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The viscous relation for the initial isotropic response of ice

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Abstract. Uni-axial compressive stress and simple shear stress experiments on ice determine the corresponding minimum strain-rates following the very small primary elastic strain or viscoelastic creep for a range of applied constant stresses. Assuming these responses are those of a non-linear incompressible simple viscous fluid, they can be related to the two response functions, each depending on two invariants, of the general viscous law, and more specifically to simpler special cases for which the response functions are determined by the uni-axial and shear responses. In particular, the customary co-axial relation with one response function of one invariant argument can only apply if there is an explicit relation between the uni-axial and shear responses. Given that experimental data shows that this relation is not satisfied, then the uni-axial and shear stress responses determine the two response functions, each depending on only one invariant, of a non-co-axial quadratic relation. Single independent strain-rate component responses can determine dependence on only one invariant. However, there is no data for both these responses, and correlations have been made with data from combined uni-axial stress and shear stress tests carried out at the University of Melbourne. Expressions for the two general response functions are derived in terms of the responses in combined uni-axial stress and shear stress tests, from which it is found that the quadratic coefficient is very significant at the majority of the data points. This implies that a co-axial relation depending on two invariants, suggested by Steinemann (1954) when his data showed that the customary co-axial relation failed, is not valid. It is also seen that the available data points cover little of the two invariants plane, so dependence on two invariants could not be deduced. The data is used to determine the two response functions of the quadratic, non-co-axial, relation, each with dependence on only one invariant, a measure of the shear strain-rate. Note, though, that the Melbourne experiments do not measure the complete strain-rate response, but instead make an assumption that the longitudinal strain-rate is zero which is not confirmed, so the present construction hinges on the validity of that approximation. The theory shows how accurate data can be used to construct a viscous law.

Key words: polar ice, viscous creep, isotropic response, constitutive law.

1 Introduction

Ice-sheet flow plays a significant role in climate change, and flow solutions involve a constitutive law for the stress dependence on ice deformation, deformation rate and temperature. To a very good approximation, ice is incompressible and the pressure is a workless constraint, not given by any constitutive law, but determined by the momentum balances and boundary conditions. On the large time-scales of ice-sheet flow, the shear response to applied stress has an initial viscous (secondary creep) response followed by tertiary creep which depends on the deformation history, described as ice fabric (microstructure) evolution which induces macroscopic anisotropy. This description neglects the very short time-scale elastic or viscoelastic (primary creep) effects. Simple solutions for idealised plane (Staroszczyk and Morland, 2000) and radially symmetric (Morland and Staroszczyk, 2006) flows have shown the significant effects of ice fabric evolution, for which the constitutive law involves the initial isotropic viscous response. Here we focus on that initial viscous response. We also assume the ice is thermorheologically simple, for which the strain-rate at a given stress has a temperature dependent rate factor.

Such a viscous law, necessarily isotropic by material frame indifference, has the general Rivlin-Ericksen quadratic representation, with alternative, but equivalent, stress and strain-rate formulations, as discussed by Morland (1979). However, it is still common practice to ignore the quadratic term and adopt a simple relation proposed by Nye (1953) in which the deviatoric stress is co-axial with the strain-rate, and which depends on only one of the two strain-rate (or deviatoric stress) invariants; specifically the second principal invariant which is a measure of the shear strain-rate or stress magnitude. The first principal strain-rate and deviatoric stress invariants are zero by incompressibility.

Glen (1958) (acknowledging F. Ursell) presented the general quadratic viscous relation for the strain-rate, but adopted the simple co-axial form proposed by Nye (1953) with dependence on one invariant, for which he assumed a power law to correlate with his uni-axial compression experimental data, since known as Glen's Law. He noted that Steinemann (1954) claimed that his compression and shear data were not consistent with a single relation of this simple co-axial form, and that Steinemann suggested retaining the co-axial form with further dependence on a second invariant - the third principal invariant - but this has not been explored. Glen's Law has been used in nearly all ice-sheet flow modelling.

We first determine the explicit relation between uni-axial and shear responses which make them both consistent with the simple form, and given that this is not satisfied by experimental data we show how they determine the two response functions depending on a single invariant in the non-co-axial quadratic form. Clearly, experimental responses which involve only one independent strain-rate component cannot determine dependence on two strain-rate invariants. It also follows that a relation between the octahedral deviatoric stress and octahedral strain-rate is only possible if the simple conventional co-axial relation applies, which experimental data has rejected.

Combined compression and shear test data involves two independent invariants. Morland (2007) analysed combined compression and shear tests for two approximations to the strain-rate configuration, and specifically that adopted for the many experiments carried out at the University of Melbourne; see, for example, Li and Jacka (1996), Warner et al. (1999), Treverrow et al. (2012), Budd et al. (2013). In particular, a universal relation is derived which is independent of the form of the response functions, and determines the longitudinal constraint stress in terms of measured variables.

Here the values of the two general response functions are determined at 21 points of the combined stress data, though data inconsistencies imply that only 15 points should be considered. That is, the response functions are determined at only 21 or only 15 points in the plane of the two invariants, see Figure 1, too sparse to define two response functions with two invariants as arguments. However, the data does determine the two response functions as functions of a single invariant argument, and specifically shows that the quadratic term is very significant at 11 of the 15 points. These points are incompatible with the Steinemann co-axial conjecture with dependence on two invariants. The above correlated response functions determine a constitutive law which in turn gives the uni-axial and shear responses. Note, though, that the Melbourne experiments do not measure the complete strain-rate response, but instead make an assumption that the longitudinal strain-rate is zero which is not confirmed, so the present construction hinges on the validity of that approximation. The theory shows how accurate data can be used to construct a viscous law.

2 The viscous relation

Ice response to stress exhibits a strong dependence on temperature T. This is assumed to be described by applying a rate factor $a(T)$ to the strain-rate, where $a(T)$ is a rapidly increasing function of T; that is, the actual strain-rate at a given stress and temperature increases rapidly with temperature. As noted by Morland (1979), this is the assumption of a thermorheologically simple response, Schwarzl and Staverman (1952) and Morland and Lee (1960), in which the same processes occur, but on a time-scale factored by $a(T)$. Smith and Morland (1981) constructed exponential representations for the rate factor $a(T)$ over different temperature ranges from the constant Mellor and Testa (1969) uni-axial stress data, and that with the widest validity is

$$a(T) = 0.7242\exp(11.9567\bar{T}) + 0.3438\exp(2.9494\bar{T}), \tag{2.1}$$

$$T = T_0 + 20^\circ\ \mathrm{K}\bar{T}, \quad T_0 = 273.15^\circ\ \mathrm{K}, \tag{2.2}$$

where T_0 is the melting point and $a(T_0) = 1.068$ is approximately unity, normalising the factor at the melt point. This ignores the dependence of the melt point on pressure which is likely to be smaller than errors in the rate factor (2.1). At 2° K below melting $a(271.15) = 0.4751$, less than half that at the melt point, and at 30° K below the melt point, a temperature magnitude found in cold ice-sheets, $a(243.15) = 0.0041$, implying much smaller strain-rates than those near melting. [2] The Melbourne data we use is based on experiments at 2° K below melting.

From the outset we use normalised dimensionless deviatoric stress $\hat{\boldsymbol{\sigma}}$ and strain-rate $\boldsymbol{D}$, the symmetric part of the velocity gradient tensor, based on a stress unit $\sigma_0 = 10^5$ Pa and strain-rate unit $D_0 = \mathrm{a}^{-1} = 3.18\times10^{-8}\ \mathrm{s}^{-1}$, where 'a' denotes year, which are the deviatoric stress and strain-rate magnitudes expected, at melt point, in ice-sheet flow. The units σ_0 and D_0 are those used by Morland and Johnson (1980), Morland (1984) and Smith and Morland (1981) to obtain a normalised dimensionless viscous relation at the melt temperature in the conventional co-axial form. We further introduce a temperature normalised strain-rate $\bar{\boldsymbol{D}}$ and principal invariants $\bar{I}_1$, $\bar{I}_2$, $\bar{I}_3$ defined by

$$\boldsymbol{D} = a(T)\ \bar{\boldsymbol{D}}, \quad I_1 = \mathrm{trace}(\boldsymbol{D}) = a(T)\bar{I}_1 = 0, \tag{2.3}$$

[2] We appreciate the advice given to us by Dr David Cole who has examined many data sets for temperature variation and judges the Mellor and Testa (1969) data to show a consistent variation with temperature, though with higher strain-rates. The above $a(T)$ variation is therefore consistent, here normalised at the melt temperature T_0.

$$I_2 = \mathrm{trace}(\boldsymbol{D}^2)/2 = [a(T)]^2 \bar{I}_2, \quad I_3 = \det(\boldsymbol{D}) = [a(T)]^3 \bar{I}_3, \tag{2.4}$$

where the vanishing of I_1 is the incompressibility condition, and $\bar{I}_2$ is a measure of the strain-rate magnitude squared. Note that this definition of I_2, used for convenience, has the opposite sign to the second principal invariant. Now let $\boldsymbol{\sigma}$ denote the Cauchy stress tensor, then

$$\hat{\boldsymbol{\sigma}} = \boldsymbol{\sigma} + p\boldsymbol{I}, \quad p = -\frac{1}{3}\mathrm{trace}(\boldsymbol{\sigma}), \quad \mathrm{trace}(\hat{\boldsymbol{\sigma}}) = 0, \tag{2.5}$$

where p is the mean pressure and $\boldsymbol{I}$ is the unit tensor. The principal invariants of $\hat{\boldsymbol{\sigma}}$ are defined by, omitting the minus sign in the second invariant for convenience,

$$J_1 = \mathrm{trace}(\hat{\boldsymbol{\sigma}}) = 0, \quad J_2 = \mathrm{trace}(\hat{\boldsymbol{\sigma}}^2)/2, \quad J_3 = \det(\hat{\boldsymbol{\sigma}}). \tag{2.6}$$

Now the most general thermorheologically simple frame-indifferent viscous law is a relation between $\hat{\boldsymbol{\sigma}}$ and $\bar{\boldsymbol{D}}$ which can be expressed in two alternative, but equivalent, forms of the Rivlin-Ericksen representation between tensors with zero trace :

$$\hat{\boldsymbol{\sigma}} = \phi_1(\bar{I}_2, \bar{I}_3)\bar{\boldsymbol{D}} + \phi_2(\bar{I}_2, \bar{I}_3)\left[\bar{\boldsymbol{D}}^2 - \frac{2}{3}\bar{I}_2\boldsymbol{I}\right], \tag{2.7}$$

$$\bar{\boldsymbol{D}} = \psi_1(J_2, J_3)\hat{\boldsymbol{\sigma}} + \psi_2(J_2, J_3)\left[\hat{\boldsymbol{\sigma}}^2 - \frac{2}{3}J_2\boldsymbol{I}\right]. \tag{2.8}$$

The vanishing of J_1 is by the definition of the deviator, so the response functions ϕ_1, ϕ_2, ψ_1, ψ_2 each depend on only two non-trivial invariants. $\bar{I}_2$ and J_2 are measures of shear strain-rate and shear stress magnitudes squared, while $\bar{I}_3$ and J_3 have no physical description. While the expansions (2.7) and (2.8) are equivalent, there is no explicit algebraic inversion. Note that $\phi_2 = 0$ implies, and is implied by, $\psi_2 = 0$.

These expansions for a simple viscous fluid are necessarily isotropic in all reference configurations, and cannot describe induced anisotropy associated with the fabric developed as the ice elements deform and crystal glide planes are re-oriented. However, this viscous response describes the initial isotropic response as ice is first formed at an ice-sheet surface, and is a crucial part of the constitutive behaviour as the ice deforms and induced anisotropy develops. It is the stress formulation (2.7) which is the constitutive law required for substitution in the momentum and energy balances of a general ice-sheet flow, so this form will be analysed here, There is an analogous analysis for the strain-rate formulation, which is required in the *reduced model*, Morland and Johnson (1980), Morland (1984), equivalent to the shallow ice approximation *SIA* formulated by Hutter (1983). Both formulations require longitudinal gradients to be very small, so are applicable only when the bed undulations are very small which is not a realistic situation in reality. The *reduced model*, *SIA*, is useful for determining solutions of idealised flows against which solutions from the more complex numerical schemes needed for the full equations can be compared for accuracy (validity !). Sadly such comparisons are not made.

The pioneering experimental work of Glen (1952, 1953, 1955, 1958) on polycrystalline (isotropic) ice measured the minimum strain-rate response to a range of applied constant uni-axial compressive stresses at different constant temperatures. This can determine only one response function of one argument, and was used to construct power laws for the simplified co-axial form of (2.8) proposed by Nye (1953), with the equivalent simplified form of (2.7), namely

$$\bar{\boldsymbol{D}} = a^{-1}(T)\ D = \psi_{c1}(J_2)\hat{\boldsymbol{\sigma}}, \quad \hat{\boldsymbol{\sigma}} = \phi_{c1}(\bar{I}_2)\bar{\boldsymbol{D}} = \phi_{c1}(\bar{I}_2)a^{-1}(T)\boldsymbol{D}. \tag{2.9}$$

In this simple form $\bar{\boldsymbol{D}}$ is co-axial with $\hat{\boldsymbol{\sigma}}$, and there is only one response function ψ_{c1} depending on only one invariant J_2, a measure of the shear stress magnitude squared, or one response function ϕ_{c1} depending on one invariant $\bar{I}_2$, a measure of the strain-rate magnitude squared. Comparisons of known data sets in 1980 were made by Smith and Morland (1981), which demonstrated wide differences. The form $(2.9)_1$ was correlated with the Glen (1955) data, showing that a three-term fifth-order polynomial representation, with finite viscosity at zero stress, was a much closer correlation than Glen's power law with infinite viscosity at zero stress. Lliboutry (1969) and Colbeck and Evans (1973) had also constructed similar polynomial representations from different data, but without error estimates. Note that in this co-axial form the rate factor $a(T)$ simply appears as a multiplying factor of $\hat{\boldsymbol{\sigma}}$, which is not the case for the general thermorheologically simple forms (2.7) and (2.8).

Simple shear stress tests also determine one response function of one argument, and Steinemann's (1954) shear tests indicated this was different to that obtained from the uni-axial compression tests. Given that the simple form (2.9) cannot satisfy both uni-axial and shear data, the most simple generalisations are

$$\hat{\boldsymbol{\sigma}} = \phi_{s1}(\bar{I}_2, \bar{I}_3)\ \bar{\boldsymbol{D}}, \quad \hat{\boldsymbol{\sigma}} = \phi_{q1}(\bar{I}_2)\bar{\boldsymbol{D}} + \phi_{q2}(\bar{I}_2)\left[\bar{\boldsymbol{D}}^2 - \frac{2}{3}\bar{I}_2\boldsymbol{I}\right], \tag{2.10}$$

with the first, proposed by Steinemann (1954), retaining the co-axial form but using dependence on two invariants, and the second losing the co-axial form but using dependence on only one invariant. Morland (2007) analysed the *Reduced Model*, *SIA* scaling with both forms to show that a leading order (in surface slope or dimensionless viscosity magnitude) simplification still follows. While a response function $\phi_{s1}(\bar{I}_2, \bar{I}_3)$ cannot be determined by only the uni-axial and shear data for which the two invariants $\bar{I}_2$ and $\bar{I}_3$ are not independent, the two response functions of one argument in the form $(2.10)_2$ can be determined by this data.

The relations $(2.9)_2$, $(2.10)_1$ and $(2.10)_2$ give the respective invariant relations

$$J_2 = \phi_{c1}^2(\bar{I}_2)\bar{I}_2, \quad J_2 = \phi_{s1}^2(\bar{I}_2, \bar{I}_3)\bar{I}_2, \tag{2.11}$$

$$J_2 = \phi_{q1}^2(\bar{I}_2)\bar{I}_2 + \phi_{q1}(\bar{I}_2)\phi_{q2}(\bar{I}_2)\mathrm{trace}(\bar{\boldsymbol{D}}^3) + \phi_{q2}^2(\bar{I}_2)[\frac{1}{2}\mathrm{trace}(\bar{\boldsymbol{D}}^4) - \frac{2}{3}\bar{I}_2^2], \tag{2.12}$$

where $\mathrm{trace}(\bar{\boldsymbol{D}}^3)$ and $\mathrm{trace}(\bar{\boldsymbol{D}}^4)$ can be expressed in terms of $\bar{I}_2$ and $\bar{I}_3$ by the Cayley-Hamilton theorem, noting that here $\bar{I}_2$ is the negative of the second principal invariant. These invariant relations are not required for the construction of a constitutive relation, but note that there is explicit dependence on $\bar{I}_3$ in $(2.11)_2$, and also in (2.12), so that neither the co-axial relation $(2.10)_1$ nor the simplified quadratic relation $(2.10)_2$, allows a $J_2 - \bar{I}_2$ relation.

Inferences from the Melbourne experiments were mainly expressed in terms of octahedral deviatoric stress τ_0 and strain-rate $\dot{\varepsilon}_0$ given by

$$\tau_0^2 = 2J_2/3, \quad \dot{\varepsilon}_0^2 = 2I_2/3, \tag{2.13}$$

so an octahedral relation is a relation between J_2 and I_2, equivalently $\bar{I}_2$, at a given temperature, which can occur only if the viscous relation is of the most simple form $(2.9)_2$. It was noted above that there is no such relation with the more general forms $(2.10)_1$ and $(2.10)_2$. That is, if $(2.9)_2$ does not apply, then there is no octahedral relation between only the second invariants of deviatoric stress and strain-rate, and such a relation cannot represent a material property. Further, an octahedral invariant relation only determines a combination of stress components, not the actual stress tensor, so is not a constitutive relation.

3 Uni-axial and shear relations

The stress and deviatoric stress tensors for a uni-axial compressive stress σ in the vertical direction are

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sigma \end{pmatrix}, \quad p = \frac{1}{3}\sigma, \quad \hat{\boldsymbol{\sigma}} = \frac{1}{3}\begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & -2\sigma \end{pmatrix}. \tag{3.1}$$

This gives rise to an axial vertical compressive strain-rate $\dot{\varepsilon}$ and equal horizontal strain-rates $\frac{1}{2}\dot{\varepsilon}$ at the melt temperature, from which the strain-rate tensor, strain-rate squared tensor and invariants are

$$\bar{\boldsymbol{D}} = \frac{1}{2}\begin{pmatrix} \dot{\varepsilon} & 0 & 0 \\ 0 & \dot{\varepsilon} & 0 \\ 0 & 0 & -2\dot{\varepsilon} \end{pmatrix}, \quad \bar{\boldsymbol{D}}^2 = \frac{1}{4}\begin{pmatrix} \dot{\varepsilon}^2 & 0 & 0 \\ 0 & \dot{\varepsilon}^2 & 0 \\ 0 & 0 & 4\dot{\varepsilon}^2 \end{pmatrix}, \tag{3.2}$$

$$\bar{I}_2 = \frac{3}{4}\dot{\varepsilon}^2, \quad \bar{I}_3 = -\frac{1}{4}\dot{\varepsilon}^3 = -2(\bar{I}_2/3)^{\frac{3}{2}}, \tag{3.3}$$

so $\bar{I}_3$ and $\bar{I}_2$ are not independent.

Given a uni-axial response $\sigma = U(\dot{\varepsilon})$ at melt temperature, determined by data, and expressing $\dot{\varepsilon}$ in terms of $\bar{I}_2$ by (3.3), the co-axial constitutive relation $(2.9)_2$ gives

$$U\left(2[\frac{\bar{I}_2}{3}]^{\frac{1}{2}}\right) = [3\bar{I}_2]^{\frac{1}{2}}\phi_{c1}(\bar{I}_2), \quad \dot{\varepsilon} = 2[\frac{\bar{I}_2}{3}]^{\frac{1}{2}}, \tag{3.4}$$

which determines $\phi_{c1}(\bar{I}_2)$ in terms of $U(\dot{\varepsilon})$. However, the quadratic constitutive relation $(2.10)_2$ gives

$$U\left(2[\frac{\bar{I}_2}{3}]^{\frac{1}{2}}\right) = [3\bar{I}_2]^{\frac{1}{2}}\phi_{q1}(\bar{I}_2) - \bar{I}_2\phi_{q2}(\bar{I}_2), \tag{3.5}$$

which relates a combination of $\phi_{q1}(\bar{I}_2)$ and $\phi_{q2}(\bar{I}_2)$ to $U(\dot{\varepsilon})$.

The set-up for the Melbourne shear experiments supposes (approximately) zero longitudinal strain-rate $D_{xx} = 0$ in comparison with the shear strain-rate, and we suppose further that approximately $D_{zz} = 0$ and hence $D_{yy} = 0$ by incompressibility. The applied shear stress $\sigma_{xz} = \tau$ gives rise to a corresponding shear strain-rate $\dot{\gamma}$ at melt, but there can additionally be axial constraint stresses which depend on the constitutive law. Thus, the stress and deviatoric stress tensors have the forms

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{xx} & 0 & \tau \\ 0 & \sigma_{yy} & 0 \\ \tau & 0 & -\sigma \end{pmatrix}, \quad p = -\frac{1}{3}(\sigma_{xx} + \sigma_{yy} - \sigma), \tag{3.6}$$

$$\hat{\boldsymbol{\sigma}} = \frac{1}{3}\begin{pmatrix} 2\sigma_{xx} - \sigma_{yy} + \sigma & 0 & 3\tau \\ 0 & 2\sigma_{yy} - \sigma_{xx} + \sigma & 0 \\ 3\tau & 0 & -2\sigma - \sigma_{xx} - \sigma_{yy} \end{pmatrix}, \tag{3.7}$$

and the strain-rate tensor, strain-rate squared tensor and invariants are

$$\bar{\boldsymbol{D}} = \begin{pmatrix} 0 & 0 & \dot{\bar{\gamma}} \\ 0 & 0 & 0 \\ \dot{\bar{\gamma}} & 0 & 0 \end{pmatrix}, \quad \bar{\boldsymbol{D}}^2 = \begin{pmatrix} \dot{\bar{\gamma}}^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \dot{\bar{\gamma}}^2 \end{pmatrix}, \tag{3.8}$$

$$\bar{I}_2 = \dot{\bar{\gamma}}^2, \quad \bar{I}_3 = 0, \tag{3.9}$$

so no dependence on $\bar{I}_3$ could be inferred. Now given a shear response at melt $\tau = S(\dot{\bar{\gamma}})$, the co-axial constitutive relation $(2.9)_2$ gives

$$S\left(\bar{I}_2^{\frac{1}{2}}\right) = \bar{I}_2^{\frac{1}{2}}\phi_{c1}(\bar{I}_2), \quad \dot{\bar{\gamma}} = \bar{I}^{\frac{1}{2}}, \tag{3.10}$$

which determines $\phi_{c1}(\bar{I}_2)$ in terms of the shear response. Comparing with (3.4) shows that the simple co-axial relation $(2.9)_2$ can only apply if

$$3^{\frac{1}{2}} S(\theta) = U(2\theta / 3^{\frac{1}{2}}), \quad 3^{\frac{1}{2}} S(3^{\frac{1}{2}}\theta / 2) = U(\theta), \tag{3.11}$$

for any positive argument θ of $S(\theta)$ and $U(\theta)$. In this case the axial constraint stresses satisfy

$$\sigma_{xx} = \sigma_{yy} = -\sigma, \tag{3.12}$$

and for determination require a further assumption; for example $\sigma_{yy} = 0$.

Accepting Steinemann's (1954) assertion that (3.11) does not apply, which also follows from the later data correlations, then, analogous to (3.5), turning to the simple quadratic generalisation $(2.10)_2$ gives

$$\sigma_{xx} = -\sigma, \quad \sigma_{yy} = -\sigma - \bar{I}_2\phi_{q2}(\bar{I}_2), \quad S(\bar{I}_2^{\frac{1}{2}}) = \bar{I}_2^{\frac{1}{2}}\phi_{q1}(\bar{I}_2), \tag{3.13}$$

which determines $\phi_{q1}(\bar{I}_2)$, the same expression as (3.10), directly in terms of the applied $S(\dot{\bar{\gamma}})$. This can now be combined with the uni-axial relation (3.5) to determine $\phi_{q2}(\bar{I}_2)$:

$$\bar{I}_2\phi_{q2}(\bar{I}_2) = 3^{\frac{1}{2}} S(\bar{I}_2^{\frac{1}{2}}) - U\left(2[\frac{\bar{I}_2}{3}]^{\frac{1}{2}}\right), \tag{3.14}$$

which is zero, if and only if, (3.11) holds. That is, given $S(\dot{\gamma})$ determines $\phi_{q1}(\bar{I}_2)$ directly, then given $U(\dot{\varepsilon})$ determines $\phi_{q2}(\bar{I}_2)$.[3] Note that monotonic increasing $S(\dot{\gamma})$ implies, and is implied by, monotonic increasing $\phi_{q1}(\bar{I}_2)$, but monotonic increasing $U(\dot{\varepsilon})$ does not imply monotonic $\phi_{q2}(\bar{I}_2)$; increasing or decreasing, and vice-versa. The above analysis shows that uni-axial and simple shear data directly determine the response functions in a viscous law of the form $(2.10)_2$.

4 Combined compression and shear responses

While the uni-axial compression and simple shear data determine both response functions in the simplified quadratic form $(2.10)_2$, they do not determine the response function in the co-axial form $(2.10)_1$, which depends on two invariants. We will now analyse the combined compression and shear relations which describe the Melbourne experiments. It is instructive to start from a more general deformation configuration, not asserting zero longitudinal strain-rate, to note the implications of this approximation:

$$x_1 = \lambda_1 X_1 + \kappa X_3, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3, \quad \lambda_1\lambda_2\lambda_3 = 1, \tag{4.1}$$

where (X_1, X_2, X_3) and (x_1, x_2, x_3) are respectively dimensionless material and spatial rectangular Cartesian co-ordinates with unit $1\,\mathrm{m}$. $\lambda_1, \lambda_2, \lambda_3$ are the stretches along the axes, with λ_1 not necessarily unity, and κ is the tangent of the shear angle in material co-ordinates. The corresponding dimensionless velocity components in spatial co-ordinates, with unit ma^{-1}, are

$$v_1 = x_1\dot{\lambda}_1/\lambda_1 + x_3(\dot{\kappa}\lambda_1 - \kappa\dot{\lambda}_1)/(\lambda_1\lambda_3), v_2 = x_2\dot{\lambda}_2/\lambda_2, v_3 = x_3\dot{\lambda}_3/\lambda_3, \tag{4.2}$$

giving the strain-rate $\bar{\boldsymbol{D}}$ and its square

$$\bar{\boldsymbol{D}} = \begin{pmatrix} \bar{D}_{11} & 0 & \dot{\gamma} \\ 0 & \bar{D}_{22} & 0 \\ \dot{\gamma} & 0 & -\dot{\varepsilon} \end{pmatrix}, \quad \bar{\boldsymbol{D}}^2 = \begin{pmatrix} \bar{D}_{11}^2 + \dot{\gamma}^2 & 0 & -\bar{D}_{22}\dot{\gamma} \\ 0 & \bar{D}_{22}^2 & 0 \\ -\bar{D}_{22}\dot{\gamma} & 0 & \dot{\varepsilon}^2 + \dot{\gamma}^2 \end{pmatrix}, \tag{4.3}$$

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[3] The published Melbourne data gives only two points for each of these responses, so cannot determine the response functions, and a wider search for both shear and uni-axial stress data has been unsuccessful.

where

$$\bar{D}_{11} = \frac{\dot{\lambda}_1}{\lambda_1}, \bar{D}_{22} = \frac{\dot{\lambda}_2}{\lambda_2}, \bar{D}_{33} = \frac{\dot{\lambda}_3}{\lambda_3} = -\dot{\bar{\varepsilon}}, \bar{D}_{13} = \bar{D}_{31} = \dot{\bar{\gamma}} = (\dot{\kappa} - \kappa\bar{D}_{11}) / (2\lambda_3), \tag{4.4}$$

with two non-trivial independent invariants, and one trivial:

$$\bar{I}_1 = \bar{D}_{11} + \bar{D}_{22} - \dot{\bar{\varepsilon}} = 0, \quad \bar{I}_2 = \dot{\bar{\gamma}}^2 + \tfrac{1}{2}(\bar{D}_{11}^2 + \bar{D}_{22}^2 + \dot{\bar{\varepsilon}}^2), \quad \bar{I}_3 = -\bar{D}_{22}(\bar{D}_{11}\dot{\bar{\varepsilon}} + \dot{\bar{\gamma}}^2). \tag{4.5}$$

Note that the shear strain-rate $\dot{\bar{\gamma}}$ is not simply given by the rate of angle change $\dot{\kappa}$ unless the approximation $\bar{D}_{11} = 0$ applies. The stress and deviatoric stress tensors have the forms given in (3.6) and (3.7).

The Melbourne longitudinally confined compression experiments assumption $\bar{D}_{11} = 0$ makes the simplifications

$$\bar{D}_{11} = 0 \to \bar{D}_{22} = \dot{\bar{\varepsilon}}, \quad \dot{\bar{\gamma}} = \dot{\kappa} / (2\lambda_3), \quad \bar{I}_2 = \dot{\bar{\gamma}}^2 + \dot{\bar{\varepsilon}}^2, \quad \bar{I}_3 = -\dot{\bar{\varepsilon}}\dot{\bar{\gamma}}^2, \tag{4.6}$$

and the first requirement $\bar{D}_{22} = \dot{\bar{\varepsilon}}$ can be confirmed, or otherwise, provided the lateral deformation is measured; this is an important property of the response which can justify, or not, the main approximation.[4] Note that the approximation $\bar{D}_{11} = 0$ and resulting $\bar{I}_2$ is different from the uni-axial relations (3.2) and (3.3), and the uni-axial data points in lines 5 and 7 in Table 1 (see the next section) are analysed here with this approximation. This configuration was analysed by Morland (2007) for the general law (2.7) to determine a universal relation between the response variables which is independent of the two viscous response functions, and which determines the longitudinal constraint stress in terms of measured response, and further to express both viscous response functions in terms of the response.

The general relation (2.7) with arbitrary response functions $\phi_1(\bar{I}_2, \bar{I}_3)$, $\phi_2(\bar{I}_2, \bar{I}_3)$ applied to the above stress and strain-rate configurations with the Melbourne approximations (4.6) now gives three independent relations

$$\sigma_{xx} = -\sigma + \phi_1\dot{\bar{\varepsilon}} - \phi_2\dot{\bar{\varepsilon}}^2, \quad \sigma_{yy} = -\sigma + 2\phi_1\dot{\bar{\varepsilon}} - \phi_2\dot{\bar{\gamma}}^2, \tag{4.7}$$

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[4] The apparatus used by Budd et al. does not allow for the direct measurement of $\bar{D}_{11}$ and $\bar{D}_{22}$. A significantly more complex experimental system would be required to enable these measurements. However, if the approximation is not good, a more general analysis allowing non-zero $\bar{D}_{11}$ would require either $\bar{D}_{11}$ or $\bar{D}_{22}$ to be measured to enable a correlation.

$$\tau = \phi_1 \dot{\bar{\gamma}} - \phi_2 \dot{\bar{\gamma}} \dot{\bar{\varepsilon}}, \tag{4.8}$$

where σ and τ are the measured applied stresses. Since the lateral extension is unconfined, the Melbourne analyses further assume that $\sigma_{yy} = 0$, which was adopted by Morland (2007), when $(4.7)_{1,2}$ give

$$\sigma_{yy} = 0 \rightarrow \sigma = 2\phi_1 \dot{\bar{\varepsilon}} - \phi_2 \dot{\bar{\gamma}}^2, \quad \sigma_{xx} = -\phi_1 \dot{\bar{\varepsilon}} + \phi_2 (\dot{\bar{\gamma}}^2 - \dot{\bar{\varepsilon}}^2), \tag{4.9}$$

and hence by (4.8) and $(4.9)_1$,

$$(\dot{\bar{\gamma}}^2 - 2\dot{\bar{\varepsilon}}^2)\phi_1 = -\dot{\bar{\varepsilon}}\sigma + \dot{\bar{\gamma}}\tau, \quad (\dot{\bar{\gamma}}^3 - 2\dot{\bar{\gamma}}\dot{\bar{\varepsilon}}^2)\phi_2 = -\dot{\bar{\gamma}}\sigma + 2\dot{\bar{\varepsilon}}\tau. \tag{4.10}$$

(4.10) are relations for the two response functions at melt strain-rates $\dot{\bar{\varepsilon}}$ and $\dot{\bar{\gamma}}$, and so at the corresponding invariants $\bar{I}_2$ and $\bar{I}_3$ using the relations (4.6), in terms of the measured applied stresses σ and τ. Eliminating ϕ_1 and ϕ_2 by (4.10) in the σ_{xx} expression $(4.7)_1$ gives a universal relation

$$\dot{\bar{\gamma}}(\sigma_{xx} + \sigma) = \dot{\bar{\varepsilon}}\tau \tag{4.11}$$

independent of the response functions, so (4.11) determines σ_{xx} directly in terms of the measured responses. It should be emphasized that (4.11) is independent of the response functions, and therefore, if σ_{xx} can be measured, provides a check on the assumption of a viscous law of the form $(2.10)_2$. Alternatively, given the response functions and strain-rate components, σ_{xx}, and other stress components, are determined by the constitutive relation (2.9).

An unconfined compression and shear configuration was also introduced in Budd et al. (2013), already analysed by Morland (2007), with inferences from experiments shown graphically, but no data points given explicitly.

5 Data correlations

We now focus on the results of the experiments performed in recent years at the University of Melbourne, where it is expected that laboratory facilities and apparatus would be improvements on those available in earlier years. Specifically, we will examine the data in Table 1 of Budd et al. (2013), which includes uni-axial stress data at 2 compressive stress levels, simple shear data at 2 shear stress levels, and combined compression and shear data at 17 stress level combinations, all at $T = 271.15\ ^\circ K$, $\bar{T} = -0.1$. First we convert to strain-rates relative to melt-temperature, $(2.9)_1$, by applying the scale factor $a^{-1}(271.15) = 2.1049$, then apply the further scaling 3.1536 to change the Melbourne unit 10^{-7} s^{-1} to our $D_0 = \mathrm{a}^{-1}$, a total scaling of 6.638. Their stress unit 10^{-6} Pa = 10 σ_0, so a stress scaling of 10 is required to convert to our units. The converted data and associated invariants, (4.10), and the consequent response functions, (4.5), are shown in our Table 1. Clearly 2 compression data points and 2 shear data points cannot determine the correlations $U(\dot{\varepsilon})$ and $S(\dot{\gamma})$ introduced in Section 3, with the consequent relations (3.4), (3.5), (3.10), (3.12), (3.13), and (3.14) for the response functions, and the criterion (3.11) which allows the conventional single response function of one invariant argument to be valid.

We will now investigate what can be determined from the combined stress experiments data. Our Table 1 shows the test lines re-ordered by increasing strain-rate invariant $\bar{I}_2$, with columns for σ, τ, $\dot{\varepsilon}$, $\dot{\gamma}$, $\bar{I}_2^{\frac{1}{6}}$, $(-\bar{I}_3)^{\frac{1}{9}}$, Φ_1, Φ_2, and $R = \Phi_2/\Phi_1$. Specifically, the two invariants are shown as the more useful magnitudes $\bar{I}_2^{\frac{1}{6}}$ and $(-\bar{I}_3)^{\frac{1}{9}}$, and the response functions as

7	15	line	test	σ	τ	$\dot{\varepsilon}$	$\dot{\gamma}$	$\bar{I}_2^{\frac{1}{6}}$	$(-\bar{I}_3)^{\frac{1}{9}}$	Φ_1	Φ_2	R
		1	13	4.90	0	1.7923	0	1.2147	0	2.4500	?	?
O	X	2	14	4.90	0.61	1.6595	0.7302	1.2194	0.9865	2.8012	1.4057	0.50
		3	8 - 11	0	2.45	0	2.5224	1.3613	0	2.4500	0	0
□	X	4	21	3.92	1.47	3.3190	1.0621	1.5161	1.1580	1.9087	-3.0600	-1.60
□	X	5	15	4.90	2.45	2.5224	9.2932	2.1275	1.8188	1.3611	-4.4954	-3.30
		6	2	1.60	4.80	17.2588	7.3018	2.6562	2.1346	-0.2569	-13.6551	53.16
		7	7	8.50	0	19.9140	0	2.7105	0	4.2500	?	?
O	X	8	4	4.90	4.20	10.6208	17.9226	2.7516	2.4692	5.0620	0.3530	0.07
□	X	9	6	9.00	1.90	24.5606	4.1156	2.9202	1.9544	4.4640	-7.1307	-1.60
		10	1, 12	0	4.90	0	30.5348	3.1256	0	4.9000	0	0
	X	11	5	7.30	3.20	31.1986	2.1905	3.1507	1.7446	3.5551	-42.2374	-11.88
		12	3	3.40	4.60	17.9226	27.2158	3.1941	2.8715	21.3098	28.7313	1.35

□	X	13	24	4.90	4.90	29.8710	19.2502	3.2877	2.8141	1.3079	-9.2052	-7.04
□	X	14	18	9.80	2.45	37.8366	4.5138	3.3651	2.0930	4.8219	-15.9732	-3.31
O	X	15	25	2.45	4.90	12.6122	36.5090	3.3803	2.9479	5.6332	1.3754	0.24
O	X	16	20	9.80	3.67	53.7678	43.1470	4.1004	3.5942	6.4815	0.7919	0.12
□	X	17	19	9.80	1.22	73.0180	3.1199	4.1810	2.0742	4.8828	-23.7177	-4.86
□	X	18	23	7.35	4.90	92.9320	15.2674	4.5497	3.0321	3.3617	-27.2243	-8.10
O	X	19	22	9.80	4.90	73.0180	73.0180	4.6915	4.1797	6.9296	0.0000	0
O	X	20	17	9.80	9.80	146.036	318.624	7.0506	6.2630	10.0702	-1.7042	-0.17
O	X	21	16	19.60	4.90	438.108	192.502	7.8218	6.3265	10.5465	-1.7850	-0.17

Table 1: Stress, strain-rate and response functions' data

$$\Phi_1 = \bar{I}_2^{-\frac{1}{2}}\phi_1(\bar{I}_2, \bar{I}_3), \quad \Phi_2 = \bar{I}_2\phi_2(\bar{I}_2, \bar{I}_3), \tag{5.1}$$

calculated from (4.10) which is applicable to general dependence on $\bar{I}_2$ and $\bar{I}_3$. However, we will be correlating data with the simplified form $(2.10)_2$ with dependence only on $\bar{I}_2$, and expressing the corresponding Φ_{q1} and Φ_{q2} as functions of $\eta = \bar{I}_2^{-\frac{1}{6}}$ by

$$\Phi_{q1}(\bar{I}_2) = \hat{\Phi}_{q1}(\eta), \quad \Phi_{q2}(\bar{I}_2) = \hat{\Phi}_{q2}(\eta), \quad \eta = \bar{I}_2^{-\frac{1}{6}}. \tag{5.2}$$

The constitutive relation $(2.10)_2$ then has the normalised form

$$\hat{\boldsymbol{\sigma}} = \hat{\Phi}_{q1}(\eta)\breve{\bar{\boldsymbol{D}}} + \hat{\Phi}_{q2}(\eta)[\breve{\bar{\boldsymbol{D}}}^2 - 2\boldsymbol{I}/3], \quad \breve{\bar{\boldsymbol{D}}} = \bar{I}_2^{-\frac{1}{2}}\bar{\boldsymbol{D}}, \tag{5.3}$$

and the final column is the ratio

$$R(\bar{I}_2) = \bar{I}_2^{\frac{1}{2}}\phi_{q2}(\bar{I}_2)/\phi_{q1}(\bar{I}_2) = \hat{\Phi}_{q2}(\eta)/\hat{\Phi}_{q1}(\eta), \tag{5.4}$$

which is a measure of the significance of the quadratic term at each strain-rate level. This is an important measure at high strain-rates where both ϕ_{q1} and ϕ_{q2} tend to zero, and this column makes it clear that the calculated ϕ_2 is significant at the majority of the data points. Thus, Steinemann's conjecture that a co-axial relation with dependence on two invariant arguments could reconcile the independent uni-axial and shear stress tests is not supported by this data. Note that the tests extend to larger stresses than those in Glen's experiments (Glen, 1955), where the maximum uni-axial stress was roughly 4×10^5 Pa and the strain-rate maximum was roughly

$18\,\mathrm{a}^{-1}$ at melt temperature, though at the same stress in a cold ice-sheet the rate factor would lower this considerably.

Tests 1 and 12 (line 10 in Table 1) are simple shears at the same $\tau = 4.90\times10^5$ Pa , but with different minimum strain-rates $\dot{\gamma} = 18.6\,\mathrm{a}^{-1}$ and $10.4\,\mathrm{a}^{-1}$ respectively, which we replace by a single data point with the average strain-rate $\dot{\gamma} = 14.5\,\mathrm{a}^{-1}$, then scale by $a^{-1}(T) = 2.1049$ to convert to our normalised strain-rate measure. The four tests 8–11 (line 3) are simple shears at the same $\tau = 2.45\times10^5$ Pa , with strain-rates $\dot{\gamma}$ ranging from 1.135 to $1.261\,\mathrm{a}^{-1}$ which we replace by a single data point with the mid-value strain-rate $\dot{\gamma} = 1.198\,\mathrm{a}^{-1}$, and then scale by 2.1049. Lines 1 and 7 are the two uni-axial compression tests, but note that while $\tau/\dot{\gamma}$ is bounded as $\tau \to 0$ with finite viscosity, $(4.10)_2$ is indeterminate, so the limit ratio for ϕ_2 must be obtained from the correlation of all the non-zero τ points. For each line of this stress and strain-rate data, the invariants $\bar{I}_2$ and $\bar{I}_3$ are calculated by (4.6), and $\hat{\Phi}_{q1}$ and $\hat{\Phi}_{q2}$ are calculated by (4.10) and (5.1), with Φ_2 indeterminate in the uni-axial stress tests, shown by the question marks '?'. Further, a uni-axial stress response is not consistent with the approximation $D_{xx} = 0$ applied to all the other tests, so the data points in lines 1 and 7 are ignored in the correlation. Since $(3.13)_3$ shows that ϕ_1 and the shear response $S(\dot{\bar{\gamma}})$ determine each other, independent of ϕ_2, the simple shear data in lines 3 and 10, implying zero $\hat{\Phi}_{q2}$, are also ignored in the correlation. Additionally, we see that lines 6 and 12 show anomalous values for both $\hat{\Phi}_{q1}$ and $\hat{\Phi}_{q2}$, so are ignored in the correlation. In summary, lines 1, 3, 6, 7, 10, 12 are ignored, leaving 15 data points for the correlations of both $\hat{\Phi}_{q1}$ and $\hat{\Phi}_{q2}$, shown in Table 1 by X in the column headed 15. Further, the 7 lines shown by O in the column headed by 7 are selected as giving a coherent variation of $\hat{\Phi}_{q1}$, and the 7 lines shown by a $\square$ as giving a coherent variation of $\hat{\Phi}_{q2}$. Note these are distinct sets of tests chosen to describe $\hat{\Phi}_{q1}$ and $\hat{\Phi}_{q2}$ respectively. They are also applied, as well as the 15 points, to determine alternative correlations for $\hat{\Phi}_{q1}$ and $\hat{\Phi}_{q2}$.

The conventional Glen's power law for $\psi_{c1}(J_2)$ with exponent n, and equivalent $\phi_{c1}(\bar{I}_2)$ Morland and Johnson, 1980; Smith and Morland, 1981), are

$$\psi_{c1}(J_2) = 1.5k(3J_2)^{\frac{n-1}{2}}, \quad \phi_{c1}(\bar{I}_2) = \frac{2}{3}k^{-(\frac{1}{n})}[\frac{4}{3}\bar{I}_2)]^{-(\frac{n-1}{2n})}, \tag{5.5}$$

with the Glen (1955) data correlation $k = 0.17$ and $n = 3.17$, though the correlation suggested a disjointed power law with different n for different stress ranges, losing the algebraic simplicity. Note that ϕ_{c1} is unbounded at $\bar{I}_2 = 0$, infinite viscosity. Smith and Morland (1981) determined a closer correlation for $\psi_{c1}(J_2)$ over the complete range with a polynomial representation yielding finite viscosity at $J_2 = 0$. Based on a dislocation theory analysis (Cole and Durell, 2001) Dr Cole suggested that the conventional assumption that in uni-axial stress the strain-rate behaves like stress to the power 3 is good at large strain-rates, which implies that both $\hat{\Phi}_{q1}$ and $\hat{\Phi}_{q2}$ behave like $(\bar{I}_2)^{\frac{1}{6}} = \eta$ as $\eta \to \infty$. Finite ϕ_1 and ϕ_2 as $\bar{I}_2$ approaches zero requires that $\hat{\Phi}_{q1}$ behaves like $\bar{I}_2^{\frac{1}{2}} = \eta^3$ and $\hat{\Phi}_{q2}$ behaves like $\bar{I}_2 = \eta^6$ as $\eta \to 0$. We have therefore examined representations with this behaviour at large and small strain-rate with $2M+2$ arbitrary coefficients:

$$\hat{\Phi}_{q1}(\eta) = \eta\sum_{m=1}^{M} c_m^2[1-\exp(-c_{M+m}^2\eta^3)] + c_{2M+1}^2\eta^3\exp(-c_{2M+2}^2\eta), \quad \mu_0 = c_{2M+1}^2/2, \tag{5.6}$$

$$\hat{\Phi}_{q2}(\eta) = -\eta\sum_{m=1}^{M} b_m^2[1-\exp(-b_{M+m}^2\eta^6)] - b_{2M+1}^2\eta^6\exp(-b_{2M+2}^2\eta). \tag{5.7}$$

μ_0 is the viscosity at $\bar{I}_2 = 0$. The correlations below are least squares minimisations.

Figure 2 first shows the 15 considered data points for $\hat{\Phi}_{q1}(\eta)$, which still reveal significant inconsistencies, and a correlation $(5.6)_1$ with $M = 1$ is shown as a continuous line; this has coefficients

$$c_1 = 1.1266, \; c_2 = 15.5063, \; c_3 = 0.0350, \; c_4 = 0.0000, \quad \mu_0 = 0.0012. \tag{5.8}$$

We have also correlated 7 selected data points which exhibit a consistent trend, marked by circles, with the correlation shown as a dashed line; this has coefficients

$$c_1 = 1.1756, \; c_2 = 2.9070, \; c_3 = 1.7906, \; c_4 = 1.1902, \quad \mu_0 = 3.2064. \tag{5.9}$$

Clearly selecting points appearing to lie on a smooth line will yield a better correlation, but such a choice is not unique, so this selection is only an example. Much more consistent data is required to be confident in a correlation.

Figure 3 shows the 15 data points for $\hat{\Phi}_{q2}(\eta)$, which still show significant inconsistencies, and a correlation $(5.7)_2$ with $M = 1$, shown as a continuous line, has coefficients

$$b_1 = 0.0000,\ b_2 = 1.3461,\ b_3 = 1.8228,\ b_4 = 1.3337. \tag{5.10}$$

We have also correlated with 7 selected data points which exhibit a consistent trend, shown by squares, with the correlation shown as a dashed line; this has coefficients

$$b_1 = 1.1431,\ b_2 = 1.6270,\ b_3 = 0.2527,\ b_4 = 0.8450. \tag{5.11}$$

Note that this selection of 7 data points shown by squares is distinct from the earlier correlation set shown by circles. Here the two correlations are widely different because the 15 data points are widely scattered. Again the 7 point correlation exhibits a closer smooth correlation, but the data inconsistency means this is only an example. While the data points are too scattered to reflect any convincing correlation, they show clearly that the calculated quadratic response function $\hat{\Phi}_{q2}$ is significant at the majority of the data points.

As illustration, we can now apply the $\hat{\Phi}_{q1}(\eta)$ and $\hat{\Phi}_{q2}(\eta)$ obtained by the above correlations to determine the consequent uni-axial stress and simple shear stress responses given by (3.5) and $(3.13)_3$. Let these at melt be denoted by $U(\dot{\varepsilon})$ and $S(\dot{\gamma})$, then

$$S(\dot{\gamma}) = \Phi_{q1}(\dot{\gamma}^2) = \hat{\Phi}_{q1}(\dot{\gamma}^{\frac{1}{3}}), \tag{5.12}$$

$$U(\dot{\varepsilon}) = 3^{\frac{1}{2}}\,\Phi_{q1}(3\dot{\varepsilon}^2/4) - \Phi_2(3\dot{\varepsilon}^2/4) = 3^{\frac{1}{2}}\,\hat{\Phi}_{q1}(\theta\dot{\varepsilon}^{\frac{1}{3}}) - \hat{\Phi}_{q2}(\theta\dot{\varepsilon}^{\frac{1}{3}}), \quad \theta = (3/4)^{\frac{1}{6}} = 0.9532. \tag{5.13}$$

The $\eta = \bar{I}_2^{\frac{1}{6}}$ maximum of 7.8218, line 21 and column 9 in the data Table 1, yields a maximum $\dot{\gamma} = \eta^3 = 479$ in (5.12) and a maximum $\dot{\varepsilon} = 2\eta^3/3^{\frac{1}{2}} = 553$ in (5.13), but the above combination of response functions leads to much higher stresses. Accordingly we have determined both $S(\dot{\gamma})$ and $U(\dot{\varepsilon})$ over the same argument range 0 – 400. Figure 4 shows the determined $S(\dot{\gamma})$ as a continuous line and a dashed line respectively for the 15 and 7 data point correlations, quite close since depending only on the close $\hat{\Phi}_{q1}$ correlations. The 2 simple shear points, lines 3 and 10 in Table 1, are marked by circles. Figure 5 shows $U(\dot{\varepsilon})$ as a continuous line with the two 15 point correlations, and a dashed line with the two 7 point correlations, with the 2 uni-axial stress points, lines 1 and 7 in Table 1, marked by squares. In addition, the melt temperature response with bounded viscosity correlated from Glen's (1955) data by Smith and Morland (1981), assuming the simple co-axial relation which in the present variables (Morland, 1984) has the inverse form

$$\dot{\varepsilon} = 0.2224 + 0.0711\sigma^2 + 0.0011\sigma^4, \tag{5.14}$$

is shown as the dash-dot line marked G.

The two correlations for the shear response are quite close as a result of the two correlations for $\Phi_1(\eta)$, shown in Figure 2, being close, since $S(\dot{\gamma})$ is independent of $\Phi_2(\eta)$. However, the two correlations for the uni-axial stress are very different because of the dependence on $\Phi_2(\eta)$, for which the correlations, shown in Figure 3, are very different. In addition, comparison with Glen's pioneering uni-axial stress experiments in the 1950's, shown by the dotted line marked G, shows a rough agreement with the 15 point correlations but much lower stress than the 7 point correlations, again due to the very different $\Phi_2(\eta)$ correlations.

Which data points should be selected or rejected, and why, is not obvious. The choices above based on "trends" give sensible looking uni-axial and shear responses, but of course are based on an analysis adopting the unconfirmed Melbourne approximation $D_{11} = 0$.

Conclusions

This analysis was prompted by the publication of data from combined compression and shear tests on ice at the University of Melbourne. It focusses on the initial isotropic viscous response, and how this data rejects the simple co-axial relation with dependence on one strain-rate invariant, or equivalently dependence on one deviatoric stress invariant, which is still the common assumption in ice-sheet flow modelling. Starting from the general frame-indifferent viscous fluid relation, the relation between the uni-axial compression response and simple shear response which must hold if a co-axial relation is valid, is determined. It is also shown that the Steinemann conjecture, that when this relation is not satisfied a co-axial relation with general dependence on two strain-rate invariants could suffice, is not valid. The Melbourne data Table 1 has only two distinct uni-axial compression and two distinct simple shear points, not sufficient to determine uni-axial and simple shear responses, and only 21 distinct data points which are too sparse to determine response functions of two invariant arguments. We then focus on the quadratic (non-co-axial) relation, but with the two response function coefficients depending on only one invariant, a measure of the shear strain-rate magnitude squared. The two response functions could be determined by sufficient uni-axial and shear data points, but we have not found such data.

The Melbourne combined stress and shear configurations determine relations for the two response functions in terms of the stress and strain-rate points, but neither the longitudinal nor lateral strain-rate was measured so we had to adopt the Melbourne approximation that the longitudinal strain-rate is (approximately) zero. Inferences and conclusions are therefore dependent on this approximation. We have expressed the Melbourne data Table 1 in dimensionless normalised variables, and re-ordered the table into data lines with increasing strain-rate magnitude. Noting immediate inconsistencies in the Melbourne data, we reduced our Table 1 to 15 lines for first correlations of the two response functions. However, the 15 point correlations show wide differences between points and correlated response functions, so separate sets of 7 points showing consistent response function variations were applied for second correlations. The correlations were made by least squares minimisation with representations using base functions with imposed behaviour at small and large strain-rates, explained in the text. Table 1 and Figure 3 show immediately that the quadratic response function in the viscous flow law $(2.10)_2$ is significant at the majority of the data points, which rejects a co-axial relation, and in consequence, as analysed, rejects any relation between the shear strain-rate invariant and shear stress invariant, and hence between the octahedral strain-rate and stress.

Finally, with the separate 15 and 7 point response function correlations for the quadratic viscous law, we calculated the consequent simple shear stress and uni-axial stress responses, shown in Figures 4 and 5 respectively. The two correlations for the shear response are quite close as a result of the two correlations for $\Phi_1(\eta)$, shown in Figure 3, being close, since $S(\dot{\gamma})$ is independent of $\Phi_2(\eta)$. However, the two correlations for the uni-axial stress are very different because of the dependence on $\Phi_2(\eta)$, for which the correlations, shown in Figure 3, are very different. In addition, comparison with Glen's pioneering uni-axial stress experiments in the 1950's, shown by the dotted line marked G, shows a rough agreement with the 15 point correlations but much lower stress than the 7 point correlations, again due to the very different $\Phi_2(\eta)$ correlations.

We must re-emphasise that the assumption of vanishing longitudinal strain-rate, instead of measurement, means that our data analysis is only reliable if this Melbourne assumption is in fact a good approximation. Further, the rejection of several anomalous data points left 15 which were still widely dispersed, but were applied in a first correlation of Φ_1 and Φ_2. Second correlations with separate subsets of 7 data points, which showed consistent variations of Φ_1 and Φ_2 respectively, were made. The consequent simple shear and uni-axial responses were determined for each of the Φ_1 and Φ_2 correlations. The 15 and 7 point correlations gave very different Φ_2 and in turn very different uni-axial stress responses. The analysis, however, shows how reliable uni-axial and simple shear responses can be correlated to determine the response functions of a viscous law of the quadratic form $(2.10)_2$, and determine the relation between uni-axial and simple shear responses which would support the simpler co-axial relation $(2.9)_2$.

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Figures

Figure 1: Data points in invariants plane defined by axes $\bar{I}_2^{\frac{1}{6}}$ and $(-\bar{I}_3)^{\frac{1}{9}}$.

Figure 2: $15\,\Phi_1(\eta)$ data points shown by $\times$ with correlation shown as continuous line, and 7 selected points shown by circles with correlation shown as dashed line.

Figure 3: $15\,\Phi_2(\eta)$ data points shown by $\times$ with correlation shown as continuous line, and 7 selected points shown by squares with correlation shown as dashed line.

Figure 4: Shear stress as function of $\dot{\gamma}$ from data correlations, continuous and dashed lines corresponding to 15 and 7 point correlations respectively; 2 Melbourne simple shear data points marked by circles.

Figure 5: Uni-axial stress as function of $\dot{\varepsilon}$ from data correlations, continuous and dashed lines corresponding to 15 and 7 point correlations respectively; 2 Melbourne uni-axial data points marked by squares. The Glen data correlation shown as a dotted line is marked G.

Highlights

- A constitutive relation describing viscous response of isotropic polar ice is constructed.
- In contrast to the customary approach in which stress tensor is a linear function of strain-rate tensor, a more general relation is considered in which stress tensor is a quadratic function of strain-rate tensor.
- The constitutive model parameters are determined by correlation with experimental results obtained in laboratory and published in 2012 and 2013.

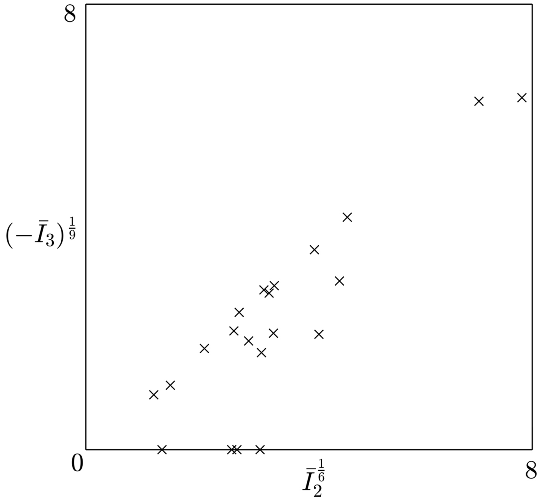

Figure 1

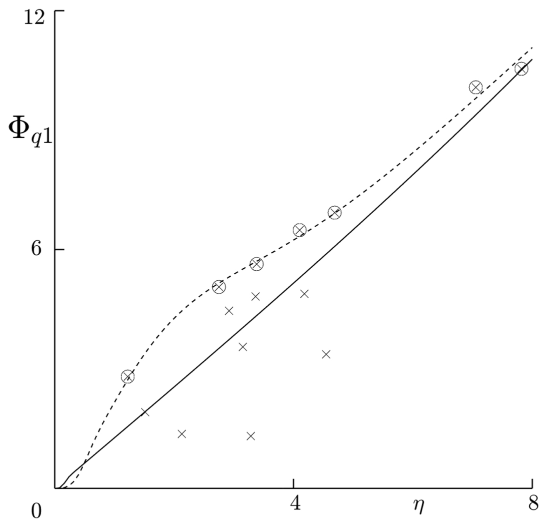

Figure 2

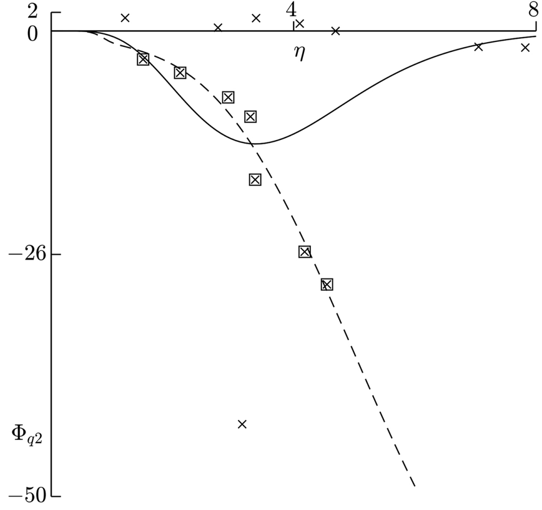

Figure 3

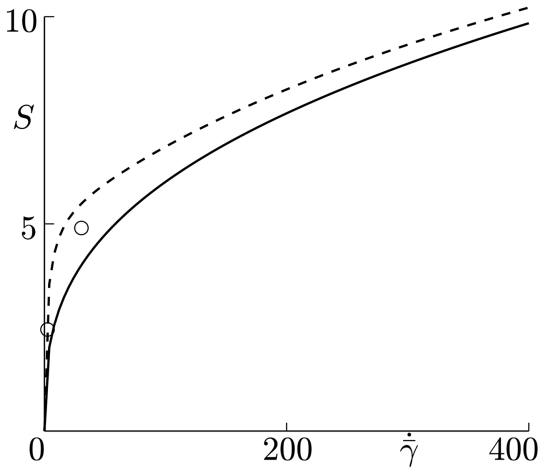

Figure 4

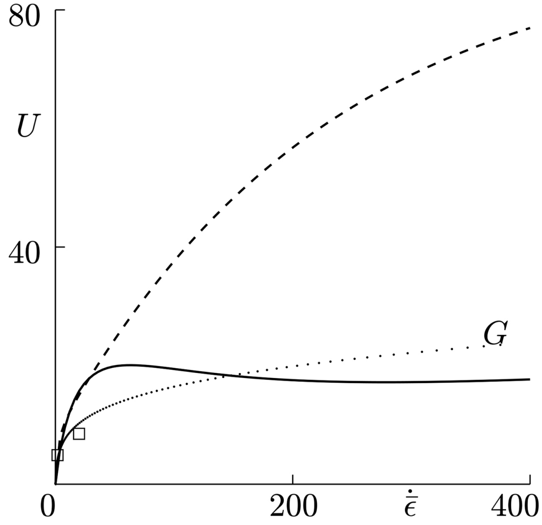

Figure 5