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Asymptotic and numerical analysis of pulse propagation in relaxation media

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We consider the case of disturbances propagating in one-dimension through a medium with multiple relaxation modes and thermoviscous diffusion. Each relaxation mode is characterized by two parameters and the evolution of the disturbance is governed by an augmented Burgers equation. We begin by considering travelling wave solutions for the propagation of a pressure step, of amplitude P, in the small viscosity limit. For a single relaxation mode, if the amplitude P is less than a certain critical value then the transition is controlled entirely by the relaxation mode whereas for larger P, a thin viscous sub-shock arises. We then consider the propagation of a rectangular pulse. We establish parameter ranges in which the waveform is described by an outer solution (obtained using characteristics) and a thin shock region. Analysis of the shock region then reveals the same richness of structure seen in the travelling wave case, with subtle changes in shock structure as the disturbance decays. This is illustrated by numerical results using a pseudospectral method. Finally, analysis of the case of two relaxation modes is presented demonstrating that in some parameter regimes the transition region consists of three separate sub-regions governed by the three different physical processes.

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INTRODUCTION

In this paper we consider a model of one-dimensional finite-amplitude acoustic disturbance propagating in relaxation medium. The propagation of the disturbance can be modelled by an augmented Burgers' equation together with a set of relaxation equations as follows (Pierce[1])

$$p_t + p p_x + \sum_{i=1}^{n} \Delta_i \, \tilde{p}_x^{(i)} = \epsilon p_{xx}, \qquad \left(1 - \tau_i \frac{\partial}{\partial x}\right) \tilde{p}^{(i)} = -\tau_i p_x. \tag{1}$$

where p represents acoustic pressure, ϵ the thermoviscous coefficient and $\tilde{p}^{(i)}$ is the partial pressure associated with i-th relaxation mode. Each relaxation mode (e.g polyatomic molecules in air or particles in dusty gases) is characterized by two parameters, a relaxation time τ and an effective concentration Δ. We consider the evolution of a step of unit amplitude, so the boundary conditions are $p, \tilde{p} \to 0$ as $x \to \infty$ and $p \to 1,\ \tilde{p} \to 0$ as $x \to -\infty$.

AUGMENTED BURGERS' EQUATION WITH SINGLE RELAXATION MODE

TRAVELLING WAVE

For a transition of unit amplitude, the travelling wave solution propagates at speed $1/2$, so we define $\theta = x - \frac{1}{2}t$, and then apply the operator $(1 - \tau \frac{\mathrm{d}}{\mathrm{d}\theta})$ to Eq. (1) to eliminate $\tilde{p}$. The travelling wave for one relaxation mode satisfies the ODE

$$2\,\tau\,\epsilon\,F_{\theta\,\theta} - 2\Big(\tau\,\big(F - \gamma\big) + \epsilon\Big)F_\theta + \Big(F - 1\Big)\,F = 0, \qquad \gamma = \frac{1}{2}\big(1 - 2\Delta\big), \tag{2}$$

with limiting conditions $F(-\infty) = 1$, $F(\infty) = 0$ and $F_\theta(\pm\infty) = 0$.

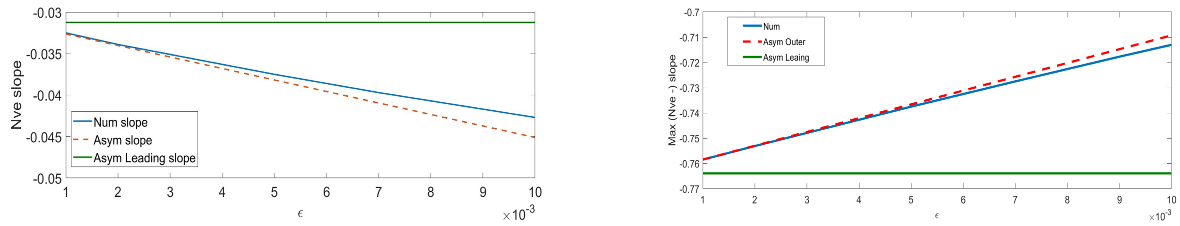
Solving the travelling wave for the limit $\epsilon \to 0$ yields an implicit equation for F

$$\mathrm{e}^{\frac{\theta}{\tau}} = \left(\frac{1 - F}{1 - F^*}\right)^{1+\phi} \left(\frac{F}{F^*}\right)^{1-\phi}, \qquad \phi = 2\Delta, \quad F^* \text{ a constant}, \tag{3}$$

which describes waveform transition from $F = 1$ to $F = 0$. We now consider the boundary conditions on the implicit solution (3) for the two cases $\phi > 1$ and $\phi < 1$ corresponding to the cases $\Delta < 1/2$ and $\Delta > 1/2$ respectively. If $\phi > 1$, we see that $F \to 1$ gives $\exp\left(\frac{\theta}{\tau}\right) \to 0$ which implies $\theta \to -\infty$, while $F \to 0$ yields $\exp\left(\frac{\theta}{\tau}\right) \to \infty$ which means $\theta \to \infty$. Thus, F satisfies the required boundary conditions when $\phi > 1$ and the waveform of the outer solution remains single-valued. This is known as a fully dispersed relaxing shock of width $O(\tau)$ (Lighthill[3]) and represents a physically realistic solution for F. In contrast if $\phi < 1$, the limits $F \to 0$ and $F \to 1$ both correspond to $\theta \to -\infty$ and it follows that both boundary conditions can not be satisfied. Thus the outer solution becomes multi-valued. In other words when $\epsilon = 0$ no travelling wave solution exists, and the waveform is known as partly dispersed. Including diffusivity ensures a single-valued solution, with an inner thermoviscous sub-shock, controlled by ϵ inserted at $\theta = \theta_0$, describing the transition from the outer solution for $\theta < \theta_0$ to zero ahead of the inner shock. In our case, the outer shock wave Eq. (3) describes the transition from $F = 1$ to $F = 1 - \phi$ and an inner sub-shock

$$f_0 = \frac{1}{2}\,(1 - \phi)\,\left[1 - \tanh\left(\frac{1 - \phi}{4\,\epsilon}\,\big(\theta - \epsilon\,\hat{\theta}_0\big)\right)\right], \tag{4}$$

describes the transition from $F = 1 - \phi$ to $F = 0$ with width $O(\epsilon/(1 - \phi))$ and $\hat{\theta}_0$ is determined and describes the $O(\epsilon)$ shift in shock center. To aid in comparisons between numeric and asymptotic results, we also obtain the $O(\epsilon)$ perturbations for both outer and inner expansions.



(a) $\phi < 1$ (b) $\phi > 1$

***Figure 1:* *Wave maximum negative slope for travelling wave numerical solution (blue) compared with the asymptotic solution at leading order (green) and with* $O(\epsilon)$ *correction (red) when* $0.001 < \epsilon < 0.01$*,* $\tau = 0.25$ *with (a)* $\phi = 0.5$ *and (b)* $\phi = 1.5$.**

The main purpose of employing matched asymptotic expansions for travelling wave is to study shock features such as shock amplitude and shock width, which is defined to be the range of the shock in the spatial variable x in which the solution decreases from 90% of its amplitude to 10% of its amplitude. An alternative characteristic of the shock structure is the value of the maximum negative wave slope, and here we focus on this method. From the two-term asymptotic series for the outer and inner solutions, the maximum negative slope is given by

$$S_m = \begin{cases} -\dfrac{\gamma^2}{2\,\epsilon}\Big(1 + \dfrac{2\,\phi\,\epsilon}{\tau\,\gamma^2}\,\log 2\Big), & \phi > 1, \\ \dfrac{1}{2\,\tau}\Big(\sqrt{\phi^2-1} - \phi\Big).\Big(1 + \epsilon\dfrac{2}{\tau\sqrt{\phi^2-1}}\Big), & \phi < 1. \end{cases} \tag{5}$$

In figure 1 comparisons are made between asymptotic predictions and numerical solutions of the full travelling wave equation. Numerical results are obtained following the method of Pierce & Kang.[2] A relationship between F and F_θ is obtained in the linearised shock tail as $\theta \to \infty$. This is then used as the initial condition for a Runge-Kutta solver of the full equation to obtain the wave form for smaller θ.

PROPAGATION OF RECTANGULAR WAVE

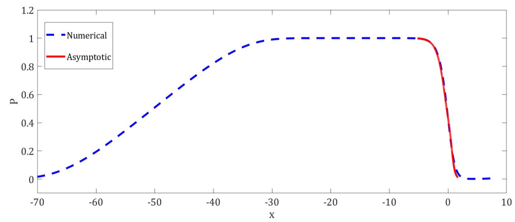
We now consider the propagation of an arbitrary initial disturbance and present an approach to solve the augmented Burgers' PDE numerically in spectral space, based on the method of integrating factors. Writing the augmented Burgers' equation in Fourier space,

$$\hat{p}_t + \frac{1}{2}\,ik\,\widehat{p^2} + k^2\Big(\epsilon + \frac{\Delta\,\tau}{1 - i\,\tau\,k}\Big)\hat{p} = 0. \tag{6}$$

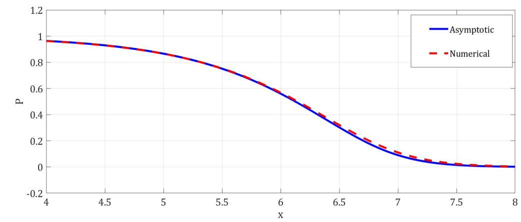
figure(5) plot(x,u) An integrating factor method is used so that the linear part of the PDE is solved exactly. We define $\hat{q} = \mathrm{e}^{f(k)\,t}\,\hat{p}$ with factor $f(k) = -k^2\Big(\epsilon + \frac{\Delta\,\tau}{1 - i\,\tau\,k}\Big)$, and use a fourth order Runge-Kutta scheme to advance the transformed equation forward in time.

We consider the evolution of a rectangular pulse of unit amplitude. For moderate times a shock of unit amplitude is present which will be either fully or partly-dispersed depending on the relaxation parameter. The two cases are illustrated in figures 3 and 2.

Figure 2 compares the numerical solution of the augmented Burgers' equation with the asymptotic travelling wave solution for a fully dispersed shock since $\Delta > 1/2$. The plot on the left is a full view of the evolving rectangular pulse and the plot on the right is a blow up of the shock region to show the correspondence of the two solutions. In figure 3 with $\Delta = 0.25$, the shock is partly dispersed and the numerical



(a)



(b)

Figure 2: Fully dispersed shock wave for the evolution of a rectangular pulse with $\epsilon = 0.01\,, \Delta = 1,\ \tau = 0.25$ ***at*** $t = 10$***. In (a) the numerical solution at*** $t = 10$ ***(blue) is compared to the asymptotic travelling wave solution (red) with (b) a blow-up of the shock region.***

solution (green) is compared to the outer asymptotic expansion (red) in the first relaxation-controlled region and to the viscous inner shock solution (blue) . If the evolution of the rectangular pulse is studied over longer

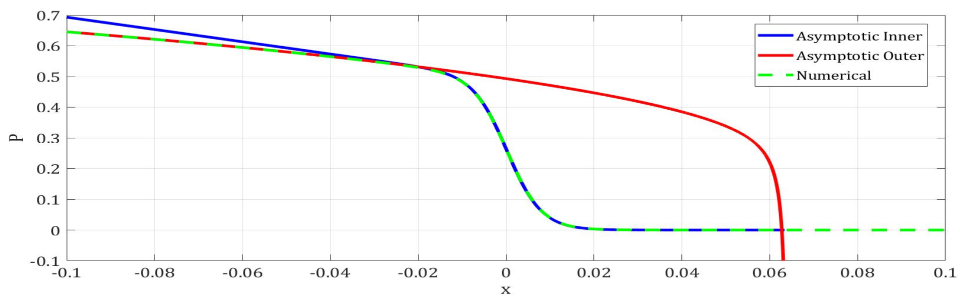


Figure 3: Partly dispersed shock wave for the evolution of a rectangular pulse with $\epsilon = 0.001\,, \Delta = 0.25,\ \tau = 0.25$ ***at*** $t = 10$***. The numerical solution (green-dashed) is compared to the first two terms of the relaxation controlled solution (red) and to the inner viscous expansion (blue).***

time intervals, the wave becomes triangular at some fixed time after which the shock amplitude decreases which changes the condition for the shock being either fully- or partly-dispersed. In figure 4 parameter values are chosen so that the shock is initially partly-dispersed. At $t = 10$ the wave reaches a triangular form after which the amplitude decreases and the shock widens. At $t \approx 39$ the amplitude has decreased to a level at which the shock changes from partly-dispersed to fully-dispersed.

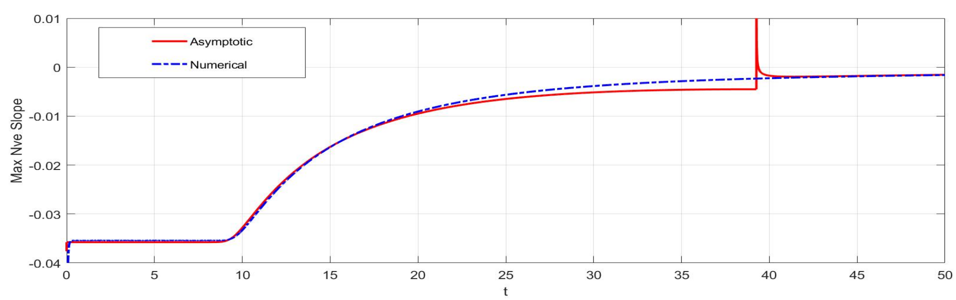


Figure 4: Comparison of wave maximum negative slope for the numerical solution (blue) of augmented Burgers equation for $\epsilon = 0.00013, \tau = 0.01, \Delta = 0.25$ ***with asymptotic solutions (red). The shock is predicted to be partly dispersed for*** $t < 39$ ***and fully dispersed for*** $t > 39$***.***

AUGMENTED BURGERS' EQUATION WITH TWO RELAXATION MODES

For propagation through air, two relaxation modes associated with nitrogen, oxygen are present. Following the same method as for one relaxation mode we obtained the travelling wave equation for two relaxation modes. Applying the operator $(1-\tau_1\frac{\mathrm{d}}{\mathrm{d}\theta}).(1-\tau_2\frac{\mathrm{d}}{\mathrm{d}\theta})$ to Eq. (1) gives

$$\left(\tau_1\frac{\mathrm{d}}{\mathrm{d}\theta}-1\right)\left(\tau_2\frac{\mathrm{d}}{\mathrm{d}\theta}-1\right)\left[\epsilon F_\theta-\frac{F}{2}\left(F-1\right)\right]=\tau_1\tau_2\left(\Delta_1+\Delta_2\right)F_{\theta\theta}-\left(\tau_1\Delta_1+\tau_2\Delta_2\right)F_\theta. \quad (7)$$

We then consider the asymptotic structure of the travelling wave when $\tau_1 \geq \tau_2 \geq \epsilon$. Once again a fully dispersed relaxing shock of width $O(\tau_1)$ arises when $\Delta_1 > \frac{1}{2}$. When $\Delta_1+\Delta_2 < \frac{1}{2}$ the waveform is covered by three regions. The first region is the transition from $F=1$ to $F=1-2\Delta_1$ of width $O(\tau_1)$. The second shock is governed by the second relaxation mode of width $O(\tau_2)$ describes the transition from $F=1-2\Delta_1$ to $F=1-2\Delta_1-2\Delta_2$. Third region is viscous sub-shock of amplitude $1-2\Delta_1-2\Delta_2$.

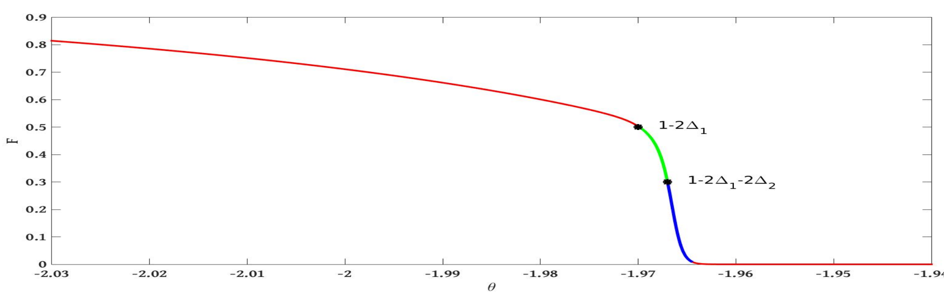


Figure 5:* *Schematic illustration of the travelling wave for two relaxation modes with $\Delta_1+\Delta_2 < \frac{1}{2}$, showing the three regions of the partly dispersed shock. First outer region (red) with transition from 1 to $1-2\Delta_1$. Second region (green) is a transition from $1-2\Delta_1$ to $1-2\Delta_1-2\Delta_2$ of width τ_2 and inner thermoviscous sub-shock (blue).

CONCLUSION

We introduce a non-dimensional model equation governing the thermoviscous non-linear propagation through a medium with $1, 2$ relaxation modes. A detailed asymptotic analysis is presented for a travelling wave controlled by a single relaxation mode, validated by numerical solutions. These asymptotic results can also be applied to the shock structure of a propagating pulse and these are validated by numerical results obtained using pseudospectral methods. Finally the asymptotic structure of shocks with two relaxation modes is discussed. to solve numerically the non-linear problem with outcomes compared.

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