrm{dom}(F)$. And a similar remark applies to the next bullet point.

[v] $\Psi_{\vec{\mathcal{E}}}``b_\alpha^F$ is of course $b_\alpha^F \restriction \min(\delta_{\vec{\mathcal{E}}})$.

Given a countable elementary substructure N of $H(\kappa)$ and a $\mathcal{P}_\beta$-condition q, for some $\beta < \kappa$, we will say that q *is potentially* $(N, \mathcal{P}_\beta)$*-generic* if and only if for every maximal antichain A of $\mathcal{P}_\beta$ such that $A \in N$ and every $q' \in \mathcal{P}_\beta$ such that $q' \leq_\beta q$ there is some $r \in A$ and some $q^* \in \mathcal{P}_\beta$ such that $q^* \leq_\beta r$ and $q^* \leq_{\beta^\dagger} q'$ for some $\beta^\dagger \geq \beta$. Note that this, even in the stronger version in which $\beta^\dagger$ is required to be β, is more general than the standard notion of $(N, \mathbb{P})$-genericity, for a forcing notion $\mathbb{P}$, which applies only if $\mathbb{P} \in N$. Indeed, in our situation $\mathcal{P}_\beta$ is of course never a member of N if $N \subseteq H(\kappa)$.

We are now ready to prove the main lemma in this section.

Lemma 3.11. *The following holds for every* $\beta \leq \kappa$:

(1) $\mathcal{P}_\beta$ *is* $\aleph_2$*-Knaster.*
(2) *If* $\beta < \kappa$*, then for every* $q \in \mathcal{P}_\beta$ *and* $N \in \mathcal{N}^q_\beta \cap \mathcal{T}_{\beta+1}$*,* q *is potentially* $(N, \mathcal{P}_\beta)$*-generic.*

Proof. We prove (1) and (2) by simultaneous induction on $\beta < \kappa$.

We start with the proof of (1). We prove that if $(q_\nu : \nu < \omega_2)$ is a sequence of $\mathcal{P}_\beta$-conditions, then there is $I \in [\omega_2]^{\aleph_2}$ such that q_{ν_0} and q_{ν_1} are compatible in $\mathcal{P}_\beta$ for all ν_0, $\nu_1 \in I$. Let M^*_ν be, for each $\nu < \omega_2$, a countable elementary submodel of $H(\kappa^+)$ such that $\vec{\Phi}_\beta$, $q_\nu \in M^*_\nu$ and let $M_\nu = M^*_\nu \cap H(\kappa)$.

By CH we may find $I \in [\omega_2]^{\aleph_2}$ and some countable R such that $M_{\nu_0} \cap M_{\nu_1} = R$ for all distinct ν_0, ν_1 in I. Again by CH, and after shrinking I if necessary, we may assume in addition that, for some n, $m < \omega$, there are, for all $\nu \in I$, enumerations $(N^\nu_i : i < n)$ and $(\xi^\nu_j : j < m)$ of $\mathcal{N}^{q_\nu}_0$ and $\mathrm{dom}(F_{q_\nu})$, respectively, such that for all $\nu_0 \neq \nu_1$ in I there is an isomorphism Ψ between $\mathcal{M}_{\nu_0}$ and $\mathcal{M}_{\nu_1}$ fixing $M_{\nu_0} \cap M_{\nu_1}$, where, given any $\nu \in I$, $\mathcal{M}_\nu$ is some canonically chosen structure with universe M_ν coding R, $(N^\nu_i : i < n)$, $\mathcal{G}_{q_\nu}$, $(\xi^\nu_j : j < m)$, $((b^{q_\nu}_{\xi^\nu_j}, d^{q_\nu}_{\xi^\nu_j}) : j < m)$ and $\vec{\Phi}_\beta \cap M_\nu$.

We may moreover assume that $(\alpha_{\nu_0}; \in, \pi_{\nu_0}``R) \cong (\alpha_{\nu_1}; \in, \pi_{\nu_1}``R)$, where $\alpha_{\nu_i} \in \omega_1$ is the Mostowski collapse of $M_{\nu_i} \cap \mathsf{Ord}$ and π_{ν_i} is the corresponding collapsing function. But then we have that Ψ is the identity on $R \cap \mathsf{Ord}$. This yields that Ψ is the identity on $R \cap H(\kappa)$ since the function $\Phi : \kappa \to H(\kappa)$ is surjective.

Let us now pick $\nu_0 \neq \nu_1$ in I. We will prove that

$$q^* := ((F_{q_{\nu_0}} + F_{q_{\nu_1}}), (\mathcal{G}_{q_{\nu_0}} \oplus \mathcal{G}_{q_{\nu_1}}) \cup \langle \{(M_{\nu_0}, \beta), (M_{\nu_1}, \beta)\} \rangle)$$

is a condition in $\mathcal{P}_\beta$ extending both q_{ν_0} and q_{ν_1}. For this, we will prove, by induction on $\alpha \leq \beta$, that

$$q^*|_\alpha := ((F_{q_{\nu_0}} + F_{q_{\nu_1}}) \restriction \alpha, (\mathcal{G}_{q_{\nu_0}} \oplus \mathcal{G}_{q_{\nu_1}})|_\alpha \cup \langle \{(M_{\nu_0}, \beta), (M_{\nu_1}, \beta)\} \rangle|_\alpha)$$

is a condition in $\mathcal{P}_\alpha$ such that $q^*|_\alpha \leq_\alpha q_{\nu_0}|_\alpha$ and $q^*|_\alpha \leq_\alpha q_{\nu_1}|_\alpha$.

Clause (1) in the definition of $\mathcal{P}_\alpha$-condition holds thanks to Lemma 2.6, together with Lemma 2.7 in the case $\alpha < \beta$. Clause (2) is trivial by construction of the function $F_{q_{\nu_0}} + F_{q_{\nu_1}}$, and (3) is true by the induction hypothesis. All subclauses of (4) except for (4)(b) are true by construction of $F_{q_0} + F_{q_1}$, and (4)(b) holds by the induction hypothesis. (6) follows from the fact that Ψ is an isomorphism between $\mathcal{M}_{\nu_0}$ and $\mathcal{M}_{\nu_1}$, and (7) is immediate from the construction of q^* and the present induction hypothesis.

Finally, for clause (5), suppose $\alpha = \alpha_0+1$. It is enough to prove that if $N \in \mathcal{N}_\alpha^{q_{\nu_0}}$, $\Xi_{\delta_N}^{q^*|_{\alpha_0+1},\alpha_0} \neq \emptyset$, $a \in N$ and $q \in \mathcal{P}_{\alpha_0}$ is such that $q \leq_{\alpha_0} q^*|_{\alpha_0}$, then there is some $q' \leq_{\alpha_0} q$ and some $M \in \mathcal{N}_{\alpha_0}^{q'} \cap \mathcal{T}_{\alpha_0+1} \cap N$ such that $a \in M$ and $q' \Vdash_{\alpha_0} \delta_M \notin \bigcup\{\dot{C}_{\delta_N}^{\bar{\alpha}} : \bar{\alpha} \in \Xi_{\delta_N}^{q^*|_{\alpha_0+1},\alpha_0}\}$.

We may assume that $\alpha_0 \in M_{\nu_0}$ (the proof when $\alpha_0 \in M_{\nu_1}$ is completely symmetrical to the proof in the present case). Let us first consider the case when $\alpha_0 \leq \Psi(\alpha_0)$. Let $q' \leq_{\alpha_0} q$ and $M \in \mathcal{N}_{\alpha_0}^{q'} \cap \mathcal{T}_{\alpha_0+1} \cap N$ such that $a \in M$ and

$$q' \Vdash_{\alpha_0} \delta_M \notin \bigcup\{\dot{C}_{\delta_N}^{\bar{\alpha}} : \bar{\alpha} \in \Xi_{\delta_N}^{(q_{\nu_0})|_{\alpha_0+1},\alpha_0}\}.$$

Such q' and M exist since, if $\Xi_{\delta_N}^{q^*|_{\alpha_0+1},\alpha_0} \setminus \Xi_{\delta_N}^{(q_{\nu_0})|_{\alpha_0+1},\alpha_0} \neq \emptyset$, then we have that $\Xi_{\delta_N}^{(q_{\nu_1})|_{\Psi(\alpha_0)+1},\Psi(\alpha_0)} \neq \emptyset$ (since $\alpha_0 \leq \Psi(\alpha_0)$), and therefore $\Xi_{\delta_N}^{(q_{\nu_0})|_{\alpha_0+1},\alpha_0} \neq \emptyset$ as Ψ is an isomorphism between $\mathcal{M}_{\nu_0}$ and $\mathcal{M}_{\alpha_1}$. Let $\bar{\alpha} \in \Xi_{\delta_N}^{q^*|_{\alpha_0+1},\alpha_0} \setminus \Xi_{\delta_N}^{(q_{\nu_0})|_{\alpha_0+1},\alpha_0}$. We will be done in this case if we can show that $q' \Vdash_{\alpha_0} \delta_M \notin \dot{C}_{\delta_N}^{\bar{\alpha}}$. Let $\alpha_* = \Psi^{-1}(\bar{\alpha})$ and let us note that $\alpha_* \leq \alpha_0$ since $\bar{\alpha} \leq \Psi(\alpha_0)$. Since also $\alpha_* \in \Xi_{\delta_N}^{(q_{\nu_0})|_{\alpha_0+1},\alpha_0}$, we have that $q' \Vdash_{\alpha_0} \delta_M \notin \dot{C}_{\delta_N}^{\alpha_*}$. Suppose now that $\alpha_* \leq \bar{\alpha}$ (the case $\bar{\alpha} < \alpha_*$ is proved similarly, by reversing the roles of M_{ν_0} and M_{ν_1} in the following argument). Now, we note that $\{(M_{\nu_0}, \alpha_*), (M_{\nu_1}, \bar{\alpha})\} \in \mathcal{G}_{q'}$ and therefore, by (2) of our induction hypothesis for $\bar{\alpha}$, $q'|_{\bar{\alpha}}$ is potentially $(M_{\nu_1}, \mathcal{P}_{\bar{\alpha}})$-generic. Hence, for every $\xi < \delta_N$, every $r \leq_{\bar{\alpha}} q'$ is $\mathcal{P}_{\bar{\alpha}^\dagger}$-compatible, for some $\bar{\alpha}^\dagger \geq \bar{\alpha}$, with some condition in M_{ν_1} deciding whether or not $\xi \in \dot{C}_{\delta_N}^{\bar{\alpha}}$.

Claim 3.12. $q' \Vdash_{\alpha_0} \dot{C}_{\delta_N}^{\alpha_*} = \dot{C}_{\delta_N}^{\bar{\alpha}}$.

Proof. Let $r \leq_{\bar{\alpha}} q'$, $\xi < \delta_N$, suppose $r \Vdash_{\alpha_0} \xi \in \dot{C}_{\delta_N}^{\bar{\alpha}}$, and let us show that $r \not\Vdash_{\alpha_0} \xi \notin \dot{C}_{\delta_N}^{\alpha_*}$ (arguing symmetrically we can show that if $r \Vdash_{\alpha_0} \xi \notin \dot{C}_{\delta_N}^{\bar{\alpha}}$, then $r \not\Vdash_{\alpha_0} \xi \in \dot{C}_{\delta_N}^{\alpha_*}$). Let $s \in M_{\nu_1}$ be a $\mathcal{P}_{\bar{\alpha}^\dagger}$-condition, for some $\bar{\alpha}^\dagger \geq \bar{\alpha}$, which is compatible with r in $\mathcal{P}_{\bar{\alpha}^\dagger}$ and decides whether or not $\xi \in \dot{C}_{\delta_N}^{\bar{\alpha}}$. Since obviously also $r \Vdash_{\bar{\alpha}^\dagger} \xi \in \dot{C}_{\delta_N}^{\bar{\alpha}}$, we must have that $s \Vdash_{\bar{\alpha}^\dagger} \xi \in \dot{C}_{\delta_N}^{\bar{\alpha}}$, and since $\dot{C}_{\delta_N}^{\bar{\alpha}}$ is a $\mathcal{P}_{\bar{\alpha}}$-name, we in fact have that $s|_{\bar{\alpha}} \Vdash_{\bar{\alpha}} \xi \in \dot{C}_{\delta_N}^{\bar{\alpha}}$. Let q'' be a common extension of $r|_{\bar{\alpha}}$ and $s|_{\bar{\alpha}}$ in $\mathcal{P}_{\bar{\alpha}}$. Since $\{(M_{\nu_0}, \alpha_*), (M_{\nu_1}, \bar{\alpha})\} \in \mathcal{G}_{q''}$, q'' extends $\Psi_{N_0,N_1}(s|_{\bar{\alpha}})$. But $\Psi_{N_0,N_1}(s|_{\bar{\alpha}}) \Vdash_{\alpha_*} \xi \in \dot{C}_{\delta_N}^{\alpha_*}$ by Lemma 3.5, from which it follows that $q'' \Vdash_{\alpha_*} \xi \in \dot{C}_{\delta_N}^{\alpha_*}$. Since $q''|_{\alpha_*} \leq_{\alpha^*} r|_{\alpha^*}$, we in particular have that $r|_{\alpha^*} \not\Vdash_{\alpha^*} \xi \notin \dot{C}_{\delta_N}^{\alpha_*}$, and therefore $r \not\Vdash_{\alpha_0} \xi \notin \dot{C}_{\delta_N}^{\alpha_*}$ (if $r \Vdash_{\alpha_0} \xi \notin \dot{C}_{\delta_N}^{\alpha_*}$, then we would have that also $r|_{\alpha_*} \Vdash_{\alpha_*} \xi \notin \dot{C}_{\delta_N}^{\alpha_*}$ since $\dot{C}_{\delta_N}^{\alpha_*}$ is a $\mathcal{P}_{\alpha_*}$-name). □

The above claim finishes the proof in this case since $q' \Vdash_{\alpha_0} \delta_M \notin \dot{C}_{\delta_N}^{\alpha_*}$.

The second case is when $\Psi(\alpha_0) < \alpha_0$. Since we may of course assume that $\Xi_{\delta_N}^{q^*|_{\alpha_0+1},\alpha_0} \setminus \Xi_{\delta_N}^{(q_{\nu_0})|_{\alpha_0+1},\alpha_0} \neq \emptyset$, we in fact have that $\Xi_{\delta_N}^{q^*|_{\alpha_0+1},\Psi(\alpha_0)} \setminus \Xi_{\delta_N}^{(q_{\nu_0})|_{\alpha_0+1},\alpha_0} \neq \emptyset$, so it makes sense to define α_1 as the maximum ordinal in $\Xi_{\delta_N}^{(q_{\nu_1})|_{\alpha_0+1},\Psi(\alpha_0)}$.

Since $\Xi_{\delta_N}^{q^*|_{\alpha_0+1},\alpha_0} \setminus \Xi_{\delta_N}^{(q_{\nu_0})|_{\alpha_0+1},\alpha_0} \neq \emptyset$, there is some $\gamma \in R$ such that (δ_N, γ) is $\mathcal{G}_{q_{\nu_0}}$-accessible from (δ_N, α_0) and $\mathcal{G}_{q_{\nu_1}}$-accessible form (δ_N, α_1). Using suitable instances of the shoulder axiom as in the proof of Lemma 2.8 we may then find sequences

$$\vec{\mathcal{E}}_0 = (\langle (N_0^{i,0}, \rho_0^{i,0}), (N_1^{i,0}, \rho_1^{i,0}) \rangle : i \leq n_0)$$

and

$$\vec{\mathcal{E}}_1 = (\langle (N_0^{i,1}, \rho_0^{i,1}), (N_1^{i,1}, \rho_1^{i,1}) \rangle : i \leq n_1)$$

such that $\langle \alpha_0, \vec{\mathcal{E}}_0 \rangle$ is a connected $\mathcal{G}_{q_{\nu_0}}$-thread with $\Psi_{\vec{\mathcal{E}}_0}(\alpha_0) = \gamma$, $\langle \gamma, \vec{\mathcal{E}}_1 \rangle$ is a connected $\mathcal{G}_{q_{\nu_1}}$-thread with $\Psi_{\vec{\mathcal{E}}_1}(\alpha_0) = \alpha_1$, $\min(\delta_{\vec{\mathcal{E}}_0}) = \delta_N$, $N_0^{0,0} = N$ and $N' := N_{n_1}^{1,1}$ is such that $\delta_{N'} = \delta_N$.[w] Letting then $\vec{\mathcal{E}}$ be the concatenation of $\vec{\mathcal{E}}_0$ and $\vec{\mathcal{E}}_1^{-1}$, we have that $\langle \alpha_0, \vec{\mathcal{E}} \rangle$ is a connected $\mathcal{G}_{q^*|_\alpha}$-thread with $\Psi_{\vec{\mathcal{E}}}(\alpha_0) = \alpha_1$. Since $N' \in \mathcal{N}_{\alpha_1+1}^{q_{\nu_1}}$, by an instance of clause (7)(b) in the definition of condition for q_{ν_1} together with Lemma 3.3, we may find $q' \leq_{\alpha_0} q$ and $M' \in \mathcal{N}_{\alpha_1}^{q'} \cap \mathcal{T}_{\alpha_1+1} \cap N'$ such that $\Psi_{\vec{\mathcal{E}}}(a) \in M'$ and

$$q'|_{\alpha_1} \Vdash_{\alpha_1} \delta_{M'} \notin \bigcup \{ \dot{C}_{\delta_N}^{\bar{\alpha}} : \bar{\alpha} \in \Xi_{\delta_N}^{(q_{\nu_1})|_{\alpha_1+1},\alpha_1} \}.$$

Let $M = \Psi_{\vec{\mathcal{E}}}^{-1}(M') \in N$ and let us note that $M \in \mathcal{N}_{\alpha_0}^{q'} \cap \mathcal{T}_{\alpha_0+1} \cap N$ and $a \in M$. It thus suffices to prove that $q' \Vdash_{\alpha_0} \delta_M \notin \dot{C}_{\delta_N}^{\bar{\alpha}}$ for every $\bar{\alpha} \in \Xi_{\delta_N}^{q^*|_{\alpha_0+1},\alpha_0}$. If $\bar{\alpha} \in \Xi_{\delta_N}^{(q_{\nu_1})|_{\alpha_0+1},\Psi(\alpha_0)}$, then we are clearly done since then $\bar{\alpha} \leq \alpha_1$. Hence, we may assume $\bar{\alpha} \in \Xi_{\delta_N}^{(q_{\nu_0})|_{\alpha_0+1},\alpha_0} \setminus \Xi_{\delta_N}^{(q_{\nu_1})|_{\alpha_0+1},\Psi(\alpha_0)}$. Let $\alpha_* = \Psi(\bar{\alpha}) \leq \alpha_1$ and let us note that $\alpha_* \in \Xi_{\delta_N}^{(q_{\nu_1})_{\alpha_1+1},\alpha_1}$. It thus follows that $q'|_{\alpha_1} \Vdash_{\alpha_1} \delta_M \notin \dot{C}_{\delta_N}^{\alpha_*}$. But now, arguing as in the proof of Claim 3.12, using the fact that $\{(M_{\nu_0}, \bar{\alpha}), (M_{\nu_1}, \alpha_*)\} \in \mathcal{G}_{q'}$ and the induction hypotheses for either $\bar{\alpha}$ or α_*, we get that $q' \Vdash_{\bar{\alpha}} \dot{C}_{\delta_N}^{\bar{\alpha}} = \dot{C}_{\delta_N}^{\alpha_*}$. This finishes the proof in this case since $q' \Vdash_{\alpha_0} \delta_M \notin \dot{C}_{\delta_N}^{\alpha_*}$.

Now that we know that $q^*|_\alpha$ is a $\mathcal{P}_\alpha$-condition, it is easy to check that it extends both $q_{\nu_0}|_\alpha$ and $q_{\nu_1}|_\alpha$ in $\mathcal{P}_\alpha$. The only point that is not completely trivial is the verification of clause (3) in the definition of the extension relation. But this clause holds thanks to the fact that q_{ν_0} and q_{ν_1} carry the same information on R.

We will now prove (2). For this, it is enough to show that if $A \in N$ is a maximal antichain of $\mathcal{P}_\beta$, then there is some $\beta^\dagger \geq \beta$ such that q is $\leq_{\beta^\dagger}$-compatible with some condition in $A \cap N$.[x] The case $\beta = 0$ follows at once from Lemma 2.3, so we will

[w]Note that we can indeed proceed here as in the proof of Lemma 2.7 (more specifically, as in the verification of the shoulder axiom at the successor stages of that construction) since the definition of pertinent function implies that α_0 and α_1 are successor ordinals.
[x]This is of course the same thing as showing that there is some $r^* \in A \cap N$ and some $q^* \in \mathcal{P}_\beta$ such that $q^* \leq_\beta r^*$ and $q^* \leq_{\beta^\dagger} q$.

assume in what follows that $\beta > 0$. By extending q if necessary we may, and will, assume that q extends some $r_0 \in A$.

Let us first consider the case that $\beta = \alpha + 1$. Suppose $\Xi^{q,\alpha}_{\delta_N} \neq \emptyset$. Let $\dot{B}$ be a $\mathcal{P}_\alpha$-name for a (partially defined) function on $\omega_1 \times A$ sending (η, r) to some condition $t \in \mathcal{P}_\beta$ with the following properties (provided there is some such t; otherwise the function is not defined at (η, r)).

(1) $t|_\alpha \in \dot{G}_\alpha$.
(2) t extends r.
(3) t extends $q \restriction N$.[y]
(4) For every $Q \in \mathcal{N}^t_{\alpha+1}$, if $\delta_Q \neq \delta_{Q'}$ for any $Q' \in \mathcal{N}^q_{\alpha+1}$, then $\delta_Q > \eta$.
(5) For every $Q \in \mathcal{N}^q_0 \cap N$, $Q \cap \mathcal{G}_q = Q \cap \mathcal{G}_t$, $Q \cap b^t_\alpha = Q \cap b^q_\alpha$ and $Q \cap d^t_\alpha = Q \cap d^q_\alpha$.

By conclusion (1) for β — which we have already proved — we know that $\mathcal{P}_\beta$ has the $\aleph_2$-c.c. and hence we may assume that $\dot{B} \in H(\kappa)$. Hence, by Lemma 3.2 and since $N \preccurlyeq (H(\kappa); \in, \Phi_{\beta+1})$ and $A \in N$, we may assume that $\dot{B} \in N$.

By an instance of clause (5) in the definition of $\mathcal{P}_\beta$-condition, together with the openness of $\bar{\delta} \backslash \dot{C}^{\bar{\alpha}}_{\bar{\delta}}$ in $V^{\mathcal{P}_\alpha}$ for all $\bar{\alpha} \leq \alpha$ and $\bar{\delta} < \omega_1$,[z] there is an extension $p \in \mathcal{P}_\alpha$ of $q|_\alpha$ for which there are $M \in \mathcal{N}^p_\alpha \cap \mathcal{T}_{\alpha+1} \cap N$ and $\eta < \delta_M$ such that

(1) A, $\dot{B}$, $q \restriction N \in M$;
(2) $p \Vdash_\alpha [\eta, \delta_N] \cap \dot{C}^{\bar{\alpha}}_{\bar{\delta}} = \emptyset$ whenever $\bar{\alpha}$ is such that $(\delta_N, \bar{\alpha})$ is $\mathcal{G}_q$-accessible from (δ_N, α) and there is $(\delta, \bar{\delta}) \in b^q_{\bar{\alpha}}$ such that $\bar{\delta} < \delta_N < \delta$ and
(3) $p \Vdash_\alpha [\eta, \delta_M] \cap \bigcup \{\dot{C}^{\bar{\alpha}}_{\delta_N} : \bar{\alpha} \in \Xi^{q,\alpha}_{\delta_N}\} = \emptyset$.

Indeed, by openness of the relevant sets $\delta \backslash \dot{C}^{\bar{\alpha}}_{\delta}$ (in the extension by $\mathcal{P}_{\bar{\alpha}}$) we can extend $q|_\alpha$ to some $p_0 \in \mathcal{P}_\alpha$ deciding some $\eta_0 < \delta_N$ such that $[\eta_0, \delta_N] \cap \dot{C}^{\bar{\alpha}}_{\delta}$ whenever $(\delta_N, \bar{\alpha})$ is $\mathcal{G}_q$-accessible from (δ_N, α) and there is $(\delta, \bar{\delta}) \in b^q_{\bar{\alpha}}$ such that $\bar{\delta} < \delta_N < \delta$ (since there only finitely many such pairs $(\delta_N, \bar{\alpha})$). Then, by an instance of clause (7)(b) in the definition of condition, this time using the openness of the relevant (finitely many) sets of the form $\delta_N \backslash \dot{C}^{\bar{\alpha}}_{\delta_N}$, we may extend p_0 to some $p \in \mathcal{P}_\alpha$ for which there is some $M \in \mathcal{N}^p_\alpha \cap \mathcal{T}_{\alpha+1} \cap N$ and some $\eta_1 < \delta_M$ such that A, $\dot{B}$, $q \restriction N$, $\eta_0 \in M$ and such that $p \Vdash_\alpha [\eta_1, \delta_M] \cap \bigcup \{\dot{C}^{\bar{\alpha}}_{\delta_N} : \bar{\alpha} \in \Xi^{q,\alpha}_{\delta_N}\} = \emptyset$. Then, letting $\eta = \max\{\eta_0, \eta_1\}$, we get the desired conclusion.

By (2) of the induction hypothesis for α there is some $u \in M \cap \mathcal{P}_\alpha$, $r^* \in M \cap A$ and $t^* \in M \cap \mathcal{P}_\beta$ such that u is $\mathcal{P}_{\alpha^\dagger}$-compatible with p for some $\alpha^\dagger \geq \alpha$ and u forces in $\mathcal{P}_\alpha$ that $\dot{B}_{\dot{G}_\alpha}(\eta, r^*)$ is defined and $\dot{B}_{\dot{G}_\alpha}(\eta, r^*) = t^*$. This is true since, in the extension of V by $\mathcal{P}_\alpha$, the existence of such a member of A is witnessed by r_0, as in turn witnessed by q, and is expressible over $(H(\kappa)^{V[\dot{G}_\alpha]}; \in, H(\kappa)^V, \dot{G}_\alpha)$ by a

[y] We note that, by the assumption that q be N-saturated below β, $q \restriction N$ is actually a $\mathcal{P}_\alpha$-condition. This, however, is not an essential point; one could in fact phrase this condition alternatively, without using the fact that $q \restriction N \in \mathcal{P}_\alpha$.
[z] Which follows from the openness of $\bar{\delta} \backslash \dot{C}^{\bar{\alpha}}_{\bar{\delta}}$ in $V^{\mathcal{P}_\alpha}$ together with Lemma 3.3.

sentence with $\dot{B}$ and η as parameters, both of which are in M). Let also $p' \in \mathcal{P}_\alpha$ be such that $p' \leq_{\alpha^\dagger} p$ and $p' \leq_{\alpha^\dagger} u$.

Let $\beta^\dagger$ be any ordinal such that $\beta^\dagger \geq \beta$ and such that $\Psi_{N_0,N_1}(\rho_0) < \beta^\dagger$ for every edge $\{(N_0,\rho_0),(N_1,\rho_1)\} \in \mathcal{G}_q$. We will now construct a condition in $\mathcal{P}_\beta$ $\leq_\beta$-extending p' and t^* and $\leq_{\beta^\dagger}$-extending q. For this, we let $q' = q \oplus p'$, $\mathcal{G}^* = \mathcal{G}_{q'} \oplus \mathcal{G}_{t^*}$, and let $F^* = \mathrm{cl}_{\mathcal{G}^*}(F_{q'} + F_{t^*})$. Let $q^* = (F^*, \mathcal{G}^*)$. We already know that $q^*|_\alpha$ is a condition in $\mathcal{P}_\alpha$, and using this fact we will show that $q^* \in \mathcal{P}_\beta$. It will then follow that $q^* \leq_\beta r^*$ (by Lemma 3.8, since $t^* \leq_\beta r^*$ and since clearly $q^* \restriction N \leq_\beta t^*$) and $q^* \leq_{\beta^\dagger} q$ (by $t^* \leq_\beta q \restriction N$ together with the fact that (5) above holds for t^*, the definition of $\mathcal{G}^*$ as $\mathcal{G}_{q'} \oplus \mathcal{G}_{t^*}$, the definition of F^* as $\mathrm{cl}_{\mathcal{G}^*}(F_{q'} + F_{t^*})$, and the choice of $\beta^\dagger$), which will finish the proof of the lemma in this case since $r^* \in N$.

Clause (1) in the definition of condition holds for q^* by Lemma 3.10 noting that, by the choice of t^*, we are indeed under the hypotheses of this lemma. As usual (2) is trivial, (3) follows from the fact that $q^*|_\alpha \in \mathcal{P}_\alpha$, and all subclauses of (4) except for (4)(b) are trivial. (4)(b) follows from our choice of η and the fact that t^* satisfies (5) with respect to η, together with Lemma 3.5 and the induction hypothesis, and (5) follows from Lemma 3.5, the induction hypothesis, and the fact that for every $Q \in \mathcal{N}^q_\beta$ such that $\delta_Q < \delta_N$ and every $\bar{\alpha} \in \Xi^{q^*,\alpha}_{\delta_Q}$ there is some $\alpha^\dagger \in \Xi^{q^*,\alpha}_{\delta_Q} \cap M$ such that $q^* \Vdash_\alpha \dot{C}^{\bar{\alpha}}_{\delta_Q} = \dot{C}^{\alpha^\dagger}_{\delta_Q}$ — by arguments as in the verification of clause (5) for the amalgamation q^* in the proof of part (1), using (2) of the induction hypothesis for α and for the relevant $\bar{\alpha}$. Finally, (6) follows from the construction of F^* as $\mathrm{cl}_{\mathcal{G}^*}(F_{q'} + F_{t^*})$, and (7) is verified in the same way as (5).

The argument when $\Xi^{q,\alpha}_{\delta_N} = \emptyset$ is exactly the same, except that in the choice of η we make sure that it satisfies (1) and (2) above, rather than (1)–(3). Also, in this case there is no need to argue in any $M \in N$; we can work in N itself.

It remains to prove the lemma in the case that β is a limit ordinal. Let $\alpha \in N \cap \beta$ be such that $\mathrm{dom}(F_q) \cap [\alpha, \beta) \cap N = \emptyset$ and let $\beta^\dagger$ be defined in the same way as in the successor case. Using (1) of the induction hypothesis for α, we may then find $r^* \in A \cap N$, $t^* \in \mathcal{P}_\beta \cap N$, $p \in \mathcal{P}_\alpha$ and $\alpha^\dagger \geq \alpha$ such that

(1) $p \leq_\alpha t^*|_\alpha$;
(2) $t^* \leq_\beta r^*$;
(3) $t^* \leq_\beta q \restriction N$;
(4) $p \leq_{\alpha^\dagger} q|_\alpha$ and
(5) for every $Q \in \mathcal{N}^q_0 \cap N$, $Q \cap \mathcal{G}_q = Q \cap \mathcal{G}_{t^*}$.

Finally, we amalgamate p, q and t^* into a condition $q^* \in \mathcal{P}_\beta$ as in the successor case; specifically, we let $q' = q \oplus p$, $\mathcal{G}^* = \mathcal{G}_{q'} \oplus \mathcal{G}_{t^*}$, $F^* = \mathrm{cl}_{\mathcal{G}^*}(F_{q'} + F_{t^*})$ and $q^* = (F^*, \mathcal{G}^*)$. The verification that q^* is a condition in $\mathcal{P}_\beta$ such that $q^* \leq_\beta t^*$ and $q^* \leq_{\beta^\dagger} q$ is contained in the corresponding proof in that case. Since $r^* \in N$, this concludes the proof in the present case, and hence the proof of the lemma. □

Corollary 3.13. *$\mathcal{P}_\kappa$ is proper.*

Proof. Let $N^* \preccurlyeq H(\kappa^+)$ be a countable model such that $\Phi \in N^*$ and let $q \in \mathcal{P}_\kappa \cap N^*$. It is enough to show that there is an extension $q^* \in \mathcal{P}_\kappa$ of q which is $(N^*, \mathcal{P}_\kappa)$-generic. Let $N = N^* \cap H(\kappa)$. By Lemma 3.7 there is an extension $q^* \in \mathcal{P}_\kappa$ of q such that $\{(N, \beta)\} \in \mathcal{G}_{q^*}$ for every $\beta \in N \cap \kappa$. Let now $A \in N^*$ be a maximal antichain of $\mathcal{P}_\kappa$ and let $q' \in \mathcal{P}_\kappa$ be such that $q' \leq_\kappa q^*$. We will show that q' is $\leq_\kappa$-compatible with a condition in $A \cap N$.

By the $\aleph_2$-c.c. of $\mathcal{P}_\kappa$ (i.e. case κ of Lemma 3.11(1)) and $\mathrm{cf}(\kappa) \geq \omega_2$, $A \in N$ and there is some ordinal $\beta \in N$ such that A is also a maximal antichain of $\mathcal{P}_\beta$. Since A is a maximal antichain of $\mathcal{P}_\kappa$ to begin with, we may assume, by picking β high enough, that $\mathrm{dom}(F_{q'}) \backslash \beta = \emptyset$. By Lemma 3.11(2) applied to β there are then $r^* \in A \cap N$, $q^* \in \mathcal{P}_\beta$ and $\beta^\dagger \geq \beta$ such that $q^* \leq_\beta r^*$ and $q^* \leq_{\beta^\dagger} q'|_\beta$. Let $\mathcal{G}_{**} = \mathcal{G}_{q^*} \oplus \mathcal{G}_{q'}$ and $F_{**} = \mathrm{cl}_{\mathcal{G}_{**}}(F_{q^*})$ and let $q^{**} = (F_{**}, \mathcal{G}_{**})$. Since $\mathrm{dom}(F_{q'}) \subseteq \beta$, it is then easy to show, by arguing as in the proof of Lemma 3.11, that q^{**} is a condition in $\mathcal{P}_\kappa$ such that $q^{**} \leq_\kappa q'$. But now we are done since also $q^{**} \leq_\kappa r^*$.

□

Remark 3.14. Our argument to prove properness does not work for $\beta < \kappa$. In fact it may not be the case that $\mathcal{P}_\beta$ is proper in general for $\beta < \kappa$.

3.2. *New reals*

The following is proved in [9, Fact 2.6].

Lemma 3.15. *$\mathcal{P}_0$ adds $\aleph_1$-many Cohen reals.*

We will now use clause (6) in the definition of condition (and the closure of $\mathcal{G}_q$ under copying whenever q is a condition) to prove Lemma 3.16, which is a counterpoint to Lemma 3.15. Lemma 3.16 shows that $\mathcal{P}_\kappa$ does not add more than $\aleph_1$-many new reals, and hence that this forcing preserves CH (cf. [9, Proof of Proposition 2.7] or the proof sketched in the introduction).

Lemma 3.16 (Few new reals). *$\mathcal{P}_\kappa$ adds not more than $\aleph_1$-many new reals.*

Proof. Suppose, towards a contradiction, that there is a $\mathcal{P}_\kappa$-condition q and a sequence $(\dot{r}_\nu)_{\nu<\omega_2}$ of $\mathcal{P}_\kappa$-names for subsets of ω such that

$$q \Vdash_\kappa \dot{r}_\nu \neq \dot{r}_{\nu'}$$

for all $\nu \neq \nu'$. We will find an extension q^* of q together with $\nu_0 \neq \nu_1$ such that $q^* \Vdash_\kappa \dot{r}_{\nu_0} = \dot{r}_{\nu_1}$, which will be a contradiction.

By $\mathcal{P}_\kappa = \bigcup_{\beta<\kappa} \mathcal{P}_\beta$, we may fix $\beta < \kappa$ such that $q \in \mathcal{P}_\beta$. Let $\nu < \omega_2$ be given. By Lemma 3.11(1) and, again, the fact that $\mathcal{P}_\kappa = \bigcup_{\beta<\kappa} \mathcal{P}_\beta$, we may assume that $\dot{r}_\nu \in H(\kappa)$ and we may find $\beta_\nu < \kappa$ above β and such that $\dot{r}_\nu$ is in fact a $\mathcal{P}_{\beta_\nu}$-name for a subset of ω.

For each $\nu < \omega_2$ let $N^*_\nu \preccurlyeq H(\kappa^+)$ be countable and containing q, Φ, $\dot{r}_\nu$ and β_ν, and let $N_\nu = N^*_\nu \cap H(\kappa)$.

Using CH we may find $\nu_0 \neq \nu_1$ in ω_2 such that

$$(N_{\nu_0}; \in, q, \dot{r}_{\nu_0}, \{\beta_{\nu_0}\}, \Phi_{\beta_{\nu_0}+1})$$

and

$$(N_{\nu_1}; \in, q, \dot{r}_{\nu_1}, \{\beta_{\nu_1}\}, \Phi_{\beta_{\nu_1}+1})$$

are isomorphic structures. In particular,

$$e = \{(N_{\nu_0}, \beta_{\nu_0} + 1), (N_{\nu_1}, \beta_{\nu_1} + 1)\}$$

is then an edge.

Let us assume that $\beta_{\nu_0} \geq \beta_{\nu_1}$. By Lemma 3.6, we may find an extension $q^* \in \mathcal{P}_{\beta_{\nu_0}}$ of q such that $e \in \mathcal{G}_{q^*}$ and $F_{q^*} = F_q$. Let now $q' \in \mathcal{P}_{\beta_{\nu_0}}$ be any extension of $q^*|_{\beta_{\nu_0}}$ and suppose, towards a contradiction, that $q' \Vdash_{\beta_{\nu_0}} n \in \dot{r}_{\nu_0} \Delta \dot{r}_{\nu_1}$ for some $n < \omega$. Let us assume that $q' \Vdash_{\beta_{\nu_0}} n \in \dot{r}_{\nu_0} \backslash \dot{r}_{\nu_1}$.

By Lemma 3.11(2), $q^*|_{\beta_{\nu_0}}$ is potentially $(N_{\nu_0}, \mathcal{P}_{\beta_{\nu_0}})$-generic. Hence, there are $\beta^\dagger_{\nu_0} \geq \beta_{\nu_0}$ and $q'' \in \mathcal{P}_{\beta_{\nu_0}}$, $q'' \leq_{\beta^\dagger_{\nu_0}} q'$, such that $q'' \leq_{\beta_{\nu_0}} p$ for some $p \in N_{\nu_0} \cap \mathcal{P}_{\beta_0}$ such that $p \Vdash_{\beta_{\nu_0}} n \in \dot{r}_{\nu_0}$. We know that $(q''|_{\beta_{\nu_0}}) \restriction N_{\nu_0} \in \mathcal{P}_{\beta_{\nu_0}}$ (by Lemma 3.9) and $(q''|_{\beta_{\nu_0}}) \restriction N_{\nu_0} \leq_{\beta_{\nu_0}} p$ (by Lemma 3.8). We then have that

$$(q''|_{\beta_{\nu_0}}) \restriction N_{\nu_0} \Vdash_{\beta_{\nu_0}} n \in \dot{r}_{\nu_0},$$

and therefore $(q''|_{\beta_{\nu_1}}) \restriction N_{\nu_1} \in \mathcal{P}_{\beta_{\nu_1}}$ and

$$(q''|_{\beta_{\nu_1}}) \restriction N_{\nu_1} \Vdash_{\beta_{\nu_1}} n \in \Psi_{N_{\nu_0}, N_{\nu_1}}(\dot{r}_{\nu_0})$$

by Lemma 3.5. Again by Lemmas 3.9 and 3.8, we have that $q''|_{\beta_{\nu_1}} \leq_{\beta_{\nu_1}} (q''|_{\beta_{\nu_1}}) \restriction N_{\nu_1}$, and therefore $q''|_{\beta_{\nu_1}} \Vdash_{\beta_{\nu_1}} n \in \Psi_{N_{\nu_0}, N_{\nu_1}}(\dot{r}_{\nu_0})$.[aa] But this yields a contradiction since $\Psi_{N_{\nu_0}, N_{\nu_1}}(\dot{r}_{\nu_0}) = \dot{r}_{\nu_1}$.

The argument in the case that $q' \Vdash_{\beta_{\nu_0}} n \in \dot{r}_{\nu_1} \backslash \dot{r}_{\nu_0}$ is symmetrical to the proof in the previous case; in that case, we take $r \in N_{\nu_0} \cap \mathcal{P}_{\beta_{\nu_0}}$ such that $r \Vdash_{\beta_{\nu_0}} n \notin \dot{r}_{\nu_0}$.[bb] $\square$

Given $\alpha < \kappa$ and a $\mathcal{P}_\kappa$-generic filter G, let

$$D^G_\alpha = \{\delta_N : N \in \mathcal{N}^G_{\alpha+1}\}.$$

Let also $\dot{D}_\alpha$ be a $\mathcal{P}_\kappa$-name for D^G_α.

We now prove the other conclusion in Theorem 2.1 involving cardinal arithmetic.

Lemma 3.17. *$\mathcal{P}_\kappa$ forces $2^{\aleph_1} = \kappa$.*

Proof. In order to prove that $\Vdash_{\mathcal{P}_\kappa} 2^{\aleph_1} \geq \kappa$, it suffices to show that $\mathcal{P}_\kappa$ forces that $\dot{D}_{\alpha_0} \backslash \dot{D}_{\alpha_1} \neq \emptyset$ for all $\alpha_0 < \alpha_1$. For this, let q be a $\mathcal{P}_\kappa$-condition, which we may

[aa] Cf. the argument in the verification of clause (5) in the definition of condition for the amalgamation q^* in the proof of $\aleph_2$-c.c. from Lemma 3.11.
[bb] Compare this proof with the proof of Claim 3.12.

assume is such that $\alpha_1 \in \mathrm{dom}(F_q)$, and let $N \in [H(\kappa)]^{\aleph_0}$ be a sufficiently correct model such that $q \in N$. By the same argument as in the proof of Lemma 3.6 we may find an extension $q' \in \mathcal{P}_\kappa$ of q such that $N \in \mathcal{N}^{q'}_{\alpha_0+1}$ and $\mathcal{N}^{q'}_{\alpha_1+1} = \mathcal{N}^{q}_{\alpha_1+1}$. Let $\delta < \delta_N$ be above δ_M for every $M \in \mathcal{N}^q_{\alpha_1+1}$ and let $q^* \in \mathcal{P}_\kappa$ be the extension of q' resulting from adding (δ, δ_N) to $d^{q'}_{\alpha_1}$. Then q^* forces that $\delta_N \in \dot{D}_{\alpha_0} \backslash \dot{D}_{\alpha_1}$. Since $q \in \mathcal{P}_\kappa$ was arbitrary, this density lemma shows that $\mathcal{P}_\kappa$ forces $\dot{D}_{\alpha_0} \backslash \dot{D}_{\alpha_1} \neq \emptyset$.

Finally, a simple counting argument of nice $\mathcal{P}_\kappa$-names for subsets of ω_1 (see [17]) using the $\aleph_2$-chain condition of $\mathcal{P}_\kappa$ and the fact that $|\mathcal{P}_\kappa|^{\aleph_1} = \kappa^{\aleph_1} = \kappa$ shows that $\mathcal{P}_\kappa$ forces $2^{\aleph_1} \leq \kappa$. □

3.3. *Measuring*

The following lemma completes the proof of Theorem 2.1.

Lemma 3.18. *$\mathcal{P}_\kappa$ forces Measuring.*

Proof. Let G be $\mathcal{P}_\kappa$-generic and let $\vec{C} = (C_\delta : \delta \in \mathrm{Lim}(\omega_1)) \in V[G]$ be a club-sequence on ω_1. We want to see that there is a club of ω_1 in $V[G]$ measuring $\vec{C}$. By $\mathcal{P}_\kappa = \bigcup_{\alpha<\kappa} \mathcal{P}_\alpha$ together with the $\aleph_2$-c.c. of $\mathcal{P}_\kappa$, we may assume that, for some $\alpha_0 < \kappa$, $\vec{C} = \dot{C}_G$ for some $\mathcal{P}_{\alpha_0}$-name $\dot{C} \in H(\kappa)$ for a club-sequence on ω_1. Again by the $\aleph_2$-c.c. of $\mathcal{P}_\kappa$ and the unboundedness of $\{\alpha \in \mathrm{Succ}(\kappa) : \Phi(\alpha) = \dot{C}\}$ in κ, we may fix some $\alpha > \alpha_0$ in $\mathrm{Succ}(\kappa)$ such that $\Phi(\alpha) = \dot{C}$. We then have that $\Phi(\alpha)$ is a $\mathcal{P}_\alpha$-name, and by Lemma 3.3 it is in fact a $\mathcal{P}_\alpha$-name for a club-sequence on ω_1. Hence, we then have that $\vec{C} = \Phi(\alpha)_G$. We will see that $(\dot{D}_\alpha)_G$ is a club of ω_1 measuring $\vec{C}$.

First of all, it is easy to see that $\dot{D}_\alpha$ is forced to be unbounded in ω_1. In fact, given any condition $q \in \mathcal{P}_\kappa$ and any sufficiently correct countable $N \preccurlyeq H(\kappa)$ such that q, $\alpha \in N$, we may find by Lemma 3.6 an extension $q^* \in \mathcal{P}_\kappa$ of q such that $N \in \mathcal{N}^{q^*}_{\alpha+1}$, and every such condition forces that $\delta_N \in \dot{D}_\alpha$.

Claim 3.19. *D^G_α is closed in ω_1.*

Proof. It suffices to prove that if $\delta \in \mathrm{Lim}(\omega_1)$ and $q \in \mathcal{P}_\kappa$ are such that q forces δ to be a limit point of $\dot{D}_\alpha$, then there is some $N \in \mathcal{N}^q_{\alpha+1}$ such that $\delta_N = \delta$.

Suppose, towards a contradiction, that $q \in \mathcal{P}_\kappa$ and $\delta \in \mathrm{Lim}(\omega_1)$ are such that q forces δ to be a limit point of $\dot{D}_\alpha$ but there is no $N \in \mathcal{N}^q_{\alpha+1}$ such that $\delta_N = \delta$. We may extend q to a condition q' obtained by adding $(\bar{\delta}, \delta)$ to d^q_α, where $\bar{\delta} < \delta$ is above δ_M for every $M \in \mathcal{N}^q_{\alpha+1}$ such that $\delta_M < \delta$, and taking copies under Ψ_{N_0,N_1} as dictated by relevant edges $\{\langle (N_0, \rho_0), (N_1, \rho_1)\rangle\} \in \mathcal{G}_q$. But that yields a contradiction since then q' forces, by clause (4)(d) in the definition of condition, that $\dot{D}_\alpha \cap \delta$ is bounded by $\bar{\delta}$. □

Given any $q \in G$ such that $\alpha \in \mathrm{dom}(F_q)$ and any limit point $\delta \in D^G_\alpha$, if $(\delta, \bar{\delta}) \in b^q_\alpha$ for some $\bar{\delta} < \delta$, then $D^G_\alpha \cap (\bar{\delta}, \delta)$ is disjoint from C_δ. Hence, in order to finish the proof of the lemma it is enough to show that if $q \in G$ is such that $\alpha \in \mathrm{dom}(F_q)$, $N \in \mathcal{N}^q_{\alpha+1}$, and there is no $q' \in G$ extending q and such that $\delta_N \in \mathrm{dom}(b^{q'}_\alpha)$, then a tail of D^G_α is contained in C_{δ_N}.

So, let q be a condition with $\alpha \in \mathrm{dom}(F_q)$ and let $N \in \mathcal{N}^q_{\alpha+1}$ be such that $\delta_N \notin \mathrm{dom}(b^{q'}_\alpha)$ for any $q' \in \mathcal{P}_\kappa$ extending q. It suffices to find an extension q^* of q in $\mathcal{P}_\kappa$ and some $\delta < \delta_N$ with the property that if $q' \in \mathcal{P}_\kappa$ extends q^* and $M \in \mathcal{N}^{q'}_{\alpha+1}$ is such that $\delta < \delta_M < \delta_N$, then $q'|_\alpha \Vdash_\alpha \delta_M \in \dot{C}^\alpha_{\delta_N}$.

We will assume that $\Xi^{q|_{\alpha+1},\alpha}_{\delta_N} \neq \emptyset$ — the proof in the case $\Xi^{q|_{\alpha+1},\alpha}_{\delta_N} = \emptyset$ is a simpler version of the proof in this case. Let $\alpha_0 = \max(\Xi^{q,\alpha}_{\delta_N})$, which is well defined since $\emptyset \neq \Xi^{q|_{\alpha+1},\alpha}_{\delta_N} \subseteq \Xi^{q,\alpha}_{\delta_N}$. As usual, we may find a sequence $\vec{\mathcal{E}} = (\langle (N^i_0, \rho^i_0), (N^i_1, \rho^1_i)\rangle : i \leq n)$ such that $\langle \alpha, \vec{\mathcal{E}}\rangle$ is a connected $\mathcal{G}_q$-thread with $\min(\delta_{\vec{\mathcal{E}}}) = \delta_N$, $\Psi_{\vec{\mathcal{E}}}(\alpha) = \alpha_0$, $N^0_0 = N$, $N^n_1 \in \mathcal{N}^q_{\alpha_0+1}$ and $\delta_{N^n_1} = \delta_N$.

Claim 3.20. *There is some extension $q_0 \in \mathcal{P}_\kappa$ of q, together with some $a \in N$, such that q_0 forces in $\mathcal{P}_\kappa$ that if $M \in \mathcal{N}^{\dot{G}_\kappa}_{\alpha_0} \cap \mathcal{T}_{\alpha_0+1} \cap N^n_1$, $\Psi_{\vec{\mathcal{E}}}(a) \in M$, and*

$$\delta_M \notin \bigcup \{\dot{C}^{\bar{\alpha}}_{\delta_N} : \bar{\alpha} \in \Xi^{q,\alpha_0}_{\delta_N}\},$$

then $\delta_M \in \dot{C}^\alpha_{\delta_N}$.

Proof. Let us assume that the conclusion fails. Given any extension q' of q and any $a \in N$, by an instance of clause (7)(b) in the definition of condition for $q|_{\alpha_0+1}$ together with Lemma 3.3, there is some $q'' \leq_\kappa q'$ and some $M \in \mathcal{N}^{q''}_{\alpha_0} \cap \mathcal{T}_{\alpha_0+1} \cap N^n_1$ such that $\Psi_{\vec{\mathcal{E}}}(a) \in M$ and

$$q''|_{\alpha_0} \Vdash_{\alpha_0} \delta_M \notin \bigcup \{\dot{C}^{\bar{\alpha}}_{\delta_N} : \bar{\alpha} \in \Xi^{q,\alpha_0}_{\delta_N}\}.$$

By our assumption, we then have that $q''|_{\alpha_0} \not\Vdash_{\alpha_0} \delta_M \in \dot{C}^\alpha_{\delta_N}$. Hence, every such q'' forces $\delta_M \notin \dot{C}^\alpha_{\delta_N}$. We have thus seen that q forces that for every $a \in N$ there is some $M \in \mathcal{N}^{\dot{G}_\kappa}_{\alpha_0} \cap \mathcal{T}_{\alpha_0+1} \cap N^n_1$ such that $\Psi_{\vec{\mathcal{E}}}(a) \in M$ and

$$\delta_M \notin \bigcup \{\dot{C}^{\bar{\alpha}}_{\delta_N} : \bar{\alpha} \in \Xi^{q,\alpha_0}_{\delta_N}\} \cup \{\dot{C}^\alpha_{\delta_N}\}.$$

Let now $\bar{\delta} < \delta_N$ be above δ_Q for every $Q \in \mathcal{N}^q_{\alpha+1}$ such that $\delta_Q < \delta_N$ and let q^* be the result of adding $(\delta_N, \bar{\delta})$ to b^q_α and closing under relevant isomorphisms Ψ_{N_0,N_1}. Then q^* is a condition in $\mathcal{P}_\kappa$ extending q (all clauses in the definition of condition except for (7)(b) are immediate, and (7)(b) follows from $\Xi^{q^*|_{\alpha+1},\alpha}_{\delta_N} \setminus \{\alpha\} = \Xi^{q|_{\alpha+1},\alpha}_{\delta_N} \subseteq \Xi^{q,\alpha}_{\delta_N}$ and the property of q we have just proved), which is a contradiction since $\delta_N \in \mathrm{dom}(b^{q^*}_\alpha)$. ∎

Let q_0 and $a \in N$ be as in Claim 3.20. Let $\delta < \delta_N$ be above δ_Q for every $Q \in \mathcal{N}^{q_0}_{\alpha+1}$ such that $\delta_Q < \delta_N$ and let q^* be the extension obtained by adding the pair (δ, a) to $d^{q_0}_\alpha$ and closing under relevant isomorphisms Ψ_{N_0,N_1}.

We now show that q^* and δ are as desired. For this, suppose $q' \in \mathcal{P}_\kappa$ extends q^* and $M \in \mathcal{N}^{q'}_{\alpha+1}$ is such that $\delta < \delta_M < \delta_N$. By an instance of (4)(d) in the definition of condition for q', we then have some $M' \in \mathcal{N}^{q'}_{\alpha+1}$ such that $\delta_{M'} = \delta_M$ and $a \in M'$. By the shoulder axiom for $\mathcal{N}^{q'}_{\alpha+1}$ there is some $N' \in \mathcal{N}^{q'}_{\alpha+1}$ such that $\delta_{N'} = \delta_N$ and $M' \in N'$. Then $M'' = \Psi_{N',N}(M') \in \mathcal{N}^{q'}_{\alpha+1} \cap N$ and $a \in M''$ since $\Psi_{N',N}(a) = a$ as $a \in N \cap N'$. Since $M'' \in \mathcal{N}^{q'}_{\alpha+1} \cap N$, we then have of course that

$$q'|_\alpha \Vdash_\alpha \delta_{M''} \notin \bigcup\{\dot{C}^{\bar{\alpha}}_{\delta_N} : \bar{\alpha} \in \Xi^{q,\alpha_0}_{\delta_N}\},$$ [cc]

from which it follows by the choice of a that $q'|_\alpha \Vdash_\alpha \delta_{M''} \in \dot{C}^{\alpha}_{\delta_N}$. This finishes the proof since $\delta_{M''} = \delta_M$. □

3.4. *On adapting the construction of Theorem* 1.2 *to other contexts*

It will be sensible to finish this section with some words addressing the issue of what goes wrong if we try to modify the present forcing so as to force CH together with Unif$(\vec{C})$, for some given ladder system $\vec{C} = (C_\delta : \delta \in \mathrm{Lim}(\omega_1))$ — as we mentioned in Sec. 1, the conjunction of these two statements cannot hold. One could in fact try to build something like a sequence of partial orders $(\mathcal{P}_\beta)_{\beta \leq \kappa}$ in our construction in such a way that, at every stage $\alpha < \kappa$, we attempt to add a uniformizing function on $\vec{C}$ for some coloring $F : \mathrm{Lim}(\omega_1) \longrightarrow \{0, 1\}$ fed to us by our book-keeping function Φ. Thus, rather than the present pairs (b, d), we would plug in conditions for a natural forcing for adding such a uniformizing function with finite conditions.

Everything would seem to go well — and in particular our construction would have the $\aleph_2$-c.c., would be proper, and would preserve CH — except that, because of the copying constraint expressed in the corresponding version of clause (6) in the definition of condition, it would not be able to force Unif$(\vec{C})$. The reason is that we would not be in a position to rule out situations in which there is a condition q with, for example, an edge $\{(N_0, \rho_0), (N_1, \rho_1)\}$ in $\mathcal{G}_q$ for which there is some $\alpha \in N_0 \cap \rho_0$ such that the color of $\dot{F}(\alpha)$ at δ_{N_0} is forced to be, say, 0, whereas the color of $\dot{F}(\bar{\alpha})$ at δ_{N_0} is forced to be 1 (where $\bar{\alpha} = \Psi_{N_0,N_1}(\alpha)$ and where $\dot{F}(\xi)$ denotes of course the name for the coloring to be uniformized at stage ξ of the construction). The requirement, imposed by the current version of clause (6), that any relevant amount of information below δ_{N_0} on the generic uniformizing function at the coordinate α be copied over to the coordinate $\bar{\alpha}$, would then make it impossible for these generic uniformizing functions to be defined on any tail of $C_{\delta_{N_0}}$. This type of problems does not arise when forcing Measuring due to the more lenient nature of the "guessing" in this case; if we cannot get the club to eventually stay outside a given C_δ, then it has to eventually get inside (see the density argument in the proof of Lemma 3.18). The fact whether one or the other is the case is determined by the specific club-sequence being measured (and by the "shape" of the surrounding condition, of course).

[cc]Note the presence in this expression of $\Xi^{q,\alpha_0}_{\delta_N}$ rather than $\Xi^{q',\alpha_0}_{\delta_N}$ or $\Xi^{q'|_{\alpha+1},\alpha}_{\delta_N}$.

It may also be worth pointing out that the type of situation described above is a source of serious obstacles towards trying to force any reasonable forcing axiom to hold together with CH using the present methods. To see this in a particularly simple case, suppose, for example, that $(\mathcal{Q}_\beta)_{\beta\leq\kappa}$ is exactly as our present construction $(\mathcal{P}_\beta)_{\beta\leq\kappa}$, except that at each stage we force with Cohen forcing. This construction enjoys all relevant nice properties that $(\mathcal{P}_\beta)_{\beta\leq\kappa}$ has. On the other hand, $\mathcal{Q}_\kappa$ cannot possibly force $\mathsf{FA}_{\aleph_1}$(Cohen), as it preserves CH. Letting $\alpha^* < \kappa$ be such that all reals in $V^{\mathcal{Q}_\kappa}$ have already appeared in $V^{\mathcal{Q}_{\alpha^*}}$, if $\alpha < \kappa$ is above α^*, then the real constructed by the generic at the coordinate α will actually fail to be Cohen-generic over $V^{\mathcal{Q}_{\alpha^*}}$; in fact, for every condition $q \in \mathcal{Q}_\kappa$ such that $\alpha \in \mathrm{dom}(F_q)$ there will be a condition q' extending q for which there is connected $\mathcal{G}_{q'}$-thread $\langle\alpha, \vec{\mathcal{E}}\rangle$ such that $\bar{\alpha} := \Psi_{\vec{\mathcal{E}}}(\alpha) < \alpha^*$. The information at the coordinate $\bar{\alpha}$ contained in any extension of q' will then have to be copied over into the coordinate α, which in this situation means that the real r_α constructed at the coordinate α is identical to the real at $\bar{\alpha}$, and this obviously prevents r_α from being Cohen-generic over $V^{\mathcal{Q}_{\alpha^*}}$.

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