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Research in Mathematics Education

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/rrme20

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To cite this article: Elena Nardi, Ioannis Kanellos & Irene Biza (02 Nov 2023): Nascent deductive proof schemes and proof readiness in adolescence, Research in Mathematics Education, DOI: 10.1080/14794802.2023.2261435

To link to this article: https://doi.org/10.1080/14794802.2023.2261435

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Published online: 02 Nov 2023.

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RESEARCH IN MATHEMATICS EDUCATION
https://doi.org/10.1080/14794802.2023.2261435

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Nascent deductive proof schemes and proof readiness in adolescence

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ABSTRACT

Using Proof-Schemes Combined, our extension of Harel and Sowder's Proof-Schemes taxonomy, we introduce the *nascent deductive proof schemes* construct (proof-scheme combinations that contain elements of deductive mathematical reasoning) and present evidence of these in written responses to ten items of 85 Year 9 (age 14–15) secondary students. We capture these students' first encounters with proof in an educational context (mixed-ability, state schools in Greece) where proof is present in lessons and key national examinations. In previous analyses, we identified evidence in the students' responses of Harel and Sowder's seven proof schemes and of the eight proof-scheme combinations from our Proof-Schemes Combined taxonomy. Here, we present evidence – found in 383 out of 850 student responses – of five of the fifteen proof schemes that were nascent deductive proof schemes. Our findings contribute to non-deficit discourses about students' mathematical reasoning potential and provide evidence in favour of proof readiness in early adolescence.

KEYWORDS

Nascent deductive proof schemes; secondary students; first encounter with mathematical proof

1. How ready are adolescents for developing competence in, and appreciation for, mathematical proof?

The role of proof and proving in mathematics – and its role in newcomers' enculturation in the nuts and bolts, and raison-d'être, of mathematical reasoning – is widely recognised as quintessential in the field (Hanna, 1991, 2000, 2007; Hanna & Barbeau, 2008; Stylianides & Stylianides, 2008). Often, particular emphasis is placed on the challenges the teaching and learning of proof poses – and, occasionally, the tone of the reporting of said challenges has been sombre and somewhat deficit (Harel, 1999, 2007; Harel & Sowder, 1998, 2007; Healy & Hoyles, 2000; Heinze, 2004; Küchemann & Hoyles, 2006; Skemp, 1976; Stylianides et al., 2004). Teachers are often found as having limited success in upholding the mantle of mathematical proof in their classrooms and students are often found as lacking in developing proving skills and appreciation for proof. This deficit take on how able and willing adolescents are to engage with, become competent in,

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and learn to appreciate, mathematical proof – and mathematical reasoning at large – is however neither universal nor comprehensive. Adolescents' ability to engage with, and construct, rigorous mathematical arguments has also been noticed (Knuth et al., 2009; Bieda & Lepak, 2014; Bieda, 2010; Mueller & Maher, 2009; Stylianides et al., 2022). Sometimes this observation has been made also for learners in their primary years (Stylianides, 2007; Knox & Kontorovich, 2022a, 2022b). As a result, a more balanced, non-deficit, proactive body of work has been steadily accumulating.

A deficit take on adolescents' mathematical reasoning has also been somewhat at odds with some of our experiences in research, teaching practice and teacher professional development. In this paper, joining the enterprise of aforementioned body of work for a more upbeat and proactive take on what may be feasible in the mathematics classroom, we present evidence that contributes to non-deficit discourses on proof readiness in adolescence.

To this aim, we deploy a diagnostic tool, Proof-Schemes Combined (Kanellos et al., 2018), that we developed from Harel and Sowder's (1998, 2007) Proof-Schemes taxonomy. We developed Proof-Schemes Combined in the course of analysing data which was collected originally for Kanellos (2014). 85 Year 9 (age 14–15) secondary students engaged with ten items that asked them to prove statements in algebra and geometry. The data captures these students' first encounters with proof and proving in a mixed-ability, state school educational context in Greece. In this context, the importance of mathematical proof is beginning to be hinted at as the students transition from primary to secondary school and is explicitly and gradually more present in algebra and geometry lessons in early secondary education. Further, proving skills are typically expected, and rewarded, in upper secondary education – from Year 10 onwards, and, eventually, in key national examinations for university entry. The analyses we present in this paper reveal that elements of deductive mathematical reasoning had notable presence in the students' writing in these early encounters with proof and proving. We draw on these analyses to make a cautious case in favour of proof readiness for mathematical proof in adolescence.

In what follows, first, we outline works on the learning and teaching of mathematical proof that have influenced our study and we present the Proof-Schemes (Harel & Sowder, 1998, 2007) and Proof-Schemes Combined (Kanellos et al., 2018) taxonomies that we deploy in our analysis. We then introduce the context, aims and methods of the study. We subsequently sample proof scheme evidence in the data with examples of student work that involve elements of deductive mathematical reasoning (we call this evidence of *nascent deductive proof schemes*) and present the aggregated table of nascent deductive proof scheme evidence across our dataset. We conclude with a brief discussion of the pedagogical implications that emerge from our findings, with a particular focus on an upbeat, non-prohibitive vision on proof readiness in adolescence, and on the role of mathematical proof in the secondary classroom.

2. Research into students' first encounters with proof and proving

In many educational contexts, students' first encounters with proof and proving take place at the post-secondary level (Hanna, 2000, 2002, 2007; Hanna & Barbeau, 2008; Harel & Sowder, 1998, 2007; Weber & Alcock, 2004) whereas some forms of mathematical argumentation and reasoning typically appear at secondary level (e.g. Healy & Hoyles, 2000;

Küchemann & Hoyles, 2006) and, sometimes, at primary level (e.g. Stylianides, 2007). Across these works (e.g. Healy & Hoyles, 2000), the students' perspectives on proof are often characterised by tension between an argument which satisfies the teacher, complies with the norms of mathematics as negotiated and understood in the classroom and receives a good mark – for example, a set of algebraic calculations that shows every step with clarity and transparency – and an empirical argument that students may find personally convincing but does not enjoy the approval and reward that a more formal argument does.

These tensions are akin to what Wertsch (1998) calls "resistance" – "irreducible tension between agent and cultural tool" (p. 30) – and can denote the "impediments experienced by students […] between a mathematical tool and its use in a situated context" (p. 280; Carlsen, 2018) such as "being bogged down in problems of recalling formulae, various expressions, algebraic syntax, and elementary mathematical calculations." (ibid.). Fiallo and Gutiérrez (2017), for example, found that at the heart of the difficulties experienced by students as they learn to prove is the tendency towards "constructive argumentations, based on perceptual elements of figures [… that] were more likely to produce empirical proofs" as opposed to "structuring argumentations on the conjectures [that] were more likely to produce deductive proofs" (p. 165). They also identified "students' inability to connect operators and representations in the geometric, algebraic, and numeric contexts" (ibid.) as a key obstacle to generating productive argumentation. Students may also lack appreciation for the role of definitions in writing a proof (Alcock & Simpson, 2002) and may be overall perplexed by what constitutes a proof (Mamona-Downs & Downs, 2005) and why it matters (Dawkins & Weber, 2017).

Concurring with Dawkins and Weber (2017) and Solomon (2006) – as well as with studies that present evidence of emerging proving skills and appreciation for proof as early as the first years of secondary education (Bieda & Lepak, 2014) or identified before undergraduates engage with a proof course upon beginning their university studies (Stylianou et al., 2015; Miller & Cadwallader-Olsker, 2020) – we problematise "the assumption that students' violation of proof norms is an indication of an innate deficit" (Dawkins & Weber, 2017, p. 138). To do so, we turn towards an educational context – Year 9, Greek compulsory education – where proof begins to be explicitly present in the textbook (Argyrakis et al., 2010), in lessons and in assessment relatively early in the students' mathematical experience. Although a formal introduction to deductive proof comes later in the curriculum (from Year 10 onwards), in Year 9 there is an expectation from students to prove simple statements in algebra and geometry. This is seen as a step towards their enculturation in deductive proof. We capture secondary students' first encounters with formal mathematical proof as the students transition from informal, empirical approaches to deductive reasoning, extending and strengthening prior reports of this transition (e.g. Küchemann & Hoyles, 2006). We aim that our analysis contributes to the debate on whether a productive start of this transition can occur in secondary school and, if so, how it can be facilitated.

One theoretical influence on our work is from studies that locate this transition – and, particularly, the emergence of the capacity to move fluently between empirical and deductive modes of mathematical reasoning – as a natural and feasible element of the shifts taking place during adolescence, e.g. in neuroscience terms (Dehaene, 2011), or, from a sociocultural standpoint, as what Smith (2020) labels the "contemporary framing of adolescence as a time of 'becoming'" (p. 174). As Watson (2010) notes, for

example, "(l)earning secondary mathematics can relate closely to the adolescent project of negotiating adulthood" (p. 133) as "the same kinds of adolescent autonomous thinking which so often lead to disaffection and rejection, not only of mathematics but of school and life more generally, can be embedded and enhanced positively within the teaching and learning of mathematics." (ibid.). In "suitable circumstances", adolescents can

> "adopt new classifications; re-theorise as a result of conflicting experience; learn from successful repetition; infer structures from multiple instances, including symbolic instances; identify relations; shift between phenomena and analogous images; imagine relations when offered new classifications; extend relations into imagined new spaces; handle new or unfamiliar entities if there is an accessible form of representation; and learn about relations between relations." (pp. 141–142)

The shifts "are not in opposition to other aspects of the adolescent project" (p. 145), resonate well with "the ways in which minds work with mathematics itself" (p. 145) and are part of what Freudenthal (1973) labels elements of "vertical mathematization", namely engaging with epistemological activity that requires re-organising the way one does mathematics, particularly in the light of new requirements and circumstances (Vergnaud, 2009). This activity is "de facto similar to the ways in which adolescents learn to negotiate with themselves, authority, and the world" (Watson, 2010, p. 145). Therefore, "abstract mathematics, like some of the creative arts, can be an arena in which the adolescent mind can have some control, can validate thinking, and can appeal to a constructed personal authority" (p. 145).

The focus of this paper was motivated by observations in Kanellos (2014) and Kanellos et al. (2018) that, on many occasions – even when compliance with the conventions of mathematical writing was imperfect or when empirical, salient features (e.g. of geometrical shapes) proliferated in a piece of student writing – there were modicums of deductive reasoning, rudimentary but non-negligible attempts at generalisation and justification. Identifying and interpreting these modicums of deductive reasoning became the focus of the diagnostic work we report here.

Essential to this work is the theoretical lens through which we analysed the data presented in this paper: Harel and Sowder's (1998, 2007) Proof-Schemes taxonomy, a framework which distinguishes variations of empirical (mostly inductive) and deductive proof and the Proof-Schemes Combined taxonomy that we introduced in Kanellos et al. (2018).

3. The proof schemes and the Proof-Schemes combined taxonomies

The choice of a Proof Schemes lens on our data originates in Kanellos (2014) and is grounded on our concurrence with Harel and Sowder's (1998, 2007) intention to frame students' proving activity within the broader social and cultural activity of arguing to convince oneself and others, including the teacher and peers in class – and to assess students' proof readiness and appreciation (e.g. Sowder & Harel, 2003).

As in Harel and Sowder (2007), we use the taxonomy as a lens through which to seek evidence of how students ascertain the truth of a mathematical statement and how they do so in a way that persuades the reader of their script (in this case, their teacher). We do so while recognising that, when students engage with a mathematical task, their priorities are not necessarily to emulate what mathematicians do (Weber et al., 2020): most likely they

respond to a mathematical task in what they see as the best of their capacity in a given moment and to meet the requirements posed by the requesting authority, their teacher. We have thus used the Proof Schemes and Proof-Schemes Combined taxonomies (Kanellos et al., 2018) to characterise written snapshots of students' mathematical activity.

Crucially, we do not use the taxonomy to characterise individual students' mathematical epistemologies overall. More than a single occurrence of responding in writing to a set of tasks, and more evidence across different modalities (e.g. verbal utterances during classroom discussions or interviews) would be needed for this. We do not claim that our analyses trace a student's proof schemes regardless of modality. Instead, we use the taxonomy to trace how a student cohort, in a given moment at these early stages of their encounters with proof and proving, responds in writing to their teacher's request. We stress that we see value in identifying proof-scheme evidence in a student's mathematical activity within a particular modality (in our case writing) that matters significantly in how students' work is assessed by their teachers and in high stake closed-book examinations that await them in Years 11 and 12.

Tracing proof-scheme evidence in students' writing is important to us for a further reason: we concur with Stylianides' (2019) observation – grounded in analyses of students' verbal and written mathematical arguments – that studies which analyse "written arguments only" may report "a less favourable picture of the potential of students' constructed proofs" than studies that "focus only on students' oral arguments" (p. 177). So, if analysis of students' writing indeed gives a "less favourable" account of their capacity to engage successfully with proof than other modalities, then tracing evidence of such capacity in this modality (writing) strengthens further the upbeat implications of our analysis (that students may indeed be readier to engage with proof at this stage of their education than often thought).

Our use of the Proof-Schemes taxonomy somewhat diverges from that of its creators (and, occasionally, others) in one aspect: the often-implicit, hierarchical assumption of superiority that deductive forms of mathematical argument seem to be invested with. As Weber et al. (e.g. 2020) have shown, the influence of external authority or disciplinary rituals is often part and parcel of what mathematicians do. As students are welcomed into the world of mathematics, we see this welcoming as including enculturation into the norms for ascertaining the truth of a mathematical statement via proof. We see learning to construct and appreciate deductive mathematical arguments as a quintessential component of learning about – and appreciating – mathematical proof. Yet, we wish this welcome to be extended to newcomers without diminishing the value of the many and varied ways in which mathematical insight can be generated in lessons and elsewhere. It is this valuing of these many and varied ways that Proof-Schemes Combined, our extension of the Proof-Schemes taxonomy (Kanellos et al., 2018), embodies.

We now briefly present the Proof Schemes and Proof-Schemes Combined taxonomies (see also Kanellos et al., 2018, pp. 278–280).

According to Harel and Sowder (2007), proof schemes describe what "constitutes ascertaining and persuading" (p. 809) in the proving activity of individuals or communities, with "ascertaining" being the removal of their own doubts and "persuading" being the removal of the doubts of others. Proof schemes can be *external conviction* proof schemes, *empirical* proof schemes and *deductive* proof schemes:

- *external conviction proof schemes* are grounded on external to the learner sources of conviction and can be
  - *authoritarian* when an authority, for instance a teacher or a book, is appealed to in support of an argument
  - *ritual* when an argument depends strictly on its "appearance […] (e.g. proofs in geometry must have a two-column format)" (p. 809)
  - *non-referential symbolic* if the argument depends "on symbol manipulations, with the symbols or the manipulations having no potential coherent system of referents" (ibid, p. 809).
- *empirical proof schemes* are grounded on experience which is deployed as evidence in the proving process and can be
  - *inductive* if there is reliance on "evidence from examples (sometimes just one example) or direct measurements of quantities, substitutions of specific numbers in algebraic expressions" (p. 809)
  - *perceptual* if there is reliance on "perceptions" (p. 809), such as on properties of a particular function as these properties appear on a graph.
- *deductive proof schemes* are grounded on deductive reasoning and can be
  - *transformational* if they share *generality* (validity in all cases, not just one or a few), *operational thought* (organised in appropriate steps to reach the final goal that completes the proof) and *logical inference* (based on logical rules)
  - *axiomatic* if they also acknowledge the axiomatic foundation of corresponding mathematical theory.

In Table 1, we summarise Harel and Sowder's Proof-Schemes taxonomy and the abbreviations for each scheme (Kanellos et al., 2018, p. 279) we have used in our analysis.

As we noted in Kanellos et al. (2018), "we see axiomatic proof schemes of limited relevance to the first – and typically not connected to sense-making of the axiomatic foundations of mathematical theory – encounters with proof of secondary students" (p. 279). We therefore focus on deductive transformational proof schemes (D.T.).

Given the taxonomy's origins (Harel & Sowder, 1998; Harel, 1999, 2007; Sowder & Harel, 2003), and uses (e.g. Housman & Porter, 2003) in post-secondary educational contexts, we were apprehensive about deploying it in the analysis of data from the secondary mathematics classroom. We were however reassured about the appropriateness of our choice by its uses at

Table 1. Harel and Sowder's (2007) Proof-schemes taxonomy and the abbreviations used in the analysis. D.T. is one of the five nascent deductive proof schemes that are the focus of this paper.

Proof schemes	Codes
External conviction proof schemes	EC.
Ritual proof schemes	EC.R.
Authoritarian proof schemes	EC.A.
Non-referential symbolic proof schemes	EC.NRS.
Empirical proof schemes	E.
Inductive proof schemes	E.I.
Perceptual proof schemes	E.P.
Deductive proof schemes	D.
Transformational proof schemes	D.T.
Axiomatic proof schemes	D.A.

the secondary level in some studies (e.g. Csíkos, 1999; Lee, 2016). We also observed in some studies (Housman & Porter, 2003; Lee, 2016; Erickson & Lockwood, 2021) something that became a central consideration in our analysis: evidence of different proof schemes may be found within the proving activity by a single student and attempts at deductive inference may co-exist with informal proof constructions in the same response.

The exploration of this observation in our data analyses led to the emergence of Proof-Schemes Combined (Kanellos et al., 2018): we noticed the high occurrence of occasions where more than one proof scheme code had been attributed to the same student response to a particular question. Combing through our data, we identified regular occurrences of the eight combinations of different proof schemes listed in Table 2. In Kanellos et al. (2018), we presented examples of each combination and made the case for the theoretical and pedagogical potency of the Proof-Schemes Combined taxonomy.

Naturally, some of the sole and some of the combined proof schemes occurred with higher frequency than others in the student scripts. We touched briefly on what these occurrences may imply for the students' emerging proof competence, confidence and appreciation in Kanellos (2014) and Kanellos et al. (2013, 2018, 2019). In this paper, we introduce a new construct (nascent deductive proof schemes) and use it to take these earlier observations further.

As we stressed in Section 2, we see evidence of one, or more, proof schemes in a student script as reflecting the ways in which the student responded to a task given by their teacher at this particular juncture of classroom activity which the teacher intends as part of the students' first encounters with mathematical proof. So, for example, we would most likely label a student response in which there is concurrent evidence of using an example and an attempt at a deductive argument as evidencing Deductive Transformational – Empirical Inductive (D.T.-E.I.) proof schemes. We recognise that, on the evidence of the student's script only – what we referred to earlier as the writing modality, particularly in response to the relevant observation by Stylianides (2019) – observing the students in class or discussing their response in an interview could perhaps shed more light on what lies behind the presence of an example in a script. The student may have included an example to gain confidence – in the affective sense of the word – in the overall argument they then put forward deductively in their script. In this case, labelling the script as evidencing Empirical Inductive (E.I.) proof schemes risks overstating the student's prioritising of empirical evidence over deductive reasoning. After all, if the student includes an example, to marshal evidence towards supporting a conjecture, which they then prove in a deductive manner, the student does what mathematicians themselves often do – and we tend to think of

Table 2. Eight combinations of proof schemes identified in the student responses and their abbreviations (Kanellos et al., 2018, p. 282). The first four proof schemes (the ones with an element of D.T.) are four of the five nascent deductive proof schemes we focus on in this paper.

Code	Proof Scheme Combination
D.T.-E.I.	Deductive Transformational – Empirical Inductive
D.T.-E.P.	Deductive Transformational – Empirical Perceptual
D.T.-EC.A.	Deductive Transformational – External Conviction Authoritarian
D.T.-EC.NRS.	Deductive Transformational – External Conviction Non-Referential Symbolic
E.I.-EC.NRS.	Empirical Inductive – External Conviction Non-Referential Symbolic
E.P.-EC.A.	Empirical Perceptual – External Conviction Authoritarian
E.P.-EC.NRS.	Empirical Perceptual – External Conviction Non-Referential Symbolic
EC.R.-EC.NRS.	External Conviction Ritual – External Conviction Non-referential symbolic

mathematicians as assuredly clear about the difference between Deductive Transformational (D.T.) and Empirical Inductive (E.I.) proof schemes. Steering clear of unduly overinterpreting the students' scripts, we simply use the labelling to note the evidence of deductive proof-schemes in their responses – either in a sole or a combined format – and thus signal where in their scripts deductive proof potency may lie.

With this clarification in mind, in this paper, we explore what we first identified in (Kanellos et al., 2018) as a potentially significant feature in the students' scripts, the "cohabitation of a range of dispositions toward proof" (p. 280). We do so by focusing on five of the fifteen (sole and combined) proof schemes that we used as codes in our earlier analysis. We call these five proof schemes *nascent deductive proof schemes* – namely, proof schemes that contain, at least in part, elements of deductive mathematical reasoning – and we exemplify each in Section 5 of this paper. We close Section 5 with a compact account of the frequencies for each of the five nascent deductive proof scheme codes (Table 5). The research question therefore that this paper explores is: What evidence, if any, of nascent deductive proof schemes can be identified in secondary students' mathematical writing?

We now briefly present the context, participants and data collection of the study (Kanellos, 2014; Kanellos et al., 2018) which the analyses presented in this paper build on.

4. A study of secondary students' proof work: context, participants, data collection, analysis

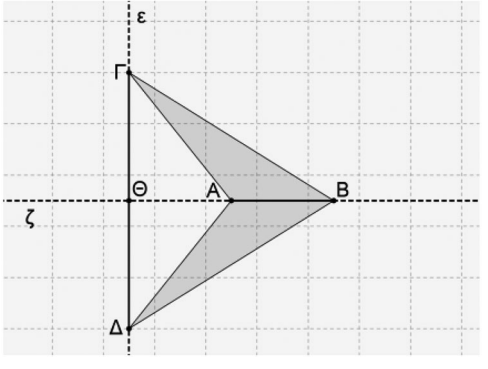
The study was conducted in Greece, where students begin to be taught proof in algebra and geometry in Year 9 (age 14–15), e.g. with proving identities such as the difference of squares in algebra and proving propositions based on the congruency criteria of triangles in geometry (Argyrakis et al., 2010). Year 9 is in this sense a turning point for students since empirical verification of mathematical statements leads gradually to argumentation based on definitions and properties of mathematical objects. Our study captures this turning point in Year 9 students' experiences. To this purpose, we worked closely with the teachers and students of a mixed ability, state-funded, Greek secondary school, and particularly one of the teachers and her four Year 9 classes (85 students), in algebra and geometry. We observed her teaching and collaborated on the construction of the tests she administered to the students throughout the year.

Our data collection included two questionnaires that contained algebra and geometry items and were administered in the beginning and at the end of the school year. In this paper, we draw on data originating in the students' responses to the end-of-year questionnaire (Table 3). While we are aware of the risks – see, for example, Vinner's (1997) observations on the "cognitive approach fallacy" – that come with relying on a singular source of evidence for the claim we put forward in this paper, we also see these 850 data entries as a sufficiently solid steppingstone towards more multi-source and more longitudinal investigations of this claim. We return to this in the concluding section of our paper.

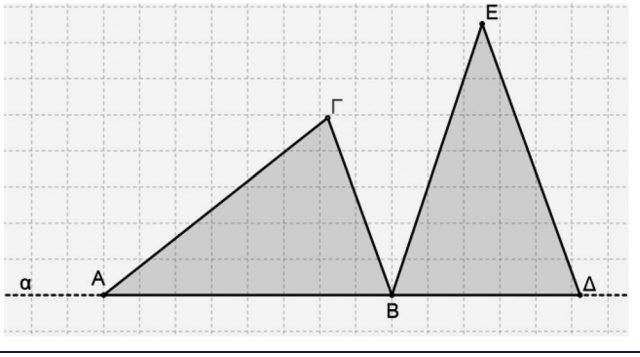
Q1–Q6 were either similar to those that students had practised across the year in algebra (Q1, Q2) and geometry lessons (Q4–Q6) or adjusted from items found in the literature on proof and proving (e.g. Q3 is inspired by a similar item in Healy & Hoyles, 2000). Q1, Q2, Q4–6 assess primarily proof fluency, whereas Q3 assesses primarily

Table 3. The proof questionnaire, translated from Greek and reproduced from Kanellos et al. (2018, p. 282).

Q1	For real numbers a and b it is given that $a^2 + b^2 = 5^2$. Prove that $\left(a\sqrt{3} + b\sqrt{2}\right)^2 + \left(a\sqrt{2} - b\sqrt{3}\right)^2 = 125$.
Q2a/b	If the difference of the squares of two natural numbers κ and λ ($\kappa > \lambda$) is equal to the sum of these numbers, then: a. Prove that the difference of these numbers is equal to one. b. Prove that $5556^2 - 5555^2 = 11111$.
Q3a/b	Two of your peers wonder how to prove that $(a + b)(a - b) = a^2 - b^2$. One of them suggests they should assign numerical values to a and b (e.g. a = 2 and b = 1), then compute both sides and check whether the right part is equal to the left part. They experiment with a number of values, and they convince themselves that the result is the same for the left part as well for the right one. a. Do you agree with your peers? If you do not agree, what would you suggest to them? b. Do you believe that their teacher would agree with them?
Q4a/b	A non-rectangular parallelogram ΑΒΓΔ is given. From the vertex A we draw a perpendicular line (α) to ΔΓ. Line (α) intersects line ΔΓ at point E. From vertex Γ we draw line (β) perpendicular to line AB. Line (β) intersects the side AB at point Z. a. Draw the figure. b. Prove that triangle ΑΔΕ is congruent to triangle ΓΒΖ.
Q5	In the figure below the following are valid: Line ζ passing through points A and B is the perpendicular bisector of the line segment ΓΔ. Prove that the triangles ΑΒΓ and ΑΒΔ are congruent.



Q6a/b	In the figure below triangles ΑΓΒ and ΕΒΔ have ΑΓ = ΕΒ, ΑΒ = ΕΔ and ΓΒ = ΒΔ. Points A, B and Δ lie on the same line (α). a. Prove that triangles ΑΓΒ and ΕΒΔ are equal. b. Prove that lines ΒΓ and ΕΔ are parallel.



proof appreciation. We outline typical proofs – namely, proofs that resonate with the textbook (Argyrakis et al., 2010), curriculum requirements and classroom activity norms that the students had experienced up to the time of participating in the study – for Q2–Q6 in the following section (these are the questions we sample student responses to in this paper). We note that, within these norms, the presence of "prove" in the

algebraic context of Q1–Q3 may trigger algebraic manipulations or application of previously established theorems or properties – or, as in the case of Q2b, straightforward calculations. For example, Q2b can be seen as a computational task (and some students did exactly that). But it can also be seen – and this is what we mainly intended to discern in the students' scripts – as being about applying a previously established theorem, namely, the converse of the claim proved in Q2a.

We designed Q1, Q2, Q4–Q6 as relatively demanding for students at this early stage of their encounter with proving tasks on purpose, to: mitigate against "yield[ing] an overoptimistic image of the students' proof fluency" (Kanellos et al., 2018, p. 281); and, "to give tasks that, although familiar to the students, somewhat challenged them and invited them to deploy their emerging proving skills, while also engaging with substantial manipulation of mathematical notation" (ibid.). The students were allowed two successive lesson periods (approximately 100 min) for producing their responses.

To analyse the student scripts, we constructed a spreadsheet with rows corresponding to participants and columns corresponding to questionnaire items. In each cell, we recorded which proof scheme (sole and/or combined) we saw as evidenced in the student response to each item. A sample of the responses was analysed by each author individually and discussed until consensus was reached. The entirety of the spreadsheet is in Kanellos (2014) and an excerpt is in Kanellos et al. (2018, p. 283).

Specifically, we aimed to establish consistency and validity in our application of the proof-scheme labelling as follows. We applied repeated cycles of triangulation in our analysis amongst the three members of our team: first, we worked on the same subset of student scripts but independently (each team member produced their proof-scheme labelling, and justification for said labelling, for each script in the subset). We then convened and discussed each team member's labelling and justifications. We did this (work independently on a subset of the data, convene to negotiate individual labelling and justifications) until there were negligible differences amongst us and the team agreed that we had built sufficient intersubjectivity in our labelling. We then shared the remaining scripts and carried out the labelling (while keeping a record of justifications for each label). The second author collated the labels produced by all team members in the aforementioned spreadsheet. To secure that individual interpretations had not slipped away from the intersubjectively agreed meanings for each label, we carried out random checks of each other's labelling and justifications throughout the process.

In Kanellos et al. (2019), we sampled our use of the original Proof Schemes and the Proof-Schemes Combined taxonomies in the data analysis for one of the items, Q6. Table 4 aggregates the student responses to Q6a and Q6b reported in (ibid.). Q6b emerged as the most challenging question for the students with just over a half not

Table 4. Proof schemes and Proof-Schemes combined occurrences in the 85 student scripts for Q6a and Q6b (Kanellos et al., 2019, p. 432).

Proof scheme	Q6a	Q6b
D.T.	41	4
D.T.-EP.	1	1
D.T.-EC.NRS.	9	11
E.P.-EC.NRS.	4	8
EC.NRS.	4	18
N.S.	26	43
Total	85	85

answering it (43 N.S., No Solution, responses). Yet, we identified evidence of deductive reasoning, even in responses to this most challenging item: evidence of nascent deductive proof schemes accounts for 67 of the 170 student responses (sum of the D.T., D.T.-EP. and D.T.-EC.NRS. entries in Table 4).

We now present evidence of nascent deductive proof schemes from the students' responses to all ten items across Q1–Q6 (Table 3).

5. Evidence of nascent deductive proof schemes in the 85 student scripts

In 5.1–5.5, we sample evidence of the five nascent deductive proof schemes (D.T. in Table 1, and the four with D.T. elements in Table 2) in the 850 student responses (85 students × 10 items across the six questions, Q1–Q6, in Table 3).

P[n] denotes "participant number n". Across Figures 1–5, where necessary, the student's writing has been translated from Greek to English.

5.1 D.T.-E.I. nascent deductive proof schemes in student responses

A typical proof for Q1 (Table 3) involves the expansion of the two brackets, simplification and then substitution of $(a^2 + b^2)$ with the given value 5^2.

$$\left(a\sqrt{3} + b\sqrt{2}\right)^2 + \left(a\sqrt{2} - b\sqrt{3}\right)^2 =$$

$$3a^2 + 2ab\sqrt{6} + 2b^2 + 2a^2 - 2ab\sqrt{6} + 3b^2 =$$

$$5a^2 + 5b^2 =$$

$$5(a^2 + b^2) =$$

$$5 \cdot 5^2 = 125$$

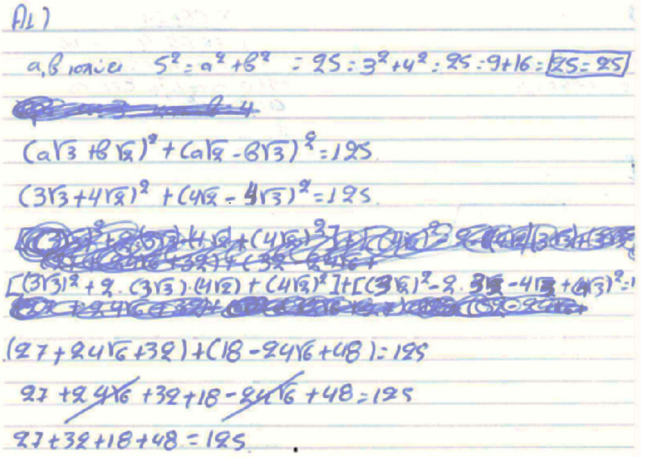


Figure 1. P[07]'s response to Q1.

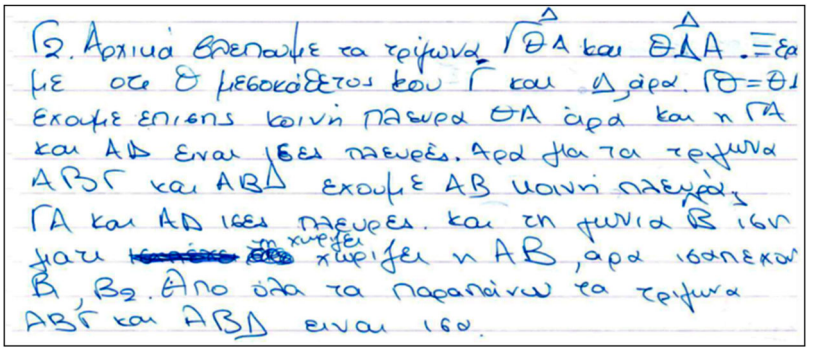


Figure 2. P[11]'s response to Q5. "We see at first the triangles ΓΘΑ and ΘΔΑ. We know that Θ is the perpendicular bisector of Γ and Δ [*sic*]. Thus, ΓΘ = ΘΔ. We have also common side ΘΑ thus also ΓΑ and ΑΔ are equal sides. Thus, for the triangles ΑΒΓ and ΑΒΔ we have ΑΒ common side, ΓΑ and ΑΔ equal sides and the angle Β equal because it is divided by ΑΒ, thus they are equidistant B_1 and B_2. From all the above the triangles ΑΒΓ and ΑΒΔ are equal."

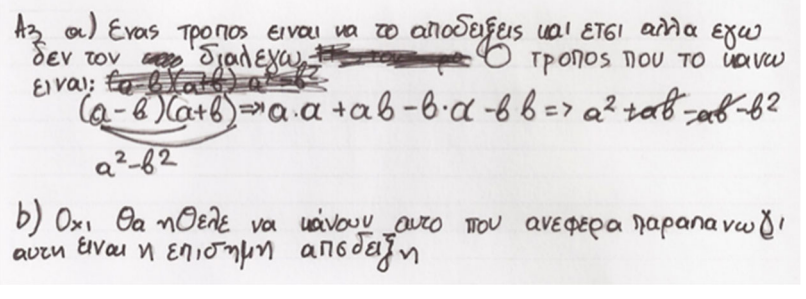


Figure 3. P[65]'s response to Q3a,b: a) This is one way to prove it but I would have not chosen this way. The way I do it is [algebraic manipulations follow]. b) No [the teacher] would want [the students] to do what I said above beca[use] this is the official proof.

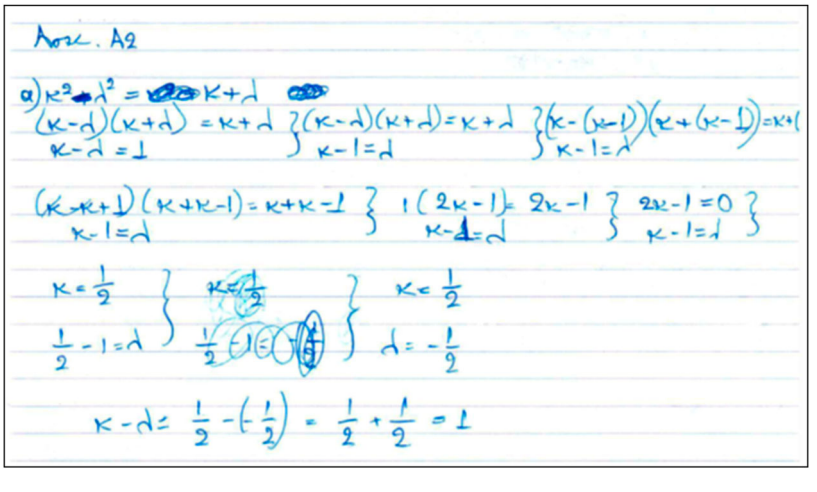


Figure 4. P[64]'s response to Q2a.

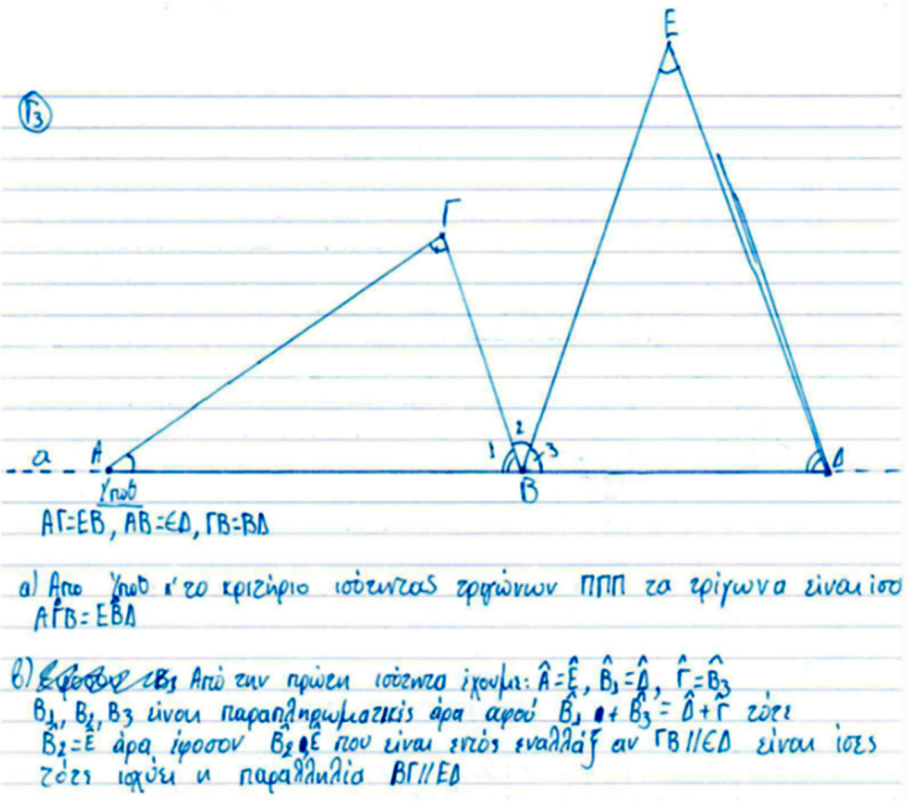


Figure 5. P[37]'s response to Q6. a) "From the givens and from the SSS triangle congruence criterion the triangles are congruent, AΓB = EBΔ". b) "From the first part [of the question] we have angles ∡A = ∡E, ∡B_1 = ∡Δ and ∡Γ = ∡B_3. B_1, B_2, B_3 are supplementary angles therefore since ∡B_1+∡B_3 = ∡Δ+∡Γ then ∡B_2 = ∡E therefore since ∡B_2 and ∡E which are alternate angles if ΓB//EΔ are equal then the parallelism of BΓ//EΔ is valid". [we have kept the sentence structure and absence of punctuation in the student script].

P[07] (Figure 1) sets out from calculating a value for a and a value for b, substitutes these values in $\left(a\sqrt{3}+b\sqrt{2}\right)^2+\left(a\sqrt{2}-b\sqrt{3}\right)^2$ and concludes with the result 125. We detect in P[07]'s response evidence of awareness that, to prove the equality in Q1, one needs to execute the calculations on the left-hand side of the equals sign in order to reach the result on the right-hand side. In recognition of the evidence of operational thought and logical inference (see D.T. proof scheme definition in Section 3), we labelled this response D.T. P[07]'s claim that a must be 3 and b must be 4 may originate in that $a^2+b^2=25$ conjures the Pythagorean triad (3, 4, 5), where indeed $3^2+4^2=5^2$. We note that we saw this claim, that a must be 3 and b must be 4, in 10 student scripts (with three of these also explicitly writing "because of the Pythagorean triad", see an example of this on p. 286 in Kanellos et al. (2018)). For this association with specific values for a and b, we labelled this response E.I., and we concluded with an overall labelling of D.T.-E.I.

5.2 D.T.-E.P. nascent deductive proof schemes in student responses

Typical proofs for Q5 (Table 3) draw on any of the three standard triangle congruency criteria that these Year 9 students are familiar with (three sides, SSS; one side and two angles, SAA; two sides and the angle between these sides, SAS). Another proof can start from noticing that line (ζ) is an axis of symmetry (even though, as the teacher noted to us, students are accustomed to seeing axes of symmetry drawn vertically, not horizontally).

P[11] (Figure 2) makes a remarkable attempt to establish the validity of the three claims that will allow drawing the conclusion that triangles ABΓ and ABΔ are congruent through, most probably, a version of the SAS criterion. The student first establishes two of these three claims: AB = AB, a common side; and, ΓΘ = ΘΔ, because line (ζ) is a perpendicular bisector of line (ε). We note that the precise implementation of the SAS criterion requires establishing the claim that angles ΓAB and BAΔ are equal. This is where coherence of the student's argument is compromised. The student opts for establishing the claim that angles ΓBA and ΔBA are equal instead (as the student has not shared the drawing they are working from, we speculate that they refer to angles ΓBA and ΔBA when they write "B_1 and B_2" and "the angle B equal"). Nonetheless, for the attempt to ground their argument in one of the congruency criteria, and for the overall flow of their writing that brims with proof-related wording and sentence structure, we labelled this response D.T. We speculate that the student sees angles ΓBA and ΔBA as equal because they seem so in the figure accompanying Q5 (Table 3). For this reason, we labelled this response also E.P. and concluded with the overall labelling D.T.-E.P.

5.3 D.T.-EC.A. nascent deductive proof schemes in student responses

Inspired by a task in (Healy & Hoyles, 2000), Q3 invites students to take a stance towards their peers' trial of numerical values as a proof for $(a+b)(a-b)=a^2-b^2$, and share their views on how their teacher would evaluate these trials. This task gave ample opportunity to us to explore the interplay of deductive and authoritarian proof schemes in the student responses, especially in terms of what they value as proof themselves (3a) and what their teacher may prioritise (3b). For instance, in the student response, we sample in (Kanellos et al., 2018, p. 287) – see also Figure 3 – P[65] writes for Q3a that trying specific values for a and b is "one way" to prove that $(a+b)(b-b)=a^2-b^2$. This led us to labelling this response E.I. However, P[65] stresses that he "would not have chosen this way" and opts for an algebraic proof instead. This led us to label the response also as evidencing D.T. proof schemes and P[65]'s response to Q3a was overall labelled D.T. – E.I.

In his response to Q3b, however, as we wrote in (Kanellos et al., 2018), P[65] writes that

> ""the teacher would want [the students] to do what I said above" (namely the deductive proof) because "this is the official proof". In this second part of the question, the student's warrant for endorsing the deductive proof is its authoritative status ("official") endorsed also by the teacher. Given the phrasing of the question, P[65]'s reference to the teacher is not unexpected; we see, however, the characterisation of deductive proof as acceptable because it is "official" as even stronger evidence of allusion to authority." (p. 287)

The co-habitation in P[65]'s response to Q3b of appreciation for deductive proof with this allusion to teacher authority led to labelling this response D.T. – EC.A.

5.4 D.T.-EC.NRS. nascent deductive proof schemes in student responses

A typical proof for Q2a (Table 3) involves expressing that "the difference of the squares of two natural numbers κ and λ ($\kappa > \lambda$) is equal to the sum of these numbers" as $\kappa^2 - \lambda^2 = \kappa + \lambda$ and concluding from there that the difference between κ and λ is 1:

$$\kappa^2 - \lambda^2 = \kappa + \lambda \Leftrightarrow (\kappa - \lambda)(\kappa + \lambda) = \kappa + \lambda$$

Given that $\kappa + \lambda > 0$, simplification leads to the conclusion that $\kappa - \lambda = 1$.

P[64]'s response (Figure 4) presents an incorrect, circular argument: the student constructs a system of two simultaneous equations. For the first equation, the student correctly constructs $\kappa^2 - \lambda^2 = \kappa + \lambda$ out of the formulation of Q2. For the second equation, the student uses $\kappa - \lambda = 1$, the relationship that the question asks to be proved. From their second equation, the student deduces that $\kappa - 1 = \lambda$ and substitutes λ in their first equation. As they remove brackets in the first equation, a mistake occurs when the equation $1(2\kappa - 1) = 2\kappa - 1$ becomes $2\kappa - 1 = 0$. This leads to the incorrect conclusion that $\kappa = \frac{1}{2}$. That $\lambda = -\frac{1}{2}$ follows also from this incorrect conclusion. Returning to the request in Q2a, the student then duly notes that the difference of κ and λ is 1 (as $\frac{1}{2} - \left(-\frac{1}{2}\right) = 1$). For the circular argument, for the computational errors and for not factoring in that κ and λ are meant as natural numbers, we saw a partial loss of coherent reference to the givens of the question that led us to labelling this response EC.NRS. Yet, we also detect in the student's response evidence of awareness that, to prove that the difference of κ and λ is 1, the given information in the question (correctly translated into the first of the two simultaneous equations) needs to be drawn on and, deductively – through a sequence of logical steps – lead to the requested conclusion. In recognition of this evidence, we labelled this response also D.T. and concluded with the overall labelling D.T. – EC.NRS.

5.5 D.T. nascent deductive proof schemes in student responses

A typical proof for Q6a (Table 3) involves deploying the SSS triangle congruence criterion (AΓ = BE, AB = EΔ, BΓ = BΔ). This is exactly what P[37] noticed (Figure 5). We labelled this response D.T.

To prove that lines BΓ and EΔ are parallel, it suffices to notice that, because of the triangle congruence proved in Q6a (AΓB = EBΔ), the respective angles ABΓ and EΔB are equal. But these angles are corresponding angles constructed by lines (α), BΓ and ΔE. Through the properties of parallelism – that these Year 9 students are in theory familiar with – this implies that lines BΓ and ΔE are parallel. We note however that the students are far more familiar with the reverse of this claim (if two lines are parallel then corresponding angles are equal). Both directions of the claim are correct but, on this occasion, the students are expected to engage with the one that they have seen far less in lessons and homework. The contrast between the student responses being labelled D.T. in the two parts of Q6 (Table 5) is telling: 41 in Q6a versus 4 in Q6b. P[37]'s response (Figure 5) is one of a handful of responses to Q6b labelled D.T. (convoluted grammatical structure of their sentence notwithstanding).

Table 5. Aggregate nascent deductive proof scheme evidence in the 85 student scripts.

Proof Scheme	Q1	Q2a	Q2b	Q3a	Q3b	Q4a	Q4b	Q5	Q6a	Q6b	Total
D.T.	12	7	27	47	41	52	12	29	41	4	272
D.T.-E.I.	10	1	0	6	3	0	0	0	0	0	20
D.T.-E.P.	0	0	0	0	0	0	6	2	1	1	10
D.T.-EC.A.	0	0	0	1	3	0	0	0	0	0	4
D.T.-EC.NRS.	11	5	6	1	0	4	16	14	9	11	77
Total (nascent)	33	13	33	55	47	56	34	45	51	16	383
Other	26	23	11	8	9	18	31	19	8	26	179
N.S.	26	49	41	22	29	11	20	21	26	43	288
Total	85	85	85	85	85	85	85	85	85	85	850

5.6 Nascent deductive proof schemes in student responses across Q1–Q6

In Table 5, we present the aggregate nascent deductive proof scheme occurrences in the student responses to the ten items of the proof questionnaire in Table 3.

We interpret overall the evidence in Table 5 as documenting significant successes in these students' attempts at proving in algebra and geometry at the end of a school year (Year 9) during which their teacher had orchestrated classroom activities aiming to ease the students' first encounters with proof (which is to become a more substantial part of their mathematical experiences from Year 10 onwards). We identified evidence of the five proof schemes clustered under the label nascent deductive proof schemes in 383 of the 850 student responses.

We acknowledge that this evidence should stand side by side with the 288 N.S. (No Solution) items – with particularly acute N.S. occurrences in Q2a, Q2b and Q6b, questions that were deliberately intended as more challenging, as we explain in 5.4 and 5.5. In considering the significance of the N.S. responses, we stress the non-mandatory nature of student participation (the teacher reassured the students that their responses would be anonymised and that the assessment of their responses would not count towards their grades). Overall, however, we see the evidence in Table 5 as supportive of the case that students may be ready for a first, imperfect yet potent, encounter with proof as early as Year 9 (age 14–15 in the educational context of our study). We conclude with summing up our case, grounded in the Proof Schemes and Proof-Schemes Combined analysis of the students' written work, and with suggesting how this analysis – deployed as a pedagogical tool and as a tool for mathematics teachers' professional development – can facilitate teachers' orchestration of this encounter.

6. Evidence of nascent deductive proof schemes in student work as evidence of proof readiness

Freudenthal's (1978) observation that

> "[if] adult mathematician[s are] entitled to […] to practise mathematics […] as an activity, to explore fields, to make mistakes and to learn from [their] mistakes, then the same privileges should be granted the learners from early childhood onwards." (p. 44)

is at the heart of what we see as the theoretical and pedagogical potency of the nascent deductive proof schemes construct. Also at the heart of our enterprise is to explore how to orchestrate young people's experience of mathematics in ways that stay close to the "many ingredients from the exquisitely structured system called mathematics"

(Sfard, 2000, p. 186). We see proof as central to these ingredients and, much like Sfard (2000), we see fear of the challenges involved in engagement with proof as a deterrent for curricular and pedagogical choices which can enable this engagement to flourish. The deficit narratives about young people's proving competence that have sometimes proliferated in the literature (as evidenced in esteemed works such as Hanna and De Villiers' (2012) edited volume on the 19th ICMI Study) have this unfortunate repercussion. We offer our analyses – enabled by the Proof Schemes (Harel & Sowder, 1998, 2007) and Proof-Schemes Combined (Kanellos et al., 2018) tools – not as contrarian to well-established prior findings that report and investigate the challenges that learners, and their teachers, face in the course of first encounters with proof and proving, but as a contribution to a body of work (see Section 1) in our field that offers an alternative, proactive and upbeat vision on what may be feasible in the secondary mathematics classroom.

We see the findings from our Proof-Schemes Combined analysis as evidencing the validity of a non-binary and nuanced narrative about adolescents' emerging proof competence. As we started noticing in Kanellos et al. (2018), this analysis has the capacity to "portray emerging elements of deductive proof behaviour […] in the midst of empirical and external conviction elements which are still clearly present in students' proving activity" (p. 292). It is this analysis that is the focus of this paper. Pedagogically, we see also the potency of Proof-Schemes Combined as a diagnostic tool for teachers "towards characterising their students' multi-faceted and evolving proving activity and thus inform their decision-making" (ibid.), especially in noticing the signals of where this "change of classroom culture" (Nardi & Knuth, 2017, p. 268) may be "on the cusp of having the capacity to occur" (Kanellos et al., 2018, p. 293).

By locating and quantifying in this paper the nascent deductive proof scheme evidence in the students' work across our dataset (samples in 5.1–5.5 and Table 5), we have elevated our speculative observation in Kanellos et al. (2018) that approximately "45% of the students gave a response that was characterised as D.T. or D.T. combined with other proof schemes" (p. 292) – notably 67.5% if we disregard N.S. responses – into a data-grounded claim.

We recognise that the grounds of our claim are 850 single-point data entries: students' written responses to six questions (ten items) collected at the end of the school year. In this paper, we draw on this dataset only – and not on other evidence of classroom activity, other questionnaires, and teacher testimonials that our collaboration with the teacher allowed us to become privy to. We have also opted for a whole-cohort analysis (the aggregate results we present in Table 5) as opposed to, say, building individual proof-scheme profiles per student through an examination of their responses across Q1–Q6. We see all these alternatives as valid ways forward that have the capacity to strengthen the validity of the claim we present here further as well as inform future research. We nonetheless see the aim that we set out to achieve in this paper – identify evidence of students' emerging deductive mathematical reasoning within the challenging modality of writing – as achievable through the analysis we present here.

Back to our promise for a glimpse into a vision of what may be feasible in the secondary mathematics classroom, we return to the kinds of pedagogic actions that Watson (2010) presents as possible when endorsing a non-deficit narrative of adolescents' capacity for substantial and meaningful mathematical thinking and, in our case, proof. Watson (2010) emphatically describes these actions as "compatible with the difficult changes of adolescence" (p. 133) and as supportive of "the shifts of mathematical

attention and perception adolescents have to make" (p. 134). Earlier, we cited the abilities that Watson (2010, pp. 141–142) lists as within the reach of adolescents in "suitable circumstances" (ibid.). We see the scope and ambition of the pedagogic actions Watson (2010) lists – ranging from helping students cope with relational complexity to expressing generalisation and abstraction of experience, enabling expansion of ideas beyond physical limitations and much else – as resonating with, and promoting "the learning developments that go on in adolescence" (p. 142).

The subtle but decisive actions of the teacher with whom we collaborated in our study – and who gave us access to her by all means unexceptional, mixed ability four classes – are akin to Watson's (2010) wish-list and to non-deficit, proactive works in our field that favour the productive enactment in the classroom of disciplinary mathematical practices ("authentic mathematics" per Stylianides et al., 2022) such as proving. While the focus in this paper is not to elaborate these pedagogical actions, we witnessed the teacher's enculturation moves in her lessons as including an emphasis on appreciating proof from as early on as possible and as a *sine qua non* for appreciating mathematics. She encouraged purposeful and multi-faceted (verbal, iconic, graphical, symbolic) engagement with tasks; she stressed connections with prior activities and learned topics; she fostered the skill to notice salient features in the tasks; she pursued and negotiated justifications for claims made by her or the students. And – even though our focus in this paper is also not on detecting how and where the curriculum, and associated resources such as textbooks, may support an appreciation for proof – we feel it is necessary to underline the presence of proof activity in the Greek secondary education curriculum and assessment (Kanellos, 2014). We hope that the evidence of young people's readiness for proof we present here will trigger interest in collecting and analysing analogous evidence in other educational contexts so that robust generalisations become possible about what this evidence may imply for mathematics curricula and pedagogy at large.

In Kanellos et al. (2019), where we hinted at the proposition we put forward in this paper (drawing on student responses to Q6 only), we pondered that, when a toddler utters their first sentences – however grammatically or syntactically imperfect these utterances may be in their first years (Dehaene, 2011) – we marvel, and aptly so; and, we orchestrate efforts to convert the toddler's utterances into productive and rewarding use of spoken or written communication. Shouldn't we perhaps muster analogous enthusiasm – and corresponding research and pedagogical action – for adolescents' first proof utterances? We see the data and analyses we present in this paper as a modest, yet non-negligible, contribution to the proposition that this undertaking may be far from Don-Quixotic, and, in fact, quite feasible.

Acknowledgements

We thank the Year 9 students and the teacher, the Mathematics Department and the School Principal who welcomed us and allowed the collection of the data that we sample from in this paper.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Statements and declarations

The authors have no competing interests to declare that are relevant to the content of this article.

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