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Planar median graphs and cubesquare-graphs

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ABSTRACT

Median graphs are connected graphs in which for all three vertices there is a unique vertex that belongs to shortest paths between each pair of these three vertices. In this paper we provide several novel characterizations of *planar* median graphs. More specifically, we characterize when a planar graph G is a median graph in terms of forbidden subgraphs and the structure of isometric cycles in G, and also in terms of subgraphs of G that are contained inside and outside of 4-cycles with respect to an arbitrary planar embedding of G. These results lead us to a new characterization of planar median graphs in terms of *cubesquare-graphs* that is, graphs that can be obtained by starting with cubes and square-graphs, and iteratively replacing 4-cycle boundaries (relative to some embedding) by cubes or square-graphs. As a corollary we also show that a graph is planar median if and only if it can be obtained from cubes and square-graphs by a sequence of "square-boundary" amalgamations. These considerations also lead to an $\mathcal{O}(n \log n)$-time recognition algorithm to compute a decomposition of a planar median graph with n vertices into cubes and square-graphs.

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1. Introduction

A median graph is a connected graph, in which, for each triple of vertices there exists a unique vertex, called the *median*, simultaneously lying on shortest paths between each pair of the triple [35]. While the term *median graph* was introduced by Nebeský [36] in 1971, they have been studied at least since the 1940s [1,9]. Today, a great deal is known about median graphs including several characterizations, see e.g. [4,32]. Median graphs naturally arise in several fields of mathematics, for example, in algebra [6], metric graph theory [4] and geometry [14], and they have practical applications

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in areas such as social choice theory [3,20], phylogenetics (where Buneman graphs are of relevance; see [23]), and forensic science [37]. It is therefore natural to develop approaches to better understand structural properties of median graphs, as well as their subclasses.

Special classes of median graphs include, for example, *trees*, *square-graphs* (see Section 2 and e.g. [5]) and *cube-free* median graphs (e.g. [10,17,18]). Interestingly, to date the class of *planar* median graphs has received relatively little attention, although it is natural to consider such graphs from both a mathematical and an application oriented perspective (see e.g. [11]). Indeed, in contrast to the plethora of characterizations available for median graphs, so far only one direct characterization of planar median graphs has been established by Peterin in [38]. In addition, the only other results concerning planar median graphs that we are aware of are an Euler-type formula for planar, cube-free median graphs [31, Corollary 5], and an algorithm for deciding in $\mathcal{O}(|V|+|E|)$ time whether or not a graph $G=(V,E)$ is a planar median graph [29, Cor. 3.4].

Before proceeding with stating our results, it is informative to briefly recall Peterin's characterization for planar median graphs. Given a graph G and a connected subgraph G' of G, the *expansion* of G with respect to G' is the graph H obtained by attaching a disjoint copy G'' of G' to G by adding edges between corresponding vertices of G' and G''. Expansions play a key role in characterizing median graphs and their relatives [34]. More specifically, defining an expansion H to be *convex* if G' is a convex subgraph of G, a graph is a median graph if and only if it can be obtained from K_1 by a series of convex expansions [33,34]. Planar median graphs can be characterized by further restricting expansions. Call H a *face expansion* if there is a planar embedding of G such that all vertices of G' are incident with the same face of G. Then a graph is a planar median graph if and only if can be obtained from an edge by a sequence of convex face expansions [38].

In this paper, we shall characterize planar median graphs in an alternative way by considering amalgamations. This has the advantage of allowing us to decompose the graph into simpler building blocks. A graph G is said to be an *amalgam* of two induced subgraphs G_1 and G_2 if their union is G and their intersection $G_1 \cap G_2$ is non-empty [7]. Amalgamation procedures differ by requiring certain properties of G_1 and G_2 as subgraphs of G and constraints imposed on their intersection $G_1 \cap G_2$. For instance, every median graph, can be obtained by successive *convex* amalgamations starting with hypercubes [30,41], i.e., G_1 and G_2 are convex subgraphs of their amalgam G along $G_1 \cap G_2$. Note that amalgamations and expansions are closely related for median graphs (see e.g. [35, Theorem 7]). Similar amalgamation results have been proven for quasi-median graphs [8, Theorem 1] and pseudo-median graphs [8, Theorem 18] (in terms of "gated" amalgamations). In this paper, we shall show that planar median graphs can be obtained by starting with cubes and square-graphs and iteratively amalgamating along the boundary of certain faces in some planar embedding of the resulting graphs. This gives new insights into the fundamental properties of planar median graphs as the structure of the basic building blocks (square-graphs and cubes) is very well-understood [5,26]. As we shall discuss below, this approach is related to the 2-face expansions that are used in [21,39] to characterize planar partial cubes.

The rest of this paper is organized as follows. After introducing the necessary notation and reviewing some relevant results from the literature in Section 2, in Section 3 we present two characterizations of median graphs amongst planar graphs. The first one (Theorem 3.5) is given in terms of forbidden subgraphs and isometric cycles of a planar graph; the second one is given by the condition that every 4-cycle or square C in an embedding of a planar graph must divide the graph into a planar median graphs that lie inside and outside of C (Theorem 3.9). The last results prompt us to introduce an operation $\circledast$ that glues graphs together at boundary squares. In particular, in Section 4, we introduce QS-graphs as those graphs that can be constructed from cubes and square-graphs by iterative application of the $\circledast$ operation. We then proceed to show that the QS-graphs are exactly the planar median graphs that are not trees (Theorem 4.9). As a corollary we then show that a graph is a planar median graph if and only if it can be obtained from cubes and square-graphs by a sequence of square-boundary amalgamations (Theorem 4.14). Section 5 is devoted to deriving an efficient algorithm for finding a sequence of $\circledast$ operations for decomposing a planar median graph into its basic pieces (i.e. cubes and square-graphs). In the last section we discuss some open problems and possible future directions.

2. Preliminaries

Graphs. We consider undirected graphs $G=(V,E)$ with finite vertex set $V(G)=V$ and edge set $E(G)=E\subseteq\binom{V}{2}$, i.e., without loops and multiple edges. If $E=\binom{V}{2}$, the graph G is *complete* and denoted by $K_{|V|}$. A *complete bipartite graph* $K_{m,n}=(V,E)$ is a graph whose vertex set V can be partitioned into two subsets V_1 and V_2 with $|V_1|=m$ and $|V_2|=n$ such that $\{v,w\}\in E$ if and only if $v\in V_i$ and $w\in V_j$ with $i\neq j$. We write $G'\subseteq G$ if G' is a subgraph of G and $G[W]$ for the subgraph in G that is induced by some subset $W\subseteq V$. The *graph union* $G_1\cup G_2$ (resp. *graph intersection* $G_1\cap G_2$) of two graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ is the graph $(V_1\cup V_2,\ E_1\cup E_2)$ (resp. $(V_1\cap V_2,\ E_1\cap E_2)$). The graph $G-X$ with $X\subseteq V$ is the graph obtained from G after removal of the vertices in X and its incident edges. A graph G is $\{G'_1,\dots,G'_m\}$-free if none of the graphs G'_i is a subgraph of G. For simplicity, we write that G is G'-free instead of $\{G'\}$-free.

A *shortest* path between v and w in G is denoted by $P^\star_G(v,w)$. The length $d_G(v,w)$ of a shortest path between two vertices v and w is called *distance of v and w (w.r.t. G)*. A subgraph G' of G is *isometric* if $d_{G'}(v,w)=d_G(v,w)$ for all vertices $v,w\in V(G')$, and $G'\subseteq G$ is *convex* if for any two vertices $v,w\in V(G')$ *every* shortest path $P^\star{}_G(v,w)$ between v and w is a subgraph of G'. Clearly, every convex subgraph of G is an isometric and induced subgraph of G. A graph G is *k-connected* (for $k\in\mathbb{N}$) if $|V(G)|>k$ and $G-X$ is connected for every set $X\subseteq V$ with $|X|<k$.

A *cycle* is a connected graph in which every vertex has degree two. The length of a cycle C is the number of edges or equivalently, the number of vertices in C. A cycle C_n of length $n \geq 3$ is called an *n-cycle*. A 4-cycle is also called a *square*. A graph that does not contain cycles is *acyclic* and, otherwise, *cyclic*. A *cogwheel* M_n consists of a cycle C_n where $n \geq 8$ is even and a "central" vertex that is adjacent to every second vertex of this cycle. A *suspended cogwheel* M_n^* is obtained from the cogwheel M_n by adding an additional vertex adjacent to the central vertex of M_n.

A connected acyclic graph $T = (V, E)$ is a *tree*. A tree is *rooted* if there is a distinguished vertex $\rho \in V$ called the *root of* T. A *(rooted) forest* is a graph whose connected components are (rooted) trees. For a rooted forest T, we say that vertex v of T, is *at level i* if the distance from the root of the connected component in T that contains v to vertex v is precisely i. Hence, all roots of the connected components of T are at level 0.

The *Cartesian product* $G_1 \Box G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph with vertex set $V(G_1 \Box G_2) = V_1 \times V_2$, and where $\{(u, u'), (v, v')\} \in E(G_1 \Box G_2)$ precisely if either $u = v$ and $\{u', v'\} \in E_2$ or $u' = v'$ and $\{u, v\} \in E_1$. The Cartesian product is associative and commutative [25], which allows us to write $\Box_{i=1}^n G_i$ for the Cartesian product of the graphs $G_1, \dots, G_n$. An n-dimensional *hypercube* Q_n is the n-fold Cartesian product $\Box_{i=1}^n K_2$. A Q_3 is called *cube*. The subgraph ${Q^-}_3$ of a cube Q_3 is obtained from this Q_3 by removing one vertex and its incident edges. A graph G is C_6^{iso}-Q_3-inferring if for each isometric C_6 in G there is a cube $Q_3 \subseteq G$ such that $C_6 \subseteq Q_3$. Analogously, a graph G is Q_3^--Q_3-inferring if for each ${Q^-}_3$ in G there is a cube $Q_3 \subseteq G$ such that ${Q^-}_3 \subseteq Q_3$.

We now provide here a simple result for later reference.

Lemma 2.1. *Let G be a K_3-free graph. Then, every 4-cycle in G is convex if and only if G is $K_{2,3}$-free.*

Proof. Let $G = (V, E)$ be a K_3-free graph. By contraposition, assume that there is a 4-cycle $C \subseteq G$ which is not convex. Thus, there are two vertices $v, w \in V(C)$ for which there is a shortest path $P_G^\star(v, w)$ that is not contained in C. Since $E \subseteq \binom{V}{2}$ and G is K_3-free, $d_G(v, w) = 2$. Consequently, the graph union $C \cup P_G^\star(v, w)$ forms a subgraph of G that is isomorphic to a $K_{2,3}$.

Conversely, assume that G is not $K_{2,3}$-free. Then, there is a square $C \subseteq K_{2,3} \subseteq G$ which is not convex. ■

Convex hull and shortest-path-extension (SPE). For a subgraph G' of G, the *convex hull* $\mathcal{H}(G')$ *of* G' *(w.r.t. G)* is the intersection of all convex subgraphs G'' of G with $G' \subseteq G''$. Note that $\mathcal{H}(G')$ is a convex subgraph of G and that $\mathcal{H}(G') = G'$ for every convex subgraph G' of G. A tool that will be useful in upcoming proof are shortest-path-extensions.

Definition 2.2. Let G' be some subgraph of G. A *shortest-path-extension (SPE) of* G' *(w.r.t. G)* is obtained by the following procedure:

1. Set ${G'}_1 := G'$, and set $i = 1$,
2. If G'_i is a convex subgraph of G, then we stop. Otherwise, there is a shortest path $P_G^\star(v, w)$ with $v, w \in V({G'}_i)$, which is not a subgraph of G'_i. In this case, we set $G'_{i+1} := G'_i \cup P_G^\star(v, w)$, increment i and repeat Step 2.

Since we have $G' = {G'}_1 \subsetneq {G'}_2 \subsetneq {G'}_3 \subsetneq \ldots \subseteq G$, and since G is finite and convex (w.r.t. G), a shortest-path-extension of G' must terminate. We call the final sequence $\mathcal{S}(G') = ({G'}_1, {G'}_2, {G'}_3, \dots, G'_m)$, with $m \geq 1$, a *SPE-sequence of* G' and the last graph G'_m in $\mathcal{S}(G')$ the *SPE-graph of* G'.

As shown next, the convex hull can be constructed by means of SPE-sequences so that, in particular, the SPE-graph is well-defined.

Lemma 2.3. *Let G be a graph and let G' be a subgraph of G. Then, the convex hull of G' w.r.t. G is equal to the SPE-graph of G' w.r.t. G, and thus, the SPE-graph is unique.*

Proof. Let $G' \subseteq G$, $\mathcal{H}(G')$ be the convex hull of G' (w.r.t. G) and $({G'}_1, {G'}_2, {G'}_3, \dots, G'_m)$ be an SPE-sequence of G' (w.r.t. G). Furthermore, let H' be some convex subgraph of G such that $G' \subseteq H'$. Note, such subgraph H' exists, since G is convex (w.r.t. G) and therefore, we may set $H' := G$. We use induction on $i \in \{1, \dots, m\}$ to show that every G'_i is a subgraph of H', and hence $G'_m \subseteq H'$. For the base case, Definition 2.2 (1) implies ${G'}_1 = G' \subseteq H'$.

Now, let us assume that $G'_k \subseteq H'$ for every $k \in \{1, \dots, i\}$, $1 \leq k < m$, and consider the graph G'_{i+1}. By Definition 2.2 (2), $G'_i \subseteq G'_{i+1}$. Since $G'_i \subseteq H'$ it remains to show that all $v \in V(G'_{i+1}) \setminus V(G'_i)$ and $e \in E(G'_{i+1}) \setminus E(G'_i)$ are also contained in H'. By definition, $G'_{i+1} = G'_i \cup P_G^\star(w, w')$ for some $w, w' \in V({G'}_i)$ where $P_G^\star(w, w')$ is a shortest path which is not a subgraph of G'_i. Let $v \in V(G'_{i+1}) \setminus V(G'_i)$ and $e \in E(G'_{i+1}) \setminus E(G'_i)$. By construction, v and e must be contained in $P_G^\star(w, w')$. Since H' is convex (w.r.t. G) and $w, w' \in V(G'_i) \subseteq V(H')$, this shortest path $P_G^\star(w, w')$ must be a subgraph of H'. Therefore, $v \in V(H')$ and $e \in E(H')$. Consequently, $V(G'_{i+1}) \subseteq V(H')$ and $E(G'_{i+1}) \subseteq E(H')$, and thus, $G'_{i+1} \subseteq H'$. By induction, we have $G'_m \subseteq H'$.

Finally, since H' was chosen arbitrarily and $\mathcal{H}(G')$ is a convex subgraph of G, we may set $H' := \mathcal{H}(G')$ and conclude that $G'_m \subseteq \mathcal{H}(G')$. By definition, the SPE-graph G'_m is a convex subgraph of G and $G' \subseteq G'_m$. Therefore, we have, by definition of the convex hull, $\mathcal{H}(G') \subseteq \mathcal{H}(G'_m) = G'_m$. In summary, $\mathcal{H}(G') = G'_m$. Since the convex hull by definition is unique, G'_m is unique as well. ■

Planar graphs, faces and boundaries. A *planar* graph G can be embedded in the plane such that its edges intersect only at their endpoints (in particular, only in case they are incident with the same endpoints). Such embeddings are called *planar embeddings* of G. A planar graph G together with a planar embedding π of G is called π*-embedded*.

Let G be a π-embedded planar graph. The connected regions in $\mathbb{R}^2$ of the complement of G are called *faces*. One of these faces is unbounded in $\mathbb{R}^2$ and is called the *outer* face, while all other faces are bounded in $\mathbb{R}^2$. These are called *inner* faces. The subgraph of G that encloses a face F is said to *bound* F and is called the *boundary* of F. If $G' \subseteq G$ bounds an inner (resp., outer) face it is called *inner* (resp., *outer*) *boundary* of G. Note, by definition, boundaries are not part of a face. However, a face is said to be *incident* with the vertices and edges of its boundary. Correspondingly, the vertices of G that are incident with the outer face are called *outer* vertices, and every other vertex, i.e., every vertex that is not incident to the outer face is called *inner* vertices. The set $\mathring{V}(G)$ denotes the set of inner vertices of G. Note that outer vertices can be incident to inner faces. If G has different faces with the same boundary, then G must be a cycle [22, Lemma 4.2.5]. Note, this is the only case where the inner and outer boundary coincide. In all other cases, different faces of a planar embedded graph, have different boundaries. A planar graph G is *outer-planar* if G can be π-embedded in such a way that all vertices of G are outer vertices (i.e., $\mathring{V}(G) = \emptyset$) [12]. In particular, cycles, trees and $K_2\Box P_n$ are outer-planar graphs.

Every planar graph has, in particular, an embedding on a 2-sphere $\mathbb{S}^2$. This observation immediately implies that every bounded region can be chosen as the outer face, see e.g. [22, Sec. 4.3] for more information. We summarize the latter in

Observation 2.4. *Let G be a π-embedded planar graph and $G' \subseteq G$ be a boundary of G. Then, there is a planar embedding of G such that G' is an inner boundary as well as a planar embedding of G such that G' is the outer boundary of G.*

It is well-known that an n-dimensional hypercube Q_n is planar if and only if $n \leq 3$, see e.g. [26]. Since every subgraph of a planar graph is planar as well, we obtain

Lemma 2.5. *For every hypercube $Q_n \subseteq G$ in a planar graph G it holds that $n \leq 3$.*

Two planar embeddings $\pi_1, \pi_2 : G \to \mathbb{S}^2$ are *equivalent* if there is a homeomorphism $h : \mathbb{S}^2 \to \mathbb{S}^2$ such that $h \circ \pi_1 = \pi_2$. We say that a graph G is *uniquely embeddable* on $\mathbb{S}^2$ (up to equivalence) if any two planar embeddings of G on $\mathbb{S}^2$ are equivalent.

Theorem 2.6 (*[42]*)**.** *Every 3-connected planar graph is uniquely embeddable on $\mathbb{S}^2$.*

In the following, every square of G that bounds a face for *some* planar embedding of G is called *square-boundary*. We will denote planar embeddings of cubes by ρ, see the graph G_d in Fig. 5 for such a ρ-embedded cube. Theorem 2.6 and the fact that cubes are 3-connected implies

Observation 2.7. *Let G be ρ-embedded cube. Then, all faces must be bounded by squares. In particular, $C \subseteq G$ is a square if and only if C is an inner or outer boundary in G w.r.t. ρ, and thus, if and only if C is a square-boundary of G.*

The following definitions are central for the presentation below. Let G be a planar graph and $G' \subseteq G$ be some subgraph of G. Fixing a planar embedding π of G and removing all edges and vertices from G that are not contained in G' yields a planar embedding of G' that is "anchored" on the planar embedding π of G. We call such an embedding of G' a (G, π)-*induced embedding*. Let C be a cycle of G and fix a planar embedding π of G. A vertex $v \in V(G)$ is *outside* (resp., *inside*) of C if v is contained in the outer (resp., inner) face bounded by C w.r.t. the (G, π)-induced embedding. By definition, $v \in V(C)$ is neither inside nor outside of this C. A vertex $v \in V(G)$ is *almost-outside* (resp., *almost-inside*) if either $v \in V(C)$ or v is outside (resp., inside) of C.

Below, we will make frequent use of the subgraphs ${G^{\mathrm{in}}}_{C,\pi}$ and ${G^{\mathrm{out}}}_{C,\pi}$ of G defined as follows. Let G be a π-embedded planar graph and $C \subseteq G$ be a square in G. Then, ${G^{\mathrm{in}}}_{C,\pi}$ (resp., ${G^{\mathrm{out}}}_{C,\pi}$) is the subgraph of G that is obtained by deleting every vertex and every edge of G that is located in the outer (resp. inner) face bounded by C w.r.t. the (G, π)-induced embedding of C. In particular, for K_3-free graphs G, the subgraph ${G^{\mathrm{in}}}_{C,\pi}$ (resp. ${G^{\mathrm{out}}}_{C,\pi}$) is induced by all vertices that are almost-inside (resp. almost-outside) of C. Note that the vertices of C are contained in both ${G^{\mathrm{in}}}_{C,\pi}$ and ${G^{\mathrm{out}}}_{C,\pi}$. Given the (G, π)-induced planar embedding of ${G^{\mathrm{in}}}_{C,\pi}$ and ${G^{\mathrm{out}}}_{C,\pi}$, the square C is the outer boundary of ${G^{\mathrm{in}}}_{C,\pi}$ and an inner boundary of ${G^{\mathrm{out}}}_{C,\pi}$.

The next result provides some insights on the location of vertices w.r.t. subgraphs C_4 of planar and $\{K_3, K_{2,3}\}$-free graphs, which we need for later reference.

Lemma 2.8. *Let G be a planar π-embedded and $\{K_3, K_{2,3}\}$-free graph, and $C, C' \subseteq G$ be squares. If $a, b \in V(C)$, then for every $c \in V(P^{\star}_G(a, b))$, we have $c \in V(C)$. Moreover, ${G^{\mathrm{in}}}_{C,\pi}$ and ${G^{\mathrm{out}}}_{C,\pi}$ as well as ${G^{\mathrm{in}}}_{C,\pi} \cap {G^{\mathrm{out}}}_{C',\pi}$ and ${G^{\mathrm{out}}}_{C,\pi} \cap {G^{\mathrm{out}}}_{C',\pi}$ and ${G^{\mathrm{in}}}_{C,\pi} \cap {G^{\mathrm{in}}}_{C',\pi}$ are convex subgraphs of G.*

Proof. Let G be a planar $K_{2,3}$-free graph, $C \subseteq G$ be a square and $a, b \in V(C)$. Hence, $d_G(a, b) \in \{0, 1, 2\}$. If $d_G(a, b) \in \{0, 1\}$, then there is nothing to show. Suppose that $d_G(a, b) = 2$. In this case, there are two vertices $c_1, c_2 \in V(C)$ on two shortest path between a and b. If there would be a third shortest path of length two, then G would contain a $K_{2,3}$, which is not possible. Hence, every vertex on the shortest paths between a and b must be part of C.

Now, let π be an arbitrary planar embedding of G and consider the (G,π)-induced embedding of a square C and let $a,b\in V(G)$ and $P^\star_G(a,b)$ be an arbitrary shortest path between a and b. Assume that a and b are almost-inside of C. Moreover, we assume for contradiction that there is some $c\in V(P^\star_G(a,b))$ that is outside of this C. Since G is planar, there must be two vertices $a',b'\in V(C)$, which are part of this $V(P^\star_G(a,b))$ such that $P^\star_G(a,b)=P^\star_G(a,a')\cup P^\star_G(a',b')\cup P^\star_G(b',b)$ and $c\in V(P^\star_G(a',b'))$, and with $P^\star_G(a,a')$, $P^\star_G(a',b')$ and $P^\star_G(b',b)$ being some shortest paths between a,a' and a',b' and b,b', respectively. Since $a',b'\in V(C)$ and $P^\star_G(a',b')$ is a shortest path with $c\in V(P^\star_G(a',b'))$, we conclude by analogous arguments as above that $c\in V(C)$. Hence, c is not outside of C; a contradiction. Thus, every $c\in V(P^\star_G(a,b))$ is almost-inside of C. Since G is K_3-free, every edge $\{v,w\}\in E(G)$ with $v,w\in V({G^{\mathrm{in}}}_{C,\pi})$ is also contained in $E({G^{\mathrm{in}}}_{C,\pi})$. Taken the last two arguments together, ${G^{\mathrm{in}}}_{C,\pi}$ is a convex subgraph of G.

Analogous arguments show that ${G^{\mathrm{out}}}_{C,\pi}$ is a convex subgraph of G. Since the intersection of convex subgraphs yields a convex subgraph (cf. e.g. [27, L. 5.2]), we conclude that ${G^{\mathrm{in}}}_{C,\pi}\cap {G^{\mathrm{out}}}_{C',\pi}$, ${G^{\mathrm{out}}}_{C,\pi}\cap {G^{\mathrm{out}}}_{C',\pi}$ and ${G^{\mathrm{in}}}_{C,\pi}\cap {G^{\mathrm{in}}}_{C',\pi}$ are convex subgraphs of G. ■

The following result is a direct consequence of Lemmas 2.3 and 2.8.

Lemma 2.9. *Let G be a planar π-embedded $\{K_3,K_{2,3}\}$-free graph which contains a square $C\subseteq G$. Moreover, let $G'\subseteq G$ be a subgraph and $\mathcal{H}(G')$ be its convex hull (w.r.t. G). If every $v\in V(G')$ is almost-inside (resp., almost-outside) of C, then every $v'\in V(\mathcal{H}(G'))$ is almost-inside (resp., almost-outside) of C, where inside and outside refer to the (G,π)-induced embedding of C.*

Note that shortest paths on isometric cycles $C'\subseteq G$ connecting its vertices must be shortest paths in the underlying graph G. By Lemma 2.8, the graphs ${G^{\mathrm{in}}}_{C,\pi}$ and ${G^{\mathrm{out}}}_{C,\pi}$ are convex subgraphs of planar π-embedded and $\{K_3,K_{2,3}\}$-free graphs. Thus, every isometric cycles of such a graph must be entirely contained in ${G^{\mathrm{in}}}_{C,\pi}$ or ${G^{\mathrm{out}}}_{C,\pi}$. We summarize the latter discussion in

Lemma 2.10. *Let G be a planar π-embedded and $\{K_3,K_{2,3}\}$-free graph, $C\subseteq G$ be a square and C' be an isometric cycle of G. Then, all vertices in $V(C')\setminus V(C)$ are either inside or outside of C w.r.t. the (G,π)-induced embedding of C.*

Lemma 2.11. *Let G be a planar graph that contains a cube Q_3 and let $u,v\in V(Q_3)$ such that $d_{Q_3}(v,w)=3$. Then, $Q_3-\{v,w\}$ results in a 6-cycle C and, for every planar embedding π of G, v is located in the inner face and w in the outer face of C or* vice versa *w.r.t. (G,π)-induced embedding of C.*

Proof. Let G be a planar graph that contains a cube Q_3 and let $v,w\in V(Q_3)$ such that $d_{Q_3}(v,w)=3$. One easily observes that $Q_3-\{v,w\}$ results in a 6-cycle C. If both v and w are inside (resp., outside) w.r.t. (G,π)-induced embedding of C, then C would be an outer (resp., inner) boundary of the cube Q_3. However, every boundary of a cube has to be a square (cf. Observation 2.7), and therefore, v must be located in the inner face and w in the outer face of C or *vice versa*. ■

Definition 2.12 (*k-FS*)**.** A graph G satisfies the *k-face-square-property (w.r.t. π)* (*k-FS*, for short) if there is a planar embedding π of G such that at least k faces are bounded by squares.

For instance, every square satisfies 2-FS and every cube satisfies 5-FS.

Median graphs and square-graphs. A vertex $x\in V(G)$ is a *median* of three vertices $u,v,w\in V(G)$ if $d_G(u,x)+d_G(x,v)=d_G(u,v)$, $d_G(v,x)+d_G(x,w)=d_G(v,w)$ and $d_G(u,x)+d_G(x,w)=d_G(u,w)$. A connected graph G is a *median graph* if every triple of its vertices has a unique median. In other words, G is a median graph if, for all $u,v,w\in V(G)$, there is a unique vertex that belongs to shortest paths between each pair of u, v and w. We denote the unique median of three vertices u, v and w in a median graph G by $\mathrm{med}_G(u,v,w)$.

For later reference, we summarize here some well-known properties of median graphs, see [32] and [33, p. 198].

Proposition 2.13. *A connected graph G is a median graph if and only if the convex hull of any isometric cycle of G is a hypercube.*

Proposition 2.14. *For every median graph $G=(V,E)$ the following statements are satisfied.*

1. *G is bipartite;*
2. *G is $K_{2,3}$-free;*
3. *(a) G is an induced subgraph of a hypercube and thus, (b) every edge $e\in E$ that lies on some cycle must be contained in some C_4;*
4. *(a) for every subgraph G' of G, the convex hull $\mathcal{H}(G')$ (w.r.t. G) is a median graph and thus, (b) every convex subgraph of G is a median graph.*

The following type of graphs will play a crucial role for our results.

Definition 2.15. A *square-graph* is a connected graph for which a planar embedding exists such that

(a) every inner boundary is a square, and
(b) every inner vertex has at least degree 4.

Such a planar embedding of a square-graph will always be denoted by σ.

Simple examples of square-graphs are trees and the 4-cycle. Further examples of σ-embedded square-graphs are shown in Fig. 2. Below, we will make use of the following results.

Lemma 2.16. *Every square-graph as well as the cube Q_3 is a planar median graph (cf. [15,16,40]). Moreover, it can be decided in $\mathcal{O}(|V(G)| + |E(G)|)$ time whether a given graph G is a square-graph or not (cf. [5, Prop. 5.3]).*

It has been shown by Soltan et al. [40] and Bandelt et al. [5, Prop. 9.1] that for every σ-embedded square-graph G every square in G is an inner boundary. This, together with the definition of square-graphs, implies

Lemma 2.17. *Let G be σ-embedded square-graph. Then, $C \subseteq G$ is a square if and only if C is an inner boundary in G w.r.t. σ. Consequently, C is a square if and only if C is a square-boundary w.r.t. σ. Moreover, every square-graph that contains k squares, satisfies k-FS (w.r.t. σ) and every cyclic square graph satisfies 1-FS (w.r.t. σ).*

Recall that every face of a planar graph can be both an inner and the outer face depending on the choice of the embedding. This, together with Observation 2.4 and the fact that in a σ-embedded square-graph and ρ-embedded cube every square is square-boundary, implies

Observation 2.18. *Let G be a square-graph with planar embedding $\pi = \sigma$ or a cube with planar embedding $\pi = \rho$. For every square C of G we can adjust π to a planar embedding π_C such that C becomes an outer boundary while all other squares distinct from C are inner boundaries w.r.t. π_C. In case G is a square-graph that contains at least two squares, there exists an inner face that is bounded by a square w.r.t. π_C.*

In particular, Bandelt et al. [5] characterized square-graphs in terms of forbidden subgraphs of median graphs (see Fig. 3).

Proposition 2.19 (*[5, Prop. 5.1 (i,ii)]*)**.** *Let G be a graph. Then, G is a square-graph if and only if G is a median graph such that G does not contain any of the following graphs as induced subgraphs (or isometric subgraphs or convex subgraphs, respectively): the cube Q_3, the book $K_2 \Box K_{1,3}$, and suspended cogwheel.*

3. Characterization of planar graphs that are median graphs

In this section, we present new characterizations for planar graphs being median graphs. To this end, we need the following

Lemma 3.1. *Let G be a planar median graph. Then, the length of every isometric cycle of G is either 4 or 6.*

Proof. Let G be a planar median graph, and let C_n with $n \geq 3$ be an isometric cycle of G. First, we show $n \leq 7$. To this end, we assume for contradiction that $n \geq 8$. Then, Proposition 2.13 implies that there is a hypercube Q_m with $C_n \subseteq Q_m \subseteq G$, and Lemma 2.5 implies that $m \leq 3$. Since $n \geq 8$, we have $C_n \subseteq Q_3$ and $n = 8$. Thus, $V(C_8) = V(Q_3)$. Since C_n contains two vertices at distance $\frac{n}{2} \geq 4$ and the diameter (i.e., the greatest distance) of Q_3 is 3, this C_n cannot be isometric; a contradiction. Hence, $n \leq 7$. Since every median graph is bipartite it cannot contain odd cycles. Therefore, $n = 4$ or $n = 6$. ∎

Lemma 3.2. *Let G be a $\{K_3, K_{2,3}\}$-free graph. Moreover, let $Q_3 \subseteq G$ be a cube, and let $v, w \in V(Q_3)$ with $d_{Q_3}(v, w) \in \{1, 2\}$. Then, for every shortest path $P^\star_G(v, w)$ in G, we have $P^\star_G(v, w) \subseteq Q_3$. Moreover, if G is additionally planar, then $P^\star_G(v, w) \subseteq Q_3$ for all $v, w \in V(Q_3)$.*

Proof. Let G be a $\{K_3, K_{2,3}\}$-free graph and $v, w \in V(Q_3)$ with $Q_3 \subseteq G$ and $d_{Q_3}(v, w) \in \{1, 2\}$. Clearly, if $d_{Q_3}(v, w) = 1$, then $P^\star_G(v, w)$ is an edge that must be contained in Q_3. Now, assume that $d_{Q_3}(v, w) = 2$. Then, it is easy to see that there is a (unique) square $C \subseteq Q_3$ with $v, w \in V(C)$. Since G is $\{K_3, K_{2,3}\}$-free, we can apply Lemma 2.1 to conclude that C is a convex subgraph of G. Hence, every shortest path $P^\star_G(v, w)$ between v and w is contained in a square $C \subseteq Q_3$.

Now, assume that G is planar in addition and let π be an arbitrary planar embedding of G. By the latter arguments, it suffices to consider vertices $v, w \in V(Q_3)$ with $d_{Q_3}(v, w) = 3$. By Lemma 2.11, $Q_3 - \{v, w\}$ results in a 6-cycle C and v is located in the inner face and w in the outer face of C or *vice versa* w.r.t. the (G, π)-induced embedding of C. This and the fact that π is a planar embedding of G implies that $P^\star_G(v, w)$ contains (at least) one vertex u of C. Hence, there are shortest paths $P^\star_G(v, u)$ and $P^\star_G(u, w)$ such that $P^\star_G(v, w) = P^\star_G(v, u) \cup P^\star_G(u, w)$. We distinguish two (mutually exclusive) cases: *(i)* $\{v, u\} \in E(Q_3) \subseteq E(G)$ and *(ii)* $\{u, w\} \in E(Q_3) \subseteq E(G)$.

In Case *(i)* we have $P^\star_G(v, u) \subseteq Q_3$ and $d_{Q_3}(u, w) = 2$. By the latter arguments, $P^\star_G(u, w) \subseteq Q_3$. Hence, $P^\star_G(v, w) = P^\star_G(v, u) \cup P^\star_G(u, w) \subseteq Q_3$. Similar arguments imply in Case *(ii)* that $P^\star_G(v, w) \subseteq Q_3$. ∎

For later reference, we show that every isometric cycle C_6 and every $Q^-{}_3$ of a median graph must be contained in a cube.

Lemma 3.3. *Every median graph is Q_3^--Q_3-inferring and C_6^{iso}-Q_3-inferring.*

Proof. Let G be a median graph. First, let $Q_3^- \subseteq G$. Consider the unique subgraph $C_6 \subseteq Q_3^-$ that is an isometric subgraph in Q_3^-. Assume, for contradiction, that this C_6 is not an isometric subgraph of G, i.e., that there are two vertices $a, b \in V(C_6)$ with $d_{C_6}(a, b) > d_G(a, b) \geq 1$ and, therefore, $d_{C_6}(a, b) \in \{2, 3\}$. If $d_{C_6}(a, b) = 2$, then $d_G(a, b) = 1$. Thus, there must be a K_3 in G; a contradiction to Proposition 2.14 (1). Moreover, if $d_{C_6}(a, b) = 3$ and $d_G(a, b) = 2$, then G must contain a K_3 or C_5; again a contradiction to Proposition 2.14 (1). Finally, assume that $d_{C_6}(a, b) = 3$ and $d_G(a, b) = 1$. Let $x, y, z \in V(C_6)$ be the three vertices that have degree 3 in Q_3^-. Among these vertices x, y, z has to be a or b; w.l.o.g. assume that $x = a$. Note, there is a vertex $v \in V(Q_3^-) \setminus V(C_6)$ that is adjacent to every vertex in $\{x, y, z\}$. Moreover, b is adjacent to every vertex in $\{x, y, z\}$ in G, since $d_G(x, b) = 1$. Hence, there is a $K_{2,3}$ with $V(K_{2,3}) = \{x, y, z, v, b\}$ in G; which is a contradiction to Proposition 2.14 (2). Thus, C_6 is an isometric subgraph of G.

Thus, by Proposition 2.13 there is a hypercube that contains C_6. Since we have $|V(C_6)| = 6$, we conclude that there is a cube $Q \subseteq G$ with $C_6 \subseteq Q$. Let w be the unique vertex in Q that is *not* adjacent to x, y and z in Q and let x', y', z' be the three vertices in C_6 that are adjacent to w in Q, where x' is adjacent to x and y, y' is adjacent to y, and z and z' is adjacent to x and z. The graph H with vertices $V(C) \cup \{v, w\}$ and edge set $\big\{\{x', x\}, \{x', y\}, \{y', y\}, \{y', z\}, \{z', x\}, \{z', z\}, \{v, x\}, \{v, y\}, \{v, z\}, \{w, x'\}, \{w, y'\}, \{w, z'\}\big\}$ is by the preceding arguments a subgraph of G and, in particular, a cube for which $H - \{w\}$ is equal to the graph Q_3^- chosen at the beginning of this proof. Hence, G is Q_3^--Q_3-inferring.

Now, let C_6 be an isometric cycle of G. Then, Proposition 2.13, together with the previous arguments, imply that there is a cube $Q_3 \subseteq G$ with $C_6 \subseteq Q_3$. Hence, G is C_6^{iso}-Q_3-inferring. ■

Lemma 3.4. *Let G be $\{K_3, K_{2,3}\}$-free, planar and C_6^{iso}-Q_3-inferring graph. Then, the convex hull of every isometric cycle C_6 in G is a Q_3. Moreover, if C_6 is an isometric cycle in a cube $Q_3 \subseteq G$, then C_6 is an isometric cycle in G.*

Proof. Let G be chosen as in the statement and let C_6 be an arbitrary isometric cycle of G. Since G is C_6^{iso}-Q_3-inferring there is a cube $Q_3 \subseteq G$ such that $C_6 \subseteq Q_3$. Moreover, let $(G'_1, G'_2, G'_3, \dots, G'_m)$ be a SPE-sequence of C_6. It is easy to verify that $Q_3 \subseteq G'_m$. Since G is $\{K_3, K_{2,3}\}$-free and planar, we can apply Lemma 3.2 to conclude that $P^{\star}_G(v, w) \subseteq Q_3$ for every shortest path $P^{\star}_G(v, w)$ with $v, w \in V(Q_3)$. Hence, $Q_3 \subseteq G'_m$ is a convex subgraph of G, and by construction of $(G'_1, \dots, G'_m)$, we have $Q_3 = G'_m$. By Lemma 2.3, it follows that Q_3 is the convex hull of C_6 (w.r.t. G).

Finally, let C_6 be an isometric cycle in a cube $Q_3 \subseteq G$. Now, let $P^{\star}_G(v, w)$ be a shortest path with $v, w \in V(C_6) \subseteq V(Q_3)$. By the same arguments as above, $P^{\star}_G(v, w) \subseteq Q_3$ and Q_3 is a convex subgraph of G. Thus, every vertex (resp., edge) of Q_3 lies on some shortest path $P^{\star}_G(v, w)$ for all $v, w \in V(C_6)$, we conclude that the convex hull $\mathcal{H}_G(C_6)$ (w.r.t. G) is this Q_3. This, together with C_6 being an isometric cycle of that $Q_3 \subseteq G$, implies C_6 is an isometric cycle in G. ■

Theorem 3.5. *Let G be a planar graph. Then, G is a median graph if and only if the following statements are satisfied:*

(1) G is connected,
(2) G is $K_{2,3}$-free,
(3) C_6^{iso}-Q_3-inferring, and
(4) every isometric cycle in G has length 4 or 6.

Proof. Let G be a planar graph. First, assume that G is a median graph. Then, by definition Statement (1) is satisfied, Proposition 2.14 (2) implies Statement (2), Lemma 3.3 implies Statement (3), and Lemma 3.1 implies Statement (4).

Conversely, assume that Statements (1)–(4) are satisfied. Now, let $C \subseteq G$ be an arbitrary isometric cycle. By Statement (4), this cycle C is either a C_4 or C_6. Note that G is K_3-free, since any K_3 would be an isometric cycle. If $C = C_4$, then we can apply Lemma 2.1 to conclude that C_4 is convex in G, and thus, the convex hull $\mathcal{H}(C_4)$ is precisely this $C_4 \simeq Q_2$. If $C = C_6$, then Lemma 3.4 implies that the convex hull of this C_6 (w.r.t. G) is a cube Q_3. Hence, in either case, the convex hull of any isometric cycle of G is a hypercube. Thus, Proposition 2.13 implies that G is a median graph. ■

Corollary 3.6. *Let G be a planar graph. Then, G is a cube-free median graph if and only if the following statements are satisfied:*

1. *G is connected,*
2. *G is $K_{2,3}$-free,*
3. *every isometric cycle in G has length 4.*

Proof. Let G be a planar and cube-free median graph. Then, by definition, Statement 1 is satisfied, and Proposition 2.14 (2) implies Statement 2. Moreover, every isometric cycle in G has length 4 or 6 (cf. Lemma 3.1). However, if there is an isometric cycle of length 6, then we can apply Theorem 3.5 (3), to conclude that G contains a cube Q_3, which is not possible by assumption. Hence, Statement 3 is satisfied.

Conversely, assume that G is a planar graph that satisfies Statements 1–3. Then, in particular, G satisfies the statements of Theorem 3.5, which implies that G is a median graph. Now, assume for contradiction that G contains a cube Q_3. This Q_3 contains an isometric cycle C_6 w.r.t. Q_3. By Lemma 3.4, C_6 is an isometric cycle in G; which is a contradiction to Statement 3. In summary, G is a cube-free median graph. ■

As a direct consequence of Lemma 2.10 and since every square in a median graph is isometric, we obtain

Corollary 3.7. *If G is a median graph containing a square $C \subseteq G$, then all squares of G are contained in the union of the squares contained in ${G^{\mathrm{in}}}_{C,\pi}$ and ${G^{\mathrm{out}}}_{C,\pi}$.*

In other words, if we have two graphs ${G^{\mathrm{in}}}_{C,\pi}$ and ${G^{\mathrm{out}}}_{C,\pi}$ such that $G = {G^{\mathrm{in}}}_{C,\pi}\ {}_C\circledast_C {G^{\mathrm{out}}}_{C,\pi}$ results in a median graph, then the only squares in G are the ones contained in ${G^{\mathrm{in}}}_{C,\pi}$ and ${G^{\mathrm{out}}}_{C,\pi}$. Corollary 3.7 can be generalized further to show that $\{K_3, K_{2,3}\}$-free planar graphs G can be characterized in terms of their subgraphs ${G^{\mathrm{in}}}_{C,\pi}$ and ${G^{\mathrm{out}}}_{C,\pi}$.

Lemma 3.8. *Let G be a π-embedded planar graph and $C \subseteq G$ be a square. Then, G is $\{K_3, K_{2,3}\}$-free if and only if ${G^{\mathrm{in}}}_{C,\pi}$ and ${G^{\mathrm{out}}}_{C,\pi}$ are $\{K_3, K_{2,3}\}$-free.*

Proof. Let G be a π-embedded planar graph, and let $C \subseteq G$ be a square. If G is $\{K_3, K_{2,3}\}$-free, then every subgraph of G is $\{K_3, K_{2,3}\}$-free, and thus, ${G^{\mathrm{in}}}_{C,\pi}$ and ${G^{\mathrm{out}}}_{C,\pi}$ must be $\{K_3, K_{2,3}\}$-free as well.

The terms "inside" and "outside" in the following refer to the (G, π)-induced embedding of C. Consider two vertices $v, w \in V(G)$, where v is outside of C and w is inside of C (and thus, in particular, $v, w \notin V(C)$). We observe that v and w cannot be adjacent in G, since in the planar embedding π an edge $\{v, w\}$ would cross edges or vertices of C.

Now, suppose that ${G^{\mathrm{in}}}_{C,\pi}$ and ${G^{\mathrm{out}}}_{C,\pi}$ are $\{K_3, K_{2,3}\}$-free. Assume first, for contradiction, that G contains a subgraph $H \simeq K_3$. This subgraph can neither be located entirely in ${G^{\mathrm{in}}}_{C,\pi}$ nor in ${G^{\mathrm{out}}}_{C,\pi}$. Hence, there are vertices $v, w \in V(H)$ such that v is outside of C and w is inside of C. Since $H \simeq K_3$ it holds that $\{v, w\} \in E(H) \subseteq E(G)$; a contradiction. Hence, G must be K_3-free.

Assume now, for contradiction, that G contains a subgraph $H \simeq K_{2,3}$. Again, this subgraph can neither be located entirely in ${G^{\mathrm{in}}}_{C,\pi}$ nor in ${G^{\mathrm{out}}}_{C,\pi}$. Hence, there are vertices $v, w \in V(H)$ such that v is outside of C and w is inside of C. Since, as argued above, $\{v, w\} \in E(H)$ is not possible, we can conclude that $d_H(v, w) = 2$, and thus, there must be (at least) two distinct paths $P_H(v, w)$ and $P'_H(v, w)$ of length 2. Let $V(P_H(v, w)) = \{v, a, w\}$ and $V(P'_H(v, w)) = \{v, b, w\}$. Since $\{v, w\} \notin E(G)$ and v is outside while w is inside of that C, the only vertices that can be adjacent to v and w are vertices of C. Hence, $a, b \in V(C)$. Note $d_H(v, w) = 2$, and since G is K_3-free, we can conclude that $d_G(a, b) = 2$. But then, the subgraph of ${G^{\mathrm{out}}}_{C,\pi}$ induced by $V(C) \cup \{v\}$ contains a subgraph isomorphic to $K_{2,3}$; a contradiction. Hence, G is $\{K_3, K_{2,3}\}$-free. ■

Recall that a connected graph is either cyclic or a tree. We are now in the position to provide an additional characterization of planar graphs that contain squares and are median graphs.

Theorem 3.9. *Let G be a π-embedded planar graph that contains a square $C \subseteq G$. Then, G is a median graph if and only if ${G^{\mathrm{in}}}_{C,\pi}$ and ${G^{\mathrm{out}}}_{C,\pi}$ are median graphs.*

Proof. Let G be a π-embedded planar graph, and let C be a square of G. By construction, $C \subseteq {G^{\mathrm{in}}}_{C,\pi}$ and $C \subseteq {G^{\mathrm{out}}}_{C,\pi}$.

First, assume that G is a median graph. Moreover, let C' be an isometric cycle of ${G^{\mathrm{in}}}_{C,\pi}$, and let $\mathcal{H}_G(C')$ be its convex hull w.r.t. G. Hence, Lemma 2.9 implies that every $v' \in V(\mathcal{H}_G(C))$ lies almost-inside of this C w.r.t. the (G, π)-induced embedding of G. Thus, by definition of ${G^{\mathrm{in}}}_{C,\pi}$, we conclude that $\mathcal{H}_G(C') \subseteq {G^{\mathrm{in}}}_{C,\pi}$. Hence, since $C' \subseteq {G^{\mathrm{in}}}_{C,\pi} \subseteq G$, we conclude that $\mathcal{H}_G(C')$ is also the convex hull of C' w.r.t. ${G^{\mathrm{in}}}_{C,\pi}$. Since G is a median graph, Proposition 2.13 implies that $\mathcal{H}_G(C')$ is a hypercube. Thus, the convex hull of an arbitrary isometric cycle C' in ${G^{\mathrm{in}}}_{C,\pi}$ is a hypercube. Hence, Proposition 2.13 implies that ${G^{\mathrm{in}}}_{C,\pi}$ is a median graph. Analogously, one can show that ${G^{\mathrm{out}}}_{C,\pi}$ is a median graph as well.

Conversely, assume that ${G^{\mathrm{in}}}_{C,\pi}$ and ${G^{\mathrm{out}}}_{C,\pi}$ are median graphs. Proposition 2.14 (1) implies that ${G^{\mathrm{in}}}_{C,\pi}$ and ${G^{\mathrm{out}}}_{C,\pi}$ are bipartite, and thus, they are K_3-free. This, together with Proposition 2.14 (2), implies that ${G^{\mathrm{in}}}_{C,\pi}$ and ${G^{\mathrm{out}}}_{C,\pi}$ are $\{K_3, K_{2,3}\}$-free. By Lemma 3.8, G is $\{K_3, K_{2,3}\}$-free. Now, let C' be an isometric cycle of G. Hence, Lemma 2.10 implies that all vertices $v \in V(C')$ lie almost-inside (resp. almost-outside) of this C. First, assume that all vertices $v \in V(C')$ lie almost-inside of the C. By definition of ${G^{\mathrm{in}}}_{C,\pi}$ and by Lemma 2.9, we conclude that the convex hull $\mathcal{H}_G(C')$ of C' w.r.t. G is equal to the convex hull $\mathcal{H}_{{G^{\mathrm{in}}}_{C,\pi}}(C')$ of C' w.r.t. ${G^{\mathrm{in}}}_{C,\pi}$. Since ${G^{\mathrm{in}}}_{C,\pi}$ is a median graph, we conclude that $\mathcal{H}_{{G^{\mathrm{in}}}_{C,\pi}}(C')$ is a hypercube. Thus, $\mathcal{H}_G(C')$ is a hypercube. Analogously, one can show that the convex hull $\mathcal{H}_G(C')$ of C' w.r.t. G is a hypercube if all vertices $v' \in V(C')$ are almost-outside of this C. Hence, in either case, the convex hull of any isometric cycle C' of G is a hypercube. Thus, Proposition 2.13 implies that G is a median graph. ■

4. Cubesquare-graphs

In this section, we establish a further characterization of planar median graphs. To this end, we provide a definition of an operator $\circledast$ to "glue" two graphs together. This definition is motivated in part by Theorem 3.9.

Definition 4.1. Let G and H be two vertex-disjoint graphs with squares $C \subseteq G$ and $C' \subseteq H$. Let φ be any isomorphism between the squares C and C'. Then, the composition $G\,{}_C\circledast_{C'}H$ is obtained from G and H by identifying the vertices and edges of C with their φ-images in $C' \subseteq H$.

We will omit the explicit reference to C and C' in ${}_C\circledast_{C'}$ whenever it is not needed. Note that $\circledast$ is not defined for graphs that do not contain squares. Since the choice of φ will not play a role here, we suppress it in our notation. Fig. 4 gives an illustrative example of Definition 4.1. There are eight different ways to define an isomorphism on squares. Therefore, there are up to eight non-isomorphic graphs $G\,{}_C\circledast_{C'}H$ obtained by gluing together G and H at the same squares with the help of different isomorphisms φ.

It is easy to see that the square C_4 serves as a unique "unit" element, that is, $G \circledast C_4 \simeq C_4 \circledast G \simeq G$. Moreover, the operator $\circledast$ is commutative, i.e. $G\,{}_C\circledast_{C'}H \simeq H\,{}_{C'}\circledast_{C}G$ for all graphs G and H. However, it is not associative, since $(G_1\,{}_C\circledast_{C'}G_2)\,{}_{C''}\circledast_{C'''}G_3$ can be well-defined, but $G_1\,{}_C\circledast_{C'}(G_2\,{}_{C''}\circledast_{C'''}G_3)$ is not; a case that in particular happens when the square C'' is part of G_1 but not of G_2; see Fig. 5 for an example.

We will use the convention that $\circledast$-composition is read from left to right, i.e.,

$$G_1 \circledast G_2 \circledast G_3 \circledast \cdots \circledast G_\ell := (\ldots((G_1 \circledast G_2) \circledast G_3) \circledast \cdots \circledast G_{\ell-1}) \circledast G_\ell \tag{1}$$

Setting $G(i) := G_1 \circledast G_2 \circledast G_3 \circledast \cdots \circledast G_i$, we therefore have $G(i+1) = G(i)\,{}_C\circledast_{C'}G_{i+1}$, where C is a square in $G(i)$ and C' is a square in G_{i+1}. Note that, by Definition 4.1, $G = G^{\text{in}}{}_{C,\pi}\,{}_C\circledast_C G^{\text{out}}{}_{C,\pi}$.

In the following, we will consider the class of cubesquare-graphs as defined below. As we shall see later, a planar median graph is either a tree or a cubesquare-graph.

Definition 4.2. A *cubesquare-graph* (or *QS-graph* for short) is defined as follows:

- **(Q1)** Every cube Q_3 and every cyclic square-graph is a QS-graph, called *basic* QS-graph.
- **(Q2)** The ordered composition $G(\ell)$ of basic QS-graphs G_i, $1 \le i \le \ell$ is a QS-graph, where $G(\ell)$ is defined recursively as $G(1) = G_1$ and $G(i) = G(i-1)\,{}_{C_{i-1}}\circledast_{C_i}G_i$, $2 \le i \le \ell$ using *square-boundaries* C_{i-1} in $G(i-1)$ and C_i in G_i.

In other words, every QS-graph can be obtained from a cube or a square-graph by iteratively replacing boundaries (w.r.t. some embedding) that are 4-cycles by cubes or square-graphs. We emphasize that, in contrast to Definition 4.1, the squares chosen in the construction of QS-graphs are not arbitrary but must be square-boundaries for some planar embedding in each iteration. We shall see below that this construction is always possible since each partial composition $G(i)$ and each basic QS-graph contains a square-boundary. An illustrative example of QS-graphs is given in Fig. 5.

In the following, we will make frequent use of the planar embeddings ρ and ρ_C of cubes (cf. Observation 2.18) and the embeddings σ and σ_C of square-graphs (cf. Definition 2.15 and Observation 2.18).

Lemma 4.3. *QS-graphs are well-defined and planar graphs that satisfy* 1*-FS.*

Proof. Recall that for $G = G_1\,{}_C\circledast_C G_2$, the square C must be a square-boundary in G_1 and G_2 but not necessarily in G, see Fig. 4. In order to show that QS-graphs are well-defined, we must, in particular, show that in each step of creating a new QS-graph at least one square-boundary remains which allows us to add another QS-graph (cf. **(Q2)**). Hence, we must show that every QS-graph satisfies 1-FS. This, in particular, implies that QS-graphs must be planar.

Let us first consider basic QS-graphs. By Lemma 2.17 and Observation 2.7, every square-graph and cube satisfies 1-FS. In particular, if a basic QS-graph contains at least two squares, then Observation 2.7 and Lemma 2.17 imply that it must satisfy 2-FS. Thus, every basic QS-graph satisfies 1-FS and every basic QS-graph containing at least two squares satisfies 2-FS.

We proceed now by induction on the number ℓ of factors to show that the ordered composition $\circledast$ of ℓ basic QS-graphs is well-defined and satisfies 1-FS which, in particular, implies that we obtain a planar QS-graph. The base case are the basic QS-graphs. Assume that $G_1 \circledast G_2 \circledast \cdots \circledast G_i$ is well-defined and results in a planar graph that satisfies 1-FS for all $1 \le i < k$. Consider now a product of k basic QS-graphs $G_1 \circledast G_2 \circledast \cdots \circledast G_k$. Set $H := G(k-1)$ and $H' := G_k$. We show first,

Claim 1. *$H\,{}_C\circledast_{C'}H'$ is well-defined and a planar graph.*

Proof of Claim 1. By assumption, H and H' satisfy 1-FS. Let C be a square-boundary of H and C' a square-boundary of H' w.r.t. some planar embedding of H, resp., H'. Now, we can use the embeddings $\pi_{C'} \in \{\sigma_{C'}, \rho_{C'}\}$ for H' depending on whether H' is a cube or a cyclic square-graph such that the square C' is the outer boundary of H'. Since C is a square-boundary in H, there is a planar embedding π_C of H such that C is an inner boundary w.r.t. π_C (cf. Observation 2.4). The outer boundary C' of H' intersects $H \subseteq H\,{}_C\circledast_{C'}H'$ only in the 4 vertices of the chosen square C by definition of $H\,{}_C\circledast_{C'}H'$. Now, consider the embedding $\kappa(\pi_C, \pi_{C'})$ of $H\,{}_C\circledast_{C'}H'$. It consists of the drawing of $H\,{}_C\circledast_{C'}H'$ based on π_C together with the "scaled" planar drawing $\pi_{C'}$ such that H' intersects H only in the vertices contained in C and the remaining vertices of H' are placed inside of the inner face of H bounded by C. Thus it yields a planar embedding of $H\,{}_C\circledast_{C'}H'$. In summary, $H\,{}_C\circledast_{C'}H'$ is well-defined and results in a planar graph. ⋄

Claim 2. *$H\,{}_C\circledast_{C'}H'$ satisfies* 1*-FS.*

Proof of Claim 2. We will make frequent use of the planar embeddings π_C of H, $\pi_{C'}$ of H' and $\kappa(\pi_C, \pi_{C'})$ of $H\,{}_C\circledast_{C'}H'$ as specified in the proof of Claim 1.

First, assume that H' contains only one square. Since H' is a basic QS-graph, it must therefore be square-graph and thus, H' is isomorphic to a C_4 to which possibly a couple of trees are attached. Let $\tilde{C}$ be the square in $H\,{}_C\circledast_{C'}H'$ that refers to the two identified cycles C and C' via the chosen subgraph isomorphism. By construction of $\kappa(\pi_C, \pi_{C'})$, $\tilde{C}$ together with these possible attached tree forms an inner boundary in $H\,{}_C\circledast_{C'}H'$ w.r.t. $\kappa(\pi_C, \pi_{C'})$. Hence, every tree that is attached to a vertex v in $\tilde{C}$ can safely be re-located in some face of $H\,{}_C\circledast_{C'}H'$ that is incident to v w.r.t. $\kappa(\pi_C, \pi_{C'})$. In this way, we obtain a new planar embedding of $H\,{}_C\circledast_{C'}H'$ such that $\tilde{C}$ is the boundary of an inner face and thus, $H\,{}_C\circledast_{C'}H'$ satisfies 1-FS.

Now, assume that H' contains more than one square. As argued above, H' together with its planar embedding $\pi_{C'} \in \{\sigma_{C'}, \rho_{C'}\}$ satisfies 2-FS w.r.t. $\pi_{C'}$ and thus in H' there are two faces bounded by a square w.r.t. $\pi_{C'}$. Since C' is an outer boundary of H' w.r.t. $\pi_{C'}$ the other face that is bounded by a square C'' must be an inner face. It is straightforward to see that C'' still bounds an inner face in $H\,{}_C\circledast_{C'}H'$ w.r.t. $\kappa(\pi_C, \pi_{C'})$. Hence, $H\,{}_C\circledast_{C'}H'$ satisfies 1-FS. ◇

In particular, $H\,{}_C\circledast_{C'}H'$ is planar and contains the required square-boundary. Thus, $G_1 \circledast G_2 \circledast \cdots \circledast G_k$ is a well-defined planar graph for all k. ■

Remark 4.4. For a QS-graph $G(i)\,{}_C\circledast_{C'}G_{i+1}$, we will use the notation π_C as well as $\pi_{C'} \in \{\rho_{C'}, \sigma_{C'}\}$ and $\kappa(\pi_C, \pi_{C'})$ for the planar embedding of $G(i)$, G_{i+1} and $G(i)\,{}_C\circledast_{C'}G_{i+1}$, respectively, as specified in the proof of Lemma 4.3.

Lemma 4.5. *Every QS-graph is a planar median graph.*

Proof. We show now, by induction on the number of factors, that every QS-graph is a median graph. As base case, we have a basic QS-graph, i.e., either a cube or a cyclic square-graph. These are planar and Lemma 2.16 implies that they are median graphs. Now, let $G = G_1 \circledast G_2 \circledast \cdots \circledast G_k = G(k-1)\,{}_C\circledast_{C'}G_k$ be the ordered composition of k basic QS-graphs and assume that the ordered composition of $i < k$ basic QS-graphs is a median graph. Since $G = G(k-1)\,{}_C\circledast_{C'}G_k$ is planar by Lemma 4.3, we can use the planar embedding $\pi := \kappa(\pi_C, \pi_{C'})$ of G (cf. Remark 4.4). It is straightforward to verify that $G(k-1) = {G^{\mathrm{out}}}_{C,\pi}$ and $G_k = {G^{\mathrm{in}}}_{C,\pi}$. Thus, Theorem 3.9 implies that G is a median graph. ■

We now want to consider the converse of Lemma 4.5. We begin with some observations.

Lemma 4.6. *Let G be a book or a suspended cogwheel. Then, for any planar embedding π of G, there is a square $C^* \subseteq G$ such that ${G^{\mathrm{in}}}_{C^*,\pi} \neq G$ and ${G^{\mathrm{out}}}_{C^*,\pi} \neq G$.*

Proof. First, let $G = K_2 \Box K_{1,3}$ be a book. Assume that $V(K_2) = \{0, 1\}$ and $V(K_{1,3}) = \{0, 1, 2, 3\}$ where 0 is the unique vertex adjacent to the remaining ones. By definition of the Cartesian product, $V(G) = \{00, 10, 01, 11, 02, 12, 03, 13\}$ and we have exactly three squares C, C', C'' in G that consist of the vertices $V(C) = \{00, 10, 02, 12\}$, $V(C') = \{00, 10, 01, 11\}$, and $V(C'') = \{00, 10, 03, 13\}$. Below, the terms "inside" and "outside" of some subgraph $G' \subseteq G$ refer to the (G, π)-induced embedding of G'. Now, consider the square C and an arbitrary embedding π of G. We have to examine the cases that k_1 vertices are inside C and k_2 are outside of C where $k_1 + k_2 = 4$, the number of remaining vertices in $V(G) \setminus V(C)$. Hence, $k_1 \in \{0, 1, 2, 3, 4\}$.

Let us start with $k_1 = 2$ and thus, $k_2 = 2$. Assume w.l.o.g. that 01 is inside of C. In this case, Lemma 2.8 implies that the second vertex inside of C must be vertex 11. Hence, the two vertices outside of C are 03 and 13. Now one readily observes that ${G^{\mathrm{in}}}_{C,\pi} \neq G$ and ${G^{\mathrm{out}}}_{C,\pi} \neq G$. The latter reasoning implies that the case $k_1 = 1$ cannot occur, since if one vertex is inside C there must also be a second vertex inside C. By symmetry, this also excludes the case $k_2 = 1$ and thus, $k_1 = 3$.

Thus, we are left with the case $k_1 \in \{0, 4\}$. Let $k_1 = 4$. Consider the square C'. By similar arguments as above, there are either 0, 2 or 4 vertices inside of C'. However, the latter case cannot occur, since all $k_1 = 4$ remaining vertices are inside C and thus, there must be vertices outside C'. If there are 2 vertices inside of C', then the vertices inside C' must be 03 and 13 since C' is almost-inside of C. Now, one easily verifies that ${G^{\mathrm{in}}}_{C',\pi} \neq G$ and ${G^{\mathrm{out}}}_{C',\pi} \neq G$. If there are no vertices inside of C', then all vertices 02, 12, 03, 13 must be outside of C'. Since 03,13 are inside of C, we can conclude that ${G^{\mathrm{in}}}_{C'',\pi} \neq G$ and ${G^{\mathrm{out}}}_{C'',\pi} \neq G$. By symmetry, the case $k_2 = 4$ and thus, $k_1 = 0$ is shown.

In summary, for all possible cases we found a square $C^* \subseteq G$ in the book G such that ${G^{\mathrm{in}}}_{C^*,\pi} \neq G$ and ${G^{\mathrm{out}}}_{C^*,\pi} \neq G$.

Now, let $G \simeq M_n^*$ be a suspended cogwheel. Let x be the vertex that is adjacent to the central vertex c of the underlying cogwheel M_n. Moreover, denote with $c_1, \ldots, c_n$ the vertices of the cycle $C_n \subseteq M_n^*$ that are distinct from x and adjacent to c. We assume that the edges of this cycle C_n are $\{c_n, c_1\}$ and $\{c_i, c_{i+1}\}$, $1 \leq i \leq n-1$. In addition, let c_k with k being even be the vertices that are adjacent to c. In what follows, all indices $j, i, i+1, i+2, \ldots$ are taken w.r.t. (mod n).

Let π be an arbitrary embedding of G. In what follows, the terms "inside" and "outside" of some subgraph $G' \subseteq G$ refer to the (G, π)-induced embedding of G'. It is easy to see that the vertex x must be located in one of the faces that is bounded by a subgraph $G' \subseteq G$ such that G' contains two vertices c_i and c_{i+2} with i being even. Thus, assume that x is in a face bounded by $G' \subseteq G$ such that $c_i, c_{i+2} \in V(G')$, i even. We continue by showing that, in this case, the square $C + x$ induced by the vertices $x, c, c_i, c_{i+1}, c_{i+2}$ must be a boundary in G. Assume, for contradiction, that this is not the case. Hence, one c_j with $j \notin \{i, i+1, i+2\}$ must be contained inside of C. But then also c_{j+1} must be contained inside of C as otherwise the edge $\{c_j, c_{j+1}\}$ would cross one of the edges or vertices of C w.r.t. the planar drawing π of G. Repeating the latter argument shows that all vertices c_j with $j \notin \{i, i+1, i+2\}$ must be located inside of C. In this case, however, x is located in a face that is bounded by some $G'' \subseteq G$ that contains the vertices $c_j, c_{j+2} \in V(G')$ where j is even and where at least one of c_j and c_{j+2} is distinct from c_i or c_{i+1}; a contradiction. Hence, $C + x$ must be a boundary in G.

Finally, observe that either ${G^{\text{in}}}_{C,\pi} = C + x$ or ${G^{\text{out}}}_{C,\pi} = C + x$ and thus, either ${G^{\text{out}}}_{C,\pi} = G - x$ or ${G^{\text{in}}}_{C,\pi} = G - x$. Hence, we found the square $C^* = C$ such that ${G^{\text{in}}}_{C^*,\pi} \neq G$ and ${G^{\text{out}}}_{C^*,\pi} \neq G$. ∎

Lemma 4.7. *If G is a π-embedded cyclic planar median graph that is not a basic QS-graph, then there is a square $C^* \subseteq G$ such that ${G^{\text{in}}}_{C^*,\pi} \neq G$ and ${G^{\text{out}}}_{C^*,\pi} \neq G$.*

Proof. Let G be a π-embedded planar median graph that is not a basic QS-graph. Proposition 2.19 implies that G must contain a cube, a book or a suspended cogwheel. Let $H \subseteq G$ be such a forbidden subgraph.

First, assume that H is a cube. Note, every face of H must be bounded by squares (cf. Observation 2.7). Since $H \subseteq G$ and G is not a cube, there must be a vertex $v \in V(G) \setminus V(H)$ that lies in a face of H that is bounded by a square $C^* \subseteq H$ in H (w.r.t. the (G, π)-induced embedding of H). Note, C^* is not necessarily a boundary in G but, of course, a subgraph of G. Let $v' \in V(H) \setminus (V(C^*) \cup \{v\})$. If v lies in the outer face of H w.r.t. the (G, π)-induced embedding, then ${G^{\text{in}}}_{C^*,\pi} \subseteq G - v \subsetneq G$ and ${G^{\text{out}}}_{C^*,\pi} \subseteq G - v' \subsetneq G$. Otherwise, if v lies in an inner face of H w.r.t. the (G, π)-induced embedding, then ${G^{\text{out}}}_{C^*,\pi} \subseteq G - v \subsetneq G$ and ${G^{\text{in}}}_{C^*,\pi} \subseteq G - v' \subsetneq G$. In either case, there is a square C^* such that ${G^{\text{in}}}_{C^*,\pi} \neq G$ and ${G^{\text{out}}}_{C^*,\pi} \neq G$.

If H is a book or a suspended cogwheel, then we can apply Lemma 4.6 to conclude that there is a square $C^* \subseteq H$ such that ${G^{\text{in}}}_{C^*,\pi} \neq H$ and ${G^{\text{out}}}_{C^*,\pi} \neq H$ for every planar embedding of H and thus, in particular, for the (G, π)-induced embedding of H. The latter immediately implies that ${G^{\text{in}}}_{C^*,\pi} \neq G$ and ${G^{\text{out}}}_{C^*,\pi} \neq G$. ∎

Lemma 4.8. *Every cyclic planar median graph is a QS-graph.*

Proof. Since squares that are possibly amalgamated with trees are square-graphs and thus, median graphs, for every integer $n \geq 4$ there is a cyclic planar median graph on $n \geq 4$ vertices. Thus, we can proceed by induction on $|V(G)|$. The square is the only cyclic median graph with $n = 4$ vertices. By definition, it is also a (basic) QS-graph, and thus serves as base case.

For the induction step consider a π-embedded planar median graph G with $n = |V(G)| > 4$ vertices and assume that every planar median graph G' with $|V(G')| < |V(G)|$ vertices is a QS-graph. If G is a cube or a square-graph then G is a QS-graph and we are done. Hence, assume that G is neither a cube nor a square-graph and let π be a planar embedding of G. By Lemma 4.7, there is a square $C^* \subseteq G$ such that ${G^{\text{in}}}_{C^*,\pi} \neq G$ and ${G^{\text{out}}}_{C^*,\pi} \neq G$. As shown in the proof of Lemma 4.7, we can find such a square C^* by taking a forbidden subgraph H, i.e., a book, a cube or suspended cogwheel, and a particular square $C^* \subseteq H$. We may assume w.l.o.g. that ${G^{\text{in}}}_{C^*,\pi}$ is either a cube or does not contain a book, a cube or a suspended cogwheel as a subgraph. Otherwise, we could iteratively replace H by such a forbidden subgraph $\tilde{H} \subseteq {G^{\text{in}}}_{C^*,\pi}$, and replace C^* by a square $\tilde{C}^*$ of $\tilde{H}$ with the property that ${G^{\text{in}}}_{\tilde{C}^*,\pi} \neq G$ and ${G^{\text{out}}}_{\tilde{C}^*,\pi} \neq G$, until we eventually obtain a forbidden subgraph H and a square C^* such that ${G^{\text{in}}}_{C^*,\pi}$ is either a cube or, otherwise, does not any longer contain a book, a cube or a suspended cogwheel.

By Theorem 3.9, ${G^{\text{in}}}_{C^*,\pi}$ and ${G^{\text{out}}}_{C^*,\pi}$ are median graphs. If ${G^{\text{in}}}_{C^*,\pi}$ is not a cube, then Proposition 2.19 and the fact that ${G^{\text{in}}}_{C^*,\pi}$ does not contain a cube, book or a suspended cogwheel, implies that ${G^{\text{in}}}_{C^*,\pi}$ is a square-graph. In either case, ${G^{\text{in}}}_{C^*,\pi}$ is a basic QS-graph. Moreover, since ${G^{\text{out}}}_{C^*,\pi}$ is a median graph with $|V({G^{\text{out}}}_{C^*,\pi})| < |V(G)| = n$, induction hypothesis implies that ${G^{\text{out}}}_{C^*,\pi}$ is a QS-graph. Hence, ${G^{\text{out}}}_{C^*,\pi}$ has an ordered composition ${G^{\text{out}}}_{C^*,\pi} = (\dots(G_1 \circledast G_2)\dots) \circledast G_k$ of basic QS-graphs. Therefore, the ordered composition $((\dots(G_1 \circledast G_2)\dots) \circledast G_k)\,{}_{C^*}\!\circledast_{C^*} {G^{\text{in}}}_{C^*,\pi}$ is well-defined and yields a QS-graph that is identical to G. ∎

As an immediate consequence of Lemmas 4.5 and 4.8 we obtain

Theorem 4.9. *A graph G is a planar median graph if and only if G is a QS-graph or a tree.*

Theorem 4.9, together with Theorem 3.9, furthermore implies the following:

Theorem 4.10. *Let G be a π-embedded planar graph and $C \subseteq G$ be a square. Then, G is a QS-graph if and only if ${G^{\text{in}}}_{C,\pi}$ and ${G^{\text{out}}}_{C,\pi}$ are QS-graphs.*

Corollary 4.11. *A planar median graph with n vertices contains $\mathcal{O}(n)$ squares.*

Proof. Let G be a planar median graph. Hence, it is a QS-graph with composition $G = G_1 \circledast \cdots \circledast G_k$ of basic QS-graphs. Let n_i be the number of vertices in G_i, $1 \leq i \leq k$. If a factor $G_i = (V_i, E_i)$ is a square-graph, then it contains $\mathcal{O}(n_i)$ squares (cf. [31, Cor. 5]) and, if it is a cube it contains $6 < n_i = 8$ squares. Hence, each factor G_i adds $\mathcal{O}(n_i)$ squares to G and therefore, G has $\mathcal{O}(\sum_i n_i) = \mathcal{O}(n)$ squares. ∎

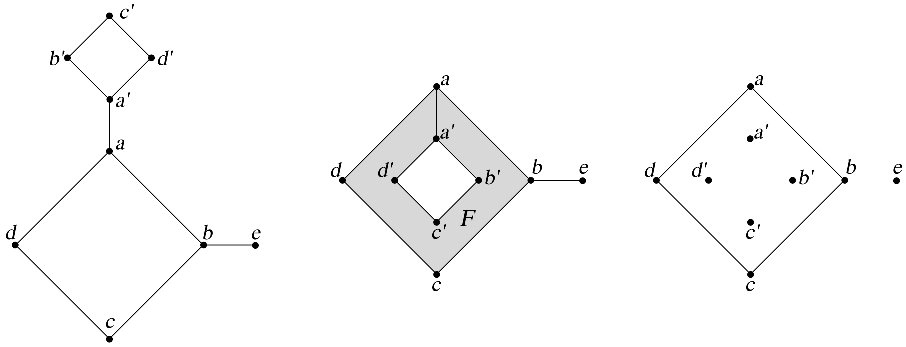

Fig. 1. Two distinct planar embeddings of a graph G (left and middle). *Left:* G is the outer boundary and every inner face is bounded by a square. Since G has no inner vertices, the conditions of Definition 2.15 are satisfied and G is a square-graph and, in particular, a planar median graph. *Middle:* $G' = G - e$ is an inner boundary which bounds the gray shaded face F. The graph induced by $\{a, b, c, d, e\}$ is the outer boundary and the 4-cycle induced by $\{a', b', c', d'\}$ an inner boundary. *Right:* Consider the embedding π of G as in the middle figure. Then, shown is the (G, π)-induced embedding of the cycle C induced by $\{a, b, c, d\}$ and additionally, all other vertices of G. W.r.t. the (G, π)-induced embedding of C, the vertices a, b, c, d are almost-inside and almost-outside C, the vertex e is outside (and, in particular, almost-outside) C and the vertices a', b', c', d' are inside (and in particular, almost-inside) of C. Placing vertex e inside C yields an embedding π_C such that the outer boundary of G is C while placing the subgraph with vertices a', b', c' and d' outside C yields an embedding π_C such that C is an inner boundary in G.

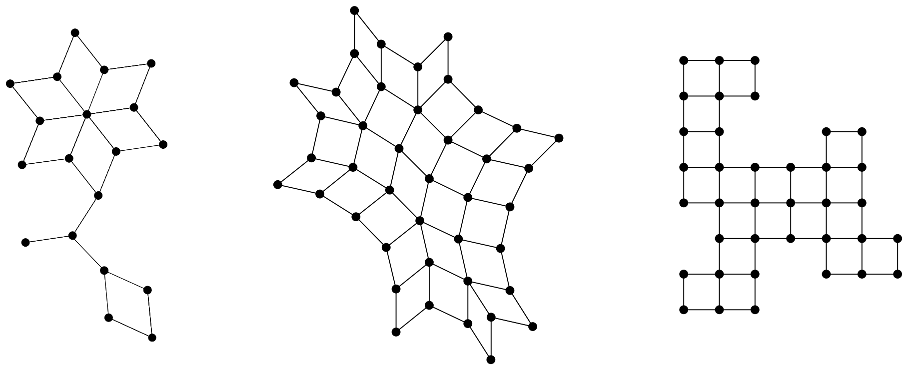

Fig. 2. *Left:* A square-graph with three articulation points. *Middle:* a 2-connected square-graph. *Right:* a so-called "polyomino". *Source:* Three examples of square-graphs adapted from [5, Fig. 1.1].

Propositions 2.14 and 2.19 and Lemma 4.6 can be used to obtain the following interesting result.

Proposition 4.12. *Let G be a planar median graph. Then, the convex hull of each boundary is a square-graph.*

Proof. Let G be a planar median graph together with some planar embedding π. Moreover, let G' be some arbitrary boundary in G. Then, Proposition 2.14 (4a) implies that the convex hull $\mathcal{H}(G')$ of G' in G is a median graph.

We continue with showing that the convex hull $\mathcal{H}(G')$ is a square-graph. To this end, assume for contradiction that there is a subgraph $H \subseteq \mathcal{H}(G')$ that is isomorphic to a cube, a book, or a suspended cogwheel.

If H is isomorphic to a book or a suspended cogwheel, then Lemma 4.6 implies that there is a square $C^* \subseteq H \subseteq \mathcal{H}(G')$ such that there is a vertex $v \in V(H)$ inside and a vertex $w \in V(H)$ outside of C^* (w.r.t. its (G, π)-induced embedding). Since G' is a boundary in G, it has to be almost-inside or almost-outside of C^*. The latter two statements together with Lemma 2.9 imply that $v \in V(H) \subseteq V(\mathcal{H}(G'))$ or $w \in V(H) \subseteq V(\mathcal{H}(G'))$ cannot be a part of $\mathcal{H}(G')$; a contradiction. Hence, H cannot be a book or a suspended cogwheel, and thus H must be a cube. However, all vertices $v \in V(G')$ have to be in some face of that cube Q_3 (w.r.t. its (G, π)-induced embedding), which has to be bounded, in particular, by a square C^* (cf. Observation 2.7). By Lemma 2.9, none of the vertices $v' \in V(Q_3) \setminus V(C^*) \neq \emptyset$ can be part of $\mathcal{H}(G')$; a contradiction. Thus, H cannot be a subgraph of $\mathcal{H}(G')$.

This together with $\mathcal{H}(G')$ being a median graph and Proposition 2.19 implies that $\mathcal{H}(G')$ is a square-graph. ■

As an illustration of Proposition 4.12, we refer to Fig. 5. Consider the graphs $G(1)$ and $G(7)$ with its outer boundary; the 8-cycle C. Here, $\mathcal{H}(C) \simeq G(1)$ is a square-graph. Note, however, not all boundaries are necessarily cycles (cf. Fig. 1).

To recall, a graph G is an amalgam of two induced subgraphs G_1 and G_2 if their union is G and their intersection is non-empty. Every finite median graph is obtained from a collection of hypercubes by successive amalgamations of

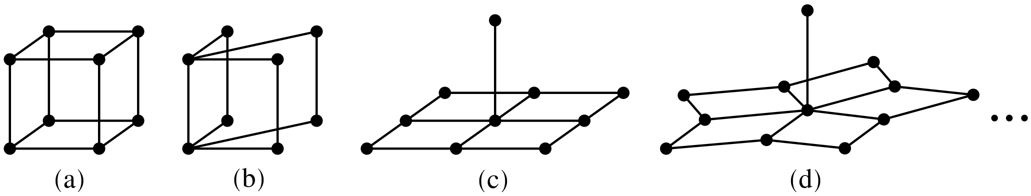

Fig. 3. (a) Cube Q_3; (b) the book $K_2 \square K_{1,3}$; (c) and (d) the first two suspended cogwheels M_8^* and M_{10}^*.
Source: Forbidden induced subgraphs of square-graphs adapted from [5, Fig. 5.1].

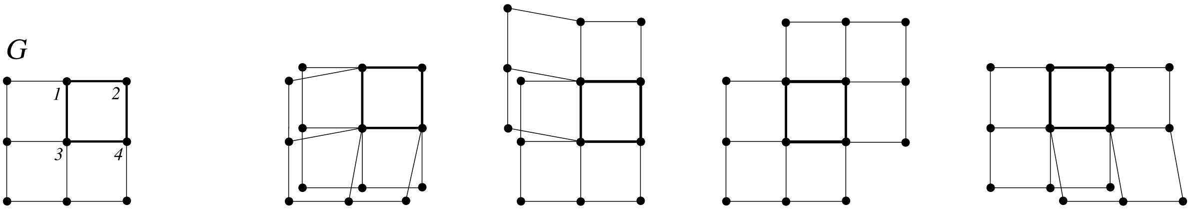

Fig. 4. There are three non-isomorphic graphs $G_C \circledast_C G$ obtained by gluing together G with a copy of itself at the square C. Shown are the four rotations of C relative to its copy. By symmetry, the four rotations of the mirror image, i.e., mapping the vertex order $(1, 3, 4, 2)$ to the order $(1, 2, 4, 3)$ in the other copy, yields the same four configurations. Furthermore, the 2nd and the 4th case result in isomorphic graphs. Note that all graphs $G_C \circledast_C G$ are planar. A generic planar embedding of the second case is provided by the drawing of the graph $G(2)$ in Fig. 5. However, only for the 3rd case graph, the square C remains a square-boundary and, in particular, the resulting graph is a square-graph.

convex subgraphs [30,41]. Every pseudo-median graph can be built up by successive amalgamations along so-called gated subgraphs of certain Cartesian products of wheels, snakes (i.e., path-like 2-trees), and complete graphs minus matchings [7]. We now explain how our results also fit into this framework. To this end, we define square-boundary amalgamations as follows. A graph G is a *square-boundary amalgam (w.r.t. C)* of two induced subgraphs G_1 and G_2, if G is an amalgam of G_1 and G_2 and the intersection $G_1 \cap G_2 := C$ is a square-boundary of both G_1 and G_2.

Observation 4.13. *G is a square-boundary amalgam of two induced subgraphs G_1 and G_2 w.r.t. C if and only of $G = G_1\, {}_C\!\circledast_C G_2$.*

Theorem 4.9 and Observation 4.13 can be used to show the following

Theorem 4.14. *A graph is a planar median graph if and only if it can be obtained from cubes and square-graphs by a sequence of square-boundary amalgamations.*

Proof. Let G be a planar median graph. By Theorem 4.9, G is a tree or a QS-graph. If G is a tree, then it is a square-graph. Otherwise, G is cyclic and thus, by definition, there is an ordered composition $G = G_1 \circledast G_2 \circledast \cdots \circledast G_k$ of basic QS-graphs, that is, cubes or cyclic square-graphs. In particular, the subgraphs $G_i \subseteq G$, $1 \le i \le k$ are induced. Hence, G is obtained from cubes and square-graphs by a sequence of square-boundary amalgamations. Conversely, if G is obtained from cubes and square-graphs by a sequence of square-boundary amalgamations, the graph G must be a tree or a QS-graph and thus, by Theorem 4.9, a planar median graph. ∎

5. Fast decomposition of planar median graphs into an ordered sequence of basic QS-graphs

Lemma 4.7 immediately implies a recursive strategy to determine an ordered composition of QS-graphs of a given planar median graph. Importantly, it is not necessary to find forbidden subgraphs as in the proof of Lemma 4.8, which we used in this proof to properly apply the induction step. First, we test if G is a planar median graph and, in the affirmative case, compute a planar embedding π of G and continue. If G is square-graph or a cube, we are done. Otherwise, G is not a basic QS-graph and thus, by Lemma 4.7, there is a square $C \subseteq G$ such that ${G^{\text{in}}}_{C,\pi} \neq G$ and ${G^{\text{out}}}_{C,\pi} \neq G$. There are two cases, either (i) both ${G^{\text{in}}}_{C,\pi}$ and ${G^{\text{out}}}_{C,\pi}$ are basic QS-graphs or (ii) at least one of them is not. In Case (i), we are done, since we found a decomposition of $G = {G^{\text{in}}}_{C,\pi}\, {}_C\!\circledast_C {G^{\text{out}}}_{C,\pi}$ into basic QS-graphs. In Case (ii), we recurse on the non-basic QS graph ${G^{\text{in}}}_{C,\pi}$ or ${G^{\text{out}}}_{C,\pi}$, resp., and repeat the latter until all such squares have been examined. In this recursion, we must, however, determine for all remaining squares C after we found a basic QS-graph H' if ${H^{\text{in}}}_{C,\pi} \neq H$ and ${H^{\text{out}}}_{C,\pi} \neq H$ where H is obtained from G by removing H' except for the square C, and, in particular, keep track of the order of the chosen factors to obtain an ordered composition of the input graph.

To address these issues, we will design a non-recursive algorithm instead. To this end, we will use a partial order on the set of all squares of G that is defined in terms of "almost-inside" w.r.t. (G, π) induced embedding

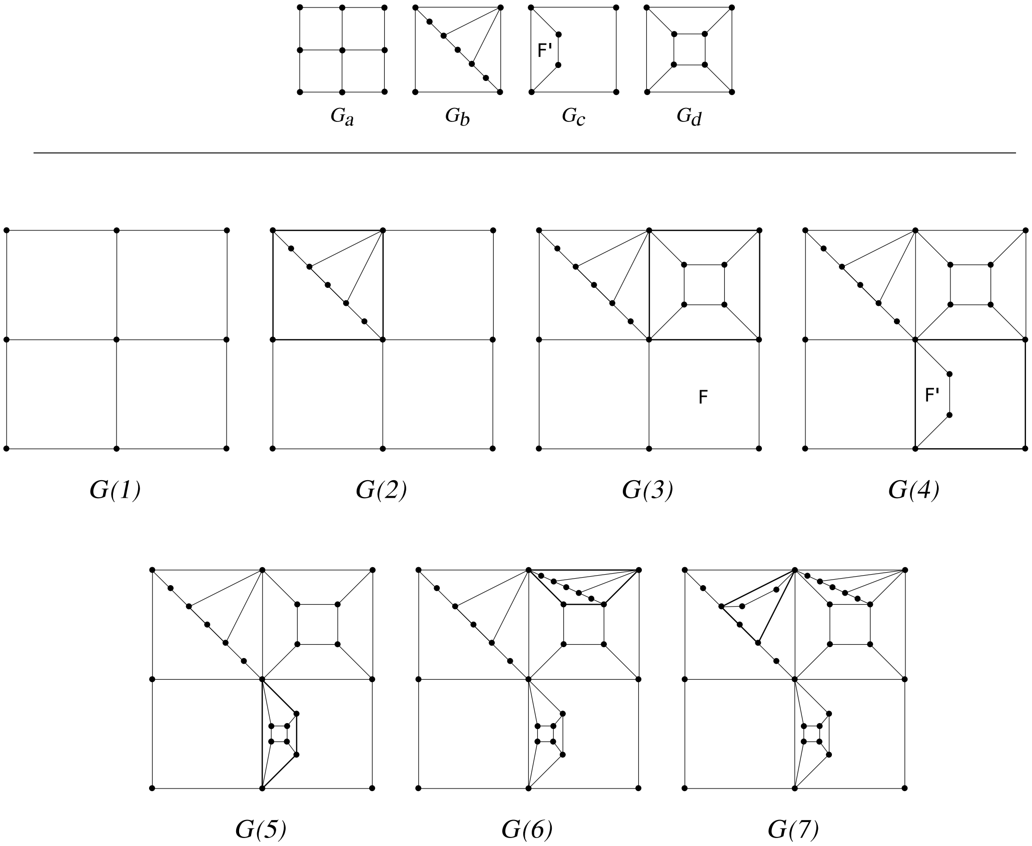

Fig. 5. In the upper part, four graphs G_a, G_b, G_c and G_d are shown. Here, $G_a \simeq G_b$ is a square-graph and G_d is a cube. Moreover, G_c is a square-graph since there is an alternative planar embedding σ such that all vertices become outer vertices, and thus, that each inner face is bounded by a square. By Definition 4.2, G_a to G_d are (basic) QS-graphs. Moreover, $G(1) = G_a$, and thus, $G(1)$ is a QS-graph. We have $G(7) = (((((G_a \circledast G_b) \circledast G_d) \circledast G_c) \circledast G_d) \circledast G_b) \circledast G_c$. Hence, every $G(i)$ with $i \in \{2, \dots, 7\}$ is precisely the graph $G(i-1) \circledast G_x$, where G_x is the appropriate QS-graph with $x \in \{b, c, d\}$ together with its shown planar embedding. Therefore, every $G(i)$ is a QS-graph. This example also shows that $\circledast$ is not associative, in general. To see this, consider the QS-graph $G(3) = (G_a\, {}_{C_a}\!\circledast_{C_b} G_b)\, {}_{C'_a}\!\circledast_{C_d} G_d$ where C_a and C'_a are the two squares in G_a that are "identified" with the squares C_b and C_d that form the outer boundary in G_b and G_d, respectively. Hence, we cannot write $G(3)$ as $G_a\, {}_{C_a}\!\circledast_{C_b} (G_b\, {}_{C'_a}\!\circledast_{C_d} G_d)$ since the square C'_a does not exist in G_b. In some cases, however, associativity is given. By way of example, consider $G(5) = (G(3)\, {}_{C}\!\circledast_{C_c} G_c)\, {}_{C_c}\!\circledast_{C_d} G_d$, where C is the square in $G(3)$ that bounds F, C_c the square that bounds F' and C_d the outer-boundary of G_d. It is easy to see that $G(5) \simeq G(3)\, {}_{C}\!\circledast_{C_c} (G_c\, {}_{C_c}\!\circledast_{C_d} G_d)$, since these constructions "overlap" on the cycle C_c.

Definition 5.1. A square $C' \subseteq G$ is *almost-inside* a square $C \subseteq G$, in symbols $C \preceq_{G,\pi} C'$, if all vertices of C' are almost-inside C w.r.t. the (G, π)-induced embedding of C. In particular, we write $C \prec_{G,\pi} C'$ if $C \preceq_{G,\pi} C'$ and $C \neq C'$.

In the following let $\mathcal{S}(G)$ denote the set of all squares contained in G.

Lemma 5.2. *For every π-embedded planar graph G, $(\mathcal{S}(G), \preceq_{G,\pi})$ is a partially ordered set.*

Proof. In the following, the terms "(almost-)inside" and "outside" refer to the (G, π)-induced embedding. By definition, $\preceq_{G,\pi}$ is reflexive. Moreover, if $C \preceq_{G,\pi} C' \preceq_{G,\pi} C''$, then all vertices of C are inside or part of C'. The same applies for C' and C''. Now, it is easy to see that $\preceq_{G,\pi}$ is transitive (i.e., $C \preceq_{G,\pi} C''$). We continue with showing that $\preceq_{G,\pi}$ is anti-symmetric. To this end, let $C, C' \in \mathcal{S}(G)$ such that $C \preceq_{G,\pi} C'$ and $C' \preceq_{G,\pi} C$. Assume, for contradiction, that $C \neq C'$. Since $C \preceq_{G,\pi} C'$ and $C \neq C'$, at least one vertex of C must be located inside of C' while all other vertices of C are almost-inside of C'. But then, at least one vertex of C' must be outside of C and thus, $C' \not\preceq_{G,\pi} C$; a contradiction. Therefore, $\preceq_{G,\pi}$ is anti-symmetric. In summary, $(\mathcal{S}(G), \preceq_{G,\pi})$ is a partially ordered set. ■

Next, we consider a condition for the nesting of squares in planar median graphs.

Lemma 5.3. *Let G be a π-embedded planar median graph and let $C, C' \in \mathcal{S}(G)$. If there is a vertex $v \in V(C)$ such that v is inside of C' w.r.t. (G, π)-induced embedding, then $C \prec_{G,\pi} C'$, i.e., C is almost-inside C' w.r.t. (G, π)-induced embedding.*

Proof. In the following, the terms "(almost-)inside" and "outside" refer to the (G, π)-induced embedding. Let $C, C' \in \mathcal{S}(G)$ and suppose that there is a vertex $v \in V(C)$ such that v is inside of C'. Assume, for contradiction, that C is not almost-inside of C'. By definition, there must be a vertex $w \in V(C)$ that is outside of C'. Hence, there is a square C that is not entirely contained in ${G^{\text{in}}}_{C',\pi}$ or ${G^{\text{out}}}_{C',\pi}$; a contradiction to Corollary 3.7. ■

Corollary 5.4. *Let G be a π-embedded planar median graph. Then, for all $C \in \mathcal{S}(G)$, there is a unique $\preceq_{G,\pi}$-maximal element $C' \in \mathcal{S}(G)$.*

Now, we define the rooted graph $\mathscr{F} := \mathscr{F}(G, \pi)$ for any given π-embedded planar median graph $G = (V, E)$. Let $W := V \setminus \left(\bigcup_{C\in\mathcal{S}(G)} V(C)\right)$ be the set of all vertices of G that are not contained in a square. The vertex set of $\mathscr{F}$ is $V(\mathscr{F}) = W \cup \mathcal{S}(G)$ and we add edges in the following cases:

- $\{C, C'\} \in E(\mathscr{F})$ with $C, C' \in \mathcal{S}(G)$ if and only if $C \prec_{G,\pi} C'$ and there is no $C'' \in \mathcal{S}(G)$ such that $C \prec_{G,\pi} C'' \prec_{G,\pi} C'$, and
- $\{C, x\} \in E(\mathscr{F})$ with $C \in \mathcal{S}(G)$ and $x \in W$ if and only if x is inside C and there is no $C' \in \mathcal{S}(G)$ such that $C' \prec_{G,\pi} C$ and x is inside C'.

To root this graph $\mathscr{F}$, observe first that if the outer boundary of G is a square C, then there must be a path from C to all other vertices in $\mathscr{F}$ and thus C is the unique $\preceq_{G,\pi}$-maximal element for all squares and vertices in $V(\mathscr{F})$. In this case, $\mathscr{F}$ must be connected and we choose C as its root. If $\mathscr{F}$ is connected but $\mathcal{S}(G) = \emptyset$, then $\mathscr{F}$ consists of a single vertex $x \in W$ which is chosen as the root. If $\mathscr{F}$ is disconnected, then every connected component T of $\mathscr{F}$ is either a single vertex $x \in W$ in which case x is chosen as the root of T, or T contains vertices in $\mathcal{S}(G)$ in which case the unique $\preceq_{G,\pi}$-maximal element of the squares that are contained in the vertex set of T is chosen as the root of T. This unique $\preceq_{G,\pi}$-maximal element exists due to Corollary 5.4.

Lemma 5.5. *Let G be a π-embedded planar median graph. Then, $\mathscr{F}(G, \pi)$ is a rooted forest. In particular, $\mathscr{F}(G, \pi)$ is a tree if the outer boundary of G w.r.t. π is a square.*

Proof. By construction and the arguments preceding this lemma, $\mathscr{F} := \mathscr{F}(G, \pi)$ is a rooted graph, i.e., all its connected components T are rooted at the unique $\preceq_{G,\pi}$-maximal element of the squares in T or, in case $V(T) = \{x\} \subseteq W$, T is rooted at x. For simplicity, we extend in this proof the partial order $\preceq_{G,\pi}$ $\mathcal{S}(G)$ to an order $\preceq^*_{G,\pi}$ on $\mathcal{S}(G) \cup W$ by putting $C \preceq^*_{G,\pi} C'$ whenever $C \preceq_{G,\pi} C'$ for all $C, C \in \mathcal{S}(G)$ and $x \preceq^*_{G,\pi} C$ for all edges $\{C, x\} \in E(\mathscr{F})$ with $C \in \mathcal{S}(G)$ and $x \in W$. It is easy to verify that $(\mathcal{S}(G) \cup W, \preceq^*_{G,\pi})$ remains a partially ordered set.

Now, we assume for contradiction that $\mathscr{F}$ is not a forest. Hence, it must contain a cycle. Let T be a connected component that contains a cycle C_T. Since $\preceq^*_{G,\pi}$ is a partial order on the set $\mathcal{S}(G) \cup W$ of all squares of G and all vertices not contained in squares, the cycle C_T must contain a vertex $v \in V(T)$, such that $v \preceq^*_{G,\pi} w$ for all $w \in C_T$. Note, v corresponds either to a square in G or a vertex that is not contained in a square. Now, consider the two vertices w and w' that are adjacent to v in C_T. Since $v \preceq^*_{G,\pi} w, w'$, the vertices w and w' coincide, by definition of $\preceq_{G,\pi}$, with two squares C and C' of G, respectively. Furthermore, we have $C \neq C'$.

However, $C \prec^*_{G,\pi} C'$ is not possible since, in this case, the edge $\{C', v\}$ would not exist in $\mathscr{F}$, by definition. Similarly, $C' \prec^*_{G,\pi} C$ is not possible. However, all vertices of one square, say C, must be almost-inside C', as otherwise, C is not entirely contained in ${G^{\text{in}}}_{C',\pi}$ and ${G^{\text{out}}}_{C',\pi}$ and we would obtain a contradiction to Corollary 3.7. But then, $C \prec^*_{G,\pi} C'$; a contradiction. Hence, $\mathscr{F}$ cannot contain cycles and is therefore, a forest.

By the arguments preceding this lemma, if the outer boundary of G is a square C, then there must be a path from C to all other vertices in $\mathscr{F}$ and thus, $\mathscr{F}$ is a tree. By construction, this tree is rooted at C. □

Since $\mathscr{F} = \mathscr{F}(G, \preceq_{G,\pi})$ is a forest and based on the definition of edges $\{C, x\} \in E(T)$ with $C \in \mathcal{S}(G)$ and $x \in W := V \setminus \left(\bigcup_{C\in\mathcal{S}(G)} V(C)\right)$, every $x \in W$ in $\mathscr{F}$ must be a leaf (i.e., it has degree one in $\mathscr{F}$) or a singleton (i.e., it is an isolated vertex in $\mathscr{F}$).

Note that the structure of $\mathscr{F}$ does not completely determine the structure of G, since it only accounts for the "hierarchy" of the nested squares. As an example, consider $\mathscr{F}(G, \preceq_{G,\pi}) = \mathscr{F}(G', \preceq_{G',\pi'}) = (\{C_1, C_2\}, \emptyset)$ for the graph G, resp., G', where all vertices are located at the outer boundary and where G, resp., G', consists precisely of two squares identified on a single vertex, resp., identified on a single edge. The forest $\mathscr{F}$, however, does determine whether or not ${G^{\text{in}}}_{C,\pi} \neq G$ for the squares $C \subseteq G$, since ${G^{\text{in}}}_{C,\pi} \neq G$ if and only if there are vertices that are contained outside of C and thus, there must be vertices or squares on the same level of C in $\mathscr{F}$ or squares above the level of C in the connected component T of in $\mathscr{F}$ that contains C. In a similar way, one can determine whether or not ${G^{\text{out}}}_{C,\pi} \neq G$.

The latter observation can also be applied to subgraphs of G in order to find basic QS-graphs as follows. Let us consider a connected induced subgraph $H \subseteq G$ where H contains only squares and vertices of G that are located in level i and $i+1$ but none of the squares and vertices of other levels j with $j < i$ and $j > i+1$. Then, for every square C in H from level i the equality ${H^{\text{in}}}_{C,\pi} = H$ holds and for every square C in H from level $i+1$ the equality ${H^{\text{out}}}_{C,\pi} = H$ holds. Hence, for all squares $C \subseteq H$ either ${H^{\text{in}}}_{C,\pi} = H$ or ${H^{\text{out}}}_{C,\pi} = H$. By Lemma 4.7, H is a basic QS-graph, provided that H is a median graph (a property that is always satisfied as shown in the proof of Lemma 5.6). Hence, in order to find a composition of a planar median graph into basis QS-graphs, we traverse $\mathscr{F}$ in top-down fashion from level to level and, in principle, use

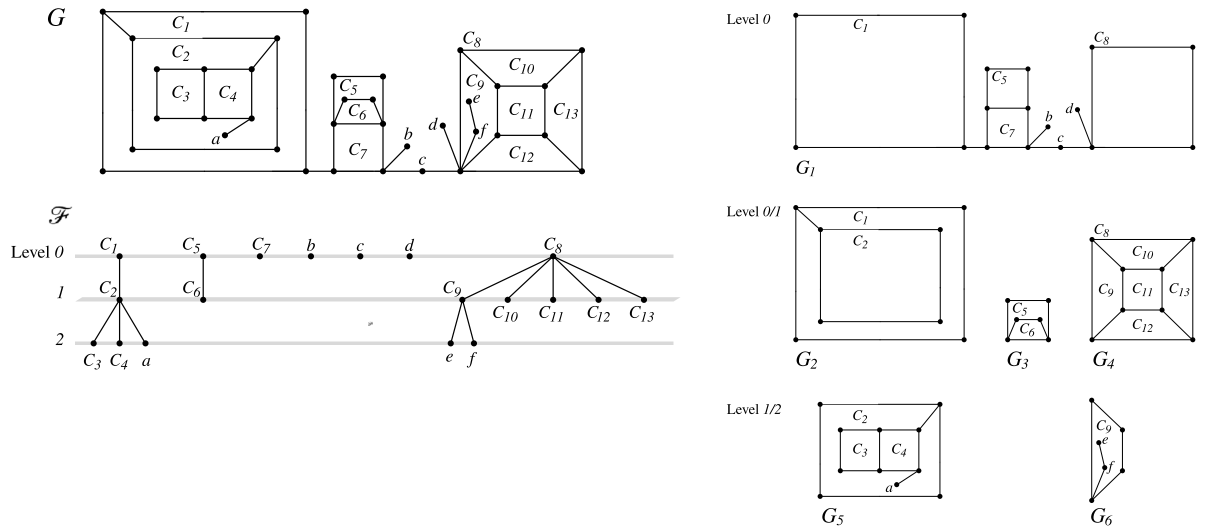

Fig. 6. A π-embedded planar median graph G and its forest $\mathscr{F} := \mathscr{F}(G,\pi)$. For better readability, set $G_i^{\text{out}}(C) := H^{\text{out}}{}_{C,\pi}$ and $G_i^{\text{in}}(C) := H^{\text{in}}{}_{C,\pi}$ for $H = G_i$. The vertices of $\mathscr{F}$ correspond to the squares of G and to the vertices of G that are not part of a square. The vertices of $\mathscr{F}$ are horizontally aligned based on the level in which they occur (highlighted by gray lines). According to Algorithm 1, we compute first the subgraph of G that is induced by the vertices (of squares) on Level 0 and we obtain the graph G_1. Since for all squares C of G_1 the equality $G_1^{\text{out}}(C) = G_1$ holds. Lemma 4.7 implies that G_1 must be a basic QS-graph. We then proceed in the next step to consider the subgraph H of G that is induced by the vertices (of squares) on Level 0 and 1 that are not isolated in $\mathscr{F}$ (cf. Line 10). Hence, H is induced by the vertices $V(G)\setminus(V(C_7)\cup\{b,c,d\})$. The connected components of H yield the factors G_2, G_3 and G_4. Here, G_2 corresponds to the subtree of $\mathscr{F}$ consisting of the edge $\{C_1, C_2\}$. According to Algorithm 1, C_1 is the cycle in level 0 which is used to identify G_1 and G_2, and we obtain $G_1\,{}_{C_1}\circledast_{C_1} G_2$. For G_2 we have $G_2^{\text{in}}(C_1) = G_2$ as well as $G_2^{\text{out}}(C_2) = G_2$ and thus, by Lemma 4.7, G_2 is a basic QS-graph. Proceeding in this manner, we obtain the final decomposition $G = ((((G_1\,{}_{C_1}\circledast_{C_1} G_2)\,{}_{C_5}\circledast_{C_5} G_3)\,{}_{C_8}\circledast_{C_8} G_4)\,{}_{C_2}\circledast_{C_2} G_5)\,{}_{C_9}\circledast_{C_9} G_6$ of G into basic QS-graphs.

Algorithm 1 `Ordered Composition of Planar Median Graphs into basic QS-graphs.`

Input: Graph $G = (V, E)$
Output: Ordered composition $G = (\dots(G_1\,{}_{C_2^*}\circledast_{C_2^*} G_2)\dots)\,{}_{C_k^*}\circledast_{C_k^*} G_k$ of basis QS-graphs, if G is a planar median graph and otherwise, return "`false`"

1: **if** G is not planar or a median graph **then return** "`false`"
2: **else if** G is a square or a tree **then return** G
3: **else** Compute planar embedding π of G
4: Compute forest $\mathscr{F} := \mathscr{F}(G,\pi)$
5: $j \leftarrow 1$
6: **if** $\mathscr{F}$ is disconnected **then**
7: $\quad V' \leftarrow$ subset of vertices of G corresponding to squares and vertices in $\mathscr{F}$ on level 0
8: $\quad G_j \leftarrow G[V']$ and $j \leftarrow j+1$
9: **for** $i = 1$ to *last_level* of $\mathscr{F}$ **do**
10: $\quad V' \leftarrow$ subset of vertices of G corresponding to non-isolated squares and vertices in $\mathscr{F}$ on level $i-1$ and i
11: $\quad$ **for** all connected components H in $G[V']$ **do**
12: $\quad\quad G_j \leftarrow H$
13: $\quad\quad C_j \leftarrow$ square from level $i-1$
14: $\quad\quad j \leftarrow j+1$
15: **return** $G = (\dots(G_1\,{}_{C_2}\circledast_{C_2} G_2)\dots)\,{}_{C_{j-1}}\circledast_{C_{j-1}} G_{j-1}$

the connected components of the subgraphs of G that are determined by the vertices x and squares C on level i and $i+1$ as basic QS-graphs, see Fig. 6 for an illustrative example. The pseudocode of this approach is summarized in Algorithm 1.

Lemma 5.6. *Algorithm 1 determines whether a graph G is a planar median graph and, in the affirmative case, it returns an ordered decomposition of G into basic QS-graphs. Unless G is a square, none of the factors is the unit element C_4.*

Proof. Let G be an arbitrary graph. Line 1 ensures that G is a planar median graph and we compute in Line 3 a respective planar embedding of G. In particular, Line 2 ensures that if G is a tree or a square, then G is returned and we obtain the trivial composition $G = G$. Else, G is a cyclic planar median graph having at least five vertices. As argued above, the forest $\mathscr{F}$ computed in Line 4 is well-defined. For better readability, we use in the following the notation $G^{\text{in}}(*)$ and $G^{\text{out}}(*)$ instead of $G^{\text{in}}{}_{*,\pi}$ and $G^{\text{out}}{}_{*,\pi}$, respectively.

Assume that $\mathscr{F}$ is disconnected (Line 6), then the outer boundary of G cannot be a square. In particular, $\mathscr{F}$ must have at least two connected components. By definition, all connected components in $\mathscr{F}$ are trees whose vertices correspond to single vertices in G or to squares $C \subseteq G$. We collect in V' all such vertices of G and the vertices of these squares of G that are in level 0 of $\mathscr{F}$ (Line 7) and put $G_1 = G[V']$. Let $C_1, \dots, C_k$ be the respective squares that are in level 0 of $\mathscr{F}$. Note, since G is a cyclic median graph, at least one such square must exist. Recall that G is planar and $\{K_3, K_{2,3}\}$-free. By construction, $G_1 = \cap_{i=1}^{k} G^{\text{out}}(C_i)$, and by Lemma 2.8, G_1 is a convex subgraph of G. Proposition 2.14 (4b) implies that G_1 is a median graph. This, together with the fact that $G_1^{\text{out}}(C) = G_1$ for all squares $C \subseteq G_1$ and Lemma 4.7, implies that G_1 is a basic QS-graphs.

Now, we proceed on all levels $i = 1$ to the last level of $\mathscr{F}$ (Line 9), which covers also the case that $\mathscr{F}$ is connected. Let us assume we are in some step i. In this case, we consider the squares and vertices in level $i-1$ and i. We collect in V' all such vertices v of G and the vertices of the squares of G that are in level i and $i-1$ of $\mathscr{F}$ (Line 10). Now, let H be some connected component of $G[V']$, and consider the respective connected of component T in $\mathscr{F}$ that corresponds to H. The vertex in level $i-1$ of T must correspond to a square C for which all vertices on level i in T are almost-inside C. We set $G_j := H$. Since C is an outer boundary of G_j, we have $G_j^{\text{in}}(C) = G_j$. All other squares $C' \neq C$ of G_j must be inner squares that satisfy $G_j^{\text{out}}(C') = G_j$ by construction. In case that G_j is a planar median graph, we can apply the contraposition of Lemma 4.7 to conclude that G_j must be a basic QS-graphs. Clearly, G_j is planar. Thus, it remains to show that G_j is a median graph. Recall that G is planar and $\{K_3, K_{2,3}\}$-free. Let $C'_1, \dots, C'_k$ be the squares of G_j that correspond to the vertices of T in level i. By construction, $G_j = G^{\text{in}}(C) \bigcap (\cap_{i=1}^{k} G^{\text{out}}(C'_i))$ and Lemma 2.8 implies that G_j is a convex subgraph of G. By Proposition 2.14 (4b), G_j is a median graph. By the aforementioned arguments, G_j is a basic QS-graph.

Thus, all graphs G_j computed by Algorithm 1 are basic QS-graphs. We traverse the connected components of $G[V']$ in Step i according to the subtrees in $\mathscr{F}$ with vertices in level $i-1$ and i. Since we consider in Line 10 only vertices that correspond to squares and vertices that are not isolated in $\mathscr{F}$ on level $i-1$ and i, for every connected component $G_j \subseteq G[V']$ that is examined in Step i in Lines 12–14, the graph G_j is based on the squares and vertices that correspond to *adjacent* vertices in $\mathscr{F}$ that are in level $i-1$ and i. Let x be a vertex in $\mathscr{F}$ on level i and assume that x corresponds to the square $C_x \subseteq G$ in G_j. Moreover, let V'' be the set of all such vertices computed in Step $i+1$ in Line 10. The square C_x will be part of some connected component $G_{j'}$ of $G[V'']$. Note, $G_{j'}$ cannot be a square, since we consider on level i and $i+1$ non-isolated vertices in $\mathscr{F}$. Otherwise, $G_{j'}$ is not the unit element and all remaining squares and vertices distinct from C_x are the adjacent vertices of x in $\mathscr{F}$ in level $i+1$. We set $C_{j'} = C_x$ (Line 13). Since $C_{j'} = C_x$ is contained in the previous factor G_j we can ensure that $(\dots)_{C_{j'}} \circledast_{C_{j'}} G_{j'}$ is well-defined in each step. The latter arguments also imply that only factors that are distinct from the unit-element are used. ■

Let G be an outer-planar median graph and π be a planar embedding such that all vertices of G are incident with the outer face. Hence, G is a tree, a square, or $\mathscr{F}(G, \pi)$ contains only vertices that are located at level 0 of $\mathscr{F}(G, \pi)$. In this case, either the graph G is returned in Line 2 or the *if*-condition in Line 6 is executed and, afterwards, G is immediately returned. In both cases, Algorithm 1 returns the basic QS-graph G. This, together with the fact that cubes are not outer-planar, implies

Corollary 5.7. *Every outer-planar median graph is a square-graph.*

Using Corollary 5.7, we can say even more. Given a 2-connected square-graph G, we let $D(G)$ be the graph with vertex set consisting of the squares in G and edge-set consisting of pairs of squares that share an edge. We call a square-graph G *arboreal* if $D(G)$ is a forest.

Theorem 5.8. *A graph G is an outer-planar median graph if and only if it is an arboreal square-graph.*

Proof. If G is an arboreal square-graph, then it is, in particular, a median graph. Hence, G is $K_{2,3}$-free and thus, every two squares share at most one edge. Assume, for contradiction, that G is not outer-planar. Consider the planar embedding σ of G. Since G is not outer-planar, G must contain inner vertices of degree $k \geq 4$ (w.r.t. σ). Now, it is straight-forward to check that $D(G)$ must contain a cycle of length k. Thus, $D(G)$ is not a forest; a contradiction. Therefore, G must be outer-planar.

Conversely, if G is an outer-planar median, then Corollary 5.7 implies that G is a squaregraph. Since G consists of outer vertices only and since every square is square-boundary (w.r.t. σ) it is straight-forward to check that $D(G)$ must be a forest and, therefore, G is arboreal. □

We proceed with investigating the running time of Algorithm 1.

Lemma 5.9. *Algorithm 1 can be implemented to run in $\mathcal{O}(|V| \log |V|)$ time for every input graph $G = (V, E)$.*

Proof. In the following, let $n = |V|$ and $m = |E|$. Then $\mathcal{O}(m + n)$ time is required to check whether a graph G is planar [19,28], and a median graph [29] in Line 1. Since a planar graph has at most $m \leq 3(n-2)$ edges for all $n \geq 3$, we have $\mathcal{O}(m) \subseteq \mathcal{O}(n)$. Testing for planarity first ensures that no graphs that violate this condition are processed further, which implies that testing whether G is a median graph can be done in $\mathcal{O}(n)$ time as well. Line 2 can be done in $\mathcal{O}(n)$ time.

All squares in a planar graph can be coded efficiently in $\mathcal{O}(m) \subseteq \mathcal{O}(n)$ time [25, Thm. 20.3 and 20.5]. Moreover, by Corollary 4.11, G has $\mathcal{O}(n)$ squares. Hence, all squares can be identified in $\mathcal{O}(n)$ time.

In Line 3, we compute a planar embedding π of G and are, in particular, interested in an embedding such that all edges are straight lines which can be done in $\mathcal{O}(n \log n)$ time [24]. The construction of a forest describing the nesting of the squares and the vertices not contained in a square can then be obtained by a modified version of the $\mathcal{O}(n \log n)$ time algorithm solving the polygon nesting problem for disjoint (not necessarily convex) polygons [2]. In brief, the non-square vertices can be treated as "polygons" consisting of a single vertex and pose no problem. In contrast to [2], we may have squares that share vertices and edges. If, during the line-sweep step, a vertex u is encountered that is contained in more than one square, one can determine the nesting of these incident squares by considering clock-wise ordering of the corresponding edges pointing to the right of the sweeping line. If multiple squares share the same first edge in this ordering, then their nesting is defined by ordering of the second edge. If both edges are shared, then the nesting is determined by the horizontal coordinate of the fourth point. For n_x squares sharing a point x, the nesting can thus be computed by sorting the $2n_x$ edges incident with x at most thrice, and thus in $\mathcal{O}(n_x \log n_x)$ time. The total effort for disentangling squares with common points thus is also bounded by $\mathcal{O}(n \log n)$.

The forest $\mathscr{F}$ has $\mathcal{O}(n)$ vertices and edges and can, thus, be traversed in $\mathcal{O}(n)$ time. Finding induced subgraphs $G[V']$ with n' vertices and m' edges can be done in $\mathcal{O}(n' + m')$ time. In each step i we have the vertices from graphs level $i-1$ and i with n_{i-1} vertices and m_{i-1} edges from level $i-1$ and n_i vertices and m_i edges from level i. The time needed to compute all the subgraphs used in the computation therefore adds up to $\mathcal{O}(2(m+n)) = \mathcal{O}(n)$.

The overall complexity is therefore $\mathcal{O}(n \log n)$. ■

Consider a π-embedded planar graph G for which $G^{\text{out}} := {G^{\text{out}}}_{C,\pi}$ and $G^{\text{in}} := {G^{\text{in}}}_{C,\pi}$ are QS-graphs. In this case, G^{out} has an ordered composition of the form $G^{\text{out}} = (\dots(G_1 \circledast G_2) \circledast \cdots G_{\ell-1}) \circledast G_\ell$ of $\ell \geq 1$ basic QS-graphs. Similarly, $G^{\text{in}} = (\dots(H_1 \circledast H_2) \circledast \cdots H_{k-1}) \circledast H_k$ is composed of $k \geq 1$ basic QS-graphs. Since this ordered composition is not associative, we cannot in general write

$$G^{\text{in}}{}_C \circledast {}_C G^{\text{out}} = (\dots((((\dots(H_1 \circledast H_2) \circledast \cdots H_{k-1}) \circledast H_k)_C \circledast {}_C\, G_1) \circledast G_2) \circledast \cdots G_{\ell-1}) \circledast G_\ell \,. \tag{2}$$

In particular, this expression is not well-defined whenever C is not contained in G_1. Algorithm 1, however, makes it possible to find ordered compositions for both G^{out} and G^{in} such that we can write G in the form of Eq. (2).

Proposition 5.10. *Let G be a connected π-embedded planar graph and $C \subseteq G$ be a square. Suppose that $G^{\text{out}} := {G^{\text{out}}}_{C,\pi}$ and $G^{\text{in}} := {G^{\text{in}}}_{C,\pi}$ are QS-graphs, and the factorization $G^{\text{out}} = (\dots(G_1 \circledast G_2) \circledast \cdots G_{\ell-1}) \circledast G_\ell$ of G^{out} and $G^{\text{in}} = (\dots(H_1 \circledast H_2) \circledast \cdots H_{k-1}) \circledast H_k$ of G^{in} has been computed with Algorithm 1 w.r.t. the (G, π)-induced planar embedding of G^{out} and G^{in}, respectively. Then, $(\dots((((\dots(H_1 \circledast H_2) \circledast \cdots H_{k-1}) \circledast H_k) \circledast\; G_1) \circledast G_2) \circledast \cdots G_{\ell-1}) \circledast G_\ell$ is well-defined and yields a factorization of G into basic QS-graphs.*

Proof. Theorem 4.10 implies that G is a QS-graph. The outer boundary of G^{in} w.r.t. (G, π)-induced embedding is the square C and thus, $\mathscr{F}(G^{\text{in}}, \pi)$ is connected and rooted at C. The square C serves an inner boundary in G^{out} w.r.t. (G, π)-induced embedding and there are no further vertices of G^{out} inside C. Hence C is a leaf in $\mathscr{F}(G^{\text{out}}, \pi)$. Now, it is easy to see that $\mathscr{F}(G, \pi)$ is identical to the forest that is obtained from $\mathscr{F}(G^{\text{out}}, \pi)$ and $\mathscr{F}(G^{\text{in}}, \pi)$ by identifying the leaf C of $\mathscr{F}(G^{\text{out}}, \pi)$ with the root C of $\mathscr{F}(G^{\text{in}}, \pi)$. In a similar fashion as in Algorithm 1, we traverse $\mathscr{F}(G, \pi)$ but use first only the vertices that are contained in $\mathscr{F}(G^{\text{out}}, \pi)$ in the same order that yields the factorization $G^{\text{out}} = (\dots(G_1 \circledast G_2) \circledast \cdots G_{\ell-1}) \circledast G_\ell$ provided by Algorithm 1 applied on G^{out} and afterwards, we traverse the subtree $\mathscr{F}(G^{\text{in}}, \pi)$ in the same order as in Algorithm 1 to obtain the factorization $G^{\text{in}} = (\dots(H_1 \circledast H_2) \circledast \cdots H_{k-1}) \circledast H_k$. This yields the factorization $G = (\dots((((\dots(H_1 \circledast H_2) \circledast \cdots H_{k-1}) \circledast H_k)_C \circledast {}_C\, G_1) \circledast G_2) \circledast \cdots G_{\ell-1}) \circledast G_\ell$. ■

The result of Algorithm 1 depends crucially on the chosen planar embedding. To see this, consider the vertex b in the graph G shown in Fig. 6. Placing b into the face bounded by the square C_7 yields an additional factor that is isomorphic to a square to which an additional vertex is attached. Another example is shown in Fig. 7.

By Theorem 4.10 $G = G_1 \circledast G_2$ is a QS-graph whenever G_1 and G_2 are QS-graphs. We say that a QS-graph G is *irreducible* if $G = G_1 \circledast G_2$ implies that G_1 or G_2 is the unit element, i.e., a square. Irreducible QS-graphs are for example the "domino" $P_3 \Box K_2$ or a square to which a single edge is attached. The composition of (G, π'') in Fig. 7 consists of irreducible QS-graphs only. The observations above imply that basic QS-graphs are neither necessarily irreducible nor that there is a unique way to decompose a planar median into basic QS-graphs. Moreover, Fig. 8 provides an example of a graph $H = (G \circledast G) \circledast (G \circledast G)$ that cannot be written as $((G \circledast G) \circledast G) \circledast G$, since the square C used in each "$\circledast$-step" is not a square-boundary in $((G \circledast G) \circledast G)$. Thus, although each factor G of H is irreducible the graph H cannot be written as ordered composition $((G \circledast G) \circledast G) \circledast G$ of irreducible graphs. The latter observation suggests to consider the following notation. A graph G that has an ordered composition $G = G_1 \circledast \cdots \circledast G_k$ of irreducible QS-graphs G_i is called *irreducible-representable*. Moreover, a graph G that has an ordered composition $G = G_1 \circledast \cdots \circledast G_k$ of irreducible or irreducible-representable QS-graphs G_i is called *almost irreducible-representable*. In addition, an ordered composition $G = G_1 \circledast \cdots \circledast G_k$ is *irreducible-well-formed* if every factor G_i is irreducible, irreducible-representable, or has a irreducible-well-formed ordered composition. By way of example, the graph $H = G_1 \circledast G_2$ with $G_1 \simeq G_2 \simeq G \circledast G$ in Fig. 8 is irreducible-well-formed as well as almost irreducible-representable since both factors G_1 and G_2 are irreducible-representable. We suspect that the following statements are true.

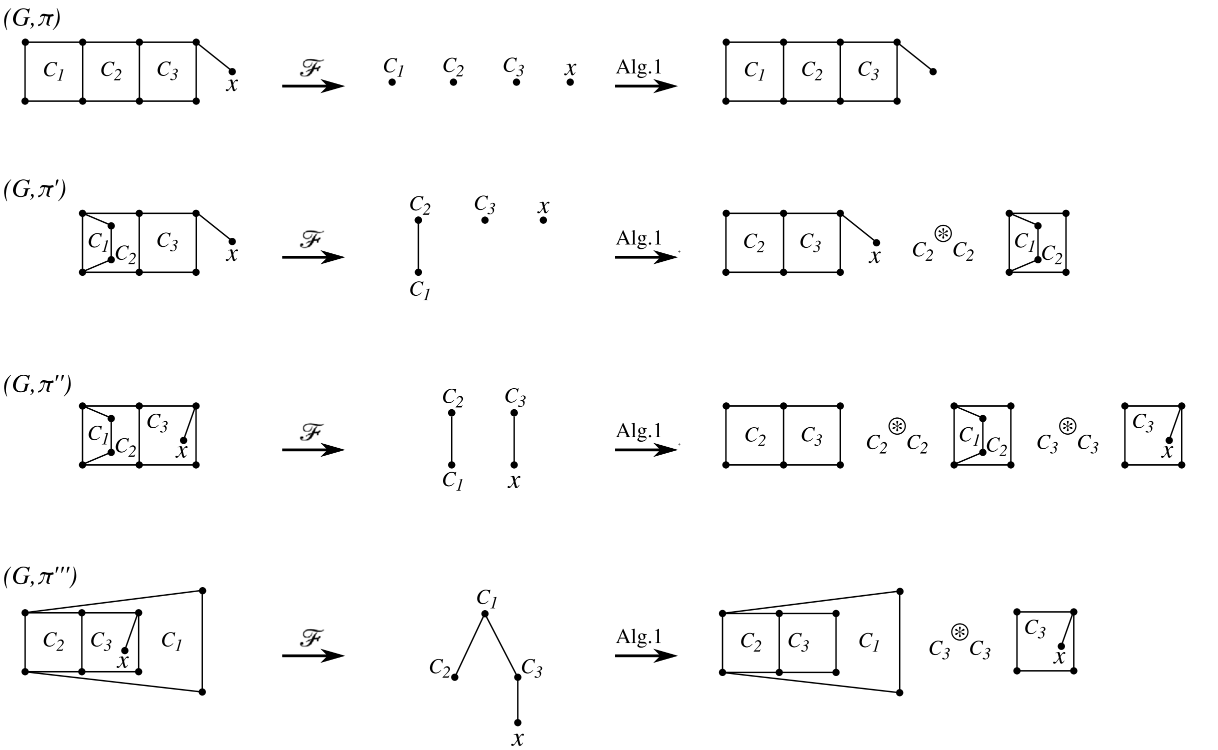

Fig. 7. Four different planar embeddings π, π', π'', and π''' of the graph G result in different forests $\mathscr{F}$. Depending on the particular embedding, Algorithm 1 returns different solutions: For (G,π) we obtain $G = G$; for (G,π') we obtain $G = G' {}_{C_2}\circledast_{C_2} H$; for (G,π'') we obtain $G = (H {}_{C_2}\circledast_{C_2} H) {}_{C_3}\circledast_{C_3} G''$ and for $(G,\pi''') = G''' {}_3\circledast_3 G''$. It can easily be verified that the factorization of (G,π'') contains irreducible factors only.

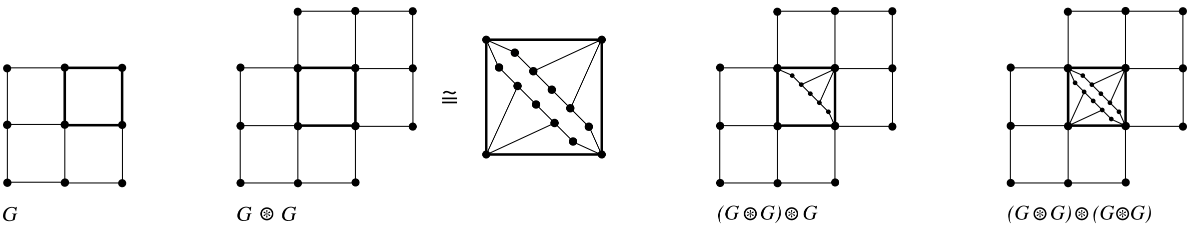

Fig. 8. Shown is a square-graph G as well as the compositions $G \circledast G$, $(G \circledast G) \circledast G$ and $(G \circledast G) \circledast (G \circledast G)$ with $\circledast := {}_C\circledast_C$. The square C that is used in each $\circledast$-step is highlighted with bold-lined edges. Note, C is not a square-boundary in $(G \circledast G) \circledast G$ and thus, $((G \circledast G) \circledast G) \circledast G$ is not well-defined.

Conjecture 1. *Every irreducible-representable planar median graph G has a unique composition $G = G_1 \circledast \cdots \circledast G_k$, $k \geq 1$ of irreducible QS-graphs G_i, $1 \leq i \leq k$ up to isomorphism and possible re-order of the factors.*

Conjecture 2. *Every almost irreducible-representable planar median graph G has a unique composition $G = G_1 \circledast \cdots \circledast G_k$, $k \geq 1$ of irreducible or irreducible-representable QS-graphs up to isomorphism and possible re-order of the factors.*

Conjecture 3. *Every planar median graph G has a unique irreducible-well-formed composition $G = G_1 \circledast \cdots \circledast G_k$ up to isomorphism and possible re-order of the factors.*

6. Summary and outlook

In this contribution, we have provided novel characterizations for planar median graph. Theorem 3.5 makes use of forbidden subgraphs and the structure of their isometric cycles. Theorem 3.9, furthermore, shows that it is sufficient to consider the induced subgraphs $G^{\text{in}}{}_{C,\pi}$ and $G^{\text{out}}{}_{C,\pi}$ almost-inside and almost-outside of an arbitrary square $C \subseteq G$. A more constructive characterization is obtained in terms of the gluing operation $\circledast$ for QS-graphs that stepwise identifies square-boundaries along which two graphs are identified. Theorem 4.9 shows that planar median graphs are exactly the union of QS-graphs and trees. The operation $\circledast$ corresponds to a specific amalgamation of graphs and provides a corresponding characterization of planar median graphs by amalgamation (cf. Theorem 4.14). The structure of QS-graphs leads to an $\mathcal{O}(n\log(n))$ time algorithm that computes an ordered composition $G = G_1 \circledast \cdots \circledast G_k$ into square-graphs and cubes for a planar median graph G with n vertices.

It would be interesting to know if these results might be extended to planar partial cubes (noting that any median graph is a partial cube). As with median graphs, partial cubes arise by isometric expansions [13], and planar partial cubes can be characterized by an expansion procedure [21]. The latter reference emphasizes that subtle constraints on the 2-fact expansions proposed in [39] are critical by showing that an additional non-crossing condition is necessary. The $\circledast$ operation is closely related to the non-crossing 2-face expansion employed in [21] used to characterize partial cubes: H is a 2-face expansion of G if G_1 and G_2 have planar embeddings such that $G' := G_1 \cap G_2$ lies on a face in both the respective embeddings. If G' is an edge $e = uv$, then u and v are trivially located on the same face in G_1 and G_2. Restricting G' to be an edge leads to a definition of a "restricted 2-face expansion" that can be expressed in terms of our gluing operation: We expand only G_1 on this edge (to get a square C) and only expand G_2 on this edge (to get a square C') and then set $H = G_1\,{}_C\!\circledast_{C'} G_2$. Every restricted 2-face expansion can therefore be expressed as "expand single edges in G_1 and G_2 and glue together G_1 and G_2 along the resulting squares via the $\circledast$ operation". It could therefore be worth while investigating if a variant of the $\circledast$ operation could be used to give new insights into the structure of planar partial cubes.

In another direction, define a planar median graph G as *irreducible* if $G = G_1 \circledast G_2$ implies that G_1 or G_2 is the unit element, i.e., a square. It would be interesting to understand whether one can decompose a given planar median graph into irreducible factors in polynomial-time. Moreover, does every planar median graph admit a unique irreducible-well-formed composition? Algorithm 1 depends crucially on the particular planar embedding of G. Does Algorithm 1 yield the same factors (possibly in a different order) if the planar embeddings differ only by the choice of the outer boundary? Answers to the latter question would possibly provide an avenue to determine the irreducible factors — at least for 3-connected planar median graphs, since all their planar embedding are equivalent. Moreover, one may ask how different forests $\mathscr{F}(G, \preceq_{G,\pi})$ and $\mathscr{F}(G, \preceq_{G,\pi'})$ are related to each other for different embeddings π and π' in the case that G has unique composition of irreducible QS-graphs?

Finally, Theorem 3.5 provides a characterization of planar median graphs in terms of forbidden subgraphs and the structure of their isometric cycles. Note, there is no forbidden subgraph characterization of planar median graphs, since the property of being a median graph is not hereditary. However, it is natural to ask whether a planar median graph can be solely characterized amongst median graphs in terms of a collection of forbidden subgraphs or minors. A good starting point for answering this question could be to understand how either 2-face expansions, convex face expansions or square-boundary amalgamations might shed new light on the forbidden subgraph theorem for square-graphs mentioned above in Proposition 2.19, with the view to extending these considerations to planar median networks.

Data availability

No data was used for the research described in the article.

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