Independence Relations in Abstract Elementary Categories

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Abstract

In model theory, a branch of mathematical logic, we can classify mathematical structures based on their logical complexity. This yields the so-called stability hierarchy. Independence relations play an important role in this stability hierarchy. An independence relation tells us which subsets of a structure contain information about each other, for example: linear independence in vector spaces yields such a relation.

Some important classes in the stability hierarchy are stable, simple and NSOP_1, each being contained in the next. For each of these classes there exists a so-called Kim-Pillay style theorem. Such a theorem describes the interaction between independence relations and the stability hierarchy. For example: simplicity is equivalent to admitting a certain independence relation, which must then be unique.

All of the above classically takes place in full first-order logic. Parts of it have already been generalised to other frameworks, such as continuous logic, positive logic and even a very general category-theoretic framework. In this thesis we continue this work.

We introduce the framework of AECats, which are a specific kind of accessible category. We prove that there can be at most one stable, simple or NSOP_1-like independence relation in an AECat. We thus recover (part of) the original stability hierarchy. For this we introduce the notions of long dividing, isi-dividing and long Kim-dividing, which are based on the classical notions of dividing and Kim-dividing but are such that they work well without compactness.

Switching frameworks, we generalise Kim-dividing in NSOP_1 theories to positive logic. We prove that Kim-dividing over existentially closed models has all the nice properties that it is known to have in full first-order logic. We also provide a full Kim-Pillay style theorem: a positive theory is NSOP_1 if and only if there is a nice enough independence relation, which then must be given by Kim-dividing.

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In memoriam

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1

Introduction

Independence relations are a central notion in model theory. They go back to Shelah's notion of forking independence in stable theories [She90], which generalises, for example, linear independence in vector spaces and algebraic independence in algebraically closed fields. In a stable first-order theory forking independence enjoys many nice properties. Later, in work of Kim and Pillay [Kim98, KP97] it was shown that forking independence also satisfies most of these nice properties in simple theories, a broader class than stable. In fact, Kim proved in [Kim01] that certain nice properties, such as symmetry and transitivity, always fail for forking independence in non-simple theories. Still, examples were known of non-simple theories that admit a relatively nice independence relation. For example infinite-dimensional vector spaces with a generic bilinear form [Gra99], ω-free PAC fields [Cha02, Cha08], and parametrised equivalence relations [CR16]. These turn out to be all NSOP$_1$ theories, a class of theories that was introduced by Džamonja and Shelah [DS04], which is more general than the class of simple theories. Inspired by ideas from Kim [Kim09], Kaplan and Ramsey developed the notion of Kim-independence [KR20]. Combined with some results following up on their original paper, [KRS17, KR19], they proved that Kim-independence in NSOP$_1$ theories satisfies all the nice properties that forking independence has in simple theories, except for one called base-monotonicity.

The classes of stable, simple and NSOP$_1$ theories form a hierarchy of increasing generality. This is sometimes referred to as the *stability hierarchy*. Or rather, a part of it, but it is the part that this thesis focuses on. Theories in each of these classes admit a nice independence relation, where moving to a more general class comes at the cost of certain properties that this independence relation has. It turns out that the existence of a nice independence relation actually characterises in which class a theory belongs. This is called a Kim-Pillay style theorem, after a result by Kim and Pillay [KP97, Theorem 4.2]. Roughly the statement is as follows:

> A theory is simple if and only if it admits an independence relation satisfying a certain list of properties. Furthermore, in this case that relation is given by forking independence.

Such a theorem tells us three things:

1. *structure theory*: assuming something about the complexity of the theory involved (in this case, simplicity), there is a nice independence relation;
2. *characterisation*: having a nice independence relation says something about the complexity of the theory (in this case that the theory is simple);
3. *canonicity*: there can be at most one such a nice independence relation.

A similar result was proved much earlier for stable theories [HH84]. For NSOP_1 theories the characterisation part was already proved in [CR16, Proposition 5.8] and it was later completed to a full Kim-Pillay style theorem in [KR20], with respect to Kim-independence.

So far we only considered the classical setting of first-order logic. However, there are many interesting classes of structures that do not fit in this framework. There are more general logical frameworks that do allow us to study these classes. For example, Banach spaces and Hilbert spaces can be studied in continuous logic [BYBHU08]. There is also positive logic, which allows us to study for example the existentially closed models of an inductive theory, such as existentially closed exponential fields [HK21]. We can also use positive logic to study hyperimaginary extensions, for example T^{heq} for a first-order theory T, see section 2.2. Finally, there is a very general category-theoretic approach via accessible categories, which subsumes all the above frameworks.

Positive logic is a proper generalisation of full first-order logic where negation is not built in, but can be added as desired. This is only slightly more general than what Shelah called model theory of kind III in [She75], where he studied stable theories. Later, work of Pillay [Pil00] and Ben-Yaacov [BY03a, BY03b] developed the theory for forking independence in positive simple theories. For this, Ben-Yaacov developed a framework called "cats" or "compact abstract theories". He proves that these are essentially positive theories [BYBHU08, Theorem 2.38].

Then there is the very general category-theoretic setting of accessible categories. For a theory T (in full first-order logic), the category of models of T with elementary embeddings forms an accessible category, but accessible categories are much more general. For example, we can also obtain a category of models from a positive theory, which will be an accessible category. Then even more general, there is Shelah's notion of AEC (abstract elementary class, see e.g. [She09]), which is a class of structures with a choice of embedding, satisfying a few axioms. Every AEC can naturally be seen as an accessible category. Going in a different direction, we can consider the category of models of a continuous theory. This will not be an AEC, but it is still an accessible category. The cats we mentioned earlier turn out to be accessible categories in practice. Even then, accessible categories are more general, they are generally the category of models of some infinitary theory with homomorphisms as arrows, see [AR94, Theorem 5.35]. In [LRV19] Lieberman, Rosický and Vasey studied stable independence relations in accessible categories and proved, among other things, a canonicity theorem.

Very roughly, the contents of this thesis and its place the aforementioned literature can be summarised in the following table, the work in this thesis being contained in the three cells on the bottom right. Each cell describes where a (partial) Kim-Pillay style theorem is developed for a given place in the stability hierarchy (the row) and in a given framework (the column).

	Full first-order logic	Positive logic	Accessible categories
Stable	Harnik, Harrington, Lascar, Shelah [Las76, HH84, She90]	Ben-Yaacov, Pillay, Shelah [She75, Pil00, BY03a, BY03b]	Lieberman, Vasey, Rosický [LRV19]
Simple	Kim, Pillay [KP97, Kim98]	Ben-Yaacov, Pillay [Pil00, BY03a, BY03b]	Theorem 5.4 (canonicity)
NSOP$_1$	Chernikov, Kaplan, Kim, Ramsey [Kim09, CR16, KR20, CKR20]	Theorem 6.79, joint with Jan Dobrowolski	Theorem 5.6 (canonicity)

The work in [LRV19] was a big source of inspiration for the category-theoretic results in this thesis. Still, our approach is slightly different. In [LRV19] an independence relation is defined as a collection of commutative squares. This has the benefit that it allows for a more category-theoretic study of the independence relation. For example, assuming transitivity of the independence relation, these squares form a category. In our approach we will define an independence relation as a relation on triples of subobjects (see section 4.1). We lose the nice way of viewing the independence relation as a category, but the benefit is that the calculus we get is more intuitive and easier to work with. Under some mild assumptions both approaches are essentially the same, in the sense that we can recover one from the other, see Remark 4.17.

1.1 Main results

For the category-theoretic part we define a specific kind of accessible category, an abstract elementary category, or AECat (Definition 3.2). This still covers all the previously mentioned examples of accessible categories, so categories of models in various logical frameworks and AECs (section 3.1). We then make sense of an independence relation in such an AECat as a relation on triples of subobjects.

We can then state the properties that such an independence relation can have in category-theoretic language (section 4.1). This results in the definition of stable, simple and NSOP_1-like independence relations (Definition 4.9), by listing the properties of independence relations that characterise those classes in the classical first-order setting.

In the classical setting of first-order logic (and also positive logic) there is also the notion of dividing, which is stronger than forking. In simple (and thus in stable) theories forking and dividing coincide. For AECats we introduce the notions of isi-forking and isi-dividing (section 4.4), which are closely related to the original notions of forking and dividing. The definitions of isi-forking and isi-dividing are such that they work well without any form of compactness. This is necessary because generally we do not have compactness in AECats.

Theorem 5.4**, paraphrased.** *Let $(\mathcal{C}, \mathcal{M})$ be an AECat with the amalgamation property. Suppose that there is a simple independence relation in $(\mathcal{C}, \mathcal{M})$. Then this independence relation is given by isi-forking and isi-dividing, which then coincide.*

As explained before, in NSOP_1 theories we have to consider a different notion of independence, namely Kim-independence. This is given by Kim-dividing, which is stronger than dividing. Again we have to sidestep any dependency on compactness, so we define a notion of long Kim-dividing (Definition 4.47).

Theorem 5.6**, paraphrased.** *Let $(\mathcal{C}, \mathcal{M})$ be an AECat with the amalgamation property. Suppose that non-isi-forking sequences exist in $(\mathcal{C}, \mathcal{M})$ and that there is an $NSOP_1$-like independence relation in $(\mathcal{C}, \mathcal{M})$. Then this independence relation is given by long Kim-dividing.*

For a discussion about the assumption on the existence of non-isi-forking sequences, we refer the reader to Example 4.50 (where this assumption has been given the name "$\mathcal{B}$-existence axiom"). All we say now is that it is a reasonable, and necessary, assumption, already in the very concrete setting of full first-order logic.

Similar to the classical setting, when isi-forking and isi-dividing coincide we have that non-isi-forking sequences exist. This allows us to put the previous two theorems together to get a canonicity statement. We recover the original stability hierarchy, because any nice enough independence relation will tell us where in the hierarchy the AECat fits and because by definition any stable independence relation is also simple, and any simple independence relation is also NSOP_1-like.

Theorem 5.7**, paraphrased.** *Let $(\mathcal{C}, \mathcal{M})$ be an AECat with the amalgamation property. Suppose that there is a simple or stable independence relation in $(\mathcal{C}, \mathcal{M})$. Then any $NSOP_1$-like independence relation will be the same relation.*

Furthermore, this unique relation is given by isi-forking, isi-dividing and long Kim-dividing, which all coincide.

The category-theoretic results are (mainly) about canonicity of certain independence relation. We do not link this independence relation to any combinatorial properties. For example, in full first-order logic being stable roughly means that we cannot find an infinite linear order in (a big enough model of) that theory. So the structure theory and characterisation parts of a Kim-Pillay style theorem are then about linking this combinatorial property to the existence of an independence relation. We do mention that [LRV19] explores such a link for stable independence relations in accessible categories, and [GMA21] explores such a link for simple-like independence relations in AECs. In this thesis we do not further explore this link for AECats, although it is an interesting question (see section 7.2).

However, in positive logic we do explore the link with the combinatorial property of being NSOP_1. This is joint work with Jan Dobrowolski. Our work can be summarised as a full Kim-Pillay style theorem (see below). So we do not only prove the canonicity part (as in AECats), but we also prove the structure theory and characterisation parts. Most of the work goes to proving that Kim-independence in a positive NSOP_1 theory satisfies all the nice properties that it is known to have in full first-order logic.

We briefly mentioned this before, but in the definition of Kim-independence the existence of non-forking sequences is essential. Kaplan and Ramsey solve this in [KR20] by only considering types over models, and thus they develop Kim-independence over models. This approach works because a type over a model always extends to a global invariant type (in any theory). In positive logic this is no longer true (see Example 2.24). However, under the very mild assumption of thickness we can always extend a type over an existentially closed model to a global Lascar-invariant type. This is enough to find the non-forking sequences we were after. Thickness was introduced in [BY03c] for positive logic and it states that being an indiscernible sequence is type-definable. Theories in full first-order logic, and their hyperimaginary extensions, are always thick.

Theorem 6.79**, paraphrased.** *Let T be a a thick positive theory. Then T is $NSOP_1$ if and only if there is an independence relation over existentially closed models satisfying a certain list of properties. Furthermore, in this case that independence relation is Kim-independence.*

The "certain list of properties" is then of course the list of properties that Kim-independence is known to have in NSOP_1 theories in full first-order logic. This is also the same list as what we call an NSOP_1-like independence relation for AECats (albeit stated slightly differently because of the different framework).

1.2 Overview

We start by discussing some preliminaries in **chapter 2**. The necessary background for positive logic and accessible categories is established there. We also dedicate section 2.2 to the construction of hyperimaginary extensions in positive logic.

In **chapter 3** we introduce the framework of abstract elementary categories, or AECats for short. We make sense of types in this setting through the notion of Galois types. We also give a definition of something we call Lascar strong Galois types, which are meant to play the role of Lascar strong types in the framework of AECats.

In **chapter 4** we define the notion of an independence relation in an AECat. We list certain properties that it can have and using these we give the definition of stable, simple and NSOP_1-like independence relations. We also introduce the notions of isi-dividing, isi-forking and long Kim-dividing in this chapter (section 4.4).

Chapter 5 is devoted to proving the canonicity theorems for independence relations in AECats. Towards the end of the chapter we study the link between Lascar strong Galois types and independence relations more closely.

We then switch frameworks. **Chapter 6** is all about developing Kim-independence in positive NSOP_1 theories. This results in a full Kim-Pillay style theorem.

Finally, in **chapter 7** we make some final remarks and summarise some questions that have been left unanswered.

2

Preliminaries

The preliminaries can be divided in two major subjects. The first is positive logic and the second is accessible categories. Both are frameworks in which we can do model theory and both are more general than the classical setting of full first-order logic.

In section 2.1 we start with the basic definitions for positive logic. Positive logic is like full first-order logic, but we do not allow the negation symbol. Through a process called Morleyisation we can always add as much negation as we want in a positive theory (Remark 2.12). So full first-order logic can be studied as a special case of positive logic.

We give all the basic definitions and discuss how we can make sense of a positive variant of the usual notions and tools that we have in full first-order logic. This includes compactness, types and a monster model. In this setup one can already do some serious model theory, see for example [Pil00, BY03b]. We also discuss some notions from [BY03a, BY03c] that are particular to positive logic, namely that of being Hausdorff, semi-Hausdorff or thick. These notions capture properties that types in full first-order logic always have, but that may not hold in a positive theory. We provide examples and counterexamples for each of these notions.

Positive logic allows us to study more classes of structures than we could with full first-order logic, for example, the class of existentially closed structures of some inductive theory (see subsection 6.10.2). Another important class arises when we add hyperimaginaries as parameters to our monster model. Then we leave the framework of full first-order logic, but we remain in the framework of positive logic. This was the main motivation in [BY03a] to study positive logic. We work out their construction for adding hyperimaginaries in far greater detail in section 2.2. This allows us to prove that certain properties are invariant under the operation of adding hyperimaginaries. We then obtain a whole class of theories where the results in chapter 6 apply (see subsection 6.10.3).

The second framework we will discuss is that of accessible categories. The idea in this approach is to study the category of models of some theory. For example, this can be a theory in full first-order logic, positive logic or continuous logic. The framework is much more general. For example, every abstract elementary class (AEC) can naturally be seen as an accessible category.

We will discuss the necessary background for accessible categories in section 2.3. We also provide concrete examples of the above mentioned cases. Later, in chapter 3, we build on this to define a specific kind of accessible category, an abstract elementary category, or AECat, which still covers all the above mentioned cases.

2.1 Positive logic

In this section we will discuss positive logic. Positive logic is a proper generalisation of full first-order logic. Most of this section already appears in literature and most of it is standard. So we will omit proofs and just refer to the relevant literature. The main references are [BY03a, PY18]. Also a lot appears in [Hod93, Chapter 8], albeit in a technically slightly less general set up, but the relevant techniques still work in the full generality of positive logic.

Incidentally, positive logic is also heavily studied in topos theory under the name of coherent logic, see for example [MM94, Joh02]. As a side project I explored this connection further from the perspective of type space functors, see [Kam22]. This has not been included in this thesis.

Definition 2.1. Fix a signature $\mathcal{L}$. A *positive existential formula* in $\mathcal{L}$ is one that is obtained from combining atomic formulas using $\wedge$, $\vee$, $\top$, $\bot$ and $\exists$. An *h-inductive sentence* is a sentence of the form $\forall x(\varphi(x) \to \psi(x))$, where $\varphi(x)$ and $\psi(x)$ are positive existential formulas. A *positive theory* is a set of h-inductive sentences.

We do not notationally distinguish between a single variable or element, or a (possibly infinite) tuple. So the x in $\varphi(x)$ generally denotes a tuple of variables.

Note that every positive existential formula $\varphi(x)$ is equivalent to something of the form $\exists y \psi(x, y)$, where $\psi(x, y)$ is positive quantifier-free. Positive existential sentences and their negations can be used as axioms in a theory, since $\forall x \varphi(x)$ and $\forall x \neg\varphi(x)$ are equivalent to $\forall x(\top \to \varphi(x))$ and $\forall x(\varphi(x) \to \bot)$ respectively.

There is a good semantic reason for Definition 2.1. Positive existential formulas are exactly those formulas whose truth is preserved by homomorphisms of structures (see Definition 2.3). The h-inductive sentences are precisely the sentences that are preserved under taking unions of chains with respect to homomorphisms (or equivalently: directed colimits). That is, if $(M_i)_{i<\lambda}$ is a chain of models of some positive theory T with homomorphisms between them then $M = \bigcup_{i<\lambda} M_i$ is again a model of T [BY03a, Lemma 1.14]. Conversely, if T is a theory in full first-order logic such that its models are closed under chains of homomorphisms then T is equivalent to a positive theory, see e.g. [Hod93, Theorem 6.5.9].

Since we will only be considering full first-order logic as a special case of positive logic, see Remark 2.12, we will make the following convention.

Convention 2.2. Whenever we say "formula" or "theory" we will mean "positive existential formula" and "positive theory" respectively, unless explicitly stated otherwise. This also means that every formula and theory we consider will be implicitly assumed to be positive (existential).

In full first-order logic we consider elementary embeddings because they preserve and reflect truth of all first-order formulas. Since we do not have negation in positive logic, there is a difference between preserving and reflecting truth of positive existential formulas.

Definition 2.3. A function $f : M \to N$ between $\mathcal{L}$-structures is called a *homomorphism* if it preserves constant symbols, functions symbols and relation symbols. That is:

(i) $f(c_M) = c_N$ for every constant symbol c;

(ii) $f(g_M(a)) = g_N(f(a))$ for every function symbol g and every tuple $a \in M$;

(iii) for any $a \in M$ and any relation symbol P we have that $M \models P(a)$ implies $N \models P(f(a))$.

In this case we also call N a *continuation* of M.

If $f : M \to N$ is a homomorphism then for every $\varphi(x)$ and every $a \in M$ we have

$$M \models \varphi(a) \implies N \models \varphi(f(a)),$$

which immediately follows from induction on the complexity of the formula.

Definition 2.4. We call a homomorphism $f : M \to N$ an *immersion* if it also reflects truth of all formulas. That is, if for every $\varphi(x)$ and every $a \in M$ we have

$$M \models \varphi(a) \iff N \models \varphi(f(a)).$$

Definition 2.5. We call a model M of T an *existentially closed model* or an *e.c. model* if the following equivalent conditions hold:

(i) every homomorphism $f : M \to N$ with $N \models T$ is an immersion;

(ii) for every $a \in M$ and $\varphi(x)$ such that there is a homomorphism $f : M \to N$ with $N \models T$ and $N \models \varphi(f(a))$, we have that $M \models \varphi(a)$;

(iii) for every $a \in M$ and $\varphi(x)$ such that $M \not\models \varphi(a)$ there is $\psi(x)$ with $T \models \neg\exists x(\varphi(x) \wedge \psi(x))$ and $M \models \psi(a)$.

See for example [PY18, Lemma 2] or [Hod93, Theorem 8.2.4] for the equivalence of the conditions in Definition 2.5.

In positive model theory we study the e.c. models of a theory. A brief category-theoretic motivation for this would be that the arrows between e.c. models preserve and reflect truth of the formulas. A brief (equivalent) logical motivation would be that e.c. models give a definite answer about the truth of any formula for any of its elements. That is, its (positive) diagram is maximal.

Remark 2.6. Some literature also uses the term *positively closed model* or *p.c. model* for Definition 2.5. It is good to be aware of the reason for this.

In older treatments of positive model theory, like [Hod93, Chapter 8] and [Pil00], the focus is on just existential formulas. So formulas of the form $\exists y \varphi(x, y)$ where $\varphi(x, y)$ is quantifier-free (possibly with negations). Then the h-inductive sentences are precisely the $\forall\exists$-sentences and homomorphisms are precisely the embeddings of $\mathcal{L}$-structures. The term "e.c. model" is then already defined in that setting, so one might want to define "p.c. model" to emphasise the difference in framework.

We can treat this setting in positive logic by introducing a relation symbol R' for every relation symbol R (including equality) in the language, and having our theory express that R' is the negation of R. This is an application of a more general technique called Moreylisation, which is discussed in more detail in Remark 2.12. Then the term "p.c. model" would specialise to "e.c. model". In line with our other terminology, we will reuse terms from more specialised settings if the new definition coincides with the original definition when specialising to that setting.

Fact 2.7. *Let T be some theory.*

(i) The union of a directed system of (e.c.) models is an (e.c.) model.

(ii) If one of $M_1 \leftarrow M \rightarrow M_2$ is an immersion then there are $M_1 \rightarrow N \leftarrow M_2$ making the relevant square commute. In particular, every e.c. model is an amalgamation base.

(iii) For every $M \models T$ there is a homomorphism $f : M \rightarrow N$, where N is an e.c. model of T.

Proof. These can all be found in [BY03a], as Lemma 1.14, Lemma 1.37 and Lemma 1.20 respectively. □

We have to be careful when using compactness. Because we are interested in e.c. models, we are only interested in realisations in e.c. models. This means that we can only use compactness for positive existential formulas. We will give the proof of this, because it is instructive.

Proposition 2.8 (Compactness). *Let T be a theory and let $\Sigma(x)$ be a set of positive existential formulas. Suppose that for every finite $\Sigma_0(x) \subseteq \Sigma(x)$ there is $M \models T$ with $a \in M$ such that $M \models \Sigma_0(a)$. Then there is an e.c. model N of T with $a \in N$ such that $N \models \Sigma(a)$.*

Proof. By the compactness theorem for full first-order logic we find a model M' of T and $a' \in M'$ such that $M' \models \Sigma(a')$. By Fact 2.7 there is a homomorphism $f : M' \to N$, where N is an e.c. model of T. Because $\Sigma(x)$ only contains positive existential formulas we have $N \models \Sigma(f(a'))$. So we set $a = f(a')$, which concludes the proof. □

To illustrate that we can generally not get more compactness, we consider the following two examples.

Example 2.9. Consider the theory T with a symbol for inequality and ω many disjoint unary predicates $P_n(x)$. Then e.c. models of T are precisely those which consist of ω-many disjoint infinite sets, one for each predicate. If we had full compactness then the set

$$\Sigma(x) = \{\neg P_n(x) : n < \omega\}$$

would have a realisation in some e.c. model, which is impossible.

Example 2.10. It could happen that there is a definable set that is infinite and bounded. This does not contradict compactness: it just means that inequality is not positively definable on that set. Such situations might arise when adding hyperimaginaries as parameters, which can be done in positive logic (see Example 2.48), but we give a simpler example here.

The signature consists of ω many constant symbols c_n. The theory T just declares all these symbols to be distinct. There is precisely one e.c. model of T (up to isomorphism), which consists of just (the interpretations of) the constant symbols. So the trivial definable set $x = x$ is bounded, but infinite. Again, with full compactness we would run into trouble because

$$\Sigma((x_i)_{i<\omega_1}) = \{x_i \neq x_j : i < j < \omega_1\}$$

would then yield a realisation with uncountably many elements.

Definition 2.11. We call a positive theory T *Boolean* if the following equivalent conditions hold:

(i) every model of T is an e.c. model;

(ii) for every positive existential formula $\varphi(x)$ there is positive existential $\psi(x)$ such that $T \models \forall x(\neg\varphi(x) \leftrightarrow \psi(x))$;

(iii) for every full first-order formula $\varphi(x)$ there is positive existential $\psi(x)$ such that $T \models \forall x(\varphi(x) \leftrightarrow \psi(x))$;

(iv) every homomorphism between models of T is an elementary embedding.

The conditions in Definition 2.11 are essentially just as in [Hod93, Theorem 8.3.1], so the proof of their equivalence there applies here as well.

The name Boolean in Definition 2.11 is because it states that for T the Lindenbaum-Tarski algebra of positive existential formulas forms a Boolean algebra. In [Hay19] the name "positively model complete" is used instead, referring to condition (iv) in Definition 2.11. While that name is also accurate, we think the name Boolean is more descriptive.

In a Boolean theory the positively definable sets and definable sets in full first-order logic coincide. So these are essentially the kind of theories that we study in full first-order model theory. The following remark describes a process to turn any theory in full first-order logic into a positive Boolean theory, showing that we can indeed study full first-order model theory as a special case of positive model theory. We will therefore often refer to Boolean theories as theories in full first-order logic.

Remark 2.12. Let us recall the process of *Morleyisation*. We extend our signature $\mathcal{L}$ to include a relation symbol $R_\varphi(x)$ for every full first-order formula $\varphi(x)$ in $\mathcal{L}$. We will construct a theory T expressing that $\forall x(R_\varphi(x) \leftrightarrow \varphi(x))$, after which any full first-order axiom can be added in the form of some R_φ. Clearly this yields a Boolean theory.

We do have to be careful, because $\varphi(x)$ might be a very complex formula in full first-order logic, so something like $\forall x(R_\varphi(x) \to \varphi(x))$ is not h-inductive. To solve this, we add axioms to express $\forall x(R_\varphi(x) \leftrightarrow \varphi(x))$ by induction on the structure of $\varphi(x)$. Only the induction step for negation is non-trivial and should also directly make clear how this approach solves the issue. So suppose that $\varphi(x)$ is of the form $\neg\psi(x)$. By induction hypothesis we have already expressed that $R_\psi(x)$ and $\psi(x)$ are equivalent. So $\varphi(x)$ is equivalent to $\neg R_\psi(x)$. It is thus sufficient to add the h-inductive sentences $\forall x(R_\varphi(x) \vee R_\psi(x))$ and $\forall x \neg(R_\varphi(x) \wedge R_\psi(x))$ to our theory.

It should be clear from the above that we can perform the Morleyisation process up to any level of complexity we wish. The application in Remark 2.6 is an example of this.

Definition 2.13. Let M be an e.c. model, $B \subseteq M$ and $a \in M$. Then the *type of a over B* is defined as:

$$\mathrm{tp}(a/B) = \{\varphi(x) \text{ with parameters in } B : M \models \varphi(a)\}.$$

In other words, it is a maximal consistent set of formulas with parameters in B. A *partial type* over B is just any consistent set of formulas over B.

For $a, a' \in M$ and $B \subseteq M$ with $\mathrm{tp}(a/B) = \mathrm{tp}(a'/B)$ we will also write $a \equiv_B a'$.

Note that in the above notation, following Convention 2.2, a (partial) type, so also $\mathrm{tp}(a/B)$, is a set of positive existential formulas. So once again we overload the notation from full first-order logic, and in Boolean theories it coincides with the original definition.

Now that we have a notion of type, the notion of indiscernible sequences naturally translates to the positive setting.

Definition 2.14. Let I be an infinite linear order, M an e.c. model, $B \subseteq M$ and $(a_i)_{i \in I}$ some sequence of compatible tuples in M. We say that $(a_i)_{i \in I}$ is *B-indiscernible* if for any $i_1 < \ldots < i_n$ and $j_1 < \ldots < j_n$ in I we have

$$a_{i_1} \ldots a_{i_n} \equiv_B a_{j_1} \ldots a_{j_n}.$$

Because e.c. models are generally not the same as just models of some theory there can be h-inductive sentences that are true in all e.c. models, but fail in some models, see Example 2.17. Such sentences can be added to the theory without changing the class of e.c. models. It will be useful to give the maximal theory that we obtain in this way a name.

Definition 2.15. Let T be a theory. We write

$$T^{\mathrm{ec}} = \{\chi \text{ an h-inductive sentence} : M \models \chi \text{ for every e.c. model } M \text{ of } T\}$$

for the theory of all h-inductive consequences of e.c. models of T.

Fact 2.16 ([PY18, Section 3.1])**.** *For any positive theory T the theories T and T^{ec} have the same e.c. models.*

Example 2.17. Even though T and T^{ec} have the same e.c. models, they may not have the same consequences and thus different models. A concrete example of this would be to consider the theory T with a symbol for inequality (and asserting that it is indeed inequality). Then models are just sets and homomorphisms are just injective functions. The e.c. models are precisely the infinite sets. So for every n we have that T^{ec} expresses "there are at least n elements", which can be done by an h-inductive sentence because we have inequality. This can clearly not be a consequence of T, because the finite sets are also models of T.

The following definitions, except for being Boolean, are taken from [BY03c]. These assumptions are very useful for developing (neo)stability theory for positive logic, while the weaker ones—thickness, and even being semi-Hausdorff—are relatively mild.

Definition 2.18. Let T be a positive theory. We call T:

- *Boolean* if every formula in full first-order logic is equivalent to some positive existential formula, modulo T (or any of the equivalent statements from Definition 2.11);
- *Hausdorff* if for any two distinct types $p(x)$ and $q(x)$ there are $\varphi(x) \notin p(x)$ and $\psi(x) \notin q(x)$ such that $T^{\text{ec}} \models \forall x(\varphi(x) \vee \psi(x))$;
- *semi-Hausdorff* if equality of types is type-definable, so there is a partial type $\Omega(x, y)$ such that for any a, b in some e.c. model M we have $\text{tp}(a) = \text{tp}(b)$ if and only if $M \models \Omega(a, b)$;
- *thick* if being an indiscernible sequence is type-definable, so there is a partial type $\Theta((x_i)_{i<\omega})$ such that for any sequence $(a_i)_{i<\omega}$ in some e.c. model M we have that $(a_i)_{i<\omega}$ is indiscernible if and only if $M \models \Theta((a_i)_{i<\omega})$.

The reason for the name Hausdorff is that this corresponds to the type spaces being Hausdorff, where formulas correspond to closed sets.

We mentioned Boolean theories in Definition 2.18 again because they fit very well in the hierarchy mentioned there, as is apparent from the following proposition.

Proposition 2.19. *Boolean implies Hausdorff implies semi-Hausdorff implies thick.*

Proof. This is already mentioned in [BY03c], but the proof is omitted. We give it here for completeness' sake.

<u>Boolean implies Hausdorff.</u> Let $p(x)$ and $q(x)$ be distinct types. Pick any $\varphi(x) \in q(x)$ such that $\varphi(x) \notin p(x)$. Because the theory is Boolean there must be $\psi(x)$ that is equivalent to $\neg\varphi(x)$, modulo the theory. So we have $\psi(x) \notin q(x)$ while also $T \models \forall x(\varphi(x) \vee \psi(x))$, so in particular $T^{\text{ec}} \models \forall x(\varphi(x) \vee \psi(x))$, as required.

<u>Hausdorff implies semi-Hausdorff.</u> Define

$$\begin{aligned}\Omega(x, y) = \{\varphi(x, y) : \text{for all } a, b \text{ in some e.c. model } M \text{ with } \text{tp}(a) = \text{tp}(b) \\ \text{we have } M \models \varphi(a, b)\}.\end{aligned}$$

Let a, b be arbitrary in some arbitrary e.c. model M. By construction we have that $\text{tp}(a) = \text{tp}(b)$ implies $M \models \Omega(a, b)$. For the other direction we prove the contrapositive. So suppose that $\text{tp}(a) \neq \text{tp}(b)$. Because the theory is Hausdorff there are $\varphi(x) \notin \text{tp}(a)$ and $\psi(x) \notin \text{tp}(b)$ such that $T^{\text{ec}} \models \forall x(\varphi(x) \vee \psi(x))$. The latter means that by definition of $\Omega(x, y)$ we then have $(\varphi(x) \wedge \varphi(y)) \vee (\psi(x) \wedge \psi(y)) \in \Omega(x, y)$. The former means that $M \not\models (\varphi(a) \wedge \varphi(b)) \vee (\psi(a) \wedge \psi(b))$, hence $M \not\models \Omega(a, b)$, as required.

<u>Semi-Hausdorff implies thick.</u> Define the partial type $\Theta((x_i)_{i<\omega})$ as:

$$\bigcup\{\Omega(x_{i_1},\ldots,x_{i_n};x_{j_1},\ldots,x_{j_n}) : n<\omega, i_1<\ldots<i_n<\omega, j_1<\ldots<j_n<\omega\}.$$

Here $\Omega(x_{i_1},\ldots,x_{i_n};x_{j_1},\ldots,x_{j_n})$ is the partial type expressing that $x_{i_1},\ldots,x_{i_n}$ and $x_{j_1},\ldots,x_{j_n}$ have the same type, which exists by the semi-Hausdorff assumption. So $\Theta((x_i)_{i<\omega})$ expresses that any two finite subsequences of $(x_i)_{i<\omega}$ of the same length have the same type, and hence it expresses indiscernibility. □

We also mention the following useful characterisation of Hausdorff theories, taken from [PY18, Section 3.5].

Definition 2.20. We say that a theory T has the *h-amalgamation property* or *APh* if for any span $M_1 \leftarrow M \rightarrow M_2$ of homomorphisms between models of T there is an amalgam of homomorphisms $M_1 \rightarrow N \leftarrow M_2$ with $N \models T$.

Fact 2.21 ([PY18, Theorem 8])**.** *A theory T is Hausdorff precisely if T^{ec} has APh.*

We now consider some examples to show that none of the implications in Proposition 2.19 are reversible.

Example 2.22. We give an example of a Hausdorff non-Boolean theory. Consider a signature $\mathcal{L}$ with four unary relation symbols: P_1, P_2, Q and Q'. Let M be an $\mathcal{L}$-structure with underlying set $\{a,b,c\}$ such that $P_1(M)=\{a,b\}$, $P_2(M)=\{b,c\}$, $Q(M)=\{b\}$ and $Q'(M)=\{a,c\}$. We take T to be the set of all h-inductive $\mathcal{L}$-sentences that are true in M. Then T specifies that there are at most three elements because it contains the sentence $\forall x_1x_2x_3x_4 \bigvee_{i\neq j} x_i = x_j$. It also specifies that there are three elements and what unary relation symbols they each satisfy, from which it follows that these three elements are distinct. For example, T contains the sentences $\exists x(P_1(x) \wedge Q(x))$ and $\forall xy(P_1(x)\wedge Q(x)\wedge P_1(y)\wedge Q(y)\rightarrow x=y)$. In other words, T determines M up to isomorphism so in particular M is an e.c. model and $T=T^{\text{ec}}$.

Because $T=T^{\text{ec}}$ determines M up to isomorphism it has APh and is thus Hausdorff by Fact 2.21. However, T is not Boolean because the set $\neg P_1(M)=\{c\}$ is not positively definable. This is easily seen by verifying that every positively existential formula is equivalent to a positive quantifier free formula (modulo T).

The above example is specifically constructed to be Hausdorff and non-Boolean. A more interesting example would be the theory of existentially closed exponential fields, studied in [HK21], see Proposition 6.98 and the discussion afterwards.

Example 2.23. We give an example of a semi-Hausdorff non-Hausdorff theory. This is essentially [PY18, Example 4]. Let our signature consist of ω many constants c_n and let T express $c_n \neq c_k$ for all $n \neq k$. There is then only one e.c. model (up to isomorphism), namely ω with c_n interpreted as n. Then for any tuples a and b we have $a \equiv b$ if and only if a and b are equal to the same tuple of constants if and only if $a = b$. So T is semi-Hausdorff. To show that T is not Hausdorff we show that APh fails for T^{ec}. By (positive) quantifier elimination one quickly sees that T^{ec} does not specify anything more than T does, so up to logical equivalence they are the same. Let M be ω together with one extra point $*$, which is then a model of T^{ec}. We define a homomorphism $f_1 : M \to \omega$ by taking the identity on ω and setting $f_1(*) = 1$. Similarly we define $f_2 : M \to \omega$ with $f_2(*) = 2$. The span $\omega \xleftarrow{f_1} M \xrightarrow{f_2} \omega$ cannot be amalgamated.

Example 2.24. We give an example of a thick theory that is not semi-Hausdorff. The construction is taken from [Poi10, section 4]. Consider the signature $\mathcal{L}$ with unary relation symbols P_n and P'_n for all $n < \omega$, and a binary relation symbol R. We define the $\mathcal{L}$-structure $M = \{a_n, b_n : n < \omega\}$ as follows. The interpretation of P_n is $\{a_n, b_n\}$ and P'_n is the complement of P_n. We take R to be the symmetric anti-reflexive relation $\{(a_n, b_n), (b_n, a_n) : n < \omega\}$, so R is the inequality relation on each P_n. Let T be the h-inductive theory of M. Then M is an e.c. model of T. There is a maximal e.c. model N of T given by $N = M \cup \{a_\omega, b_\omega\}$, where $N \models P'_n(a_\omega) \wedge P'_n(b_\omega)$ for all $n < \omega$ and also $N \models R(a_\omega, b_\omega) \wedge R(b_\omega, a_\omega)$.

Since N is maximal, the only indiscernible sequences are the constant ones. So T is a thick theory. However, T is not semi-Hausdorff. By [BY03c, Lemma 3.11] every type over an e.c. model M in a semi-Hausdorff theory extends to a global M-invariant type. It was observed by Rosario Mennuni that in T the type $\text{tp}(a_\omega/M)$ does not extend to a global M-invariant type, hence T cannot be semi-Hausdorff. To see this, suppose that there is a global M-invariant extension $q \supseteq \text{tp}(a_\omega/M)$. Then either $q = \text{tp}(a_\omega/N)$ or $q = \text{tp}(b_\omega/N)$. So since $a_\omega \equiv_M b_\omega$ we must have by M-invariance that $R(x, a_\omega)$ and $R(x, b_\omega)$ are both in q. The only realisations possible of q are a_ω and b_ω, so this contradicts the fact that $T \models \forall x \neg R(x, x)$.

Example 2.25. In [BY03b, Example 4.3] the theory of ultrametric spaces with distances in $\mathbb{N}$ is discussed. We will not work out the details, but this is an example of theory that is not thick. An explanation of why this is, is given at the start of section 1 in [BY03c].

In full first-order logic we usually work with complete theories. This translates to the following notion in positive logic.

Definition 2.26. We say that a theory T has the *joint embedding property* or *JEP* if the following equivalent conditions hold:

(i) for any two models M_1 and M_2 there are homomorphisms $M_1 \to N \leftarrow M_2$ with $N \models T$;

(ii) if $T \models \neg\varphi \vee \neg\psi$ then $T \models \neg\varphi$ or $T \models \neg\psi$.

See for example [PY18, Section 3.6] for more details. For a Boolean theory T the joint embedding property is equivalent to the theory being complete.

In model theory it is convenient to work in a *monster model* (some authors use the term *universal domain*). As is usual, we leave the notion of 'small' a bit vague and up to the set-theoretic framework that the reader wants to assume. For example, one could fix some inaccessible cardinal κ and then 'small' means '$< \kappa$'. A different approach would be to fix a cardinal larger than any model or set involved. A monster model $\mathfrak{M}$ is then an e.c. model that is:

- *very homogeneous*: any partial immersion $f : \mathfrak{M} \to \mathfrak{M}$ with small domain and codomain extends to an automorphism on all of $\mathfrak{M}$;
- *very saturated*: any finitely satisfiable small set of formulas over $\mathfrak{M}$ is satisfiable in $\mathfrak{M}$.

Assuming JEP, it directly follows that for any small model M there is a homomorphism $M \to \mathfrak{M}$, which is of course an immersion if M is an e.c. model. The exact construction of such a monster model depends on the notion of smallness that the reader chooses, but the usual proofs go through in positive logic as well.

Fact 2.27. *If T has JEP then there is a monster model $\mathfrak{M}$.*

The JEP assumption is merely convenient. We can always 'refine' a theory to one that has JEP and then just consider a monster model for each such 'refinement'.

For the rest of this section we will work in some monster model $\mathfrak{M}$, and thus follow the following convention. This implicitly assumes that the theory T has JEP.

Convention 2.28. Whenever we work in a monster model $\mathfrak{M}$ of T, we will simplify notation as follows. All sets and tuples will assumed to be 'small' with respect to $\mathfrak{M}$. We will use lowercase Latin letters $a, b, \ldots$ for (possibly small infinite) tuples inside the monster model and uppercase Latin letters $A, B, \ldots$ for (small) parameter sets inside the monster model. We will use letters M and N when these sets are e.c. models. We will also omit $\mathfrak{M}$ from the notation. For example, we write $\models \varphi(a)$ instead of $\mathfrak{M} \models \varphi(a)$.

Definition 2.29. Let B be some set of parameters. We write $\mathrm{Aut}(\mathfrak{M}/B)$ for the group of automorphisms on $\mathfrak{M}$ over B. That is, automorphisms $f : \mathfrak{M} \to \mathfrak{M}$ that fix B pointwise.

By homogeneity of the monster model we have $\mathrm{tp}(a/B) = \mathrm{tp}(a'/B)$ if and only if there is $f \in \mathrm{Aut}(\mathfrak{M}/B)$ such that $f(a) = a'$.

Definition 2.30. Let a, a' be two tuples and let B be any parameter set. We write $\mathrm{d}_B(a, a') \leq n$ if there are $a = a_0, a_1, \ldots, a_n = a'$ such that a_i and a_{i+1} are on a B-indiscernible sequence for all $0 \leq i < n$. The minimal n such that $\mathrm{d}_B(a, a') \leq n$ is called the *Lascar distance* over B between a and a'. If there is no such n, we say that the Lascar distance is infinite.

Fact 2.31 ([BY03c, Proposition 1.5])**.** *A theory is thick if and only if the property "$\mathrm{d}_B(x, x') \leq n$" is type-definable over B for all B and $n < \omega$.*

In full first-order logic having the same type over a model implies that the Lascar distance is at most two. In fact, the same holds in semi-Hausdorff theories.

Fact 2.32 ([BY03c, Proposition 3.13])**.** *In semi-Hausdorff theories we have that if $a \equiv_M a'$, where M is an e.c. model, then $\mathrm{d}_M(a, a') \leq 2$.*

In thick theories this is no longer necessarily true, as we will see in Example 2.33 below. The solution is to work over models that are saturated enough, as we will prove in Proposition 2.39. Before we do that we first need to recall a few definitions and tools.

Example 2.33. We continue Example 2.24. We have there that M is an e.c. model and that $a_\omega \equiv_M b_\omega$. Since the only indiscernible sequences are the constant ones, we have that the Lascar distance between a_ω and b_ω (over M) is infinite. So we see that outside the semi-Hausdorff setting having the same type over an e.c. model no longer guarantees having finite Lascar distance.

There are some subtle differences in possible definitions of saturatedness, see for example [PY18, Section 2.4]. We are only interested in e.c. models, so for us it will mean the following. Constructing models of a certain level of saturation is then standard (see e.g. [TZ12, Lemma 6.1.2]).

Definition 2.34. Let M be an e.c. model of some theory T. We say that M is *κ-saturated* if for every $A \subseteq M$ with $|A| < \kappa$ we have that a set $\Sigma(x)$ of formulas over A is satisfiable in M if and only if it is finitely satisfiable in M.

Fact 2.35. *For any $\kappa \geq |A| + |T|$ there is a κ^+-saturated $N \supseteq A$ with $|N| \leq 2^\kappa$.*

We will actually make a slight improvement by considering a weaker notion of saturation.

Definition 2.36. We call an e.c. model M *finitely λ-saturated* if for every finite tuple $a \in M$ there is a λ-saturated e.c. model $M_0 \subseteq M$ with $a \in M_0$.

Clearly every λ-saturated e.c. model is also finitely λ-saturated. The point of this definition is that the class of finitely λ-saturated models is closed under unions of chains. This will be relevant later on, see Remark 3.21.

Fact 2.37 ([BY03b, Lemma 1.2])**.** *Let B be any parameter set, κ any cardinal and $\lambda = \beth_{(2^{|T|+|B|+\kappa})^+}$. Then for any sequence $(a_i)_{i<\lambda}$ of κ-tuples there is a B-indiscernible sequence $(a'_i)_{i<\omega}$ such that for all $n < \omega$ there are $i_1 < \ldots < i_n < \lambda$ with $a'_1 \ldots a'_n \equiv_B a_{i_1} \ldots a_{i_n}$.*

Definition 2.38. For a theory T we write $\lambda_T = \beth_{(2^{|T|})^+}$.

The significance of λ_T is that given any sequence of length λ_T of finite tuples and any finite set of parameters B, we can find a B-indiscernible sequence based on it by Fact 2.37, which we will directly use in the following proposition.

Proposition 2.39. *Let T be a thick theory and let M be a finitely λ_T-saturated e.c. model. Then $a \equiv_M a'$ implies $\mathrm{d}_M(a, a') \leq 2$.*

Proof. By thickness, $\mathrm{d}_M(x, y) \leq 1$ is M-type-definable. Let $\varphi(x, y)$ be a finite conjunction of formulas in $\mathrm{d}_M(x, y) \leq 1$. It is enough to show that $\varphi(x, a) \wedge \varphi(x, a')$ is satisfiable, because then the partial type "$\mathrm{d}_M(x, a) \leq 1$ and $\mathrm{d}_M(x, a') \leq 1$" is finitely satisfiable.

Since φ is just a formula, we may as well assume a and a' to be finite. Let m denote the (finite) part of M that appears in φ. Let $M_0 \subseteq M$ be such that M_0 is λ_T-saturated and $a, a', m \in M_0$. Then there is a sequence $(a_i)_{i<\lambda_T}$ in M_0 such that $a_i(a_j)_{j<i} \equiv_m a(a_j)_{j<i}$ for all $i < \lambda_T$. Using Fact 2.37 we then find m-indiscernible $(a'_i)_{i<\omega}$ based on $(a_i)_{i<\lambda_T}$. So $\models \varphi(a'_0, a'_1)$, and thus there are $i_0 < i_1 < \lambda_T$ such that $\models \varphi(a_{i_0}, a_{i_1})$. By construction we have that $a_{i_1} a_{i_0} \equiv_m a a_{i_0}$, so $\models \varphi(a_{i_0}, a)$. Since $a \equiv_M a'$ and $a_{i_0} \in M_0 \subseteq M$ we also have $\models \varphi(a_{i_0}, a')$, and we are done. $\square$

Definition 2.40. Let T be a thick theory. We say that a and a' have the same *Lascar strong type* over B, and write $a \equiv^{\mathrm{Ls}}_B a'$ if the following equivalent conditions hold:

(i) $\mathrm{d}_B(a, a') \leq n$ for some $n < \omega$;

(ii) for each bounded B-invariant equivalence relation $E(x, y)$ we have $E(a, a')$;

(iii) for some $n < \omega$, there are finitely λ_T-saturated e.c. models $M_1, \ldots, M_n$, each containing B, and $a = a_0, \ldots, a_n = a'$ such that $a_i \equiv_{M_{i+1}} a_{i+1}$ for all $0 \leq i < n$.

We write $\mathrm{Lstp}(a/B)$ for the $\equiv^{\mathrm{Ls}}_B$-equivalence class of a.

Conditions (i) and (ii) from Definition 2.40 are in fact equivalent in any positive theory, see for example [Pil00, Lemma 3.15] or [BY03b, Lemma 1.38]. The implication (i) $\implies$ (iii) also holds in every positive theory. To see this, let a and a' be on some B-indiscernible sequence I. Let M' be any finitely λ_T-saturated e.c. model containing B. By Fact 2.37 and an automorphism we find $M \equiv_B M'$ such that I is M-indiscernible. The converse (iii) $\implies$ (i) follows from Proposition 2.39. By Fact 2.32 we can delete "finitely λ_T-saturated" from (iii) in semi-Hausdorff theories.

Definition 2.41. Let $\mathrm{Aut}_f(\mathfrak{M}/B)$ be the group generated by

$$\bigcup\{\mathrm{Aut}(\mathfrak{M}/M) : M \text{ is a finitely } \lambda_T\text{-saturated model and } B \subseteq M\}.$$

We call its elements *Lascar strong automorphisms*.

In a thick theory we have $a \equiv^{\mathrm{Ls}}_B a'$ if and only if there is $f \in \mathrm{Aut}_f(\mathfrak{M}/B)$ such that $f(a) = a'$.

2.2 Hyperimaginaries

We have already seen different examples of positive theories. There is one important class of examples, namely theories that arise from adding hyperimaginaries as real elements. It is well known that by doing so we leave the framework of full first-order logic, for example because we can get a bounded infinite definable set (Example 2.48). However, we do stay within the framework of positive logic. We show that adding hyperimaginaries as real elements does not essentially change anything. So working with hyperimaginaries in positive logic requires no special treatment.

The construction in this section is based on [BY03a, Example 2.16], but we work things out in far greater detail. This then allows us to prove that certain properties are invariant under adding hyperimaginaries, see for example Theorem 2.46 and Theorem 6.102.

We fix the following things throughout the rest of this section. A positive theory T in a signature $\mathcal{L}$ with monster model $\mathfrak{M}$. For simplicity we assume $\mathcal{L}$ is single sorted (extending this to the multi-sorted setting is straightforward). Let $\mathcal{E}$ be a set of partial types (over $\emptyset$) $E(x, y)$, where x and y are (possibly infinite, but small) tuples of variables, such that each E defines an equivalence relation in $\mathfrak{M}$.

Definition 2.42. We define the *hyperimaginary language* $\mathcal{L}_\mathcal{E}$ as a multi-sorted extension of $\mathcal{L}$. The sort of $\mathcal{L}$ will be called the *real sort* and is denoted by S_{real}. Then for each $E \in \mathcal{E}$ we add a sort S_E, called a *hyperimaginary sort*.

For a variable y of sort S_E we denote by y_r a tuple of variables of the real sort, matching the length of the representatives of the E-equivalence classes.

For all $E_1, \ldots, E_n \in \mathcal{E}$ we add a relation symbol $R_\varphi(x, y_1, \ldots, y_n)$ of sort $S_{\text{real}}^{|x|} \times S_{E_1} \times \ldots \times S_{E_n}$ for each $\mathcal{L}$-formula $\varphi(x, y_{1,r}, \ldots, y_{n,r})$.

In the above definition, not all variables in $\varphi(x, y_{1,r}, \ldots, y_{n,r})$ need to actually appear in the formula. In particular, it is not problem for the $y_{i,r}$ to be infinite tuples. Similarly, when we write something like $\exists y_r \varphi(y_r)$, then we really only quantify over the variables that actually appear in φ.

Definition 2.43. We extend $\mathfrak{M}$ to an $\mathcal{L}_\mathcal{E}$-structure $\mathfrak{M}^\mathcal{E}$ as follows. The real sort S_{real} is just $\mathfrak{M}$, and for each $E \in \mathcal{E}$ the sort S_E is $\mathfrak{M}^\alpha/E$, where α is the length of the tuples of free variables in E. From now on we will use the shorthand notation $\mathfrak{M}/E$ and not mention α. For $E_1, \ldots, E_n \in \mathcal{E}$ and $\varphi(x, y_{1,r}, \ldots, y_{n,r})$ we interpret the relation symbol R_φ as follows. We let $\mathfrak{M}^\mathcal{E} \models R_\varphi(a, c_1, \ldots, c_n)$ if and only if there are representatives $b_1, \ldots, b_n$ of $c_1, \ldots, c_n$ such that $\mathfrak{M} \models \varphi(a, b_1, \ldots, b_n)$.

For a real tuple b and some $E \in \mathcal{E}$ we will write $[b]$ for the corresponding hyperimaginary in $\mathfrak{M}/E$. To prevent cluttering of notation, we will actually also use the notation $[b]$ for a tuple of hyperimaginaries. This notation leaves implicit which sort(s) [b] belongs to, but that should not be a problem in what follows.

Definition 2.44. We define the $\mathcal{L}_\mathcal{E}$-theory $T^\mathcal{E}$ as the set of all h-inductive $\mathcal{L}_\mathcal{E}$-sentences true in $\mathfrak{M}^\mathcal{E}$.

We will prove the following results about this construction. Along the way we also develop some technical tools that will turn out to be useful in chapter 6. Let us first just state these results, to make it clear what we are working towards.

Theorem 2.45. *The structure $\mathfrak{M}^\mathcal{E}$ is a monster model of $T^\mathcal{E}$.*

Theorem 2.46. *The following properties of T are preserved when adding hyperimaginaries:*

- *Hausdorff,*
- *semi-Hausdorff,*
- *thick.*

That is, if T has the property then $T^\mathcal{E}$ has it as well.

Lemma 2.47. *Let $\Gamma(x, y)$ be a set of $\mathcal{L}_\mathcal{E}$-formulas, where x is a tuple of real variables and y is a tuple of hyperimaginary variables. Then there is a set of $\mathcal{L}$-formulas $\Sigma_\Gamma(x, y_r)$ such that $\mathfrak{M} \models \Sigma_\Gamma(a, b)$ if and only if $\mathfrak{M}^\mathcal{E} \models \Gamma(a, [b])$.*

We set up our construction in such a way that we can add any set $\mathcal{E}$ of hyperimaginaries. If we wish to study $\mathfrak{M}^{\text{heq}}$, where we have added all hyperimaginaries, we would have to add a proper class of hyperimaginaries. We can formalise this by taking $\mathcal{E}$ to be the set of all equivalence relations $E(x, y)$ where $|x| \leq |T|$. Then, by [BY03c, Corollary 3.3], every possible hyperimaginary is interdefinable with a set of hyperimaginaries in $\mathcal{E}$. So we can take $\mathfrak{M}^{\text{heq}}$ and T^{heq} to be $\mathfrak{M}^{\mathcal{E}}$ and $T^{\mathcal{E}}$.

Before we prove the above statements we will first discuss one example. After the example the rest of the section is devoted to the proofs of the above statements.

Example 2.48. In this example we will consider the first order theory RCF of the real closed field. We take $\mathcal{E}$ to consist of one equivalence relation $E(x, y)$ that says $|x - y| \leq \frac{1}{n}$ for all $n \in \mathbb{N}$. Following Definition 2.43 we then have a relation symbol $R_{\leq}(x, y)$, where x and y are of sort S_E. So we have $R_{\leq}([a], [b])$ iff there are a' and b' that are arbitrarily close to a and b respectively such that $a' \leq b'$. It is straightforward to check that $R_{\leq}$ defines a linear order on $\mathfrak{M}/E$. So $R_{\leq}([0], x) \wedge R_{\leq}(x, [1])$ then essentially defines the closed unit interval of standard real numbers and is thus an example of a bounded infinite definable set. We see that even though RCF is a theory in full first-order logic, $\mathsf{RCF}^{\mathcal{E}}$ does no longer fit in the framework of full first-order logic.

Lemma 2.49. *Let $\varphi(x, y)$ be an $\mathcal{L}_E$-formula, where x is a tuple of real variables and y is a tuple of hyperimaginary variables. Then there is a set of $\mathcal{L}$-formulas $\Sigma_\varphi(x, y_r)$ such that $\mathfrak{M} \models \Sigma_\varphi(a, b)$ if and only if $\mathfrak{M}^{\mathcal{E}} \models \varphi(a, [b])$.*

Proof. We first assume that $\varphi(x, y)$ is of the form

$$\exists wz \left(\psi(x, w) \wedge \varepsilon(y, z) \wedge \bigwedge_{i \in I} R_{\chi_i}(x, w, y, z) \right).$$

Here w is a tuple of real variables and z a tuple of hyperimaginary variables. The formula $\psi(x, w)$ is an $\mathcal{L}$-formula and $\varepsilon(y, z)$ is a conjunction of equalities of hyperimaginaries.

We define the partial type Γ_φ as follows. For each $i \in I$ we introduce tuples of real variables y_i and z_i matching y_r and z_r respectively. We let $E_\varepsilon(y_r, z_r)$ be the union of partial types in $\mathcal{E}$ expressing $\varepsilon([y_r], [z_r])$, and we close E_ε under conjunctions. Then we set

$$\begin{aligned}
&\Gamma_\varphi(x, y_r, w, z_r, (y_i)_{i\in I}, (z_i)_{i\in I}) && = \\
&\left\{\psi(x,w) \wedge \epsilon(y_r, z_r) \wedge \bigwedge_{i\in I} \chi_i(x, w, y_i, z_i) : \epsilon \in E_\varepsilon\right\} && \cup \qquad (2.2.1)\\
&\bigcup\{E_y(y_r, y_i) : i \in I\} && \cup \qquad (2.2.2)\\
&\bigcup\{E_z(z_r, z_i) : i \in I\}. && \qquad (2.2.3)
\end{aligned}$$

Here E_y and E_z are the equivalence relations corresponding to the hyperimaginary variables y and z respectively.

Let $\Sigma_\varphi(x, y_r)$ express the following:

$$\exists w z_r (y_i)_{i\in I}(z_i)_{i\in I}\Gamma_\varphi(x, y_r, w, z_r, (y_i)_{i\in I}, (z_i)_{i\in I}).$$

Now suppose that a, b are such that $\mathfrak{M} \models \Sigma_\varphi(a, b)$. Then we find realisations such that

$$\mathfrak{M} \models \Gamma_\varphi(a, b, c, d, (b_i)_{i\in I}, (d_i)_{i\in I}).$$

Then (2.2.2) and (2.2.3) tell us that $[b] = [b_i]$ and $[d] = [d_i]$ for all $i \in I$, while (2.2.1) guarantees that $\mathfrak{M}^{\mathcal{E}} \models \varphi(a, [b])$. This proves the forward direction and the converse is straightforward by just taking representatives of the hyperimaginaries that are involved.

We assumed φ to be of a particular form. Since every formula can be written as a disjunction of regular formulas (i.e. formulas built using conjunction and existential quantification), we are only left an induction step for disjunction. So let $\varphi_1(x, y)$ and $\varphi_2(x, y)$ with $\Sigma_{\varphi_1}(x, y_r)$ and $\Sigma_{\varphi_2}(x, y_r)$ be given. We define $\Sigma_{\varphi_1\vee\varphi_2}(x, y_r)$ as

$$\{\psi_1 \vee \psi_2 : \psi_1 \in \Sigma_{\varphi_1}, \psi_2 \in \Sigma_{\varphi_2}\}.$$

One easily checks that $\mathfrak{M} \models \Sigma_{\varphi_1\vee\varphi_2}(a, b)$ precisely when $\mathfrak{M} \models \Sigma_{\varphi_1}(a, b)$ or $\mathfrak{M} \models \Sigma_{\varphi_2}(a, b)$ or both, and the result follows. □

Proof of Lemma 2.47. Define

$$\Sigma_\Gamma(x, y_r) = \bigcup_{\varphi\in\Gamma} \Sigma_\varphi(x, y_r),$$

where Σ_φ is as in Lemma 2.49. □

Lemma 2.50. *If* $\mathrm{tp}(a[b]) = \mathrm{tp}(a'[b'])$ *then there is* b'' *such that* $\mathrm{tp}(ab) = \mathrm{tp}(a'b'')$ *and* $[b'] = [b'']$.

Proof. Define

$$\Sigma(x, y) = \mathrm{tp}_{\mathcal{L}}(ab) \cup E(b', y).$$

It is enough to prove that $\Sigma(a', y)$ is finitely satisfiable. Let $\varphi(x, y) \in \mathrm{tp}_{\mathcal{L}}(ab)$. Then $\mathfrak{M}^{\mathcal{E}} \models R_\varphi(a, [b])$, so $\mathfrak{M}^{\mathcal{E}} \models R_\varphi(a', [b'])$. So there is $b'' \in \mathfrak{M}$ with $\mathfrak{M} \models E(b', b'')$ and $\mathfrak{M} \models \varphi(a', b'')$, as required. □

Lemma 2.51. *For every tuple of hyperimaginary variables y there is a partial $\mathcal{L}_{\mathcal{E}}$-type $\Xi(y_r, y)$ such that $\mathfrak{M}^{\mathcal{E}} \models \Xi(a, [a'])$ if and only if $[a] = [a']$.*

Proof. We define

$$\Xi(y_r, y) = \{R_\varepsilon(y_r, y) : \varepsilon \in E\},$$

where E is the equivalence relation corresponding to y. The right to left direction is clear. For the forward direction we suppose $\mathfrak{M}^{\mathcal{E}} \models \Xi(a, [a'])$. Consider the partial type

$$\Gamma(y_r) = E(a, y_r) \cup E(y_r, a').$$

For any $\varepsilon(a, y_r) \in E(a, y_r)$ we have $\mathfrak{M}^{\mathcal{E}} \models R_\varepsilon(a, [a'])$. So there must be $a^* \in \mathfrak{M}$ such that $[a^*] = [a']$ and $\mathfrak{M} \models \varepsilon(a, a^*)$. Thus $\mathfrak{M} \models \varepsilon(a, a^*) \wedge E(a^*, a')$. We thus see that Γ is finitely satisfiable, so there is a realisation a''. We conclude that $[a] = [a''] = [a']$. □

Lemma 2.52. *Any automorphism $f : \mathfrak{M} \to \mathfrak{M}$ uniquely extends to an automorphism $f^{\mathcal{E}} : \mathfrak{M}^{\mathcal{E}} \to \mathfrak{M}^{\mathcal{E}}$ by setting $f^{\mathcal{E}}([b]) = [f(b)]$.*

Proof. It is straightforward to check that $f^{\mathcal{E}}$ is well-defined and bijective. We need to show that $f^{\mathcal{E}}$ preserves and reflects truth of the new relation symbols in $\mathcal{L}_{\mathcal{E}}$ (preservation of equality is just saying that $f^{\mathcal{E}}$ is well-defined). Suppose that $\mathfrak{M}^{\mathcal{E}} \models R_\varphi(a, [b])$. By definition there is b' such that $[b'] = [b]$ and $\mathfrak{M} \models \varphi(a, b')$. Then $\mathfrak{M} \models \varphi(f(a), f(b'))$ and hence $\mathfrak{M}^{\mathcal{E}} \models R_\varphi(f(a), [f(b')])$ which is just $\mathfrak{M}^{\mathcal{E}} \models R_\varphi(f^{\mathcal{E}}(a), f^{\mathcal{E}}([b]))$. The converse follows in a similar way.

Finally we check uniqueness of $f^{\mathcal{E}}$. Suppose that $g : \mathfrak{M}^{\mathcal{E}} \to \mathfrak{M}^{\mathcal{E}}$ also extends f. For $[b] \in \mathfrak{M}^{\mathcal{E}}$ we have $\mathfrak{M}^{\mathcal{E}} \models \Xi(b, [b])$ by Lemma 2.51. So if g is an automorphism we must have $\mathfrak{M}^{\mathcal{E}} \models \Xi(g(b), g([b]))$, which means that $g([b]) = [g(b)] = [f(b)]$, as required. □

Theorem 2.45**, repeated.** *The structure $\mathfrak{M}^{\mathcal{E}}$ is a monster model of $T^{\mathcal{E}}$.*

Proof. We prove that $\mathfrak{M}^{\mathcal{E}}$ is e.c. and is just as saturated and homogeneous as $\mathfrak{M}$. So let κ be such that $\mathfrak{M}$ is κ-saturated and κ-homogeneous. Note that this means that κ is definitely bigger than the length of any tuple representing a hyperimaginary.

Existentially closed. We will use (iii) from Definition 2.5. Suppose that $\mathfrak{M}^{\mathcal{E}} \not\models \varphi(a, [b])$. Then $\mathfrak{M} \not\models \Sigma_\varphi(a, b)$, where Σ_φ is from Lemma 2.47. So there is $\psi(x, y_r) \in \Sigma_\varphi(x, y_r)$ such that $\mathfrak{M} \not\models \psi(a, b)$. Because $\mathfrak{M}$ is e.c. we find $\chi(x, y_r)$ with $T \models \neg\exists x y_r(\psi(x, y_r) \wedge \chi(x, y_r))$ and $\mathfrak{M} \models \chi(a, b)$. We thus have $\mathfrak{M}^{\mathcal{E}} \models R_\chi(a, [b])$. We will conclude by proving that

$\mathfrak{M}^{\mathcal{E}} \models \neg\exists xy(\varphi(x,y) \wedge R_\chi(x,y))$. Suppose for a contradiction that there are a' and b' such that $\mathfrak{M}^{\mathcal{E}} \models \varphi(a',[b']) \wedge R_\chi(a',[b'])$. Then there is b'' with $[b'] = [b'']$ and $\mathfrak{M} \models \chi(a',b'')$. So $\mathfrak{M}^{\mathcal{E}} \models \varphi(a',[b''])$ and thus $\mathfrak{M} \models \Sigma_\varphi(a',b'')$. We then get that $\mathfrak{M} \models \psi(a',b'') \wedge \chi(a',b'')$, which cannot happen.

Saturation. Let $\Gamma(x,y,c,[d])$ be a finitely satisfiable partial $\mathcal{L}_{\mathcal{E}}$-type with $|c[d]| < \kappa$. Let $\Sigma_\Gamma(x,y,c,d)$ be the set of $\mathcal{L}$-formulas from Lemma 2.47. By the construction there we have

$$\Sigma_\Gamma(x,y,c,d) = \bigcup_{\varphi\in\Gamma} \Sigma_\varphi(x,y,c,d),$$

where Σ_φ is as in Lemma 2.49. So finite satisfiability of $\Gamma(x,y,c,[d])$ implies finite satisfiability of $\Sigma_\Gamma(x,y,c,d)$. We thus find $a,b \in \mathfrak{M}$ with $\mathfrak{M} \models \Sigma_\Gamma(a,b,c,d)$ and hence $\mathfrak{M}^{\mathcal{E}} \models \Gamma(a,[b],c,[d])$.

Homogeneity. If $\mathrm{tp}(a[b]) = \mathrm{tp}(a'[b'])$, then by Lemma 2.50 there is b'' such that $[b''] = [b']$ and $\mathrm{tp}(ab) = \mathrm{tp}(a'b'')$. Let $f : \mathfrak{M} \to \mathfrak{M}$ be an automorphism with $f(ab) = a'b''$. Then by Lemma 2.52 we find $f^{\mathcal{E}} : \mathfrak{M}^{\mathcal{E}} \to \mathfrak{M}^{\mathcal{E}}$ with $f^{\mathcal{E}}(a[b]) = f(a)[f(b)] = a'[b''] = a'[b']$, as required. □

Lemma 2.53. *A sequence $(a_i[b_i])_{i\in I}$ is indiscernible if and only if there are representatives b'_i of $[b_i]$ such that $(a_i b'_i)_{i\in I}$ is indiscernible.*

Proof. We first prove the left to right direction. By compactness we may assume I to be long enough. We can find indiscernible $(a^*_i b^*_i)_{i\in I}$ based on $(a_i b_i)_{i\in I}$. Let $p((x_i y_{i,r})_{i\in I}) = \mathrm{tp}((a^*_i b^*_i)_{i\in I})$ and define the following type

$$\Gamma = p((a_i y_{i,r})_{i\in I}) \cup \{\Xi(y_{i,r},[b_i]) : i \in I\}.$$

Then a realisation of Γ is precisely what we need, so we prove that Γ is finitely satisfiable. That is, for $i_1 < \ldots < i_n \in I$ we will produce a realisation of Γ restricted to the variables $y_{i_1,r} \ldots y_{i_n,r}$ and parameters $a_{i_1},\ldots,a_{i_n},[b_{i_1}],\ldots,[b_{i_n}]$. By construction there are $j_1 < \ldots < j_n \in I$ such that $\mathrm{tp}(a^*_{i_1} b^*_{i_1} \ldots a^*_{i_n} b^*_{i_n}) = \mathrm{tp}(a_{j_1} b_{j_1} \ldots a_{j_n} b_{j_n})$. As $\mathrm{tp}(a_{i_1}[b_{i_1}] \ldots a_{i_n}[b_{i_n}]) = \mathrm{tp}(a_{j_1}[b_{j_1}] \ldots a_{j_n}[b_{j_n}])$, by Lemma 2.50 we can find $b'_{i_1} \ldots b'_{i_n}$ with $\mathrm{tp}(a_{i_1} b'_{i_1} \ldots a_{i_n} b'_{i_n}) = \mathrm{tp}(a_{j_1} b_{j_1} \ldots a_{j_n} b_{j_n})$ while also $[b'_{i_k}] = [b_{i_k}]$ for all $1 \leq k \leq n$. So $b'_{i_1} \ldots b'_{i_n}$ is the desired realisation of Γ restricted to $y_{i_1,r} \ldots y_{i_n,r}$ and $a_{i_1},\ldots,a_{i_n},[b_{i_1}],\ldots,[b_{i_n}]$.

For the right to left direction we note that for any $i_1 < \ldots < i_n \in I$ and $j_1 < \ldots < j_n \in I$ we have

$$\Sigma_{\mathrm{tp}(a_{i_1}[b_{i_1}]\ldots a_{i_n}[b_{i_n}])} \subseteq \mathrm{tp}(a_{i_1} b'_{i_1} \ldots a_{i_n} b'_{i_n}) = \mathrm{tp}(a_{j_1} b'_{j_1} \ldots a_{j_n} b'_{j_n}).$$

So $\mathrm{tp}(a_{i_1}[b_{i_1}] \ldots a_{i_n}[b_{i_n}]) \subseteq \mathrm{tp}(a_{j_1}[b_{j_1}] \ldots a_{j_n}[b_{j_n}])$, and the claim follows by maximality of types. □

Theorem 2.46**, repeated.** *The following properties of T are preserved when adding hyperimaginaries:*

- *Hausdorff,*
- *semi-Hausdorff,*
- *thick.*

That is, if T has the property then $T^{\mathcal{E}}$ has it as well.

Proof. Hausdorff. Let $a[b] \not\equiv a'[b']$. Then there is $\varphi \in \mathrm{tp}(a[b])$ such that $\varphi \notin \mathrm{tp}(a'[b'])$. So there is a negation $\psi \in \mathrm{tp}(a'[b'])$ of φ. By Lemma 2.47 we have Σ_φ and Σ_ψ are consistent while $\Sigma_\varphi \cup \Sigma_\psi$ is inconsistent.

Fix some type q of T such that $\Sigma_\psi \subseteq q$. We will produce formulas α_q and β_q such that $\Sigma_\varphi \cup \{\alpha_q\}$ is inconsistent, $\beta_q \notin q$ and $T \models \forall x y_r(\alpha_q(x, y_r) \vee \beta_q(x, y_r))$. Let $p \supseteq \Sigma_\varphi$ be a type of T. Then because T is Hausdorff there are formulas χ_p and θ_p such that $\chi_p \notin p$ and $\theta_p \notin q$, while $T \models \forall x y_r(\chi_p(x, y_r) \vee \theta_p(x, y_r))$. Then $\Sigma_\varphi \cup \{\chi_p : p \supseteq \Sigma_\varphi\}$ is inconsistent, so there are $p_1, \ldots, p_n$ such that $\Sigma_\varphi \cup \{\chi_{p_1} \wedge \ldots \wedge \chi_{p_n}\}$ is inconsistent. We can now take α_q to be $\chi_{p_1} \wedge \ldots \wedge \chi_{p_n}$ and β_q to be $\theta_{p_1} \vee \ldots \vee \theta_{p_n}$.

Now $\Sigma_\psi \cup \{\beta_q : q \supseteq \Sigma_\psi\}$ is inconsistent. So there are $q_1, \ldots, q_k$ such that $\Sigma_\psi \cup \{\beta_{q_1} \wedge \ldots \wedge \beta_{q_k}\}$ is inconsistent. We set $\beta = \beta_{q_1} \wedge \ldots \wedge \beta_{q_k}$ and $\alpha = \alpha_{q_1} \vee \ldots \vee \alpha_{q_n}$. We then also have that $\Sigma_\varphi \cup \{\alpha\}$ is inconsistent and $T \models \forall x y_r(\alpha(x, y_r) \vee \beta(x, y_r))$.

Now consider the formulas $R_\alpha(x, y)$ and $R_\beta(x, y)$. By construction we have $T^{\mathcal{E}} \models \forall xy(R_\alpha(x, y) \vee R_\beta(x, y))$. We claim that $R_\alpha \notin \mathrm{tp}(a[b])$. Suppose for a contradiction that $\mathfrak{M}^{\mathcal{E}} \models R_\alpha(a, [b])$. Then there is b^* with $[b^*] = [b]$ such that $\mathfrak{M} \models \alpha(a, b^*)$. Since $\varphi \in \mathrm{tp}(a[b]) = \mathrm{tp}(a[b^*])$ we also have $\mathfrak{M} \models \Sigma_\varphi(a, b^*)$, contradicting that $\Sigma_\varphi \cup \{\alpha\}$ is inconsistent. So indeed $R_\alpha \notin \mathrm{tp}(a[b])$. Analogously we get that $R_\beta \notin \mathrm{tp}(a'[b'])$, which concludes the proof that $T^{\mathcal{E}}$ is Hausdorff.

Semi-Hausdorff. Suppose that equality of $\mathcal{L}$-types is type-definable by a partial $\mathcal{L}$-type Ω. Then for a tuple x of real variables and a tuple y of hyperimaginary variables, we consider the partial $\mathcal{L}_{\mathcal{E}}$-type $\Omega^{\mathcal{E}}(xy, x'y')$ that expresses the following:

$$\exists y_r y'_r(\Xi(y_r, y) \wedge \Xi(y'_r, y') \wedge \Omega(x y_r, x' y'_r)).$$

We claim that $\Omega^{\mathcal{E}}$ expresses equality of $\mathcal{L}_{\mathcal{E}}$-types.

If $\mathfrak{M}^{\mathcal{E}} \models \Omega^{\mathcal{E}}(a[b], a'[b'])$ then we find c, c' such that $\mathfrak{M}^{\mathcal{E}} \models \Xi(c, [b]) \wedge \Xi(c', [b']) \wedge \Omega(ac, a'c')$. By Lemma 2.51 we have that $[c] = [b]$ and $[c'] = [b']$. Hence $\varphi \in \mathrm{tp}(a[b]) = \mathrm{tp}(a[c])$ iff $\Sigma_\varphi \subseteq \mathrm{tp}(ac) = \mathrm{tp}(a'c')$ iff $\varphi \in \mathrm{tp}(a'[c']) = \mathrm{tp}(a'[b'])$. So $\mathrm{tp}(a[b]) = \mathrm{tp}(a'[b'])$, as required.

Conversely, if $\mathrm{tp}(a[b]) = \mathrm{tp}(a'[b'])$ then by Lemma 2.50 we find b'' such that $[b''] = [b']$ and $\mathrm{tp}(ab) = \mathrm{tp}(a'b'')$. Hence $\models \Xi(b, [b]) \wedge \Xi(b'', [b']) \wedge \Omega(ab, a'b'')$.

Thick. Let Θ express indiscernibility of a sequence of real tuples, then

$$\exists (y_{i,r})_{i<\omega} \left(\Theta((x_i y_{i,r})_{i<\omega}) \wedge \bigwedge_{i<\omega} \Xi(y_{i,r}, y_i) \right)$$

expresses indiscernibility of $(x_i y_i)_{i<\omega}$ in $T^{\mathcal{E}}$. Here we use that a sequence in $\mathfrak{M}^{\mathcal{E}}$ is indiscernible if and only if there is an indiscernible sequence of real representatives, see Lemma 2.53. □

2.3 Accessible categories

The general idea of the accessible categories approach to model theory is to consider some category of models and then work in there. For example, we can consider the category of models of some theory in full first-order logic, with elementary embeddings as arrows. Due to the Löwenheim-Skolem theorem every object in that category can be 'built' from 'small' objects. This is the main idea behind accessible categories. We will treat the basics in this section. A great reference for accessible categories is [AR94].

Convention 2.54. Throughout, κ, λ and μ will denote regular cardinals.

We will first concern ourselves with defining a notion of size in category-theoretic language. This will allow us to make 'small' precise.

Definition 2.55. A poset P is called λ-*directed* if every subset $A \subseteq P$ with $|A| < \lambda$ has an upper bound. A λ-*directed diagram* in some category $\mathcal{C}$ is a functor $F : P \to \mathcal{C}$, where P is a λ-directed poset (considered as a category). If $X = \operatorname{colim}_{i \in I} X_i$ for some λ-directed diagram X_i then we call X the λ-*directed colimit* of $(X_i)_{i \in I}$. If $\lambda = \aleph_0$ we will drop it from the notation and just say *directed diagram* and *directed colimit*.

Example 2.56. In this example we consider the category **Set** of sets. Recall that for a set X we write $[X]^{<\lambda}$ for the set of all subsets of X of cardinality $< \lambda$. Ordered by inclusion, $[X]^{<\lambda}$ forms a λ-directed diagram whose colimit is X.

These diagrams can be used to get information about the cardinality of a set. That is, we have that $|X| < \lambda$ if and only if for every λ-directed diagram $(Y_i)_{i \in I}$ with colimit Y and every function $f : X \to Y$ we have that f factors essentially uniquely through $(Y_i)_{i \in I}$.

Suppose that $|X| < \lambda$ and let $f : X \to Y$ and $(Y_i)_{i \in I}$ be as in the statement. Write $g_i : Y_i \to Y$ for the coprojections. For every $x \in X$ there is $i_x \in I$ such that $f(x) \in g_i(Y_i)$. By λ-directedness there is an upper bound $j \in I$ of $\{i_x : x \in X\}$. So f factors through g_j.

Conversely, we use that X is the λ-directed colimit of $[X]^{<\lambda}$. So the identity function Id_X factors through the diagram $[X]^{<\lambda}$ and $|X| < \lambda$ follows.

The characterisation of cardinality in Example 2.56 can be stated in purely category-theoretic terms, see Definition 2.57 below. When applying this definition to other categories we see that it coincides with some notion of size using the same technique from Example 2.56.

Definition 2.57. An object X in a category $\mathcal{C}$ is called *λ-presentable* if the following equivalent conditions hold:

(i) $\mathrm{Hom}(X, -)$ preserves λ-directed colimits;

(ii) for every λ-directed colimit $Y = \mathrm{colim}_{i \in I} Y_i$ every arrow $f : X \to Y$ factors essentially uniquely through the diagram $(Y_i)_{i \in I}$.

If $\lambda = \aleph_0$ we will also say that X is *finitely presentable*.

For completeness we recall the precise meaning of "factors essentially uniquely through the diagram $(Y_i)_{i \in I}$". The diagram is actually a functor $F : I \to \mathcal{C}$ where $F(i) = Y_i$. The colimit Y comes equipped with coprojections $g_i : Y_i \to Y$. Then saying that $f : X \to Y$ factors through $(Y_i)_{i \in I}$ means that there is $j \in I$ and $f' : X \to Y_j$ such that $f = g_j f'$. To say that this factorisation is essentially unique means that if there is another arrow $f^* : X \to Y_j$ with $f = g_j f^*$ then there is $k \geq j$ in I such that $F(j \leq k) f' = F(j \leq k) f^*$.

Fact 2.58 ([AR94, Proposition 1.16])**.** *A colimit of a diagram with less than λ arrows consisting of λ-presentable objects is again λ-presentable. In particular, if $\kappa < \lambda$ and $(X_i)_{i<\kappa}$ is a diagram of shape κ where each X_i is λ-presentable then $X = \mathrm{colim}_{i<\kappa} X_i$ (if it exists) is λ-presentable.*

Now that we have a category-theoretic notion of size we can define accessible categories, making precise the idea sketched in the beginning of this section. Namely that every object can be 'built' from 'small' objects.

Definition 2.59. A category $\mathcal{C}$ is called *λ-accessible* if

(i) $\mathcal{C}$ has λ-directed colimits;

(ii) there is a set $\mathcal{A}$ of λ-presentable objects, such that every object in $\mathcal{C}$ can be written as a λ-directed colimit of objects in $\mathcal{A}$.

A category is called *accessible* if it is λ-accessible for some λ.

Recall that a *chain* is a diagram of shape δ for some ordinal δ. In model theory it is often useful to have colimits of chains. Clearly we have colimits of chains if we have directed colimits, but the converse is also true.

Fact 2.60 ([AR94, Corollary 1.7])**.** *A category $\mathcal{C}$ has colimits of chains if and only if it has directed colimits. For such $\mathcal{C}$ a functor $F : \mathcal{C} \to \mathcal{D}$ preserves colimits of chains if and only if it preserves directed colimits.*

We also recall some terminology concerning chains. A chain $(X_i)_{i<\delta}$ is called *continuous* if for every limit ordinal $\ell < \delta$ we have that $X_\ell = \operatorname{colim}_{i<\ell} X_i$. A *chain bound* for $(X_i)_{i<\delta}$ is just a cocone.

Definition 2.61. Given a span $Y_1 \leftarrow X \rightarrow Y_2$ in some category $\mathcal{C}$ we call a cospan $Y_1 \rightarrow Z \leftarrow Y_2$ an *amalgam* if the relevant square commutes. We call an object X an *amalgamation base* if every span $Y_1 \leftarrow X \rightarrow Y_2$ admits an amalgam. We say that $\mathcal{C}$ has the *amalgamation property* or *AP* if every object is an amalgamation base.

We close out this section with some examples of accessible categories. We will refer back to those examples in chapter 3, as applications of the framework we develop there.

Example 2.62. Let T be some positive theory. We write $\mathbf{Mod}(T)$ for the category of e.c. models of T with immersions as arrows. Recall from Remark 2.12 that this subsumes the usual construction for full first-order logic, where we take models of some theory T and elementary embeddings between them.

By Löwenheim-Skolem and a similar argument as in Example 2.56 we have for $\lambda > |T|$ that any $M \models T$ is λ-presentable in $\mathbf{Mod}(T)$ if and only if $|M| < \lambda$. Up to isomorphism there is only a set of models of cardinality $\leq |T|$. So there is a set $\mathcal{A}$ of $|T|^+$-presentable models and every model is a $|T|^+$-directed colimit of models in $\mathcal{A}$. We thus see that $\mathbf{Mod}(T)$ is $|T|^+$-accessible.

More is true. We mention a few properties that will be useful later on. Clearly every arrow in $\mathbf{Mod}(T)$ is a monomorphism. By Fact 2.7 we also see that $\mathbf{Mod}(T)$ has directed colimits and has AP.

Example 2.63. In this example we consider continuous logic in the sense of [BYBHU08]. Let T be a continuous theory. We write $\mathbf{MetMod}(T)$ for the category of metric models. The arrows are elementary embeddings, in the continuous sense. The right notion of size in this category is that of *density character*: the smallest cardinality of a dense subset in the space. Denote the density character of a space X by $\operatorname{density}(X)$. Write $|T|$ for the cardinality of the signature of T. Then for all $\lambda > |T|$ we have that object M in $\mathbf{MetMod}(T)$ is λ-presentable if and only if $\operatorname{density}(M) < \lambda$.

Many of the tools we have in full first-order logic and positive logic also exist in continuous logic. In particular we have compactness and Löwenheim-Skolem. So we can follow the same arguments to see that $\mathbf{MetMod}(T)$ is $|T|^+$-accessible, has all directed colimits and AP and every arrow is a monomorphism.

Example 2.64. Let $\mathcal{K}$ be an *AEC* (*abstract elementary class*). We refer to [She09] for an extensive treatment of AECs. We can naturally view $\mathcal{K}$ as an accessible category, and we will indeed do so, in the following way. The objects are just the elements of $\mathcal{K}$. The arrows are $\mathcal{K}$-embeddings. That is, functions

$f : M \to N$ such that $f(M) \preceq_{\mathcal{K}} N$ and f is an isomorphism of M onto $f(M)$. The Tarski-Vaught chain axioms are then precisely saying that $\mathcal{K}$ has colimits of chains, and hence all directed colimits by Fact 2.60. Writing $\mathrm{LS}(\mathcal{K})$ for the Löwenheim-Skolem number, we get that $\mathcal{K}$ is $\mathrm{LS}(\mathcal{K})^+$-accessible.

3

Abstract Elementary Categories

In this chapter we introduce the framework of *abstract elementary categories*, or *AECats*. These are a specific kind of accessible category, that is still general enough to cover all the examples we discussed in section 2.3. In some applications we would like to have access to the subsets of models, so the framework is made flexible enough to also fit something like the category of subsets of models. In section 3.1 we give the definition and provide the motivating examples for AECats, arising from full first-order logic, positive logic, continuous logic and AECs.

Even though the framework of AECats is very close to that of AECs, it is still more general. Some settings are hard to handle as AECs, but naturally fit the category-theoretic framework of AECats. For example, the class of metric models of a continuous theory in the sense of [BYBHU08] is not an AEC, but it does form an AECat (see Example 3.7). Of course, there are *metric AECs* as introduced in [HH09], but AECats provide a unifying approach.

AECats do not have syntax, but we can still make sense of a notion of types through the idea of Galois types, which we do in section 3.2. Since we do not have access to single elements in our category, we instead consider (tuples of) arrows, keeping in mind that each arrow can actually represent an entire tuple of elements. In later chapters we will also need a substitute for Lascar strong types, for which we introduce Lascar strong Galois types in section 3.4.

An interesting property for Galois types is being finitely short, which says that the Galois type of a tuple is determined by the Galois types of its finite subtuples (Definition 3.31). We do not need this property in the rest of this thesis (except for section 4.5, where a further connection is explored), and in fact there are interesting AECats that are not finitely short (see Example 3.35). We mention finite shortness because it provides a nice link to existing frameworks. One of the interesting consequences of finite shortness is that we can recover some compactness.

It is standard in model theory to work with monster models. In the general category-theoretic setting this would still be possible. For example, in [LR14] it is shown that such monster objects exist in any accessible category with directed colimits and the amalgamation property. This assumes some additional set theory, namely that there is a proper class of cardinals λ such that $\lambda = \lambda^{<\lambda}$. We choose not to work with monster objects. This might come at some notational cost, but it keeps everything within the standard set theory.

3.1 Definition and examples

Before we give the definition of an abstract elementary category, or AECat, we will first introduce the motivating example. We have already seen $\mathbf{Mod}(T)$ for a theory T in Example 2.62 and we will see that this gives us an AECat. Sometimes we would want to have access to (certain) subsets of models. For this we introduce the category of subsets of models.

Definition 3.1. Let T be some positive theory. We write $\mathbf{SubMod}(T)$ for the category of subsets of models of T. Its objects are pairs (A, M) where $A \subseteq M$ and M is an e.c. model of T. An arrow $f : (A, M) \to (B, N)$ is an immersion $f : A \to B$. That is, for every $a \in A$ and every $\varphi(x)$ we have $M \models \varphi(a)$ if and only if $N \models \varphi(f(a))$.

The role of the model M in a pair (A, M) is just to make sense of the evaluation of formulas with parameters in A. These formulas may contain quantifiers that refer to the rest of M.

There is a full and faithfull embedding $\mathbf{Mod}(T) \hookrightarrow \mathbf{SubMod}(T)$ by sending M to (M, M). So we consider $\mathbf{Mod}(T)$ as a full subcategory of $\mathbf{SubMod}(T)$.

In $\mathbf{SubMod}(T)$ we have for any λ that (A, M) is λ-presentable precisely when $|A| < \lambda$. Directed colimits exist in $\mathbf{SubMod}(T)$. They are calculated by taking the union in a big enough model. So we see that $\mathbf{SubMod}(T)$ is an accessible category.

The idea of an AECat is now to allow for categories like $\mathbf{SubMod}(T)$, but we need to keep track of which objects are considered to be models.

Definition 3.2. An *AECat*, short for *abstract elementary category*, consists of a pair $(\mathcal{C}, \mathcal{M})$ where $\mathcal{C}$ and $\mathcal{M}$ are accessible categories and $\mathcal{M}$ is a full subcategory of $\mathcal{C}$ such that:

(i) $\mathcal{M}$ has directed colimits, which the inclusion functor into $\mathcal{C}$ preserves;

(ii) all arrows in $\mathcal{C}$ (and thus in $\mathcal{M}$) are monomorphisms.

The objects in $\mathcal{M}$ are called *models*. We say that $(\mathcal{C}, \mathcal{M})$ has the *amalgamation property* (or *AP*) if $\mathcal{M}$ has the amalgamation property.

The name "abstract elementary category" has been used before in [BR12, Definition 5.3] for a very similar concept. As noted there as well, the name was used even before that in an unpublished note by Jonathan Kirby [Kir08].

Note that if $(\mathcal{C}, \mathcal{M})$ is an AECat then $(\mathcal{M}, \mathcal{M})$ is an AECat as well.

Example 3.3. As seen in the discussion before, both $(\mathbf{Mod}(T), \mathbf{Mod}(T))$ and $(\mathbf{SubMod}(T), \mathbf{Mod}(T))$ are AECats with AP. These are the prototypical examples of AECats to keep in mind.

We will only be interested in AECats with AP in the rest of this thesis.

To help with intuition that objects in $\mathcal{C}$ play the role of subsets of models, the reader may assume that for every object A in $\mathcal{C}$, there is an arrow $A \to M$ with M in $\mathcal{M}$. This is in fact true in all examples we consider and any object in $\mathcal{C}$ we will consider in this thesis will always come with an arrow into some model anyway.

Remark 3.4. By Fact 2.60 we could replace (i) in Definition 3.2 by: "$\mathcal{M}$ has colimits of chains, which the inclusion functor into $\mathcal{C}$ preserves".

Remark 3.5. If $(\mathcal{C}, \mathcal{M})$ is an AECat then $\mathcal{C}$ and $\mathcal{M}$ may be accessible for different cardinals. By [AR94, Corollary 2.14] and [AR94, Theorem 2.19] there are arbitrarily large κ such that both $\mathcal{C}$ and $\mathcal{M}$ are κ-accessible and the inclusion $\mathcal{M} \hookrightarrow \mathcal{C}$ preserves κ-presentable objects. In fact, in such a situation the inclusion functor preserves λ-presentable objects for all $\lambda \geq \kappa$. This follows from [BR12, Proposition 4.3], because $\mathcal{M}$ has directed colimits and the inclusion functor preserves those. In their statement $\mathcal{C}$ would be required to have directed colimits as well. However, this is not necessary if we are just interested in preserving λ-presentability: for that part their proof goes through.

It will be useful to give the above situation a name.

Definition 3.6. We call an AECat $(\mathcal{C}, \mathcal{M})$ a *κ-AECat* if $\mathcal{C}$ and $\mathcal{M}$ are both κ-accessible and the inclusion functor preserves κ-presentable objects.

So we can restate Remark 3.5 as: "for any AECat $(\mathcal{C}, \mathcal{M})$ there are arbitrarily large κ such that $(\mathcal{C}, \mathcal{M})$ is a κ-AECat".

We close out this section with a few examples of AECats arising from continuous logic, AECs, compactes abstract theories and quasiminimal excellent classes.

Example 3.7. Fix some continuous theory T in the sense of [BYBHU08]. Similar to Definition 3.1 we define a category of subsets of metric models $\mathbf{SubMetMod}(T)$. The objects are pairs (A, M) where M is a metric model of T and $A \subseteq M$ is a closed subset. An arrow $f : (A, M) \to (B, N)$ is then what is called an "elementary map" from A to B in [BYBHU08, Definition 4.3(3)].

For any λ we have that (A, M) is λ-presentable in $\mathbf{SubMetMod}(T)$ if and only if density$(A) < \lambda$. We also have directed colimits: they are calculated by taking the closure of the union in a big enough model. We thus see that $\mathbf{SubMetMod}(T)$ is accessible.

Recall $\mathbf{MetMod}(T)$ from Example 2.63. We have that $(\mathbf{MetMod}(T), \mathbf{MetMod}(T))$ and $(\mathbf{SubMetMod}(T), \mathbf{MetMod}(T))$ are AECats with AP.

Example 3.8. Let $\mathcal{K}$ be an AEC. We define the category $\mathbf{SubSet}(\mathcal{K})$ as follows. Its objects are pairs (A, M) where $M \in \mathcal{K}$ and $A \subseteq M$. An arrow $f : (A, M) \to (B, N)$ is then a $\mathcal{K}$-embedding $f : M \to N$ such that $f(A) \subseteq B$.

Note that this is slightly different from the approach of $\mathbf{SubMod}(T)$, where an arrow $(A, M) \to (B, N)$ does not need to extend to an immersion $M \to N$. This does happen when N is $|M|$-saturated, so it should be clear that the model theory considered in both constructions remains the same. In [Kam20, Example 2.11] we consider a slightly different construction for $\mathbf{SubSet}(\mathcal{K})$ that directly generalises $\mathbf{SubMod}(T)$. However, some assumptions on $\mathcal{K}$ are needed there, like the amalgamation property and some type shortness. These assumptions hold in $\mathbf{SubMod}(T)$, but generally make the construction more complicated.

We also get a slightly different notion of size in $\mathbf{SubSet}(\mathcal{K})$. That is, for $\lambda > \mathrm{LS}(\mathcal{K})$ we have that an object (A, M) is λ-presentable if and only if $|M| < \lambda$. On the other hand, directed colimits are easier to compute: we just take the union in each component. So we still end up with an accessible category $\mathbf{SubSet}(\mathcal{K})$. Then $(\mathcal{K}, \mathcal{K})$ and $(\mathbf{SubSet}(\mathcal{K}), \mathcal{K})$ are AECats, and they have AP exactly when $\mathcal{K}$ has AP.

Example 3.9. In [BY03a] the concept of a *compact abstract theory*, or *cat*, is introduced. Although no formal definition is given, it turns out that in practice such a cat is in fact an AECat with AP. See also Example 3.33.

Example 3.10. In this example we consider Zilber's quasiminimal excellent classes. We use the terminology from [Kir10]. Let $\mathcal{C}$ be a quasiminimal excellent class, also satisfying axiom IV, which states that $\mathcal{C}$ is closed under unions of chains and has an infinite dimensional model. Then $\mathcal{C}$ together with strong embeddings is a finitely accessible category, where $M \in \mathcal{C}$ is κ-presentable precisely when M has dimension $< \kappa$. So $(\mathcal{C}, \mathcal{C})$ is an AECat with AP.

We have now covered how existing frameworks can be placed in the framework of AECats. In section 3.6 we will do the converse. There we discuss how, under some additional assumptions, AECats can be placed in existing frameworks.

3.2 Galois types

In [She87, Definition II.1.9] types are considered as the orbit of a tuple under some automorphism group. Later this idea was generalised by replacing the automorphisms by embeddings into a bigger model, and the name Galois type was introduced (see [Gro02]). We use this idea, replacing elements by arrows.

Definition 3.11. Let M be a model in an AECat. An *extension* of M is an arrow $M \to N$, where N is some model.

Convention 3.12. Usually, there will be only one relevant extension of models. So to prevent cluttering of notation we will not give such an extension a name. Given such an extension $M \to N$ and some arrow $a : A \to M$ we will then denote the arrow $A \xrightarrow{a} M \to N$ by a as well.

Definition 3.13. Let $(\mathcal{C}, \mathcal{M})$ be an AECat with AP. We will use the notation $((a_i)_{i \in I}; M)$ to mean that the a_i are arrows into M and that M is a model. We will denote the domain of a_i by A_i, unless specified otherwise.

We say that two tuples $((a_i)_{i \in I}; M)$ and $((a'_i)_{i \in I}; M')$ have the same *Galois type*, and write

$$\mathrm{gtp}((a_i)_{i \in I}; M) = \mathrm{gtp}((a'_i)_{i \in I}; M'),$$

if there is a common extension $M \to N \leftarrow M'$ such that, for all $i \in I$, $\mathrm{dom}(a_i) = \mathrm{dom}(a'_i)$ and a_i and a'_i give the same arrow into N. That is, the following commutes for all $i \in I$:

$$\begin{tikzcd}
& N & \\
M \arrow[ur] & & M' \arrow[ul] \\
& A_i \arrow[ul, "a_i"] \arrow[ur, "a'_i"'] &
\end{tikzcd}$$

Note that AP ensures that having the same Galois type is an equivalence relation. For this reason, we will only be interested in AECats with AP in the rest of this thesis.

We consider tuples of arrows in Galois types, rather than just a single arrow, even though an arrow itself can already represent a tuple of elements. When reading through the proofs and constructions it should be clear why we do this, but we will give a quick summary of the reasons here. First of all, our AECat may not have unions. For example, if we work in an AECat of the form $(\mathbf{Mod}(T), \mathbf{Mod}(T))$ and we have models M and M' (in some bigger model N) then there is generally not an arrow (into N) representing the union MM'. This would be problematic in dealing with parameters, because saying that M and M' have the same type over some third model M_0 amounts to saying that MM_0 and $M'M_0$ have the same type, see also Example 3.14. Finally, we will often want to consider types of sequences, and a sequence is just a tuple of arrows (see Section 4.3), just as a sequence classically is just a tuple of tuples.

Example 3.14. Let $(\mathcal{C}, \mathcal{M})$ be $(\mathbf{SubMod}(T), \mathbf{Mod}(T))$ for some theory T. Then Galois types in $(\mathcal{C}, \mathcal{M})$ coincide with syntactic types in the following sense. Let M and M' be two models of T, we will abbreviate the object (M, M) in $\mathbf{SubMod}(T)$ to M and similarly for M'. Suppose we are given some arrows $\{a_i : (A_i, M_i) \to M\}_{i \in I}$ and $\{a'_i : (A_i, M_i) \to M'\}_{i \in I}$ for some index set I. For each $i \in I$ fix some enumeration of A_i. The image of that enumeration under $a_i : A_i \to M$ then yields a tuple $\bar{a}_i \in M$. Similarly we find a tuple $\bar{a}'_i \in M'$. We write $\mathrm{tp}^M((\bar{a}_i)_{i \in I})$ for the type of $(\bar{a}_i)_{i \in I}$ in M, i.e. the set of all formulas satisfied by $(\bar{a}_i)_{i \in I}$ in M. Then:

$$\mathrm{tp}^M((\bar{a}_i)_{i \in I}) = \mathrm{tp}^{M'}((\bar{a}'_i)_{i \in I}) \quad \iff \quad \mathrm{gtp}((a_i)_{i \in I}; M) = \mathrm{gtp}((a'_i)_{i \in I}; M').$$

A similar statement holds for $(\mathbf{SubMetMod}(T), \mathbf{MetMod}(T))$, where T is a continuous theory.

We can also deal with parameters. Let M be a model of T. Fix some parameter set $B \subseteq M$ and tuples $\bar{a}, \bar{a}' \in M$. We also fix some enumeration $\bar{b}$ of B. Then:

$$\mathrm{tp}^M(\bar{a}/B) = \mathrm{tp}^M(\bar{a}'/B) \quad \iff \quad \mathrm{tp}^M(\bar{a}\bar{b}) = \mathrm{tp}^M(\bar{a}'\bar{b}).$$

Let a and a' be arrows into M representing $\bar{a}$ and $\bar{a}'$ and we let $b : B \to M$ be the inclusion then the above is further equivalent to $\mathrm{gtp}(a, b; M) = \mathrm{gtp}(a', b; M)$. So fixing a parameter set in a model corresponds to fixing an arrow in the categorical setting.

Lemma 3.15. *Let $M \to N$ be any extension, then for any tuple $((a_i)_{i \in I}; M)$:*

$$\mathrm{gtp}((a_i)_{i \in I}; M) = \mathrm{gtp}((a_i)_{i \in I}; N).$$

Proof. This is a good example of Convention 3.12. A more precise statement would be to give the extension $M \to N$ a name, say f, then for any $((a_i)_{i \in I}; M)$ we have that $\mathrm{gtp}((a_i)_{i \in I}; M) = \mathrm{gtp}((f a_i)_{i \in I}; N)$. To see that the latter statement holds, note that the diagram below commutes for all $i \in I$:

$$\begin{tikzcd}
& N \arrow[dr, equal] & \\
M \arrow[ur, "f"] & & N \\
& A_i \arrow[ul, "a_i"] \arrow[ur, "f a_i"'] &
\end{tikzcd}$$

□

Proposition 3.16. *If $\mathrm{gtp}((a_i)_{i \in I}; M) = \mathrm{gtp}((a'_i)_{i \in I}; M')$ then:*

(i) (restriction) we have $\mathrm{gtp}((a_i)_{i \in I_0}; M) = \mathrm{gtp}((a'_i)_{i \in I_0}; M')$ for any $I_0 \subseteq I$;

(ii) (monotonicity) given an arrow $b_i : B_i \to \mathrm{dom}(a_i)$ *for each* $i \in I$*, then*

$$\mathrm{gtp}((a_i)_{i\in I}, (a_i b_i)_{i\in I}; M) = \mathrm{gtp}((a'_i)_{i\in I}, (a'_i b_i)_{i\in I}; M')$$

and thus $\mathrm{gtp}((a_i b_i)_{i\in I}; M) = \mathrm{gtp}((a'_i b_i)_{i\in I}; M')$*;*

(iii) (extension) for any $(b; M)$ *there is an extension* $M' \to N$ *and some* $(b'; N)$ *such that* $\mathrm{gtp}(b, (a_i)_{i\in I}; M) = \mathrm{gtp}(b', (a'_i)_{i\in I}; N)$*.*

Proof. For (i) and (ii) the common extension witnessing the original equality will also witness the new equality. The last claim in (ii) follows from (i).

For (iii) let $M \xrightarrow{f} N \xleftarrow{g} M'$ be witnesses of $\mathrm{gtp}((a_i)_{i\in I}; M) = \mathrm{gtp}((a'_i)_{i\in I}; M')$. We define $b' = fb$, so that:

$$\mathrm{gtp}(b, (a_i)_{i\in I}; M) = \mathrm{gtp}(fb, (fa_i)_{i\in I}; N) = \mathrm{gtp}(b', (ga'_i)_{i\in I}; N).$$

Then the result follows directly if we take the extension $M' \to N$ to be g, so that we would write the right-hand side as $\mathrm{gtp}(b', (a'_i)_{i\in I}; N)$. $\square$

Proposition 3.17. *Suppose we have* $(a, b; M)$*, such that* $a = bi$ *for some arrow* i*. If then* $(a', b'; M')$ *is such that*

$$\mathrm{gtp}(a, b; M) = \mathrm{gtp}(a', b'; M'),$$

then a' *factors through* b' *in the same way:* $a' = b'i$*.*

Proof. From $\mathrm{gtp}(a, b; M) = \mathrm{gtp}(a', b'; M')$ we get extensions $M \to N \leftarrow M'$ and a diagram

$$\begin{tikzcd}
& N & \\
M \arrow[ur, "f"] & B \arrow[l, "b"] \arrow[r, "b'"'] & M' \arrow[ul, "g"'] \\
& A \arrow[u, "i"] \arrow[ul, bend left, "a"] \arrow[ur, bend right, "a'"'] &
\end{tikzcd}$$

where everything commutes by definition except for possibly the bottom right triangle (i.e. the triangle $a' = b'i$). So we have $ga' = fa = fbi = gb'i$ and so $a' = b'i$ because g is a monomorphism. $\square$

3.3 Galois type sets, an analogue of Stone spaces

In full first-order logic we can collect all types of a fixed arity in a topological space called the Stone space. The topology is given by the logical structure. In this section we do something similar for AECats: we show that all Galois types with some fixed domains can be collected in a set. Putting a sensible topology on this set would amount to recovering the logical structure of the AECat, which can be hard in full generality. While this would be an interesting problem to study, it is outside the scope of this thesis and so we will not go into it. The main use of this section is that the Galois types with some fixed domains actually form a set, see Proposition 3.19.

Definition 3.18. Let $(\mathcal{C}, \mathcal{M})$ be an AECat with AP. For a tuple $(A_i)_{i \in I}$ of objects in $\mathcal{C}$, let $S((A_i)_{i \in I})$ be the collection of all tuples $((a_i)_{i \in I}; M)$ such that $\operatorname{dom}(a_i) = A_i$. We define the *Galois type set* $\mathrm{S}_{\mathrm{gtp}}((A_i)_{i \in I})$ as:

$$\mathrm{S}_{\mathrm{gtp}}((A_i)_{i \in I}) = S((A_i)_{i \in I}) / \sim_{\mathrm{gtp}},$$

where $\sim_{\mathrm{gtp}}$ is the equivalence relation of having the same Galois type.

An AECat is generally a large category. So $S((A_i)_{i \in I})$ will generally be a proper class. Below we prove that $\mathrm{S}_{\mathrm{gtp}}((A_i)_{i \in I})$ is small (i.e. a set), so the name is justified.

The above notation clashes with standard notation. That is, classically one would expect the notation $S_n(T)$ for the n-types of a theory T. Then, fixing some parameter set A, we would write $S_n(A)$ for the set of n-types with parameters in A. However, for our Galois types the difference between domain and parameters fades, see Example 3.14. So the only relevant information about a Galois type set is the domains that we fix, that is the $(A_i)_{i \in I}$.

Definition 3.18 allows us to talk about $\mathrm{gtp}((a_i)_{i \in I}; M)$ as an object in itself: it is one of the equivalence classes in $\mathrm{S}_{\mathrm{gtp}}((A_i)_{i \in I})$.

Proposition 3.19. *Let $(\mathcal{C}, \mathcal{M})$ be an AECat with AP. Then for any tuple $(A_i)_{i \in I}$ of objects $\mathrm{S}_{\mathrm{gtp}}((A_i)_{i \in I})$ is a set.*

Proof. We prove that there is a subset $S'((A_i)_{i \in I}) \subseteq S((A_i)_{i \in I})$, such that for every tuple $((a_i)_{i \in I}; M) \in S((A_i)_{i \in I})$, there is some $((a'_i)_{i \in I}; M') \in S'((A_i)_{i \in I})$ with $\mathrm{gtp}((a_i)_{i \in I}; M) = \mathrm{gtp}((a'_i)_{i \in I}; M')$.

Let λ be such that every A_i is λ-presentable, $\lambda > |I|$ and the inclusion functor $\mathcal{M} \hookrightarrow \mathcal{C}$ is λ-accessible and preserves λ-presentable objects. Such a λ must exist since each object in an accessible category is presentable by [AR94, Proposition 1.16], and by Remark 3.5.

Let $\mathcal{M}_\lambda$ be (a skeleton of) all the models that are λ-presentable. Then $\mathcal{M}_\lambda$ is a set (see the remark after [AR94, Definition 1.9]). We define:

$$S'((A_i)_{i \in I}) = \coprod_{M \in \mathcal{M}_\lambda} \prod_{i \in I} \mathrm{Hom}(A_i, M)$$

We check that $S'((A_i)_{i \in I})$ has the required property. Let $((a_i)_{i \in I}; M) \in S((A_i)_{i \in I})$. Then because $\mathcal{M}$ is λ-accessible, M is a λ-directed colimit of λ-presentable objects $(M_j)_{j \in J}$. That is, objects in $\mathcal{M}_\lambda$. Since the inclusion functor $\mathcal{M} \hookrightarrow \mathcal{C}$ preserves directed colimits, we still have $M = \mathrm{colim}_{j \in J} M_j$ in $\mathcal{C}$. As A_i is λ-presentable for each $i \in I$, we have that each a_i factors through some M_{j_i}. Then since $\lambda > |I|$, there is $j \in J$ such that every a_i factors through M_j. Write this factorisation as $A_i \xrightarrow{a'_i} M_j \xrightarrow{m_j} M$, where m_j is the coprojection from the colimit. Then by construction $((a'_i)_{i \in I}; M_j) \in S'((A_i)_{i \in I})$ and $\mathrm{gtp}((a_i)_{i \in I}; M) = \mathrm{gtp}((a'_i)_{i \in I}; M_j)$. $\square$

3.4 Lascar strong Galois types

In this section we will give a definition of Lascar strong Galois type. This will coincide with the usual notion of Lascar strong type in $(\mathbf{Mod}(T), \mathbf{Mod}(T))$ or $(\mathbf{SubMod}(T), \mathbf{Mod}(T))$ when T is a theory in full first-order logic. Even for reasonable positive theories T this will coincide, see Remark 3.22 and Definition 2.40. This notion will be useful later in the property Independence Theorem for independence relations, see Definition 4.8.

To place the following definition in context, we quickly recall a possible definition for Lascar strong types in full first-order logic. Tuples a and a' have the same Lascar strong type over B if there are $a = a_0, \ldots, a_n = a'$ and models $M_1, \ldots, M_n$, each containing B, such that $\mathrm{tp}(a_i/M_{i+1}) = \mathrm{tp}(a_{i+1}/M_{i+1})$ for all $0 \leq i < n$.

Definition 3.20. Let $(\mathcal{C}, \mathcal{M})$ be an AECat with AP and fix some $((b_j)_{j \in J}; M)$. We write $((a_i)_{i \in I}/(b_j)_{j \in J}; M) \sim_{\mathrm{Lgtp}} ((a'_i)_{i \in I}/(b_j)_{j \in J}; M)$ if there is some extension $M \to N$ and some $m_0 : M_0 \to N$, where M_0 is a model, such that b_j factors through m_0 for all $j \in J$ and $\mathrm{gtp}((a_i)_{i \in I}, m_0; N) = \mathrm{gtp}((a'_i)_{i \in I}, m_0; N)$.

We write

$$\mathrm{Lgtp}((a_i)_{i \in I}/(b_j)_{j \in J}; M) = \mathrm{Lgtp}((a'_i)_{i \in I}/(b_j)_{j \in J}; M)$$

for the transitive closure of $\sim_{\mathrm{Lgtp}}$ and we say that $((a_i)_{i \in I}; M)$ and $((a'_i)_{i \in I}; M)$ have the same *Lascar strong Galois type* over $(b_j)_{j \in J}$.

Remark 3.21. Let T be a semi-Hausdorff theory (recall this includes theories in full full first-order logic, see Remark 2.12). Let $\mathcal{C}$ be either $\mathbf{SubMod}(T)$ or $\mathbf{Mod}(T)$ and let $\mathcal{M}$ be $\mathbf{Mod}(T)$. Then having the same Lascar strong Galois type in $(\mathcal{C}, \mathcal{M})$ coincides with have the same Lascar strong type in the usual sense, see Fact 2.32. A similar statement is true for continuous theories.

If T is a thick theory we need to slightly adjust the category of models. This is because in a thick theory, having the same type over an e.c. model is generally not enough to guarantee having the same Lascar strong type. See Example 2.33. So following Proposition 2.39 we can take $\mathcal{M}$ to be the category of finitely λ_T-saturated models. Then Lascar strong Galois types once again coincide with

Lascar strong types. Here the notion of being *finitely* λ_T-saturated, rather than being λ_T-saturated, is important to guarantee that $\mathcal{M}$ has directed colimits.

Remark 3.22. In the usual definition for Lascar strong types we have two more equivalent conditions, see Definition 2.40.

One condition states that a and b have the same Lascar strong type if they can be connected by indiscernible sequences. Outside the finitely short setting indiscernible sequences may not be the right tool to work with. They are usually so nice because we can make them arbitrarily long and they are highly homogeneous, but both of these properties require at least finite shortness (see section 4.5).

The other condition states that a and b have the same Lascar strong type if they are equivalent under every bounded invariant equivalence relation. This can be made sense of in general AECats, see Definition 5.12. However, it is hard to prove any properties for this, like Proposition 3.24 or Proposition 3.25.

So the main missing link is a good replacement of homogeneous enough sequences that can be extended to arbitrary lengths. In section 5.3 we show that a nice enough independence relation provides this link. In particular we get the familiar equivalent characterisations of Lascar strong types for Lascar strong Galois types, replacing indiscernible sequences by something we call strongly 2-indiscernible sequences.

The following proposition shows that having the same Lascar strong Galois type is preserved under having the same Galois type. For ease of notation we prove this for single arrows, but the proof goes through word for word if we replace those by tuples of arrows.

Proposition 3.23. *Suppose that* $\mathrm{gtp}(a_1, a_2, b; M) = \mathrm{gtp}(a_1', a_2', b'; M')$*. Then we have* $\mathrm{Lgtp}(a_1/b; M) = \mathrm{Lgtp}(a_2/b; M)$ *if and only if* $\mathrm{Lgtp}(a_1'/b'; M') = \mathrm{Lgtp}(a_2'/b'; M')$*.*

Proof. It suffices to prove that $(a_1/b; M) \sim_{\mathrm{Lgtp}} (a_2/b; M)$ implies $(a_1'/b'; M') \sim_{\mathrm{Lgtp}} (a_2'/b'; M')$. Let $M \to N$ with $M_0 \leq N$ witness $(a_1/b; M) \sim_{\mathrm{Lgtp}} (a_2/b; M)$. So we have that b factors through M_0, that is $b = m_0 b^*$ for some representative m_0 of M_0 and $b^* : B \to M_0$, and $\mathrm{gtp}(a_1, m_0; N) = \mathrm{gtp}(a_2, m_0; N)$. Let $N \to N' \leftarrow M'$ witness $\mathrm{gtp}(a_1, a_2, b; N) = \mathrm{gtp}(a_1, a_2, b; M) = \mathrm{gtp}(a_1', a_2', b'; M')$. Then we get the following commuting diagram:

$$\begin{tikzcd}
 & N' & \\
N \arrow[ur] & & M' \arrow[ul] \\
M \arrow[u] & M_0 \arrow[ul] & \\
A_1 \arrow[u] \arrow[uurr, bend right] & B \arrow[u] \arrow[ul] \arrow[uur, bend right] & A_2 \arrow[uu] \arrow[ull]
\end{tikzcd}$$

So we have $M_0 \leq N'$ and b' factors though M_0. This follows from the fact that the above diagram commutes, so $b' : B \to M' \to N'$ and $B \to M_0 \to N \to N'$ are the same arrow. Furthermore, we have that $\text{gtp}(a_1, m_0; N') = \text{gtp}(a_2, m_0; N')$ and because a_1 and a_2 are the same arrows into N' as a'_1 and a'_2 respectively, we get $\text{gtp}(a'_1, m_0; N') = \text{gtp}(a'_2, m_0; N')$. So we conclude $(a'_1/b'; M') \sim_{\text{Lgtp}} (a'_2/b'; M')$, as required. $\square$

Having the same Lascar strong Galois type has a bounded number of equivalence classes. We will again give a proof for single arrows, which again also works for tuples of arrows.

Proposition 3.24. *Given objects A and B there is λ such that for any $b : B \to M$ the relation of having the same Lascar strong Galois type over b partitions $\text{Hom}(A, M)$ into at most λ many equivalence classes.*

Proof. We will first prove the following claim: for any $b : B \to M$ there is λ_b such that for any $b' : B \to M'$ with $\text{gtp}(b'; M') = \text{gtp}(b; M)$ there are at most λ_b many equivalence classes of Lascar strong Galois types over b' in $\text{Hom}(A, M')$. By Proposition 3.19 the collection $\text{S}_{\text{gtp}}(A, M)$ is a set. We pick $\lambda_b = |\,\text{S}_{\text{gtp}}(A, M)|$. Now let $M \to N \leftarrow M'$ witness $\text{gtp}(b'; M') = \text{gtp}(b; M)$. For any two arrows $a, a' : A \to M'$ we have that $\text{gtp}(a, m; N) = \text{gtp}(a', m; N)$ implies that $\text{Lgtp}(a/b'; M') = \text{Lgtp}(a'/b'; M')$, by Proposition 3.23. The claim then follows by choice of λ_b.

By the claim we can take λ to be the supremum of λ_b, where b ranges over the representatives of the Galois types in $\text{S}_{\text{gtp}}(B)$. $\square$

Proposition 3.25. *If* $\text{Lgtp}((a_i)_{i\in I}/b; M) = \text{Lgtp}((a'_i)_{i\in I}/b; M)$ *then:*

(i) (restriction) we have $\text{Lgtp}((a_i)_{i\in I_0}/b; M) = \text{Lgtp}((a'_i)_{i\in I_0}/b; M)$ *for any* $I_0 \subseteq I$*;*

(ii) (monotonicity) given an arrow $c_i : C_i \to \text{dom}(a_i)$ for each $i \in I$, then

$$\text{Lgtp}((a_i)_{i\in I}, (a_i c_i)_{i\in I}/b; M) = \text{Lgtp}((a'_i)_{i\in I}, (a'_i c_i)_{i\in I}/b; M)$$

and thus $\mathrm{Lgtp}((a_i c_i)_{i \in I}/b; M) = \mathrm{Lgtp}((a'_i c_i)_{i \in I}/b; M)$*;*

(iii) (extension) for any $(c; M)$ *there is an extension* $M \to N$ *and some* $(c'; N)$ *such that* $\mathrm{Lgtp}(c, (a_i)_{i \in I}/b; N) = \mathrm{Lgtp}(c', (a'_i)_{i \in I}/b; N)$*.*

Proof. This is essentially the same Proposition 3.16, but then for Lascar strong Galois types. To prove it, apply the definition of Lascar strong Galois types to reduce to some equality of Galois types and then apply Proposition 3.16. □

3.5 Subobjects

Later we will define independence relations as a relation on subobjects. In this section we recall the definition and some basic properties about subobjects. We also briefly explain how they interact with Galois types.

The intuition is that subobjects correspond to just subsets, while arrows correspond to indexed tuples. For example, if we consider the inclusion $i : \mathbb{Z} \to \mathbb{R}$ then this enumerates the subset $\mathbb{Z} \subseteq \mathbb{R}$. The arrow $f : \mathbb{Z} \to \mathbb{R}, x \mapsto -x$ enumerates the same subset but in a different way. So from the point of view of subsets i and f are 'equivalent', the idea is then that a subobject is an equivalence class of 'equivalent' arrows.

Definition 3.26. Let $\mathcal{C}$ be an arbitrary category and fix some object X. Let $f : A \to X$ and $g : B \to X$ be two monomorphisms. We write $f \leq g$ if f factors through g. That is, there is some $h : A \to B$ such that $f = gh$.

$$\begin{tikzcd} & X & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] & & B \arrow[ul, "g"'] \end{tikzcd}$$

It is easy to check that $\leq$ defines a pre-order on the collection of monomorphisms into X. So we can define an equivalence relation $\sim$ on this collection by saying that $f \sim g$ if and only if $f \leq g$ and $g \leq f$. An equivalence class of $\sim$ is called a *subobject of* X. We write $\mathrm{Sub}(X)$ for the collection of subobjects of X, and the pre-order on the monomorphisms induces a partial order $\leq$ on $\mathrm{Sub}(X)$.

Fact 3.27. *Let everything be as in Definition* 3.26*.*

(i) If $f \leq g$ *then the comparison arrow* h *is a monomorphism.*

(ii) If $f \sim g$ *then the comparison arrow* h *is an isomorphism.*

Proof. For (i): let $p, q : Y \to A$ be arrows such that $hp = hq$. Then $fp = ghp = ghq = fq$. So $p = q$ because f is a monomorphism. For (ii): let $h' : B \to A$ be such that $fh' = g$. We claim that h' is the inverse of h. We have that $fId_A = f = gh = fh'h$, so $h'h = Id_A$ because f is a monomorphism. Similarly we find $hh' = Id_B$. □

Convention 3.28. We make the following notational conventions about subobjects in AECats.

(i) Any arrow in an AECat is a monomorphism, and is thus a representative of some subobject. Whenever we have an unnamed arrow $A \to B$ we also write A for the subobject represented by it.

(ii) Due to Fact 3.27(i) we have that whenever $A, B \in \mathrm{Sub}(X)$ and $A \leq B$ then we may also consider A as a subobject of B, that is $A \in \mathrm{Sub}(B)$. So we use the notation $A \leq X$ to also mean $A \in \mathrm{Sub}(X)$.

(iii) In line with Convention 3.12, if we have a subobject $A \leq M$ and an extension $M \to N$ then we will consider A as a subobject of N as well (by composing with the extension).

(iv) Due to Fact 3.27(ii) we have that whenever f and g represent the same subobject A then their domains are isomorphic. So it makes sense to talk about the presentability of a subobject. More precisely, when we say that a subobject A is λ-presentable then we mean that the domain of some (equivalently: every) representative is λ-presentable.

Fact 3.29 ([AR94, Chapter 1])**.** *In any accessible category the collection* $\mathrm{Sub}(X)$ *is small, meaning that it is a set. In particular this is true in any AECat.*

It would be natural to ask whether we can make sense of Galois types for subobjects. This is in fact possible and was done in [Kam20], see the example below. However, we have to be careful with such a definition, as is illustrated in that example. The definition turned out to be a source of confusion and it is easy to work around it by explicitly picking representatives of subobjects whenever necessary. So ultimately we opted out of such a definition for this thesis and only work with Galois types of arrows.

Example 3.30. In classical model theory we would say that two sets A and A' have the same type if there is an enumeration a of A and a' of A' such that a and a' have the same type. So given a tuple of subobjects $(A_i)_{i \in I}$ of some model M and a tuple $(A'_i)_{i \in I}$ of some model M', we could define $\mathrm{gtp}((A_i)_{i \in I}; M) = \mathrm{gtp}((A'_i)_{i \in I}; M')$ to mean that there are representatives $(a_i)_{i \in I}$ of $(A_i)_{i \in I}$ and $(a'_i)_{i \in I}$ of $(A'_i)_{i \in I}$ such that $\mathrm{gtp}((a_i)_{i \in I}; M) = \mathrm{gtp}((a'_i)_{i \in I}; M')$. This turns out to be equivalent to having extensions $M \to N \leftarrow M'$ such that $A_i = A'_i$ as subobjects of N for all $i \in I$. The latter formulation is in fact [Kam20, Definition 3.5], and the equivalence between the two formulations is [Kam20, Proposition 3.6].

We now illustrate why we have to be careful in applying the above definition. Consider the category of infinite sets with injective functions. This is easily seen to be an AECat with AP if we take $\mathcal{M}$ to be the entire category. Alternatively, this is precisely $\mathbf{Mod}(T_{\text{inf}})$, where T_{inf} is the theory of infinite sets, and is thus an AECat with AP as discussed in Example 3.3.

Let $f : \mathbb{N} \to \mathbb{N}$ be the bijection that swaps the odd and even numbers. So $f(0) = 1$, $f(1) = 0$, $f(2) = 3$, and so on. Denote by $2\mathbb{N}$ the set of even numbers and let $e : 2\mathbb{N} \to \mathbb{N}$ be the inclusion. So we have the following commuting diagram:

$$\begin{tikzcd}
& \mathbb{N} & \\
\mathbb{N} \arrow[ur, bend left, "Id_{\mathbb{N}}"] \arrow[ur, bend right, "f"'] \arrow[loop left, "f"] & & 2\mathbb{N} \arrow[ul, "e"']
\end{tikzcd}$$

We denote by $[Id_{\mathbb{N}}]$ the subobject represented by $Id_{\mathbb{N}}$, and likewise for f and e. Then $[Id_{\mathbb{N}}] = [f]$, so we definitely have

$$\text{gtp}([Id_{\mathbb{N}}], [e]; \mathbb{N}) = \text{gtp}([f], [e]; \mathbb{N}).$$

However, we cannot have

$$\text{gtp}(Id_{\mathbb{N}}, e; \mathbb{N}) = \text{gtp}(f, e; \mathbb{N}).$$

If this equality of Galois types were to hold then there would be injective functions $g, h : \mathbb{N} \to X$, where X is some infinite set, such that $g = hf$ and $ge = he$. However, this would mean that $g(2) = ge(2) = he(2) = h(2) = hf(3) = g(3)$, contradicting injectivity of g.

Intuitively what happens is that e plays the role of fixing a parameter set, because we use the same arrow on both sides of the equality of Galois types. The arrows $Id_{\mathbb{N}}$ and f both play the role of enumerating $\mathbb{N}$, but do this in different ways. What we have shown is that these different enumerations yield different types over the set of even numbers $2\mathbb{N}$. In classical model-theoretic notation we would write $\text{tp}(0, 1, 2, 3, \ldots/2\mathbb{N}) \neq \text{tp}(1, 0, 3, 2, \ldots/2\mathbb{N})$.

So subobjects are like sets without a specific enumeration, while picking a representative of such a subobject is like picking an enumeration of the set. This example shows that we cannot just pick any representatives of subobjects when considering Galois types, in the same way that we have to be careful when picking enumerations of sets in syntactic types.

3.6 Finitely short AECats

In this section we discuss an important property that connects AECats with existing frameworks and allows us to recover a bit of compactness, see Proposition 3.38. Nothing in this section will be used in the rest of this thesis, except in section 4.5, where a further connection to indiscernible sequences is explored. However, that section too is used nowhere else in this thesis.

This property is to have some locality for Galois types (inspired by [GV06]): the Galois type of an infinite tuple should be determined by the Galois types of all of its finite subtuples.

Definition 3.31. We say that an AECat is *finitely short* if for any two (infinite) tuples $((a_i)_{i \in I}; M)$ and $((a'_i)_{i \in I}; M')$ such that for all finite $I_0 \subseteq I$

$$\text{gtp}((a_i)_{i \in I_0}; M) = \text{gtp}((a'_i)_{i \in I_0}; M'),$$

we have that

$$\text{gtp}((a_i)_{i \in I}; M) = \text{gtp}((a'_i)_{i \in I}; M').$$

Example 3.32. The AECats $(\mathbf{Mod}(T), \mathbf{Mod}(T))$ and $(\mathbf{SubMod}(T), \mathbf{Mod}(T))$ from Example 3.3 are both finitely short (recall that this includes the full first-order case), because Galois types coincide with the usual syntactic types.

For the same reasons, for a continuous theory T, $(\mathbf{MetMod}(T), \mathbf{MetMod}(T))$ and $(\mathbf{SubMetMod}(T), \mathbf{MetMod}(T))$ from Example 3.7 are finitely short.

An AEC $\mathcal{K}$ with AP that is fully $< \aleph_0$-type short over the empty set yields AECats $(\mathcal{K}, \mathcal{K})$ and $(\mathbf{SubSet}(\mathcal{K}), \mathcal{K})$, as per Example 3.8, which are both finitely short.

Example 3.33. In Example 3.9 we mentioned cats from [BY03a]. One definition there allows for a nice comparison to AECats, namely that of an *elementary category (with amalgamation)* [BY03a, Definition 2.27]. This is a concrete category $\mathcal{C}$ that satisfies a few additional assumptions, similar to the axioms of an AEC. Every such elementary category $\mathcal{C}$ will form an AECat with AP as $(\mathcal{C}, \mathcal{C})$, if we additionally assume $\mathcal{C}$ to be accessible[1].

Conversely, given an AECat $(\mathcal{C}, \mathcal{M})$ we can make it into a concrete category using a version of the Yoneda embedding. Let λ be such that $\mathcal{C}$ is λ-accessible and let $\mathcal{A}$ be the full subcategory of λ-presentable objects in $\mathcal{C}$. Then there is a fully faithful canonical functor $E : \mathcal{C} \to \mathbf{Set}^{\mathcal{A}^{\text{op}}}$ that preserves λ-directed colimits, see [AR94, 1.25 and 2.8]. If $(\mathcal{C}, \mathcal{M})$ has AP then taking the image of $\mathcal{M}$ under E, we obtain an elementary category with amalgamation.

In [BY03a, Definition 2.32] a few properties are defined for the Galois types:

- *type boundedness*: this is always true in an AECat, see Proposition 3.19;
- *type locality*: this is precisely what we called being finitely short;

[1] Technically, [BY03a, Definition 2.27] does not require the existence of directed colimits but something slightly weaker called the "elementary chain property". However, it is likely that actually directed colimits are meant and in practice this is what we have.

- *weak compactness*: this is what we call "compactness for Galois types", see Definition 3.36, which holds for example in categories obtained from a full first-order, positive or continuous theory.

Note that Example 3.33 does generally not yield an AEC. For example, take $\mathcal{C} = \mathcal{M}$ to be the category of infinite sets with injective functions. If we make this category into a concrete category through the functor $E : \mathcal{C} \to \mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$ then $E(\omega + \omega)$ contains the arrow $f : \omega \to \omega + \omega$ where $f(n) = \omega + n$. If this would be an AEC then $E(\omega + \omega) = \bigcup_{n<\omega} E(\omega + n)$, but f is not in $E(\omega + n)$ for any n. So the Tarski-Vaught chain axiom for AECs fails. The point is of course that a directed colimit can be more than just the union of underlying sets.

In [BR12, Corollary 5.7] a characterisation is given of those accessible categories that are equivalent to an AEC. It also describes how to construct an AEC from such an accessible category, through a construction very similar to Example 3.33.

Example 3.34. Let $(\mathcal{C}, \mathcal{M})$ be a finitely short AECat with AP. Suppose furthermore that $\mathcal{M}$ is connected. That is, it has the joint embedding property: for any two models M_1 and M_2 there is a third model N with arrows $M_1 \to N \leftarrow M_2$. Then there is a strong connection with homogeneous model theory [BL03]. We sketch the construction and would like to thank an anonymous referee of [Kam20] for pointing this out.

As discussed in Example 3.33, we can turn $\mathcal{M}$ into a concrete category. So using the usual tools we can build a monster model $\mathfrak{M}$, which we will fit in the framework of [BL03]. The elements in $\mathfrak{M}$ are arrows in $\mathcal{C}$, and having the same Galois type corresponds to having the same orbital type in $\mathfrak{M}$. For every Galois type of a finite tuple we add a relation symbol, and we close these under finite conjunctions and disjunctions. Then two (infinite) tuples of elements are in the same orbit of $\mathfrak{M}$ iff they have the same Galois type iff their finite subtuples have the same Galois type iff the finite subtuples satisfy the same relation symbols.

The constructions in Example 3.33 and Example 3.34 do not change our category, they only add data to make it into a concrete category. So any notion that is defined on just the objects and arrows in our category is preserved by this operation. In particular independence relations, as we define in chapter 4, are preserved. It would be interesting to study these connections further, but that is beyond the scope of this thesis.

Quasiminimal excellent classes, as discussed in Example 3.10, are generally not finitely short. The motivating example, Zilber's exponential field, is not finitely short. We treat an easier example.

Example 3.35. We consider the quasiminimal excellent class in [Kir10, Example 1.2(4)]. Models are are sets with an equivalence relation, where each equivalence class has cardinality $\aleph_0$. For a subset A of some model M we write $\mathrm{cl}_M(A)$ for the closure of A, which is the union of all equivalence classes in M that A intersects. We define $\mathcal{C}$ to be the category of subsets of models as follows. Objects are pairs (A, M), where $A \subseteq M$ and M is a model. An arrow $f : (A, M) \to (B, N)$ is then an injection $f : A \to B$ such that:

(i) it preserves and reflects the equivalence relation: $a \sim a'$ iff $f(a) \sim f(a')$;

(ii) if $\mathrm{cl}_M(a) \subseteq A$ then $f(\mathrm{cl}_M(a)) = \mathrm{cl}_N(f(a))$.

We let $\mathcal{M}$ be the full subcategory of (objects isomorphic to) objects of the form (M, M). One easily checks that $\mathcal{C}$ is ω_1-accessible and that (A, M) is λ-presentable precisely when $|A| < \lambda$, for $\lambda \geq \omega_1$. It is then straightforward to verify that $(\mathcal{C}, \mathcal{M})$ is an AECat with AP.

To see that finite shortness fails, we consider $\mathbb{N}$ as a model with just one equivalence class. For each $n \in \mathbb{N}$ we consider arrows $a_n, b_n : (\{n\}, \mathbb{N}) \to (\mathbb{N}, \mathbb{N})$ given by $a_n(n) = n$ and $b_n(n) = 2n$. For any finite $n_1, \ldots, n_k \in \mathbb{N}$ we let $h : \mathbb{N} \to \mathbb{N}$ be a bijection such that $h(2n_i) = n_i$ for all $1 \leq i \leq k$. Since h is a bijection, it is in fact an arrow $h : (\mathbb{N}, \mathbb{N}) \to (\mathbb{N}, \mathbb{N})$. So the following commutes:

$$\begin{tikzcd}
& (\mathbb{N}, \mathbb{N}) & \\
(\mathbb{N}, \mathbb{N}) \arrow[ur, equal] & & (\mathbb{N}, \mathbb{N}) \arrow[ul, "h"'] \\
(\{n_1\}, \mathbb{N}) \arrow[u, "a_{n_1}"] \arrow[urr, bend left, "b_{n_1}"] & \ldots & (\{n_k\}, \mathbb{N}) \arrow[u, "b_{n_k}"'] \arrow[ull, bend right, "a_{n_k}"']
\end{tikzcd}$$

We thus have $\mathrm{gtp}(a_{n_1}, \ldots, a_{n_k}; (\mathbb{N}, \mathbb{N})) = \mathrm{gtp}(b_{n_1}, \ldots, b_{n_k}; (\mathbb{N}, \mathbb{N}))$. If we would have finite shortness, then this would imply $\mathrm{gtp}((a_n)_{n \in \mathbb{N}}; (\mathbb{N}, \mathbb{N})) = \mathrm{gtp}((b_n)_{n \in \mathbb{N}}; (\mathbb{N}, \mathbb{N}))$. However, this cannot happen, because then we would find $f, g : (\mathbb{N}, \mathbb{N}) \to (\mathbb{N}, \mathbb{N})$ such that $fa_n = gb_n$ for all $n \in \mathbb{N}$. By property (ii) of the arrows, both f and g must be surjective. Set $m = g(1)$ and let n be such that $f(n) = m$. So we have $g(2n) = gb_n(n) = fa_n(n) = m$, but then $g(2n) = g(1)$ which contradicts injectiveness of g.

As suggested by Jonathan Kirby we can easily generalise this example by replacing $\aleph_0$ by any infinite cardinal. So we get equivalence classes of cardinality κ for some fixed infinite κ. This is no longer a quasiminal excellent class, but the rest of this example goes through. That is, the category $\mathcal{C}$ of subsets of these models is κ^+-accessible and (A, M) is λ-presentable precisely when $|A| < \lambda$ for $\lambda > \kappa$. We again have that $(\mathcal{C}, \mathcal{M})$ is an AECat with AP, where $\mathcal{M}$ is the category of models. In this case, the AECat even fails to be $< \kappa$-type short, by a similar argument as above. That is, there are two tuples that have different Galois types, while all subtuples of cardinality $<\kappa$ have the same Galois type.

In the remainder of this section we show how we can recover some compactness from finite shortness. The argument is standard, just translated to the framework of AECats. See for example [BY03a, Remark 2.34]

Definition 3.36. Let $(A_i)_{i \in I}$ be some (infinite) tuple of objects in some AECat with AP. Given a subset $I_0 \subseteq I$, we call a tuple $((a_{I_0,i})_{i \in I_0}; M_{I_0})$ an *interpretation for $(A_i)_{i \in I_0}$ in M_{I_0}*.

Let $\mathcal{I} \subseteq \mathcal{P}(I)$ be a downwards closed set of subsets of I. Denote by $\bar{\mathcal{I}}$ the ideal generated by $\mathcal{I}$ (i.e. close it under finite unions). Then a *system of satisfiability* for $\mathcal{I}$ consists of an interpretation for each element of $\bar{\mathcal{I}}$, such that for all $I_0 \subseteq I_1$, with $I_0 \in \mathcal{I}$ and $I_1 \in \bar{\mathcal{I}}$, we have

$$\mathrm{gtp}((a_{I_0,i})_{i \in I_0}; M_{I_0}) = \mathrm{gtp}((a_{I_1,i})_{i \in I_0}; M_{I_1}).$$

If $\mathcal{I}$ is the set of all finite subsets of I, then we call such a system a *system of finitary satisfiability*. Note that in that case $\mathcal{I} = \bar{\mathcal{I}}$.

A *realisation* for a system of satisfiability is an interpretation for all of I, such that for each $I_0 \in \mathcal{I}$ we have

$$\mathrm{gtp}((a_{I,i})_{i \in I_0}; M_I) = \mathrm{gtp}((a_{I_0,i})_{i \in I_0}; M_{I_0}).$$

For such a realisation we will drop the subscript I, so the notation becomes

$$\mathrm{gtp}((a_i)_{i \in I_0}; M) = \mathrm{gtp}((a_{I_0,i})_{i \in I_0}; M_{I_0}).$$

We say that we have *directed compactness for Galois types* if every system of finitary satisfiability admits a realisation. We say that we have *compactness for Galois types* if every system of satisfiability admits a realisation.

Example 3.37. Let $(\mathcal{C}, \mathcal{M}) = (\mathbf{SubMod}(T), \mathbf{Mod}(T))$ for some theory T. We will sketch how one can see that $(\mathcal{C}, \mathcal{M})$ has compactness for Galois types. Recall that in our terminology a *type* is a maximal consistent set of formulas, see Definition 2.13.

Let $(x_i)_{i \in I}$ be some tuple of variables indexed by I and let $\mathcal{I} \subseteq \mathcal{P}(I)$ be a downwards closed set of subsets of I. A system of satisfiability then consists of a type $p_{I_0}((x_i)_{i \in I_0})$ for each $I_0 \in \bar{\mathcal{I}}$ such that for each $I_0 \subseteq I_1$ with $I_0 \in \mathcal{I}$ and $I_1 \in \bar{\mathcal{I}}$ we have that $p_{I_0} \subseteq p_{I_1}$. A realisation of this system is then precisely a realisation of

$$p((x_i)_{i \in I}) = \bigcup_{I_0 \in \mathcal{I}} p_{I_0}((x_i)_{i \in I_0}).$$

By compactness, Proposition 2.8, it is enough to show that p is finitely satisfiable. So let $\varphi_1, \ldots, \varphi_n \in p$. Then there are $I_1, \ldots, I_n \in \mathcal{I}$ such that $\varphi_k \in p_{I_k}$ for all $1 \leq k \leq n$. Take $J = I_1 \cup \ldots \cup I_n$, so $J \in \bar{\mathcal{I}}$. Hence $\varphi_1, \ldots, \varphi_n \in p_J$ for all

$1 \leq k \leq n$. So since p_J is satisfiable, we conclude that $\{\varphi_1, \ldots, \varphi_n\}$ is satisfiable, as required.

Proposition 3.38. *A finitely short AECat with AP has directed compactness for Galois types.*

Proof. Suppose we have a tuple $(A_i)_{i \in I}$ of objects. We prove by induction on the cardinality $|I|$ that every system of finitary satisfiability for $(A_i)_{i \in I}$ has a realisation. The case where I is finite is trivial.

For the induction step we assume that every system of finitary satisfiability of cardinality $< \kappa$ has a realisation. So suppose we are given some system of finitary satisfiability, using the same notation as in Definition 3.36, with $|I| = \kappa$. Then we may actually assume $I = \kappa$ and thus write $(A_i)_{i<\kappa}$. By induction we will construct a chain of models $(N_i)_{i<\kappa}$ and $a_i : A_i \to N_{i+1}$, such that for all $\alpha < \kappa$ the tuple $(a_i)_{i<\alpha}$ is a realisation for $(A_i)_{i<\alpha}$ in N_α.

Base case. We just take N_0 to be $M_\emptyset$.

Successor step. Suppose we have constructed $(N_i)_{i \leq \alpha}$ and $(a_i)_{i<\alpha}$. Since $|\alpha+1| < \kappa$, we can use the induction hypothesis on κ to find a realisation $(a'_i)_{i<\alpha+1}$ in some M', of our system of finitary satisfiability restricted to $(A_i)_{i<\alpha+1}$. Then for all $i_1 < \ldots < i_n < \alpha$ we have

$$\mathrm{gtp}(a_{i_1}, \ldots, a_{i_n}; N_\alpha) = \mathrm{gtp}(a'_{i_1}, \ldots, a'_{i_n}; M'),$$

because both are realisations of the same (restricted) system of finitary satisfiability. Being finitely short implies

$$\mathrm{gtp}((a_i)_{i<\alpha}; N_\alpha) = \mathrm{gtp}((a'_i)_{i<\alpha}; M').$$

Then apply Proposition 3.16(iii) to a'_α to find $N_\alpha \to N_{\alpha+1}$ and $a_\alpha : A_\alpha \to N_{\alpha+1}$.

Limit step. For $\ell < \kappa$ a limit, we just take $N_\ell = \mathrm{colim}_{i<\ell} N_i$.

Now take $N_\kappa = \mathrm{colim}_{i<\kappa} N_i$, so $(a_i)_{i<\kappa}$ in N_κ is the required realisation. □

Remark 3.39. Suppose we have a system of finitary satisfiability where everything is interpreted in the same model M. Suppose furthermore that all interpretations of some $P \subseteq I$ are fixed. That is, $a_p = a_{I_0,p}$ does not depend on $I_0 \subseteq I$. If we assume that the AECat is finitely short, then for a realisation $((a_i)_{i \in I}; N)$:

$$\mathrm{gtp}((a_p)_{p \in P}; N) = \mathrm{gtp}((a_p)_{p \in P}; M).$$

So we may assume N to be an extension of M, and the realisation of $p \in P$ to be given by $A_p \xrightarrow{a_p} M \to N$.

4

Independence relations

In this chapter we discuss independence relations in AECats. In section 4.1 we define an abstract independence relation $\mathop{\smile\hskip-0.9em\raise0.3em\hbox{$\vert$}}\,$ as a ternary relation on triples of subobjects of models. We then formulate the properties that such a relation can have. Classically, in theories (e.g. in full first-order logic) the properties that an abstract independence relation has, if it exists, can be used to characterise that theory as stable, simple or NSOP_1. Following this we define which properties make an independence relation in an AECat stable, simple or NSOP_1-like.

In section 4.2 we study the connection between three properties that an independence relation can have: Independence Theorem, 3-amalgamation and Stationarity. Roughly summarised we prove that the first two properties are equivalent, and that the last one implies the first two.

In stable and simple theories in full first-order (or positive) logic, independence is given by dividing. We introduce the notions of long dividing and isi-dividing in section 4.4. These are closely related to dividing, but they are better suited for settings that lack compactness or even settings that are not finitely short. In NSOP_1 theories in full first-order logic—and also in positive theories, see chapter 6—independence is given by Kim-dividing. Accordingly, we also introduce the notion of long Kim-dividing, also in section 4.4. Again, this is closely related, but better suited for AECats in general.

In the last section of this chapter, section 4.5, we continue the exploration of the link between finitely short AECats and existing frameworks. The contents of this section are not needed anywhere else in this thesis. We prove that the finite shortness assumption allows us to create and manipulate indiscernible sequences as usual in more concrete settings.

4.1 Independence relations in AECats

Definition 4.1. In an AECat with AP, an *independence relation* is a relation on triples of subobjects of models. If such a triple (A, B, C) of subobjects of a model M is in the relation, we call it *independent* and denote this by:

$$A \mathop{\downarrow}_{C}^{M} B.$$

This notation should be read as "A is independent from B over C (in M)".

We also allow each of the subobjects in the notation to be replaced by an arrow representing them. For example, if a is an arrow representing the subobject A then $a \downarrow_C^M B$ means $A \downarrow_C^M B$.

We may want to restrict the objects that can appear in the base of the independence relation. For example, in chapter 6, we will develop independence only over e.c. models.

Definition 4.2. Let $(\mathcal{C}, \mathcal{M})$ be an AECat with AP and let $\mathcal{B}$ be a collection of objects in $\mathcal{C}$ with $\mathcal{M} \subseteq \mathcal{B}$. Then we call $\mathcal{B}$ a *base class*. An independence relation $\downarrow$ is called an *independence relation over* $\mathcal{B}$ if it only allows subobjects with their domain in $\mathcal{B}$ in the base. That is, $A \downarrow_C^M B$ implies that the domain of C is in $\mathcal{B}$. We will also say that $\mathcal{B}$ is *the base class of* $\downarrow$, written as $\mathcal{B} = \mathrm{base}(\downarrow)$.

Convention 4.3. For a base class $\mathcal{B}$ and some subobject $C \leq M$ we will also write $C \in \mathcal{B}$ to mean that the domain of C is in $\mathcal{B}$, and similarly for $C \notin \mathcal{B}$.

Definition 4.4. We call an independence relation $\downarrow$ a *basic independence relation* if it satisfies the following properties.

INVARIANCE $a \downarrow_c^M b$ and $\mathrm{gtp}(a, b, c; M) = \mathrm{gtp}(a', b', c'; M')$ implies $a' \downarrow_{c'}^{M'} b'$.

MONOTONICITY $A \downarrow_C^M B$ and $A' \leq A$ implies $A' \downarrow_C^M B$.

TRANSITIVITY $A \downarrow_B^M C$ and $A \downarrow_C^M D$ with $B \leq C$ implies $A \downarrow_B^M D$.

SYMMETRY $A \downarrow_C^M B$ implies $B \downarrow_C^M A$.

EXISTENCE $A \downarrow_C^M C$ for all $A, C \leq M$ with $C \in \mathrm{base}(\downarrow)$.

EXTENSION If $a \downarrow_c^M b$ and $(b'; M)$ is such that b factors through b' then there is an extension $M \to N$ with some $(a'; N)$ such that $\mathrm{gtp}(a', b, c; N) = \mathrm{gtp}(a, b, c; M)$ and $a' \downarrow_c^N b'$.

UNION Let $(B_i)_{i \in I}$ be a directed system with a cocone into some model M, and suppose $B = \mathrm{colim}_{i \in I} B_i$ exists. Then if $A \downarrow_C^M B_i$ for all $i \in I$, we have $A \downarrow_C^M B$.

Before we define some additional properties for independence relations, we first need to translate the notion of a club set to categorical language.

Definition 4.5. Let $(\mathcal{C}, \mathcal{M})$ be an AECat. For a model M and a regular cardinal κ we write $\mathrm{Sub}^{\kappa}_{\mathcal{M}}(M)$ for the poset of κ-presentable subobjects of M in $\mathcal{M}$. That is, $\mathrm{Sub}^{\kappa}_{\mathcal{M}}(M)$ is the set of $M' \leq M$ such that M' is κ-presentable.

Note that if we are given a chain $(M_i)_{i<\theta}$ in $\mathrm{Sub}^{\kappa}_{\mathcal{M}}(M)$ with $\theta < \kappa$ then its join in $\mathrm{Sub}^{\kappa}_{\mathcal{M}}(M)$ exists and is given by $\mathrm{colim}_{i<\theta} M_i$. This is the reason why we restrict ourselves to $\mathcal{M}$, because there we have directed colimits. If $\mathcal{C}$ has directed colimits as well then all these definitions would make sense for $\mathcal{C}$ as well.

Definition 4.6. Let $\mathcal{F} \subseteq \mathrm{Sub}^{\kappa}_{\mathcal{M}}(M)$ be a nonempty set.

(i) We call $\mathcal{F}$ *unbounded* if for every $M_0 \in \mathrm{Sub}^{\kappa}_{\mathcal{M}}(M)$ there is $M_1 \in \mathcal{F}$ such that $M_0 \leq M_1$.

(ii) We call $\mathcal{F}$ *closed* if for any chain $(M_i)_{i<\theta}$ in $\mathcal{F}$ with $\theta < \kappa$ its join $\mathrm{colim}_{i<\theta} M_i$ is again in $\mathcal{F}$.

(iii) We call $\mathcal{F}$ a *club set* if it is closed and unbounded.

Fact 4.7. *The following two facts are standard.*

(i) The intersection of two club sets on $\mathrm{Sub}^{\kappa}_{\mathcal{M}}(M)$ *is again a club set.*

(ii) If $M = \mathrm{colim}_{i<\kappa} M_i$*, where* $(M_i)_{i<\kappa}$ *is a continuous chain of* κ*-presentable models, then* $\{M_i : i < \kappa\}$ *is a club set on* $\mathrm{Sub}^{\kappa}_{\mathcal{M}}(M)$*.*

Proof. Fact (i) is standard, see for example [Jec03, Theorem 8.2]. We just apply the argument to the poset $\mathrm{Sub}^{\kappa}_{\mathcal{M}}(M)$ instead of to a cardinal considered as a poset. Fact (ii) is just unfolding definitions. The chain $(M_i)_{i<\kappa}$ is unbounded because it is κ-directed, so any κ-presentable $M' \leq M$ will factor through the chain, and continuity is precisely saying that the chain is a closed set. $\square$

Definition 4.8. We also define the following properties for an independence relation.

Base-Monotonicity $A \mathop{\smile\!\!\!\!\!\!\mid}^{M}_{C} B$ and $C \leq C' \leq B$ with $C' \in \mathrm{base}(\mathop{\smile\!\!\!\!\!\!\mid})$ implies $A \mathop{\smile\!\!\!\!\!\!\mid}^{M}_{C'} B$.

Club Local Character For every regular cardinal λ there is a regular cardinal $\Upsilon(\lambda)$ such that the following holds for all regular $\kappa \geq \Upsilon(\lambda)$. Let $A, M \leq N$, with A λ-presentable and M a model. Then there is a club set $\mathcal{F} \subseteq \mathrm{Sub}^{\kappa}_{\mathcal{M}}(M)$ such that for all $M_0 \in \mathcal{F}$ we have $A \mathop{\smile\!\!\!\!\!\!\mid}^{N}_{M_0} M$.

Stationarity If $\mathrm{gtp}(a, m; N) = \mathrm{gtp}(a', m; N)$, where the domain of m is a model, then $a \mathop{\smile\!\!\!\!\!\!\mid}^{N}_{m} b$ and $a' \mathop{\smile\!\!\!\!\!\!\mid}^{N}_{m} b$ implies $\mathrm{gtp}(a, m, b; N) = \mathrm{gtp}(a', m, b; N)$.

INDEPENDENCE THEOREM Suppose we have $a \mathop{\smile\!\!\!\!\!\!\downarrow}^M_c b$, $a' \mathop{\smile\!\!\!\!\!\!\downarrow}^M_c b'$, $b \mathop{\smile\!\!\!\!\!\!\downarrow}^M_c b'$ and $\mathrm{Lgtp}(a/c; M) = \mathrm{Lgtp}(a'/c; M)$. Then there is an extension $M \to N$ with $(a^*; N)$ such that $a^* \mathop{\smile\!\!\!\!\!\!\downarrow}^N_c M$, $\mathrm{Lgtp}(a^*/c, b; N) = \mathrm{Lgtp}(a/c, b; N)$ and $\mathrm{Lgtp}(a^*/c, b'; N) = \mathrm{Lgtp}(a'/c, b'; N)$.

Definition 4.9. Let $\mathop{\smile\!\!\!\!\!\!\downarrow}$ be a basic independence relation.

- We call $\mathop{\smile\!\!\!\!\!\!\downarrow}$ a *stable independence relation* if it additionally satisfies BASE-MONOTONICITY, CLUB LOCAL CHARACTER, STATIONARITY and INDEPENDENCE THEOREM.
- We call $\mathop{\smile\!\!\!\!\!\!\downarrow}$ a *simple independence relation* if it additionally satisfies BASE-MONOTONICITY, CLUB LOCAL CHARACTER and INDEPENDENCE THEOREM.
- We call $\mathop{\smile\!\!\!\!\!\!\downarrow}$ an *NSOP$_1$-like independence relation* if it additionally satisfies CLUB LOCAL CHARACTER and INDEPENDENCE THEOREM;

Remark 4.10. A few remarks about the properties for independence relations.

1. The SYMMETRY property is part of the definition of a basic independence relation. So, for example, we can apply MONOTONICITY to both sides. That is, if $A \mathop{\smile\!\!\!\!\!\!\downarrow}^M_C B$ and $A' \leq A$, then $A' \mathop{\smile\!\!\!\!\!\!\downarrow}^M_C B$. We will often suppress mentions of SYMMETRY when using the symmetric version of another property. If the independence relation does not have SYMMETRY, one would have to distinguish between "left" and "right" versions (e.g. LEFT-MONOTONICITY and RIGHT-MONOTONICITY).

2. The UNION property is our version of what is usually known as "finite character". In a concrete setting it follows directly from finite character, but this formulation is more suited for our category-theoretic setting. In the setting of AECs one often sees the name "$(< \aleph_0)$-witness property", which implies UNION.

3. In the statement of UNION: we can view B as a subobject of M because the universal property of the colimit guarantees an arrow $B \to M$, which must be a monomorphism because all arrows are monomorphisms in an AECat. If every B_i is a model, then the colimit B always exists and is a model. We will only need to apply UNION to directed systems of models.

4. In a more traditional definition of local character, such as in simple theories, one would just require that for $A, M \leq N$ as above there is some $\Upsilon(\lambda)$-presentable $M_0 \leq M$ such that $A \mathop{\smile\!\!\!\!\!\!\downarrow}^N_{M_0} M$. This is (almost) the definition that was used in [Kam20], where we also have access to BASE-MONOTONICITY. We then get CLUB LOCAL CHARACTER by

considering the club set $\mathcal{F} = \{M_0 \in \mathrm{Sub}^{\kappa}_{\mathcal{M}}(M) : M_0 \leq M\}$. In NSOP$_1$-like settings we do generally not have Base-Monotonicity. So Club Local Character then still gives us a good amount of Base-Monotonicity, namely on a club set. These ideas are due to [KRS17].

5. Stationarity is sometimes also called "uniqueness".

Example 4.11. A natural question would be to ask whether there are examples that are not finitely short, but where there still is an independence relation satisfying Union. Quasiminimal excellent classes as discussed in Example 3.10 are such an example: the pregeometry there yields a stable independence relation. We discussed one such example in detail in Example 3.35, where models are equivalence relation with equivalence classes of cardinality $\aleph_0$. In fact, the independence relation there admits an easy description: we set $A \mathop{\smile\!\!\!\!\!\!\!\;\vert}_C^M B$ if $\mathrm{cl}_M(A) \cap \mathrm{cl}_M(B) \subseteq \mathrm{cl}_M(C)$. It is easy to check that this is indeed a stable independence relation.

Other examples can be found in AECs with intersections. For example [Vas17, Appendix C] discusses how to find a stable independence relation in such AECs. There is no direct mention of the Union property there, but the "$(< \aleph_0)$-witness property" directly implies Union. Also [GMA21, Section 8.2] comes close to giving a simple example, rather than stable, although no explicit examples are given and they do not get the full $(< \aleph_0)$-witness property.

Proposition 4.12 (Strong extension). *Let $\mathop{\smile\!\!\!\!\!\!\!\;\vert}$ be a basic independence relation and suppose that $a \mathop{\smile\!\!\!\!\!\!\!\;\vert}_c^M b$. Then for any $(d; M)$ there is an extension $M \to N$ and $(d'; N)$ such that $\mathrm{Lgtp}(d'/b, c; N) = \mathrm{Lgtp}(d/b, c; N)$ and $a \mathop{\smile\!\!\!\!\!\!\!\;\vert}_c^N d'$.*

Proof. We first apply Extension to find $M \to N_1$ with $m' : M \to N_1$ such that $a \mathop{\smile\!\!\!\!\!\!\!\;\vert}_c^{N_1} m'$ and $\mathrm{gtp}(m', b, c; N_1) = \mathrm{gtp}(m, b, c; N_1)$. In particular this means that b and c factor through m'. We apply Extension again to find an extension $n_1 : N_1 \to N$ and $n_1' : N_1 \to N$ with $a \mathop{\smile\!\!\!\!\!\!\!\;\vert}_c^N n_1'$ and $\mathrm{gtp}(n_1', m'; N) = \mathrm{gtp}(n_1, m'; N)$. We define d' to be the composition $D \xrightarrow{d} M \to N_1 \xrightarrow{n_1'} N$. By Monotonicity we then have $a \mathop{\smile\!\!\!\!\!\!\!\;\vert}_c^N d'$. We also have $\mathrm{gtp}(d', m'; N) = \mathrm{gtp}(d, m'; N)$, so since b and c factor through m', and m' has a model as domain, we indeed get $\mathrm{Lgtp}(d'/b, c; N) = \mathrm{Lgtp}(d/b, c; N)$. $\square$

Corollary 4.13. *Let $\mathop{\smile\!\!\!\!\!\!\!\;\vert}$ be a basic independence relation and suppose that $a \mathop{\smile\!\!\!\!\!\!\!\;\vert}_c^M b$. Then for any $(d; M)$ there is an extension $M \to N$ and $(a'; N)$ such that $\mathrm{Lgtp}(a'/b, c; N) = \mathrm{Lgtp}(a/b, c; N)$ and $a' \mathop{\smile\!\!\!\!\!\!\!\;\vert}_c^N d$.*

Proof. Apply Proposition 4.12 to find $M \to N'$ with $(d'; N')$ such that $a \mathop{\smile\!\!\!\!\!\!\!\;\vert}_c^{N'} d'$ and $\mathrm{Lgtp}(d'/b, c; N) = \mathrm{Lgtp}(d/b, c; N)$. Then just pick $(a'; N)$ in an extension $N' \to N$ such that $\mathrm{Lgtp}(a', d/b, c; N) = \mathrm{Lgtp}(a, d'/b, c; N)$. $\square$

Convention 4.14. We call the class function Υ for Club Local Character a *local character function*. For an object A we write $\Upsilon(A)$ for $\Upsilon(\lambda)$ where λ is the least regular cardinal such that A is λ-presentable.

Lemma 4.15 (Chain local character)**.** *Let $\mathop{\smile\!\!\!\!\!\mid}$ be an independence relation in a λ-AECat $(\mathcal{C}, \mathcal{M})$ with AP, satisfying* Club Local Character. *Let $A \leq N$ and $\kappa \geq \Upsilon(A) + \lambda$. Suppose that we are given a continuous chain $(M_i)_{i<\kappa}$ of κ-presentable models with $M = \operatorname{colim}_{i<\kappa} M_i \leq N$. Then there is $i_0 < \kappa$ such that $A \mathop{\smile\!\!\!\!\!\mid}^N_{M_{i_0}} M$.*

Proof. Let $\mathcal{F} \subseteq \mathrm{Sub}^{\kappa}_{\mathcal{M}}(M)$ be the club set from Club Local Character. By Fact 4.7(ii) the chain $(M_i)_{i<\kappa}$ forms a club set on $\mathrm{Sub}^{\kappa}_{\mathcal{M}}(M)$. So by Fact 4.7(i) $\{M_i : i < \kappa\} \cap \mathcal{F}$ is nonempty. $\square$

Remark 4.16. For all our results we only need chain local character. That is, the conclusion of Lemma 4.15. In particular the canonicity theorems in section 5.1 go through even if we would just assume chain local character.

Given that we actually only need chain local character, as per Remark 4.16, it is natural to ask whether the converse of Lemma 4.15 holds. That is, if chain local character implies Club Local Character. This is not so clear, so we leave it at this.

Remark 4.17. As opposed to [LRV19] we have defined an independence relation here on triples of subobjects, while they define it as a relation on commuting squares. Their notion has the advantage of the independent squares forming an accessible category, and allowing for a more category-theoretic study of the independence relation itself (see also [LRV20]). Our approach has the benefit that the calculus we get from it is more intuitive and easier to work with.

In an AECat of the form $(\mathcal{C}, \mathcal{C})$, these two notions are essentially the same. That is, assuming basic properties on the relevant independence relations, one can be recovered from the other and vice versa. This is done as follows. Suppose we have an independence relation $\mathop{\smile\!\!\!\!\!\mid}$ on subobjects and a commutative square

$$\begin{tikzcd} A \arrow[r, "a"] & M \\ C \arrow[u] \arrow[r] \arrow[ur, dashed, "c"] & B \arrow[u, "b"'] \end{tikzcd}$$

where the dashed arrow is just to give the arrow $C \to M$ a name. Then we declare this square to be independent if $a \mathop{\smile\!\!\!\!\!\mid}^M_c b$. Conversely, given subobjects $A, B, C \leq M$ with representatives a, b, c respectively, we set $A \mathop{\smile\!\!\!\!\!\mid}^M_C B$ if there are dashed arrows as in the diagram below, such that the diagram below commutes and the outer square is independent.

$$\begin{tikzcd}
A' \arrow[rrr, dashed] & & & N \\
& A \arrow[ul, dashed] \arrow[r, "a"] & M \arrow[ur, dashed] & \\
& & B \arrow[u, "b"] \arrow[dr] & \\
C \arrow[uuu, dashed] \arrow[uurr, "c"] \arrow[rrr, dashed] & & & B' \arrow[uuu, dashed]
\end{tikzcd}$$

In [LRV19] the 3-amalgamation is not mentioned, but as we will see in Theorem 4.21 we get that for free given the rest of the properties of a stable independence relation.

4.2 Independence theorem, 3-amalgamation and stationarity

It is well known that the property Independence Theorem can also be formulated as an amalgamation property of some independent system. This allows for a more categorical statement without any mention of Lascar strong Galois types. However, we need to restrict ourselves to work only over models. We will give this property its own name and prove its equivalence to Independence Theorem, modulo some basic properties, in Theorem 4.20.

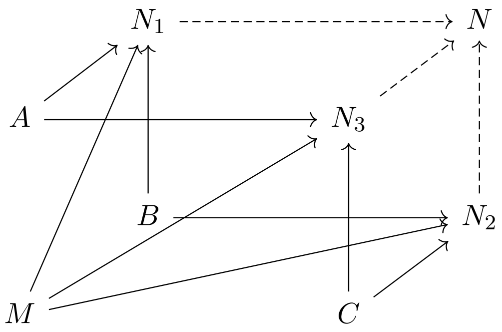
Definition 4.18. An independence relation $\mathop{\smile\hspace{-0.9em}\mid}$ has 3-amalgamation if the following holds. Suppose that we have

$$A \underset{M}{\overset{N_1}{\mathop{\smile\hspace{-0.9em}\mid}}} B, \quad B \underset{M}{\overset{N_2}{\mathop{\smile\hspace{-0.9em}\mid}}} C, \quad C \underset{M}{\overset{N_3}{\mathop{\smile\hspace{-0.9em}\mid}}} A,$$

overloading notation for subobjects of different models. Suppose furthermore that M is a model and that

$$\begin{aligned}
\mathrm{gtp}(a, m; N_1) &= \mathrm{gtp}(a, m; N_3), \\
\mathrm{gtp}(b, m; N_1) &= \mathrm{gtp}(b, m; N_2), \\
\mathrm{gtp}(c, m; N_2) &= \mathrm{gtp}(c, m; N_3),
\end{aligned}$$

where a, b, c and m are representatives for the subobjects A, B, C and M respectively (again, overloading notation for different models). Then we can find extensions from N_1, N_2 and N_3 to some N such that the diagram we obtain in that way commutes:

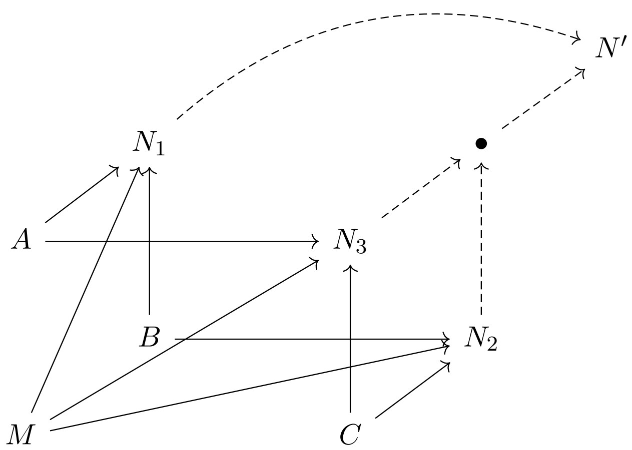


Furthermore, these extensions are such that $A \mathop{\perp}^{N}_{M} N_2$.

Remark 4.19. For the canonicity theorem for simple independence relations, Theorem 5.4, we only need 3-AMALGAMATION. Or equivalently, by Theorem 4.20, INDEPENDENCE THEOREM over models. Even if base($\perp$) is more than just $\mathcal{M}$, e.g. base($\perp$) $= \mathcal{C}$. See also Remark 5.5

Theorem 4.20. *Let $\perp$ be a basic independence relation. If $\perp$ satisfies* INDEPENDENCE THEOREM *then it also satisfies* 3-AMALGAMATION*. Conversely, if $\perp$ satisfies* 3-AMALGAMATION *then it satisfies* INDEPENDENCE THEOREM *over models (i.e. we require the base C to be a model).*

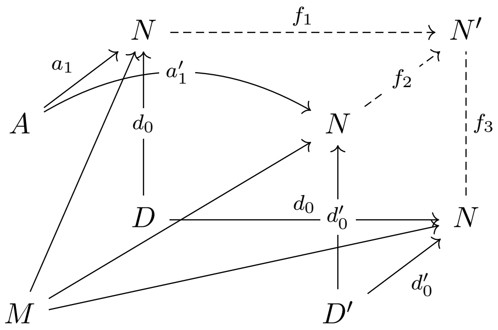
Proof. We first prove that INDEPENDENCE THEOREM implies 3-AMALGAMATION. Let the set up be as in Definition 4.18. In the diagram below we find the dashed arrows by first using $\mathrm{gtp}(c, m; N_2) = \mathrm{gtp}(c, m; N_3)$ and then $\mathrm{gtp}(b, m; N_1) = \mathrm{gtp}(b, m; N_2)$.



We write a_1 for the arrow $A \to N_1 \to N'$ and a_3 for the arrow $A \to N_3 \to N'$. Then we have $\mathrm{Lgtp}(a_1/m; N') = \mathrm{Lgtp}(a_3/m; N')$. We can thus apply INDEPENDENCE THEOREM to find some extension $N' \to N^*$ with some $a^* : A \to N^*$ such that $\mathrm{Lgtp}(a^*/m, b; N^*) = \mathrm{Lgtp}(a_1/m, b; N^*)$,

$\mathrm{Lgtp}(a^*/m, c; N^*) = \mathrm{Lgtp}(a_3/m, c; N^*)$ and $a^* \mathop{\smile\hspace{-0.8em}|}_M^{N^*} N'$. So in particular we have $a^* \mathop{\smile\hspace{-0.8em}|}_M^{N^*} N_2$ by MONOTONICITY. Using $\mathrm{gtp}(a, b, m; N_1) = \mathrm{gtp}(a_1, b, m; N^*) = \mathrm{gtp}(a^*, b, m; N^*)$ and $\mathrm{gtp}(a, c, m; N_3) = \mathrm{gtp}(a_3, c, m; N^*) = \mathrm{gtp}(a^*, c, m; N^*)$ after each other we find an extension $N^* \to N$ together with extensions from N_1 and N_3 to N and we just forget about the previous extensions from N_1 and N_3 to N^*. These two new extensions, together with $N_2 \to N^* \to N$, then form the solution to our 3-AMALGAMATION problem.

Now we prove the converse. So we assume 3-AMALGAMATION and we prove INDEPENDENCE THEOREM over models. So suppose that $a \mathop{\smile\hspace{-0.8em}|}_m^N b$, $a' \mathop{\smile\hspace{-0.8em}|}_m^N c$ and $b \mathop{\smile\hspace{-0.8em}|}_m^N c$ and $\mathrm{Lgtp}(a/m; N) = \mathrm{Lgtp}(a'/m; N)$. We will again go through a few extensions of N, and by replacing N by these extensions each time we assume everything we find already lives in N. Let $(d; N)$ be such that both a and b factor through d. Then we can apply strong extension, Proposition 4.12, to $b \mathop{\smile\hspace{-0.8em}|}_m^N c$ to find $(d_0; N)$ such that $d_0 \mathop{\smile\hspace{-0.8em}|}_m^N c$ and $\mathrm{Lgtp}(d_0/m, b; N) = \mathrm{Lgtp}(d/m, b; N)$. Now let $(d'; N)$ be such that both a and c factor through d' and find d'_0 such that $d_0 \mathop{\smile\hspace{-0.8em}|}_m^N d'_0$ and $\mathrm{Lgtp}(d'_0/m, c; N) = \mathrm{Lgtp}(d_0/m, c; N)$. Applying the dual of strong extension, Corollary 4.13, we find $(a_1; N)$ such that $a_1 \mathop{\smile\hspace{-0.8em}|}_m^N d_0$ and $\mathrm{Lgtp}(a_1/m, b; N) = \mathrm{Lgtp}(a/m, b; N)$. Similarly we find $(a'_1; N)$ with $a'_1 \mathop{\smile\hspace{-0.8em}|}_m^N d'_0$ with $\mathrm{Lgtp}(a'_1/m, c; N) = \mathrm{Lgtp}(a'/m, c; N)$. We can now fit all this in the diagram below, where 3-AMALGAMATION gives us the dashed arrows.



We can indeed apply 3-AMALGAMATION because $\mathrm{gtp}(a_1, m; N) = \mathrm{gtp}(a, m; N) = \mathrm{gtp}(a', m; N) = \mathrm{gtp}(a'_1, m; N)$. To see that this indeed gives the solution required for INDEPENDENCE THEOREM we take the extension $N \to N'$ to be f_3. Then a^* is $f_1 a_1 = f_2 a'_1$. So indeed $a^* \mathop{\smile\hspace{-0.8em}|}_m^{N'} f_3$ and hence $a^* \mathop{\smile\hspace{-0.8em}|}_m^{N'} N$ because we chose the extension $N \to N'$ to be f_3, so f_3 is a representative of the subobject N. We will conclude the proof by showing that $\mathrm{Lgtp}(a^*/m, b; N') = \mathrm{Lgtp}(a/m, b; N')$, while $\mathrm{Lgtp}(a^*/m, c; N') = \mathrm{Lgtp}(a'/m, c; N')$ follows analogously. Write a_0 for the composition $A \to D \xrightarrow{d_0} N'$. Then by construction

$$\mathrm{Lgtp}(a_0/m, b; N') = \mathrm{Lgtp}(a/m, b; N') = \mathrm{Lgtp}(a_1/m, b; N').$$

From the commutativity of the above diagram we get $\mathrm{gtp}(a^*, d_0, m; N') = \mathrm{gtp}(a_1, d_0, m; N')$. By construction b factors through d_0, so we find $\mathrm{gtp}(a^*, a_0, b, m; N') = \mathrm{gtp}(a_1, a_0, b, m; N')$. The result then follows by applying this to the above string of equalities of Lascar strong Galois types. $\square$

The following is a standard argument (see e.g. [Hru06, Lemma 4.1]) translated into the framework of AECats. For example, it is helpful in comparing our notion of stable independence relation to that of [LRV19], see Remark 4.17.

Theorem 4.21. *Let* $\mathop{\smile\!\!\!\!\!\!|}$ *be a basic independence relation satisfying* STATIONARITY *then it also satisfies* 3-AMALGAMATION.

Proof. By Theorem 4.20 it is enough to prove INDEPENDENCE THEOREM over models, not concerning ourselves with Lascar strong Galois types. So suppose that

$$a_1 \mathop{\smile\!\!\!\!\!\!|}_{m}^{N} b, \; a_2 \mathop{\smile\!\!\!\!\!\!|}_{m}^{N} c, \; b \mathop{\smile\!\!\!\!\!\!|}_{m}^{N} c,$$

and $\mathrm{gtp}(a_1, m; N) = \mathrm{gtp}(a_2, m; N)$, where m has a model as domain. Apply EXTENSION twice to find an extension $n : N \to N'$ together with copies $n_1 : N \to N'$ and $n_2 : N \to N'$ such that $a_1 \mathop{\smile\!\!\!\!\!\!|}_{m}^{N'} n_1$, $a_2 \mathop{\smile\!\!\!\!\!\!|}_{m}^{N'} n_2$, $\mathrm{gtp}(n_1, b, m; N') = \mathrm{gtp}(n, b, m; N')$ and $\mathrm{gtp}(n_2, c, m; N') = \mathrm{gtp}(n, c, m; N')$. So we have $\mathrm{gtp}(n_2, m; N') = \mathrm{gtp}(n_1, m; N')$, and we thus find $N' \to N''$ together with $(a_2'; N'')$ such that $\mathrm{gtp}(a_2', n_1, m; N'') = \mathrm{gtp}(a_2, n_2, m; N'')$. In particular $a_2' \mathop{\smile\!\!\!\!\!\!|}_{m}^{N''} n_1$ and $\mathrm{gtp}(a_2', m; N'') = \mathrm{gtp}(a_2, m; N'') = \mathrm{gtp}(a_1, m; N'')$.

We can thus apply STATIONARITY to obtain $\mathrm{gtp}(a_1, n_1, m; N'') = \mathrm{gtp}(a_2', n_1, m; N'') = \mathrm{gtp}(a_2, n_2, m; N'')$. So there is an extension $N'' \to N^*$ with some $(a^*; N^*)$ such that:

$$\mathrm{gtp}(a^*, n, m; N^*) = \mathrm{gtp}(a_1, n_1, m; N'') = \mathrm{gtp}(a_2, n_2, m; N'').$$

By INVARIANCE we then have $a^* \mathop{\smile\!\!\!\!\!\!|}_{m}^{N^*} N$ and by restricting Galois types we have $\mathrm{gtp}(a^*, m, b; N^*) = \mathrm{gtp}(a_1, m, b; N^*)$ and $\mathrm{gtp}(a^*, m, c; N^*) = \mathrm{gtp}(a_2, m, c; N^*)$, as required. $\square$

4.3 Sequences and isi-sequences

Sequences play a big role in independence relations. They are essential in defining various notions of dividing, see Section 4.4, which in turn yield various independence relations. We generally only consider sequences where any two elements are compatible in some sense. For example, in classical model theory we would require all the tuples in the sequence to have the same length. In the setting of AECats elements are replaced by arrows in some model, and the compatibility translates to requiring that these arrows all have the same domain.

Definition 4.22. A *sequence* is a tuple $((a_i)_{i \in I}; M)$ where every a_i has the same domain and I is a linear order.

We will only be interested in ordinal-shaped sequences. So from now on I will be an ordinal.

We will often need to treat an initial segment of a sequence as one object. The following definition makes sense of this in a category-theoretic setting.

Definition 4.23. A *chain of initial segments* for a sequence $(a_i)_{i<\kappa}$ in some M is a continuous chain $(M_i)_{i<\kappa}$ of models with chain bound M, such that for all $i < \kappa$ we have that a_i factors (necessarily uniquely) through M_{i+1}.

If an arrow $c : C \to M$ factors as $C \to M_0 \to M$, then we say that c *embeds in* $(M_i)_{i<\kappa}$.

We have required the objects in the chain of initial segments to be models. This allows us to have a continuous chain, because an AECat always has directed colimits of models. Another reason is more technical, but chains of initial segments will interact with various independence relations in later proofs (for example, as witnesses of independence, see Definition 4.27). The fact that the objects involved are models will then allow us to use things such as 3-AMALGAMATION.

Convention 4.24. For a chain of initial segments $(M_i)_{i<\kappa}$ for some sequence $(a_i)_{i<\kappa}$ in M we will also view a_i as an arrow into M_j for $i < j$. Similarly, if c embeds in $(M_i)_{i<\kappa}$, we view c as an arrow into M_i for all $i < \kappa$. Unless explicitly stated otherwise, we will denote the extension $M_i \to M$ by m_i.

Below is a picture of a sequence $(a_i)_{i<\kappa}$ in some model M with chain of initial segments $(M_i)_{i<\kappa}$ and c embedded in the chain of initial segments.

$$\begin{tikzcd}
C \arrow[r] & M_0 \arrow[r] & M_1 \arrow[r] & M_2 \arrow[r] & \dots \arrow[r] & M \\
& A \arrow[ur, "a_0"] & A \arrow[ur, "a_1"] & A \arrow[ur, "a_2"] & \dots &
\end{tikzcd}$$

Definition 4.25. We call a sequence $(a_i)_{i<\kappa}$ in M, together with a chain of initial segments $(M_i)_{i<\kappa}$, an *isi-sequence* (short for *initial segment invariant*) if for all $i \leq j < \kappa$ we have:

$$\mathrm{gtp}(a_i, m_i; M) = \mathrm{gtp}(a_j, m_i; M).$$

For $c : C \to M$ we say this is an *isi-sequence over* c if c embeds in $(M_i)_{i<\kappa}$.

So under the intuition that M_i plays the role of the initial segment $a_{<i}$ we should think of the above condition as saying that a_i and a_j have the same Galois type over $ca_{<i}$.

Classically, indiscernible sequences are used a lot in the context of independence relations. Indiscernible sequences are very homogeneous, in the sense that any two subsequences with the same order type have the same (Galois) type. However, to construct and manipulate indiscernible sequences one often needs some extra assumptions, such as finite shortness (see Section 4.5). The idea of isi-sequences is to have something weaker than indiscernible sequences so that we can work with them in more general settings, while still keeping enough homogeneity.

In finitely short settings any indiscernible sequence is indeed an isi-sequence, see Proposition 4.58. The converse is not true, even in very nice settings such as full first-order logic, see the example below.

Example 4.26. In this example we consider the theory (in full first-order logic) of the random graph and we work in a monster model. The theory states that for any two disjoint finite sets of vertices A and B there is a vertex c such that c has an edge to every vertex in A and no edge to any vertex in B. This property makes compactness arguments very easy, which we will (implicitly) use repeatedly below. It is well known that the theory of the random graph has quantifier elimination (see e.g. [TZ12, Exercise 3.3.1]), so two tuples have the same type if and only if they are isomorphic as graphs.

Fix some (infinite) cardinal κ, we will inductively construct a sequence $(a_i)_{i<\kappa}$ with chain of initial segments $(M_i)_{i<\kappa}$ forming an isi-sequence, but such that $(a_i)_{i<\kappa}$ is not indiscernible. Each a_i will just be a single element and will be so that there is an edge between a_i and a_j if and only if $\min(i, j)$ is an even ordinal. At stage i we will have constructed M_i, which will contain $(a_j)_{j<i}$.

We start by taking M_0 to be any model. Then having constructed M_i we let a_i be outside of M_i as follows: for $b \in M_i$ there is an edge between a_i and b if and only if $b = a_j$ for some even $j < i$. Then we take M_{i+1} to be any model containing $a_i M_i$. At limit stages we take the union of the models constructed so far.

To see that $(a_i)_{i<\kappa}$ together with $(M_i)_{i<\kappa}$ forms an isi-sequence we let $i \leq j < \kappa$ and we need to prove that $\text{tp}(a_i/M_i) = \text{tp}(a_j/M_i)$. Let $b \in M_i$, then by construction there is an edge between a_i and b if and only if $b = a_k$ for some even $k < \kappa$ if and only if there is an edge between a_j and b. So $(a_i)_{i<\kappa}$ together with $(M_i)_{i<\kappa}$ does indeed form an isi-sequence.

However, $(a_i)_{i<\kappa}$ is not indiscernible. For example, we have $\text{tp}(a_1 a_2) \neq \text{tp}(a_2 a_3)$, because there is no edge between a_1 and a_2 while there is an edge between a_2 and a_3. Generally, the type $\text{tp}(a_i a_j)$ for $i < j < \kappa$ depends on the parity of i.

Note that we can find an indiscernible subsequence. For example, if we take $I = \{i < \kappa : i \text{ is even}\}$ then $(a_i)_{i \in I}$ is an indiscernible subsequence of length λ. This phenomenon can never really be ruled out, because if there would be a Ramsey cardinal λ then any sequence of length λ has a cofinal indiscernible subsequence (see Lemma 4.56).

Definition 4.27. Suppose we have an independence relation $\downfree$. Let $(a_i)_{i<\kappa}$ be a sequence in some M and let $c : C \to M$ be an arrow. Suppose that $(M_i)_{i<\kappa}$ is a chain of initial segments for $(a_i)_{i<\kappa}$ and that c embeds in the chain. Then we call $(M_i)_{i<\kappa}$ *witnesses of* $\downfree_c$*-independence* for $(a_i)_{i<\kappa}$ if

$$a_i \mathop{\downfree}_c^M M_i$$

for all $i < \kappa$. We say that a sequence is $\downfree_c$*-independent* if it admits a chain of witnesses of $\downfree_c$-independence.

The following proposition is the standard argument showing that we can find arbitrarily long independent sequences, assuming very few properties for our independence relation (see e.g. [Kim14, Proposition 2.2.4]). The proposition after that shows that if we additionally assume UNION we can actually get arbitrarily long independent isi-sequences.

Proposition 4.28. *Let* $\downfree$ *be an independence relation satisfying* INVARIANCE, EXISTENCE *and* EXTENSION. *Then for any* $(a, c; M)$ *with* $\mathrm{dom}(c) \in \mathrm{base}(\downfree)$ *and any* κ *there is some extension* $M \to N$ *containing a* $\downfree_c$*-independent sequence* $(a_i)_{i<\kappa}$ *with* $\mathrm{gtp}(a_i, c; N) = \mathrm{gtp}(a, c; M)$ *for all* $i < \kappa$.

Proof. We construct the witnesses of independence $(M_i)_{i<\kappa}$ and sequence $(a_i)_{i<\kappa}$ by induction. At stage i we will construct a_i and M_{i+1}. By EXISTENCE we have $a \downfree_c^M c$, and so we will have $a \downfree_c^{M_i} c$ for all $i < \kappa$. At every stage we will apply EXTENSION to the latter.

Base case. Set $M_0 = M$ and use EXTENSION to find a_0 and $M \to M_1$ with $\mathrm{gtp}(a_0, c; M_1) = \mathrm{gtp}(a, c; M)$ and $a_0 \downfree_c^{M_1} M_0$.

Successor step. By EXTENSION we find $M_{i+1} \to M_{i+2}$ and a_{i+1} such that $a_{i+1} \downfree_c^{M_{i+2}} M_{i+1}$ and $\mathrm{gtp}(a_{i+1}, c; M_{i+2}) = \mathrm{gtp}(a, c; M)$.

Limit step. For limit $\ell < \kappa$ let $M_\ell = \mathrm{colim}_{i<\ell} M_i$. We use EXTENSION to find $M_\ell \to M_{\ell+1}$ and a_ℓ with $\mathrm{gtp}(a_\ell, c; M_{\ell+1}) = \mathrm{gtp}(a, c; M)$ and $a_\ell \downfree_c^{M_{\ell+1}} M_\ell$.

We finish the construction by taking $N = \mathrm{colim}_{i<\kappa} M_i$. ∎

Proposition 4.29. *Suppose that* $\mathop{\smile\!\!\!\!\!\!|}$ *is an independence relation satisfying* Invariance, Existence, Extension *and* Union. *Then given* $(a, c; M)$ *with* $\mathrm{dom}(c) \in \mathrm{base}(\mathop{\smile\!\!\!\!\!\!|})$ *and any* κ, *there is a* $\mathop{\smile\!\!\!\!\!\!|}_c$*-independent isi-sequence* $(a_i)_{i<\kappa}$ *over* c *in some extension* $M \to N$ *such that* $\mathrm{gtp}(a_i, c; N) = \mathrm{gtp}(a, c; M)$ *for all* $i < \kappa$.

Proof. We inductively build chains of models $(M_i)_{i<\kappa}$ and $(N_i)_{i<\kappa}$ together with arrows $a_i : A \to M_{i+1}$, where N_0 is an extension of M and c embeds in $(M_i)_{i<\kappa}$, such that:

(i) there is an extension $M_i \to N_i$, and this is natural in the sense that

$$\begin{tikzcd} N_j \arrow[r] & N_i \\ M_j \arrow[u] \arrow[r] & M_i \arrow[u] \end{tikzcd}$$

commutes for all $j < i$;

(ii) $a \mathop{\smile\!\!\!\!\!\!|}_c^{N_i} M_i$;

(iii) for successor $i = j + 1$, we have $\mathrm{gtp}(a_j, m_j; N_i) = \mathrm{gtp}(a, m_j; N_i)$.

<u>*Base case.*</u> By Existence we have $a \mathop{\smile\!\!\!\!\!\!|}_c^M c$, so we can apply Extension to find $M \to N_0$ and $m_0 : M \to N_0$ with $\mathrm{gtp}(m_0, c; N_0) = \mathrm{gtp}(m, c; M)$ and $a \mathop{\smile\!\!\!\!\!\!|}_c^{N_0} m_0$. We take the extension $M_0 \to N_0$ to be m_0.

<u>*Successor step.*</u> We use the induction hypothesis to apply Extension to find and extension $N_i \to N_{i+1}$ and $m_{i+1} : N_i \to N_{i+1}$ such that $a \mathop{\smile\!\!\!\!\!\!|}_c^{N_{i+1}} m_{i+1}$ and $\mathrm{gtp}(m_{i+1}, m_i; N_{i+1}) = \mathrm{gtp}(Id_{N_i}, m_i; N_i)$. We take the extension $M_{i+1} \to N_{i+1}$ to be m_{i+1}. Properties (i) and (ii) follow directly. We had an arrow $a : A \to N_i$, and M_{i+1} is the same object as N_i. So we have an arrow $a_{i+1} : A \to M_{i+1}$. Applying monotonicity of Galois types to $\mathrm{gtp}(m_{i+1}, m_i; N_{i+1}) = \mathrm{gtp}(Id_{N_i}, m_i; N_i)$ then shows that property (iii) holds.

<u>*Limit step.*</u> For limit $\ell < \kappa$ we let $M_\ell = \mathrm{colim}_{i<\ell} M_i$ and $N_\ell = \mathrm{colim}_{i<\ell} N_i$. For every $i < \ell$ we can compose $M_i \to N_i$ with the coprojection $N_i \to N_\ell$. By property (i) this makes N_ℓ into a cocone for $(M_i)_{i<\ell}$. So the universal property gives us an arrow $M_\ell \to N_\ell$, clearly satisfying property (i). Property (ii) follows from Union.

Having finished the inductive construction, we set $N = \mathrm{colim}_{i<\kappa} N_i$. Then property (iii) ensures that $(a_i)_{i<\kappa}$ with chain of initial segments $(M_i)_{i<\kappa}$ is an isi-sequence over c. Since c embeds in $(M_i)_{i<\kappa}$, we also see that

$\operatorname{gtp}(a_i, c; N) = \operatorname{gtp}(a, c; M)$ for all $i < \kappa$. Finally, the $(M_i)_{i<\kappa}$ are witnesses of independence, which follows from combining (ii) and (iii). $\square$

Lemma 4.30. *Let $\downarrow$ be an independence relation in $(\mathcal{C}, \mathcal{M})$ satisfying* Right-Monotonicity *and let $(a_i)_{i<\kappa}$ be a $\downarrow_c$-independent sequence in some M. If κ is such that* $\operatorname{dom}(a_i)$ *(which is the same for all i) and* $\operatorname{dom}(c)$ *are κ-presentable and $(\mathcal{C}, \mathcal{M})$ is a κ-AECat then there is a chain $(M_i)_{i<\kappa}$ of witnesses of independence such that each M_i is κ-presentable.*

Proof. Let $(M'_i)_{i<\kappa}$ be a chain of witnesses of independence. We build a chain of initial segments $(M_i)_{i<\kappa}$ by induction, such that $M_i \leq M'_i$ for all $i < \kappa$. The fact that these are witnesses of independence then follows by Right-Monotonicity.

Base case. We have that $c : C \to M'_0$ factors as $C \to M_0 \to M'_0$ for some κ-presentable M_0, because C is κ-presentable.

Successor step. Having constructed M_i we write $M'_{i+1} = \operatorname{colim}_{j \in J} N_j$ for some κ-directed diagram of κ-presentable objects $(N_j)_{j \in J}$. Then there is $j \in J$ such that a_i, as an arrow into M'_{i+1}, and $M_i \to M'_{i+1}$ both factor through N_j. Set $M_{i+1} = N_j$.

Limit step. For limit ℓ set $M_\ell = \operatorname{colim}_{i<\ell} M_i$. Then by the universal property of the colimit $M_\ell \leq M'_\ell$. $\square$

4.4 Long dividing and isi-dividing

In this section we introduce various notions of dividing, each yielding its own independence relation. These notions are based on the classical notion of dividing, as we know it from full first-order logic. For the convenience of the reader, and to compare it to the new definitions, we recall the classical definition of dividing for full first-order logic.

Definition 4.31. In the setting of full first-order logic, we say that a type $p(x, b) = \operatorname{tp}(ab/C)$ *divides over* C if there is a C-indiscernible sequence $(b_i)_{i<\omega}$ such that $\operatorname{tp}(b_i/C) = \operatorname{tp}(b/C)$ for all $i < \omega$ and $\bigcup_{i<\omega} p(x, b_i)$ is inconsistent.

In many proofs, to use this definition, one has to apply compactness in one way or another. This is generally an issue in AECats, because we do not have compactness there. The following are some places where compactness is useful in combination with Definition 4.31.

1. Finding an indiscernible sequence is often done by constructing some very long sequence and then use some combinatorial tools (such as the

Erdoös-Rado theorem) together with compactness to extract an indiscernible sequence. In fact, finite shortness would be enough here, see Lemma 4.54, but we generally also do not have that.

2. Whenever $(b_i)_{i<\omega}$ is a C-indiscernible sequence that witnesses dividing of $p(x, b)$ there is some $k < \omega$ such that for any $i_1 < \ldots < i_k < \omega$ we have that $p(x, b_{i_1}) \cup \ldots \cup p(x, b_k)$ is inconsistent. This follows directly from compactness and indiscernibility.

3. As a consequence of the previous point we can use compactness to elongate $(b_i)_{i<\omega}$ to $(b_i)_{i<\kappa}$ for any cardinal κ, while keeping the same properties. In fact, finite shortness is again enough, see Lemma 4.53.

To solve these issues we introduce the notion of long dividing. The name is due to the fact that we consider arbitrarily long sequences in the definition, something that we would normally have to use compactness for. Based on that we also introduce a notion of isi-dividing, which uses isi-sequences to have some homogeneity in the sequences involved. Of course, indiscernible sequence would be even more homogeneous (see Example 4.26), but the little bit that isi-sequences offer us turns out to be enough.

Definition 4.32. Let $(\mathcal{C}, \mathcal{M})$ be an AECat with AP and fix some $(a, b, c; M)$.

1. Suppose that we also have some sequence $((b_i)_{i \in I}; M)$ such that $\text{gtp}(b_i, c; M) = \text{gtp}(b, c; M)$ for all $i \in I$. We say that $\text{gtp}(a, b, c; M)$ is *consistent* for $(b_i)_{i \in I}$ if there is an extension $M \to N$ and an arrow $(a'; N)$ such that
$$\text{gtp}(a, b, c; M) = \text{gtp}(a', b_i, c; N)$$
for all $i \in I$. We call a' a *realisation* of $\text{gtp}(a, b, c; M)$ for $(b_i)_{i \in I}$.

   Being inconsistent is the negation of the above. So we say that $\text{gtp}(a, b, c; M)$ is *inconsistent* for $(b_i)_{i \in I}$ if there is no extension of M with a realisation a' of $\text{gtp}(a, b, c; M)$ for $(b_i)_{i \in I}$.

2. We say that $\text{gtp}(a, b, c; M)$ *long divides* over c if there is μ such that for every $\lambda \geq \mu$ there is a sequence $(b_i)_{i<\lambda}$ in some extension $M \to N$ with $\text{gtp}(b_i, c; N) = \text{gtp}(b, c; M)$ for all $i < \lambda$, such that for some $\kappa < \lambda$ and every $I \subseteq \lambda$ with $|I| = \kappa$ we have that $\text{gtp}(a, b, c; M)$ is inconsistent for $(b_i)_{i \in I}$.

3. We say that $\text{gtp}(a, b, c; M)$ *isi-divides* if it long divides with respect to isi-sequences over c. That is, we require the sequence $(b_i)_{i<\lambda}$ to be an isi-sequence over c.

We already discussed how long dividing and isi-dividing are inspired by dividing. In fact, the classical definition we gave of dividing, Definition 4.31, works for positive logic and homogeneous model theory as well (see [Pil00, BY03b] and [BL03]). A natural question would be to ask whether or not long dividing and isi-dividing are actually the same in those settings. We give a partial answer in two propositions below and leave the rest as a question (Question 4.37).

All of the settings mentioned above are finitely short AECats, see Section 3.6. So to be precise we formulate the propositions below in terms of finitely short AECats. Part of the tools and definitions necessary are developed later in Section 4.5, but the reader is welcome to read the propositions as if they were written for a more concrete case such as full first-order logic, and then these tools are standard consequences of compactness.

Definition 4.33. Let $(\mathcal{C}, \mathcal{M})$ be a finitely short AECat with AP. We say that $\text{gtp}(a, b, c; M)$ *divides* over c if there is some extension $M \to N$ an infinite c-indiscernible sequence $(b_i)_{i<\lambda}$ in N, with $\text{gtp}(b_i, c; N) = \text{gtp}(b, c; M)$ for all $i < \lambda$, such that $\text{gtp}(a, b, c; M)$ is inconsistent for $(b_i)_{i<\lambda}$.

Proposition 4.34. *Let $(\mathcal{C}, \mathcal{M})$ be a finitely short AECat with AP. If $\text{gtp}(a, b, c; M)$ divides over c then it isi-divides over c, and hence also long divides over c.*

Proof. Let $(b_i)_{i<\kappa}$ be an infinite c-indiscernible sequence, in some extension $M \to N$, witnessing that $\text{gtp}(a, b, c; M)$ divides over c. The μ in the definition of isi-dividing will be κ^+. By Lemma 4.53 we can elongate the sequence $(b_i)_{i<\kappa}$ to $(b_i)_{i<\lambda}$ for any $\lambda \geq \mu = \kappa^+$. We can inductively construct a chain of initial segments that make $(b_i)_{i<\lambda}$ into an isi-sequence, as is done in Proposition 4.58. Take any $I \subseteq \lambda$ of cardinality κ. Write δ for its order-type, so $\kappa \leq \delta < \kappa^+$. We then have that $\text{gtp}((b_i)_{i<\delta}; N') = \text{gtp}((b_i)_{i\in I}; N')$, by indiscernibility and finite shortness. We thus see that $\text{gtp}(a, b, c; M)$ is inconsistent for $(b_i)_{i\in I}$, because it is inconsistent for $(b_i)_{i<\kappa}$ and hence for $(b_i)_{i<\delta}$. We have thus shown that for every cardinal $\lambda \geq \mu$ there is an isi-sequence $(b_i)_{i<\lambda}$ such that $\text{gtp}(a, b, c; M)$ is inconsistent for every subsequence of cardinality κ and by construction $\kappa < \lambda$, so we conclude that $\text{gtp}(a, b, c; M)$ isi-divides. The final claim follows because isi-dividing implies long dividing, which is direct from the definition. □

Proposition 4.35. *Let $(\mathcal{C}, \mathcal{M})$ be a finitely short AECat with AP and assume the existence of a proper class of Ramsey cardinals. If $\text{gtp}(a, b, c; M)$ long divides over c then it divides over c. Consequently, isi-dividing also implies dividing.*

Proof. Pick a big enough Ramsey cardinal λ and let $(b_i)_{i<\lambda}$ be a sequence in some extension $M \to N$ witnessing long dividing. So there is $\kappa < \lambda$ such that for every $I \subseteq \lambda$ with $|I| \geq \kappa$ we have that $\text{gtp}(a, b, c; M)$ is inconsistent for $(b_i)_{i\in I}$. Because we chose λ big enough we can apply Lemma 4.56 and find $I \subseteq \lambda$ of order-type λ such that $(b_i)_{i \in I}$ is a c-indiscernible subsequence. So $\operatorname{gtp}(a, b, c; M)$ is inconsistent for $(b_i)_{i \in I}$ and we conclude that $\operatorname{gtp}(a, b, c; M)$ divides over c. The final claim follows because isi-dividing implies long dividing, which is direct from the definition. $\square$

Corollary 4.36. *Let $(\mathcal{C}, \mathcal{M})$ be a finitely short AECat with AP and assume the existence of a proper class of Ramsey cardinals. Then dividing, long dividing and isi-dividing all coincide.*

The question that remains is whether the use of large cardinals is necessary.

Question 4.37. Do long dividing and isi-dividing imply dividing in finitely short AECats?

In the full first-order and positive logic setting, dividing in a simple theory will give what we call a simple independence relation (see Example 5.8). Then by our canonicity theorem for simple independence relations, Theorem 5.4, this must coincide with the independence relation given by isi-dividing. So in simple theories dividing and isi-divding will coincide, even without assuming the existence of large cardinals.

The point of introducing these various notions of dividing is that they yield independence relations. We can already prove some basic properties about these independence relations in arbitrary AECats, similar to the basic properties that dividing always has. The first thing we prove is that the dividing notions are invariant under taking different representatives of subobjects (Proposition 4.38), and so it is really a property of the subobjects. This corresponds to the fact that classically dividing is invariant under changing the enumeration of the tuples involved, and it is thus really a property of the sets involved. This then allows for Definition 4.39, where we define an independence relation on subobjects.

Proposition 4.38. *Let $A, B, C \leq M$ be subobjects and let $(a, b, c; M)$ and $(a', b', c'; M)$ be two sets of representatives. Then $\operatorname{gtp}(a, b, c; M)$ long divides over c if and only if $\operatorname{gtp}(a', b', c'; M)$ long divides over c'. The same statement holds for isi-dividing.*

Proof. This comes down to checking all the definitions, which is lengthy to do in detail. However, there is only one trick that we repeatedly use, and that is Proposition 3.16(ii). We recall for the convenience of the reader that this means the following. Suppose we have some $((d_i)_{i \in I}; N)$ and $((d'_i)_{i \in I}; N')$ such that $\operatorname{gtp}((d_i)_{i \in I}; N) = \operatorname{gtp}((d'_i)_{i \in I}; N')$. Write $D_i = \operatorname{dom}(d_i) = \operatorname{dom}(d'_i)$ and suppose we have $f_i : E_i \to D_i$ for every $i \in I$. Then we also have $\operatorname{gtp}((d_i f_i)_{i \in I}; N) = \operatorname{gtp}((d'_i f_i)_{i \in I}; N')$.

To apply this trick we let f, g, h be isomorphisms such that $a' = af$, $b' = bg$ and $c' = ch$. Then, using the above trick, we easily see that for any sequence

$(b_i)_{i<\lambda}$ witnessing long dividing of $\operatorname{gtp}(a, b, c; M)$ we have that $(b_i g)_{i<\lambda}$ witnesses long dividing for $\operatorname{gtp}(a, bg, c; M) = \operatorname{gtp}(a, b', c; M)$. Similarly we can replace c by $ch = c'$ and a by $af = a'$. The same holds for isi-dividing, noting that any isi-sequence over c is also an isi-sequence over $ch = c'$. □

Definition 4.39. For subobjects $A, B, C \leq M$ we write $A \mathop{\smile\!\!\!\!\!\!\!\mid}_C^{\mathrm{ld},M} B$ if $\operatorname{gtp}(a, b, c; M)$ does not long divide for all (equivalently: some) representatives a, b, c of A, B, C. Similarly, we write $A \mathop{\smile\!\!\!\!\!\!\!\mid}_C^{\text{isi-d},M} B$ if $\operatorname{gtp}(a, b, c; M)$ does not isi-divide.

Proposition 4.40. *Long dividing and isi-dividing always satisfy the following properties:* Invariance, Left-Monotonicity, Existence *and* Base-Monotonicity. *In addition, long dividing also satisfies* Right-Monotonicity

Proof. Everything is direct from the definition, except for Right-Monotonicity for long dividing and Base-Monotonicity. For both we will prove the contrapositive.

For Base-Monotonicity let $(a, b, c, c'; M)$ be such that $\operatorname{gtp}(a, b, c'; M)$ long divides over c' and $C \leq C' \leq B$, where C, C', B are the subobjects represented by c, c', b respectively. Let μ be as in the definition of long dividing and let $\lambda \geq \mu$. Then there is $(b_i)_{i<\lambda}$ in some N that witnesses long dividing of $\operatorname{gtp}(a, b, c'; M)$ over c'. We will prove that it also witnesses long dividing of $\operatorname{gtp}(a, b, c; M)$. Indeed we have for all $i < \lambda$ that $\operatorname{gtp}(b_i, c'; N) = \operatorname{gtp}(b, c'; M)$ and thus $\operatorname{gtp}(b_i, c; N) = \operatorname{gtp}(b, c; M)$, because $C \leq C'$. Let $\kappa < \lambda$ be such that for $I \subseteq \lambda$ with $|I| = \kappa$ we have that $\operatorname{gtp}(a, b, c'; M)$ is inconsistent for $(b_i)_{i \in I}$. We claim that for such I we also have that $\operatorname{gtp}(a, b, c; M)$ is inconsistent for $(b_i)_{i \in I}$. Suppose that there would be a realisation $(a'; N')$ for some extension $N \to N'$, then $\operatorname{gtp}(a', b_i, c; N') = \operatorname{gtp}(a, b, c; M)$ for all $i \in I$. Since c' factors through b and b_i in the same way for all $i < \lambda$, we then have $\operatorname{gtp}(a', b_i, c'; N') = \operatorname{gtp}(a, b, c'; M)$ for all $i \in I$, contradicting that $\operatorname{gtp}(a, b, c'; M)$ is inconsistent for $(b_i)_{i \in I}$. This proves Base-Monotonicity for long dividing. We have shown that the same sequences that witness long dividing of $\operatorname{gtp}(a, b, c'; M)$ also witness long dividing of $\operatorname{gtp}(a, b, c; M)$. As any isi-sequence over c' is an isi-sequence over c, the same proof shows that isi-dividing has Base-Monotonicity.

Now we prove Right-Monotonicity for long dividing. Let $(a, b, b', c; M)$ be such that $\operatorname{gtp}(a, b, c; M)$ long divides over c and b factors through b'. For any sequence $(b_i)_{i<\lambda}$ in some N witnessing long dividing we can form $(b'_i)_{i<\lambda}$ by letting b'_i be such that $\operatorname{gtp}(b'_i, b_i, c; N) = \operatorname{gtp}(b', b, c; M)$ for all $i < \lambda$ (possibly replacing N by an extension in the process). Then for $I \subseteq \lambda$ a realisation of $\operatorname{gtp}(a, b', c; M)$ for $(b'_i)_{i \in I}$ would also be a realisation of $\operatorname{gtp}(a, b, c; M)$ for $(b_i)_{i \in I}$. So if we let $\kappa < \lambda$ be such that every $I \subseteq \lambda$ with $|I| = \kappa$ we have that $\operatorname{gtp}(a, b, c; M)$ is inconsistent for $(b_i)_{i \in I}$, we also get that $\mathrm{gtp}(a, b', c; M)$ is inconsistent for $(b'_i)_{i \in I}$ for any such I. We conclude that $\mathrm{gtp}(a, b', c; M)$ long divides over c. □

We note that in the above proof we did not have to change the sequence involved for BASE-MONOTONICITY, which was why the proof directly works for isi-dividing as well. In the proof of RIGHT-MONOTONICITY we had to build a new sequence, which might not be an isi-sequence again. This is why that proof only works for long dividing.

An important property that misses in Proposition 4.40 is EXTENSION. Classically this is fixed by considering forking instead of dividing. Basically this forces the EXTENSION property as follows. Suppose that EXTENSION is fails for some type $p = \mathrm{tp}(a/Cb)$. Then there is some set, say D, such that every extension of p to D divides over C. In other words, p implies a disjunction of types over D such that every type in that disjunction divides over C. Classically we could even further reformulate this by using compactness and having p actually imply a disjunction of dividing formulas, but that is not necessary and we want to avoid compactness in our definitions. So our definition of isi-forking will be the semantical way of saying "implies a (possibly infinite) disjunction of types that each isi-divide".

Definition 4.41. We say that $\mathrm{gtp}(a, b, c; M)$ *isi-forks* over c if there is some extension $M \to N$ with $((a_j)_{j \in J}, (d_j)_{j \in J}; N)$ such that:

(i) $\mathrm{gtp}(a_j, d_j, c; N)$ isi-divides over c for each $j \in J$;

(ii) given an extension $N \to N'$ with some $(a'; N')$ such that $\mathrm{gtp}(a', b, c; N') = \mathrm{gtp}(a, b, c; N)$ there is $j \in J$ such that $\mathrm{gtp}(a', d_j, c; N') = \mathrm{gtp}(a_j, d_j, c; N)$.

Note that we do not require that b actually factors through the d_j (i.e. as subobjects they do not have to extend each other). This is because we also want to force in RIGHT-MONOTONICITY, which we now get for free.

Of course, one could also define a notion of *long forking* by replacing isi-dividing by long dividing in the above. However, we will have no use for this.

Remark 4.42. The definition of isi-forking is just the semantical way of saying "$\mathrm{gtp}(a, b, c; M)$ implies a (possibly infinite) disjunction of Galois types that each isi-divide over c". In the full first-order setting and in the positive setting (see [Pil00]) forking has been defined and can be formulated as follows: a type forks over C if it implies a (possibly infinite) disjunction of types that each divide over C. It should then be clear that forking implies isi-forking. This uses the fact that dividing implies isi-dividing, see Proposition 4.34. If isi-dividing and dividing coincide then the converse is true, so isi-forking would then imply forking. This happens if there is a proper class of Ramsey cardinals, or in simple theories already without the large cardinal assumption, see Proposition 4.35 and the discussion after Question 4.37.

As before, we will prove various basic properties of isi-forking. Again we start with the fact that isi-forking is invariant under taking different representatives of the same subobject, so that we can again define an independence relation based on isi-forking (Definition 4.44).

Proposition 4.43. *Let $A, B, C \leq M$ be subobjects and let $(a, b, c; M)$ and $(a', b', c'; M)$ be two sets of representatives. Then* $\mathrm{gtp}(a, b, c; M)$ *isi-forks over c if and only if* $\mathrm{gtp}(a', b', c'; M)$ *isi-forks over c'.*

Proof. We use the same trick as we did in Proposition 4.38. Let f be the isomorphism such that $a' = af$. Let $M \to N$ be an extension with $((a_j)_{j \in J}, (d_j)_{j \in J}; N)$ witnessing isi-forking of $\mathrm{gtp}(a, b, c; M)$. Then using Proposition 4.38 we have that $\mathrm{gtp}(a_j f, d_j, c'; N)$ isi-divides over c' for all $j \in J$. We claim that $((a_j f)_{j \in J}, (d_j)_{j \in J}; N)$ witnesses isi-forking of $\mathrm{gtp}(a', b', c'; M)$, for which we are now left to check (ii) from Definition 4.41.

Let g and h be isomorphisms such that $b' = bg$ and $c' = ch$. Let $N \to N^*$ be an extension with some $(a^*; N^*)$ such that $\mathrm{gtp}(a^*, b', c'; N^*) = \mathrm{gtp}(a', b', c'; N)$. Then

$$\begin{aligned} \mathrm{gtp}(a^* f^{-1}, b, c; N^*) &= \mathrm{gtp}(a^* f^{-1}, b' g^{-1}, c' h^{-1}; N^*) \\ &= \mathrm{gtp}(a' f^{-1}, b' g^{-1}, c' h^{-1}; N) \\ &= \mathrm{gtp}(a, b, c; N), \end{aligned}$$

so there is $j \in J$ with $\mathrm{gtp}(a^* f^{-1}, d_j, c; N^*) = \mathrm{gtp}(a_j, d_j, c; N)$. Hence $\mathrm{gtp}(a^*, d_j, c'; N^*) = \mathrm{gtp}(a^* f^{-1} f, d_j, ch; N^*) = \mathrm{gtp}(a_j f, d_j, c'; N)$. □

Definition 4.44. For subobjects $A, B, C \leq M$ we write $A \mathop{\smile\!\!\!\!\!\!|}_{C}^{\text{isi-f},M} B$ if $\mathrm{gtp}(a, b, c; M)$ does not isi-fork for all (equivalently: some) representatives a, b, c of A, B, C.

Proposition 4.45. *Isi-forking satisfies the following properties:* Invariance, Monotonicity *on both sides,* Extension *and* Base-Monotonicity.

Proof. The properties Invariance and Right-Monotonicity are direct from the definition. We prove the contrapositive of the remaining three.

For Left-Monotonicity suppose that $\mathrm{gtp}(a', b, c; M)$ isi-forks over c and let $(a; M)$ be such that a' factors through a. Let $((a'_j)_{j \in J}, (d_j)_{j \in J}; N)$ in some extension $M \to N$ witness the isi-forking. Let f be such that $af = a'$. The following is a set by Proposition 3.19:

$$\begin{aligned} F = \{\mathrm{gtp}(a^*, d_j, c; N^*) : {}& N^* \text{ is an extension of } N \text{ and} \\ & \mathrm{gtp}(a^*, b, c; N^*) = \mathrm{gtp}(a, b, c; M) \text{ and} \\ & \mathrm{gtp}(a^* f, d_j, c; N^*) = \mathrm{gtp}(a'_j, d_j, c; N)\}. \end{aligned}$$

By LEFT-MONOTONICITY of isi-dividing, every Galois type in F isi-divides over c. By inductively amalgamating things we find one extension $N \to N^*$ with $((a_k)_{k\in K}, (d_k)_{k\in K}; N^*)$ such that every Galois type in F is realised by $(a_k, d_k, c; N^*)$ for some $k \in K$. This then witnesses isi-forking of $\text{gtp}(a, b, c; M)$ over c.

For EXTENSION let $(a, b, b', c; M)$ be such that b factors through b' and for every $(a'; N)$ in some extension $M \to N$ with $\text{gtp}(a', b, c; N) = \text{gtp}(a, b, c; M)$ we have that $\text{gtp}(a', b', c; N)$ isi-forks over c. So the conclusion of the EXTENSION property for $\text{gtp}(a, b, c; M)$ fails. We have to prove that then $\text{gtp}(a, b, c; M)$ isi-forks over c. By Proposition 3.19 and the definition of isi-forking, the following is a set:

$$\begin{aligned} F = \{\, &\text{gtp}(a', d, c; N) : N \text{ is an extension of } M \text{ and} \\ &\quad \text{gtp}(a', b, c; N) = \text{gtp}(a, b, c; M) \text{ and} \\ &\quad \text{gtp}(a', d, c; N) \text{ is one of the witnesses of isi-forking of } \text{gtp}(a', b', c; N)\}. \end{aligned}$$

By inductively amalgamating things we find one extension $N \to N^*$ with $((a_j)_{j\in J}, (d_j)_{j\in J}; N^*)$ such that every Galois type in F is realised by $(a_j, d_j, c; N^*)$ for some $j \in J$. This then witnesses isi-forking of $\text{gtp}(a, b, c; M)$ over c.

Finally, for BASE-MONOTONICITY let $(a, b, c, c'; M)$ be such that $\text{gtp}(a, b, c'; M)$ isi-forks over c' and $C \leq C' \leq B$, where C, C', B are the subobjects represented by c, c', b respectively. Let $((a_j)_{j\in J}, (d_j)_{j\in J}; N)$ witness this in some extension $M \to N$. We claim that this also witnesses isi-forking of $\text{gtp}(a, b, c; M)$ over c. Indeed, let $a' : A \to N'$ for some extension $N \to N'$ be such that $\text{gtp}(a', b, c; N') = \text{gtp}(a, b, c; N)$. We have $C' \leq B$, so $\text{gtp}(a', b, c'; N') = \text{gtp}(a, b, c'; N)$. So there must be some $j \in J$ such that $\text{gtp}(a', d_j, c'; N') = \text{gtp}(a_j, d_j, c'; N)$. As $C \leq C'$ this restricts to $\text{gtp}(a', d_j, c; N') = \text{gtp}(a_j, d_j, c; N)$, which concludes the proof. $\square$

Proposition 4.46. *For any $A, B, C \leq M$ we always have*

$$A \underset{C}{\overset{\text{isi-f},M}{\mathop{\perp\!\!\!\!\perp}}} B \implies A \underset{C}{\overset{\text{isi-d},M}{\mathop{\perp\!\!\!\!\perp}}} B.$$

The converse holds if and only if isi-dividing satisfies RIGHT-MONOTONICITY *and* EXTENSION.

Proof. The first implication is just the contrapositive of the trivial statement that isi-dividing implies isi-forking. If the converse of this implication holds, then isi-dividing and isi-forking coincide and so isi-dividing satisfies RIGHT-MONOTONICITY and EXTENSION by Proposition 4.45.

We are left to prove that if isi-dividing satisfies RIGHT-MONOTONICITY and

EXTENSION that then isi-forking implies isi-dividing. Suppose for a contradiction that $\mathrm{gtp}(a, b, c; M)$ isi-forks over c but does not isi-divide over c. Let $((a_j)_{j \in J}, (d_j)_{j \in J}; N)$ for some extension $M \to N$ witness the isi-forking of $\mathrm{gtp}(a, b, c; M)$. By EXTENSION for isi-dividing we find an extension $N \to N'$ with $(a'; N')$ such that $\mathrm{gtp}(a', b, c; N') = \mathrm{gtp}(a, b, c; N)$ and $\mathrm{gtp}(a', N, c; N')$ does not isi-divide. By isi-forking, there must be $j \in J$ such that $\mathrm{gtp}(a', d_j, c; N')$ isi-divides over c contradicting RIGHT-MONOTONICITY of isi-dividing. □

When considering NSOP_1-theories in full first-order logic (and in positive logic, see chapter 6) the useful notion of independence is given by Kim-dividing, see for example [KR20]. The idea is to only consider dividing with respect to Morley sequences, that is, with respect to indiscernible nonforking sequences. We adapt that definition to our earlier ideas as follows.

Definition 4.47. We say that $\mathrm{gtp}(a, b, c; M)$ *long Kim-divides* over c if it long divides over c with respect to $\mathop{\smile\!\!\!\!\!\mid}_c^{\text{isi-f}}$-independent sequences. That is, the definition is exactly as long dividing, but we require the sequence $(b_i)_{i<\lambda}$ to be $\mathop{\smile\!\!\!\!\!\mid}_c^{\text{isi-f}}$-independent. We write $A \mathop{\smile\!\!\!\!\!\mid}_C^{\text{lK},M} B$ if $\mathrm{gtp}(a, b, c; M)$ does not long Kim-divide over c for all (equivalently: some) representatives a, b, c of the subobjects A, B, C.

We implicitly used a combination of Proposition 4.38 and Proposition 4.43 to conclude that long Kim-dividing is invariant under taking different representatives of subobjects.

We have defined $\mathop{\smile\!\!\!\!\!\mid}_C^{\text{lK}}$-independence using $\mathop{\smile\!\!\!\!\!\mid}_C^{\text{isi-f}}$-independent sequences, but these may not exist. For this we define the following axiom, from which the existence of such sequences follows.

Definition 4.48. Let $(\mathcal{C}, \mathcal{M})$ be an AECat and let $\mathcal{B}$ be a base class. We say that $(\mathcal{C}, \mathcal{M})$ satisfies the *$\mathcal{B}$-existence axiom* if $\mathop{\smile\!\!\!\!\!\mid}^{\text{isi-f}}$ with its base restricted to $\mathcal{B}$ satisfies EXISTENCE. That is, for all $A, C \leq M$ with $C \in \mathcal{B}$ we have $A \mathop{\smile\!\!\!\!\!\mid}_C^{\text{isi-f},M} C$.

Corollary 4.49. *If $(\mathcal{C}, \mathcal{M})$ satisfies the $\mathcal{B}$-existence axiom then for any $(a, c; M)$ with $\mathrm{dom}(c) \in \mathcal{B}$ and any κ there is some extension $M \to N$ containing a $\mathop{\smile\!\!\!\!\!\mid}_c^{\text{isi-f}}$-independent sequence $(a_i)_{i<\kappa}$ with $\mathrm{gtp}(a_i, c; N) = \mathrm{gtp}(a, c; M)$.*

Proof. Combine Proposition 4.28 and Proposition 4.45. □

Example 4.50. We discuss some examples of the $\mathcal{B}$-existence axiom. These are either settings where we have the axiom, or where it is natural to assume the axiom.

(i) If T is a semi-Hausdorff positive theory (this includes theories in full first-order logic) then any type over an e.c. model M can be extended to a global M-invariant type, see [BY03c, Lemma 3.11]. This can be used in a standard argument to show that such types do not fork over M, see for example Proposition 6.17. We can use the same technique to show that such types do not isi-fork over M. So for any e.c. model M we have that any $\mathrm{tp}(a/M)$ does not isi-fork over M. We thus have the $\mathbf{Mod}(T)$-existence axiom.

In fact, we can also make this argument work if we just assume T to be thick. In that case we can extend $\mathrm{tp}(a/M)$ to what we call a global M-Ls-invariant type, see Lemma 6.13. This can be used in a similar way to show non-isi-forking. This is done in Lemma 6.91, which is stated in terms of long dividing but also works for isi-dividing and isi-forking.

(ii) Analogous to the previous point, for a continuous theory T we have the $\mathbf{MetMod}(T)$-existence axiom.

(iii) For an NSOP_1 theory T in full first-order logic it is common to assume the existence axiom for forking. It is still an open problem whether or not the existence axiom for forking holds in every NSOP_1 theory T, but it has been proved in many particular instances, see [DKR19, Fact 2.14].

If for such T we take $(\mathcal{C}, \mathcal{M}) = (\mathbf{SubMod}(T), \mathbf{Mod}(T))$ then we are very close to having the $\mathcal{C}$-existence axiom. The only difference is that we work with isi-forking, see Remark 4.42 for a comparison. In particular, the $\mathcal{C}$-existence axiom implies the existence axiom for forking. Furthermore, if there is a proper class of Ramsey cardinals then isi-forking and forking coincide and so the converse would hold as well. Additionally, it is quite likely that techniques to prove existence for forking also work for isi-forking. For example, in [DKR19, Remark 2.15] it is shown that in the theory of parametrised equivalence relations any type over any set A can be extended to a global A-invariant type. Following point (i) we then see that such a type does not isi-fork over A.

(iv) If $(\mathcal{C}, \mathcal{M})$ is an AECat with a simple independence relation $\mathop{\smile\!\!\!\!\!\mid}$ then it will satisfy the base($\mathop{\smile\!\!\!\!\!\mid}$)-existence axiom. This follows from canonicity, Theorem 5.4, because then $\mathop{\smile\!\!\!\!\!\mid} = \mathop{\smile\!\!\!\!\!\mid}^{\text{isi-f}}$ over base($\mathop{\smile\!\!\!\!\!\mid}$). This mirrors the fact that simple theories in full first-order logic (and even simple thick positive theories, see [BY03c]) have the existence axiom for forking, see also the previous point.

Remark 4.51. The usual definition of Kim-dividing states that a type Kim-divides if it divides with respect to non-forking Morley sequences, see e.g. [DKR19]. To compare this to long Kim-dividing we first note that by Remark 4.42 any $\mathop{\smile\!\!\!\!\!\mid}^{\text{isi-f}}$-independent sequence is also a forking-independent sequence, and the converse is true if isi-dividing coincides with dividing. As before, if we assume that there is a proper class of Ramsey cardinals then long

Kim-dividing and Kim-dividing coincide, using the same arguments as in Proposition 4.34 and Proposition 4.35.

If we do not want to assume large cardinals then we can again use canonicity, this time Theorem 5.6, to see that long Kim-dividing and Kim-dividing coincide in NSOP_1-theories where it has been developed. See Example 5.9 for a more detailed discussion.

4.5 Indiscernible sequences in finitely short AECats

In this section we will continue exploring the connections of AECats to existing frameworks, as we did in section 3.6. We mention again that nothing in this section is needed elsewhere in this thesis.

To be more precise, in this section we will show that we can create and manipulate indiscernible sequences in finitely short AECats. We close out by proving that any indiscernible sequence can be made into an isi-sequence.

Nothing in this section is really a new insight. We just verify that the standard constructions and arguments go through in finitely short AECats.

Definition 4.52. Let $(\mathcal{C}, \mathcal{M})$ be a finitely short AECat with AP and let $((c_k)_{k \in K}; M)$ be a tuple of arrows into a model. Then we say that a sequence $(a_i)_{i \in I}$ in M is $(c_k)_{k \in K}$-*indiscernible* if for all $i_1 < \ldots < i_n$ and $j_1 < \ldots < j_n$ in I we have:

$$\mathrm{gtp}(a_{i_1}, \ldots, a_{i_n}, (c_k)_{k \in K}; M) = \mathrm{gtp}(a_{j_1}, \ldots, a_{j_n}, (c_k)_{k \in K}; M),$$

The following is just a standard compactness argument for extending indiscernible sequences.

Lemma 4.53. *Let $(\mathcal{C}, \mathcal{M})$ be a finitely short AECat with AP, and let $(a_i)_{i \in I}$ be an infinite $(c_k)_{k \in K}$-indiscernible sequence in M. Then given any linear order $J \supseteq I$, there is an extension $M \to N$ and a $(c_k)_{k \in K}$-indiscernible sequence $(a_j)_{j \in J}$ in N extending $(a_i)_{i \in I}$.*

Proof. We will use directed compactness for Galois types to construct $(a_j)_{j \in J}$. Let A be the common domain of the $(a_i)_{i \in I}$. We construct a system of finitary satisfiability for $(C_k)_{k \in K}$ together with a copy A_j of A for each $j \in J$.

Every interpretation will be in M, and we always interpret C_k as c_k. For finite $J_0 \subseteq J$, we enumerate J_0 as $j_1 < \ldots < j_n$ and fix some $i_1 < \ldots < i_n$ in I. Then we let the interpretations $a_{J_0, j_1}, \ldots, a_{J_0, j_n}$ be $a_{i_1}, \ldots, a_{i_n}$ respectively. It follows from $(c_k)_{k \in K}$-indiscernibility of $(a_i)_{i \in I}$ that this indeed forms a system of finitary satisfiability.

Applying compactness for Galois types, we find a realisation $(a'_j)_{j \in J}$ and $(c'_k)_{k \in K}$ in some extension $M \to N$. By construction of our system of finitary satisfiability and because $(\mathcal{C}, \mathcal{M})$ is finitely short, $(a'_j)_{j \in J}$ is $(c'_k)_{k \in K}$-indiscernbile. Furthermore:

$$\mathrm{gtp}((a'_i)_{i \in I}, (c'_k)_{k \in K}; N) = \mathrm{gtp}((a_i)_{i \in I}, (c_k)_{k \in K}; M).$$

So we may indeed assume that c'_k is just c_k (composed with the extension to N), for each $k \in K$, and that $(a'_j)_{j \in J}$ is an extension of $(a_i)_{i \in I}$. □

The following lemmas (Lemma 4.54 and Lemma 4.56) show that we can find an indiscernible sequence based on a long enough sequence. The first one is a standard argument and appears for example in [BY03a, BL03, Pil00]. The second one is stronger and much shorter, but assumes the existence of large cardinals (this trick is also well-known).

Lemma 4.54. *Let $(\mathcal{C}, \mathcal{M})$ be a finitely short AECat with AP, and A and $(C_k)_{k \in K}$ be objects. Then there is λ (depending on those objects) such that the following holds. Given a sequence $(a_i)_{i \in I}$ in some M and arrows $\{c_k : C_k \to M\}_{k \in K}$, with $|I| \geq \lambda$ and $\mathrm{dom}(a_i) = A$ for all $i \in I$, there is a $(c_k)_{k \in K}$-indiscernible sequence $(a'_i)_{i < \omega}$ in some extension $M \to N$ such that for all $n < \omega$ there are $i_1 < \ldots < i_n$ in I with*

$$\mathrm{gtp}(a'_1, \ldots, a'_n, (c_k)_{k \in K}; N) = \mathrm{gtp}(a_{i_1}, \ldots, a_{i_n}, (c_k)_{k \in K}; M).$$

The proof of Lemma 4.54 relies on a combination of compactness and the Erdős-Rado theorem (see e.g. [Jec03, Theorem 9.6]).

Theorem 4.55 (Erdős-Rado)**.** *For all infinite cardinals μ we have*

$$\beth_n^+(\mu) \to (\mu^+)_\mu^{n+1}.$$

Recall that the notation $\kappa \to (\lambda)_\mu^n$ means that for every function $f : [\kappa]^n \to \mu$ we can find a subset $X \subseteq \kappa$ with $|X| = \lambda$ such that f is constant on $[X]^n$.

Proof of Lemma 4.54. Let τ be such that $|\mathrm{S}_{\mathrm{gtp}}(A^n, (C_k)_{k \in K})| < \tau$ for all $n < \omega$, where A^n denotes n copies of A. Take λ to be $\beth_{\tau^+}$. Then λ has the following properties:

(i) $\mathrm{cf}(\lambda) > \tau$;

(ii) for all $\kappa < \lambda$ and $n < \omega$, there is some $\kappa' < \lambda$ such that $\kappa' \to (\kappa)_\tau^n$.

Property (i) should be clear, and (ii) follows from the Erdős-Rado theorem.

Let $(a_i)_{i \in I}$ in M and $((c_k)_{k \in K}; M)$ be as in the statement. By induction we will build $I_n \subseteq I$, for all $n < \omega$, such that

(1) $|I_n| = n$;

(2) for all $m \leq n$ and $I'_n \subseteq I_n$ with $|I'_n| = m$ we have:

$$\mathrm{gtp}((a_i)_{i \in I'_n}, (c_k)_{k \in K}; M) = \mathrm{gtp}((a_i)_{i \in I_m}, (c_k)_{k \in K}; M);$$

(3) for all $\kappa < \lambda$ there is some $I' \subseteq I$ with $|I'| = \kappa$ such that for any $I'' \subseteq I'$ of size n we have

$$\mathrm{gtp}((a_i)_{i \in I_n}, (c_k)_{k \in K}; M) = \mathrm{gtp}((a_i)_{i \in I''}, (c_k)_{k \in K}; M).$$

The base case, where $n = 0$ is easy. We just take $I_0 = \emptyset$. Property (2) is vacuous and (3) becomes trivial.

So suppose we have constructed I_n, we will construct I_{n+1}. Let $\kappa < \lambda$ be arbitrary. Then by property (ii) of λ, there is $\kappa' < \lambda$ such that $\kappa' \to (\kappa)^{n+1}_\tau$. Property (3) from the induction hypothesis gives us $I' \subseteq I$ with $|I'| = \kappa'$, such that for all $I'' \subseteq I'$ of size n we have

$$\mathrm{gtp}((a_i)_{i \in I_n}, (c_k)_{k \in K}; M) = \mathrm{gtp}((a_i)_{i \in I''}, (c_k)_{k \in K}; M).$$

We define $f : [I']^{n+1} \to \mathrm{S}_{\mathrm{gtp}}(A^{n+1}, (C_k)_{k \in K})$ by $f(J) = \mathrm{gtp}((a_i)_{i \in J}, (c_k)_{k \in K}; M)$. From how we chose I' we find a subset $I_\kappa \subseteq I' \subseteq I$ with $|I_\kappa| = \kappa$ and such that for all $J, J' \subseteq I_\kappa$ of size $n + 1$ we have

$$\mathrm{gtp}((a_i)_{i \in J}, (c_k)_{k \in K}; M) = \mathrm{gtp}((a_i)_{i \in J'}, (c_k)_{k \in K}; M).$$

So we can associate a single Galois type in $n + 1$ copies of A and $(c_k)_{k \in K}$ to I_κ.

Since $\kappa < \lambda$ was arbitrary, we can construct such I_κ for all $\kappa < \lambda$. By property (i) of λ there must be cofinally many κ that are associated to the same Galois type. We will take I_{n+1} to be any subset of size $n + 1$ of such an I_κ. More precisely, let K be this cofinal subset of λ. Pick any $\kappa^* \in K$ and let I_{n+1} be any subset of I_{κ^*} of size $n + 1$. Property (1) then holds by construction, and (3) follows from K being cofinal, so we check (2).

We constructed I_{κ^*} as the subset of some $I' \subseteq I$, where any $I'' \subseteq I'$ of size n satisfies

$$\mathrm{gtp}((a_i)_{i \in I_n}, (c_k)_{k \in K}; M) = \mathrm{gtp}((a_i)_{i \in I''}, (c_k)_{k \in K}; M).$$

So in particular, this is true for any $I'' \subseteq I_{n+1}$ of size n. Then the statement for all $m \leq n$ follows from the induction hypothesis for I_n, and by restriction of Galois types. This proves property (2).

This finishes the inductive construction of the I_n. We now claim that we can use the I_n to form a system of finitary satisfiability. We consider the tuple $(A_n)_{n<\omega}$ where $A_n = A$ for all $n < \omega$, together with $(C_k)_{k \in K}$. We will interpret everything in M and we will always interpret C_k as c_k for $k \in K$. Then for any finite $J \subseteq \omega$ we let $n = |J|$, and we interpret $(A_j)_{j \in J}$ as $(a_i)_{i \in I_n}$. Property (2) from the induction hypothesis then guarantees that this indeed is a system of finitary satisfiability.

Using directed compactness for Galois types, we find a realisation $(a'_i)_{i<\omega}$ in some extension $M \to N$. By Remark 3.39 we may assume the realisation of $(C_k)_{k \in K}$ to be $(c_k)_{k \in K}$. Then for any finite $J_0, J_1 \subset \omega$ of size n, we have

$$\mathrm{gtp}((a'_i)_{i \in J_0}, (c_k)_{k \in K}; N) = \mathrm{gtp}((a_i)_{i \in I_n}, (c_k)_{k \in K}; M) = \mathrm{gtp}((a'_i)_{i \in J_1}, (c_k)_{k \in K}; N),$$

which proves both the claim about the existence of $i_1 < \ldots < i_n$ (take $J_0 = \{1, \ldots, n\}$ and let $i_1 < \ldots < i_n$ enumerate I_n) and indiscernibility over $(c_k)_{k \in K}$. $\square$

We recall that a *Ramsey cardinal* is a cardinal λ such that $\lambda \to (\lambda)^{<\omega}_\kappa$ for all $\kappa < \lambda$. That is, for every function $f : [\lambda]^{<\omega} \to \kappa$, there is some subset $X \subseteq \lambda$ with $|X| = \lambda$ such that for every $n < \omega$ we have that f is constant on $[X]^n$.

Lemma 4.56. *Let $(\mathcal{C}, \mathcal{M})$ be an AECat with AP. Let A and $(C_k)_{k \in K}$ be objects and suppose that there is a Ramsey cardinal λ such that $|\mathrm{S}_{\mathrm{gtp}}(A^\omega, (C_k)_{k \in K})| < \lambda$, where A^ω denotes ω many copies of A. Then given a sequence $(a_i)_{i<\lambda}$ in some M there is a subset $I \subseteq \lambda$ with $|I| = \lambda$ such that the subsequence $(a_i)_{i \in I}$ is $(c_k)_{k \in K}$-indiscernible.*

Proof. We define

$$\begin{aligned} f : [\lambda]^{<\omega} &\to \bigcup_{n<\omega} \mathrm{S}_{\mathrm{gtp}}(A^n, (C_k)_{k \in K}), \\ I_0 &\mapsto \mathrm{gtp}((a_i)_{i \in I_0}, (c_k)_{k \in K}; M). \end{aligned}$$

Then because λ is a Ramsey cardinal, we find $I \subseteq \lambda$ with $|I| = \lambda$ such that f is constant on $[I]^n$ for each $n < \omega$. Then by the definition of f, we have that $(a_i)_{i \in I}$ is $(c_k)_{k \in K}$-indiscernible. $\square$

By combining Lemma 4.53 and Lemma 4.54 we can make any indiscernible sequence indiscernible over a parameter set of any desired shape.

Corollary 4.57. *Let $(\mathcal{C}, \mathcal{M})$ be a finitely short AECat with AP. Let $(a_i)_{i<\kappa}$ be some $(c_k)_{k \in K}$-indiscernible sequence in M and let $d : D \to M$ be any arrow. Then there is an extension $M \to N$ and some arrow $d' : D \to N$ such that $(a_i)_{i<\kappa}$ is d'-indiscernible in N and $\mathrm{gtp}(d', (c_k)_{k \in K}; N) = \mathrm{gtp}(d, (c_k)_{k \in K}; M)$.*

Proof. By Lemma 4.53 we may assume κ to be big enough to apply Lemma 4.54. This then yields an extension $M \to N'$ with a $d(c_k)_{k \in K}$-indiscernible sequence $(a'_i)_{i<\omega}$ such that for all $n < \omega$ there are $i_1 < \ldots < i_n$ in I with

$$\mathrm{gtp}(a'_1, \ldots, a'_n, (c_k)_{k \in K}; N') = \mathrm{gtp}(a_{i_1}, \ldots, a_{i_n}, (c_k)_{k \in K}; M).$$

By Lemma 4.53 we can elongate $(a'_i)_{i<\omega}$ to a $d(c_k)_{k\in K}$-indiscernible sequence $(a'_i)_{i<\kappa}$ in some extension $N' \to N''$. In particular this sequence is $(c_k)_{k\in K}$-indiscernible and all finite subtuples of $(a_i)_{i<\kappa}$ and $(a'_i)_{i<\kappa}$ have the same Galois type over $(c_k)_{k\in K}$. So finite shortness yields

$$\mathrm{gtp}((a_i)_{i<\kappa}, (c_k)_{k\in K}; M) = \mathrm{gtp}((a'_i)_{i<\kappa}, (c_k)_{k\in K}; N'').$$

Using Proposition 3.16(iii) we then find an extension $N'' \to N$ and $d' : D \to N$ such that

$$\mathrm{gtp}(d', (a_i)_{i<\kappa}, (c_k)_{k\in K}; N) = \mathrm{gtp}(d, (a'_i)_{i<\kappa}, (c_k)_{k\in K}; N''),$$

which is then clearly the d' we had to construct. □

We close out by showing that any indiscernible sequence can be made into an isi-sequence. Basically what happens is that we inductively apply Corollary 4.57 to build a chain of initial segments. Such that each time the tail is indiscernible over the relevant link in the chain.

Proposition 4.58. *Let $(\mathcal{C}, \mathcal{M})$ be a finitely short AECat with AP. Then given a c-indiscernible sequence $(a_i)_{i<\kappa}$ in M, there is an extension $M \to N$ and a chain of initial segments $(M_i)_{i<\kappa}$ in N witnessing that $(a_i)_{i<\kappa}$ is an isi-sequence over c.*

Proof. We will use transfinite induction to construct chains of models $(M_i)_{i<\kappa}$ and $(N_i)_{i<\kappa}$, with the following induction hypothesis:

(i) there is an extension $M \to N_0$;

(ii) we have an extension $m_i : M_i \to N_i$, and these are natural in the sense that for all $j \leq i$, the square

$$\begin{array}{ccc} N_j & \longrightarrow & N_i \\ {\scriptstyle m_j}\uparrow & & \uparrow{\scriptstyle m_i} \\ M_j & \longrightarrow & M_i \end{array}$$

of extensions commutes;

(iii) for successor $i = j+1$, the arrow a_j (as an arrow into N_i) factors through M_i;

(iv) the tail segment $(a_j)_{i\leq j<\kappa}$ in N_i is m_i-indiscernible.

Once we have constructed such chains, we can take $N = \mathrm{colim}_{i<\kappa} N_i$. Then this gives us the required extension $M \to N$ and chain of initial segments $(M_i)_{i<\kappa}$ (we check this in more detail at the end of the proof).

Base case. We take M_0 to be M. Then we apply Corollary 4.57 to $(a_i)_{i<\kappa}$ as a c-indiscernible sequence in M, where we take D to be M_0 (so d is the identity arrow). Then we find an extension $M \to N_0$ and an arrow $m_0 : M_0 \to N_0$ such that $(a_i)_{i<\kappa}$ is m_0-indiscernible (note that m_0 is not the same as the extension $M \to N_0$) and $\text{gtp}(m_0, c; N_0) = \text{gtp}(m, c; N_0)$. So c will embed in the chain $(M_i)_{i<\kappa}$. Properties (i) and (iv) now hold by construction, and properties (ii) and (iii) are trivial.

Successor step. Suppose we have constructed $(M_i)_{i\leq\alpha}$ and $(N_i)_{i\leq\alpha}$. Then by the induction hypothesis (iv) we have that $(a_i)_{\alpha\leq i<\kappa}$ is m_α-indiscernible, so $(a_i)_{\alpha+1\leq i<\kappa}$ is $m_\alpha a_\alpha$-indiscernible. We can thus apply Corollary 4.57 to $(a_i)_{\alpha+1\leq i<\kappa}$ with N_α in the role of M, M_α and A_α in the role of $(C_k)_{k\in K}$ and in the role of D we also take N_α. We then obtain an extension $N_\alpha \to N_{\alpha+1}$, and some $d' : N_\alpha \to N_{\alpha+1}$ such that $(a_i)_{\alpha+1\leq i<\kappa}$ is indiscernible over d'. We take $m_{\alpha+1} : M_{\alpha+1} \to N_{\alpha+1}$ to be d'.

This directly takes care of (iv) in the induction hypothesis. The application of Corollary 4.57 also gives us the following fact:

$$\text{gtp}(m_{\alpha+1}, m_\alpha, a_\alpha; N_{\alpha+1}) = \text{gtp}(n_\alpha, m_\alpha, a_\alpha; N_{\alpha+1}).$$

This means that a_α and m_α factor through N_α and $M_{\alpha+1}$ in the same way. This takes care of properties (ii) and (iii). Finally, property (i) says nothing about this stage.

Limit step. Let $\ell < \kappa$ be a limit. Set $M_\ell = \text{colim}_{i<\ell} M_i$ and $N_\ell = \text{colim}_{i<\ell} N_i$. For every $i < \ell$ we have an extension $M_i \to N_\ell$ by composing $m_i : M_i \to N_i$ with the coprojection $N_i \to N_\ell$. This makes N_ℓ in the vertex of a cocone for $(M_i)_{i<\ell}$, and so by the universal property of colimits we find an extension $m_\ell : M_\ell \to N_\ell$. This takes care of property (ii). Properties (i) and (iii) are vacuous.

That leaves property (iv). By the induction hypothesis we have that for each $i_0 < \ell$ the tail $(a_i)_{\ell\leq i<\kappa}$ is m_{i_0}-indiscernible. So if we let $I_0, I_1 \subseteq \{i : \ell \leq i < \kappa\}$ be two finite subsets, then we have

$$\text{gtp}((a_i)_{i\in I_0}, m_{i_0}; N_\ell) = \text{gtp}((a_i)_{i\in I_1}, m_{i_0}; N_\ell),$$

and thus

$$\text{gtp}((a_i)_{i\in I_0}, (m_j)_{j\leq i_0}; N_\ell) = \text{gtp}((a_i)_{i\in I_1}, (m_j)_{j\leq i_0}; N_\ell).$$

Then because $(\mathcal{C}, \mathcal{M})$ is finitely short we can conclude that

$$\text{gtp}((a_i)_{i\in I_0}, (m_i)_{i<\ell}; N_\ell) = \text{gtp}((a_i)_{i\in I_1}, (m_i)_{i<\ell}; N_\ell),$$

and so we see that $(a_i)_{\ell \leq i < \kappa}$ is $(m_i)_{i<\ell}$-indiscernible. We can thus apply Corollary 4.57 to $(a_i)_{\ell \leq i < \kappa}$ in N_ℓ with $(M_i)_{i<\ell}$ in the role of $(C_k)_{k \in K}$ and M_ℓ in the role of D. Then we obtain $m' : M_\ell \to N$ for some extension $N_\ell \to N$, such that $(a_i)_{\ell \leq i < \kappa}$ is indiscernible over m' in N and $\mathrm{gtp}(m', (m_i)_{i<\ell}; N) = \mathrm{gtp}(m_\ell, (m_i)_{i<\ell}; N_\ell)$. The latter means that m' is a morphism of cocones from $M_\ell = \mathrm{colim}_{i<\ell} M_i$ to N. By the universal property of the colimit, this morphism is unique, so we have $m' = m_\ell$ (composed with the extension $N_\ell \to N$). We can thus conclude that $(a_i)_{\ell \leq i < \kappa}$ is m_ℓ-indiscernible.

This finishes the construction of $(M_i)_{i<\kappa}$ and $(N_i)_{i<\kappa}$. As mentioned before, we get the required extension $M \to N$ by setting $N = \mathrm{colim}_{i<\kappa} N_i$, using property (i) that gives us an extension $M \to N_0 \to N$. Then properties (ii) and (iii) guarantee that $(M_i)_{i<\kappa}$ actually forms a chain of initial segments for $(a_i)_{i<\kappa}$ in N. Finally, (iv) together with the fact that c embeds into $(M_i)_{i<\kappa}$ (as discussed in the base case) then gives us that this chain actually witnesses $(a_i)_{i<\kappa}$ being an isi-sequence over c. $\square$

5

Canonicity of independence

This chapter contains the main results for AECats: canonicity of independence. These results are contained in section 5.1. The first main theorem, Theorem 5.4, states that there can only be one simple independence relation in an AECat. The second main theorem, Theorem 5.6, states the same thing for NSOP$_1$-like independence relations, under the assumption of the $\mathcal{B}$-existence axiom. These can then nicely be put together to rebuild part of the classical stability hierarchy, which is done in Theorem 5.7. In the second section of this chapter, section 5.2, we describe the link with known results and how these theorems can be applied in more concrete settings such as positive logic (and thus full first-order logic) and continuous logic.

In positive logic (and thus also in full first-order logic) Lascar strong types interact heavily with nice enough independence relations. In section 5.3 we explore this interaction in the setting of AECats and prove very similar results. We originally defined Lascar strong Galois types in one particular way, but classically there are a few equivalent definitions, e.g. as the finest bounded invariant equivalence relation. We recover these equivalent definitions in an arbitrary AECat, given a nice enough independence relation.

5.1 Canonicity

Theorem 5.1. *Let $(\mathcal{C}, \mathcal{M})$ be an AECat with AP and let $\mathop{\smile\!\!\!\!\!\!|}\;$ be a basic independence relation that also satisfies* Club Local Character*. Then $A \mathop{\smile\!\!\!\!\!\!|}\;^{\text{isi-d},M}_C B$ implies $A \mathop{\smile\!\!\!\!\!\!|}\;^M_C B$ for any $C \in \operatorname{base}(\mathop{\smile\!\!\!\!\!\!|}\;)$.*

If $\mathop{\smile\!\!\!\!\!\!|}\;$ satisfies the same assumptions, except possibly Union*, then we still have that $A \mathop{\smile\!\!\!\!\!\!|}\;^{\text{ld},M}_C B$ implies $A \mathop{\smile\!\!\!\!\!\!|}\;^M_C B$ for any $C \in \operatorname{base}(\mathop{\smile\!\!\!\!\!\!|}\;)$.*

Proof. Suppose that $\operatorname{gtp}(a, b, c; M)$ does not isi-divide over c. Let $\kappa \geq \Upsilon(A)$ such that $(\mathcal{C}, \mathcal{M})$ is a κ-AECat and $\operatorname{dom}(a)$ and $\operatorname{dom}(c)$ are κ-presentable. By Proposition 4.29 we find a long enough $\mathop{\smile\!\!\!\!\!\!|}\;_c$-independent isi-sequence $(b_i)_{i<\lambda}$ over c in some $M \to N$ with $\lambda > \kappa$ and $\operatorname{gtp}(b_i, c; N) = \operatorname{gtp}(b, c; M)$ for all $i < \lambda$. Let $(M_i)_{i<\lambda}$ be witnesses of independence. Since $\operatorname{gtp}(a, b, c; M)$ does not isi-divide over c there is $I \subseteq \lambda$ with $|I| = \kappa$ such that $\operatorname{gtp}(a, b, c; M)$ is consistent for

$(b_i)_{i \in I}$. Let a' be a realisation for $(b_i)_{i \in I}$, which for convenience we may assume to be in N. By possibly deleting an end segment from I we may assume that I has the order type of κ. Using Lemma 4.30 we may assume that each object in the chain $(M_i)_{i \in I}$ is κ-presentable. Then by chain local character, Lemma 4.15, we find $i_0 \in I$ such that $a' \mathop{\smile\!\!\!\!\!\!|}^{N}_{M_{i_0}} M_I$ where $M_I = \operatorname{colim}_{i \in I} M_i$. By MONOTONICITY and SYMMETRY we then have

$$b_{i_0} \mathop{\smile\!\!\!\!\!\!|}^{N}_{M_{i_0}} a'.$$

We also have

$$b_{i_0} \mathop{\smile\!\!\!\!\!\!|}^{N}_{c} M_{i_0}.$$

So by TRANSITIVITY we have $b_{i_0} \mathop{\smile\!\!\!\!\!\!|}^{N}_{c} a'$ and the result follows by SYMMETRY and because $\operatorname{gtp}(a', b_{i_0}, c; N) = \operatorname{gtp}(a, b, c; M)$.

For the final claim we just note that if we do not have UNION we can still apply Proposition 4.28 instead of Proposition 4.29 to get an arbitrarily long $\mathop{\smile\!\!\!\!\!\!|}_c$-independent sequence. It might just not be an isi-sequence. Then the rest of the proof goes through as written. □

The following lemma generalises the INDEPENDENCE THEOREM property to independent sequences of any length. The original INDEPENDENCE THEOREM can be viewed as just considering an independent sequence of length two.

Lemma 5.2 (Generalised independence theorem)**.** *Suppose that $\mathop{\smile\!\!\!\!\!\!|}$ is a basic independence relation satisfying* INDEPENDENCE THEOREM. *Let δ be any (possibly finite) ordinal. Suppose we have $(a, b, c; N)$ such that $a \mathop{\smile\!\!\!\!\!\!|}^{N}_{c} b$ and a $\mathop{\smile\!\!\!\!\!\!|}_c$-independent sequence $(b_i)_{i<\delta}$ in N with $\operatorname{Lgtp}(b_i/c; N) = \operatorname{Lgtp}(b/c; N)$ for all $i < \delta$. Then there is an extension $N \to N'$ with $(a'; N')$ such that $a' \mathop{\smile\!\!\!\!\!\!|}^{N'}_{c} N$ and $\operatorname{Lgtp}(a', b_i/c; N') = \operatorname{Lgtp}(a, b/c; N')$ for all $i < \delta$.*

In particular, $\operatorname{gtp}(a, b, c; N)$ is consistent for $(b_i)_{i<\delta}$.

Proof. Let $(M_i)_{i<\delta}$ be witnesses of independence for $(b_i)_{i<\delta}$. We will add one more link M_δ to the chain. If δ is a limit ordinal we set $M_\delta = \operatorname{colim}_{i<\delta} M_i$. If δ is a successor ordinal we set $M_\delta = N$.

We will by induction construct a chain $(N_i)_{i \le \delta}$ with N_0 extending N, together with extensions $\{m'_i : M_i \to N_i\}_{i \le \delta}$ and an arrow $(a''; N_0)$ such that $m'_0 = m_0$ and $\operatorname{Lgtp}(a''/c; N_0) = \operatorname{Lgtp}(a/c; N_0)$ while at stage i we have:

(i) the extensions $\{m'_j : M_j \to N_j\}_{j \le i}$ are natural in the sense that

$$\begin{tikzcd}
N_j \arrow[r] & N_i \\
M_j \arrow[u, "m'_j"] \arrow[r] & M_i \arrow[u, "m'_i"']
\end{tikzcd}$$

commutes for all $j \leq i$;

(ii) if i is a successor, say $i = j+1$, then $\mathrm{Lgtp}(a'', b'_j/c; N_i) = \mathrm{Lgtp}(a, b/c; N_i)$, where b'_j is the composition $B \xrightarrow{b_j} M_i \xrightarrow{m'_i} N_i$;

(iii) $a'' \mathop{\smile\!\!\!\!\!\!|}\,_c^{N_i} m'_i$.

<u>Base case.</u> By Existence we have $a \mathop{\smile\!\!\!\!\!\!|}\,_c^{N} c$, so we can apply strong extension (Corollary 4.13) to find $N \to N_0$ and $(a''; N_0)$ with $a'' \mathop{\smile\!\!\!\!\!\!|}\,_c^{N_0} M_0$ and $\mathrm{Lgtp}(a''/c; N_0) = \mathrm{Lgtp}(a/c; N_0)$.

<u>*Successor step.*</u> Suppose we have constructed N_i and m'_i. By (i) and since $m'_0 = m_0$ have $\mathrm{gtp}(m'_i, m_0; N_i) = \mathrm{gtp}(m_i, m_0; N_i)$, so $\mathrm{Lgtp}(m'_i/c; N_i) = \mathrm{Lgtp}(m_i/c; N_i)$. We can thus find $(a^*, m^*_{i+1}; N^*)$ for some $N_i \to N^*$ such that $\mathrm{Lgtp}(m^*_{i+1}, m'_i/c; N^*) = \mathrm{Lgtp}(m_{i+1}, m_i/c; N^*)$ and $\mathrm{Lgtp}(a^*, b^*_i/c; N^*) = \mathrm{Lgtp}(a, b/c; N^*)$, where b^*_i is given by $B \xrightarrow{b_i} M_{i+1} \xrightarrow{m^*_{i+1}} N^*$. For this last construction we used that $\mathrm{Lgtp}(b_i/c; N) = \mathrm{Lgtp}(b/c; N)$ and that b_i factors through m_{i+1}. Then $a'' \mathop{\smile\!\!\!\!\!\!|}\,_c^{N^*} m'_i$, $a^* \mathop{\smile\!\!\!\!\!\!|}\,_c^{N^*} b^*_i$ and $b^*_i \mathop{\smile\!\!\!\!\!\!|}\,_c^{N^*} m'_i$. So we can apply Independence Theorem to find $N^* \to N_{i+1}$ and $(a^{**}; N_{i+1})$ with $a^{**} \mathop{\smile\!\!\!\!\!\!|}\,_c^{N_{i+1}} N^*$. By Monotonicity we get $a^{**} \mathop{\smile\!\!\!\!\!\!|}\,_c^{N_{i+1}} m^*_{i+1}$. We have $\mathrm{Lgtp}(a^{**}/c, m'_i; N_{i+1}) = \mathrm{Lgtp}(a''/c, m'_i; N_{i+1})$, so we find $m'_{i+1} : M_{i+1} \to N_{i+1}$ (possibly after replacing N_{i+1} by an extension) such that $\mathrm{Lgtp}(a^{**}, m^*_{i+1}/c, m'_i; N_{i+1}) = \mathrm{Lgtp}(a'', m'_{i+1}/c, m'_i; N_{i+1})$. We verify the induction hypothesis:

(i) we have by construction that $\mathrm{gtp}(m'_{i+1}, m'_i; N_{i+1}) = \mathrm{gtp}(m^*_{i+1}, m'_i; N_{i+1}) = \mathrm{gtp}(m_{i+1}, m_i : N_{i+1})$, so m'_i factors through m'_{i+1} in the same way that m_i factors through m_{i+1}, and naturality follows;

(ii) $\mathrm{Lgtp}(a'', b'_i/c; N_{i+1}) = \mathrm{Lgtp}(a^{**}, b^*_i/c; N_{i+1}) = \mathrm{Lgtp}(a^*, b^*_i/c; N_{i+1}) = \mathrm{Lgtp}(a, b/c; N_{i+1})$;

(iii) this follows from $a^{**} \mathop{\smile\!\!\!\!\!\!|}\,_c^{N_{i+1}} m^*_{i+1}$ and Invariance.

<u>*Limit step.*</u> For limit ℓ let $N_\ell = \mathrm{colim}_{i<\ell} N_i$. By (i) from the induction hypothesis the arrows m'_i composed with the coprojections $N_i \to N_\ell$ form a cocone on $(M_i)_{i<\ell}$. By continuity $M_\ell = \mathrm{colim}_{i<\ell} M_i$, so there is a universal arrow $m'_\ell : M_\ell \to N_\ell$. This directly establishes (i). Property (ii) is vacuous. Property (iii) follows from the induction hypothesis and Union.

Having finished the inductive construction, we have two arrows $M_\delta \to N_\delta$, namely $m_\delta : M_\delta \to N \to N_\delta$ and the m'_δ we just constructed. By (i) from the induction hypothesis we have $\mathrm{gtp}(m_\delta, m_0; N_\delta) = \mathrm{gtp}(m'_\delta, m_0; N_\delta)$. So we find an extension $N_\delta \to N'$ and some $(a'; N')$ such that $\mathrm{gtp}(a', m_\delta, m_0; N') = \mathrm{gtp}(a'', m'_\delta, m_0; N_\delta)$. Using that c factors through m_0 and (ii) from the induction hypothesis, we find that for any $i < \delta$ we have $\mathrm{Lgtp}(a', b_i/c; N') = \mathrm{Lgtp}(a'', b'_i/c; N') = \mathrm{Lgtp}(a, b/c; N')$. By (iii) from the induction hypothesis we also have $a' \mathop{\smile\!\!\!\!\!\!|}\,_c^{N'} M_\delta$. So if δ was a successor ordinal we had $M_\delta = N$ and we are done. Otherwise we can just apply Extension and relabel things to get $a' \mathop{\smile\!\!\!\!\!\!|}\,_c^{N'} N$.

The final claim follows because a' is a realisation of $\mathrm{gtp}(a, b, c; N)$ for $(b_i)_{i<\delta}$.

□

Remark 5.3. In the context of Lemma 5.2 if C is a model then there is no need to concern ourselves with Lascar strong Galois types. That is, the proof as written then goes through if we replace "Lascar strong Galois type" by just "Galois type" everywhere. We also only apply Independence Theorem with C in the base. So if C is a model then it would be enough to just have Independence Theorem over models. Or equivalently, to have 3-amalgamation, see Theorem 4.20.

A slightly weaker version of the following theorem has been published in [Kam20].

Theorem 5.4 (Canonicity of simple independence)**.** *Let $(\mathcal{C}, \mathcal{M})$ be an AECat with AP, and suppose that $\mathop{\smile\!\!\!\!\!\!|}$ is a simple independence relation. Then $\mathop{\smile\!\!\!\!\!\!|} = \mathop{\smile\!\!\!\!\!\!|}^{\text{isi-d}} = \mathop{\smile\!\!\!\!\!\!|}^{\text{isi-f}}$ over* $\mathrm{base}(\mathop{\smile\!\!\!\!\!\!|})$.

Proof. The implication $\mathop{\smile\!\!\!\!\!\!|}^{\text{isi-d}} \implies \mathop{\smile\!\!\!\!\!\!|}$ is already given by Theorem 5.1. For the converse we will assume that $A \mathop{\smile\!\!\!\!\!\!|}\,_C^M B$ and we will prove that $A \mathop{\smile\!\!\!\!\!\!|}\,_C^{\text{isi-d},M} B$. Pick some representatives a, b, c of A, B, C. Let μ be such that $(\mathcal{C}, \mathcal{M})$ is a μ-AECat and let $\lambda > \Upsilon(B) + \mu$. Let $(b_i)_{i<\lambda}$ be an isi-sequence over c in some $M \to N$, with chain of initial segments $(M_i)_{i<\lambda}$ and $\mathrm{gtp}(b_i, c; N) = \mathrm{gtp}(b, c; M)$ for all $i < \lambda$. Let $\Upsilon(B) + \mu \leq \kappa < \lambda$. By a similar argument as Lemma 4.30 (we can just ignore the independence relation there) we may assume that M_i is κ-presentable for all $i < \kappa$. We can thus apply chain local character, Lemma 4.15, to find $i_0 < \kappa$ such that $b_\kappa \mathop{\smile\!\!\!\!\!\!|}\,_{M_{i_0}}^N M_\kappa$. We will aim to show that $\mathrm{gtp}(a, b, c; M)$ is consistent for $(b_i)_{i_0 \leq i < \kappa}$. We use $\mathrm{gtp}(b_{i_0}, c; N) = \mathrm{gtp}(b, c; M)$ to find a common extension $M \to N' \leftarrow N$ where $b = b_{i_0}$ as arrows into N'. By applying Extension to the assumption $a \mathop{\smile\!\!\!\!\!\!|}\,_c^M b$ we then find $(a'; N')$ (possibly after replacing N' by an extension) such that $a' \mathop{\smile\!\!\!\!\!\!|}\,_c^{N'} N$ and $\mathrm{gtp}(a', b, c; N') = \mathrm{gtp}(a, b, c; M)$. Then by Base-Monotonicity and Monotonicity we find $a \mathop{\smile\!\!\!\!\!\!|}\,_{M_{i_0}}^{N'} b$. For any $i_0 \leq i < \kappa$ we have $\mathrm{gtp}(b_i, m_i, m_{i_0}; N') = \mathrm{gtp}(b_\kappa, m_i, m_{i_0}; N')$ because $(b_i)_{i<\lambda}$ is an isi-sequence. So by Monotonicity and Invariance and the earlier fact that

$b_\kappa \mathop{\smile\!\!\!\!\!\!|}\,^N_{M_{i_0}} M_\kappa$, we find

$$b_i \mathop{\smile\!\!\!\!\!\!|}_{M_{i_0}}^{N'} M_i$$

for all $i_0 \leq i < \kappa$. So $(b_i)_{i_0 \leq i < \kappa}$ is a $\mathop{\smile\!\!\!\!\!\!|}_{M_{i_0}}$-independent sequence. We can thus apply the generalised independence theorem, Lemma 5.2, to conclude that $\mathrm{gtp}(a', b, c; N') = \mathrm{gtp}(a, b, c; M)$ is indeed consistent for $(b_i)_{i_0 \leq i < \kappa}$. As κ was arbitrarily large below λ, λ itself was arbitarily large and $(b_i)_{i_0 \leq i < \kappa}$ is a subsequence of an arbitrary isi-sequence of length λ we conclude that indeed $A \mathop{\smile\!\!\!\!\!\!|}_C^{\text{isi-d},M} B$.

Finally, the claim $\mathop{\smile\!\!\!\!\!\!|}^{\text{isi-d}} = \mathop{\smile\!\!\!\!\!\!|}^{\text{isi-f}}$ follows from Proposition 4.46 because $\mathop{\smile\!\!\!\!\!\!|}^{\text{isi-d}} = \mathop{\smile\!\!\!\!\!\!|}$ has Extension and Right-Monotonicity. □

Remark 5.5. We follow up on Remark 4.19. In the proof of Theorem 5.4 we only applied the Independence Theorem indirectly through Lemma 5.2. The base, i.e. C in that lemma, is by construction always a model. So by Remark 5.3 it would be enough to only assume 3-amalgamation instead of Independence Theorem.

Theorem 5.6 (Canonicity of NSOP$_1$-like independence)**.** *Let $(\mathcal{C}, \mathcal{M})$ be an AECat with AP and let $\mathcal{B}$ be some base class. Suppose that $(\mathcal{C}, \mathcal{M})$ satisfies the $\mathcal{B}$-existence axiom and suppose that there is an NSOP$_1$-like independence relation $\mathop{\smile\!\!\!\!\!\!|}$ over $\mathcal{B}$. Then $\mathop{\smile\!\!\!\!\!\!|} = \mathop{\smile\!\!\!\!\!\!|}^{\text{lK}}$ over $\mathcal{B}$.*

Proof. Suppose that $A \mathop{\smile\!\!\!\!\!\!|}_C^M B$ with $C \in \mathcal{B}$ and pick some representatives a, b, c of A, B, C. There is a bound μ on the cardinality of the set of Lascar strong Galois types compatible with $(b, c; M)$, see Proposition 3.24. Let $\lambda > \mu$ and let $(b_i)_{i<\lambda}$ be a $\mathop{\smile\!\!\!\!\!\!|}_c^{\text{isi-f}}$-independent sequence in some $M \to N$ with $\mathrm{gtp}(b_i, c; N) = \mathrm{gtp}(b, c; M)$ for all $i < \lambda$. Then $(b_i)_{i<\lambda}$ is also $\mathop{\smile\!\!\!\!\!\!|}_c$-independent, by Theorem 5.1 and Proposition 4.46. We have to show that for every $\kappa < \lambda$ there is $I \subseteq \lambda$ with $|I| = \kappa$ such that $\mathrm{gtp}(a, b, c; M)$ is consistent for $(b_i)_{i \in I}$. So let $\kappa < \lambda$. Then by the choice of μ and λ there must be some $I \subseteq \lambda$ with $|I| = \kappa$ such that $\mathrm{Lgtp}(b_i/c; N) = \mathrm{Lgtp}(b_j/c; N)$ for all $i, j \in I$. Pick some $i_0 \in I$ and let $(a'; N')$ for some extension $N \to N'$ be such that $\mathrm{gtp}(a', b_{i_0}, c; N') = \mathrm{gtp}(a, b, c; M)$. Then we can apply the generalised independence theorem, Lemma 5.2, to see that $\mathrm{gtp}(a', b_{i_0}, c; N') = \mathrm{gtp}(a, b, c; M)$ is consistent for $(b_i)_{i \in I}$. We conclude that indeed $A \mathop{\smile\!\!\!\!\!\!|}_C^{\text{lK},M} B$.

For the other direction, suppose that $A \mathop{\smile\!\!\!\!\!\!|}_C^{\text{lK},M} B$ with $C \in \mathcal{B}$. Let $\kappa \geq \Upsilon(A)$ be such that $(\mathcal{C}, \mathcal{M})$ is a κ-AECat and A and C are κ-presentable. Let $(b_i)_{i<\lambda}$ be a long enough $\mathop{\smile\!\!\!\!\!\!|}_c^{\text{isi-f}}$-independent sequence in some extension $M \to N$, with $\lambda > \kappa$, witnesses of independence $(M_i)_{i<\lambda}$ and $\mathrm{gtp}(b_i, c; N) = \mathrm{gtp}(b, c; M)$ for all $i < \lambda$. Such a sequence exists because we assumed the $\mathcal{B}$-existence axiom, so we can apply Corollary 4.49. As before, this is also a $\mathop{\smile\!\!\!\!\!\!|}_c$-independent sequence. By definition of long Kim-dividing there is $I \subseteq \lambda$ with $|I| = \kappa$ such that $\mathrm{gtp}(a, b, c; N)$ is consistent for $(b_i)_{i \in I}$. Let a' be a realisation for this (we may assume a' is an arrow into N). By possibly deleting an end segment from I we may assume that I has the order type of κ. Using Lemma 4.30 we may assume that each object in the chain $(M_i)_{i \in I}$ is κ-presentable. Then by chain local character, Lemma 4.15, we find $i_0 \in I$ such that $a' \mathop{\downfree}^N_{M_{i_0}} M_I$ where $M_I = \mathrm{colim}_{i \in I} M_i$. So by Monotonicity and Symmetry we have

$$b_{i_0} \mathop{\downfree}^{N}_{M_{i_0}} a'.$$

Furthermore, we have

$$b_{i_0} \mathop{\downfree}^{N}_{C} M_{i_0}.$$

So by Transitivity we have $b_{i_0} \mathop{\downfree}^N_C a'$. The result then follows by Symmetry and the fact that $\mathrm{gtp}(a', b_{i_0}, c; N) = \mathrm{gtp}(a, b, c; M)$. $\square$

By definition any stable independence relation is also simple, and any simple independence relation is also NSOP$_1$-like. The canonicity theorems then tell us that these are indeed unique in a given AECat with AP and with what notion of dividing they coincide. We make this precise in the following theorem.

Theorem 5.7. *Let $(\mathcal{C}, \mathcal{M})$ be an AECat with AP and suppose that $\downfree$ is a stable or a simple independence relation in $(\mathcal{C}, \mathcal{M})$. Suppose furthermore that $\downfree^*$ is an NSOP$_1$-like independence relation in $(\mathcal{C}, \mathcal{M})$ with* $\mathrm{base}(\downfree) = \mathrm{base}(\downfree^*)$. *Then*

$$\downfree = \overset{*}{\downfree} = \overset{\text{isi-d}}{\downfree} = \overset{\text{isi-f}}{\downfree} = \overset{\text{lK}}{\downfree}.$$

Proof. This follows directly from Theorem 5.4 and Theorem 5.6. To apply the latter we need the $\mathrm{base}(\downfree)$-existence axiom. This is automatic, as $\downfree = \downfree^{\text{isi-f}}$ over $\mathrm{base}(\downfree)$ by Theorem 5.4 and we have Existence by assumption. $\square$

We can classify AECats based on the existence of certain independence relations, just as we can classify theories in full first-order logic in that way. For example, suppose that we have an AECat $(\mathcal{C}, \mathcal{M})$ with AP and an NSOP$_1$-like independence relation $\downfree$ where Base-Monotonicity fails. Then we can never find a simple independence relation in $(\mathcal{C}, \mathcal{M})$ (with the same base class). Because if we would have such a simple independence relation $\downfree'$ then by Theorem 5.7 we would have $\downfree = \downfree'$, but that is impossible because a simple independence relation must satisfy Base-Monotonicity. So we can classify $(\mathcal{C}, \mathcal{M})$ as NSOP$_1$, but non-simple. See Example 6.101 for a concrete example of this.

5.2 Relationship with known results

In this section we discuss how this work extends and brings together previously known results. We also describe precisely how to apply the canonicity theorems, Theorem 5.4 and Theorem 5.6, in these more concrete settings.

Example 5.8. Let T be a thick theory (recall that this includes theories in full first-order logic, see Remark 2.12). Let $\mathcal{C}$ be either $\mathbf{SubMod}(T)$ or $\mathbf{Mod}(T)$. Following Remark 3.21 we take $\mathcal{M}$ to be the category of finitely λ_T-saturated models. If T is semi-Hausdorff we can instead just take $\mathcal{M} = \mathbf{Mod}(T)$.

If T is stable or simple then there is respectively a stable or simple independence relation $\mathop{\smile\!\!\!\!\!\!|}$ in $(\mathcal{C}, \mathcal{M})$ with $\text{base}(\mathop{\smile\!\!\!\!\!\!|}) = \mathcal{C}$. This follows from a combination of [BY03b, BY03c]. So Theorem 5.4 applies.

The Stationarity property in a stable theory follows from [BY03b, Theorem 2.8]. In their statement the base model M is assumed to be $|T|^+$-saturated. This is necessary for only two reasons. The first reason is that types over M should coincide with Lascar strong types over M, but by our choice of $\mathcal{M}$ and the thickness assumption this happens for all $M \in \mathcal{M}$. The second reason is that types over M should be what they call extendible, but in a simple thick theory every type is extendible, see [BY03c, Theorem 1.15].

Example 5.9. Let $(\mathcal{C}, \mathcal{M})$ be an AECat based on some semi-Hausdorff or thick theory T as in Example 5.8. Then by Example 4.50(i) we have the $\mathbf{Mod}(T)$-existence axiom. If T is NSOP_1 then it has an NSOP_1-like independence relation $\mathop{\smile\!\!\!\!\!\!|}$ with $\text{base}(\mathop{\smile\!\!\!\!\!\!|}) = \mathbf{Mod}(T)$, see chapter 6. So Theorem 5.6 applies. See also section 6.10 for more concrete examples.

Here we had to restrict the base class to e.c. models, simply because Kim-independence in positive logic has only been developed over e.c. models. It is quite likely that this work can be extended to arbitrary base sets, see Question 6.105. For theories in full first-order logic this has already been done, see [CKR20, DKR19]. To make this work we need to assume the existence axiom, see also Example 4.50(iii). So let T be an NSOP_1 theory in full first-order logic and set $(\mathcal{C}, \mathcal{M}) = (\mathbf{SubMod}(T), \mathbf{Mod}(T))$, which satisfies the $\mathcal{C}$-existence axiom. Then the aforementioned sources show that there is an NSOP_1-like independence relation $\mathop{\smile\!\!\!\!\!\!|}$ with $\text{base}(\mathop{\smile\!\!\!\!\!\!|}) = \mathcal{C}$ and so Theorem 5.6 applies.

Finally we note that there is a Kim-Pillay style theorem in [CKR20, Theorem 5.1] for Kim-independence over arbitrary sets. They still rely on a syntactical property "strong finite character", which could be replaced by just "finite character". Theorem 5.6 gives us just the canonicity part. To conclude that a theory with such an independence relation is NSOP_1, without using strong finite character, we can restrict ourselves to work over models and use the proof from Theorem 6.79.

Example 5.10. Let T be a continuous theory, in the sense of [BYBHU08]. Let $\mathcal{C}$ be either $\mathbf{SubMetMod}(T)$ or $\mathbf{MetMod}(T)$, and let $\mathcal{M}$ be $\mathbf{MetMod}(T)$. Then $(\mathcal{C}, \mathcal{M})$ is an AECat with AP, as discussed in Example 3.7.

If T is stable or simple then there is respectively a stable or simple independence relation $\mathop{\smile\!\!\!\!\!\!|}$ in $(\mathcal{C}, \mathcal{M})$ with $\operatorname{base}(\mathop{\smile\!\!\!\!\!\!|}) = \mathcal{C}$. Every continuous theory is in particular a Hausdorff compact abstract theory, and so the machinery of [BY03b, BY03c] applies. This shows we can indeed find an appropriate independence relation in any simple or stable continuous theory. There is also [BYBHU08, section 14] for a further discussion about stability specifically in continuous theories. So Theorem 5.4 applies.

In [BYBHU08] some examples of stable continuous theories and their corresponding independence relations are given, including Hilbert spaces and atomless probability spaces.

Example 5.11. In this example we consider the continuous theory T_N of Hilbert spaces with a distance function to a random subset, as studied in [BHV18]. They prove that this theory has TP_2 and can thus not be simple. They also define an independence relation $\mathop{\smile\!\!\!\!\!\!|}^*$ over arbitrary sets that has all the properties of an NSOP_1-like independence relation. Except that they do not prove the full Independence Theorem, but enough for 3-amalgamation. So setting $\mathcal{C} = \mathbf{SubMetMod}(T_N)$ and $\mathcal{M} = \mathbf{MetMod}(T_N)$, and taking $\operatorname{base}(\mathop{\smile\!\!\!\!\!\!|}^*) = \mathbf{MetMod}(T)$, we have that $\mathop{\smile\!\!\!\!\!\!|}^*$ is an NSOP_1-like independence relation in $(\mathcal{C}, \mathcal{M})$. By Example 4.50(ii) we also have the $\mathbf{MetMod}(T)$-existence axiom. So Theorem 5.6 applies.

5.3 More on Lascar strong Galois types

In this section we will show that in the presence of a nice enough independence relation there are some equivalent definitions of Lascar strong Galois types. This includes the usual definition of being the finest bounded invariant equivalence relation.

It is well known that Lascar strong types heavily interact with independence relations in full first-order logic. For example, independence relations can be used to show that having the same Lascar strong type is type-definable in any simple theory (we say that the theory is "G-compact"), see [Kim14, Proposition 5.1.11]. The same technique applies to any NSOP_1 theory in full first-order logic that satisfies the existence axiom [DKR19, Corollary 5.9]. We essentially adapt this technique in this section, while at the same time using independence relations to build what we call "strongly 2-indiscernible" sequences (Definition 5.15), which take the role of the usual indiscernible sequences in settings with compactness.

Throughout this section we will work with single arrows a, b, c and objects A and C, where $\operatorname{dom}(a) = \operatorname{dom}(b) = A$ and $\operatorname{dom}(c) = C$. This leads to cleaner notation and when working with independence relations we can only work with single arrows anyway (i.e. the sides and base of an independence relation do not allow tuples of arrows in our definition). However, it is not too difficult to extend the main result of this section (Theorem 5.17) to arbitrary tuples, see Remark 5.23.

Definition 5.12. Let $(\mathcal{C}, \mathcal{M})$ be an AECat with AP and fix some objects A and C. Suppose that for each M and each $c : C \to M$ we are given an equivalence relation $\equiv_{c,M}$ on $\mathrm{Hom}(A, M)$. Then we say that the family $\equiv$ is an *equivalence relation over* C.

We call $\equiv$ a *bounded equivalence relation* if there is λ such that $\equiv_{c,M}$ has at most λ many equivalence classes for any M and $c : C \to M$.

We call $\equiv$ an *invariant equivalence relation* if it is invariant under equality of Galois types over C. That is, if $\mathrm{gtp}(a, b, c; M) = \mathrm{gtp}(a', b', c'; M')$ then we have that $a \equiv_{c,M} b$ if and only if $a' \equiv_{c',M'} b'$.

Convention 5.13. We will only deal with invariant equivalence relations. To further simplify the notation we will drop the M from the notation. So we write $a \equiv_c b$ instead of $a \equiv_{c,M} b$. This makes sense because of invariance, because then it does not matter if we consider a and b as arrows into M or as arrows into an extension of M.

Example 5.14. We give some familiar examples.

(i) Taking just equality as equivalence relation is an equivalence relation over any C. This relation is invariant, but generally not bounded because $\mathrm{Hom}(A, M)$ may become arbitrarily large.

(ii) The trivial equivalence where everything is equivalent is a bounded invariant equivalence relation over any C.

(iii) Having the same Galois type is a bounded invariant equivalence relation over any C. That is, we define $\equiv$ as $a \equiv_c b$ if and only if $\mathrm{gtp}(a, c; M) = \mathrm{gtp}(b, c; M)$. Clearly $\equiv$ is invariant, and by Proposition 3.19 it is bounded.

(iv) Having the same Lascar strong Galois type is a bounded invariant equivalence relation. Similar to the previous point we define $\equiv$ as $a \equiv_c b$ if and only if $\mathrm{Lgtp}(a/c; M) = \mathrm{Lgtp}(a'/c; M)$. This is invariant by Proposition 3.23 and bounded by Proposition 3.24.

(v) In any theory T, any hyperimaginary yields an invariant equivalence relation (see section 2.2). That is, if $E(x, y)$ is a set of formulas that defines an equivalence relation modulo T then we can define an invariant equivalence relation $\equiv^E$ over $\emptyset$ as follows: for tuples $a, b \in M$ we set $a \equiv^E b$ iff $M \models E(a, b)$.

Usually bounded invariant equivalence relations are linked to Lascar strong types using indiscernible sequences (see Definition 2.40). This requires some compactness, see section 4.5, which we generally do not have. To solve this we will adapt the idea of strongly indiscernible sequences from [HL06].

Definition 5.15. We call sequence $(a_i)_{i<\kappa}$ in some M *2-c-indiscernible* if for any $i_1 < i_2 < \kappa$ and $j_1 < j_2 < \kappa$ we have $\mathrm{gtp}(a_{i_1}, c; M) = \mathrm{gtp}(a_{i_2}, c; M)$ and $\mathrm{gtp}(a_{i_1}, a_{i_2}, c; M) = \mathrm{gtp}(a_{j_1}, a_{j_2}, c; M)$. We call such a sequence *strongly 2-c-indiscernible* if it can be extended to a 2-c-indiscernible sequence (possibly in some extension model) of arbitrary length.

Given an independence relation $\mathop{\smile\!\!\!\!\!\!|}\,$ we define a *2-$\mathop{\smile\!\!\!\!\!\!|}_c$-Morley sequence* to be a 2-c-indiscernible sequence that is also $\mathop{\smile\!\!\!\!\!\!|}_c$-independent. Such a sequence is called a *strong 2-$\mathop{\smile\!\!\!\!\!\!|}_c$-Morley sequence* if it can be extended to a 2-$\mathop{\smile\!\!\!\!\!\!|}_c$-Morley sequence (possibly in some extension model) of arbitrary length.

Definition 5.16. For $(a, b, c; M)$ we write $a \sim^2_c b$ if a and b are on some strongly 2-c-indiscernible sequence. We write $\equiv^2_c$ for the transitive closure of $\sim^2_c$. Similarly, given an independence relation $\mathop{\smile\!\!\!\!\!\!|}\,$, we write $a \sim^{\mathop{\smile\!\!\!\!\!\!|}}_c b$ if a and b are on some strong 2-$\mathop{\smile\!\!\!\!\!\!|}_c$-Morley sequence and $\equiv^{\mathop{\smile\!\!\!\!\!\!|}}_c$ for its transitive closure. Finally, we write $a \equiv^B_c b$ if a and b are equivalent for every bounded invariant relation over c.

One easily verifies that $\equiv^2$ and $\equiv^B$ are equivalence relations over any C. For $\equiv^{\mathop{\smile\!\!\!\!\!\!|}}$ we may generally not have reflexivity, but we will have that in the situations we are interested in. In particular, Lemma 5.21 shows that $\equiv^{\mathop{\smile\!\!\!\!\!\!|}}$ has reflexivity over models.

Theorem 5.17. *Let $(\mathcal{C}, \mathcal{M})$ be an AECat with AP, and suppose that $\mathop{\smile\!\!\!\!\!\!|}\,$ is a basic independence relation that also satisfies* 3-AMALGAMATION. *Then the following are equivalent for any $(a, b, c; M)$:*

(i) $\mathrm{Lgtp}(a/c; M) = \mathrm{Lgtp}(b/c; M)$;

(ii) $a \equiv^B_c b$, so a and b are equivalent under every bounded invariant equivalence relation over C;

(iii) $a \equiv^2_c b$, so a and b can be connected by strongly 2-c-indiscernible sequences.

We note that the above conditions are equivalent in any thick positive theory, without assuming the existence of any independence relation, see Definition 2.40 and the discussion afterwards. In this very general setting we need the independence relation as a replacement for where we would usually use compactness. More concretely, we need it to build strong 2-Morley sequences (which are then strongly 2-indiscernible by definition), see Lemma 5.20. Proving the equivalence of the above conditions without a nice independence relation seems a lot harder, if not impossible in this generality.

Remark 5.18. In Theorem 5.17 we only required $\mathop{\smile\!\!\!\!\!\!|}\,$ to have 3-AMALGAMATION. So in the assumptions of the theorem do not mention anything about Lascar strong Galois types. That means that, in the presence of such an independence relation, we can take any of the equivalent conditions in Theorem 5.17 as the definition for Lascar strong Galois types, without any circularity in the definitions.

This might be relevant if one would then want to prove the full INDEPENDENCE THEOREM, which does mention Lascar strong Galois types, from the rest of the properties of $\mathop{\smile\!\!\!\!\!\!|}\,$. For example, something like this is done in [DKR19]. There they use the fact that NSOP_1 theories in full first-order logic satisfy the independence theorem over models, and they used that to prove the independence theorem over arbitrary sets.

The remainder of this section is devoted to proving Theorem 5.17.

Lemma 5.19. *For any $(a, b, c; M)$ and any independence relation $\mathop{\smile\!\!\!\!\!\!|}\,$ we always have*

$$a \equiv_c^{\mathop{\smile\!\!\!\!\!\!|}} b \quad\Longrightarrow\quad a \equiv_c^2 b \quad\Longrightarrow\quad a \equiv_c^B b \quad\Longrightarrow\quad \operatorname{Lgtp}(a/c; M) = \operatorname{Lgtp}(b/c; M).$$

Proof. The first implication follows because any strong 2-$\mathop{\smile\!\!\!\!\!\!|}_c$-Morley sequence is in particular strongly 2-c-indiscernible. The final implication follows because having the same Lascar strong Galois type is a bounded invariant equivalence relation, see Example 5.14(iv). We prove the middle implication. So let $\equiv$ be a bounded invariant equivalence relation over C. It is enough to prove that $a \sim_c^2 b$ implies $a \equiv_c b$. Let κ be the bound of $\equiv$. Since $a \sim_c^2 b$ we find a 2-c-indiscernible sequence $(a_i)_{i<\kappa^+}$ in some extension N of M with a and b on it. Without loss of generality we may assume $a_0 = a$ and $a_1 = b$. By boundedness we find $i < j < \kappa^+$ such that $a_i \equiv_c a_j$. By 2-c-indiscernibility we have $\operatorname{gtp}(a, b, c; N) = \operatorname{gtp}(a_i, a_j, c; N)$. So $a \equiv_c b$ follows from invariance. $\square$

Lemma 5.20. *Suppose that $\mathop{\smile\!\!\!\!\!\!|}\,$ is a basic independence relation that also satisfies* 3-AMALGAMATION. *Let m be an arrow with a model as domain and let $\delta \geq 2$ be any ordinal (possibly finite). Then any 2-$\mathop{\smile\!\!\!\!\!\!|}_m$-Morley sequence $(a_i)_{i<\delta}$ is a strong 2-$\mathop{\smile\!\!\!\!\!\!|}_m$-Morley sequence. In particular $a \mathop{\smile\!\!\!\!\!\!|}_m^N b$ and $\operatorname{gtp}(a, m; N) = \operatorname{gtp}(b, m; N)$ implies $a \sim_m^{\mathop{\smile\!\!\!\!\!\!|}} b$.*

Proof. By Remark 5.3 we can apply the generalised independence theorem, Lemma 5.2, while avoiding referring to Lascar strong Galois types. We can thus inductively apply Lemma 5.2 to elongate $(a_i)_{i<\delta}$ to any length we want. We do this by letting a_0, a_1 and m play the roles of b, a and c respectively. $\square$

Lemma 5.21. *Suppose that $\mathop{\smile\!\!\!\!\!\!|}$ is a basic independence relation that also satisfies* 3-AMALGAMATION. *If* $\mathrm{gtp}(a, m; N) = \mathrm{gtp}(b, m; N)$*, where* $\mathrm{dom}(m)$ *is a model, then* $a \equiv^{\mathop{\smile\!\!\!\!\!\!|}}_m b$.

Proof. By EXISTENCE we have $a \mathop{\smile\!\!\!\!\!\!|}^N_m m$, so we can apply EXTENSION to find $N \to N'$ and some $(a'; N')$ such that $a' \mathop{\smile\!\!\!\!\!\!|}^{N'}_m N$ while $\mathrm{gtp}(a', m; N') = \mathrm{gtp}(a, m; N')$. By MONOTONICITY we then have $a' \mathop{\smile\!\!\!\!\!\!|}^{N'}_m a$ and $a' \mathop{\smile\!\!\!\!\!\!|}^{N'}_m b$. So by Lemma 5.20 we have $a \sim^{\mathop{\smile\!\!\!\!\!\!|}}_m a' \sim^{\mathop{\smile\!\!\!\!\!\!|}}_m b$, and we are done. □

Proof of Theorem 5.17. By Lemma 5.19 we only need to prove that $\mathrm{Lgtp}(a/c; M) = \mathrm{Lgtp}(b/c; M)$ implies $a \equiv^2_c b$. It is enough to prove that $(a/c; M) \sim_{\mathrm{Lgtp}} (b/c; M)$ implies $a \equiv^2_c b$. So let $M \to N$ be an extension and let $m_0 : M_0 \to N$ be such that c factors through m_0, M_0 is a model and $\mathrm{gtp}(a, m_0; N) = \mathrm{gtp}(b, m_0; N)$. Then by Lemma 5.21 we get $a \equiv^{\mathop{\smile\!\!\!\!\!\!|}}_{m_0} b$. So by Lemma 5.19 we have $a \equiv^2_{m_0} b$. Any strongly 2-m_0-indiscernible sequence is also strongly 2-c-indiscernible, because c factors through m_0. So we conclude that indeed $a \equiv^2_c b$. □

Remark 5.22. The proof of Theorem 5.17 also shows that if $\mathrm{dom}(c)$ is a model then $\mathrm{Lgtp}(a/c; M) = \mathrm{Lgtp}(b/c; M)$ is further equivalent to $a \equiv^{\mathop{\smile\!\!\!\!\!\!|}}_c b$.

Remark 5.23. We have stated Theorem 5.17 for single arrows, rather than for tuples of arrows. We briefly sketch how we can get the result for tuples of arrows as well. That is, if we replace a, b and c by $(a_i)_{i \in I}$, $(b_i)_{i \in I}$ and $(c_j)_{j \in J}$ respectively.

First we extend the definitions of (strongly) 2-indiscernible, $\equiv^2$ and $\equiv^B$ to tuples of arrows in a straightforward way. Then following the same proof as in Lemma 5.19 we get:

$$
\begin{aligned}
&(a_i)_{i \in I} \equiv^2_{(c_j)_{j \in J}} (b_i)_{i \in I} && \Longrightarrow \\
&(a_i)_{i \in I} \equiv^B_{(c_j)_{j \in J}} (b_i)_{i \in I} && \Longrightarrow \\
&\mathrm{Lgtp}((a_i)_{i \in I}/(c_j)_{j \in J}; M) = \mathrm{Lgtp}((b_i)_{i \in I}/(c_j)_{j \in J}; M).
\end{aligned}
$$

So we are left to prove that $\mathrm{Lgtp}((a_i)_{i \in I}/(c_j)_{j \in J}; M) = \mathrm{Lgtp}((b_i)_{i \in I}/(c_j)_{j \in J}; M)$ implies $(a_i)_{i \in I} \equiv^2_{(c_j)_{j \in J}} (b_i)_{i \in I}$. It is enough to prove that $((a_i)_{i \in I}/(c_j)_{j \in J}; M) \sim_{\mathrm{Lgtp}} ((b_i)_{i \in I}/(c_j)_{j \in J}; M)$ implies $(a_i)_{i \in I} \equiv^2_{(c_j)_{j \in J}} (b_i)_{i \in I}$. So let $M \to N$ be an extension and let $m_0 : M_0 \to N$ be such that all arrows in $(c_j)_{j \in J}$ factor through m_0, M_0 is a model and $\mathrm{gtp}((a_i)_{i \in I}, m_0; N) = \mathrm{gtp}((b_i)_{i \in I}, m_0; N)$. Pick some $d : D \to N$ such that every arrow in $(a_i)_{i \in I}$ factors through d. We then find an extension $N \to N'$ and $d' : D \to N'$ such that every arrow in $(b_i)_{i \in I}$ factors through d' and $\mathrm{gtp}(d, m_0; N') = \mathrm{gtp}(d', m_0; N')$. Now we can apply the original result Theorem 5.17 to obtain $d \equiv^2_{m_0} d'$ and hence $(a_i)_{i \in I} \equiv^2_{(c_j)_{j \in J}} (b_i)_{i \in I}$, as required.

Remark 5.24. In Remark 5.18 we explained why we only assumed 3-AMALGAMATION. If we also assume INDEPENDENCE THEOREM together with base$(\mathop{\smile\!\!\!\!\!\!|}\,) = \mathcal{C}$, as is for example true in any simple thick theory where $\mathop{\smile\!\!\!\!\!\!|}\,$ is he usual dividing independence (see Example 5.8). We would then get a little bit more, namely that $\mathrm{Lgtp}(a/c; M) = \mathrm{Lgtp}(b/c; M)$ is further equivalent to $a \equiv_c^{\mathop{\smile\!\!\!\!\!\!|}} b$.

The proof of this is largely the same as the proof in this section. We sketch where some changes would need to be made. We adjust Lemma 5.20 as follows: any 2-$\mathop{\smile\!\!\!\!\!\!|}_c$-Morley sequence $(a_i)_{i<\delta}$ such that $\mathrm{Lgtp}(a_i/c; M) = \mathrm{Lgtp}(a_j/c; M)$ for all $i < j < \delta$ is a strong 2-$\mathop{\smile\!\!\!\!\!\!|}_c$-Morley sequence. Here the extra assumption "$\mathrm{Lgtp}(a_i/c; M) = \mathrm{Lgtp}(a_j/c; M)$" is necessary to still apply Lemma 5.2, and so the proof goes through. Then Lemma 5.21 can be restated as $\mathrm{Lgtp}(a/c; M) = \mathrm{Lgtp}(b/c; M)$ implies $a \equiv_c^{\mathop{\smile\!\!\!\!\!\!|}} b$. The only change in the proof is that we need to apply strong extension Corollary 4.13. This then already concludes the proof.

This also tells us that in this case we will need at most two strong 2-$\mathop{\smile\!\!\!\!\!\!|}_c$-Morley sequences to connect a and b. That is, there is some a' in an extension of M such that $a \sim_c^{\mathop{\smile\!\!\!\!\!\!|}} a' \sim_c^{\mathop{\smile\!\!\!\!\!\!|}} b$. In particular this also means that we need at most two strongly 2-c-indiscernible sequences to connect a and b, because strong 2-$\mathop{\smile\!\!\!\!\!\!|}_c$-Morley sequences are in particular strongly 2-c-indiscernible. This is closely related to the notion of G-compactness (in settings where that makes sense), which is equivalent to saying that having the same Lascar strong type is type-definable. More precisely, for thick theories we get G-compactness in the above situation. To see this, note that being on a strongly 2-c-indiscernible sequence is the same as being on a c-indiscernible sequence, because we can elongate the 2-c-indiscernible sequence to something long enough to then base an indiscernible sequence on (this works in any finitely short AECat by Lemma 4.54). Furthermore, in a thick theory having the same Lascar strong Galois type coincides with having the same Lascar strong type, see Remark 3.21. So by the above we have $a \equiv_c^{\mathrm{Ls}} b$ if and only if $\mathrm{d}_c(a, b) \leq 2$ and the latter is type-definable in a thick theory.

6

Kim-independence in positive logic

This chapter is joint work with Jan Dobrowolski and the work is also contained in [DK21].

While forking and dividing have been developed for simple theories in positive logic [Pil00, BY03b], Kim-dividing for NSOP_1 theories has only been developed in full first-order logic. This work was started by Kaplan and Ramsey in [KR20], inspired by ideas from Kim [Kim09]. In this chapter we generalise this work on Kim-independence in NSOP_1 theories to thick positive theories. We start by recalling basic notions of forking, dividing, heirs and coheirs for positive logic in section 6.1. Then in section 6.2 we give the definition for NSOP_1 in positive logic. We have to be careful here not to rely on any negations that are implicit in the definition for full first-order logic. This can be solved by an idea of [HK21], see also Remark 6.7.

In section 6.4 we give the definition of Kim-dividing and we prove some basic properties. In [KR20] Kim-dividing is defined as dividing with respect to a Morley sequence in some global invariant type. This would work in positive logic as well, if these global invariant types exist. In semi-Hausdorff theories they do exist [BY03c, Lemma 3.11], but in thick theories they might not (see subsection 6.10.1). We solve this by using global Lascar-invariant types, which we discuss one section earlier, in section 6.3.

Section 6.5 contains mostly technical tools. One of the technical challenges in adapting the results from [KR20, KR19] to the positive setting is that the tree modelling property [KKS14, Theorem 4.3], on which most of the constructions there rely, is not available in the positive setting. This forced us in particular to work only with trees of finite height, which turns out to be enough due to compactness and a careful choice of the global types with which we work. Consequently, we substitute the notion of a tree Morley sequence used in [KR20] with a weaker notion of a CR-Morley sequence, see Definition 6.48.

In the next few sections we prove the main properties of Kim-independence, one in each section. That is: symmetry (section 6.6), the independence theorem (section 6.7) and transitivity (section 6.8).

In section 6.9 we prove a Kim-Pillay style theorem, characterising which thick positive theories are NSOP_1 by the existence of a certain independence relation. Furthermore, this independence relation must then be the same as Kim-independence. In this section we also verify that Kim-independence has (club) local character.

Finally, in section 6.10, we discuss three examples. The first example, subsection 6.10.1, is a thick non-semi-Hausdorff NSOP$_1$ theory, which shows that the generality of thick theories in which we work is really more general than semi-Hausdorff theories. The next example, subsection 6.10.2, is that of existentially closed exponential fields, as studied in [HK21]. This is an example of a Hausdorff NSOP$_1$ theory that does not fit in the framework of full first-order logic. We use our Kim-Pillay style theorem, Theorem 6.79, to verify that the independence relation given in [HK21] is indeed Kim-independence. The third example, subsection 6.10.3, continues the study of hyperimaginaries from section 2.2. So this is not so much an example of one concrete theory, but rather an entire class of theories to which our work applies. The main result is that being NSOP$_1$ is preserved when moving to a hyperimaginary extension, so in particular our work applies to T^{heq} for every theory T in full first-order logic.

Throughout this entire chapter, when we consider a theory T we will implicitly assume that it has JEP and work in a monster model $\mathfrak{M}$ of T.

6.1 Forking, dividing, heirs and coheirs

In this section we discuss some definitions about various notions of independence in positive logic, namely those of forking, dividing, heirs and coheirs. Throughout we work in (a monster model of) some positive theory T.

The definition of dividing in positive theories is the same as in full first-order logic [Pil00, BY03b]. Following [Pil00] we have to adjust forking to allow infinite disjunctions because compactness can no longer guarantee disjunctions to be finite.

Definition 6.1. We say that a partial type $\Sigma(x, b)$ *divides over* C if there is a C-indiscernible sequence $(b_i)_{i<\omega}$ with $b_0 \equiv_C b$ such that $\bigcup_{i<\omega} \Sigma(x, b_i)$ is inconsistent.

We say that $\Sigma(x, b)$ *forks over* C if there is a (possibly infinite) set of formulas $\Phi(x)$ with parameters, each of which divides over C, such that $\Sigma(x, b)$ implies $\bigvee \Phi(x)$.

We write $a \mathop{\smile\!\!\!\!\!\!|}\,^d_C b$ (or $a \mathop{\smile\!\!\!\!\!\!|}\,^f_C b$) if $\text{tp}(a/Cb)$ does not divide (fork) over C.

Remark 6.2. We have that $\text{tp}(a/Cb)$ divides over C if and only if there is a formula $\varphi(x, b) \in \text{tp}(a/Cb)$ that divides over C. This follows directly from compactness. Note that for forking this is no longer necessarily true, because the disjunction may be infinite so we cannot apply compactness.

Definition 6.3. For a type p over a set B and a subset $A \subseteq B$, the restriction of p to A is a type over A which we denote by $p|_A$.

Definition 6.4. Let $M \subseteq B$ and let $p = \text{tp}(a/B)$ be a type over B. We say that p is a *coheir* of $p|_M$, and write $a \mathop{\smile\!\!\!\!\!\mid}^u_M B$, if p is finitely satisfiable in M. We say that p is an *heir* of $p|_M$ if for every formula $\varphi(x, y)$, with parameters in M, and every $b \in B$ such that $\varphi(x, b) \in p$ there is some $b' \in M$ such that $\varphi(x, b') \in p$. In this case we write $a \mathop{\smile\!\!\!\!\!\mid}^h_M B$.

Remark 6.5. It is direct from the definition that we always have $a \mathop{\smile\!\!\!\!\!\mid}^u_M M$ and $a \mathop{\smile\!\!\!\!\!\mid}^h_M M$. It also directly follows from the definition that $a \mathop{\smile\!\!\!\!\!\mid}^h_M b$ if and only if $b \mathop{\smile\!\!\!\!\!\mid}^u_M a$.

In Proposition 6.17 we compare the above notions of independence further.

6.2 NSOP$_1$

The goal of this chapter is to study Kim-independence in positive logic. In simple theories Kim-independence coincides with forking independence, and thus with dividing independence as well. The class of NSOP$_1$ theories is more general than the class of simple theories. In this class Kim-independence is still well-behaved, while forking independence may not be as well-behaved any more. Note that all the claims in this introduction follow easily from existing Kim-Pillay style theorems (e.g. [BY03b]) for stable and simple theories, together with the one that we prove for NSOP$_1$ theories (Theorem 6.79).

In this section we give the definition of (N)SOP$_1$ and a useful lemma, Lemma 6.8, for showing that a theory has SOP$_1$. It is really that lemma that we will use the most.

We recall that $2^{<\omega}$ is the set of all finite sequences of zeroes and ones. For $\eta, \nu \in 2^{<\omega}$ we write $\eta \preceq \nu$ if ν continues the sequence η. We write $\eta^\frown\nu$ for concatenation, so for example $\eta^\frown 0$ is the sequence η with a 0 concatenated to it.

Definition 6.6. Let T be a theory and let $\varphi(x, y)$ be a formula. We say that $\varphi(x, y)$ has *SOP$_1$* if there are $\psi(y_1, y_2)$ and $(a_\eta)_{\eta \in 2^{<\omega}}$ such that:

(i) for every $\sigma \in 2^\omega$ the set $\{\varphi(x, a_{\sigma|n}) : n < \omega\}$ is consistent;

(ii) $\psi(y_1, y_2)$ implies that $\varphi(x, y_1) \wedge \varphi(x, y_2)$ is inconsistent, that is

$$T \models \forall y_1 y_2 \neg[\psi(y_1, y_2) \wedge \exists x(\varphi(x, y_1) \wedge \varphi(x, y_2))];$$

(iii) for every $\eta, \nu \in 2^{<\omega}$ such that $\eta^\frown 0 \preceq \nu$ we have $\models \psi(a_{\eta^\frown 1}, a_\nu)$.

We say that T is *NSOP$_1$* if no formula has SOP$_1$.

Remark 6.7. The idea of introducing the inconsistency witness $\psi(y_1, y_2)$ is due to Haykazyan and Kirby, [HK21]. In a theory in full first-order logic we can just take $\psi(y_1, y_2)$ to be $\neg\exists x(\varphi(x, y_1) \wedge \varphi(x, y_2))$, so we see that the definitions coincide there. The point of having ψ is that the inconsistency in (iii) is again definable by a single formula for all relevant η and ν. This enables us to apply compactness to make the tree $(a_\eta)_{\eta \in 2^{<\omega}}$ as big as we wish.

The following lemma gives us a condition that implies SOP_1. It is the contrapositive that will actually be useful to us: if, in an NSOP_1 theory, we have two sequences that are 'parallel to each other' in a certain way then we can transfer consistency for a formula along one sequence to the other.

Lemma 6.8 (Parallel sequences lemma). *Suppose that $\varphi(x, y)$ is a formula, and $(\bar{c}_i) = (c_{i,0}, c_{i,1})_{i \in I}$ is an indiscernible sequence satisfying:*

(i) $c_{i,0} \equiv_{\bar{c}_{<i}} c_{i,1}$ *for all* $i \in I$*;*

(ii) $\{\varphi(x; c_{i,0}) : i \in I\}$ *is consistent;*

(iii) $\{\varphi(x; c_{i,1}) : i \in I\}$ *is inconsistent.*

Then T has SOP_1.

Proof. This is the same as [KR20, Lemma 2.3] and that proof mostly goes through. We sketch a few small changes that are needed. Obviously we already start with an indiscernible sequence and by compactness we can freely change the order type of I preserving properties (i)–(iii). Then in the claim in that proof we need to make the array $(a_{i,0}, a_{i,1})$ sufficiently long. This can easily be done by elongating the original indiscernible sequence $(\bar{c}_i)$. Then we can find an indiscernible sequence based on $(\bar{a}_i) = (a_{i,0}, a_{i,1})$. Note that properties (1)–(3) in that claim are preserved by this operation. The reason for all this is because we need to start with an indiscernible sequence in [KR20, Lemma 2.2] as well. Then the rest of that proof goes through. Finally, inconsistency of $\{\varphi(x, c_{l,1}), \chi(x, d_{l',0})\}$ should be witnessed by some formula (similarly for [KR20, Lemma 2.2]), but the existence of such a witness easily follows from the construction of χ. $\square$

6.3 Global Lascar-invariant types

In this section we define the notion of a global Lascar-invariant type. These are like global invariant types, but we work with Lascar strong types everywhere instead of normal types. The reason for this is that in a thick theory not every type over an e.c. model has a global invariant extension (see subsection 6.10.1), while this is crucial in the definition of Kim-dividing. In this section we prove that global Lascar-invariant extensions do exist in thick theories, and that they can be used to define Morley sequences in a similar way to global invariant types. This turns out to be enough for the definition of Kim-dividing.

Convention 6.9. Recall that a *global type* is a type over the monster model $\mathfrak{M}$. Building on Convention 2.28 about the monster model, we will use lowercase Greek letters $\alpha, \beta, \ldots$ for realisations of global types (in a bigger monster).

Definition 6.10. A global type q is called *A-Ls-invariant*, short for *A-Lascar-invariant*, if for a realisation $\alpha \models q$ we have that $b \equiv_A^{\mathrm{Ls}} b'$ implies $\alpha b \equiv_A^{\mathrm{Ls}} \alpha b'$.

Note that this definition does not depend on the choice of α. If α' is any other realisation of q, then $\alpha \equiv_{\mathfrak{M}} \alpha'$. So there is an automorphism f of the bigger monster over $\mathfrak{M}$ with $f(\alpha) = \alpha'$. So if $b \equiv_A^{\mathrm{Ls}} b'$ then $\alpha b \equiv_A^{\mathrm{Ls}} \alpha b'$ and hence $f(\alpha)f(b) \equiv_{f(A)}^{\mathrm{Ls}} f(\alpha)f(b')$ which is just $\alpha' b \equiv_A^{\mathrm{Ls}} \alpha' b'$, since f fixes $\mathfrak{M}$.

Remark 6.11. Let q be any global type in a thick theory, $\alpha \models q$ and let A be any (small) parameter set. Then there is $a \in \mathfrak{M}$ with $a \equiv_A^{\mathrm{Ls}} \alpha$. To see this, let $M \supseteq A$ be a λ_T-saturated model, and take any $a \models q|_M$.

Lemma 6.12. *Suppose that q is a global A-Ls-invariant type in a thick theory. Then:*

(i) for any $f \in \mathrm{Aut}(\mathfrak{M}/A)$ the type $f(q)$ is A-Ls-invariant;

(ii) for any $B \supseteq A$, q is also B-Ls-invariant.

Proof. Point (i) is straightforward, we prove (ii). Let $\alpha \models q$ and $b \equiv_B^{\mathrm{Ls}} b'$. Then there are λ_T-saturated models $M_1, \ldots, M_n$, all containing B, and $b = b_0, \ldots, b_n = b'$ such that $b_i \equiv_{M_{i+1}} b_{i+1}$ for all $0 \leq i < n$. Let $0 \leq i < n$, it is enough to show $\alpha b_i \equiv_{M_{i+1}} \alpha b_{i+1}$. We have $b_i M_{i+1} \equiv_A^{\mathrm{Ls}} b_{i+1} M_{i+1}$, so by A-Ls-invariance $\alpha b_i M_{i+1} \equiv_A^{\mathrm{Ls}} \alpha b_{i+1} M_{i+1}$, which implies the desired result. $\square$

Lemma 6.13. *Let T be thick and let $p = \mathrm{tp}(a/B)$ be a coheir over $M \subseteq B$. Then there is a global M-Ls-invariant type extending p.*

Proof. Define

$$\Gamma(x) = p(x) \cup \bigcup \{\mathrm{d}_M(xc, xc') \leq 1 : c, c' \in \mathfrak{M} \text{ with } \mathrm{d}_M(c, c') \leq 1\}.$$

We claim that $\Gamma(x)$ is consistent. For finite $p_0(x) \subseteq p(x)$ there is $d \in M$ such that $d \models p_0$. Then for any c, c' with $\mathrm{d}_M(c, c') \leq 1$ we have that $\mathrm{d}_M(dc, dc') \leq 1$ because d is in M. Any maximal extension of $\Gamma(x)$ will be a desired global M-Ls-invariant type. $\square$

Definition 6.14. For $A \subseteq B$ we say that $\mathrm{Lstp}(c/B)$ *extends* $\mathrm{Lstp}(c'/A)$ if $c \equiv_A^{\mathrm{Ls}} c'$.

Corollary 6.15. *In a thick theory we have that* $\mathrm{Lstp}(a/M)$ *extends to a global M-Ls-invariant type for any a and M.*

Proof. By Lemma 6.13 we have that $p = \mathrm{tp}(a/M)$ extends to some global M-Ls-invariant type q. For $\alpha \models q$ let $a' \equiv_M^{\mathrm{Ls}} \alpha$. Then there is $f \in \mathrm{Aut}(\mathfrak{M}/M)$ such that $f(a') = a$. So by Lemma 6.12(i) $f(q)$ is global M-Ls-invariant and is exactly what we need. □

Definition 6.16. For a type $p = \mathrm{tp}(a/Cb)$ write $a \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_C^{iLs} b$ if there is a global C-Ls-invariant extension of p.

Proposition 6.17. *In any thick theory T we have*

$$a \mathop{\smile\!\!\!\!\!\!\mid}\limits_C^{u} b \implies a \mathop{\smile\!\!\!\!\!\!\mid}\limits_C^{iLs} b \implies a \mathop{\smile\!\!\!\!\!\!\mid}\limits_C^{f} b \implies a \mathop{\smile\!\!\!\!\!\!\mid}\limits_C^{d} b.$$

Proof. Standard, but we write out the arguments to check they hold with the slightly changed definitions for positive logic. The first implication is precisely Lemma 6.13, while the last implication is direct from the definition of dividing and forking.

We prove the middle implication. Assume $a \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_C^{iLs} b$ and suppose for a contradiction that $p(x) = \mathrm{tp}(a/Cb)$ forks over C. Let $\Phi(x)$ be a set of formulas that all divide over C, such that $p(x)$ implies $\bigvee \Phi(x)$. Let q be a global C-Ls-invariant extension of p and let $\alpha \models q$. Then there must be $\varphi(x, d) \in \Phi(x)$ such that $\models \varphi(\alpha, d)$. Let $(d_i)_{i<\omega}$ be C-indiscernible with $d_0 = d$. For all $i < \omega$ we have $d \equiv_C^{\mathrm{Ls}} d_i$ and thus $\alpha d \equiv_C^{\mathrm{Ls}} \alpha d_i$. So in particular $\alpha \models \{\varphi(x, d_i) : i < \omega\}$, which contradicts that $\varphi(x, d)$ divides over C. □

In the remainder of this section we will develop tensoring of global Ls-invariant types. This comes down to verifying that the usual constructions for global invariant types (see e.g. [Sim15, Section 2.2.1]) work when we carefully replace types by Lascar strong types everywhere.

Lemma 6.18. *Suppose T is thick, q a global A-Ls-invariant type and $p = \mathrm{Lstp}(a^*/A)$. Then, for $\beta \models q$, the set*

$$R_{p,q}(A) = \{(a, b) \in \mathfrak{M} : a \equiv_A^{\mathrm{Ls}} a^* \text{ and } b \equiv_{Aa}^{\mathrm{Ls}} \beta\}$$

is (the set of realisations of) a Lascar strong type over A.

Proof. Clearly this does not depend on the choice of a^* or β. The set is non-empty, as for any $b \equiv_{Aa^*}^{\mathrm{Ls}} \beta$ we have $(a^*, b) \in R_{p,q}(A)$.

Let $(a, b), (a', b') \in R_{p,q}(A)$. Then $a \equiv_A^{\mathrm{Ls}} a^* \equiv_A^{\mathrm{Ls}} a'$, so by A-Ls-invariance $ab \equiv_A^{\mathrm{Ls}} a\beta \equiv_A^{\mathrm{Ls}} a'\beta \equiv_A^{\mathrm{Ls}} a'b'$. Conversely, suppose $(a, b) \in R_{p,q}(A)$ and $ab \equiv_A^{\mathrm{Ls}} a'b'$. Then $a' \equiv_A^{\mathrm{Ls}} a \equiv_A^{\mathrm{Ls}} a^*$. Furthermore, by A-Ls-invariance $\beta ab \equiv_A^{\mathrm{Ls}} \beta a'b'$, so applying an automorphism to $b \equiv_{Aa}^{\mathrm{Ls}} \beta$ we get $b' \equiv_{Aa'}^{\mathrm{Ls}} \beta$ and conclude that $(a', b') \in R_{p,q}(A)$. □

Theorem 6.19. *Suppose T is thick with global A-Ls-invariant types q and r. Then there is a unique global A-Ls-invariant type $q \otimes r$ such that for any $\alpha \models q$, $\beta \models r$ and $(\alpha', \beta') \models q \otimes r$ the following are equivalent for all $B \supseteq A$:*

(i) $ab \equiv^{\mathrm{Ls}}_B \alpha'\beta'$,

(ii) $a \equiv^{\mathrm{Ls}}_B \alpha$ *and* $b \equiv^{\mathrm{Ls}}_{Ba} \beta$.

In particular this implies that also $\alpha' \models q$ and $\beta' \models r$.

Proof. Throughout, let $\alpha \models q$ and $\beta \models r$. For $B \supseteq A$, denote by q_B the Lascar strong type $\mathrm{Lstp}(\alpha/B)$. By Lemma 6.12(ii) and Lemma 6.18, we have a well-defined Lascar strong type $R_{q_B,r}(B)$.

Claim. For $A \subseteq B \subseteq C$ we have $R_{q_C,r}(C) \subseteq R_{q_B,r}(B)$.
Proof of claim. Let $(a, b) \in R_{q_C,r}(C)$. Then $a \equiv^{\mathrm{Ls}}_C \alpha$ and $b \equiv^{\mathrm{Ls}}_{Ca} \beta$. Hence $a \equiv^{\mathrm{Ls}}_B \alpha$ and $b \equiv^{\mathrm{Ls}}_{Ba} \beta$, so $(a, b) \in R_{q_B,r}(B)$.

For $M \supseteq A$ a λ_T-saturated model $R_{q_M,r}(M)$ corresponds to the usual syntactic type over M. So viewing $R_{q_M,r}(M)$ as a set of formulas over M, we get by the claim that the following is a well-defined global type:

$$q \otimes r := \bigcup \{R_{q_M,r}(M) : M \text{ is a } \lambda_T\text{-saturated model and } A \subseteq M\}.$$

First we verify that $q \otimes r$ satisfies the universal property we claimed. So let $(\alpha', \beta') \models q \otimes r$ and $B \supseteq A$. Let $M \supseteq B$ be a λ_T-saturated model and pick $a'b' \equiv^{\mathrm{Ls}}_M \alpha'\beta'$. Then by construction $(a', b') \in R_{q_M,r}(M)$ and so by the claim $(a', b') \in R_{q_B,r}(B)$. So for any a, b we have $ab \equiv^{\mathrm{Ls}}_B \alpha'\beta'$ if and only if $ab \equiv^{\mathrm{Ls}}_B a'b'$ if and only if $(a, b) \in R_{q_B,r}(B)$ if and only if $a \equiv^{\mathrm{Ls}}_B \alpha$ and $b \equiv^{\mathrm{Ls}}_{Ba} \beta$.

Uniqueness follows because any global type satisfying this universal property must restrict to $R_{q_M,r}(M) = (q \otimes r)|_M$ for all λ_T-saturated $M \supseteq A$.

Finally we prove A-Ls-invariance. Let $d \equiv^{\mathrm{Ls}}_A d'$, and pick a, b in $\mathfrak{M}$ such that $ab \equiv^{\mathrm{Ls}}_{Add'} \alpha'\beta'$. So $a \equiv^{\mathrm{Ls}}_{Add'} \alpha'$ and thus by A-Ls-invariance of q:

$$ad \equiv^{\mathrm{Ls}}_A \alpha' d \equiv^{\mathrm{Ls}}_A \alpha' d' \equiv^{\mathrm{Ls}}_A ad'.$$

Then A-Ls-invariance of r gives us $\beta' ad \equiv^{\mathrm{Ls}}_A \beta' ad'$. From the universal property we get $b \equiv^{\mathrm{Ls}}_{Add'a} \beta'$, so $abd \equiv^{\mathrm{Ls}}_A abd'$. Because, by assumption, $ab \equiv^{\mathrm{Ls}}_{Add'} \alpha'\beta'$, we conclude that $\alpha'\beta' d \equiv^{\mathrm{Ls}}_A \alpha'\beta' d'$ and we are done. $\square$

Lemma 6.20. *For any global A-Ls-invariant types p, q, r in a thick theory we have:*

(i) associativity: $(p \otimes q) \otimes r = p \otimes (q \otimes r)$;

(ii) monotonicity: for any $q'(x_0) = q(x_0, x_1)|_{x_0} \subseteq q(x_0, x_1)$ and $r'(y_0) = r(y_0, y_1)|_{y_0} \subseteq r(y_0, y_1)$, we have $q' \otimes r' \subseteq q \otimes r$.

Proof. (i) Let $(\alpha, \beta, \gamma) \models (p \otimes q) \otimes r$ and $(\alpha', \beta', \gamma') \models p \otimes (q \otimes r)$. We will prove that $\alpha\beta\gamma \equiv^{\mathrm{Ls}}_B \alpha'\beta'\gamma'$ for all $B \supseteq A$. Let $abc \equiv^{\mathrm{Ls}}_B \alpha\beta\gamma$, then $b \equiv^{\mathrm{Ls}}_{Ba} \beta$ and $c \equiv^{\mathrm{Ls}}_{Bab} \gamma$. So we have $bc \equiv^{\mathrm{Ls}}_{Ba} \beta'\gamma'$. Since also $a \equiv^{\mathrm{Ls}}_B \alpha$ we thus conclude that $abc \equiv^{\mathrm{Ls}}_B \alpha'\beta'\gamma'$.

(ii) Let $(\alpha, \beta) = ((\alpha_0, \alpha_1), (\beta_0, \beta_1)) \models q \otimes r$ and let $ab \equiv^{\mathrm{Ls}}_B \alpha\beta$, where $B \supseteq A$ is arbitrary. Then in particular $a_0 \equiv^{\mathrm{Ls}}_B \alpha_0$ and $b_0 \equiv^{\mathrm{Ls}}_{Ba_0} \beta_0$. So if we let $(\alpha', \beta') \models q' \otimes r'$ then $\alpha_0\beta_0 \equiv^{\mathrm{Ls}}_B a_0b_0 \equiv^{\mathrm{Ls}}_B \alpha'\beta'$. So $(\alpha_0, \beta_0) \models q' \otimes r'$ and we are done. $\square$

Definition 6.21. For a global A-Ls-invariant type, we define $q^{\otimes\delta}$ for an ordinal $\delta \geq 1$ by induction as follows.

- $q^{\otimes 1} = q$,
- $q^{\otimes \delta+1} = q^{\otimes\delta} \otimes q$,
- $q^{\otimes\delta} = \bigcup_{\gamma<\delta} q^{\otimes\gamma}$ when δ is a limit.

A *Morley sequence in q (over A)* is a sequence $(a_i)_{i<\delta}$ such that $(a_i)_{i<\delta} \equiv^{\mathrm{Ls}}_A (\alpha_i)_{i<\delta}$, where $(\alpha_i)_{i<\delta} \models q^{\otimes\delta}$.

Lemma 6.22. *Suppose that q is a global A-Ls-invariant type and let $(\alpha_i)_{i<\delta} \models q^{\otimes\delta}$. Then for any strictly increasing sequence $(i_\eta)_{\eta<\gamma}$ in δ we have that $(\alpha_{i_\eta})_{\eta<\gamma} \models q^{\otimes\gamma}$.*

Proof. From the construction of $q^{\otimes\delta}$ it is clear that for $\gamma < \delta$ and $(\alpha_i)_{i<\delta} \models q^{\otimes\delta}$ we have $(\alpha_i)_{i<\gamma} \models q^{\otimes\gamma}$.

We prove the lemma by induction to γ. The base case and the limit step are easy, so we prove the successor step. So suppose $(\alpha_{i_\eta})_{\eta<\gamma} \models q^{\otimes\gamma}$. We will prove $(\alpha_{i_\eta})_{\eta<\gamma}\alpha_{i_\gamma} \equiv^{\mathrm{Ls}}_B \alpha_{<\gamma}\alpha_\gamma$ for all $B \supseteq A$. Let $a_{\leq i_\gamma} \equiv^{\mathrm{Ls}}_B \alpha_{\leq i_\gamma}$, then in particular $(a_{i_\eta})_{\eta<\gamma} \equiv^{\mathrm{Ls}}_B (\alpha_{i_\eta})_{\eta<\gamma}$ and $a_{i_\gamma} \equiv^{\mathrm{Ls}}_{B(a_{i_\eta})_{\eta<\gamma}} \alpha_{i_\gamma}$. By the induction hypothesis and the universal property this means $(a_{i_\eta})_{\eta<\gamma}a_{i_\gamma} \equiv^{\mathrm{Ls}}_B \alpha_{<\gamma}\alpha_\gamma$, which concludes the successor step. $\square$

By Lemma 6.22 we have that $(a_i)_{i<\delta} \models q^{\otimes\delta}|_A$ if and only if $(a_{i_1}, \ldots, a_{i_n}) \models q^{\otimes n}|_A$ for all $i_1 < \ldots < i_n < \delta$. From this perspective it makes sense to make the following convention, even though we technically have not defined $q^{\otimes I}$ for arbitrary linear orders I.

Convention 6.23. Let I be any linear order and let q be a global A-Ls-invariant type. Then by $(a_i)_{i\in I} \models q^{\otimes I}|_A$ we mean that for any $i_1 < \ldots < i_n$ in I we have $(a_{i_1}, \ldots, a_{i_n}) \models q^{\otimes n}|_A$.

Proposition 6.24. *For any Morley sequence $(a_i)_{i<\delta}$ in a global A-Ls-invariant type q the following hold:*

(i) *for all* $i < \delta$, $a_i \equiv^{\mathrm{Ls}}_{Aa_{<i}} \alpha$, *where* $\alpha \models q$;

(ii) $(a_i)_{i<\delta}$ *is* A*-indiscernible.*

Proof. We first prove (i). Let $(\alpha_i)_{i<\delta} \models q^{\otimes \delta}$ and $i < \delta$. Then $a_{<i}a_i \equiv^{\mathrm{Ls}}_A \alpha_{<i}\alpha_i$. As $\alpha_{<i}\alpha_i \models q^{\otimes i} \otimes q$, the universal property yields $a_i \equiv^{\mathrm{Ls}}_{Aa_{<i}} \alpha_i$, as required.

For (ii), consider any $i_1 < \ldots < i_n < \delta$. By Lemma 6.22 we have that $\alpha_{i_1} \ldots \alpha_{i_n} \equiv_{\mathfrak{M}} \alpha_1 \ldots \alpha_n$, so in particular $\alpha_{i_1} \ldots \alpha_{i_n} \equiv^{\mathrm{Ls}}_A \alpha_1 \ldots \alpha_n$. As $(a_i)_{i<\delta} \equiv_A (\alpha_i)_{i<\delta}$, we conclude that $a_{i_1} \ldots a_{i_n} \equiv_A a_1 \ldots a_n$. □

6.4 Kim-dividing

The idea of Kim-dividing is to restrict dividing witnesses to non-forking Morley sequences. Proving the existence of such sequences over arbitrary sets turns out to be difficult, and is in fact an open problem for NSOP$_1$ theories in full first-order logic, see [DKR19, Remark 2.6, Question 6.6]. In [KR20] this is solved by using Morley sequences in some global invariant type. In first-order logic any type over a model extends to a global invariant type. In positive logic we need to assume the theory to be semi-Hausdorff to find global invariant extensions [BY03c, Lemma 3.11], because they may not exist otherwise (see Subsection 6.10.1). In the more general setting of thick positive theories we can always find global Ls-invariant extensions and the notion of a Morley sequence makes sense in such a global Ls-invariant type, see Section 6.3. Since we can generally only extend types over e.c. models to global Ls-invariant types, we will consider Kim-dividing only over e.c. models (cf. Question 6.105).

Definition 6.25. Let $\Sigma(x, b)$ be a partial type in a thick theory, possibly with parameters in M, and let q be a global M-Ls-invariant extension of $\mathrm{tp}(b/M)$. We say that $\Sigma(x, b)$ *q-divides over M* if for any (equivalently: some) Morley sequence $(b_i)_{i<\omega}$ in q (over M) the set $\bigcup_{i<\omega} \Sigma(x, b_i)$ is inconsistent.

By compactness q-dividing does not depend on the length of the Morley sequence, as long as it is infinite.

Proposition 6.26. *Let T be thick, let q be a global M-Ls-invariant extension of* $\mathrm{tp}(b/M)$ *and write* $p(x, y) = \mathrm{tp}(ab/M)$. *Then the following are equivalent.*

(i) *The type $p(x, b)$ does not q-divide.*

(ii) *For any* $f \in \mathrm{Aut}(\mathfrak{M}/M)$ *the type $p(x, b)$ does not $f(q)$-divide.*

(iii) *For any (equivalently some)* $(b_i)_{i<\omega} \models q^{\otimes \omega}|_M$ *the set* $\bigcup_{i<\omega} p(x, b_i)$ *is consistent.*

(iv) *There is an Ma-indiscernible sequence* $(b_i)_{i<\omega} \models q^{\otimes \omega}|_M$ *with* $b_0 = b$.

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) This follows because consistency of $\bigcup_{i<\omega} p(x, b_i)$ only depends on $\mathrm{tp}((b_i)_{i<\omega}/M)$, together with the fact that given a Morley sequence $(b_i)_{i<\omega}$ in q we have that $(f(b_i))_{i<\omega}$ is a Morley sequence in $f(q)$.

(i) $\implies$ (iv) Let $(b_i)_{i<\lambda}$ be a Morley sequence in q for big enough λ. Let a^* realise $\bigcup_{i<\lambda} p(x, b_i)$ and let $(b'_i)_{i<\omega}$ be Ma^*-indiscernible, based on $(b_i)_{i<\lambda}$. So there is $i < \lambda$ such that $a^* b'_0 \equiv_M a^* b_i \equiv_M ab$. Let $(b''_i)_{i<\omega}$ with $b''_0 = b$ be such that $a(b''_i)_{i<\omega} \equiv_M a^*(b'_i)_{i<\omega}$. Then $(b''_i)_{i<\omega}$ is Ma-indiscernible. Furthermore, since $(b_i)_{i<\lambda}$ was already M-indiscernible, we have $(b''_i)_{i<\omega} \equiv_M (b'_i)_{i<\omega} \equiv_M (b_i)_{i<\omega}$, so $(b''_i)_{i<\omega} \models q^{\otimes\omega}|_M$.

(iv) $\implies$ (iii) For such an Ma-indiscernible sequence $(b_i)_{i<\omega}$ we have $ab = ab_0 \equiv_M ab_i$ for all $i < \omega$. So a realises $\bigcup_{i<\omega} p(x, b_i)$. $\square$

Proposition 6.27. *Let T be thick, let $\Sigma(x, b)$ be a partial type with parameters in M and let q be a global M-Ls-invariant extension of* $\mathrm{tp}(b/M)$. *If $\Sigma(x, b)$ does not q-divide over M then there is a complete $p(x, b) \supseteq \Sigma(x, b)$ that does not q-divide over M.*

Proof. Let $(b_i)_{i<\lambda} \models q^{\otimes\lambda}|_M$ with $b_0 = b$. Then there is some $a \models \bigcup_{i<\lambda} \Sigma(x, b_i)$. Then, assuming we chose λ large enough, there is some $i_0 < \lambda$ such that for infinitely many $i < \lambda$ we have $ab_i \equiv_M ab_{i_0}$. Set $p(x, y) = \mathrm{tp}(ab_{i_0}/M)$, then $p(x, b_{i_0})$ does not q-divide while also $\Sigma(x, b_{i_0}) \subseteq p(x, b_{i_0})$. By invariance $p(x, b)$ does not q-divide. $\square$

The following lemma is the core of the connection between Kim-dividing and NSOP$_1$ theories. It tells us that q-dividing does not depend on the global Lascar-invariant type q. More discussion on the origins of this lemma can be found in [KR20]. Briefly put: Kim proved that in simple theories a formula divides with respect to every Morley sequence if and only if it divides with respect to some Morley sequence [Kim98, Proposition 2.1]. The lemma below is an analogue of that for NSOP$_1$ theories.

Proposition 6.28 (Kim's lemma)**.** *If T is thick NSOP$_1$, then q-dividing does not depend on q. That is, if q and r are global M-invariant types extending* $\mathrm{tp}(b/M)$ *then a partial type $\Sigma(x, b)$ q-divides if and only if it r-divides.*

Proof. This is essentially the proof of [KR20, Proposition 3.15], adapted to the thick positive logic setting. By Proposition 6.26(ii) we may assume that q and r extend $\mathrm{Lstp}(b/M)$. Suppose that $\Sigma(x, b)$ does not q-divide while it r-divides. We will prove that T has SOP$_1$. Let $(\bar{b}_i)_{i<\omega} = (b_{i,0}, b_{i,1})_{i<\omega}$ be a Morley sequence in $q \otimes r$. By Lemma 6.20(ii) and induction, $(b_{i,0})_{i<\omega}$ and $(b_{i,1})_{i<\omega}$ are Morley sequences in q and r respectively.

Since $\Sigma(x, b)$ r-divides, the set $\bigcup_{i<\omega} \Sigma(x, b_{i,1})$ is inconsistent. So by compactness there is an M-formula $\varphi(x, y) \in \Sigma(x, y)$ such that

$\{\varphi(x, b_{i,1}) : i < \omega\}$ is inconsistent. Because $\Sigma(x, b)$ does not q-divide we have that $\{\varphi(x, b_{i,0}) : i < \omega\}$ is consistent.

We wish to apply the parallel sequences lemma, Lemma 6.8, to $\varphi(x, y)$ and $(\bar{b}_i)_{i<\omega^{\text{op}}}$ where ω^{op} carries the opposite order of ω. So we are left to prove that $b_{i,0} \equiv_{M\bar{b}_{>i}} b_{i,1}$ for all $i < \omega$. We do this by proving that $b_{i,0}(\bar{b}_i)_{i<j<n} \equiv_M b_{i,1}(\bar{b}_i)_{i<j<n}$ for all $i < n < \omega$. Let $(\bar{\beta}_i)_{i<\omega} \models (q \otimes r)^{\otimes \omega}$. By Lemma 6.20(i) we have $(q \otimes r)^{\otimes n} = (q \otimes r)^{\otimes i+1} \otimes (q \otimes r)^{\otimes n-i-1}$. So we have $\bar{\beta}_{<n} \models (q \otimes r)^{\otimes i+1} \otimes (q \otimes r)^{\otimes n-i-1}$ and as $\bar{b}_{<n} \equiv^{\text{Ls}}_M \bar{\beta}_{<n}$ we have $(\bar{b}_j)_{i<j<n} \equiv^{\text{Ls}}_{M\bar{b}_{\leq i}} (\bar{\beta}_j)_{i<j<n}$. As $b_{i,0} \equiv^{\text{Ls}}_M b \equiv^{\text{Ls}}_M b_{i,1}$ we get by M-Ls-invariance that $b_{i,0}(\bar{\beta}_j)_{i<j<n} \equiv^{\text{Ls}}_M b_{i,1}(\bar{\beta}_j)_{i<j<n}$. Putting the two together yields the required result. $\square$

Definition 6.29. We say $\Sigma(x, b)$ *Kim-divides (over M)* if it q-divides for some global M-Ls-invariant q that extends $\text{tp}(b/M)$. We write $a \mathop{\smile\!\!\!\!\!\!\mid\;}^K_M b$ when $\text{tp}(a/Mb)$ does not Kim-divide over M and call this *Kim-independence*.

Remark 6.30. By Lemma 6.13 we can extend any type over an e.c. model M in a thick theory to a global M-Ls-invariant type. So assuming NSOP_1, we have by Proposition 6.28 that $\text{tp}(a/Mb)$ Kim-divides if and only if it q-divides for *any* global M-invariant extension q of $\text{tp}(b/M)$.

In some constructions it will be necessary to stay within the same Lascar strong type. For this we introduce the technical tool of q-Ls-dividing.

Definition 6.31. Let T be thick and let q be a global M-Ls-invariant extension of $\text{Lstp}(b/M)$. We say that $\text{Lstp}(a/Mb)$ *does not q-Ls-divide (over M)* if there is a Morley sequence $(b_i)_{i<\omega}$ in q with $b_0 = b$ that is Ma-indiscernible.

Remark 6.32. The length of the Morley sequence does not matter in Definition 6.31, as long as it is infinite. However, the argument here takes a little more care than for q-dividing.

One direction is clear: if there is an Ma-indiscernible Morley sequence $(b_i)_{i<\delta}$ in q for some $\delta \geq \omega$, then we can just take an initial segment. For the other direction we let $N \supseteq M$ be λ_T-saturated and $(b_i)_{i<\omega} \models q^{\otimes \omega}|_N$. Then $(b_i)_{i<\omega}$ is a Morley sequence in q. Applying a Lascar strong automorphism we find $a'b_0 \equiv^{\text{Ls}}_M ab$ such that $(b_i)_{i<\omega}$ is Ma'-indiscernible. Let n be such that $\text{d}_M(a'b_0, ab) \leq n$. Consider the set of formulas

$$q^{\otimes \delta}|_N((y_i)_{i<\delta}) \cup \text{“}(xy_i)_{i<\delta} \text{ is } M\text{-indiscernible”} \cup \text{d}_M(xy_0, ab) \leq n.$$

This set is finitely satisfiable, hence it has a realisation. So we find an Ma''-indiscernible Morley sequence $(b'_i)_{i<\delta}$ in q with $a''b'_0 \equiv^{\text{Ls}}_M ab$. The result follows by applying a Lascar strong automorphism.

Lemma 6.33. *Let T be thick and let q be a global M-Ls-invariant extension of* $\mathrm{Lstp}(b/M)$. *A type $p = \mathrm{tp}(a/Mb)$ does not q-divide if and only if there is a realisation $a' \models p$ such that* $\mathrm{Lstp}(a'/Mb)$ *does not q-Ls-divide.*

Proof. The right to left direction is clear by Proposition 6.26(iv). For the other direction we let $(b'_i)_{i<\omega}$ be a Morley sequence in q with $b'_0 = b$. By Proposition 6.26(iv) there is $(b_i)_{i<\omega} \models q^{\otimes\omega}|_M$ that is Ma-indiscernible with $b_0 = b$. Pick a' such that $a'(b'_i)_{i<\omega} \equiv_M a(b_i)_{i<\omega}$ and we are done. $\square$

Corollary 6.34. *Let T be thick and let q be a global M-Ls-invariant extension of* $\mathrm{Lstp}(b/M)$. *Suppose that there is $M \subseteq N \subseteq b$ such that N is λ_T-saturated. Then* $\mathrm{tp}(a/Mb)$ *does not q-divide if and only if* $\mathrm{Lstp}(a/Mb)$ *does not q-Ls-divide.*

Proof. By Lemma 6.33 we only need to prove the left to right direction. So suppose that $\mathrm{tp}(a/Mb)$ does not q-divide. Then there is a' with $a' \equiv_{Mb} a$ such that $\mathrm{Lstp}(a'/Mb)$ does not q-Ls-divide. In particular we have that $a'b \equiv_N ab$, so $a'b \equiv^{\mathrm{Ls}}_M ab$. It follows that $\mathrm{Lstp}(a/Mb)$ does not q-Ls-divide. $\square$

Proposition 6.35. *In a thick NSOP$_1$ theory Kim-independence always satisfies the following properties.*

(i) Strong finite character: if $a \not\mathop{\smile}\hspace{-0.9em}\mid\hspace{0.5em}^K_M b$ then there is a formula $\varphi(x, b, m) \in$ $\mathrm{tp}(a/Mb)$ *such that for any $a' \models \varphi(x, b, m)$ we have $a' \not\mathop{\smile}\hspace{-0.9em}\mid\hspace{0.5em}^K_M b$.*

(ii) Existence over models: $a \mathop{\smile}\hspace{-0.9em}\mid\hspace{0.5em}^K_M M$.

(iii) Monotonicity: $aa' \mathop{\smile}\hspace{-0.9em}\mid\hspace{0.5em}^K_M bb' \implies a \mathop{\smile}\hspace{-0.9em}\mid\hspace{0.5em}^K_M b$.

Proof. All follow directly from the definitions, using compactness for (i). $\square$

Remark 6.36. Let T be a thick theory. Then Kim-dividing implies dividing because any Morley sequence in some q is in particular an indiscernible sequence. So by Proposition 6.17:

$$a \mathop{\smile}\hspace{-0.9em}\mid\hspace{0.5em}^{u}_{M} b \implies a \mathop{\smile}\hspace{-0.9em}\mid\hspace{0.5em}^{iLs}_{M} b \implies a \mathop{\smile}\hspace{-0.9em}\mid\hspace{0.5em}^{f}_{M} b \implies a \mathop{\smile}\hspace{-0.9em}\mid\hspace{0.5em}^{d}_{M} b \implies a \mathop{\smile}\hspace{-0.9em}\mid\hspace{0.5em}^{K}_{M} b$$

The following lemma is a slightly stronger version of obtaining an indiscernible sequence from a very long sequence (Fact 2.37). By assuming that the input sequence is A-indiscernible and assuming thickness, we can find a B-indiscernible sequence that has the same Lascar strong type over A.

Lemma 6.37. *Let T be a thick theory. Let $B \supseteq A$, κ any cardinal and set $\lambda = \lambda_{|T|+|B|+\kappa}$. Then for any A-indiscernible sequence $(a_i)_{i<\lambda}$ of κ-tuples, there is B-indiscernible $(a'_i)_{i<\lambda}$ based on $(a_i)_{i<\lambda}$ such that* $\mathrm{d}_A((a_i)_{i<\lambda}, (a'_i)_{i<\lambda}) \leq 1$.

Proof. By Fact 2.37 there is B-indiscernible $(b_i)_{i<\omega}$ based on $(a_i)_{i<\lambda}$. Prolong this to B-indiscernible $(b_i)_{i<\lambda}$. Define

$$\Sigma((x_i)_{i<\lambda}) = \mathrm{tp}((b_i)_{i<\lambda}/B) \cup \text{“}\,\mathrm{d}_A((x_i)_{i<\lambda}, (a_i)_{i<\lambda}) \leq 1\text{”},$$

and let $\Sigma_0(x_{i_1}, \ldots, x_{i_n}) \subseteq \Sigma((x_i)_{i<\lambda})$ be finite, only mentioning parameters in B and $a_{i_1}, \ldots, a_{i_n}$. Let $j_1 < \ldots < j_n < \lambda$ be such that $a_{j_1} \ldots a_{j_n} \equiv_B b_1 \ldots b_n \equiv_B b_{i_0} \ldots b_{i_n}$. It follows from the proof of Fact 2.37 that we may choose j_1 to be arbitrarily large below λ, so we may assume $j_1 > i_n$. Then $a_{j_1} \ldots a_{j_n}$ realises Σ_0. By compactness we find the required $(a_i')_{i<\lambda}$ as a realisation of Σ. $\square$

Proposition 6.38. *Let T be a thick theory, M an e.c. model of T, and let a, b, c be tuples. Let also $q(x, y)$ be a global M-Ls-invariant extension of* $\mathrm{Lstp}(bc/M)$ *and write $r(x) = q|_x$. If* $\mathrm{Lstp}(a/Mb)$ *does not r-Ls-divide then there is $c^*b \equiv^{\mathrm{Ls}}_M cb$ such that* $\mathrm{Lstp}(a/Mbc^*)$ *does not q-Ls-divide.*

Proof. Let $(b_i, c_i)_{i<\lambda}$ be a Morley sequence over M in q for some big enough λ. Since $(b_i)_{i<\lambda}$ is a Morley sequence over M in r and $\mathrm{Lstp}(a/Mb)$ does not r-divide there is a' with $a'b_0 \equiv^{\mathrm{Ls}}_M ab$ such that $(b_i)_{i<\lambda}$ is Ma'-indiscernible.

Let $f \in \mathrm{Aut}_f(\mathfrak{M}/M)$ be such that $f(a'b_0) = ab$ and put $(b_i', c_i') = (f(b_i), f(c_i))$. Then $b_0' = b$, $(b_i')_{i<\lambda}$ is Ma-indiscernible and $(b_i', c_i')_{i<\lambda}$ is a Morley sequence over M in q.

Let $M' \supseteq Ma$ be λ_T-saturated and use Lemma 6.37 to find M'-indiscernible $(b_i'', c_i'')_{i<\lambda}$ based on $(b_i', c_i')_{i<\lambda}$ and such that $\mathrm{d}_M((b_i'', c_i'')_{i<\lambda}, (b_i', c_i')_{i<\lambda}) \leq 1$. In particular $(b_i'', c_i'')_{i<\lambda}$ is a Morley sequence over M in q. Let $i < \lambda$ be such that $b_0'' \equiv_{M'} b_i'$ then $b_0'' \equiv^{\mathrm{Ls}}_{Ma} b_i' \equiv^{\mathrm{Ls}}_{Ma} b_0' = b$. So there is $g \in \mathrm{Aut}_f(\mathfrak{M}/Ma)$ such that $g(b_0'') = b$. Set $c^* = g(c_0'')$, so $bc^* \equiv^{\mathrm{Ls}}_M b_0''c_0'' \equiv^{\mathrm{Ls}}_M b_0'c_0' \equiv^{\mathrm{Ls}}_M b_0c_0 \equiv^{\mathrm{Ls}}_M bc$. Finally, since $(g(b_i''), g(c_i''))_{i<\lambda}$ is a Morley sequence over M in q starting with bc^* that is Ma-indiscernible, we conclude that $\mathrm{Lstp}(a/Mbc^*)$ does not q-Ls-divide. $\square$

Corollary 6.39 (Extension)**.** *In a thick NSOP$_1$ theory we have that if $a \mathop{\smile\!\!\!\!\!\mid}^K_M b$ then for any c there is $c' \equiv^{\mathrm{Ls}}_{Mb} c$ such that $a \mathop{\smile\!\!\!\!\!\mid}^K_M bc'$.*

Proof. We first prove a weaker version where we conclude $c' \equiv_{Mb} c$ instead of $c' \equiv^{\mathrm{Ls}}_{Mb} c$.

Let $q(x, y)$ be an M-Ls-invariant extension of $\mathrm{Lstp}(bc/M)$ and write $r(x) = q|_x$, where x matches b. Since $a \mathop{\smile\!\!\!\!\!\mid}^K_M b$ there is $a'b \equiv_M ab$ such that $\mathrm{Lstp}(a'/Mb)$ does not r-Ls-divide. By Proposition 6.38 we thus find $bc^* \equiv^{\mathrm{Ls}}_M bc$ such that $\mathrm{Lstp}(a'/Mbc^*)$ does not q-Ls-divide. Letting c' be such that $abc' \equiv_M a'bc^*$ then satisfies $a \mathop{\smile\!\!\!\!\!\mid}^K_M bc'$ and furthermore we have $bc' \equiv_M bc^* \equiv_M bc$.

Now we use the weaker version to prove the full version. Let $N \supseteq Mb$ be some λ_T-saturated model. By the above we can find $N' \equiv_{Mb} N$ such that $a \mathop{\smile\!\!\!\!\!\mid}^K_M N'$. Then using the above again we find $c' \equiv_{N'} c$ such that $a \mathop{\smile\!\!\!\!\!\mid}^K_M N'c'$. Since $Mb \subseteq N'$ we thus get $c' \equiv^{\mathrm{Ls}}_{Mb} c$ and $a \mathop{\smile\!\!\!\!\!\mid}^K_M bc'$, as required. $\square$

6.5 EM-modelling and CR-Morley sequences

In this section we will introduce some tools which will be useful later in certain tree constructions.

Definition 6.40 ([KKS14, Defintion 2.1])**.** The *Shelah language*

$$L_s = \{\trianglelefteq, \wedge, <_{lex}, (P_\alpha)_{\alpha<\omega}\}$$

consists of binary relation symbols $\trianglelefteq, <_{lex}$, a binary function symbol $\wedge$, and unary relation symbols P_α. We will consider a tree $\omega^{\leq k}$ (with $k < \omega$) as an L_s-structure, where $\trianglelefteq$ is interpreted as the containment relation, $<_{lex}$ as the lexicographic order, $\wedge$ as the meet function and P_α as the α-th level of the tree.

Definition 6.41 ([KKS14, Definition 3.7])**.** Let I be an arbitrary index structure and C an arbitrary set of parameters. The *EM-type* of a tuple $A = (a_i)_{i\in I}$ over C is the partial type in variables $(x_i)_{i\in I}$, consisting of all the formulas of the form $\varphi(x_{\bar{i}})$ over C (where $\bar{i}$ is a tuple in I) satisfying the following property: $\models \varphi(a_{\bar{j}})$ holds whenever $\bar{j}$ is a tuple in I with $\mathrm{qftp}_I(\bar{j}) = \mathrm{qftp}_I(\bar{i})$. We let $EM_I(A/C)$ denote this partial type.

In particular, we will write $EM_s(A/C)$ [respectively, $EM_<(A/C)$] for $EM_I(A/C)$ where I is considered as an L_s-structure [respectively, a $\{<\}$-structure].

Definition 6.42. Let I be an index structure and let $A = (a_i)_{i\in I}$ and $B = (b_i)_{i\in I}$ be I-indexed tuples of compatible parameters. We will say that A is *EM_I-based on B over C* if $EM_I(A/C) \supseteq EM_I(B/C)$.

Corollary 6.43. *If A is any set of parameters, then for any compatible sequence $(a_i)_{i<\omega}$ there is an A-indiscernible sequence $(b_i)_{i<\omega}$ which is $EM_<$-based on $(a_i)_{i<\omega}$ over A.*

Proof. By compactness there is a sequence $(a'_i)_{i<\lambda_{|T|+|A|+|a_0|}}$ which is $EM_<$-based on $(a_i)_{i<\omega}$ over A. Then by Fact 2.37 there is an A-indiscernible sequence $(b_i)_{i<\omega}$ which is $EM_<$-based on $(a'_i)_{i<\lambda_{|T|+|A|+|a_0|}}$ over A, hence $EM_<$-based on $(a_i)_{i<\omega}$ over A. $\square$

In what follows we consider $\omega^{\leq k}$ as an L_s-structure (see Definition 6.40). We will only work with trees of width ω, as we will only need those, but everything naturally works for arbitrary (infinite) widths.

Definition 6.44. We call a tree $(a_\eta)_{\eta\in\omega^{\leq k}}$ *s-indiscernible over C* if for any $\bar{\eta}, \bar{\nu} \subseteq \omega^{\leq k}$ such that $\bar{\eta} \equiv_{qf} \bar{\nu}$ we have that $a_{\bar{\eta}} \equiv_C a_{\bar{\nu}}$.

Lemma 6.45. *Suppose $\bar{\eta} = (\eta_0, \ldots, \eta_{n-1}) \equiv_{qf} \bar{\nu} = (\nu_0, \ldots, \nu_{n-1})$ are tuples of elements of $\omega^{\leq k}$ for some $k < \omega$. Then there exists a sequence I of n-tuples of elements of $\omega^{\leq k}$ such that $\bar{\eta} \frown I$ and $\bar{\nu} \frown I$ are qf-indiscernible sequences in $\omega^{\leq k}$.*

Proof. Let $l < \omega$ be such that $\bar{\eta}, \bar{\nu} \subseteq \{\emptyset\} \cup \{\xi \in \omega^{\leq k} \backslash \{\emptyset\} : \xi(0) < l\}$. For every $0 < m < \omega$ choose a tuple $\bar{\chi}^m \subseteq \{\emptyset\} \cup \{\xi \in \omega^{\leq k} \backslash \{\emptyset\} : ml \leq \xi(0) < (m+1)l\}$ such that $\bar{\chi}^m \equiv_{qf} \bar{\eta} \equiv_{qf} \bar{\nu}$ (for example, for every $n' < n$ put $\chi^m_{n'}(0) = \eta_{n'}(0) + ml$ and $\chi^m_{n'}(i) = \eta_{n'}(i)$ for every $0 < i \leq k$). Finally, put $I = (\bar{\chi}^m)_{0<m<\omega}$. □

Corollary 6.46. *If T is thick then s-indiscernibility is type-definable, i.e. for every $k < \omega$ and a tuple of variables y there is a partial type $\pi((x_\eta)_{\eta \in \omega^{\leq k}}, y)$ over $\emptyset$ such that $((a_\eta)_{\eta \in \omega^{\leq k}}, D) \models \pi$ if and only if $(a_\eta)_{\eta \in \omega^{\leq k}}$ is s-indiscernible over D. More specifically, $\pi((x_\eta)_{\eta \in \omega^{\leq k}}, y)$ can be taken as the type expressing that $(x_{\eta_0}, \dots, x_{\eta_{n-1}})$ and $(x_{\nu_0}, \dots, x_{\nu_{n-1}})$ are at Lascar distance at most 2 over y for any $(\eta_0, \dots, \eta_{n-1}) \equiv_{qf} (\nu_0, \dots, \nu_{n-1})$.*

Proof. Let π be as above and consider arbitrary $(a_\eta)_{\eta \in \omega^{\leq k}}$ and D. If $((a_\eta)_{\eta \in \omega^{\leq k}}, D) \models \pi$ then $(a_\eta)_{\eta \in \omega^{\leq k}}$ is indiscernible over D as being at Lascar distance at most 2 over D implies equality of types over D.

Conversely, if $((a_\eta)_{\eta \in \omega^{\leq k}}, D)$ is s-indiscernible over D and $\bar{\eta} = (\eta_0, \dots, \eta_{n-1}) \equiv_{qf} \bar{\nu} = (\nu_0, \dots, \nu_{n-1})$, then with $I = (\bar{\chi}^m)_{0<m<\omega}$ given by Lemma 6.45 we have that $a_{\bar{\eta}} \frown (a_{\bar{\chi}^m})_{0<m<\omega}$ and $a_{\bar{\nu}} \frown (a_{\bar{\chi}^m})_{0<m<\omega}$ are both indiscernible sequences over D, so $a_{\bar{\eta}}$ and $a_{\bar{\nu}}$ are at Lascar distance at most 2 over D. □

We now adapt the proof of [KKS14, Theorem 4.3] to obtain the EM_s-modeling property for positive logic.

Proposition 6.47. *Suppose T is thick and consider arbitrary set of parameters D and $k < \omega$. Then for any tree $A = (a_\eta)_{\eta \in \omega^{\leq k}}$ of compatible tuples there is an s-indiscernible over D tree $C = (c_\eta)_{\eta \in \omega^{\leq k}}$ which is EM_s-based on A over D.*

Proof. We proceed by induction on k. The case $k = 0$ is trivial. Suppose the assertion holds for some k and consider any $A = (a_\eta)_{\eta \in \omega^{\leq k+1}}$. For any $i < \omega$ consider an $\omega^{\leq k}$-indexed tree $A_i := (a_{i \frown \eta})_{\eta \in \omega^{\leq k}}$. Using the inductive hypothesis we choose inductively for each $i < \omega$ a tree $B_i = (b^i_\eta)_{\eta \in \omega^{\leq k}}$ which is s-indiscernible over $Da_\emptyset B_{<i} A_{>i}$ and EM_s-based on A_i over $Da_\emptyset B_{<i} A_{>i}$. Let $B = (b_\eta)_{\eta \in \omega^{\leq k+1}}$ where $b_\emptyset = a_\emptyset$ and $b_{i \frown \xi} = b^i_\xi$ for every $i < \omega$ and $\xi \in \omega^{\leq k}$.

Claim 1. B_i is s-indiscernible over $Db_\emptyset B_{\neq i}$ for every $i < \omega$.

Fix $i < \omega$. We will show by induction on j that B_i is s-indiscernible over $Db_\emptyset B_{<i} B_{i+1} \dots B_{j-1} A_{\geq j}$ for every $j > i$, which is enough by Corollary 6.46. For $j = i + 1$ this follows directly from the choice of B_i. Now suppose the assertion holds for some $j > i$. By Corollary 6.46 there is a type $\pi((x_\eta)_{\eta \in \omega^{\leq k}}, \bar{y})$ over $D' := Db_\emptyset B_{<i} B_{i+1} \dots B_{j-1} A_{>j}$, where $\bar{y} = (y_\eta)_{\eta \in \omega^{\leq k}}$, expressing that $(x_\eta)_{\eta \in \omega^{\leq k}}$ is s-indiscernible over $D'\bar{y}$. Then $B_i A_j \models \pi$. Note that the type $\pi(B_i, \bar{y})$ is invariant under all permutations of $\bar{y}$, hence if $\varphi(y_{\eta_0}, \dots, y_{\eta_{n-1}}) \in \pi(B_i, \bar{y})$ then $\varphi(y_{\nu_0}, \dots, y_{\nu_{n-1}}) \in \operatorname{tp}(A_j / D' B_i)$ for all $\nu_0, \dots, \nu_{n-1} \in \omega^{\leq k}$. In particular,

$\pi(B_i, \bar{y}) \subseteq EM_s(A_j/D'B_i)$. Thus, by the choice of B_j, we have that $\pi(B_i, \bar{y}) \subseteq EM_s(B_j/D'B_i)$, so in particular $B_iB_j \models \pi$. Hence B_i is indiscernible over $D'B_j = Db_\emptyset B_{<i}B_{i+1}\dots B_jA_{\geq j+1}$, as required.

Claim 2. B is EM_s-based on A over D.

Consider any $i < \omega$ and the trees $E = (e_\eta)_{\eta\in\omega^{\leq k+1}}$ and $F = (f_\eta)_{\eta\in\omega^{\leq k+1}}$ given by $e_\emptyset = f_\emptyset = a_\emptyset$, $e_{j\frown\eta} = \begin{cases} b_{j\frown\eta} & \text{for } j < i \\ a_{j\frown\eta} & \text{for } j \geq i \end{cases}$, and $f_{j\frown\eta} = \begin{cases} b_{j\frown\eta} & \text{for } j \leq i \\ a_{j\frown\eta} & \text{for } j > i \end{cases}$. We will prove that $\pi_0 := EM_s(E/D) \subseteq EM_s(F/D) =: \pi_1$ which clearly is sufficient to prove the claim. Let $\bar{x} = (x_\eta)_{\eta\in\omega^{\leq k+1}}$ be a tuple of variables compatible with a_η's. We naturally view π_0 and π_1 as partial types in the variable $\bar{x}$. Consider any formula $\varphi(x_{\eta_0}, \dots, x_{\eta_l}, x_{\eta_{l+1}}, \dots, x_{\eta_{l'}}) \in \pi_0$ over D with $\eta_0, \dots, \eta_l \in K_i := \{i \frown \xi : \xi \in \omega^{\leq k}\}$ and $\eta_{l+1}, \dots, \eta_{l'} \in \omega^{\leq k+1}\backslash K_i$. We will be done if we show that $\models \varphi(f_{\eta_0}, \dots, f_{\eta_{l'}})$. Write $\eta_t = i \frown \xi_t$ for $t = 0, 1, \dots, l$. For any $\xi'_0, \dots, \xi'_l \in \omega^{\leq k}$ with $\mathrm{qftp}_{L_s}(\xi'_0, \dots, \xi'_l) = \mathrm{qftp}_{L_s}(\xi_0, \dots, \xi_l)$ we have $\mathrm{qftp}_{L_s}(\eta_0, \dots, \eta_{l'}) = \mathrm{qftp}_{L_s}(i \frown \xi_0, \dots, i \frown \xi_l, \eta_{l+1}, \dots, \eta_{l'}) = \mathrm{qftp}_{L_s}(i \frown \xi'_0, \dots, i \frown \xi'_l, \eta_{l+1}, \dots, \eta_{l'})$, hence, as $\varphi \in \pi_0$, we get that $\models \varphi(e_{i\frown\xi'_0}, \dots, e_{i\frown\xi'_l}, e_{\eta_{l+1}}, \dots, e_{\eta_{l'}})$. This shows that $\varphi(y_{\xi_0}, \dots, y_{\xi_l}, e_{\eta_{l+1}}, \dots, e_{\eta_{l'}}) \in EM_s(A_i/a_\emptyset A_{<i}B_{>i})$ where A_i is naturally indexed by $\omega^{\leq k}$, so, by the choice of B_i we get that $\models \varphi(b^i_{\xi_0}, \dots, b^i_{\xi_l}, e_{\eta_{l+1}}, \dots, e_{\eta_{l'}})$. As $(b^i_{\xi_0}, \dots, b^i_{\xi_l}, e_{\eta_{l+1}}, \dots, e_{\eta_{l'}}) = (f_{i\frown\xi_0}, \dots, f_{i\frown\xi_l}, f_{\eta_{l+1}}, \dots, f_{\eta_{l'}}) = (f_{\eta_0}, \dots, f_{\eta_{l'}})$, this means that $\models \varphi(f_{\eta_0}, \dots, f_{\eta_{l'}})$, as required.

By Fact 2.37 we find a sequence $(C_i)_{i<\omega} = ((c^i_\eta)_{\eta\in\omega^{\leq k}})_{i<\omega}$ which is $EM_<$-based on $(B_i)_{i<\omega}$ over $Db_\emptyset$ and indiscernible over $Db_\emptyset$. Let $C = (c_\eta)_{\eta\in\omega^{\leq k+1}}$ be given by $c_\emptyset = b_\emptyset$ and $c_{i\frown\xi} = c^i_\xi$ for any $\xi \in \omega^{\leq k}$ and $i < \omega$. By Claim 1 and Corollary 6.46 we get that C_i is s-indiscernible over $C_{\neq i}Dc_\emptyset$ for every $i < \omega$, which, together with $Dc_\emptyset$-indiscernibility of $(C_i)_{i<\omega}$ easily gives that C is s-indiscernible over D (as in [KKS14]). It is left to prove:

Claim 3. C is EM_s-based on B (and hence on A) over D.

Consider any formula $\varphi(x_{i_1\frown\xi_1}, \dots, x_{i_l\frown\xi_l}, x_\emptyset) \in EM_s(B/D)$ with $i_1, \dots, i_l \in \omega$ and $\xi_1, \dots, \xi_l \in \omega^{\leq k}$. Then for every $j_1, \dots, j_l \in \omega$ with $\mathrm{qftp}_{\{<\}}(j_1, \dots, j_l) = \mathrm{qftp}_{\{<\}}(i_1, \dots, i_l)$ we have that $\mathrm{qftp}_{L_s}(j_1 \frown \xi_1, \dots, j_l \frown \xi_l, \emptyset) = \mathrm{qftp}_{L_s}(i_1 \frown \xi_1, \dots, i_l \frown \xi_l, \emptyset)$, so $\models \varphi(b_{j_1\frown\xi_1}, \dots, b_{j_l\frown\xi_l}, b_\emptyset)$. This means that

$$\varphi(x_{i_1\frown\xi_1}, \dots, x_{i_l\frown\xi_l}, b_\emptyset) \in EM_<((B_i)_{i<\omega}/b_\emptyset D),$$

hence by the choice of C we have $\models \varphi(c_{i_1\frown\xi_1}, \dots, c_{i_l\frown\xi_l}, c_\emptyset)$ so $\varphi(x_{i_1\frown\xi_1}, \dots, x_{i_l\frown\xi_l}, x_\emptyset) \in EM_s(C/D)$, as required. □

Definition 6.48. Let I be a linearly ordered set. For a global M-Ls-invariant type q, we will call a sequence $(a_i)_{i\in I}$ a *CR-Morley sequence in q over M* (CR stands for the Chernikov-Ramsey criterion on SOP_1), if there is some $(b_i)_{i\in I} \models q^{\otimes I}|_M$ such that the pair (a_i, b_i) starts an $Ma_{>i}b_{>i}$-indiscernible sequence for every $i \in I$. We will say that $(a_i)_{i\in I}$ is a CR-Morley sequence in $\mathrm{tp}(a/M)$ if it is a CR-Morley sequence in some global M-Ls-invariant type $q \supseteq \mathrm{tp}(a/M)$.

In the semi-Hausdorff case we can replace the condition "(a_i, b_i) starts an $Ma_{>i}b_{>i}$-indiscernible sequence" by "$a_i \equiv_{Ma_{>i}b_{>i}} b_i$". The reason for which we need the stronger condition in thick theories is that equality of types is not necessarily type-definable there, so some of the compactness arguments below would not work with the weaker condition.

We slightly reformulate the parallel sequences lemma, Lemma 6.8:

Lemma 6.49. *Let T be thick and suppose $\varphi(x, y)$ is a formula and $(c_{i,0}, c_{i,1})_{i\in I}$ is an infinite sequence of pairs with $(c_{i,1})_{i\in I}$ indiscernible, such that:*

(i) for every $i \in I$, the pair $(c_{i,0}, c_{i,1})$ starts a $c_{>i,0}c_{>i,1}$-indiscernible sequence;

(ii) $\{\varphi(x; c_{i,0}) : i \in I\}$ is consistent;

(iii) $\{\varphi(x; c_{i,1}) : i \in I\}$ is inconsistent.

Then T has SOP_1.

Proof. We may assume the tuples $c_{i,0}$ and $c_{i,1}$ to be finite. As $(c_{i,1})_{i\in I}$ is indiscernible and $\{\varphi(x, c_{i,1}) : i \in I\}$ is inconsistent, there is some $\psi(y_1, \ldots, y_k)$ that implies $\neg\exists x(\varphi(x, y_1) \wedge \ldots \wedge \varphi(x, y_k))$ such that for any $i_1 < \ldots < i_k \in I$ we have $\models \psi(c_{i_1,1}, \ldots, c_{i_k,1})$. Call this ψ-inconsistent. By compactness there is a sequence of pairs $(\bar{c}'_i)_{i<\lambda_T} = (c'_{i,0}, c'_{i,1})_{i<\lambda_T}$ such that $(c'_{i,0}, c'_{i,1})$ starts a $\bar{c}'_{>i}$-indiscernible sequence for every $i < \lambda_T$, $\{\varphi(x, c'_{i,0}) : i < \lambda_T\}$ is consistent and $\{\varphi(x, c'_{i,1}) : i < \lambda_T\}$ is ψ-inconsistent. Then an indiscernible sequence based on $(\bar{c}'_i)_{i<\lambda_T}$ will satisfy the assumptions of Lemma 6.8, so T has SOP_1. □

By Kim's Lemma (Proposition 6.28) and Lemma 6.49 we easily get the following.

Corollary 6.50. *Suppose T is thick $NSOP_1$ with an e.c. model M, $\Sigma(x, b)$ is a partial type, I is an infinite linearly ordered set, and $(b_i)_{i\in I}$ a CR-Morley sequence in $\mathrm{tp}(b/M)$. If $\bigcup\{\Sigma(x, b_i) : i \in I\}$ is consistent then $\Sigma(x, b)$ does not Kim-divide over M. If $(b_i)_{i\in I}$ is indiscernible over M, then the converse also holds.*

Definition 6.51. Let M be an e.c. model and q a global M-Ls-invariant type.

(i) We say that a tree $(c_\eta)_{\eta\in\omega^{\le k}}$ is *q-spread-out over M* if for any $\eta_1 \in \omega^1, \eta_2 \in \omega^2, \ldots, \eta_k \in \omega^k$ such that $\eta_1 >_{lex} \eta_2 >_{lex} \cdots >_{lex} \eta_k$ and $(\forall l < l' \le k)(\eta_{l'} \wedge \eta_l \in \omega^{l-1})$ we have that $(c_{\eta_k}, \ldots, c_{\eta_1})$ is a Morley sequence in q over M.

(ii) We will say that $(c_\eta)_{\eta\in\omega^{\leq k}}$ is *weakly q-spread-out over M* if $(c_{\eta_k},\dots,c_{\eta_1}) \models q^{\otimes k}|_M$ for η_i's as in (i).

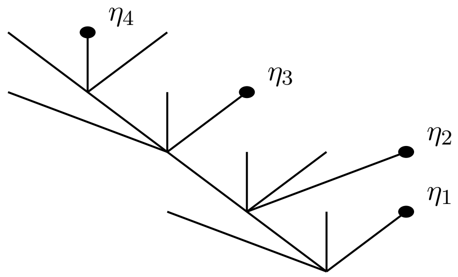


Figure 6.5.1: An example of η_i's from Definition 6.51

Clearly q-spread-outness implies weak q-spread-outness. We will freely use the above definition for trees of parameters indexed by trees naturally isomorphic to trees of the form $\omega'^{\leq k'}$, e.g. subtrees of $\omega^{\leq k}$ consisting of all nodes extending a fixed node.

The point of the conditions on the η_i's in Definition 6.51 is that this is quantifier-free definable by an L_s-formula. This is useful for preservation when EM_s-basing trees on one another, as we do in the following lemma.

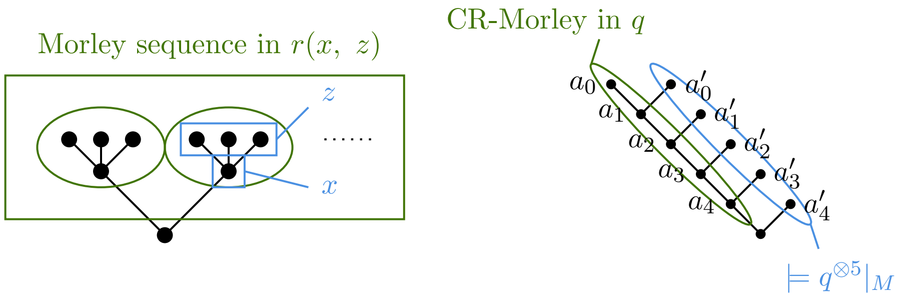
Lemma 6.52. *Let k be a natural number, M an e.c. model and q a global M-Ls-invariant type.*

(i) If $((c_{i\frown\eta})_{\eta\in\omega^{\leq k-1}})_{i<\omega}$ is a Morley sequence in a global M-Ls-invariant type $r(x,z) \supseteq q(x)$ over M, where x corresponds to the elements c_i and $(c_{0\frown\eta})_{\eta\in\omega^{\leq k-1}}$ is q-spread-out over M then also $(c_\eta)_{\eta\in\omega^{\leq k}}$ is q-spread-out over M for any choice of root $c_\emptyset$.

(ii) If $(c_\eta)_{\eta\in\omega^{\leq k}}$ is weakly q-spread-out over M and $(c'_\eta)_{\eta\in\omega^{\leq k}} \models EM_s((c_\eta)_{\eta\in\omega^{\leq k}}/M)$, then also $(c'_\eta)_{\eta\in\omega^{\leq k}}$ is weakly q-spread-out over M.

(iii) If $(c_\eta)_{\eta\in\omega^{\leq k}}$ is weakly q-spread-out over M and s-indiscernible over M, then for $a_i = c_{0^{k-i}}$ we have that $(a_i)_{i<k}$ is a CR-Morley sequence in q over M.

Proof. (i) Let $\eta_k \in \omega^k,\dots,\eta_1\in\omega^1$ be such that $\eta_1 >_{lex} \dots >_{lex} \eta_k$ and $(\forall l < l' \leq k)(\eta_{l'} \wedge \eta_l \in \omega^{l-1})$. We will prove that $(c_{\eta_k},\dots,c_{\eta_1})$ is a Morley sequence in q. For each $\ell \geq 2$ let $\beta_\ell \in \omega^1$ be such that $\eta_\ell \trianglerighteq \beta_\ell$. For every $\ell > 2$ we have by assumption that $\eta_2 \wedge \eta_\ell = \eta_2|_1 = \beta_2$, hence $\beta_\ell = \beta_2 =: \beta$ (and $\eta_1 >_{\text{lex}} \beta$ as $\eta_1 >_{\text{lex}} \eta_2$). In particular $(c_{\eta_k},\dots,c_{\eta_2})$ is contained in $(c_{\beta\frown\eta})_{\eta\in\omega^{\leq k-1}}$, which has the same Lascar strong type over M as $(c_{0\frown\eta})_{\eta\in\omega^{\leq k-1}}$. So, as $(c_{0\frown\eta})_{\eta\in\omega^{\leq k-1}}$ is q-spread-out by assumption, $(c_{\eta_k},\dots,c_{\eta_2})$ is a Morley sequence in q. As $((c_{i\frown\eta})_{\eta\in\omega^{\leq k-1}})_{i<\omega}$ is a Morley sequence in r, we have that $(c_{\eta_1\frown\eta})_{\eta\in\omega^{\leq k-1}}$, which contains c_{η_1}, has



(a) Part (i) (b) Part (iii)

Figure 6.5.2: Illustrations of Lemma 6.52

the same Lascar strong type over $M(c_{\beta\frown\eta})_{\eta\in\omega^{\leq k-1}}$, which contains $Mc_{\eta_k}\dots c_{\eta_2}$, as some realisation of r. Since $q(x) = r|_x$ we see that c_{η_1} has the same Lascar strong type over $Mc_{\eta_k},\dots,c_{\eta_2}$ as some realisation of q. So we conclude that $(c_{\eta_k},\dots,c_{\eta_1})$ is indeed a Morley sequence in q.
(ii) This holds because the condition on $(\eta_1,\dots,\eta_k)$ in the definition of weak q-spread-outness is expressible by a quantifier-free L_s-formula.
(iii) Put $a'_i := c_{0^{k-i-1}\frown 1}$ for $i < k$. Then $(a'_i)_{i<k} \models q^{\otimes k}|_M$ by weak q-spread-outness, and (a_i, a'_i) starts an $M_{a_{>i}a'_{>i}}$-indiscernible sequence for each $i < k$ by s-indiscernibility. $\square$

6.6 Symmetry

Lemma 6.53 (Chain lemma)**.** *Let T be a thick NSOP$_1$ theory and let M be an e.c. model. Let $(b_i)_{i<\kappa}$ be a Morley sequence in some global M-Ls-invariant $q(x)$. If $(b_i)_{i<\kappa}$ is Ma-indiscernible then $a \mathop{\smile\!\!\!\!\!\mid}^K_M (b_i)_{i<\kappa}$.*

Proof. We will prove that $a \mathop{\smile\!\!\!\!\!\mid}^K_M b_{i_1}\dots b_{i_k}$ for all $i_1 < \dots < i_k < \kappa$. This is indeed enough by finite character. By Ma-indiscernibility of $(b_i)_{i<\kappa}$ we may assume $\{i_1,\dots,i_k\} = \{0,\dots,k-1\}$.

We have $(b_i)_{i<\omega} \equiv^{\mathrm{Ls}}_M (\beta_i)_{i<\omega}$ for some $(\beta_i)_{i<\omega} \models q^{\otimes\omega}$. Define the tuple $\gamma_i = (\beta_{ik}, \beta_{ik+1},\dots,\beta_{ik+k-1})$ for all $i<\omega$. Then $(\gamma_i)_{i<\omega} \models (q^{\otimes k})^{\otimes\omega}$ by associativity of tensoring (Lemma 6.20). We let $c_i = (b_{ik}, b_{ik+1},\dots,b_{ik+k-1})$ for all $i<\omega$. Then $(c_i)_{i<\omega} \equiv^{\mathrm{Ls}}_M (\gamma_i)_{i<\omega}$. So $(c_i)_{i<\omega}$ is a Morley sequence in $q^{\otimes k}$ over M and $(c_i)_{i<\omega}$ is Ma-indiscernible. So $\mathrm{tp}(a/Mc_0) = \mathrm{tp}(a/Mb_0\dots b_{k-1})$ does not $q^{\otimes k}$-divide, and thus $a \mathop{\smile\!\!\!\!\!\mid}^K_M b_0\dots b_{k-1}$ as required. $\square$

Definition 6.54. Suppose M is an e.c. model, q a global type extending $\mathrm{Lstp}(a/M)$ and λ a cardinal. We will say that the extension $q \supseteq \mathrm{Lstp}(a/M)$ satisfies $(*)_\lambda$ if for every c with $|c| \leq \lambda$ there is a global M-Ls-invariant type $r(x,y) \supseteq \mathrm{Lstp}(ac/M)$ extending $q(x)$ (in particular, q is M-Ls-invariant).

Lemma 6.55. *For any e.c. model M, tuple a and cardinal λ there is $q \supseteq \mathrm{Lstp}(a/M)$ satisfying $(*)_\lambda$.*

Proof. Let M, a and λ be as in the statement. Choose a small tuple d such that for any c with $|c| \leq \lambda$ there is some $d' \subseteq d$ with $\mathrm{Lstp}(ad'/M) = \mathrm{Lstp}(ac/M)$ (this is possible as the number of Lascar types of tuples of fixed length over M is bounded). Now take a global M-Ls-invariant extension $r(x, y)$ of $\mathrm{Lstp}(ad/M)$, where x corresponds to a. Then $q := r|_x$ is an extension of $\mathrm{Lstp}(a/M)$ satisfying $(*)_\lambda$. □

Remark 6.56. If $q \supseteq \mathrm{Lstp}(a/M)$ is finitely satisfiable in M then it satisfies $(*)_\lambda$ for any cardinal λ, see [Men20, Lemma 3.4]. However, finitely satisfiable extensions may not exist in thick theories.

Theorem 6.57 (Symmetry)**.** *In a thick NSOP$_1$ theory $a \mathop{\smile\!\!\!\!\!\!|}\,^K_M b$ implies $b \mathop{\smile\!\!\!\!\!\!|}\,^K_M a$.*

Proof. We may assume that b enumerates a λ_T-saturated model containing M. If this is not the case let $N \supseteq Mb$ be a λ_T-saturated model. By extension, Corollary 6.39, we find $N' \equiv_{Mb} N$ such that $a \mathop{\smile\!\!\!\!\!\!|}\,^K_M N'$. Now we replace b by N' and we continue the proof.

Set $\lambda = |ab|$. By Lemma 6.55 we can choose a global extension $q \supseteq \mathrm{Lstp}(a/M)$ satisfying $(*)_\lambda$. Let $p(y, a) = \mathrm{tp}(b/Ma)$. We will show that there is a CR-Morley sequence $(a_i)_{i<\omega}$ in q over M such that $\bigcup_{i<\omega} p(y, a_i)$ is consistent, which is enough by Corollary 6.50. All the properties we wish $(a_i)_{i<\omega}$ to have are type-definable. It is thus enough to find such a sequence of length k for every $k < \omega$.

So fix any $k < \omega$. By backward induction on $k' = k+1, k, \dots, 1$ we will define trees $(c_\eta)_{\eta \in S_{k'}}$ where $S_{k'} = \{\xi \in \omega^{\leq k+1} : 0^{k'-1} \trianglelefteq \xi\}$. We will write $S^*_{k'}$ for $S_{k'}$ without the root, so $S^*_{k'} = S_{k'} - \{0^{k'-1}\}$. For each k' the tree $(c_\eta)_{\eta \in S_{k'}}$ will satisfy the following conditions:

(A1)$_{k'}$ $c_\eta c_\nu \equiv^{\mathrm{Ls}}_M ab$ for all $\nu \triangleright \eta \in S_{k'}$ with $\nu \in \omega^{k+1}$ and $\eta \in \omega^{\leq k}$;

(A2)$_{k'}$ $(c_\eta)_{\eta \in S_{k'} \cap \omega^{\leq k}}$ is q-spread-out over M;

(A3)$_{k'}$ (the root is independent from the rest) we have $c_{0^{k'-1}} \mathop{\smile\!\!\!\!\!\!|}\,^K_M (c_\eta)_{\eta \in S^*_{k'}}$.

For $k' = k+1$ we let t be a global M-Ls-invariant extension of $\mathrm{Lstp}(b/M)$. Since $a \mathop{\smile\!\!\!\!\!\!|}\,^K_M b$ we have that $\mathrm{tp}(a/Mb)$ does not t-divide. By Corollary 6.34 and our assumption on b this means that $\mathrm{Lstp}(a/Mb)$ does not t-Ls-divide. So we find an Ma-indiscernible Morley sequence $(c_{0^k \frown \alpha})_{\alpha<\omega}$ in t with $c_{0^{k+1}} = b$. By Lemma 6.53 we have that $a \mathop{\smile\!\!\!\!\!\!|}\,^K_M (c_{0^k \frown \alpha})_{\alpha<\omega}$. So we pick $c_{0^k} = a$ and directly satisfy (A3)$_{k'}$. Condition (A2)$_{k'}$ is vacuous and (A1)$_{k'}$ follows directly from Ma-indiscernibility of $(c_{0^k \frown \alpha})_{\alpha<\omega}$ and the fact that $c_{0^{k+1}} = b$.

For the inductive step, suppose that we have constructed $(c_\eta)_{\eta \in S_{k'}}$. By (A1)$_{k'}$ there is a tuple d such that $c_{0^{k'-1}}(c_\eta)_{\eta \in S^*_{k'}} \equiv^{\mathrm{Ls}}_M ad$. So, by $(*)_\lambda$ there is a global M-Ls-invariant type $r(x,z) \supseteq q(x)$ extending $\mathrm{Lstp}(c_{0^{k'-1}}(c_\eta)_{\eta \in S^*_{k'}}/M)$. By $(\mathrm{A3})_{k'}$ we have that $c_{0^{k'-1}} \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_M (c_\eta)_{\eta \in S^*_{k'}}$. So since $b \subseteq (c_\eta)_{\eta \in S^*_{k'}}$ and using our assumption on b we have by Corollary 6.34 that $\mathrm{Lstp}(c_{0^{k'-1}}/M(c_\eta)_{\eta \in S^*_{k'}})$ does not $r|_z$-Ls-divide. By extension for Ls-dividing, Proposition 6.38, we find c such that $c(c_\eta)_{\eta \in S^*_{k'}} \equiv^{\mathrm{Ls}}_M c_{0^{k'-1}}(c_\eta)_{\eta \in S^*_{k'}}$ and $\mathrm{Lstp}(c/M(c_\eta)_{\eta \in S_{k'}})$ does not r-Ls-divide. So there is an Mc-indiscernible Morley sequence $((d_{\eta,i})_{\eta \in S_{k'}})_{i<\omega}$ in r such that $(d_{\eta,0})_{\eta \in S_{k'}} = (c_\eta)_{\eta \in S_{k'}}$. We set $c_{0^{k'-2}} = c$ and $c_{0^{k'-2} \frown i \frown \zeta} = d_{0^{k'-1} \frown \zeta, i}$. Again, using Lemma 6.53 we directly get $(\mathrm{A3})_{k'-1}$.

Now $(\mathrm{A2})_{k'-1}$ follows from Lemma 6.52(i). We verify $(\mathrm{A1})_{k'-1}$. Everything above the root consists of copies (via a Lascar strong automorphism over M) of $(c_\eta)_{\eta \in S_{k'}}$, so we only need to check that $c_{0^{k'-2}} c_\nu \equiv^{\mathrm{Ls}}_M ab$ for all $\nu \in S_{k'-1} \cap \omega^{k+1}$. By indiscernibility we may assume $\nu \in S_{k'} \cap \omega^{k+1}$. Then $(\mathrm{A1})_{k'-1}$ follows from $(\mathrm{A1})_{k'}$ and the fact that $c_{0^{k'-2}}(c_\eta)_{\eta \in S^*_{k'}} \equiv^{\mathrm{Ls}}_M c_{0^{k'-1}}(c_\eta)_{\eta \in S^*_{k'}}$.

Thus the inductive step, and hence the construction of the tree $(c_\eta)_{\eta \in \omega^{k+1}} = (c_\eta)_{\eta \in S_1}$, is completed.

Consider the following condition:

(A1')$_1$ $c_\eta c_\nu \equiv_M ab$ for all $\nu \rhd \eta$ with $\nu \in \omega^{k+1}$ and $\eta \in \omega^{\leq k}$;

which is clearly implied by $(\mathrm{A1})_1$ as it is seen by the EM_s-type of $(c_\eta)_{\eta \in \omega^{\leq k+1}}$ over M. Letting $(c'_\eta)_{\eta \in \omega^{\leq k+1}}$ be an s-indiscernible tree which is EM_s-based on $(c_\eta)_{\eta \in \omega^{\leq k+1}}$ over M, we get that $(c'_\eta)_{\eta \in \omega^{\leq k+1}}$ satisfies (A1')$_1$, and $(c'_\eta)_{\eta \in \omega^{\leq k}}$ is weakly q-spread-out over M by Lemma 6.52(ii).

Put $a_i = c'_{0^{k+1-i}}$. Then $(a_1, \ldots, a_k)$ is a CR-Morley sequence in q over M by Lemma 6.52(iii), and by (A1')$_1$ we have that $\bigcup_{1 \leq i \leq k} p(y, a_i)$ is consistent because it is realised by $c'_{0^{k+1}}$. This completes the proof. □

Lemma 6.58. *Let T be a thick theory. Suppose that $\varphi(x,y)$ has SOP_1, witnessed by $\psi(y_1,y_2)$. Then there is an e.c. model M and b_1, b_2, c_1, c_2 such that $c_1 \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^u_M c_2$, $c_1 \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^u_M b_1$, $c_2 \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^u_M b_2$ and $b_1 c_1 \equiv^{\mathrm{Ls}}_M b_2 c_2$ and $\models \varphi(b_1,c_1) \wedge \varphi(b_2,c_2) \wedge \psi(c_1,c_2)$.*

Proof. The proof is mostly the same as [HK21, Proposition A.7] but we have to adjust a few things throughout to get equality of Lascar strong types rather than just equality of types. As in that proof, we will use a Skolemisation technique for positive logic [HK21, Lemma A.6]. In such a Skolemised theory the positively definable closure of any set is an e.c. model and the reduct of an e.c. model (to the original language) is an e.c. model (of the original theory). It is not directly clear whether this Skolemisation construction preserves thickness, but that is not a problem. Ultimately we are interested in Lascar strong types in our original theory. So even though we technically work in a Skolemised theory the (type-definable) predicate $\mathrm{d}(x,y) \leq 1$ should be taken as in our original theory.

Let κ be any cardinal. By compactness we find parameters $(a_\eta)_{\eta \in 2^{<\kappa}}$ such that:

(i) for every $\sigma \in 2^\kappa$ the set $\{\varphi(x, a_{\sigma|i}) : i < \kappa\}$ is consistent,

(ii) for every $\eta, \nu \in 2^{<\kappa}$ such that $\eta^\frown 0 \preceq \nu$ we have $\models \psi(a_{\eta^\frown 1}, a_\nu)$.

For a big enough cardinal λ, we construct by induction a sequence $(\eta_i, \nu_i)_{i<\lambda}$ with $\eta_i, \nu_i \in 2^{<\kappa}$ such that:

(1) $\eta_i \trianglelefteq \eta_j$ and $\eta_i \trianglelefteq \nu_j$ for all $i < j < \lambda$;

(2) $\eta_i \trianglerighteq (\eta_i \wedge \nu_i) \frown 0$, $\nu_i = (\eta_i \wedge \nu_i) \frown 1$, and (a_{η_i}, a_{ν_i}) starts an $a_{\eta_{<i}} a_{\nu_{<i}}$-indiscernible sequence for every $i < \lambda$.

Assume $(\eta_j, \nu_j)_{j<i}$ has been constructed and set $\eta = \bigcup_{j<i} \eta_j$. If we chose κ to be large enough then, by applying Fact 2.37 to $(a_{\eta^\frown 0^\alpha \frown 1})_{\alpha>0}$, it follows that there are $0 < \alpha < \beta < \kappa$ such that $(a_{\eta^\frown 0^\alpha \frown 1}, a_{\eta^\frown 0^\beta \frown 1})$ starts an $\{\eta_j, \nu_j : j < i\}$-indiscernible sequence. We set $\nu_i = \eta^\frown 0^{\alpha\frown} 1$ and $\eta_i = \eta^\frown 0^{\beta\frown} 1$.

By (i) and (1), there is b_2 realising $\{\varphi(x, a_{\eta_i}) : i < \lambda\}$. Now let $(e_i, d_i)_{i<\omega+2}$ be indiscernible over b_2 based on $(a_{\eta_i}, a_{\nu_i})_{i<\lambda}$.

Let M be the positively definable closure of $\{e_i, d_i : i < \omega\}$. As discussed, we may assume M to be an e.c. model. Set $c_1 = d_\omega$ and $c_2 = e_{\omega+1}$. Then $c_1 \mathop{\smile\!\!\!\!\!\mid}^u_{\{e_i,d_i:i<\omega\}} c_2$ and $c_2 \mathop{\smile\!\!\!\!\!\mid}^u_{\{e_i,d_i:i<\omega\}} b_2$ by indiscernibility. So $c_1 \mathop{\smile\!\!\!\!\!\mid}^u_M c_2$, $c_2 \mathop{\smile\!\!\!\!\!\mid}^u_M b_2$ and $\models \varphi(b_2, c_2)$. By construction $c_1 c_2 = d_\omega e_{\omega+1} \equiv a_{\nu_{i_0}} a_{\eta_{i_1}}$ for some $i_0 < i_1 < \lambda$ and thus $\models \psi(c_1, c_2)$ by (ii), (1), and (2).

To find b_1 we first claim that $\mathrm{d}_M(e_\omega, d_\omega) \leq 1$. By compactness it suffices to prove that $\mathrm{d}_A(e_\omega, d_\omega) \leq 1$ for all finite $A \subseteq M$. By how we constructed M it then suffices to prove that (e_ω, d_ω) starts an indiscernible sequence over $\{e_i, d_i : i < n\}$ for all $n < \omega$. To prove this last statement we let $i_0 < \ldots < i_{n+1} < \lambda$ be such that

$$e_0 d_0 \ldots e_n d_n e_\omega d_\omega \equiv a_{\eta_{i_0}} a_{\nu_{i_0}} \ldots a_{\eta_{i_n}} a_{\nu_{i_n}} a_{\eta_{i_{n+1}}} a_{\nu_{i_{n+1}}}.$$

By how we constructed $(\eta_i, \nu_i)_{i<\lambda}$ we have $(a_{\eta_{i_{n+1}}}, a_{\nu_{i_{n+1}}})$ starts an indiscernible sequence over $\{a_{\eta_{i_0}} a_{\nu_{i_0}} \ldots a_{\eta_{i_n}} a_{\nu_{i_n}}\}$. So the claim follows after applying the automorphism.

Now we leave the Skolemised theory and work in the original theory, so $\mathrm{d}(x, y) \leq 1$ corresponds to actually having Lascar distance one. We have $c_2 = e_{\omega+1} \equiv^{\mathrm{Ls}}_M e_\omega \equiv^{\mathrm{Ls}}_M d_\omega = c_1$, so there is $f \in \mathrm{Aut}_f(\mathfrak{M}/M)$ such that $f(c_2) = c_1$. Let $b_1 = f(b_2)$. Then $c_2 b_2 \equiv^{\mathrm{Ls}}_M c_1 b_1$, hence also $\models \varphi(b_1, c_1)$ and $c_1 \mathop{\smile\!\!\!\!\!\mid}^u_M b_1$, as required. $\square$

Theorem 6.59. *Let T be a thick theory. The following are equivalent:*

(i) T is NSOP$_1$;

(ii) (symmetry) $a \mathop{\smile\!\!\!\!\!\mid}^K_M b$ implies $b \mathop{\smile\!\!\!\!\!\mid}^K_M a$;

(iii) (weak symmetry) $a \mathop{\smile\!\!\!\!\!\mid}^{iLs}_M b$ implies $b \mathop{\smile\!\!\!\!\!\mid}^K_M a$.

Proof. Theorem 6.57 is precisely (i) $\implies$ (ii). For (ii) $\implies$ (iii) we just note that $a \perp\!\!\!\perp^{iLs}_M b$ implies $a \perp\!\!\!\perp^K_M b$. Finally, for (iii) $\implies$ (i) we proceed is as in [KR20, Proposition 3.22] replacing their reference to [CR16] by Lemma 6.58 and being careful about using global Ls-invariant types instead of just global invariant types.

We prove the contrapositive, so assume T has SOP_1. Then by Lemma 6.58 there is an e.c. model M and b_1, b_2, c_1, c_2 such that $c_1 \perp\!\!\!\perp^u_M c_2$, $c_1 \perp\!\!\!\perp^u_M b_1$, $c_2 \perp\!\!\!\perp^u_M b_2$ and $b_1 c_1 \equiv^{\mathrm{Ls}}_M b_2 c_2$. Furthermore, for $p(x, c_1) = \mathrm{tp}(b_1 c_1/M)$, we have that $p(x, c_1) \cup p(x, c_2)$ is inconsistent. In particular we have that $\mathrm{Lstp}(c_1/Mc_2)$ extends to a global M-Ls-invariant q. Then as $c_1 \equiv^{\mathrm{Ls}}_M c_2$ there is a Morley sequence $(d_i)_{i<\omega}$ in q with $d_0 d_1 = c_2 c_1$. We thus have that $\bigcup\{p(x, d_i) : i < \omega\}$ is inconsistent. So $b_2 \not\perp\!\!\!\perp^K_M c_2$. Since also $c_2 \perp\!\!\!\perp^u_M b_2$ and thus $c_2 \perp\!\!\!\perp^{iLs}_M b_2$ we see that weak symmetry fails and this concludes our proof. $\square$

6.7 Independence theorem

We recall the following facts. The first is the same as [KR20, Lemma 7.4] and the second is the same as the claim in [DKR19, Lemma 5.3]. Their proofs work in our setting as well.

Fact 6.60. *The following hold in any thick $NSOP_1$ theory.*

(i) If $a \perp\!\!\!\perp^d_M bc$ and $b \perp\!\!\!\perp^K_M c$ then $ab \perp\!\!\!\perp^K_M c$.

(ii) If $a \perp\!\!\!\perp^K_M b$ and $a \perp\!\!\!\perp^K_M c$ then there is c' with $ac' \equiv_M ac$ such that $a \perp\!\!\!\perp^K_M bc'$.

For the following lemma we borrow a trick from [DKR19, Lemma 5.4].

Lemma 6.61. *Let T be thick $NSOP_1$ and let $a \equiv^{\mathrm{Ls}}_M a'$, $a \perp\!\!\!\perp^K_M b$ and $a' \perp\!\!\!\perp^K_M c$. Then there is c' such that $ac' \equiv^{\mathrm{Ls}}_M a'c$ and $a \perp\!\!\!\perp^K_M bc'$.*

Proof. Let c^* be such that $ac^* \equiv^{\mathrm{Ls}}_M a'c$, so $a \perp\!\!\!\perp^K_M c^*$. Let $N' \supseteq M$ be λ_T-saturated and let q be a global M-Ls-invariant extension of $\mathrm{Lstp}(N'/M)$. Let N realise $q|_{Mabc^*}$, so we have $N \perp\!\!\!\perp^{iLs}_M abc^*$. By Fact 6.60(i) we then have $Na \perp\!\!\!\perp^K_M b$ and $Na \perp\!\!\!\perp^K_M c^*$. So by fact Fact 6.60(ii) we find c' with $Nac' \equiv_M Nac^*$ and $Na \perp\!\!\!\perp^K_M bc'$. We thus have $ac' \equiv^{\mathrm{Ls}}_M ac^* \equiv^{\mathrm{Ls}}_M a'c$, as required. $\square$

Definition 6.62. We write $b \perp\!\!\!\perp^*_M c$ to mean that $\mathrm{Lstp}(b/Mc)$ extends to a global M-Ls-invariant type $\mathrm{tp}(N/\mathfrak{M})$ for some $\beth_\omega(\lambda_T + |Mbc|)$-saturated model $N \supseteq M$. Extending $\mathrm{Lstp}(b/Mc)$ here means that there is some $\beta \in N$ with $\beta \equiv^{\mathrm{Ls}}_{Mc} b$.

The point of the enormous cardinal $\beth_\omega(\lambda_T + |Mbc|)$ is that we will want to find a λ_T-saturated model M' containing M and a copy of b in N, and then again some λ_T-saturated $M'' \supseteq M'$ inside N. By Fact 2.35 we can choose these

λ_T-saturated models small enough so that this process can be repeated any finite number of times.

We easily see that $\mathop{\smile\!\!\!\!\!\!|}^*$ is invariant under automorphisms and, assuming thickness, that $b \mathop{\smile\!\!\!\!\!\!|}^*_M M$ for all M.

Lemma 6.63. *We have that* $\mathop{\smile\!\!\!\!\!\!|}^*$ *satisfies the following extension properties.*

(i) (left extension) If $b \mathop{\smile\!\!\!\!\!\!|}^*_M c$ *and* $|d| < \beth_\omega(\lambda_T + |Mbc|)$*, then there is* $d' \equiv^{\mathrm{Ls}}_{Mb} d$ *such that* $bd' \mathop{\smile\!\!\!\!\!\!|}^*_M c$*.*

(ii) (right extension) If $b \mathop{\smile\!\!\!\!\!\!|}^*_M c$ *and* $|d| < \beth_\omega(\lambda_T + |Mbc|)$*, then there is* $d' \equiv^{\mathrm{Ls}}_{Mc} d$ *such that* $b \mathop{\smile\!\!\!\!\!\!|}^*_M cd'$*.*

Proof. In both cases we assume $b \mathop{\smile\!\!\!\!\!\!|}^*_M c$. So let $q = \mathrm{tp}(N/\mathfrak{M})$ be a global M-Ls-invariant extension of $\mathrm{Lstp}(b/Mc)$ for some $\beth_\omega(\lambda_T + |Mbc|)$-saturated $N \supseteq M$.

We first prove left extension. Let $N' \equiv^{\mathrm{Ls}}_{Mc} N$ be in $\mathfrak{M}$. By moving things by a Lascar strong automorphism over Mc we may assume $b \in N'$. By Fact 2.35 there is $Mb \subseteq M' \subseteq N'$ where M' is λ_T-saturated and of cardinality $\leq 2^{\lambda_T + |Mb|}$. Let d' realise $\mathrm{tp}(d/M')$ in N'. So $d' \equiv^{\mathrm{Ls}}_{Mb} d$ while q also extends $\mathrm{Lstp}(bd'/Mc)$, so indeed $bd' \mathop{\smile\!\!\!\!\!\!|}^*_M c$.

Now we prove right extension. Let $\beta \in N$ be such that $\beta \equiv^{\mathrm{Ls}}_{Mc} b$. Pick $b' \in \mathfrak{M}$ such that $b' \equiv^{\mathrm{Ls}}_{Mcd} \beta$. Then clearly $b' \mathop{\smile\!\!\!\!\!\!|}^*_M cd$. We finish the proof by picking d' such that $bd' \equiv^{\mathrm{Ls}}_{Mc} b'd$. $\square$

Proposition 6.64 (Weak independence theorem)**.** *Let T be thick NSOP$_1$. Suppose that* $a \equiv^{\mathrm{Ls}}_M a'$, $a \mathop{\smile\!\!\!\!\!\!|}^K_M b$, $a' \mathop{\smile\!\!\!\!\!\!|}^K_M c$ *and* $b \mathop{\smile\!\!\!\!\!\!|}^*_M c$*. Then there is* a'' *with* $a'' \equiv^{\mathrm{Ls}}_{Mb} a$ *and* $a'' \equiv^{\mathrm{Ls}}_{Mc} a'$ *such that* $a'' \mathop{\smile\!\!\!\!\!\!|}^K_M bc$*.*

Proof. We may assume that b and c both enumerate a λ_T-saturated model containing M. If this is not the case let $N \supseteq Mb$ be λ_T-saturated and such that $|N| < \beth_\omega(\lambda_T + |Mbc|)$. By left extension from Lemma 6.63 we then find $N' \equiv^{\mathrm{Ls}}_{Mb} N$ with $N' \mathop{\smile\!\!\!\!\!\!|}^*_M c$. By Corollary 6.39 we find a_0 with $a_0 \equiv^{\mathrm{Ls}}_{Mb} a$ and $a_0 \mathop{\smile\!\!\!\!\!\!|}^K_M N'$. Now we can replace a by a_0 and b by N' and continue the proof. The case for c is analogous.

By Lemma 6.61 there is c' such that $ac' \equiv^{\mathrm{Ls}}_M a'c$ and $a \mathop{\smile\!\!\!\!\!\!|}^K_M bc'$. Apply left extension from Lemma 6.63 to $b \mathop{\smile\!\!\!\!\!\!|}^*_M c$ and c' to find $c'' \equiv^{\mathrm{Ls}}_{Mb} c$ with $bc' \mathop{\smile\!\!\!\!\!\!|}^*_M c''$. Let b^* be such that $b^*c'' \equiv^{\mathrm{Ls}}_M bc'$ and apply right extension from Lemma 6.63 to $bc' \mathop{\smile\!\!\!\!\!\!|}^*_M c''$ and b^* to find $b'' \equiv^{\mathrm{Ls}}_{Mc''} b^*$ with $bc' \mathop{\smile\!\!\!\!\!\!|}^*_M b''c''$. In particular we have $b''c'' \equiv^{\mathrm{Ls}}_M bc'$ and $\mathrm{Lstp}(bc'/Mb''c'')$ extends to a global M-Ls-invariant type q. So there is a Morley sequence $(b_ic_i)_{i<\omega}$ in q with $(b_0, c_0) = (b'', c'')$ and $(b_1, c_1) = (b, c')$. As $a \mathop{\smile\!\!\!\!\!\!|}^K_M bc'$, we can find a^* with $a^*b''c'' \equiv_M abc'$ such that $(b_ic_i)_{i<\omega}$ is Ma^*-indiscernible. By construction we had $c'' \equiv^{\mathrm{Ls}}_{Mb} c$, so there is a Lascar strong automorphism σ over Mb such that $\sigma(c'') = c$. Set $a'' = \sigma(a^*)$, we check that this is indeed the a'' we are looking for.

By the chain lemma (Lemma 6.53) we have $a^* \mathop{\smile\!\!\!\!\!\mid}\nolimits^K_M (b_i c_i)_{i<\omega}$, so we have $a^* \mathop{\smile\!\!\!\!\!\mid}\nolimits^K_M bc''$ and $a'' \mathop{\smile\!\!\!\!\!\mid}\nolimits^K_M bc$ then follows by invariance. By Ma^*-indiscernibility we have $a''b \equiv_M a^*b \equiv_M a^*b'' \equiv_M ab$. We assumed b to enumerate a λ_T-saturated model, so indeed $a'' \equiv^{\mathrm{Ls}}_{Mb} a$. By construction of c' we have $a''c \equiv_M a^*c'' \equiv_M ac' \equiv_M a'c$. We assumed c to enumerate a λ_T-saturated model, so indeed $a'' \equiv^{\mathrm{Ls}}_{Mc} a'$, which concludes the proof. □

Fact 6.65. *In a thick theory, if $N \supseteq M$ is $(2^{|M|+\lambda_T})^+$-saturated and q and r are global M-Ls-invariant types with $q|_N = r|_N$ then $q = r$.*

Proof. By Fact 2.35 there is $M \subseteq M' \subseteq N$ where M' is a λ_T-saturated model and $|M'| < (2^{|M|+\lambda_T})^+$. Let $\varphi(x, b)$ be any formula with parameters b. Let $b' \in N$ realise $\mathrm{tp}(b/M')$. Then $b \equiv^{\mathrm{Ls}}_M b'$. By M-Ls-invariance and $q|_N = r|_N$ we have

$$\varphi(x, b) \in q \quad \Leftrightarrow \quad \varphi(x, b') \in q \quad \Leftrightarrow \quad \varphi(x, b') \in r \quad \Leftrightarrow \quad \varphi(x, b) \in r,$$

which concludes the proof. □

Theorem 6.66 (Independence theorem)**.** *Let T be a thick NSOP$_1$ theory. Suppose that $a \equiv^{\mathrm{Ls}}_M a'$, $a \mathop{\smile\!\!\!\!\!\mid}\nolimits^K_M b$, $a' \mathop{\smile\!\!\!\!\!\mid}\nolimits^K_M c$ and $b \mathop{\smile\!\!\!\!\!\mid}\nolimits^K_M c$. Then there is a'' with $a'' \equiv^{\mathrm{Ls}}_{Mb} a$, $a'' \equiv^{\mathrm{Ls}}_{Mc} a'$ and $a'' \mathop{\smile\!\!\!\!\!\mid}\nolimits^K_M bc$.*

Proof. We may assume that b and c both enumerate a λ_T-saturated model containing M. If this is not the case let $N \supseteq Mb$ be λ_T-saturated. By extension (Corollary 6.39) and symmetry then find $N' \equiv^{\mathrm{Ls}}_{Mb} N$ with $N' \mathop{\smile\!\!\!\!\!\mid}\nolimits^K_M c$. Applying extension again we find a_0 with $a_0 \equiv^{\mathrm{Ls}}_{Mb} a$ and $a_0 \mathop{\smile\!\!\!\!\!\mid}\nolimits^K_M N'$. Now we can replace a by a_0 and b by N' and continue the proof. The case for c is analogous.

Let $N_0 \supseteq M$ be $(2^{|M|+\lambda_T})^+$-saturated and let κ be a big enough cardinal (depending only on $|N_0 bc|$). Pick some global M-Ls-invariant type $q(y, z)$ extending $\mathrm{Lstp}(bc/M)$ such that q also extends to a global M-Ls-invariant type $\mathrm{tp}(N/\mathfrak{M})$ for some saturated enough $N \supseteq M$ (depending only on κ). So there is β realising $q|_y$ with $\beta \equiv^{\mathrm{Ls}}_M b$. Let $(b_i c_i)_{i<\kappa}$ be a Morley sequence in q with $b_0 = b$ and let $b_\kappa \equiv^{\mathrm{Ls}}_{M(b_i c_i)_{i<\kappa}} \beta$. Then we have $b_i c_i \mathop{\smile\!\!\!\!\!\mid}\nolimits^*_M b_{<i} c_{<i}$ for all $i < \kappa$ and $b_\kappa \mathop{\smile\!\!\!\!\!\mid}\nolimits^*_M (b_i c_i)_{i<\kappa}$.

We will inductively construct a sequence $(b'_i)_{i \leq \kappa}$ with $b'_0 = b$ such that at step i:

(i) $c \mathop{\smile\!\!\!\!\!\mid}\nolimits^K_M b'_{\leq i}$,

(ii) $cb'_i \equiv^{\mathrm{Ls}}_M cb$,

(iii) $b'_{\leq i} \equiv^{\mathrm{Ls}}_M b_{\leq i}$.

The base case is already fixed: $b'_0 = b$. So suppose we have constructed $b'_{\leq i}$. By induction hypothesis (iii) we can find $b^* b'_{\leq i} \equiv^{\mathrm{Ls}}_M b_{i+1} b_{\leq i}$. So $b^* \mathop{\smile\!\!\!\!\!\mid}\nolimits^*_M b'_{\leq i}$. Let

c^* be such that $c^*b^* \equiv^{\mathrm{Ls}}_M cb$, so $c^* \mathop{\smile\!\!\!\!\!\!\;|}^K_M b^*$. So also using (i) from the induction hypothesis we can apply the weak independence theorem (Proposition 6.64) to find c' such that $c' \mathop{\smile\!\!\!\!\!\!\;|}^K_M b'_{\leq i}b^*$, $c' \equiv^{\mathrm{Ls}}_{Mb^*} c^*$ and $c' \equiv^{\mathrm{Ls}}_{Mb'_{\leq i}} c$. We now pick b'_{i+1} to be such that $cb'_{i+1} \equiv^{\mathrm{Ls}}_{Mb'_{\leq i}} c'b^*$. Then indeed $c \mathop{\smile\!\!\!\!\!\!\;|}^K_M b'_{\leq i+1}$. We also have $b'_{\leq i}b'_{i+1} \equiv^{\mathrm{Ls}}_M b'_{\leq i}b^* \equiv^{\mathrm{Ls}}_M b_{\leq i}b_{i+1}$. Finally, $cb'_{i+1} \equiv^{\mathrm{Ls}}_M c'b^* \equiv^{\mathrm{Ls}}_M c^*b^* \equiv^{\mathrm{Ls}}_M cb$. So this concludes the successor step. For the limit stage we assume we have constructed $b'_{<i}$. We then have $c \mathop{\smile\!\!\!\!\!\!\;|}^K_M b'_{<i}$ by finite character. We also have $b'_{\leq j} \equiv^{\mathrm{Ls}}_M b_{\leq j}$ for all $j < i$. So we have $b'_{<i} \equiv_M b_{<i}$. We assumed b to enumerate a λ_T-saturated model containing M, so because $b'_0 = b = b_0$ we do in fact have $b'_{<i} \equiv^{\mathrm{Ls}}_M b_{<i}$. We then construct b'_i in an analogous way to the successor step.

We let $(c'_i)_{i<\kappa}$ be such that $b'_\kappa(b'_ic'_i)_{i<\kappa} \equiv^{\mathrm{Ls}}_M b_\kappa(b_ic_i)_{i<\kappa}$. So by M-Ls-invariance of $q|_y$ we have $\beta b'_\kappa(b'_ic'_i)_{i<\kappa} \equiv^{\mathrm{Ls}}_M \beta b_\kappa(b_ic_i)_{i<\kappa}$ and thus by how we chose b_κ we have $b'_\kappa \equiv^{\mathrm{Ls}}_{M(b'_ic'_i)_{i<\kappa}} \beta$.

Since $q \subseteq \mathrm{tp}(N/\mathfrak{M})$ for some saturated enough N we can find $\beta\gamma(\beta_i,\gamma_i)_{i<\kappa} \equiv^{\mathrm{Ls}}_M b'_\kappa c(b'_i, c'_i)_{i<\kappa}$ in N, where $\beta\gamma \models q$. Here we used the fact that $b'_\kappa c \equiv^{\mathrm{Ls}}_M bc$. Set $q'((y_i,z_i)_{i<\kappa}, y, z) = \mathrm{tp}((\beta_i,\gamma_i)_{i<\kappa}\beta\gamma/\mathfrak{M})$. Then q' is global M-Ls-invariant because $\mathrm{tp}(N/\mathfrak{M})$ is global M-Ls-invariant. By Fact 6.65 and our choice of κ we get that some global M-Ls-invariant type $q'|_{y_iz_iyz}$ occurs for κ many i (modulo identifying the the variables for different i's). We now focus on a subsequence of length ω such that (after relabelling) $q'|_{y_iz_iyz}$ does not depend on i, and we forget about κ. We also relabel b'_κ to b'.

Claim 1. In summary, we have just constructed the following.

(i) A Morley sequence $(b'_ic'_i)_{i<\omega}$ in q, where q is a global M-Ls-invariant extension of $\mathrm{Lstp}(bc/M)$.

(ii) For every $i < \omega$ we have $b'_ic \equiv^{\mathrm{Ls}}_M b'c \equiv^{\mathrm{Ls}}_M bc$.

(iii) Let $\beta \models q|_y$ then $b' \equiv^{\mathrm{Ls}}_{M(b'_ic'_i)_{i<\omega}} \beta$.

(iv) $q(y,z) \subseteq q'((y_i,z_i)_{i<\omega}, y, z)$ and q' is global M-Ls-invariant and extends $\mathrm{Lstp}((b'_i,c'_i)_{i<\omega}b'c/M)$.

(v) There is some sufficiently saturated N such that $q' \subseteq \mathrm{tp}(N/\mathfrak{M})$ and $\mathrm{tp}(N/\mathfrak{M})$ is M-Ls-invariant.

(vi) The type $q'|_{y_iz_iyz}$ does not depend on i, modulo identifying variables for different i's.

Claim 2. For every $k < \omega$ there are $g_0h_0g_1h_1\dots g_{k-1}h_{k-1}g_k$, $g'_0h'_0g'_1h'_1\dots g'_{k-1}h'_{k-1}$ and $h''_0g''_1h''_1\dots g''_{k-1}h''_{k-1}g''_k$ such that:

(i) $(g'_ih'_i)_{i<k} \models (q'|_{y_0,z})^{\otimes k}|_M$,

(ii) $(h''_i g''_{i+1})_{i<k} \models (q'|_{z_0,y})^{\otimes k}|_M$,

(iii) $(g_i h_i, g'_i h'_i)$ starts an $Mg_{>i}h_{>i}g'_{>i}h'_{>i}$-indiscernible sequence for every $i < k$,

(iv) $(h_i g_{i+1}, h''_i g''_{i+1})$ starts an $Mh_{>i}g_{>i+1}h''_{>i}g''_{>i+1}$-indiscernible sequence for every $i < k$.

We first prove that the theorem follows from Claim 2. We set $p_0(x,y) = \mathrm{tp}(ab/M)$ and $p_1(x,z) = \mathrm{tp}(a'c/M)$. We will prove that $p_0(x,b) \cup p_1(x,c)$ does not Kim-divide over M. This is enough, because by Proposition 6.27 we can then extend it to a complete type that does not Kim-divide over M. Since we assumed b and c to enumerate λ_T-saturated models containing M, any realisation a'' of that complete type is then what we needed to construct.

By compactness we can find M-indiscernible $(g_i h_i g'_i h'_i g''_i h''_i)_{i\in\mathbb{Z}}$ such that $(g'_i h'_i)_{i\in\mathbb{Z}} \models (q'|_{y_0,z})^{\otimes\mathbb{Z}}|_M$ and $(h''_i g''_{i+1})_{i\in\mathbb{Z}} \models (q'|_{z_0,y})^{\otimes\mathbb{Z}}|_M$. Furthermore, we can make it so that for every $i \in \mathbb{Z}$ we have $g_i h_i \equiv_{Mg_{>i}h_{>i}g'_{>i}h'_{>i}} g'_i h'_i$ and $h_i g_{i+1} \equiv_{Mh_{>i}g_{>i+1}h''_{>i}g''_{>i+1}} h''_i g''_{i+1}$. We have that $q'|_{y,z_0} \supseteq \mathrm{tp}(b'c'_0/M)$, by Claim 1(iv). So by Claim 1(iii) and (v) we have that $b' \mathop{\smile\!\!\!\!\!\mid}\nolimits^*_M c'_0$. Then by Proposition 6.64 we have that $p_0(x, g''_1) \cup p_1(x, h''_0)$ does not Kim-divide. Then because $(h''_i g''_{i+1})_{i\geq n} \models (q'|_{z_0,y})^{\otimes\omega}|_M$ for all $n \in \mathbb{Z}$, we get that $\bigcup_{i\in\mathbb{Z}} p_0(x, g''_{i+1}) \cup p_1(x, h''_i)$ is consistent. By the parallel sequences lemma, Lemma 6.8, we thus have that $\bigcup_{i\in\mathbb{Z}} p_0(x, g_{i+1}) \cup p_1(x, h_i)$ is consistent. This is the same set as $\bigcup_{i\in\mathbb{Z}} p_0(x, g_i) \cup p_1(x, h_i)$. So again by Lemma 6.8 we get that $\bigcup_{i\in\mathbb{Z}} p_0(x, g'_i) \cup p_1(x, h'_i)$ is consistent. By Claim 1(ii) and (iii) we have that $q'|_{y_0,z}$ extends $\mathrm{Lstp}(bc/M)$. So we conclude that $p_0(x,b) \cup p_1(x,c)$ does not Kim-divide over M, as required.

We are left to verify Claim 2. We fix k and by backwards induction on $k' = 2k, 2k-1, \ldots, 1$ we will define trees $(d_\eta e_\eta)_{\eta\in S_{k'}}$ where $S_{k'} = \{\xi \in \omega^{\leq 2k+1} : 0^{k'-1} \trianglelefteq \xi\}$ such that for each k' the tree $(d_\eta e_\eta)_{\eta\in S_{k'}}$ satisfies the following condition:

(P)$_{k'}$ For every $\eta \in \omega^{\leq 2k-1}$ and $i < \omega$ such that $\eta \frown i \in S_{k'}$ we have that:

$$(d_{\eta\frown i\frown j} e_{\eta\frown i\frown j})_{j<\omega} d_{\eta\frown i} e_{\eta\frown i} \equiv^{\mathrm{Ls}}_{M(d_{\trianglerighteq\eta\frown i'} e_{\trianglerighteq\eta\frown i'})_{i'<i}} (\beta_j\gamma_j)_{j<\omega}\beta\gamma.$$

Recall that $q' = \mathrm{tp}((\beta_j\gamma_j)_{j<\omega}\beta\gamma/\mathfrak{M})$. So in particular $(d_{\eta\frown j} e_{\eta\frown j})_{j<\omega} d_\eta e_\eta \equiv^{\mathrm{Ls}}_M (\beta_j\gamma_j)_{j<\omega}\beta\gamma$ for all $\eta \in \omega^{\leq 2k} \cap S_{k'}$.

For $k' = 2k$ we let $(d_\eta e_\eta)_{\eta\in S_{2k}}$ just be $(b'_i c'_i)_{i<\omega} b'c$. Suppose now that we have constructed $(d_\eta e_\eta)_{\eta\in S_{k'}}$. By (P)$_{k'}$ we have that $(d_{0^{k'-1}\frown i} e_{0^{k'-1}\frown i})_{i<\omega} d_{0^{k'-1}} e_{0^{k'-1}} \equiv^{\mathrm{Ls}}_M (\beta_i\gamma_i)_{i<\omega}\beta\gamma$. So by Claim 1(v) there is global M-Ls-invariant $r \supseteq q'$ such that r also extends $\mathrm{Lstp}((d_\eta e_\eta)_{\eta\in S_{k'}}/M)$. Here we match $(d_{0^{k'-1}\frown i} e_{0^{k'-1}\frown i})_{i<\omega} d_{0^{k'-1}} e_{0^{k'-1}}$ with the variables in q'. Let $((d_{\eta,i} e_{\eta,i})_{\eta\in S_{k'}})_{i<\omega}$ be a Morley sequence in r with $(d_{\eta,0} e_{\eta,0})_{\eta\in S_{k'}} = (d_\eta e_\eta)_{\eta\in S_{k'}}$. We set $d_{0^{k'-2}\frown i\frown\xi} e_{0^{k'-2}\frown i\frown\xi} = d_{0^{k'-1}\frown\xi,i} e_{0^{k'-1}\frown\xi,i}$ for all $i < \omega$ and $\xi \in \omega^{\leq 2k+2-k'}$. We directly get (P)$_{k'-1}$ for $\eta \in S_{k'} - \{0^{k'-2}\}$ by virtue of $((d_{\eta,i} e_{\eta,i})_{\eta\in S_{k'}})_{i<\omega}$

being a Morley sequence. By Claim 1(iv) we have that $(d_{0^{k'-2}\frown i}e_{0^{k'-2}\frown i})_{i<\omega}$ is a Morley sequence in q. So we can find $d_{0^{k'-2}}e_{0^{k'-2}}$ such that $(d_{0^{k'-2}\frown i}e_{0^{k'-2}\frown i})_{i<\omega}d_{0^{k'-2}}e_{0^{k'-2}} \equiv^{\mathrm{Ls}}_M (\beta_i\gamma_i)_{i<\omega}\beta\gamma$ and that concludes the construction of $(d_\eta e_\eta)_{\eta\in S_{k'-1}}$.

Similarly as in the proof of Lemma 6.52, we will now show by induction on $n \leq k$ that the following holds.

(Q)$_n$ Let $\eta_{2k-2m} \in \omega^{2k-2m}$ and $\nu_{2k-2m+1} \in \omega^{2k-2m+1}$ for $0 \leq m \leq n$. Suppose that $\eta_{2k-2m} \triangleleft \nu_{2k-2m+1}$ for all $0 \leq m \leq n$, $\eta_{2k} >_{\mathrm{lex}} \eta_{2k-2} >_{\mathrm{lex}} \ldots >_{\mathrm{lex}} \eta_{2k-2n}$ and for all $0 \leq m' < m \leq n$ we have that $\eta_{2k-2m} \wedge \eta_{2k-2m'} \in \omega^{2k-2m-1}$. Then $(d_{\nu_{2k-2m+1}}e_{\nu_{2k-2m+1}}d_{\eta_{2k-2m}}e_{\eta_{2k-2m}})_{m\leq n}$ is a Morley sequence in $q'|_{y_0z_0yz}$.

For $n = 0$ this follows immediately from (P)$_1$ and Claim 1(vi). So suppose (Q)$_n$ holds for some $n < k$ and let $\eta_{2k-2m} \in \omega^{2k-2m}$ and $\nu_{2k-2m+1} \in \omega^{2k-2m+1}$ for $0 \leq m \leq n+1$ be as in the statement of (Q)$_{n+1}$. For any $m < n$ we have that $\eta_{2k-2m} \wedge \eta_{2k-2n-2} = \eta_{2k-2n-2}|_{2k-2n-3}$. So we can write $\eta_{2k-2n-2} = \xi \frown i$ for some $\xi \in \omega^{2k-2n-3}$ and $i < \omega$. We then have $\eta_{2k-2m} \trianglerighteq \xi \frown i'$ for some $i' < i$ for all $m \leq n$. So it follows from (P)$_1$, Claim 1(vi) and the induction hypothesis that $(d_{\nu_{2k-2m+1}}e_{\nu_{2k-2m+1}}d_{\eta_{2k-2m}}e_{\eta_{2k-2m}})_{m\leq n+1}$ is a Morley sequence in $q'|_{y_0z_0yz}$.

By exactly the same argument we also have the following condition. It differs from (Q)$_n$ in that the levels have been shifted by one (so we only consider it for $n < k$).

(Q')$_n$ Let $\eta_{2k-2m-1} \in \omega^{2k-2m-1}$ and $\nu_{2k-2m} \in \omega^{2k-2m}$ for $0 \leq m \leq n$. Suppose that $\eta_{2k-2m-1} \triangleleft \nu_{2k-2m}$ for all $0 \leq m \leq n$, $\eta_{2k-1} >_{\mathrm{lex}} \eta_{2k-3} >_{\mathrm{lex}} \ldots >_{\mathrm{lex}} \eta_{2k-2n-1}$ and for all $0 \leq m' < m \leq n$ we have that $\eta_{2k-2m-1} \wedge \eta_{2k-2m'-1} \in \omega^{2k-2m-2}$. Then $(d_{\nu_{2k-2m}}e_{\nu_{2k-2m}}d_{\eta_{2k-2m-1}}e_{\eta_{2k-2m-1}})_{m\leq n}$ is a Morley sequence in $q'|_{y_0z_0yz}$.

Now let $(d'_\eta e'_\eta)_{\eta\in\omega^{2k+1}}$ be an s-indiscernible over M tree which is EM_s-based on $(d_\eta e_\eta)_{\eta\in\omega^{2k+1}}$ over M. We put $g_i = d'_{0^{2(k-i)+1}}$ for $i \leq k$, and for $i < k$ we put $h_i = e'_{0^{2(k-i)}}$, $g'_i = d'_{0^{2(k-i)-1}\frown 1\frown 0}$, $h'_i = e'_{0^{2(k-i)-1}\frown 1}$, $g''_{i+1} = d'_{0^{2(k-i-1)}\frown 1}$ and $h''_i = e'_{0^{2(k-i-1)}\frown 1\frown 0}$, see Figure 6.7.1. Then conditions (i) and (ii) from Claim 2 follow from (Q)$_k$ and (Q')$_{k-1}$, while conditions (iii) and (iv) follow from s-indiscernibility. □

Now that we have proved the independence theorem, we first note some useful immediate consequences in Corollary 6.69. After that, the rest of this section will be devoted to proving a stronger version of the independence theorem, Theorem 6.74.

Definition 6.67. Let I be a linear order. We will say that $(a_i)_{i\in I}$ is a $\mathop{\smile\!\!\!\!\!\!|}^K_M$-*independent sequence* if $a_i \mathop{\smile\!\!\!\!\!\!|}^K_M a_{<i}$ for every $i \in I$. We will say that $(a_i)_{i\in I}$ is $\mathop{\smile\!\!\!\!\!\!|}^K_M$-*Morley* if it is $\mathop{\smile\!\!\!\!\!\!|}^K_M$-independent and M-indiscernible.

Lemma 6.68. *Let T be thick NSOP$_1$ with an e.c. model M, and let a, b, c be any tuples of parameters and x a tuple of variables. Then there exists a (partial) type*

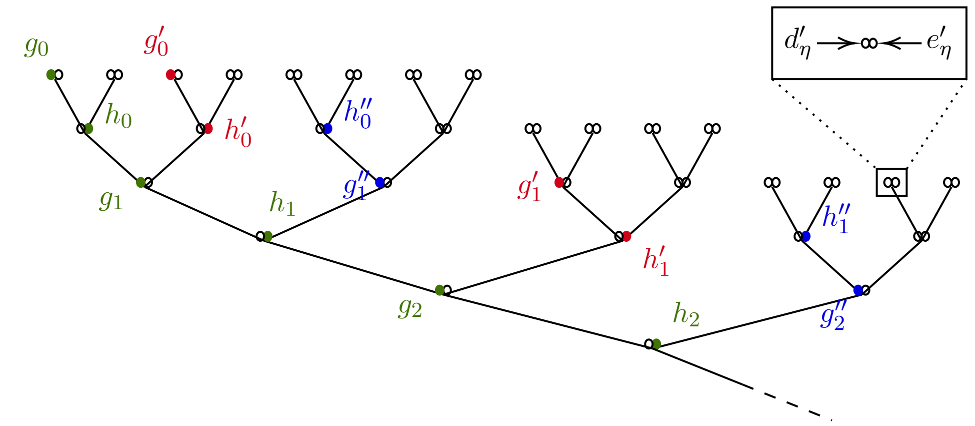


Figure 6.7.1: Choice of the $g_i h_i g_i' h_i' g_i'' h_i''$.

$\Sigma(x, y)$ over Mab such that for any x, y we have that

$$\models \Sigma(x, y) \iff (y \equiv_{Mb} c) \wedge (xa \mathop{\smile\!\!\!\!\!\!\mid}^{K}_{M} yb).$$

In particular, taking $y = \emptyset$, we get that the condition $xa \mathop{\smile\!\!\!\!\!\!\mid}^{K}_{M} b$ is type definable over Mab in the variable x.

Proof. Let $q(y, z)$ be a global M-Ls-invariant type extending $\mathrm{tp}(cb/M)$. Then, by Kim's Lemma, for any $y \equiv_{Mb} c$ and any x, the condition $xa \mathop{\smile\!\!\!\!\!\!\mid}^{K}_{M} yb$ is equivalent to:

$$\exists (y_i z_i)_{i<\omega} \left(q^{\otimes\omega}|_M((y_i z_i)_{i<\omega}) \text{ and } y_0 z_0 = yb \text{ and } (y_i z_i)_{i<\omega} \text{ is } Max\text{-indiscernible} \right),$$

which is clearly a type-definable over Mab condition by thickness. $\square$

In particular, we get that being an $\mathop{\smile\!\!\!\!\!\!\mid}^{K}_{M}$-independent sequence in a fixed type over M is type-definable over M in thick NSOP$_1$ theories. That is, for a linear order I we can use the type

$$\bigcup_{i \in I} \Sigma(x_{<i}, x_i),$$

where Σ is as in Lemma 6.68. Then by symmetry, Theorem 6.57, this (partial) type expresses exactly what we wanted.

Corollary 6.69. *Suppose T is thick NSOP$_1$ with an e.c. model M.*

(i) If $a \mathop{\smile\!\!\!\!\!\!\mid}^{K}_{M} b$ and $a \equiv^{\mathrm{Ls}}_{M} b$ then there exists an infinite M-indiscernible sequence starting with (a, b).

(ii) If $a \equiv^{\mathrm{Ls}}_{M} b$ then a and b are at Lascar distance at most 2 over M. In particular, Lascar equivalence over e.c. models is type-definable.

(iii) (Generalised independence theorem) Let $(a_i)_{i<\kappa}$ an $\mathop{\smile\!\!\!\!\!\!\mid}^K_M$-independent sequence. Suppose $b_i \equiv^{\mathrm{Ls}}_M b$ and $b_i \mathop{\smile\!\!\!\!\!\!\mid}^K_M a_i$ for every $i < \kappa$. Then there exists b' such that $b'a_i \equiv^{\mathrm{Ls}}_M b_i a_i$ for every $i < \kappa$ and $b' \mathop{\smile\!\!\!\!\!\!\mid}^K_M (a_i)_{i<\kappa}$.

Proof. (i) We can inductively find a sequence $(c_i)_{i<\omega}$ such that $c_0c_1 = ab$, $c_i \equiv^{\mathrm{Ls}}_M b$, $c_i \mathop{\smile\!\!\!\!\!\!\mid}^K_M c_{<i}$ and $c_ic_j \equiv_M ab$ for all $i < j < \omega$: indeed, if we have constructed $c_{\leq i}$ then by the independence theorem we can choose c_{i+1} such that $c_{i+1} \equiv^{\mathrm{Ls}}_{Mc_{<i}} c_i$, $c_ic_{i+1} \equiv^{\mathrm{Ls}}_M ab$ and $c_{i+1} \mathop{\smile\!\!\!\!\!\!\mid}^K_M c_{\leq i}$.

By compactness we can find a sequence $(c'_i)_{i<\lambda_{|T|+|Ma|}}$ with $c'_ic'_j \equiv_M ab$ for all $i < j < \lambda_{|T|+|Ma|}$. Choose an M-indiscernible sequence $(d_i)_{i<\omega}$ based on $(c'_i)_{i<\lambda_{|T|+|Ma|}}$ over M. Then $d_0d_1 \equiv_M ab$, so we conclude that the pair (a, b) starts an M-indiscernible sequence.

(ii) By extension (Corollary 6.39) we can choose $c \equiv^{\mathrm{Ls}}_M a$ with $c \mathop{\smile\!\!\!\!\!\!\mid}^K_M ab$. By (i) we get that (a, c) and (b, c) both start M-indiscernible sequences.

(iii) We choose inductively a sequence $(b'_j)_{j\leq\kappa}$ such that $b'_ja_i \equiv^{\mathrm{Ls}}_M b_ia_i$ for every $i < j$ and $b'_j \mathop{\smile\!\!\!\!\!\!\mid}^K_M (a_i)_{i<j}$, so that we can put $b' := b_\kappa$. The successor step follows directly by the independence theorem, and the limit step follows by type-definability of Lascar equivalence over M, Lemma 6.68 and compactness. □

Definition 6.70. We will say that a tree $(c_\eta)_{\eta\in\omega^{\leq k}}$ is *spread-out over M* if $(c_{\trianglerighteq\eta^\frown i})_{i<\omega}$ is a Morley sequence in some global M-Ls-invariant type for every $\eta \in \omega^{\leq k-1}$.

There are two differences between being spread-out over M and being q-spread-out over M (see Definition 6.51 for the latter). In the latter the global M-Ls-invariant type involved has to be q, while the former just requires some global M-Ls-invariant type. The second difference is in the sequence in the tree that is required to be a Morley sequence. In the former we consider a sequence of subtrees above some fixed node, all at the same level. In the latter we consider a sequence of nodes in the tree, one in every level (except for the root), as pictured in Figure 6.5.1.

The following lemma follows from the independence theorem exactly as in [KR20, Lemma 6.2/Remark 6.3], so we omit the proof.

Fact 6.71. *Suppose T is thick NSOP$_1$, M an e.c. model, $a \mathop{\smile\!\!\!\!\!\!\mid}^K_M b$, $(b_\eta)_{\eta\in\omega^{\leq k}}$ (with $k < \omega$) is a spread-out over M tree such that $b_\eta \mathop{\smile\!\!\!\!\!\!\mid}^K_M b_{\triangleright\eta}$ and $b_\eta \equiv^{\mathrm{Ls}}_M b$ for every $\eta \in \omega^{\leq k}$. Then, writing $p(x, b) = tp(a/Mb)$, there exists $a' \models \bigcup_{\eta\in\omega^{\leq k}} p(x, b_\eta)$ with $a' \mathop{\smile\!\!\!\!\!\!\mid}^K_M (b_\eta)_{\eta\in\omega^{\leq k}}$ and $a' \equiv^{\mathrm{Ls}}_M a$.*

Lemma 6.72. *Suppose T is thick NSOP$_1$, M an e.c. model, $b \equiv^{\mathrm{Ls}}_M b'$, $b \mathop{\smile\!\!\!\!\!\!\mid}^K_M b'$ and I is a linear order with two distinct elements 0 and 1. Then there is a $\mathop{\smile\!\!\!\!\!\!\mid}^K_M$-Morley CR-Morley in $\mathrm{tp}(b/M)$ sequence $(b_i)_{i\in I}$ with $b_0 = b$ and $b_1 = b'$.*

Proof. By extension (Corollary 6.39) there is a λ_T-saturated model $N \supseteq Mb$ with $N \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_M^K b'$. Then there is a λ_T-saturated model $N' \supseteq Mb'$ with $N' \equiv_M^{\mathrm{Ls}} N$. Hence, again by extension, we can find $N'' \equiv_{Mb'}^{\mathrm{Ls}} N'$ with $N \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_M^K N''$. So replacing b and b' by N and N'' we may assume without loss of generality that b and b' are λ_T-saturated models containing M. Put $\lambda = |b|$ and (using Lemma 6.55) choose a global M-Ls-invariant extension q of $\mathrm{Lstp}(b'/M)$ satisfying $(*)_\lambda$.

We claim that it is enough to show that for any $1 < k < \omega$ there is a $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits_M^K$-independent CR-Morley sequence $(a_i)_{i<k}$ in q over M with $a_i \equiv_M^{\mathrm{Ls}} b'$ and $a_i a_j \equiv_M bb'$ for all $i < j < k$: indeed, if we show this, then, as all these conditions are type-definable by Lemma 6.68 and Corollary 6.69(ii), we can find by compactness a $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K$-independent over M CR-Morley sequence $(a_i)_{i<\lambda_{|T|+|b|}}$ in q over M with $a_i a_j \equiv_M bb'$ for each $i < j$, and then taking an M-indiscernible sequence indexed by I which is based on $(a_i)_{i<\lambda_{|T|+|Mb|}}$ over M and moving it by an automorphism to guarantee that $b_0 b_1 = bb'$ (note this may change q) will do the job.

So fix any $1 < k < \omega$ and put $p = tp(b'/Mb)$. By backward induction on $k' = k+1, k, \ldots, 1$ we will define trees $(c_\eta)_{\eta \in S_{k'}}$ where $S_{k'} := \{\xi \in \omega^{\leq k} : 0^{k'-1} \trianglelefteq \xi\}$ such that for each k' the tree $(c_\eta)_{\eta \in S_{k'}}$ is spread-out over M and satisfies the following conditions:

$(\mathrm{A1})_{k'}$ $c_\eta c_\nu \equiv_M bb'$ for any $\nu, \eta \in S_{k'}$ with $\nu \triangleleft \eta$ and $c_\eta \equiv_M^{\mathrm{Ls}} b'$ for any $\eta \in S_{k'}$;

$(\mathrm{A2})_{k'}$ $(c_\eta)_{\eta \in S_{k'}}$ is q-spread-out over M;

$(\mathrm{A3})_{k'}$ $c_\eta \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_M^K c_{\triangleright \eta}$ for every $\eta \in S_{k'}$.

For $k' = k+1$ putting $c_{0^k} = b'$ works. Now suppose we are done for some $k' \leq k+1$. By Fact 6.71 we can find $c' \models \bigcup_{\eta \in S_{k'}} p(x, c_\eta)$ with $c' \equiv_M^{\mathrm{Ls}} b'$ and $c' \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_M^K (c_\eta)_{\eta \in S_{k'}}$. By $(\mathrm{A1})_{k'}$ there is a tuple d such that $c_{0^{k'-1}} (c_\eta)_{\eta \in S^*_{k'}} \equiv_M^{\mathrm{Ls}} b'd$. Now, by $(*)_\lambda$ there is some global M-Ls-invariant type $r(x,z) \supseteq q(x)$ which extends $\mathrm{Lstp}(b'd/M) = \mathrm{Lstp}(c_{0^{k'-1}} (c_\eta)_{\eta \in S^*_{k'}}/M)$. Also, as $c' \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_M^K (c_\eta)_{\eta \in S_{k'}}$ and c_η's are λ_T-saturated models (as b' is), we get by Corollary 6.34 that $\mathrm{Lstp}(c'/M(c_\eta)_{\eta \in S_{k'}})$ does not $r(x,z)$-Ls-divide over M. Hence, there is an Mc'-indiscernible Morley sequence $I := ((c_{\eta,i})_{\eta \in S_{k'}})_{i<\omega}$ in $r(x,z)$ over M with $c_{\eta,0} = c_\eta$ for each $\eta \in S_{k'}$. By the chain condition Lemma 6.53 we have that $c' \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_M^K I$. Thus, putting $c_{0^{k'-2} \frown i \frown \zeta} := c_{0^{k'-1} \frown \zeta, i}$ for all $i < \omega$, $\zeta \in \omega^{\leq k+1-k'}$, and $c_{0^{k'-2}} := c'$, we immediately get that the tree $(c_\eta)_{\eta \in S_{k'-1}}$ satisfies $(\mathrm{A3})_{k'-1}$. $(\mathrm{A1})_{k'-1}$ follows from $(\mathrm{A1})_{k'}$, the choice of c' and Mc' indiscernibility of I. $(\mathrm{A2})_{k'-1}$ follows from $(\mathrm{A2})_{k'}$ and Lemma 6.52(i). This completes the inductive construction.

Letting $(c'_\eta)_{\eta \in \omega^{\leq k}}$ be an s-indiscernible over M tree which is EM_s-based on $(c_\eta)_{\eta \in \omega^{\leq k}}$ over Mb', we get that $(c'_\eta)_{\eta \in \omega^{\leq k}}$ satisfies $(\mathrm{A1})_1$ and $(\mathrm{A3})_1$ (by Lemma 6.68 and Corollary 6.69(ii)) and is weakly q-spread-out over M by Lemma 6.52(ii).

Put $a_i := c'_{0^{k-i}}$ for $i < k$. Then by Lemma 6.52(iii) we have that $(a_i)_{i<k}$ is CR-Morley in q over M. Also, $a_i a_j \equiv_M bb'$ for all $i < j < k$ by $(\mathrm{A1})_1$, and $(a_i)_{i<k}$ is $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_M$-independent over M by $(\mathrm{A3})_1$. This completes the proof. □

Lemma 6.73 (Chain lemma for $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K$-Morley sequences)**.** *Suppose T is thick NSOP$_1$ with an e.c. model M, $(d_i)_{i \in I}$ is an infinite $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_M$-Morley sequence and $a \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_M d_{i_0}$ for some $i_0 \in I$. Then there exists $a^* d_{i_0} \equiv^{\mathrm{Ls}}_M a d_{i_0}$ such that $(d_i)_{i \in I}$ is indiscernible over Ma^* and $a^* \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_M (d_i)_{i \in I}$.*

Proof. By compactness there is a $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_M$-Morley sequence $(d''_i)_{i<\lambda}$ such that $(d_i)_{i \in I} \frown (d''_i)_{i<\lambda}$ is M-indiscernible, where $\lambda = \lambda_{|T|+|Mad_0|+|I|}$. As $d_{i_0} \equiv^{\mathrm{Ls}}_M d''_0$, $a \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_M d_{i_0}$ and $(d''_i)_{i<\lambda}$ is $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K$-independent over M , we get by Corollary 6.69(iii) that there exists a' with $a' d''_i \equiv^{\mathrm{Ls}}_M a d_{i_0}$ for every $i < \lambda$ and $a' \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_M (d''_i)_{i<\lambda}$. Let $(d'_i)_{i \in I}$ be an Ma'-indiscernible sequence based on $(d''_i)_{i<\lambda}$ over $Maa'(d_i)_{i \in I}$. Then (by finite character and invariance of $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K$) $a' \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_M (d'_i)_{i \in I}$, $(d'_i)_{i \in I} \equiv^{\mathrm{Ls}}_M (d_i)_{i \in I}$ (as $(d_i)_{i \in I} \frown (d'_i)_{i \in I}$ is indiscernible over M), and $a' d'_{i_0} \equiv^{\mathrm{Ls}}_M a d_{i_0}$. Hence, letting f be a Lascar strong automorphism over M sending $(d'_i)_{i \in I}$ to $(d_i)_{i \in I}$ and putting $a^* = f(a')$ we get that $a^* \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_M (d_i)_{i \in I}$ and $(d_i)_{i \in I}$ is Ma^*-indiscernible. Also $a^* d_{i_0} \equiv^{\mathrm{Ls}}_M a' d'_{i_0} \equiv^{\mathrm{Ls}}_M a d_{i_0}$ as required. □

The following is a stronger version of the independence theorem, Theorem 6.66. The assumptions are the same, but in the conclusion we get two extra instances of independence.

Theorem 6.74 (Strong independence theorem)**.** *Suppose T is thick NSOP$_1$ with an e.c. model M, $a_0 \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_M b$, $a_1 \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_M c$, $b \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_M c$, and $a_0 \equiv^{\mathrm{Ls}}_M a_1$. Then there is a such that $a \equiv^{\mathrm{Ls}}_{Mb} a_0$, $a \equiv^{\mathrm{Ls}}_{Mc} a_1$, $a \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_M bc$, $b \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_M ac$, $c \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_M ab$.*

Proof. By a similar trick as at the start of the proof of Theorem 6.66 we may assume that b and c enumerate λ_T-saturated models containing M.

By the independence theorem there is a_2 with $a_2 \equiv^{\mathrm{Ls}}_{Mb} a_0$, $a_2 \equiv^{\mathrm{Ls}}_{Mc} a_1$ and $a_2 \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_M bc$. By extension (Corollary 6.39) there is $b' \equiv^{\mathrm{Ls}}_{Mc} b$ such that $b \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_M b'c$, so $b'c \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_M b$ by symmetry. By extension again, there is $c' \equiv^{\mathrm{Ls}}_{Mb} c$ with $b'c \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_M bc'$. As $b'c \equiv^{\mathrm{Ls}}_M bc \equiv^{\mathrm{Ls}}_M bc'$, we get by Lemma 6.72 that there is a $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_M$-Morley CR-Morley in $\mathrm{tp}(bc/M)$ sequence $I = (b_i, c_i)_{i \in \mathbb{Z}}$ with $b_0 c_0 = bc'$ and $b_1 c_1 = b'c$. As $a_2 \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_M bc$, we get by Lemma 6.73 that there is some a such that $abc' \equiv^{\mathrm{Ls}}_M a_2 bc$, I is Ma-indiscernible and $a \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_M I$.

Then by monotonicity $a \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_M bc$. We also have $ab \equiv^{\mathrm{Ls}}_M a_2 b \equiv^{\mathrm{Ls}}_M a_0 b$, and, by indiscernibility, $ac \equiv^{\mathrm{Ls}}_M ac' \equiv^{\mathrm{Ls}}_M a_2 c \equiv^{\mathrm{Ls}}_M a_1 c$. Since b and c were assumed to enumerate λ_T-saturated models we get $a \equiv^{\mathrm{Ls}}_{Mb} a_0$ and $a \equiv^{\mathrm{Ls}}_{Mc} a_1$. Also, $(b_i)_{i \leq 0}$ is an Mac-indiscernible CR-Morley sequence in $\mathrm{tp}(b/M)$ with $b_0 = b$, which gives $b \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_M ac$ by Corollary 6.50. Similarly, as $(c_i)_{i \geq 1}$ is an Mab-indiscernible CR-Morley sequence in $\mathrm{tp}(c/M)$ with $c_1 = c$, we get that $c \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_M ab$. □

6.8 Transitivity

Lemma 6.75. *If $M \subseteq N$ are e.c. models of a thick NSOP$_1$ theory, $a \mathop{\smile\!\!\!\!\!\!|}^K_M N$, and μ is a small cardinal, then there is a CR-Morley in* $\mathrm{tp}(a/N)$ *sequence $(a_i)_{i\in\mu}$ with $a_0 = a$ such that $a_i \mathop{\smile\!\!\!\!\!\!|}^K_M Na_{<i}$ for every $i < \mu$.*

Proof. Put $\lambda = |Na| + \aleph_0$ and (using Lemma 6.55) choose a global N-Ls-invariant extension q of $\mathrm{Lstp}(a/N)$ satisfying $(*)_\lambda$.

By Lemma 6.68, compactness, finite character of Kim-independence, and an automorphism, it is enough to find for any given $k < \omega$ a CR-Morley sequence $(a_i)_{i<k}$ in q over N such that $a_i \mathop{\smile\!\!\!\!\!\!|}^K_M Na_{<i}$ for every $i < k$.

So fix any $k < \omega$. By backward induction on $k' = k+1, k, \dots, 1$ we will construct trees $(c_\eta)_{\eta\in S_{k'}}$, where $S_{k'} := \{\xi \in \omega^{\leq k} : 0^{k'-1} \trianglelefteq \xi\}$, such that for each k' the tree $(c_\eta)_{\eta\in S_{k'}}$ satisfies the following conditions:

(A1)$_{k'}$ For any $\eta \in S_{k'}$ we have $c_\eta \mathop{\smile\!\!\!\!\!\!|}^K_M Nc_{\triangleright\eta}$ and $c_\eta \equiv^{\mathrm{Ls}}_N a$;

(A2)$_{k'}$ $(c_\eta)_{\eta\in S_{k'}}$ is q-spread-out over N.

For $k' = k+1$ we let $c_{0^k} = a$. For the inductive step, suppose we are done for some k'. By (A1)$_{k'}$ we have $c_{0^{k'-1}} \equiv^{\mathrm{Ls}}_N a$, so by $(*)_\lambda$ there is a global N-Ls-invariant type $r(x,y) \supseteq q(x)$ extending $\mathrm{Lstp}(c_{0^{k'-1}}, (c_\eta)_{\eta\in S^*_{k'}}/N)$ where x corresponds to $c_{0^{k'-1}}$. Choose a Morley sequence $I := ((c_{\eta,i})_{\eta\in S_{k'}})_{i<\omega}$ in $r(x,y)$ over N with $c_{\eta,0} = c_\eta$ for each $\eta \in S_{k'}$. By extension (Corollary 6.39) there is $c' \equiv^{\mathrm{Ls}}_N a$ with $c' \mathop{\smile\!\!\!\!\!\!|}^K_M NI$. Put $c_{0^{k'-2}\frown i\frown\zeta} := c_{0^{k'-1}\frown\zeta,i}$ for all $i < \omega$, $\zeta \in \omega^{\leq k+1-k'}$, and $c_{0^{k'-2}} := c'$. Then (A2)$_{k'-1}$ follows by Lemma 6.52(i), whereas (A1)$_{k'-1}$ with $\eta \in S^*_{k'-1}$ follows by invariance of Kim-independence, and (A1)$_{k'-1}$ with $\eta = 0^{k'-2}$ follows by the choice of $c_{0^{k'-2}} = c'$. Thus the inductive step, and hence the construction of the tree $(c_\eta)_{\eta\in\omega^{\leq k}} = (c_\eta)_{\eta\in S_1}$, is completed.

Letting $(c'_\eta)_{\eta\in\omega^{\leq k}}$ be an s-indiscernible over N tree which is EM_s-based on $(c_\eta)_{\eta\in\omega^{\leq k}}$ over Na, we get that $(c'_\eta)_{\eta\in\omega^{\leq k}}$ satisfies (A1)$_1$ by Lemma 6.68 and Corollary 6.69(ii), and is weakly q-spread-out over N by Lemma 6.52(ii). Thus, by Lemma 6.52(iii), putting $a_i = c'_{0^{k-i}}$ for $i < k$ we get a CR-Morley sequence $(a_i)_{i<k}$ in q over N satisfying the requirements. $\square$

Lemma 6.76. *Suppose T is thick NSOP$_1$ and $M \subseteq N$ are e.c. models of T. If $a \mathop{\smile\!\!\!\!\!\!|}^K_M N$ and $c \mathop{\smile\!\!\!\!\!\!|}^K_M N$ then there is $c' \equiv^{\mathrm{Ls}}_N c$ such that $ac' \mathop{\smile\!\!\!\!\!\!|}^K_M N$ and $a \mathop{\smile\!\!\!\!\!\!|}^K_N c'$.*

Proof. By Lemma 6.68 there is a type $\Gamma(x; N, a)$ equivalent to the condition $ax \mathop{\smile\!\!\!\!\!\!|}^K_M N$. By Lemma 6.75 there is a CR-Morley in $\mathrm{tp}(a/N)$ sequence $(a_i)_{i<\lambda_T}$ with $a_0 = a$ such that $a_i \mathop{\smile\!\!\!\!\!\!|}^K_M Na_{<i}$ for every $i < \lambda_T$. Replacing $(a_i)_{i<\lambda_T}$ with an N-indiscernible sequence based on it over N and moving by an automorphism (to keep $a_0 = a$), we may assume $(a_i)_{i<\lambda_T}$ is N-indiscernible.

Claim 4. $\bigcup_{i<\lambda_T} \Gamma(x; N, a_i)$ has a realisation c'' such that $c'' \equiv^{\mathrm{Ls}}_N c$.

Proof. By induction on $n < \omega$ we will find $c_n \equiv^{\mathrm{Ls}}_N c$ such that $c_n \mathop{\smile\!\!\!\!\!\mid}^K_M Na_{<n}$ and $c_n \models \bigcup_{i<n} \Gamma(x; N, a_i)$, which is enough by compactness, N-indiscernibility of $(a_i)_{i<\lambda_T}$ and Corollary 6.69(ii). For $n = 0$ put $c_0 = c$. Assume we have found c_n and find by extension (Corollary 6.39) some $c' \equiv^{\mathrm{Ls}}_M c$ be such that $c' \mathop{\smile\!\!\!\!\!\mid}^K_M a_n$. By Theorem 6.74 there is c_{n+1} with $c_{n+1}a_{<n} \equiv^{\mathrm{Ls}}_N c_n a_{<n}$, $c_{n+1}a_n \equiv^{\mathrm{Ls}}_M c'a_n$, $c_{n+1} \mathop{\smile\!\!\!\!\!\mid}^K_M Na_{<n+1}$ and $a_n c_{n+1} \mathop{\smile\!\!\!\!\!\mid}^K_M Na_{<n}$. In particular $c_{n+1} \equiv^{\mathrm{Ls}}_N c_n \equiv^{\mathrm{Ls}}_N c$ and $c_{n+1} \models \bigcup_{i<n+1} \Gamma(x; N, a_i)$. □

Let c'' be given by the claim, and let $(a'_i)_{i<\omega}$ be an Nc''-indiscernible sequence based on $(a_i)_{i<\lambda_T}$ over $Nc''a$. Then $a'_0 \equiv^{\mathrm{Ls}}_N a$ (as $a_i \equiv^{\mathrm{Ls}}_N a$ for every $i < \lambda_T$) so there is a Lascar strong automorphism f over N sending a'_0 to $a = a_0$. Put $c' := f(c'')$. Then $(f(a'_i))_{i<\omega}$ is an Nc'-indiscernible CR-Morley sequence in $\mathrm{tp}(a/N)$ starting with a, so $c' \mathop{\smile\!\!\!\!\!\mid}^K_N a$ by Corollary 6.50. Also, $c' \models \Gamma(x; N, a)$, so $ac' \mathop{\smile\!\!\!\!\!\mid}^K_M N$ by the choice of Γ, and we are done. □

Lemma 6.77. *Suppose T is thick NSOP$_1$ with e.c. models $M \subseteq N$ and $a \mathop{\smile\!\!\!\!\!\mid}^K_M N$. Then there is a $\mathop{\smile\!\!\!\!\!\mid}^K_N$-Morley CR-Morley in $\mathrm{tp}(a/M)$ sequence $(a_i)_{i<\omega}$ with $a = a_0$.*

Proof. By extension (Corollary 6.39) we may assume that a is a λ_T-saturated model extending M. By Lemma 6.55 there is a global M-Ls-invariant extension $q(x) \supseteq \mathrm{tp}(a/M)$ satisfying the property $(*)_\lambda$ with $\lambda = |a| + \aleph_0$. We claim that it is enough to find for any given $k < \omega$ a CR-Morley sequence $(a_i)_{i<k}$ in q over M such that $a_i \mathop{\smile\!\!\!\!\!\mid}^K_N a_{<i}$ and $a_i \equiv_N a$ for every $i < k$: indeed, if we prove this, then, since the condition $(a_i \equiv_N a) \wedge (a_i \mathop{\smile\!\!\!\!\!\mid}^K_N a_{<i})$ is type-definable by Lemma 6.68, we can find by compactness such a sequence of length $\lambda_{|T|+|Na|}$. Then taking an N-indiscernible sequence based on $(a_i)_{i<\lambda_{|T|+|Na|}}$ over N and moving it by an automorphism we obtain a desired sequence.

So fix any $k < \omega$. By backward induction on $k' = k+1, k, \ldots, 1$ we will define trees $(c_\eta)_{\eta \in S_{k'}}$, where $S_{k'} := \{\xi \in \omega^{\leq k} : 0^{k'-1} \trianglelefteq \xi\}$, such that for each k' the tree $(c_\eta)_{\eta \in S_{k'}}$ satisfies the following conditions:

$(\mathrm{A1})_{k'}$ For any $\eta \in S_{k'}$ we have $c_\eta \mathop{\smile\!\!\!\!\!\mid}^K_N c_{\triangleright\eta}$ and $c_\eta \equiv^{\mathrm{Ls}}_N a$;

$(\mathrm{A2})_{k'}$ $(c_\eta)_{\eta \in S_{k'}}$ is q-spread-out over M;

$(\mathrm{A3})_{k'}$ $(c_\eta)_{\eta \in S_{k'}} \mathop{\smile\!\!\!\!\!\mid}^K_M N$.

For $k' = k+1$ we let $c_{0^k} = a$. For the inductive step, suppose we are done for some k'. By $(*)_\lambda$ and $(\mathrm{A1})_{k'}$ there is a global M-invariant type $r(x, y)$ extending $\mathrm{Lstp}(c_{0^{k'-1}}, (c_\eta)_{\eta \in S^*_{k'}}/M)$ and $q(x)$. As c_η's are λ_T-saturated models, we get by $(\mathrm{A3})_{k'}$ and Corollary 6.34 that $\mathrm{Lstp}(N/(c_\eta)_{\eta \in S_{k'}})$ does not r-Ls-divide over M. Thus there is an N-indiscernible Morley sequence $I = ((c_{\eta,i})_{\eta \in S_{k'}})_{i<\omega}$ in $r(x, y)$ over M with $c_{\eta,0} = c_\eta$ for each $\eta \in S_{k'}$ and $I \mathop{\smile\!\!\!\!\!\mid}^K_M N$. By Lemma 6.76 there is $a' \equiv^{\mathrm{Ls}}_N a$ such that $a' \mathop{\smile\!\!\!\!\!\mid}^K_N I$ and $a'I \mathop{\smile\!\!\!\!\!\mid}^K_M N$. Put $c_{0^{k'-2}\frown i \frown \zeta} := c_{0^{k'-1}\frown\zeta, i}$

for all $i < \omega$, $\zeta \in \omega^{\leq k+1-k'}$, and $c_{0^{k'-2}} := a'$. Then we get $(\mathrm{A2})_{k'-1}$ by Lemma 6.52(i), $(\mathrm{A1})_{k'-1}$, using that $a' \mathop{\smile\!\!\!\!\!\!\mid}^K_N I$, and $(\mathrm{A3})_{k'-1}$ holds as $a'I \mathop{\smile\!\!\!\!\!\!\mid}^K_M N$. Thus the inductive step, and hence the construction of the tree $(c_\eta)_{\eta \in \omega^{\leq k}} = (c_\eta)_{\eta \in S_1}$, is completed.

Letting $(c'_\eta)_{\eta \in \omega^{\leq k}}$ be an s-indiscernible over N tree which is EM_s-based on $(c_\eta)_{\eta \in \omega^{\leq k}}$ over Na, we get that $(c'_\eta)_{\eta \in \omega^{\leq k}}$ is weakly q-spread-out over M by Lemma 6.52(ii) and satisfies $(\mathrm{A1})_1$ by Lemma 6.68 and Corollary 6.69(ii). Thus putting $a_i = c'_{0^{k-i}}$ for $i < k$ we get by Lemma 6.52(iii) a CR-Morley sequence $(a_i)_{i<k}$ in q over M satisfying the requirements. □

Theorem 6.78 (Transitivity)**.** *Suppose T is thick NSOP$_1$ with e.c. models $M \subseteq N$. If $a \mathop{\smile\!\!\!\!\!\!\mid}^K_M N$ and $a \mathop{\smile\!\!\!\!\!\!\mid}^K_N c$, then $a \mathop{\smile\!\!\!\!\!\!\mid}^K_M Nc$.*

Proof. By Lemma 6.77 there is a $\mathop{\smile\!\!\!\!\!\!\mid}^K_N$-Morley CR-Morley in $\mathrm{tp}(a/M)$ sequence $I = (a_i)_{i<\omega}$ with $a_0 = a$. As $a \mathop{\smile\!\!\!\!\!\!\mid}^K_N c$, we get by Lemma 6.73 an Nc-indiscernible sequence $I' = (a'_i)_{i<\omega} \equiv_{Na} I$. As I' is also CR-Morley in $\mathrm{tp}(a/M)$ and $a'_0 = a$, we get by Corollary 6.50 that $Nc \mathop{\smile\!\!\!\!\!\!\mid}^K_M a$, so by symmetry we are done. □

6.9 Kim-Pillay style theorem

We can characterise thick NSOP$_1$ theories and Kim-independence by a Kim-Pillay style theorem, see Theorem 6.79 below. This has some overlap with the general canonicity theorem for NSOP$_1$-like independence relations in AECats, Theorem 5.6. However, there are some important differences. The most important difference is that the theorem below links the existence of a certain (NSOP$_1$-like) independence relation to the combinatorial property of being NSOP$_1$, for thick positive theories. That is, it states that a thick positive theory is NSOP$_1$ if and only if there is a nice enough independence relation. This link is not present in the theorem for AECats, we do not even have a combinatorial definition of being NSOP$_1$ in that generality (see also section 7.2). Another subtle difference is that in the theorem below we show that the independence relation must be given by Kim-dividing instead of long Kim-dividing. See Remark 4.51 for a more detailed comparison between those two notions.

Theorem 6.79. *Let T be a thick positive theory. Then T is NSOP$_1$ if and only if there is an automorphism invariant ternary relation $\mathop{\smile\!\!\!\!\!\!\mid}$ on small subsets of the monster model, only allowing e.c. models in the base, satisfying the following properties:*

FINITE CHARACTER *if $a \mathop{\smile\!\!\!\!\!\!\mid}_M b_0$ for all finite $b_0 \subseteq b$ then $a \mathop{\smile\!\!\!\!\!\!\mid}_M b$.*

EXISTENCE *$a \mathop{\smile\!\!\!\!\!\!\mid}_M M$ for any model M.*

Monotonicity *$aa' \mathop{\smile\!\!\!\!\!\!|}_M bb'$ implies $a \mathop{\smile\!\!\!\!\!\!|}_M b$.*

Symmetry *$a \mathop{\smile\!\!\!\!\!\!|}_M b$ implies $b \mathop{\smile\!\!\!\!\!\!|}_M a$.*

Chain Local Character *let a be a finite tuple and $\kappa > |T|$ be regular then for every continuous chain $(M_i)_{i<\kappa}$, with $|M_i| < \kappa$ for all i, there is $i < \kappa$ such that $a \mathop{\smile\!\!\!\!\!\!|}_{M_i} M$, where $M = \bigcup_{i<\kappa} M_i$.*

Independence Theorem *if $a \mathop{\smile\!\!\!\!\!\!|}_M b$, $a' \mathop{\smile\!\!\!\!\!\!|}_M c$ and $b \mathop{\smile\!\!\!\!\!\!|}_M c$ with $a \equiv^{\mathrm{Ls}}_M a'$ then there is a'' such that $a''b \equiv^{\mathrm{Ls}}_M ab$, $a''c \equiv^{\mathrm{Ls}}_M a'c$ and $a'' \mathop{\smile\!\!\!\!\!\!|}_M bc$.*

Extension *if $a \mathop{\smile\!\!\!\!\!\!|}_M b$ then for any c there is $a' \equiv_{Mb} a$ such that $a' \mathop{\smile\!\!\!\!\!\!|}_M bc$.*

Transitivity *if $a \mathop{\smile\!\!\!\!\!\!|}_M N$ and $a \mathop{\smile\!\!\!\!\!\!|}_N b$ with $M \subseteq N$ then $a \mathop{\smile\!\!\!\!\!\!|}_M Nb$.*

Furthermore, in this case $\mathop{\smile\!\!\!\!\!\!|} = \mathop{\smile\!\!\!\!\!\!|}^K$.

The properties in Theorem 6.79 are not as strong as they could be. For example, we actually proved the strong independence theorem for $\mathop{\smile\!\!\!\!\!\!|}^K$, see Theorem 6.74. The slightly simpler formulation of the properties in Theorem 6.79 is easier to verify for an arbitrary independence relation $\mathop{\smile\!\!\!\!\!\!|}$. Then it follows immediately from $\mathop{\smile\!\!\!\!\!\!|} = \mathop{\smile\!\!\!\!\!\!|}^K$ that such an independence relation $\mathop{\smile\!\!\!\!\!\!|}$ also satisfies the stronger formulations.

Remark 6.80. In the existing Kim-Pillay style theorems for full first-order logic [KR20, Theorem 9.1], [KR19, Theorem 6.11] and [CKR20, Theorem 5.1] there are still various properties that mention syntax. Our Theorem 6.79 is completely syntax-free. One syntax-dependent property is mentioned in all of the above theorems, and is called Strong Finite Character: if $a \not\mathop{\smile\!\!\!\!\!\!|}_M b$ then there is $\varphi(x, b, m) \in \mathrm{tp}(a/Mb)$ such that for any $a' \models \varphi(x, b, m)$ we have $a' \not\mathop{\smile\!\!\!\!\!\!|}_M b$.

We could replace Finite Character and Chain Local Character in Theorem 6.79 by Strong Finite Character. Obviously Strong Finite Character implies Finite Character and modulo the other properties it also implies Chain Local Character by Lemma 6.86 and Lemma 6.87.

Remark 6.81. To conclude that a theory is NSOP_1 it is enough to find an independence relation with the properties Strong Finite Character, Existence, Monotonicity, Symmetry and Independence Theorem, see [HK21, Theorem 6.4]. However, that does not guarantee that the independence relation is also Kim-independence, see [KR20, Remark 9.39] for an example (already in full first-order logic). We also point out that [HK21, Theorem 6.4] says nothing about the properties that Kim-independence generally has in NSOP_1 theories. Finally, our proof is also different because we do not rely on the syntactic property Strong Finite Character.

Remark 6.82. We point out a minor difference between Theorem 6.66 and INDEPENDENCE THEOREM in Theorem 6.79. In the former we get $a'' \equiv^{\mathrm{Ls}}_{Mb} a$, which is generally stronger than the $a''b \equiv^{\mathrm{Ls}}_{M} ab$ in the latter (and similar for c). Again, the reason is that the latter is easier to verify. Definitely in semi-Hausdorff theories, because then $a''b \equiv^{\mathrm{Ls}}_{M} ab$ is equivalent to $a''b \equiv_M ab$, so we do not have to worry about Lascar strong types. For a concrete example of this, see Fact 6.97(i). The only place where INDEPENDENCE THEOREM is used, namely to get consistency along a certain sequence, we only need this weaker version.

We have proved that $\mathop{\smile\!\!\!\!\!\mid}^K$ has all properties in Theorem 6.79, except for CHAIN LOCAL CHARACTER. We will actually prove a stronger version, namely CLUB LOCAL CHARACTER, to really show that it is actually an NSOP$_1$-like independence relation as defined for AECats in Definition 4.9. We translate the necessary definitions into the more concrete setting that we work in here. We already mentioned it in Remark 4.10(4), but we mention it here again: these ideas about using club sets are due to [KRS17].

Definition 6.83. Let κ be a regular cardinal and X any set. We write $[X]^{<\kappa} = \{Y \subseteq X : |Y| < \kappa\}$. We call a family of subsets $\mathcal{F} \subseteq [X]^{<\kappa}$ ...

- ... *unbounded* if for every $Z \in [X]^{<\kappa}$ there is $Y \in \mathcal{F}$ with $Z \subseteq Y$.
- ... *closed* if for every chain $(Y_i)_{i<\gamma}$ in $\mathcal{F}$ (i.e. $i \leq j < \gamma$ implies $Y_i \subseteq Y_j$) with $\gamma < \kappa$ we have that $\bigcup_{i<\gamma} Y_i \in \mathcal{F}$.
- ... *club* if $\mathcal{F}$ is closed and unbounded.

Definition 6.84. We say that an independence relation $\mathop{\smile\!\!\!\!\!\mid}$ has CLUB LOCAL CHARACTER if there is a regular cardinal μ such that for any finite a, any e.c. model M and any regular $\kappa \geq \mu$ there is a club set of e.c. models $\mathcal{F} \subseteq [M]^{<\kappa}$ with $a \mathop{\smile\!\!\!\!\!\mid}_{M_0} M$ for all $M_0 \in \mathcal{F}$.

Remark 6.85. There is a slight difference between the two statements of CLUB LOCAL CHARACTER. In the concrete statement (Definition 6.84) we talk about the existence of one cardinal μ and we require a to be finite, while in the AECat version (Definition 4.8) the cardinality of a can be anything and we require there to be a regular cardinal $\Upsilon(\lambda)$ that works whenever $|a| < \lambda$. So a priori it seems like Definition 6.84 only defines CLUB LOCAL CHARACTER for $\lambda = \aleph_0$ with $\Upsilon(\aleph_0) = \mu$. However, the full version follows from this, if we also assume FINITE CHARACTER.

To see this, we set $\Upsilon(\lambda) = \lambda + \mu$. Now let a be any infinite tuple with $|a| < \lambda$, let M be any e.c. model and let $\kappa \geq \Upsilon(\lambda)$. For every finite $a_0 \subseteq a$ there is some club set of e.c. models $\mathcal{F}_{a_0} \subseteq [M]^{<\kappa}$ such that $a_0 \mathop{\smile\!\!\!\!\!\mid}_{M_0} M$ for all $M_0 \in \mathcal{F}_{a_0}$. So $\{F_{a_0} : a_0 \in [a]^{<\omega}\}$ is a family of club sets and

$$|\{F_{a_0} : a_0 \in [a]^{<\omega}\}| \leq |a|^{<\omega} = |a| < \lambda \leq \Upsilon(\lambda) \leq \kappa.$$

By [Jec03, Theorem 8.22] the intersection of $< \kappa$ club sets in $[M]^{<\kappa}$ is again a club set. So $\mathcal{F} = \bigcap \{F_{a_0} : a_0 \in [a]^{<\omega}\}$ is a club set of e.c. models in $[M]^{<\kappa}$. Now for any $M_0 \in \mathcal{F}$ we have by construction that $a_0 \mathop{\smile\!\!\!\!\!\!\mid}_{M_0} M$ for all finite $a_0 \subseteq a$, and hence $a \mathop{\smile\!\!\!\!\!\!\mid}_{M_0} M$ by Finite Character. So we do indeed have the full version of Club Local Character, as stated in Definition 4.8.

Lemma 6.86. *Let* $\mathop{\smile\!\!\!\!\!\!\mid}$ *satisfy* Strong Finite Character, Existence, Monotonicity *and* Symmetry. *Then* $a \mathop{\smile\!\!\!\!\!\!\mid}^u_M b$ *implies* $a \mathop{\smile\!\!\!\!\!\!\mid}_M b$.

Proof. Exactly as in [CR16, Proposition 5.8]. □

Lemma 6.87. *Let* $\mathop{\smile\!\!\!\!\!\!\mid}$ *be as in Lemma* 6.86. *Then it satisfies satisfies* Club Local Character, *and thus in particular* Chain Local Character.

Proof. By Lemma 6.86 the proof from [KRS17, Theorem 3.2] applies. The proof is actually quite elegant and instructive, so we repeat it here for completeness sake.

Let a be finite and M be any e.c. model. We will prove that for regular $\kappa \geq |T|^+$ there is a club set $\mathcal{F} \subseteq [M]^{<\kappa}$ of e.c. models such that for all $M_0 \in \mathcal{F}$ we have $a \mathop{\smile\!\!\!\!\!\!\mid}_{M_0} M$.

Write $p(x) = \text{tp}(a/M)$. We define

$$\mathcal{F} = \{M_0 \in [M]^{<\kappa} : M_0 \text{ is an e.c. model and } p \text{ is an heir of } p|_{M_0}\}.$$

We first claim that $\mathcal{F}$ is a club set in $[M]^{<\kappa}$. The fact that $\mathcal{F}$ is closed follows directly from the definition of being and heir, together with the fact that the union of a chain of e.c. models is again an e.c. model.

To prove unboundedness we let $A \subseteq M$ with $|A| < \kappa$. We expand the structure M in a way that forces p to be definable. That is, we expand our language $\mathcal{L}$ to $\mathcal{L}_p$ as follows: for every $\mathcal{L}$-formula $\varphi(x, y)$ we add a relation $R_\varphi(y)$. Then we expand M to and $\mathcal{L}_p$-structure M_p by interpreting every such $R_\varphi(y)$ as $\{b \in M : \varphi(x, b) \in p(x)\}$. This does not change the Löwenheim-Skolem number, so there must be an $\mathcal{L}_p$-elementary substructure N_p of M_p that contains A with $|N_p| < \kappa$. We let N be the $\mathcal{L}$-reduct of N_p. Then we have that p is an heir of $p|_N$. Finally, any elementary substructure of and e.c. model is again an e.c. model, so $N \in \mathcal{F}$ while also $A \subseteq N$. This completes the proof of our claim that $\mathcal{F}$ is a club set.

We are now left to show that $a \mathop{\smile\!\!\!\!\!\!\mid}_{M_0} M$ for all $M_0 \in \mathcal{F}$. So let $M_0 \in \mathcal{F}$. We have by construction that $a \mathop{\smile\!\!\!\!\!\!\mid}^h_{M_0} M$. By the duality between heirs and coheirs, see Remark 6.5, we have $M \mathop{\smile\!\!\!\!\!\!\mid}^u_{M_0} a$. It then follows from Lemma 6.86 that $M \mathop{\smile\!\!\!\!\!\!\mid}_{M_0} a$, and thus $a \mathop{\smile\!\!\!\!\!\!\mid}_{M_0} M$ by Symmetry.

The final sentence follows just as in Lemma 4.15. □

Corollary 6.88 (Local Character)**.** *In a thick NSOP*$_1$ *theory* $\mathop{\smile\!\!\!\!\!\!\mid}^K$ *satisfies* Club Local Character *and hence also* Chain Local Character.

The notion of long dividing, Definition 4.32, will actually be useful in the proof of Theorem 6.79. The reason for this is that it allows us to directly work with a subsequence of some very long sequence, rather than just with sequence that is based on that very long sequence. In particular, if we have a chain of models that witnesses independence of the very long sequence (see Definition 6.90) then we can use a subchain as witnesses of independence for a subsequence. This will be relevant in Proposition 6.93 and we discuss this further after that proposition. Even though we have defined these notions before, we will recall them here and directly translate them into this more concrete setting.

Definition 6.89. We say that a type $p(x, b) = \mathrm{tp}(a/Cb)$ *long divides* over C if there is μ such that for every $\lambda \geq \mu$ there is a sequence $(b_i)_{i<\lambda}$ with $b_i \equiv_C b$ for all $i < \lambda$, such that for some $\kappa < \lambda$ and every $I \subseteq \lambda$ with $|I| \geq \kappa$ we have that $\bigcup_{i \in I} p(x, b_i)$ is inconsistent. We write $a \mathop{\smile\!\!\!\!\!\!|}\,_C^{\mathrm{ld}} b$ if $\mathrm{tp}(a/Cb)$ does not long divide over C.

We also recall the notion of witnesses of independence from Definition 4.27, and directly translate it into this more concrete setting.

Definition 6.90. Let $\mathop{\smile\!\!\!\!\!\!|}$ be some independence relation and let $(a_i)_{i<\kappa}$ be some sequence. Suppose furthermore that there is a continuous chain $(M_i)_{i<\kappa}$ of e.c. models, with $M \subseteq M_0$, such that $a_{<i} \subseteq M_i$ and $a_i \mathop{\smile\!\!\!\!\!\!|}\,_M M_i$ for all $i < \kappa$. Then we call $(M_i)_{i<\kappa}$ *witnesses of* $\mathop{\smile\!\!\!\!\!\!|}\,_M$*-independence*.

Lemma 6.91. *We have that* $a \mathop{\smile\!\!\!\!\!\!|}\,_C^{iLs} b$ *implies* $a \mathop{\smile\!\!\!\!\!\!|}\,_C^{\mathrm{ld}} b$.

Proof. Let $p(x, y) = \mathrm{tp}(ab/C)$ and let λ be any regular cardinal bigger than the number of Lascar strong types over C (compatible with b). Let $(b_i)_{i<\lambda}$ be any sequence in $\mathrm{tp}(b/C)$. By choice of λ there must be $I \subseteq \lambda$ such that $b_i \equiv_C^{\mathrm{Ls}} b_j$ for all $i, j \in I$ and $|I| = \lambda$. Pick some $i_0 \in I$ and let a' be such that $a' b_{i_0} \equiv_C ab$. By assumption $a \mathop{\smile\!\!\!\!\!\!|}\,_C^{iLs} b$, so $a' \mathop{\smile\!\!\!\!\!\!|}\,_C^{iLs} b_{i_0}$. Let $q \supseteq \mathrm{tp}(a'/Cb_{i_0})$ be a global C-Ls-invariant extension and let $\alpha \models q$. Then $\alpha b_i \equiv_C^{\mathrm{Ls}} \alpha b_{i_0}$ for all $i \in I$, so $\bigcup\{p(x, b_i) : i \in I\}$ is consistent. $\square$

Remark 6.92. Let $\mathop{\smile\!\!\!\!\!\!|}$ be an independence relation satisfying EXISTENCE and EXTENSION, let a be any tuple and let M be any model. Then as usual we can inductively build arbitrarily long sequences $(a_i)_{i<\kappa}$ together with witnesses $(M_i)_{i<\kappa}$ of $\mathop{\smile\!\!\!\!\!\!|}\,_M$-independence, such that $a \equiv_M a_i$ for all $i < \kappa$. In fact, this is exactly Proposition 4.28.

Proposition 6.93. *Let* $\mathop{\smile\!\!\!\!\!\!|}$ *be as in Theorem* 6.79 *then* $a \mathop{\smile\!\!\!\!\!\!|}\,_M^{\mathrm{ld}} b$ *implies* $a \mathop{\smile\!\!\!\!\!\!|}\,_M b$.

Proof. We want to apply Theorem 5.1. We have to be a little bit careful, because we only have CHAIN LOCAL CHARACTER instead of CLUB LOCAL CHARACTER and only when the left side of the independence relation, in this case a, is finite.

However, the proof of Theorem 5.1 only uses Chain Local Character and only applies it with an isomorphic copy of a on the left side. So we only need to show that we can reduce to the case where a is finite.

Indeed, this is not hard to do. This is because $\mathop{\smile\!\!\!\!\!\!\mid}\,^{\mathrm{ld}}$ has Left-Monotonicity, see Proposition 4.40. It is then enough to check only for finite a, by Finite Character and Symmetry of $\mathop{\smile\!\!\!\!\!\!\mid}$. □

We note that in the above proof (so actually, the proof of Theorem 5.1) it is relevant that we work with long dividing instead of dividing. This is because the application of Chain Local Character only really makes sense if the chain consists of e.c. models, as we only allow e.c. models in the base. At the same time we need those e.c. models to be witnesses of independence for the rest of the proof to work. If we would try to follow the same proof just for dividing then we would have to work with indiscernible sequences. Finding an indiscernible $\mathop{\smile\!\!\!\!\!\!\mid}$-independent sequence is not an issue. This can be done as usual: we first build a very long $\mathop{\smile\!\!\!\!\!\!\mid}$-independent sequence and then base an indiscernible sequence on it. This preserves being $\mathop{\smile\!\!\!\!\!\!\mid}$-independent due to Finite Character, but it does not carry over the chain of witnesses of independence. In long dividing this is not an issue, because we work directly with the very long sequence we constructed. So any 'decorations', such as the chain of witnesses of independence, are then at our disposal.

The following lemma and its proof are a weaker version of the chain lemma for $\mathop{\smile\!\!\!\!\!\!\mid}^K$-Morley sequences (Lemma 6.73) that works for long enough $\mathop{\smile\!\!\!\!\!\!\mid}^K$-independent sequences.

Lemma 6.94. *Let T be a thick NSOP$_1$ theory. Suppose that $a \mathop{\smile\!\!\!\!\!\!\mid}^K_M b$. Let $(b_i)_{i<\kappa}$ be an $\mathop{\smile\!\!\!\!\!\!\mid}^K_M$-independent sequence, where κ is a regular cardinal larger than the number of Lascar strong types over M (compatible with b) and where $b \equiv_M b_i$ for all $i < \kappa$. Then there is $I \subseteq \kappa$ with $|I| = \kappa$ such that $\bigcup_{i \in I} p(x, b_i)$ does not Kim-divide (and is thus consistent), where $p(x, y) = \mathrm{tp}(ab/M)$.*

Proof. By the choice of κ there is $I \subseteq \kappa$ with $|I| = \kappa$ such that $b_i \equiv^{\mathrm{Ls}}_M b_j$ for all $i, j \in I$. We conclude by the generalised independence theorem (Corollary 6.69(iii)). □

Proof of Theorem 6.79. We already proved that $\mathop{\smile\!\!\!\!\!\!\mid}^K$ has all the listed properties if T is NSOP$_1$. So now we assume that we have an abstract independence relation $\mathop{\smile\!\!\!\!\!\!\mid}$ satisfying the listed properties and we prove that $\mathop{\smile\!\!\!\!\!\!\mid} = \mathop{\smile\!\!\!\!\!\!\mid}^K$ and that T is NSOP$_1$.

The direction $a \mathop{\smile\!\!\!\!\!\!\mid}_M b \implies a \mathop{\smile\!\!\!\!\!\!\mid}^K_M b$. This proof is based on the proof of the same direction in [KR20, Theorem 9.1]. Let $p(x, b) = \mathrm{tp}(a/Mb)$ and let q be any global M-Ls-invariant extension of $\mathrm{tp}(b/M)$. Then a Morley sequence $(b_i)_{i<\omega}$ in q is a $\mathop{\smile\!\!\!\!\!\!\mid}_M$-Morley sequence by Lemma 6.91 and Proposition 6.93. By the standard INDEPENDENCE THEOREM argument we thus find that $\bigcup_{i<\omega} p(x, b_i)$ is consistent, hence $a \mathop{\smile\!\!\!\!\!\!\mid}^K_M b$.

The theory T is NSOP$_1$. We prove weak symmetry as in Theorem 6.59. So suppose that $a \mathop{\smile\!\!\!\!\!\!\mid}^{iLs}_M b$. Then combining Lemma 6.91 and Proposition 6.93 again we get $a \mathop{\smile\!\!\!\!\!\!\mid}_M b$. So by SYMMETRY we have $b \mathop{\smile\!\!\!\!\!\!\mid}_M a$ and then $b \mathop{\smile\!\!\!\!\!\!\mid}^K_M a$ follows from the above.

The direction $a \mathop{\smile\!\!\!\!\!\!\mid}^K_M b \implies a \mathop{\smile\!\!\!\!\!\!\mid}_M b$. This proof is based on the proof of the same direction in [CKR20, Theorem 5.1]. By Remark 6.92 we obtain a long enough sequence $(b_i)_{i<\kappa}$ with witnesses of $\mathop{\smile\!\!\!\!\!\!\mid}_M$-independence $(M_i)_{i<\kappa}$ and $b_i \equiv_M b$ for all $i < \kappa$. By the above the $(M_i)_{i<\kappa}$ are also witnesses of $\mathop{\smile\!\!\!\!\!\!\mid}^K_M$-independence. So by Lemma 6.94 there is $I \subseteq \kappa$ with order type κ such that $\bigcup_{i\in I} p(x, b_i)$ is consistent, where $p(x, b) = \mathrm{tp}(a/Mb)$. Let a' be a realisation of this set. By deleting an end-segment, MONOTONICITY and downward Löwenheim-Skolem we may assume that $(M_i)_{i\in I}$ is a continuous chain with $|M_i| \leq |T| + |M|$ for all $i \in I$ and I has order type $(|T| + |M|)^+$. By CHAIN LOCAL CHARACTER there is $i_0 \in I$ such that $a' \mathop{\smile\!\!\!\!\!\!\mid}_{M_{i_0}} M_I$, where $M_I = \bigcup_{i\in I} M_i$, and thus $a' \mathop{\smile\!\!\!\!\!\!\mid}_{M_{i_0}} b_{i_0}$. We also have $b_{i_0} \mathop{\smile\!\!\!\!\!\!\mid}_M M_{i_0}$ so by SYMMETRY and TRANSITIVITY we get $a' \mathop{\smile\!\!\!\!\!\!\mid}_M b_{i_0}$, hence $a \mathop{\smile\!\!\!\!\!\!\mid}_M b$. $\square$

6.10 Examples

In this section we present some examples of thick NSOP$_1$ theories. First, we recall Poizat's example of a thick non-semi-Hausdorff theory (which is bounded hence NSOP$_1$). Next, we look at (the JEP refinements of) the positive theory of existentially closed exponential fields, which was shown to be NSOP$_1$ in [HK21] by constructing a suitable independence relation. We deduce from the known results that this theory is Hausdorff (hence thick), and then we show that Kim-independence coincides in it with the independence relation studied in [HK21]. Finally, we show that NSOP$_1$ is preserved under taking hyperimaginary extensions; in particular, the hyperimaginary extension of an arbitrary NSOP$_1$ theory in full first-order logic is a Hausdorff NSOP$_1$ theory.

6.10.1 A thick, non-semi-Hausdorff theory

In Example 2.24 we discussed an example of a theory T that is thick, but not semi-Hausdorff. The e.c. models of T are bounded, so T is NSOP$_1$. In that example we also show that there is a type over some e.c. model M that does not extend to a global M-invariant type. This shows that, in the definition of Kim-independence in thick theories, it is necessary to work with Ls-invariant types rather than just invariant types.

6.10.2 Existentially closed exponential fields

In [HK21] the class of existentially closed exponential fields is studied using positive logic. They prove that this is NSOP_1 by providing a nice enough independence relation. We verify that this independence relation is indeed Kim-independence.

Definition 6.95. An *exponential field* or *E-field* is a field of characteristic zero with a group homomorphism E from the additive group to the multiplicative group. We call such a field an *EA-field* if it is also an algebraically closed field. We can axiomatise EA-fields by a positive theory and call this theory $T_{\text{EA-field}}$. The existentially closed exponential fields are then the e.c. models of $T_{\text{EA-field}}$.

Our definition is slightly different from [HK21] where they consider the class of e.c. models of just the theory of E-fields. However, these classes of e.c. models coincide, see [HK21, Proposition 3.3] and the discussion after it.

There are also many different JEP-refinements, see [HK21, Corollary 4.6]. To work in a monster model we need to fix one such JEP-refinement. This is not an issue, since everything we discuss here works in any JEP-refinement.

Definition 6.96 ([HK21, Definition 5.1])**.** For any set A write $\langle A \rangle^{\mathrm{EA}}$ for the smallest EA-subfield containing A. We define an independence notion $\mathop{\smile\!\!\!\!\!\!|}$ as follows

$$A \mathop{\smile\!\!\!\!\!\!|}_C B \iff \langle AC \rangle^{\mathrm{EA}} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}_{\langle C \rangle^{\mathrm{EA}}} \langle BC \rangle^{\mathrm{EA}},$$

where $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}$ is the usual independence relation in algebraically closed fields.

Note that the independence relation $\mathop{\smile\!\!\!\!\!\!|}$ actually makes sense over arbitrary sets. It would be interesting to compare this once Kim-independence over arbitrary sets has been developed in positive logic (see Question 6.105 below). For now we will restrict ourselves to working over e.c. models.

Fact 6.97. *We recall the following facts about* $T_{\text{EA-field}}$.

(i) The independence relation $\mathop{\smile\!\!\!\!\!\!|}$ *satisfies* Strong Finite Character, Existence, Monotonicity, Symmetry, Independence Theorem.

(ii) Any span $F_1 \leftarrow F \rightarrow F_2$ of embeddings of EA-fields can be amalgamated in such a way that, after embedding the result into the monster model, $F_1 \mathop{\smile\!\!\!\!\!\!|}_F F_2$.

(iii) For EA-fields F_1 and F_2, if $\mathrm{qftp}(F_1) = \mathrm{qftp}(F_2)$ *then* $\mathrm{tp}(F_1) = \mathrm{tp}(F_2)$.

Proof. (i) This is [HK21, Theorem 6.5]. They do not mention Lascar strong types in their formulation of Independence Theorem. However, as we will see in Proposition 6.98, the theory is Hausdorff, so the types over e.c. models are Lascar strong types.

(ii) This is [HK21, Theorem 4.3]. The fact that $F_1 \mathop{\smile\!\!\!\!\!\!|}_F F_2$ is not mentioned in [HK21, Theorem 4.3], but it is direct from their proof.

(iii) This follows directly from (ii). $\square$

To apply our theorem, Theorem 6.79, we need to verify a few more things.

Proposition 6.98. *The theory $T_{\text{EA-field}}$ is Hausdorff.*

Proof. By Fact 6.97(ii) the models of $T_{\text{EA-field}}$ are already amalgamation bases, so the models of $T^{\text{ec}}_{\text{EA-field}}$ are in particular also amalgamation bases. We can thus apply Fact 2.21 to conclude that $T_{\text{EA-field}}$ is Hausdorff. $\square$

Note that Hausdorff is the best we can get, because [HK21, Corollary 3.8] tells us that $T_{\text{EA-field}}$ cannot be Boolean. They prove this by showing that in every e.c. model F of $T_{\text{EA-field}}$ we have for all $a \in F$ that

$$a \in \mathbb{Z} \iff F \models \forall x(E(x) = 1 \to E(ax) = 1),$$

so if the theory were Boolean this would contradict compactness.

Proposition 6.99. *The independence relation* $\mathop{\smile\!\!\!\!\!\!|}$ *in* $T_{\text{EA-field}}$ *satisfies* Extension *and* Transitivity.

Proof. We first prove Transitivity. Let $A \mathop{\smile\!\!\!\!\!\!|}_B C$ and $A \mathop{\smile\!\!\!\!\!\!|}_C D$ with $B \subseteq C$. So we have $\langle AB\rangle^{\text{EA}} \mathop{\smile\!\!\!\!\!\!|}^{\text{ACF}}_{\langle B\rangle^{\text{EA}}} \langle BC\rangle^{\text{EA}}$, which is just $\langle AB\rangle^{\text{EA}} \mathop{\smile\!\!\!\!\!\!|}^{\text{ACF}}_{\langle B\rangle^{\text{EA}}} \langle C\rangle^{\text{EA}}$. We also have $\langle AC\rangle^{\text{EA}} \mathop{\smile\!\!\!\!\!\!|}^{\text{ACF}}_{\langle C\rangle^{\text{EA}}} \langle CD\rangle^{\text{EA}}$ and thus by monotonicity of ACF-independence $\langle AB\rangle^{\text{EA}} \mathop{\smile\!\!\!\!\!\!|}^{\text{ACF}}_{\langle C\rangle^{\text{EA}}} \langle CD\rangle^{\text{EA}}$. Then by transitivity of ACF-independence the result follows.

Now we prove Extension. Let $a \mathop{\smile\!\!\!\!\!\!|}_C b$ and let d be arbitrary, so from the definition we directly get $a \mathop{\smile\!\!\!\!\!\!|}_C Cb$. We apply Fact 6.97(ii) to $\langle Cab\rangle^{\text{EA}} \supseteq \langle Cb\rangle^{\text{EA}} \subseteq \langle Cbd\rangle^{\text{EA}}$, and we can embed the amalgamation in the monster in such a way that $\langle Cbd\rangle^{\text{EA}}$ remains the same. So we get some EA-field F with $\text{qftp}(F/\langle Cb\rangle^{\text{EA}}) = \text{qftp}(\langle Cab\rangle^{\text{EA}}/\langle Cb\rangle^{\text{EA}})$ and $F \mathop{\smile\!\!\!\!\!\!|}_{\langle Cb\rangle^{\text{EA}}} \langle Cbd\rangle^{\text{EA}}$, which simplifies to $F \mathop{\smile\!\!\!\!\!\!|}_{Cb} Cbd$. By Fact 6.97(iii) and restricting ourselves to the copy $a' \in F$ of a we thus have $\text{tp}(a'/Cb) = \text{tp}(a/Cb)$. So we get $a' \mathop{\smile\!\!\!\!\!\!|}_C Cb$ and $a' \mathop{\smile\!\!\!\!\!\!|}_{Cb} Cbd$, and $a' \mathop{\smile\!\!\!\!\!\!|}_C bd$ follows from Transitivity and Monotonicity. $\square$

Corollary 6.100. *The independence relation* $\mathop{\smile\!\!\!\!\!\!|}$ *in* $T_{\text{EA-field}}$ *is the same as Kim-independence over e.c. models.*

Proof. This is a direct application of Theorem 6.79, using Remark 6.80 to replace Local Character by Strong Finite Character. $\square$

In [HK21] it is shown that $T_{\text{EA-field}}$ cannot be simple. They do this by showing that it has a certain combinatorial property, called TP_2, which implies that the theory cannot be simple (see [HK21, Propositions 6.2 and A.5]). Here we present another way to conclude that the theory cannot be simple, using independence relations. Much like the reasoning after Theorem 5.7, we will aim to show that Base-Monotonicity fails for $\mathop{\smile\!\!\!\!\!\!\mid}$ in $T_{\text{EA-field}}$, see Example 6.101 below. Then if $T_{\text{EA-field}}$ were simple, we would have that dividing yields a simple independence relation by work of Ben-Yaacov [BY03b, BY03c]. By canonicity—here we can either use the AECat version, Theorem 5.7, or the positive logic version, Theorem 6.79—we would then have that this simple independence relation coincides with $\mathop{\smile\!\!\!\!\!\!\mid}$. That would mean that $\mathop{\smile\!\!\!\!\!\!\mid}$ satisfies Base-Monotonicity, a contradiction.

Example 6.101. This example is due to Jonathan Kirby. We construct a counterexample to Base-Monotonicity in $T_{\text{EA-field}}$.

Let C be any EA-field. Let F be the field $F = C(a, d, b_1, b_2)^{\text{alg}}$, where a, d, b_1, b_2 are algebraically independent over C. We consider various algebraically closed subfields of F, and will make them into EA-fields. Let $A = C(a)^{\text{alg}}$ and $D = C(d)^{\text{alg}}$, and choose any exponential maps on them extending that on C to make them EA-field extensions of C. Let $B = D(b_1, b_2)^{\text{alg}}$, and choose any exponential map making it an EA-field extension of D.

Let $t = ab_1 + b_2 \in F$. Then t is transcendental over $A \cup D$, and transcendental over B. Let $E = A(d, t)^{\text{alg}}$. The point $ad \in E$ is not in the $\mathbb{Q}$-linear span $A + B$. Indeed, as a $\mathbb{Q}$-vector subspace of F, E is of the form $A + D + V$ where $V \cap B = D$. We have $ad \in V$. So we can extend the exponential map from $A + B$ to an exponential map on E such that $\exp(ad) = t$. Then we choose any exponential map on F extending that on $E + B$. The EA-closure of $A \cup D$ in F is then E.

We have the following diagram of EA-fields, with transcendence degrees of each extension as given.

$$
\begin{tikzcd}
 & & F & \\
 & E \arrow[ur, "1"] & & B \arrow[ul, "1"'] \\
A \arrow[ur, "2"] & & D \arrow[ul, "2"'] \arrow[ur, "2"'] & \\
 & C \arrow[ul, "1"] \arrow[ur, "1"'] & &
\end{tikzcd}
$$

Now we have $C \subseteq D \subseteq B$ and by considering transcendence degrees, we see that $A \mathop{\smile\!\!\!\!\!\!\mid}_C^{\text{ACF}} B$ and thus $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ but $E \not\mathop{\smile\!\!\!\!\!\!\mid}_D^{\text{ACF}} B$, that is, $A \not\mathop{\smile\!\!\!\!\!\!\mid}_D B$ becase $E = \langle AD \rangle^{\text{EA}}$. So Base-Monotonicity fails.

6.10.3 Hyperimaginaries

We continue the study of hyperimaginaries from section 2.2. We stress once more that in what follows T is just any positive theory. However, the particular case where T is a theory in full first-order logic is worth discussing. From section 2.2 we already know that if we start with a theory T in full first-order logic and add hyperimaginaries to obtain $T^{\mathcal{E}}$ we do remain in the framework of positive logic and obtain a Hausdorff theory (Theorem 2.46). So in particular $T^{\mathcal{E}}$ is thick and our work in this chapter applies. We can say even more, namely that being NSOP$_1$ is preserved by adding hyperimaginaries, see Theorem 6.102 below. So if we start with a first-order NSOP$_1$ theory T then $T^{\mathcal{E}}$ is a Hausdorff NSOP$_1$ theory and all the work on Kim-dividing in this chapter applies.

Theorem 6.102. *The theory T is NSOP$_1$ if and only if $T^{\mathcal{E}}$ is NSOP$_1$.*

The technique in the proof of Theorem 6.102 can also be applied to other combinatorial properties, such as the order property, TP, TP$_2$, IP, etc. Of course, to do this, one first needs to write down a proper definition of these properties for positive logic, such as Definition 6.6 for SOP$_1$ or [HK21, Definition 6.1] for TP$_2$.

Proof. One direction is trivial: if T has a formula with SOP$_1$, then so has $T^{\mathcal{E}}$.

We prove the other direction: suppose that $T^{\mathcal{E}}$ has a formula with SOP$_1$, we will show that T already has a formula with SOP$_1$. So let $\varphi(x, y; w, z)$ be an $\mathcal{L}_{\mathcal{E}}$-formula with SOP$_1$. Here x and w are tuples of real variables, and y and z are tuples of hyperimaginary variables. Let $(a_\eta[b_\eta] : \eta \in 2^{<\omega})$ and $\psi(w_1, z_1; w_2, z_2)$ be witnesses of SOP$_1$. Let $\Sigma_\varphi(x, y_r; w, z_r)$ and $\Sigma_\psi(w_1, z_{1,r}; w_2, z_{2,r})$ be as in Lemma 2.49. Then

$$\Sigma_\psi(w_1, z_{1,r}; w_2, z_{2,r}) \cup \Sigma_\varphi(x, y_r, w_1; z_{1,r}) \cup \Sigma_\varphi(x, y_r, w_2; z_{2,r})$$

is inconsistent. Hence there are finite $\varphi' \in \Sigma_\varphi$ and $\psi' \in \Sigma_\psi$ that are inconsistent with each other. That is, the following is a consequence of T:

$$\neg\exists x y_r w_1 z_{1,r} w_2 z_{2,r} (\psi'(w_1, z_{1,r}, w_2, z_{2,r}) \wedge \varphi'(x, y_r, w_1, z_{1,r}) \wedge \varphi'(x, y_r, w_2, z_{2,r})). \tag{6.10.1}$$

As usual, any variables not actually appearing in the formulas should be ignored in the existential quantifier. We claim that φ' has SOP$_1$, which is witnessed by $(a_\eta b_\eta : \eta \in 2^{<\omega})$ and ψ'. We check the items in Definition 6.6.

(i) Let $\sigma \in 2^\omega$, then $\{\varphi(x, y, a_{\sigma|_n}, [b_{\sigma|_n}]) : n < \omega\}$ is consistent. So there are c and $[d]$ such that $\models \varphi(c, [d], a_{\sigma|_n}, [b_{\sigma|_n}])$ for all $n < \omega$. That is, we have $\Sigma_\varphi(c, d, a_{\sigma|_n}, b_{\sigma|_n})$ for all $n < \omega$. In particular $\{\varphi'(x, y_r, a_{\sigma|_n}, b_{\sigma|_n}) : n < \omega\}$ is consistent.

(ii) By construction, see (6.10.1).

(iii) Let $\eta, \nu \in \omega^{<\omega}$ such that $\eta^\frown 0 \preceq \nu$. Then $\models \psi(a_{\eta^\frown 1}, [b_{\eta^\frown 1}], a_\nu, [b_\nu])$, so $\models \Sigma_\psi(a_{\eta^\frown 1}, b_{\eta^\frown 1}, a_\nu, b_\nu)$ and in particular $\models \psi'(a_{\eta^\frown 1}, b_{\eta^\frown 1}, a_\nu, b_\nu)$.

□

To develop Kim-dividing over arbitrary sets one needs to assume the existence axiom for forking, see Definition 6.103 below. In full first-order logic this has been successfully done in for example [DKR19, CKR20]. For theories in full first-order logic this is a natural assumption, because it has been verified for many NSOP_1 theories, see [DKR19, Fact 2.14]. Below we show that this assumption is preserved when adding hyperimaginaries. We also note that it is closely related to the $\mathcal{B}$-existence axiom for AECats, see Example 4.50(iii).

Definition 6.103. We say that a theory satisfies the *existence axiom for forking* if $\text{tp}(a/B)$ does not fork over B for any a and B.

Theorem 6.104. *The theory T satisfies the existence axiom for forking if and only if $T^{\mathcal{E}}$ satisfies the existence axiom for forking.*

Proof. One direction is immediate: anything witnessing forking in T will also be in $T^{\mathcal{E}}$. We prove the other direction. So assume that there is $\text{tp}(a[b]/C[D])$ that forks over $C[D]$. That is, it implies a (possibly infinite) disjunction $\bigvee_{i \in I} \varphi_i(xy, e^i[f^i])$ with $\varphi_i(xy, e^i[f^i])$ dividing over $C[D]$ for each $i \in I$. For each $i \in I$ we let $(e^i_j[f^i_j])_{j \in J}$ be a long enough $C[D]$-indiscernible sequence with $e^i_0[f^i_0] = e^i[f^i]$ such that $\{\varphi_i(xy, e^i_j[f^i_j]) : j \in J\}$ is inconsistent. By Lemma 2.50 we may assume that $e^i_j f^i_j \equiv e^i f^i$ for every $j \in J$. We claim that $\Sigma_{\varphi_i}(x, y_r, e^i, f^i)$ (see Lemma 2.49) divides over CD for all $i \in I$. Note that Σ_{φ_i} possibly contains parameters from CD.

To prove the claim let k be such that $\{\varphi_i(xy, e^i_j[f^i_j]) : j \in J_0\}$ is inconsistent for all $J_0 \subseteq J$ with $|J_0| = k$. So $\bigcup_{j \in J_0} \Sigma_{\varphi_i}(x, y_r, e^i_j, f^i_j)$ is inconsistent for all such J_0. Let $(e_n f_n)_{n<\omega}$ be a CD-indiscernible sequence based on $(e^i_j f^i_j)_{j \in J}$ over CD. Then there are $j_1 < \ldots < j_k \in J$ such that $e_1 f_1 \ldots e_k f_k \equiv_{CD} e^i_{j_1} f^i_{j_1} \ldots e^i_{j_k} f^i_{j_k}$, so $\bigcup_{n<\omega} \Sigma_{\varphi_i}(x, y_r, e_n, f_n)$ is inconsistent. We conclude that $\Sigma_{\varphi_i}(x, y_r, e^i, f^i)$ divides over CD, as claimed.

By the claim there is $\psi_i(x, y_r, e^i, f^i)$ that is implied by $\Sigma_{\varphi_i}(x, y_r, e^i, f^i)$ such that $\psi_i(x, y_r, e^i, f^i)$ divides over CD, for all $i \in I$. Let $p = \text{tp}(a[b]C[D])$, then $\Sigma_p(x, y_r, C, D)$ implies $\bigvee_{i \in I} \Sigma_{\varphi_i}(x, y_r, e^i, f^i)$. We thus have that $\Sigma_p(x, y_r, C, D)$ implies $\bigvee_{i \in I} \psi_i(x, y_r, e^i, f^i)$. So $\Sigma_p(x, y_r, C, D)$ forks over CD. □

In the discussion following Definition 4.1 in [Kim21] it is stated that one may produce results for Kim-independence for the hyperimaginary extension M^{heq} of a first-order structure M parallel with those for first-order structures, provided that M^{heq} satisfies the existence axiom for forking (which, by the above theorem, is equivalent to the assumption that T satisfies this axiom). More generally, one can ask if our results on Kim-independence over models in thick NSOP$_1$ theories can be extended to arbitrary base sets assuming the existence axiom for forking:

Question 6.105. Suppose T is a thick positive NSOP$_1$ theory satisfying the existence axiom for forking. Can $\mathop{\smile\!\!\!\!\!\!|}^{K}$ be extended to an automorphism-invariant ternary relation between arbitrary small sets which satisfies the properties listed in Theorem 6.79?

7

Final remarks

We make some final remarks and summarise the questions that have been left unanswered. In section 7.1 we discuss the connection between long dividing, isi-dividing and dividing, and the notions that are based on that. In section 7.2 we ask if we can link the existence of certain nice independence relations in AECats to combinatorial properties. Finally, section 7.3 is about developing Kim-independence in positive logic over arbitrary (small) sets.

7.1 Long dividing, isi-dividing and dividing

In section 4.4 we introduced the notions of long dividing, isi-dividing, isi-forking and long Kim-dividing. These are based on the classical notions of dividing, dividing (again), forking and Kim-dividing respectively. These classical notions are defined using indiscernible sequences, and indiscernible sequences make sense in the context of a finitely short AECats (see section 4.5). So we will state the questions in that context, but they would already be interesting in the more concrete settings of positive logic or even full first-order logic.

Due to Proposition 4.34 we have that dividing implies both long dividing and isi-dividing. The main question is then the converse.

Question 4.37**, repeated.** *Do long dividing and isi-dividing imply dividing in finitely short AECats?*

If the answer to the above question is "yes" then dividing, long dividing and isi-dividing all coincide (in finitely short AECats). This would then directly imply that forking and isi-forking coincide (see Remark 4.42). Long Kim-dividing is defined with respect to non-isi-forking sequences, while Kim-dividing is defined with respect to indiscernible non-forking sequences (i.e. Morley sequences). So if isi-forking and forking coincide we only need to take care of the indiscernibility to prove that Kim-dividing and long Kim-dividing coincide. This is exactly the same difficulty as we have in showing that dividing and long dividing (or isi-dividing) coincide. So it is highly likely, although not automatic, that the same technique applies and that we also get that Kim-dividing and long Kim-dividing coincide.

We note once more that in the presence of a proper class of Ramsey cardinals all these notions coincide. That is, Proposition 4.35 tells us that in that case dividing, long dividing and isi-dividing all coincide. So forking and isi-forking also coincide. Then following Remark 4.51 we also get that Kim-dividing and long Kim-dividing coincide.

We also note that in specific AECats we can use the canonicity theorems to prove that various notions of forking and dividing coincide, without assuming the existence of large cardinals. For example, by the results in chapter 6 we know that in a thick NSOP_1 theory Kim-dividing gives an NSOP_1-like independence relation, which must thus coincide with long Kim-dividing by canonicity, Theorem 5.6.

Outside the finitely short setting there seems to be very little hope. One problem is that generally indiscernible sequences can no longer be stretched to arbitrary lengths.

Question 7.1. Is there an AECat, possibly one that is not finitely short, where dividing does not coincide with long dividing and/or isi-dividing.

7.2 AECats and combinatorial properties

The main results for AECats in this thesis are about the canonicity of certain independence relations. That is, they state there can be only one nice enough independence relation and they tell us what it must be (e.g. it must be given by isi-dividing). In a full Kim-Pillay style theorem the existence of such an independence relation is linked to a combinatorial property, which often asserts the absence of a certain configuration. For example, stability is the same as not having the order property, which roughly states that there can be no infinite linear order. Simplicity means not having the tree property. For NSOP_1 we actually gave a precise definition in this thesis: Definition 6.6. These combinatorial properties are more natural to formulate when there are underlying sets, and formulating them in the generality of AECats is non-trivial. So we have not defined any of these combinatorial properties for AECats, let alone studied possible links with independence relations. This brings us to the following very broad question.

Question 7.2. Can we link stable, simple or NSOP_1-like independence relations to combinatorial properties in AECats? This can go in two directions:

1. assuming a combinatorial property, can we find a nice independence relation?
2. assuming that we have a nice independence relation, can we prove that the AECat satisfies a good combinatorial property?

This question may sound a bit vague, so we give an example of what an answer may look like. One could try to define a notion of being NSOP_1 for AECats, for example one that states that certain tree configurations do not exist in the AECat. Then an answer to the first part of the question could be of the form "in an NSOP_1 AECat long Kim-dividing induces an NSOP_1-like independence relation". An answer to the second part could be "an AECat with an NSOP_1-like independence relation is NSOP_1".

There are already (partial) answers to Question 7.2. These are in different, but very similar, frameworks. In [LRV19] the existence of a stable independence relation in an accessible category is linked (in both directions) to a certain order property. In [GMA21] simple-like independence relations are studied in AECs, and they prove that having a simple-like independence relation implies the failure a certain tree property.

7.3 Kim-independence over arbitrary sets in positive logic

In chapter 6 we only developed Kim-independence over e.c. models. That is, the base in the relation $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K$ always has to be an e.c. model. This is because, assuming thickness, Morley sequences always exist in types over e.c. models. These Morley sequences are then necessary to even define Kim-dividing (see also the discussion at the start of section 6.4). In full first-order logic Kim-independence has also been developed over arbitrary (small) sets (see e.g. [DKR19]), by simply assuming that these Morley sequences exist in any type. This assumption is called the existence axiom (for forking), see Definition 6.103. In full first-order logic this is a reasonable assumption for NSOP_1 theories, because this axiom has been verified for all known NSOP_1 theories. Theorem 6.104 tells us that a theory T has the existence axiom for forking if and only if its hyperimaginary extension T^{heq} has it. So this already gives us many positive theories with the existence axiom for forking. The following question is then natural.

Question 6.105**, repeated.** *Suppose T is a thick positive $NSOP_1$ theory satisfying the existence axiom for forking. Can $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K$ be extended to an automorphism-invariant ternary relation between arbitrary small sets which satisfies the properties listed in Theorem* 6.79*?*

So far we do not have an example of a thick non-simple positive theory that satisfies the existence axiom for forking, but is not a hyperimaginary extension of some theory in full first-order logic. This is simply because we have not verified it yet. The theory of existentially closed exponential fields—studied in [HK21], see subsection 6.10.2—seems like a good candidate.

Question 7.3. Does the theory of existentially closed exponential fields (i.e. $T_{\text{EA-field}}$ from Definition 6.95) satisfy the existence axiom for forking?

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