Aggregation of opinions in networks of individuals and collectives*

Hervé Crès† **Mich Tvede**‡

Abstract

We study the formation of opinions in a bipartite network of firms' boards and directors theoretically. A director and a board are connected provided the director is a board member. Opinions are sets of beliefs about the likelihood of different states of the world tomorrow. Our basic assumption is that boards as well as directors aggregate opinions of each other: a production plan is better than another for a board (director) provided every director (board of which she is a member) finds it better. Opinions are stable provided aggregation does not result in revision of opinions. We show that for connected networks: opinions are stable if and only if they are unambiguous and identical; and repeated aggregation leads to stable opinions. Hence, there will eventually be a single society-wide intersubjective "truth".

Keywords Boards · Directors · Networks · Pareto principle

JEL Classification D2 · D5 · D6 · D8

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†New York University in Abu Dhabi, PO Box 129188 Abu Dhabi, United Arab Emirates, herve.cres@nyu.edu.

‡University of East Anglia, Norwich Research Park, Norwich, NR4 7TJ, United Kingdom, m.tvede@uea.ac.uk

1 Introduction

In the present paper we develop a formal theory of formation of opinions in networks. Our theory uses connectedness of networks to explain consensus in these networks.

Motivation: We have looked at all 800 decisions taken in 2014 in general assembly meetings of the 40 largest French corporations (CAC 40) leaving out six decisions that were put on vote without the approval of the boards and obtained very little support (2-16%). On average every decision was supported by 94.9% of the voters. For a *consensus index* defined as the highest x for which $x\%$ of the decisions were supported by at least $x\%$ of the votes the number is 87.6%. Decisions on collective issues have an index of 89.4%, decisions on individual compensation an index of 78.8% and decisions on appointments an index of 87.5%. On average general assembly meetings are well attended. Indeed for the period 2014-2017 on average about 1000 shareholders were present and further about 13600 shareholders were represented by proxies.

Unless markets for goods and assets are believed to be perfectly competitive and complete much less consensus should be expected in assemblies of shareholders. Indeed for anybody familiar with the social choice literature, these numbers are striking. An illustrative example of disagreement is m individuals sharing a cake where everybody just cares about the size of her own slice: for all allocations of the cake $m-1$ individuals would agree to split the slice of the last individual; and the stability index would be zero.

The boards of corporations in CAC 40 have a total of 501 directors of which 57 are members of two boards, 14 are members of three boards and nobody is a member of four or more boards. In Figure 1 we show the network with corporations being nodes and common board members being links, so there are 40 nodes and 99 links. A closer look at the boards in the CAC 40 corporations reveals that they are all connected directly or indirectly. Since the average number of links per firm is just below 2.5, connectedness of the network is striking.

We develop a theory of opinion formation in bipartite networks. Our theory uses one of the striking features, namely the connectedness of the network, to explain the other striking feature, namely the very high consensus index. Perhaps it would have been more natural

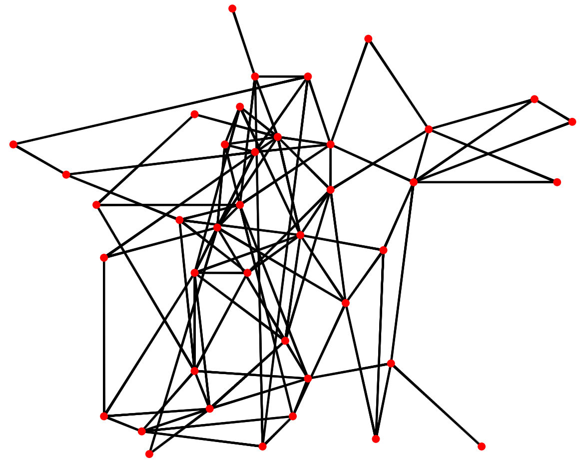

Figure 1: ***CAC 40***

to focus on either general assembly meetings or boards, but as general assembly meetings appoint boards and boards put up proposals to general assembly meetings we suggest the high consensus in general assemblies carries over to boards. A suggestion supported by casual evidence.

Several studies of networks of boards and directors have found that the networks tend to be connected (Burt, 2006; Davis, 1996). The median firm in Fortune 500 in the 1980s shared board members with seven other firms and some firms shared board members with 40 or more firms (Davis, 1996). It has been pointed out that firms benefit from being connected because thereby expertise and opinions of board members in other firms becomes indirectly available (Davis et al., 2002; Mace, 1971).

Overview of the paper: In our economy decisions in firms have to be made today about production tomorrow. The state tomorrow is unknown so production is uncertain. In every firm a decision about production has to be taken. These decisions are equivalent to choosing beliefs for which firms should maximize profit. There are two types of agents, namely boards and directors. The decision in a firm is taken by the board that consists of some directors who can be members of one board or several boards. Every director has an opinion about how likely the different states are. These opinions can be unambiguous, in which case the director has a unique subjective belief about the likelihood of the different states, or ambiguous, in which case the director has multiple subjective beliefs.

Despite differences in opinions among directors, boards have to take decisions. We impose a rather mild condition, namely the Pareto principle, on aggregation processes of opinions in a board: If every director considers one production plan at least as good as another, and a least one director considers it better, then the board finds it better too. The opinions of boards are intersubjective in that they are formed by a group of individuals, namely the directors, rather than by a single individual. Therefore the decisions taken in boards have a tinge of objectivity. We assume directors are affected by the decisions taken in boards: opinions of directors are outcomes of aggregation processes of the decisions in which they are involved. Like in boards, we impose the Pareto principle on aggregation processes within directors, but we include an element of reflexivity: If every board, in which a director is a member, *and the director herself* consider one production plan at least as good as another, and at least one board *or the director herself* considers it better, then the director finds it better too.

Since boards aggregate opinions of directors and directors aggregate opinions of boards, mutually interdependent aggregation processes take place in our model. Networks can be used to represent connections between firms and directors: a firm and a director are directly connected when the director is in the board of the considered firm. A firm and a director are indirectly connected when the director is in the board of a firm in which another director is a member of the board of another firm in which... another director is in the board of the considered firm. We focus on connected networks in which every firm and every director are directly or indirectly connected. However at the end of the paper we briefly discuss how our findings extend to disconnected networks.

Stability is the notion of equilibrium for opinions. Indeed, opinions are *stable* provided they satisfy the Pareto principle with respect to aggregation. Our main results concern stable opinions. We find that opinions are stable with respect to aggregation if and only if they are identical and unambiguous (Theorem 1). Hence stability of opinions is equivalent to a single society-wide intersubjective "truth" or a shared belief. Intuitively the Pareto principle implies aggregated opinions are convex combinations of opinions and every opinion is given positive weight. Because every agent has an aggregated opinion of other aggregated opinions and so on, stable opinions are identical and unambiguous.

Since directors meet other directors in boards it could be argued that they should be affected solely by the opinions of other directors rather than decisions in boards. We extend the characterization of stable opinions to include the case where directors are affected by other directors rather than boards (Corollary 1) as well as the dual case where boards are affected by other boards rather than directors (Corollary 2).

We have modelled our economy as an Actor-Network. It is flat: boards are formed by directors and directors are formed by boards; and, there is no aggregate level above them. Theorem 1 strengthens such an interpretation of our world. Opinions of agents depend on the opinions of their connections whose opinions depend on the opinions of their connections and so on: Directors and boards are their connections. Consequently it does not make sense to study an agent in isolation. Agents are treated symmetrically. Boards and directors aggregate opinions they experience from the other type of agent. They can be seen as direct or indirect collectives of collectives (Breiger, 1974; Simmel, 1955): directors are collectives of boards which are collectives of directors which are collectives of boards and so on. However, rather than applying Actor-Network Theory to a specific topic like decisions in boards of the CAC 40 corporations we construct a formal Actor-Network aimed at capturing the mutually interdependent aggregation processes in boards and directors. And rather than going into the specifics of how directors and boards interact and react, we merely assume that aggregation processes satisfy the Pareto principle.

The characterization of stable opinions is non-constructive like existence theorems (Debreu, 1959), it is merely saying that nobody feels the need to revise her opinion as a result of social interaction if and only if there is a shared belief. There is no explanation of how opinions become stable. Therefore we consider dynamic mutually interdependent aggregation processes where boards revise their opinions as a consequence of directors revising their opinions and directors revise their opinions as a consequence of boards revising their opinions and so on. We find that dynamic mutually interdependent aggregation processes converge to stable opinions (Theorem 2). As a converse we show that if opinions are stable, then there are mutually interdependent dynamic aggregation processes that converge to these opinions (Theorem 3). Hence stable opinions and limits of mutually interdependent dynamic aggregation processes are identical.

Naturally aggregation can be interpreted as deliberative processes in boards and directors. Therefore stable opinions can be seen as reflective equilibria (Rawls 1971; Rorty, 1990; Daniels, 2011). In the context of reflective equilibrium, Theorems 1 and 2 are extremely strong. Deliberation leads to a shared belief so deliberation removes all ambiguity and resolves all conflicts. However, in our world opinions are completely flexible in that agents are willing and able to revise them based on their experiences. And revision of opinions occurs in accordance with the Pareto principle so revision pushes toward agglomeration rather than polarization. As mentioned previously we discuss aggregation in disconnected networks at the end of the paper.

Related Literature: The seminal model of influence and aggregation of information and opinions in networks is DeGroot (1974). It is a discrete time dynamic model where a group of agents starts with initial opinions and then periodically updates them by taking weighted averages of the opinions of their neighbors according to a fixed vector of weights. Interactions between agents are hence captured through a matrix λ whose (i, j)-entry is λ_{ij}, the degree of confidence of agent i in the opinion of agent j. Conditions of convergence of the updating process are thoroughly studied in Jackson (2008) and Golub & Jackson (2010).[1]

DeMarzo, Vayanos & Zwiebel (2003) propose a microfoundation of DeGroot's model as a repeated naive maximum likelihood estimation procedure of an underlying parameter that captures a form of persuasion bias. The interpretation gave rise to a growing literature on aggregation of information on networks, relaxing the linearity in the naive-updating rules of the agents. Some recent contributions, like Molavi, Tahbaz-Salehi & Jadbabaie (2018) consider social learning where agents both receive signals about an underlying state of the world and naively combine the beliefs of their neighbors; others, like Cerreia-Vioglioa, Corrao & Lanzani (2020), study the long-run opinions as the size of the society grows to infinity. Both papers take an axiomatic approach, postulating some properties of the opinion aggregators.

Our model differs from this literature in two ways. First, the opinions of our agents

[1]For a thorough and comprehensive review of this literature see Acemoglu & Ozdaglar, 2011, Golub & Sadler, 2016, Mobius & Rosenblat, 2014, and the references therein.

(whether individual or collective) do not rely on ‘information’ in a strict sense: they do not receive signals about some ‘true’ state of the world; their opinions are purely subjective. Second, although our approach is axiomatic, we do not use aggregators, but just postulate the Pareto principle on how opinions are updated. The fact that our graph is bipartite, with boards aggregating the opinions of directors, and conversely directors aggregating the opinions of boards, introduces some additional structure that allows connectedness to result in a unique unambiguous opinion.[2]

Plan of the paper: In Section 2 we outline the structure with time and uncertainty and describe the agents, boards and directors, and their relations. In Section 3 we introduce the Pareto principle for aggregation. In Section 4 we introduce our notion of stability of opinions and characterize stable opinions. In Section 5 we characterize the outcomes of repeated social interaction and discuss a weaker version of the Pareto principle. In Section 6 we discuss our results and some extensions. Proofs are gathered in the Appendix.

2 Setup

The structure is outlined. We consider an economy with two dates $t \in \{0,1\}$, no uncertainty at the first date and uncertainty at the second date. There are two types of agents, firms and directors. At the first date directors interact in boards of firms and boards interact through directors. At the second date production takes place.

Production and beliefs

There are S states at the second date $\{1,\ldots,S\}$ with one commodity in every state.

There are J firms in the economy with $\mathbb{J} = \{1,\ldots,J\}$. Every firm j is characterized by its production set $Y_j \subset \mathbb{R}^S$ that is assumed to be non-empty, closed and convex. A production

[2]In our model, convergence to a single prior comes out as the result of the aggregation process of basically risk-neutral agents; this is to be distinguished from the literature on aggregation of multiple prior opinions where for a large class of preferences, aggregation is shown to be impossible unless the set of priors is a singleton (Gajdos, Tallon & Vergnaud, 2008). See Section 2 for a discussion of the literature on ambiguity.

plan $y_j \in Y_j$ is efficient provided $Y_j \cap (\{y_j\} + \mathbb{R}^S_+) = \{y_j\}$. For all efficient production plans there is a belief $\nabla \in \mathbb{R}^S_+$ with $\nabla \neq 0$ such that $\nabla \cdot y_j \geq \nabla \cdot y'_j$ for all production plans $y'_j \in Y_j$. Therefore the choice of a production plan corresponds to the choice of a belief. The set of possible beliefs is the unit simplex $\triangle$ in $\mathbb{R}^S$,

$$\triangle = \left\{ \nabla \in \mathbb{R}^S_+ \mid \sum_{s=1}^{S} \nabla^s = 1 \right\}.$$

Directors, boards and opinions

There are I potential directors in the economy with $\mathbb{I} = \{1, \ldots, I\}$. Every firm j is governed by a board of directors $\mathbb{I}_j \subset \mathbb{I}$. Let $\mathbb{J}_i$ be the set of firms that have director i in their boards so $j \in \mathbb{J}_i$ if and only if $i \in \mathbb{I}_j$. Directors and boards have opinions in form of sets of beliefs that they use to assess production plans. There is no ambiguity in opinions consisting of single beliefs and there is ambiguity in opinions consisting of multiple beliefs.

Consider a set of vectors $V \subset \mathbb{R}^S$. The *convex hull* of the set V is denoted $\operatorname{co} V$. The *relative interior* of the set V is denoted $\operatorname{ri} V$. There is no need to distinguish between opinions and their convex hulls. Consequently, opinions are assumed to non-empty, closed and convex sets where $F_i \subset \triangle$, respectively $G_j \subset \triangle$, is the opinion of director i, respectively board j.

Rationale for considering ambiguous agents

The problem we have in mind is that of business ventures whose success depend on uncertain events, such as the average temperature of the globe rising by 0.5 or 1 degree Celsius within the next two decades. To take a decision, board j needs to assess the probability of such an event, and to gather the opinions of the directors.

Different directors have different opinions though. Phenomena like global warming are rare and take a very long time; moreover the conditions that prevailed in past episodes of global warming are not sufficiently similar to the current ones for simple empirical frequencies to make sense. Sophisticated econometric techniques could be mobilized, but the directors would probably disagree on the underlying assumptions. Hence, the directors do not agree, and the board cannot assume an 'objective' probability for the event considered.

A benchmark case is one in which all directors are Bayesian. Let ∇_i be the subjective probability distribution on events according to director $i \in \mathbb{I}_j$. A natural aggregation rule is to take a weighted average of the directors' opinions. For any vector of non-negative weights $(\lambda_i)_{i\in\mathbb{I}_j}$ summing up to one, we can define

$$\nabla_j \;=\; \sum_{i\in\mathbb{I}_j} \lambda_i \nabla_i$$

This rule for aggregation of probabilistic assessments has been dubbed *linear opinion pool* by Stone (1961), and is attributed to Laplace (see Genest and Zidek, 1986, for a survey).

The weight λ_i can be interpreted as the degree of confidence board j has in director i. Another interpretation is that the board believes that one director has the ability to assess the "correct" probability, and gives probability λ_i to the event that director i has access to the "truth".[3]

The Bayesian paradigm relies on powerful axiomatic foundations laid by Ramsey (1931), de Finetti (1931, 1937), and Savage (1954). Yet, it is not exempt of critiques, both on descriptive and normative grounds. Following the seminal contributions of Knight (1921), Ellsberg (1961), Schmeidler (1986, 1989), Gilboa & Schmeidler (1989) and Bewley (2002) a large body of literature has dealt with more general models representing uncertainty.[4] In particular, it has been argued that, if the directors fail to agree on a probability, how can the board be so sure that there is one probability that is 'correct'? Maybe it's safer to allow for a *set* of possible probabilities, rather than pick only one (Gilboa, Postlewaite & Schmeidler, 2009, 2012).

Applied to our context, board j can for example be very cautious and take the convex hull of the probability assessments of the directors as its set of priors or it can take some strictly smaller set, for instance, the board can consider

$$G_j \;=\; \left\{ \nabla_j \in \triangle \mid \nabla_j = \sum_{i\in\mathbb{I}_j} \lambda_i \nabla_i \text{ for some } (\lambda_i)_{i\in\mathbb{I}_j} \in L_j \right\}$$

[3]The averaging of probabilities resembles the averaging of utilities in social choice theory, derived from a Pareto condition à la Harsanyi (1955), when all directors share the same utility function.

[4]See Gilboa (2009) for a survey of axiomatic foundations of the Bayesian approach, their critiques, and alternative models.

where L_j is some subset of weights. As argued in Crès, Gilboa & Vieille (2011), a non-singleton subset L_j of weights allows the board to: assign different weights to different directors; take into account how many directors provided certain assessments; and, leave room for a healthy degree of doubt. The fact that ultimately a production plan is implemented does not seem inconsistent with the decision maker having multiple priors as in Gilboa & Schmeidler (1989).

If we argue that rationality or scientific caution can favor a set of probabilities over a single one, then directors too should provide their assessments as sets of probabilities, rather than probabilities; *especially in our model, where directors form their opinion by aggregation of opinions*. The procedure suggested above extends naturally to the case of non-Bayesian directors: board j has beliefs given by the set

$$G_j = \left\{ \nabla_j \in \triangle \mid \forall i \in \mathbb{I}_j \exists \nabla_i \in F_i : \nabla_j = \sum_{i \in \mathbb{I}_j} \lambda_i \nabla_i \text{ for some } (\lambda_i)_{i \in \mathbb{I}_j} \in L_j \right\}.$$

These considerations are all the more prevalent in the context of opinion aggregation in a social network as the structure of interdependence is more complex. Empirical evidence suggest that in many social networks individuals do not adjust their opinions in a Bayesian fashion.[5] Hence non-Bayesian models of aggregation of opinions might offer better descriptions of how opinions evolve.

3 The Pareto principle

As discussed in the introduction both types of agents aggregate opinions of each other. The Pareto principle for aggregation in boards and directors is introduced and discussed. It is a mild but fundamental condition on aggregation procedures. Indeed in Arrow (1951, 1963) the weak version of the Pareto principle (a mere unanimity condition) is one of the conditions used to obtain the General Possibility Theorem. Moreover, it is a basic condition in the literature on joint aggregation of beliefs and tastes (Hylland and Zeckhauser, 1979; Mongin,

[5] See Breza, Chandrasekhar & Tahbaz-Salehi , 2018, and Chandrasekhar, Larreguy & Xandri, 2020, and the references therein. For a review of the difficulties of Bayesian modelling of social networks see Breza, Chandrasekhar, Golub & Parvathaneni, 2019.

1995). Finally it is an important condition in the literature on aggregation of judgements and logical aggregation theory (Kornhauser and Sager, 1986; List and Pettit, 2002; Mongin, 2012).

The Pareto principle for boards

Aggregation in boards of opinions of directors is assumed to satisfy the Pareto principle.

Definition 1 *For opinions* $((F_i)_{i\in\mathbb{I}}, (G_j)_{j\in\mathbb{J}})$*:*

- G_j *respects the* ***Pareto principle across directors*** *(PPD) provided that for all* $\Delta y \in \mathbb{R}^S$*,* $\nabla \cdot \Delta y \geq 0$ *for all* $\nabla \in \cup_{i\in\mathbb{I}_j} F_i$ *with* $>$ *for some* $\nabla \in \cup_{i\in\mathbb{I}_j} F_i$ *implies* $\nabla_j \cdot \Delta y > 0$ *for all* $\nabla_j \in G_j$*.*
- $(G_j)_{j\in\mathbb{J}}$ *is* ***Pareto stable across directors*** *(PSD) provided that for every* j*,* G_j *respects PPD.*

Suppose *every* director i considers a change of production at least as good as no change for *all* beliefs $\nabla_i \in F_i$. The Pareto principle implies that if *some* director k considers the change of production better than no change for *some* belief $\nabla_k \in F_k$, then the board j finds the change better than no change for *all* beliefs $\nabla_j \in G_j$.

Our version of the Pareto principle gives weight to all beliefs of all directors as shown in Lemma 1 below. The motivation is based on democracy and diversity: everybody should be given positive weight; and, all their beliefs should be considered. We discuss a weaker version of the Pareto principle in Section 5 acknowledging that "implies $\nabla_j \cdot \delta y > 0$ for all $\nabla_j \in G_j$" is somewhat demanding.

In Definition 1 all possible changes of production plans are considered: $\triangle y \in \mathbb{R}^S$. However, some changes of efficient production plans are not feasible. Indeed, for all efficient production plans y_j there is a vector $\nabla_j \in \triangle$ such that for $L_j^- \subset \mathbb{R}^S$ defined by $L_j^- = \{\triangle y \in \mathbb{R}^S \mid \nabla_j \cdot \triangle y \leq 0\}$, if $y_j + \triangle y \in Y_j$, then $\triangle y \in L_j^-$. For director i let the sets $K_i^-, K_i^+ \subset \mathbb{R}^S$ be defined by

$$\begin{cases} K_i^- &= \{\triangle y \in \mathbb{R}^S \mid \forall\, \nabla_i \in F_i : \nabla_i \cdot \triangle y \leq 0\} \\ K_i^+ &= \{\triangle y \in \mathbb{R}^S \mid \forall\, \nabla_i \in F_i : \nabla_i \cdot \triangle y \geq 0\} \end{cases}$$

where according to director i, K_i^- (resp. K_i^+) is the set of changes that under no (resp. all) circumstances are beneficial. Then K_i^- is the polar cone of F_i and $K_i^+ = -K_i^-$. Moreover, F_i is the intersection of the polar cone of K_i^- and $\triangle$ according to Theorem 14.1 in Rockafellar (1970) and $K_i^- = (K_i^- \cap L_j^-) \cup (-(K_i^+ \cap L_j^-))$. Therefore, discussions in boards allow directors to reveal their beliefs even in case these discussions are limited to whether different feasible changes are beneficial under no circumstances and under all circumstances. Strict inequalities in the definitions of L_j^-, K_i^- and K_i^+ can be handled by considering closures of these sets. Consequently, all possible changes are considered.

The Pareto principle across directors is equivalent to beliefs of boards being convex combinations of beliefs of directors.

Lemma 1 *For $((F_i)_{i\in\mathbb{I}}, (G_j)_{j\in\mathbb{J}})$, the opinion G_j respects PPD if and only if for all $\nabla_j \in G_j$ there is $(\nabla_i)_{i\in\mathbb{I}_j}$ with $\nabla_i \in ri\,co\,F_i$ for every i such that $\nabla_j \in ri\,co\,\{(\nabla_i)_{i\in\mathbb{I}_j}\}$.*

Since the weight on every belief of every director is positive, the Pareto principle implies minorities, even down to individual directors, have a say on the decisions of boards; and every belief counts.

The Pareto principle for directors

Since the opinions of boards are intersubjective, directors are affected by opinions of boards. Therefore directors, just like boards, aggregate opinions. And like in boards, aggregation within directors of opinions of boards is assumed to satisfy the Pareto principle.

We introduce an important difference though: directors aggregate opinions of boards as well as their own opinions. Indeed if every board *and the concerned director herself* consider one production plan at least as good as another for all beliefs, and at least one board *or the concerned director herself* considers it better for at least one belief, then the concerned director finds it better too for all beliefs.

Definition 2 *For opinions $((F_i)_{i\in\mathbb{I}}, (G_j)_{j\in\mathbb{J}})$:*

- *F_i respects the **Pareto principle across boards** (PPB) provided that for all $\Delta y \in \mathbb{R}^S$,*

$\nabla \cdot \Delta y \geq 0$ for all $\nabla \in \cup_{j \in \mathbb{J}_i} G_j \cup F_i$ with $>$ for some $\nabla \in \cup_{j \in \mathbb{J}_i} G_j \cup F_i$ implies $\nabla_i \cdot \Delta y > 0$ for all $\nabla_i \in F_i$.

- *$(F_i)_{i \in \mathbb{I}}$ is **Pareto stable across boards** (PSB) provided that for every i, F_i respects PPB.*

Hence directors keep track of their own former opinions whereas boards do not. Thereby an element of reflexivity is present in how directors aggregate opinions making the Pareto principle less demanding. Board members are solely affected by opinions of the boards at the margin. Indeed, board members are affected to the extent that unanimity of boards at most changes an indifference into a strict preference.

Despite the important difference in aggregation of opinions in boards and directors, the characterization of opinions respecting the Pareto principle is identical: the Pareto principle across boards is equivalent to beliefs of directors being convex combinations of beliefs of boards with the belief of every board being in the relative interior and the weight being positive on the belief of every board. And no board is disregarded.

Lemma 2 *For $((F_i)_{i \in \mathbb{I}}, (G_j)_{j \in \mathbb{J}})$, the opinion F_i respects PPB if and only if for all $\nabla_i \in F_i$ there is $(\nabla_j)_{j \in \mathbb{J}_i}$ with $\nabla_j \in ri\, G_j$ for every j such that $\nabla_i \in ri\, co\, \{(\nabla_j)_{j \in \mathbb{J}_i}\}$.*

The Pareto principle for aggregation within directors can be seen as a formalization of the idea that directors seek to have positive view on the decisions they experience in boards and that, by revising their opinions, directors try to minimize the tension between their own opinions and the opinions they experience in boards.

4 Stable opinions

Stability is the notion of equilibrium for opinions where opinions of directors and firms are stable provided they satisfy the Pareto principle.

Definition 3 *Opinions $((F_i)_{i \in \mathbb{I}}, (G_j)_{j \in \mathbb{J}})$ are **stable** provided $(F_i)_{i \in \mathbb{I}}$ is PSB and $(G_j)_{j \in \mathbb{J}}$ is PSD.*

Opinions are stable provided the opinions of everybody are not contradicted by the opinions of everybody else. Therefore, nobody feels the need to revise her opinion as a result of social interaction. Consequently, there are no changes of production that are supported by everybody.

A single society-wide intersubjective "truth"

Consider a network obtained by the bipartite graph $\mathscr{A}$ with vertices $\mathbb{I} \cup \mathbb{J}$ and edges between i and j if and only if $i \in \mathbb{I}_j$ or equivalently $j \in \mathbb{J}_i$ and no other edges.

For connected graphs, where every director and every board are connected directly or indirectly, opinions are stable if and only if they are identical and unambiguous.[6]

Theorem 1 *Assume the graph $\mathscr{A}$ is connected. Then opinions $((F_i)_{i \in \mathbb{I}}, (G_j)_{j \in \mathbb{J}})$ are stable if and only if there is a belief $\nabla^* \in \triangle$ such that*

$$((F_i)_{i \in \mathbb{I}}, (G_j)_{j \in \mathbb{J}}) = \{\nabla^*\}^{I+J}.$$

For stable opinions Lemmas 1 and 2 imply opinions of directors are in the relative interior of the convex hull of the opinions of boards and opinions of boards are in the relative interior of the convex hull of the opinions of directors. As an implication of the contracting nature of the aggregation according to Lemmas 1 and 2, stable opinions are identical and unambiguous. Social interaction will confirm the opinion of everybody by showing everybody that everybody else shares her unambiguous opinion. In other words, there is a single society-wide intersubjective "truth" or shared belief.

Theorem 1 offers theoretical support to empirical findings that connected boards create an embeddedness for board decisions (Davis, 1996; Granovetter, 1985). Opinions in one firm or director becomes the starting point for opinions in other firms and directors and so

[6]As mentioned in the introduction several studies of networks of boards and directors have found that the networks tend to be connected. Davis (1996): "the median Fortune 500 firm during the 1980s collectively sat on the boards of seven other Fortune firm boards, and, some firms [...] shared directors ('interlocked') with 40 or more large firms. The aggregate result is the creation of an interlocking directorate linking virtually all large American firms into a single network based on shared board members.".

on. The structure of the network as well as how firms and directors aggregate opinions will determine how opinions evolve.

Boards as mere places for deliberation

The characterization of stability in Theorem 1 allows for another interpretation of the model according to which directors take into account the opinions of the other directors they meet in boards rather than the opinions of the boards. Consequently boards are merely places for deliberation and exchange of opinions. Formally for every $j \in \mathbb{J}$,

$$G_j \;=\; \mathrm{co} \bigcup_{i \in \mathbb{I}_j} F_i.$$

It is still the case that if directors aggregate opinions across boards according to the Pareto principle, then opinions are stable provided there is a shared belief.

Corollary 1 *Assume the graph $\mathscr{A}$ is connected. Suppose $(F_i)_{i\in\mathbb{I}}$ is PSB and $G_j = co \cup_{i\in\mathbb{I}_j} F_i$ for every $j \in \mathbb{J}$. Then there is $\nabla^* \in \triangle$ such that*

$$((F_i)_{i\in\mathbb{I}}, (G_j)_{j\in\mathbb{J}}) \;=\; \{\nabla^*\}^{I+J}.$$

According to Lemma 2 opinions of directors are in the relative interior of the convex hull of opinions of boards. By assumption opinions of boards are in the convex hull of opinions of directors. As an implication of the contracting nature of the aggregation according to Lemma 2, stable opinions are identical and unambiguous. Naturally, Corollary 1 is true provided G_i is contained in co $\cup_{i\in\mathbb{J}_j} F_i$ for every j.

An alternative account for the present interpretation can be proposed. It consists in modifying the Pareto principle to deal only with individual opinions: If every director, whom director i meets in some board, considers one production plan at least as good as another for all her beliefs and some director considers it better for some of her beliefs, then director i finds it better too for all her beliefs. Let $\mathbb{I}_i$ be the set of directors whom director i meets in some board, $\mathbb{I}_i = \{\, k \in \mathbb{I} \mid \exists j \in \mathbb{J} : i,k \in \mathbb{I}_j \,\}$ so $i \in \mathbb{I}_i$.

The modified Pareto principle becomes: for all $\Delta y \in \mathbb{R}^S$, $\nabla \cdot \Delta y \geq 0$ for all $\nabla \in \cup_{k\in\mathbb{I}_i} F_k$ with $>$ for some $\nabla \in \cup_{k\in\mathbb{I}_i} F_k$ implies $\nabla_i \cdot \Delta y > 0$ for all $\nabla_i \in F_i$. Lemma 2 and its proof can be modified to show that F_i respects the modified Pareto principle provided: for all $\nabla_i \in F_i$ and $k \in \mathbb{I}_i$ there is $\nabla_k \in \operatorname{ri} F_k$ such that $\nabla_i \in \operatorname{ri}\operatorname{co}\{(\nabla_k)_{k\in\mathbb{I}_i}\}$.

Directors as mere gatherers of opinions

The characterization of stability in Theorem 1 allows for the dual interpretation of the model according to which an individual director is appointed in a board to channel her experience from other boards.[7] Directors are perceived to bring knowledge from other boards.[8] Hence directors are merely channeling opinions between boards for deliberation and decision making in these boards. Formally for every $i \in \mathbb{I}$,

$$F_i \;=\; \operatorname{co} \bigcup_{j\in\mathbb{J}_i} G_j.$$

It is still the case that, if boards aggregate according to the Pareto principle, then opinions are stable provided there is a shared belief.

Corollary 2 *Assume the graph $\mathscr{A}$ is connected. Suppose $F_i = co \cup_{j\in\mathbb{J}_i} G_j$ for every $i \in \mathbb{I}$ and $(G_j)_{j\in\mathbb{J}}$ is PSD. Then there is $\nabla^* \in \triangle$ such that*

$$\left((F_i)_{i\in\mathbb{I}}, (G_j)_{j\in\mathbb{J}}\right) \;=\; \{\nabla^*\}^{I+J}.$$

According to Lemma 1 opinions of boards are in the relative interior of the convex hull of opinions of directors. By assumption opinions of directors are in the convex hull of opinions of boards. As an implication of the contracting nature of the aggregation according to Lemma 1, stable opinions are identical and unambiguous. Naturally, Corollary 2 is true provided F_i is contained in $\operatorname{co} \cup_{j\in\mathbb{I}_i} G_j$ for every i.

[7] As mentioned in the introduction several studies point out that firms benefit from being connected. Mace (1971) p.197: “The opportunity to learn through exposure to other companies’ operations is something of value that might be useful in their own situation [...].”.

[8] Davis et al. (2002), p. 305: “[...] board interlocks may be a fortuitous by-product of board preferences for recruiting experienced directors [...] The prior experience of directors is part of the raw material of board decision making, and it is unsurprising that a director who has been involved in acquisitions, alliances [...] or any other board-level decision (including recruiting other directors) would bring that expertise to bear.”.

Corollary 2 like Theorem 1 seems to provide theoretical support to empirical findings that interlocking directorates create “an informational and normative context - an ‘embeddedness’ - for board decision (Granovetter, 1985). Decisions at one board in turn become part of the raw material for decisions at other boards. In the aggregate, the structure of the network [...] will influence how the field as a whole evolves [...]” (Davis 1996, p.154). However the theoretical support in Corollary 2 is extreme in that directors are merely channeling opinions between boards.

5 Convergence of opinions

Consider some structure on revisions of opinions, so there are revision functions mapping opinions into aggregated opinions. Agents are characterized solely by their initial opinions and how they revise their opinions, but not by their final opinions. Therefore it can be argued that these functions are the closest we come to selves in our setup. Everything is revised as a result of social interaction, but there is structure in the aggregation principle underlying the revision. A position at one end of the spectrum is the position often taken in economics according to which preferences should be fixed and therefore not subject to any kind of revision (Becker, 1976). A position at another end of the spectrum is that everything is constructed and therefore the result of revision (Gergen, 2015). Both positions leave little room for exploring the formation of opinions: according to the former nothing changes and according to the latter everything changes in a completely unstructured manner. Our revision processes reflect a middle position according to which opinions can change as a result of social interaction, but changes have some structure.

To formalize the middle position implicit in our setup two ingredients are needed. The first ingredient is functions mapping opinions into revised opinions. For directors these functions represent how directors process other opinions and consequently how they form their selves. For boards these functions can represent everything from mechanical aggregation like majority voting to deliberation. The second ingredient is a topology on opinions where the topology allows us to have a notion of the distance between two different opinions.

For opinions $((F_i)_{i\in\mathbb{I}},(G_j)_{j\in\mathbb{J}})$ let $\phi_i((G_j)_{j\in\mathbb{J}_i},F_i)$ be the revised opinion of director i and

$\psi_j((F_i)_{i\in\mathbb{I}_j})$ the revised opinion of board j. Revised opinions are supposed to be non-empty, closed and convex subsets of beliefs.

Let $\mathbb{K}$ be the set of closed and convex subsets of $\triangle$. The set $\mathbb{K}$ is endowed with the Hausdorff distance $\rho : \mathbb{K}\times\mathbb{K}\to\mathbb{R}_+$ defined by

$$\rho(F,G) \;=\; \min\left\{ \varepsilon \geq 0 \mid \max_{v\in F}\min_{w\in G}\|v-w\|, \max_{w\in G}\min_{v\in F}\|w-v\| \leq \varepsilon \right\}.$$

Then $\mathbb{K}$ is compact. Intuitively the Hausdorff distance between two sets is equal to the maximum of how close it is possible to be in one of the sets from the other set. Let the distance between $((F_i)_{i\in\mathbb{I}},(G_j)_{j\in\mathbb{J}})\in\mathbb{K}^{I+J}$ and $((A_i)_{i\in\mathbb{I}},(B_j)_{j\in\mathbb{J}})\in\mathbb{K}^{I+J}$ be defined as $\max\{\max_i \rho(F_i,A_i),\max_j \rho(G_j,B_j)\}$.

It is assumed that $\phi_i : \mathbb{K}^{\mathbb{J}_i\cup\{i\}}\to\mathbb{K}$ is continuous for every i and $\psi_j : \mathbb{K}^{\mathbb{I}_j}\to\mathbb{K}$ is continuous for every j. Moreover, it is assumed that ϕ_i respects PPB for every i and ψ_j respects PPD for every j: $\phi_i((G_j)_{j\in\mathbb{J}_i},F_i)\subset \mathrm{ri\,co}\cup_{j\in\mathbb{J}_i} G_j\cup F_i$ for every i; and $\psi_j((F_i)_{i\in\mathbb{I}_j})\subset \mathrm{ri\,co}\cup_{i\in\mathbb{I}_j} F_i$ for every j.

Repeated revision of opinions leads to elimination of ambiguity independently of initial opinions and a shared belief.

Theorem 2 *Assume the graph $\mathscr{A}$ is connected. Then for all opinions $((F_i)_{i\in\mathbb{I}},(G_j)_{j\in\mathbb{J}})\in\mathbb{K}^{I+J}$ and revision functions $((\phi_i)_{i\in\mathbb{I}},(\psi_j)_{j\in\mathbb{J}})$ there is a belief $\nabla^*\in ri\,co\{(F_i)_{i\in\mathbb{I}},(G_j)_{j\in\mathbb{J}}\}$ such that*

$$\lim_{n\to\infty}((\phi_i)_{i\in\mathbb{I}},(\psi_j)_{j\in\mathbb{J}})^n((F_i)_{i\in\mathbb{I}},(G_j)_{j\in\mathbb{J}}) \;=\; \{\nabla^*\}^{I+J}.$$

The combination of the Pareto principle and reflexivity in how directors aggregate opinions are both needed to ensure that repeated revision of opinions makes opinions converge to some shared belief. The Pareto principle implies revised opinions of agents are in the relative interior of the convex hull of the opinions they aggregate. Reflexivity implies agents are not swapping opinions because directors aggregate opinions of boards in which they are members and their own opinions. Indeed, consider one firm and a director, both with unambiguous opinions. If the director is non-reflexive, then revision of opinions amounts to the firm and the director swapping opinions and nothing else. However, if the director is reflexive, then revision by the director leads to opinions contained in the relative interior of the convex hull of the opinions of the firm and the director.

As a converse to Theorem 2 we show that for all beliefs in the relative interior of the convex hull of opinions there are revision functions such that repeated revision of opinions leads to the belief being established as the shared belief.

Theorem 3 *Assume the graph $\mathscr{A}$ is connected. Then for all opinions $((F_i)_{i\in\mathbb{I}},(G_j)_{j\in\mathbb{J}}) \in \mathbb{K}^{I+J}$ and beliefs $\nabla^* \in ri\,co\,\{(F_i)_{i\in\mathbb{I}},(G_j)_{j\in\mathbb{J}}\}$ there are revision functions $((\phi_i)_{i\in\mathbb{I}},(\psi_j)_{j\in\mathbb{J}})$ such that*

$$\lim_{n\to\infty}((\phi_i)_{i\in\mathbb{I}},(\psi_j)_{j\in\mathbb{J}})^n((F_i)_{i\in\mathbb{I}},(G_j)_{j\in\mathbb{J}}) \;=\; \{\nabla^*\}^{I+J}.$$

With linear revision functions as in DeGroot (1974) some beliefs in the interior of the convex hull of beliefs need not be possible as outcomes of repeated revision of opinions. Indeed, consider one firm and one director with unambiguous opinions and linear revision functions. Then the shared belief is a convex combination of the opinions of the board and the director where the weight on the opinion of the director is between one half and one. The weight on opinion of the director will be larger than the weight on the opinion of the board, because the director is reflexive and the board is not. With non-linear revision functions, the director could have a weight close to one on the opinion of the board in the first revision and a weight close to one on the opinion of herself in subsequent revisions. Since revision of opinions in the board amounts to swapping its opinion for the opinion of the director, repeated revision would lead to the weight on the opinion of the board being larger than the weight on the opinion of the director. The weights in the revision function of the director can be chosen to depend continuously on the opinions being aggregated.

Theorem 3 shows that with ambiguous opinions and non-linear revision functions all beliefs in the interior of the convex hull of beliefs are possible as the shared belief. Therefore with ambiguous opinions and non-linear revision functions the specific structure of the graph $\mathscr{A}$ can become unimportant.

The weak Pareto principle and revision of opinions

It can be argued that the Pareto principle reflects that indifference is overlooked in the presence of ambiguity. Indeed suppose every agent is ambiguous and considers a change in production at least as good as no change for all beliefs and one agent considers the change better than no change for some beliefs. Then the Pareto principle implies that the concerned agent finds the change better than no change for all beliefs. Therefore we propose a weaker Pareto principle according to which the concerned agent finds the change at least as good as no change for all beliefs and better for some beliefs. We briefly discuss its implications for convergence of opinions and show that generically opinions respecting the weaker Pareto principle satisfy the Pareto principle. Consequently opinions converge to a shared belief generically.

On the one hand the weak Pareto principle introduced below is immune to the argument that indifference is overlooked. On the other hand moving from the Pareto principle to the weak Pareto principle generically does not change any of our results. The intuition comes from standard consumer theory where convexity assumptions have a big impact on properties of demand. Indeed, strict convexity of preferences imply demand is a function and convexity of preferences imply demand is a correspondence, but generically convex preferences are strictly convex as shown in Mas-Colell (1985).

First the weak Pareto principle for boards.

Definition 4 *For opinions* $((F_i)_{i\in\mathbb{I}}, (G_j)_{j\in\mathbb{J}})$*:*

- G_j *respects the* ***weak Pareto principle across directors*** *(wPPD) provided that for all* $\Delta y \in \mathbb{R}^S$, $\nabla\cdot\Delta y \geq 0$ *for all* $\nabla \in \cup_{i\in\mathbb{I}_j} F_i$ *with* $>$ *for some* $\nabla \in \cup_{i\in\mathbb{I}_j} F_i$ *implies* $\nabla_j \cdot \Delta y \geq 0$ *for all* $\nabla_j \in G_j$ *with* $>$ *for some* $\nabla_j \in G_j$*.*

- $(G_j)_{j\in\mathbb{J}}$ *is* ***weakly Pareto stable across directors*** *(wPSD) provided that for every* j*,* G_j *respects wPPD.*

Theorems 1 and 2 are true even if opinions of boards respect wPPD and opinions of directors respect PPB. Voting in boards respects wPPD, but not necessarily PPD. Indeed, voting could lead to opinions in the convex hulls of the opinions of majorities of directors, but not necessarily in the relative interior of the convex hull of the opinions of directors.

Second the weak Pareto principle for directors.

Definition 5 *For opinions* $((F_i)_{i\in\mathbb{I}},(G_j)_{j\in\mathbb{J}})$*:*

- F_i *respects the* ***weak Pareto principle across boards*** *(wPPB) provided that for all* $\Delta y \in \mathbb{R}^S$*,* $\nabla\cdot\Delta y \geq 0$ *for all* $\nabla \in \cup_{j\in\mathbb{J}_i} G_j \cup F_i$ *with* $>$ *for some* $\nabla \in \cup_{j\in\mathbb{J}_i} G_j \cup F_i$ *implies* $\nabla_i \cdot \Delta y \geq 0$ *for all* $\nabla_i \in F_i$ *with* $>$ *for some* $\nabla_i \in F_i$*.*

- $(F_i)_{i\in\mathbb{I}}$ *is* ***weakly Pareto stable across boards*** *(wPSB) provided that for every i,* F_i *respects wPPB.*

Theorems 1 and 2 are not true for opinions of boards respecting PPD and opinions of directors respecting wPPB. With wPPB instead of PPB, opinions of directors could be independent of opinions in the boards in which they are members so stable opinions would not need to be a shared belief and repeated revision of opinions would not lead to a shared belief. It is the combination of wPPB and reflexivity that allow opinions of directors to be independent of opinions in boards in which they are members.

Third weak stability of opinions.

Definition 6 *Opinions* $((F_i)_{i\in\mathbb{I}},(G_j)_{j\in\mathbb{J}})$ *are* ***weakly stable*** *provided* $(F_i)_{i\in\mathbb{I}}$ *is wPSB and* $(G_j)_{j\in\mathbb{J}}$ *is wPSD.*

Weak stability does not imply there is a shared belief. Indeed, if agents have identical opinions, independent of whether they are unambiguous or not, then these opinions are stable for the weak Pareto principle. However, below we show that in the set of revision functions respecting the weak Pareto principle, the Pareto principle is a generic property. Consequently, the existence of a shared belief is a generic property of stability and a generic outcome of repeated revision of opinions.

In order to study generic properties of the set of revision functions respecting the weak Pareto principle the diameter of a set needs to be defined and the set of revisions respecting the weak Pareto principle needs to be endowed with a topology. Let $\delta : \mathbb{K} \to \mathbb{R}_+$ be the diameter $\delta(K) = \max_{v,w \in K} \|v - w\|$. For two continuous function μ and ν from $\mathbb{K}^m$ to $\mathbb{K}$ let a metric be defined by

$$d(\mu, \nu) \;=\; \sup_{H \in \mathbb{K}^m} \rho(\mu(H), \nu(H)).$$

The set of revision functions respecting the weak Pareto principle is endowed with the topology induced by the metric d.

First sets of revision functions, that respect the Pareto principle in case there is sufficient variation in beliefs and the weak Pareto principle in case there is not sufficient variation in beliefs, are defined.

Definition 7 *A function $\mu : \mathbb{K}^m \to \mathbb{K}$ respects the **ε-Pareto principle** provided for all $(K_a)_a$, it respects the weak Pareto principle and $\delta(co \cup_a K_a) \geq \varepsilon$ implies it respects the Pareto principle.*

Second it is shown that the set of revision functions respecting the Pareto principle in case there is sufficient variation in beliefs is an open and dense set of the set of revision functions satisfying the weak Pareto principle.

Lemma 3 *For all $\varepsilon > 0$ an open and dense set of revision functions $((\phi_i)_{i \in \mathbb{I}}, (\psi_j)_{j \in \mathbb{J}})$ respect the ε-Pareto principle.*

Third, it is shown that the set of revision functions respecting the Pareto principle is dense in the set of revision functions respecting the weak Pareto principle. Therefore revision functions respecting the weak Pareto principle can be approximated by revision functions respecting the Pareto principle.

Theorem 4 *Suppose revision functions $((\phi_i)_{i \in \mathbb{I}}, (\psi_j)_{j \in \mathbb{J}})$ respect the weak Pareto principle. Then there is a sequence of revision functions $((\phi_i^n)_{i \in \mathbb{I}}, (\psi_j^n)_{j \in \mathbb{J}})$ respecting the Pareto principle for every n such that*

$$\lim_{n \to \infty} \left(\sum_{i \in \mathbb{I}} d(\phi_i, \phi_i^n) + \sum_{j \in \mathbb{J}} d(\psi_j, \psi_j^n) \right) \;=\; 0.$$

Consider the set of revision functions satisfying the weak Pareto principle. Theorems 1 and 4 imply that generically opinions are stable if and only if there is a shared belief. Theorems 2 and 4 imply that generically repeated revision results in initial opinions converging to a shared belief. Therefore, generically there should be a shared belief.

6 Discussion of our results and some extensions

Coming back to CAC 40 and our consensus index of 87.6% for decisions made in general assembly meetings, we have used connectedness of networks of firms and directors to explain the widespread consensus in these networks. Indeed, our results show that the consensus index eventually becomes 100% as shown in Theorems 1 and 2.

We have combined two ideas, namely that social interaction between boards and directors lead them to revise their opinions and that revisions are compatible with the Pareto principle. Thereby we have obtained a foundation for DeGroot learning (DeGroot, 1974) as shown in Lemmas 1 and 2 and Theorem 2. The outcome of repeated revision of opinions is that opinions become identical and unambiguous as shown in Theorems 1 and 2. Hence social interaction establishes a single society-wide intersubjective "truth" or shared belief provided all members of society are connected. However, non-linear revision functions can enlarge the set of possible shared beliefs significantly compared to linear revision functions as shown in Theorem 3. Indeed, the specific structure of the graph becomes unimportant.

The Pareto principle within boards is quite standard. It is simply a formalization of the idea that aggregation of opinions in boards should not lead to opinions at odds with the opinion of every board member. The Pareto principle within directors is not standard, at least not in economics. However it is a formalization of the ideas that individuals seek to have positive views on what they experience and that individuals try to minimize tension between the opinions they experience and their own opinions.

If the graph $\mathscr{A}$ is not connected, then it consists of a number of connected components $\mathscr{A} = \cup_b \mathscr{A}_b$. Our findings for connected graphs generalize immediately to every connected component. However stable opinions across components need not be identical. But they differ exactly because there is no social interaction across connected components. Hence agents in a component are not exposed to opinions of agents in other components.

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Appendix: proofs

Proof of Lemma 1:

Let ∇_i^a be the average belief of director i. Then $\nabla_i^a \in \mathrm{ri}\, F_i$ for every i. The opinion G_j in board j respects PPD if and only if for all $\nabla_j \in G_j$ there is no $\Delta y \in \mathbb{R}^S$ such that

$$\begin{aligned} \nabla \cdot \Delta y &\geq 0 \text{ for all } \nabla \in \cup_{i \in \mathbb{I}_j} F_i \\ \textstyle\sum_{i \in \mathbb{I}_j} \nabla_i^a \cdot \Delta y &> 0 \\ \nabla_j \cdot \Delta y &\leq 0. \end{aligned}$$

Since Δy is unbounded in all directions the opinion G_j respects PDD if and only if for all $\nabla_j \in G_j$ and all $\delta > 0$ there is no $\Delta y \in \mathbb{R}^S$ such that

$$\begin{aligned} \nabla \cdot \Delta y &\geq 0 \text{ for all } \nabla \in \cup_{i \in \mathbb{I}_j} F_i \\ \textstyle\sum_{i \in \mathbb{I}_j} \nabla_i^a \cdot \Delta y - \delta &\geq 0 \\ \nabla_j \cdot \Delta y &\leq 0. \end{aligned}$$

Suppose G_j respects PPD. Then for all $\nabla_j \in G_j$ there is no $\Delta y \in \mathbb{R}^S$ such that

$$\begin{aligned} \nabla \cdot \Delta y &> 0 \text{ for all } \nabla \in \cup_{i \in \mathbb{I}_j} F_i \\ -\nabla_j \cdot \Delta y &> 0 \end{aligned}$$

Therefore according to Theorem 21.3 in Rockafellar (1970) either there is $\Delta y \in \mathbb{R}^S$ such that

$$\begin{aligned} -\nabla \cdot \Delta y &\leq 0 \text{ for all } \nabla \in \cup_{i \in \mathbb{I}_j} F_i \\ -\textstyle\sum_{i \in \mathbb{I}_j} \nabla_i^a \cdot \Delta y + \delta &\leq 0 \\ \nabla_j \cdot \Delta y &\leq 0. \end{aligned}$$

or there is $((\alpha_{i\nabla_i})_{i \in \mathbb{I}_j, \nabla_i \in F_i}, \beta, \gamma)$ with $\alpha_{i\nabla_i} \geq 0$ for every i and all ∇_i and $\alpha_{i\nabla_i} > 0$ for finitely many (i, ∇_i) and $\beta, \gamma \geq 0$ such that for some $\varepsilon > 0$ and all $v \in \mathbb{R}^S$,

$$-\sum_{i \in \mathbb{I}_j, \nabla_i \in F_i} \alpha_{i\nabla_i} \nabla_i \cdot v - \beta \sum_{i \in \mathbb{I}_j} \nabla_i^a \cdot v + \beta\delta + \gamma \nabla_j \cdot v \;\geq\; \varepsilon.$$

The inequality is satisfied for all v if and only if

$$\gamma\nabla_j = \sum_{i\in\mathbb{I}_j,\nabla_i\in F_i}\alpha_{i\nabla_i}\nabla_i+\beta\sum_{i\in\mathbb{I}_j}\nabla_i^a$$

and $\beta>0$. Hence $\gamma>0$ so

$$\begin{aligned}\nabla_j &= \sum_{i\in\mathbb{I}_j}\left(\sum_{\nabla_i\in F_i}\frac{\alpha_{i\nabla_i}}{\gamma}\nabla_i+\frac{\beta}{\gamma}\nabla_i^a\right)\\ &= \sum_{i\in\mathbb{I}_j}\frac{\sum_{\nabla_i\in F_i}\alpha_{i\nabla_i}+\beta}{\gamma}\left(\sum_{\nabla_i\in F_i}\frac{\alpha_{i\nabla_i}}{\sum_{\nabla_i\in F_i}\alpha_{i\nabla_i}+\beta}\nabla_i+\frac{\beta}{\sum_{\nabla_i\in F_i}\alpha_{i\nabla_i}+\beta}\nabla_i^a\right).\end{aligned}$$

Since $\beta>0$ and $\nabla_i^a\in\operatorname{ri}F_i$ for every i,

$$\sum_{\nabla_i\in F_i}\frac{\alpha_{i\nabla_i}}{\sum_{\nabla_i\in F_i}\alpha_{i\nabla_i}+\beta}\nabla_i+\frac{\beta}{\sum_{\nabla_i\in F_i}\alpha_{i\nabla_i}+\beta}\nabla_i^a \in \operatorname{ri}F_i \text{ for every } i\in\mathbb{I}_j.$$

Therefore there is $(\nabla_i)_{i\in\mathbb{I}_j}$ with $\nabla_i\in\operatorname{ri}F_i$ for every i such that $\nabla_j\in\operatorname{ri}\operatorname{co}\{(\nabla_j)_{j\in\mathbb{J}_i}\}$.

Proof of Lemma 2:

Let ∇_i^a be the average belief of director i and ∇_j^a the average belief of board j. Then $\nabla_i^a\in\operatorname{ri}F_i$ for every i and $\nabla_j^a\in\operatorname{ri}G_j$ for every j. The opinion F_i of director i respects PPB if and only if for all $\nabla_i\in F_i$ there is no $\Delta y\in\mathbb{R}^S$ such that

$$\begin{aligned}\nabla\cdot\Delta y &\geq 0 \text{ for all } \nabla\in\cup_{j\in\mathbb{J}_i}G_j\cup F_i\\ (\textstyle\sum_{j\in\mathbb{J}_i}\nabla_j^a+\nabla_i^a)\cdot\Delta y &> 0\\ \nabla_i\cdot\Delta y &\leq 0.\end{aligned}$$

Since Δy is unbounded in all directions the opinion F_i of director i respects PPB if and only if for all $\nabla_i\in F_i$ and all $\delta>0$ there is no $\Delta y\in\mathbb{R}^S$ such that

$$\begin{aligned}\nabla\cdot\Delta y &\geq 0 \text{ for all } \nabla\in\cup_{j\in\mathbb{J}_i}G_j\cup F_i\\ (\textstyle\sum_{j\in\mathbb{J}_i}\nabla_j^a+\nabla_i^a)\cdot\Delta y-\delta &\geq 0\\ \nabla_i\cdot\Delta y &\leq 0.\end{aligned}$$

Suppose F_i respects PPB. Then for all $\nabla_i \in F_i$ there is no $\Delta y \in \mathbb{R}^S$ such that

$$\begin{aligned} \nabla \cdot \Delta y &> 0 \text{ for all } \nabla \in \cup_{j\in\mathbb{J}_i} G_j \cup F_i \\ -\nabla_j \cdot \Delta y &> 0. \end{aligned}$$

Therefore according to Theorem 21.3 in Rockafellar (1970) either there is $\Delta y \in \mathbb{R}^S$ such that

$$\begin{aligned} -\nabla \cdot \Delta y &\le 0 \text{ for all } \nabla \in \cup_{j\in\mathbb{J}_i} G_j \cup F_i \\ -(\textstyle\sum_{i\in\mathbb{I}_j} \nabla_i^a + \nabla_i^a) \cdot \Delta y + \delta &\le 0 \\ \nabla_i \cdot \Delta y &\le 0. \end{aligned}$$

or there is $((\alpha_{j\nabla_j})_{j\in\mathbb{J}_i, \nabla_j\in G_j}, (\alpha_{\nabla_i})_{\nabla_i\in F_i}, \beta, \gamma)$ with $\alpha_{j\nabla_j} \ge 0$ for every j and $\alpha_{j\nabla_j} > 0$ for finitely many (i, ∇_i), $\alpha_{\nabla_i} \ge 0$ for all ∇_i and $\alpha_{\nabla_i} > 0$ for finitely many ∇_i and $\beta, \gamma \ge 0$ such that for some $\varepsilon > 0$ and all $v \in \mathbb{R}^S$,

$$-\sum_{j\in\mathbb{J}_i, \nabla_j\in G_j} \alpha_{j\nabla_j} \nabla_j \cdot v - \sum_{\nabla_i'\in F_i} \alpha_{\nabla_i'} \nabla_i' \cdot v - \beta(\sum_{j\in\mathbb{J}_i} \nabla_j^a + \nabla_i^a) \cdot v + \beta\delta + \gamma\nabla_j \cdot v \ \ge\ \varepsilon.$$

The inequality is satisfied for all v if and only if

$$\gamma\nabla_i \ =\ \sum_{j\in\mathbb{J}_i, \nabla_j\in G_j} \alpha_{j\nabla_j} \nabla_j + \sum_{\nabla_i'\in F_i} \alpha_{\nabla_i'} \nabla_i' + \beta(\sum_{j\in\mathbb{J}_i} \nabla_j^a + \nabla_i^a)$$

and $\beta > 0$. Hence $\gamma > 0$ so

$$\begin{aligned} \nabla_i &= \sum_{j\in\mathbb{J}_i} \left(\sum_{\nabla_j\in G_i} \frac{\alpha_{j\nabla_j}}{\gamma} \nabla_j + \frac{\beta}{\gamma} \nabla_j^a \right) + \sum_{\nabla_i'\in F_i} \frac{\alpha_{\nabla_i'}}{\gamma} \nabla_i' + \frac{\beta}{\gamma} \nabla_i^a \\ &= \sum_{j\in\mathbb{J}_i} \frac{\sum_{\nabla_j\in G_j} \alpha_{j\nabla_j} + \beta}{\gamma} \left(\sum_{\nabla_j\in G_j} \frac{\alpha_{j\nabla_j}}{\sum_{\nabla_j\in G_j} \alpha_{j\nabla_j} + \beta} \nabla_j + \frac{\beta}{\sum_{\nabla_j\in G_j} \alpha_{j\nabla_j} + \beta} \nabla_j^a \right) \\ &\quad + \frac{\sum_{\nabla_i\in F_i} \alpha_{\nabla_i} + \beta}{\gamma} \left(\sum_{\nabla_i'\in F_i} \frac{\alpha_{\nabla_i'}}{\sum_{\nabla_i\in F_i} \alpha_{\nabla_i} + \beta} \nabla_i' + \frac{\beta}{\sum_{\nabla_i\in F_i} \alpha_{\nabla_i} + \beta} \nabla_i^a \right) \end{aligned}$$

Since $\beta > 0$ and $\nabla_j^a \in \operatorname{ri} G_j$ for every j and $\nabla_i^a \in \operatorname{ri} F_i$,

$$\begin{aligned} \sum_{\nabla_j\in G_j} \frac{\alpha_{j\nabla_j}}{\sum_{\nabla_j\in G_j} \alpha_{j\nabla_j} + \beta} \nabla_j + \frac{\beta}{\sum_{\nabla_j\in G_j} \alpha_{j\nabla_j} + \beta} \nabla_j^a &\in \operatorname{ri} G_j \text{ for every } j \in \mathbb{J}_i \\ \sum_{\nabla_i'\in F_i} \frac{\alpha_{\nabla_i'}}{\sum_{\nabla_i\in F_i} \alpha_{\nabla_i} + \beta} \nabla_i' + \frac{\beta}{\sum_{\nabla_i\in F_i} \alpha_{\nabla_i} + \beta} \nabla_i^a &\in \operatorname{ri} F_i. \end{aligned}$$

Therefore there are $(\nabla_j)_{j\in\mathbb{J}_i}$ with $\nabla_j \in \mathrm{ri}\, G_j$ for every j and $\nabla'_i \in F_i$ such that $\nabla_i \in \mathrm{ri}\,\mathrm{co}\{(\nabla_j)_{j\in\mathbb{J}_i}, \nabla'_i\}$.

If $F_i \setminus \mathrm{ri}\,\mathrm{co}\{(\nabla_j)_{j\in\mathbb{J}_i}\} \neq \emptyset$, then there is an extreme point ∇'_i of F_i with $\nabla'_i \in F_i \setminus \mathrm{ri}\,\mathrm{co}\{(\nabla_j)_{j\in\mathbb{J}_i}\}$ contradicting there are $(\nabla_j)_{j\in\mathbb{J}_i}$ with $\nabla_j \in \mathrm{ri}\, G_j$ for every j and $\nabla'_i \in \mathrm{ri}\, F_i$ such that $\nabla_i \in \mathrm{ri}\,\mathrm{co}\{(\nabla_j)_{j\in\mathbb{J}_i}, \nabla'_i\}$. Hence $F_i \in \mathrm{ri} \cup_{j\in\mathbb{J}_i} G_j$ so there are $(\nabla_j)_{j\in\mathbb{J}_i}$ with $\nabla_j \in \mathrm{ri}\, G_j$ for every j such that $\nabla_i \in \mathrm{ri}\,\mathrm{co}\{(\nabla_j)_{j\in\mathbb{J}_i}\}$.

Proof of Theorem 1:

Let $P_{\mathbb{I}}$ be the convex hull of $\cup_{i\in\mathbb{I}} F_i$ and $P_{\mathbb{J}}$ the convex hull of $\cup_{j\in\mathbb{J}} G_j$. Then Lemmas 1 and 2 imply $P_{\mathbb{I}} = P_{\mathbb{J}}$.

Suppose p is an extreme point of $P_{\mathbb{I}} = P_{\mathbb{J}}$ and let $\mathbb{I}(p) = \{\, i \in \mathbb{I} \mid p \in F_i \,\}$ and $\mathbb{J}(p) = \{\, j \in \mathbb{J} \mid p \in G_j \,\}$. Then $\mathbb{I}(p), \mathbb{J}(p) \neq \emptyset$ by construction. Moreover for every $i \in \mathbb{I}(p)$, $\mathbb{J}_i \subset \mathbb{J}(p)$ with $G_k = \{p\}$ for every $k \in \mathbb{J}_i$ according to Lemma 2 and for every $j \in \mathbb{J}(p)$, $\mathbb{I}_j \subset \mathbb{I}(p)$ with $F_k = \{p\}$ for every $k \in \mathbb{I}_j$ according to Lemma 1 because p is an extreme point of $P_{\mathbb{I}} = P_{\mathbb{J}}$. Therefore $F_i = \{p\}$ for every $i \in \mathbb{I}(p)$ according to Lemma 2 and $G_j = \{p\}$ for every $j \in \mathbb{J}(p)$ according to Lemma 1. Since $\mathscr{A}$ is connected, $\mathbb{I}(p) = \mathbb{I}$ and $\mathbb{J}(p) = \mathbb{J}$.

Proof of Theorem 2:

Let $\mathbb{L}$ be the set of non-empty and closed subsets L of $\triangle$ so $\mathbb{K} \subset \mathbb{L}$. The set $\mathbb{L}$ is endowed with the Hausdorff distance $\rho : \mathbb{L}\times\mathbb{L} \to \mathbb{R}_+$. Then $\mathbb{L}$ is compact. Let $\delta : \mathbb{L} \to \mathbb{R}_+$ be the diameter: $\delta(L) = \max_{v,w\in L} \|v - w\|$.

Let $\Gamma : \mathbb{K}^{I+J} \to \mathbb{K}^{I+J}$ be defined by $\Gamma_k((F_i)_i, (G_j)_j) = \phi_k((G_j)_{j\in\mathbb{J}_i}, F_k)$ for $k \in \mathbb{I}$ and $\Gamma_{k+I}((F_i)_i, (G_j)_j) = \psi_k((F_i)_{i\in\mathbb{I}_k})$ for $k \in \mathbb{J}$. Then $\cup_k \Gamma_k((F_i)_i, (G_j)_j) \subset \mathrm{co} \cup_i F_i \cup_j G_j$ because ϕ_i respects PPB for every i and ψ_j respects PPD for every j so

$$\delta(\cup_k \Gamma_k((F_i)_i, (G_j)_j)) \;\leq\; \delta(\cup_i F_i \cup_j G_j).$$

Suppose $\nabla \in \triangle$ is an extreme point of $\cup_i F_i \cup_j G_j$. If $\{\nabla\} = \cup_i F_i \cup_j G_j$, then $\Gamma((F_i)_i, (G_j)_j) = \{\nabla\}$. If $\{\nabla\} \neq \cup_i F_i \cup_j G_j$, then there is a director such that $\nabla \in \cup_{j\in\mathbb{J}_i} G_j \cup F_i$ and $\{\nabla\} \neq \cup_{j\in\mathbb{J}_i} G_j \cup F_i$. Therefore, $\nabla \notin \Gamma_i((F_i)_i, (G_j)_j)$ so $\cup_k \Gamma^n_k((F_i)_i, (G_j)_j) \subset \mathrm{ri}\,\mathrm{co} \cup_i F_i \cup_j G_j$ for every $n \geq I+J$. Since $\mathbb{K}^{I+J}$ is compact and Γ and δ are continuous, there is $\gamma \in [0,1[$ such that for all $((F_i)_i,(G_j)_j)\in\mathbb{K}^{I+J}$ and every $n\ge I+J$,

$$\delta(\cup_k\Gamma_k^n((F_i)_i,(G_j)_j))\ \le\gamma\delta(\cup_iF_i\cup_j G_j).$$

Hence $\lim_{n\to\infty}\delta(\cup_k\Gamma_k^n((F_i)_i,(G_j))=0$ so there is $\nabla^*\in\operatorname{co}\cup_i F_i\cup_j G_j$ such that

$$\lim_{n\to\infty}\cup_k\Gamma_k^n((F_i)_i,(G_j))=\{\nabla^*\}.$$

Proof of Theorem 3:

For a closed and convex set $V\in\triangle$ and a positive number $\varepsilon\ge 0$ let V_ε be the set of points in V whose distance from an extreme point of V is at least ε. Then V_ε is closed and possible empty with $V_\varepsilon=V$ for $\varepsilon=0$ and $V_\varepsilon\subset\operatorname{ri}V$ for $\varepsilon>0$.

For opinions $((F_i)_{i\in\mathbb{I}},(G_j)_{j\in\mathbb{J}})$ let $F_i^0=F_i$ for every i and $G_j^0=G_j$ for every j. Moreover, let a list of opinions $((F_i^n)_{i\in\mathbb{I}},(G_j^n)_{j\in\mathbb{J}})_{n\in\{1\dots,I+j\}}$ be defined by

$$\begin{aligned}F_i^n &= (\operatorname{co}(\cup_{j\in\mathbb{J}_i}G_j^{n-1}\cup F_i^{n-1}))_{\varepsilon_i}\\ G_j^n &= (\operatorname{co}(\cup_{i\in\mathbb{I}_j}F_i^{n-1}))_{\varepsilon_j}.\end{aligned}$$

Suppose $\varepsilon_i=\varepsilon_j=0$ for every i and every j. Then for every i and j, $n=I+J$ implies $F_i^n=G_i^n=\operatorname{co}\cup_iF_i\cup_jG_j$ provided $\varepsilon_i=\varepsilon_j=0$. However, the list of opinions $((F_i^n)_i,(G_j^n)_j)_{n\in\{0,\dots,I+J\}}$ does not satisfy PSD and PSB because $\varepsilon_i=\varepsilon_j=0$ for every i and every j. For $N_i\subset\{0,1,\dots,I+J\}$ defined by $n\in N_i$ if and only if $F_i^n\ =\ \operatorname{co}\cup_i F_i\cup_j G_j$ let $n_i=\min_{n\in N_i}n$. For $N_j\subset\{0,1,\dots,I+J\}$ defined by $n\in N_j$ if and only if $G_j^n\ =\ \operatorname{co}\cup_i F_i\cup_j G_j$ let $n_j=\min_{n\in N_j}n$.

For all $\nabla^*\in\operatorname{ri}\operatorname{co}\cup_i F_i\cup_j G_j$ there there are $\varepsilon_i,\varepsilon_j>0$ such that the sets in $(F_i^n)_{n\in\{0,\dots,n_i\}}$ are different and $\nabla^*\in\operatorname{ri}F_i^{n_i}$ for every i and the sets in $(G_j^n)_{n\in\{0,\dots,n_j\}}$ are different and $\nabla^*\in\operatorname{ri}G_j^{n_j}$ for every j. For every i and every $n\in\{0,\dots,n_i\}$ let $U_i^n\subset\mathbb{K}^{J_i+1}$ be an open ball with center $((G_i^{n-1})_{j\in\mathbb{J}_i},F_i^{n-1})$ and radius $\delta>0$. For every j and every $n\in\{0,\dots,n_j\}$ let $V_j^n\subset\mathbb{K}^{I_j}$ be an open ball with center $(F_i^{n-1})_{i\in\mathbb{I}_j}$ and radius $\delta>0$.

For every i, if $\delta>0$ is sufficiently small, then for every $n\in\{1,\dots,n_i\}$ and all $((B_j)_{j\in\mathbb{J}_i},A_i)\in U_i^{n-1}$, $F_i^n\subset\operatorname{ri}\operatorname{co}\cup_{j\in\mathbb{J}_i}B_j^{n-1}\cup A_i^{n-1}$, and, for $n=n_i$ and all $((B_j)_{j\in\mathbb{J}_i},A_i)\in U_i^n$, $\nabla^*\in\operatorname{ri}\operatorname{co}\cup_{j\in\mathbb{J}_i}B_j\cup A_i$. For every j, if $\delta>0$ is sufficiently small, then for every $n\in\{1,\dots,n_j\}$ and all $(A_i)_{i\in\mathbb{I}_j}\in V_j^{n-1}$, $G_j^n\subset\operatorname{ri}\operatorname{co}\cup_{i\in\mathbb{I}_j}A_i^{n-1}$, and, for $n=n_j$ and all $(A_i)_{i\in\mathbb{I}_j}\in V_j^{n-1}$, $\nabla^*\in\operatorname{ri}\operatorname{co}\cup_{i\in\mathbb{I}_j}A_i$.

For every i let the correspondence $\phi_i^n : U_i^n \to \mathbb{K}$ be defined by

$$\phi_i^n((B_j)_{j\in\mathbb{J}_j}, A_i) \;=\; \begin{cases} F_i^{n+1} & \text{for } n \in \{0,\dots,n_i-1\} \\ \{\nabla^*\} & \text{for } n = n_i. \end{cases}$$

Then ϕ_i^n respects PPB for every i and every n. Suppose the correspondence $\chi_i : \mathbb{K}^{J_i+1} \to \mathbb{K}$ respect PPB for every i. For every j let the correspondence $\psi_j^n : V_j^n \to \mathbb{K}$ be defined by

$$\psi_i^n((A_i)_{i\in\mathbb{I}_j}) \;=\; \begin{cases} G_i^{n+1} & \text{for } n \in \{0,\dots,n_j-1\} \\ \{\nabla^*\} & \text{for } n = n_j. \end{cases}$$

Then ψ_j^n respects PPD for every j and every n. Suppose the correspondence $\omega_j : \mathbb{K}^{I_j} \to \mathbb{K}$ respects PPD for every j.

Consider an open cover of $\mathbb{K}^{J_i+1}$, $((U_i^n)_{n\in\{0,\dots,n_i\}}, X_i)$ where $X_i = \mathbb{K}^{J_i+1} \setminus \cup_{n\in\{0,\dots,n_i\}}\{((G_j^n)_{j\in\mathbb{J}_i}, F_i^n)\}$. Consider a partition of unity $((h_i^n)_{n\in\{0,\dots,n_i\}}, h_i)$ subordinate to $((U_i^n)_{n\in\{0,\dots,n_i\}}, W_i)$. Then $\phi_i : \mathbb{K}^{J_i+1} \to \mathbb{K}$ defined by

$$\phi_i((B_j)_{j\in\mathbb{J}_i}, A_i) \;=\; \sum_{n=0}^{n_i} h_i^n((B_j)_{j\in\mathbb{J}_i}, A_i)\phi_i^n((B_j)_{j\in\mathbb{J}_i}, A_i) + h_i((B_j)_{j\in\mathbb{J}_i}, A_i)\chi_i((B_j)_{j\in\mathbb{J}_i}, A_i)$$

respects PPB. Consider an open cover of $\mathbb{K}^{I_j}$, $((V_j^n)_{n\in\{0,\dots,n_j\}}, Y_j)$ where $Y_j = \mathbb{K}^{I_j} \setminus \cup_{n\in\{0,\dots,n_j\}}\{(F_i^n)_{i\in\mathbb{I}_j}\}$. Consider a partition of unity $((h_j^n)_{n\in\{0,\dots,n_j\}}, h_j)$ subordinate to $((U_i^n)_{n\in\{0,\dots,n_i\}}, V_i)$. Then $\psi_j : \mathbb{K}^{I_j} \to \mathbb{K}$ defined by

$$\psi_j((A_i)_{j\in\mathbb{I}_j}) \;=\; \sum_{n=0}^{n_j} h_j^n((A_i)_{i\in\mathbb{I}_j})\psi_j^n((A_i)_{i\in\mathbb{I}_j}) + h_j((A_i)_{i\in\mathbb{I}_j})\omega_j((A_i)_{i\in\mathbb{I}_j})$$

respects PPB. Moreover, $((\phi_i)_i, (\psi_j)_j)^{I+J}((F_i)_i, (G_j)_j) = \{\nabla^*\}^{I+J}$ so opinions become ∇^* after finitely many revisions.

Proof of Lemma 3:

Suppose the revision functions $((\phi_i)_i, (\psi_j)_j)$ respecting the ε-Pareto principle. Since revision functions are continuous, for all $\varepsilon > 0$ there is $\alpha > 0$ such that $\delta(\mathrm{co}((G_j)_{j\in\mathbb{J}_i}, F_i)) \geq \varepsilon$ implies

$$\max_{v\in\mathrm{co}((G_j)_{j\in\mathbb{J}_i}, F_i)} \; \min_{w\in\phi_i((G_j)_{j\in\mathbb{J}_i}, F_i)} \|v - w\| \geq \alpha$$

for every i and $\delta(\mathrm{co}\,(F_i)_{i\in\mathbb{I}_j}) \geq \varepsilon$ implies

$$\max_{v\in \mathrm{co}\,(F_i)_{i\in\mathbb{I}_j}} \min_{w\in\psi_j((F_i)_{i\in\mathbb{I}_j})} \|v-w\| \geq \alpha$$

for every j. Therefore there is no sequence of revision functions not respecting the ε-Pareto principle converging to $((\phi_i)_i,(\psi_j)_j)$.

Suppose the revision functions $((\phi_i)_i,(\psi_j)_j)$ respect wPPB and wPPD. For the continuous function $m:\mathbb{K}\to\triangle$ defined by

$$m(K) \;=\; \left\{ v\in\triangle \mid \forall x\in\triangle : \max_{w\in K}\|v-w\| \leq \max_{w\in K}\|x-w\| \right\}$$

let the sequence of revision functions $((\phi_i^n)_i,(\psi_j^n)_j)_{n\in\mathbb{N}}$ be defined by

$$\begin{aligned} \phi_i^n((G_j)_{j\in\mathbb{J}_i},F_i) &= \frac{1}{n}\{m(\mathrm{co}\cup_{j\in\mathbb{J}_i} G_j\cup F_i)\} + \frac{n-1}{n}\phi_i((G_j)_{j\in\mathbb{J}_i},F_i) \\ \psi_i^n((F_i)_{i\in\mathbb{I}_j}) &= \frac{1}{n}\{m(\mathrm{co}\cup_{i\in\mathbb{I}_j} F_i)\} + \frac{n-1}{n}\psi_i((F_i)_{i\in\mathbb{I}_j}). \end{aligned}$$

Then $((\phi_i^n)_i,(\psi_j^n)_j)$ respects PPB and PPD for every n and $((\phi_i^n)_i,(\psi_j^n)_j)_{n\in\mathbb{N}}$ converges to $((\phi_i)_i,(\psi_j)_j)$.

Proof of Theorem 4:

Since the set of revision functions satisfying the ε-Pareto principle is open and dense for all ε, the intersection is dense according to the Baire Category Theorem. The set of revision functions satisfying the Pareto principle is the intersection of sets of revision functions respecting the ε-Pareto principle for every $n\in\mathbb{N}$. Indeed consider the sequence of revision functions $((\phi_i^n)_i,(\psi_j^n)_j)_n$ defined in the proof of Lemma 3, then $((\phi_i^n)_i,(\psi_j^n)_j)$ respects the Pareto principle for every n and the sequence converges to $((\phi_i)_i,(\psi_j)_j)$.