The Vertical Mode Method in a Problem of Hydroelastic Waves Generated by an External Load

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Abstract. Two-dimensional problem of the response of an ice cover to an applied external load is considered. The ice cover is modeled as a thin elastic plate. The fluid under the plate is incompressible, inviscid and is of finite depth. The external load has a given shape and does not move. Amplitude of the load oscillates at a given frequency. The deflections of the ice have the form of standing waves far away from the load. The problem is solved using the Green's function and the method of vertical modes. The eigenvalues of the vertical modes are the roots of the dispersion relation for hydroelastic waves propagating along the plate. The contribution of each type of roots to the formation of the ice deflections is studied. It is shown that the ice deflections for an elastic plate approximate the ice deflections for a porous plate with low porosity.

INTRODUCTION

The interaction of gravitational waves with thin porous structures is of considerable interest [1, 2]. There are many examples of mathematical models of ocean waves interacting with porous structures. In particular, problems related to reducing the wave force acting on engineering structures is of great importance. These problems are widely occur in applications to coastal engineering, where structures are often used to dissipate wave energy to protect the coast. Examples of such structures are vertical breakwaters and floating or submerged horizontal plates. Mathematical formulation in this case often incudes the theory of thin plates and research focuses on analyzing the characteristics of these structures to increase scattering of the wave energy [3, 4, 5, 6]. In the work [6] the last problem was solved in a three-dimensional formulation, and a method for matching eigenfunctions was developed for the problem of linear scattering of water waves by a round floating porous elastic plate. It was found that the dissipation of wave energy due to porosity of the plate initially increases as the plate becomes more porous, reaches a maximum, and then slowly decreases as the porosity further increases. In works on the dynamics of thin plates the large class of problems is related to the behavior of a thin ice cover. The ice, in general, is a porous medium. In many studies the porosity of ice is not taken into account. In [7] unsteady oscillations of the poroelastic ice cover under the action of the applied periodic pressure were investigated. It was found that the increase in the porosity increases the deflections of the poroelastic plate in the region of applied pressure, reduces the ice deflections at a distance from the load and reduces the observed disturbed area of the ice. It was shown that porosity increases the length of the hydroelastic wave propagating along an ice cover from the applied pressure. This study is a continuation of the research reported in [7]. In [7] the question about the convergence of the ice deflections when the porosity tends to zero remains open. The method of the solution in the considered work does not allow calculating the ice deflections in the case of zero porosity. In this paper the problem of oscillations of an elastic ice plate under the action of an applied pressure with a periodically oscillating amplitude is considered. The problem is solved by the method of vertical modes, in which it is necessary to calculate the complex roots of the corresponding dispersion relation. The contribution of different roots to the profile of the ice deflections is studied and the result is compared with the ice deflections for a plate with small porosity.

FORMULATION OF THE PROBLEM

The oscillations of an ice cover under the action of an external load are considered. The problem is two-dimensional and formulated within the linear theory of hydroelasticity. The ice cover is modeled as a thin elastic plate. The

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liquid under the ice cover is inviscid, incompressible and has a finite depth H, $(-H < y < 0)$. The flow caused by the ice deflections is potential. The liquid and the ice cover are not limited horizontally. The vertical displacement (deflections) of the plate from the equilibrium position satisfies the equation of a thin elastic beam, the velocity potential of the flow satisfies the Laplace equation. The load is modeled by a smooth localized pressure with an amplitude oscillating at a constant frequency. The scheme of the problem is shown in Fig. 1.

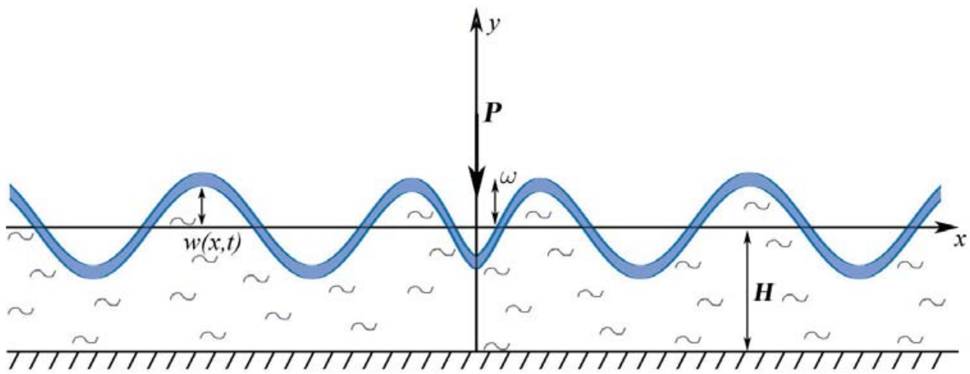

FIGURE 1. The scheme of the problem.

We are concerned with the case where the ice deflections has the form of standing waves at a distance from the load, periodically oscillating with the frequency of oscillation of the external load's amplitude. In this case, the governing system of equations are

$$Mw_{tt} + EJw_{xxxx} = p(x,0,t) - P_{ext}(x)\cos(\omega t), \tag{1}$$

$$\phi_{xx} + \phi_{yy} = 0 \quad (-H < y < 0), \tag{2}$$

$$\phi_y = 0 \quad (y = -H), \quad \phi_y = w_t \quad (y = 0), \tag{3}$$

$$p(x,0,t) = -\rho\phi_t - \rho g w \quad (y = 0), \tag{4}$$

where $p(x,0,t)$ is the pressure of the liquid at the ice/liquid interface, defined by the linearized Bernoulli integral, $M = \rho_i h_i$ is the mass of the ice per unit area, ρ_i is the density of ice, h_i is the thickness of ice, $P_{ext}(x)$ is the function describing the shape of the external load, $EJ = \frac{Eh_i^3}{12(1-\nu^2)}$ is the bending rigidity of the plate, E is the Young's modulus, ν is the Poisson's ratio, $w_{xxxx} = \frac{d^4w}{dx^4}$, ω is the frequency of the oscillations of the amplitude of the load.

The solution of the problem (1) – (4) are sought in the form

$$w(x,t) = \mathrm{Re}[W(x)e^{i\omega t}], \qquad \phi(x,y,t) = \mathrm{Re}[i\omega\Phi(x,y)e^{i\omega t}], \qquad p(x,0,t) = \mathrm{Re}[P(x)e^{i\omega t}].$$

Substituting these function to the system (1) – (4), we governing equations read

$$-M\omega^2 W + EJW^{IV} = \rho\omega^2\Phi(x,0) - \rho g W - P_{ext}(x) \quad (-\infty < x < +\infty), \tag{5}$$

$$\Phi_{xx} + \Phi_{yy} = 0 \quad (-H < y < 0), \tag{6}$$

$$\Phi_y = 0 \quad (y = -H), \quad \Phi_y = W \quad (y = 0), \tag{7}$$

The absence of porosity, viscosity and other damping effects leads to the condition for the ice deflections at a distance from the load

$$W \sim W_\infty e^{-ik|x|} \quad (|x| \to 0), \tag{8}$$

where k is the wavenumber and W_∞ is the amplitude of the hydroelastic waves far away from the load. The problem (5) – (8) is solved in dimensionless variables. We introduce

$$y = H\widetilde{y}, \quad x = H\widetilde{x}, \quad P_{ext} = P_0\widetilde{P}_e(\widetilde{x}), \quad W = W_{sc}\widetilde{W}(\widetilde{x}), \quad \Phi = HW_{sc}\widetilde{\Phi},$$

where P_0 is the maximum amplitude of the external load, $W_{sc} = P_0H^4/[EJ]$. In the dimensionless variables (the sign $\sim$ is omitted) the system of equation (5) – (8) has the form

$$W^{IV} + \delta_0 W = \gamma\Phi - P_e \ \ (-\infty < x < +\infty), \quad \Delta_2\Phi = 0 \ \ (-\infty < x < +\infty, -1 < y < 0), \tag{9}$$

$$\Phi_y = 0 \ \ (y = -1), \quad \Phi_y = W(x) \ \ (y = 0), \quad W \sim W_\infty e^{-ik|x|} \quad (|x| \to 0), \tag{10}$$

where

$$\Delta_2 = \partial/\partial x^2 + \partial/\partial y^2, \quad \delta_0 = \left(1 - \frac{M\omega^2}{\rho g}\right)\left(\frac{H}{L_c}\right)^4, \quad \gamma = \frac{\omega^2 H}{g}\left(\frac{H}{L_c}\right)^4, \quad L_c = \left(\frac{EJ}{\rho g}\right)^{1/4}.$$

We shall find $W(x)$ and then determine the shape of the ice deflections near and far away from the load and comapre the result with the ice deflection of the plate with small porosity.

METHOD OF THE SOLUTION

The boundary-value problem (9) – (10) is solved using the Green's function

$$W(x) = \int_{-\infty}^{+\infty} P_{ext}(x_0)G(x,x_0)dx_0, \quad \Phi(x,y) = \int_{-\infty}^{+\infty} P_{ext}(x_0)\Psi(x,x_0,y)dx_0,$$

The equations for G and Ψ are derived from the system of the equations (5) – (8). The differential operator in (5) has constant coefficients, so we can write $G(x,x_0) = G_1(x - x_0)$, where $G_1(-x) = G_1(x)$, and $\Psi(x,x_0,y) = \Psi_1(x - x_0, y)$, where $\Psi_1(-x,y) = \Psi_1(x,y)$. This allows to find the functions G and Ψ for $x \in [0,+\infty)$ and then extend the solutions for negative x. In this case the ice deflection $W(x)$ and the potential $\Phi(x,y)$ are calculated as integrals of the product of the shape of the load and Green's functions

$$W(x) = \int_{-\infty}^{+\infty} P_{ext}(x_0)G(|x - x_0|)dx_0, \quad \Phi(x,y) = \int_{-\infty}^{+\infty} P_{ext}(x_0)\Psi(|x - x_0|, y)dx_0. \tag{11}$$

It can be shown that the boundary conditions for $G_1(x)$ are $G_1'(0) = 0$, $G_1'''(0) = -1/2$ and symmetry condition for $\Psi_1(x,y)$ is $\Psi_1'(0,y) = 0$. Using the relation between G_1 and Ψ_1 the original boundary-value problem can be reduced to the problem for Ψ_1

$$\Psi_1^V + \delta_0\Psi_1^{IV} = \gamma\Psi_1 \quad (y = 0), \quad \Psi_{1,xx} + \Psi_{1,yy} = 0 \quad (-1 < y < 1),$$

$$\Psi_{1,y} = 0 \quad (y = -1), \quad \Psi_1 \sim \Psi_{1\infty}e^{-ik|x|} \quad (|x| \to \infty),$$

This problem is solved using the vertical mode method [8]

$$\Psi_1(x,y) = \sum_{n=-2,\, n\neq 0}^{\infty} C_n e^{-i\kappa_n x} f_n + C_0 e^{i\kappa_0 x} f_0,$$

where κ_n is the dimensionless wavenumber of hydroelastic waves propagating along the plate.

Functions $f_n(y) = \cosh(\kappa_n(1+y))/[\kappa_n \sinh(\kappa_n)]$ are called vertical modes. They are normalized and orthogonal in a special sense

$$< F, G >= \int_{-1}^{0} F(z)G(z)dz + \frac{1}{\gamma}(F'''(0)G'(0) + F'(0)G'''(0)), \quad f_n'(0) = 1.$$

The principal coordinated of the vertical modes are found from the boundary conditions for G_1 and the definition of the last orthogonal product

$$C_k = \frac{1}{2\gamma}\frac{1}{i\kappa_k Q_k}, \quad C_0 = -\frac{1}{2\gamma}\frac{1}{i\kappa_0 Q_0}, \quad Q_k = \frac{1}{2\kappa_n^2\gamma^2}[\kappa_n^2(\kappa_n^4+\delta_0)^2 + \gamma(5\kappa_n^4+\delta_0-\gamma)],$$

The Green's function for the ice deflections is represented as the sum of three functions due to the three types of roots of the dispersion relation

$$G_1(x) = C_{1R}(x) + G_{1S}(x) + G_{1C}(x),$$

where the first term gives the contribution of purely imaginary roots, the second gives the contribution of complex roots and the third gives the contribution of a real root. For the purely imaginary roots $\kappa_n = i\tilde{\kappa}_n$

$$G_{1R}(x) = \gamma\sum_{n=1}^{\infty}\frac{\tilde{\kappa}_n e^{-\tilde{\kappa}_n x}}{\gamma(5\tilde{\kappa}_n^4+\delta_0-\gamma) - \tilde{\kappa}_n^2(\tilde{\kappa}_n^4+\delta_0)^2}.$$

For the complex roots $\kappa_{-1} = a_0 + ib_0$ and $\kappa_{-2} = -a_0 + ib_0$

$$G_{1S}(x) = -\frac{e^{-b_0 x}}{\gamma}\mathrm{Re}\left[i\frac{e^{ia_0 x}}{(a_0+ib_0)Q_{-1}}\right].$$

For the real root κ_0

$$G_{1C}(x) = \frac{ie^{-i\kappa_0 x}}{2\gamma\kappa_0 Q_0}.$$

After the wavenumbers are calculated the ice deflection is determined by the equation (11).

NUMERICAL RESULTS AND DISCUSSION

The calculations of the ice deflections under the action of the periodic load are performed for $H = 2$ m, $E = 4.2\cdot 10^9$ Pa, $\nu = 0.3$, $h_i = 0.1$ m, $\rho = 1024$ kg/m^3, $\rho_i = 917$ kg/m^3, $g = 9.8$ m/s^2, $\omega = 1\mathrm{c}^{-1}$, $t = 0$ s. The function describing the load $P_{ext}(x)$ in our calculations has the form $P(x) = P_0 P_1(x)$, $P_1(x) = (\cos(\pi c x)+1)/2 \quad (c|x| < 1), \quad P_1(x) = 0$ $(c|x| > 1)$ with $c = 1/2$, $P_0 = 1000$ Pa.

For a complete solution of the problem, it is necessary to calculate both real and complex κ_n, which are the roots of the dispersion relation

$$(\kappa^4 + \delta_0)tanh(\kappa) = \gamma.$$

This dispersion relation has an infinite number of solutions: 2 real roots $\pm\kappa_0$, a countable number of imaginary roots $\kappa_n = i\tilde{\kappa}_n$, $\tilde{\kappa}_n > 0$ and 4 complex roots $\pm a_0 \pm ib_0$, $a_0 > 0, b_0 > 0$. Calculation of the last roots is difficult. The dispersion relation after the substitution of these roots is divided into 2 implicit equations for a_0 and b_0. These equations are the real and imaginary parts of the dispersion relation respectively. The graphics of these equations are shown in Fig. 2. We need to take into account not all roots of the dispersion relation, but only the roots giving different solutions and damping of the ice deflections as $x \to \infty$. These roots are 1 positive real root, 2 complex conjugate roots, and a countable number of imaginary roots. All roots are calculated numerically. Real and purely imaginary roots are calculated by the iteration method. The algorithm of the calculation of the complex root is: first, the area with root on the Oxy plane (the neighborhood of the intersection of blue and red lines in Fig. 2) is cut to a square 1x1, then the implicit equations for the real and imaginary parts are constructed numerically, and the point of their intersection is found. All roots are calculated up to 10^{-5}. The root values depend on the δ_0 and γ parameters.

Note that $w(x,0) = W(x)$ for $t = 0$ s. The corresponding values of the function $W(x)$ are shown in Figs. 3 and 4. All results are shown in dimensional variables. Contributions to the formation of the ice deflections with purely imaginary and complex roots of the dispersion relation are shown in Fig. 3. The contributions with these roots are concentrated in a local area near the load. Contribution of the real root of the dispersion relation κ_0 is shown in Fig. 4a. This contribution is the most significant. For the considered case $\kappa_0 = 0.4474$. It can be shown that the

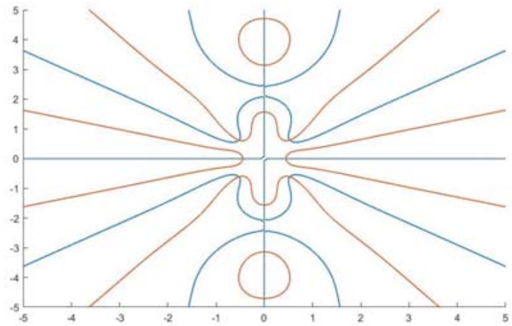

FIGURE 2. Implicit functions that are real and imaginary parts of the equation $(\kappa^4 + \delta_0)\tanh(\kappa) = \gamma$, blue line – the imaginary part, red line – the real part.

corresponding dimensional wavenumber is the limit of the wavenumbers of hydroelastic waves propagating from the load in a porous plate with decreasing porosity. Wavenumbers for a porous plate can be estimated numerically, see [7]. The ice deflections of the elastic plate with zero porosity (solid line) and the ice deflections of the plate with low porosity (markers) are shown in Fig. 4b. The ice deflections match with visual accuracy in both cases. The solution of the problem with porous plate, as well as the discussion of the choice of the porosity value, is presented in [7].

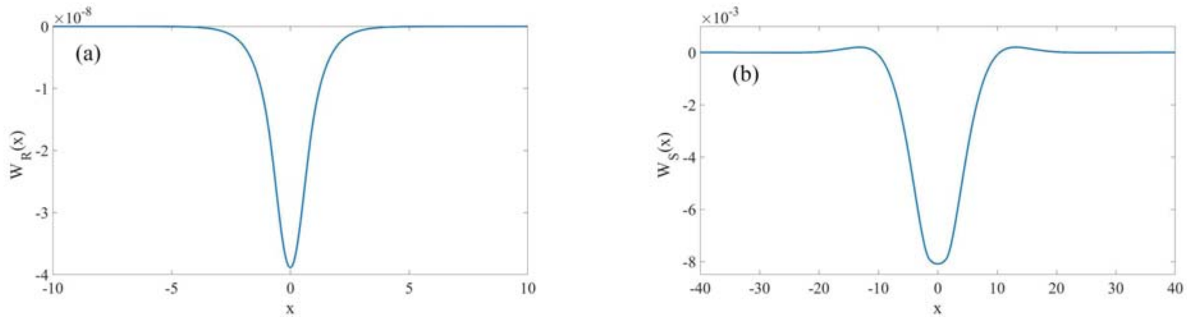

FIGURE 3. Contribution to the formation of the ice deflections of the terms with purely imaginary roots of the dispersion relation (a), with the complex roots (b).

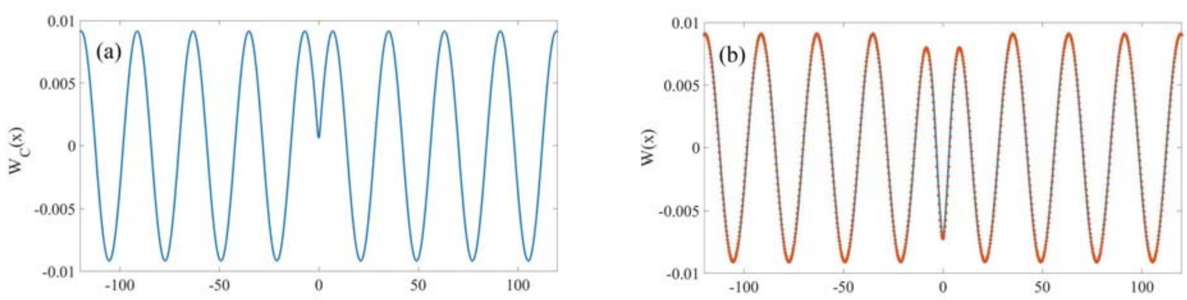

FIGURE 4. Contribution to the formation of the ice deflections of the term with the real root of the dispersion relation (a). The ice deflections in the considered case (solid blue line) and in the case of the plate with low porosity (red markers) (b).

CONCLUSION

The two-dimensional problem of oscillations of a thin elastic plate under the action of an external load is considered. The external load does not move and has a periodic amplitude. The problem is solved using the Green's function and the method of vertical modes, in which the solution is written as a sum of functions that are orthogonal in a special sense. The eigenvalues of these functions are the roots of the dispersion relation of periodic waves propagating along the plate. For a complete solution, it is necessary to take into account both real and complex roots of the dispersion relation. The form of the waves propagating from the load is determined and its wavenumber is calculated. This wavenumber is the limit of the wavenumbers of waves propagating from the load in the porous plate with decreasing porosity of the plate. It is shown that terms with complex roots and purely imaginary roots give local contributions near the load; a significant contribution is for a term with real root. It is shown that the shape of the ice deflections in the problem for a porous plate with decreasing porosity converges to the shape of the ice deflections for a plate with zero porosity. Further development of this problem involves the construction of the vertical mode method for a porous ice cover.

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REFERENCES

1. S. Zhenga, M. Meylan, G. Zhua, D. Greavesa, and G. Iglesias, "Wave scattering from multiple circular floating porous elastic plates," The 35th International Workshop on Water Waves and Floating Bodies (2020).
2. D. Mondal and S. Banerjea, "Scattering of water waves by an inclined porous plate submerged in ocean with ice cover," Quarterly Journal of Mechanics and Applied Mathematics **69**, 195–213 (2016).
3. I. Cho and M. Kim, "Wave absorbing system using inclined perforated plates," Journal Fluid Mechanics **608**, 1 – 20 (2008).
4. S. Koley, R. Kaligatla, and T. Sahoo, "Oblique wave scattering by a vertical flexible porous plate," Stud. Appl. Math. **135(1)**, 1 – 34 (2015).
5. H. Behera and T. Sahoo, "Hydroelastic analysis of gravity wave interaction with submerged horizontal flexible porous plate," J. Fluids Struct. **54**, 643 – 660 (2015).
6. M. Meylan, L. Bennetts, and M. Peter, "Water-wave scattering and energy dissipation by a floating porous elastic plate in three dimensions," Wave Motion **70**, 240–250 (2017).
7. K. N. Zavyalova, K. A. Shishmarev, and A. A. Korobkin, "The response of a poroelastic ice plate to an external pressure," Journal of Siberian Federal University. Mathematics and Physics **14(1)**, 87–97 (2021).
8. A. Korobkin, S. Malenica, and T. Khabakhpasheva, "The vertical mode method in the problems of flexural-gravity waves diffracted by a vertical cylinder," Applied Ocean Research **84**, 111–121 (2019).