Calculus as a discursive bridge for Algebra, Geometry and Analysis: The case of tangent line

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Introduction

The tangent line to a curve is one of the mathematical topics that appear in different domains of mathematics very often with different uses. For example, we meet the tangent line in Geometry (e.g., tangent to a circle); in Algebra (e.g., tangent to parabola and other Cartesian curves); in Calculus (e.g., tangent to a function graph at a point where the function is differentiable) or in Analysis (e.g., tangent to a function graph as a line with the limit of the difference quotient as its slope). Research reports students' challenges with the tangent line to a function graph (e.g., Biza & Zachariades, 2010; Vinner, 1991). These challenges have been attributed, inter alia, to students' experiences with tangents in different domains of mathematics – for example, the tangent to a circle influences how students deal with tangents to function graphs.

In this short paper, I use the case of the tangent line to investigate the origins of students' difficulties when they learn topics in different mathematical domains including Calculus. To this aim, I draw on the commognitive framework proposed by Sfard (2008) that sees mathematics as a discourse and learning of mathematics as a communication act within this discourse. I start from the viewpoint that different mathematical domains endorse different discourses, namely they call for the use of different notation, are governed by different rules, and apply different definitions. Thus, if we see the tangent line as an object established in these different mathematical domains, most likely we speak about tangent line as a different discursive object in each domain. Very often, students are invited to learn and work with tangents while they engage with (and shift between) these discourses without being aware of these underpinning differences.

In my previous research on students' perspectives about tangent line (e.g., Biza & Zachariades, 2010), correct/incorrect characterisation of students' responses to tasks involving tangent lines led to a classification in groups with different perspectives (*analytical local*, *geometrical global* and *intermediate* between geometrical and analytical). In this paper, I return to the data from the same group of first year university mathematics students (Biza & Zachariades, 2010) in order to examine not only the correctness of the students' responses but also how they justify their choices and how different discourses are present in these justifications, regardless of their correctness or not – see a preliminary analysis in Biza (2017). I see Calculus as a crossroads between Geometry, Algebra and Analysis and argue that the lack of awareness of the differences in the transitions across these domains and their underpinning discourses can explain students' challenges with tangent line that research has reported repeatedly. I conclude the paper by highlighting potentialities of this analysis for teaching mathematical topics that are present in different mathematical domains, including Calculus, towards bridging the different mathematical discourses of these domains.

Tangent line from the commognitive perspective: An object in different mathematical discourses

According to the commognitive framework (Sfard, 2008), mathematics is seen as a discourse which is established within a certain community. Mathematical discourse includes objects (e.g. the tangent line) and "discourses-about discourse" (p. 161), which are meta- rules about the use of these objects (e.g. what makes a line a tangent line). A mathematical discourse is defined by four characteristics: *word use*, *visual mediators*, *narratives* and *routines*. Word use includes the use of mathematical terms (e.g., tangent, derivative or direction coefficient) as well as everyday words with a specific meaning within mathematics (such as touch, region or point). Visual mediators are "visible object that are operated upon as part of the process of communication" (p. 133) and include mediators of mathematical meaning (e.g., function graphs, geometrical figures or symbols) as well as physical objects. Narratives include texts, written or spoken, which describe objects and processes as well as relationships among those (e.g., definitions, theorems or proofs), and are subject to endorsement, modification or rejection according to rules defined by a community (e.g., 'a tangent line is a line that has one common point with a curve' is an endorsed narrative for tangents in Euclidean Geometry but not in Analysis). Routines include regularly employed and well-defined practices that are used in distinct, characteristic ways by a community (such as defining, conjecturing, proving, estimating, generalising and abstracting). For example, identifying a tangent to a circle at a point A in Geometry means drawing a line, which is vertical to the radius at this point, whereas in Algebra this identification involves using the formula of the circle in the Cartesian plane and calculating the tangent line equation. In the commognitive frame, learning is seen as the development of discourse either at object-level (e.g., expansion of an existing discourse with new words and routines) or at meta-level, (e.g., changes in meta-rules). Very often, the teacher moves fluently between different narratives without communicating these differences explicitly (e.g., Park, 2015). Students, unaware of these differences, may be reluctant to change routines that worked well for them for new ones without seeing a reason for doing so. This reason is less transparent when teaching emphasises the *how* in the mathematical discourse, by mostly focusing on practical actions resulting in changing objects (e.g. how we calculate the formula of a tangent line), and with less attention on the *when* an existing or a new routine should be used. However, the *when* is exactly the aspect of an object and associated routines that can expand these routines in new ones or change them.

Methodology and Context

Data reported in this paper were collected with a questionnaire administered to 182 first year university students (97 female) from two Greek mathematics departments. All participants had been taught about the tangent line in Euclidean Geometry (from Year 7), in Algebra (from Year 10) and in a Calculus with elements of Analysis course (in Year 12), but not yet at university as the study took place at the beginning of their first year. The questionnaire, inspired by previous works such as Vinner (1991), consisted of eight tasks in which the students were asked to: explain the tangent line in their own words (Q1); describe its properties (Q2); identify if a drawn line is a tangent line of a given curve (Q3); construct the tangent line, if it exists, of a given curve through a specific point (Q4 and Q5); provide the definition (Q6), write the formula (Q7), and apply the formula on specific functions (Q8) (Biza & Zachariades, 2008).

Students' definition of the tangent line to the function graph

The analysis of students' justifications identified engagement with the different domains/discourses they had met tangents in: *Geometry*; *Algebra*; *Calculus*; and, *Analysis* as well as *Geometry-Local*, a *hybrid* discourse that endorses geometry narratives together with local meta-rules (e.g., "the line has one common point with the curve in a region of the tangency point"). Although this *hybridisation* was not in the curriculum, it did appear in student responses. Table 1, summarises these discourses with response examples. I note that students often engaged with more than one discourses in the same or across questions. For example: "$A(x_0,f(x_0))$ $y-y_0=\lambda(x-x_0)$ $f'(x_0)=\lambda$ direction coefficient. It is a unique line with only one common point near to A" can be seen as both *Calculus* and *Geometry-Local.*

Discourse	Example (data have been translated from Greek)
Geometry	"No [it is not a tangent], the line has 2 points in common with the function graph"
Algebra	"Yes, the line ε is tangent at A, the slope equals to the direction coefficient [the coefficient m in $y=mx+b$, that indicates the slope of a line] of the line"
Calculus	"A function which is differentiable at a point $A(x_0,f(x_0))$ has a tangent at this point […] its formula is $y-f(x_0)=f'(x_0)(x-x_0)$"
Analysis	"The line has formula ε: $y=\lambda x+\beta$ at the point $A(x_0, y_0)$ the point A satisfies this formula [sic] and $\lim\limits_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=\lambda$"
Geometry-Local	"[It is a tangent, b]ecause if we consider a small region (κ, γ) around the point A where [the line] ε is tangent we can see that [the line] ε does not touch any other point"

Table 1: Discourses identified in student responses with examples.

For some students, working across discourses compromised the correctness of their responses. Other students, navigated across and within discourses with success. For example, S[149], a student who performed well in the questionnaire, writes in Q1: "[The tangent] is [the line] that has 'two' [his emphasis] common points with C_f the distance of which is infinitely small and thus we consider that it [C_f] has a double point". Then, in Q2, he adds: "$f'(x_A)=\lambda$ the direction coefficient. $f'(x_A)=\lim\limits_{x\to x_0}\frac{f(x)-f(x_A)}{x-x_A}$. At this point it [the line] has one 'double' [his emphasis] common point with C_f. It can have other common points with C_f, $x\neq x_A$" [Figure 1]. In Q3.6, (inflection point) he writes: "The [line] ε is [tangent] because f is differentiable at x_A and ε has one (double) common point with C_f in the region $(x_A-\kappa, x_A+\kappa)$, $\kappa>0$ and very small". In Q3.7 (corner point where the function is not differentiable), he rejects the line because "C_f has two tangent semi-lines at A which, however, do not have the same slope". We see in S[149]'s responses a mixture of *words* from Geometry (common points), Algebra (double point), Calculus (derivative) and Analysis (limits). A recurring endorsed narrative in his responses is the analytical definition of the tangent line through the limit of the difference quotient or/and the secants. All these words and narratives have been subsumed in the Analysis discourse, e.g., the *double point* is not seen as the algebraic solution but as the limiting position of two points that approach each other ("infinitely small"). As a result, the words are still used but this use is different – and is not contradictory. For S[149], drawing on the meta-rule of convergence (in Analysis) has shifted his meaning of common points, double point and derivative as discursive objects.

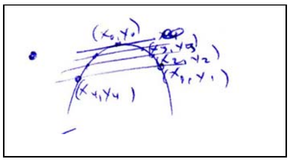

Figure 1: S[149]'s response to Q2

Conclusions with teaching suggestions

The discursive analysis of students' responses indicated engagement with a range of discourses, from Geometry, Algebra, Calculus and Analysis and with a combination of discourses (see Table 1). Through this analysis, student responses that might at first have seemed incoherent (and very often incorrect), were explained, rationalised and demystified when seen in the context of student activities and experiences. Although previous studies on students' cognitive processes have created plausible explanations of students' thinking about tangents, a closer, commognitive look at students' justifications reveals potential origins of the challenges students face in the transition across mathematical domains. These challenges may originate, for example, in: the *applicability of routines* (Sfard, 2008, p. 215: a well-established routine may be evoked even if it is not appropriate) and d*ifferences in discursive objects* (ibid, p.161: discursive objects may keep the same name but may have different uses and different meta-rules in different mathematical discourses).

Mapping out students' discursive activity through a commognitive lens suggests the potency of rethinking how we address students' difficulties, especially for topics students meet in different mathematical domains. First, considering the differences of these discourses is key in demystifying and addressing the challenges students often face. Second, conflicts between discourses is a significant part in students' learning and not a contingency that teachers may ignore or avoid. Third, not seeing mathematics as a homogeneous discourse and raising awareness of different discourses is essential in resolving such conflicts. Fourth, teaching with emphasis on mathematical definitions without discussing the rules on which these definitions are grounded may obstruct students from moving between discourses. Finally, engaging with a substantial range of examples in which a mathematical object is realised – in our case, tangency – is central to raising awareness of the different discourses in which this object is present. Appropriately selected examples act as catalysts between students' and teachers' discourses, can generate and resolve conflicts and offer a platform on which to discuss not only the *how* but also the *when* in mathematics, especially in cases such as Calculus which lies at a crossroads between Geometry, Algebra and Analysis.

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