Generalisations And Specifications In The Categorification Of Representation Theory

A thesis submitted to the School of Mathematics at the University of East Anglia in partial fulfilment of the requirements for the degree of Doctor of Philosophy

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11/2020

Abstract

We extend the 2-representation theory of finitary 2-categories in two separate fashions. For the first, we examine certain 2-categories with infinitely many objects, called locally finitary 2-categories, and for the second we examine certain 2-categories with infinitely many isomorphism classes of indecomposable 1-morphisms, called (locally) wide finitary 2-categories. In both cases, we extend various classification results relating to transitive and simple transitive 2-representations to the new setting, and provide examples where this new theory applies. Most prominently, we generalise the classification of simple transitive cell 2-representations of fiat 2-categories by cell 2-representations to the locally finitary setting (and further extend it to the weakly fiat case), and we generalise to both settings the classification of all transitive 2-representations of weakly fiat 2-categories as equivalent to 2-representations associated to coalgebra 1-morphisms.

Acknowledgements

Dedicated to Duncan Macpherson. You were an amazing father, always wise and always supportive. I miss you.

My deep thanks go to my supervisor for the thesis, Vanessa Miemietz, for her ever-constant support and encouragement, and her analytical dissection of my mistakes. I would also like to thank my second supervisor, Robert Gray, for being willing to assist if I ever needed help.

I would like to thank Rebecca, Cristina, Virryna and everyone from the office for their friendship and random assistance over the years, as well as for valued distractions when I needed them. Specific thanks go to Tom and Dane, who also constructed the LaTeX template used to compile this thesis.

Finally, I would like to thank my mum, who has been a constant source of love and support, always providing advice when needed (and much appreciated lifts in her car as well).

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Introduction

The study of the 2-representation theory of finitary and fiat 2-categories, pioneered by Mazorchuk and Miemietz in [MM11] through [MM16b] and further explored by those authors and others in [MMMT16], [CM19a] etc., is a powerful new tool in representation theory. There are important applications of this theory to certain quotients of 2-Kac-Moody algebras (see [MM16c]) and to Soergel bimodules (see for example [MMM+19]).

However, while powerful, the setup used to date in this theory has multiple restrictions, primarily relating to finiteness conditions. Specifically, the theory considers 2-categories which have only finitely many objects and whose hom-categories have finitely many isomorphism classes of indecomposable 1-morphisms and finite-dimensional spaces of 2-morphisms. The relaxation of these restrictions would enable the study of a much wider class of examples using techniques analogous to those for 2-representations of finitary 2-categories.

This thesis examines some relaxations of these finiteness conditions. The first approach is to allow countably many objects in the 2-categories, which we call 'locally finitary' 2-categories. While this may seem a comparatively mild generalisation, it already enables the study of multiple interesting examples that were previously inaccessible, including a much wider class of quotients of 2-Kac-Moody algebras. The second approach combines this with allowing countably many isomorphism classes of indecomposable 1-morphisms, which we call '(locally) wide finitary' 2-categories. This is a powerful generalisation, though this thesis is only an initial step along this route.

In the first part of this thesis, we give the generalisation of multiple finitary results to the locally finitary case. Of specific note, in Theorem 3.3.5 we construct for any transitive 2-representation of a locally weakly fiat 2-category an equivalent 'internal' 2-representation of comodule 1-morphism categories, analagously to a major result in [MMMT16]. In Theorem 3.4.32, a generalisation of the primary result in [MM16c], we further classify all simple transitive 2-representations of strongly regular locally weakly fiat 2-categories as being equivalent to cell 2-representations. We then utilise the latter result to classify all simple transitive 2-representations of cyclotomic 2-Kac-Moody algebras in Corollary 3.5.41.

We follow this with a specialisation of the locally finitary setup to the case where the 2-categories have an additional graded structure. In this setup, we show in Theorem 4.3.9 and Corollary 4.3.10 that the previously constructed internal 2-representations associated to a transitive 2-representation can be viewed as a 'degree zero' construction in a canonical fashion. We use this result by considering again cyclotomic 2-Kac-Moody algebras, demonstrating in Theorem 4.4.5 that any simple transitive 2-representation is in fact a graded 2-representation.

Proceeding this, we move to considering locally wide finitary setup. After defining a more general environment to work in, we again prove in Theorem 5.6.14 the existence of the internal 2-representation of comodule 1-morphism categories equivalent to transitive 2-representations. We also provide two classes of examples of this theory. First, we examine locally wide finitary 2-categories associated to infinite dimensional bound path algebras, where we show the coalgebra 1-morphisms underlying the comodules 1-morphism categories are particularly pleasant. Second, we demonstrate that the theory of locally wide finitary 2-categories applies to 2-categories of singular Soergel bimodules for any Coxeter system with finitely many simple reflections.

The structure of the thesis is as follows. After this introduction, Chapter 2 provides an overview of various concepts from category theory and algebra that we will be using, most prominently an overview of the 2-representation theory of finitary categories of [MM11] through [MM16b] (starting in Section 2.3), as well as the theory of coalgebra and comodule 1-morphisms from [MMMT16] (starting in Section 2.5).

In Chapter 3, we move on to considering locally finitary 2-categories. We begin with definitions of the basic concepts, before moving on to the generalisation of the initial sections of [MMMT16]. The first notable results are Theorem 3.3.5 as well as Theorem 3.3.9, which classifies the simple transitive 2-representations for locally finitary 2-categories associated to certain infinite dimensional algebras. Along the way, we give some minor results demonstrating that the cell structure of the 2-category remains pleasant in this generalisation.

The second half of Chapter 3 focusses on generalising results from the Mazorchuk-Miemietz series of papers, particularly [MM11] and [MM16c]. The eventual goal of this section is Theorem 3.4.32. Section 3.5 reviews the theory of 2-Kac-Moody algebras before presenting an application of Theorem 3.4.32 by demonstrating that cyclotomic 2-Kac-Moody algebras of given weights are locally weakly fiat 2-categories, and thus submit to the aforementioned theorem.

This chapter is followed by the closely related Chapter 4, where we examine the specialisation to locally G-finitary 2-categories for some countable abelian group G. We construct a degree zero 2-category associated to such a 2-category, and use it to construct a degree zero coalgebra 1-morphism for a given graded transitive 2-representation of the original 2-category. This setup allows us to prove Theorem 4.3.9 and Corollary 4.3.10. Finally, we apply this to the cyclotomic 2-Kac-Moody categories of given weights, showing that their cell 2-representations are all graded transitive 2-representations, leading to Theorem 4.4.5.

Chapter 5 moves on to the locally wide finitary portion of the thesis. It begins by explaining the categorical construction of pro-categories (from [GV72]) and Adelman abelianisation (from [Ade73]), before utilising them to derive larger 2-categories in which internal comodule 1-morphism categories live. This allows us to prove Theorem 5.6.14.

We then present two applications of this theory. The first, found in Section 5.7, considers (locally) wide finitary 2-categories associated to bound path algebras. In this situation, we prove in Corollary 5.7.13 that the coalgebra 1-morphisms underlying the internal 2-representations live not just in the pro-category of the Adelman abelianisation of the wide finitary 2-category, as the base theory proves, but legitimately within the wide finitary 2-category itself. For the second application, found in Section 5.8, we demonstrate that the 2-category of (singular) Soergel bimodules associated to an arbitrary Coxeter system with finitely many simple reflections is a locally wide finitary 2-category. This allows the aforementioned theory to be applied to it, and is a much wider class of examples than has previously been studied using finitary 2-representation theory. Specifically, papers such as [MMM+19] only consider finite Coxeter systems.

2

The Basics

2.1 General Category Theoretic Definitions

We begin by recalling some definitions in category theory that we will find useful. We draw the following from [ASS06] unless otherwise stated, though that book calls $\Bbbk$-linear categories '$\Bbbk$-categories'.

Definition 2.1.1. A category $\mathscr{C}$ is *additive* if:

- For objects $\mathtt{i}, \mathtt{j} \in \mathscr{C}$, the hom-set $\mathrm{Hom}_{\mathscr{C}}(\mathtt{i}, \mathtt{j})$ is an abelian group $(\mathrm{Hom}_{\mathscr{C}}(\mathtt{i}, \mathtt{j}), +)$ such that composition of morphisms is bilinear; that is, $f \circ (g + h) = f \circ g + f \circ h$ and $(f + g) \circ h = f \circ h + g \circ h$.
- $\mathscr{C}$ has all finite biproducts - that is, given any finite set B of objects of $\mathscr{C}$, the direct sum and direct product of B exist and are equal, such that the composition of the injection and projection morphisms is identity on the elements of B. We denote the biproduct of C and D as $C \oplus D$.
- There is a zero object $0 \in \mathscr{C}$ such that id_0 is the zero element of $\mathrm{Hom}_{\mathscr{C}}(0, 0)$.

Definition 2.1.2. A functor $F : \mathscr{C} \to \mathscr{C}'$ between additive categories is called *additive* if it preserves biproducts and the abelian group structure of the hom-sets. Explicitly, for objects C, D of $\mathscr{C}$ and morphisms f, g of $\mathscr{C}$, $F(C \oplus D) = F(C) \oplus F(D)$ and $F(f + g) = F(f) + F(g)$.

Definition 2.1.3. Given a field $\Bbbk$, an additive category $\mathscr{C}$ is a $\Bbbk$-*linear category* if each $\mathrm{Hom}_{\mathscr{C}}(\mathtt{i}, \mathtt{j})$ has the structure of a $\Bbbk$-vector space such that composition is $\Bbbk$-

bilinear. Explicitly, for $k_1, k_2, k_3 \in \Bbbk$ and morphisms $f, g, h \in \mathscr{C}$, $k_1 f \circ (k_2 g + k_3 h) = k_1 k_2 f \circ g + k_1 k_3 f \circ h$, and $(k_1 f + k_2 g) \circ k_3 h = k_1 k_3 f \circ h + k_2 k_3 g \circ h$.

If $\mathscr{C}$ is a $\Bbbk$-linear category and $\mathtt{i} \in \mathscr{C}$, then $\mathrm{Hom}_{\mathscr{C}}(\mathtt{i}, \mathtt{i})$ is a $\Bbbk$-algebra with composition as multiplication.

Definition 2.1.4. Given a field $\Bbbk$, an additive functor $F : \mathscr{C} \to \mathscr{C}'$ of $\Bbbk$-linear categories is $\Bbbk$-*linear* if it also preserves the $\Bbbk$-linear structure of the hom-sets. Explicitly, for $k \in \Bbbk$ and f a morphism in $\mathscr{C}$, $F(kf) = kF(f)$.

Definition 2.1.5. An object A in an additive category $\mathscr{C}$ is *indecomposable* if whenever $A \cong B \oplus C$ as a biproduct, either $B \cong 0$ or $C \cong 0$.

Definition 2.1.6. An additive category $\mathscr{C}$ is *Krull-Schmidt* if every object is a direct sum of finitely many indecomposable objects and if each indecomposable object has a local endomorphism ring.

Definition 2.1.7 (via [Swa06] II.1)**.** Given an additive category $\mathscr{C}$, the *split Grothendieck group* $[\mathscr{C}]$ of $\mathscr{C}$ is the abelian group generated by the isomorphism classes $[A]$ of $\mathscr{C}$ modulo the relation $[A \oplus B] = [A] + [B]$.

Definition 2.1.8 (via [ASS06])**.** Let $\mathscr{C}$ be an additive $\Bbbk$-linear category. A class $\mathscr{I}$ of morphisms in $\mathscr{C}$ is a *two-sided ideal* of $\mathscr{C}$ if:

- $\mathscr{I}$ contains the zero morphism 0_X for all objects $X \in \mathscr{C}$.
- If $f, g : \mathtt{i} \to \mathtt{j} \in \mathscr{I}$ and $\lambda, \mu \in \Bbbk$, then $\lambda f + \mu g \in \mathscr{I}$.
- If $f \in \mathscr{I}$ and g and h are morphisms of $\mathscr{C}$, then $g \circ f \circ h \in \mathscr{I}$ whenever this is defined.

We set $\mathrm{Hom}_{\mathscr{I}}(\mathtt{i}, \mathtt{j}) = \{f \in \mathrm{Hom}_{\mathscr{C}}(\mathtt{i}, \mathtt{j}) | f \in \mathscr{I}\}$. Each $\mathrm{Hom}_{\mathscr{I}}(\mathtt{i}, \mathtt{j})$ is a $\Bbbk$-subspace of $\mathrm{Hom}_{\mathscr{C}}(\mathtt{i}, \mathtt{j})$. We thus define the *quotient category* $\mathscr{C}/\mathscr{I}$ whose objects are the same as $\mathscr{C}$, and whose hom-sets are defined as $\mathrm{Hom}_{\mathscr{C}/\mathscr{I}}(\mathtt{i}, \mathtt{j}) = \mathrm{Hom}_{\mathscr{C}}(\mathtt{i}, \mathtt{j}) / \mathrm{Hom}_{\mathscr{I}}(\mathtt{i}, \mathtt{j})$. Composition is given by $[g] \circ [f] = [gf]$.

Definition 2.1.9. Let S be a set of objects in an additive category $\mathscr{C}$. The *additive closure* $\operatorname{add} S$ is the smallest full subcategory of $\mathscr{C}$ containing S that is closed under direct sums and direct summands.

Definition 2.1.10. Given a $\Bbbk$-algebra A, we let $A\text{-Mod}$ denote the category of (left) A-modules with module homomorphisms, and $A\text{-mod}$ the category of finite dimensional (left) A-modules with module homomorphisms. Similarly, given $\Bbbk$-algebras A and B, we let $(A\text{-}B)\text{-biMod}$ denote the category of $(A\text{-}B)$-bimodules with bimodule homomorphisms, and $(A\text{-}B)\text{-bimod}$ the category of finite dimensional $(A\text{-}B)$-bimodules with bimodule homomorphisms.

Definition 2.1.11. If $(\mathscr{K}, \otimes, I)$ is a monoidal category with tensor product $\otimes$ and tensor unit I, a category $\mathscr{C}$ *enriched over* $\mathscr{K}$ has:

- A set of objects $\mathrm{Ob}(\mathscr{C})$.
- For each pair of objects C and D, a *hom-object* $\mathscr{C}(C, D) \in \mathscr{K}$. For our purposes, we will generally be using monoidal categories whose objects are sets with extra structure, so that we can refer to these hom-objects as e.g. hom-(vector) spaces.
- Families of $\mathscr{K}$-morphisms $\circ_{C,D,E} : \mathscr{C}(D, E) \otimes \mathscr{C}(C, D) \to \mathscr{C}(C, E)$ and $\mathrm{id}_C : I \to \mathscr{C}(C, C)$ that follow the standard axioms for composition and identity in a category.

2.2 General Algebraic Definitions

In this section, we will be reviewing various standard definitions from the general theory of algebras that will be used at various points in later chapters, as well as some useful results. These definitions are primarily taken from [ASS06].

Definition 2.2.1. An algebra A over a field $\Bbbk$ is *self-injective* if it is injective as a module over itself.

Definition 2.2.2. A self-injective algebra A over a field $\Bbbk$ is *weakly-symmetric* if, for any projective A-module P, $\operatorname{top} P \cong \operatorname{soc} P$.

Definition 2.2.3. Two idempotents e and f in an algebra A are *orthogonal* when $ef = fe = 0$. An idempotent e is *primitive* if it cannot be written $e = e_1 + e_2$ where e_1 and e_2 are non-zero orthogonal idempotents. A set $\{e_i | i \in I\}$ of idempotents is *complete* if $\sum_{i \in I} e_i = 1$.

Definition 2.2.4. Let A be an algebra over a field $\Bbbk$ with a complete set of primitive orthogonal idempotents $\{e_i | i \in I\}$. We say that A is *basic* if $i \neq j$ implies $Ae_i \ncong Ae_j$.

If M is an A-module and S is a simple A-module, we let $[M : S]$ denote the multiplicity of S in the composition series of M.

Definition 2.2.5. Given a $\Bbbk$-algebra A and an A-module M, a pair of sets

$$\{x_i \in M | i \in I\} \text{ and } \{f_i \in \operatorname{Hom}(M, A) | i \in I\}$$

for some set I is a *dual basis* for M if for all $m \in M$:

- $f_i(m) = 0$ for all but finitely many i;
- $m = \sum_{i \in I} f_i(m) x_i$.

We now state a useful result about projective modules, often called the *Dual Basis* theorem:

Proposition 2.2.6 ([Lam99], Lemma 2.9)**.** *An A-module P has a dual basis if and only if it is projective.*

This allows us to derive the following two results.

Lemma 2.2.7. *If P is a finitely generated projective A-module, then every dual basis of P is finite.*

Proof. If $p \in P$ then $p = \sum_{i=1}^{n} a_i p_i$ for some generating set $\{p_1 \ldots, p_n\}$. Take a dual basis $\{x_j | j \in J\}$, $\{f_j | j \in J\}$. Then for $p \in P$,

$$p = \sum_{j \in J} f_j(p)x_j = \sum_{j \in J} (f_j(\sum_{i=1}^{n} a_i p_i)x_j) = \sum_{j \in J} (\sum_{i=1}^{n} a_i f_j(p_i))x_j.$$

For each p_i there are only finitely many j such that $f_j(p_i) \neq 0$. It follows that the set of these over all the p_i is also finite, and thus there is a finite set $F = \{f_{i_1}, \ldots, f_{i_n}\}$ such that if $f_j \notin F$, then $f_j(p) = 0$ for all $p \in P$. The result follows. □

Lemma 2.2.8. *If A is a $\Bbbk$-algebra, P is a finitely generated projective A-module and M is some other A-module, then $\mathrm{Hom}_A(P, M) \cong \mathrm{Hom}_A(P, A) \otimes_A M$.*

Proof. By Lemma 2.2.7 P has a finite dual basis $\{x_1, \ldots, x_n\}$, $\{f_1, \ldots, f_n\}$. Thus if $\varphi \in \mathrm{Hom}_A(P, M)$, for any $p \in P$ we have

$$\varphi(p) = \varphi(\sum_{i=1}^{n} f_i(p)x_i) = \sum_{i=1}^{n} f_i(p)\varphi(x_i)$$

due to φ being a module homomorphism. We define maps

$$\alpha : \mathrm{Hom}_A(P, M) \to \mathrm{Hom}_A(P, A) \otimes_A M$$

and

$$\beta : \mathrm{Hom}_A(P, A) \otimes_A M \to \mathrm{Hom}_A(P, M)$$

by

$$\alpha(\varphi) = \alpha(\sum_{i=1}^{n} f_i \varphi(x_i)) = \sum_{i=1}^{n} f_i \otimes \varphi(x_i)$$

and

$$\beta(\sum_{j=1}^{m} g_j \otimes n_j) = \sum_{j=1}^{m} g_j n_j.$$

We wish to show that α and β are mutually inverse module homomorphisms.

It is easy to see that $\beta\alpha = \mathrm{id}_{\mathrm{Hom}_A(P,M)}$. Let

$$\sum_{j=1}^{m} g_j \otimes n_j \in \mathrm{Hom}_A(P, A) \otimes_A M$$

and let $p = \sum_{i=1}^{n} f_i(p)x_i \in P$. Then

$$\begin{aligned}
\beta(\sum_{j=1}^{m} g_j \otimes n_j)(p) &= \sum_{j=1}^{m} g_j(p)n_j \\
&= \sum_{j=1}^{m} g_j(\sum_{i=1}^{n} f_i(p)x_i)n_j \\
&= \sum_{j=1}^{m} \sum_{i=1}^{n} f_i(p)g_j(x_i)n_j \\
&= \sum_{i=1}^{n} f_i(p)(\sum_{j=1}^{m} g_j n_j)(x_i).
\end{aligned}$$

If follows that

$$\begin{aligned}
\alpha\beta(\sum_{j=1}^{m} g_j \otimes n_j)(p) &= \alpha(\sum_{i=1}^{n} f_i(p)(\sum_{j=1}^{m} g_j n_j)(x_i)) \\
&= \sum_{i=1}^{n} f_i \otimes (\sum_{j=1}^{m} g_j(x_i)n_j) \\
&= \sum_{j=1}^{m}(\sum_{i=1}^{n} f_i g_j(x_j)) \otimes n_j.
\end{aligned}$$

But for $p \in P$, $\sum_{i=1}^{n} f_i(p)g_j(x_i) = g_j(p)$ by the definition of the dual basis, and thus $\alpha\beta(\sum_{j=1}^{m} g_j \otimes n_j) = \sum_{j=1}^{m} g_j \otimes n_j$, hence $\alpha\beta = \mathrm{id}_{\mathrm{Hom}_A(P,A)\otimes_A M}$, and the result follows. □

A rich vein of algebras we will be tapping for examples in this thesis are what are called *bound path algebras of a quiver*.

Definition 2.2.9. A *quiver* is an ordered quadruple $\Gamma = (\Gamma_0, \Gamma_1, s, t)$, where:

- Γ_0 is a set, whose elements we call *vertices*;

- Γ_1 is a set whose elements we call *arrows*;
- $s, t : \Gamma_1 \to \Gamma_0$ are functions that pick out the *source* and *target* vertices (or *domain* and *codomain* vertices respectively) of each arrow.

In essence, a quiver is a directed multigraph with loops.

Definition 2.2.10. Given vertices a and b of a quiver Γ, we say that a *path* p of length l from a to b is a set of l arrows $\{p_1, \ldots, p_l\}$ such that $s(p_{i+1}) = t(p_i)$ for $1 \leq i < l$, $s(p_1) = a$ and $t(p_l) = b$. That is, a path is a sequence of consecutive arrows that begins at a and ends at b. We define $s(p) = s(p_1)$ and $t(p) = t(p_l)$. We also associate to each $a \in \Gamma_0$ a path of length 0 which we denote e_a.

If we define composition of paths p and q as their concatenation $q * p$ whenever $t(p) = s(q)$, and let R be the set of paths of Γ, then (Γ_0, R) forms a category in the obvious fashion.

Definition 2.2.11. Given a quiver Γ and a field $\Bbbk$, the *path algebra* $\Bbbk\Gamma$ of Γ over $\Bbbk$ is the algebra formed as the free vector space over the set of paths of Γ with multiplication being defined on paths as $q \circ p = \begin{cases} q * p & \text{if } t(p) = s(q) \\ 0 & \text{otherwise} \end{cases}$ and extending linearly.

Definition 2.2.12. Given a quiver Γ and a path algebra $\Bbbk\Gamma$, let $\Bbbk\Gamma_i$ denote the ideal generated by all paths of Γ of length at least i. A *bound path algebra* B of $\Bbbk\Gamma$ is a quotient $B = \Bbbk\Gamma/I$, where I is an ideal of $\Bbbk\Gamma$ such that there exists some k with $\Bbbk\Gamma_k \subseteq I \subseteq \Bbbk\Gamma_2$.

Definition 2.2.13. A *multisemigroup* S consists of a set S and an associative binary operation $* : \mathscr{P}(S) \times \mathscr{P}(S) \to \mathscr{P}(S)$. The binary operation is generally written with infix notation, and for $a, b \in S$ we commonly write $a * b$ for $\{a\} * \{b\}$. We require the the binary operation to satisfy $(\bigcup_{i \in I} P_i) * Q = \bigcup_{i \in I}(P_i * Q)$ and $Q * (\bigcup_{i \in I} P_i) = \bigcup_{i \in I}(Q * P_i)$ for any indexing set I and any subsets $P_i, Q \subseteq S$. Informally, a multisemigroup resembles a semigroup, but the composition of two elements is a subset of the multisemigroup rather than a single element. This operation is then extended to subsets via the obvious unions.

2.3 Finitary 2-Categories and their 2-Representations

The majority of this thesis will be concerned with generalisations and specifications of the representation theory of certain types of 2-category, named finitary and (weakly) fiat 2-categories, which were defined and initially studied by Mazorchuk and Miemietz in their series of papers [MM14] through [MM16b], and then by those authors and others in later papers such as [MMMT16] and [CM19b]. Although we will rarely be considering these basic constructions directly, they will be useful tools for proving many results later in the thesis, and we will thus be recalling the definitions and various results below for reference.

2.3.1 Bicategories and 2-Categories

We start by giving the definition of a 2-category, as well as those of 2-functors and 2-natural transformations. These definitions are drawn from those in [Lei98].

Definition 2.3.1. A *bicategory* $\mathscr{C}$ is defined with the following data:

- A collection of *objects*, which we denote by $\mathtt{i}$, $\mathtt{j}$, $\ldots$.
- For each pair of items $\mathtt{i}, \mathtt{j} \in \mathscr{C}$, a category $\mathscr{C}(\mathtt{i}, \mathtt{j})$ of morphisms between them. The objects of $\mathscr{C}$ are called *1-morphisms*, which we generally denote $X, Y, \ldots$ or $F, G, \ldots$, and the arrows are called *2-morphisms*, which we generally denote $\alpha, \beta, \ldots$.
- Functors $c_{\mathtt{ijk}} : \mathscr{C}(\mathtt{j}, \mathtt{k}) \times \mathscr{C}(\mathtt{i}, \mathtt{j}) \to \mathscr{C}(\mathtt{i}, \mathtt{k})$ where $c_{\mathtt{ijk}}((Y, X)) =: Y \circ X$ and $c_{\mathtt{ijk}}((\beta, \alpha)) =: \beta \circ_H \alpha$. We also have functors $\mathbb{1}_{\mathtt{i}} : \mathbf{1} \to \mathscr{C}(\mathtt{i}, \mathtt{i})$; that is, a 1-morphism $\mathbb{1}_{\mathtt{i}} : \mathtt{i} \to \mathtt{i}$ for every $\mathtt{i}$. We often write YX for $Y \circ X$.

- Natural isomorphisms defined by the commutative diagrams:

$$\begin{array}{ccc}
\mathscr{C}(\mathtt{k},\mathtt{l}) \times \mathscr{C}(\mathtt{j},\mathtt{k}) \times \mathscr{C}(\mathtt{i},\mathtt{j}) & \xrightarrow{1\times c_{\mathtt{ijk}}} & \mathscr{C}(\mathtt{k},\mathtt{l}) \times \mathscr{C}(\mathtt{i},\mathtt{k}) \\
\downarrow{\scriptstyle c_{\mathtt{jkl}}\times 1} & & \downarrow{\scriptstyle c_{\mathtt{ikl}}} \\
\mathscr{C}(\mathtt{j},\mathtt{l}) \times \mathscr{C}(\mathtt{i},\mathtt{j}) & \xrightarrow[c_{\mathtt{ijl}}]{} & \mathscr{C}(\mathtt{i},\mathtt{l})
\end{array}$$

$$\begin{array}{ccc}
\mathscr{C}(\mathtt{i},\mathtt{j}) \times \mathbf{1} & & \\
\downarrow{\scriptstyle 1\times \mathbb{1}_{\mathtt{i}}} & \searrow{\scriptstyle \sim} & \\
\mathscr{C}(\mathtt{i},\mathtt{j}) \times \mathscr{C}(\mathtt{i},\mathtt{i}) & \xrightarrow[c_{\mathtt{iij}}]{} & \mathscr{C}(\mathtt{i},\mathtt{j})
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{1} \times \mathscr{C}(\mathtt{i},\mathtt{j}) & & \\
\downarrow{\scriptstyle \mathbb{1}_{\mathtt{j}}\times 1} & \searrow{\scriptstyle \sim} & \\
\mathscr{C}(\mathtt{j},\mathtt{j}) \times \mathscr{C}(\mathtt{i},\mathtt{j}) & \xrightarrow[c_{\mathtt{ijj}}]{} & \mathscr{C}(\mathtt{i},\mathtt{j})
\end{array}$$

We thus have 2-morphisms

$$a_{ZYX} : (ZY)X \xrightarrow{\sim} Z(YX)$$

$$r_X : X\mathbb{1}_{\mathtt{i}} \xrightarrow{\sim} X$$

$$l_X : \mathbb{1}_{\mathtt{j}}X \xrightarrow{\sim} X.$$

We also require as axioms that the following two diagrams commute:

$$\begin{array}{ccccc}
 & ((WZ)Y)X & \xrightarrow{a*1} & (W(ZY))X & \\
\swarrow{\scriptstyle a} & & & & \searrow{\scriptstyle a} \\
(WZ)(YX) & & & & W((ZY)X) \\
 & \searrow{\scriptstyle a} & & \swarrow{\scriptstyle 1*a} & \\
 & & W(Z(YX)) & &
\end{array}$$

$$\begin{array}{ccc}
(Y\mathbb{1})X & \xrightarrow{a} & Y(\mathbb{1}X) \\
 & {\scriptstyle r*1}\searrow \quad \swarrow{\scriptstyle 1*l} & \\
 & YX &
\end{array}$$

For notational aesthetics, we refer to the hom-set between two 1-morphisms $F, G : \mathtt{i} \to \mathtt{j}$ as $\mathrm{Hom}_{\mathscr{C}(\mathtt{i},\mathtt{j})}(X,Y)$ rather than the uglier $\mathscr{C}(\mathtt{i},\mathtt{j})(X,Y)$. In addition, we denote the vertical composition of 2-morphisms α and β (that is, when they are both in the same hom-category) as $\beta \circ_V \alpha$, and their horizontal composition (that is, the image of the pair (α,β) under the functors $c_{\mathtt{ijk}}$ defined earlier) as $\beta \circ_H \alpha$.

Definition 2.3.2. If the a, l and r as defined above are identities, that is when $(ZY)X = Z(YX)$ and $\mathbb{1}X = X = X\mathbb{1}$, we call $\mathscr{C}$ a *2-category*, or occasionally a *strict 2-category* when addition clarity is needed.

Definition 2.3.3. Given bicategories $\mathscr{C}$ and $\mathscr{C}'$, a *bifunctor* $F = (F, \varphi)$ is given by the following data:

- A function $F : \mathrm{Ob}\,\mathscr{C} \to \mathrm{Ob}\,\mathscr{C}'$.
- Functors $F_{\mathtt{ij}} : \mathscr{C}(\mathtt{i}, \mathtt{j}) \to \mathscr{C}'(F\mathtt{i}, F\mathtt{j})$.
- Natural isomorphisms defined by the commutative diagrams:

$$\begin{array}{ccc} \mathscr{C}(\mathtt{j},\mathtt{k}) \times \mathscr{C}(\mathtt{i},\mathtt{j}) & \xrightarrow{\quad c \quad} & \mathscr{C}(\mathtt{i},\mathtt{k}) \\ {\scriptstyle F_{\mathtt{j},\mathtt{k}} \times F_{\mathtt{i},\mathtt{j}}}\downarrow & & \downarrow{\scriptstyle F_{\mathtt{ik}}} \\ \mathscr{C}'(F\mathtt{j},F\mathtt{k}) \times \mathscr{C}'(F\mathtt{i},F\mathtt{j}) & \xrightarrow[\quad c' \quad]{} & \mathscr{C}'(F\mathtt{i},F\mathtt{k}) \end{array}$$

$$\begin{array}{ccc} \mathbf{1} & \xrightarrow{\mathbb{1}_{\mathtt{i}}} & \mathscr{C}(\mathtt{i},\mathtt{i}) \\ \Vert & & \downarrow{\scriptstyle F_{\mathtt{ii}}} \\ \mathbf{1} & \xrightarrow[\mathbb{1}'_{F\mathtt{i}}]{} & \mathscr{C}'(F\mathtt{i},F\mathtt{i}) \end{array} \quad .$$

We thus get 2-isomorphisms

$$\varphi_{XY} : FY \circ FX \cong F(Y \circ X)$$

$$\varphi_{\mathtt{i}} : \mathbb{1}'_{F\mathtt{i}} \cong F\mathbb{1}_{\mathtt{i}}.$$

We also require the following diagrams to commute:

$$\begin{array}{ccccc} (FZ \circ FY) \circ FX & \xrightarrow{\varphi * 1} & F(Z \circ Y) \circ FX & \xrightarrow{\varphi} & F((Z \circ Y) \circ X) \\ {\scriptstyle a'}\downarrow & & & & \downarrow{\scriptstyle Fa} \\ FZ \circ (FY \circ FX) & \xrightarrow[1 * \varphi]{} & FZ \circ F(Y \circ X) & \xrightarrow[\varphi]{} & F(Z \circ (Y \circ X)) \end{array}$$

$$\begin{array}{ccccc} FX \circ \mathbb{1}'_{F\mathtt{i}} & \xrightarrow{1 * \varphi} & FX \circ F\mathbb{1}_{\mathtt{i}} & \xrightarrow{\varphi} & F(X \circ \mathbb{1}_{\mathtt{i}}) \\ {\scriptstyle r'}\downarrow & & & & \downarrow{\scriptstyle Fr} \\ FX & = & = & = & FX \end{array}$$

$$\begin{array}{ccccc} \mathbb{1}'_{F\mathtt{j}} \circ FX & \xrightarrow{\varphi * 1} & F\mathbb{1}_{\mathtt{j}} \circ FX & \xrightarrow{\varphi} & F(\mathbb{1}_{\mathtt{j}} \circ X) \\ {\scriptstyle l'}\downarrow & & & & \downarrow{\scriptstyle Fl} \\ FX & = & = & = & FX. \end{array}$$

Definition 2.3.4. If $\mathscr{C}$ and $\mathscr{C}'$ are 2-categories and the above φ are identities, we call F a *2-functor*.

We will also be using the 2-categorical equivalent of a natural transformation, specifically for 2-functors:

Definition 2.3.5. Given two 2-categories $\mathscr{C}$ and $\mathscr{B}$ and 2-functors $F, G : \mathscr{C} \to \mathscr{B}$, a *2-natural transformation* $\sigma : F \to G$ is defined by the following data:

- 1-morphisms $\sigma_X : FX \to GX$ for $C \in \mathscr{C}$.
- Natural transformations

$$\begin{array}{ccc} \mathscr{C}(C,B) & \xrightarrow{F_{CB}} & \mathscr{B}(FC,FB) \\ {\scriptstyle G_{CB}}\downarrow & \nearrow^{\sigma_{CB}} & \downarrow{\scriptstyle (\sigma_B)_*} \\ \mathscr{B}(GC,GB) & \xrightarrow[(\sigma_C)^*]{} & \mathscr{B}(FC,GB) \end{array}$$

  where given a 1-morphism $h : X \to Y$ we notate the natural induced functors by $h_* : \mathscr{C}(C,X) \to \mathscr{C}(C,Y)$ and $h^* : \mathscr{C}(Y,B) \to \mathscr{C}(X,B)$. Thus by the diagram we get 2-morphisms $\sigma_f : Gf \circ \sigma_C \to \sigma_B \circ Ff$.

We further require that $(\sigma_g \circ_H \mathrm{id}) \circ_V (\mathrm{id} \circ_H \sigma_f) = \sigma_{gf}$ and that $G\mathbb{1}_C \circ \sigma_C = \sigma_C \circ F\mathbb{1}_C$.

We also mention here opposite 2-categories. It is possible to reverse 1-morphisms, 2-morphisms or both, and in general different notation is used for each. However, for the purposes of this thesis, we only require the situation where both 1-morphisms and 2-morphisms are reversed, hence:

Definition 2.3.6. For a 2-category $\mathscr{C}$, we define the 2-category $\mathscr{C}^{\mathrm{op}}$ to have the same objects as $\mathscr{C}$, with $\mathscr{C}^{\mathrm{op}}(\mathtt{i}, \mathtt{j}) = \mathscr{C}(\mathtt{j}, \mathtt{i})^{\mathrm{op}}$.

2.3.2 Finitary 2-Categories and 2-Representations

We now move on to defining 2-representations of 2-categories, following the ideas in [MM14], [MM16a] etc.. To begin, we define some specific 2-categories that will be the targets for 2-representations, similar to how $\mathrm{GL}(V)$ is the target for a representation in classical representation theory.

Let $\Bbbk$ be an algebraically closed field. We denote by $\mathfrak{A}_\Bbbk$ the 2-category whose objects are small $\Bbbk$-linear Krull-Schmidt categories, whose 1-morphisms are $\Bbbk$-linear additive functors and whose 2-morphisms are natural transformations. We further denote by $\mathfrak{A}_{\Bbbk}^{f}$ the full sub-2-category of $\mathfrak{A}_{\Bbbk}$ with objects those categories $\mathscr{A}$ such that $\mathscr{A}$ has only finitely many isomorphism classes of indecomposable objects, and such that $\dim \mathrm{Hom}_{\mathscr{A}}(\mathtt{i},\mathtt{j}) < \infty$ for all $\mathtt{i},\mathtt{j}$. We finally define $\mathfrak{R}_{\Bbbk}$ as the full sub-2-category of $\mathfrak{A}_{\Bbbk}$ whose objects are equivalent to A-mod for some finite dimensional associative $\Bbbk$-algebra A.

Definition 2.3.7. An object of $\mathfrak{A}_{\Bbbk}^{f}$ is called a *finitary* category.

We now state a fairly strong finiteness condition we need to impose on a 2-category $\mathscr{C}$ to allow the following theory to apply. The bulk of this thesis will be concerned with relaxing or removing parts of this restriction.

Definition 2.3.8. A 2-category $\mathscr{C}$ is *finitary* over $\Bbbk$ when it has finitely many objects, and when $\mathscr{C}(\mathtt{i},\mathtt{j}) \in \mathfrak{A}_{\Bbbk}^{f}$ for every $\mathtt{i}$ and $\mathtt{j}$. We further require that horizontal composition is additive and $\Bbbk$-linear, and that $\mathbb{1}_{\mathtt{i}}$ is indecomposable for all $\mathtt{i}$.

Definition 2.3.9. A finitary 2-category $\mathscr{C}$ is *weakly fiat* if:

- It has a weak object-preserving anti-autoequivalence $(-)^*$; that is a bifunctor $(-)^* : \mathscr{C} \to \mathscr{C}^{\mathrm{op}}$ such that $\mathtt{i}^* = \mathtt{i}$ for an object $\mathtt{i}$ and such that $(FG)* \cong G^*F^*$ for 1-morphisms F and G (the latter isomorphism giving the nomenclature of 'weak' to the anti-autoequivalence. Note that this 'weak' is unrelated to the 'weakly' in 'weakly fiat' - the latter stems from the absence of an involutive structure, as defined below).
- For any 1-morphism $F \in \mathscr{C}(\mathtt{i},\mathtt{j})$ there exist 2-morphisms $\alpha : F \circ F^* \to \mathbb{1}_{\mathtt{j}}$ and $\beta : \mathbb{1}_{\mathtt{i}} \to F^* \circ F$ such that the internal adjunction axioms

$$(\alpha \circ_H \mathrm{id}_F) \circ_V (\mathrm{id}_F \circ_H \beta) = \mathrm{id}_F$$

  and

$$(\mathrm{id}_{F^*} \circ_H \alpha) \circ_V (\beta \circ_H \mathrm{id}_{F^*}) = \mathrm{id}_{F^*}$$

  hold.

We let ${}^*(-)$ denote the inverse of $(-)^*$.

Definition 2.3.10. If $\mathscr{C}$ is a weakly fiat 2-category such that $(-)^*$ is a weak involution, then we say $\mathscr{C}$ is a *fiat* 2-category.

We now define a 2-representation or, more precisely, various types of 2-representation of a 2-category. These are 2-functors into 2-categories whose objects are 1-categories. In pleasant cases, this allows us to study properties of the 2-category via studying the properties of these 1-categories.

Definition 2.3.11. Given a finitary 2-category $\mathscr{C}$, a 2-functor $\mathbf{M} : \mathscr{C} \to \mathbf{Cat}$ is a *2-representation*. Similarly, a 2-functor $\mathbf{M} : \mathscr{C} \to \mathfrak{A}_{\Bbbk}$ is an *additive 2-representation*, a 2-functor $\mathbf{M} : \mathscr{C} \to \mathfrak{A}_{\Bbbk}^{f}$ is a *finitary* 2-representation and a 2-functor $\mathbf{M} : \mathscr{C} \to \mathfrak{R}_{\Bbbk}$ is an *abelian* 2-representation.

Definition 2.3.12. If $\mathscr{C}$ is a fiat 2-category whose objects are small fully additive $\Bbbk$-linear categories, whose 1-morphisms are additive $\Bbbk$-linear functors and whose 2-morphisms are natural transformations, then we define the *defining additive 2-representation* $\mathbf{I} : \mathscr{C} \to \mathfrak{A}_{\Bbbk}$ as the natural injection of $\mathscr{C}$ into $\mathfrak{A}_{\Bbbk}$. Similar definitions exist for the finitary and abelian cases.

The defining 2-representation is only useful in limited circumstances. However, we introduce below the principal 2-representations, which retain some of the uses of the defining 2-representation (e.g. for forming further 2-representations), but are more generally applicable.

Definition 2.3.13. For a finitary 2-category $\mathscr{C}$ and $\mathtt{i} \in \mathscr{C}$ an object of $\mathscr{C}$, we define the $\mathtt{i}$*th principal additive 2-representation* $\mathbf{P}_{\mathtt{i}} : \mathscr{C} \to \mathfrak{A}_{\Bbbk}$ as $\mathbf{P}_{\mathtt{i}} = \mathscr{C}(\mathtt{i}, -)$; that is, it takes an object $\mathtt{j} \in \mathscr{C}$ to the category $\mathscr{C}(\mathtt{i}, \mathtt{j})$, the 1-morphism F to the functor defined by post-composition with F, and the 2-morphism α to the natural transformation defined by horizontal composition with α. Since $\mathscr{C}$ is finitary, this is a finitary 2-representation.

Definition 2.3.14. Two 2-representations $\mathbf{M}, \mathbf{N} : \mathscr{C} \to \mathbf{Cat}$ of a finitary 2-category are *equivalent* if there is a 2-natural transformation $\Phi : \mathbf{M} \to \mathbf{N}$ such that $\Phi_{\mathtt{i}}$ is an equivalence for each $\mathtt{i}$.

Given a 2-representation $\mathbf{M}$ of a finitary 2-category $\mathscr{C}$, we define for notational purposes $\mathscr{M} = \coprod_{\mathtt{i}\in\mathscr{C}} \mathbf{M}(\mathtt{i})$. This coproduct is taken in **Cat**.

2.3.3 Cells and Ideals

By definition, the collection of indecomposable 1-morphisms in a finitary 2-category $\mathscr{C}$ splits into finitely many isomorphism classes. We denote by $S(\mathscr{C})$ this finite set. We identify the isomorphism class $[F]$ of a 1-morphism F with F itself in the following sections for ease of notation. Per [MM11], we can introduce three partial orders $\leq_{\mathscr{L}}$, $\leq_{\mathscr{R}}$ and $\leq_{\mathscr{J}}$ on $S(\mathscr{C})$ as follows: $F \leq_{\mathscr{L}} G$ precisely when there exists some 1-morphism $H \in \mathscr{C}$ such that G is a direct summand of $H \circ F$. Similarly, $F \leq_{\mathscr{R}} G$ precisely when G is a direct summand of $F \circ H$ for some 1-morphism H, and $F \leq_{\mathscr{J}} G$ precisely when G is a summand of $H \circ F \circ K$ for some 1-morphisms H and K.

Definition 2.3.15. We call the equivalence classes of $\leq_{\mathscr{L}}$ the *$\mathscr{L}$-cells* (or *left cells*) of $\mathscr{C}$, those of $\leq_{\mathscr{R}}$ the *$\mathscr{R}$-cells* (or *right cells*) of $\mathscr{C}$ and those of $\leq_{\mathscr{J}}$ the *$\mathscr{J}$-cells* (or *two-sided cells*) of $\mathscr{C}$. We notate these equivalence relations as $\mathscr{L}$, $\mathscr{R}$ and $\mathscr{J}$ respectively.

As a note, [MM14] among other papers phrases this differently, by considering $S(\mathscr{C})$ as a multisemigroup with unit with $F \circ G = \{H \in S(\mathscr{C}) | H \text{ is a summand of } F \circ G\}$. Letting $T \circ F = \bigcup_i T_i \circ F$ for $T = \{T_i\}_i$, we say that $F \leq_{\mathscr{L}} G$ if $S(\mathscr{C}) \circ G \subseteq S(\mathscr{C}) \circ F$. Similarly, $F \leq_{\mathscr{R}} G$ if $G \circ S(\mathscr{C}) \subseteq F \circ S(\mathscr{C})$ and $F \leq_{\mathscr{J}} G$ if $S(\mathscr{C}) \circ G \circ S(\mathscr{C}) \subseteq S(\mathscr{C}) \circ F \circ S(\mathscr{C})$. It is not hard to see that these two definitions are equivalent.

Definition 2.3.16. We say a $\mathscr{J}$-cell $\mathcal{J}$ is *regular* if any two $\mathscr{L}$-cells contained in $\mathcal{J}$ are not comparable by $\leq_{\mathscr{L}}$. We say it is *strongly regular* if we further have that the intersection of any $\mathscr{L}$-cell in $\mathcal{J}$ and any $\mathscr{R}$-cell in $\mathcal{J}$ is a single element.

If F and G are in the same $\mathscr{L}$-cell, then let $F : \mathtt{i} \to \mathtt{j}$ and $G : \mathtt{k} \to \mathtt{l}$. There is some $H : \mathtt{j} \to \mathtt{m}$ such that G is a direct summand of $H \circ F : \mathtt{i} \to \mathtt{m}$. Thus we must have $\mathtt{i} = \mathtt{k}$. Similarly, if they share a $\mathscr{R}$-cell, we must have $\mathtt{l} = \mathtt{j}$.

Definition 2.3.17. Let $\mathscr{C}$ be a finitary 2-category and let $\mathbf{M}$ be a finitary 2-representation of $\mathscr{C}$. Then we define the *annihilator* $\mathrm{Ann}_{\mathscr{C}}(\mathbf{M})$ of $\mathbf{M}$ to be the two-sided-ideal of $\mathscr{C}$ consisting of all 2-morphisms of $\mathscr{C}$ annihilated by $\mathbf{M}$ (this exists by [MM11] Section 4.2).

We now quote a definition and a result from [CM19a]:

Definition 2.3.18. A $\mathscr{J}$-cell $\mathfrak{J}$ is *idempotent* if there exist $F, G, H \in \mathfrak{J}$ such that H is a direct summand of FG.

Lemma 2.3.19 ([CM19a] Lemma 1)**.** *Let $\mathscr{C}$ be a finitary category and $\mathbf{M}$ a finitary 2-representation of $\mathscr{C}$. We let $\mathscr{C}_{\mathbf{M}}$ denote the finitary 2-category $\mathscr{C}/\mathrm{Ann}_{\mathscr{C}}(\mathbf{M})$, and let $\mathscr{S}(\mathscr{C}_{\mathbf{M}})$ denote the poset of $\mathscr{J}$-cells of $\mathscr{C}_{\mathbf{M}}$ under $\leq_{\mathscr{J}}$. Then $\mathscr{S}(\mathscr{C}_{\mathbf{M}})$ has a unique maximum element. This $\mathscr{J}$-cell is idempotent.*

Also note that any element of $\mathscr{S}(\mathscr{C}_{\mathbf{M}})$ corresponds to a $\mathscr{J}$-cell of $\mathscr{C}$.

Definition 2.3.20. The unique $\mathscr{J}$-cell of $\mathscr{C}$ corresponding to the $\mathscr{J}$-cell of $\mathscr{C}_{\mathbf{M}}$ identified above is called the *apex* of $\mathbf{M}$.

We now present two useful alterations to the concept of an ideal of a category:

Definition 2.3.21. If $\mathscr{C}$ is a $\Bbbk$-linear 2-category, we define a *left 2-ideal* $\mathcal{J}$ of $\mathscr{C}$ to have the same objects as $\mathscr{C}$, and for each pair $\mathtt{i}, \mathtt{j}$ of objects an ideal $\mathcal{J}(\mathtt{i}, \mathtt{j})$ of the 1-category $\mathscr{C}(\mathtt{i}, \mathtt{j})$ which is stable under left (horizontal) multiplication with 1- and 2-morphisms of $\mathscr{C}$. We can then similarly define *right 2-ideals* and *two-sided 2-ideals*. We call the latter simply *2-ideals*.

Definition 2.3.22. For a $\Bbbk$-linear 2-category $\mathscr{C}$ and a 2-ideal $\mathscr{I}$ of $\mathscr{C}$, we can define the *quotient 2-category* $\mathscr{C}/\mathcal{J}$ as follows: the objects and 1-morphisms of $\mathscr{C}/\mathcal{J}$ are the same as those of $\mathscr{C}$; the hom-sets between 1-morphisms are defined as

$$\mathrm{Hom}_{\mathscr{C}/\mathcal{J}(\mathtt{i},\mathtt{j})}(F, G) = \mathrm{Hom}_{\mathscr{C}(\mathtt{i},\mathtt{j})}(F, G)/\,\mathrm{Hom}_{\mathcal{J}(\mathtt{i},\mathtt{j})}(F, G).$$

Horizontal composition of 2-morphisms is given in the standard fashion: $[\alpha] \circ_H [\beta] = [\alpha \circ_H \beta]$.

Definition 2.3.23. Given a 2-category $\mathscr{C}$ and a 2-representation $\mathbf{M}$ of $\mathscr{C}$, an *ideal* $\mathcal{I}$ of $\mathbf{M}$ is a collection of ideals $\mathcal{I}(\mathtt{i}) \subseteq \mathbf{M}(\mathtt{i})$ which is closed under the action of $\mathscr{C}$ in that for any morphism $f \in \mathcal{I}(\mathtt{i})$ and any 1-morphism $F \in \mathscr{C}$, $\mathbf{M}(F)(f)$ is a morphism in $\mathcal{I}$ if it is defined.

Definition 2.3.24. Given a finitary 2-representation $\mathbf{M}$ of a finitary 2-category $\mathscr{C}$ and an ideal $\mathcal{I}$ of $\mathbf{M}$, we define the *quotient 2-representation* $\mathbf{M}/\mathcal{I}$ by setting $(\mathbf{M}/\mathcal{I})(\mathtt{j}) = \mathbf{M}(\mathtt{j})/\mathcal{I}(\mathtt{j})$ for any object $\mathtt{j} \in \mathscr{C}$. The definition of an ideal of a 2-representation immediately gives that $\mathbf{M}/\mathcal{I}$ has a canonical structure of a 2-representation of $\mathscr{C}$ given on morphisms by $(\mathbf{M}/\mathcal{I})(F)([f]) = [\mathbf{M}(F)(f)]$ for a morphism $f \in \mathscr{M}$ and a 1-morphism $F \in \mathscr{C}$.

Definition 2.3.25. Let $\mathscr{C}$ be a finitary 2-category and let $\mathcal{J}$ be a $\mathscr{J}$-cell in $\mathscr{C}$. We say that $\mathcal{J}$ is *non-trivial* if it contains some non-identity 1-morphism. Otherwise, $\mathcal{J}$ is *trivial*. We similarly define trivial and non-trivial $\mathscr{L}$- and $\mathscr{R}$-cells.

Definition 2.3.26. A 2-category $\mathscr{C}$ with a $\mathscr{J}$-cell $\mathcal{J}$ is $\mathcal{J}$*-simple* if, for any non-trivial 2-ideal $\mathscr{I} \subseteq \mathscr{C}$, there exists $F \in \mathcal{J}$ such that $\mathrm{id}_F \in \mathscr{I}$.

We have a useful theorem from [MM14]:

Theorem 2.3.27 ([MM14] Theorem 15)**.** *Let $\mathscr{C}$ be a fiat 2-category and $\mathcal{J}$ a non-zero $\mathscr{J}$-cell of $\mathscr{C}$. There there is a unique 2-ideal $\mathscr{I}$ of $\mathscr{C}$ such that $\mathscr{C}/\mathscr{I}$ is $\mathcal{J}$-simple.*

2.3.4 (Simple) Transitive 2-Representations and Cell 2-Representations

In classical representation theory, a lot of information about the representations of a group can be gleaned from studying its irreducible representations. In 2-representation theory, we desire a similar construction, which turns out to be the concept of a simple transitive 2-representation. We start by defining transitive 2-representations, taking these definitions primarily from [MM16c].

Definition 2.3.28. Given a 2-representation $\mathbf{M}$ of a 2-category $\mathscr{C}$ and a collection of objects $\{X_k\}_{k\in K}$ in $\mathscr{M}$, we define the $\mathbf{M}$-*span* of the X_k, $\mathbf{G_M}(\{X_k\})$, to be

$$\mathbf{G_M}(\{X_k\}) = \mathrm{add}\{\mathbf{M}(F)X_k | k \in K, F \in \mathscr{C}(\mathtt{i},\mathtt{j}) \text{ for some } \mathtt{i},\mathtt{j} \in \mathscr{C}\}.$$

Defining

$$\mathbf{G_M}(\{X_j\})(\mathtt{k}) = \mathrm{add}\{\mathbf{M}(F)X_j | j \in J, F \in \mathscr{C}(\mathtt{i},\mathtt{k}) \text{ for some } \mathtt{i} \in \mathscr{C}\},$$

it is immediate from the definition that $\mathbf{G_M}(\{X_j\})$ is a sub-2-representation of $\mathbf{M}$.

Definition 2.3.29. Let $\mathscr{C}$ be a finitary 2-category and let $\mathbf{M}$ be a finitary 2-representation of $\mathscr{C}$. We say that $\mathbf{M}$ is *transitive* if for any $\mathtt{i} \in \mathscr{C}$ and any non-zero $X \in \mathbf{M}(\mathtt{i})$, $\mathbf{G_M}(\{X\})$ is equivalent to $\mathbf{M}$.

To move on to simple transitive 2-representations, we first need the following Lemma from [MM16c]:

Lemma 2.3.30 ([MM16c] Lemma 4)**.** *Let $\mathbf{M}$ be a transitive 2-representation of a finitary 2-category $\mathscr{C}$. There exists a unique maximal ideal $\mathscr{I}$ of $\mathbf{M}$ such that $\mathscr{I}$ does not contain any identity morphisms apart from for the zero object.*

Definition 2.3.31. We say that a transitive 2-representation $\mathbf{M}$ is *simple transitive* if the maximal ideal of $\mathbf{M}$ given in Lemma 2.3.30 is the zero ideal.

To see that this concept does indeed reflect that of irreducible representations (i.e. simple modules when viewing the theory from a module-theoretic perspective), let $\mathbf{M}$ be a simple transitive 2-representation of a finitary 2-category $\mathscr{C}$, and let $\mathscr{I}$ be an ideal of $\mathbf{M}$. If $\mathscr{I}$ does not contain any identity morphisms for non-zero objects, it is zero by the definition of a simple transitive 2-representation. Assume $\mathrm{id}_M \in \mathscr{I}$ for some object M. Since $\mathbf{M}$ is transitive, for any other object $N \in \mathscr{M}$, there exists some 1-morphism F of $\mathscr{C}$ such that N is a summand of $\mathbf{M}(F)M$. But since $\mathscr{I}$ is closed under the action of $\mathscr{C}$, $\mathbf{M}(F)(\mathrm{id}_M) = \mathrm{id}_{\mathbf{M}(F)M} \in \mathscr{I}$, and since $\mathscr{I}$ is closed under pre- and post-composition, we can compose $\mathrm{id}_{\mathbf{M}(F)M}$ with the injection and projection morphisms for N to derive that $\mathrm{id}_N \in \mathscr{I}$. But then for any $f : N \to N'$ in $\mathbf{M}$, $f = f \circ \mathrm{id}_N$, and thus $f \in \mathscr{I}$ and $\mathscr{I}$ contains all morphisms of $\mathscr{M}$. Therefore $\mathbf{M}$ has no 'proper' ideals, which gives a direct analogue of the definition of a simple module.

While in general there exist transitive 2-representations that are not simple transitive, we can always quotient out by the maximal ideal to derive a simple transitive 2-representation:

Definition 2.3.32. Given any transitive 2-representation $\mathbf{M}$ of a finitary 2-category $\mathscr{C}$, let $\mathscr{I}$ be the unique maximal ideal previously defined, and denote by $\mathbf{M}_{\mathscr{I}}$ the quotient of $\mathbf{M}$ by $\mathscr{I}$, called the *simple transitive quotient* of $\mathbf{M}$.

The most important example of simple transitive 2-representations that we will be considering throughout this thesis is that of cell 2-representations. We give here an equivalent definition to that given in [MM16c].

Let $\mathscr{C}$ be a finitary 2-category with a $\mathscr{J}$-cell $\mathcal{J}$ and an $\mathscr{L}$-cell $\mathcal{L} \subseteq \mathcal{J}$. We let $\mathtt{i} = \mathtt{i}_{\mathcal{L}}$ be the object of $\mathscr{C}$ which is the domain of every $F \in \mathcal{L}$. This is a single object by the note at the end of Subsection 2.3.3. For each $j \in \mathscr{C}$, we define a subset $\mathbf{N}_{\mathcal{L}}(\mathtt{j})$ of $\mathbf{P}_{\mathtt{i}}(\mathtt{j})$ by $\mathbf{N}_{\mathcal{L}}(\mathtt{j}) = \mathrm{add}\{FX | F \in \coprod_{\mathtt{k} \in \mathscr{C}} \mathscr{C}(\mathtt{k}, \mathtt{j}), X \in \mathcal{L}\}$. $\mathbf{N}_{\mathcal{L}}$ thus defines a map from the object set of $\mathscr{C}$ into $\mathfrak{A}$. To make this a 2-representation, of $\mathscr{C}$, we take a 1-morphism F to the functor defined by left composition with F, and a 2-morphism α is taken to the natural transformation defined by left horizontal composition with α.

By a similar proof to that of Lemma 2.3.30, $\mathbf{N}_{\mathcal{L}}$ has a unique maximal ideal not containing any identity morphisms for non-zero objects, and we can take its simple transitive quotient:

Definition 2.3.33. The simple transitive quotient of this 2-representation, viewed as a representation of $\mathscr{C}$ via quotients and restriction, is the *cell 2-representation* of $\mathscr{C}$ corresponding to the $\mathscr{L}$-cell $\mathcal{L}$. We generally denote this as $\mathbf{C}_{\mathcal{L}}$.

Definition 2.3.34. A *trivial* cell 2-representation is a cell 2-representation corresponding to a trivial $\mathscr{L}$-cell.

2.4 Abelianisations of Additive Categories

We now define a process of obtaining from a finitary (2-)category an associated abelian (2-)category. In fact, we will examine two related processes, with the first originating in [Fre66].

Definition 2.4.1. Let $\mathscr{C}$ be an additive category. We define the *injective classical Freyd abelianisation* $\underline{\underline{\mathscr{C}}}$ of $\mathscr{C}$ as a quotient of the arrow category of $\mathscr{C}$ given as follows:

- Objects of $\underline{\underline{\mathscr{C}}}$ are morphisms $f : X \to Y$ of $\mathscr{C}$.
- Morphisms of $\underline{\underline{\mathscr{C}}}$ are commutative squares $\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ X' & \xrightarrow[f']{} & Y' \end{array}$ modulo those squares for which there exists a morphism $p : Y \to X'$ such that $g = qf$.

The *projective classical Freyd abelianisation* $\overline{\overline{\mathscr{C}}}$ is defined dually.

By (the dual of) [Fre66] Theorem 1.4, $\underline{\underline{\mathscr{C}}}$ is an abelian category if and only if $\mathscr{C}$ has weak cokernels (technically that source also requires weak finite coproducts, but since an additive category contains all finite biproducts by definition this requirement is immediately satisfied). For our purposes, we use that this is true if $\mathscr{C}$ is a finitary category, and thus in that case $\underline{\underline{\mathscr{C}}}$ is indeed abelian.

While we may wish to extend this directly to abelianising 2-categories, the resulting abelianised '2-category' will only be a bicategory. Instead, we turn to a variant of the classical Freyd abelianisation first deployed in [MMMT16] Section 3.2 specifically to fix this problem. That paper uses finitary (2-)categories as its setting, but we present below a more general definition.

Definition 2.4.2. Given an additive category $\mathscr{C}$, the *injective fan Freyd abelianisation* $\underline{\mathscr{C}}$ of $\mathscr{C}$ is an additive category that is defined as follows:

- Objects of $\underline{\mathscr{C}}$ are equivalence classes of tuples of the form $(X, k, Y_i, f_i)_{i\in\mathbb{Z}^+}$ where X and the Y_i are objects of $\mathscr{C}$, the f_i are morphisms of $\mathscr{C}$, and k is a non-negative integer such that $Y_i = 0$ for $i > k$. Two tuples are equivalent if they only differ in the value of k.

- A morphism from (X, k, Y_i, f_i) to (X', k', Y'_i, f'_i) is an equivalence class of tuples $(g, h_{ij})_{i,j\in\mathbb{Z}^+}$ where $g : X \to X'$ and $h_{ij} : Y_i \to Y'_j$ are morphisms of $\mathscr{C}$ such that $f'_i g = \sum_j h_{ji} f_j$ for each i, modulo the (g, h_{ij}) such that there exist $q_i : Y_i \to X'$ with $\sum_i q_i f_i = g$.

- Identity morphisms are $(\mathrm{id}_X, \delta_{ij}\mathrm{id}_{Y_i})$ and composition is given by $(g', h'_{ij}) \circ (g, h_{ij}) = (g'g, \sum_k h'_{kj} h_{ik})$.

The *projective fan Freyd abelianisation* is defined dually.

As noted in [MMMT16], $\underline{\mathscr{C}}$ is indeed an additive category, with biproducts given by

$$(X, k, Y^i, f^i) \oplus (X', k', Y'^i, f'^i) = (X \oplus X', \max\{k, k'\}, Y^i \oplus Y'^i, f^i \oplus f'^i),$$

and is equivalent to $\underline{\underline{\mathscr{C}}}$. In particular, $\underline{\mathscr{C}}$ is thus an abelian category when $\mathscr{C}$ has weak cokernels.

We expand the definition to 2-categories with additive hom-categories such that composition respects additivity (with locally finitary 2-categories being an example). Given such a 2-category $\mathscr{C}$, we define $\underline{\mathscr{C}}$ as follows:

- The objects of $\underline{\mathscr{C}}$ are the same as those of $\mathscr{C}$.

- $\underline{\mathscr{C}}(\mathtt{i}, \mathtt{j}) = \underline{\mathscr{C}(\mathtt{i}, \mathtt{j})}$.

- Composition of 1-morphisms is defined as follows:

$(F, G_i, k, \alpha_i) \circ (F', G'_i, k', \alpha'_i) = (FF', H_i, k + k', \beta_i)$, where:

$$H_i = \begin{cases} F \circ G'_i, & i = 1, \dots, k' \\ G_{i-k'} \circ F', & i = k' + 1, \dots, k' + k \\ 0, & \text{else} \end{cases}$$

and

$$\beta_i = \begin{cases} \mathrm{id}_F \circ_H \alpha'_i, & i = 1, \dots, k' \\ \alpha_{i-k'} \circ_H \mathrm{id}_{F'}, & i = k' + 1, \dots, k' + k \\ 0, & \text{else.} \end{cases}$$

- Identity 1-morphisms are $(\mathbb{1}_{\mathtt{i}}, 0, 0, 0)$.
- Horizontal composition of 2-morphisms is defined component-wise.

Let $\mathscr{C}$ be an additive 2-category and $\mathbf{M}$ an additive 2-representation of $\mathscr{C}$. The injective abelianisation $\underline{\mathbf{M}}$ of $\mathbf{M}$ is defined such that $\underline{\mathbf{M}}(\mathtt{i}) = \underline{\mathbf{M}(\mathtt{i})}$. This is easily seen to be a 2-representation of $\mathscr{C}$ with component-wise action. Further, it has the structure of a $\underline{\mathscr{C}}$ 2-representation with the action defined by

$$(F, G_i, k, \alpha_i) \circ (M, N_i, k', f_i) = (FM, H_i, k + k', g_i),$$

where:

$$H_i = \begin{cases} FN_i. & i = 1, \dots, k' \\ G_{i-k'}M, & i = k' + 1, \dots, k' + k \\ 0, & \text{else;} \end{cases}$$

$$g_i = \begin{cases} Ff_i, & i = 1, \dots, k' \\ (\alpha_{i-k'})_M, & i = k' + 1, \dots, k' + k \\ 0, & \text{else.} \end{cases}$$

The action of 2-morphisms is defined component-wise.

2.5 Coalgebra 1-Morphisms and Comodule Categories

An important tool we will be using is the concept of coalgebra 1-morphisms associated to a 2-representation. These were initially developed in [MMMT16] based on material from [EGNO16], though we lean more on the somewhat less opaque presentation found in [CM19b].

Definition 2.5.1. Let $\mathscr{C}$ be a 2-category. A *coalgebra 1-morphism* of $\mathscr{C}$ is an ordered triple (C, μ_C, ϵ_C) where $C : \mathtt{i} \to \mathtt{i}$ is a 1-morphism of $\mathscr{C}$ and $\mu_C : C \to C \circ C$ and $\epsilon_C : C \to \mathbb{1}_{\mathtt{i}}$ are 2-morphisms of $\mathscr{C}$. We require them to satisfy the axioms

$$(\mathrm{id}_C \circ_H \mu_C) \circ_V \mu_C = (\mu_C \circ_H \mathrm{id}_C) \circ_V \mu_C$$

and

$$(\mathrm{id}_C \circ_H \epsilon_C) \circ_V \mu_C = \mathrm{id}_C = (\epsilon_C \circ_H \mathrm{id}_C) \circ_V \mu_C.$$

Definition 2.5.2. Let $\mathscr{C}$ be a 2-category with a coalgebra 1-morphism (C, μ_C, ϵ_C). A *(left) comodule 1-morphism* of C is an ordered pair (M, ρ_M) where $M : \mathtt{i} \to \mathtt{j}$ is a 1-morphism of $\mathscr{C}$ and $\rho_M : M \to C \circ M$ is a 2-morphism of $\mathscr{C}$. We require them to satisfy the axioms

$$(\mathrm{id}_C \circ_H \rho_M) \circ_V \rho_M = (\mu_C \circ_H \mathrm{id}_C) \circ_V \rho_M$$

and

$$(\epsilon_C \circ_H \mathrm{id}_C) \circ_V \rho_M = \mathrm{id}_M.$$

Let $\mathscr{C}$ be a finitary 2-category and let $\mathbf{M}$ be a finitary 2-representation of $\mathscr{C}$. Let $S \in \mathbf{M}(\mathtt{i})$ for some object $\mathtt{i}$ of $\mathscr{C}$. For the category $\underline{\mathscr{C}}(\mathtt{i}) = \coprod_{\mathtt{j} \in \mathscr{C}} \underline{\mathscr{C}}(\mathtt{i}, \mathtt{j})$ we have the evaluation functor $\mathrm{ev}_S : \underline{\mathscr{C}}(\mathtt{i}) \to \underline{\mathscr{M}}$ given by $\mathrm{ev}_S(F) \mapsto FS$ for a 1-morphism $F : \mathtt{i} \to \mathtt{j}$ and $\mathrm{ev}_S(\alpha) = \alpha_S$ for a 2-morphism $\alpha : F \to G$. As [CM19b] notes, this functor has a left adjoint.

Definition 2.5.3. We refer to this left adjoint as the *internal cohom-functor* (at S), and denote it by $[S, -] : \underline{\mathscr{M}} \to \underline{\mathscr{C}}(\mathtt{i})$.

From [MMMT16] Lemmas 4.3 and 4.4, $[S,S]$ is a coalgebra 1-morphism and $[S,T]$ is an $[S,S]$-comodule 1-morphism for any $M \in \mathscr{M}$. We denote the category of comodule 1-morphisms of $[S,S]$ by $\mathrm{comod}_{\underline{\mathscr{C}}}([S,S])$ and its subcategory of injective comodules by $\mathrm{inj}_{\underline{\mathscr{C}}}([S,S])$. These can be considered instead as 2-representations of $\mathscr{C}$ and $\underline{\mathscr{C}}$ in a natural fashion; when considering them as such we denote them as $\textbf{comod}_{\underline{\mathscr{C}}}([S,S])$ and $\textbf{inj}_{\underline{\mathscr{C}}}([S,S])$ respectively.

Theorem 2.5.4 ([MMMT16] Theorem 4.7)**.** *Let* $\mathbf{M}$ *be a transitive 2-representation of* $\mathscr{C}$ *and let* $S \in \mathbf{M}(\mathtt{i})$ *be non-zero. Then there is an equivalence of 2-representations of* $\underline{\mathscr{C}}$ *between* $\underline{\mathbf{M}}$ *and* $\textbf{comod}_{\underline{\mathscr{C}}}([S,S])$ *which restricts to an equivalence of 2-representations of* $\mathscr{C}$ *between* $\mathbf{M}$ *and* $\textbf{inj}_{\underline{\mathscr{C}}}([S,S])$.

3

The Extension to Infinitely Many Objects

As was mentioned previously, the definition of finitary 2-categories includes multiple finiteness restrictions, and we aim to relax some of these over the following thesis. Initially, we will examine the most straightforward generalisation, that of allowing infinitely many objects in our 2-category. To avoid having to consider size issues, we will always assume that our 2-categories have countably many objects.

Many of the results in this chapter have proofs that work essentially identically to the finitary case. However, we have needed to construct novel proofs when the finitary proofs would fail to generalise. Once we have laid out the definitions, we will indicate at the beginning of each (sub)section whether the proofs found there are novel.

3.1 Locally Finitary 2-Categories

Definition 3.1.1. A 2-category $\mathscr{C}$ is *locally finitary* over $\Bbbk$ when:

- $\mathscr{C}$ has countably many objects.
- $\mathscr{C}(\mathtt{i},\mathtt{j}) \in \mathfrak{A}_{\Bbbk}^{f}$ for every $\mathtt{i}$ and $\mathtt{j}$.
- Horizontal composition is additive and $\Bbbk$-linear.
- $\mathbb{1}_{\mathtt{i}}$ is indecomposable for all $\mathtt{i}$.

This terminology follows the standard practise of referring to a 2-category as being 'locally [property] 2-category' when each of its hom-categories is a [property] category (e.g. locally abelian 2-category, locally triangulated 2-category etc.).

Definition 3.1.2. Given a set of objects $\underline{\mathtt{i}} = \{\mathtt{i}_1, \ldots, \mathtt{i}_n\}$, we denote the sub-2-category generated by the $\mathtt{i}_j$ (i.e. with objects $\mathtt{i}_1, \ldots, \mathtt{i}_n$, and with hom-categories $\mathscr{C}(\mathtt{i}_j, \mathtt{i}_k)$ for all j and k) by $\mathscr{C}_{\underline{\mathtt{i}}}$, and call it a *full finitary sub-2-category*.

Much of the language and approach of finitary 2-categories carries over immediately to the locally finitary case, using these full finitary sub-2-categories.

Definition 3.1.3. A locally finitary 2-category $\mathscr{C}$ is *locally weakly fiat* if every full finitary sub-2-category $\mathscr{C}_{\underline{\mathtt{i}}}$ of $\mathscr{C}$ is weakly fiat. Similarly, a locally finitary 2-category $\mathscr{C}$ is *locally fiat* if every full finitary sub-2-category of $\mathscr{C}$ is fiat. Equivalently, $\mathscr{C}$ is locally weakly fiat if there exists a weak object preserving anti-autoequivalence $(-)^* : \mathscr{C} \to \mathscr{C}$ that obeys the same axioms as in Definition 2.3.9, and it is fiat if $(-)^*$ is an involution.

Definition 3.1.4. Given a locally finitary 2-category $\mathscr{C}$, a 2-functor $\mathbf{M} : \mathscr{C} \to \mathfrak{A}_{\Bbbk}$ is an *additive* 2-representation of $\mathscr{C}$, a 2-functor $\mathbf{M} : \mathscr{C} \to \mathfrak{A}_{\Bbbk}^{f}$ is a *finitary* 2-representation of $\mathscr{C}$ and a 2-functor $\mathbf{M} : \mathscr{C} \to \mathfrak{R}_{\Bbbk}$ is an *abelian* 2-representation.

Definition 3.1.5. For a locally finitary 2-category $\mathscr{C}$ and $\mathtt{i} \in \mathscr{C}$ an object of $\mathscr{C}$, we define the $\mathtt{i}$*th principal (additive) 2-representation* $\mathbf{P}_{\mathtt{i}} : \mathscr{C} \to \mathscr{A}_{\Bbbk}$ as $\mathbf{P}_{\mathtt{i}} = \mathscr{C}(\mathtt{i}, -)$ in a similar fashion to before. Since $\mathscr{C}$ is a locally finitary 2-category, this is a finitary 2-representation.

We present an important example of a locally finitary 2-category. Let $A = \{A_{\mathtt{i}}\}_{\mathtt{i} \in I}$ be a countable collection of basic self-injective connected finite dimensional $\Bbbk$-algebras. We define the locally finitary 2-category $\mathscr{C}_A$ as follows:

- The object set of of $\mathscr{C}_A$ is the set I given above. We associate each $\mathtt{i}$ with (small categories equivalent to) $A_{\mathtt{i}}\text{-mod}$ for the purpose of defining 1- and 2-morphisms below.

- The 1-morphisms in $\mathscr{C}_A(\mathtt{i},\mathtt{j})$ are taken to be direct sums of functors from $A_\mathtt{i}\text{-mod}$ to $A_\mathtt{j}\text{-mod}$ which have summands isomorphic to sums of the identity functor and functors given by tensoring with projective $(A_\mathtt{j}\text{-}A_\mathtt{i})$-bimodules.
- The 2-morphisms of $\mathscr{C}_A(\mathtt{i},\mathtt{j})$ are all natural transformations of the functors that form the 1-morphisms in $\mathscr{C}_A(\mathtt{i},\mathtt{j})$.

This is indeed a locally finitary 2-category by a similar argument to that found in [MM11] Section 7.3, and indeed is actually a locally weakly fiat 2-category (again, by a similar argument to that found in [MM16c] Section 5.1).

Take A as before. Let $Z_\mathtt{i}$ denote the centre of $A_\mathtt{i}$. We can identify $Z_\mathtt{i}$ with $\mathrm{End}_{\mathscr{C}_A}(\mathbb{1}_\mathtt{i})$, and we denote by $Z'_\mathtt{i}$ the subalgebra of $Z_\mathtt{i}$ that is generated by $\mathrm{id}_{\mathbb{1}_\mathtt{i}}$ and all elements of $Z_\mathtt{i}$ which factor through 1-morphisms given by tensoring with projective $(A_\mathtt{i}\text{-}A_\mathtt{i})$-bimodules.

We now choose subalgebras $X_\mathtt{i}$ of $Z_\mathtt{i}$ containing $Z'_\mathtt{i}$, and let $X = (X_1, \dots)$. We can then define a sub-2-category $\mathscr{C}_{A,X}$ of $\mathscr{C}_A$ which has the same objects and 1-morphisms, and the same 2-morphisms except that $\mathrm{End}_{\mathscr{C}_{A,X}}(\mathbb{1}_\mathtt{i}) = X_\mathtt{i}$. We present the generalisation of Lemma 12 in [MM16a]:

Lemma 3.1.6. *$\mathscr{C}_{A,X}$ is well-defined and locally weakly fiat.*

Proof. We mirror the proof of [MM16a] Lemma 12, with extra detail to clarify for our situation. To show that $\mathscr{C}_{A,X}$ is well-defined, we need to show that it is closed under horizontal and vertical composition. First, $\mathrm{End}_{\mathscr{C}_{A,X}}(\mathbb{1}_\mathtt{i}) = X_\mathtt{i}$ is a $\Bbbk$-algebra, from which it follows immediately that $\mathscr{C}_{A,X}$ is closed under vertical composition. For horizontal composition, since we already have that $\mathscr{C}_A$ is well-defined, we only need to check morphisms involving the $\mathbb{1}_\mathtt{i}$. If we have a composition $F \circ G : \mathtt{i} \to \mathtt{i}$ with F and G indecomposable, then $\mathbb{1}_\mathtt{i}$ is only a direct summand of $F \circ G$ if both F and G are isomorphic to $\mathbb{1}_\mathtt{i}$. Any horizontal composition of two 2-morphisms $\phi \circ_H \psi \in X_\mathtt{i}$ can be decomposed so that without loss of generality ϕ and ψ both act on indecomposable 1-morphisms. Then we must have by the previous reasoning that

ϕ and ψ and both members of $X_{\mathtt{i}}$. But then as $A_{\mathtt{i}} \cong A_{\mathtt{i}} \otimes_{A_{\mathtt{i}}} A_{\mathtt{i}}$, it follows from $X_{\mathtt{i}}$ being a subalgebra that it must also be closed under horizontal composition as required.

We already know that $\mathscr{C}_A$ is locally weakly fiat. As the only difference between $\mathscr{C}_A$ and $\mathscr{C}_{A,X}$ is the endomorphisms algebras of the $\mathbb{1}_{\mathtt{i}}$, which are 2-morphisms, $\mathscr{C}_{A,X}$ still contains F^* for any given F. However, we still need to justify that the adjunction 2-morphisms are contained in $\mathscr{C}_{A,X}$, i.e. that F and F^* remain adjuncts. The adjunction 2-morphisms have some $\mathbb{1}_{\mathtt{i}}$ as either source or target. The adjunction 2-morphism from $\mathbb{1}_{\mathtt{i}}$ to $\mathbb{1}_{\mathtt{i}}$ for some $\mathtt{i}$ is $\mathrm{id}_{\mathbb{1}_{\mathtt{i}}}$, which is in $\mathscr{C}_{A,X}$ by definition. Any other adjunction 2-morphism goes between $\mathbb{1}_{\mathtt{i}}$ and some 1-morphism that is a direct sum of indecomposables not including $\mathbb{1}_{\mathtt{i}}$, and is thus contained within $\mathscr{C}_{A,X}$ by definition, giving the result. □

There are times where we wish to consider the situation where the $A_{\mathtt{i}}$ are not necessarily basic. Given a non-basic algebra $A_{\mathtt{i}}$, let $\{e^1_{\mathtt{i}1}, \ldots, e^{n_1}_{\mathtt{i}1}, e^1_{\mathtt{i}2}, \ldots e^{n_m}_{\mathtt{i}m}\}$ be a complete set of primitive idempotents of $A_{\mathtt{i}}$ such that $A_{\mathtt{i}}e^k_{\mathtt{i}j} \cong A_{\mathtt{i}}e^p_{\mathtt{i}l}$ if and only if $j = l$, and otherwise $e^k_{\mathtt{i}j}e^p_{\mathtt{i}l} = 0$. Letting $e^b = e^1_{\mathtt{i}1} + e^1_{\mathtt{i}2} + \cdots + e^1_{\mathtt{i}m}$, we define the *basic algebra* $A^b_{\mathtt{i}}$ associated to $A_{\mathtt{i}}$ as $A^b_{\mathtt{i}} = e^b A_{\mathtt{i}} e^b$ (see [ASS06] Section I.6 for further discussion). Note that if $A_{\mathtt{i}}$ is basic, then $A^b_{\mathtt{i}} = A_{\mathtt{i}}$.

Given a countable collection $A = \{A_{\mathtt{i}}\}_{\mathtt{i}\in I}$ of not-necessarily basic self-injective connected finite dimensional $\Bbbk$-algebras, we define $A^b = \{A^b_{\mathtt{i}}\}_{\mathtt{i}\in I}$ and define $\mathscr{C}_A = \mathscr{C}_{A^b}$, the latter as defined previously.

3.2 Cells, Ideals and Multisemigroups

The definitions for (2-)ideals are mutatis mutandis those for finitary 2-categories, as they are defined in an entirely local manner. However, things are more complicated when it comes to cells, and require some care compared to the finitary setup. To explain in detail, we need to first define the Green's relations for multisemigroups (see

Definition 2.2.13 for a defiition of a multisemigroup). These are originally defined for multisemigroups in [KM12], based on the original definition for semigroups in [Gre51]. To recap, if $(S, *)$ is a multisemigroup with $x \in S$, then the *principal left ideal* of x is the set $L_x = Sx \cup \{x\}$, the *right principal ideal* is $R_x = xS \cup \{x\}$, and the *two-sided principal ideal* is $J_x = SxS \cup Sx \cup xS \cup \{x\}$. This gives rise to equivalence relations $\mathscr{L}$, $\mathscr{R}$ and $\mathscr{J}$ where e.g. $x \sim_{\mathscr{L}} y$ if $L_x = L_y$. For the multisemigroup $S(\mathscr{C})$ of a locally finitary 2-category, this is precisely the same definition as for $\mathscr{L}$-, $\mathscr{R}$- and $\mathscr{J}$-cells given for finitary 2-categories.

However, there are two additional Green's relations. One is $\mathscr{H} = \mathscr{L} \cap \mathscr{R}$, and the other is $\mathscr{D}$, defined as the meet of $\mathscr{L}$ and $\mathscr{R}$ in the poset of equivalence relations - that is, it is the smallest equivalence relation to contain both $\mathscr{L}$ and $\mathscr{R}$. It is not always true that $\mathscr{D} = \mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L}$. The best we can say is the following:

Proposition 3.2.1. $\mathscr{D} = \bigcup_{i \in \mathbb{N}} (\mathscr{L} \circ \mathscr{R})^{\circ i}$.

Proof. Since $\mathscr{L}, \mathscr{R} \subseteq \mathscr{L} \circ \mathscr{R}$, and since $\mathscr{D}$ is the meet of $\mathscr{L}$ and $\mathscr{R}$, it follows that $\mathscr{D} \subseteq \bigcup_{i \in \mathbb{N}} (\mathscr{L} \circ \mathscr{R})^{\circ i}$. However if $\mathscr{L}$ and $\mathscr{R}$ are contained in an equivalence relation M, then it follows from transitivity that $(\mathscr{L} \circ \mathscr{R})^{\circ i} \subseteq M$ for all i. Thus $\bigcup_{i \in \mathbb{N}} (\mathscr{L} \circ \mathscr{R})^{\circ i} \subseteq \mathscr{D}$, and the result follows. □

We cannot assume that $\mathscr{D} = \mathscr{J}$ for multisemigroups, and thus we need to apply care when considering a locally finitary 2-category. That said, we are able to recover useful results, as follows:

For the rest of the subsection, let $\mathscr{C}$ be a locally finitary 2-category with multisemigroup of indecomposables $S(\mathscr{C})$, on which we have Green's relations $\mathscr{L}_{\mathscr{C}}$, $\mathscr{R}_{\mathscr{C}}$, $\mathscr{J}_{\mathscr{C}}$, $\mathscr{D}_{\mathscr{C}}$ and $\mathscr{H}_{\mathscr{C}}$. Given a full finitary sub-2-category $\mathscr{B}$ of $\mathscr{C}$, we have that $S(\mathscr{B}) \subseteq S(\mathscr{C})$. Let $\mathscr{L}_{\mathscr{B}}$, $\mathscr{R}_{\mathscr{B}}$, $\mathscr{J}_{\mathscr{B}}$, $\mathscr{D}_{\mathscr{B}}$ and $\mathscr{H}_{\mathscr{B}}$ denote the Green's relations of $S(\mathscr{B})$, which we consider as equivalence relations on $S(\mathscr{B})$. We can extend these to equivalence relations on $S(\mathscr{C})$ by setting $\overline{\mathscr{X}_{\mathscr{B}}} = \mathscr{X}_{\mathscr{B}} \cup \Delta_{S(\mathscr{C})}$ for $\mathscr{X}$ one of the Green's relations and Δ the diagonal equivalence relation. We first show that all the Green's relations on $\mathscr{C}$ barring $\mathscr{H}_{\mathscr{C}}$ are determined by the Green's relations on the full finitary sub-2-categories.

Proposition 3.2.2. *Let χ denote the set of full finitary sub-2-categories of $\mathscr{C}$. Then $\bigcup_{\mathscr{B}\in\chi} \overline{\mathscr{L}_{\mathscr{B}}} = \mathscr{L}_{\mathscr{C}}$. A similar result holds for $\mathscr{R}_{\mathscr{C}}$ and $\mathscr{J}_{\mathscr{C}}$.*

Proof. All three results have similar proofs - we will give the full proof for $\mathscr{L}_{\mathscr{C}}$. When $(F,G) \in \bigcup_{\mathscr{B}\in\chi} \overline{\mathscr{L}_{\mathscr{B}}}$, either $(F,G) \in \Delta_{S(\mathscr{C})}$ and $F = G$ or there exists some $\mathscr{B} \in \chi$ such that $(F,G) \in \mathscr{L}_{\mathscr{B}}$. If $F = G$ then $(F,G) \in \mathscr{L}_{\mathscr{C}}$. If $(F,G) \in \mathscr{L}_{\mathscr{B}}$ then there exist $H, K \in S(\mathscr{B})$ such that F is a direct summand of HG and G is a direct summand of KF. But then it follows that $F \sim_{\mathscr{L}_{\mathscr{C}}} G$ and thus $(F,G) \in \mathscr{L}_{\mathscr{C}}$ and $\bigcup_{\mathscr{B}\in\chi} \overline{\mathscr{L}_{\mathscr{B}}} \subseteq \mathscr{L}_{\mathscr{C}}$.

Conversely, if $(F,G) \in \mathscr{L}_{\mathscr{C}}$, then there exist some $H, K \in S(\mathscr{C})$ such that F is a direct summand of HG and G is a direct summand of KF. Now let $\mathscr{B}$ be a full finitary sub-2-category of $\mathscr{C}$ that contains F, G, H and K. Thus $(F,G) \in \mathscr{L}_{\mathscr{B}}$, and hence $(F,G) \in \bigcup_{\mathscr{B}\in\chi} \overline{\mathscr{L}_{\mathscr{B}}}$. Therefore $\mathscr{L}_{\mathscr{C}} \subseteq \bigcup_{\mathscr{B}\in\chi} \overline{\mathscr{L}_{\mathscr{B}}}$, and $\mathscr{L}_{\mathscr{C}} = \bigcup_{\mathscr{B}\in\chi} \overline{\mathscr{L}_{\mathscr{B}}}$ as required. □

Proposition 3.2.3. *In the same setup as above, $\bigcup_{\mathscr{B}\in\chi} \overline{\mathscr{D}_{\mathscr{B}}} = \mathscr{D}_{\mathscr{C}}$.*

Proof. If $(F,G) \in \mathscr{L}_{\mathscr{C}}$ then as by Proposition 3.2.2 $\mathscr{L}_{\mathscr{C}} = \bigcup_{\mathscr{B}\in\chi} \overline{\mathscr{L}_{\mathscr{B}}}$, without loss of generality $(F,G) \in \mathscr{L}_{\mathscr{B}}$ for some $\mathscr{B} \in \chi$. By the definition of $\mathscr{D}_{\mathscr{B}}$, $(F,G) \in \mathscr{D}_{\mathscr{B}}$. Thus $(F,G) \in \bigcup_{\mathscr{B}\in\chi} \overline{\mathscr{D}_{\mathscr{B}}}$, and hence $\mathscr{L}_{\mathscr{C}} \subseteq \bigcup_{\mathscr{B}\in\chi} \overline{\mathscr{D}_{\mathscr{B}}}$. Similarly, $\mathscr{R}_{\mathscr{C}} \subseteq \bigcup_{\mathscr{B}\in\chi} \overline{\mathscr{D}_{\mathscr{B}}}$. But then as $\mathscr{D}_{\mathscr{C}}$ is the join of $\mathscr{L}_{\mathscr{C}}$ and $\mathscr{R}_{\mathscr{C}}$, we must have that $\mathscr{D}_{\mathscr{C}} \subseteq \bigcup_{\mathscr{B}\in\chi} \overline{\mathscr{D}_{\mathscr{B}}}$.

For the opposite inclusion, let $(F,G) \in \bigcup_{\mathscr{B}\in\chi} \overline{\mathscr{D}_{\mathscr{B}}}$. Then from Proposition 3.2.1 $\mathscr{D}_{\mathscr{B}} = \bigcup_{i\in\mathbb{N}} (\mathscr{L}_{\mathscr{B}} \circ \mathscr{R}_{\mathscr{B}})^{\circ i}$. Therefore there exists some $n \in 2\mathbb{N}$ and $H_i \in S(\mathscr{B})$ for $0 \leq i \leq n$ such that

$$F = H_0 \sim_{\mathscr{L}_{\mathscr{B}}} H_1 \sim_{\mathscr{R}_{\mathscr{B}}} H_2 \sim_{\mathscr{L}_{\mathscr{B}}} \cdots \sim_{\mathscr{L}_{\mathscr{B}}} H_{n-1} \sim_{\mathscr{R}_{\mathscr{B}}} H_n = G.$$

Then by Proposition 3.2.2,

$$(H_i, H_{i+1}) \in \mathscr{L}_{\mathscr{B}} \Rightarrow (H_i, H_{i+1}) \in \mathscr{L}_{\mathscr{C}},$$

and

$$(H_i, H_{i+1}) \in \mathscr{R}_{\mathscr{B}} \Rightarrow (H_i, H_{i+1}) \in \mathscr{R}_{\mathscr{C}}.$$

Therefore $(H_i, H_{i+1}) \in \mathscr{D}_{\mathscr{C}}$ for all i. But as $\mathscr{D}_{\mathscr{C}}$ is an equivalence relation it follows that $(F, G) \in \mathscr{D}_{\mathscr{C}}$. Thus $\bigcup_{\mathscr{B} \in \chi} \overline{\mathscr{D}_{\mathscr{B}}} \subseteq \mathscr{D}_{\mathscr{C}}$ and the result follows. □

We can also give a useful result for sufficiently nice $\mathscr{J}$-cells of $\mathscr{C}$:

Theorem 3.2.4. *Let $\mathcal{J}$ be a $\mathscr{J}$-cell of $\mathscr{C}$ such that every $\mathscr{H}$-cell of $\mathcal{J}$ is non-empty. Let $\mathscr{L}_{\mathcal{J}}$, $\mathscr{R}_{\mathcal{J}}$, $\mathscr{D}_{\mathcal{J}}$ and $\mathscr{J}_{\mathcal{J}}$ denote the restrictions of the Green's relations of $\mathscr{C}$ to $\mathcal{J}$. Then $\mathscr{L}_{\mathcal{J}} \circ \mathscr{R}_{\mathcal{J}} = \mathscr{R}_{\mathcal{J}} \circ \mathscr{L}_{\mathcal{J}} = \mathscr{D}_{\mathcal{J}} = \mathscr{J}_{\mathcal{J}}$.*

Proof. The proof of [MM11] Proposition 28 b) is local and generalises immediately, proving that $\mathscr{L}_{\mathcal{J}} \circ \mathscr{R}_{\mathcal{J}} = \mathscr{R}_{\mathcal{J}} \circ \mathscr{L}_{\mathcal{J}} = \mathscr{J}_{\mathcal{J}}$. Finally, it is immediate from the definitions that $\mathscr{L}_{\mathcal{J}} \circ \mathscr{R}_{\mathcal{J}} \subseteq \mathscr{D}_{\mathcal{J}} \subseteq \mathscr{J}_{\mathcal{J}}$, and thus the remaining equality follows directly from the prior equalities. □

From this result, we can define strongly regular $\mathscr{J}$-cells in the same fashion as the finitary case:

Definition 3.2.5. A $\mathscr{J}$-cell $\mathcal{J}$ of $\mathscr{C}$ is *strongly regular* if any two $\mathscr{L}$-cells of $\mathcal{J}$ are not comparable under $\leq_{\mathscr{L}}$, and each $\mathscr{H}$-cell of $\mathcal{J}$ contains precisely one isomorphism class. If only the first condition holds, then $\mathcal{J}$ is called *regular*.

We will show later that being (strongly) regular in the locally fiat setup will give a much more pleasant structure to the Green's relations than the general case, but this will need more structure.

We also define transitive 2-representations identically to the finitary case. Explicitly, a finitary 2-representation $\mathbf{M}$ of $\mathscr{C}$ is *transitive* if for any object $N \in \mathscr{M}$, $\mathbf{G}_{\mathbf{M}}(N)$ is equivalent to $\mathbf{M}$. This leads to the first relevant result for locally finitary 2-categories.

Lemma 3.2.6. *Let $\mathbf{M}$ be a transitive 2-representation of $\mathscr{C}$. There exists a unique maximal ideal $\mathscr{I}$ of $\mathbf{M}$ such that $\mathscr{I}$ does not contain any identity morphisms apart from the zero object.*

Proof. We mirror the proof for the finitary 2-category case given for [MM16c] Lemmas 3 and 4. Since $\mathscr{M} = \coprod_{\mathtt{i}\in\mathscr{C}} \mathbf{M}(\mathtt{i})$ is a coproduct of additive categories, $\mathscr{I}$ is uniquely determined by its morphisms between indecomposable objects. In particular, if $\mathrm{id}_Y \in \mathscr{I}$ for some Y and some ideal $\mathscr{I}$, then we can pre- and post-compose this with injection and surjection morphisms to find that $\mathrm{id}_X \in \mathscr{I}$ for each indecomposable summand X of Y. If $X \in \mathscr{M}$ is indecomposable, then we know that $\mathrm{End}_{\mathscr{M}} X$ is a local algebra. Further, by definition, $\mathrm{id}_X \notin \mathscr{I}$, and therefore $\mathscr{I} \cap \mathrm{End}_{\mathscr{M}} X$ is a proper ideal of $\mathrm{End}_{\mathscr{M}} X$, and is thus contained in its radical. Further, the sum of any two subspaces of a radical is still contained in the radical. Thus the sum of all ideals of $\mathscr{M}$ which do not contain id_X for any (indecomposable) X still has this property. Since we have only countably many isomorphism classes of indecomposable objects in $\mathscr{M}$, this argument still holds when we consider ideals that do not contain id_X for any indecomposable X (and hence do not contain any identity morphisms for non-zero objects), and the result follows. □

Definition 3.2.7. A transitive 2-representation $\mathbf{M}$ is *simple transitive* if the maximal ideal of $\mathbf{M}$ given in Lemma 3.2.6 is the zero ideal.

The definition and notation of a cell 2-representation is mutatis mutandis that found in Definition 2.3.33.

3.2.1 Example: $\mathscr{C}_{A,X}$

We now examine the cell structure and cell 2-representations of $\mathscr{C}_A$ and $\mathscr{C}_{A,X}$ in detail. Up to isomorphism, the non-identity indecomposable 1-morphisms of $\mathscr{C}_A$ or $\mathscr{C}_{A,X}$ are of the form

$$A_{\mathtt{j}} e_{\mathtt{j}l} \otimes_{\Bbbk} e_{\mathtt{i}k} A_{\mathtt{i}} \otimes_{A_{\mathtt{i}}} -$$

where $\mathtt{i}, \mathtt{j} \in I$ and $e_{\mathtt{i}1}, \ldots, e_{\mathtt{i}n_{\mathtt{i}}}$ (respectively $e_{\mathtt{j}1}, \ldots, e_{\mathtt{j}n_{\mathtt{j}}}$) are a complete set of primitive orthogonal idempotents of $A_{\mathtt{i}}$ (respectively $A_{\mathtt{j}}$). For notational compactness, we define

$$F_{\mathtt{j}l}^{\mathtt{i}k} = A_{\mathtt{j}} e_{\mathtt{j}l} \otimes_{\Bbbk} e_{\mathtt{i}k} A_{\mathtt{i}} \otimes_{A_{\mathtt{i}}} -$$

in $\mathscr{C}_A$ or $\mathscr{C}_{A,X}$. We also denote $A_{\mathtt{j}l}^{\mathtt{i}k} = A_{\mathtt{j}} e_{\mathtt{j}l} \otimes_{\Bbbk} e_{\mathtt{i}k} A_{\mathtt{i}}$ for the corresponding bimodule.

Excluding the trivial $\mathscr{J}$-cells, there is a single $\mathscr{J}$-cell of $\mathscr{C}_{A,X}$ consisting of all the $A_{\mathtt{i}} e_{\mathtt{i}k} \otimes_{\Bbbk} e_{\mathtt{j}l} A_{\mathtt{j}}$. The $\mathscr{L}$-cells are of the form

$$\mathscr{L}_{\mathtt{i}k} = \{F_{\mathtt{j}l}^{\mathtt{i}k} | \mathtt{j} \in I, l = 1, \dots, n_{\mathtt{j}}\}$$

and the $\mathscr{R}$-cells are of the form

$$\mathscr{R}_{\mathtt{j}l} = \{F_{\mathtt{j}l}^{\mathtt{i}k} | \mathtt{i} \in I, k = 1, \dots, n_{\mathtt{j}}\}.$$

Let $\mathscr{L}_{\mathtt{i}k}$ be an $\mathscr{L}$-cell as defined above. The corresponding 2-representation $\mathbb{N}_{\mathtt{i}k} = \mathbb{N}_{\mathscr{L}_{\mathtt{i}k}}$ has as indecomposable objects $F_{\mathtt{j}l}^{\mathtt{i}k} \in \mathbf{N}_{\mathtt{i}k}(\mathtt{j})$ for $l = 1, \dots n_{\mathtt{j}}$. A bimodule homomorphism $\sigma : A_{\mathtt{j}l}^{\mathtt{i}k} \to A_{\mathtt{j}m}^{\mathtt{i}k}$ is defined by its image on $e_{\mathtt{j}l} \otimes e_{\mathtt{i}k}$, and is thus a $\Bbbk$-linear combination of homomorphisms of the form $\varphi_{a,b} : A_{\mathtt{j}l}^{\mathtt{i}k} \to A_{\mathtt{j}m}^{\mathtt{i}k}$, where $\varphi_{a,b}(e_{\mathtt{j}l} \otimes e_{\mathtt{i}k}) = a \otimes b$ for $a \in e_{\mathtt{j}l} A_{\mathtt{j}} e_{\mathtt{j}m}$ and $b \in e_{\mathtt{i}k} A_{\mathtt{i}} e_{\mathtt{i}k}$. Additionally, $\mathrm{id}_{F_{\mathtt{j}l}^{\mathtt{i}m}} = \varphi_{e_{\mathtt{i}m}, e_{\mathtt{j}l}}$ (abusing notation to let $\varphi_{a,b}$ refer both to the bimodule homomorphism and the corresponding 2-morphism in $\mathscr{C}_{A,X}$).

Proposition 3.2.8. *Let $\mathscr{I}$ be the unique maximal ideal of $\mathbf{N}_{\mathtt{i}k}$ not containing any identity morphisms for non-zero objects (so that $\mathbf{N}_{\mathtt{i}k}/\mathscr{I}$ is a cell 2-representation). Then its components $\mathscr{I}(\mathtt{j}) \subseteq \mathbf{N}_{\mathtt{i}k}(\mathtt{j})$ are matrices of $\Bbbk$-linear combinations of morphisms of the form $\varphi_{a,b}$ with $a \in A_{\mathtt{j}}$ and $b \in R = \operatorname{rad} e_{\mathtt{i}k} A_{\mathtt{i}} e_{\mathtt{i}k}$.*

Proof. Since the $\mathbf{N}_{\mathtt{i}k}(\mathtt{j})$ are additive categories, composing elements of $\coprod_{\mathtt{j}} \mathscr{I}(\mathtt{j})$ with biproduct injections and projections it is sufficient to consider the elements of $\coprod_{\mathtt{j}} \mathscr{I}(\mathtt{j})$ that are morphisms between indecomposable objects. We refer to this process as an injection-projection argument. First, given any $a, \alpha, \gamma \in A_{\mathtt{j}}$, $\beta, \delta \in A_{\mathtt{i}}$ and $b, b' \in \operatorname{rad} e_{\mathtt{i}k} A_{\mathtt{i}} e_{\mathtt{i}k}$, $\varphi_{\alpha,\beta} \varphi_{a,b} \varphi_{\gamma,\delta} = \varphi_{\gamma a \alpha, \beta b \delta}$ and $\varphi_{a,b} + \varphi_{a,b'} = \varphi_{a,b+b'}$. Since

R is a two-sided ideal of $e_{\mathtt{i}k}A_{\mathtt{i}}e_{\mathtt{i}k}$, this implies that $\mathscr{J}$ is indeed an ideal of $\mathscr{N}_{\mathtt{i}k}$. Further, if $\mathrm{id}_{F^{\mathtt{i}k}_{\mathtt{j}l}} \in \mathscr{J}(\mathtt{j})$ then $e_{\mathtt{i}k} \in R$, which is a contradiction as R is a proper ideal of $e_{\mathtt{i}k}A_{\mathtt{i}}e_{\mathtt{i}k}$. Hence $\mathscr{J}$ does not contain id_X for non-zero $X \in \mathscr{N}_{\mathtt{i}k}$.

To show that $\mathscr{J}$ is $\mathscr{C}_{A,X}$-stable, it is again sufficient by an injection-projection argument to consider the action of indecomposable 1-morphisms of $\mathscr{C}_{A,X}$. Stability under the action of any $\mathbb{1}_{\mathtt{j}}$ is clear by definition. Let $F^{\mathtt{j}m}_{\mathtt{1}n} \in \mathscr{C}_{A,X}(\mathtt{j},\mathtt{1})$ and $\varphi_{a,b} : F^{\mathtt{i}k}_{\mathtt{j}s} \to F^{\mathtt{i}k}_{\mathtt{j}t} \in \mathscr{J}(\mathtt{j})$. Then

$$\mathbf{N}_{\mathtt{i}k}(F^{\mathtt{j}m}_{\mathtt{1}n})(\varphi_{a,b}) = \varphi_{e_{\mathtt{1}n},e_{\mathtt{j}m}} \otimes \varphi_{a,b} : A^{\mathtt{j}m}_{\mathtt{1}n} \otimes_{A_{\mathtt{j}}} A^{\mathtt{i}k}_{\mathtt{j}s} \to A^{\mathtt{j}m}_{\mathtt{1}n} \otimes_{A_{\mathtt{j}}} A^{\mathtt{i}k}_{\mathtt{j}t}.$$

Under the isomorphism

$$A^{\mathtt{j}m}_{\mathtt{1}n} \otimes_{A_{\mathtt{j}}} A^{\mathtt{i}k}_{\mathtt{j}t} \cong A_{\mathtt{1}n} \otimes_{\Bbbk} e_{\mathtt{j}m}A_{\mathtt{j}}e_{\mathtt{j}t} \otimes_{\Bbbk} e_{\mathtt{i}k}A_{\mathtt{i}} \cong (A^{\mathtt{i}k}_{\mathtt{1}n})^{\oplus \dim e_{\mathtt{j}m}A_{\mathtt{j}}e_{\mathtt{j}t}},$$

the element

$$(\varphi_{e_{\mathtt{1}n},e_{\mathtt{j}m}} \otimes \varphi_{a,b})(e_{\mathtt{1}n} \otimes e_{\mathtt{j}m} \otimes e_{\mathtt{j}s} \otimes e_{\mathtt{i}k}) = e_{\mathtt{1}n} \otimes e_{\mathtt{j}m} \otimes a \otimes b$$

maps first to $e_{\mathtt{1}n} \otimes e_{\mathtt{j}m}a \otimes b$ and then to $\bigoplus_{x=1}^{\dim e_{\mathtt{j}m}A_{\mathtt{j}}e_{\mathtt{j}t}} v_x e_{\mathtt{1}n} \otimes b$, for some $v_x \in \Bbbk$. Since $b \in R$, this implies that $\mathbf{N}_{\mathtt{i}k}(F^{\mathtt{j}m}_{\mathtt{1}n})(\varphi_{a,b}) \in \mathscr{J}(\mathtt{1})$, giving $\mathscr{C}_{A,X}$-stability.

It remains to show that $\mathscr{J}$ is the unique maximal such ideal. If it is maximal, then by Lemma 3.2.6 it is immediate that it is unique. Thus assume for contradiction that there exists some other ideal $\mathscr{K} \supset \mathscr{J}$ such that $\mathscr{K}$ does not contain id_F for any non-zero F. Choose some $\sigma \in \mathscr{K} \setminus \mathscr{J}$. By injection-projection arguments we may assume that σ is a morphism from $F^{\mathtt{i}k}_{\mathtt{j}l}$ and $F^{\mathtt{i}k}_{\mathtt{j}m}$. Thus $\sigma = \sum_{v=1}^{t} \varphi_{a_v,b_v}$ for some t with $a_v \in e_{\mathtt{j}m}A_{\mathtt{j}}e_{\mathtt{j}l}$ and $b_v \in e_{\mathtt{i}k}A_{\mathtt{i}}e_{\mathtt{i}k}$.

If $b_v \in R$ for some v, then by definition $\varphi_{a_v,b_v} \in \mathscr{J} \subset \mathscr{K}$, and thus without loss of generality $b_v \notin R$ for all v. But then by [ASS06] Lemma I.4.6, $e_{\mathtt{i}k} - b_v \in R$ for all

v, and hence

$$\sigma + \sum_{v=1}^{t} \varphi_{a_v, e_{\mathtt{i}k} - b_v} = \sum_{v=1}^{t} \varphi_{a_v, e_{\mathtt{i}k}} = \varphi_{\sum_v a_v, e_{\mathtt{i}k}} \in \mathscr{K}.$$

But since $\mathscr{K}$ is $\mathscr{C}_{A,X}$-stable, by tensoring $\varphi_{\sum a_v, e_{\mathtt{i}k}}$ with $\varphi_{e_{\mathtt{j}m}, e_{tt\mathtt{i}k}}$ similarly to above and composing with injection and projection morphisms, we derive that $z\varphi_{e_{\mathtt{j}m}, e_{\mathtt{i}k}} \in \mathscr{K}$ for some non-zero $z \in \Bbbk$. But this implies $\mathrm{id}_{F_{\mathtt{j}m}^{\mathtt{i}k}} \in \mathscr{K}$, a contradiction. Thus $\mathscr{J}$ is indeed maximal, and the result follows.

□

3.3 Coalgebra and Comodule 1-Morphisms for Locally Finitary 2-Categories

We now present the generalisation of a collection of constructions and results in [MMMT16] to the locally finitary situation. Most of the proofs are straightforward generalisations, but Lemma 3.3.8 and Theorem 3.3.9 are novel proofs, for reasons explained later in this section. We initially take $\mathscr{C}$ to be a locally finitary 2-category. The definition(s) of abelianisation given in Section 2.4 are sufficiently general to already cover the locally finitary case. Indeed, since each hom-category $\mathscr{C}(\mathtt{i},\mathtt{j})$ of $\mathscr{C}$ is a finitary category, its abelianisation $\underline{\mathscr{C}}(\mathtt{i},\mathtt{j})$ is indeed abelian.

Given any $S \in \mathbf{M}(\mathtt{i})$, we define an evaluation functor $\mathrm{ev}_S : \coprod_{\mathtt{j}\in\mathscr{C}} \underline{\mathscr{C}}(\mathtt{i},\mathtt{j}) \to \coprod_{\mathtt{j}\in\mathscr{C}} \underline{\mathbf{M}}(\mathtt{j})$ which takes F to FS and $\alpha : F \to G$ to $\alpha_S : FS \to GS$. Since ev_S maps each $F \in \underline{\mathscr{C}}(\mathtt{i},\mathtt{j})$ to an object in $\underline{\mathbf{M}}(\mathtt{j})$, it follows that this functor has a left adjoint if and only if each of the component evaluation functors $\mathrm{ev}_{S,\mathtt{j}} : \underline{\mathscr{C}}(\mathtt{i},\mathtt{j}) \to \underline{\mathbf{M}}(\mathtt{j})$ does so. But the latter case is the finitary case, where such adjoints exist by e.g. [MMMT16] Section 4.1, and thus there exists some left adjoint $[S,-] : \coprod_{\mathtt{j}\in\mathscr{C}} \underline{\mathbf{M}}(\mathtt{j}) \to \coprod_{\mathtt{j}\in\mathscr{C}} \underline{\mathscr{C}}(\mathtt{i},\mathtt{j})$ (we choose this notation to be suggestive of an internal (co)hom). We then have the following generalisation of [MMMT16] Lemma 5.

Lemma 3.3.1. *With the above notation, $[S,S]$ has the structure of a coalgebra*

1-morphism in $\underline{\mathscr{C}}(\mathtt{i},\mathtt{i})$.

Proof. We mirror the proof of [MMMT16] Lemma 5. The image of $\mathrm{id}_{[S,S]}$ under the adjunction isomorphism $\mathrm{Hom}_{\underline{\mathscr{C}}}([S,S],[S,S]) \to \mathrm{Hom}_{\underline{\mathbf{M}}}(S,[S,S]S)$ gives a non-zero morphism $\mathrm{coev}_S : S \to [S,S]S$. This gives the composition $S \xrightarrow{\mathrm{coev}_S} [S,S]S \xrightarrow{[S,S]\,\mathrm{coev}_S} [S,S][S,S]S$ from S to $[S,S][S,S]S$. But then as

$$\mathrm{Hom}_{\underline{\mathbf{M}}}(S,[S,S][S,S]S) \cong \mathrm{Hom}_{\underline{\mathscr{C}}}([S,S],[S,S][S,S]),$$

again by the adjunction isomorphism, this gives us a non-zero comultiplication 2-morphism $[S,S] \to [S,S][S,S]$.

For the counit morphism, we again use the adjunction isomorphism to derive that $\mathrm{Hom}_{\underline{\mathbf{M}}}(S,S) = \mathrm{Hom}_{\underline{\mathbf{M}}}(S,\mathbb{1}_{\mathtt{i}}S) \cong \mathrm{Hom}_{\underline{\mathscr{C}}}([S,S],\mathbb{1}_{\mathtt{i}})$, and thus choose the (non-zero) image of id_S under this isomorphism. We denote the comultiplication 2-morphism by Δ_S and the counit 2-morphism by ϵ_S. We will now show that the axioms for a coalgebra hold.

We first wish to show that the morphism $(\epsilon_S \circ_H \mathrm{id}_{[S,S]}) \circ_V \Delta_S = \mathrm{id}_{[S,S]}$. We note that

$$\mathrm{Hom}_{\underline{\mathscr{C}}}([S,S],[S,S]) \supseteq \mathrm{Hom}_{\underline{\mathscr{C}}}([S,S][S,S],[S,S]) \circ_V \mathrm{Hom}_{\underline{\mathscr{C}}}([S,S],[S,S][S,S]),$$

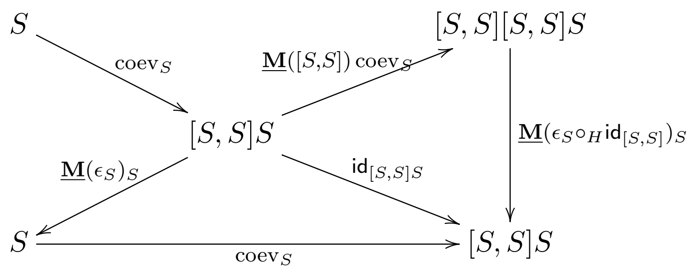
and using the adjunction between ev_S and $[S,-]$ and the representation isomorphism on the second hom set $(\epsilon_S \circ_H \mathrm{id}_{[S,S]}) \circ_V \Delta_S$ corresponds to

$$\underline{\mathbf{M}}(\epsilon_S \circ_H \mathrm{id}_{[S,S]})_S \circ (\underline{\mathbf{M}}([S,S])\,\mathrm{coev}_S \circ \mathrm{coev}_S).$$

Since $\mathrm{id}_{[S,S]}$ corresponds under the isomorphism to coev_S, it is sufficient to show that

$$\underline{\mathbf{M}}(\epsilon_S \circ_H \mathrm{id}_{[S,S]})_S \circ (\underline{\mathbf{M}}([S,S])\,\mathrm{coev}_S) \circ \mathrm{coev}_S = \mathrm{coev}_S\,.$$

Diagrammatically, we wish to show that the upper right half of the diagram



commutes. But $\underline{\mathbf{M}}(\epsilon_S \circ_H \mathsf{id}_{[S,S]}) = \underline{\mathbf{M}}(\epsilon_S)_{[S,S]S}$, and by the definition of a natural transformation

$$\underline{\mathbf{M}}(\epsilon_S)_{[S,S]S} \circ \underline{\mathbf{M}}([S,S]) \operatorname{coev}_S = \operatorname{coev}_S \circ \underline{\mathbf{M}}(\epsilon_S)_S.$$

We thus wish to show that

$$\operatorname{coev}_S \circ \underline{\mathbf{M}}(\epsilon_S)_S \circ \operatorname{coev}_S = \operatorname{coev}_S,$$

which is the lower left half of the above diagram. But as $\operatorname{coev}_S \in \operatorname{Hom}_{\underline{\mathbf{M}}}(S, [S,S]S)$, we can consider $\underline{\mathbf{M}}(\epsilon_S)_S$ to be the action of ϵ_S on the functor $\operatorname{Hom}_{\underline{\mathbf{M}}}(S, [S,S]-)$ in a similar fashion to above. Thus using the isomorphism of functors as before,

$$\underline{\mathbf{M}}(\epsilon_S)_S \circ \operatorname{coev}_S \mapsto \epsilon_S \circ_V \mathsf{id}_{[S,S]} = \epsilon_S,$$

and hence $\underline{\mathbf{M}}(\epsilon_S)_S \circ \operatorname{coev}_S = \mathsf{id}_S$, giving the required result.

For the other half of the counit axiom, $\underline{\mathbf{M}}(\mathsf{id}_{[S,S]} \circ_H \epsilon_S)_S = \underline{\mathbf{M}}([S,S])\underline{\mathbf{M}}(\epsilon_S)_S$. Therefore

$$\begin{aligned}
\underline{\mathbf{M}}(\mathsf{id}_{[S,S]} \circ_H \epsilon_S)_S \circ \underline{\mathbf{M}}([S,S]) \operatorname{coev}_S \circ \operatorname{coev}_S \\
= \underline{\mathbf{M}}([S,S])(\underline{\mathbf{M}}(\epsilon_S)_S \circ \operatorname{coev}_S) \circ \operatorname{coev}_S \\
= \underline{\mathbf{M}}([S,S])(\mathsf{id}_S) \circ \operatorname{coev}_S \\
= \operatorname{coev}_S
\end{aligned}$$

which gives us the desired result, with the middle equality following from the above paragraph.

We also need to show that

$$(\mathsf{id}_{[S,S]} \circ_H \Delta_S) \circ_V \Delta_S \cong (\Delta_S \circ_H \mathsf{id}_{[S,S]}) \circ_V \Delta_S.$$

Using a similar method to above, this is equivalent to showing

$$\begin{aligned} &\underline{\mathbf{M}}(\mathsf{id}_{[S,S]} \circ_H \Delta_S)_S \circ (\underline{\mathbf{M}}([S,S]) \operatorname{coev}_S) \circ \operatorname{coev}_S \\ = &\underline{\mathbf{M}}(\Delta_S \circ_H \mathsf{id}_{[S,S]})_S \circ (\underline{\mathbf{M}}([S,S]) \operatorname{coev}_S) \circ \operatorname{coev}_S . \end{aligned}$$

Diagramatically, this means showing that the diagram

$$\begin{array}{ccccc} S & \xrightarrow{\operatorname{coev}_S} & [S,S]S & \xrightarrow{\underline{\mathbf{M}}([S,S]) \operatorname{coev}_S} & [S,S][S,S]S \\ \downarrow{\scriptstyle \operatorname{coev}_S} & & & & \\ [S,S]S & & & & \downarrow{\scriptstyle \underline{\mathbf{M}}(\Delta_S \circ_H \mathsf{id}_{[S,S]})_S} \\ \downarrow{\scriptstyle \underline{\mathbf{M}}([S,S]) \operatorname{coev}_S} & & & & \\ [S,S][S,S]S & & \xrightarrow[\underline{\mathbf{M}}(\mathsf{id}_{[S,S]} \circ_H \Delta_S)_S]{} & & [S,S][S,S][S,S]S \end{array}$$

is commutative.

Using the principles outlined above,

$$\begin{aligned} &\underline{\mathbf{M}}(\Delta_S \circ_H \mathsf{id}_{[S,S]})_S \circ \underline{\mathbf{M}}([S,S]) \operatorname{coev}_S \circ \operatorname{coev}_{S,S} \\ =&\underline{\mathbf{M}}(\Delta_S)_{[S,S]S} \circ \underline{\mathbf{M}}([S,S]) \operatorname{coev}_S \circ \operatorname{coev}_S \\ =&\underline{\mathbf{M}}([S,S][S,S]) \operatorname{coev}_S \circ \underline{\mathbf{M}}(\Delta_S)_S \circ \operatorname{coev}_S, \end{aligned}$$

while

$$\underline{\mathbf{M}}(\mathsf{id}_{[S,S]} \circ_H \Delta_S)_S \circ \underline{\mathbf{M}}([S,S]) \operatorname{coev}_S \circ \operatorname{coev}_S = \underline{\mathbf{M}}([S,S])(\underline{\mathbf{M}}(\Delta_S)_S \circ \operatorname{coev}_S) \circ \operatorname{coev}_S .$$

But applying the isomorphism of 2-representations given above,

$$\underline{\mathbf{M}}(\Delta_S)_S \circ \operatorname{coev}_S \mapsto \Delta_S \circ_V \mathsf{id}_{[S,S]} = \Delta_S \mapsto \underline{\mathbf{M}}([S,S]) \operatorname{coev}_S \circ \operatorname{coev}_S .$$

Therefore

$$\begin{aligned}
&\underline{\mathbf{M}}([S,S][S,S])\,\mathrm{coev}_S \circ \underline{\mathbf{M}}(\Delta_S)_S \circ \mathrm{coev}_S \\
=&\underline{\mathbf{M}}([S,S][S,S])\,\mathrm{coev}_S \circ \underline{\mathbf{M}}([S,S])\,\mathrm{coev}_S \circ \mathrm{coev}_S,
\end{aligned}$$

while

$$\begin{aligned}
&\underline{\mathbf{M}}([S,S])(\underline{\mathbf{M}}(\Delta_S)_S \circ \mathrm{coev}_S) \circ \mathrm{coev}_S \\
=&\underline{\mathbf{M}}([S,S])(\underline{\mathbf{M}}([S,S])\,\mathrm{coev}_S \circ \mathrm{coev}_S) \circ \mathrm{coev}_S \\
=&\underline{\mathbf{M}}([S,S])\,\mathrm{coev}_S \circ \underline{\mathbf{M}}([S,S])\,\mathrm{coev}_S \circ \mathrm{coev}_S
\end{aligned}$$

and the result follows. □

We define $\mathrm{comod}_{\underline{\mathscr{C}}}([S,S])$, $\textbf{comod}_{\underline{\mathscr{C}}}([S,S])$, $\mathrm{inj}_{\underline{\mathscr{C}}}([S,S])$ and $\textbf{inj}_{\underline{\mathscr{C}}}([S,S])$ as in Section 2.5. Each $[S,T]$ can be considered as an $[S,S]$-comodule 1-morphism in a canonical fashion. First define a map $\mathrm{coev}_{S,T} : T \to [S,T]S$ in $\underline{\mathscr{M}}$ by taking the image of $\mathrm{id}_{[S,T]}$ under the adjunction isomorphism

$$\mathrm{Hom}_{\underline{\mathscr{C}}}([S,T],[S,T]) \cong \mathrm{Hom}_{\underline{\mathscr{M}}}(T,[S,T]S).$$

This gives a morphism

$$\underline{\mathbf{M}}([S,T])(\mathrm{coev}_S) : [S,T]S \to [S,T][S,S]S,$$

and we take the image of the composition $\underline{\mathbf{M}}([S,T])(\mathrm{coev}_S) \circ \mathrm{coev}_{S,T}$ under the adjunction isomorphism

$$\mathrm{Hom}_{\underline{\mathscr{M}}}(T,[S,T][S,S]S) \cong \mathrm{Hom}_{\underline{\mathscr{C}}}([S,T],[S,T][S,S])$$

to be the canonical coaction 2-morphism. It is straightforward to check that the comodule axioms hold, and we denote this 2-morphism by $\rho_{[S,T]}$, or ρ_T when there is no confusion.

For the rest of the section we will assume that $\mathscr{C}$ is a locally weakly fiat 2-category unless otherwise noted. Using the previous construction, we can now define a functor $\Theta : \mathscr{M} \to \mathrm{comod}_{\underline{\mathscr{C}}}([S,S])$ by setting $\Theta(T) = ([S,T], \rho_T)$ and by $\Theta(f) = [S,f]$. While this is a functor between categories, we in fact have the following generalisation of [MMMT16] Lemma 4.4:

Lemma 3.3.2. *The functor Θ (weakly) commutes with the action of $\mathscr{C}$ and defines a morphism of 2-representations.*

Proof. The proof given in [MMMT16] proves that $[S, XT] \cong X[S,T]$ in $\underline{\mathscr{C}}$ for any 1-morphism X in $\mathscr{C}$ by referring to only hom-categories between three objects of $\mathscr{C}$. The proof is therefore entirely local and generalises immediately. □

We can also present generalisations of Lemmas 4.5 and 4.6 in [MMMT16]:

Lemma 3.3.3. *For any 1-morphism X in $\mathscr{C}$ and any $C \in \mathrm{comod}_{\underline{\mathscr{C}}}([S,S])$ there is an isomorphism $\mathrm{Hom}_{\mathrm{comod}_{\underline{\mathscr{C}}}([S,S])}(C, X[S,S]) \cong \mathrm{Hom}_{\underline{\mathscr{C}}}(C, X)$.*

Proof. The proof given in [MMMT16] is an entirely local proof, and thus generalises immediately. □

Lemma 3.3.4. *Θ factors over the inclusion $\mathrm{inj}_{\underline{\mathscr{C}}}([S,S]) \hookrightarrow \mathrm{comod}_{\underline{\mathscr{C}}}([S,S])$.*

Proof. We mirror the proof given for [MMMT16] Lemma 4.6, with some extra detail to clarify it for our situation. We first consider the case where $T = XS$ for some 1-morphism $X \in \mathscr{C}(\mathtt{i}, \mathtt{j})$. By Lemma 3.3.2, we have that $[S,T] = [S, XS] \cong X[S,S]$. By the definition of a comodule, $[S,S]$ is injective in $\mathrm{comod}_{\underline{\mathscr{C}}}([S,S])$. We claim that, because $\mathscr{C}$ is fiat, $X[S,S]$ is also injective. To see this, the existence of internal adjunctions in $\mathscr{C}$ gives that $\mathrm{Hom}_{\mathrm{comod}_{\underline{\mathscr{C}}}([S,S])}(-, X[S,S]) \cong \mathrm{Hom}_{\mathrm{comod}_{\underline{\mathscr{C}}}([S,S])}({}^*X-, [S,S])$, and as the latter is exact by the injectivity of $[S,S]$ and ${}^*X$ having both left and right adjoints, so is the former. But now as $\mathbf{M}$ is transitive, any T is isomorphic to a direct summand of some XS. The result follows. □

We can now give the generalisation of Theorem 4.7 in [MMMT16].

Theorem 3.3.5. *Let $\mathbf{M}$ be a transitive 2-representation of $\mathscr{C}$ and let $S \in \mathbf{M}(\mathtt{i})$ be non-zero. Letting Θ be the functor defined above, Θ induces an equivalence of 2-representations between $\underline{\mathbf{M}}$ and* **comod**$_{\underline{\mathscr{C}}}([S,S])$. *This restricts to an equivalence between $\mathbf{M}$ and* **inj**$_{\underline{\mathscr{C}}}([S,S])$.

Proof. The proof given in [MMMT16] generalises without issue to the locally finitary case. The references to [MMMT16] Lemmas 4.4, 4.5 and 4.6 in that proof are replaced by Lemma 3.3.2, Lemma 3.3.3 and Lemma 3.3.4 here. □

If we are working in a locally fiat 2-category, we can use the involution to get the dual result. For an algebra 1-morphism A in $\overline{\mathscr{C}}$, we can denote by $\mathrm{mod}_{\overline{\mathscr{C}}}(A)$ the category of right A-module 1-morphisms, and by $\mathrm{proj}_{\overline{\mathscr{C}}}(A)$ the full subcategory of projective A-module 1-morphisms, with **mod**$_{\overline{\mathscr{C}}}(A)$ and **proj**$_{\overline{\mathscr{C}}}(A)$ the respective 2-representations. Then the proof of Corollary 4.8 in [MMMT16] generalises immediately to the locally finitary case, and we get the following:

Corollary 3.3.6. *There exists a algebra 1-morphism $\|S,S\|$ in $\overline{\mathscr{C}}$ and an equivalence of 2-representations between $\overline{\mathbf{M}}$ and* **mod**$_{\overline{\mathscr{C}}}(\|S,S\|)$, *which restricts to an equivalence between $\mathbf{M}$ and* **proj**$_{\overline{\mathscr{C}}}(\|S,S\|)$.

Returning to the locally weakly fiat case, Corollary 4.10 in [MMMT16] is an explicitly local corollary, and thus also generalises immediately:

Corollary 3.3.7. *For $\mathtt{i} \in \mathscr{C}$, consider the endomorphism 2-category $\mathscr{A}_{\mathtt{i}}$ of $\mathtt{i}$ in $\mathscr{C}$. There is a bijection between the equivalence classes of simple transitive representations on $\mathscr{A}_{\mathtt{i}}$ and simple transitive representations in $\mathscr{C}$ which are non-zero at $\mathtt{i}$.*

We can use this to provide a simple proof of the general case of Theorem 15 in [MM16c]. Indeed, we need a new proof of this general case: the [MM16c] proof applies the Perron-Frobenius theorem to a matrix whose rows and columns are indexed by elements of $S(\mathscr{C})$. The Perron-Froberius theorem, however, only applies to finite square matrices, and in the locally finitary case $S(\mathscr{C})$ can have infinitely many elements.

Before the general proof, we will give a simple proof of the connected version of that Theorem. Even this proof has an advantage over that of the connected proof given in [MM16c] Theorem 15, in that it does not utilise the (fan) Freyd abelianisation. This allows it to be more easily generalised to settings where a different version of abelianisation is used (e.g. the setup in Chapter 5 below).

Lemma 3.3.8. *Let A be a connected basic self-injective finite dimensional $\Bbbk$-algebra. Then for every simple transitive 2-representation of $\mathscr{C}_A$ there is an equivalent cell 2-representation.*

Proof. We consider a larger fiat 2-category $\mathscr{C}_{A\times\Bbbk}$. This category has two objects $*$ and $*_\Bbbk$. We identify the former with a small category $\mathscr{A}$ equivalent to $A\text{-mod}$ and the latter with a small category equivalent to $\Bbbk\text{-mod}$. The category $\mathscr{C}_{A\times\Bbbk}(*,*)$ is taken to be $\mathscr{C}_A(*,*)$ and the category $\mathscr{C}_{A\times\Bbbk}(*_\Bbbk,*_\Bbbk)$ is taken to be $\mathscr{C}_\Bbbk(*_\Bbbk,*_\Bbbk)$. The 1-morphisms between $*$ and $*_\Bbbk$ (respectively $*_\Bbbk$ and $*$) are direct summands of direct sums of functors isomorphic to tensoring with projective ($\Bbbk$-A)-bimodules (respectively projective (A-$\Bbbk$)-bimodules). The 2-morphisms are natural transformations.

The endomorphism 2-category of $*$ is equivalent to $\mathscr{C}_A$. Further, if we have an indecomposable 1-morphism $Ae_i \otimes_\Bbbk e_jA \otimes_A -$ in $\mathscr{C}_{A\times\Bbbk}(*,*)$, then Ae_i is an (A-$\Bbbk$)-bimodule and e_jA is a ($\Bbbk$-A)-bimodule, and hence this factors over $*_\Bbbk$.

We denote by $\mathscr{C}_\Bbbk$ the endomorphism 2-category of $*_\Bbbk$, and claim that any simple transitive 2-representation of it is equivalent to the cell 2-representation. Let $\mathbf{M}$ be a simple transitive 2-representation of $\mathscr{C}_\Bbbk$. As the 1-morphisms in $\mathscr{C}_\Bbbk$ are all of the form $\mathbb{1}_{*_\Bbbk}^{\oplus m}$ for some $m \in \mathbb{Z}_0^+$, let $N \in \mathbf{M}(*_\Bbbk)$ be indecomposable. Then as $\mathbf{M}$ is transitive and $\mathbf{M}(*_\Bbbk)$ is idempotent complete, $M \cong \mathrm{id}^{\oplus n}(N) \cong N^{\oplus n}$ for any $M \in \mathbf{M}(*_\Bbbk)$ and for some $n \in \mathbb{Z}_0^+$. It follows from Proposition 3.2.8 that for the $\mathscr{L}$-cell $\mathscr{L}_\Bbbk = \{\mathbb{1}_{*_\Bbbk}\}$

of $\mathscr{C}_{\Bbbk}$, $\mathbf{P}_{*_{\Bbbk}} = \mathbf{N}_{\mathscr{L}_{\Bbbk}} = \mathbf{C}_{\mathscr{L}_{\Bbbk}}$ as $\operatorname{rad}\Bbbk = 0$. Let $\Phi : \mathbf{P}_{*_{\Bbbk}} \to \mathbf{M}$ be a morphism of 2-representations defined on objects by $\Phi(F) = \mathbf{M}(F)(N)$ and on morphisms by $\Phi(f) = \mathbf{M}(f)_N$. We can abuse notation and equate Φ with $\Phi_{*_{\Bbbk}} : \mathbf{P}_{*_{\Bbbk}}(*_{\Bbbk}) \to \mathbf{M}(*_{\Bbbk})$. It is immediate from the prior discussion that Φ is essentially surjective on objects and faithful.

To show that Φ is full, let $\mathscr{K}$ be the $\mathscr{C}$-stable ideal of $\mathbf{M}(*_{\Bbbk})$ generated by $\operatorname{rad}\operatorname{End}_{\mathbf{M}}(N)$. Since N is indecomposable, we can apply similar reasoning to the proof of [ASS06] I.4.8 to show that $\operatorname{End}_{\mathbf{M}}(N)$ is local and $\operatorname{rad}\operatorname{End}_{\mathbf{M}}(N)$ is the unique maximal ideal. Assume for contradiction that $\mathrm{id}_M \in \mathscr{K}$ for some $M \in \mathbf{M}(*_{\Bbbk})$. Then by standard injection-projection arguments $\mathrm{id}_N \in \mathscr{K}$. But since any morphism $f : N^{\oplus m} \to N^{\oplus n}$ is an $m \times n$ matrix of elements of $\operatorname{End}_{\mathbf{M}}(N)$, this implies that $\mathrm{id}_N = \sum_{i=1}^{n} f_i k_i g_i$ where $f_i, g_i \in \operatorname{End}_{\mathbf{M}}(N)$ for all i and $k_i \in \operatorname{rad}\operatorname{End}_{\mathbf{M}}(N)$ for all i, i.e. that $\mathrm{id}_N \in \operatorname{rad}\operatorname{End}_{\mathbf{M}}(N)$, a contradiction. Hence $\mathscr{K}$ does not contain id_M for any $M \in \mathbf{M}(*_{\Bbbk})$. But since $\mathbf{M}$ is simple transitive by assumption, this implies that $\mathscr{K} = 0$ and thus that $\operatorname{End}_{\mathbf{M}}(N) \cong \Bbbk$. It follows immediately that Φ is full, and thus an equivalence of 2-representations as we wished to show.

Returning to the main aim of the proof, if $A = \Bbbk$ then we are done by the above work. Hence assume that $A \neq \Bbbk$. Using the previous paragraph and Corollary 3.3.7, we know that there is a unique equivalence class of simple transitive 2-representations of $\mathscr{C}_{A\times\Bbbk}$ that is non-zero on $*_{\Bbbk}$. We now claim that if a 2-representation $\mathbf{N}$ of $\mathscr{C}_{A\times\Bbbk}$ is non-zero on $*$, then it is either non-zero on $*_{\Bbbk}$ or equivalent to the trivial cell 2-representation on $\mathscr{C}_A$.

If $Ae_i \otimes_{\Bbbk} e_jA$ acts in a non-zero fashion on $\mathbf{N}(*)$ for some i or j, then as it factors through $\mathbf{N}(*_{\Bbbk})$, we must have that $\mathbf{N}(*_{\Bbbk})$ is non-zero. Therefore assume that $Ae_i \otimes_{\Bbbk} e_jA$ acts as the zero functor for every i and j. Then the only 1-morphisms in $\mathscr{C}_A(*,*)$ that act non-trivially on $\mathbf{N}(*)$ are direct sums of $\mathbb{1}_*$. In particular, if $N \in \mathbf{N}(*)$ is indecomposable, then for any $M \in \mathbf{N}(*)$, $M \cong N^{\oplus n}$ for some n. But this is equivalent to the cell 2-representation for the trivial $\mathscr{L}$-cell by a similar argument to above.

It follows by Corollary 3.3.7 that there is only one equivalence class of simple transitive 2-representations on $\mathscr{C}_A$ that is not equivalent to the identity cell 2-representation, and as we know that $\mathscr{C}_A$ has a cell 2-representation for the maximal $\mathscr{J}$-cell (which by assumption is distinct from $\{[\mathbb{1}_*]\}$) the result follows. $\square$

This leads to the main theorem of this section:

Theorem 3.3.9. *i) Let $A = \{A_\mathtt{i} | \mathtt{i} \in I\}$ be a countable collection of basic self-injective connected finite dimensional $\Bbbk$-algebras and let $X = \{X_i | i \in I\}$ be a collection of subalgebras $X_i \subseteq A_i$ as defined in Subsection 3.2.1. Then any non-zero simple transitive 2-representation of $\mathscr{C}_{A,X}$ is equivalent to a cell 2-representation.*

ii) For any $\mathscr{L}$-cells $\mathscr{L}_{\mathtt{i}k}$ and $\mathscr{L}_{\mathtt{i}l}$ in $\mathscr{C}_{A,X}$, $\mathbf{C}_{\mathscr{L}_{\mathtt{i}k}}$ and $\mathbf{C}_{\mathscr{L}_{\mathtt{j}l}}$ are equivalent 2-representations.

Proof. We first prove i) and ii) in the case where $X_i = A_i$. If $A = \{\Bbbk\}$ then we are done by the proof of Lemma 3.3.8. Assume that $A \neq \{\Bbbk\}$. Let $\mathbf{M}$ be a simple transitive 2-representation of $\mathscr{C}_A$. Assume that there is some $\mathtt{j}$ such that $\mathbf{M}(\mathtt{j}) = 0$ and let $\mathtt{i}$ be such that $\mathbf{M}(\mathtt{i}) \neq 0$. Then

$$A_\mathtt{i} e_{\mathtt{i}k} \otimes_\Bbbk e_{\mathtt{j}l} A_\mathtt{j} \otimes_{A_\mathtt{j}} A_\mathtt{j} e_{\mathtt{j}m} \otimes_\Bbbk e_{\mathtt{i}n} A_\mathtt{i} \otimes_{A_\mathtt{i}} -$$

is the zero map for any primitive idempotents $e_{\mathtt{i}k}$ and $e_{\mathtt{i}n}$. But in particular

$$A_\mathtt{i} e_{\mathtt{i}k} \otimes_\Bbbk e_{\mathtt{j}m} A_\mathtt{j} \otimes_{A_\mathtt{j}} A_\mathtt{j} e_{\mathtt{j}m} \otimes_\Bbbk e_{\mathtt{i}n} A_\mathtt{i} \cong (A_\mathtt{i} e_{\mathtt{i}k} \otimes_\Bbbk e_{\mathtt{i}n} A_\mathtt{i})^{\oplus \dim e_{\mathtt{j}m} A_\mathtt{j} e_{\mathtt{j}m}},$$

and this implies that $A_\mathtt{i} e_{\mathtt{i}j} \otimes_\Bbbk e_{\mathtt{i}m} A_\mathtt{i} \otimes_{A_\mathtt{i}} -$ is the zero map on $\mathbf{M}(\mathtt{i})$ for any j and m. But by a similar argument to Lemma 3.3.8, this means $\mathbf{M}$ is equivalent to a cell 2-representation for an identity cell. Thus if $\mathbf{M}$ is not equivalent to a trivial cell 2-representation, it must follow that $\mathbf{M}$ is non-zero on every $\mathtt{i}$.

Assume that $\mathbf{M}(\mathtt{j}) \neq 0$ for all $\mathtt{j} \in \mathscr{C}_A$, and choose some $\mathtt{i} \in \mathscr{C}_A$. By Lemma 3.3.8, every simple transitive 2-representation of $\mathscr{C}_{A_\mathtt{i}}$ is equivalent to a cell 2-representation, and in particular there is only one equivalence class of simple transitive 2-representations that is not equivalent to the trivial cell 2-representation. But as every simple transitive 2-representation of $\mathscr{C}_A$ not equivalent to a trivial cell 2-representation is non-zero when it restricts down to $\mathscr{C}_{A_\mathtt{i}}$ it follows from Corollary 3.3.7 that there is only a single equivalence class of simple transitive 2-representations of $\mathscr{C}_A$ not equivalent to a trivial cell 2-representation. This gives claim ii) in this case. Since $\mathscr{C}_A$ has a cell 2-representation for the maximal $\mathscr{J}$-cell, claim i) in this case follows.

For the general case, the arguments above and in the proof of Lemma 3.3.8 do not depend on the endomorphism of $\mathbb{1}_\mathtt{i}$, and thus generalise immediately. □

3.4 Locally Weakly Fiat and Strongly Regular $\mathscr{J}$-Cells

We now move on to extending results from the Mazorchuk–Miemietz series of papers to the locally finitary case, with the eventual aim of generalising [MM16c] Theorem 18 to show that any simple transitive 2-representation of a strongly regular locally weakly fiat 2-category is equivalent to a cell 2-representation. Here we are using the projective Freyd abelianisation rather than the injective equivalent as we did above. The projective case is in general preferable, but we were forced to use the injective case above to derive left exactness for various constructs, as is discussed in [MMMT16].

3.4.1 General Properties of the Abelianisation

We start by giving some general properties of the action of 1-morphisms of a locally weakly fiat 2-category $\mathscr{C}$ on its $\mathtt{i}$-th abelian principal 2-representation $\overline{\mathbf{P}_\mathtt{i}}$. The isomorphism classes of indecomposable projectives and simples are indexed by the isomorphism classes if indecomposables in $\mathscr{C}$, and we denote them as P_F and L_F

respectively. The proofs in this subsection are generally straightforward generalisations, with occasional adaptations for moving from fiat proofs to weakly fiat ones.

Proposition 3.4.1. *Let F, G be indecomposable 1-morphisms of $\mathscr{C}$. Then $FL_G \neq 0$ is equivalent to $F \leq_{\mathscr{L}} G^*$.*

Proof. This is a generalisation of [MM11] Lemma 12. Though the proof is similar, there are alterations needed due to how adjunctions work in the (locally) weakly fiat case.

We assume that $G \in \mathscr{C}(\mathtt{i}, \mathtt{j})$ and $F \in \mathscr{C}(\mathtt{j}, \mathtt{k})$. We first claim that $FL_G \neq 0$ if and only if there exists some indecomposable $H \in \mathscr{C}(\mathtt{i}, \mathtt{k})$ such that $\mathrm{Hom}_{\overline{\mathscr{C}(\mathtt{i},\mathtt{k})}}(P_H, FL_G) \neq 0$. The 'if' direction is immediate, as if such an H exists we have a non-zero mapping, which must go to a non-zero object. Conversely, if $FL_G \neq 0$, then it must have a non-zero projective presentation, from which the claim follows.

We then use the adjointness between F and ${}^*F$ to get

$$0 \neq \mathrm{Hom}_{\overline{\mathscr{C}(\mathtt{i},\mathtt{k})}}(P_H, FL_G) \cong \mathrm{Hom}_{\overline{\mathscr{C}(\mathtt{i},\mathtt{j})}}({}^*F \circ HP_{\mathbb{1}_\mathtt{i}}, L_G),$$

where we use that $P_H = HP_{\mathbb{1}_\mathtt{i}}$ by construction of the abelianisation. As ${}^*F \circ HP_{\mathbb{1}_\mathtt{i}}$ is projective and L_G is simple, this inequality is equivalent to saying that $P_G = GP_{\mathbb{1}_\mathtt{i}}$ is a direct summand of ${}^*F \circ HP_{\mathbb{1}_\mathtt{i}}$, i.e. G is a direct summand of ${}^*F \circ H$. This gives that ${}^*F \leq_{\mathscr{R}} G$, and applying $-^*$ gives us the result. □

We will now present the generalisations of [MM11] Lemma 13 and Corollary 14, whose proofs are entirely and explicitly local and thus generalise immediately, with similar minor adjustments to accommodate for moving from the fiat to the weakly fiat setup. We recall from Section 2.2 the notation $[M : S]$ for the multiplicity of a simple module S in the composition series of M.

Lemma 3.4.2. *For $F, H, K \in S(\mathscr{C})$, $[FL_K : L_H] \neq 0$ implies $H \leq_{\mathscr{L}} K$. If $H \leq_{\mathscr{L}} K$, there exists some $M \in S(\mathscr{C})$ such that $[ML_K : L_H] \neq 0$.*

Corollary 3.4.3. *Let $F, G, H \in S(\mathscr{C})$. If L_F occurs in the top or socle of HL_G, then $F \in \mathcal{L}_G$.*

Proposition 3.4.4. *Let $\mathcal{L}$ be an $\mathscr{L}$-cell of $\mathscr{C}$ with domain $\mathtt{i}$.*

- *There is a unique submodule $K = K_{\mathcal{L}}$ of $P_{\mathbb{1}_{\mathtt{i}}}$ such that every simple subquotient of $P_{\mathbb{1}_{\mathtt{i}}}/K$ is annihilated by any $F \in \mathcal{L}$ and such that K has simple top $L_{G_{\mathcal{L}}}$ for some $G_{\mathcal{L}} \in \mathcal{L}$ such that $FL_{G_{\mathcal{L}}} \neq 0$ for any $F \in \mathcal{L}$.*
- *For any $F \in \mathcal{L}$, $FL_{G_{\mathcal{L}}}$ has simple top L_F.*

Proof. These are parts a) and b) of [MM11] Proposition 17. The proof given there for those sections is local and does not depend on * being an involution, and so generalises immediately. □

In keeping with the finitary case, we denote $G_{\mathcal{L}}$ as the *Duflo involution* of $\mathcal{L}$. One of the most useful results regarding the Duflo involution (justifying calling it an 'involution') is:

Proposition 3.4.5. $G_{\mathcal{L}}, G_{\mathcal{L}}^* \in \mathcal{L}$.

Proof. This is the generalisation of claims c) and e) of [MM11] Proposition 17. That section of the proof is again local and does not utilise * being an involution, and hence also generalises directly. □

From this, if $\mathcal{J}$ is strongly regular and $\mathcal{L} \subseteq \mathcal{J}$, then $\mathcal{L} \cap {}^*\mathcal{L} = \{G_{\mathcal{L}}\}$.

Proposition 3.4.6. *For a maximal, strongly regular $\mathscr{J}$-cell $\mathcal{J}$ of $\mathscr{C}$ and for $F, H \in \mathcal{J}$, there exists some integer $\mathbf{m}_{F,H}$ such that $H^* \circ F \cong \mathbf{m}_{F,H} K$, where $\{K\} = \mathcal{R}_{H^*} \cap \mathcal{L}_F$.*

Proof. This is a direct consequence of $\mathcal{J}$ being strongly regular and maximal, and of $\mathscr{L}$-cells being closed under indecomposable direct summands of left 1-composition and $\mathscr{R}$-cells under indecomposable direct summands of right 1-composition. □

Proposition 3.4.7. *For any $F \in S(\mathscr{C})$, $F^* \sim_{\mathscr{J}} F$.*

Proof. This is a generalisation of [MM11] Lemma 26, and the proof given there only involves morphisms between at most three objects of $\mathscr{C}$ and uses a result generalised as Proposition 3.4.5, and thus generalises to the locally finitary case without issue. □

We now taker $\mathscr{C}$ to be a locally weakly fiat 2-category and consider the 2-representation $\mathbf{M} = \overline{\mathbf{C}_{\mathcal{L}}}$, the abelianisation of the cell 2-representation for some $\mathscr{L}$-cell $\mathcal{L}$. We use P_F, I_F and L_F to refer to projectives, injectives and simples in $\mathscr{M}$ respectively. For the remainder of this section and the following two, by quotienting $\mathscr{C}$ by the 2-ideal generated by all id_F such that $F \not\leq \mathcal{J}$ (for $\mathcal{J}$ the $\mathscr{J}$-cell containing $\mathcal{L}$), we can assume without loss of generality that $\mathcal{J}$ is the unique maximal $\mathscr{J}$-cell of $\mathscr{C}$ (see Subsection 3.4.3 for more details). We now generalise the first parts of [MM16b] Proposition 30:

Proposition 3.4.8. *The projective object P_F is injective for any $F \in \mathcal{L}$.*

Proof. We mirror the proof for the above citation, with extra clarifying details. By adjunction,

$$\mathrm{Hom}_{\overline{\mathscr{M}}}(\mathrm{L}_{G_{\mathcal{L}}}, F^*\mathrm{L}_F) \cong \mathrm{Hom}_{\overline{\mathscr{M}}}(F\mathrm{L}_{G_{\mathcal{L}}}, \mathrm{L}_F).$$

By the comment at the end of Proposition 3.4.6, L_F is the simple top of $F\mathrm{L}_{G_{\mathcal{L}}}$ and hence the latter space is non-zero and one-dimensional. Since $\mathrm{L}_{G_{\mathcal{L}}}$ is simple, this therefore implies that $\mathrm{L}_{G_{\mathcal{L}}}$ injects into $F^*\mathrm{L}_F$.

Let I be an injective object in some $\overline{\mathbf{C}_{\mathcal{L}}}(\mathtt{i})$ and let L_K be one of its simple quotients with $K \in \mathcal{L}$. Then $\mathrm{L}_{G_{\mathcal{L}}}$ is a subquotient of the object K^*I which is injective as K^* is exact. Using Proposition 3.4.1 and the strong regularity of $\mathcal{J}$ we have that $G_{\mathcal{L}}L_H = 0$ unless $H \cong G_{\mathcal{L}}$. Therefore $G_{\mathcal{L}}\mathrm{L}_H = 0$ unless $H \cong G_{\mathcal{L}}$. By Proposition 3.4.4, $G_{\mathcal{L}}\mathrm{L}_{G_{\mathcal{L}}}$ has simple top $\mathrm{L}_{G_{\mathcal{L}}}$ and $\mathrm{L}_{G_{\mathcal{L}}}$ appears in the top of the object GK^*I, which is injective as G is exact. It follows that P_F appears as a quotient, and thus a direct summand, of FGK^*I, which is injective as F is exact. The result follows. □

Proposition 3.4.9. *For any $F \in \mathcal{J}$, $F^*\mathrm{L}_F \cong I_{G_{\mathcal{L}}}$.*

Proof. We mirror the proof of [MM11] Proposition 38 a). By the proof of Proposition 3.4.8 $\mathrm{L}_{G_{\mathcal{L}}}$ injects into $F^*\mathrm{L}_F$, and hence by strong regularity, $F^*\mathrm{L}_F$ has simple socle $\mathrm{L}_{G_{\mathcal{L}}}$. In particular, it follows that $F^*\mathrm{L}_F$ is indecomposable. But as $\mathrm{L}_{G_{\mathcal{L}}}$ injects into $F^*\mathrm{L}_F$, it follows that $F^*\mathrm{L}_F$ is a direct summand of $I_{G_{\mathcal{L}}}$, giving the result. □

3.4.2 The Regularity Condition and some Other Results

We move on to examining a property of strongly regular $\mathscr{J}$-cells called the regularity condition. The proofs in this subsection are direct generalisations of the finitary setup. We quote a 1-categorical result, [MM16c] Lemma 13:

Proposition 3.4.10. *Let B be a finite dimensional $\Bbbk$-algebra and G an exact endofunctor of $B\text{-mod}$. Assume that G sends each simple object to a projective object. Then G is a functor isomorphic to tensoring with a projective bimodule (which we call a* projective functor*).*

Before stating the regularity condition result, we give a lemma we will need for it.

Proposition 3.4.11. *Let $\mathcal{J}$ be a strongly regular $\mathscr{J}$-cell of a locally weakly fiat 2-category $\mathscr{C}$, and let $\mathbf{m} : \mathcal{J} \to \mathbb{Z}^+$ be defined as ${}^*F \circ F \cong \mathbf{m}_F H \oplus K$, with no indecomposable direct summand of K belonging to $\mathcal{J}$. Then $\mathbf{m}$ is constant on $\mathscr{R}$-cells of $\mathcal{J}$.*

Proof. This is a generalisation of [MM16b] Proposition 1, and we mirror the proof found therein. Let $\mathcal{L}$ be an $\mathscr{L}$-cell in $\mathcal{J}$ and let $\mathbf{C}_{\mathcal{L}}$ and $\overline{\mathbf{C}_{\mathcal{L}}}$ be the corresponding cell 2-representation and its abelianisation respectively. For $F, H \in \mathcal{L}$, by Proposition 3.4.1 and Proposition 3.4.5 we can immediately derive that $F\mathrm{L}_H \cong \mathrm{P}_F$ if $H \cong G_{\mathcal{L}}$ and zero otherwise. Since every element of $\mathcal{L}$ has the same source object, we can apply Proposition 3.4.10 which gives that, for each $F \in \mathcal{L}$, $\overline{\mathbf{C}_{\mathcal{L}}}(F)$ is an indecomposable projective functor.

For each $\mathtt{j} \in \mathscr{C}$, let $A_{\mathtt{j}}$ denote the basic algebra such that $\overline{\mathbf{C}_{\mathscr{L}}}(\mathtt{j})$ is equivalent to $A_{\mathtt{j}}\text{-mod}$. Let $\{e_{\mathtt{j}1}, \ldots, e_{\mathtt{j}n_{\mathtt{j}}}\}$ be a complete set of pairwise orthogonal primitive idempotents in $A_{\mathtt{j}}$. Then without loss of generality, each $\overline{\mathbf{C}_{\mathscr{L}}}(F)$ is the projective functor $A_{\mathtt{j}}e_{\mathtt{j}s} \otimes_{\Bbbk} e_{\mathtt{i}1}A_{\mathtt{i}} \otimes_{A_{\mathtt{i}}} -$ for some $s \in \{1, \ldots, n_{\mathtt{j}}\}$.

It follows from Proposition 3.4.8 that $A_{\mathtt{j}}$ is self-injective when $\mathcal{J}$ is maximal. There is thus some permutation $\sigma_{\mathtt{j}} \in S_{n_{\mathtt{j}}}$ such that $(e_{\mathtt{j}s}A_{\mathtt{j}})^* \cong A_{\mathtt{j}}e_{\mathtt{j}\sigma(s)}$. It follows that $\overline{\mathbf{C}_{\mathscr{L}}}(F^*)$ is the projective functor $A_{\mathtt{i}}e_{\mathtt{i}\sigma_{\mathtt{i}}(1)} \otimes_{\Bbbk} e_{\mathtt{j}s}A_{\mathtt{j}} \otimes_{A_{\mathtt{j}}} -$. By taking the tensor product, $\mathbf{m}_{F^*} = \dim(e_{\mathtt{i}1}A_{\mathtt{i}}e_{\mathtt{i}\sigma_{\mathtt{i}}(1)})$ which is independent of the choice of $F \in \mathscr{L}$. Since $F \mapsto F^*$ is a bijection from $\mathscr{L}$ to the $\mathscr{R}$-cell of $\mathcal{J}$ containing $G^*_{\mathscr{L}}$, $\mathbf{m}$ is constant on the $\mathscr{R}$-cell. But by Proposition 3.4.5 every $\mathscr{R}$-cell contains a Duflo involution, and hence the result follows. □

We can use this to derive the following:

Proposition 3.4.12. *For $\mathcal{J}$ a strongly regular maximal $\mathscr{J}$-cell in $\mathscr{C}$ and any $F \in \mathscr{L}$ $FG_{\mathscr{L}} \cong \mathbf{m}_{G_{\mathscr{L}}}F$.*

Proof. This is a generalisation of [MM16b] Proposition 29, and the proof there generalises immediately to this setup. □

3.4.3 Restricting to Smaller 2-Categories

In this subsection we examine cell 2-representations of quotients of locally weakly fiat 2-categories, using entirely novel proofs. Let $\mathscr{C}$ be a locally weakly fiat 2-category and let $\mathcal{J}$ be a strongly regular $\mathscr{J}$-cell of $\mathscr{C}$. We construct a 2-ideal $\mathscr{I}_{\not\leq\mathcal{J}}$ of $\mathscr{C}$ that is generated by 2-morphisms id_F for $F \not\leq_{\mathscr{J}} \mathcal{J}$; that is, $\mathscr{I}_{\not\leq\mathcal{J}}(\mathtt{i}, \mathtt{j})$ consists of 2-morphisms between 1-morphisms in $\mathscr{C}(\mathtt{i}, \mathtt{j})$ that factor through a direct sum of 1-morphisms F with $F \not\leq_{\mathscr{J}} \mathcal{J}$. We also define $\mathfrak{I}_{\leq\mathcal{J}}$ to be the maximal 2-ideal of $\mathscr{C}$ such that $\mathrm{id}_F \notin \mathfrak{I}_{\leq\mathcal{J}}$ for any $F \in \mathcal{J}$. We define 2-categories $\mathscr{C}_{\not\leq\mathcal{J}} = \mathscr{C}/\mathfrak{I}_{\not\leq\mathcal{J}}$ and $\mathscr{C}_{\leq\mathcal{J}} = \mathscr{C}/\mathfrak{I}_{\leq\mathcal{J}}$. We also define $\mathscr{C}^{\mathcal{J}}_{\not\leq\mathcal{J}}$ to be the sub-2-category of $\mathscr{C}_{\not\leq\mathcal{J}}$ closed under direct sums and isomorphisms and generated by the $\mathbb{1}_{\mathtt{i}}$ for $\mathtt{i} \in \mathscr{C}$ and by $F \in \mathcal{J}$.

Proposition 3.4.13. $\mathscr{I}_{\not\leq \mathcal{J}} \subseteq \mathscr{I}_{\leq \mathcal{J}}$.

Proof. By the definition of $\mathscr{I}_{\leq \mathcal{J}}$ it suffices to show that $\mathsf{id}_F \notin \mathscr{I}_{\not\leq \mathcal{J}}$ for any $F \in \mathcal{J}$. Assume otherwise for contradiction. Then for some $F \in \mathcal{J}$, $\mathsf{id}_F = \sum_{k=1}^{n} f_k \mathsf{id}_{G_k} g_k$ where $G_k \not\leq \mathcal{J}$, $g_k : F \to G_k$, $f_k : G_k \to F$ for all k.

Since F is indecomposable, by using similar arguments to the proof of [ASS06] Corollary I.4.8, for each k either $f_k g_k$ is nilpotent or it is an automorphism. If $f_k g_k$ is nilpotent for all k, then as nilpotent morphisms form an ideal so is id_F, a contradiction. Therefore there exists some k such that $f_k g_k$ is an automorphism, say with inverse h. But then $(hf_k)g_k = \mathsf{id}_F \Rightarrow F \cong G_k$ and by construction $F \in \mathcal{J}$ and $G_k \notin \mathcal{J}$, a contradiction. The result follows. ☐

Proposition 3.4.14. *The image of $\mathcal{J}$ remains a $\mathscr{J}$-cell in $\mathscr{C}^{\mathcal{J}}_{\not\leq \mathcal{J}}$.*

Proof. Let $\mathcal{L}$ be an $\mathscr{L}$-cell in $\mathcal{J}$ and let $F \in \mathcal{L}$. First, by Proposition 3.4.13, $\mathsf{id}_F \notin \mathscr{I}_{\not\leq \mathcal{J}}$ for any $F \in \mathcal{J}$, and so $\mathrm{End}_{\mathscr{C}^{\mathcal{J}}_{\not\leq \mathcal{J}}}(F) \neq 0$, and hence $F \neq 0$ in $\mathscr{C}^{\mathcal{J}}_{\not\leq \mathcal{J}}$. For the Duflo involution $G_{\mathcal{L}}$ of $\mathcal{L}$ we know by strong regularity of $\mathcal{J}$ that $FG_{\mathcal{L}} \cong F^{\oplus m}$ in $\mathscr{C}$ for some positive m. Since the composition of the injection 2-morphism $F \to FG_{\mathcal{L}}$ with the projection 2-morphism $FG_{\mathcal{L}} \to F$ is id_F, the direct sum structure is preserved in $\mathscr{C}^{\mathcal{J}}_{\not\leq \mathcal{J}}$ and so $G_{\mathcal{L}} \leq_{\mathscr{L}} F$ in $\mathscr{C}^{\mathcal{J}}_{\not\leq \mathcal{J}}$.

Conversely, again by strong regularity of $\mathcal{J}$, $F^*F \cong G_{\mathcal{L}}^{\oplus n}$ for some positive n in $\mathscr{C}$, and consequently by a similar argument to above $F \leq_{\mathscr{L}} G_{\mathcal{L}}$ in $\mathscr{C}_{\not\leq \mathcal{J}}$ and $\mathcal{L}$ is contained in an $\mathscr{L}$-cell in $\mathscr{C}_{\not\leq \mathcal{J}}$. But it is immediate from the definitions that if $F \leq_{\mathscr{L}} H$ in $\mathscr{C}_{\not\leq \mathcal{J}}$, then $F \leq_{\mathscr{L}} H$ in $\mathscr{C}$ and thus $\mathcal{L}$ is precisely an $\mathscr{L}$-cell in $\mathscr{C}_{\not\leq \mathcal{J}}$. Applying adjunctions gives the corresponding result for $\mathscr{R}$-cells and the result follows. ☐

For an $\mathscr{L}$-cell $\mathcal{L}$ of $\mathcal{J}$ we recall the finitary 2-representation $\mathbf{N}_{\mathcal{L}} : \mathscr{C} \to \mathfrak{A}^f_{\Bbbk}$ given by $\mathbf{N}_{\mathcal{L}}(\mathtt{j}) = \mathrm{add}\{FX | F \in \coprod_{\mathtt{k} \in \mathscr{C}} \mathscr{C}(\mathtt{k}, \mathtt{j}), X \in \mathcal{L}\}$ and set $\mathcal{N}_{\mathcal{L}} = \coprod_{\mathtt{j} \in \mathscr{C}} \mathbf{N}(\mathtt{j})$, with class of morphisms $\mathrm{Ar}(\mathcal{N}_{\mathcal{L}})$.

We define two 2-representations of $\mathscr{C}_{\not\leq \mathcal{J}}$. First let $\mathbf{N}^{\not\leq \mathcal{J}}_{\mathcal{L}} : \mathscr{C}_{\not\leq \mathcal{J}} \to \mathfrak{A}^f_{\Bbbk}$ be defined by $\mathbf{N}_{\mathscr{L}}^{\not\leq\mathcal{J}}(\mathtt{j}) = \mathrm{add}\{FX | F \in \coprod_{\mathtt{k}\in\mathscr{C}} \mathscr{C}_{\not\leq\mathcal{J}}(\mathtt{k},\mathtt{j}), X \in \mathscr{L}\}$. Second, we define the 2-representation $\mathbf{N}_{\mathscr{L}}/\mathcal{I}_{\not\leq\mathcal{J}}$ by setting

$$(\mathbf{N}_{\mathscr{L}}/\mathcal{I}_{\not\leq\mathcal{J}})(\mathtt{j}) = \mathbf{N}_{\mathscr{L}}(\mathtt{j})/(\mathrm{Ar}(\mathcal{N}_{\mathscr{L}}) \cap \mathcal{I}_{\not\leq\mathcal{J}}(\mathtt{i},\mathtt{j})),$$

where $\mathtt{i}$ is the source object of $\mathscr{L}$, with the obvious induced action of $\mathscr{C}_{\not\leq\mathcal{J}}$.

Proposition 3.4.15. *$\mathbf{N}_{\mathscr{L}}^{\not\leq\mathcal{J}}$ and $\mathbf{N}_{\mathscr{L}}/\mathcal{I}_{\not\leq\mathcal{J}}$ are equivalent as 2-representations of $\mathscr{C}_{\not\leq\mathcal{J}}$.*

Proof. By construction there is a bijection between objects of $\mathcal{N}_{\mathscr{L}}^{\not\leq\mathcal{J}}$ and $\mathcal{N}_{\mathscr{L}}/\mathcal{I}_{\not\leq\mathcal{J}}$ and it suffices to show that for $F, G \in \mathbf{N}_{\mathscr{L}}(\mathtt{j})$,

$$\mathrm{Hom}_{\mathbf{N}_{\mathscr{L}}^{\not\leq\mathcal{J}}}(F,G) \cong \mathrm{Hom}_{\mathbf{N}_{\mathscr{L}}/\mathcal{I}_{\not\leq\mathcal{J}}}(F,G).$$

But

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{N}_{\mathscr{L}}/\mathcal{I}_{\not\leq\mathcal{J}}}(F,G) &= \mathrm{Hom}_{\mathcal{N}_{\mathscr{L}}}(F,G)/(\mathrm{Hom}_{\mathcal{N}_{\mathscr{L}}}(F,G) \cap \mathcal{I}_{\not\leq\mathcal{J}}(\mathtt{i},\mathtt{j}) \\
&= \mathrm{Hom}_{\mathscr{C}}(F,G)/(\mathrm{Hom}_{\mathscr{C}}(F,G) \cap \mathcal{I}_{\not\leq\mathcal{J}}(\mathtt{i},\mathtt{j})) \\
&= \mathrm{Hom}_{\mathscr{C}_{\not\leq\mathcal{J}}}(F,G) \\
&= \mathrm{Hom}_{\mathbf{N}_{\mathscr{L}}^{\not\leq\mathcal{J}}}(F,G)
\end{aligned}$$

as required. □

By Proposition 3.4.14 $\mathcal{J}$ descends to a $\mathscr{J}$-cell of $\mathscr{C}_{\not\leq\mathcal{J}}^{\mathcal{J}}$, which we will also denote by $\mathcal{J}$. We can thus define the 2-representation $\mathbf{N}_{\mathscr{L}}^{\mathcal{J}}$ of $\mathscr{C}_{\not\leq\mathcal{J}}^{\mathcal{J}}$ in the standard fashion. We can consider $\mathbf{N}_{\mathscr{L}}^{\not\leq\mathcal{J}}$ as a 2-representation of $\mathscr{C}_{\not\leq\mathcal{J}}^{\mathcal{J}}$ by restriction. We define the 2-representation $(\mathbf{N}_{\mathscr{L}})^{\mathcal{J}}$ of $\mathscr{C}_{\not\leq\mathcal{J}}^{\mathcal{J}}$ as the full sub-2-representation of $\mathbf{N}_{\mathscr{L}}^{\not\leq\mathcal{J}}$ generated by $F \in \mathcal{J}$ and closed under isomorphism.

Proposition 3.4.16. *$\mathbf{N}_{\mathscr{L}}^{\mathcal{J}}$ is equivalent to $(\mathbf{N}_{\mathscr{L}})^{\mathcal{J}}$ as 2-representations of $\mathscr{C}_{\not\leq\mathcal{J}}^{\mathcal{J}}$.*

Proof. By construction, if we have 1-morphisms $F, G \in \mathscr{C}_{\not\leq\mathcal{J}}^{\mathcal{J}}$ such that $F, G \in \mathcal{N}_{\mathscr{L}}^{\mathcal{J}}$

and $F, G \in (\mathcal{N}_{\mathscr{L}})^{\mathcal{J}}$ then

$$\mathrm{Hom}_{\mathcal{N}_{\mathscr{L}}^{\mathcal{J}}}(F, G) = \mathrm{Hom}_{(\mathcal{N}_{\mathscr{L}})^{\mathcal{J}}}(F, G)$$

and thus it suffices to demonstrate an essential bijection between objects in the component categories of the 2-representations.

If $F \in \mathbf{N}_{\mathscr{L}}^{\mathcal{J}}(\mathtt{j})$ is indecomposable, then F is a direct summand of GX for some $X \in \mathscr{L}$ and some $G \in \coprod_{\mathtt{k}\in\mathscr{C}} \mathscr{C}_{\not\leq \mathcal{J}}^{\mathcal{J}}(\mathtt{k}, \mathtt{j})$. But then G is a direct sum of elements of $\mathcal{J}$ and thus $F \in \mathcal{J}$. As we can also consider G to be in $\coprod_{\mathtt{k}\in\mathscr{C}} \mathscr{C}_{\not\leq \mathcal{J}}(\mathtt{k}, \mathtt{j})$, it follows that $F \in (\mathbf{N}_{\mathscr{L}})^{\mathcal{J}}(\mathtt{j})$.

Conversely, let F be an indecomposable object in $(\mathbf{N}_{\mathscr{L}})^{\mathcal{J}}(\mathtt{j})$. Then $F \in \mathcal{J}$ and F is a direct summand of GX for some $X \in \mathscr{L}$ and $G \in \coprod_{\mathtt{k}\in\mathscr{C}} \mathscr{C}_{\not\leq \mathcal{J}}(\mathtt{k}, \mathtt{j})$. Hence $X \leq_{\mathscr{L}} F$. But by the definition of a strongly regular $\mathscr{J}$-cell, different $\mathscr{L}$-cells of $\mathcal{J}$ are incomparable under $\leq_{\mathscr{L}}$. Thus we must have that $F \sim_{\mathscr{L}} X$ and $F \in \mathscr{L}$. But then F is a direct summand of $\mathbb{1}_{\mathtt{j}} F$ and since $\mathbb{1}_{\mathtt{j}} \in \mathscr{C}_{\not\leq \mathcal{J}}^{\mathcal{J}}$, the result follows. □

We now consider the cell 2-representations. Let $\mathbf{C}_{\mathscr{L}}$ denote the 2-representation of $\mathscr{C}$ corresponding to $\mathscr{L}$. This corresponds to the quotient of $\mathbf{N}_{\mathscr{L}}$ by the maximal $\mathscr{C}$-stable ideal $\mathscr{K}_{\mathscr{L}}$ not containing id_F for any $F \in \mathscr{L}$. We define similar ideals $\mathscr{K}_{\mathscr{L}}^{\not\leq \mathcal{J}}$ and $\mathscr{K}_{\mathscr{L}}^{\mathcal{J}}$ of $\mathbf{N}_{\mathscr{L}}^{\not\leq \mathcal{J}}$ and $\mathbf{N}_{\mathscr{L}}^{\mathcal{J}}$ respectively. We let $\mathbf{C}_{\mathscr{L}}^{\not\leq \mathcal{J}}$ and $\mathbf{C}_{\mathscr{L}}^{\mathcal{J}}$ denote the respective cell 2-representations.

Proposition 3.4.17. $\mathrm{Ar}(\mathbf{N}_{\mathscr{L}}(\mathtt{j})) \cap \mathcal{I}_{\not\leq \mathcal{J}}(\mathtt{i}, \mathtt{j}) \subseteq \mathscr{K}_{\mathscr{L}}$.

Proof. By the construction of $\mathscr{K}_{\mathscr{L}}$, it suffices to show that $\mathrm{id}_F \notin \mathrm{Ar}(\mathbf{N}_{\mathscr{L}}(\mathtt{j})) \cap \mathcal{I}_{\not\leq \mathcal{J}}(\mathtt{i}, \mathtt{j})$ for any $F \in \mathscr{L}$. But if $\mathrm{id}_F \in \mathrm{Ar}(\mathbf{N}_{\mathscr{L}}(\mathtt{j})) \cap \mathcal{I}_{\not\leq \mathcal{J}}(\mathtt{i}, \mathtt{j})$ for some $F \in \mathscr{L} \subseteq \mathcal{J}$, then in particular $\mathrm{id}_F \in \mathcal{I}_{\not\leq \mathcal{J}}(\mathtt{i}, \mathtt{j})$ for some $F \in \mathcal{J}$, which we showed was a contradiction in the proof of Proposition 3.4.13, and we are done. □

Corollary 3.4.18. $\mathbf{C}_{\mathscr{L}}$ *has a natural structure of a* $\mathscr{C}_{\not\leq \mathcal{J}}$ *2-representation, and further is equivalent to* $\mathbf{C}_{\mathscr{L}}^{\not\leq \mathcal{J}}$ *as 2-representations of* $\mathscr{C}_{\mathscr{L}}^{\not\leq \mathcal{J}}$.

Proof. By Proposition 3.4.17, quotienting $\mathcal{N}_{\mathcal{L}}$ by $\mathcal{K}_{\mathcal{L}}$ factors through quotienting by $\mathcal{N}_{\mathcal{L}} \cap \mathcal{I}_{\not\leq \mathcal{J}}$, giving the first statement. For the second, by Proposition 3.4.15 $\mathbf{N}_{\mathcal{L}}/\mathcal{I}_{\not\leq\mathcal{J}}$ is equivalent to $\mathbf{N}_{\mathcal{J}}^{\not\leq\mathcal{J}}$. It thus suffices to show that $\mathcal{K}_{\mathcal{L}}^{\not\leq\mathcal{J}}$ is the image under this equivalence of the image of $\mathcal{K}_{\mathcal{L}}$ in the quotient.

Let $\sigma : \mathcal{N}_{\mathcal{L}} \to \mathcal{N}_{\mathcal{L}}/\mathcal{I}_{\not\leq\mathcal{J}}$ denote the canonical quotient functor. It is straightforward to see that the preimage $\mathbb{Q}$ of $\mathcal{K}_{\mathcal{L}}^{\not\leq\mathcal{J}}$ is a $\mathscr{C}$-stable ideal of $\mathcal{N}_{\mathcal{L}}$. We will show that $\mathbb{Q} \subseteq \mathcal{K}_{\mathcal{L}}^{\not\leq\mathcal{J}}$. Assume for contradiction that $\mathrm{id}_F \in \mathbb{Q}$ for some $F \in \mathcal{L}$. Since $\mathrm{id}_F \notin \mathcal{I}_{\not\leq\mathcal{J}}$ by the proof of Proposition 3.4.13, this implies that $\mathrm{id}_F \in \mathcal{K}_{\mathcal{L}}^{\not\leq\mathcal{J}}$, a contradiction. Therefore $\mathbb{Q}$ does not contain id_F for any $F \in \mathcal{L}$, and thus $\mathbb{Q} \subseteq \mathcal{K}_{\mathcal{L}}$. Hence $\mathcal{K}_{\mathcal{L}}^{\not\leq\mathcal{J}} \subseteq \sigma(\mathcal{K}_{\mathcal{L}})$. But by definition $\mathrm{id}_F \notin \sigma(\mathcal{K}_{\mathcal{L}})$ for any $F \in \mathcal{L}$. Therefore by definition $\sigma(\mathcal{K}_{\mathcal{L}}) \subseteq \mathcal{K}_{\mathcal{L}}^{\not\leq\mathcal{J}}$ and the result follows. □

The cell 2-representation $\mathbf{C}_{\mathcal{L}}^{\not\leq\mathcal{J}}$ has the structure of a 2-representation of $\mathscr{C}_{\not\leq\mathcal{J}}^{\mathcal{J}}$ by restriction, and we can take the full sub-2-representation $(\mathbf{C}_{\mathcal{L}})^{\mathcal{J}}$ where the generating objects in the component categories are those in $\mathcal{J}$.

Proposition 3.4.19. *$(\mathbf{C}_{\mathcal{L}})^{\mathcal{J}}$ is equivalent to $\mathbf{C}_{\mathcal{L}}^{\mathcal{J}}$ as 2-representations of $\mathscr{C}_{\not\leq\mathcal{J}}^{\mathcal{J}}$.*

Proof. By Proposition 3.4.16 it suffices to prove that the restriction $(\mathcal{K}_{\mathcal{L}})^{\mathcal{J}}$ of $\mathcal{K}_{\mathcal{L}}^{\not\leq\mathcal{J}}$ to $(\mathbf{N}_{\mathcal{L}})^{\mathcal{J}}$ is equal to $\mathcal{K}_{\mathcal{L}}^{\mathcal{J}}$. By construction $\mathrm{id}_F \notin (\mathcal{K}_{\mathcal{L}})^{\mathcal{J}}$ for any $F \in \mathcal{L}$, and thus $(\mathcal{K}_{\mathcal{L}})^{\mathcal{J}} \subseteq \mathcal{K}_{\mathcal{L}}^{\mathcal{J}}$. It remains to show that $\mathcal{K}_{\mathcal{L}}^{\mathcal{J}} \subseteq (\mathcal{K}_{\mathcal{L}})^{\mathcal{J}}$.

Let $\mathbb{Q}$ denote the $\mathscr{C}$-stable ideal of $\mathbf{N}_{\mathcal{L}}^{\not\leq\mathcal{J}}$ generated by $\mathcal{K}_{\mathcal{L}}^{\mathcal{J}}$. We will show that $\mathbb{Q} \subseteq \mathcal{K}_{\mathcal{L}}^{\not\leq\mathcal{J}}$. Assume for contradiction that $\mathrm{id}_F \in \mathbb{Q}$ for some $F \in \mathcal{L}$. Then using a similar component argument to previous proofs, we have that $\mathrm{id}_F = \beta\mathbf{N}_{\mathcal{L}}^{\not\leq\mathcal{J}}(K)(\gamma)\alpha$, where $\gamma : G \to H$ is in $\mathcal{K}_{\mathcal{L}}^{\mathcal{J}}$, $K \leq_{\mathscr{J}} \mathcal{J}$, $\alpha : F \to KG$ and $\beta : KH \to F$. We immediately see that F is a direct summand of KG and KH.

Let $S \in \mathcal{J}$ be a 1-morphism such that $SF \neq 0$ in $\mathscr{C}_{\not\leq\mathcal{J}}$, which exists as $\mathcal{J}$ is strongly regular. Then $\mathrm{id}_{SF} = S(\mathrm{id}_F) = S(\beta)SK(\gamma)S(\alpha)$. But for a indecomposable summand V of SF, $V \geq_{\mathscr{J}} S$, and as $SK \neq 0$, it follows that $V \in \mathcal{J}$. Hence by pre- and post-composing with injection and projection 2-morphisms it follows that

$\mathrm{id}_V = \beta' SK(\gamma)\alpha'$ for some β' and α'. Without loss of generality $V \in \mathcal{L}$ (e.g. by taking $S = G_{\mathcal{L}}$), and by a similar argument to before every indecomposable summand of SK is in $\mathcal{J}$. Hence $\mathrm{id}_V \in \mathcal{K}_{\mathcal{L}}^{\mathcal{J}}$, a contradiction. Hence $\mathbb{Q} \subseteq \mathcal{K}_{\mathcal{L}}^{\not\leq\mathcal{J}}$, and thus $\mathcal{K}_{\mathcal{L}}^{\mathcal{J}} \subseteq (\mathcal{K}_{\mathcal{L}})^{\mathcal{J}}$ and the result follows.

□

Corollary 3.4.20. *The restriction of the cell 2-representation* $\mathbf{C}_{\mathcal{L}}$ *of* $\mathscr{C}$ *to* $\mathscr{C}_{\not\leq\mathcal{J}}^{\mathcal{J}}$ *is the corresponding 2-representation.*

Proof. This is a direct consequence of combining Corollary 3.4.18 and Proposition 3.4.19 given Proposition 3.4.14. □

3.4.4 The Action of Indecomposables on Simples

Our aim in this section is to prove the following theorem:

Theorem 3.4.21. *Let* $\mathscr{C}$ *be a locally weakly fiat 2-category with* $\mathcal{J}$ *a strongly regular* $\mathscr{J}$*-cell in* $\mathscr{C}$ *and* $\mathcal{L}$ *an* $\mathscr{L}$*-cell in* $\mathcal{J}$*. Then for* $F \in \mathcal{L}$ *and* $H \in \mathcal{J}$*,* $H\mathrm{L}_F$ *is an indecomposable projective in* $\coprod \overline{\mathbf{C}_{\mathcal{L}}}(\mathtt{i})$*.*

To prove this, we first give a supplementary lemma.

Lemma 3.4.22. *Let* $F, H \in \mathcal{J}$*. Then* $H\mathrm{L}_F$ *is either zero or injective-projective in* $\overline{\mathbf{C}_{\mathcal{L}}}$*.*

Proof. If $H\mathrm{L}_F \neq 0$, then by a variant of Proposition 3.4.1, F^* and H are in the same $\mathscr{L}$-cell, and so $H\mathrm{L}_F$ is a direct summand of $KF^*\mathrm{L}_F$ for some K by strong regularity of $\mathcal{J}$, which by Proposition 3.4.8 and Proposition 3.4.9 is projective-injective. □

To prove the theorem given the above, it suffices to prove that $H\mathrm{L}_F$ is indecomposable. We mirror the proof given for [MM16b] Proposition 30, using the references given above. Let $\mathcal{R}_F$ denote the $\mathscr{R}$-cell in $\mathcal{J}$ containing F. By strong regularity of $\mathcal{J}$, there is a unique $\hat{G} \in \mathcal{R}_F$ such that $\hat{G}\mathrm{L}_F \neq 0$. But this implies that

$\hat{G}L_F \neq 0$, and thus $\hat{G}L_{\hat{G}} \neq 0$ as F and $\hat{G}$ share a $\mathscr{R}$-cell. It thus follows from $\mathcal{L} \cap {}^*\mathcal{L} = \{G_{\mathcal{L}}\}$ that $\hat{G}$ is a Duflo involution.

As we have already proved $\hat{G}\mathrm{L}_F$ is projective, it follows that $\hat{G}\mathrm{L}_F \cong k\mathrm{P}_F$ for some positive integer k. We now compute $F^*G_{\mathcal{L}}\mathrm{L}_F$ in two separate fashions. First,

$$F^*G\mathrm{L}_F \cong kF^*\mathrm{P}_F \cong kF^*F\mathrm{L}_F \cong k\mathbf{m}_F\mathrm{P}_{G^*_{\mathcal{L}}}.$$

However by construction F^* is in the same $\mathscr{L}$-cell as $\hat{G}^*$, and as $\hat{G}$ is a Duflo involution, it follows that F^* is in the same $\mathscr{L}$-cell as $\hat{G}$. Thus $F^*\hat{G}\mathrm{L}_F \cong \mathbf{m}_{\hat{G}}\mathrm{P}_{G^*}$. But by Proposition 3.4.11, $\mathbf{m}$ is constant on $\mathscr{R}$-cells, and so $k = 1$ and the result follows for this specific case.

The general case generalises immediately from the proof given in [MM16b] Proposition 30, using Proposition 3.4.11, and thus the result follows.

3.4.5 $\mathcal{J}$-Simple, $\mathcal{J}$-Full and Almost Algebra 2-Categories

Definition 3.4.23. Let $\mathscr{C}$ be a locally finitary 2-category and $\mathbf{M}$ a 2-representation of $\mathscr{C}$. We say that $\mathbf{M}$ is *2-full* if for any 1-morphisms in $\mathscr{C}$ the representation map $\mathrm{Hom}_{\mathscr{C}}(F,G) \to \mathrm{Hom}_{\mathfrak{R}_{\Bbbk}}(\mathbf{M}F, \mathbf{M}G)$ is surjective. For a $\mathscr{J}$-cell $\mathcal{J}$ of $\mathscr{C}$, we say $\mathbf{M}$ is *$\mathcal{J}$-2-full* if for every $F, G \in \mathcal{J}$, the representation map is surjective.

For this section, we will assume that $\mathscr{C}$ is a locally weakly fiat 2-category that contains a unique non-trivial strongly regular $\mathscr{J}$-cell $\mathcal{J}$ and that $\mathscr{C}$ is $\mathcal{J}$-simple. The proofs within this section are semi-straightforward generalisations, though moving from fiat proofs to the weakly fiat case requires some more work than previously.

We now state our initial main theorem for this section, a generalisation of [MM16a] Theorem 13. Let $\mathbf{M} = \overline{\mathbf{C}_{\mathcal{L}}}$, the abelian cell 2-representation for some $\mathscr{L}$-cell $\mathcal{L} \subseteq \mathcal{J}$. We recall that for a countable collection $A = \{A_i | i \in I\}$ of basic self-injective connected finite dimensional $\Bbbk$-algebras, with $X = \{X_i | i \in I\}$ a set of subalgebras X_i of A_i distinguished by certain properties as defined in Section 3.1, there is an associated locally weakly fiat 2-category $\mathscr{C}_{A,X}$.

Theorem 3.4.24. *There exist some A and X such that $\mathscr{C}$ is biequivalent to $\mathscr{C}_{A,X}$.*

Proof. We start by generalising the proof of [MM16a] Theorem 13, with some extra detail for clarity. For $\mathtt{i} \in \mathscr{C}$, let $A_\mathtt{i}$ be a finite dimensional connected basic $\Bbbk$-algebra such that $\mathbf{M}(\mathtt{i})$ is equivalent to $A_\mathtt{i}\text{-mod}$. Letting $Z_\mathtt{i}$ be the centre of $A_\mathtt{i}$, we define $X_\mathtt{i} = \mathbf{M}(\mathrm{End}_{\mathscr{C}}(\mathbb{1}_\mathtt{i})) \subseteq Z_\mathtt{i}$. We can define an action of $\mathbf{M}(F)$ on $\mathscr{C}_{A,X}$ for $F \in \mathcal{J}$ using the equivalence between $\mathbf{M}(\mathtt{i})$ and $A_\mathtt{i}\text{-mod}$. Then by the definition of the cell 2-representation, each $\mathbf{M}(F)$ for $F \in \mathcal{J}$ is a projective functor in $\mathrm{End}(\mathscr{M})$, and since each $\mathbf{M}(\mathbb{1}_\mathtt{i})$ acts as the identity, this implies that $\mathbf{M}$ factors through $\mathscr{C}_{A,X}$, and thus $\mathbf{M}$ corestricts to a 2-functor from $\mathscr{C}$ to $\mathscr{C}_{A,X}$. By construction $\mathbf{M}$ is surjective up to equivalence on objects (and indeed is bijective on objects).

We will show that each

$$\mathbf{M}_{\mathtt{i},\mathtt{j}} : \mathscr{C}(\mathtt{i},\mathtt{j}) \to \mathscr{C}_{A,X}(\mathtt{i},\mathtt{j})$$

is an equivalence. Since $\mathscr{C}$ is $\mathcal{J}$-simple it follows that $\mathbf{M}_{\mathtt{i},\mathtt{j}}$ is faithful. To show that $\mathbf{M}_{\mathtt{i},\mathtt{j}}$ is essentially surjective on 1-morphisms, by construction a 1-morphism in $\mathscr{C}_{A,X}$ is equivalent to tensoring with a projective $(A_\mathtt{i}\text{-}A_\mathtt{j})$-bimodule. In particular, any indecomposable $(A_\mathtt{i}\text{-}A_\mathtt{j})$-bimodule will take a simple module to either zero or to an indecomposable projective module. But by the construction of $\mathscr{C}_{A,X}$ and Theorem 3.4.21 these are precisely $\mathbf{M}(F)$ for F indecomposable, giving essential surjectivity.

By the construction of $\mathscr{C}_{A,X}$, $\mathbf{M}$ is surjective when applied to $\mathrm{End}_{\mathscr{C}}(\mathbb{1}_\mathtt{i})$. For any other hom-space $\mathrm{Hom}_{\mathscr{C}}(F,G)$ with $F,G \neq \mathbb{1}_\mathtt{i}$, by the definition of $\mathcal{J}$-2-fullness, it clearly suffices to show that $\mathbf{M}$ is $\mathcal{J}$-2-full. We will do this and show the remaining cases as a three step process: we will show that

$$\mathrm{Hom}_{\mathscr{C}}(G_{\mathscr{L}}, \mathbb{1}_\mathtt{i}) \to \mathrm{Hom}_{\mathbf{M}(\mathtt{i})}(\mathbf{M}(G_{\mathscr{L}}), \mathbf{M}(\mathbb{1}_\mathtt{i}))$$

is surjective for the Duflo involution $G_{\mathscr{L}}$, that this implies surjectivity for any

$$\mathrm{Hom}_{\mathscr{C}}(F, \mathbb{1}_{\mathtt{j}}) \to \mathrm{Hom}_{\mathbf{M}(\mathtt{j})}(\mathbf{M}(F), \mathbf{M}(\mathbb{1}_{\mathtt{j}}))$$

and then derive that each $\mathbf{M}_{\mathtt{i},\mathtt{j}}$ is indeed full.

These three steps are the generalisations of [MM16a] Theorem 9 (specifically the proof of that Theorem), Proposition 6 and Corollary 8 respectively. To give the generalisations, we thus first need to generalise [MM16a] Lemma 7. While this is a 1-categorical statement, the way in which we use it requires us to give a slight generalisation and thus manipulate the proof a little:

Lemma 3.4.25. *Let A be a countable product of finite dimensional connected $\Bbbk$-algebras and let e and f be primitive idempotents of A. Assume that F is an exact endofunctor of A-mod such that $FL_f \cong Ae$ and $FL_g = 0$ for any other simple $L_g \ncong L_f$. Then F is isomorphic to the functor F' given by tensoring with the bimodule $Ae \otimes_{\Bbbk} fA$, and moreover*

$$\mathrm{Hom}_{\mathfrak{R}_{\Bbbk}}(F, id_{A\text{-mod}}) \cong \mathrm{Hom}_A(Ae, Af).$$

Proof. As e and f are primitive idempotents, they each belong to A_e and A_f for finite dimensional connected components A_e and A_f of A. Thus without loss of generality we can restrict F to A'-mod, where $A' = A_e \times A_f$, which is a finite dimensional connected algebra. Hence we can apply the original form of the lemma in [MM16a] and the result follows. □

Using Lemma 3.4.25 and Proposition 3.4.4 we can generalise the proof of [MM16a] Theorem 9 directly, since it is a local proof which does not use any properties of involutions. We give our version of that result:

Proposition 3.4.26. *The representation map*

$$\mathrm{Hom}_{\mathscr{C}}(G_{\mathscr{L}}, \mathbb{1}_{\mathtt{i}}) \to \mathrm{Hom}_{\mathbf{M}(\mathtt{i})}(\mathbf{M}(G_{\mathscr{L}}), \mathbf{M}(\mathbb{1}_{\mathtt{i}}))$$

is surjective.

However, the proofs of [MM16a] Proposition 6 and Corollary 8 do use that $-^*$ is an involution in that paper. For Proposition 6 we will give an adaptation of the whole proof, reworked to avoid the involution issues.

Proposition 3.4.27. *Assuming that the representation map*

$$\mathrm{Hom}_{\mathscr{C}}(F, \mathbb{1}_{\mathtt{j}}) \to \mathrm{Hom}_{\mathbf{M}(\mathtt{j})}(\mathbf{M}(F), \mathbf{M}(\mathbb{1}_{\mathtt{j}}))$$

is surjective for $F = G_{\mathscr{L}}$ and $\mathtt{j} = \mathtt{i}$, then it is surjective for any F and $\mathtt{j}$.

Proof. Without loss of generality $F \in \mathscr{C}(\mathtt{j}, \mathtt{j})$. Let $H, K \in \mathscr{L}$ for some $\mathscr{L}$-cell $\mathscr{L}$ of $\mathscr{J}$ and assume that $H, K \in \mathscr{C}(\mathtt{j}, \mathtt{k})$. By strong regularity $HK^* \cong aX$ for some $X \in \mathscr{J}$ and some non-negative integer a. Since $\mathscr{J}$ consists of a single $\mathscr{D}$-cell, we can vary H and K over $\mathscr{L}$ to get any element of $\mathscr{J}$, and in particular we can choose H and K such that $HK^* \cong aF$ for some non-negative integer a. To show that $HK^* \neq 0$, note $K^*\mathrm{L}_K \cong \mathtt{I}_{G_{\mathscr{L}}} \in \overline{\mathbf{C}_{\mathscr{L}}}(\mathtt{j})$ by Proposition 3.4.9 (which still applies to the cell 2-representation case) and further $H\mathtt{I}_{G_{\mathscr{L}}} \neq 0$ as $H\mathrm{L}_{G_{\mathscr{L}}} \neq 0$ by Proposition 3.4.4. It follows that $HK^* \neq 0$.

Similarly, ${}^*KH \cong bG_{\mathscr{L}}$ for some non-negative integer b. In addition, since $-^*$ is an anti-auto-equivalence, it follows that

$$\mathrm{Hom}_{\mathscr{C}}(H, K) \cong \mathrm{Hom}_{\mathscr{C}}(K^*, H^*).$$

Applying adjunctions, we have that

$$\mathrm{Hom}_{\mathscr{C}}(H, K) \cong b\,\mathrm{Hom}_{\mathscr{C}}(G_{\mathscr{L}}, \mathbb{1}_{\mathtt{i}}), \qquad \mathrm{Hom}_{\mathscr{C}}(K^*, H^*) \cong a\,\mathrm{Hom}_{\mathscr{C}}(F, \mathbb{1}_{\mathtt{j}}).$$

Evaluating $\mathrm{Hom}_{\mathscr{C}}(H, K)$ at $\mathrm{L}_{G_{\mathscr{L}}}$ is surjective, and thus

$$\mathrm{Hom}_{\mathbf{M}(\mathtt{j})}(H\mathrm{L}_{G_{\mathscr{L}}}, K\mathrm{L}_{G_{\mathscr{L}}}) \cong b\,\mathrm{Hom}_{\mathbf{M}(\mathtt{i})}(G_{\mathscr{L}}\mathrm{L}_{G_{\mathscr{L}}}, \mathrm{L}_{G_{\mathscr{L}}}).$$

Applying Proposition 3.4.4 gives that $G_{\mathscr{L}}\mathrm{L}_{G_{\mathscr{L}}}$ has simple top $\mathrm{L}_{G_{\mathscr{L}}}$. Therefore the space $\mathrm{Hom}_{\mathbf{M}(\mathtt{i})}(G_{\mathscr{L}}\mathrm{L}_{G_{\mathscr{L}}}, \mathrm{L}_{G_{\mathscr{L}}})$ is one-dimensional and

$$b = \dim \mathrm{Hom}_{\mathbf{M}(\mathtt{j})}(H\mathrm{L}_{G_{\mathscr{L}}}, K\mathrm{L}_{G_{\mathscr{L}}}).$$

Let $\mathrm{L}_{\mathtt{j}}$ denote a multiplicity-free direct sum of all simple modules in $\mathbf{M}(\mathtt{j})$. By adjunction $\mathrm{Hom}_{\mathbf{M}(\mathtt{i})}(K^*\mathrm{L}_{\mathtt{j}}, H^*\mathrm{L}_{\mathtt{j}}) \cong a\,\mathrm{Hom}_{\mathbf{M}(\mathtt{j})}(F\mathrm{L}_{\mathtt{j}}, \mathrm{L}_{\mathtt{j}})$. By Proposition 3.4.1 $K^*\mathrm{L}_Q \neq 0$ for $Q \in \mathscr{L}$ if and only if Q is in the same $\mathscr{R}$-cell as K. But then by strong regularity $K \cong Q$. Thus by Proposition 3.4.9 $K^*\mathrm{L}_{\mathtt{j}} \cong \mathrm{I}_{G_{\mathscr{L}}}$. By a similar argument $H^*\mathrm{L}_{\mathtt{j}} \cong \mathrm{I}_{G_{\mathscr{L}}}$ and the left side of the above isomorphism is isomorphic to $\mathrm{End}_{\mathbf{M}(\mathtt{i})}(\mathrm{I}_{G_{\mathscr{L}}})$.

As F is a direct summand of HK^*, it follows that L_K is the only summand of $\mathrm{L}_{\mathtt{j}}$ not annihilated by F. By Theorem 3.4.21 $F\mathrm{L}_K$ is an indecomposable projective in $\mathbf{M}(\mathtt{j})$, and thus by strong regularity we must have $F\mathrm{L}_K \cong \mathrm{P}_H$. Therefore $\dim \mathrm{Hom}_{\mathbf{M}(\mathtt{j})}(F\mathrm{L}_{\mathtt{j}}, \mathrm{L}_{\mathtt{j}}) = 1$ and $a = \dim \mathrm{End}_{\mathbf{M}(\mathtt{i})}(\mathrm{I}_{G_{\mathscr{L}}})$.

From Lemma 3.4.22 it follows that $H\mathrm{L}_{G_{\mathscr{L}}}$ is an indecomposable projective in $\mathscr{M}$ with simple top L_H, and is thus isomorphic to P_H. It follows that $\mathrm{I}_{G_{\mathscr{L}}} \cong \mathrm{P}_{G_{\mathscr{L}}^*}$. Using Proposition 3.4.26 we can apply Lemma 3.4.25 to $\mathrm{Hom}_{\mathscr{C}}(G_{\mathscr{L}}, \mathbb{1}_{\mathtt{i}})$, and thus $\mathrm{Hom}_{\mathscr{C}}(G_{\mathscr{L}}, \mathbb{1}_{\mathtt{i}}) \cong \mathrm{End}_{\mathbf{M}(\mathtt{i})}(\mathrm{P}_{G_{\mathscr{L}}})$.

We now show that $\dim \mathrm{End}_{\mathbf{M}(\mathtt{i})}(\mathrm{P}_{G_{\mathscr{L}}}) = \dim \mathrm{End}_{\mathbf{M}(\mathtt{i})}(\mathrm{P}_{G_{\mathscr{L}}^*})$. Take some finite dimensional $\Bbbk$-algebra A such that $\mathbf{M}(\mathtt{i})$ is equivalent to A-mod. We can thus consider $\mathrm{P}_{G_{\mathscr{L}}}$ to be isomorphic to $Ae_{G_{\mathscr{L}}}$ for some idempotent $e_{G_{\mathscr{L}}}$ of A. Hence $\mathrm{End}_{\mathbf{M}(\mathtt{i})}(\mathrm{P}_{G_{\mathscr{L}}}) \cong \mathrm{End}_{A\text{-}\mathrm{mod}}(Ae_{G_{\mathscr{L}}}) \cong e_{G_{\mathscr{L}}}Ae_{G_{\mathscr{L}}}$. But on the other hand

$$\mathrm{End}_{A\text{-}\mathrm{mod}}(Ae_{G_{\mathscr{L}}}) \cong \mathrm{End}_{\mathrm{mod}\text{-}A}(e_{G_{\mathscr{L}}}A) \cong \mathrm{End}_{A\text{-}\mathrm{mod}}(Ae_{\sigma(G_{\mathscr{L}})}),$$

where σ is the permutation defined by the weakly fiat structure on $\mathscr{C}$. But by this same structure $e_{\sigma(G_{\mathscr{L}})} = e_{G_{\mathscr{L}}^*}$, i.e. $Ae_{\sigma(G_{\mathscr{L}})}$ is isomorphic to $P_{G_{\mathscr{L}}^*}$.

We thus have that

$$\dim \operatorname{Hom}_{\mathscr{C}}(G, \mathbb{1}_{\mathtt{i}}) = \dim \operatorname{End}_{\mathbf{M}(\mathtt{i})}(P_{G_{\mathscr{L}}}) = \dim \operatorname{End}_{\mathbf{M}(\mathtt{i})}(P_{G^*_{\mathscr{L}}}).$$

Using the above results and Lemma 3.4.25, we have

$$\begin{aligned}\dim \operatorname{Hom}_{\mathscr{C}}(H, K) &= \dim \operatorname{Hom}_{\mathbf{M}(\mathtt{j})}(H\mathtt{L}_{G_{\mathscr{L}}}, K\mathtt{L}_{G_{\mathscr{L}}}) \dim \operatorname{End}_{\mathbf{M}(\mathtt{i})}(P_{G^*_{\mathscr{L}}}) \\ &= \dim \operatorname{Hom}_{\mathbf{M}(\mathtt{j})}(\mathtt{P}_H, \mathtt{P}_K) \dim \operatorname{End}_{\mathbf{M}(\mathtt{i})}(\mathtt{P}_{G^*_{\mathscr{L}}})\end{aligned}$$

and

$$\begin{aligned}\dim \operatorname{Hom}_{\mathscr{C}}(K^*, H^*) &= \dim \operatorname{Hom}_{\mathscr{C}}(F, \mathbb{1}_{\mathtt{i}}) \dim \operatorname{End}_{\mathbf{M}(\mathtt{i})}(\mathtt{I}_{G_{\mathscr{L}}}) \\ &= \dim \operatorname{Hom}_{\mathscr{C}}(F, \mathbb{1}_{\mathtt{i}}) \dim \operatorname{End}_{\mathbf{M}(\mathtt{i})}(\mathtt{P}_{G^*_{\mathscr{L}}}).\end{aligned}$$

As $\mathscr{C}$ is $\mathscr{J}$-simple,

$$\dim \operatorname{Hom}_{\mathscr{C}}(F, \mathbb{1}_{\mathtt{j}}) \leq \dim \operatorname{Hom}_{\mathfrak{R}_{\Bbbk}}(\mathbf{M}(F), \mathbf{M}(\mathbb{1}_{\mathtt{j}}))$$

and applying Lemma 3.4.25 we see the latter is equal to $\dim \operatorname{Hom}_{\mathbf{M}(\mathtt{j})}(\mathtt{P}_H, \mathtt{P}_K)$. Dividing by $\operatorname{End}_{\mathbf{M}(\mathtt{i})}(\mathtt{P}_{G^*_{\mathscr{L}}})$,

$$\begin{aligned}\dim \operatorname{Hom}_{\mathbf{M}(\mathtt{j})}(\mathtt{P}_H, \mathtt{P}_K) &= \dim \operatorname{Hom}_{\mathscr{C}}(F, \mathbb{1}_{\mathtt{i}}) \\ &\leq \dim \operatorname{Hom}_{\mathfrak{R}_{\Bbbk}}(\mathbf{M}(F), \mathbf{M}(\mathbb{1}_{\mathtt{j}})) \\ &= \dim \operatorname{Hom}_{\mathbf{M}(\mathtt{j})}(\mathtt{P}_H, \mathtt{P}_K)\end{aligned}$$

where the last equality follows by applying Lemma 3.4.25. Therefore

$$\dim \operatorname{Hom}_{\mathscr{C}}(F, \mathbb{1}_{\mathtt{i}}) = \dim \operatorname{Hom}_{\mathfrak{R}_{\Bbbk}}(\mathbf{M}(F), \mathbf{M}(\mathbb{1}_{\mathtt{i}})).$$

Since the representation map is injective by $\mathscr{J}$-simplicity of $\mathscr{C}$, we get surjection and the result is proved. □

Proposition 3.4.28. *Let* $H, K \in \mathcal{J} \cap \mathscr{C}(\mathtt{j}, \mathtt{k})$. *If the representation map*

$$\mathrm{Hom}_{\mathscr{C}}(G_{\mathcal{L}}, \mathbb{1}_{\mathtt{i}}) \to \mathrm{Hom}_{\mathbf{M}(\mathtt{i})}(\mathbf{M}(G_{\mathcal{L}}), \mathbf{M}(\mathbb{1}_{\mathtt{i}}))$$

is surjective, then so is the representation map

$$\mathrm{Hom}_{\mathscr{C}}(H, K) \to \mathrm{Hom}_{\mathbf{M}}(\mathbf{M}(H), \mathbf{M}(K)).$$

Proof. The proof is mostly an immediate generalisation of the one for [MM16a] Corollary 8, except that ${}^*KH$ needs to be read for K^*H. □

From this it follows immediately that $\mathbf{M}$ is $\mathcal{J}$-2-full and each $\mathbf{M}_{\mathtt{ij}}$ is full. Therefore the main theorem is proven. □

3.4.6 Reducing to the $\mathcal{J}$-Simple Case

The above result is useful because cell 2-representations classify all simple transitive 2-representations for $\mathscr{C}_{A,X}$ by Theorem 3.3.9. However, as it stands, it limits the categories we can expand this to. This section aims to bypass this restriction. We will assume for this section that $\mathscr{C}$ is a strongly regular locally weakly fiat 2-category. The proofs in this section are straightforward generalisations.

Lemma 3.4.29. *Let* $\mathbf{M}$ *be a simple transitive 2-representation of* $\mathscr{C}$. *Then there exists some* $\mathscr{J}$*-cell* $\mathcal{J}$ *such that* $\mathbf{M}$ *factors over* $\mathscr{C}_{\not\leq \mathcal{J}}$ *and the restriction* $\mathbf{M}^{\mathcal{J}}_{\not\leq \mathcal{J}}$ *of this to* $\mathscr{C}^{\mathcal{J}}_{\not\leq \mathcal{J}}$ *is still simple transitive.*

Proof. The first part of the proof of [MM16c] Theorem 18, which generalises immediately to our setting, gives that there is a unique maximal $\mathscr{J}$-cell of $\mathscr{C}$ that does not annihilate $\mathbf{M}$. We choose $\mathcal{J}$ to be this unique maximal $\mathscr{J}$-cell. This is (the generalisation of) the apex as defined in 2.3.20. If F is a 1-morphism of $\mathscr{C}$ such that F is not annihilated by $\mathbf{M}$, then consider the $\mathscr{J}$-cell $\mathcal{K}$ containing F. As

$\mathscr{J}$-cells are partially ordered by $\leq_J$, we must have that $\mathscr{K} \leq_J \mathcal{J}$. Therefore passing to $\mathscr{C}_{\not\leq \mathcal{J}}$ we can assume that $\mathcal{J}$ is the unique maximal $\mathscr{J}$-cell of $\mathscr{C}$.

We note that $\mathbf{M}$ restricts to a 2-representation $\mathbf{M}^{\mathcal{J}}_{\not\leq \mathcal{J}}$ of $\mathscr{C}^{\mathcal{J}}_{\not\leq \mathcal{J}}$. The argument given in the proof of [MM16c] Theorem 18 generalises immediately to the locally weakly fiat case, and it follows that $\mathbf{M}^{\mathcal{J}}_{\not\leq \mathcal{J}}$ is simple transitive as required. □

By construction, $\mathscr{C}^{\mathcal{J}}_{\not\leq \mathcal{J}}$ has $\mathcal{J}$ as its unique maximal $\mathscr{J}$-cell. We now give the following generalisation of [MM14] Lemma 18:

Lemma 3.4.30. *There is a unique 2-ideal $\mathscr{I}$ of $\mathscr{C}^{\mathcal{J}}_{\not\leq \mathcal{J}}$ such that $\mathscr{C}^{\mathcal{J}}_{\not\leq \mathcal{J}}/\mathscr{I}$ is $\mathcal{J}$-simple.*

Proof. The proof of this result generalises immediately from that of [MM14] Lemma 18. □

We let $\mathscr{C}_{\mathcal{J}}$ denote this quotient. We denote by $\mathbf{M}_{\mathcal{J}}$ the restriction of $\mathbf{M}$ to $\mathscr{C}_{\mathcal{J}}$ (with $\mathscr{M}_{\mathcal{J}}$ the corresponding coproduct category). We claim that $\mathbf{M}_{\mathcal{J}}$ is a transitive 2-representation of $\mathscr{C}_{\mathcal{J}}$. To see this, since $\mathcal{J}$ is the unique maximal $\mathscr{J}$-cell not annihilated by $\mathbf{M}$, it follows immediately that $\ker(\mathbf{M}) \subseteq \mathcal{J}$. Second, let $N \in \mathscr{M}_{\mathcal{J}}$, and let $F \in \mathcal{J}$. Since $\mathbf{M}$ is a transitive 2-representation, any $M \in \mathscr{M}$ is isomorphic to a direct summand of GFN for some 1-morphism $G \in \mathscr{C}$. But by the construction of $\mathscr{J}$-cells, all indecomposable summands of GF are in $\mathcal{J}$, and thus $GF \in \mathscr{C}_{\mathcal{J}}$ and hence $\mathbf{M}_{\mathcal{J}}$ is indeed transitive.

As $\mathscr{C}_{\mathcal{J}}$ is a $\mathcal{J}$-simple category with a unique non-trivial two-sided ideal, it is biequivalent to $\mathscr{C}_{A,X}$ for some A and X. By a simple generalisation of a previous result, any simple transitive 2-representation of any $\mathscr{C}_{A,X}$ is equivalent to a cell 2-representation. We thus have that $\mathbf{M}_{\mathcal{J}}$ is equivalent to $(\mathbf{C}_{\mathscr{L}})_{\mathcal{J}}$ for some $\mathscr{L}$-cell $\mathscr{L}$ of $\mathcal{J}$. We now provide a lemma that, along with Lemma 3.4.29, will allow us to generalise [MM16c] Theorem 18:

Lemma 3.4.31. *If $\mathbf{M}$ is a simple transitive 2-representation of $\mathscr{C}$ such that $\mathbf{M}_{\mathcal{J}}$ is equivalent to some cell 2-representation of $\mathscr{C}_{\mathcal{J}}$, then $\mathbf{M}$ is equivalent to some cell 2-representation of $\mathscr{C}$.*

Proof. The second half of the proof of [MM16c] Theorem 18 generalises immediately. □

Hence we have:

Theorem 3.4.32. *Any simple transitive 2-representation of $\mathscr{C}$ is equivalent to a cell 2-representation of $\mathscr{C}$.*

Proof. This is a direct consequence of applying Lemma 3.4.31 to the result of Lemma 3.4.29. □

3.5 An Application: 2-Kac-Moody Algebras

We present an immediate application of even this initial generalisation of finitary 2-representation theory to the locally finitary setup, which is the classification of the simple transitive 2-representations of 2-Kac-Moody algebras.

3.5.1 Classical Kac-Moody Algebras

We begin by defining (1-)Kac-Moody algebras. We take our definitions, results and notation as a mixture of those given in [KK12] and [HK02], since each has some benefits and drawbacks. We work over some (algebraically closed) field $\Bbbk$ unless otherwise stated.

Definition 3.5.1. For a finite index set I, a square matrix $A = (a_{ij})_{i,j\in I}$ over $\mathbb{Z}$ is a *generalised Cartan matrix* if it satisfies:

i) $a_{ii} = 2$;

ii) $a_{ij} \leq 0$ if $i \neq j$;

iii) $a_{ij} = 0$ if and only if $a_{ji} = 0$;

If A is symmetrisable by a diagonal matrix, we denote by $D = \mathrm{diag}(d_i | i \in I)$ the symmetrising matrix.

Definition 3.5.2. A *Cartan datum* is an ordered quintuple $(A, P, \Pi, P^\vee, \Pi^\vee)$ where:

i) A is a symmetrisable generalised Cartan matrix,

ii) P is a free abelian group of finite rank, called the *weight lattice*,

iii) $\Pi = \{\alpha_i \in P | i \in I\}$ is a set of elements of P called the *simple roots*,

iv) $P^\vee = \mathrm{Hom}(P, \mathbb{Z})$ is called the *dual weight lattice*, and

v) $\Pi^\vee = \{h_i | i \in I\} \subseteq P^\vee$ is the set of *simple coroots*.

We require these to satisfy the following properties:

i) $\langle h_i, \alpha_j \rangle = a_{ij}\ \forall i, j \in I$,

ii) Π is linearly independent, and

iii) $\forall i \in I\ \exists \Lambda_i \in P$ such that $\langle h_j, \Lambda_i \rangle = \delta_{ij}\ \forall i, j \in I$.

The Λ_i are called *fundamental weights*.

Definition 3.5.3. We define the set of *dominant integral weights* to be

$$P^+ = \{\lambda \in P | \langle h_i, \lambda \rangle \in \mathbb{Z}_0^+ \quad \forall i \in I\}.$$

Definition 3.5.4. We define a free abelian group $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$, called the *root lattice*, and let $Q^+ = \sum_{i \in I} \mathbb{Z}_0^+ \alpha_i$. If $\alpha = \sum_{i \in I} k_i \alpha_i \in Q^+$, we define the *height* of α to be $|\alpha| = \sum k_i$.

We define $\mathfrak{h} = \Bbbk \otimes_{\mathbb{Z}} P^\vee$. As A is symmetrisable, there is a symmetric bilinear form $(\cdot|\cdot)$ on $\mathfrak{h}^*$ that satisfies $(\alpha_i | \alpha_j) = d_i a_{ij}$ and $\langle h_i, \lambda \rangle = \frac{2(\alpha_i|\lambda)}{(\alpha_i|\alpha_i)}$ for any $\lambda \in \mathfrak{h}^*$ and $i \in I$. We also have a partial ordering on $\mathfrak{h}^*$ defined by $\lambda \geq \mu$ if and only if $\lambda - \mu \in Q^+$.

Definition 3.5.5. The *Kac-Moody algebra* $\mathfrak{g}$ associated with the Cartan datum $(A, P, \Pi, P^\vee, \Pi^\vee)$ is a Lie algebra generated over $\Bbbk$ by the elements e_i, f_i $(i \in I)$ and $h \in P^\vee$ subject to the Lie bracket relations:

- $[h, h'] = 0$ for $h, h' \in P^\vee$;
- $[e_i, f_j] = \delta_{ij} h_i$;
- $[h, e_i] = \langle h, \alpha_i \rangle e_i$ for $h \in P^\vee$;
- $[h, f_i] = -\langle h, \alpha_i \rangle f_i$ for $h \in P^\vee$;
- $(\mathrm{ad}_{e_i})^{1-a_{ij}} e_j = 0$ if $i \neq j$;
- $(\mathrm{ad}_{f_i})^{1-a_{ij}} f_j = 0$ if $i \neq j$.

Here $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$, $x \mapsto \mathrm{ad}_x$ is defined as $\mathrm{ad}_x(y) = [x, y]$

We define $\mathfrak{g}_+$ (respectively $\mathfrak{g}_-$) as the subalgebra of $\mathfrak{g}$ generated by the e_i (respectively the f_i), and for $\alpha \in Q$, we let

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} | [h, x] = \langle h, \alpha \rangle x \; \forall h \in \mathfrak{h}\}.$$

It can be shown that:

Proposition 3.5.6 ([HK02] Proposition 2.1.4)**.** *There are vector space decompositions*

- $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{h} \oplus \mathfrak{g}_-$ *(the* triangular decomposition*);*
- $\mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha$ *with* $\dim \mathfrak{g}_\alpha < \infty \; \forall \alpha \in Q$ *(the* root space decomposition*).*

Definition 3.5.7. If $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$, then we justify the name of the above decomposition by calling α a *root* of $\mathfrak{g}$, $\mathfrak{g}_\alpha$ the *root space* attached to α and $\dim \mathfrak{g}_\alpha$ the *root multiplicity* of α.

Since $\mathfrak{g}$ is a Lie algebra, we can also consider its universal enveloping algebra:

Definition 3.5.8 (Proposition 2.1.6, [HK02])**.** The *universal enveloping algebra* $U(\mathfrak{g})$ for a Kac-Moody algebra $\mathfrak{g}$ is the associative unital algebra over $\Bbbk$ (with unit) generated by e_i, f_i $(i \in I)$ and some $\Bbbk$-basis H of $\mathfrak{h}$ such that:

- $hh' = h'h$ for all $h, h' \in H$;
- $e_i f_j - f_j e_i = \delta_{ij} h_i$ for $i, j \in I$;
- $he_i - e_i h = \langle h, \alpha_i \rangle e_i$ for $h \in \mathfrak{h}$, $i \in I$;
- $hf_i - f_i h = -\langle h, \alpha_i \rangle f_i$ for $h \in \mathfrak{h}$, $i \in I$;
- $\sum\limits_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} e_i^{1-a_{ij}-k} e_j e_i^k = 0$ for $i \neq j$;
- $\sum\limits_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} f_i^{1-a_{ij}-k} f_j f_i^k = 0$ for $i \neq j$.

Definition 3.5.9. Similarly to the Lie algebra itself, we define the subalgebras $U^+(\mathfrak{g})$, $U^0(\mathfrak{g})$ and $U^-(\mathfrak{g})$ of $U(\mathfrak{g})$ to be those generated by the e_i, $\mathfrak{h}$ and the f_i respectively. We also define the *root spaces* to be

$$U_\beta = U_\beta(\mathfrak{g}) = \{u \in U | hu - uh = \langle h, \beta \rangle u \text{ for all } h \in \mathfrak{h}\} \text{ for } \beta \in Q;$$

$$U_\beta^\pm = U_\beta^\pm(\mathfrak{g}) = \{u \in U | hu - uh = \langle h, \beta \rangle u \text{ for all } h \in \mathfrak{h}\} \text{ for } \beta \in Q_\pm.$$

Proposition 3.5.10 (Proposition 2.1.7, [HK02])**.** *1. As a vector space,*

$$U(\mathfrak{g}) \cong U^+(\mathfrak{g}) \otimes U^0(\mathfrak{g}) \otimes U^-(\mathfrak{g}).$$

2. *As a vector space,* $U(\mathfrak{g}) = \bigoplus\limits_{\beta \in Q} U_\beta$.
3. *As vector spaces,* $U^\pm(\mathfrak{g}) = \bigoplus\limits_{\beta \in Q} U_\beta^\pm$.

We now examine the representation theory of Kac-Moody algebras, and in particular we focus on a type of representation called weight modules.

Definition 3.5.11. A module M of a Kac-Moody algebra $\mathfrak{g}$ is a *weight module* if it has a *weight space decomposition* $V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu$ with

$$V_\mu = \{v \in V | hv = \mu(h)v \quad \forall h \in \mathfrak{h}\}.$$

Such a vector v is called a *weight vector* of weight μ. We denote the set of μ such that $V_\mu \neq 0$ as $\mathrm{wt}(V)$.

Definition 3.5.12. For $\lambda \in \mathfrak{h}^*$, we let $D(\lambda) = \{\mu \in \mathfrak{h}^* | \mu \leq \lambda\}$. Then we can define a category $\mathcal{O}(\mathfrak{g})$ which consists of weight modules over $\mathfrak{g}$ where every weight space V_μ has finite dimension, and where there exist $\lambda_1, \dots, \lambda_n \in \mathfrak{h}^*$ such that $\mathrm{wt}(V) \subseteq \bigcup_{i=1}^{n} D(\lambda_i)$. We take the morphisms to be $\mathfrak{g}$-module homomorphisms. Note that $\mathcal{O}(\mathfrak{g})$ is closed under finite direct sums, finite products and quotients.

As is standard for such situations, we ideally wish to classify these representations. We can indeed classify all irreducible $\mathfrak{g}$-modules in $\mathcal{O}(\mathfrak{g})$.

Definition 3.5.13. A weight module V of $\mathfrak{g}$ is a *highest weight module* with highest weight $\lambda \in \mathfrak{h}^*$ if there exists a nonzero $v_\lambda \in V$, the *highest weight vector* such that

- $e_i v_\lambda = 0 \quad \forall i \in I$;
- $hv_\lambda = \langle h, \lambda \rangle v_\lambda \ \forall h \in \mathfrak{h}$;
- $V = U(\mathfrak{g}) v_\lambda$.

We note that $\dim V_\lambda = 1$ and $\dim V_\mu < \infty$ for any $\mu \in \mathrm{wt}(V)$, and $V = \bigoplus_{\mu \leq \lambda} V_\mu$. Hence a highest weight module is an element of $\mathcal{O}(\mathfrak{g})$ for any λ.

We now wish to construct an irreducible highest weight module.

Definition 3.5.14. Fix $\lambda \in \mathfrak{h}^*$ and let $J(\lambda)$ be the left ideal of $U(\mathfrak{g})$ generated by $h - \lambda(h)1$ for $h \in \mathfrak{h}$ and all the e_i. We denote the quotient of $U(\mathfrak{g})$ by $J(\lambda)$ as $M(\lambda)$, and call it the *Verma module*.

Proposition 3.5.15 ([HK02] Proposition 2.3.3)**.** *1. $M(\lambda)$ is a highest weight $\mathfrak{g}$-module with highest weight λ and highest weight vector $1 + J(\lambda)$.*

2. *Every highest weight $\mathfrak{g}$-module with highest weight λ is a homomorphic image of $M(\lambda)$.*

3. *As a $U(\mathfrak{g})^-$-module, $M(\lambda)$ is free of rank 1, generated by the highest weight vector v_λ.*

4. *$M(\lambda)$ has a unique maximal submodule.*

We denote by $N(\lambda)$ the maximal submodule of $M(\lambda)$, and set $V(\lambda) = M(\lambda)/N(\lambda)$. We then get the following result:

Proposition 3.5.16 ([HK02] Proposition 2.3.4)**.** *Every irreducible $\mathfrak{g}$-module in $\mathcal{O}(\mathfrak{g})$ is isomorphic to $V(\lambda)$ for some $\lambda \in \mathfrak{h}^*$.*

3.5.2 Quantum Deformations of Kac-Moody Algebras

Our application, rather than being based directly on categorifications of Kac-Moody algebras, bases itself off the categorification of a quantum deformation of a Kac-Moody Algebra. We present the theory for this here, again pulling from [KK12] and [HK02].

Definition 3.5.17. Let q be an indeterminate and set $q_i = q^{\frac{(\alpha_i|\alpha_i)}{2}} = q^{d_i}$. For $m, n \in \mathbb{Z}_0^+$, we set $[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}$, $[n]_i! = \prod_{k=1}^n [k]_i$, $\begin{bmatrix} m \\ n \end{bmatrix}_i = \frac{[m]_i!}{[m-n]_i![n]_i!}$.

The concept of a quantum deformation is replacing integers with these 'quantum integers'.

Definition 3.5.18. The *quantum group* $U_q(\mathfrak{g})$ associated with the Kac-Moody algebra $\mathfrak{g}$ (equivalently with the Cartan datum $(A, P, \Pi, P^\vee, \Pi^\vee)$) is an associative algebra over $\Bbbk(q)$ generated by the e_i, f_i and K_h ($h \in P^\vee$) subject to:

- $K_0 = 1$, $K_h K_{h'} = K_{h+h'}$ for $h, h' \in P^\vee$;
- $K_h e_i K_{-h} = q^{\langle h, \alpha_i \rangle} e_i$; $K_h f_i K_{-h} = q^{\langle h, \alpha_i \rangle} f_i$ for $h \in P^\vee$;
- $e_i f_j - f_i e_j = \delta_{ij} \frac{K_{d_i h_i} - K_{-d_i h_i}}{q_i - q_i^{-1}}$;

- $\sum_{k=0}^{1-a_{ij}} (-1)^k \left[\begin{smallmatrix}1-a_{ij} \\ k\end{smallmatrix}\right]_i e_i^{1-a_{ij}-k} e_j e_i^k = 0$ if $i \neq j$;
- $\sum_{k=0}^{1-a_{ij}} (-1)^k \left[\begin{smallmatrix}1-a_{ij} \\ k\end{smallmatrix}\right]_i f_i^{1-a_{ij}-k} f_j f_i^k = 0$ for $i \neq j$.

If we define $U_q^+(\mathfrak{g})$ and $U_q^-(\mathfrak{g})$ as the subalgebras of $U_q(\mathfrak{g})$ generated by the e_i and f_i respectively, and $U_q^0(\mathfrak{g})$ the subalgebra generated by the K_h, then we again have a triangular decomposition $U_q(\mathfrak{g}) \cong U_q^-(\mathfrak{g}) \otimes U_q^0(\mathfrak{g}) \otimes U_q^+(\mathfrak{g})$, and if we define

$$U_q(\mathfrak{g})_\alpha = \{x \in U_q(\mathfrak{g}) | K_h x K_{-h} = q^{\langle h,\alpha \rangle} x \forall h \in P^\vee\},$$

we further get the weight space decomposition $U_q(\mathfrak{g}) = \bigoplus_{\alpha \in Q} U_q(\mathfrak{g})_\alpha$.

We will wish to work with the idempotent completion $\dot{U}_q(\mathfrak{g})$ of $U_q(\mathfrak{g})$. As defined in [Lus10] 23.1.1, for any $\alpha, \beta \in Q$, we define

$${}_\alpha U_q(\mathfrak{g})_\beta = U_q(\mathfrak{g}) / (\sum_{h \in P^\vee} (K_h - q^{\langle h,\alpha \rangle}) U_q(\mathfrak{g}) + \sum_{h \in P^\vee} U_q(\mathfrak{g})(K_h - q^{\langle h,\beta \rangle})),$$

and set $\dot{U}_q(\mathfrak{g}) = \bigoplus_{\alpha,\beta \in Q} {}_\alpha U_q(\mathfrak{g})_\beta$. The local identities are the image of $1_{U_q(\mathfrak{g})}$ under the natural projection maps.

We now move on to the representation theory of the quantum variants, drawing from both [KK12] and [HK02]. This will be important for defining the categorification of the Kac-Moody algebra.

Definition 3.5.19. For a Kac-Moody algebra $\mathfrak{g}$ and its associated quantum group $U_q(\mathfrak{g})$, a $U_q(\mathfrak{g})$-module M is a *weight module* if it decomposes as $M = \bigoplus_{\mu \in P} M_\mu$ where $M_\mu = \{v \in M | K_h v = q^{\langle h,\mu \rangle} v \ \forall h \in P^\vee\}$.

Definition 3.5.20. Similarly to above, we define a *highest weight module* M of $U_q(\mathfrak{g})$ of *highest weight* Λ to be a $U_q(\mathfrak{g})$-module with a *highest weight vector* $v_\Lambda \in M$ such that:

1. $e_i v_\Lambda = 0 \ \forall i \in I$
2. $K_h v_\Lambda = q^{\langle h,\Lambda \rangle} v_\Lambda \ \forall h \in P^\vee$

3. $M = U_q(\mathfrak{g})v_\Lambda$.

Again, we have a unique irreducible highest weight module $V_q(\Lambda)$ for each $\Lambda \in P$.

Definition 3.5.21. A weight module M is *integrable* if for each $m \in M$ there exists some positive integer s such that $e_i^s m = f_i^s m = 0$ for any $i \in I$. We say that the e_i and f_i act on M *locally nilpotently* if this holds.

Proposition 3.5.22. *[[KK12] Proposition 2.3]*

a) If M is an integrable highest weight module of weight Λ, then $M \cong V_q(\Lambda)$.

b) For v_Λ a highest weight vector in $V_q(\Lambda)$, $f_i^{\langle h_i,\Lambda\rangle+1}v_\Lambda = 0\ \forall i \in I$.

Let $\mathbf{A} = \mathbb{Z}[q, q^{-1}]$, and $e_i^{(n)} = \frac{e_i^n}{[n]_i!}$ and $f_i^{(n)} = \frac{f_i^n}{[n]_i!}$. We define $U_{\mathbf{A}}(\mathfrak{g})$ to be the $\mathbf{A}$-subalgebra of $U_q(\mathfrak{g})$ generated by the $e_i^{(n)}$, the $f_i^{(n)}$ and K_h for $h \in P^\vee$.

Definition 3.5.23. We define the $\mathbf{A}$-*form* $V_{\mathbf{A}}(\Lambda)$ of $V_q(\Lambda)$ as $V_{\mathbf{A}}(\Lambda) = U_{\mathbf{A}}(\mathfrak{g})v_\Lambda$.

As with the classical case, we can find the (quantum) Verma module and use that to derive the irreducible highest weight module of weight Λ. In this case, let the left ideal $J_q(\Lambda) \lhd U_q(\mathfrak{g})$ be generated by the e_i and by $K_h - q^{\langle h,\Lambda\rangle}1$ for $h \in P^\vee$. We then let $M_q(\Lambda) = U_q(\mathfrak{g})/J_q(\Lambda)$. As before ([HK02] Proposition 3.2.2), $M_q(\Lambda)$ is a highest weight module with a unique maximal submodule $N_q(\Lambda)$, and the quotient is $V_q(\Lambda)$.

3.5.3 Khovanov-Lauda-Rouquier Algebras

We now examine Khovanov-Lauda-Rouquier algebras, and particularly their cyclotomic algebras, and show that we can use them to construct an object that decategorifies to the $V_{\mathbf{A}}(\Lambda)$ defined above. We follow [KK12] throughout the following two subsections.

We work with a Cartan datum $(A, P, \Pi, P^\vee, \Pi^\vee)$ with indexing set I as we did with Kac-Moody algebras. We also use a (non-negatively graded) base ring $\Bbbk = \bigoplus_{n\in\mathbb{Z}_{\geq 0}} \Bbbk_n$.

We first define a matrix $Q(u,v)$ over $\Bbbk[u,v]$ such that $Q_{ij}(u,v) = Q_{ji}(v,u)$ and

$$Q_{ij}(u,v) = \begin{cases} 0 & \text{if } i = j \\ \sum\limits_{p,q\geq 0} t_{i,j;p,q}u^p v^q & \text{if } i \neq j \end{cases}$$

where $t_{i,j;p,q} \in \Bbbk_{-2(\alpha_i|\alpha_j)-(\alpha_i|\alpha_i)p-(\alpha_j|\alpha_j)q}$ and $t_{i,j;-a_{ij},0} \in \Bbbk_0^\times$. In some sense, Q is a matrix that measures the degree and fashion that symmetry is broken in the definition below. Later, we will specify to a trivially graded field, so that $t_{i,j;p,q}$ is zero whenever $p + \frac{d_j}{d_i}q \neq -a_{ij}$. Despite this trivial grading, this still allows for non-trivial definitions of Q.

As a quick notational note, we let S_n be the symmetric group on n letters, and we denote by s_i the transposition $(i, i+1)$.

Definition 3.5.24. Given a Cartan datum $(A, P, \Pi, P^\vee, \Pi^\vee)$ and a matrix (Q_{ij}) as defined above, the *Khovanov-Lauda-Rouquier (KLR) algebra* $R(n)$ of degree n as the associative algebra over $\Bbbk$ generated by $e(\nu)$ for $\nu \in I^n$, x_k for $1 \leq k \leq n$ and τ_l for $1 \leq l \leq n-1$ subject to the following:

- $e(\nu)e(\nu') = \delta_{\nu,\nu'}e(\nu)$, $\sum\limits_{\nu\in I^n} e(\nu) = 1$;
- $x_k x_l = x_l x_k$, $x_k e(\nu) = e(\nu)x_k$;
- $\tau_l e(\nu) = e(s_l\nu)\tau_l$, $\tau_k\tau_l = \tau_l\tau_k$ if $|k-l| > 1$;
- $\tau_k^2 e(\nu) = Q_{\nu_k,\nu_{k+1}}(x_k, x_{k+1})e(\nu)$;
- $(\tau_k x_l - x_{s_k(l)}\tau_k)e(\nu) = \begin{cases} -e(\nu) & \text{if } l = k, \nu_k = \nu_{k+1} \\ e(\nu) & \text{if } l = k+1, \nu_k = \nu_{k+1} \\ 0 & \text{otherwise;} \end{cases}$
- $(\tau_{k+1}\tau_k\tau_{k+1} - \tau_k\tau_{k+1}\tau_k)e(\nu) =$
$\begin{cases} \frac{Q_{\nu_k,\nu_{k+1}}(x_k,x_{k+1}) - Q_{\nu_{k+2},\nu_{k+1}}(x_{k+2},x_{k+1})}{x_k - x_{k+2}} & \text{if } \nu_k = \nu_{k+2} \\ 0 & \text{otherwise.} \end{cases}$

The $e(\nu)$ form a complete set of orthogonal idempotents. Further, this can be considered in some sense a deformation $\Bbbk[x_1, \dots, x_n] \otimes \Bbbk S_n$. Then the matrix Q measures the amount by which this deviates from $\Bbbk S_n$ - specifically, in S_n, $s_i^2 = 1$, $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ and $s_i s_j = s_j s_i$ if $|i-j| > 1$, and Q determines how the τ_l deviate from this.

As some particular and universal examples, $R(0) \cong \Bbbk$, and $R(1) \cong (\bigoplus_{i \in I} \Bbbk e(i))[x_1]$. We can apply a $\mathbb{Z}$-grading on $R(n)$ via $\deg e(\nu) = 0$, $\deg x_k e(\nu) = (\alpha_{\nu_k} | \alpha_{\nu_k})$ and $\deg \tau_l e(\nu) = -(\alpha_{\nu_l} | \alpha_{\nu_{l+1}})$. Here the α_i are the simple roots.

We now define some operators that will be useful later on. We extend our notation for S_n from earlier to let $s_{a,b} \in S_n$ be the transposition switching a and b, and we can thus let S_n act on $\bigoplus_{\nu \in I^n} \Bbbk[x_1, \dots, x_n] e(\nu)$ via transposing the x_i. Also, we let $e_{a,b} = \sum_{\nu \in I^n, \nu_a = \nu_b} e(\nu)$. We can now define the operator $\partial_{a,b}$ on $\bigoplus_{\nu \in I^n} \Bbbk[x_1, \dots, x_n] e(\nu)$ as

$$\partial_{ab} f = \frac{s_{ab} f - f}{x_a - x_b} e_{a,b},$$

and for compactness of notation we write $\partial_a = \partial_{a,a+1}$.

As we are wanting a comparison to the Kac-Moody case for categorification, we would like to be able to consider this from the case of a single weight. We split I into subsets by defining $I^\beta = \{\nu = (\nu_1, \dots, \nu_n) \in I^n | \alpha_{\nu_1} + \dots + \alpha_{\nu_n} = \beta\}$. We then define $e(\beta) = \sum_{\nu \in I^\beta} e(\nu)$, and can then define the following:

Definition 3.5.25. The *Khovanov-Lauda-Rouquier algebra at* β for β a root of weight n is $R(\beta) = R(n)e(\beta) = \bigoplus_{\nu \in I^\beta} R(n) e(\nu)$. This is precisely the elements of $R(n)$ that can be written to end with $e(\nu)$ for some $\nu \in I^\beta$.

There is some further useful notation, namely

$$e(\beta, i) = \sum_{\nu \in I^{\beta + \alpha_i},\, \nu_{n+1} = i} e(\nu) \in R(\beta + \alpha_i),$$

$$e(i, \beta) = \sum_{\nu \in I^{\beta + \alpha_i}, \nu_1 = 1} e(\nu) \in R(\beta + \alpha_i),$$

and similar definitions for $e(n,i)$ and $e(i,n)$. To give some useful intuition for this, fixing $\nu_{n+1} = i$ in some sense gives a copy of the 'identity' $\sum_{\nu \in I^\beta} e(\nu)$ of $R(\beta)$ inside $R(\beta+\alpha_i)$. Notably, we can consider $e(\beta,i)R(\beta+\alpha_i)$ as a left $R(\beta)$ module, as if $e(\nu) \in I^{\beta+\alpha_i}$ with $\nu_{i+1} = i$, then $\alpha_{\nu_1} + \cdots + \alpha_{\nu_n} = \beta + \alpha_i - \alpha_{\nu_{n+1}} = \beta + \alpha_i - \alpha_i = \beta$ which allows an inclusion map from $R(\beta)$ and thus a left action by multiplication. Similarly, $R(\beta+\alpha_i)e(\beta,i)$ has the structure of a right $R(\beta)$-module.

A lot of our work here will be with various types of modules over the $R(\beta)$. However, as our elements of $R(n)$ have a grading, we also wish to place a grading on the modules:

Definition 3.5.26. For $\beta \in Q^+$, we let $R(\beta)\text{-}\mathrm{Mod}_{\mathbb{Z},0}$ denote the (abelian) category of $\mathbb{Z}$-graded (left) $R(\beta)$-modules with morphisms homogeneouns bimodule homomorphisms of degree zero. Here we let $[\![n]\!]$ denote the n-fold composite of the grade shift endofunctor on $R(\beta)\text{-}\mathrm{Mod}_{\mathbb{Z},0}$: for a $\mathbb{Z}$-graded $R(\beta)$-module $M = \sum_{k \in \mathbb{Z}} M_k$, we define $M[\![n]\!]$ by $M[\![n]\!]_k = M_{k-n}$.

3.5.4 Cyclotomic KLR Algebras

While standard KLR algebras are useful, they have issues that mean they are not suitable for our purposes. In particular, they are not in general finitely generated as $\Bbbk$-modules. We instead turn to quotients of them that are finitely generated. To begin, let Λ be a dominant integral weight; that is, $\Lambda \in P^+$.

For each $i \in I$, take some monic polynomial $a_i^\Lambda(u) = \sum_{k=0}^{\langle h_i,\Lambda\rangle} c_{i,k} u^{\langle h_i,\Lambda\rangle - k}$ of degree $\langle h_i, \Lambda\rangle$ where $c_{i,k} \in \Bbbk_{k(\alpha_i|\alpha_i)}$ and $c_{i,0} = 1$. For our purposes, we can take $a_i^\Lambda(u)$ to simply be $a_i^\Lambda(u) = u^{\langle h_i,\Lambda\rangle}$. For $1 \le k \le n$, we can now define $a^\Lambda(x_k) = \sum_{\nu \in I^n} a_{\nu_k}^\Lambda(x_k)e(\nu) \in R(n)$.

Definition 3.5.27. The *cyclotomic Khovanov-Lauda-Rouquier algebra* $R^\Lambda(\beta)$ of weight β at Λ is defined as the quotient algebra $R^\Lambda(\beta) = \frac{R(\beta)}{R(\beta)a^\Lambda(x_1)R(\beta)}$, with $R^\Lambda(0) = \Bbbk$. We note that $R^\Lambda(n) = \bigoplus_{|\beta|=n} R^\Lambda(\beta) = \frac{R(n)}{R(n)a^\Lambda(x_1)R(n)}$.

These are much more amenable for our purposes:

Theorem 3.5.28 ([KK12] Corollary 4.4)**.** *For any $\beta \in Q^+$, $R^\Lambda(\beta)$ is a finitely generated $\Bbbk$-module.*

The $R^\Lambda(\beta)$ inherit the $\mathbb{Z}$-grading of $R(\beta)$, since we are quotienting out by a homogeneous polynomial in $R(\beta)$.

We now wish to construct functors between the $R^\Lambda(\beta)\text{-}\mathrm{Mod}$ that will correspond to the e_i and f_i in the Kac-Moody case when we apply the decategorification process.

Definition 3.5.29. For each $i \in I$, we define functors

$$E_i^\Lambda : R^\Lambda(\beta + \alpha_i)\text{-}\mathrm{Mod} \to R^\Lambda(\beta)\text{-}\mathrm{Mod}$$

$$F_i^\Lambda : R^\Lambda(\beta)\text{-}\mathrm{Mod} \to R^\Lambda(\beta + \alpha_i)\text{-}\mathrm{Mod}$$

by

$$E_i^\Lambda(N) = e(\beta, i)N = e(\beta, i)R^\Lambda(\beta + \alpha_i) \otimes_{R^\Lambda(\beta+\alpha_i)} N$$

$$E_i^\Lambda(f) = e(\beta, i)f = \mathrm{id}_{e(\beta,i)R^\Lambda(\beta+\alpha_i)} \otimes f$$

and

$$F_i^\Lambda(M) = R^\Lambda(\beta + \alpha_i)e(\beta, i) \otimes_{R^\Lambda(\beta)} M$$

$$F_i^\Lambda(f) = \mathrm{id}_{R^\Lambda(\beta+\alpha_i)e(\beta,i)} \otimes f.$$

We notate $e(\beta, i)R^\Lambda(\beta + \alpha_i)$ as $\mathfrak{e}_i$ and $R^\Lambda(\beta + \alpha_i)e(\beta, i)$ as $\mathfrak{f}_i$.

Theorem 3.5.30 ([KK12] Theorem 4.5)**.** *$R^\Lambda(\beta + \alpha_i)e(\beta, i)$ is a projective right $R^\Lambda(\beta)$-module, and $e(\beta, i)R^\Lambda(\beta + \alpha_i)$ is a projective left $R^\Lambda(\beta)$-module.*

Corollary 3.5.31 ([KK12] Corollary 4.6)**.**
1. *E_i^Λ sends finitely generated projective modules to finitely generated projective modules.*
2. *F_i^Λ is exact.*

We will actually be doing the decategorification with a grade shift of the E_i^Λ functor: we will use $E_i^\Lambda[\![d_i(1 - \langle h_i, \Lambda - \beta\rangle)]\!]$ to line things up properly.

We now mention the analogues of the $e_i f_j - f_j e_i$ equation in the definition of quantum groups.

Theorem 3.5.32 ([KK12] Theorem 5.1)**.** *For $j \neq i$, there exists a natural isomorphism $F_j^\Lambda E_i^\Lambda [\![-(\alpha_i|\alpha_j)]\!] \xrightarrow{\sim} E_i^\Lambda F_j^\Lambda$.*

Theorem 3.5.33 ([KK12] Theorem 5.2)**.** *Let $\lambda = \Lambda - \beta$. Then we have natural isomorphisms as follows:*

1. *If $\langle h_i, \lambda\rangle \geq 0$, we have an isomorphism*

$$F_i^\Lambda E_i^\Lambda [\![-2d_i]\!] \oplus \bigoplus_{k=0}^{\langle h_i,\lambda\rangle -1} [\![2kd_i]\!] \xrightarrow{\sim} E_i^\Lambda F_i^\Lambda.$$

2. *If $\langle h_i, \lambda\rangle \leq 0$, we have an isomorphism*

$$F_i^\Lambda E_i^\Lambda [\![-2d_i]\!] \xrightarrow{\sim} E_i^\Lambda F_i^\Lambda \oplus \bigoplus_{k=1}^{-\langle h_i \lambda\rangle -1} [\![-2(k+1)d_i]\!].$$

We can now talk precisely about the (de)categorification. Given $\beta \in Q^+$, we denote by $R^\Lambda(\beta)\text{-proj}$ the category of finitely generated projective graded $R^\Lambda(\beta)$-modules with morphisms given by tensoring with projective $(R^\Lambda(\beta), R^\Lambda(\beta))$-bimodules. By Theorem 3.5.30 and Corollary 3.5.31, the following diagram has exact arrows:

$$R^\Lambda(\beta) - \text{proj} \underset{E_i^\Lambda[\![1-\langle h_i,\Lambda-\beta\rangle]\!]}{\overset{F_i^\Lambda}{\rightleftarrows}} R^\Lambda(\beta+\alpha_i) - \text{proj}$$

These thus descend to isomorphisms $\mathtt{F}_i$ and $\mathtt{E}_i$ on the Grothendieck group $[R^\Lambda\text{- proj}] = \bigoplus_{\beta\in Q^+} [R^\Lambda(\beta)\text{- proj}]$. Further, we have the result:

Lemma 3.5.34 ([KK12] Lemma 6.1)**.** *For all $i, j \in I$,*

$$[\mathtt{E}_i, \mathtt{F}_j] = \delta_{ij} \sum_{s=1}^{\langle h_i,\Lambda-\beta\rangle} [\![(\langle h_i, \Lambda-\beta\rangle - 2s + 1)d_i]\!].$$

We can examine $[R^\Lambda\text{-proj}]$ as a $U_{\mathbf{A}}(\mathfrak{g})$ module by letting the e_i and f_i act as $\mathtt{E}_i$ and $\mathtt{F}_i$ respectively.

We state the main result of [KK12]:

Theorem 3.5.35 ([KK12] Theorem 6.2)**.** *As* $U_{\mathbf{A}}(\mathfrak{g})$ *modules,* $[R^{\Lambda}\text{-}\mathrm{proj}] \cong V_{\mathbf{A}}(\Lambda)$.

3.5.5 2-Kac-Moody Algebras

We now have enough structure to define a 2-Kac-Moody algebra from an algebraic standpoint. First, let $U_q(\mathfrak{g})$ be the quantum group associated to a Kac-Moody algebra, and let $V(\Lambda)$ be its irreducible highest weight module for some $\Lambda \in P^+$. Let $\dot{U}_q(\mathfrak{g})$ be the Lusztig idempotent completion. We then categorify the endomorphisms of its highest-weight module for this following [Web17] as the 2-category $\mathscr{U}_\Lambda$:

- The objects of $\mathscr{U}_\Lambda$ are the weights λ such that $V(\Lambda)_\lambda \neq \{0\}$. Writing $\lambda = \Lambda - \beta$ for some β, we identify these with (small categories equivalent to) the module categories $R^\Lambda(\beta)$-proj.
- The 1-morphisms of $\mathscr{U}_\Lambda$ are direct summands of direct sums of the identity 1-morphisms and of compositions of 1-morphisms isomorphic to functors formed by tensoring with tensor products of (grade-shifts of) the $\mathfrak{e}_i$ and $\mathfrak{f}_i$ as defined in Definition 3.5.29 (at any weight β below Λ). Following that section, we denote the functor given by tensoring with $\mathfrak{e}_i[\![g]\!]$ by $E_i^\Lambda[\![g]\!]$ and the functor given by tensoring with $\mathfrak{f}_i[\![g]\!]$ as $F_i^\Lambda[\![g]\!]$.
- The 2-morphisms are the bimodule homomorphisms between the bimodules that correspond to the 1-morphisms. This implies that, for any $\mathscr{U}_\Lambda(\lambda, \mu)$, the spaces of 2-morphisms are finite dimensional.

Definition 3.5.36. We call this construction the *cyclotomic 2-Kac-Moody category of weight* Λ associated to a Kac-Moody algebra $U(\mathfrak{g})$.

Theorem 3.5.37. $\mathscr{U}_\Lambda$ *is a locally finitary 2-category.*

Proof. We wish to show $\mathscr{U}_\Lambda(\lambda, \mu) \in \mathfrak{A}_{\Bbbk}^f$ for all weights λ and μ. We already have that the (2-)morphisms form a finite dimensional space and as $R^\Lambda(\beta)$ is indecomposable for all β, $\mathbb{1}_\lambda$ is indecomposable for all λ. It thus remains to show that there are only finitely many isomorphism classes of indecomposable objects. The objects for this category are generated by products and direct summands of the $E_i\mathbb{1}_\zeta$ and the $F_i\mathbb{1}_\zeta$ for arbitrary weights ζ. Let $Q : \lambda \to \mu$ be a general 1-morphism. We wish to show that $Q \in \mathrm{add}(\{F_{i_1} \dots F_{i_l}\mathbb{1}_\lambda\} \cup \{F_{j_1} \dots F_{j_m} E_{k_1} \dots E_{k_n}\mathbb{1}_\lambda\} \cup \{\delta_{\lambda\mu}\mathbb{1}_\lambda\})$ for some choice of i_x, j_y and k_z.

If Q is of the form $M_1 E_i^\Lambda F_j^\Lambda M_2 \xrightarrow{\sim} M_1 E_i^\Lambda F_j^\Lambda \mathbb{1}_\epsilon M_2$ for some products M_1 and M_2 and some weight ϵ, then if $i \neq j$, $M_1 E_i^\Lambda F_j^\Lambda M_2 \xrightarrow{\sim} M_1 F_j^\Lambda E_i^\Lambda M_2$. If $i = j$ then we can use Theorem 3.5.32 and the fact that composition distributes over direct sums to get one of the two following cases, depending on ϵ:

- If $\langle h_i, \epsilon\rangle \geq 0$,

$$M_1 E_i^\Lambda F_i^\Lambda M_2 \xrightarrow{\sim} M_1 F_i^\Lambda E_i^\Lambda M_2[\![-(\alpha_i|\alpha_i)]\!] \oplus \bigoplus_{k=0}^{\langle h_i,\epsilon\rangle - 1} M_1 M_2[\![k(\alpha_i|\alpha_i)]\!].$$

- If $\langle h_i, \epsilon\rangle \leq 0$,

$$M_1 E_i^\Lambda F_i^\Lambda M_2 \oplus \bigoplus_{k=0}^{-\langle h_i,\epsilon\rangle - 1} M_1 M_2[\![-(k-1)(\alpha_i|\alpha_i)]\!] \xrightarrow{\sim} M_1 F_i^\Lambda E_i^\Lambda M_2[\![-(\alpha_i|\alpha_i)]\!].$$

In the second case, $Q = M_1 E_i^\Lambda F_i^\Lambda M_2 \in \mathrm{add}(M_1 F_i^\Lambda E_i^\Lambda M_2)$, while in the first case $Q \in \mathrm{add}(\{M_1 F_i^\Lambda E_i^\Lambda M_2, M_1 M_2\})$.

Let R be some formal product of the E_i^Λ and the F_i^Λ, which we will notate as $R = T_{j_0} E_{i_1}^\Lambda T_{j_1} E_{i_2}^\Lambda \dots E_{i_n}^\Lambda T_{j_n}$, where each T_{j_k} is a possibly empty product of the F_i^Λ. If $T_{j_k} = F_{q_1}^\Lambda \dots F_{q_l}^\Lambda$, we let $|T_{j_q}| = l$. Define the finite non-negative integer $\mathrm{Len}(R) = \sum_{m=1}^{n} \sum_{q=m}^{n} |T_{j_q}|$, the length of R. Note that if $T_{j_k} = 0$ for all $k > 0$, which corresponds to $R = F_{i_1}^\Lambda \dots F_{i_n}^\Lambda E_{j_1}^\Lambda \dots E_{j_m}^\Lambda$ for $m, n \geq 0$, then $\mathrm{Len}(R) = 0$. Further,

$$\mathrm{Len}(Q) = \mathrm{Len}(M_1 E_i^\Lambda F_j^\Lambda M_2) = \mathrm{Len}(M_1 F_j^\Lambda E_i^\Lambda M_2) + 1$$

and $\mathrm{Len}(Q) > \mathrm{Len}(M_1M_2)$. Finally, if $\mathrm{Len}(Q) > 0$, there exists a subproduct $E_i^\Lambda F_j^\Lambda$ somewhere in Q, and we can apply one of the above operations to it. If $\mathrm{Len}(Q) > 0$, Q is thus contained in the additive closure of finitely many 1-morphisms of strictly lesser length, and as length is non-negative, proceeding recursively will terminate in finite time, giving the claim.

We claim that there are only finitely many $F_{i_1}^\Lambda \dots F_{i_l}^\Lambda \mathbb{1}_\lambda : \lambda \to \mu$ and only finitely many $F_{j_1}^\Lambda \dots F_{j_m}^\Lambda E_{k_1}^\Lambda \dots E_{k_n}^\Lambda \mathbb{1}_\lambda : \lambda \to \mu$. For the first case, given $F_i^\Lambda \mathbb{1}_\lambda$ takes λ to $\lambda - \alpha_i$, a simple combinatorial argument shows there can only be finitely many of them from λ to μ. Specifically, choose a_i such that $\mu = \lambda - \sum_{i=1}^{n} a_i\alpha_i$. If $\bar{a}_i = \max\{a_i, 0\}$, the number of such $F_{i_1} \dots F_{i_l}$ is equal to the number of distinct ordered tuples containing $\bar{a}_i$ many α_i for all i. Indeed, in the poset of weights below Λ where Λ is the highest weight, if $\mu \not\leq \lambda$, then none of these exist.

For the $F_{j_1}^\Lambda \dots F_{j_m}^\Lambda E_{k_1}^\Lambda \dots E_{k_n}^\Lambda \mathbb{1}_\lambda$, write $\lambda = \Lambda - \sum_i b_i\alpha_i$, where all the b_i are non-negative. Then as the E_i^Λ move up the poset of weights, by a similar argument to the previous one, there are only finitely many possibly non-zero products $E_{k_1}^\Lambda \dots E_{k_n}^\Lambda \mathbb{1}_\lambda : \lambda \to \delta$ with $\Lambda \geq \delta \geq \lambda$. Then by another similar argument, for any such δ there are only finitely many possible products $F_{j_1}^\Lambda \dots F_{j_m}^\Lambda \mathbb{1}_\delta : \delta \to \mu$. Thus there are in total only finitely many $F_{j_1}^\Lambda \dots F_{j_m}^\Lambda E_{k_1}^\Lambda \dots E_{k_n}^\Lambda \mathbb{1}_\lambda : \lambda \to \mu$ as we required.

Each of these morphisms has a finite number of indecomposable direct summands. We thus find that $\mathscr{U}(\lambda, \mu)$ can only have finitely many isomorphism classes of indecomposable objects.

□

We can in fact say more than this.

Proposition 3.5.38. *$\mathscr{U}_\Lambda$ is a locally fiat 2-category.*

Proof. We prove that the 1-morphisms E_i and F_i have adjoints, and the other 1-morphisms will be an immediate consequence through composition. We claim that the adjoint of $E_i^\Lambda \mathbb{1}_\lambda [\![z]\!]$ is $F_i^\Lambda \mathbb{1}_{\lambda-\alpha_i} [\![z - \frac{(\alpha_i,\alpha_i)}{2}(1 + \langle \alpha_i, \lambda \rangle)]\!]$ and that the adjoint of

$F_i^\Lambda \mathbb{1}_\lambda[\![z]\!]$ is $E_i^\Lambda \mathbb{1}_{\lambda+\alpha_i}[\![z - \frac{(\alpha_i,\alpha_i)}{2}(1-\langle\alpha_i,\lambda\rangle)]\!]$.

To see this, consider $F_i^\Lambda = R^\Lambda(\beta+\alpha_i)e(\beta,i)\otimes_{R^\Lambda(\beta)} -$ and ignore grading for the moment. The right adjoint of this is therefore $\mathrm{Hom}_{R^\Lambda(\beta+\alpha_i)}(R^\Lambda(\beta+\alpha_i)e(\beta,i),-)$. But since $R^\Lambda(\beta+\alpha_i)e(\beta,i)$ is projective over $R^\Lambda(\beta+\alpha_i)$, this is isomorphic to

$$\mathrm{Hom}_{R^\Lambda(\beta+\alpha_i)}(R^\Lambda(\beta+\alpha_i)e(\beta,i), R^\Lambda(\beta+\alpha_i))\otimes_{R^\Lambda(\beta+\alpha_i)} -.$$

This is isomorphic to

$$\mathrm{Hom}_\Bbbk(R^\Lambda(\beta+\alpha_i)e(\beta,i),\Bbbk)\otimes_{R^\Lambda(\beta+\alpha_i)} -$$

because $R^\Lambda(\beta+\alpha_i)$ is symmetric. But by another application of this symmetric property, this is then isomorphic to $e(\beta,i)R^\Lambda(\beta+\alpha_i)\otimes_{R^\Lambda(\beta+\alpha_i)} - = E_i^\Lambda$. Finally, the grading is a consequence of the comment before Theorem 3.5.32. □

Further, this 2-category will turn out to be strongly regular. However, to prove this we need to extend our definition of a cyclotomic 2-Kac-Moody algebra to a wider setup. This definition is a generalisation of a construction from [MM16c] Section 7.2

Definition 3.5.39. Choose a set of positive weights $\underline{\Lambda} = \{\Lambda_1,\ldots,\Lambda_n\} \subseteq P^+$. Without loss of generality we assume that $\Lambda_i \not\leq \Lambda_j$ for $i \neq j$. We define a 2-category $\mathscr{U}_{\underline{\Lambda}}$, the *truncated cyclotomic 2-Kac-Moody algebra*, as follows:

1. The objects of $\mathscr{U}_{\underline{\Lambda}}$ are ordered pairs (β,i) where $\beta\in Q^+$ and $1\leq i\leq n$, modulo an equivalence relation where $(\beta,i)\sim(\gamma,j)$ if $\Lambda_i-\beta=\Lambda_j-\gamma$.
2. The 1-morphisms of $\mathscr{U}_{\underline{\Lambda}}$ are the additive closure of (grade shifts of) the identity 1-morphisms and morphisms of the form E_i^Λ and F_i^Λ, as in the cyclotomic 2-Kac-Moody algebra case, with identical relations to that situation.
3. The 2-morphisms are identical to the single weight case.

This 2-category is well-defined. First, if $(\beta,i)\sim(\gamma,j)$, then if $(\beta\pm\alpha_k,i)$ and $(\gamma\pm\alpha_k,j)$ are both objects in the 2-category, $\Lambda_i-(\beta\pm\alpha_k)=\Lambda_j-(\gamma\pm\alpha_k)$, so

$(\beta \pm \alpha_k, i) \sim (\gamma \pm \alpha_k, j)$. Second, at a given object (β, i), the entwining relations for $F_k E_k$ and $E_k F_k$ depend on the value of $\langle h_k, \Lambda_i - \beta \rangle$, so if $(\gamma, j) \sim (\beta, i)$ then $\langle h_k, \Lambda_j - \gamma \rangle = \langle h_k, \Lambda_i - \beta \rangle$.

A further useful point is that, using the interchange structure and Theorem 3.5.37 for distinct F_i and E_j, if (β, i) and (γ, j) are objects of $\mathscr{C}$ such that there does not exist Λ_k with both $\Lambda_i - \beta \leq \Lambda_k$ and $\Lambda_j - \gamma \leq \Lambda_k$, then $\mathscr{U}_{\underline{\Lambda}}((\beta, i), (\gamma, j)) = \mathscr{U}_{\underline{\Lambda}}((\gamma, j), (\beta, i)) = 0$. We also define the notation $\underline{\Lambda}^p$ as $\underline{\Lambda}^p = \{\lambda | \exists i, \exists \beta, \lambda = \Lambda_i - \beta\}$, the set of weights below at least one of the Λ_i.

That this 2-category is locally weakly fiat follows immediately from the above considerations, since the internal adjoint 1-morphism and adjunction 2-morphisms will remain identical to the traditional case. We will combine this with the following result (recall the definition of 'strongly regular' from Definition 3.2.5):

Theorem 3.5.40. *For any $\underline{\Lambda}$, $\mathscr{U}_{\underline{\Lambda}}$ is strongly regular.*

Proof. We mirror the proof for [MM16c] Theorem 21. We first consider the $\mathscr{J}$-cell $\mathcal{J}_{\Lambda_i}$ containing $\mathbb{1}_{\Lambda_i}$ for some $\Lambda_i \in \underline{\Lambda}$. If we quotient out by the maximal 2-ideal in $\mathscr{U}_{\underline{\Lambda}}$ which contains $\mathrm{id}_{\mathbb{1}_{\Lambda_i}}$ but not any identity 2-morphisms for a 1-morphism not in $\mathcal{J}_{\Lambda_i}$, the resulting 2-category is equivalent to one of the form $\mathscr{U}_{\underline{\Theta}}$, where $\underline{\Theta}$ is the unique set of highest weights such that $\underline{\Theta}^p = \underline{\Lambda}^p \setminus \{\Lambda_i\}$ (see [DG17] Section 9 for more details). It is thus sufficient to prove that $\mathcal{J}_{\Lambda_i}$ is strongly regular.

Let $\mathcal{L}$ denote the $\mathscr{L}$-cell of $\mathbb{1}_{\Lambda_i}$. From the proof of Theorem 3.5.37, any element of $\mathcal{L}$ is in the additive closure of 1-morphisms of the form $F_1^{\Lambda} \dots F_n^{\Lambda}$ and $F_1^{\Lambda} \dots F_n^{\Lambda} E_1^{\Lambda} \dots E_m^{\Lambda}$. But since any element of $\mathcal{L}$ must have source object Λ_i and any morphism of the form $F_1^{\Lambda} \dots F_n^{\Lambda} E_i^{\Lambda} \dots E_m^{\Lambda}$ with source object Λ_i must necessarily be zero (as Λ_i is a highest weight in the 2-category), it follows that $\mathcal{L}$ consists of direct summands of products of the F_i.

Let L be an indecomposable object in $R_0^{\Lambda_i}\text{-proj}$. Since $R_0^{\Lambda_i} \cong \Bbbk$, we have that $L \cong \Bbbk$. By [Rou08] Theorem 5.7 and [VV11] Theorem 4.4, the mapping that takes an $F \in \mathcal{L}$ to FL induces a bijection between $\mathcal{L}$ and the set of isomorphism classes of indecomposable objects in $\prod_{n\geq 0} R_n^{\Lambda_i}\text{-}\operatorname{proj}$. We define two algebras, $A = \bigoplus_{n\geq 0} R_n^{\Lambda_i}$ and $B = \bigoplus_{n\geq 1} R_n^{\Lambda_i}$. Since every element $X \in \mathscr{L}$ can be expressed as $X\mathbb{1}_{\Lambda_i}$ and as $\mathbb{1}_{\Lambda_i}M = 0$ for any $M \in B\text{-proj}$, it follows that $XM = 0$ for any $X \in \mathscr{L}$. We now consider the (projective) abelianisation $\overline{\mathbf{C}_{\mathscr{L}}}$ of the cell 2-representation for $\mathscr{L}$.

By the construction of the abelianisation, $\overline{\mathbf{C}_{\mathscr{L}}}(X)$ can be considered as a functor from $\Bbbk\text{-mod}$ to $R^{\Lambda_i}(\beta) - \text{mod}$ for some positive weight β. This can consequently be considered as an endofunctor of $\Bbbk \times R^{\Lambda_i}(\beta)\text{-mod}$. Further, since the only projective it is non-zero on is L, which it must take to an indecomposable projective, it follows from [MM16c] Lemma 13 that $\overline{\mathbf{C}_{\mathscr{L}}}(X)$ is an indecomposable projective endofunctor. By consideration of sub-categories of the domain and of the range where this functor acts trivially or does not map to respectively, we can indeed say that $\overline{\mathbf{C}_{\mathscr{L}}}(X)$ is an indecomposable projective functor from $\Bbbk\text{-mod}$ to $A\text{-mod}$. But by this projectivity, for any $Y \in \mathscr{L}$, $\overline{\mathbf{C}_{\mathscr{L}}}(X \circ Y^*)$ is also indecomposable.

We claim that this implies that $X \circ Y^*$ is itself indecomposable. For assume that $X \circ Y^* \cong V \oplus W$ for non-zero V and W. Then without loss of generality $\overline{\mathbf{C}_{\mathscr{L}}}(W) = 0$. But since $\mathcal{J}$ is a maximal 2-sided cell, we must have for every indecomposable summand W' of W that $W' \in \mathcal{J}$. But then by construction, $\overline{\mathbf{C}_{\mathscr{L}}}(W') \neq 0$ and hence $\overline{\mathbf{C}_{\mathscr{L}}}(W) \neq 0$, a contradiction. The claim follows.

Hence if $X \neq X'$ then $XY^* \neq X'Y^*$, and if $Y \neq Y'$, $XY^* \neq XY'^*$. Further, the set $\{X \circ Y^*\}$ is a set of indecomposables that forms a $\mathscr{D}$-cell by construction and hence by Theorem 3.2.4 is a $\mathscr{J}$-cell that contains $\mathbb{1}_{\Lambda_i}$, and thus is equal to $\mathcal{J}$. Fixing X and varying Y clearly gives a $\mathscr{R}$-cell in $\mathcal{J}$, and fixing Y and varying X gives an $\mathscr{L}$-cell, and therefore this process must exhaust all such $\mathscr{L}$- and $\mathscr{R}$-cells. In particular, the intersection of any $\mathscr{L}$-cell with any $\mathscr{R}$-cell is thus a unique element. Thus $\mathcal{J}$ is strongly regular, and the result follows. □

Corollary 3.5.41. *Every simple transitive 2-representation of a truncated cyclotomic 2-Kac-Moody algebra is equivalent to a cell 2-representation.*

Proof. By Theorem 3.5.37, Proposition 3.5.38 and Theorem 3.5.40, a truncated cyclotomic 2-Kac-Moody algebra is a strongly regular locally weakly fiat 2-category. Therefore applying Theorem 3.4.32 gives the result immediately. □

4

A Specialisation to Graded 2-Categories

4.1 Initial Definitions

During this chapter, we always take G to be a countable abelian group unless otherwise stated. We start by defining G-graded (2-)categories and their G-envelopes, following the ideas in [BD17] Section 3.5.

Definition 4.1.1. Let A be a $\Bbbk$-algebra and let G be a countable abelian group. We say that A is *G-graded* if we have a decomposition $A = \bigoplus_{g \in G} A_g$ of A into a direct sum of vector spaces such that $A_g A_h \subseteq A_{g+h}$.

For notational purposes, we say that if A is a G-graded vector space and $g \in G$, then $A[\![g]\!]$ is a G-graded vector space isomorphic to A such that $A[\![g]\!]_h \cong A_{h-g}$.

Definition 4.1.2. Let A be a G-graded $\Bbbk$-algebra. We say that A is *G-graded-finite dimensional* if it has a G-grading $A = \bigoplus_{g \in G} A_g$ such that each A_g is finite dimensional as a $\Bbbk$-vector space. In particular, this implies that A_0 is a finite dimensional $\Bbbk$-algebra.

Definition 4.1.3. The category $\Bbbk\text{-}\mathrm{Mod}_G$ has as objects the G-graded $\Bbbk$-vector spaces, and as morphisms finite linear combinations of homogeneous linear maps of arbitrary degree $g \in G$. We define the full subcategory $\Bbbk\text{-}\mathrm{Mod}_G^{\mathrm{gf}}$ to contain the G-graded $\Bbbk$-vector spaces that are G-graded-finite dimensional. We let $\Bbbk\text{-}\mathrm{Mod}_{G,0}$ and $\Bbbk\text{-}\mathrm{Mod}_{G,0}^{\mathrm{gf}}$ denote the subcategories of the above categories with the same objects but with morphisms only those homogeneous linear maps of degree zero.

Definition 4.1.4. We define a G*-graded category* to be a category enriched over the monoidal category $\Bbbk\text{-Mod}_{G,0}$. Explicitly, if $\mathscr{C}$ is a G-graded category, it has G-graded hom-spaces of morphisms such that composition respects degree (and hence all identity morphisms are of degree zero).

To give an explanation for why we take $\Bbbk\text{-Mod}_{G,0}$ to enrich over as opposed to $\Bbbk\text{-Mod}_G$ (and similarly with $\Bbbk\text{-Mod}^{\text{gf}}_{G,0}$ below), composition in an enriched category is given by a collection of morphisms $\cdot_{A,B,C} : \text{Hom}(B, C) \times \text{Hom}(A, B) \to \text{Hom}(A, C)$ (in the category that is being enriched over) that respect certain axioms. In the case of G-graded vector spaces, a homogeneous element of $V \times W$ is a pair (v, w) where v and w are homogeneous in V and W respectively, and $\deg(v, w) = \deg v + \deg w$. We wish for composition in our enriched category to respect degree; that is, if f and h are arbitrary homogeneous morphisms, $\deg(fg) = \deg(f) + \deg(g)$. That is, $\deg \cdot_{A,B,C}((f, h)) = \deg((f, h))$. This implies that the $\cdot_{A,B,C}$ have to be homogeneous of degree zero, hence the construction.

Definition 4.1.5. We define a G*-graded finitary category* to be an additive idempotent complete category enriched over the monoidal category $\Bbbk\text{-Mod}^{\text{gf}}_{G,0}$ with a finite set of isomorphism classes of indecomposable objects.

When $G = \{e\}$ is the trivial group, the above definition is precisely that of a finitary category first defined in [MM11].

Definition 4.1.6 ([BD17] Section 3.5)**.** Let $\mathscr{C}$ be a G-graded category. We define its G*-envelope* $\tilde{\mathscr{C}}$ as a category with objects defined formally as symbols of the form $X[\![g]\!]$ where X is an object of $\mathscr{C}$ and $g \in G$. We set hom-spaces as

$$\text{Hom}_{\tilde{\mathscr{C}}}(X[\![g]\!], Y[\![h]\!]) \cong \text{Hom}_{\mathscr{C}}(X, Y)[\![h - g]\!],$$

and composition of morphisms is given by the obvious inheritance from $\mathscr{C}$. We also use the notation $X[\![g]\!][\![h]\!] = X[\![g + h]\!]$. If $\mathscr{C}$ is a G-graded finitary category, we call $\tilde{\mathscr{C}}$ a G*-finitary category.*

We denote by $\text{id}_{X,g} : X \to X[\![g]\!]$ the canonical shift of the identity isomorphism for any object $X \in \mathscr{C}$ and any $g \in G$, which is homogeneous of degree g with inverse $\mathrm{id}_{X[\![g]\!],-g}$.

Definition 4.1.7. Let $\mathscr{C}$ and $\mathscr{D}$ be G-finitary categories. We say that a functor $F : \mathscr{C} \to \mathscr{D}$ is a *G-graded* functor if it respects the structure of the grading and the envelope. Explicitly, this means the following:

- For X an object of $\mathscr{C}$ and $g \in G$, $F(X[\![g]\!]) = F(X)[\![g]\!]$.
- For X and Y objects in $\mathscr{C}$, $F_{X,Y} : \mathrm{Hom}(X,Y) \to \mathrm{Hom}(FX,FY)$ is homogeneous of degree zero; that is, $\deg(F(\alpha)) = \deg(\alpha)$ for any homogeneous morphism α.

Definition 4.1.8. Let $\Bbbk$-$\mathbf{Cat}_G$ denote the category whose objects are G-graded categories and whose morphisms are all G-homogeneous functors between them. Let $\Bbbk$-$\mathbf{Cat}_G^{\mathrm{gf}}$ denote the category whose objects are G-graded finitary categories and whose morphisms are all G-homogeneous functors between them. This is a monoidal category where the tensor product is products of categories and the tensor unit is the G-graded category with one object whose only morphisms are scalar multiples of the identity.

Definition 4.1.9. We define a *G-graded 2-category* as a category enriched over $\Bbbk$-$\mathbf{Cat}_G$. Explicitly, it has G-graded hom-spaces of 2-morphisms such that horizontal and vertical composition both respect degree. We define a *locally G-graded finitary 2-category* to be a category with countably many objects enriched over $\Bbbk$-$\mathbf{Cat}_G^{\mathrm{gf}}$ such that each identity 1-morphism is indecomposable.

Definition 4.1.10. Let $\mathscr{C}$ be a G-graded 2-category. We define the *G-envelope* 2-category $\tilde{\mathscr{C}}$ of $\mathscr{C}$ by taking the same objects as $\mathscr{C}$, and defining each hom-category $\tilde{\mathscr{C}}(\mathtt{i},\mathtt{j})$ as the G-envelope of the category $\mathscr{C}(\mathtt{i},\mathtt{j})$. We further require that composition respects the envelope; that is, for 1-morphism $X[\![g]\!]$ and $Y[\![h]\!]$, $X[\![g]\!] \circ Y[\![h]\!] = (X \circ Y)[\![g+h]\!]$ wherever this makes sense. We also define horizontal and vertical composition of 2-morphisms as the obvious induction from composition in $\mathscr{C}$. If $\mathscr{C}$ is a locally G-graded finitary 2-category, then we say that its G-envelope is a *locally G-finitary 2-category*.

Again, if we take $G = \{e\}$ to be trivial, a locally G-graded finitary or locally G-finitary 2-category is a locally finitary 2-category in the sense of Chapter 3.

Definition 4.1.11. Let $\mathscr{C}$ be a locally G-finitary 2-category. If there exists a weak equivalence $-^* : \mathscr{C} \to \mathscr{C}^{\mathrm{op}}$ such that for any 1-morphism $X \in \mathscr{C}(\mathtt{i}, \mathtt{j})$ there are natural homogeneous 2-morphisms $\alpha : X \circ X^* \to \mathbb{1}_{\mathtt{i}}$ and $\beta : \mathbb{1}_{\mathtt{j}} \to X \circ X^*$ of degree zero such that $(\alpha \circ_H \mathrm{id}_X) \circ_V (\mathrm{id}_X \circ_H \beta) = \mathrm{id}_X$ and $(\mathrm{id}_{X^*} \circ_H \alpha) \circ_V (\beta \circ_H \mathrm{id}_{X^*}) = \mathrm{id}_{X^*}$, then we say that $\mathscr{C}$ is a *locally weakly G-fiat* 2-category. If $-^*$ is an involution, we say that $\mathscr{C}$ is *locally G-fiat*.

4.2 2-Representations and Ideals

We first examine 2-functors for the graded setup, again following [BD17].

Definition 4.2.1. Let $\mathscr{C}$ and $\mathscr{B}$ be locally G-finitary 2-categories. We say that a 2-functor $F : \mathscr{C} \to \mathscr{B}$ is a *G-graded* 2-functor if each $F_{\mathtt{i},\mathtt{j}} : \mathscr{C}(\mathtt{i}, \mathtt{j}) \to \mathscr{B}(F\mathtt{i}, F\mathtt{j})$ is a G-graded functor.

Definition 4.2.2. Let $\mathscr{C}$ and $\mathscr{B}$ be locally G-finitary 2-categories, and let $P, Q : \mathscr{C} \to \mathscr{B}$ be graded 2-functors. We say that a 2-natural transformation $\alpha : P \to Q$ is a *G-graded* 2-natural transformation if, for each 1-morphism $X \in \mathscr{C}$, the associated 2-isomorphism α_X is of degree zero.

We denote by $\mathfrak{A}_{\Bbbk}^{G\text{-gf}}$ the 2-category which has as objects G-finitary categories, as 1-morphisms $\Bbbk$-linear additive graded functors, and as 2-morphisms natural transformations of these.

We now recall the definition of a 2-representation from [MM11], and give its specification to this setup.

Definition 4.2.3. Let $\mathscr{C}$ be a locally G-finitary 2-category. A *G-finitary 2-representation* is a G-graded 2-functor from $\mathscr{C}$ to $\mathfrak{A}_{\Bbbk}^{G\text{-gf}}$. An *abelian* 2-representation is a 2-functor from $\mathscr{C}$ to the category $\mathfrak{R}_{\Bbbk}$. We denote 2-representations as $\mathbf{M}$, $\mathbf{N}$ etc., except for the $\mathtt{i}$th *principal representation*

$\mathbf{P}_{\mathtt{i}} = \mathscr{C}(\mathtt{i}, -)$. We denote the 2-category with objects G-finitary 2-representations of $\mathscr{C}$, 1-morphisms graded 2-natural transformations and 2-morphism modifications by $\mathscr{C}_G\text{-afmod}$. We say that two G-finitary 2-representations $\mathbf{M}$ and $\mathbf{N}$ are *equivalent* if there exists a graded 2-natural transformation $\Phi : \mathbf{M} \to \mathbf{N}$ such that $\Phi_{\mathtt{i}}$ is an equivalence for each $\mathtt{i}$.

There are multiple concepts that we have previously defined that still apply to our case. We retain the concepts of $\mathscr{L}$-, $\mathscr{R}$-, $\mathscr{D}$- and $\mathscr{J}$-orders and cells, as well as the concepts of strongly-regular and $\mathscr{D}$-strongly regular cells. Ideals in 2-representations and 2-ideals also apply with no issues.

4.3 Degree Zero Sub-2-Categories

We wish to generalise the idea of a coalgebra 1-morphism in the 2-category and the related theory to locally G-finitary 2-categories. However, there turn out to be significant obstacles to this approach - in general, locally G-finitary 2-categories are not particularly pleasant structures to work with for 2-representation theory. Most prominently, the method of abelianisation given in [MMMT16] is not guaranteed to give an abelian category. Explicitly, as was shown in [Fre66], the process of injective (respectively projective) abelianisation given in section 3 of [MMMT16] results in an abelian (2-)category if and only if the original (2-)category has weak kernels (respectively weak cokernels). This is a criterion that fails for many cases of locally G-finitary 2-categories. Later in this thesis we will be considering a more powerful abelianisation, but there is use in the following specialisation.

We instead consider locally *restricted* G-finitary 2-categories; that is, locally G-finitary 2-categories where the hom-spaces of 2-morphisms are not only graded-finite dimensional, but also finite dimensional in totality. In this case, they are simply locally finitary 2-categories, but with extra structure on the hom-spaces of 2-morphisms. We will use this extra structure to prove facts about the coalgebras constructed in [MMMT16] is the case of G-finitary 2-representations of the

2-category. To do this, however, will require introducing a specific sub-2-category.

Let $\mathscr{C}$ be a G-finitary category. We define the subcategory $\mathscr{C}_0$ by taking the objects of $\mathscr{C}_0$ to be the same as the objects of $\mathscr{C}$, but taking the morphisms to be only those morphisms of $\mathscr{C}$ that are homogeneous of degree zero. Though this construction removes most of the morphisms of $\mathscr{C}$, it does not lose any information: if $p : X \to Y$ is a homogeneous morphism of $\mathscr{C}$ of degree g, then there is a corresponding morphism $p_{-g} : X[\![g]\!] \to Y$ that is homogeneous of degree zero, and is thus in $\mathscr{C}_0$.

Let $\mathscr{C}$ be a locally G-finitary 2-category with a G-finitary 2-representation $\mathbf{M}$. We define a sub-2-category $\mathscr{C}_0$ to have the same objects as $\mathscr{C}$, and we set the hom-categories to be $\mathscr{C}_0(\mathtt{i},\mathtt{j}) = (\mathscr{C}(\mathtt{i},\mathtt{j}))_0$ for all objects $\mathtt{i},\mathtt{j} \in \mathscr{C}$. This implies that the 1-morphisms of $\mathscr{C}_0$ are also the same as those of $\mathscr{C}$. Further, it is still the case that $\mathbb{1}_\mathtt{i}$ is an indecomposable 1-morphism for each object $\mathtt{i} \in \mathscr{C}$. However, $\mathscr{C}_0$ is *not* a locally finitary 2-category - since we can no longer guarantee that $F \cong F[\![g]\!]$ for any non-zero g as the canonical isomorphism $\mathrm{id}_{F,g}$ is of non-zero degree, in general $\mathscr{C}_0$ has infinitely many isomorphism classes of indecomposable 1-morphisms. However, in the restricted case, this will turn our to be a surmountable problem.

Similarly, we define the 2-representation $\mathbf{M}_0$ of $\mathscr{C}_0$ to by setting $\mathbf{M}_0(\mathtt{i}) = (\mathbf{M}(\mathtt{i}))_0$ for all objects $\mathtt{i}$ of $\mathscr{C}_0$. This is naturally a 2-representation of $\mathscr{C}_0$, as given a 1-morphism $F[\![g]\!] \in \mathscr{C}_0$, we have the functor $\mathbf{M}_0(F[\![g]\!])$ being defined as the restriction of $\mathbf{M}(F[\![g]\!])$ to $\mathbf{M}_0(\mathtt{i})$, since by the definition of a 2-representation, $\mathbf{M}(F[\![g]\!])_{M,N} : \mathrm{Hom}_\mathbf{M}(M,N) \to \mathrm{Hom}_\mathbf{M}(FM[\![g]\!], FN[\![g]\!])$ is a homogeneous map of degree zero, and so restricts to a morphism between the degree zero subspaces. Further, as horizontal and vertical composition in $\mathscr{C}$ are also defined to preserve degree, the restriction of $\mathbf{M}(\alpha)$ for $\alpha : F \to G$ a 2-morphism of degree zero to $\mathbf{M}_0(\alpha) : \mathbf{M}_0(F) \to \mathbf{M}_0(G)$ is also well defined. We also notate $\mathscr{M}_0 = \coprod_{\mathtt{i} \in \mathscr{C}_0} \mathbf{M}_0(\mathtt{i})$ in a similar fashion to $\mathscr{M}$.

Assume that $\mathbf{M}$ is transitive.

Proposition 4.3.1. *Let $\mathscr{C}$ be a locally G-finitary 2-category and let $\mathbf{M}$ be a* *transitive 2-representation of $\mathscr{C}$. The 2-representation $\mathbf{M}_0$ of $\mathscr{C}_0$ is also a transitive 2-representation.*

Proof. Let $M, N \in \mathscr{M}_0$ with N indecomposable. It is sufficient for the proof to find $F \in \mathscr{C}$ such that N is isomorphic to a summand of FM in $\mathscr{M}_0$, i.e. via an isomorphism that is homogeneous of degree zero. Since $\mathbf{M}$ is transitive, we have some $\bar{F} \in \mathscr{C}$ such that there exists an isomorphism $\bar{\varphi} : \bar{F}M \to N \oplus N''$ in $\mathscr{M}$ for some $N'' \in \mathscr{M}$. We therefore have morphisms $\iota : N \to \bar{F}M$ and $\sigma : \bar{F}M \to N$ such that $\sigma\iota = \mathrm{id}_N$.

Setting the homogeneous decompositions $\iota = \sum_{g\in G} \iota^g$ and $\sigma = \sum_{g\in G} \sigma^g$, we see from comparison of degree with id_N that for $g \neq 0$, $\sum_{h\in G} \sigma^h \iota^{g-h} = 0$ while $\sum_{h\in G} \sigma^h \iota^{-h} = \mathrm{id}_N$. Since N is indecomposable, by standard nilpotent arguments there exists a $g \in G$ such that $\sigma^g \iota^{-g}$ is an automorphism.

We thus have some $\rho \in \mathrm{End}_{\mathscr{M}}(N)$ such that $\rho\sigma^g\iota^{-g} = \mathrm{id}_N$. But as id_N and $\sigma^g\iota^{-g}$ are homogeneous of degree zero, so too must ρ be. Thus we have homogeneous morphisms $\iota^{-g} : N \to \bar{F}M$ and $\rho\sigma^g : \bar{F}M \to N$ in $\mathscr{M}$ such that $\rho\sigma^g\iota^{-g} = \mathrm{id}_N$. We now set $F = \bar{F}[\![g]\!]$. We thus have the corresponding morphisms $\iota_g^{-g} : N \to FM$ and $\rho\sigma_{-g}^g : FM \to N$ that are homogeneous of degree zero and such that $\rho\sigma_{-g}^g\iota_g^{-g} = \mathrm{id}_N$. The result follows. □

Corollary 4.3.2. *If $\mathbf{M}$ is a simple transitive 2-representation of $\mathscr{C}$, then $\mathbf{M}_0$ is a simple transitive 2-representation of $\mathscr{C}_0$.*

Proof. Assume that $\mathbf{M}$ is simple transitive, and let $\mathscr{J}_0$ be a $\mathscr{C}_0$-ideal of $\mathbf{M}_0$. $\mathscr{J}_0$ therefore generates a $\mathscr{C}$-ideal $\mathscr{J}$ of $\mathbf{M}$. If $\mathscr{J} = 0$ then $\mathscr{J}_0 = 0$. Therefore assume that $\mathscr{J} \neq 0$. As $\mathbf{M}$ is simple transitive, there exists some $X \in \mathscr{M}$ such that $\mathrm{id}_X \in \mathscr{J}$, and by standard injection/projection arguments we may assume that X is indecomposable. Then $\mathrm{id}_X = \sum_{i=1}^{n} \alpha_i\gamma_i\beta_i$ for $\gamma_i \in \mathscr{J}_0$ for all i and $\alpha_i, \beta_i \in \mathscr{M}$ for all i. Then by a standard nilpotency argument there exists some i and some morphisms $\alpha', \beta' \in \mathscr{M}$ such that $\mathrm{id}_X = \alpha'\gamma_i\beta' = \sum_{g,h\in G} \alpha'_g\gamma_i\beta'_h$, with the latter sum being the decomposition of α' and β' into homogeneous components. Again by a nilpotency argument, there exist some $k, l \in G$ such that $\mathrm{id}_X = \alpha'_k \gamma_i \beta'_l$. But id_X and γ_i are both homogeneous of degree zero, and therefore $k = -l$. But then $\beta'_{-k,-k} : X[\![k]\!] \to X$ and $\alpha'_{k,k} : X \to X[\![k]\!]$ are morphisms in $\mathscr{M}_0$ such that $\mathrm{id}_{X[\![k]\!]} = \beta'_{-k,-k}\gamma_i\alpha'_{k,k} \in \mathscr{I}_0$. Hence if $\mathscr{I}_0$ is nonzero it contains some non-zero identity morphism, and the result follows. □

A further result we wish to have is that if $\mathscr{C}$ is a locally restricted G-finitary 2-category, then the (injective) Freyd abelianisation of $\mathscr{C}_0$, $\underline{\mathscr{C}_0}$, is indeed a locally abelian 2-category. In fact, this is true for the projective Freyd abelianisation as well, and by [Fre66], it is sufficient to show the following:

Proposition 4.3.3. *Let $\mathscr{C}$ be a restricted G-finitary category. Then $\mathscr{C}_0$ has weak kernels and weak cokernels.*

Proof. Let $p : X \to Y$ be a morphism in $\mathscr{C}_0$. Consider the full subcategory $\mathscr{C}_{0,p}$ of $\mathscr{C}_0$ closed under isomorphisms and generated by

$$\mathrm{add}\{X, Y, H[\![g]\!] | H \in \mathscr{C}_0 \text{ indecomposable}, g \in G, \mathrm{Hom}_{\mathscr{C}_0}(Y, H[\![g]\!]) \neq 0\}.$$

Since the total dimension of hom-spaces in $\mathscr{C}$ is finite, given any indecomposable $H \in \mathscr{C}$ there are only finitely many $g \in G$ such that $\mathrm{Hom}^g_{\mathscr{C}}(Y, H) \neq 0$, and hence only finitely many $g \in G$ such that $\mathrm{Hom}_{\mathscr{C}_0}(Y, H[\![g]\!]) \neq 0$. Since $\mathscr{C}$ has only finitely many G-orbits of isomorphism classes of indecomposables and since X and Y each have only finitely many indecomposable summands, $\mathscr{C}_{0,p}$ contains only finitely many isomorphism classes of indecomposable 1-morphisms. It is additive and idempotent complete by construction, and as a subcategory of a category with finite dimensional hom-spaces it also has finite dimensional hom-spaces. Since it also inherits being $\Bbbk$-linear, $\mathscr{C}_{0,p}$ is actually a finitary category.

There thus exists a weak cokernel $\mathrm{wcoker}\, p : Y \to L$ of p in $\mathscr{C}_{0,p}$. We claim that $\mathrm{wcoker}\, p$ is a weak cokernel of p in $\mathscr{C}_0$. For let $m : Y \to K$ be a morphism in $\mathscr{C}_0$ such that $mp = 0$. If $m = 0$, then we clearly have the zero morphism $L \to K$

satisfying the weak cokernel diagram. If $m \neq 0$, then by the definition of $\mathscr{C}_{0,p}$ we have $m \in \mathscr{C}_{0,p}$. Thus as $\operatorname{wcoker} p$ is a weak cokernel in $\mathscr{C}_{0,p}$, we have a morphism $q : L \to K$ satisfying the weak cokernel diagram, which also satisfies the diagram in $\mathscr{C}_0$. Hence p does indeed have a weak cokernel and thus $\mathscr{C}_0$ has weak cokernels. The weak kernel case is precisely dual to the above argument, and the result follows. □

Corollary 4.3.4. *Let $\mathscr{C}$ be a locally restricted G-finitary 2-category. Then $\mathscr{C}_0$ has weak cokernel 2-morphisms and weak kernel 2-morphisms.*

Corollary 4.3.5. *Let $\mathscr{C}$ be a locally restricted G-finitary 2-category with a G-finitary 2-representation $\mathbf{M}$. Then $\underline{\mathscr{C}_0}$ is a locally abelian 2-category and $\underline{\mathscr{M}_0}$ is an abelian category.*

Proof. This is a direct consequence of applying the preproof to Theorem 4 in [Fre66] to Corollary 4.3.4. □

4.3.1 Grading Coalgebras

For this section let $\mathscr{C}$ be a locally restricted G-finitary 2-category and let $\mathbf{M}$ be a G-finitary 2-representation of $\mathscr{C}$. Choose $T \in \mathbf{M}(\mathtt{j})$ and $S \in \mathbf{M}(\mathtt{i})$. Following [MMMT16] and Section 3.3 we construct a functor (which we denote Γ) from $\mathscr{C}(\mathtt{i},\mathtt{j})$ to $\Bbbk\text{-mod}$ which takes F to $\operatorname{Hom}_{\mathscr{M}}(T, FS)$, and a functor Γ_0 from $\mathscr{C}_0(\mathtt{i},\mathtt{j})$ to $\Bbbk\text{-mod}$ which takes F to $\operatorname{Hom}_{\mathscr{M}_0}(T, FS)$. We can extend these uniquely to left-exact functors $\underline{\Gamma}$ and $\underline{\Gamma_0}$ from $\underline{\mathscr{C}}(\mathtt{i},\mathtt{j})$ and $\underline{\mathscr{C}_0}(\mathtt{i},\mathtt{j})$ respectively to $\Bbbk\text{-mod}$.

We now introduce the equivalent of the representative 1-morphisms in [MMMT16] Section 4.1. Since $\underline{\Gamma_0}$ is left exact, by [GV72] Section 8, it is pro-representable; that is, a small filtered colimit of representable functors. However by definition $\underline{\mathscr{C}_0}(\mathtt{i},\mathtt{j})$ has enough injectives and the functor category is closed under small filtered colimits, and thus the functor is in fact representable. We denote this representative by $[S,T]_0$.

We have the following analogy of a result in [MMMT16] (Lemma 4.2) that we will find useful:

Lemma 4.3.6. *For any* $H \in \coprod_{\mathtt{i} \in \mathscr{C}_0} \underline{\mathscr{C}_0}(\mathtt{i}, \mathtt{j})$,

$$\mathrm{Hom}_{\underline{\mathbf{M}_0}}(T, HS) \cong \mathrm{Hom}_{\underline{\mathscr{C}_0(\mathtt{i},\mathtt{j})}}([S,T]_0, H)$$

in $\Bbbk\text{-mod}$.

Proof. Take $H \in \underline{\mathscr{C}_0}(\mathtt{i}, \mathtt{j})$. Then H has an exact sequence $H \hookrightarrow F_1 \to F_2$ where $F_i \in \mathscr{C}_0(\mathtt{i}, \mathtt{j})$ for both i. We have left exact functors $\mathrm{Hom}_{\underline{\mathbf{M}_0}}(T, -S)$ and $\mathrm{Hom}_{\underline{\mathscr{C}}}([S,T]_0, -)$ and a diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\underline{\mathbf{M}_0}}(T, HS) & & \mathrm{Hom}_{\underline{\mathscr{C}_0}}([S,T]_0, H) \\
\big\downarrow & & \big\downarrow \\
\mathrm{Hom}_{\mathbf{M}_0}(T, F_1 S) & \xrightarrow{\sim} & \mathrm{Hom}_{\underline{\mathscr{C}_0}}([S,T]_0, F_1) \\
\big\downarrow & & \big\downarrow \\
\mathrm{Hom}_{\mathbf{M}_0}(T, F_2 S) & \xrightarrow{\sim} & \mathrm{Hom}_{\underline{\mathscr{C}_0}}([S,T]_0, F_2)
\end{array}$$

of vector spaces. The result then follows from an application of the five lemma. □

The functor $\underline{\Gamma}$ is representable by [MMMT16], with representative 1-morphism $[S,T]$.

Proposition 4.3.7. *If* $S = T$, *then* $\mathtt{i} = \mathtt{j}$ *and* $A^S = [S,S]$ *has the structure of a coalgebra 1-morphism in* $\underline{\mathscr{C}}(\mathtt{i},\mathtt{i})$ *and* $A_0^S = [S,S]_0$ *has the structure of a coalgebra 1-morphism in* $\underline{\mathscr{C}_0}(\mathtt{i},\mathtt{i})$.

Proof. The first result is Lemma 3.3.1 and the second result is mutatis mutandis the first.

□

4.3.2 The Main Results

Let $\mathscr{C}$, $\mathscr{C}_0$, A^S and A_0^S be as in the previous section. The reason we wish to study A_0^S is that we do not a priori know anything about the component degrees of the internal 2-morphisms for the counit and comultiplication 2-morphisms for A^S. However, we know by definition the corresponding component 2-morphisms for A_0^S have to be homogeneous of degree zero. We will show that we can take A^S to be precisely A_0^S with its counit and comultiplication 2-morphisms. We use the natural inclusion of $\mathscr{C}_0$ into $\mathscr{C}$ to state the following lemma:

Lemma 4.3.8. *For any* $F \in \mathscr{C}(\mathtt{i},\mathtt{i})$ *and* $H \in \underline{\mathscr{C}_0}(\mathtt{i},\mathtt{i})$*, in* $\Bbbk\text{-mod}$

$$\mathrm{Hom}_{\underline{\mathscr{C}}}(H,F) \cong \bigoplus_{g\in G} \mathrm{Hom}_{\underline{\mathscr{C}_0}}(H,F[\![g]\!]).$$

Proof. Let $H = (X,k,Y_i,\alpha_i)$, using the Freyd abelianisation notation from Definition 2.4.2. Then as $F = (F,0,0,0)$, a morphism from H to F in $\underline{\mathscr{C}}$ is of the form $[(p,0)]$ with $p : X \to F$ a morphism in $\mathscr{C}$ and the equivalence relation is spanned by those p such that there exist $q_i : Y_i \to F$ with $\sum q_i\alpha_i = p$. We let $p = \sum_{g\in G} p_g$ for some finite sum. Assume that $[(\tilde{p},0)]$ is equivalent to $[0]$ with morphisms q_i as specified. Since $H \in \underline{\mathscr{C}_0}(\mathtt{i},\mathtt{i})$, the α_i are homogeneous of degree zero. In particular, if we decompose each q_i as $q_i = \sum_{h\in G} q_{i,h}$, then by comparison of degree if follows that $\tilde{p}_g = \sum_i q_{i,g}\alpha_i$. Therefore if $[(p,0)]$ is a general morphism from H to F, then $[(p,0)] = \sum_{g\in G}[(p_g,0)]$ where the equivalence relation is spanned by those p_g such that there exist $q_{g,i} : Y_i \to F$ homogeneous of degree g such that $p_g = \sum_i q_{i,g}\alpha_i$.

We thus define the map

$$\bigoplus_{g\in G} \mathrm{Hom}_{\underline{\mathscr{C}_0}}(H,F[\![g]\!]) \to \mathrm{Hom}_{\underline{\mathscr{C}}}(H,F)$$

by taking $\sum_{g\in G}[(p_g,0)]$ to $[(\sum_{g\in G} p_g,0)]$. This is clearly a vector space homomorphism, and the above working shows that this map is well-defined and injective. If we have some $[(p,0)] \in \mathrm{Hom}_{\underline{\mathscr{C}}}(H,F)$, then as $[(p,0)] = \sum_{g\in G}[(p_g,0)]$, and as any $p_g : H \to F$ corresponds to a degree zero 2-morphism $p_g : H \to F[\![g]\!]$, the assocaited 2-morphism $\sum_{g\in G}[(p_g,0)] \in \bigoplus_{g\in G}\mathrm{Hom}_{\underline{\mathscr{C}_0}}(H,F[\![g]\!])$ maps to $[(p,0)]$, the map is surjective and we have the required isomorphism. □

This isomorphism 'commutes' with composition in the following fashion: if $\alpha \in \mathrm{Hom}_{\underline{\mathscr{C}_0}}(K,H)$ for some 1-morphism K, and we abuse notation by also letting α

refer to the equivalent 2-morphism in $\underline{\mathscr{C}}$, then for any $p \in \mathrm{Hom}_{\underline{\mathscr{C}}}(H, F)$, $p \circ \alpha = [\sum_{g \in G} (p_g, 0)] \circ \alpha = \sum_{g \in G} [(p_g, 0)] \circ \alpha$, with a similar process up to some grading shifts occurs with post-composition.

We can now give the main result of the chapter, which we split into a theorem and a corollary for readability.

Theorem 4.3.9. *$A^S \cong A_0^S$ in $\underline{\mathscr{C}}$.*

Proof. For $F \in \mathscr{C}(\mathtt{i}, \mathtt{i})$ or $\mathscr{C}_0(\mathtt{i}, \mathtt{i})$,

$$\mathrm{Hom}_{\underline{\mathbf{M}}}(S, FS) \cong \mathrm{Hom}_{\mathbf{M}}(S, FS)$$

and similarly

$$\mathrm{Hom}_{\underline{\mathbf{M}_0}}(S, FS) \cong \mathrm{Hom}_{\mathbf{M}_0}(S, FS).$$

By the definition of the grading on $\mathbf{M}$,

$$\mathrm{Hom}_{\mathbf{M}}(S, FS) \cong \bigoplus_{g \in G} \mathrm{Hom}_{\mathbf{M}_0}(S, F[\![g]\!]S).$$

But by the definition of A_0^S,

$$\bigoplus_{g \in G} \mathrm{Hom}_{\mathbf{M}_0}(S, F[\![g]\!]S) \cong \bigoplus_{g \in G} \mathrm{Hom}_{\underline{\mathscr{C}_0}}(A_0^S, F[\![g]\!]).$$

Applying Lemma 4.3.8 for $H = A_0^S$, we have that $\mathrm{Hom}_{\mathbf{M}}(S, FS) \cong \mathrm{Hom}_{\underline{\mathscr{C}}}(A_0^S, F)$ for all $F \in \mathscr{C}(\mathtt{i}, \mathtt{i})$. But by the definition of the representative,

$$\mathrm{Hom}_{\mathbf{M}}(S, FS) \cong \mathrm{Hom}_{\underline{\mathscr{C}}}(A^S, F)$$

and A^S is unique with this property up to isomorphism, hence $A^S \cong A_0^S$ as required. □

Corollary 4.3.10. *We can choose a representative of the isomorphism class of A^S in $\underline{\mathscr{C}}$ such that, when considered as a coalgebra 1-morphism, its coalgebra and comultiplication 2-morphisms have components homogeneous of degree zero.*

Proof. We take the representative A_0^S of the isomorphism class. We have the following diagram of vector spaces:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{M}_0}(S,S) & \xrightarrow{\sim} & \mathrm{Hom}_{\underline{\mathscr{C}_0}}(A_0^S, \mathbb{1}_{\mathtt{i}}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathbf{M}}(S,S) & \xrightarrow{\sim} & \mathrm{Hom}_{\underline{\mathscr{C}}}(A_0^S, \mathbb{1}_{\mathtt{i}}) \end{array}$$

where the vertical arrows are the natural inclusions and the horizontal arrows are the representation isomorphisms. By choice of A_0^S, this diagram is strictly commutative. Taking id$_S$ in $\mathrm{Hom}_{\mathbf{M}_0}(S,S)$, its image along the top path is the image of ϵ_0 under the natural injection (i.e. ϵ_0 considered as a 2-morphism in $\underline{\mathscr{C}}$) while the image under the lower path is the counit of A_0^S as a coalgebra in $\underline{\mathscr{C}}$. It follows that this counit is equal to ϵ_0, and thus has components (or more accurately, non-zero component) homogeneous of degree zero. By constructing similar diagrams for the coevaluation and hence comultiplication 2-morphisms, the result follows. □

4.4 Comodules

We briefly study the structure of comodule 1-morphisms of this 'homogeneous' coalgebra 1-morphism. Let $\mathrm{comod}_{\underline{\mathscr{C}}}([S,S])$ denote the category of A^N-comodule 1-morphisms in $\underline{\mathscr{C}}$, and denote its full subcategory of injectives by $\mathrm{inj}_{\underline{\mathscr{C}}}(A^S)$. We have the following proposition, which is identical to the locally finitary case but whose proof we give here for notational purposes.

Proposition 4.4.1. $[S,T]$ *is a right* $[S,S]$*-comodule 1-morphism for any* $T \in \coprod_{\mathtt{j}\in\mathscr{C}} \mathbf{M}(\mathtt{j})$.

Proof. To find the comodule 2-morphism, we first note that

$$\mathrm{Hom}_{\underline{\mathbf{M}(\mathtt{j})}}(T, [S,T]S) \cong \mathrm{Hom}_{\underline{\mathscr{C}}}([S,T],[S,T]),$$

and so we let $\delta : T \to [S,T]S$ be the (non-zero) image of id$_{[S,T]}$ under this isomorphism. We thus have a composite

$$T \xrightarrow{\delta} [S,T]S \xrightarrow{[S,T]\,\mathrm{coev}_{S,S}} [S,T][S,S]S.$$

But we again have that

$$\mathrm{Hom}_{\underline{\mathbf{M}}}(T,[S,T][S,S]S) \cong \mathrm{Hom}_{\underline{\mathscr{C}}}([S,T],[S,T][S,S])$$

and we denote the image of the composite under this isomorphism by

$$\rho : [S,T] \to [S,T][S,S].$$

It is straightforward to verify that ρ satisfies the comodule axioms. □

Similar to the previous section, we can consider $[S,T]_0$ and construct δ_0 and ρ_0. These are definable because $\mathrm{id}_{[S,T]_0}$ exists, and as $\mathrm{coev}_{S,S}$ has components homogeneous of degree zero by Corollary 4.3.10, so does $\underline{\mathbf{M}}_0([S,T]_0)\,\mathrm{coev}_{S,S}$. The proof that $[S,T]_0 \cong [S,T]$ in $\underline{\mathscr{C}}$ is mutatis mutandis to that given in the previous chapter for $A^S \cong A^S_0$, as is the proof that we can choose $[S,T]$ to have a coevaluation 2-morphism with components homogeneous of degree zero.

4.4.1 An Application: 2-Kac-Moody Algebras

Consider the Khovanov-Lauda-Rouquier algebras we examined in Subsection 3.5.3 and their cyclotomic quotients examined in Subsection 3.5.4. As noted in these sections, there is a $\mathbb{Z}$-grading on KLR algebras, which is preserved in the cyclotomic quotient, since the quotient ideal is generated by a homogeneous polynomial. Considering then the definition of the locally fiat 2-category $\mathscr{U}_\Lambda$ found in Subsection 3.5.5, the following is an immediate consequence:

Proposition 4.4.2. *$\mathscr{U}_\Lambda$ is a locally (restricted) $\mathbb{Z}$-finitary 2-category.*

We set $R^\Lambda = \bigoplus_{n\geq 0} R^\Lambda(n)$, where $R^\Lambda(n)$ is the cyclotomic KLR algebra of degree n

as defined in Subsection 3.5.4.

Proposition 4.4.3. *The indecomposable 1-morphisms of the form $Q_1 \mathbb{1}_\Lambda Q_2$, where the Q_i are 1-morphisms in $\mathscr{U}_\Lambda$, form a maximal $\mathscr{J}$-cell in $\mathscr{U}_\Lambda$.*

Proof. Since the hom-categories of $\mathscr{U}_\Lambda$ are idempotent complete, an indecomposable 1-morphism of the form $Q_1 \mathbb{1}_\Lambda Q_2$ is isomorphic to a direct summand of a functor of the form $M_1 e_1 \otimes_{\Bbbk} e_2 M_2 \otimes_{R^\Lambda} -$, where the e_i are primitive idempotents in R^Λ and M_1 and M_2 are products of the $\mathfrak{e}_i$ and $\mathfrak{f}_i$. In particular, we have that $M_1 e_1 = R^\Lambda e_1$ and $e_2 M_2 = e_2 R^\Lambda$. It follows that such a bimodule is a projective R^Λ-R^Λ-bimodule.

Similarly, given some 1-morphism Q that corresponds to a functor $M \otimes_{R^\Lambda} -$, we can choose primitive idempotents e and f of R^Λ such that $eMf \neq 0$. Then for any $R^\Lambda e' \otimes_{\Bbbk} f' R^\Lambda$ (e' and f' primitive),

$$R^\Lambda e' \otimes_{\Bbbk} e R^\Lambda \otimes_{R^\Lambda} M \otimes_{R^\Lambda} R^\Lambda f \otimes_{\Bbbk} f' R^\Lambda \cong (R^\Lambda e' \otimes_{\Bbbk} f' R^\Lambda)^{\oplus m},$$

where $m = \dim eMf$. Thus $M \otimes_{R^\Lambda} - \leq_{\mathscr{J}} R^\Lambda e' \otimes_{\Bbbk} f' R^\Lambda \otimes_{R^\Lambda} -$. This shows in particular that any indecomposable 1-morphism isomorphic to a summand of a functor of the form $R^\Lambda e \otimes_{\Bbbk} f R^\Lambda \otimes_{R^\Lambda} -$ (for e and f primitive) is $\mathscr{J}$-equivalent to any other 1-morphism isomorphic to a functor of the same form.

It is immediate from the previous paragraph that the $\mathscr{J}$-cell containing (the indecomposable summands of) functors of the form $R^\Lambda e \otimes_{\Bbbk} f R^\Lambda \otimes_{R^\Lambda} -$, for e and f primitive, is maximal, and it remains to show that these functors exhaust the isomorphism classes of members of the $\mathscr{J}$-cell. By construction $e(\beta, i) R^\Lambda(\beta + \alpha_i)$ is a projective right $R^\Lambda(\beta + \alpha_i)$-module (and hence a projective right R^Λ-module), while by Theorem 3.5.30 it is a projective left $R^\Lambda(\beta)$-module (and hence a projective left R^Λ-module). A similar argument gives that $R^\Lambda(\beta + \alpha_i) e(\beta, i)$ is a projective left and projective right R^Λ-module. As a consequence, every 1-morphism in $\mathscr{U}_\Lambda$ is both left projective and right projective.

Let M be some $(R^\Lambda$-$R^\Lambda)$-bimodule with $Q = M \otimes_{R^\Lambda} -$ such that there exist primitive idempotents e and f with $Q \geq_{\mathscr{J}} R^\Lambda e \otimes_{\Bbbk} f R^\Lambda \otimes_{R^\Lambda} -$ in $\mathscr{U}_\Lambda$. This means that there are some 1-morphisms $T \otimes_{R^\Lambda} -$ and $S \otimes_{R^\Lambda} -$ in $\mathscr{U}_\Lambda$ such that M is a direct summand of

$$T \otimes_{R^\Lambda} R^\Lambda e \otimes_{\Bbbk} f R^\Lambda \otimes_{R^\Lambda} S \cong Te \otimes_{\Bbbk} fS.$$

Since T is left projective and S is right projective, M thus decomposes over $\Bbbk$ and is a summand of a bimodule of the form $R^\Lambda e' \otimes_{\Bbbk} f' R^\Lambda$ for some primitive idempotents e' and f'. Both of the remaining claims follow immediately, and the result is proved. □

Lemma 4.4.4. *Every cell 2-representation of $\mathscr{U}_\Lambda$ is a graded simple transitive 2-representation.*

Proof. Let $\mathcal{J}$ be a $\mathscr{J}$-cell in $\mathscr{U}_\Lambda$. By considering the 2-category $\mathscr{U}_{\Lambda,\mathscr{J}}$ as defined in Subsection 3.4.6, without loss of generality $\mathcal{J} =: \mathcal{J}_\Lambda$ is the highest $\mathscr{J}$-cell of $\mathscr{U}_\Lambda$, which by Proposition 4.4.3 is the indecomposable 1-morphisms that factor over Λ. Since these all correspond to tensoring with a projective $(R^\Lambda\text{-}R^\Lambda)$-bimodule, we can embed $\mathcal{J}_\Lambda$ into the 2-category $\mathscr{C}_R$ associated to R^Λ (c.f. Subsection 3.2.1; since the $R^\Lambda(\beta)$ are not necessarily basic, see specifically the definition at the end of that section). In fact, we claim that this embedding is an equivalence between $\mathscr{C}_R$ and $\mathscr{U}_{\Lambda,\mathcal{J}}$.

To see this, by [VV11] Theorem 4.4 $\mathcal{J}$ contains $R^\Lambda e \otimes_{\Bbbk} \Bbbk$ for all primitive idempotents e of R^Λ. But then as $\mathcal{J}$ is strongly regular, it is closed under adjunctions and hence contains $\Bbbk \otimes_{\Bbbk} f R^\Lambda$ for all primitive idempotents f. But then as $\mathcal{J}$ is closed under direct summands of compositions, it also contains $R^\Lambda e \otimes_{\Bbbk} f R^\Lambda$ for all primitive idempotents e and f. Therefore $\mathcal{J}$ not only embeds into $\mathscr{C}_R$, but also essentially surjects, giving the required equivalence.

This means that any $\mathscr{L}$-cell of $\mathcal{J}_\Lambda$ will give an equivalent cell 2-representation by Theorem 3.3.9, and we choose a particularly useful one. Consider the $\mathscr{L}$-cell $\mathscr{L}_{\Bbbk}$ which embeds into $\mathscr{C}_R$ as (the finite dimensional elements of) $\mathrm{add}\{R^\Lambda \otimes_{\Bbbk} \Bbbk\}$. To construct the cell 2-representation, we first construct the transitive (but not necessarily simple transitive) 2-representation $\mathbf{N}_{\mathscr{L}_{\Bbbk}}$ (c.f. Definition 2.3.33). This is a graded 2-representation by construction, and thus to show the cell 2-representation is graded it suffices to show that the ideal $\mathscr{I}$ of the 2-representation we quotient by to form the simple transitive quotient is homogeneous. But given some indecomposable $R^\Lambda e \otimes_{\Bbbk} \Bbbk$ for some idempotent e, we recall from Proposition 3.2.8 that $\mathscr{I}$ is generated by morphisms of the form $\varphi_{a,b} : R^\Lambda e \otimes_{\Bbbk} \Bbbk \to R^\Lambda e \otimes_{\Bbbk} \Bbbk$ where $\varphi_{a,b}(e \otimes 1) = eae \otimes b$, with $b \in \operatorname{rad} \Bbbk$. But $\operatorname{rad} \Bbbk = 0$, and hence $\mathscr{I} = 0$, which is trivially a homogeneous ideal. The result follows. □

This gives us the following result:

Theorem 4.4.5. *Any simple transitive 2-representation of $\mathscr{U}_\Lambda$ is in fact a graded 2-representation, and is equivalent to a cell 2-representation.*

Proof. This is a direct consequence of combining Lemma 4.4.4 and Corollary 3.5.41.

□

5

Wide Finitary 2-Categories

The final chapter of this thesis concerns a more complicated generalisation, where we relax the requirement that each hom-category in the 2-category contains only finitely many isomorphism classes of indecomposable 1-morphisms. While doing so, we will also relax the requirement that the hom-spaces of 2-morphisms are finite dimensional. To do this, we will need a construction that we have not used previously, that of the pro-category (and our own definition of a pro-2-category).

5.1 Pro-(2-)Categories and Ind-(2-)Categories

Definitions and results in this section are drawn from [GV72], primarily Section 8, unless otherwise specified.

Definition 5.1.1. A non-empty category I is *filtered* when:

- For all objects $i, j \in I$ there is another object k and morphisms $i \to k$, $j \to k$.
- For any pair of morphisms $f, g : i \to j$ there is a morphism $h : j \to k$ such that $hf = hg$.

The dual of a filtered category is a *cofiltered* category. Explicitly:

- For all objects $i, j \in I$ there is another object k and morphisms $k \to i$, $k \to j$.

- For any pair of morphisms $f, g : i \to j$ there is a morphism $h : k \to i$ such that $fh = gh$.

This allows us to give one definition of the pro- and ind-categories of a category.

Definition 5.1.2. Let $\mathscr{C}$ be a category. The *pro-category* $\mathrm{Pro}(\mathscr{C})$ of $\mathscr{C}$ is a category whose objects are cofiltered diagrams of $\mathscr{C}$ (i.e. functors $I \to \mathscr{C}$ where I is a cofiltered category). We denote this as $X = (X_i)_{i\in I}$ or $X = \varprojlim_{i\in I} X_i$ (dropping the labelling category where there is no confusion). Morphism sets are defined as $\mathrm{Hom}_{\mathrm{Pro}(\mathscr{C})}(X, Y) = \lim_j \mathrm{colim}_i \mathrm{Hom}_{\mathscr{C}}(X_i, Y_j)$ (with the limit and colimit taken in **Set**).

Dually, the *ind-category* $\mathrm{Ind}(\mathscr{C})$ of $\mathscr{C}$ is the category whose objects are filtered diagrams of $\mathscr{C}$, denoted $X = (X_i)_{i\in I}$ or $X = \varinjlim_{i\in I} X_i$ (again dropping the labelling category where there is no confusion). Morphism sets are defined as $\mathrm{Hom}_{\mathrm{Ind}(\mathscr{C})}(X, Y) = \lim_i \mathrm{colim}_j \mathrm{Hom}_{\mathscr{C}}(X_i, Y_j)$.

There is an alternate way to define these which we will also use when convenient. For a category $\mathscr{C}$, let $\mathscr{C}^\wedge$ denote the *presheaf category* of $\mathscr{C}$; that is, the category of contravariant functors from $\mathscr{C}$ to **Set**. $\mathscr{C}$ embeds into $\mathscr{C}^\wedge$ via the functor $h : \mathscr{C} \to \mathscr{C}^\wedge$ given by $h(X) = \mathrm{Hom}_{\mathscr{C}}(-, X)$. Any presheaf that is isomorphic to such an $h(X)$ is called a *representable presheaf*. We have the following standard result:

Proposition 5.1.3 ([GV72] Proposition 3.4)**.** *Any presheaf is (isomorphic to) a colimit of representable presheaves.*

This allows us to give the following definition, which is equivalent to the one given above (see [GV72] Section 8.2).

Definition 5.1.4. Given a category $\mathscr{C}$, the *ind-category* $\mathrm{Ind}(\mathscr{C})$ is the full subcategory of $\mathscr{C}^\wedge$ whose objects are those presheaves which are isomorphic to a filtered limit of representable presheaves.

The equivalence is taking a filtered diagram $X = \varinjlim_i X_i$ in $\mathscr{C}$ to its colimit $L(X)$ in $\mathscr{C}^\wedge$, with the natural bijection from $\mathrm{Hom}_{\mathrm{Ind}(\mathscr{C})}(X, Y) \cong \mathrm{Hom}_{\mathscr{C}^\wedge}(L(X), L(Y))$.

Dually, we let $\mathscr{C}^\vee$ denote the category of covariant functors from $\mathscr{C}$ to **Set**. Since this means $\mathscr{C}^\vee \cong (\mathscr{C}^{\mathrm{op}})^\wedge$, any such covariant functor is a colimit of 'corepresentable' functors, i.e. functors isomorphic to ones of the form $\mathrm{Hom}_{\mathscr{C}}(X, -)$ for $X \in \mathscr{C}$. Hence we have a similar definition:

Definition 5.1.5. Given a category $\mathscr{C}$, the *pro-category* $\mathrm{Pro}(\mathscr{C})$ is the full subcategory of $\mathscr{C}^\vee$ whose objects are those covariant functors which are isomorphic to a cofiltered limit of corepresentable functors.

The equivalence is given by taking a cofiltered diagram $X = \varprojlim_i X_i$ in $\mathscr{C}$ to its limit $L(X)$ in $\mathscr{C}^\vee$.

Given a functor $F : \mathscr{C} \to \mathscr{D}$, it is possible to extend it to a functor $\mathrm{Pro}(F) : \mathrm{Pro}(\mathscr{C}) \to \mathrm{Pro}(\mathscr{D})$. The action of $\mathrm{Pro}(F)$ on objects is straightforward: for an object $X = \varprojlim_i X_i$, $\mathrm{Pro}(F)(X) = \varprojlim_i F(X_i)$. For morphisms, given any hom-set $\mathrm{Hom}_{\mathscr{C}}(P, Q)$ in $\mathscr{C}$, F induces a morphism in **Set** $F_{P,Q} : \mathrm{Hom}_{\mathscr{C}}(P, Q) \to \mathrm{Hom}_{\mathscr{D}}(FP, FQ)$. Therefore given any $X = (X_i)_{i \in I}$, $Y = (Y_j)_{j \in J}$ in $\mathrm{Pro}(\mathscr{C})$, we have maps

$$F_{i,j} : \mathrm{Hom}_{\mathscr{C}}(X_i, Y_j) \to \mathrm{Hom}_{\mathscr{D}}(F(X_i), F(Y_j)).$$

These therefore induce a map

$$\mathrm{Pro}(F)_{X,Y} : \mathrm{colim}_i \lim_j \mathrm{Hom}_{\mathscr{C}}(X_i, Y_j) \to \mathrm{colim}_i \lim_j \mathrm{Hom}_{\mathscr{D}}(F(X_i), F(Y_j)),$$

which is the required map from $\mathrm{Hom}_{\mathrm{Pro}(\mathscr{C})}(X, Y)$ to $\mathrm{Hom}_{\mathrm{Pro}(\mathscr{D})}(FX, FY)$.

5.1.1 The 2-Categorical Construction

Let $\mathscr{C}$ be a bicategory. We will construct a pro-bicategory $\mathrm{Pro}(\mathscr{C})$. The objects of $\mathrm{Pro}(\mathscr{C})$ as the same as the objects of $\mathscr{C}$. The 1-morphisms of $\mathrm{Pro}(\mathscr{C})$ are cofiltered diagrams of 1-morphisms of $\mathscr{C}$. We will construct the 2-morphisms of $\mathrm{Pro}(\mathscr{C})$ explicitly and carefully, and then use those constructions to define horizontal and vertical composition of 2-morphisms.

As in the 1-categorical case, for 1-morphisms $X = \varprojlim_{i \in I} X_i$ and $Y = \varprojlim_{j \in J} Y_j$, we define $\mathrm{Hom}_{\mathrm{Pro}(\mathscr{C})}(X,Y) = \lim_j \mathrm{colim}_i \mathrm{Hom}_{\mathscr{C}}(X_i, Y_j)$, with the limit and colimit taken in **Set**. Explicitly, for a fixed j, an element of $\mathrm{colim}_i \mathrm{Hom}_{\mathscr{C}}(X_i, Y_j)$ is some $[\alpha] \in (\coprod_{i \in I} \mathrm{Hom}_{\mathscr{C}}(X_i, Y_j))/\sim$, where for $\alpha_i : X_i \to Y_j$ and $\alpha_k : X_k \to Y_j$, $[\alpha_i] = [\alpha_k]$ if there exist 2-morphisms $\beta_i : X_l \to X_i$ and $\beta_k : X_l \to X_k$ in the diagram of X such that $\alpha_i \circ_V \beta_i = \alpha_k \circ_V \beta_k$. Therefore,

$$\lim_j \mathrm{colim}_i \mathrm{Hom}_{\mathscr{C}}(X_i, Y_j) = \{([\alpha_j])_j \in \prod_{j \in J}((\coprod_{i \in I} \mathrm{Hom}_{\mathscr{C}}(X_i, Y_j))/\sim) | \\ \forall \varphi : Y_j \to Y_k, [\varphi \circ_V \alpha_j] = [\alpha_k]\},$$

where the φ are 2-morphisms in the diagram of Y.

The vertical composition of two 2-morphisms is given by the standard construction for pro-categories. This construction is somewhat complicated, but is as follows: Let $X = \varprojlim_{i \in I} X_i$, $Y = \varprojlim_{j \in J} Y_j$ and $Z = \varprojlim_{k \in K} Z_k$ be 1-morphisms in $\mathrm{Pro}(\mathscr{C})$. Considering the X_i and Z_j as 1-morphisms in $\mathrm{Pro}(\mathscr{C})$ by the natural embedding, we first have that

$$\mathrm{Hom}_{\mathrm{Pro}(\mathscr{C})}(X_i, Y) = \{(\alpha_k) \in \prod_{k \in K} \mathrm{Hom}_{\mathscr{C}}(X_i, Y_k) | \forall \varphi : Y_k \to Y_l, \varphi \circ_V \alpha_k = \alpha_l\},$$

while $\mathrm{Hom}_{\mathrm{Pro}(\mathscr{C})}(Y, Z_j) = \coprod_{k \in K} \mathrm{Hom}_{\mathscr{C}}(Y_k, Z_j)/\sim$, with the previously defined equivalence relation.

We thus define a function

$$\mathrm{Hom}_{\mathrm{Pro}(\mathscr{C})}(X_i, Y) \times \mathrm{Hom}_{\mathrm{Pro}(\mathscr{C})}(Y, Z_k) \to \mathrm{Hom}_{\mathrm{Pro}(\mathscr{C})}(X_i, Z_j)$$

by $((\alpha_k), [\beta]) \mapsto \beta \circ_V \alpha_p$, where we have $\beta : Y_p \to Z_j$. We claim that this is a well defined map. For assume $\beta \sim \gamma$ for some $\gamma : Y_q \to Z_j$. Then there exists some 2-morphisms $\delta : Y_m \to Y_p$ and $\epsilon : Y_m \to Y_q$ in the diagram of Y such that

$\beta \circ_V \delta = \gamma \circ_V \epsilon$. But then by the definition of the α_k we have that

$$\beta \circ_V \alpha_p = \beta \circ_V \delta \circ_V \alpha_m = \gamma \circ_V \epsilon \circ_V \alpha_m = \gamma \circ_V \alpha_q,$$

as we require.

From here, we will proceed by first taking the colimit for X and then the limit for Z. Thus, we first define a function

$$\mathrm{Hom}_{\mathrm{Pro}(\mathscr{C})}(X, Y) \times \mathrm{Hom}_{\mathrm{Pro}(\mathscr{C})}(Y, Z_k) \to \mathrm{Hom}_{\mathrm{Pro}(\mathscr{C})}(X, Z_j)$$

by $(([\alpha_k])_k, [\beta]) = [\beta \circ_V \alpha_p]$ for $\beta : Y_p \to Z_j$ and $\alpha_k : X_{m_k} \to Y_k$. We claim that this is well defined: assume that there is some set of 2-morphisms $\delta_k : X_{n_k} \to Y_k$ such that $\alpha_k \sim \delta_k$ for all $k \in K$. Then there exist some 2-morphisms $\sigma_k : X_{z_k} \to X_{m_k}$ and $\pi : X_{z_k} \to X_{m_k}$ such that $\alpha_k \sigma_k = \delta_k \pi_k$ for all k. Then $\beta \alpha_p \sigma_p = \beta \delta_p \pi_p$, i.e. $\beta \alpha_p \sim \beta \delta_p$ and $[\beta \alpha_p] = [\beta \delta_p]$ in $\mathrm{Hom}_{Pro(\mathscr{C})}(X, Z_j)$ and combining this with the reasoning in the prior paragraph gives the required result.

Finally, we define a function

$$\mathrm{Hom}_{\mathrm{Pro}(\mathscr{C})}(X, Y) \times \mathrm{Hom}_{\mathrm{Pro}(\mathscr{C})}(Y, Z) \to \mathrm{Hom}_{\mathrm{Pro}(\mathscr{C})}(X, Z)$$

by $(([\alpha_k])_k, ([\beta_l])_l) \mapsto ([\beta_l \circ_V \alpha_k])_l$. By the previous paragraphs this is well defined, and we claim that this is an element of $\mathrm{Hom}_{\mathrm{Pro}(\mathscr{C})}(X, Z)$. For if $\varphi : Z_l \to Z_m$ is a 2-morphism in the diagram of Z, then by the definition of the β_l, $[\varphi \circ_V \beta_l] = [\beta_m]$. Therefore $[\varphi \circ_V (\beta_l \circ_V \alpha_k)] = [\beta_m \circ_V \alpha_k]$, and the result follows. We also note that the identity 2-morphism on X is $([\mathrm{id}_{X_j}])$.

For composition of 1-morphisms, given two composable 1-morphisms $X = \varprojlim_{i \in I} X_i$ and $Z = \varprojlim_{k \in K} Z_k$, we define the composite as $X \circ Z = \varprojlim_{i,k} X_i \circ Z_k$; that is, the diagram with 1-morphisms $X_i \circ Z_k$ and 2-morphisms $\alpha \circ_H \beta : X_i \circ Z_k \to X_j \circ Z_l$, where $\alpha : X_i \to X_j$ is a 2-morphism in the diagram of X and $\beta : Z_k \to Z_l$ a 2-morphism in the diagram of Z. In particular, the identity 1-morphism of $\mathtt{i}$ in $\mathrm{Pro}(\mathscr{C})$

is the trivial diagram consisting of $\mathbb{1}_{\mathtt{i}}$ and $\mathrm{id}_{\mathbb{1}_{\mathtt{i}}}$.

It remains to define the horizontal composition of 2-morphisms. Given 1-morphisms $X = \varprojlim_{i\in I} X_i$, $Y = \varprojlim_{j\in J} Y_j$, $Z = \varprojlim_{k\in K} Z_k$ and $W = \varprojlim_{l\in L} W_l$, and 2-morphisms $([\alpha_k])_k : X \to Z$ and $([\beta_l])_l : Y \to W$, we define the horizontal composition $([\alpha_k]) \circ_H ([\beta_l]) : XY \to ZW$ to be $([\alpha_k]) \circ_H ([\beta_l]) = ([\alpha_k \circ_H \beta_l])$.

This is well-defined: first, if $\alpha_k \sim \gamma_k$ and $\beta_l \sim \delta_l$, then there exist some 2-morphisms a, b, c, d such that $a \circ_V \alpha_k = c \circ_V \gamma_k$ and $b \circ_V \beta_l = d \circ_V \delta_l$. Then using the interchange law,

$$
\begin{aligned}
(a \circ_H b) \circ_V (\alpha_k \circ_H \beta_l) = (a \circ_V \alpha_k) \circ_H (b \circ_V \beta_l) &= (c \circ_V \gamma_k) \circ_H (d \circ_V \delta_H) \\
&= (c \circ_H d) \circ_V (\gamma_k \circ_H \delta_l)
\end{aligned}
$$

and thus $(\alpha_k \circ_H \beta_l) \sim (\gamma_k \circ_H \delta_l)$. Second, for a fixed Y_k and W_l (considered as trivial cofiltered diagrams), $[\alpha_k \circ_H \beta_l] = [\alpha_k] \circ_H [\beta_l]$ by definition. Therefore if $\varphi : Z_k \to Z_m$ is a 2-morphism in the diagram of Y and $\sigma : W_l \to W_n$ is a 2-morphism in the diagram of W, we have that

$$
\begin{aligned}
[(\varphi \circ_H \sigma) \circ_V (\alpha_k \circ_H \beta_l)] = [(\varphi \circ_V \alpha_k) \circ_H (\sigma \circ_V \beta_l)] &= [\varphi \circ_V \alpha_k] \circ_H [\sigma \circ_V \beta_l] \\
&= [\alpha_m] \circ_H [\beta_n] = [\alpha_m \circ_H \beta_n].
\end{aligned}
$$

We now check the coherence and identity axioms for the new bicategory. Let $X = \varprojlim_{i\in I} X_i$, $Y = \varprojlim_{j\in J} Y_j$ and $Z = \varprojlim_{k\in K} Z_k$ be 1-morphisms in $\mathrm{Pro}(\mathscr{C})$. The diagram for the 1-morphism $(X \circ Y) \circ Z$ has as component 1-morphisms $(X_i \circ Y_j) \circ Z_k$, ranging over all $i \in I$, $j \in J$ and $k \in K$. For the 2-morphisms, for any $f : X_i \to X_{i'}$ in the diagram of X, $g : Y_j \to Y_{j'}$ in the diagram of Y and $h : Z_k \to Z_{k'}$ in the diagram of Z there is a 2-morphism $(f \circ_H g) \circ_H h$. Similarly, $X \circ (Y \circ Z)$ has as component 1-morphisms $X_i \circ (Y_j \circ Z_k)$ and component 2-morphisms $f \circ_H (g \circ_H h)$.

We let $a_{X_i,Y_j,Z_k} : (X_i \circ Y_j) \circ Z_k \to X_i \circ (Y_j \circ Z_k)$ denote the associativity 2-isomorphism in $\mathscr{C}$. We claim that the associativity 2-isomorphism $(X \circ Y) \circ Z \to X \circ (Y \circ Z)$ is

$([a_{X_i,Y_j,Z_k}])$. That this is well defined and is indeed a 2-morphism in $\mathrm{Pro}(\mathscr{C})$ follows directly from the definition of the associativity 2-isomorphisms in $\mathscr{C}$. We first claim that it is a 2-isomorphism with inverse $([a^{-1}_{X_i,Y_j,Z_k}])$:

$$([a_{X_i,Y_j,Z_k}]) \circ_V ([a^{-1}_{X_i,Y_j,Z_k}]) = ([a_{X_i,Y_j,Z_k} \circ_V a^{-1}_{X_i,Y_j,Z_k}]) = ([\mathsf{id}_{(X_i \circ Y_j)\circ Z_k}])$$

as required.

It remains to prove the pentagon axiom. That is, to prove that

$$\begin{aligned}&([a_{X_i,Y_j,(ZW)_{kl}}]) \circ_V ([a_{(XY)_{ij},Z_k,W_l}])\\ =&(([\mathsf{id}_{X_i}]) \circ_H ([a_{Y_j,Z_k,W_l}])) \circ_V ([a_{X_i,(YZ)_{jk},Z_l}]) \circ_V (([a_{X_i,Y_j,Z_k}]) \circ_H ([\mathsf{id}_{W_l}])).\end{aligned}$$

But

$$\begin{aligned}&(([\mathsf{id}_{X_i}]) \circ_H ([a_{Y_j,Z_k,W_l}])) \circ_V ([a_{X_i,(YZ)_{jk},Z_l}]) \circ_V (([a_{X_i,Y_j,Z_k}]) \circ_H ([\mathsf{id}_{W_l}]))\\ =&([a_{X_i,Y_j,Z_k} \circ_H \mathsf{id}_{W_l}]) \circ_V ([a_{X_i,(YZ)_{jk},W_l}]) \circ_V ([\mathsf{id}_{X_i} \circ_H a_{Y_j,Z_k,W_l}])\\ =&([(a_{X_i,Y_j,Z_k} \circ_H \mathsf{id}_{W_l}) \circ_V a_{X_i,(YZ)_{jk},W_l} \circ_V (\mathsf{id}_{X_i} \circ_H a_{Y_j,Z_k,W_l})])\\ =&([a_{X_i,Y_j,(ZW)_{kl}} \circ_V a_{(XY)_{ij},Z_k,W_l}])\\ =&([a_{X_i,Y_j,(ZW)_{kl}}]) \circ_V ([a_{(XY)_{ij},Z_k,W_l}])\end{aligned}$$

as required with the first, second and fourth equalities coming from our prior definitions of horizontal and vertical composition of 2-morphisms, and the third coming from the pentagon axiom for $\mathscr{C}$. In a similar fashion, we can show that if $\rho_{X_i} : \mathbb{1}_{\mathtt{i}} X_i \to X_i$ and $\iota_{X_i} : X_i \mathbb{1}_{\mathtt{i}} \to X_i$ are the right and left unital 2-morphisms in $\mathscr{C}$, then $([\rho_{X_i}])_i : \mathbb{1}_{\mathtt{i}} X \to X$ and $[(\iota_{X_i})] : X\mathbb{1}_{\mathtt{i}} \to X$ are the right and left unital 2-morphisms in $\mathrm{Pro}(\mathscr{C})$.

Proposition 5.1.6. *In the setup of the prior paragraphs, if $\mathscr{C}$ is a strict 2-category, $\mathrm{Pro}(\mathscr{C})$ is a strict 2-category.*

Proof. If $a_{X_i,Y_j,Z_k} = \mathsf{id}_{X_iY_jZ_k}$ for all i, j and k, then $([a_{X_i,Y_j,Z_k}]) = ([\mathsf{id}_{X_iY_jZ_k}]) =$

id_{XYZ} as required, and similarly for the left and right unital 2-morphisms. □

When $\mathscr{C}$ is a strict 2-category, we call $\mathrm{Pro}(\mathscr{C})$ a *pro-2-category*. There are different definitions of pro-2-categories found in the literature, for example [DD14]. The root of the difference between that paper and this thesis is that, since our focus is on 2-categories stemming from categorification of 1-categories, we focus our attention on the hom-categories, with the objects being often comparable to indexing of components. We therefore consider the pro structure in the sense of cofiltered diagrams of 1-morphisms. In comparison, [DD14] consider the whole 2-category and hence, for example, constructs the pro-2-category such that the objects are co-2-filtered diagrams of objects in the base category.

5.2 Wide Finitary 2-Categories

We now define the main 2-categories we will be studying in this chapter.

Definition 5.2.1. A category $\mathscr{C}$ is *wide finitary* if it is an additive $\Bbbk$-linear Krull-Schmidt category with countably many isomorphism classes of indecomposable objects and where the morphism sets are $\Bbbk$-vector spaces of countable dimension. We define the 2-category $\mathfrak{A}_{\Bbbk}^{wf}$ to have as objects wide finitary categories, as 1-morphisms $\Bbbk$-linear functors, and as 2-morphisms natural transformations.

Definition 5.2.2. A 2-category $\mathscr{C}$ is *locally wide finitary* if:

- $\mathscr{C}$ has countably many objects.
- For any objects $\mathtt{i}, \mathtt{j} \in \mathscr{C}$, $\mathscr{C}(\mathtt{i}, \mathtt{j}) \in \mathfrak{A}_{\Bbbk}^{wf}$.
- Horizontal composition is biadditive and $\Bbbk$-linear.
- For each object $\mathtt{i} \in \mathscr{C}$, the identity 1-morphism $\mathbb{1}_{\mathtt{i}}$ is indecomposable.

If the 2-category only has finitely many objects, we refer to it just as a *wide finitary* 2-category.

Definition 5.2.3. A wide finitary category $\mathscr{C}$ is *sparse at an indecomposable object* $F \in \mathscr{C}$ if the set $\{H \in \mathscr{C} | H \text{ indecomposable}, \mathrm{Hom}_{\mathscr{C}}(F, H) \neq 0\}$ contains only finitely many isomorphism classes of objects. $\mathscr{C}$ is *cosparse at an indecomposable object* $F \in \mathscr{C}$ if the set $\{K \in \mathscr{C} | K \text{ indecomposable}, \mathrm{Hom}_{\mathscr{C}}(K, F) \neq 0\}$ contains only finitely many isomorphism classes of objects. A wide finitary category $\mathscr{C}$ is *sparse* if it is sparse at every indecomposable object, and is *cosparse* if it is cosparse at every indecomposable object.

As an immediate observation, since any object in $\mathscr{C}$ is a direct sum of finitely many indecomposable objects, this is equivalent to saying that the condition holds for any object in $\mathscr{C}$.

Definition 5.2.4. A locally wide finitary 2-category $\mathscr{C}$ is *sparse* (resp. *cosparse*) if $\mathscr{C}(\mathtt{i}, \mathtt{j})$ is sparse (resp. cosparse) at every non-identity indecomposable 1-morphism in $\mathscr{C}(\mathtt{i}, \mathtt{j})$ for every pair of objects $\mathtt{i}, \mathtt{j} \in \mathscr{C}$.

Proposition 5.2.5. *Let $\mathscr{C}$ be a locally G-finitary 2-category. Then $\mathscr{C}_0$ is a locally wide finitary 2-category, is both sparse and cosparse, and has finite dimensional hom-spaces.*

Proof. The former claim is shown in the discussion immediately following the definition of $\mathscr{C}_0$; for the latter claim, see the proof of Proposition 4.3.3. □

Definition 5.2.6. Let $\mathscr{C}$ be a locally wide finitary 2-category. If there exists a weak equivalence $-^* : \mathscr{C} \to \mathscr{C}^{\mathrm{op}}$ such that for any 1-morphism $X \in \mathscr{C}(\mathtt{i}, \mathtt{j})$ there are natural 2-isomorphisms $\alpha : X \circ X^* \to \mathbb{1}_{\mathtt{i}}$ and $\beta : \mathbb{1}_{\mathtt{j}} \to X^* \circ X$ with equalities $(\alpha \circ_H \mathrm{id}_X) \circ_V (\mathrm{id}_X \circ_H \beta) = \mathrm{id}_X$ and $(\mathrm{id}_{X^*} \circ_H \alpha) \circ_V (\beta \circ \mathrm{id}_{X^*}) = \mathrm{id}_{X^*}$, then we say that $\mathscr{C}$ is a *locally wide weakly fiat 2-category*. We notate the inverse of $-^*$ by ${}^*-$. If $-^*$ is an involution, then we say that $\mathscr{C}$ is *locally wide fiat*.

5.3 2-Representations, Cells and Ideals

Definition 5.3.1. Let $\mathscr{C}$ be a locally wide finitary 2-category. A *2-representation* of $\mathscr{C}$ is a strict 2-functor from $\mathscr{C}$ to **Cat**. A *wide finitary* 2-representation of $\mathscr{C}$ is a strict 2-functor from $\mathscr{C}$ to $\mathfrak{A}_{\Bbbk}^{\mathsf{wf}}$. A *abelian* 2-representation is a strict 2-functor from $\mathscr{C}$ to the 2-category $\mathbf{AbCat}$ of abelian categories with additive functors and natural transformations.

We can again define left, right and two-sided orders ($\leq_L$, $\leq_R$ and $\leq_J$ respectively), and hence left, right and $\mathscr{J}$-cells in the standard fashion. Since we are no longer necessarily working with only finitely many isomorphism classes of indecomposable 1-morphisms, we can no longer assume that cells are finite in size or that there are only finitely many of them, even within a single hom-category. Indeed, we present a somewhat pathological example below:

Consider first a 2-category $\mathscr{D}$ defined as follows: $\mathscr{D}$ has only one object $*$. $\mathscr{D}(*,*)$ is additive and $\Bbbk$-linear, with indecomposable 1-morphisms consisting of $\mathbb{1}_*$ and $F[\![z]\!]$ for $z \in \mathbb{Z}$, with composition defined by $F[\![z]\!] \circ F[\![y]\!] = F[\![z+y]\!] \oplus F[\![z+y+1]\!]$. For 2-morphisms we set $\mathrm{Hom}_{\mathscr{D}}(F[\![z]\!], F[\![y]\!]) \cong \delta_{yz}\Bbbk$ and extend additively. This clearly gives us a wide finitary 2-category. In addition, it is clearly sparse and cosparse. However, $F \circ F[\![z]\!] \cong F[\![z]\!] \oplus F[\![z+1]\!]$, while $F[\![-1]\!] \circ F[\![z+1]\!] \cong F[\![z]\!] \oplus F[\![z+1]\!]$. Therefore, $F[\![z]\!] \sim_L F[\![z+1]\!]$ for any z, and thus we only have two $\mathscr{L}$-cells, one containing the identity 1-morphism and an infinitely large one containing all the $F[\![z]\!]$. We get similar results for $\mathscr{R}$- and $\mathscr{J}$-cells.

Now construct the category $\mathscr{B}$ as a disjoint union of countably many copies of $\mathscr{D}$. It is clearly a locally wide finitary 2-category, and is still sparse and cosparse. However, by construction, it now has infinitely many infinitely large cells.

We might assume that there is an analogue of strongly regular $\mathscr{J}$-cells, and that a strongly regular locally wide fiat 2-category might display more pleasant structure. It is certainly true that the proof [MM11] Proposition 28 b) is powerful enough to generalise to the locally wide finitary setting, giving the following result analogous to Theorem 3.2.4 and allowing us to define strongly regular $\mathscr{J}$-cells and strongly regular locally wide finitary 2-categories in a similar fashion to previously.

Theorem 5.3.2. *Let $\mathscr{C}$ be a locally wide finitary 2-category and let $\mathcal{J}$ be a $\mathscr{J}$-cell of $\mathscr{C}$ such that every $\mathscr{H}$-cell of $\mathcal{J}$ is non-empty. Let $\mathscr{L}_\mathcal{J}$, $\mathscr{R}_\mathcal{J}$, $\mathscr{D}_\mathcal{J}$ and $\mathscr{J}_\mathcal{J}$ denote the restrictions of Green's relations to $\mathcal{J}$. Then $\mathscr{L}_\mathcal{J} \circ \mathscr{R}_\mathcal{J} = \mathscr{R}_\mathcal{J} \circ \mathscr{L}_\mathcal{J} = \mathscr{D}_\mathcal{J} = \mathscr{J}_\mathcal{J}$.*

Definition 5.3.3. A locally wide finitary 2-category $\mathscr{C}$ is *strongly regular* if each $\mathscr{H}$-cell of $\mathcal{J}$ contains precisely one isomorphism class of indecomposables.

Unfortunately, beyond the obvious restriction on the size of $\mathscr{H}$-cells, even being strongly regular does not induce any further size limits on $\mathscr{L}$-, $\mathscr{R}$- or $\mathscr{J}$-cells. Consider a 2-category $\mathscr{D}$ with a single object, and whose indecomposable 1-morphisms consist of the identity 1-morphism $\mathbb{1}$, and of 1-morphisms F_{ij} for $i, j \in \mathbb{Z}$, where composition is defined by $F_{ij} \circ F_{kl} = F_{il}$. We have a $\mathscr{J}$-cell containing only the identity 1-morphism, which is trivially strongly regular, and then a $\mathscr{J}$-cell containing all the F_{ij}. Its $\mathscr{L}$-cells are of the form $\mathscr{L}_i = \{F_{ix} | x \in \mathbb{Z}\}$, and its $\mathscr{R}$-cells are of the form $\mathscr{R}_j = \{F_{yj} | y \in \mathbb{Z}\}$. The $\mathscr{L}$-cells are clearly incomparable under the left order, and $\mathscr{L}_i \cap \mathscr{R}_j = \{F_{ij}\}$, and therefore the $\mathscr{J}$-cell in indeed strongly regular. However, the $\mathscr{J}$-cell and all its component $\mathscr{L}$-and $\mathscr{R}$-cells are infinite in size. We can again take $\mathscr{B}$ to be countably many disjoint copies of $\mathscr{D}$, to form a strongly regular locally wide finitary 2-category with infinitely many infinitely large $\mathscr{J}$-cells. For those interested in semi-group theory, $\mathscr{D}$ is the 2-category induced by the rectangular band on $\mathbb{Z} \times \mathbb{Z}$ (see [Cli54]).

We take the standard definitions of a 2-ideal of a 2-category and an ideal of a 2-representation. We can also give some miscellaneous results that generalise with minimal changes from the (locally) finitary proof. We begin with the generalisations of [MM14] Lemma 16 i), Lemma 18 and Theorem 15.

Lemma 5.3.4. *Let $\mathscr{C}$ be a locally wide finitary 2-category and $\mathscr{I}$ a 2-ideal of $\mathscr{C}$. If $\mathrm{id}_F \in \mathscr{I}$ for some indecomposable 1-morphism F, then $\mathrm{id}_G \in \mathscr{I}$ for any indecomposable 1-morphism G such that $F \leq_\mathscr{J} G$.*

Proof. The proof of [MM14] generalises to the locally wide finitary setting without issue. □

Lemma 5.3.5. *Let $\mathscr{C}$ be a locally wide finitary 2-category with a unique maximal $\mathscr{J}$-cell $\mathcal{J}$. Then there is a unique 2-ideal $\mathscr{I}$ of $\mathscr{C}$ such that $\mathscr{C}/\mathscr{I}$ is $\mathcal{J}$-simple.*

Proof. Let $F \in \mathscr{C}(\mathtt{i}, \mathtt{j})$ be an indecomposable 1-morphism of $\mathscr{C}$. Since $\mathscr{C}(\mathtt{i}, \mathtt{j})$ is a Krull-Schmidt category, $\mathrm{End}_{\mathscr{C}}(F)$ is local. Therefore, any proper ideal of $\mathrm{End}_{\mathscr{C}}(F)$ is contained in $\mathrm{rad}\,\mathrm{End}_{\mathscr{C}}(F)$. The proof of [MM14] Lemma 18 therefore generalises to the locally wide finitary setting. □

Theorem 5.3.6. *Let $\mathscr{C}$ be a locally wide finitary 2-category and let $\mathcal{J}$ be a $\mathscr{J}$-cell of $\mathscr{C}$. Then there is a unique 2-ideal $\mathscr{I}$ of $\mathscr{C}$ such that $\mathscr{C}/\mathscr{I}$ is $\mathcal{J}$-simple.*

Proof. The proof of [MM14] Theorem 15 generalises immediately given Lemma 5.3.5. □

5.4 (Simple) Transitive and Cell 2-Representations

For the duration of this section, we take $\mathscr{C}$ to be a locally wide finitary 2-category.

Definition 5.4.1. A wide finitary 2-representation $\mathbf{M}$ of $\mathscr{C}$ is *transitive* if for any $M \in \mathcal{M}$, the 2-representation induced by $\mathbf{G}_{\mathbf{M}}(M)$ is equivalent to $\mathbf{M}$. Equivalently, $\mathbf{M}$ is transitive if for any $M, N \in \mathcal{M}$, N is isomorphic to a direct summand of $\mathbf{M}(F)(M)$ for some 1-morphism F of $\mathscr{C}$.

We present another iteration of [MM16c] Lemma 4, for our specific setting.

Lemma 5.4.2. *Let $\mathscr{C}$ be a locally wide finitary 2-category and let $\mathbf{M}$ be a transitive finitary 2-representation. There exists a unique maximal ideal $\mathcal{I}$ of $\mathbf{M}$ which does not contain the identity morphism of any non-zero object.*

Proof. We adapt the proof given in [MM16c] (and the generalisation for Lemma 3.2.6) for our situation. The $\mathbf{M}(\mathtt{i})$ are still additive categories and we can still form the coproduct$\mathscr{M} = \coprod_{\mathtt{i} \in \mathscr{C}} \mathbf{M}(\mathtt{i})$ in **Cat**. Further, the structure of ideals of $\mathbf{M}$ does not depend on the number of indecomposable objects in any $\mathbf{M}(\mathtt{i})$, and so given an ideal $\mathbf{K}$ of $\mathbf{M}$ that does not contain any identity morphisms of non-zero objects, the coproduct $\mathscr{K} = \coprod_{\mathtt{i} \in \mathscr{C}} \mathbf{K}(\mathtt{i})$ is still an ideal of $\mathscr{M}$.

Further, the $\mathbf{M}(\mathtt{i})$ are Krull-Schmidt, and thus by definition for X an indecomposable object, $\mathrm{End}_{\mathbf{M}}(X)$ a local algebra, and $\mathscr{K} \cap \mathrm{End}_{\mathbf{M}}(X)$ is still a proper ideal of $\mathrm{End}_{\mathbf{M}}(X)$. As the argument given in [MM16c] is a 'pointwise' argument that considers the endomorphism ring of each indecomposable X separately, and since an infinite sum of ideals is still an ideal, the proof generalises.

□

Definition 5.4.3. Let $\mathbf{M}$ be a transitive 2-representation of $\mathscr{C}$, and let $\mathscr{I}$ denote the maximal ideal as given in Lemma 5.4.2. If $\mathscr{I} = 0$, then we say that $\mathbf{M}$ is a *simple transitive* 2-representation. Given a transitive 2-representation $\mathbf{M}$ and its ideal $\mathscr{I}$, we can form the simple transitive quotient 2-representation $\mathbf{M}^S = \mathbf{M}/\mathscr{I}$, called the *simple transitive quotient* of $\mathbf{M}$.

We define the *cell 2-representations* of $\mathscr{C}$ is a similar fashion to previous settings. Given the pathological examples given above, in general the cell 2-representations are wide finitary 2-representations - the $\mathscr{L}$-cells, regardless of any other comparable 1-morphisms, may contain infinitely many isomorphism classes of indecomposable 1-morphisms. As before, we notate the cell 2-representation of $\mathscr{C}$ corresponding to a $\mathscr{L}$-cell $\mathcal{L}$ by $\mathbf{C}_{\mathcal{L}}$.

Definition 5.4.4. Let $\mathbf{M}$ be a wide finitary 2-representation of a locally wide finitary 2-category $\mathscr{C}$. Then $\mathbf{M}$ is *sparse* (*cosparse*) if $\mathbf{M}(\mathtt{i})$ is sparse (cosparse) for each $\mathtt{i}$ in $\mathscr{C}$.

5.5 Adelman Abelianisation

5.5.1 1-Categorical Construction

While the fan Freyd abelianisation presented previously has been useful for our needs, it only constructs a legitimately abelian category when the original category has weak kernels or weak cokernels (depending on whether projective or injective abelianisation is performed). We present below a more powerful version of abelianisation. The basic construction is due to [Ade73], but we will again present an equivalent but more complicated version to ensure we retain a 2-category once finished.

Definition 5.5.1. Let $\mathscr{C}$ be an additive category. We construct the *Adelman abelianisation* $\tilde{\mathscr{C}}$ as follows:

- The objects of $\tilde{\mathscr{C}}$ are quintuples (Y, X, Z, α, β), where X, Y, Z are objects of $\mathscr{C}$ and $\alpha : Y \to X$ and $\beta : X \to Z$ are morphisms of $\mathscr{C}$.
- Morphisms of $\tilde{\mathscr{C}}$ are equivalence classes of triples $(s, r, t) : (Y, X, Z, \alpha, \beta) \to (Y', X', Z', \alpha', \beta')$ where $s : Y \to Y'$, $r : X \to X'$ and $t : Z \to Z'$ are morphisms of $\mathscr{C}$, modulo those triples (s, r, t) that satisfy a homotopy relation, explicitly triples (s, r, t) for which there exist morphisms $p : X \to Y'$ and $q : Z \to X'$ such that $\alpha' p + q\beta = r$.
- Composition of triples is given by $(s, r, t) \circ (s', r', t') = (ss', rr', tt')$.
- Identity morphisms are of the form $(\mathrm{id}_Y, \mathrm{id}_X, \mathrm{id}_Z)$.

Definition 5.5.2. Let $\mathscr{C}$ be an additive category. We construct the *fan Adelman abelianisation* $\widehat{\mathscr{C}}$ as follows:

- The objects of $\widehat{\mathscr{C}}$ are equivalence classes of sextuples of the form $(Y_i, X, Z_j, \alpha_i, \beta_j, k)_{i,j \in \mathbb{Z}^+}$ with $Y_i, X, Z_j \in \mathscr{C}$ and $\alpha_i : Y_i \to X$, $\beta_j : X \to Z_j$ morphisms in $\mathscr{C}$. We require that for all $i, j > k$, $Y_i = Z_j = 0$. Two sextuples are equivalent if they only differ in the value of k.

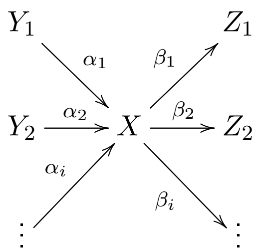
- Morphisms in $\widehat{\mathscr{C}}$ from $(Y_i, X, Z_j, \alpha_i, \beta_j, k)$ to $(Y'_i, X', Z'_j, \alpha'_i, \beta'_j, k')$ are equivalence classes of triples $(s_{ij}, r, t_{mn})_{i,j,m,n\in\mathbb{Z}^+}$ where $r : X \to X'$, $s_{ij} : Y_i \to Y'_j$ and $t_{mn} : Z_m \to Z'_n$ are morphisms in $\mathscr{C}$, with the equivalence relation being spanned by triples that satisfy a homotopy relation, explicitly triples (s_{ij}, r, t_{mn}) such that there exist $p_i : X \to Y'_i$ and $q_j : Z_j \to X'$ such that $\sum\limits_i \alpha'_i p_i + \sum\limits_j q_j\beta_j = r$.

- Composition of triples is given by

$$(s'_{ij}, r', t'_{mn}) \circ (s_{ij}, r, t_{mn}) = (\sum_l s'_{lj}s_{il}, r'r, \sum_z t'_{zn}t_{mz}).$$

- Identity morphisms are of the form $(\delta_{ij}\mathrm{id}_{Y_i}, \mathrm{id}_X, \delta_{mn}\mathrm{id}_{Z_m})_{i,j,m,n\in\mathbb{Z}^+}$.

Similarly to the fan Freyd abelianisation from [MMMT16], this can be thought of as a variation of the traditional Adelman abelianisation with multiple objects at the left and right, as in the diagram:



While eventually both the Y_i and the Z_j will be zero, the minimal i and j where this occurs will in general not be identical. However, giving a single bound that is the larger of the two simplifies the already somewhat unwieldy notation.

In a similar fashion to [MMMT16], $\widehat{\mathscr{C}}$ is additive and is equivalent to the traditional Adelman abelianisation via the assignment

$$(Y_i, X, Z_j, \alpha_i, \beta_j, k) \mapsto \oplus Y_i \overset{\oplus\alpha_i}{\to} X \overset{\oplus\beta_j}{\to} \oplus Z_j$$

and is hence abelian. Further, $\mathscr{C}$ embeds into $\widehat{\mathscr{C}}$ via the assignment $X \mapsto (0, X, 0, 0, 0, 0)$ and $f : X \to Y \mapsto (0, f, 0)$. As mentioned in [Ade73], its image is the full subcategory of injective-projectives of $\widehat{\mathscr{C}}$. Let $I_{\mathscr{C}} : \mathscr{C} \to \widehat{\mathscr{C}}$ denote this canonical embedding of $\mathscr{C}$ into $\widehat{\mathscr{C}}$.

An important consequence of this construction is the following theorem due to [Ade73], using the aforementioned equivalence of categories between $\widehat{\mathscr{C}}$ and the original Adelman abelianisation:

Theorem 5.5.3 ([Ade73] Theorem 1.14)**.** *Let $\mathscr{C}$ be an additive 2-category, and let $F : \mathscr{C} \to \mathscr{A}$ be a additive functor into an abelian category $\mathscr{A}$. Then there is a unique (up to natural equivalence) exact functor $F^\diamond : \widehat{\mathscr{C}} \to \mathscr{A}$ such that $F^\diamond I_{\mathscr{C}} \cong F$.*

If $\mathscr{C}$ has finite dimensional hom-spaces, then the dimension of $\mathrm{Hom}_{\widehat{\mathscr{C}}}((Y_i, X, Z_j, \alpha_i, \beta_j, k), (Y'_i, X', Z'_j, \alpha'_i, \beta'_j, k'))$ is bounded above by

$$\prod_{i \leq k, m \leq k'} \dim \mathrm{Hom}_{\mathscr{C}}(Y_i, Y'_m) \cdot \dim \mathrm{Hom}_{\mathscr{C}}(X, X) \cdot \prod_{j \leq k, n \leq k'} \dim \mathrm{Hom}_{\mathscr{C}}(Z_j, Z'_n),$$

and thus $\widehat{\mathscr{C}}$ also has finite dimensional hom-spaces. More generally, if $\mathscr{C}$ is a small additive category, then so is $\widehat{\mathscr{C}}$.

5.5.2 2-Categorical Construction

Let $\mathscr{C}$ be either a locally wide finitary 2-category or a locally G-finitary 2-category. We define the *fan Adelman abelianisation* $\widehat{\mathscr{C}}$ of $\mathscr{C}$ as follows:

- The objects of $\widehat{\mathscr{C}}$ are the same as those of $\mathscr{C}$.
- For any objects $\mathtt{i}, \mathtt{j} \in \mathscr{C}$, $\widehat{\mathscr{C}}(\mathtt{i}, \mathtt{j}) = \widehat{\mathscr{C}(\mathtt{i}, \mathtt{j})}$, i.e. the hom-categories of $\widehat{\mathscr{C}}$ are the fan Adelman abelianisations of the hom-categories of $\mathscr{C}$.
- Composition of 1-morphisms is defined by

$$(Y_i, X, Z_j, \alpha_i, \beta_j, k) \circ (Y'_i, X', Z'_j, \alpha'_i, \beta'_j, k') = (V_i, XX', W_j, \gamma_i, \delta_j, k + k'),$$

where:

$$V_i = \begin{cases} Y_i \circ X', & i = 1, \dots, k \\ X \circ Y'_{i-k}, & i = k+1, \dots, k+k' \\ 0, & \text{else,} \end{cases}$$

$$W_j = \begin{cases} X \circ Z'_j, & j = 1, \dots, k' \\ Z_{j-k'} \circ X', & j = k'+1, \dots, k'+k \\ 0, & \text{else,} \end{cases}$$

$$\gamma_i = \begin{cases} \alpha_i \circ \mathrm{id}_{X'}, & i = 1, \dots, k \\ \mathrm{id}_X \circ \alpha'_{i-k}, & i = k+1, \dots, k+k' \\ 0, & \text{else,} \end{cases}$$

$$\delta_j = \begin{cases} \mathrm{id}_X \circ \beta'_j, & j = 1, \dots, k' \\ \beta_{j-k'} \circ \mathrm{id}_{X'}, & j = k'+1, \dots, k'+k \\ 0, & \text{else.} \end{cases}$$

- Identity 1-morphisms are $(0, \mathbb{1}_{\mathtt{i}}, 0, 0, 0, 0)$.
- Horizontal composition of 2-morphisms is defined component-wise.

The embedding of each 1-category $\mathscr{C}(\mathtt{i}, \mathtt{j})$ as the projective-injectives of $\widehat{\mathscr{C}}(\mathtt{i}, \mathtt{j})$ leads to the embedding of $\mathscr{C}$ as a sub-2-category of $\widehat{\mathscr{C}}$.

If $\mathscr{C}$ is a locally wide finitary 2-category or a locally G-finitary 2-category, let $\mathbf{M}$ be a wide finitary 2-representation or a G-finitary 2-representation respectively. We define the *fan Adelman abelianisation* $\widehat{\mathbf{M}}$ of $\mathbf{M}$ by setting $\widehat{\mathbf{M}}(\mathtt{i}) = \widehat{\mathbf{M}(\mathtt{i})}$ for each object $\mathtt{i}$ of $\mathscr{C}$. This has the natural structure of a 2-representation of $\mathscr{C}$ by component-wise action.

In addition, similar to [MMMT16], we can make $\widehat{\mathbf{M}}$ a $\widehat{\mathscr{C}}$ 2-representation by setting $\widehat{\mathbf{M}}((Y_i, X, Z_j, \alpha_i, \beta_j, k))(N_i, M, P_j, f_i, g_j, k') = (S_i, R, T_j, h_i, l_j, k+k')$, where:

$$R_i = \begin{cases} \mathbf{M}(Y_i)M, & i = 1, \dots, k \\ \mathbf{M}(X)N_{i-k}, & i = k+1, \dots, k+k' \\ 0, & \text{else,} \end{cases}$$

$$T_j = \begin{cases} \mathbf{M}(X)P_j, & j = 1, \dots, k' \\ \mathbf{M}(Z_{j-k'})M, & j = k'+1, \dots, k'+k \\ 0, & \text{else,} \end{cases}$$

$$h_i = \begin{cases} \mathbf{M}(\alpha_i)_M, & i = 1, \dots, k \\ \mathbf{M}(X)f_i, & i = k+1, \dots, k+k' \\ 0, & \text{else,} \end{cases}$$

$$l_j = \begin{cases} \mathbf{M}(X)g_j, & j = 1, \dots, k' \\ \mathbf{M}(\beta_{j-k'})_M, & j = k'+1, \dots, k'+k \\ 0, & \text{else.} \end{cases}$$

5.5.3 Beligiannis Abelianisation

While we will mostly be using the fan Adelman abelianisation due to it being (relatively) simple to explicitly construct, there is another 'universal' abelianisation that we will occasionally be referring to due to the process of its construction, originally defined in [Bel00].

Definition 5.5.4. Given an additive category $\mathscr{C}$, we define the *(fan) Beligiannis abelianisation* $\overline{\underline{\mathscr{C}}} = \underline{(\overline{\mathscr{C}})} (= \overline{(\underline{\mathscr{C}})})$ to be formed by taking the (fan) injective abelianisation of the (fan) projective abelianisation of $\mathscr{C}$.

By [Bel00] Theorem 6.1 (1), this is equivalent to taking the (fan) projective abelianisation of the (fan) injective abelianisation of $\mathscr{C}$. Further, by [Bel00]

Theorem 6.1 (4) this has the same universal property as $\widehat{\mathscr{C}}$ does in Theorem 5.5.3, and hence $\widehat{\mathscr{C}}$ and $\overline{\underline{\mathscr{C}}}$ are canonically equivalent.

5.6 Constructing the Coalgebra 1-Morphism

While we have a method of producing an abelianisation of a (locally) wide finitary (2-)category, we cannot immediately generalise the construction of the coalgebra 1-morphism from [MMMT16], or indeed the algebra morphisms from [EGNO16] Section 7.8, as these both assume finiteness conditions that we lack. We thus need to expand to a larger setting.

Let $\mathscr{C}$ be a locally wide finitary 2-category and let $\mathbf{M}$ be a transitive 2-representation of $\mathscr{C}$. Choose $S \in \mathbf{M}(\mathtt{i})$. We define $\mathscr{C}_{\mathtt{i}} := \coprod_{\mathtt{j}\in\mathscr{C}} \mathscr{C}(\mathtt{i},\mathtt{j})$. We also define the notation $\mathscr{C}_{\mathtt{i}}(\mathtt{j}) = \mathscr{C}(\mathtt{i},\mathtt{j})$. We define a functor $\mathrm{ev}_S : \mathscr{C}_{\mathtt{i}} \to \mathscr{M}$ by $\mathrm{ev}_S(F) = FS$ and for $\alpha : F \to G$, $\mathrm{ev}_S(\alpha) = \alpha_S$. We let $\mathrm{ev}_{S,\mathtt{j}}$ denote the restriction and corestriction of ev_S to $\mathscr{C}_{\mathtt{i}}(\mathtt{j})$ and $\mathbf{M}(\mathtt{j})$ respectively. By composing with the natural injection of $\mathscr{M}$ into $\widehat{\mathscr{M}}$, we can consider $\mathrm{ev}_{S,\mathtt{j}}$ to be a functor from $\mathscr{C}_{\mathtt{i}}(\mathtt{j})$ to $\widehat{\mathbf{M}(\mathtt{j})}$. Since $\widehat{\mathbf{M}(\mathtt{j})}$ is an abelian category, we can use the universal property of the Adelman abelianisation to extend $\mathrm{ev}_{S,\mathtt{j}}$ to an exact functor $\widehat{\mathrm{ev}_{S,\mathtt{j}}} : \widehat{\mathscr{C}_{\mathtt{i}}}(\mathtt{j}) \to \widehat{\mathbf{M}(\mathtt{j})}$. These then combine to give us a functor $\widehat{\mathrm{ev}_S} : \widehat{\mathscr{C}_{\mathtt{i}}} \to \widehat{\mathscr{M}}$.

Proposition 5.6.1. *We can take $\widehat{\mathrm{ev}_S}$ to be evaluation at S.*

Proof. We will show that evaluation at S is an exact functor from $\widehat{\mathscr{C}}(\mathtt{i},\mathtt{j})$ to $\widehat{\mathbf{M}}(\mathtt{j})$ for any $\mathtt{j}$. Then since $\widehat{\mathrm{ev}_{S,\mathtt{j}}}$ is the unique up to equivalence exact extension of $\mathrm{ev}_{S,\mathtt{j}}$, it must be equivalent to evaluation at S. For simplicity of notation we work in the non-fan Adelman abelianisation, since we are in essence dealing with a pair of 1-categories, and thus it is equivalent to the fan case.

Consider some short exact sequence in $\widehat{\mathscr{C}_{\mathtt{i}}}(\mathtt{j})$,

$$\begin{array}{ccccc} 0 & & 0 & & 0, \\ \downarrow & & \downarrow & & \downarrow \\ X_2 & \xrightarrow{x_1} & X_1 & \xrightarrow{x_2} & X_3 \\ {\scriptstyle f_2}\downarrow & & {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_3} \\ Y_2 & \xrightarrow{y_1} & Y_1 & \xrightarrow{y_2} & Y_3 \\ {\scriptstyle g_2}\downarrow & & {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle g_3} \\ Z_2 & \xrightarrow{z_1} & Z_1 & \xrightarrow{z_2} & Z_3 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 \end{array}$$

which we also notate as $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ for brevity. Since $\widehat{\mathscr{C}}_{\mathtt{i}}(\mathtt{j})$ is an abelian category and f is monic, $\operatorname{Im} f = Y$ and so $f = \ker g$. But by [Ade73] Theorem 1.1, we thus have an explicit construction for $\ker g$ (up to isomorphism), namely

$$\begin{array}{ccccc} Y_2 \oplus Z_2 & \xrightarrow{\begin{pmatrix} y_1 & 0 \\ g_2 & -1 \end{pmatrix}} & Y_1 \oplus Z_2 & \xrightarrow{\begin{pmatrix} g_1 & -z_1 \\ y_2 & 0 \end{pmatrix}} & Z_1 \oplus Y_3 \\ {\scriptstyle (1,0)}\downarrow & & {\scriptstyle (1,0)}\downarrow & & \downarrow{\scriptstyle (0,1)} \\ Y_2 & \xrightarrow{y^1} & Y_1 & \xrightarrow{y_2} & Y_3. \end{array}$$

But the evaluation of this at S is

$$\begin{array}{ccccc} Y_2S \oplus Z_2S & \xrightarrow{\begin{pmatrix} (y_1)_S & 0 \\ (g_2)_S & (-1)_S \end{pmatrix}} & Y_1S \oplus Z_2S & \xrightarrow{\begin{pmatrix} (g_1)_S & (-z_1)_S \\ (y_2)_S & 0 \end{pmatrix}} & Z_1S \oplus Y_3S \\ {\scriptstyle (1,0)}\downarrow & & {\scriptstyle (1,0)}\downarrow & & \downarrow{\scriptstyle (0,1)} \\ Y_2S & \xrightarrow{(y_1)_S} & Y_1S & \xrightarrow{(y_2)_S} & Y_3S \end{array}$$

and since $\widehat{\mathbf{M}}(\mathtt{j})$ is an Adelman abelianisation, this is the kernel of g_S. Thus f_S is monic and since $\widehat{\mathbf{M}}(\mathtt{j})$ is abelian, $\operatorname{im} f_S = \ker g_S$.

It remains to show that g_S is epic. But g is epic, and since $\widehat{\mathscr{C}}(\mathtt{j})$ is abelian, it is thus a cokernel of some morphism $h : W \to Y$. But again using [Ade73] Theorem 1.1 and a similar procedure to above, we derive that g_S is the cokernel of $h_S : WS \to YS$, and thus is epic. Therefore f_S is monic, g_S is epic and $\operatorname{im} f_S = \ker g_S$, and hence the sequence $0 \to XS \xrightarrow{f_S} YS \xrightarrow{g_S} ZS \to 0$ is exact in $\widehat{\mathbf{M}}(\mathtt{j})$, and the result follows. $\square$

Before continuing, we wish to define the action of the pro-2-category $\operatorname{Pro}(\widehat{\mathscr{C}})$ on $\operatorname{Pro}(\widehat{\mathscr{M}})$. We define this component-wise: since we have the action of $\widehat{\mathscr{C}}(\mathtt{i}, \mathtt{j})$ on

$\widehat{\mathbf{M}}(\mathtt{i})$, i.e. a bifunctor $\mathrm{ev}_{-}(-) : \widehat{\mathscr{C}}(\mathtt{i},\mathtt{j}) \times \widehat{\mathbf{M}}(\mathtt{i}) \to \widehat{\mathbf{M}}(\mathtt{j})$, we can take the pro-functor $\mathrm{Pro}(\mathrm{ev}_{-}(-))$ as the action by using a similar process to the definition of a pro-2-category in Subsection 5.1.1. In particular, keeping S as above and taking $X = \varprojlim_i X_i \in \mathrm{Pro}(\mathscr{C}_{\mathtt{i}}(\mathtt{j}))$, we have that $XS = \varprojlim_i X_i S \in \mathrm{Pro}(\widehat{\mathbf{M}}(\mathtt{j}))$.

Proposition 5.6.2. *$\mathrm{Pro}(\widehat{\mathrm{ev}_S})$ is evaluation at S.*

Proof. Let $X = \varprojlim_i X_i \in \mathrm{Pro}(\widehat{\mathscr{C}_{\mathtt{i}}}(\mathtt{j}))$. Then

$$\mathrm{Pro}(\widehat{\mathrm{ev}_{S,\mathtt{j}}})(X_i) = \varprojlim_i(\widehat{\mathrm{ev}_{S,\mathtt{j}}}(X_i)) = \varprojlim_i X_i S = XS$$

by Proposition 5.6.1, and the result follows. ☐

We recall here a pair of results from [GV72], though we give the dual versions thereof:

Lemma 5.6.3 ([GV72] Proposition 8.11.4)**.** *Let $\mathscr{C}$ be a category equivalent to a small category and let $\mathscr{D}$ be a category. Then a functor $F : \mathscr{C} \to \mathscr{D}$ has a pro-adjoint if and only if it is right exact.*

Lemma 5.6.4 ([GV72] Proposition 8.11.2)**.** *Let $\mathscr{C}$ and $\mathscr{D}$ be categories. Then a functor $F : \mathscr{C} \to \mathscr{D}$ has a pro-adjoint if and only if $\mathrm{Pro}(F)$ has a left adjoint.*

Proposition 5.6.5. *$\mathrm{Pro}(\widehat{\mathrm{ev}_S})$ has a left adjoint, denoted $[S,-] : \mathrm{Pro}(\widehat{\mathscr{M}}) \to \mathrm{Pro}(\widehat{\mathscr{C}_{\mathtt{i}}})$.*

Proof. By construction $\widehat{\mathrm{ev}_S}$ is an exact functor, which by Lemma 5.6.3 is equivalent to $\widehat{\mathrm{ev}_S}$ having a pro-adjoint, which by Lemma 5.6.4 is equivalent to $\mathrm{Pro}(\widehat{\mathrm{ev}_S})$ having a left adjoint, as required. ☐

We are especially interested in $[S,S]$, since we will show that $[S,S]$ has the structure of a coalgebra 1-morphism in $\mathrm{Pro}(\widehat{\mathscr{C}_{\mathtt{i}}}(\mathtt{i}))$ when $\mathscr{C}$ is sufficiently pleasant. An element m of $\mathrm{Hom}_{\mathrm{Pro}(\widehat{\mathscr{C}}(\mathtt{i}))}(X,Y)$ acts as a function $\mathrm{Hom}_{\mathrm{Pro}(\widehat{\mathscr{C}}(\mathtt{i}))}([S,S],X) \to \mathrm{Hom}_{\mathrm{Pro}(\widehat{\mathscr{C}}(\mathtt{i}))}([S,S],Y)$ via composition. But using the adjunction isomorphism, this can also be considered as a function $\mathrm{Hom}_{\mathrm{Pro}(\widehat{\mathbf{M}}(\mathtt{i}))}(S,XS) \to \mathrm{Hom}_{\mathrm{Pro}(\widehat{\mathbf{M}}(\mathtt{i}))}(S,YS)$ via composition with m_S.

Proposition 5.6.6. *If $\mathscr{C}$ is a locally wide weakly fiat 2-category and $G, H \in \mathrm{Pro}(\widehat{\mathscr{C}})$, $\mathrm{Hom}_{\mathrm{Pro}(\widehat{\mathscr{C}})}(FG, H) \cong \mathrm{Hom}_{\mathrm{Pro}(\widehat{\mathscr{C}})}(G, F^*H)$ for any $F \in \mathscr{C}$.*

Proof. For a 1-morphism $F \in \mathscr{C}(\mathtt{i}, \mathtt{j})$, the evaluation and coevaluation 2-morphisms $\alpha : FF^* \to \mathbb{1}_{\mathtt{j}}$ and $\beta : \mathbb{1}_{\mathtt{i}} \to F^*F$ still exist in $\mathrm{Pro}(\widehat{\mathscr{C}})$ and $\mathrm{End}_{\mathrm{Pro}(\widehat{\mathscr{C}})}(G) = \mathrm{End}_{\mathscr{C}}(G)$ for any $G \in \mathscr{C}$, we still have that $(\alpha \circ_H \mathrm{id}_F) \circ_V (\mathrm{id}_F \circ_H \beta) = \mathrm{id}_F$ and $(\mathrm{id}_{F^*} \circ_H \alpha) \circ_V (\beta \circ_H \mathrm{id}_{F^*}) = \mathrm{id}_{F^*}$, and thus F and F^* still form an internal adjunction as required. □

An immediate consequence is the following:

Corollary 5.6.7. *Let $\mathscr{C}$ be a locally wide weakly fiat 2-category, $\mathbf{M}$ a wide finitary 2-representation of $\mathscr{C}$ and $X, Y \in \mathrm{Pro}(\widehat{\mathscr{M}})$. Then*

$$\mathrm{Hom}_{\mathrm{Pro}(\widehat{\mathscr{M}})}(FX, Y) \cong \mathrm{Hom}_{\mathrm{Pro}(\widehat{\mathscr{M}})}(X, F^*Y)$$

for any $F \in \mathscr{C}$

Proposition 5.6.8. *$[S, S]$ has the structure of a coalgebra 1-morphism in $\mathrm{Pro}(\widehat{\mathscr{C}}(\mathtt{i}))$.*

Proof. We construct comultiplication and counit 2-morphisms for $[S, S]$ analogously to the proof of Lemma 3.3.1. For the comultiplication, by construction of the adjunction we have an isomorphism of hom-spaces

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Pro}(\widehat{\mathscr{C}}(\mathtt{i}))}([S, S], [S, S]) &\cong \mathrm{Hom}_{\mathrm{Pro}(\widehat{\mathbf{M}}(\mathtt{i}))}(S, \mathrm{Pro}(\widehat{\mathrm{ev}_S})[S, S]) \\ &= \mathrm{Hom}_{\mathrm{Pro}(\widehat{\mathbf{M}}(\mathtt{i}))}(S, [S, S]S),\end{aligned}$$

with the equality following from Proposition 5.6.2. Let $\mathrm{coev}_S : S \to [S, S]S$ be the image of $\mathrm{id}_{[S,S]}$ under this isomorphism. We can thus form the composition

$$([S, S]\,\mathrm{coev}_S) \circ \mathrm{coev}_S \in \mathrm{Hom}_{\mathrm{Pro}(\widehat{\mathbf{M}}(\mathtt{i}))}(S, [S, S][S, S]S).$$

But again by the adjunction isomorphism,

$$\mathrm{Hom}_{\mathrm{Pro}(\widehat{\mathbf{M}}(\mathtt{i}))}(S,[S,S][S,S]S) \cong \mathrm{Hom}_{\mathrm{Pro}(\widehat{\mathscr{C}}(\mathtt{i}))}([S,S],[S,S][S,S]).$$

We take the coevaluation 2-morphism δ_S to be the image of $([S,S]\,\mathrm{coev}_S)\circ \mathrm{coev}_S$.

For the counit 2-morphisms, we have the adjunction isomorphism

$$\mathrm{Hom}_{\mathrm{Pro}(\widehat{\mathbf{M}}(\mathtt{i}))}(S,S) = \mathrm{Hom}_{\mathrm{Pro}(\widehat{\mathbf{M}}(\mathtt{i}))}(S,\mathbb{1}_{\mathtt{i}}S) \cong \mathrm{Hom}_{\mathrm{Pro}(\widehat{\mathscr{C}}(\mathtt{i}))}([S,S],\mathbb{1}_{\mathtt{i}}).$$

We thus take the counit 2-morphism ϵ_S to be the image of id_S under this isomorphism.

Showing that δ_S and ϵ_S satisfy the coalgebra axioms is mutatis mutandis the arguments given in Lemma 3.3.1, giving the result. □

Proposition 5.6.9. *Let $T \in \mathbf{M}(\mathtt{j})$. Then $[S,T]$ is a comodule 1-morphism over $[S,S]$.*

Proof. The proof is directly analogous to the discussion found directly after Lemma 3.3.1. □

We denote the category of comodule 1-morphisms over $[S,S]$ by $\mathrm{comod}_{\mathrm{Pro}(\widehat{\mathscr{C}})}([S,S])$, which we abbreviate to $\mathrm{comod}([S,S])$ when it does not cause confusion. Similarly to Section 2.5, we denote the corresponding 2-representation of $\mathrm{Pro}(\widehat{\mathscr{C}})$ by $\mathbf{comod}_{\mathrm{Pro}(\widehat{\mathscr{C}})}([S,S])$ or $\mathbf{comod}([S,S])$. This allows us to define a functor $\Theta : \mathscr{M} \to \mathrm{comod}_{\mathrm{Pro}(\widehat{\mathscr{C}})}([S,S])$ given on objects by $T \mapsto [S,T]$ and on morphisms by $f \mapsto [S,f]$. We denote by $\mathrm{Forg}_S : \mathrm{comod}_{\mathrm{Pro}(\widehat{\mathscr{C}})}([S,S]) \to \mathrm{Pro}(\widehat{\mathscr{C}})$ the canonical forgetful functor.

Proposition 5.6.10. *Θ is indeed a functor.*

Proof. It is sufficient to show that $[S,f] : [X,T] \to [S,T']$ is a morphism in $\mathrm{comod}_{\mathrm{Pro}(\widehat{\mathscr{C}})}([S,S])$ for any morphism $f : T \to T'$ in $\mathscr{M}$. Specifically, the diagram

$$\begin{array}{ccc} [S,T] & \xrightarrow{\quad\rho_T\quad} & [S,T][S,S] \\ {\scriptstyle [S,f]}\downarrow & & \downarrow{\scriptstyle [S,f]\circ_H \mathsf{id}_{[S,S]}} \\ [S,T'] & \xrightarrow[\quad\rho_{T'}\quad]{} & [S,T'][S,S] \end{array}$$

needs to commute. We will show this by showing that the images of the sides under the adjunction isomorphism are equal.

For notation, let $\mathrm{coev}_{S,T}$ denote the image of $\mathsf{id}_{[S,T]}$ under its adjunction isomorphism. Letting η be the unit of the adjunction and σ the counit, we have that $\eta_T = \mathrm{coev}_{S,T}$, and given $\alpha \in \mathrm{Hom}_{\mathrm{Pro}(\widehat{\mathscr{C}})}([S,T],F)$, the image of α under the adjunction isomorphism is given by $\alpha_S \circ \eta_T = \alpha_S \circ \mathrm{coev}_{S,T}$. Similarly, the image of $f \in \mathrm{Hom}_{\mathrm{Pro}(\widehat{\mathscr{M}})}(T,FS)$ is $\eta_{[S,T]} \circ_V [S,f]$.

Under the transferral of the action in the previous paragraph, $([S,f]\circ_H \mathsf{id}_{[S,S]}) \circ_V \rho_T$ maps to

$$([S,f]\circ_H \mathsf{id}_[S,S])_S \circ [S,T]\,\mathrm{coev}_S \circ \mathrm{coev}_{S,T} = [S,f]_{[S,S]S} \circ [S,T]\,\mathrm{coev}_S \circ \mathrm{coev}_{S,T}\,.$$

We wish to show that the diagram

$$\begin{array}{ccccc} T & \xrightarrow{\mathrm{coev}_{S,T}} & [S,T]S & \xrightarrow{[S,T]\,\mathrm{coev}_S} & [S,T][S,S]S \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle [S,f]_S} & & \downarrow{\scriptstyle [S,f]_{[S,S]S}} \\ T' & \xrightarrow[\mathrm{coev}_{S,T'}]{} & [S,T']S & \xrightarrow[{[S,T']\,\mathrm{coev}_S}]{} & [S,T'][S,S]S \end{array}$$

commutes. The right hand square does in fact commute, since $\mathbf{M}([S,f])$ is a natural transformation. For the left hand square, the image of $\mathrm{coev}_{S,T'} \circ f = \eta_{T'} \circ f$ under the adjunction isomorphism is $\sigma_{[S,T']} \circ_V [S, \eta_{T'} \circ f]$. But by the triangle identities for adjunctions, $\sigma_{[S,T']} \circ_V [S, \eta_{T'} \circ -] = [S,-]$, and therefore the image of $\mathrm{coev}_{S,T'} \circ f$ is $[S,f]$, and the reverse isomorphism takes $[S,f]$ to $[S,f]_S \circ \mathrm{coev}_{S,T}$, giving the required isomorphism.

Thus

$$[S,f]_{[S,S]S} \circ [S,T]\,\mathrm{coev}_S \circ \mathrm{coev}_{S,T} = [S,T']\,\mathrm{coev}_S \circ \mathrm{coev}_{S,T'} \circ f.$$

Similarly, $\rho_{T'} \circ_V [S,f]$ maps to $(\rho_{T'})_S \circ [S,f]_S \circ \mathrm{coev}_{S,T}$, which again by the commutativity of the left square above is equal to $(\rho_{T'})_S \circ \mathrm{coev}_{S,T'} \circ f$. We thus wish to show that

$$(\rho_{T'})_S \circ \mathrm{coev}_{S,T'} \circ f = [S,T']\,\mathrm{coev}_S \circ \mathrm{coev}_{S,T'} \circ f.$$

But using the adjunction isomorphism and the definition of $\rho_{T'}$, we have that

$$[S,T']\,\mathrm{coev}_S \circ \mathrm{coev}_{S,T'} \mapsto \rho'_T \mapsto (\rho_{T'})_S \circ \mathrm{coev}_{S,T'},$$

and the result follows.

□

Definition 5.6.11. We define the *image of $\mathscr{M}$ in* $\mathrm{comod}([S,S])$ to be the full subcategory $[S,\mathscr{M}]$ of $\mathrm{comod}([S,S])$ with objects (those objects isomorphic to) $\{([S,T],\rho_T) \in \mathrm{comod}([S,S]) | T \in \mathscr{M}\}$. Similarly, we define $[S,\mathscr{M}(\mathtt{i})]$ to be the full subcategory of $\mathrm{comod}([S,S])$ with objects (those objects isomorphic to) $\{([S,T],\rho_T) \in \mathrm{comod}([S,S]) | T \in \mathscr{M}(\mathtt{i})\}$.

By definition the image of Θ is contained in $[S,\mathscr{M}]$. Further, we can give $[S,\mathscr{M}]$ the structure of a locally wide finitary 2-representation $[S,\mathbf{M}]$ of $\mathscr{C}$ by setting $[S,\mathbf{M}](\mathtt{i}) = [S,\mathscr{M}(\mathtt{i})]$, $([S,\mathbf{M}](F))([S,T]) = [S,\mathbf{M}(F)T]$ and $([S,\mathbf{M}](\alpha))_T = [S,\mathbf{M}(\alpha)_T]$. For the rest of this section, assume that $\mathscr{C}$ is a locally wide weakly fiat 2-category.

Proposition 5.6.12. Θ *defines a morphism of 2-representations of* $\mathscr{C}$ *between* $\mathbf{M}$ *and* **comod**$([S,S])$.

Proof. We mirror the proof of [MMMT16] Lemma 4.4. We will first show that $[S,FT] \cong F[S,T]$ for any $F \in \mathscr{C}(\mathtt{i},\mathtt{j})$. Indeed, for any $G \in \mathrm{Pro}(\widehat{\mathscr{C}})(\mathtt{i},\mathtt{j})$ we have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Pro}(\widehat{\mathscr{C}})}([S,FT],G) &\cong \mathrm{Hom}_{\mathrm{Pro}(\widehat{\mathscr{M}})}(FT,GS)\\ &\cong \mathrm{Hom}_{\mathrm{Pro}(\widehat{\mathscr{M}})}(T,F^*GS)\\ &\cong \mathrm{Hom}_{\mathrm{Pro}(\widehat{\mathscr{C}})}([S,T],F^*G)\\ &\cong \mathrm{Hom}_{\mathrm{Pro}(\widehat{\mathscr{C}})}(F[S,T],G)\end{aligned}$$

from which the claim follows.

The rest of the proof is a direct generalisation of the remainder of the proof of [MMMT16] Lemma 4.4. □

Proposition 5.6.13. *For any $F \in \mathrm{Pro}(\widehat{\mathscr{C}}(\mathtt{i}))$, $[S, FS] \cong F[S,S]$ is a cofree $[S,S]$-coalgebra; that is, for any comodule $X \in \mathrm{comod}([S,S])$,*

$$\mathrm{Hom}_{\mathrm{comod}([S,S])}(X, F[S,S]) \cong \mathrm{Hom}_{\mathrm{Pro}(\widehat{\mathscr{C}})}(X, F).$$

Proof. We construct the relevant adjunction. We again define $\mathrm{Forg} : \mathrm{comod}([S,S]) \to \mathrm{Pro}(\widehat{\mathscr{C}}(\mathtt{i}))$ to be the forgetful functor. Let $\mathrm{ev}_{[S,S]} : \mathrm{Pro}(\widehat{\mathscr{C}}(\mathtt{i})) \to \mathrm{comod}([S,S])$ be defined for a 1-morphism $F \in \mathrm{Pro}(\widehat{\mathscr{C}}(\mathtt{i}))$ as $\mathrm{ev}\,[S,S](F) = (F[S,S], \mathsf{id}_F \circ_H \delta_S)$ and for a 2-morphism $\beta : F \to G$ as $\mathrm{ev}_{[S,S]}(\beta) = \beta \circ_H \mathsf{id}_{[S,S]}$. This is indeed a functor because the diagram

$$\begin{array}{ccc} F[S,S] & \xrightarrow{\mathsf{id}_F \circ_H \delta_S} & F[S,S][S,S] \\ \downarrow{\scriptstyle \beta \circ_H \mathsf{id}_{[S,S]}} & & \downarrow{\scriptstyle \beta \circ_H \mathsf{id}_{[S,S][S,S]}} \\ G[S,S] & \xrightarrow[\mathsf{id}_G \circ_H \delta_S]{} & G[S,S][S,S] \end{array}$$

clearly commutes. We claim that $\mathrm{ev}_{[S,S]}$ is a right adjoint of Forg, from which the result follows immediately.

To see this is indeed an adjunction, we will define the unit and counit. The unit $\eta : \mathrm{Id}_{\mathrm{comod}([S,S])} \to \mathrm{ev}_{[S,S]} \circ \mathrm{Forg}$ is defined by $\eta_{(X,\rho_X)} = \rho_X$, and the counit by $\sigma : \mathrm{Forg} \circ \mathrm{ev}_{[S,S]} \to \mathrm{Id}_{\mathrm{Pro}(\widehat{\mathscr{C}})}$ by $\sigma_F = \mathsf{id}_F \circ_H \epsilon_S$. The left triangle identity is thus expressed for $X \in \mathrm{comod}([S,S])$ by $X \xrightarrow{\mathrm{Forg}(\rho_X)} X[S,S] \xrightarrow{\mathsf{id}_X \circ_H \epsilon_S} X$, which is the composite $(\mathsf{id}_X \circ_H \epsilon_S) \circ \rho_X$, which is id_X by the comodule axioms. The right triangle identity is expressed for $F \in \mathrm{Pro}(\widehat{\mathscr{C}}(\mathtt{i}))$ by

$$F[S,S] \xrightarrow{\mathsf{id}_F \circ_H \delta_S} F[S,S][S,S] \xrightarrow{\mathsf{id}_F \circ_H \epsilon_S \circ_H \mathsf{id}_{[S,S]}} F[S,S],$$

i.e. the composite

$$(\mathsf{id}_F \circ_H (\epsilon_S \circ_H \mathsf{id}_{[S,S]})) \circ_V (\mathsf{id}_F \circ_H \delta_S) = \mathsf{id}_F \circ_H ((\epsilon_S \circ_H \mathsf{id}_{[S,S]}) \circ_V \delta_S),$$

which is equal to $\mathsf{id}_F \circ_H \mathsf{id}_{[S,S]}$ by the coalgebra axioms. We are thus done. □

Theorem 5.6.14. Θ *define an equivalence of 2-representations of* $\mathscr{C}$ *between* $\mathbf{M}$ *and* $[S, \mathbf{M}]$.

Proof. We mirror the proof of [MMMT16] Theorem 4.7. By definition, Θ is essentially surjective when corestricted to $[S, \mathscr{M}]$, and it remains to show that it is fully faithful. To start, consider $FS, GS \in \mathscr{M}$ for $F, G \in \mathscr{C}$. Then we have

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{comod}([S,S])}(\Theta(FS), \Theta(GS)) &\cong \mathrm{Hom}_{\mathrm{comod}([S,S])}(F[S,S], G[S,S]) \\
&\cong \mathrm{Hom}_{\mathrm{Pro}(\widehat{\mathscr{C}})}(F[S,S], G) \\
&\cong \mathrm{Hom}_{\mathrm{Pro}(\widehat{\mathscr{C}})}([S,S], F^*G) \\
&\cong \mathrm{Hom}_{\mathscr{M}}(S, F^*GS) \\
&\cong \mathrm{Hom}_{\mathscr{M}}(FS, GS).
\end{aligned}$$

But now since $\mathbf{M}$ is a transitive 2-representation, for any $T_1, T_2 \in \mathscr{M}$ there exist some 1-morphisms $F, G \in \mathscr{C}$ such that T_1 is a direct summand of FS and T_2 is a direct summand of GS. Thus by pre- and post-composing with injection and projection morphisms, and using that Θ preserves biproducts, we derive that $\mathrm{Hom}_{\mathrm{comod}([S,S])}(\Theta(T_1), \Theta(T_2)) \cong \mathrm{Hom}_{\mathscr{M}}(T_1, T_2)$. Hence Θ is fully faithful, and the result follows. □

5.7 Application: Bound Path Algebras

Let A be a connected non-unital self-injective bound path algebra over $\Bbbk$. We notate by V_A the set of vertices of the underlying quiver of A, and without loss of generality we notate that $V_A \subseteq \mathbb{Z}$. For our purposes, we also assume that every element of V_A has finite total vertex degree. In this case, A has the following property: A has an orthogonal set of primitive idempotents $\{e_i | i \in V_A\}$ such that $A = \bigoplus_{i,j \in V_A} e_i A e_j$, and such that Ae_i and $e_j A$ are finite dimensional over $\Bbbk$ for all $i, j \in V_A$. Without loss of generality the e_i are the paths of length 0.

As an example, consider the algebra A_{zig} stemming from the infinite zigzag quiver;

that is, the quiver with vertices labelled by the integers and arrows $a_i : i \to i+1$ and $b_i : i+1 \to i$ such that $a^2 = b^2 = 0$ and $ab = ba$. In this case, $V_A = \mathbb{Z}$, and A_{zig} is generated by e_i, a_i and b_i subject to the following conditions:

- $\{e_i | i \in \mathbb{Z}\}$ is a set of idempotents fitting the above property;
- $a_i \in e_{i+1} A_{\text{zig}} e_i$; $b_i \in e_i A_{\text{zig}} e_{i+1}$ for all i;
- $a^2 = b^2 = 0$ and $ab = ba$.

We have a pleasant classification of the finite dimensional projective modules of A:

Proposition 5.7.1. *Every indecomposable projective module of $A\text{-mod}$ is of the form Ae_i for some $i \in A$.*

Proof. It is clear that each Ae_i is an indecomposable projective module, and thus it suffices to show that they exhaust the indecomposable projective modules. Let M be some finite dimensional A module. Since it is finite dimensional, it has a composition series $\{0\} = M_0 \subset M_1 \subset \cdots \subset M_n = M$. Let $L_t = M_t / M_{t-1}$ be the tth (simple) composition factor. Let $i_t \in V_A$ be such that $e_{i_t} L_t \neq 0$. Then we have surjections $Ae_{i_t} \twoheadrightarrow L_t$ and $M_t \twoheadrightarrow L_t$. By the universal property of projective modules, this implies there is a homomorphism $Ae_{i_t} \to M_t$ making the resulting diagram commutative.

These morphisms compile to form a surjection $q : \bigoplus_{t=1}^{n} Ae_{i_t} \twoheadrightarrow M$. In particular, if M is bijective, then q is a split epimorphism by the universal property of projective modules (applied to M), and hence M is a direct summand of $\bigoplus_{t=1}^{n} Ae_{i_t}$. The result follows. □

As an example, in $A_{\text{zig}}\text{-mod}$, the indecomposable projective $A_{\text{zig}} e_i$ is four-dimensional as a $\Bbbk$-vector space, with a canonical basis $\{e_i, a_i, b_{i-1}, a_i b_i = b_{i-1} a_{i-1}\}$.

From the prior paragraphs, it follows that the indecomposable projective $(A\text{-}A)$-bimodules are of the form $Ae_i \otimes_{\Bbbk} e_j A$ for $i, j \in V_A$. For compactness of notation, we set $A_{ij} = Ae_i \otimes_{\Bbbk} e_j A$. This allows us to construct a 2-category $\mathscr{C}_A$ as follows:

- $\mathscr{C}_A$ has one object, which we associate with (a small category equivalent to) $A\text{-mod}$.
- 1-morphisms are isomorphic to direct summands of direct sums of the identity and of functors isomorphic to tensoring with $Ae_i \otimes_{\Bbbk} e_jA$ for $i, j \in V_A$. For compact notation, we set $F_{ij} = Ae_i \otimes_{\Bbbk} e_jA \otimes_A - = A_{ij} \otimes_A -$.
- 2-morphisms between F_{ij} and F_{mn} are considered to be bimodule homomorphisms. For 2-morphisms to or from the identity, we consider the identity as tensoring with A, and take bimodule homomorphisms in this case (these are technically bimodule homomorphisms in $(A\text{-}A)\text{-biMod}$, as A is not necessarily a finite dimensional A-module).

We will not be working directly with $\mathscr{C}_A$ because the endomorphism hom-space of the identity is not in general necessarily finite dimensional, and indeed is not necessarily of countable dimension. For example, let Γ be a quiver on $\mathbb{Z}$ where there is one arrow $i \to i$ for each $i \in \mathbb{Z}$ and no other arrows. Define $A = \Bbbk\Gamma/\Bbbk\Gamma_2$. Then the image of each e_i under a bimodule homomorphism $\varphi : A \to A$ is independent of the image of any other e_i, and as $\dim e_iAe_i = 2$, this implies that $\dim \mathrm{End}_{\mathscr{C}_A}(\mathbb{1}_{\mathtt{i}}) \geq 2^{|\mathbb{Z}|}$. While this can be viewed as a somewhat degenerate example, we will also show later that the same inequality holds for $\mathrm{End}_{\mathscr{C}_{A_{\mathrm{zig}}}}(\mathbb{1}_{\mathtt{i}})$, though the reasoning is more complicated.

To fix this, we introduce a generalisation of the $\mathscr{C}_{A,X}$ definition given in Section 3.1:

Definition 5.7.2. Let $\mathscr{C}_A$ be as defined above. Let Z denote the subalgebra of $\mathrm{End}_{\mathscr{C}_A}(\mathbb{1}_*)$ generated by $\mathrm{id}_{\mathbb{1}_*}$ and by any 2-morphism that factors over a non-identity 1-morphism. Let X denote a local subalgebra of $\mathrm{End}_{\mathscr{C}_A}(\mathbb{1}_*)$ containing Z. The 2-category $\mathscr{C}_{A,X}$ is defined to have the same objects, 1-morphisms and 2-morphisms as $\mathscr{C}_A$ with the exception that $\mathrm{End}_{\mathscr{C}_{A,X}}(\mathbb{1}_*) = X$.

Proposition 5.7.3. *$\mathscr{C}_{A,X}$ is a Krull-Schmidt 2-category.*

Proof. $\mathscr{C}_{A,X}$ is an additive idempotent complete 2-category by definition. Since F_{ij} is indecomposable and $\mathrm{End}_{\mathscr{C}_{A,X}}(F_{ij})$ is finite dimensional, $\mathrm{End}_{\mathscr{C}_{A,X}}(F_{ij})$ is local by a standard argument. We chose $\mathrm{End}_{\mathscr{C}_{A,X}}(\mathbb{1}_*)$ to be local, completing the proof. $\square$

We will show that such $\mathscr{C}_{A,X}$ exist for certain classes of path algebras.

Proposition 5.7.4. *Let A be a bound path algebra with underlying quiver Γ_A. Then $\operatorname{rad} A$ is generated as an ideal by the equivalence classes of all paths in Γ_A of length at least 1.*

Proof. Fix a basis $\mathscr{B}$ of A such that $e_i \in \mathscr{B}$ for all $i \in V_A$. The classification of $\operatorname{rad} A$ in the statement is equivalent to saying that $\operatorname{rad} A = \bigcap_{i \in \mathbb{Z}} A \setminus E_i$, where E_i is the set of all elements of A that have a multiple of e_i as a summand. It is thus sufficient to show that the $A \setminus E_i$ are precisely the maximal proper left ideals of A.

If $A \setminus E_i$ is a left ideal of A, it is maximal since if $A \setminus E_i \subset J$ for some ideal J, then there is some $e_i + a \in J$ with e_i not a summand of a. But then by definition $a \in A \setminus E_i$, and thus $a \in J$ and hence $e_i \in J$, and we must have that $J = A$. It remains to show that $A \setminus E_i$ is a left ideal.

Let $a \in A$ and $r \in A \setminus E_i$. $A \setminus E_i$ is clearly a subspace of A. Assume for contradiction that $ar \notin A \setminus E_i$. Then $ar = e_i + b$ for some b. Set $r = \sum_{j,k \in \mathbb{Z}} r_{jk}$ with $r_{jk} \in e_j A e_k$, and $a = \sum_{l,m \in \mathbb{Z}} a_{lm}$ with $a_{lm} \in e_l A e_m$. Then $e_i + b = \sum_{j,k,l,m \in \mathbb{Z}} a_{lm} r_{jk}$. Hence e_i is a summand of some $a_{lm} r_{jk}$. This is only possible if $k = j = l = m = i$. But e_i is not a summand of r_{ii}, and since $A = \Bbbk\Gamma_A / I$ for $I \subseteq (\Bbbk\Gamma_A)_2$, it follows that e_i cannot be a summand of $a_i r_i$, a contradiction. Hence $ar \in A \setminus E_i$ and hence $A \setminus E_i$ is a left ideal of A.

Finally, let I be some maximal proper left ideal of A. If $e_i \in I$ for all i, then for any $a \in A$, $a = a \sum_{j=1}^{n} e_{i_j}$ for some finite collection of the e_i, and hence $I = A$, a contradiction. But then there is at least one i such that $e_i \notin I$, and from the above reasoning it follows that $I \subseteq A \setminus E_i$, and hence $I = A \setminus E_i$ as I is maximal. The result follows. $\square$

Corollary 5.7.5. *Let A be a bound path algebra with underlying quiver Γ_A. Then $\operatorname{rad}^k A$ is generated as an ideal by the equivalence classes of all paths in Γ_A of length k. In particular, there is some integer m such that $\operatorname{rad}^m A = 0$.*

Proof. For the first claim, we proceed by induction. Proposition 5.7.4 provides the base case. Assume that $\operatorname{rad}^{k-1} A$ is generated as an ideal by the equivalence classes of all paths of length $k-1$. It immediately follows that $\operatorname{rad}^k A$ contains the equivalence classes of every path of length k. Conversely, let $a \in \operatorname{rad}^k A$. Then $a = \sum_{i=1}^{m} r_i b_i$ for some m, where $r_i \in \operatorname{rad} A$ and $b_i \in \operatorname{rad}^{k-1} A$ for all i. But then a is a sum of elements of the form $xp_1 y p_{k-1} z$, where $x, y, z \in A$, p_1 is (an equivalence class of) a path of length 1 and p_{k-1} is (an equivalence class of) a path of length $k-1$. In particular, this summand can further be written as a sum of elements of the form $vp_k w$, where p_k is (an equivalence class of) a path of length k. The first result follows.

For the second statement, we note from the definition of A that $A = \Bbbk\Gamma_A/I$ with $(\Bbbk\Gamma_A)_k \subseteq I$ for some finite k. That is, the equivalence class of every path of length at least k is zero, and the statement follows. □

Proposition 5.7.6. *Let A be a bound path algebra and let $Z \subseteq \operatorname{End}_{(A\text{-}A)\text{-bimod}}(A)$ be as defined in Definition 5.7.2. Then Z is a local $\Bbbk$-algebra.*

Proof. If V_A is finite then this is the finitary case which has already been proved in [MM16a] Section 4.5. Therefore assume $V_A = \mathbb{Z}$. We claim that the subspace I generated by all bimodule endomorphisms that factor over some A_{ij} is a maximal proper left ideal of Z. If I is a proper left ideal, it is immediately maximal - if $J \supset I$, then there must be an element of the form $\mathrm{id}_A + v \in J$ for $v \in I$. But then $v \in I \Rightarrow v \in J$, and therefore $\mathrm{id}_A \in J$ and $J = Z$. That I is an ideal is clear - by definition it is closed under addition, and composing an endomorphism that factors over some A_{ij} with another endomorphism still results in an endomorphism that factors over the same A_{ij}, and hence I is closed under composition with elements of Z. It remains to show that I is proper.

Assume for contradiction that id_A is a member of I. Then without loss of generality $\mathrm{id}_A = \sum_{l=1}^{s} \tau_l \sigma_l$ for $\sigma_l : A \to A_{i_l j_l}$ and $\tau_l : A_{i_l j_l} \to A$ for $i_l, j_l \in \mathbb{Z}$ and s finite. We can therefore construct the column morphism $\sigma = (\sigma_l)_{l=1,\ldots,s} : A \to \bigoplus_{l=1}^{s} A_{i_l j_l}$

and the row morphism $\tau = (\tau_l)_{l=1,\dots,s} : \bigoplus_{l=1}^{s} A_{i_l j_l} \to A$ such that $\tau\sigma = \mathrm{id}_A$. In particular, this implies that σ_l is a split monomorphism, and thus in particular a monomorphism. This is a contradiction since A is infinite dimensional and $\bigoplus_{l=1}^{s} A_{i_l j_l}$ is finite dimensional. Therefore I is indeed proper.

We now claim that it is the unique maximal left ideal. For let J be a maximal proper left ideal of Z. If $J \neq I$, then it must contain an element of the form $\mathrm{id}_A + v$ for $v \in I$. But since v is nilpotent (say with nilpotency degree k), $(\sum_{j=0}^{k-1}(-v)^j)(\mathrm{id}_A + v) = \mathrm{id}_A + (-1)^k v^k = \mathrm{id}_A \in J$, a contradiction. Hence I is the unique maximal proper left ideal of Z and we are done. □

Proposition 5.7.7. *Assume that* $A\text{-mod}$ *is a Frobenius category. Then* $\mathscr{C}_{A,X}$ *is a (locally) wide weakly fiat 2-category.*

Proof. If V_A is finite, then the statement has been proved in [MM11] Lemma 45 and [MM16c] Section 4.1. Therefore asssume $V_A = \mathbb{Z}$. Let $*$ denote the unique object in $\mathscr{C}_{A,X}$. It is immediate from the definitions that $\mathscr{C}_{A,X}(*,*)$ is additive and $\Bbbk$-linear. Since the isomorphism classes of F_{ij} are in bijection with $\mathbb{Z}\times\mathbb{Z}$ and there is a single isomorphism class of identity 1-morphisms, there are countably many indecomposable 1-morphisms up to isomorphism. We proved in Proposition 5.7.3 that $\mathscr{C}_{A,X}$ is Krull-Schmidt, and thus $\mathscr{C}_{A,X}$ is a locally wide finitary 2-category.

To show that $\mathscr{C}_{A,X}$ is weakly fiat it is sufficient to show that each F_{ij} is part of an internal adjunction. Since $A\text{-mod}$ is a Frobenius category, by Proposition 5.7.1 it follows that the dual of any Ae_i in $A\text{-mod}$ is isomorphic to $Ae_{\sigma(i)}$ for some permutation σ of $\mathbb{Z}$, which we refer to as the Nakayama bijection. We claim that the right internal adjoint of F_{ij} is $F_{\sigma(j)i}$. To show this, we mirror the proof of

[MM11] Lemma 45, adjusted to our setup. Given some $M \in A\text{-mod}$, we have that

$$\begin{aligned}
\mathrm{Hom}_{A\text{-}\mathrm{mod}}(Ae_i \otimes_{\Bbbk} e_j A, M) &\cong \mathrm{Hom}_{\Bbbk\text{-}\mathrm{mod}}(e_j A, \mathrm{Hom}_{A\text{-}\mathrm{mod}}(Ae_i, M)) \\
&\cong \mathrm{Hom}_{\Bbbk\text{-}\mathrm{mod}}(e_j A, e_i M) \\
&\cong \mathrm{Hom}_{\Bbbk\text{-}\mathrm{mod}}(e_j A, e_i A \otimes_A M) \\
&\cong \mathrm{Hom}_{\Bbbk\text{-}\mathrm{mod}}(e_j A, \Bbbk) \otimes_{\Bbbk} e_i A \otimes_A M \\
&\cong (e_j A)^* \otimes_{\Bbbk} e_i A \otimes_A M.
\end{aligned}$$

But as noted, $(e_j A)^* \cong Ae_{\sigma(j)}$, giving the claim. □

By [Dub17] Remark 4, $A_{\mathrm{zig}}\text{-mod}$ is a Frobenius category. We consider the 2-category $\mathscr{C}_{A_{zig}}$ explicitly by examining the bimodule homomorphisms:

A morphism $\varphi : F_{ij} \to F_{mn}$ must take $e_i \otimes e_j$ to an element of $e_i A_{\mathrm{zig}} e_m \otimes_{\Bbbk} e_n A_{\mathrm{zig}} e_j$. Thus, $\mathrm{Hom}_{\mathscr{C}_{A_{\mathrm{zig}}}}(F_{ij}, F_{mn}) = 0$ whenever $|i - m| \geq 2$ or $|j - n| \geq 2$. If $|i - m| = |j - n| = 1$, then the hom-space is 1-dimensional, if $|i - m| + |j - n| = 1$ it is 2-dimensional and if $|i - m| = |j - n| = 0$ then the hom-space is 4-dimensional. Similarly, consider a bimodule homomorphism $\varphi : A_{\mathrm{zig}} \to A_{\mathrm{zig}} e_i \otimes_{\Bbbk} e_j A_{\mathrm{zig}}$. Given some idempotent e_k, we have that $e_k \varphi(e_k) = \varphi(e_k) = \varphi(e_k) e_k$, and thus $\varphi(e_k) \in e_k A_{\mathrm{zig}} e_i \otimes_{\Bbbk} e_j A_{\mathrm{zig}} e_k$. Thus if $|i - j| > 2$, at least one of $e_k A_{\mathrm{zig}} e_i$ and $e_j A_{\mathrm{zig}} e_k$ is zero, and hence $\mathrm{Hom}_{\mathscr{C}_{A_{\mathrm{zig}}}}(\mathbb{1}_*, A_{\mathrm{zig}} e_i \otimes_{\Bbbk} e_j A_{\mathrm{zig}}) = 0$. If $|i - j| = 2$ the hom-space is 1-dimensional, if $|i - j| = 1$ the hom-space is 4-dimensional and if $i = j$ the hom-space is 6-dimensional.

A bimodule homomorphism $\varphi : A_{\mathrm{zig}} e_i \otimes_{\Bbbk} e_j A_{\mathrm{zig}} \to A_{\mathrm{zig}}$ is determined by its image on $e_i \otimes e_j$, and since $e_i \varphi(e_i \otimes e_j) e_j = \varphi(e_i \otimes e_j)$, $\varphi(e_i \otimes e_j) \in e_i A_{\mathrm{zig}} e_j$. Thus if $|i - j| \geq 2$, $\mathrm{Hom}_{\mathscr{C}_{A_{\mathrm{zig}}}}(F_{ij}, \mathbb{1}_{\mathtt{i}}) = 0$. If $|i - j| = 1$ the hom-space is 1-dimensional, and if $i = j$ the hom-space is 2-dimensional.

Finally, we examine $\mathrm{End}_{(A_{\mathrm{zig}}\text{-}A_{\mathrm{zig}})\text{-}\mathrm{biMod}}(A_{\mathrm{zig}})$ and Z_{zig}. If we have a bimodule endomorphism φ such that $\varphi(e_i) = e_i$ for some i, then $\varphi(a_i) = a_i \varphi(e_i) = a_i$, but $\varphi(a_i) = \varphi(e_{i+1}) a_i$ which implies that $\varphi(e_{i+1}) = e_{i+1}$. A similar argument applies for b_i and e_{i-1}, and therefore by bidirectional induction we derive that $\varphi(e_i) = e_i$ for all $i \in \mathbb{Z}$, and hence $\varphi = \mathrm{id}_A$.

If $\varphi(e_i) = b_i a_i$, then $\varphi(a_i) = 0$ and thus $\varphi(e_{i+1})a_i = 0$, and thus $\varphi(e_{i+1})$ is either (a scalar multiple of) $b_{i+1}a_{i+1}$ or 0. However, this choice can be made freely, and it follows that a basis of $\mathrm{End}_{(A_{\mathrm{zig}}\text{-}A_{\mathrm{zig}})\text{-}\mathrm{biMod}}(A_{\mathrm{zig}})$ consists of the identity and of homomorphisms of the form φ_I, where $I \subseteq \mathbb{Z}$ and $\varphi_I(e_i) = b_i a_i$ if $i \in I$, and 0 otherwise. The set of these φ_I is thus in bijection with the powerset of $\mathbb{Z}$, and hence $\mathrm{End}_{\mathscr{C}_{A_{\mathrm{zig}}}}(\mathbb{1}_{\mathtt{i}})$ has uncountable dimension.

Regarding Z_{zig}, we can write $\varphi_{\{i\}}$ as $\sigma_i \tau_i$, where $\tau_i : A_{\mathrm{zig}} \to A_{\mathrm{zig}}e_{i+1} \otimes_{\Bbbk} e_{i+1}A_{\mathrm{zig}}$ is given by $\tau_i(e_j) = \delta_{ij} b_i \otimes a_i$, and $\sigma_i : A_{\mathrm{zig}}e_{i+1} \otimes_{\Bbbk} e_{i+1}A_{\mathrm{zig}} \to A_{\mathrm{zig}}$ given by $\sigma_i(e_{i+1} \otimes e_{i+1}) = e_{i+1}$. Consequently, Z_{zig} is generated by id_A and by those φ_I where I is a finite subset of $\mathbb{Z}$. By Proposition 5.7.6, Z is a local algebra and therefore $\mathscr{C}_{A_{\mathrm{zig}},Z}$ is a locally wide weakly fiat 2-category.

In general,

$$A_{ij} \otimes_A A_{kl} \cong Ae_i \otimes_{\Bbbk} e_j A e_k \otimes_{\Bbbk} e_l A \cong (A_{il})^{\oplus \dim e_j A e_k}.$$

It follows that $F_{ij} \circ F_{kl} \cong F_{il}^{\oplus \dim e_j A e_k}$. In particular, the $\mathscr{L}$-cells of $\mathscr{C}_A$ are of the form $\mathscr{L}_j = \{F_{ij} | i \in \mathbb{Z}\}$ (and $\mathscr{L}_* = \{\mathbb{1}_*\}$) while the $\mathscr{R}$-cells are of the form $\mathscr{R}_i = \{F_{ij} | j \in \mathbb{Z}\}$ (and $\mathscr{R}_* = \{\mathbb{1}_*\}$). It thus follows that $F_{ij} \sim_{\mathscr{L}\circ\mathscr{R}} F_{mn}$ for any $i, j, m, n \in \mathbb{Z}$, and hence there are two $\mathscr{D}$-cells: $\mathscr{D}_* = \{\mathbb{1}_*\}$ and $\mathscr{D}_{\mathbb{Z}} = \{F_{ij} | i, j \in \mathbb{Z}\}$. But since it is clear that $\mathbb{1}_* >_{\mathscr{J}} F_{ij}$ for any $i, j \in \mathbb{Z}$, it follows that, on $\mathscr{C}_{A,X}$, the $\mathscr{J}$ partial order agrees with the $\mathscr{D}$ partial order, and the $\mathscr{J}$-cells are precisely the $\mathscr{D}$-cells. Further, since $\mathscr{L}_j \cap \mathscr{R}_i = \{F_{ij}\}$, both $\mathscr{J}$-cells are strongly regular.

This allows us to construct the cell 2-representations corresponding to $\mathscr{C}_{A,X}$. Choose some $j \in \mathbb{Z}$ and consider the $\mathscr{L}$-cell $\mathscr{L}_j$. Then

$$\mathrm{add}\{FX | F \in \mathscr{C}, X \in \mathscr{L}_j\} = \mathrm{add}\{F_{mn}F_{ij} | m, n, i \in \mathbb{Z}\} = \mathrm{add}\,\mathscr{L}_j.$$

We denote this 2-representation of $\mathscr{C}_A$ by $\mathbf{N}_j$. To recall some notation, we define the bimodule homomorphism $\varphi_{a,b} : A_{ij} \to A_{kl}$ by $\varphi_{a,b}(e_i \otimes e_j) = a \otimes b$. These $\varphi_{a,b}$ span $\mathrm{Hom}_{(A\text{-}A)\text{-}\mathrm{bimod}}(A_{ij}, A_{kl})$. We have the following useful result:

Lemma 5.7.8. *The maximal ideal $\mathcal{I}$ of $\mathbf{N}_j$ that is $\mathscr{C}_{A,X}$-stable and does not contain any identity morphism for non-zero objects is generated as a collection of $\Bbbk$-vector spaces by the set $\{\varphi_{a,b} | b \in \mathrm{rad}\, e_j A e_j\}$.*

Proof. The proof of this result generalises mutatis mutandis from the proof given for Proposition 3.2.8. □

In the case of $\mathscr{C}_{A_{\mathrm{zig}}}$, $\mathrm{rad}\, e_j A e_j$ is a 1-dimensional vector space spanned by $a_j b_j$.

We denote the cell 2-representation corresponding to $\mathscr{L}_j$ by $\mathbf{C}_j$. In this case, we will show that we can give a stronger result that Proposition 5.6.8. Specifically, we will show that for an object $S \in \mathbf{C}_j(\mathtt{i})$, $[S,S]$ is a coalgebra 1-morphism in $\mathscr{C}_{A,X}$ (or more precisely, its image under the forgetful functor $\mathrm{Forg}_S : \mathrm{comod}([S,S]) \to \mathrm{Pro}(\widehat{\mathscr{C}_{A,X}})$ lives in the image of $\mathscr{C}_{A,X}$ under the canonical injection 2-functor). Let $\mathcal{C}_{A,X} = \mathscr{C}_{A,X}(*,*)$.

Proposition 5.7.9. *The functor $\mathrm{ev}_j : \mathcal{C}_{A,X} \to \mathbf{C}_j(*)$ given by $\mathrm{ev}_j(F) = FF_{jj}$ and $\mathrm{ev}_j(\alpha) = \alpha \circ_H id_{F_{jj}}$ is right adjoint to the functor $\mathrm{Forg} : \mathbf{C}_j(*) \to \mathcal{C}_{A,X}$ given by $\mathrm{Forg}(F) = F$ and $\mathrm{Forg}(\alpha) = \alpha$.*

Proof. We will prove the adjunction by constructing the unit and counit adjunctions. By injection-projection arguments, it is sufficient to consider the components of the counit and the unit on indecomposable 1-morphisms/indecomposable objects. By a similar argument to the working in the proof of Proposition 3.2.8, $\mathrm{Hom}_{\mathscr{C}_{A,X}}(F_{ij}, \mathbb{1}_*)$ consists of homomorphisms of the form $\varphi_a : A_{ij} \to A$ where $\varphi_a(e_i \otimes e_j) = a \in e_i A e_j$. On the other hand, $\mathrm{Hom}_{\mathbf{C}_j}(F_{ij}, F_{jj})$ consists of homomorphisms of the form $\varphi_{a,e_j} : A_{ij} \to A_{jj}$ where $\varphi_{a,e_j}(e_i \otimes e_j) = a \otimes e_j$ for $a \in e_i A e_j$. Similarly, the morphism space $\mathrm{Hom}_{\mathbf{C}_j}(F_{ij}, F_{kl}F_{jj})$ consists of linear combinations of morphisms of the form φ_{a,b,e_j} where $\varphi_{a,b,e_j}(e_i \otimes e_j) = a \otimes b \otimes e_j$ for $a \in e_i A e_k$ and $b \in e_k A e_j$. In addition, again by a similar argument to the working in the proof of Proposition 3.2.8, the morphism space $\mathrm{Hom}_{\mathscr{C}_{A,X}}(F_{ij}, F_{kl})$ consists of linear combinations of morphisms of the form $\varphi_{a,b} : A_{ij} \to A_{kl}$ where $\varphi_{a,b}(e_i \otimes e_j) = a \otimes b$ with $a \in e_i A e_k$ and $b \in e_l A e_j$.

We define the unit morphisms $\eta_{F_{ij}} \in \mathrm{Hom}_{\mathbf{C}_j(*)}(F_{ij}, F_{ij}F_{jj})$ as $\eta_{kl} = \varphi_{e_i,e_j,e_j}$. For the counit ϵ, we first define $\epsilon_{\mathbb{1}_*} \in \mathrm{Hom}_{\mathscr{C}_{A,X}}(F_{jj}, \mathbb{1}_*)$ as $\epsilon_{\mathbb{1}_*} = \varphi_{e_j}$. We then define $\epsilon_{F_{kl}} \in \mathrm{Hom}_{\mathscr{C}_{A,X}}(F_{kl}F_{jj}, F_{kl})$ as $\epsilon_{F_{kl}} = \mathrm{id}_{F_{kl}} \circ_H \epsilon_{\mathbb{1}_*}$. It is straightforward to show that these satisfy the unit/counit axioms, and the results follows. □

Corollary 5.7.10. *Let $\mathbf{C}_j$ be a cell 2-representation of $\mathscr{C}_{A,X}$. Then there exists some object $S \in \coprod_{\mathtt{i} \in \mathscr{C}_{A,X}} (\mathbf{C}_j(\mathtt{i}))$ such that the restriction of Forg_S to $[S, \coprod_{\mathtt{i} \in \mathscr{C}} (\mathbf{C}_j(\mathtt{i}))]$ factors over $\coprod_{\mathtt{j} \in \mathscr{C}_{A,X}} \mathscr{C}_{A,X}(\mathtt{i}, \mathtt{j})$.*

Proof. By Theorem 5.6.14 for any $S \in \mathscr{C}_j$ there is an equivalence of 2-representations between $\mathbf{C}_j$ and $[S, \mathbf{M}]$, where $[S, -]$ is the right adjoint of $\mathrm{Pro}(\widehat{\mathrm{ev}_S})$ as defined previously. By Proposition 5.6.2, $\mathrm{Pro}(\widehat{\mathrm{ev}_S})$ is evaluation at S. If we set $S = F_{jj}$, then it follows by Proposition 5.7.9 that $[S, \mathscr{C}_j] = \mathrm{Forg}(\mathscr{C}_j) \subseteq \mathscr{C}_{A,X}$ as required. □

We can say more. To begin, we present the generalisation of [MMMT16] Corollary 4.10 to the locally wide fiat case:

Proposition 5.7.11. *Let $\mathscr{C}$ be a locally wide weakly fiat 2-category, and let $\mathtt{i} \in \mathscr{C}$. Denoting by $\mathscr{C}_\mathtt{i}$ the endomorphism 2-category of $\mathtt{i}$ in $\mathscr{C}$, there is a bijection between equivalence classes of simple transitive 2-representations of $\mathscr{C}_\mathtt{i}$ and equivalence classes of simple transitive 2-representations of $\mathscr{C}$ that have a non-trivial value at $\mathtt{i}$.*

Proof. The (injective version of the) proof given in [MMMT16] assumes only the existence of the equivalence of 2-representations between $\mathbf{M}$ and $[S, \mathbf{M}]$, given here in Theorem 5.6.14. Beyond that we only need that for $S \in \mathbf{M}(\mathtt{i})$, $[S, S]$ lives in $\mathrm{Pro}(\widehat{\mathscr{C}_{A,X}(\mathtt{i}, \mathtt{i})})$, which remains true by definition. □

We can now generalise Lemma 3.3.8 to our setup:

Theorem 5.7.12. *Any simple transitive 2-representation of $\mathscr{C}_{A,X}$ is equivalent to a cell 2-representation.*

Proof. We again consider the larger 2-category $\mathscr{C}_{A\times\Bbbk,X\times\Bbbk}$, defined mutatis mutandis as in the proof of Lemma 3.3.8. The endomorphism 2-category $\mathscr{C}_{\Bbbk}$ of $*_{\Bbbk}$ is identical to that found in the referenced proof, and thus in particular all simple transitive 2-representations of it are equivalent to the cell 2-representation on it. Finally, any 1-morphism $Ae_i \otimes_{\Bbbk} e_j A$ of $\mathscr{C}_{A,X}$ still factors over $*_{\Bbbk}$, and hence the rest of the proof of Lemma 3.3.8 generalises without issue. □

Corollary 5.7.13. *Let $\mathbf{M}$ be a simple transitive 2-representation of $\mathscr{C}_{A,X}$. Then there exists some object $S \in \mathscr{M}$ such that the restriction of Forg_S to $[S, \mathscr{M}]$ factors over $\mathscr{C}$.*

Proof. This is a direct consequence of combining Corollary 5.7.10 and Theorem 5.7.12

□

5.8 Application: Soergel Bimodules

We give a second application of this theory by demonstrating that the 2-category associated to a collection of Soergel bimodules is in fact a (locally) wide finitary 2-category. For the purposes of this section, assume that $\Bbbk$ is an algebraically closed field of characteristic 0.

5.8.1 Soergel Bimodules: the Definitions

We begin by defining Soergel bimodules, which were originally defined in [Soe92], though we draw our definitions here more from the summary paper [Wil11], with some alterations for our specific case.

Definition 5.8.1. Given $\mathbb{Z}$-graded algebras A and B, we denote by $(A\text{-}B)\text{-}_{\mathbb{Z}}\mathrm{biMod}_{\mathbb{Z},0}$ the category whose objects are graded $(A\text{-}B)$-bimodules and whose morphisms are homogeneous graded bimodule homomorphisms of degree zero.

Definition 5.8.2. A *(finite) Coxeter matrix* M is a symmetric square matrix with entries in $\mathbb{Z}^+ \cup \{\infty\}$ where the diagonal entries are 1 and the non-diagonal entries are at least 2.

Definition 5.8.3. A *Coxeter system* is a pair (W, S) where W is a group and S is a finite subset such there is a presentation of W of the form

$$\langle s \in S | (sr)^{m_{sr}} = e \text{ whenever } m_{ij} \text{ is finite.} \rangle$$

where $M = (m_{sr})_{s,r \in S}$ is a Coxeter matrix. The elements of S are called *simple reflections* and any element $w \in W$ that is conjugate to some $s \in S$ is called a *reflection*.

There does exist theory for the generalisation where S may not be of finite cardinality which could be of interest in the 2-representation setting, but that is outside the scope of this thesis.

Definition 5.8.4. Let (W, S) be a Coxeter system and let $x \in W$. An *expression* of x is a tuple $(s_1, s_2, \ldots, s_n) \in S^n$ for some finite n such that $x = s_1 s_2 \ldots s_n$. This expression is *reduced* if n is minimal, and in this case we call it the *length* of x, denoted $l(x)$.

Definition 5.8.5. Given a subset $I \subset S$, we denote by W_I the subgroup of W generated by I. This is generally called the *parabolic subgroup* associated to I. By construction, (W_I, I) is a Coxeter system. If W_I is a finite subgroup, we say that I is *subgroup-finite*. If I is subgroup-finite, we denote by w_I the unique longest element of W_I.

Note that [Wil11] calls subgroup-finite subsets 'finitary' subsets, but this presents obvious confusion issues for this thesis, leading to the alternate nomenclature.

Definition 5.8.6. For subsets $I, J \subseteq S$ let $W_I \setminus W / W_J$ denote the set of double cosets of W with respect to W_I and W_J. For a double coset $p \in W_I \setminus W/W_J$, we let p_- denote the unique element of minimal length in p.

Definition 5.8.7. Let V be a ($\Bbbk$-)representation of W. For a subset $X \subset W$, we let V^X denote the subspace of V invariant under every element of X. We generally notate $V^{\{w\}} =: V^w$ for simplicity.

Definition 5.8.8. Let V be a finite dimensional representation of W. We say V is a *reflection faithful* representation of W if:

- The representation is faithful.
- $\operatorname{codim} V^w = 1$ if and only if w is a reflection.

By [Soe07] Proposition 2.1, any Coxeter system has a reflection faithful representation. Let $R = S(V^*)$ be the symmetric algebra on V^*, graded such that V^* is in degree 2. There is a natural action of W on R given by $w(f(v)) = f(w^{-1}v)$ for any $w \in W$, $f \in R$ and $v \in V$. Thus given any subset $I \subset S$ or element $w \in W$ we can define R^w and R^I to be the subalgebras of R invariant under w and W_I respectively. We note that, since $\Bbbk$ is of characteristic 0, R is a graded-free R^I-module for any $I \subseteq S$. We recall the following result from [Wil11]:

Proposition 5.8.9 ([Wil11] Lemma 4.1.3)**.** *For $I \subseteq J$ subgroup-finite subsets of S, R^I is a finitely generated graded free R^J-module. In addition,*

$$\operatorname{Hom}_{R^J\text{-}_{\mathbb{Z}}\mathrm{Mod}}(R^I[\![l(w_I) - l(w_J)]\!], R^J) \cong R^I[\![l(w_I) - l(w_J)]\!].$$

Definition 5.8.10. Let $I, J \subseteq S$ and let $p \in W_I \setminus W/W_J$. Set $K = I \cap p_- J {p_-}^{-1}$. The *standard module indexed by* (I, p, J), which we notate ${}^I R_p^J$, is an object of $(R^I\text{-}R^J)\text{-}_{\mathbb{Z}}\mathrm{biMod}_{\mathbb{Z},0}$ that as a ring is equal to R^W. The bimodule actions are given by:

- $r.m = rm$ for $r \in R^I$ and $m \in {}^I R_p^J$;

- $mr = m(p\_r)$ for $r \in r^J$ and $m \in {}^I R^J_p$.

Of primary interest for us is the case where the double coset p contains the identity e. In this case, we drop the p from the notation and simply write ${}^I R^J$.

Definition 5.8.11. Let $I, J, K \subseteq S$ be subgroup-finite sets. We define two functors based on J and K, both (following [Wil11]) notated as ${}^J\vartheta^K$ based on the following conditions:

- If $J \subset K$ then we have

$$ {}^J\vartheta^K : (R^I\text{-}R^J)\text{-}\,_{\mathbb{Z}}\mathrm{biMod}_{\mathbb{Z},0} \to (R^I\text{-}R^K)\text{-}\,_{\mathbb{Z}}\mathrm{biMod}_{\mathbb{Z},0} $$

  given on objects by ${}^J\vartheta^K(M) = M_{R^K}[\![l(w_J) - l(w_K)]\!]$ with the obvious action on morphisms.

- If $K \subset J$ then we have

$$ {}^J\vartheta^K : (R^I\text{-}R^J)\text{-}\,_{\mathbb{Z}}\mathrm{biMod}_{\mathbb{Z},0} \to (R^I\text{-}R^K)\text{-}\,_{\mathbb{Z}}\mathrm{biMod}_{\mathbb{Z},0} $$

  given on objects by ${}^J\vartheta^K(M) = M \otimes_{R^J} R^K$ and on morphisms by ${}^J\vartheta^K(f) = f \otimes \mathrm{id}_{R^K}$.

- If $K = J$, we set ${}^J\vartheta^K$ to be the identity functor on $(R^I\text{-}R^J)\text{-}\,_{\mathbb{Z}}\mathrm{biMod}_{\mathbb{Z},0}$.

Since the definition of ${}^J\vartheta^K$ has mutually exclusive components, there is never any ambiguity in its use, and this multi-part definition allows for more compact definitions later.

This allows us to define the bicategory of singular Soergel bimodules:

Definition 5.8.12. The *bicategory of singular Soergel bimodules* $\mathscr{B}_b = \mathscr{B}_{(W,S),b}$ has objects enumerated by the subgroup-finite subsets I of S. We abuse notation by also referring to the object associated to $I \subseteq S$ as I. The hom-categories $\mathscr{B}(I, J)$ are the smallest additive subcategories of $(R^J\text{-}R^I)\text{-}\,_{\mathbb{Z}}\mathrm{biMod}_{\mathbb{Z},0}$ subject to the following:

1. $\mathscr{B}_b(I,J)$ is closed in $(R^J\text{-}R^I)\text{-}_{\mathbb{Z}}\mathrm{biMod}_{\mathbb{Z},0}$ under taking isomorphisms, direct summands and $\mathbb{Z}$-grading shifts.

2. ${}^I R^I$ is a 1-morphism in $\mathscr{B}_b(I,I)$ for all objects $I \in \mathscr{B}_b$.

3. If $B \in \mathscr{B}_b(I,J)$, then ${}^J\vartheta^K(B) \in \mathscr{B}_b(I,K)$ whenever this is defined (i.e. when K is subgroup-finite with either $K \subseteq J$ or $J \subseteq K$).

We take composition of 1-morphisms and horizontal composition of 2-morphisms to be tensor products over then common algebra (i.e. for $B \in \mathscr{B}(I,J)$ and $C \in \mathscr{B}(J,K)$, $C \circ B = B \otimes_{R^J} C \in \mathscr{B}(I,K)$). We note that $\mathbb{1}_I = R^I$.

As noted in Section 1 of [Wil11], this is equivalent to setting $\mathscr{B}_b(I,J)$ as the smallest full additive sub-category of $(R^I\text{-}R^J)\text{-}_{\mathbb{Z}}\mathrm{biMod}_{\mathbb{Z},0}$ containing all objects isomorphic to direct summands of shifts of objects of the form

$$R^{I_1} \otimes_{R^{J_1}} R^{I_2} \otimes_{R^{J_2}} \cdots \otimes_{R^{J_{n-1}}} R^{I_n}$$

with $I = I_1 \subset J_1 \supset I_2 \subset \cdots \subset J_{n-1} \supset I_n = J$ subgroup-finite subsets.

Definition 5.8.13. The *2-category of singular Soergel bimodules* $\mathscr{B} = \mathscr{B}_{(WS)}$ is defined as a 2-category biequivalent to $\mathscr{B}_b$.

The endomorphism sub-2-category $\mathscr{B}_\varnothing$ of the object $\varnothing$ is referred to as the 2-category of *Soergel bimodules*.

5.8.2 Soergel Bimodules: the Structure

We now demonstrate that the 2-category of singular Soergel bimodules is in fact a locally wide fiat 2-category. We first give a special case of [EW16] Lemma 6.24, adapted to the language used in this thesis:

Lemma 5.8.14. *The hom-category $\mathscr{B}(\varnothing,\varnothing)$ is a Krull-Schmidt category.*

The proof of [EW16] Lemma 6.24 adapts without issue to any other hom-category in $\mathscr{B}$, giving the following lemma:

Lemma 5.8.15. *For any objects I and J of $\mathscr{B}$, $\mathscr{B}(I, J)$ is a Krull-Schmidt category.*

Lemma 5.8.16. *The 2-category $\mathscr{B}$ is a locally wide finitary 2-category.*

Proof. Since we took the Coxeter system (W, S) such that S is a finite set, $\mathscr{B}$ has finitely many objects and thus certainly at most countably many objects. Since for any subgroup-finite I and J $\mathscr{B}_b(I, J)$ is a sub-2-category of $(R^I\text{-}R^J)\text{-}\,_{\mathbb{Z}}\mathrm{biMod}_{\mathbb{Z},0}$ it is certainly additive and $\Bbbk$-linear with countable dimension hom-spaces of 2-morphisms, and hence so is $\mathscr{B}(I, J)$.

There are only countably many $(R^I\text{-}R^J)$-bimodules of the form

$$R^{I_1} \otimes_{R^{J_1}} R^{I_2} \otimes_{R^{J_2}} \cdots \otimes_{R^{J_{n-1}}} R^{I_n}$$

for $I = I_1 \subset J_1 \supset I_2 \subset \cdots \subset J_{n-1} \supset I_n = J$ subgroup-finite subsets, and hence there are only countably many grade-shifts of these. Since each of these has only finitely many indecomposable direct summands, it follows that $\mathscr{B}_b(I, J)$ (and hence $\mathscr{B}(I, J)$) has countably many isomorphism classes of indecomposable 1-morphisms. By construction the identity 1-morphisms are indecomposable, and finally by Lemma 5.8.15 the hom-categories are Krull-Schmidt. This completes the proof. □

In fact, we can say more:

Lemma 5.8.17. *The 2-category $\mathscr{B}$ is a locally wide fiat 2-category.*

Proof. It is a consequence of [Soe07] Proposition 5.10 that $R \otimes_{R^s} R \otimes_R - \in \mathscr{B}(\varnothing, \varnothing)$ is self-adjoint for any $s \in S$, and it thus follows that the endomorphism 2-category of $\varnothing$ is locally wide fiat. That the auto-involution extends to the whole of $\mathscr{B}$ comes from using a straightforward generalisation of [Soe07] Proposition 5.10 to the singular Soergel bimodule setup. □

This is a much wider array of Soergel bimodule 2-categories that can be studied than under simply finitary 2-representation theory. To give a clearer view of this, we will relate the Soergel bimodules to the Coxeter-Dynkin diagrams associated to the Coxeter groups. We will not discuss Coxeter-Dynkin diagrams in detail (see [GB85] Chapter 5 for a detailed study), but briefly they are graphs whose edges are labelled with positive integers which fully classify Coxeter groups.

Original finitary 2-representation theory can only cover finite Coxeter groups. These are classified by three infinite families of Coxeter-Dynkin diagrams (called A_n, B_n and D_n), as well as a small finite set of exceptional diagrams. The wide finitary theory detailed herein instead applies to any Coxeter-Dynkin diagram with finitely many nodes, including not only the affine Coxeter-Dynkin diagrams, but also a wide array of wild Coxeter-Dynkin diagrams.

Theorem 5.8.18. *Let $\mathscr{B} = \mathscr{B}_{(W,S)}$ be the 2-category of singular Soergel bimodules associated to a Coxeter system (W, S) with S a finite set. Let $\mathbf{M}$ be a transitive 2-representation of $\mathscr{B}$. Then there exists a coalgebra 1-morphism C in $\mathrm{Pro}(\widehat{\mathscr{B}})$ such that $\mathbf{M}$ is equivalent as a 2-representation to a subcategory of the category of comodule 1-morphisms over C.*

Proof. Lemma 5.8.17 gives that $\mathscr{B}$ is a locally wide fiat 2-category, and we can thus apply Theorem 5.6.14 to get the result immediately. □

Bibliography

[Ade73] Murray Adelman. Abelian categories over additive ones. *Journal of Pure and Applied Algebra*, 3(2):103–117, 1973.

[ASS06] Ibrahim Assem, Andrzej Skowronski, and Daniel Simson. *Elements of the Representation Theory of Associative Algebras: Volume 1: Techniques of Representation Theory*. Cambridge University Press, 2006.

[BD17] Jonathan Brundan and Nicholas Davidson. Categorical actions and crystals. *Categorification and Higher Representation Theory*, 684:105, 2017.

[Bel00] Apostolos Beligiannis. On the Freyd categories of an additive category. *Homology, Homotopy and Applications*, 2(1):147–185, 2000.

[Cli54] Alfred H Clifford. Bands of semigroups. *Proceedings of the American Mathematical Society*, 5(3):499–504, 1954.

[CM19a] Aaron Chan and Volodymyr Mazorchuk. Diagrams and discrete extensions for finitary 2-representations. *Mathematical Proceedings of the Cambridge Philosophical Society*, 166(2):325–352, 2019.

[CM19b] Aaron Chan and Vanessa Miemietz. Coidempotent subcoalgebras and short exact sequences of finitary 2-representations, 2019. arXiv:1901.04685.

[DD14] M. Emilia Descotte and Eduardo J. Dubuc. A theory of 2-pro-objects (with expanded proofs), 2014. arXiv:1406.5762.

[DG17] Stephen Doty and Anthony Giaquinto. Cellular bases of generalised q-Schur algebras. *Mathematical Proceedings of the Cambridge Philosophical Society*, 162(3):533–560, 2017.

[Dub17] Brendan Dubsky. Koszulity of some path categories. *Communications in Algebra*, 45(9):4084–4092, 2017.

[EGNO16] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. *Tensor categories*. American Mathematical Soc., 2016.

[EW16] Ben Elias and Geordie Williamson. Soergel calculus. *Representation Theory of the American Mathematical Society*, 20(12):295–374, 2016.

[Fre66] Peter Freyd. Representations in abelian categories. In *Proceedings of the Conference on Categorical Algebra*, pages 95–120. Springer, 1966.

[GB85] LC Grove and CT Benson. *Finite Reflection Groups*. Springer, 1985.

[Gre51] James A Green. On the structure of semigroups. *Annals of Mathematics*, pages 163–172, 1951.

[GV72] A Grothendieck and JL Verdier. Expose 1: Prefaisceaux. In *Théorie des topos et cohomologie étale des schémas*. Springer, 1972.

[HK02] Jin Hong and Seok-Jin Kang. *Introduction to quantum groups and crystal bases*, volume 42. American Mathematical Soc., 2002.

[KK12] Seok-Jin Kang and Masaki Kashiwara. Categorification of highest weight modules via Khovanov-Lauda-Rouquier algebras. *Inventiones mathematicae*, 190(3):699–742, 2012.

[KM12] Ganna Kudryavtseva and Volodymyr Mazorchuk. On multisemigroups. *Portugaliae Mathematica*, 72(1):47–80, 2012.

[Lam99] Tsit Yuen Lam. *Lectures on modules and rings*, volume 189. Springer-Verlag, New York, 1999.

[Lei98] Tom Leinster. Basic bicategories. arXiv:math/9810017, 1998.

[Lus10] George Lusztig. *Introduction to quantum groups*. Springer Science & Business Media, 2010.

[MM11] Volodymyr Mazorchuk and Vanessa Miemietz. Cell 2-representations of finitary 2-categories. *Compositio Mathematica*, 147(05):1519–1545, 2011.

[MM14] Volodymyr Mazorchuk and Vanessa Miemietz. Additive versus abelian 2-representations of fiat 2-categories. *Moscow Mathematical Journal*, 14(3):595–615, 2014.

[MM16a] Volodymyr Mazorchuk and Vanessa Miemietz. Endomorphisms of cell 2-representations. *International Mathematics Research Notices*, 2016(24):7471–7498, 2016.

[MM16b] Volodymyr Mazorchuk and Vanessa Miemietz. Isotypic faithful 2-representations of $\mathcal{J}$-simple fiat 2-categories. *Mathematische Zeitschrift*, 282(1-2):411–434, 2016.

[MM16c] Volodymyr Mazorchuk and Vanessa Miemietz. Transitive 2-representations of finitary 2-categories. *Transactions of the American Mathematical Society*, 368(11):7623–7644, 2016.

[MMM$^+$19] Marco Mackaay, Volodymyr Mazorchuk, Vanessa Miemietz, Daniel Tubbenhauer, and Xiaoting Zhang. 2-representations of Soergel bimodules, 2019. arXiv:1906.11468.

[MMMT16] M Mackaay, V Mazorchuk, V Miemietz, and D Tubbenhauer. Simple transitive 2-representations via (co) algebra 1-morphisms. *Indiana University Mathematics Journal*, 68(1):1–33, 2016.

[Rou08] Raphaël Rouquier. 2-Kac-Moody algebras, 2008. arXiv:0812.5023.

[Soe92] Wolfgang Soergel. The combinatorics of Harish-Chandra bimodules. *Journal für die reine und angewandte Mathematik*, 1992(429):49–74, 1992.

[Soe07] Wolfgang Soergel. Kazhdan-Lusztig-polynome und unzerlegbare bimoduln über polynomringen. *Journal of the Institute of Mathematics of Jussieu*, 6(3):501–525, 2007.

[Swa06] Richard G Swan. *Algebraic K-theory*, volume 76. Springer, 2006.

[VV11] Michela Varagnolo and Eric Vasserot. Canonical bases and KLR-algebras. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2011(659):67–100, 2011.

[Web17] Ben Webster. *Knot invariants and higher representation theory*. Number 1191. American Mathematical Society, 2017.

[Wil11] Geordie Williamson. Singular Soergel bimodules. *International Mathematics Research Notices*, 2011(20):4555–4632, 2011.