Introducing Learning Study Approach to the Teaching and Learning of Mathematics in Kenyan Secondary Schools

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Abstract

Recent reports ([KNEC], 2014) have shown that students' performance in the national mathematics examinations in Kenya is weak and have expressed concerns about the pedagogical methods adopted by teachers. The Kenyan Government has made some interventions in the past, including the initiation of in-service training for mathematics and science teachers in secondary schools. However, performance has not improved significantly ([KNEC], 2014).

The purpose of this study is to explore the effect of Learning Study (LS) approach in the teaching and learning of quadratic expressions and equations – one of the topics of concern ([KNEC], 2014). The LS approach promotes collaborative work between teachers. Firstly, they prepare a lesson together, then one of them teaches the lesson while the others observe, and they later meet to reflect on and revise the lesson. In this study and in tandem with the recommendations from the Kenyan Ministry of Education ([MoEST], 2012) for more student-centred learning, the lessons were organised with the students participating in small group discussions followed by a whole-class discussion.

The participants of the study included three teachers teaching two Form 3 (16-18 years) classes, and 79 students. I applied a LS design (Lo, 2012) and collected qualitative data from students' pre-and post-lesson tests, classroom observations, individual interviews with the teachers, and a group interview of eight students. Lesson observation data was analysed using a Variation Theory framework (Lo, 2012) and interview data was analysed using thematic analysis (Braun and Clarke, 2006).

The findings show that: students adequately learned the topic, experienced positive changes in their attitudes towards mathematics, improved participation and communication in mathematics lessons, and increased their confidence when solving mathematical problems. The teachers appreciated the LS approach, saying that teamwork improved their teaching of the topic and helped them learn from each other.

KEY WORDS: Quadratic Expression and Equation, Learning Study, Variation Theory, Collaboration, Group work.

Table of Contents

List of Tables

List of Figures

Dedication

I dedicate this Thesis to: my wife Roseline; my children Philadelphia, Kennedy, Winnie, Gilbert, Paula, Nicholas, Linda and Mildred; and, my late father John and my late mother Beldina.

Acknowledgement

I would, firstly, like to take the opportunity to sincerely thank my supervisors, Professor Elena Nardi and Professor Tim Rowland, for their unrelenting support and guidance which they provided during my study, including the production of a conference paper based on my study. The data from the first pair of lesson observations (discussed in Chapter 4), was presented at the 10th Conference of the European Research in Mathematics Education (CERME 10) held in Dublin, Republic of Ireland, in February 2017. My paper qualified and was published in conference proceedings.

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I would also like to thank my entire family for their endurance, patience, and support, while missing a husband, a father and a son for a long time.

Above all, may I take this chance to thank the Almighty God, for giving me good health, and an open mind to understand the course, throughout this journey.

Abbreviations

ALPS	Active Learning through Professional Support
ASEI	Activity, Student, Experiment, Improvisation
B.Ed. Sc.	Bachelor of Education, Science
CAT	Continuous Assessment Test
CEMASTEA	Centre for Mathematics, Science and Technology Education in Africa
CERME	Conference of European Research in Mathematics Education
CDE	County Director of Education
CPD	Continuous Professional Development
DEO	District Education Officer
DT	District Trainer
ECDE	Early Childhood Development and Education
ESQAC	Education Standards and Quality Assurance Commission
GoJ	Government of Japan
GoK	Government of Kenya
HoD	Head of Department
HOTS	High Order Thinking Skills
INSET	In-Service Education and Training
JICA	Japan International Cooperation Agency
JOOUST	Jaramogi Oginga Odinga University of Science and Technology
KCPE	Kenya Certificate of Primary Education
KCSE	Kenya Certificate of Secondary Education
KICD	Kenya Institute of Curriculum Development
KIE	Kenya Institute of Education
KLB	Kenya Literature Bureau
KNEC	Kenya National Examinations Council
KQ	Knowledge Quartet
KSTC	Kenya Science Teachers College
LCM	Least Common Multiple

LHS	Left Hand Side
LS	Learning Study
MoEST	Ministry of Education, Science and Technology
NACOSTI	National Commission for Science Technology and Innovation
NT	National Trainer
PDSI	Plan, Do, See, Improve
PGR	Post Graduate Research
PhD	Doctor of Philosophy
PPD	Personal and Professional Development
RHS	Right Hand Side
RME	Research in Mathematics Education
SMASSE	Strengthening of Mathematics and Science in Secondary Education
TIMSS	Third International Mathematics and Science Survey
TA	Thematic Analysis
TL	Team Leader
TSC	Teachers Service Commission
UEA	University of East Anglia
UK	United Kingdom
USA	United States of America
VT	Variation Theory
VITAL	Variation for Improvement of Teaching and Learning

Chapter 1 – Background of the Study

1.1 Introduction

In this study I analyse the effects of Learning Study (LS) approach to the teaching and learning of mathematics in a Kenyan secondary school. The teaching and learning is centred around the topic of quadratic expressions and equations in Form 3 (16-18 years of age). This is one of the topics reported by the Kenya National Examinations Council ([KNEC], 2014) as a topic of concern, related to students' performance in the Kenya Certificate of Secondary Education (KCSE) national examination. Apart from the (KNEC] (2014) report, this topic is also internationally reported as a topic of concern (Clement, 1982; Clement et al., 1981; Didis & Erbas, 2015; Stacey and MacGregor, 2000; Vaiyavutjamai & Clements, 2006). This topic was also chosen because, at the intended time of my data collection, it was the topic being taught according to the Kenyan secondary mathematics curriculum (Kenya Institute of Education [KIE], 2002).

In section 2.5 I discuss, in detail, the LS approach, which involves collaborative work among teachers teaching a mathematics class. Firstly, a team of teachers prepares a lesson, then one of them teaches the lesson (observed by the other teachers and myself (the researcher)), after which the whole team meets for a post-lesson reflection session (Lo, 2012; Marton, 2015; Pang, 2006 & 2008). The group of three teachers worked together preparing lessons that were taught in two Form 3 classes. After the classes had been taught, I interviewed the teachers and eight students who represented the rest from both classes. Data was collected at each of the aforementioned stages and analysed using two different approaches. The data from classroom observations were analysed using the theoretical framework of Variation Theory, as explained in section 2.7. The data from the interviews were analysed using a thematic data analysis, which is discussed in section 3.7.2.

In order to explain why this study is important to me, and to the country of Kenya, I would like to discuss my professional background. I trained as a secondary school teacher, to teach mathematics and physics, graduating with a Bachelor of Education

(Science) (B.Ed, Sc). Secondary school teachers in Kenya are trained to teach two subjects. I taught in public schools for 15 years (from 1988 to 2003) before being transferred to a national in-service teacher training centre (Centre for Mathematics, Science and Technology Education in Africa (CEMASTEA)). I worked at the centre for 10 years (from 2003-2013), with mathematics and science teachers, training them in improvement of the teaching and learning of mathematics and sciences in secondary schools. Since 2013, I have been employed as a lecturer in the School of Education, in one of the 31 public universities in Kenya - the Jaramogi Oginga Odinga University of Science and Technology (JOOUST).

While teaching in secondary school, the Kenya National Examinations Council (KNEC) – (the body mandated by the Government to assess curricula for primary schools, secondary schools and tertiary institutions, excluding Universities), appointed me as an examiner to mark the Kenya Certificate of Secondary Education (KCSE) mathematics examinations. KCSE examination is undertaken by Form 4 students (17-19 years) at the end of secondary school education. I marked the examinations for 23 years (from 1989 to 2012), the last ten years as a senior examiner – a position referred to as Team Leader (TL).

While at CEMASTEA, the Kenya Institute of Curriculum Development (KICD), formerly known as the Kenya Institute of Education (KIE) (the body that develops the curricula that are assessed by KNEC), engaged me to help in the development of another mathematics curriculum for secondary schools referred to as Alternative B (see section 1.3). This curriculum was developed mainly for students in non-formal secondary schools and some sub-County schools (see section 1.3).

While I was at CEMASTEA, I did my Master study at Syracuse University in the USA (2010 to 2012). My course of study was Master of Science (MSc) in teaching and curriculum, with emphasis on mathematics education. In addition, while at CEMASTEA, I had two short-study visits to Japan. During the first visit (August to October 2005) I went on a three-month course on the teaching and learning of mathematics at Hiroshima University. This course involved discussions on different approaches to the teaching and learning of mathematics, as well as classroom observations. One of the approaches I participated in was lesson study. During the second visit (August/September 2013), I went on a one-month seminar on the application of lesson study in the teaching and learning of mathematics at Tokyo Gakugei University. The seminar involved observation of a lesson study, conducted as a research entity, and observations of normal classes taught in a lesson study approach via problem-solving. Those of us from different countries who attended the seminar also formed groups where we planned and taught lessons in a lesson study approach. I drew my inspiration from these experiences in order to conduct this study.

My experience as a teacher, examiner and teacher-trainer made me realise that, perhaps, the teaching approach in Kenya, referred to as a traditional approach (Mulala, 2015), could be reviewed in order to try other approaches such as lesson study or learning study (LS), both lauded as helping students perform well in mathematics (Pang, 2008; Stigler & Hiebert, 1999).

While teaching in high school I had encountered many students performing dismally in the national mathematics examinations. This prompted the Government of Kenya, in collaboration with the Government of Japan, to initiate an In-Service Education and Training (INSET) project, for teachers of mathematics and sciences, called Strengthening of Mathematics and Science in Secondary Education (SMASSE) (see section 1.3.1). This project was meant to improve the students' performance in the mathematics and sciences national examinations. However, at the end of the project, the students' performance had not improved significantly ([KNEC], 2014).

As I mentioned in the sixth paragraph of this introduction, I had observed lessons taught using the lesson study approach to teaching and learning, and I also read success stories of Japanese students' performance in mathematics international comparisons such as Third International Mathematics and Science Survey ([TIMSS], 1999). Because of this I felt that I wanted to explore the lesson study approach to the teaching and learning of mathematics in the Kenyan secondary schools. However, after reading articles and studies in Learning Study (LS) approach, I realised that LS has clear guidelines in monitoring the teaching and learning process, given that it is backed by a theory of learning (Variation Theory), as opposed to lesson study approach which is implicitly backed by the constructivist theory (Elliot, 2014) (see sections 2.3, 2.4 and 2.5). Therefore, I changed my mind and decided to use the LS approach for this study.

The outcome of this study is significant to me as its application in Kenya is within a different culture from where the approach had originally been applied. The outcome of the study could also be significant to the people of Kenya as it applies a teaching and learning technique which is different from the usual approach(es) that they have been used to.

In the next section, I discuss the Kenyan education system in the context of the problem that led to this study. I also briefly discuss the Kenyan administrative structure, explaining the link between it and the education system. This should also help the reader to understand the reason behind the selection of the school for this study, as I discussed in section 3.2.

1.2 The Kenyan Education System

I begin this section by briefly discussing the Kenyan administrative structure. The current Kenyan administrative structure has two levels of Governments, a National Government (headed by an elected President), and 47 County Governments (each headed by an elected Governor) (The Constitution, 2010). Figure 1 shows a map of Kenya with its 47 counties indicated. Kenya is a country with multi-ethnic communities comprising about 44 tribes (The Constitution, 2010). Each of the tribes speaks its own language and has different cultural practices. Most of the counties are ethnically homogeneous, except for a few counties which contain cities and major towns, such as Nairobi (which is the capital city), Mombasa, Kisumu, Nakuru and Eldoret, and some counties close to the cities, which have heterogeneous communities.

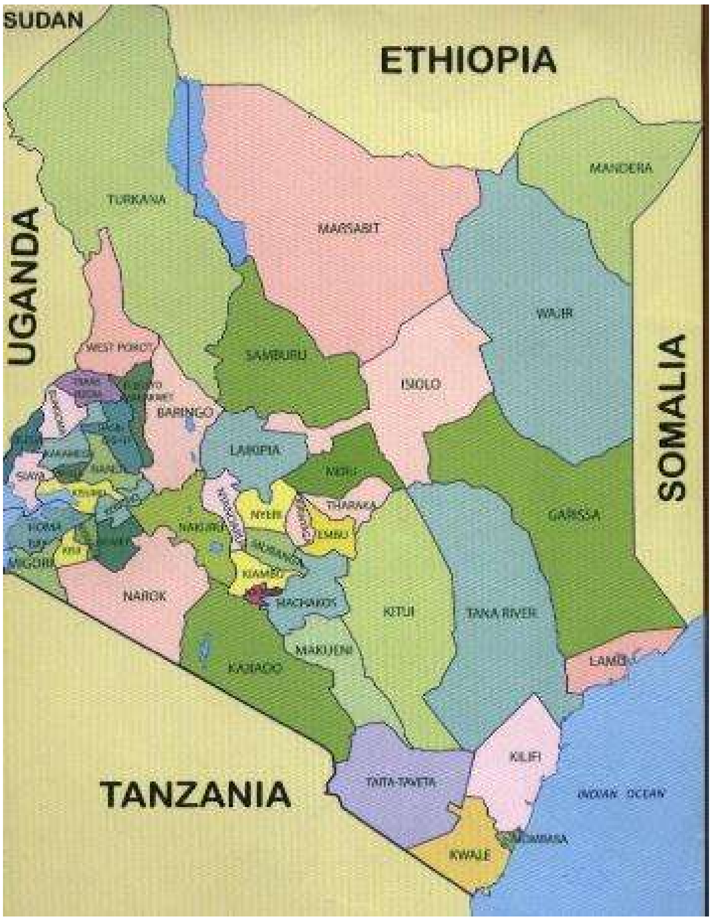


***Figure 1*: The administrative map of Kenya showing the 47 counties and the neighbouring countries adapted from: (https://www.google.co.uk/url?sa=i&rct=j&q=&esrc=s&source=images&cd=&cad=rja&uact=8&ved=2ahUKEwjvydHKvrXdAhUrDsAKHdV0AqwQjRx6BAgBEAU&url=https%3A%2F%2Finformationcradle.com%2Fkenya%2Fcounties-in-kenya%2F&psig=AOvVaw00Shp7DtLQhP8_Cv-77-rF&ust=1536842480185872)**

The ministries that are involved in the governance of the country are shared between the two levels of Government, with the National Government controlling the ministries which are responsible for creating harmony among the multi-ethnic communities (such as Ministry of Education, Science and Technology). However, the department of Early Childhood Development and Education (ECDE), which promotes learning in the mother tongue at an early level, is devolved to the County Government.

Due to its multi-ethnic communities, the Kenyan Constitution has adopted English and Kiswahili as official languages, with the latter being the national language (Constitution of Kenya, 2010). However, teaching is done in English from Class 1 (6-8 years of age) for all the subjects except Kiswahili, and other foreign languages such as French and German which are taught in the respective languages. These languages are taught in high schools as elective subjects ([MoEST], 2012).

The current Kenyan education system is referred to as eight-four-four (8 – 4 – 4) and was adopted in 1985, on the recommendation of a commission of the Presidential Working Party (Mackay report, 1981). The commission, which is popularly known as the Mackay commission – after its chairperson Dr. Colin Mackay – recommended the formation of a second university in Kenya, eight years of primary education, four years of secondary education, and four years of basic degree programmes at the Universities. The current system changed from the old system of education, which was adopted in 1964 immediately after independence in 1963. The former system had been recommended by the first post-colonial commission – Ominde Commission – which recommended seven years of primary education, four years of secondary education (ordinary level (O-Level)), two years of higher level of secondary education (Advanced Level (A-Level)), and three years of basic University degree programmes (7-4-2-3). The new system abolished the A-level and distributed the two years between the first and last cycles of education.

In order to move from one cycle of education to the next cycle, students have to sit national examinations. For example, to move from primary school to secondary school, students sit for an examination (at the end of eight years in primary school), called the Kenya Certificate of Primary Education (KCPE) examinations. Students' admission into secondary schools depends on their performance in the KCPE.

Public secondary schools in Kenya are categorised into National schools, Extra-County schools, County schools and Sub-County schools by the Kenya National Examinations Council ([KNEC, 2015]). National, Extra-County and County schools are boarding schools, while Sub-County schools are day schools. National schools and Extra-County schools have better learning facilities, such as laboratories, libraries, books, and many have more Teachers Service Commission (TSC) employed teachers than the other categories of schools.

Students are admitted into the different categories of secondary schools based on their performance in the KCPE examinations and on the given proportions and quotas from their respective counties ([KNEC], 2015; [MoEST], 2012). National schools admit students from across the country, based on their performance in KCPE and their county's quota: extra-county schools admit 20% from the host sub-county, 40% from the host county, and 40% nationally; county schools admit 20% from the host sub-county, and 80% from the whole county; sub-county schools admit 100% from the sub-county (because they are day schools) (Onderi & Makori, 2014). These proportional distributions are done in order to ensure that students from different counties, and by extension ethnic-communities, learn together to "foster nationalism, patriotism, and promote national unity and respect for diverse cultures" ([MoEST], 2012, p. 15).

At the end of the secondary school cycle, (the end of the fourth year of secondary education), students sit for the KCSE examinations and, depending on a student's performance, he/she is admitted to the universities to pursue degree programmes, or in the tertiary colleges for either diploma or certificate courses. In the KCSE examinations students are tested in eight subjects, but graded in seven subjects, in the following combination. Three compulsory subjects that include English, Kiswahili and Mathematics; at least two science subjects from biology, physics and chemistry; at least one humanity subject, and any additional subject from either a science, humanity or technical subject. Technical subjects include computer science, home science, agriculture, business studies, or foreign languages such as French and German.

1.3 Mathematics Performance in KCSE and Measures taken by the Government

Performance in mathematics in the KCSE examinations has been consistently below average, that is below 50%, with more than 70% of students scoring low grades of D, D- and E – as observed in the KNEC reports (Miheso-O'Connor, 2011). For instance, in the years 1979, 1983, 2002 and 2006, the percentage of students who attained these low grades were 73, 73, 72 and 79 respectively (Miheso-O'Connor, 2011). [KNEC] (2006) reported that about 40% of the students who sat for the KCSE examination in 2005 scored grade E (the lowest grade in the grading system).

Based on these reports, the Government of Kenya initiated some measures with an intention of improving mathematics performance. However, these interventions do not seem to have worked well, as performance is still below average – as shown by the [KNEC] (2014) report in Table 1. Although the table indicates some improvement in performance from 2009 to 2012, with a drop in 2013, all the performances are below 30%.

Table 1: ***Mathematics performance in KCSE from 2009-2013 for students of Alternative A curriculum***

Year	2009	2010	2011	2012	2013
% Mean Score	21	23	25	29	28

The Government of Kenya, through the Teachers Service Commission (TSC), ensured that only trained mathematics teachers were employed to teach mathematics. This action was taken to ensure that only qualified teachers, with content and pedagogy, taught mathematics. The training of pre-service teachers, teaching in high schools in Kenya, is done at two levels ([MoEST], 2012). One level is the teachers trained at the diploma teachers' training colleges for three years who qualify with a Diploma of Education. The other level is teachers trained at the universities for four years who qualify with a Bachelor of Education degree. The university training comprises two models ([MoEST], 2012). The first model, which is the common model in Kenya, is a concurrent one, where trainees spend four years studying the subjects' contents, as well as the pedagogical knowledge of teaching, together with other education courses such as psychology of education, school administration, philosophy of education and curriculum development. The second model, called the consecutive model, is where the trainees initially spend four years studying the subjects' content areas and graduate with either Bachelor of Arts or Bachelor of Science degree, and later undertake a nine-month postgraduate diploma training on the pedagogy knowledge of teaching and other educational courses. The Government agreed to employ only these two categories of teachers to teach mathematics.

In 1995 the Government awarded a salary increase to the teachers of mathematics, as an extrinsic motivation, so that the teachers would put in more effort to help students to improve their mathematics performance at the KCSE examination ([TSC], 1997).

1n 1998, the Government of Kenya (GoK) through MoEST, in conjunction with the Government of Japan (GoJ) through the Japanese International Cooperation Agency (JICA), initiated an In-service Education and Training (INSET) project for trained teachers of mathematics and science subjects in secondary schools called Strengthening of Mathematics and Science in Secondary Education (SMASSE). The initiative was meant to help teachers improve the teaching and learning of mathematics and sciences in secondary schools. I explain more about SMASSE in section 1.3.1.

In addition, the GoK, through the Kenya Institute of Curriculum Development (KICD)[1], developed an Alternative B mathematics curriculum in 2008 ([MoEST], 2008) in which some topics such as three-dimensional geometry, latitude and longitude, and Calculus, (which are in the main curriculum, referred to as Alternative A) were removed. The Alternative B curriculum was developed mainly for non-formal schools. Many of the non-formal schools are found among the pastoralist communities who do not have

[1] KICD is a semi-autonomous body affiliated to Ministry of Education Science and Technology in charge of developing curriculum for primary and secondary schools as well as tertiary colleges offering diploma and certificate courses. In addition, it vets books published for these curricula.

permanent homes and keep moving with their animals in search of pastures. Some non-formal schools are also found within the slums in urban areas ([MoEST], 2008). Many of these schools do not have any organised learning programmes, neither do they have consistent teachers, but they do register for KCSE examinations. Volunteers, many of whom are not trained teachers, offer to teach the students. However, the Government also allows some formal schools, especially the sub-county schools, to offer the Alternative B curriculum. In practice, very few schools offer it ([KNEC], 2014) and the performance is not good, as shown in Table 2. The GoK took the initiative to improve the performance of mathematics, in both Alternatives A and B, in the KCSE examination. Despite this, there is very little improvement in Alternative A mathematics, as shown in Table 1, while there is a consistent decline in performance for Alternative B, as shown in Table 2.

Table 2: ***Mathematics performance in KCSE examinations from 2010-2013 for students of Alternative B curriculum***

Year	2010	2011	2012	2013
% Mean Score	19	13	10	9

1.3.1 The SMASSE Project

As mentioned in section 1.3, the SMASSE project was initiated in 1998 with the broad aim of improving the capability of young Kenyans in mathematics and science subjects. The project was implemented for 10 years, in two phases of five years (Nui & Wahome, 2006). The first phase was a pilot covering nine districts, out of the then 71 districts, and implemented from 1998 to 2003.

Before the implementation of the first phase, the team (who were mainly lecturers from the Kenya Science Teachers College (KSTC)), carried out a baseline survey in the nine districts. It was from the outcome of this survey that the team developed the training curriculum. Among the data collection instruments was a mathematics test for Form 4 (17-19 years) students. Out of 4,243 students who did the test, 3446 (81%) scored D+ or below, which confirmed the low grades that had been observed in the previous KCSE examinations. In addition, the survey revealed many challenges, which the team narrowed down to the ones that they thought could be addressed through the INSET. The identified challenges were: (1) Negative attitudes towards the teaching and learning of mathematics and sciences by both teachers and students. (2) Inappropriate teaching methodology. (3) Inadequate content mastery by both teachers and students on certain topics, which they called topics of concern. (4) Inadequate assignments. (5) Few or no interactive forums for teachers to share ideas. (6) Missing link between primary and secondary school levels (Nui & Wahome, 2006 p. 48-49).

Based on these outcomes, the team developed a four-cycle (module) INSET curriculum with identified themes for each cycle. The theme of the first cycle was attitude change, the theme of the second cycle was hands-on activities, the theme of the third cycle was actualisation and the theme of the fourth cycle was monitoring and evaluation (Kiige & Atina, 2016).

These themes of training guided the nature of the tasks for group work. For example, during the hands-on activity cycle, groups discussed various ways of teaching a mathematics topic using a practical activity in order to help the students enjoy the learning process, develop their interest in mathematics, and learn the content of the topic. During the actualisation cycle, the trainees prepared the lessons in groups on the topics taught in the INSET training centres, which were the topics taught in schools at that time according to the curriculum. One of the group members taught the lesson while others observed. The trainee teachers did not necessarily teach their own students, since they taught schools around the INSET Centre, and some of them came from other parts of the country. The teachers and their trainers then converged for a reflection session at the INSET Centres.

The training was conducted in a two-tier cascade model. National trainers (NT) based at the national training centre in Nairobi (CEMASTEA), trained selected teachers from the districts, referred to as district trainers (DTs). The DTs then trained the rest of the mathematics and science teachers at District INSET Centres. The duration of the training for each cycle was two weeks, and each cycle was implemented once per year. DTs organised their training during the school holidays.

Kenya follows a three-term arrangement in its education system with the first term running from early January to early April, the second term running from early May to early August, and the third term running from early September to Mid-November. The DTs conducted their training either in April or August, as teachers are engaged to mark KCSE examinations during the November/December holiday.

The National Trainers coined the following acronyms ASEI (Activity, Student, Experiment, Improvisation) and PDSI (Plan, Do, See, Improve) to be the SMASSE outcome, which was explained as a paradigm shift from teacher-centred teaching to student-centred learning. ASEI stands for activity-based teaching, with student-centred learning, carrying out an experiment where necessary, and encouraging improvisation in the absence of a conventional device (such as a clinometer), or when linking the teaching with the everyday environment (such as used boxes for packaging when teaching the measurement of solids). To do this, teachers needed to Plan, Do, See and Improve (PDSI). 'The Plan' is the usual lesson planning, 'Do' refers to the teaching, 'See' refers to the evaluation of the lesson (i.e. looking back at the lesson to see what has worked and what has not worked well) and Improve refers to lesson improvement.

After the pilot phase, the GoK (and the GoJ) applied the project to the remaining 62 districts for a further five years from 2003 to 2008. During this second phase, dubbed the SMASSE national INSET, more national trainers were recruited, and I was one of them. The same curriculum as used in the pilot phase was used for the training.

At the expiry of the SMASSE project, the GoK adopted the SMASSE INSET as one of the ways to help the Government to achieve its Vision 2030 objectives ([MoEST], 2012). The Government developed the Vision 2030 document in 2007. It aims to transform Kenya into "a newly-industrialising, middle-income country providing a high quality of life to all its citizens in a clean and secure environment by the year 2030" (Ministry of Planning and National Development, 2007, p. 1). Vision 2030 aims to provide students with "a better learning environment, including improved teaching skills and more textbooks" (p. 99). In addition, the Vision 2030 objectives require students to be active participants during learning and to embrace creativity and reasoning.

After the SMASSE project, some studies were conducted to determine the impact of the SMASSE INSET project in schools. Kiige and Atina (2016) conducted a study on the effectiveness of the SMASSE INSET project on KCSE examination performance in mathematics and chemistry subjects. They conducted the study in one of the pilot districts (Kikuyu district), which is currently called Kikuyu Sub-county of Kiambu County. This Sub-County's mathematics performance in the KCSE examination from 2004 to 2007 is shown in Table 3.

Table 3: ***The mean score of mathematics performance in KCSE examination in Kikuyu sub-county from 2004 to 2007***

Year	2004	2005	2006	2007
% mean score	18	16	19	19

Although this study does not show the KCSE examination performance for the period before the SMASSE project, it was reported that there was no significant difference between the pre-SMASSE results and the post-SMASSE results. They reported that the teachers agreed that "the skills they learned in the SMASSE INSET project are effective and are applicable to the teaching of mathematics" (p. 60). However, they were not implementing the skills, claiming that the SMASSE INSET project demands were burdensome and time consuming, especially the ASEI lesson plan.

Makewa, Role and Biego (2011) conducted a study on teachers' attitudes towards the SMASSE INSET project in Nandi Central Sub-County, part of the SMASSE phase two project. From their findings, teachers showed a positive attitude towards SMASSE INSET stating that "SMASSE added some knowledge to their teaching of mathematics and it had helped them solve some problems they encountered in the field" (p. 15).

However, they reported that teachers did not practice the ASEI/PDSI in the classroom. The teachers stated that:

> They did not enjoy using ASEI/ PDSI pedagogy during a mathematics lesson. ASEI/PDSI approach is cumbersome and requires a lot of time to prepare and to execute a lesson. The allocated time of 40 minutes per lesson was not enough to cover the syllabus and realize results if ASEI/PDSI pedagogy was to be fully implemented (p. 16).

Furthermore, the teachers claimed that they could not apply the ASEI/PDSI approach to teach all the topics in mathematics, as it was difficult to find appropriate teaching and learning activities in some topics.

From the findings, teachers were in agreement that the SMASSE INSET project was useful and could improve their ways of teaching mathematics. However, teachers were not applying what they learned into their classroom teaching. They cited challenges such as more time required to prepare ASEI lesson, and lack of activities for some of the mathematics topics and syllabus coverage, as some of the reasons they are not implementing the SMASSE INSET project.

My observations of the students' weak performance in the mathematics national examinations, the teaching initiatives that the Government of Kenya implemented, and the teachers' reports of not implementing what they learned from the SMASSE INSET project, became the basis of my study. I felt that, potentially, the problem with the students' performance in mathematics national examinations could be addressed by implementing teaching and learning approaches that would allow the students to be active participants during the lesson.

Although ASEI/PDSI allowed students to be active participants during the teaching and learning of mathematics, reports indicated that the teachers were not implementing the approach. In view of this, I felt that the LS approach would be effective in the Kenyan classroom as the teachers could work collaboratively within their own schools and implement it in their own classes.

1.4 Purpose of the Study

As shown in Tables 1 and 2, the [KNEC] (2014) report, which considered the whole population of students that sat the examinations, indicated that students' performance in the mathematics KCSE examination was below 50%. In addition, the report indicated the topics where students consistently performed badly. Some of the topics included: proportional division of a line under the topic of geometrical constructions (taught in Form 1 (14-16 years)), trigonometry (taught in Form 2 and Form 3), and quadratic expressions and equations (taught in Form 2 and Form 3) ([KIE], 2002). Looking at the Kenyan secondary schools' mathematics curriculum schedule, I realised that the quadratic expressions and equations' schedules (for both classes) would fit within my data collection timeframe. The Form 2 aspect was to be taught in the Third term, while the Form 3 aspect was to be taught in the First term, the terms within which I scheduled my data collection. Therefore, I decided to observe the teaching and learning of quadratic expressions and equations, and then interview the teachers, and some volunteers among the students from both classes.

Having observed the steps that the GoK had taken when trying to improve the performance of mathematics, and the outcomes of those interventions, I realised that I needed to reflect on possible reasons that might be a hindrance to the expected improvement. One of the reasons I identified was the fact that the ASEI/PDSI approach in the SMASSE INSET project did not have a theoretical backing – this could have presented a clear road map on how to implement it within the classroom. The traditional teaching and learning approach, which has existed as a classroom culture for a long time, needs to be changed – perhaps that was why the teachers found it difficult to implement. In addition, the introduction of the Alternative B curriculum, without a change or improvement to the teaching approach, might have not been very effective in improving the students' performance.

As previously stated in section 1.1, the LS approach is backed by a theory (Variation Theory) which gives guidelines on how to monitor the teaching and learning process in the classroom and had been reported as being successful in some studies (Marton and Pang, 2014; Pang, 2008). The purpose of this study is therefore to find out how LS approach could contribute to the teaching and learning of mathematics in Kenyan secondary schools, a country with a different culture from where the approach has previously been applied. In addition, it is to find out the teachers' and students' perception on the application of LS approach to teach and learn mathematics.

1.5 Research Questions

Based on the purpose of my study cited in section 1.4, which addressed the topic of quadratic expressions and equations, I arrived at the following questions to guide this study:

1. What is the outcome when a learning study (LS) approach is applied to the teaching and learning of mathematics in a Kenyan cultural context?
2. What are the teachers' views on the application of a LS approach in the teaching and learning of the topic of quadratic expressions and equations, and with a possibility of extending the same to other topics?
3. What are the students' perceptions and experiences on the application of LS in the teaching and learning of the topic of quadratic expressions and equations?

1.6 Structure of the Thesis

This thesis in presented in eight chapters.

Chapter 2 discusses the relevant literature that shaped the study. I explain the historical background of quadratic equation before reviewing studies on the students' performance in the topic. I discuss lesson study in detail, explaining its historical background in Japan, before presenting its spread to other countries outside Japan along with the challenges the teachers in those countries encountered when trying to implement it. Then I discuss learning study (LS), explaining its similarities and differences to lesson study, and the Variation Theory, which is the theory behind the application of the LS approach. Since LS has not been applied in Kenya before, I discuss cultural practices, especially with regard to classroom culture, which is applicable to the Kenyan context. I conclude the chapter by discussing the application of Variation Theory as a theoretical framework to be used to analyse a lesson.

Chapter 3 discusses the methodology applied in this study. I present the research site, the participants and their selection, and I discuss the research design and approach that defines this study. In addition, I discuss the data collection instruments applied, the procedures used to collect data, and the data analyses processes. Different approaches were used to analyse the data collected through classroom observation and the data collected during interviews. I conclude the chapter by discussing the ethical considerations for this study.

Chapters 4, 5 and 6 present the data analyses of the six lessons which I observed, discussing them in pairs, one pair per chapter. In each chapter the two lessons are presented separately before the Variation Theory framework analysis.

Chapter 7 discusses the data collected from the teachers' and students' interviews. The data is organised and analysed using a thematic data analysis.

Chapter 8 presents the conclusion of this study. This chapter harmonises the findings of each of the analyses chapters, before discussing the teachers' and students' experiences with the new teaching approach in the Kenyan cultural context. In addition, some limitations of the study are covered, along with some proposed recommendations. Finally, I give a reflection on my whole PhD journey and a suggested way forward.

Chapter 2 – Literature Review Chapter

2.1 Overview

The literature review chapter comprises six sections namely:

- Quadratic expressions and equations
- Lesson study
- Variation theory
- Learning study
- Cultural issues in classroom practice
- Variation theory as a theoretical framework for lesson analysis

I begin the chapter by presenting the topic ‘quadratic expressions and equations’, the teaching and learning of which I have observed during this research study. In the discussion of the topic, I firstly justify its inclusion for observation in my research. Secondly, I will give a brief history of the origin of the topic before discussing the challenges that students face when solving quadratic equations. This research is on Learning Study (LS) approach, and I will discuss lesson study and Variation Theory before discussing LS. This is because LS draws its organisational structure from lesson study, and it applies the theoretical framework of Variation Theory in its classroom practice. In the discussion of lesson study, I will explain how it originated from Japan followed by an explanation on how it spread to other countries outside Japan. After that, I will explain some of the challenges faced by the implementers outside Japan. Next, I will discuss Variation Theory by explaining the aspects that make it applicable as a framework for monitoring the learning process in the classroom.

I will discuss LS by explaining its connection with lesson study within the teaching and learning process. Then I will explain the LS cycle as it incorporates the Variation Theory aspects in its organisational structure. Since I applied the LS approach in Kenya, which has a different classroom culture from the cultures where LS has been applied in the past, I will also discuss cultural issues in classroom teaching. I will look at different teaching approaches as they are applied in some other countries and discuss the challenges faced when changing classroom cultures.

The LS approach to teaching and learning includes a 'student task' during the teaching and learning process. In most of the studies reported, students usually carry out the task individually. However, in this research, I incorporated small group discussions. I did so because LS was an unfamiliar approach for the students and I judged from my teaching experience that they would not be able to do the task individually. I conclude the chapter by discussing Variation Theory as a theoretical framework for lesson analysis. In this discussion I include cases where the framework has also been used with non-LS lessons.

2.2 Quadratic Expressions and Equations

Quadratic expressions and equations is separated into two topics in the Kenyan mathematics curriculum for secondary schools. The first topic is stated as Quadratic expressions and equations and is taught in Form 2 (14-16 years) while the second, stated as Quadratic expressions and equations (2) is taught in Form 3 (15-17 years) ([KIE], 2002). In Form 2, students are taught the following subtopics: (1) Expansion of algebraic expressions such as $(p + 2)(p + 3)$. (2) Expansions of algebraic expressions of the form $(p + q)^2$, $(p - q)^2$ and $(p + q)(p - q)$, which are called "the three quadratic identities" ([KIE], 2002 p. 22). (3) Use of the three quadratic identities. (4) Factorisation of quadratic expressions. (5) Solution of quadratic equations by factor method. (6) Formation and solution of quadratic equations.

In Form 3, the following sub-topics are taught: (1) Perfect squares. (2) Completion of the square. (3) Solution of quadratic equations by completing the square. (4) Derivation of the quadratic formula. (5) Solution of quadratic equations using the formula. (6) Formation of quadratic equations and how to solve them. (7) Tables of values for a given quadratic relation. (8) Graphs of quadratic functions. (9) Simultaneous equations – one linear and one quadratic. (10) Applications to real life situations.

I observed the teaching and learning of quadratic expressions and equations for two main reasons. Firstly, as I mentioned in section 1.3, the reports from the KNEC usually mention the topic as one in which students do not perform well in the Kenya Certificate of Secondary Education (KCSE) ([KNEC], 2014). The report highlighted some questions that students found difficult to answer such as:

(1) Factorise $9a^2 - \frac{16}{b^2c^2}$, which is an application of the identity of difference of two squares.

(2a) Complete the table below (Table 4) for the equation $y = 3x^2 + 5x - 2$

Table 4: ***Table to be filled for the quadratic function $y = 3x^2 + 5x - 2$ for $-4 \leq x \leq 2$***

x	**-4**	**-3**	**-2**	**-1**	**0**	**1**	**2**
3*x*²	48			3	0		12
5*x*	-20		-10		0		10
-2	-2	-2	-2	-2	-2	-2	-2
y	26				-2		

(2b) On the grid provided draw the graph of $y = 3x^2 + 5x - 2$ for $-4 \leq x \leq 2$

(2c) Use your graph in (b) to estimate the roots of the equation $3x^2 + 5x - 2 = 0$

The report indicates that, in the first question, many students did not realise that the question was testing their understanding on the factorisation of the difference of two squares, which is taught under the three quadratic identities. Concerning the second question, the report indicates that many students could not fill in the table correctly, draw a smooth curve, and could, therefore, not determine the roots of the equation correctly. Also, this topic is internationally considered as a topic of concern as will be seen in the later sections of this chapter.

Secondly, amongst the topics mentioned by the [KNEC] (2014) report, the topic of quadratic expressions and equations was due for teaching and learning during my proposed data collection period, according to the Kenya secondary schools' mathematics curriculum and the Ministry of Education schools' term dates ([KIE], 2002; [MoEST], 2013).

I will begin the discussion of the topic with a brief look at the history of algebra and the inception of the topic of quadratic expressions and equations. Algebra spread to Europe, and to other parts of the world, from the work of Muhammad Ibn Musa Al-Khwarizmi, an Arab from Persia (Iran), as translated by Abraham bar Hiyya (Katz, 2009). The name 'Algebra' came from the word *al-Jabar*, which was published in Al-Khuwarizmi's Arabian science book named *al-Kitab al-mukhtasar fi hisab al-Jabar was-mu qabala* (Gandz, 1937). However, the word *al-Jabar* also has a Babylonian translation meaning 'Equation, Confrontation', or, the confrontation between two equal sides (Gandz, 1937, p. 409). Although Al-Khuwarizmi had written about algebra, especially the equations leading to the solution of quadratic equations, in his book, the equations seem to have originated from Babylon. His book had only three equations and these were called the three fundamental types of quadratic equations:

(1) $x^2 + ax = b$ $$x = \sqrt{\left(\frac{a}{2}\right)^2 + b} - \frac{a}{2}$$

(2) $x^2 + b = ax$ $$x = \frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 - b}$$

(3) $x^2 = ax + b$ $$x = \sqrt{\left(\frac{a}{2}\right)^2 + b} + \frac{a}{2}$$

These equations were part of the nine Babylonian equations, in which they tried to find the sides of a rectangle referred to as the rectangular equations, as shown in the next paragraph. The Babylonians worked with the two variables x and y representing the sides (length and breadth) of a rectangle:

(1) $x + y = a;\ xy = b$ $$\left.\begin{matrix} x \\ y \end{matrix}\right\} = \frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 - b}$$

(2) $x - y = a;\ xy = b$ $$\left.\begin{matrix} x \\ y \end{matrix}\right\} = \sqrt{\left(\frac{a}{2}\right)^2 + b} \pm \frac{a}{2}$$

(3) $x + y = a; x^2 + y^2 = b$ $$\left.\begin{matrix} x \\ y \end{matrix}\right\} = \frac{a}{2} \pm \sqrt{\frac{b}{2} - \left(\frac{a}{2}\right)^2}$$

(4) $x - y = a; x^2 + y^2 = b$ $$\left.\begin{matrix} x \\ y \end{matrix}\right\} = \sqrt{\frac{b}{2} - \left(\frac{a}{2}\right)^2} \pm \frac{a}{2}$$

(5) $x + y = a; x^2 - y^2 = b$ $$\left.\begin{matrix} x \\ y \end{matrix}\right\} = \frac{a}{2} \pm \frac{b}{2a} = \frac{1}{2}.\frac{a^2 \pm b}{a}$$

(6) $x - y = a; x^2 - y^2 = b$ $$\left.\begin{matrix} x \\ y \end{matrix}\right\} = \frac{b}{2a} \pm \frac{a}{2} = \frac{1}{2}.\frac{b \pm a^2}{a}$$

(7) $x^2 + ax = b$ $$x = \sqrt{\left(\frac{a}{2}\right)^2 + b} - \frac{a}{2}$$

(8) $x^2 - ax = b$ $$x = \sqrt{\left(\frac{a}{2}\right)^2 + b} + \frac{a}{2}$$

(9) $x^2 + b = ax$ $$x = \frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 - b}$$

Although the Babylonians listed the nine equations, they only worked with the first eight equations, referred to here as B1 – B8. They avoided the ninth equation due to the dual nature of the solution, which comes from the positive and negative square roots of a number. The Babylonians did not know the dual nature of the square root despite the fact that some of their calculations had room for two solutions. These two solutions were purely for finding the values of x and y, but not two values of the same quantity. The equations (1) and (2) are concerned with the application of the area to find the sides of the rectangle, while equations (3) – (6) make use of the diagonals to calculate the sides of the rectangle. However, the constructions of the questions were such that the equations were reduced to perfect squares when working out the solutions (Gandz, 1937).

From the solution to Al-Khwarizmi's equation (2), it appears as if he knew the dual nature of the square root, but later it was confirmed that he did not. The confirmation of Al-Khuwarizmi's equations, commonly referred to as the Arabic type of quadratic equations, was later done through the method of completing the square. The three Arabic equations are the ones that first reached Europe through Greece before spreading out to other parts of the world.

In Greece, Euclid confirmed the Babylonian equations through the geometrical approach and confirmed that the equations had been reduced to the three Arabic equations (Gandz, 1937). The Babylonians, and the entire community of European mathematicians, avoided the solutions to equation (2). They regarded the equation impossible to solve because it would sometimes lead to a negative number, considered embarrassing since the solutions represented the lengths of a rectangle. Bhaskara, a Hindu mathematician, announced the existence of the negative root. He affirmed that the square root of a number is twofold, positive and negative.

This short history of algebra, and solutions to quadratic equations in particular, shows that quadratic equations developed because of the need for geometrical problems to work out the sides of rectangles. There was a purpose to calculating the solutions and hence the link between algebra and geometry. Current textbooks, such as the Kenya Secondary Mathematics Pupil's Book ([KLB], 2003), do not explain this link and simply require the students to solve abstract quadratic equations applying given methods. Perhaps this explains why students generally find the topic difficult.

International studies have shown that students have difficulty in forming algebraic expressions (modelling) from word, statements, and diagrams (Clement, 1982; Clement et al., 1981; Didis & Erbas, 2015; Runesson, 2013). These studies have also shown that students have difficulty solving algebraic equations across all levels of schooling from primary and secondary through to tertiary institutions. Runesson (2013) carried out a study with year 4 and year 5 students in exploring the teaching and learning of sentence conversion to algebraic expressions. She found that students had difficulty expressing word statements such as “An ice-cream costs 5 kronor more than a coke” (p. 175), as algebraic expressions. Didis and Erbas (2015) worked with 10th-grade students on a study that looked at students’ performance in solving quadratic equations. They found that one of the difficulties the students met was with the formation of quadratic equations from word statements, which I will elaborate on shortly.

Stacey and MacGregor (2000) conducted research on the formation of algebraic equations from statements with high school students (13 to 16 years) in their third or fourth year. One of the questions showed a triangle drawn with its sides marked as x cm, $2x$ cm, and 14 cm, and the perimeter stated as 44 cm, and the students were asked to write an algebraic equation connecting the sides of the triangle with its perimeter. Sixty two percent of the students did not reach the correct equation.

Clement (1982) gave some tests to first-year university engineering students. The students were asked to form algebraic equations from statements. One such test question stated “Write an equation using the variables S and P to represent the following statement: there are six times as many students as professors at this university. Use S for the number of students and P for the number of professors” (p. 17). Clement found that about 40% of the students did not answer the question correctly, and many of these students expressed their equation as $6S = P$. He did a follow-up with these students where he asked them to represent the information in a diagram form, which they did as shown in Figure 2. However, the majority still wrote the equation $6S = P$.

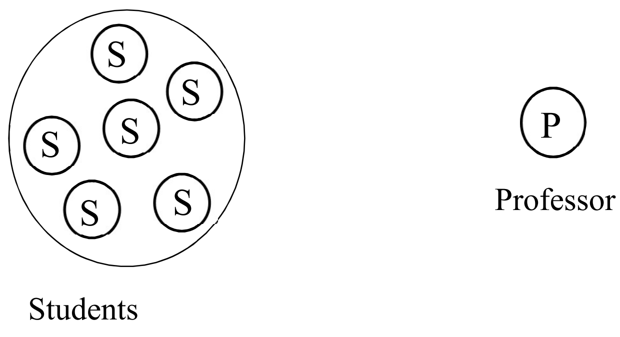


***Figure 2*: Diagrammatic representation of the statement, adapted from Clement (1982, p. 21)**

The examples in section 2.2 suggest that students may have difficulty in comprehending the statements. Didis and Erbas (2015) interviewed the students who could not solve word problems in quadratic equations and found that the problems were threefold: "(1) There were students who did not fully comprehend the word statements. (2) There were students who understood the problem; however, they did not know how to represent the information as a quadratic equation. (3) There were students who understood the problem and represented the information as a quadratic equation. However, they had difficulty solving the problem and thus also with interpreting it" (p. 1145). The findings of Didis and Erbas (2015) reveal that students have different problems with quadratic equations at different stages. The first case points to the command of language by students, to comprehend the statements and think of relationships in the statements such as 'the length is twice the breadth'. However, the first two stages seem to affect algebra in general. Clement's (1982) and Runesson's (2013) cases appear to fall under the second stage. However, Clement's (1982) students could represent the information correctly on the diagram but still failed to interpret the algebraic relation, a confirmation that students have problems in forming algebraic relations.

Other studies have also indicated that students have difficulty in solving quadratic equations (Didis & Erbas, 2015; Saglam & Alacaci, 2012; Stacey & MacGregor, 2000 Vaiyavutjamai & Clements, 2006). Stacey and MacGregor (2000) note that it is not only the learning of quadratic equations that is a problem to the students but also learning to solve other algebraic problems in general. They argue that the problems arise from the students' prior experience in solving arithmetic problems. They manifest in: (1) the meaning students give to 'the unknown', (2) their interpretation of what an equation is, and (3) the methods they choose to solve equations (p. 149). Vaiyavutjamai and Clements (2006) confirmed Stacey and MacGregor's assertion on the meaning that students give to the unknown. They realised that students were able to solve the equation $(x - 3)(x - 5) = 0$ and obtain the solution as $x = 3$ and $x = 5$. However, they insisted that the two x's have different values. When the students were asked to confirm their answers, they substituted 3 for x in the first bracket and 5 for x in the second bracket.

Concerning the choice of the method, Didis and Erbas (2015) noticed that the majority of students in their study solved the quadratic equations using the quadratic formula. In some cases, students expanded the already factorised equation such as $(x - 3)(x - 5) = 0$ and applied the quadratic formula to solve the equation. The observation supported the finding of Tall et al. (2014), which showed that none of the students solved the already factorised equation $(y - 3)(y - 2) = 0$, by the factorisation method. All the students who attempted the equation opened up the factors on the left-hand of the equation and applied the quadratic formula to solve the equation. Tall et al. (2014) argue that students lack a conceptual understanding of quadratic equations and that is why the majority of them resort to quadratic formulas, which are procedural in solving the equations.

Didis and Erbas (2015) observed that a few students from their study applied the completing square method to solve some problems such as $x^2 + 2x - 1 = 0$. However, many of them did not calculate the problem correctly. The students' mistakes arose from: "(i) failure to add numbers on both sides correctly and (ii) failure to convert the left-hand side of the equation to its squared form" (p. 1144).

However, as Stacey & MacGregor (2000) explained, students' prior knowledge of arithmetic influences their solution of algebraic equations. Didis and Erbas (2015) observed that many students failed to solve the problems correctly, due to arithmetic errors. The students either "(i) Computed the discriminant incorrectly because of calculation errors or could not compute it at all. (ii) Computed the discriminant correctly, but applied the quadratic formula incorrectly, since they had misremembered it. (iii) Computed the discriminant incorrectly but they applied the quadratic formula correctly" (p. 1143).

Tall et al. (2014) noted in their study that some students applied the difference of two square methods to solve the equation $9x^2 - 25 = 0$, but they ended up with only one solution. The students failed to recognise the negative value when working out the square root of 25. Other students resorted to the method of factorisation to solve the equation $x^2 - 5x + 6 = 0$, but many of them failed to obtain the correct solution; the most common mistake was a failure to obtain the necessary factors of the constant term 6 that sum to -5. Most of them indicated the factors as -6 and 1.

Tall et al. (2014) confirmed Stacey and MacGregor's (2000) assertion that students' difficulties in solving algebraic equations arose from their interpretation of an equation. Tall et al. (2014) reported that when they asked the students "What is an equation?", many students responded that "it is a calculation in mathematics" (p. 6). Some students who were unable to remember methods of solving quadratic equations still worked out some solutions from their interpretation of the equation – one student worked out the equation $3l^2 - l = 0$ as shown in Figure 3.

$3l^2 - l = 0$

$9l - l = 0$ – squaring 3 instead of l

$8l = 0$

$l = \frac{8}{0}$ – dividing by 0

$l = 8$ – perhaps thinking that zero has no effect

***Figure 3*: Solution to a quadratic equation by high school students 16 years old. Adapted from Tall et al. (2014, p. 8)**

This student's work shows that he/she had the zeal to obtain a solution. Arising from their explanation that an equation is a calculation, the student was determined to work out a solution. The statement reinforces Stacey and MacGregor's (2000) finding on the effect of prior arithmetic thinking in understanding algebra, where students have a compulsion, in the calculation, to obtain a solution. However, De Bock, Van Dooren, Janssens and Verschaffel (2002) call this type of solution, "a misuse of linearity in a non-linear situation – sometimes referred to as the illusion of linearity (or proportionality)" (p. 312).

Vaiyavutjamai and Clements (2006) note that there is a limited number of studies in the field of mathematics education concerning quadratic equations. A few foci mostly on the students' difficulty in solving quadratic equations, while others focus on the analysis of the amount of time allocated to quadratic expressions and equations in various curricula. They also observed that there is scarce research on the teaching and learning of quadratic equations. Many of the studies confirm that teaching of quadratic equations follows a traditional approach. Vaiyavutjamai and Clements (2006) argue that a traditional method uses a set order of "review of previous lessons' work, introduction, model example, seatwork and a summary of the lesson" (p. 47). The teacher leads in the talking with many chorus answers. Gray and Thomas (2001) reported on a study of the use of graphical calculators as an aid to learning quadratic functions and graphs, but the teaching followed a traditional method. The outcome of their work did not reveal any improvement in the fluency and understanding of the quadratic functions, which they had expected.

In all of the articles reviewed, the authors concluded by suggesting that there is a need for a teaching methodology that involves the students in the learning of the topic. Vaiyavutjamai and Clements (2006) asked "Are there realistically feasible forms of teaching that will result in students, and not just high-achieving students, learning quadratic equations, and other mathematics topics, in a relational way?" (p. 73). Stacey and MacGregor (2000), argue that some teachers promote non-algebraic methods to be applied by students when solving quadratic equations, claiming that they are easy for students to understand. However, they suggested that teachers should provide opportunities for students to embrace the use of algebra, and to learn methods that are more powerful. Saglam and Alacaci (2012) suggested that more time should be allowed for the teaching and learning of quadratic expressions and equations in various curricula.

Having looked at the suggestions and conclusions of these studies, which are international, discussed in section 2.2, I designed my research on a Learning Study (LS) approach with a conceptual framework that allowed students to learn through small group discussions, followed by a whole class discussion. The approach was meant to involve learners in the learning process of quadratic expressions. LS allows at least two teachers to interact during the lesson preparation, as well as obtaining students' views before lesson preparation. However, before discussing LS and the conceptual framework, I first discuss lesson study (considered a precursor to LS), whose organisational structure is applied by LS.

2.3 Lesson Study

Lesson study originated in Japan and is expressed in Japanese as *Jugyokenkyu – jugyo* means *lesson* and *kenkyu* mean *study* or *research*. It began in the 19^{th} century (early 1870s), in two schools in Tokyo. The schools served as laboratory schools for student teaching and studying, as well as places for experimenting with new teaching methods (Ono & Ferreira, 2010). The practice became popular in Japan after 1960, which is when the Japanese education system adopted it. Lesson study is a type of classroom research conducted and directed by classroom teachers. A group of teachers collaborate to prepare a research lesson, one of them teaches the lesson, with the remaining teachers (or other people) observing the lesson, and then the whole group converges to reflect on the lesson. Lesson study is organised on selected topics, concepts or skills considered as areas of concern by education stakeholders, teachers and students (Fernandez, 2002; Ono & Ferreira, 2010). Research lessons are sometimes triggered by changes within the curriculum especially when a new topic area is added to a curriculum, for example, a topic such as environmental science (Lewis & Tsuchida, 1998).

In Japan, the organisation of lesson studies is at various levels including: school, prefecture (local region), and regional and National (Lewis & Tsuchida, 1998). At all levels, the research team comprises four to six teachers, working together for two to four weeks, preparing the research lesson. One of the teachers implements a prepared research lesson to his/her students. As I have mentioned, teachers within the school (and some guests), observe the lesson during teaching and learning. Sometimes the research lesson is open to teachers from a town, district, prefecture, region or national level

(Lewis &Tsuchida, 1998). Some of the guests are lesson study experts invited to make final remarks on the lesson. Lesson study has four main parts; Formulation of research study goals, research-lesson planning, implementation, and reflection (debriefing) after the lesson as shown in Figure 4.

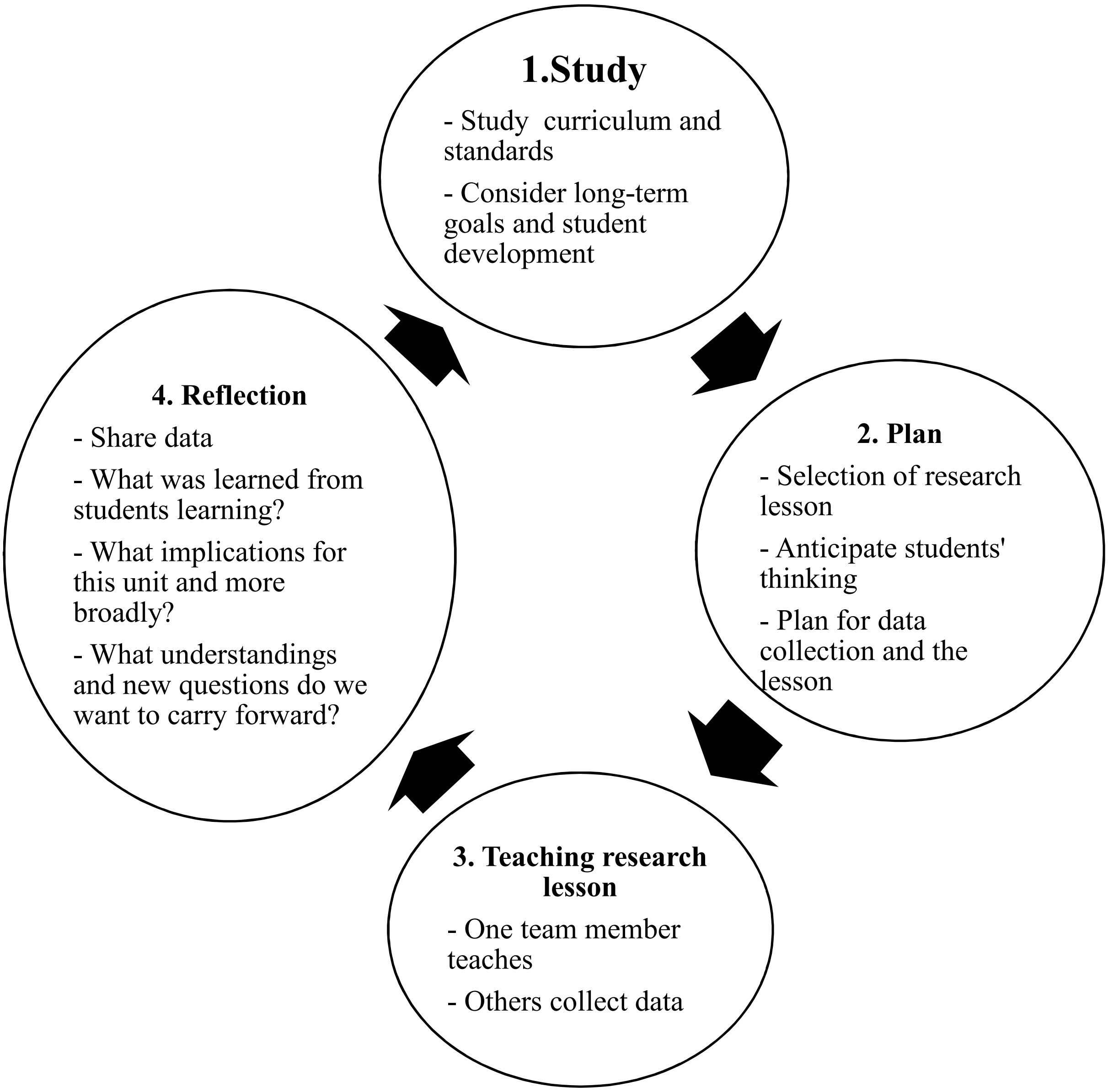


***Figure 4*: Lesson study cycle, adapted from Lewis, 2009, p. 97**

Research study goals are broadly stated and include such statements as: helping students to take the initiative as learners, be active problem-solvers, becoming more involved in learning mathematics from each other, developing their perspectives and ways of thinking (Lewis, 2009; Lewis & Tschuda, 1998). These goals are in line with the Japanese national curriculum. Sometimes teachers identify research goals from current debates in the education cycle. For example, a discussion such as mathematics learning, should foster high order thinking skills (HOTS) in students. In such a case, the broad objective of a lesson study would help learners to acquire HOTS through the learning of mathematics.

Upon identification of the study goals, the team works collaboratively by consulting various materials that include curriculum/syllabus, textbooks, the internet, research papers and other printed articles (Takahashi & McDougal, 2016; Lewis, 2009). The team studies the materials in detail for some time before embarking on the development of a lesson plan - a process referred to as *Kyouzai kenkyuu* (the study of materials for teaching) (Takahashi & McDougal, 2016; Yoshida, 2012). The teachers then plan the research lesson. Among the things they consider when planning are: (1) Decision on a key question for the lesson and a plan for the anticipated students' responses to the key question, and (2) observation of the lesson, including key areas to be observed in line with the theme of the lesson (Yoshida, 2012; Wake, Swan & Foster, 2016). During this planning time, the team usually receive some support from "knowledgeable others". These are experts in lesson study who understand the content of the subject. They provide some advice to the team on matters pertaining to the research lesson.

The observers comprise research lesson team members, administrators from the school, the subject teachers from other schools, university lecturers from departments/schools concerned with the research subject and knowledgeable others/experts. In mathematics lessons, teachers usually apply a problem-solving approach to teaching. They pose the key question and allow the students some time to work on the problem. The teacher and the observers walk around the class to check students' progress, and the teacher asks some students – representing different approaches – to present their work for whole-class discussion (Takahashi, 2009). The observers use a lesson plan checklist to observe the lesson and make comments. They also observe students' working and behaviour during the lesson. It is important to note that, during the research lesson, all the work shown on the board remains from the beginning to the end, as Takahashi (2009) explains,

> it is to keep a record of the lesson, help students to remember what they need to do and to think about. Help students see the connections between different parts of the lesson and the progression of the lesson. Compare, contrast, and discuss ideas that students present. Help to organise students' thinking and discovery of new ideas and foster organised student note-taking skills by modeling good organisation. (p. 6-7)

Usually, the debriefing (reflection) session takes place in the same classroom where teaching took place. This is to help the observing team refer to notes on the chalk board. The invited knowledgeable person gives his/her key comments last. If, in the opinion of the observers, there is a need to re-teach the lesson, the research lesson team revises the lesson plan and another member of the team re-teaches the modified version of the lesson to his/her class. Otherwise, the research team reports and documents the research findings. Re-teaching a research lesson is not a common practice in the Japanese lesson study cycle, and usually the observers' comments are only used to enrich the report (Takahashi & McDougal, 2016).

Essentially, Japanese lesson study is a broad-based, teacher-led system for improvement of teaching and learning (Cerbin & Kopp, 2006; Wood, 2014). Due to the frequent implementation of research lessons in Japan, lesson study has been adopted by the Ministry of Education as a continuous professional development (CPD) programme for newly recruited teachers and teachers who have been in the field for five years (Fernandez, 2002).

In addition, it is being adapted in Japanese classrooms as a teaching approach, especially in mathematics (Fernandez, Cannon & Chokshi, 2003; Groves, Doig, Widjala, Garner & Palmer, 2013; Lewis, 2009; Shimizu, 1999). During the lesson, the teacher applies problem-solving teaching strategy, which Stigler and Hiebert (1999) refer to as “structured problem-solving” (p. 36). The teacher poses the key question and students individually work on the problem. Some students with different approaches to the solution are asked to present their work for a whole-class discussion. Usually, observers eagerly wait to witness what is called “the critical moment of the lesson” during whole class discussion. It is the moment when students experience learning of the content and they express their feelings by such chorus expressions as “Ooh” or “yes” to signal the understanding of the content. The exclamation is what Banes (2000) calls the “Aha! Experience”, which is the sudden flash of understanding of a problem.

I attended a one-month seminar in Japan (August/September 2013), as discussed in section 1.1, where the group observed a series of mathematical problem-solving teaching in a lesson study approach. Other observers included a team of mathematics education professors, mainly from Japan, USA, and Australia. In most of the lessons we witnessed the “Aha! Experience” (critical moment) during whole class discussion. However, in one of the lessons, no critical moment of learning (Aha! Experience) was witnessed, and there was no debriefing/reflection session convened after that. The fact that the observers did not witness the critical moment may not amount to a lack of understanding of the content by the students. There may be a need to adopt a different way of monitoring the learning process.

Following the report by Stigler and Hiebert (1999) on the “teaching gap”, which came about as a result of the outcome of the Third International Mathematics and Science Study (TIMSS), where Japanese students outperformed those in other countries, lesson study expanded worldwide beyond Japan. The argument was that perhaps the lesson study approach to teaching practice in Japanese classrooms was what brought about the good performance (Stigler & Hiebert, 1999; Wood, 2014). The practice has since spread to North America, Europe (especially the United Kingdom), Australia, Asia, and, to a small extent, Africa (Dudley, 2011; Fernandez, 2002; Groves et al., 2013; Huang, Su & Xu, 2014; Lewis, 2009; Lewis et al., 2006; Ono & Ferreira, 2010; Stigler & Hiebert, 1999). Chen and Yang (2013) observed that in China, lesson study had been in existence for over 100 years. However, the implementation had challenges due to shifts in focus and purpose. At one point the focus was on teachers' behaviour, but this later changed to students' learning.

Lewis, Perry, Hurd and O'Connell (2006) reported a successful implementation of a Japanese lesson plan in one of the schools in the United States. It all started when the coordinator of a cluster of schools in the district was looking for a professional development model that would support sustained teacher-led improvement of classroom instruction. Mathematics teachers in one of the schools in the district took up the initiative, and prepared research lessons which they taught, refined, and discussed with other teachers within the district. In the following year the coordinator became the principal of the school and she entrenched the practice in the school and spread it to other subjects.

The authors reported some insights experienced by the teachers of the school such as: (i) Lesson study is about teacher learning, not just about lessons, therefore, lesson study is a teacher's research, not improvement, of a lesson plan. (ii) Effective lesson study focuses on skillful observation and subsequent discussion. (iii) Turning to outside sources of knowledge enhances the quality of lesson study. In the beginning, teachers felt that they were knowledgeable enough and did not need to consult with others outside their circle. However, later they consulted widely with district subject specialists, educationalists, and others who were knowledgeable about lesson study, during research lesson preparation. (iv) Phases of the lesson study cycle are balanced and integrated. Initially, teachers were taking too much time discussing a research lesson (as if it was a final product), but they later learned that one research lesson acted as a catalyst for further studies and improvements.

Fernandez (2012) notes that generally the purpose of conducting lesson study is to enhance the teachers' capacity in areas such as content knowledge, pedagogical content knowledge, and curriculum knowledge, which would systemically improve instruction and student learning in the classroom. However, many of the countries aside from Japan, are experiencing some challenges with the implementation of lesson study. Chokshi and Fernandez (2004), Fernandez (2012), and Takahashi and Yoshida (2004) highlighted some of the challenges experienced by US teachers in implementing lesson study.

They broadly grouped the challenges into five categories: (i) Misunderstanding and/or lack of understanding of lesson study. Many practitioners (teachers) and administrators learned about lesson study by reading research studies such as The Teaching Gap (Stigler and Hiebert, 1999). They thought that the idea of lesson study was to develop exemplary lesson plans to be implemented by teachers in the future. (ii) Insufficient content and pedagogical knowledge of teachers. Teachers should acquire strong content knowledge and pedagogical knowledge and skills. These will help them anticipate student learning behaviours as well as develop appropriate content and its methodology. (iii) Lack of support and resources to conduct high-quality lesson study. For teachers to conduct quality lesson study, they need to have in-depth *kyouzai kenkyu*. Teachers need to mobilize necessary and relevant materials capable of helping them improve their content and pedagogical content knowledge. (iv) Non-systematic approach to conducting effective lesson study. Lesson study is about teacher collaboration in order to enhance learning. Teachers need to exchange ideas with other colleagues within the school, district, and nationally, and to allow other people to observe and critique their lessons. (v) Short-sightedness in planning for improvement and lack of time for professional development. Many US classroom teachers do not have much time outside classroom work to prepare a lesson study. Preparation requires enough time to have meaningful *kyouzai kenkyu*. All administrators need awareness about lesson study so as to ensure continuity with the practice, even when the school acquires a new administrator. Doig, Groves and Fujii (2011), and Ono and Ferreira (2010) reported similar challenges from their studies in Australia and South Africa respectively.

Elliot (2014), and Wood (2014) argue that the problem with the implementation of lesson study outside Japan is the lack of an explicit theoretical framework that guides its implementation, although they suggest that Japanese lesson study appears to be implicitly guided by constructivist theory. Perhaps this suggestion arises from the interactive nature of lesson study in which teachers work collaboratively to prepare and execute a lesson. Perhaps it is this lack of explicit theory that causes many countries to face challenges with the implementation, as has been narrated, especially with regards to monitoring the learning process. The teachers in Japan have internalised lesson study and are able to identify the "Aha! Experience" in the lesson, but this may not be enough to confirm that learning has taken place. Sometimes only one or two students in the class may exclaim to signal the critical moment of the lesson, while other students may not have understood the concept. In other classroom cultures, students may not express their understanding of a concept by an exclamation. In such circumstances, it would be difficult to ascertain how much learning has taken place.

Due to these unresolved arguments, especially the lack of an explicit theoretical framework, I decided to situate this study in a Learning Study (LS) approach, guided by the Variation Theory of learning. The theory has structures that help to monitor the classroom learning process. In fact, Elliot (2012) suggested that "Lesson study when informed by an explicit learning theory, such as Variation Theory, provides a strong basis for the development of a practitioner-based science of teaching" (p. 108). Before discussing LS, I will present Variation Theory in the next section.

2.4 Variation Theory

This theory is attributed to the work of Ference Marton, a professor of education at the Gothenburg University in Sweden. He is a Swedish educational psychologist best known internationally for introducing the distinction between deep and surface approaches to learning, and developing phenomenography as a methodology for educational research (Lo, 2012; Marton, 2015). "Phenomenography is interested in the 'qualitatively' different ways in which people experience the same thing or phenomenon" (Lo, 2012, p. 18). It is this idea of people experiencing things in different ways that Marton explored when developing the concept of Variation Theory as a learning theory (Lo, 2012; Marton & Booth, 1997). Lo (2012) explains that one cannot know what something is without knowing what it is not, which is the idea of Variation Theory. For example, "one cannot understand base-ten system without having come across other number systems with other bases" (p. 5).

Pang and Marton (2003) explain that "When you educate people you want to prepare people for a future, which is entirely or partly unknown. Moreover, you try to prepare people by helping them learn what is known" (p. 181). In other words, you prepare others to learn from known to unknowns. According to Marton, Runesson and Tsui (2004) "Learning is the process of becoming capable of doing something ('doing' in the wide sense) as a result of having had certain experiences (of doing something or of something happening)" (p. 5).

They (Marton et al. (2004)) explain that Variation Theory, which is a theory of learning, proposes that learning is always directed towards an object, which is the content, and could be a skill or a concept referred to as the object of learning (Lo, 2012). The object of learning differs from the educational learning objectives, which from their statements point to the end of the process of learning. They relate to what students can do at the end of the lesson. Learning objectives suggest that the result of learning is predetermined. However, "the object of learning refers to what the students need to learn to achieve the desired learning objectives. So, in a sense, it points to the starting point of the learning journey rather than to the end of the learning process" (Lo, 2012, p. 43).

There are two aspects of the object of learning namely, the specific (direct) object of learning and general (indirect) object of learning (Marton & Booth, 1997). The direct object of learning refers to the content to be learned, while the indirect object of learning refers to what the learner is supposed to become capable of doing with the content (Lo, 2012). For example, when teaching the concept of ratio to high school students, the presentation should be in such a way that the learning of it helps the students to apply the concept in an actual life environment. The students should be able to explain how the concept of ratio, for example, is applicable in the consumption of goods and commodities produced.

Marton (2015) explains that Variation Theory of learning "enables learners to deal with the future novel situations in powerful ways" (p. 26). The learners have to notice some aspects that they have not been able to learn. These aspects are called critical aspects and critical features of the object of learning. "Critical aspect refers to a dimension of variation, whereas critical feature is a value of that dimension of variation (Lo, 2012, p. 65). Pang and Marton (2003) explain, "to discern a particular feature; the individual must experience variation in the dimension corresponding to the feature in question" (p. 181).

The critical feature and the critical aspect are not mutually exclusive, and they are discerned simultaneously during the lesson. For example, suppose that helping students to solve simultaneous linear equations by elimination is the object of learning. The elimination method becomes the critical aspect (dimension of variation), and the critical feature becomes the collating of the equations so that one variable has the same numerical coefficient in both equations. When we think of having variables in an equation with the same numerical coefficient, the dimension of elimination method comes to mind. The critical feature is embedded in the critical aspect, and the two are discerned simultaneously. Therefore, when teachers prepare the lessons, they avail necessary conditions for learning, so that students are able to discern the object of learning by focusing on the critical feature(s).

During learning, teachers focus on both direct and indirect objects of learning by creating necessary conditions for the students to learn the content in a way that they can apply it in a new environment. The approach means that the object of learning is dynamic, that is, it can change in the process of learning depending on students' reactions to the conditions of learning availed (Lo, 2012; Marton, 2015).

The prepared lesson on the critical aspect/critical feature that the student should discern is called the *intended* object of learning. The intended object of learning is the teachers' view of the lesson. The presentation of the critical aspect that makes it possible for discernment by students is the *enacted* object of learning. It is the observers' view of the lesson. The critical aspect that the students discern after the lesson is the *lived* object of learning. It is the object of learning as experienced and learned by the students.

These categories of the object of learning help in monitoring the learning process during teaching, and they also form the basis of lesson analysis. However, as teachers provide necessary conditions of learning as the enacted object of learning, they vary some components of critical aspects as others remain invariant. Marton, Runesson, and Tsui (2004) explain that "No conditions of learning ever cause learning. They only make it possible for learners to learn certain things" (p. 22-23). Therefore, teachers simply set scenes conducive for learning with a hope that, in the end, learning takes place. These necessary conditions of learning that teachers provide are the *patterns* of variation and invariance (Lo, 2012; Marton, 2015; Marton, Runesson & Tsui, 2004). Lo (2012) explains that "To discern the critical features of novel situations, or to discern previously taken for granted features of familiar situations, learners must experience for themselves certain patterns of variation and invariance of these features" (p. 83). Sometimes, the patterns of variations and invariance may not be explicitly expressed, but only implied, in the enacted object of learning. Marton (2015) explains:

> As learning, to a great extent, is about separating and bringing together aspects, as parts and wholes of the world around us, and as the experience of certain patterns of variation and invariance is a necessary condition for that to happen, the fact that learning is taking place implies that the learners are experiencing certain patterns of variation and invariance. (p. 175)

According to this statement, patterns of variation and invariance can be observed even in non-Variation Theory design lessons. However, Lo (2012) explains that students can learn new aspects and construct new meanings when there is a deliberate action to vary certain aspects/features while others remain systematically invariant. Marton (2015) explains that the simplest pattern of variation and invariance involves two aspects in which one is brought to focus (critical aspect), and the other is not highlighted. Both Lo

(2012) and Marton (2015) identify four patterns of variation and invariance namely: contrast, generalisation, separation, and fusion.

(1) **Contrast**. Marton (2015) describes contrast as awareness brought about by experiencing the difference (variation) between two values. For example, to discern the concept of cardinal number four in a decimal base, students must experience other numbers that are not four, against an invariant number four. That is, four is the critical feature and brought to focus while other numbers in the same dimension (aspect), such as three or five, are varied.

(2) **Generalisation**. Consider the example given under contrast of discerning the concept of the number four. For students to understand the concept fully, they must experience varying appearances of the concept. That is, they need different forms of four items, such as four books, four pens, and four oranges. The critical feature (four) remains invariant as the forms vary. The students would be able to generalise the cardinality of the number four as representing magnitude.

(3) **Separation**. Lo (2012) describes separation as a situation where:

> The learner previously treated this object as an undivided whole, but after becoming aware of the value (feature) and its dimension of variation (aspect) is capable of focusing on the value independently […] The value becomes visible by opening up the dimension of variation in which it is a value. In this way, the value is separated from the object of which it is a feature. (p. 90)

For example, a student would initially treat all quadratic equations as being the same. However, after learning the factor method as solutions to quadratic equations, students would be able to use the conditions to separate quadratic equations that are solvable by the factor method. The students open up the dimension of variation (aspect), which are solutions of quadratic equations and separate the feature, which is the factor method. They can now learn the factor method independently from the general solutions of quadratic equations.

(4) **Fusion**. Marton (2015) defines fusion as the relationship between two or more critical aspects that vary simultaneously. There are cases where the learner needs to

consider some aspects of variation simultaneously in order to aid their learning. For example, when teaching the comparison of fractions, both the numerator and denominators vary. However, during the learning process, a teacher may teach the lesson by first keeping the denominator invariant while varying the numerator, for example $\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}$. Students would be able to discern that as the numerator increases, the fraction becomes bigger. After that, the teacher keeps the numerator invariant while varying the denominator, for example $\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5}$. Students would be able to discern that as the denominator increases, the fraction becomes smaller. To compare two fractions such as $\frac{3}{4}$ and $\frac{5}{8}$, the students have to work out the equivalent fractions of the two simultaneously with the same denominator in order to be able to compare the fractions. For example, the LCM of 4 and 8 is equal to 8. The students would work out the equivalent fraction of $\frac{3}{4}$, to obtain $\frac{6}{8}$, hence they would be able to compare $\frac{6}{8}$ and $\frac{5}{8}$. Both separation and fusion patterns of variation and invariance apply an arrangement of part-whole approach to discern the object of learning.

By applying the patterns of variations and invariance during lesson implementations (enacted object of learning) teachers create necessary conditions of learning to help students experience the variations that, in the end, help them to discern the object of learning. The proponents of the Variation Theory assert that there can be no learning without discernment, and there can be no discernment without experiencing variation (Lo, 2012; Marton, 2015; Marton & Booth, 1997). In this research, as discussed in Chapters 4, 5 and 6, the teachers applied various patterns of variation and invariance, as necessary, to discern the objects of learning.

Variation Theory of learning has been applied remotely in general classrooms, especially in the application of the patterns of variations. Many lessons, not designed in a Variation Theory approach to teaching and learning, have been analysed using the

Variation Theory framework (see Section 2.7). However, LS, as a teaching approach, applies Variation Theory as the theory informing its classroom implementation. In this research, I applied a LS approach in teaching the factorisation of quadratic expressions and solutions of quadratic equations. In the next section, I discuss LS as the teaching approach of this research.

2.5 Learning Study

Learning Study (LS) originated in Hong Kong in 2000 according to Marton and Runesson (2014). They explained that the Government of Hong Kong shifted their education system from the elitist approach to mass education, and was concerned with how teachers dealt with the diverse range of talents and abilities of their students. The Government commissioned and funded five projects to address the disparity. One of the projects was LS, undertaken by lecturers from the University of Hong Kong and the Hong Kong Institute of Education (Marton & Runesson, 2014). The project marked the inception of LS and Ference Marton first publicly presented it through a lecture in 2001.

The introduction of LS to Sweden was through a research project in 2003 funded by the Swedish Research Council (Marton & Runesson, 2014). The implementation of LS in Hong Kong was in the context of school development, while in Sweden it was in the context of research. LS has not significantly spread to other countries. The studies published about LS mostly come from either Hong Kong or Sweden, with a few from other countries such as the UK, Singapore, the USA and South Africa. Marton (2015) hoped, that by the end of 2015, there would be 1,000 completed studies on LS, as he states:

> The learning study model spread in Hong Kong and also in some other parts of China, and subsequently in Sweden and quite a few other countries. I would guess that by the time this book is published, the number of completed learning studies will approach 1,000 altogether. (p. 276)

The statement shows that LS, as a research entity, is still in its formative stage and more improvement on its practice may still need to be envisaged.

Marton and Runesson (2014) explain that:

> A LS is simply a study of the relationship between learning and the conditions of learning, carried out by a group of teachers, with the double aim of boosting the participating teachers' ability to help their students to learn, on the one hand, and to produce new insights into learning and teaching that can also be shared with teachers who do not participate in the study, on the other hand. (p. 104)

From the explanation, it may mean that LS supports teachers' professional development as much as it enhances students' learning process. In addition, Marton and Runesson (2014) explain that LS is a methodological concept and not a theoretical concept. By that they mean that LS can apply some theories to classroom practices. In many studies, the LS approach in the classroom practices has applied Variation Theory as its theoretical framework. However, that does not rule out the application of other learning theories in the LS classroom practice.

LS draws its organisation structure from lesson study, where a group of teachers prepares a lesson based on information gathered from the students, and resource materials such as textbooks, syllabus/curriculum, the internet and other research papers. One of the teachers teaches the lesson, others observe the lesson, and research data is collected before they all converge after the lesson for a reflection session (Marton & Runesson, 2014; Pang & Marton, 2005; Pang, 2008; Runesson, 2013).

In addition, teachers apply the LS structure by systematically applying Variation Theory practice in a classroom situation (Runesson, 2013; Wood, 2014). LS uses the components of Variation Theory to guide the process on when and how each component is applied during the learning process. Teachers use the theory as a tool and resource to design lessons, and to contribute practical ideas and classroom experiences in the innovative lesson preparation and implementation (Pang, 2008; Pang & Marton, 2005). Lo (2012) groups the variations concerned with the LS framework into 3 categories namely V_1, V_2 and V_3.

(1) **V_1**: Variation in students' ways of understanding the object of learning. That is, the identification of students' existing understanding of the object of learning so that any difference (gap) is properly organised, addressed and utilised in teaching.

(2) **V_2**: Variation in the teachers' understanding and ways of dealing with the object of learning. That is, teachers' experience, pedagogical content knowledge and knowledge of the students. Through collaboration, teachers and researchers discuss different ideas and approaches to teaching a given topic.

(3) **V_3**: Using variation as a guiding principle of pedagogical design. That is, with the knowledge gained from V_1 and V_2, teachers identify the critical feature and they design patterns of variation and invariance that can bring about the desired outcomes (p. 31-32).

In LS approach, the lesson is prepared on the premise of the object of learning, which is dynamic and can change in the process of learning as teachers and students interact (Lo, 2012). The object of learning is established by gathering information about the intended content/topic by consulting with the students on their prior knowledge, learning difficulties and their conceptions about the content, and as well as perusing through the syllabus, textbooks, research article(s) and other related resources. For the ease of monitoring the learning process, the object of learning is further categorised into: lived object of learning 1 and 2, intended object of learning, and enacted object of learning. Figure 5 shows the LS cycle, indicating parts where various objects of learning are applicable.

The group discusses the topic, mostly based on their experience with the topic's difficulty to teach or to learn. They isolate the concept or skill to be addressed and discuss how the learning should be made possible by considering the necessary conditions for learning the concept.

The concept or the skill thus becomes the object of learning. The group prepares a well thought about diagnostic pre-test or interview, to be given to the students in order to obtain their prior conception of the object of learning and to identify any gap that may exist within their understanding. The outcome of the pre-test/interview becomes the lived object of learning 1.

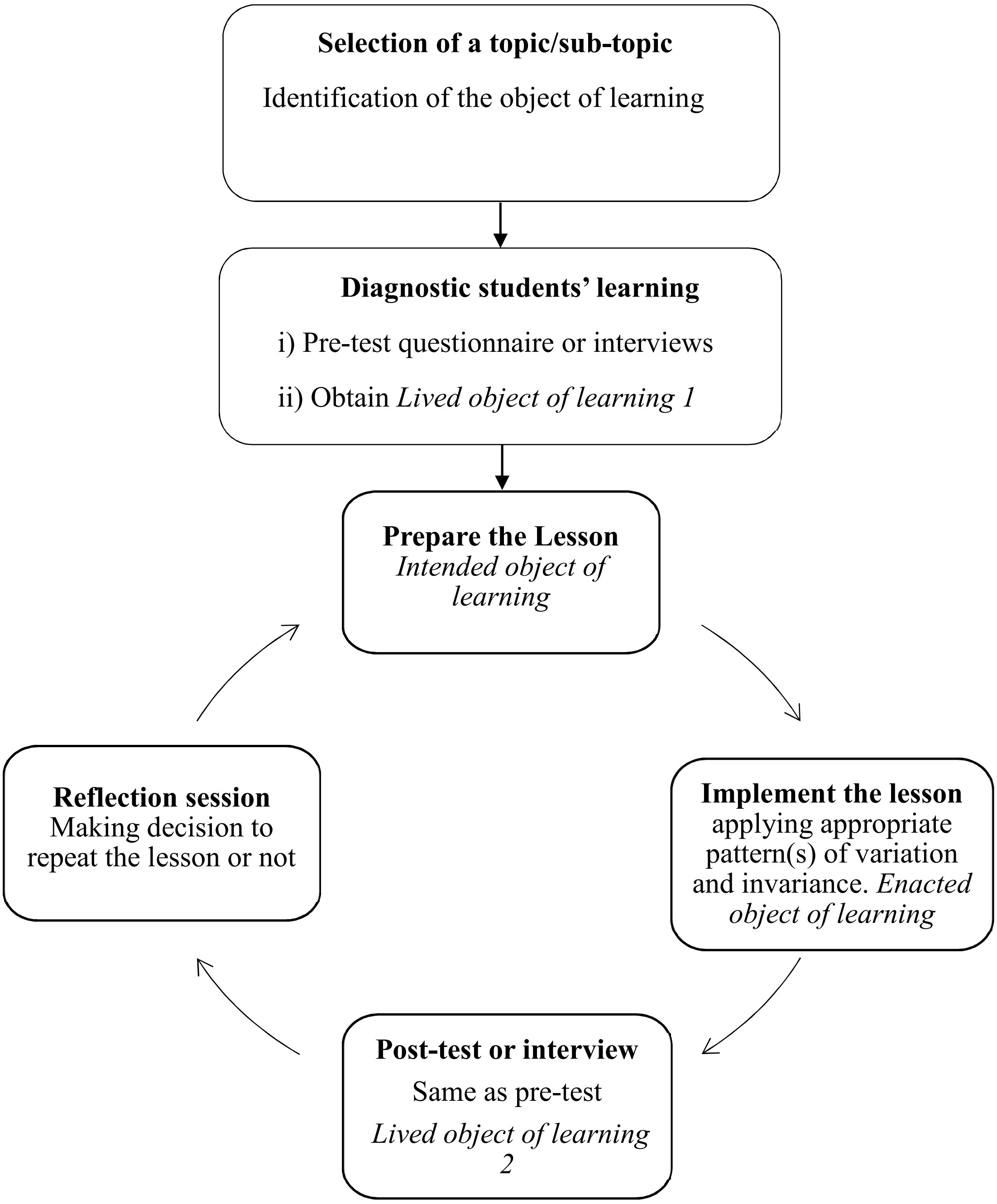


***Figure 5*: Learning Study cycle, adapted from Runesson, 2013, p. 173**

Based on the outcome from the students' pre-test and the teachers' past experiences, as well as the knowledge of the subject matter, the team discusses and identifies a critical feature to be focused on in order to discern the object of learning. The discernment of the object of learning leads to the achievement of the learning objective(s) as specified in the syllabus (Marton, 2015).

After deciding on the critical feature, the team comes together to prepare a lesson intended to help discern the object of learning, hence the name "intended object of learning". The preparation of the lesson (which usually involves three to four teachers, meeting about three times, for two to three hours (Marton, 2015)), involves in-depth discussion on the materials gathered about the concept of study. It is at this preparation stage that the team decides on the type(s) of patterns of variation and invariance to apply during the lesson.

Thereafter, one teacher teaches the lesson, while other teachers and the researcher(s) observe the lesson and collect data. This session is the main data collection session. The object of learning may be modified, within the process of learning, depending on how the lesson unfolds. The modification would be done in order to accommodate emerging issues, and this is the practicability of the lesson. The actual lesson implemented is the enacted object of learning.

After the lesson, in most cases, a post-test (the same as the pre-test) is given to the students. The outcome of the post-test is part of the lived object of learning 2. It shows the students' experienced awareness and discernment of the object of learning during the lesson (Marton & Booth, 1997). Marton (2015) states:

> The main question in a LS is the relation between the learning that is made possible on the one hand (enacted object of learning) and what is actually learned on the other (lived object of learning). This is usually done by having the students repeat at the end of the lesson the test they did at the beginning of the lesson. The difference between results from the pre- and post-lesson test is an indication of what the students have learned during the lesson (p. 264).

The post-lesson test is supposed to reveal the contribution of the lesson to the understanding of the intended knowledge or skill of the lesson, beyond the prior knowledge the students held before the lesson.

The team comprising teachers and the researcher, converge in a post-lesson reflection session, where the team discusses their observations of the lesson, together with the post-lesson test outcome. In this study, the teacher of the lesson was firstly given the opportunity to explain his experience with the lesson, and whether he made any adjustment to the intended object of learning. This was to help the teacher to settle down emotionally.

The reflection session forms part of the process of data collection. During this session the team judges whether the lesson met the intended object of learning. Then they proceed and write a report, or, decide if the lesson needs to be revised, improved and re-taught in a different class by another teacher (who must be one of the participating teachers). However, in a LS approach, Marton (2015) and Runesson (2013) report that they prefer the teaching to be done more than once to boost the collection of data.

Pang and Marton (2003) assert that LS is a learning experience in three senses:

> First, the students […] are expected to learn about the object of learning and to learn better than they otherwise would have done. Second, the teachers […] are expected to learn about handling the object of learning, not only the specific object but the object of learning in general. Third, the researchers […] are expected to learn about how the theory works, because every learning study is based on a particular theory and that theory is put to the test. The learning study is expected to be a bridge between theory and practice and between basic research and developmental work. (p. 180)

The argument by Pang and Marton (2003) shows that LS is supposed to improve students' learning through the stages of interaction. It is also supposed to improve teachers' professional development, as well as preparing teachers to be researchers. Also, it is supposed to improve the students' effective learning. On the side of the researcher, it is supposed to help them improve on the theory applied to the LS teaching and learning. The argument also points at applying LS as action research.

As an example, I report here a study that Pang (2008) presented at the 11th International Conference on Mathematics Education (ICME) on a LS approach of a lesson to teach the concept of slope as the object of learning. To discern the object of learning, the teachers identified vertical height, horizontal distance and angle of inclination with the *x*-axis as the critical features.

To obtain the lived object of learning 1, students responded to the situations as follows: (1) Students were asked to identify a hill that was difficult to climb, from two hills of the same height but with different horizontal distances. (2) Students were asked to explain the triangle with the steepest hypotenuse from two right-angled triangles with the same horizontal distances but different vertical distances. (3) Students were asked to differentiate the triangle with the steepest hypotenuse from two similar triangles with different vertical distances and different horizontal distances. (4) Students were asked to explain the steepest line from two lines of the same lengths with angles of inclinations with the x – axis as 60° and 120° respectively, measured from the x-axis in an anticlockwise direction.

Through some interaction with the students, the teachers identified some of their conceptions about the slope that required explanations:

i) Some students think the formula to find the slope is $\frac{(y_2-y_1)}{(x_1-x_2)}, \frac{(y_1-y_2)}{(x_2-x_1)}$ or $\frac{(x_1-x_2)}{(y_1-y_2)}$.

ii) Some students perceive the slope to be the angle of inclination with the x-axis

iii) If two straight lines are parallel in the Cartesian coordinate system, then some students think that the longer one will have a steeper slope than the shorter one

iv) Some students conceive that straight lines with big values of the negative slope are steeper than those with small values

Based on these findings from the diagnostic pre-tests, teachers prepared the intended object of learning. Table 5 shows the patterns of variation and invariance applied during the first lesson:

Table 5: ***Application of patterns of variation and invariance adapted from (Pang, 2008, p.8)***

	Vertical distance	**Horizontal distance**	**Angle of inclination**	**Discernment**
Vertical distance	Invariant	Varied	Varied	Slope depends on the horizontal distance and angle of inclination
Horizontal distance	Varied	Invariant	Varied	Slope depends on the vertical distance and angle of inclination
Angle of inclination	Varied	Varied	Invariant	Slope remains constant for a given angle of inclination

Students responded to the post-tests after the lesson, and the enacted object of learning. These outcomes of pre- and post-lesson tests are shown in Table 6:

Table 6: ***Scores from pre-tests and post-tests (Pang, 2008, p.13)***

	Learning Study group	
	Pre-test	Post-test
Question 1	4.8%	40.4%
Question 2	11.5%	44.2%
Question 3	0.9%	9.6%
Question 4	0.0%	8.7%

From the results shown in Table 6, and from the observations made during the lesson, the team decided, during the reflection session, to re-teach the lesson. One of the observations made was that there was not enough time to discuss all the identified critical features of the lesson. The team's recommendation was to re-teach the lesson in a double period of 80 minutes. The pre- and post-lesson test outcomes suggest that a LS approach may improve learning. Pang (2008) concluded, "the findings of the learning study presented here seem to suggest that collaboration among the teachers in a learning study that is grounded in the Variation Theory of learning is quite effective in improving students' mathematical understanding" (p. 19).

Elliot and Yu (2008) on their evaluation report of Variation for Improvement of Teaching and Learning (VITAL) project found that teachers have challenges in applying patterns of variations in their classrooms. One of the centre's staff reported that, "[…] Then even if they cannot use V_3 and they only learn from V_1 and V_2, it will still help them a lot in their professional development. We are trying to do more in V_3" (p. 182).

Although the teachers in the VITAL project went through training on Variation Theory and LS before applying them, they still had some challenges in identifying the appropriate patterns of variation and invariance to apply in their lessons. One academic consultant confessed that *"I still cannot understand the V_1, V_2. I still do not have a good mastery of the use of V_2 as a teaching strategy in the English subject"* (p. 184). Therefore, teachers require time to internalise the theory and learn how best to put it into practice.

LS approach is a teacher-driven concept, and the inclusion of Variation Theory for its practice makes it possible to be evaluated. It may help teachers to develop research skills and apply the practice in their teaching to help students' performance to improve in the subjects.

However, as has been explained in this section 2.5, the LS approach to teaching and learning has not spread to many other countries. Kenya is one such country where LS has not been applied. Therefore, its application in a country with different societal culture, and perhaps different classroom culture from where it has been applied, requires that I briefly discuss some issues around societal and classroom cultures and how to adapt the practice. In the next section I discuss cultural issues in classroom practices.

2.6 Cultural Issues in Classroom Practices

Culture seems to be a very elusive term to define, as Mitchel (1995) explains, "Culture is an incredibly slippery term" (p. 104). Attempts have been made to define the term culture: Mitchel explains that Duncan (1980) in his article "The Superorganic in American Cultural Geography", viewed culture as "Socially constructed, actively maintained by social actors and people in its engagement with other 'spheres' of human life and activity" (Mitchel, 1995, p. 102). Cosgrove and Jackson (1987) explained culture as "The medium through which people transform the mundane phenomenon of the material world into a world of significant symbols to which they give meaning and attach value." (p. 99). Jackson (1989) suggested, as a working definition for culture, "The level at which social groups develop distinct patterns of life, called cultures which themselves are *maps of meaning* through which the world is made intelligible." (p. 2). Hofstede, a Dutch social psychologist, defined culture as "the collective programming of the mind that distinguishes the members of one group or category of people from others" (Li, 2017, p. 2). A dictionary definition from the Cambridge dictionary, defines culture as "The way of life, especially the general customs and beliefs, of a particular group of people at a particular time."

Hofstede categorised culture into five value dimensions of: power distance, group attachment, gender association, uncertainity avoidance and time orientation (Mai, 2015). However, in all these dimensions, it is power distance that applies mostly to both the society and school levels. "Power distance is the extent to which the less poweful members of the society accept and expect that power is distributed unequally" (Mai, 2015, p. 3). Hofstede says "universally everybody accepts that some people are more powerful than others" (Hofstede & Hofstede, 2005, p. 46). However, he explains that the existence of power distance, from different cultures, differs in what he refers to as high, average or low power distance. These levels refer to how people in a culture collectively accept the inequality that creates the power distance. Although Mai (2015) has not specified the countries individually, he says that high power distance is mostly in countries at lower latitudes (with tropical climate); while low power distance is mostly in countries at higher altitudes (with moderate and cold climates).

At the school level, the role pair shifts from parent-child to teacher-student. Respect for teachers is universal but it is more so in high power distance cultures where students may stand up for teachers when they enter a class or bow when they pass. “Everywhere, teachers control a classroom’s communication, but in cultures with high Power Distance this becomes a strict order with students speaking up only when invited and teachers are almost never publicly contradicted or criticised” (Mai, 2015, p. 4). Kenya is one of such countries with high power distance.

Drawing from the attempts made to explain what culture is in the first paragraph of section 2.6, it appears that a system of education can be determined by these definitions as a culture; classroom teaching can also be a culture since there are ways in which the society views education and what people believe education can achieve. Bruner (1996) and Stigler and Hiebert (1999) view classroom teaching as a cultural activity from the idea of societal expectation and beliefs. Bruner (1996) explains that:

> Education in schools is designed to cultivate skills and abilities, to impart knowledge of facts and theories, and to cultivate an understanding of the beliefs and intentions of those nearby and far away. (p. 63)

Bruner (1996) further explains that the classroom is situated in a broader culture of education; it is in the classroom where “teachers and pupils come together to effect that crucial but mysterious interchange that we so glibly call education” (p. 44). He groups teaching into four models of pedagogy, which he suggests are relationships between minds and culture:

1. Seeing children as imitative learners: that is the acquisition of “know-how”. This is a model where a teacher demonstrates a skill to the learner then the learner practises the skill, commonly applied in vocational training.

2. Seeing children as learning from didactic exposure: that is the acquisition of propositional knowledge. This is a model based on the notion that learners need to be presented with facts, principles, and rules, which should be learned, remembered and applied. It is a procedural knowledge and perhaps the most practised model.
3. Seeing children as thinkers: that is the development of intersubjective interchange. In this model, the teacher views the child as a thinker who can construct knowledge and explain the finding. The learner is, therefore, an active participant in the learning process. This model encourages collaborative learning and interaction among the learners.
4. Seeing children as knowledgeable: that is the management of objective knowledge. This model suggests that teaching should help children grasp the distinction between personal knowledge, on the one side, and 'what is taken to be known' by the culture, on the other. However, apart from the distinction, the learner should also understand the basis, as it were, in the history of knowledge (p 53-61).

Stigler and Hiebert (1999), in their account of teaching as a cultural activity, explain that cultural activities are learned through implicit observations and participation without studying them deliberately. They argue that, although teachers learn how to teach in college (in teacher-training programmes), teaching is also learned through informal participation over a long period. People within a culture share a mental picture of what teaching is. That means, if a particular model of pedagogy, as explained by Bruner (1996), is continuously applied then parents (who were once students), teachers (who are the implementors of the pedagogy), and the students, would understand teaching to be that particular model. Stigler and Hiebert (1999) argue that these models are systems of teaching which are cultural in nature depending on how long they are applied. They are "understood about the cultural beliefs and assumptions that surround them" (p. 88). They demonstrated this by comparing the models of teaching applied by the teachers in the USA and the teachers in Japan, to teach mathematics.

Their findings show that some of the American teachers were applying the second model of teaching, as explained by Bruner (1996) with the "belief that school mathematics is a set of procedures… whose use for students, in the end, is a set of procedures for solving problems" (p. 89). The students in the USA were aware of the approach and waited for their teachers to explain the procedures, then they practised and applied the same to other sums. Stigler and Hiebert (1999) explain that the teachers in the American schools believed that there should not be any confusion or frustration when students were solving the problems. They observed that whenever the teachers noted any confusion, they would quickly move to explain to the students what the procedure was for solving the problem. When they asked the teachers to explain what the main thing was that they wanted the students to learn, the response was, "They wanted the students to be able to perform a procedure, solve a particular kind of problem, and so on" (p. 89).

On the other hand, their findings showed that the Japanese teachers applied the third model in their lessons; teachers started their lessons by asking challenging questions and moving around to help the students to understand the problem and to start working on a solution. As students worked on the solution, the teachers monitored the solution methods so that they could discuss them later in a whole-class discussion arrangement. When Stigler and Hiebert (1999) asked the teachers to explain what they wanted their students to learn, the responses were, "they wanted their students to learn how to think about things in a new way, such as to see new relationships between mathematical ideas" (p. 90).

The responses, indicating what the American teachers and the Japanese teachers wanted to achieve in their approaches, show the kind of classroom cultures the two countries have adopted. The teachers expressed the belief held in their respective countries classroom culture about mathematics teaching. Most likely that is what they went through as students, and that is the societal expectation.

The different cultural approach to the teaching of mathematics could perhaps be one of the reasons why other countries find it difficult to implement lesson study in their classrooms. Chokshi and Fernandez (2004), Fernandez (2012), Lewis (2009), and

Takahashi and Yoshida (2004) explained some challenges that teachers outside Japan met when implementing lesson study in their countries. Those reasons might be as a result of cultural differences adopted by those countries in teaching mathematics. The teachers outside Japan may be applying some techniques in the lesson study approach to a different model of pedagogy, or they leave out those techniques altogether – hence the challenges they encounter. For example, the lesson study approach requires teachers to develop students' discussion tasks and students' anticipated responses to those tasks. This requirement may not fit in a procedural model culture, where students expect teachers to lead them through the procedures of solving the mathematical problems.

The procedural model, as Bruner (1996) hinted, seems to be popular in many classroom cultures. A study by Stephens (2007) on culture in education and development in some selected schools in Ghana and Uganda revealed that the dominant classroom teaching in these countries was teacher-centred and content-driven, reminiscent of Bruner's (1996) model two. The Ghana study was on "Girls and Basic Education in Ghana" (p. 96). One of the aims of the study was "to address issues of access and gender in schooling within one national context from a cultural perspective". One of their findings under the domain of the culture of the school revealed:

> a professional climate in which lessons are almost completely teacher-centred, content driven, the assessment is in the form of oral response to a teacher's question, much time spent by pupils copying work into 'neat books' with little pair discussion or group work. (p. 96)

The Uganda study was on "Children and Health Education in Uganda: issues of culture, language, and the curriculum" (p. 111). The Ugandan case was an evaluation study of a child-to-child programme. It was an innovative approach to health education that was launched in 1979. The main aim of the programme was to encourage child-centred learning about health education. Schools were asked to identify a local health issue, and explain the issue to the students through the teachers. The children, with the help of their teachers, would decide on the action to be taken to develop the topic, they then would take action on the issue together, or individually, beyond the classroom including informing their parents. The child-to-child programme was later adopted as an approach for teaching and learning in schools. The study stated, as one of the findings, "the research reported little change in teaching methodology, with teachers relying upon chalk and talk, which is reasonable enough given the ever-present pressure of examinations" (p. 111). In the Ugandan case, the examination pressure has been cited as a possible reason for not changing the teaching methodology. Kenyan schools also follow didactic teaching culture, as explained in section 1.2, similar to the teaching cultures as discussed in the USA, Ghana and Uganda cases.

As the dictionary definition of culture "as a way of life of a particular group of people at a particular time" implies, culture is dynamic and is capable of changing, perhaps depending on a situation or circumstance. This means that a teaching approach can change and be sustained as a cultural practice. In any case, most of the school cultures, practised by the countries that were earlier colonised, adopted their systems from their colonisers. Stephens (2007) reported on an evaluation study in Indonesia about a project called "Active Learning through Professional Support" (ALPS) (p. 145). The project had two main aims: "(1) to change classroom practice at primary level; (2) to develop a professional support system that would, in the long term, sustain innovative behaviour in the classroom" (p. 145). The implementation of the project involved organizing workshops for teachers, head teachers, and supervisors in some established training centres in different provinces called in-country training centres with visits to the UK for further training. In the workshops, the educators discussed how to change classroom practices from "didactic pedagogy and rote-learning characteristics of traditional Indonesian education to a child-centred approach, based on activities designed to help children work and learn together, often divided into small groups" (145). Initially, the facilitators were lecturers from the UK. However, later, those who had received training then cascaded the training to the rest of the teachers. These trained teachers formed a Teachers' Club in which they would meet regularly during school hours to exchange experiences, share ideas and make plans for classroom activities. The study reports a successful story about the ALPS project explaining that the practice built on the

Indonesian culture, encourages “the maintenance of consensus and harmony” (p. 146). The Indonesian study show a changed classroom culture.

My research design effectively proposes a shift in teaching culture from the didactic approach to a child-centred approach categorised by Bruner (1996) as the third model.

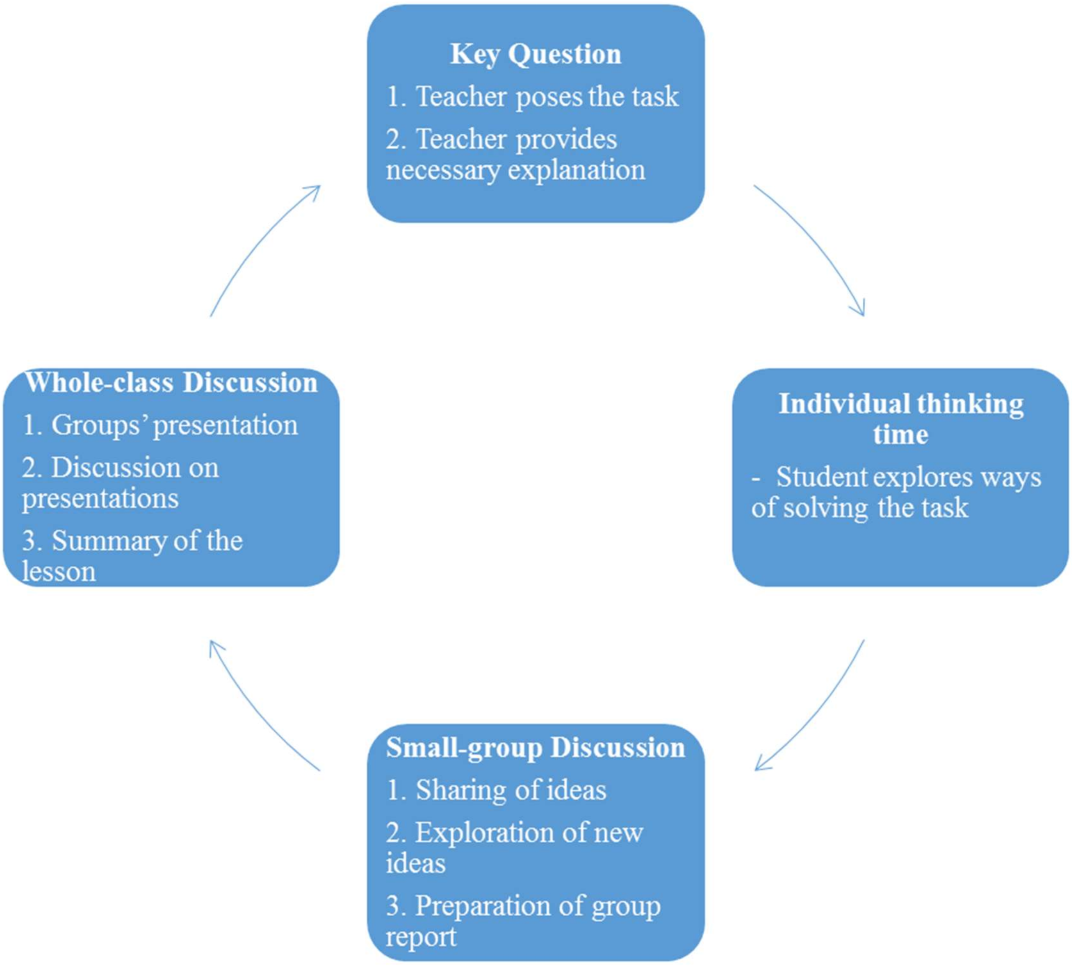


***Figure 6*: Conceptual framework adapted from a description of Yackel and Cobb’s sociocultural learning (1996, p. 460)**

In this approach, teachers prepared discussion tasks, which the students discussed in small groups during the lessons in a conceptual framework as shown in Figure 6.

Yackel and Cobb (1996) note that students get an opportunity to learn when they try to make sense of the explanations they receive from others. They wrote, “We noted earlier that additional learning opportunities arise when children attempt to make sense of explanations given by others, to compare others’ solutions to their own...” (p. 466).

Although Yackel and Cobb (1996) applied it as a framework to address socio-mathematical norms present in the classroom, I adapted it purposely to organise classroom discussion during the lesson. I noted that students asked very sensible questions whenever different groups presented their work.

Apart from changes in students’ learning approach, the design also included another classroom change of observing a teacher while teaching. There are generally only a few times when other people sit in a teacher’s class in Kenya; these include: (1) When one trains to be a teacher and his/her lecturers from the training institution come to assess his/her teaching. (2) When the principal of the school, who is the employer’s agent in the school, receives complaints from the students about the teacher. In such a case, either the principal or the deputy principal of the school would observe the teacher teaching in class ([MoEST], 2012). (3) When there is a general inspection of the school by the Ministry of Education Science and Technology (Education Act, 2013) through the Education Standards and Quality Assurance Commission (ESQAC) department. During the inspection process the ESQAC officers randomly choose classes for observation.

Although these three categories of classroom observations have different objectives, they are all aimed at checking a teacher’s performance in the classroom. The last two categories of classroom observations are rare occurrences. It means that, once teachers graduate from the training colleges, they teach their classes without other teachers observing. As a result, teachers would definitely be uneasy with any classroom observation, such as the ones used during this research, although, as I explain in section 3.9, I asked the teachers to try to continue as normal.

In the next section, I discuss the theoretical framework of Variation Theory as it is applied in the analysis of classroom learning of mathematics.

2.7 Variation Theory as a Theoretical Framework for Lesson Analysis

Although Variation Theory is the theory informing the implementation of the LS teaching approach, its application as a theoretical framework to analyse lessons is not confined to LS approaches only. Lo (2012) notes that "Learning study is applied in practice with Variation Theory and tested in research lessons in authentic classroom situations so that important information and feedback can be obtained to develop the theory further" (p. 143). However, he notes that the Variation Theory framework can be applied in analysing all lessons, not only LS lessons. Marton (2015) notes that every lesson has an object of learning, which is the lens through which it is seen. He argues that:

> Quite obviously, the teachers and the students in the class create the patterns of variation and invariance that are necessary for learning, all the time, in addition to being created by authors of textbooks and exercise books, or of other pedagogical resources (p. 175).

Marton's (2015) argument is that a teacher does not need to understand what patterns of variation and invariance are in order for him/her to apply them in the lessons. They do exist at any time in the lesson, implicitly or explicitly. He explains that "Whenever or wherever learning is taking place, there are patterns of variation and invariance" (p. 175). He continues to explain that learning mostly involves separating some aspects at some point and bringing them together at some other points, that is the part whole arrangement. This requires patterns of variation and invariance as a necessary condition for learning. Marton (2015) gives a summary by saying that "in addition to planning teaching, the theory can be used for analysing teaching that is not at all based on the theory" (p. 176). In that case, the question should be "To what extent has it been made possible for the students to learn this or that?" (p. 176). It means that the lesson would be analysed based on the teacher's action to make learning possible and on the students' responses to indicate whether learning took place or not. Other studies, not conducted in an LS approach, have been analysed using the Variation Theory framework (Ko & Marton, 2004; Olteanu & Holmqvist, 2012; Runesson, 2005).

Ko and Marton (2004) reported on a lesson that was taught by an expert teacher in the People's Republic of China. It was a reading lesson on semantics where the teacher "introduced several linguistic concepts on semantics, including the scope of meaning of words; the level of generality of words; homonyms, synonyms, and antonyms, and their usage" (p. 46). The teacher introduced the lesson by giving a story of a hairdresser who asked a customer some incomplete questions. The hairdresser asked, "Do you want your eyebrows?" The customer replied, "Of course! Why ask!" The hairdresser shaved the customer and gave him the hairs from the eyebrows. The hairdresser's action astonished the customer. The customer had a well-kept beard, and the hairdresser again asked the customer, "Do you want your beard?" The customer replied, "My beard? No, no!" The hairdresser shaved the customer's beard. On checking his face in a mirror, the customer was not happy that the hairdresser shaved off his well-kept beard. The teacher used the hairdresser-customer story to teach homonyms from the word 'want'.

Using the Variation Theory lens to analyse this first part of the lesson, we realise that the teacher applied patterns of variation and invariance to help the students learn the meaning of homonyms. The meaning varied as the word was kept invariant. In the first instance, the hairdresser used the word 'want' in the context meaning of giving. In the second case, the hairdresser used the word in the context of keeping, while the customer was still thinking about the hairdresser's first use of the word.

Olteanu and Holmqvist (2012) applied the framework to analyse the learning of second-degree equations (quadratic equations) as applied in different classrooms. They observed two classes taught by teachers, Maria and Anne. The students had learned the quadratic formula by solving the equation $x^2 + px + q = 0$ leading to $x = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$. In both classes, the teachers wrote down the following equations and asked the students to compare the parts of those equations with the general form in order to help them solve the equations:

(1) $x^2 + 2x - 15 = 0$, (2) $2x - 3x^2 = -1$. In these equations, the teachers varied the relationship between the equations' parts on the general form while the formula remained invariant. Students were able to compare the equations with the general form and solve them. In the end, they developed a full general quadratic formula with the x^2-coefficient other than one that is $ax^2 + px + q = 0$, leading to the solution

$$x = \frac{-p \pm \sqrt{p^2 - 4aq}}{2a}.$$

Runesson (2005) applied the framework as an alternative approach to analysing Jaworski's (1991, 1994) studies which reported learning from a constructivist point of view, and Cobb et al.'s (1997a, 1997b) studies which reported teaching and learning as discourse and interaction. In Jaworski's (1994) study, there was an argument between a teacher and his/her students concerning the different shapes of a cuboid. According to the students, all cuboids have the same shape. The teacher challenged them by providing different cuboids that included a tea packet, an electric bulb packet, and a metre rule and asked, "Are they of the same shape?" (p. 75). One student responded that "Well, no-o, They have all got six separate sides though" (p. 75). The students appeared unsure. The teacher compared the metre rule with the bulb packet and responded that, "I would not say that they are of the same shape." The students mumbled some words like *long* and *size.* The teacher then brought two more cuboids of cereal packets, one bigger than the other, and put them side by side and then asked, "would you say that those two are different shapes?"

Runesson (2005) analysed this conversation between the teacher and the students through the Variation Theory lens and selected the shape as the critical aspect of that lesson and then asked, "What does 'the same shape' mean?" From this classroom interaction, the teacher kept sides of the different shapes invariant as he/she varied the lengths of the sides. The lengths, breadths, and heights were not proportional in all of the objects. By guiding students to discuss the sizes of proportional sides of similar objects simultaneously, the teacher explained the concept of the same shape.

The three cases reported in this section 2.7, analysed the lesson at the classroom level, which is the enacted object of learning about Variation Theory. Marton, Runesson and Tsui (2004) argue that "the enacted object of learning is the researcher's description of whether, to what extent, and in what forms the necessary conditions of a particular object of learning appear in a certain setting" (p. 4-5). However, when using the Variation Theory to analyse a lesson designed in an LS approach, the analysis takes care of the whole component of the object of learning, which is the intended object of learning, the enacted object of learning and the lived object of learning. Lo (2012) argues that "the advantage of using an LS research lesson for the analysis is that there is always evidence of students' learning outcomes in the form of pre-test and post-test results and interview data, which provide evidence to question, support and inform the analysis" (p. 144).

Lo (2012) reported some lessons from an LS approach that were analysed using the Variation Theory framework. One of the lessons was on the teaching of the concept of equal sharing in fractions. Two teachers, teacher A and teacher B, approached the teaching of the lesson differently. Teacher A displayed 10 oranges and asked the students, "How many oranges do we have if we take $\frac{1}{2}$?" The teacher then divided the oranges into two equal heaps and informed the students that five is $\frac{1}{2}$ of 10. He then put the oranges together and re-grouped them into two with one group having six and the other having four. He asked, pointing at one heap "Is this $\frac{1}{2}$?" He continued by reducing the number of oranges and repeating the process on grouping the oranges. Teacher B, who also had 10 oranges, approached the lesson by picking one orange, two oranges, three oranges … at a time. Each time he asked the students to state the fraction of the picked orange over the total. Also, they were to state the fraction of the remaining oranges over the total. For example, in the first picking they recorded ($\frac{1}{10}$ and $\frac{9}{10}$).

Table 7 and Table 8 show the patterns of variation designed by teacher A and teacher B respectively:

Table 7: ***Patterns of variation designed by teacher A, from Lo (2012, p. 191)***

Invariant	Varied	Discernment
Fraction (always one half)	Equal sharing/unequal sharing	The whole must be divided into two equal parts
Numerator always one	Denominator (from 10 parts to 8, 6 etc.)	The denominator represents how many equal parts the whole must be divided into

Table 8: ***Patterns of variation designed by teacher B, from Lo (2012, p. 192)***

Invariant	Varied	Discernment
Denominator (10 oranges)	Numerator (the oranges being taken away and the remaining oranges)	The value of a fraction depends on the numerator (the value being taken away or remaining)

Table 7 and Table 8 represent the analysis of the enacted object of learning in both classes. The teachers also analysed the outcomes of the diagnostic pre-test and post-test for their classes, which was partly the analysis of the intended object of learning and partly the lived object of learning. The outcome of the diagnostic pre-test influenced the preparation of the intended object of learning and the post-test outcome evaluated the same.

Figure 7 below shows the summary of teachers' actions and students' actions on the components of the object of learning. These actions form the basis of analysis of the LS lessons using the Variation Theory framework. I developed the summary to guide me on the areas of focus when analysing the lessons' observation data.

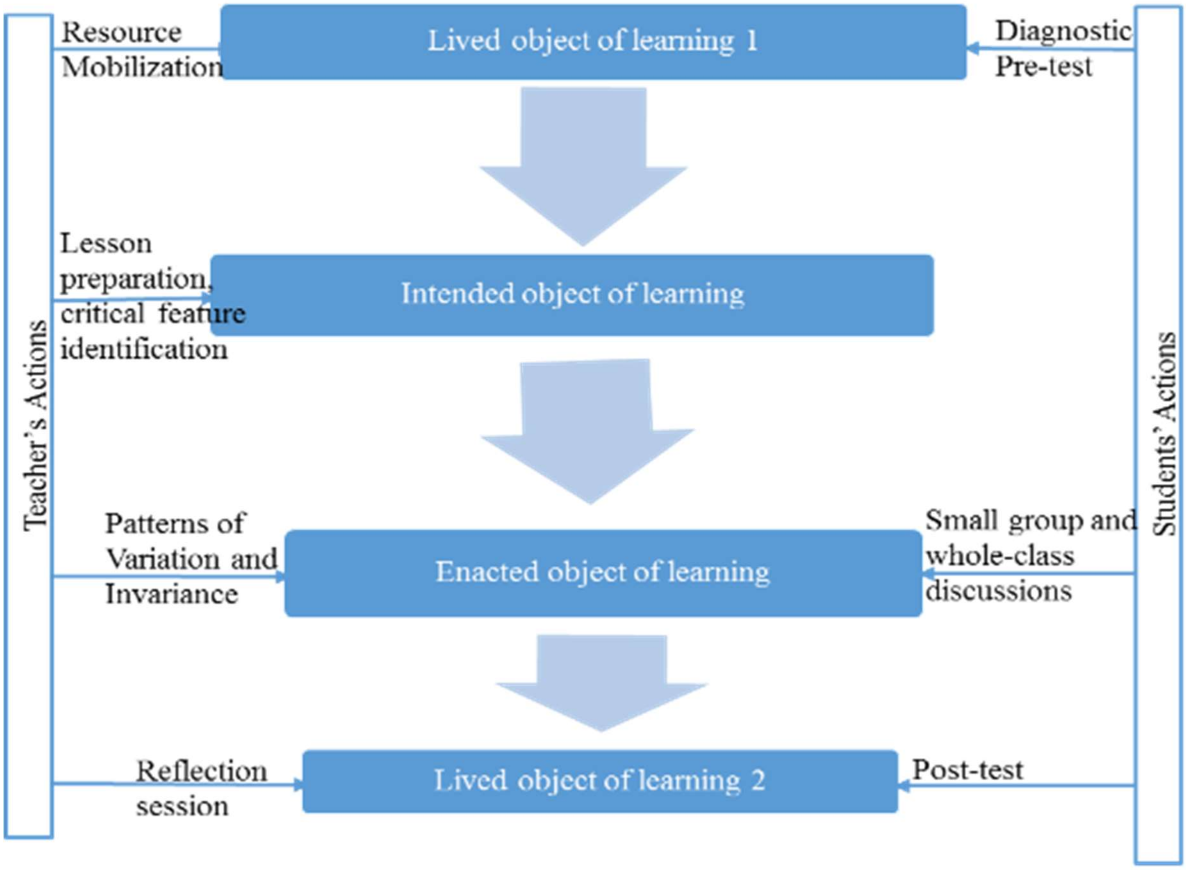


***Figure 7*: Summary of the application of Variation Theory to analyse LS lessons**

The pre-lesson test outcome adds more information to the resources and materials and helps the teachers to identify the critical feature(s) of the lesson and to prepare the intended object of learning. Preparation of the lesson includes the identification of the patterns of variation and invariance, which are crucial for the enacted object of learning. Students' discussion of the tasks is supposed to help them discern the critical feature, which is the area of focus that helps them to learn the intended concept. The post-test outcome helps the teachers to evaluate the extent of achievement of the intended object of learning. Therefore, based on the outcome of the post-test, together with the discussion of the lesson during the reflection session, the teachers then make a decision on the lesson, regarding whether to modify the lesson and re-teach or to continue to the next lesson.

2.8 Conclusion

This chapter presented various sub-sections of the literature review that shaped this study, by highlighting the gaps that this study proposes to address. While writing this chapter, I realised that quadratic expressions and equations is a challenging topic, in almost all its subtopics. However, there seems to be no study that has addressed different teaching approaches to this topic. Many articles reviewed concluded their studies by recommending a study on a teaching approach that could possibly help students overcome the challenges they face with the topic. This study attempted to address the concern by applying a learning study (LS) approach to the teaching and learning of the topic.

This chapter has revealed the link between lesson study and learning study (LS) by describing how the LS approach uses a lesson study structure of teachers' collaborative work when planning, teaching and discussing a lesson. However, lesson study is found to lack an explicit theoretical backing that can help monitor the learning process. In view of that, LS was preferred in this study since it is backed by a Variation Theory of learning, which provides a clear guideline in monitoring the learning process through the intended, the enacted and the lived objects of learning.

Whereas the LS approach does not suggest the kind of student involvement during the lesson, this research suggests the use of small group discussions. This was because of different societal and classroom cultures compared to countries where the design has been applied before and where this research has been practised. This was meant to encourage the students to share their individual thoughts with the rest of the class for the benefit of all.

In addition, this chapter revealed that many analyses of the reviewed literature on LS studies focused on the enacted and the lived objects of learning. However, this research has considered all aspects of the object of learning, as explained in Section 2.7.

The next chapter will discuss the methodology applied in this research. It will explain links between the literature reviewed and data collection procedures. In addition, it will discuss the site of the research, the participants, and their selection criteria, taking into consideration all the ethical policies laid down for such forms of research.

Chapter 3 – Research Methodology

3.1 Introduction

This research was conducted in a secondary school in Kenya, where I observed a series of six lessons in two Form 3 classes (16-18 years old). Observations were followed by interviews with the teachers and the students. Participants were the teachers of the Form 3 classes, their students, and the head of the mathematics department. Because LS was new to the teachers, I organised three orientation sessions with the teachers during which I explained the LS approach and discussed its implementation. The research followed qualitative research approach with an LS research design where teachers worked together in the three stages of the lesson (before, during, and after the lesson). I then collected data through classroom observations, and interviews with teachers and a group of students. I analysed the data using two different approaches. The data from the classroom observations were analysed using the Variation Theory framework, as discussed in section 2.7, while the data from the interviews were analysed using a thematic data analysis approach (see section 3.7.2). I also discuss some limitations of the study and the ethical considerations for this study, including gaining access to the school.

In this chapter I will discuss the research site and participants, and then explain the role of the researcher. Thereafter, I will discuss the orientation sessions with the teacher-participants and will then present the research design, discuss the instrumentation and data collection procedures, followed by the analysis, and then the limitations of the research. I will conclude by discussing ethical considerations.

3.2 Research Site, Participants and Selection Criteria

3.2.1 Research Site

I conducted the research in Siaya County, one of the 47 counties in Kenya (as shown in Figure 8). Since the KNEC report ([KNEC], 2014) on students' difficulty with the solution of quadratic equations was not specific to any county, I chose Siaya as it is conveniently, my county of residence, and this helped to minimise the cost of data collection (Trochim & Donelly, 2006). There is a disadvantage when doing convenience sampling as, potentially, it is not representative of the entire population – especially in a country such as Kenya where the counties are mostly ethnic based. I mitigated this flaw by sampling the Extra-County and County schools that admit students from across the counties, as explained in section 1.2.

***Figure 8*: Map of Siaya County, where the current study was conducted, showing its Sub-Counties and their Headquarters adapted from: (http://maps.maphill.com/kenya/nyanza/siaya/3d-maps/silver-style-map/silver-style-3d-map-of-siaya.jpg)**

Siaya County has six sub-counties namely: Bondo, Rarieda, Siaya, Gem, Ugunja and Ugenya. On the map of Siaya County above, the sub-counties are indicated by their Headquarters. Three of the six Headquarters are named after the sub-counties, that is, Rarieda, Bondo and Ugunja. Other sub-counties have separate names for their Headquarters. These are Boro for Siaya sub-county, Yala for Gem sub-county, and Ukwala for Ugenya sub-county.

The current study was conducted in a secondary school in the sub-county of Bondo. As I mention in section 1.2, secondary schools in Kenya are categorised into national schools, extra-county schools, county schools and sub-county schools. Siaya County has 222 secondary schools of which two are national schools, 10 are extra-county schools, 26 are county schools and 184 are sub-county schools ([KNEC], 2016). I collected data from one of the county schools. Before deciding on the school, I sampled 12 out of the 36 extra-county and county schools. I did so for two reasons: (1) These schools have at least two streams[2] per form, thereby allowing the teachers to teach a lesson in one stream and teach the modified lesson in the other stream. (2) The 12 schools included two schools from each of the six sub-counties. Conducting research from a school involves obtaining an access letter from the County Director of Education (CDE), therefore I decided to involve all the sub-counties in order to help me convince the CDE that the study covered the whole county.

The selection of two schools from each of the six sub-counties was a non-proportional stratified random sampling (Trochim & Donelly, 2006). My sampling procedure was non-proportional since I had not based my selection of the two schools per sub-county on their number of extra-county and county schools. I based my stratification on the sub-counties, thereby ensuring that I had selected schools from all the sub-counties – I grouped the extra-county and county schools from each sub-county together before randomly picking two schools.

I obtained 12 letters from the CDE, addressed to the principals of each of the schools, which enabled me to gain access to them. The principals of the schools arranged a meeting involving the Form 3 mathematics teachers, the head of the mathematics department, and myself. The principals did not attend the meetings. In the meetings I explained the nature of this study to the teachers. I then gave them a consent form and

[2] The streams in Kenyan schools are mostly random-based like the school where I conducted my study. They are meant to increase students' access to secondary education in the relatively few schools available (MoE, 2012). However, there are a few cases in which streaming is done, according to students' choices on elective subjects.

asked those who were willing to participate, to indicate so on the form. At least three teachers (including a head of department), in five out of the twelve schools, expressed a willingness to participate in the study. I chose one of these five schools for this study, taking into account the proximity of the school to my residence.

3.2.2 Participants

The school I chose had two streams in Form 3 (16-18 years old); each stream was taught mathematics by a different teacher. The head of the mathematics department did not teach mathematics in either of these streams. I involved three teachers: Dominic (head of the department), and Peter and John (Form 3 mathematics teachers). These names are all pseudonyms. Dominic had had 10 years of teaching experience and was also teaching physics as a second teaching subject. Peter had had five years of teaching experience and also taught chemistry in other classes, while John had had four years of teaching experience and was also teaching business studies in other classes (see section 1.2). The two teachers taught mathematics to these same classes in Form 2 and had continued with them to Form 3. In addition, I involved 79 students from the two Form 3 classes, Form 3 East and Form 3 West. Form 3 East had 46 students, while Form 3 West had 33. However, there were fluctuations in the number of students in attendance during the observed lessons, as shown in sections 4.2, 5.2 and 6.2.

The criteria for selecting the teachers was that they had to be trained teachers, with at least two of them teaching in different streams of Form 3, and a head of department who was not teaching either of the two streams. The rationale for having at least three teachers originated from the requirements of the LS approach; at least two teachers should teach different classes where one of them would teach a lesson in their class, with the other one teaching a modified lesson in their class. The head of the mathematics department (HoD) was involved for two reasons. The first reason was in order to add to the number of teachers assisting in lesson observation, as well as data collection. (This was a safety measure in case one of the teachers was not able to observe a lesson, thereby ensuring that there would still be another teacher to assist with classroom observation and data collection). The second reason was to make the HoD aware of the LS approach so that, together with the other teachers, they could discuss and continue its use after this study was completed.

The choice of Form 3 was due to the topic of quadratic expression and equations. This is a Form 3 topic and was due for teaching at the proposed time of data collection. I chose this topic because the KNEC reported it as one of the topics that students consistently performed poorly in during the Kenya Certificate of Secondary Education (KSCE) examinations ([KNEC], 2006; [KNEC], 2014).

The selection of student participants followed by virtue of them being members of the classes whose teachers were participating in the study.

In a LS approach the researcher also plays a key role. As Pang (2008) puts it, "The primary role of the researcher(s) in a learning study is to have a professional dialogue with the teachers and to provide professional support when necessary" (p. 21). In addition, in a lesson study, a pre-cursor to LS, there is a provision for a "knowledgeable other in the discussion of the lesson to improve the quality of *kyouzai Kenkyuu*" (discussion of the learning material) (Yoshida, 2012, p. 10). In the next section I discuss my role in the study.

3.3 The Role of the Researcher in a LS

The 'knowledgeable other' in an LS approach is an expert who guides the preparation of the lesson by discussing the materials (*Kyouzai Kenkyuu*) as well as providing comment during post-lesson discussions (Takahashi, 2013). In this study, I played the role of 'knowledgeable other' due to my experience of lesson and learning studies. I have read relevant literature on these studies, observed classes where these studies are practised, and have taken part in reflection sessions. However, regarding one of the research questions on 'the teacher's view of the application of the LS approach to teaching and learning', I was only involved with lesson preparation at the initial stage of data collection. My involvement in the later stages, mainly during post-lesson reflection, was minimal.

I introduced the teachers to the meanings and applications of lesson study, LS and Variation Theory, with respect to classroom teaching. I helped them to come up with the first activity for the first pair of lessons, and I provided some professional support during the reflection sessions. Before the students signed the consent forms I explained to them about the kind of arrangement we would have during learning, their participation in the pre- lesson and post-lesson tests and the involvement of other teachers in observing the lessons.

As I have mentioned in the first paragraph of this section, I gradually withdrew my involvement during lesson preparation for the second and third pairs of lessons. The teachers then prepared the activities and pre-lesson and post-lesson tests on their own. I did this in order to allow the teachers to internalise the process, thus ensuring that they could freely explain their experiences with the new approach, as they had made their own decisions regarding class activities. I remained as an observer during the lesson observations. During the reflection sessions I offered to explain any unusual aspects which I noticed during the lessons. For example, during the second lesson in the second pair of lessons, the students asked John some questions, but they felt dissatisfied with the answers that he gave. John noticed this and commented on it during the reflection session; he appeared not to know its cause. I offered my view on the problem, explaining how such problem could cause a misconception in algebra (see section 5.2.2).

I set up orientation sessions with the teachers, in order for them to understand the nature of my research and all the new terms involved in a LS approach (lesson study, LS, object of learning and Variation Theory among others). The next section discusses these sessions.

3.4 Orientation Sessions with the Teachers

In Kenya, the school annual academic calendar is in a three-term format, from January to December, as I explained in section 1.3.1. I planned my data collection period from the Third term 2015 to the First term 2016. I wanted to have three orientation sessions, of two hours each, with the teachers in the Third term 2015, which was beginning on 31st August 2015. However, the teachers' union called for a nationwide strike due to a salary dispute with the Government. This delayed the re-opening of schools until 5th October 2015.

In the original arrangement, I planned to observe participant teachers in this research in their teaching approach before introducing the LS approach to them. This was meant to help me understand their teaching approach including the nature of classroom involvement. Thereafter, I planned to introduce them to the LS approach to the teaching and learning of mathematics and observe some lessons (as pilot lessons) in the new arrangement prior to the scheduled data collection lessons on the topic of quadratic expressions and equations. This arrangement was meant to allow me to interact with the teachers, obtain their views about the new approach and correct/amend some areas that might have not worked well during the pilot sessions. This might have helped improve some areas like the diagnostic pre-lesson test questions and lesson activities that I noted had a few problems, such as construction of the questions and having same lesson activity as pre-post lesson tests. After orientation sessions, I planned to start data collection in Third term of 2015 when the students were being taught first part of quadratic expressions and equations in Form 2, and to continue with the exercise in the following First term 2016 while the students were in Form 3. This was not possible as I have explained in this paragraph and I had to fully collect data in First term 2016. However, due to the strike, the two parts of the topic of quadratic expressions and equations ([KIE], 2002) were taught in Form 3, which was advantageous to my study.

After 5th October 2015, I collected the 12 letters for the principals of the selected schools, from the CDE's office. I then obtained consent from the Form 3 teachers in the week starting 12th October 2015. Due to this change to my original plan, I was only able to organise one orientation session in the Third term, this was on 29th October 2015. I conducted the remaining two orientation sessions in the First term 2016, utilising the third session as a preparation session for the first pair of lessons.

During the session with the teachers on 29th October 2015, I discussed the lesson study and the LS approach, explaining the similarities and differences. I also discussed the Variation Theory and its connection with the LS approach. At the end of the session I gave the teachers some handouts to read about the LS approach and its application to teaching and learning, copies of which are included in Appendix 1. One of the teachers' main concern was on the diagnostic pre-lesson test. Peter asked:

> Peter: How can we obtain learners' views on what they have not been taught?

I considered this as a genuine concern, but I explained that the diagnostic pre-lesson test was to help the teachers understand the students' prior knowledge concerning the topic at large. As Marton (2015) explains, the pre-test should be helpful in finding the critical aspects to develop the intended object of learning. He further explains that the aim is to find out whether the students discern certain aspects (dimensions of variation). I explained that the students' answers are expected to help the teachers plan the lesson by showing them what the students' prior knowledge is. In addition, I informed the teachers that part of what I was investigating was their views about the functionality of the LS approach to teaching and learning of mathematics and that one of the components of the approach is the diagnostic pre-lesson test. Thereafter, the teachers and I scheduled the next meeting for 12th January 2016.

In this next meeting the teachers and I discussed the class activities, during which students would work in groups on tasks and then report their findings for whole class discussion. Using the Strengthening of Mathematics and Science in Secondary Education's (SMASSE's) categorization of activities into hands-on and minds-on, the team discussed the two categories. I referred to the SMASSE categorization because these teachers had attended SMASSE In-service Education and Training (INSET) as explained in section 1.3.1, so they were aware of the categorization. According to SMASSE, hands-on activities deal with manipulative skills that include the use of concrete objects while the minds-on activities are concerned with the abstract discussions of the contents.

Zahorik (1996) explains hands-on activity as:

> A range of activities in which the student is an active participant rather than a passive listener. The term includes the use of manipulatives such as pattern blocks in mathematics; playing games of all kinds; participating in simulations, role playing, and drama […] (p. 555).

This explanation shows that the term is applied to a wide range of activities that involve students in active participation during the learning process. Pedersen and McCurdy (1990) explain that a heavy stress was placed on hands-on activities in the laboratories. All these studies agree that hands-on activities, or application of concrete objects, become effective for learning when minds-on activities are incorporated. Clements (1999) states, “Good manipulatives are those that aid students in building, strengthening, and connecting various representations of mathematical ideas” (p. 49).

In view of these explanations, I discussed with the teachers the need to identify activities (hands-on or minds-on), that would help the students discern the object of learning. The teachers then brainstormed on various activities that would help them introduce the factorisation of a quadratic expression. As I explained in section 3.2.2, apart from being one of the topics of concern according to [KNEC] (2014), this subject was also chosen as it was the first topic to be taught in Form 3 First term, according to the [KIE] (2002) syllabus. This is during the period when I was scheduled to collect data.

During this discussion, Dominic said that the topic of quadratic expressions and equations is a “dry topic”, and it was difficult to think of any activity that would help them teach factorisation of a quadratic expression without telling the students in advance what to do. I understood the term “dry topic” to mean that there were no hands-on activities for this topic.

I explained to the teachers that, in the history of mathematics, quadratic equations emerged from the calculation of areas of rectangles, as discussed in section 2.2. Therefore, the topic may not be “dry” as there are hands-on activities that could be used to introduce the factorisation of a quadratic expression such as $x^2 + 5x + 6$. The teachers became interested in this and I asked them to find a sheet of paper and cut out pieces representing the areas of each term of the expression then join them to form a whole rectangle. This was an activity for the teachers to discuss in the next orientation session scheduled for 14th January 2016.

During the third orientation session Peter represented the group when explaining the activity and how it would be used to teach factorisation of the expression $x^2 + 5x + 6$. He stated that students would be given the pieces of paper, asked to form rectangles, then use algebraic expressions representing the sides of the rectangle to calculate its area. During the discussion, after Peter’s presentation, John acknowledged that it took them some time to form a rectangle from the pieces of paper. I asked them, “What do you think can happen if the activity is given to students during a lesson?” Dominic responded that:

Dominic: […] for teaching in class, it might take a lot of time and students may fail to form a rectangle. We took some time to form the rectangle.

However, Peter had a different view:

Peter: Yes, it is a good hands-on activity. Our students are not used to such practical activities in mathematics. Apart from taking a lot of time, it can raise their curiosity because we saw that after forming the rectangle it is easy to see the “sum and product” concept.

Peter’s suggestion of raising students’ curiosity is supported by Zahorik’s (1996) study, which claims that “Although teachers establish students’ interest in a number of ways […] the main method teachers employ is hands-on activities” (p. 560). Although Peter said “curiosity”, simply because it was going to be the first time the students would have a practical activity, in the end the activities would help students develop an interest in mathematics. It was after this discussion that the teachers agreed to adapt the activity for the first pair of lessons.

3.5 Research Design

The study followed an approach of qualitative research with a LS approach as a research design. Lodico, Spaulding and Voegtle (2006) explain that qualitative research in education, which is always interpretive, is adapted to an educational setting from sociology and anthropology disciplines. Lodico et al. (2006) cite the following as key characteristics of qualitative research:

- Studies are carried out in a naturalistic setting.
- Researchers ask broad research questions designed to explore, interpret, or understand the social context.
- Participants are selected through non-random methods based on whether the individuals have information vital to the question being asked.
- Data collection techniques involve observation and interviewing that bring the researcher in close contact with the participants.
- The researcher is likely to take an interactive role where she or he gets to know the participants and the social context in which they live.
- The study reports the data in narrative form (p. 264).

Creswell (2008) adds that “qualitative researchers gather multiple forms of data, such as interviews, observations and documents…” (p. 175). He continues to say that “researchers often use theoretical lenses to view their studies…” (p. 176).

In this study I interacted with the participants and collected data through classroom observation and interviews with the participants. I also used the Kenyan secondary schools’ mathematics curriculum to explain the position of the topic of quadratic expressions and equations in the curriculum, and to highlight its relevant pre-requisite topics. The current study uses a theoretical lens of Variation Theory to interpret and analyse data.

However, Tracy (2010) on her work on qualitative quality, notes that qualitative research needs to adhere to some criteria of quality for qualitative methodological research. She proposes such criteria as “worthy topic, rich rigor, sincerity, credibility, resonance, significant contribution, ethics and meaningful coherence” (p. 839). In this study I chose LS approach as my research design and I have addressed the aforementioned criteria in the analysis chapters as well as chapters 3 and 8.

The LS approach has the advantages of involving both the teachers and the students at every stage of a lesson, beginning with the preparation, during the enactment, and after (as shown in Figure 9). Before each lesson the teachers gathered materials, including students’ pre-lesson tests responses. These helped the teachers decide on the lesson’s object of learning, together with the critical feature(s), and they then prepared the lesson together as a team. This whole process of lesson preparation, along with the decisions on possible ways of enacting the lesson, is the intended object of learning as discussed in sections 2.4 and 2.5. During the preparation, teachers proposed patterns of variations and invariances to be enacted during the lesson, as stipulated in the Variation Theory of learning, as discussed in section 2.4.

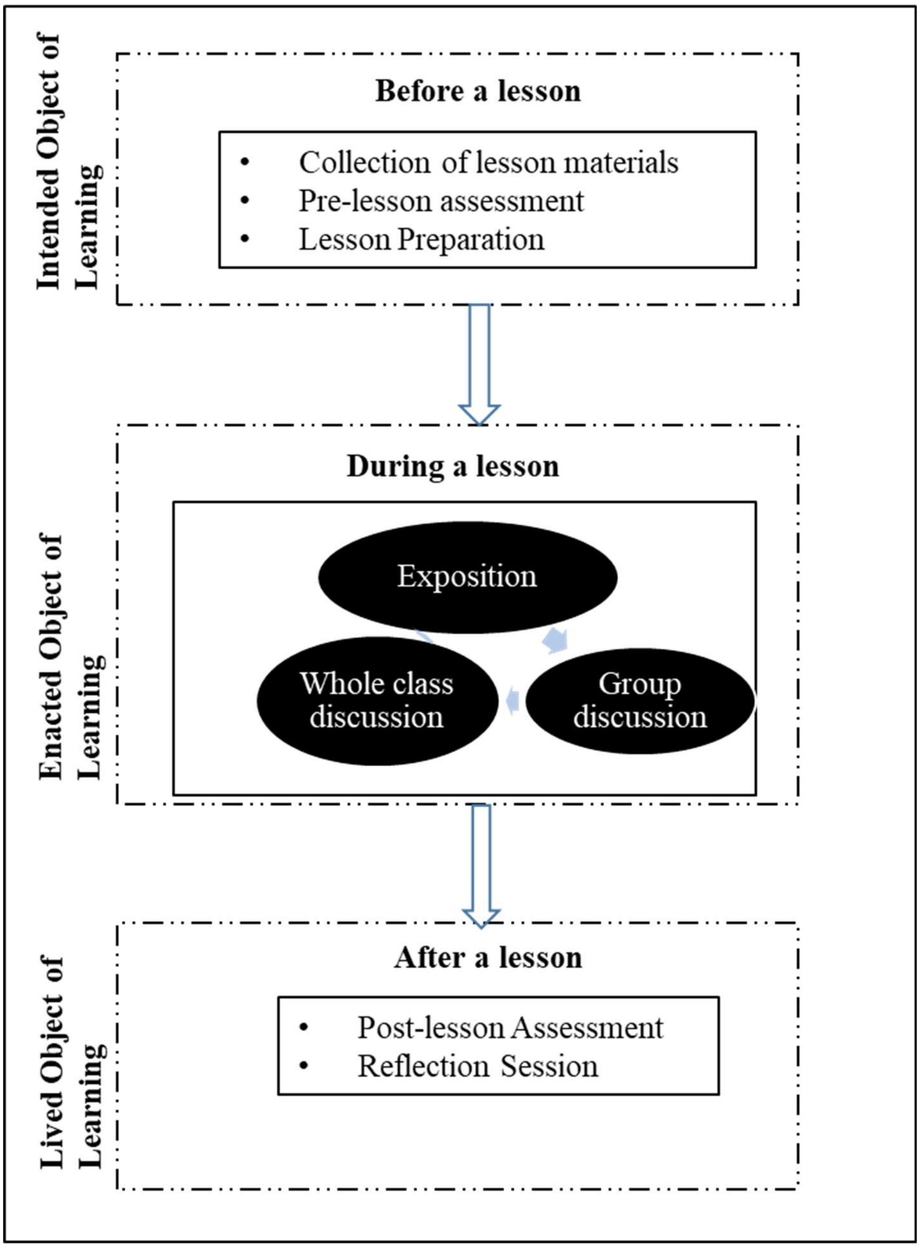


***Figure 9*: The LS research design used in this study**

During the lesson (called the enacted object of learning), the teachers started with a brief exposition that included discussions based on the outcomes of the pre-lesson tests, instructions to the students on how to engage with the activities in small groups, and how to report their solutions in plenary class discussion. Pang (2008) explains, "The lesson is then analysed in terms of whether the object of learning was made attainable through the actual patterns of variation and invariance by the teachers" (p. 4). The teacher's instructions helped the students do the activities in steps, by implementing the patterns of variation and invariance as proposed in the lesson.

Occasionally the teachers took contingency measures, such as adjusting the time for discussion due to the students' reactions to the activities. Rowland, Thwaites and Jared (2011) note that "Mathematics teaching rarely proceeds according to plan, if ever" (p. 73). They explain that one of the reasons for lesson interruption is what they call a contingent situation "in which a teacher encounters something unexpected, requiring them to think on their feet" (p. 73). Such situations were also observed in this research as will be observed in Chapters 4, 5 and 6. Rowland et al. (2011) separate these situations into three types of "responding to students' ideas, a consequence of teacher insight and when the teacher is responding to the (un) availability of tools and resources*"* (p. 75). Such situations occurred during this research as will be seen in Chapters 4, 5 and 6. The most common situation was the first type. The measures taken by the teachers were meant to help as many students as possible complete the tasks so as to learn the content. However, in some cases, especially at the initial stages, the teachers ignored the students' ideas, as was also observed by Rowland et al. (2011) concerning novice teachers.

At the end of the activities the teachers picked groups with different solutions or approaches to present their work, which was then discussed with the whole class. The students were actively involved in the lesson by working in groups. This allowed them to explore possible ways of handling the activities. Sometimes the groups failed to solve the tasks but then suggested other possible approaches that the whole class discussed. Marton (2015) explains:

> […] if you do not solve a problem and you eventually see how it is solved by someone else (the teacher or a classmate), there will be a contrast between the canonical way of solving it and your own. Your own, perhaps less elegant or even failed attempt, will enable you to see the solution much more clearly. It will have a particular meaning for you (p. 183).

The approach gave the students an opportunity to understand their solutions, or the solutions from those who had helped them, including the teacher in circumstances that did not obtain their own solutions. The teachers explained the solutions while summarising the activities at the end of each lesson. Marton (2015) notes that “when you are told how to solve the problem, the one who tells you makes all the distinctions that have to be made” (p. 183). By allowing group discussions, and then reporting their findings for a whole class discussion, the students made intra-group and inter-group decisions on their solutions. This helped the teachers refer to the students’ work as they concluded the lessons.

After each lesson the teachers and I met for a reflection session and deliberated on the lesson. In the meeting the team discussed the lesson, taking into consideration what worked well and what did not work, and proposed modifications for the subsequent lessons. The students also answered the post-lesson tests. The observed learning during the lesson, together with the reflection sessions and the outcomes of pre- and post-lesson tests, constituted the lived object of learning. Pang (2006) presents the lived object of learning in two parts, namely: lived object of learning 1, and lived object of learning 2, as explained in section 2.4. Apart from the feedback from the pre- and post-lesson tests, the teachers were able to reflect on the extent to which the students, during the group presentations, had mastered the object of learning.

3.6 Instrumentation and Data Collection Procedures

3.6.1 Data Collection Instruments

When collecting the data, I used diagnostic pre-lesson and post-lesson tests, a lesson observation checklist, and semi-structured interview schedules. Recording instruments included a video camera, a digital camera, and an audio voice recorder.

The diagnostic pre-lesson test is considered part of the LS lesson preparation as it helps the teachers to decide on the materials, activities and structures used in the lesson (Marton, 2015). The questions are expected to be phrased in everyday words since they are asked prior to teaching the content. In this study the questions were such that the students had to give some statements in their answers, as opposed to solving an equation for an answer. The responses helped the teachers decide on the lesson activities, as well as introductory remarks to clarify some students' misunderstanding. However, the hands-on activity of the cut pieces of paper was decided upon before giving the diagnostic pre-test. This was because the teachers had done the activity during the orientation session and they felt that it was interesting enough to be given to the students during small group discussion, in order to help the teaching and learning of the factorisation of a quadratic expression (see section 4.2). The post-lesson tests were the same as the diagnostic pre-lesson tests so as to help teachers assess the students' discernment of the object of learning.

I soon realised that the development of test items posed some challenges, as the students' prior knowledge of the content should be tested before they learn the content, as Peter pointed out in section 3.4. The observed lessons were introducing new content, as discussed in the next section. This made it a bit difficult for the teachers to find pre-lesson questions that would test the previous knowledge that links up with the new content.

Eriksson and Lindberg (2016) have also raised some issues with respect to diagnostic pre-lesson tests. They report on a comparative study in two PhD theses that used LS approaches in their studies. They found that the approach to diagnostic pre-lesson tests varied from traditional paper and pen tests to semi-structured interviews. They noted that they could not guarantee the validity of the questions, especially for the paper and pen tests. In addition, they found that the purposes for the use of the tests varied. While, in some studies, the focus is on the outcome of the learning in order to measure the effects after the lesson, in others, the focus is on changing the teaching in order to enhance learning. Lo (2012) and Marton (2015) propose that the pre- and post-lesson information should be used for both purposes. The pre-lesson responses could be used to prepare the lesson (to enhance the learning of the content) while the post-lesson responses could help teachers assess whether these learning outcomes have been achieved.

The observations from these authors suggest that there is a need for more discussion on the development of the instruments for the pre- and post-lesson tests. Perhaps it is due to the fact that the studies on LS approach to teaching and learning are still relatively new in the field of research, as it was first introduced in 2001 (Marton & Runesson, 2014). Both Lo (2012) and Marton (2015) suggest the use of either a questionnaire approach or interview. The researchers could, perhaps, try different approaches to find a suitable method of collecting information from students, prior to the preparation of a lesson.

In this study I discussed the two possibilities (questionnaires or interviews) with the teachers and we agreed to use pre- and post-lesson questionnaires in order to gather information from the students. As I explained in section 3.4, the teachers questioned the idea of testing students on what has not been taught. The students also raised the same issue, as discussed in section 7.3.1.

I worked with the teachers to develop the first set of questions for the first pair of lessons, and the teachers worked without me when developing the questions for the remaining lessons. In a LS approach the teachers are supposed to use their prior experiences with the content to develop the pre- and post-lesson questions, because they can recall the difficulties students normally have. The information from the students' responses help the teachers decide on the critical features of the lesson.

Concerning the lesson observation checklist, I adapted the one used by the Centre for Mathematics Science and Technology Education in Africa (CEMASTEA) in Kenya, to observe the in-service teachers' lessons during their teaching practice. The checklist is in three columns and three rows. The first column is in three parts: (1) activities in the introduction stage, (2) activities during the lesson development stage and (3) the activities in the conclusion stage. The other two columns concern the teacher's activities and the students' activities. In each of the three parts of the first column there are teacher's activities as well as students' activities. During a lesson, observers use the checklist to record the teacher's and students' activities at every stage. A copy of the checklist is included in Appendix 2 (2a).

I used a video camera to record the lessons, to complement the information recorded in the checklist. I used a digital camera to take photographs of the students working in groups and during presentations. I used an audio recorder during the reflection sessions, and the interviews. I later transcribed these audio recordings.

Table 9: ***Summary of the research questions and the instruments to address them***

Number	Research Question	Instrument
1	What is the outcome when a learning study approach is applied to the teaching and learning of mathematics in a Kenyan cultural context?	(a) Pre-post-test questions (b) Observation checklist (c) Interview Schedules
2	What are the teachers' views on the application of learning study approach in the teaching and learning of the topic of quadratic expressions and equations, and with a possibility of extending the same to other topics?	Teachers' interview schedule
3	What are the students' perceptions and experiences on the application of LS in the teaching and learning of the topic of quadratic expressions and equations?	(a) Students' interview schedules (c) Observation checklist

I developed two different semi-structured interview schedules for interviews with the teachers and the students, and copies of these are included in Appendix 2 (2b) and (2c) respectively. The schedules were guides so that I could maintain consistency, especially with the teachers whom I interviewed individually. I also added questions which arose from the participants' responses in the course of the interviews. The instruments helped in answering the research questions as shown in Table 9.

3.6.2 Data Collection Procedures

Data collection focused on the preparation, observation, and evaluation of lessons in the topic of quadratic expressions and equations. The Kenyan secondary schools' mathematics curriculum presents this topic in two parts ([KIE], 2002). The first part, which is the 38th topic out of the 68 topics of the Kenyan secondary schools' mathematics curriculum, referred to as "Quadratic Expressions and Equations" ([KIE], 2002, p. 22), is scheduled for teaching in the Third term of Form 2 (15-17 years). The second part, which is the 44th topic, referred to as "Quadratic Expressions and Equations (2)" ([KIE], 2002, p. 26), is usually the first topic in the First term of Form 3 (16-18 years) – this is the term in which I collected the data. However, due to the teachers' strike that took place in the Third term of 2015, the teachers did not teach the first part in Form 2 (15-17 years). The delay had an unexpected benefit for my study as I was able to observe the teaching of the whole topic within a term, since both aspects were taught in Form 3.

The first step of the data collection was through pre- and post-lesson tests. The teachers gave the diagnostic pre-lesson tests to the students in both classes, a day before the lesson for the First and the Third pairs of lessons. The pre-lesson test was given two days prior to the lesson for the Second pair of lessons. They did this so that they would have time to consider the pre-test outcomes during the preparation of the lessons. Immediately after each lesson the teachers gave the students a post-lesson test.

The second step of the data collection was classroom observation. Two teachers out of the three, and I, observed each lesson and collected data by using the lesson observation checklist, video recordings and photographs of the students' work. The third teacher taught the lesson. We observed the teaching of three sub-topics (contents) of this topic during six lessons, meaning that each sub-topic was observed in a pair of lessons. All the three sub-topics were observed in their introductory stages. The first sub-topic was the introduction of the factorisation of a quadratic expression with a unit coefficient of x^2 in the form $x^2 + bx + c$, where b and c are constants. The second sub-topic was the solution of a quadratic equation by completing the square, for equations in the form $x^2 + bx + c = 0$. The third sub-topic was on graphs of quadratic functions in the form $y = \pm x^2 + bx + c$. A summary of the observed lessons is shown in Table 10. Further applications of the contents continued in subsequent lessons, but these were not observed.

Table 10: ***Summary of the observed lessons***

Pairs of lessons	**Date**	**Content**	**Lessons**	**Teachers**	**Observers**
First pair	19/01/2016	Factorisation of quadratic expressions	Lesson 1	Peter	John, Dominic, Fred
			Lesson 2	John	Peter, Dominic, Fred
Second pair	29/01/2016	Solutions of quadratic equations by completing the square	Lesson 1	Peter	John, Dominic, Fred
			Lesson 2	John	Peter, Dominic, Fred
Third pair	19/02/2016	Graphs of quadratic functions	Lesson 1	Peter	John, Dominic, Fred
			Lesson 2	John	Peter, Dominic, Fred

The lessons were taught on the shown dates so as to allow enough time for the teachers to secure the materials needed, discuss the activities they would use, and to prepare for the lesson. This is in line with the LS requirements of allowing teachers enough time to extensively discuss and prepare for the lesson (Yoshida, 2012). Meanwhile the teachers continued to teach other sub-topics in between the lessons. I chose not to interfere with the school's timetable arrangement and so I fitted the teaching of the lessons within the stipulated times, according to the curriculum and the school's timetable. We ended up observing all the first lessons in Peter's class and all the second lessons in John's class because these were the days when all the three teachers were free to observe the lessons and Peter's lessons appeared first on the timetable.

The third step of the data collection was during the reflection sessions after each lesson. As was mentioned earlier, at the end of each lesson, the three teachers and I met in a reflection session to discuss the lesson. I audio recorded the discussions, which I later transcribed. During the reflection sessions, the teacher who had taught the class expressed his views first. I adopted this arrangement to allow the teacher of the lesson to first discuss their observations from the lesson and to help the other teachers then feel free to add their observations to the views already expressed. I was always the last participant to comment on the lesson. I mainly pointed out areas where they could improve the application of the LS approach to teaching and learning.

The fourth, and final, step of the data collection was the interviews. I interviewed each teacher individually for about 30 minutes, and I interviewed one group of eight students (from both classes) for about 40 minutes. The students volunteered to be interviewed. I interviewed Dominic, John and the students on the same day (one week after the last pair of lessons), and I interviewed Peter three weeks later as he was away when I interviewed the rest of the participants.

3.7 Data Analysis

As described in section 3.6.2, I collected data through four procedures. I then organised and analysed the data that I had collected through the first three procedures (pre- and post-lesson tests, classroom observations and reflection sessions) using Variation

Theory as the analytical framework. I organised and analysed the data that I had collected through the teachers' and students' interviews using a thematic data analysis.

3.7.1 Variation Theory as an Analytical Framework

I organised and analysed the data in terms of the components of the object of learning: intended, enacted and lived. As I discussed in section 3.6.2 (summarised in Table 10), there were six lessons grouped in three pairs. I analysed each pair of lessons, with the analysis of each pair constituting one of chapters 4, 5 and 6. In each chapter I present the two lessons, giving the results of the pre- and post-lesson tests, an account of the lessons and the reflection sessions, and an overall summary of the lessons. I present a detailed description of each lesson with some interpretations and suggestions. I explain how the teachers incorporated the pre-lesson test outcomes into their lessons and how they applied their lesson plans in every section of the lesson.

Thereafter, I did a detailed analysis of the lessons in each pair, using the lens of Variation Theory. Both the teachers' and the students' actions/activities in each component (the intended, the enacted and the lived) were analysed and discussed, incorporating the relevant literature. The analysis of the intended object of learning focused on the lesson preparation, including patterns of variation and invariance that the teachers planned to apply, and how they planned to implement them. Marton (2015) states, "Whenever and wherever learning is taking place, there are patterns of variation and invariance" (p. 175). These patterns are the necessary conditions that the teachers plan to use in their lessons.

In the enacted object of learning, the analyses focus on how the teachers applied the patterns of variation and invariance, explaining what varied and what was kept invariant. The students' participation in the lessons, which included small group and whole-class discussions, and the outcomes from those participations, were also analysed.

In the lived object of learning the analyses focus on the outcomes from the activities, as discussed in groups and then presented for whole-class discussion. The students' reactions were gleaned from the teachers' summary of the lessons, and the outcomes of the post-lesson tests, as well as the observations made by the teachers during post-lesson reflection sessions. I conclude each chapter with a reflection on the students' and the teachers' learning experiences from the lessons.

3.7.2 Thematic Analysis

For the qualitative data collected through the teachers' and students' interviews, I adapted a thematic data analysis, as presented by Braun and Clarke (2006, 2013). I did not continue with the Variation Theory framework for this section of data analysis because some of the teachers' and students' views of events outside the classroom, such as examination pressure and syllabus coverage, made it difficult to use the framework. I needed a broader framework for this part of the data.

Braun and Clarke (2006) define thematic analysis (TA) as "a method of identifying, analysing and reporting patterns (themes) within data" (p. 79). They explain that TA is applied through six phases as shown in Table 11.

Braun and Clarke (2006) explain that "a theme captures something important about the data in relation to the research question and represents some level of patterned response or meaning within the data set" (p. 82). Themes emerge from a coded data set, as explained in Table 11. The initial themes combine to form basic themes, which are then grouped together to summarise a more abstract principle called organising themes. The organising themes encapsulate into a broader theme called global theme (Attride-Stirling, 2001, Braun & Clarke, 2006, 2013). These categories of themes form what Attride-Stirling (2001) calls a thematic network, while Braun and Clarke (2006, 2013) refer to it as a thematic map.

Themes can be identified either through an inductive (bottom-up) or deductive (top-down) process. In an inductive approach, themes are identified from the data set depending on the prevalence of the information, as mentioned by the participants, or through observation. Therefore, the themes are data driven. A deductive approach to theme identification is “driven by the researcher’s theoretical or analytic interest in the area and is thus more explicitly analyst-driven” (Braun & Clarke, 2006, p. 84).

Table 11: ***Phases of thematic analysis, adapted from Braun and Clarke (2006, p. 87)***

Phase	Description of the process
1. Familiarising yourself with your data:	Transcribing data (if necessary), reading and re-reading the data, noting down initial ideas.
2. Generating initial codes:	Coding interesting features of the data in a systematic fashion across the entire data set, collating data relevant to each code.
3. Searching for themes:	Collating codes into potential themes, gathering all data relevant to each potential theme
4. Reviewing themes:	Checking if the themes work in relation to the coded extracts (Level 1) and the entire data set (Level 2), generating a thematic ‘map’(network) of the analysis.
5. Defining and naming themes:	Ongoing analysis to refine the specifics of each theme, and the overall story the analysis tells, generating clear definitions and names for each theme.
6. Producing the report:	The final opportunity for analysis. Selection of vivid, compelling extract examples, final analysis of selected extracts, relating back to the analysis to research question and literature, producing a scholarly report of the analysis.

In this research, themes identify largely with the deductive category since the teachers and the students answered the questions with reference to the lessons taught. This gave the interview an evaluative tone. Teachers mostly explained their views with respect to their collaborative work during the planning and enactment of the lesson, as well as students' behaviour during and after the lessons. Similarly, students explained their experience with the teaching and learning approach with regard to their classroom activities and their behaviour after the lessons.

However, I developed codes from the data set with regards to the way the students answered the questions. These codes were collated to form basic themes which were further collated to organising themes and finally, to global theme. The summarised categories of themes with relevant data set as suggested in step 5 of Table 11 is appended as Appendix 3. Analysis of Chapter 7 was based on these themes as summarised in Figure 10.

Therefore, the main theme (which Attride-Stirling (2001) and Braun and Clarke (2006) would refer to as a global theme), was the teachers' and the students' experiences of the teaching and learning of the topic of quadratic expressions and equations in a LS approach. I separated the main theme into two organising themes called Strengths and Challenges (as shown in Figure 10). The organising themes were further separated into basic themes (Attride-Stirling, 2001; Braun & Clarke, 2006), *'teachers' professional development through classroom practices and students' learning'* under Strengths; and *'cultural changes, national examination pressure and teacher shortage'* under Challenges.

My analysis along these themes and sub-themes incorporated relevant literature, with some observations from the lessons. The analysis collated the information from the teachers, students, and classroom observations, together with the relevant literature.

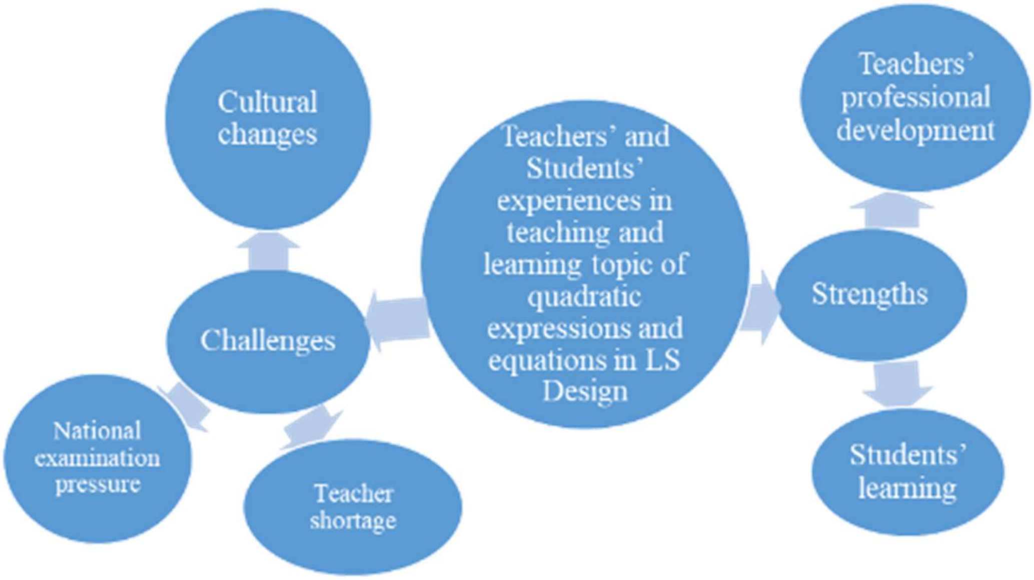


Figure 10**: Thematic network showing main themes and sub-themes. Adapted from Attride-Stirling (2001, p. 388)**

3.8 Limitations of my research

I discuss the limitations to my study from two perspectives – those that challenged the ideal implementation of the LS approach, and the areas that the research did not investigate.

The LS approach is a relatively new approach and is certainly novel in Kenya, so the teachers needed time to familiarise themselves with its components, as well as its application. I planned three orientation meetings in the term preceding the data collection, but as explained in section 3.4, I only managed one meeting. These earlier meetings would have given the teachers more time to learn the approach, have some peer teaching and teaching the classes as pilot preparation, before applying it in class during data collection. However, the team bridged the gap by choosing the topic that they would later teach as the first pair of the lessons and planning a lesson which they demonstrated during the last two meetings. This action helped them to learn the approach as they practised it.

In the usual set-up of the LS approach, when collecting data, the teachers usually have a day or two to modify the lesson before teaching it. In this case, the two classes had mathematics lessons every day, so the teachers had to modify the lessons and teach them on the same day. However, they had at least three hours between the lessons in which to reflect on the first lesson and to modify the second one. Same day modification may have had the advantage of teachers clearly remembering areas that required improvement.

The data collection was limited to classroom observations and interviews with the teachers and a group of students, therefore the findings were based on these data. There were also data from pre- and post-lesson tests, which supported the preparation and the evaluation of the lessons. Future study may include a component of a retention test some time after the study in order to check students' performance after implementation.

3.9 Ethical Considerations

Prior to embarking on the data collection, I obtained an approval letter from the University of East Anglia (UEA) School of Education research ethics committee (appendix 4(a)). Before I received this approval, I had to confirm to the committee that I had read the University research ethics policy together with the British Educational Research Association's Revised Ethical Guidelines for Educational Research. I committed to behave in a professional manner and agreed to not put the lives of participants and my own at any risk, such as through disclosure of their identities both in written or pictorial form. Any reference to the participants would be done by use of pseudonyms. The participants' identities were protected according to the Data Protection Act (1998) and Freedom of Information Act (2005). The data collected was confidentially treated, kept in safe custody, and only used for the purpose of the study and any future publications that may come from it. I carried out the research organising mutually convenient times, and in a way that sought to minimise disruption to schedules and burdens on the participants. In this regard I observed the lessons in the usual periods, as stipulated in the timetable, and at the topic's scheduled time in the curriculum, in order to reduce any extra work that the teachers would be required to do.

I explained to the participants that data collected would be strictly confidential, kept safe and only seen by my supervisors and myself. Their participation would be purely voluntary, and they could opt out any time during the data collection period without any prejudice. I also explained to the students that I would respect their dignity and interest, and I gave them official University phone numbers to communicate directly with my supervisors whenever there was a need to do so.

Before going to the school to begin data collection, I obtained an approval letter to conduct research in Kenya from the National Commission for Science, Technology and Innovation (NACOSTI), a body charged with the responsibility of vetting all research carried out in Kenya (see appendix (4b)). To gain access to the schools I also obtained a letter from the County Director of Education (CDE), Siaya, (appendix (4c)), addressed to the principals of the schools, as explained in section 3.2.1.

I explained the nature of my study to the principal of the school and the participants. I informed the participants that I would confidentially handle the data and the only other person who would have access to the data would be my supervisor. I also informed participants that any reference to them in the research would be through pseudonyms. I explained all this to the teachers and the students separately. Thereafter, each participant signed the individual consent forms. Since the data collection involved asking teachers to do some work above their usual daily work, such as preparing the lessons together, observing each other's lessons, and giving out students' tests, I discussed timings with the participants and allowed them to decide on the most convenient times to do these tasks.

As other people, besides the class teacher, would be attending the lessons (especially myself, whom they considered the more knowledgeable participant in the intervention), we agreed, as the observing team, that nobody apart from the teacher teaching the lesson would comment or help the students during those lessons. We made this decision to secure consistency across all sessions and to help the teacher relax and settle emotionally.

At the end of the study I thanked all the participants for their active participation in this research, and the principal of the school for allowing me to use the school's facilities during the data collection. I talked to the students of Forms 3 and 4, upon the request of the principal, advising them to work hard in mathematics and pass their Kenya Certificate of Secondary Education (KCSE) examinations. I also promised to go back to the school after my graduation to inform them about my findings.

Chapter 4 – First Pair of Lessons: Factorisation of Quadratic Expressions

4.1 Overview

This chapter is the first of four chapters relating to the analysis of my research. The first three of these (chapters 4-6) present the analyses of and discussions about data collected through lesson observations. Chapter 7 presents the analysis and discussion of data collected through interviews.

I begin the discussion of this chapter by outlining the topics that I consider prerequisite to the topic of quadratic expressions and equations. Afterwards, I discuss the preparation and implementation of the first pair of lessons – Factorisation of quadratic expressions – in line with the requirements of a LS approach (Lo, 2012, Marton, 2015). In the discussion, I give an account of how the teachers taught the lessons, beginning with the identification of the object of learning and its critical feature. I continue the discussion to show how each of the two teachers incorporated the planned patterns of variation and invariance within the enacted object of learning as explained by Marton (2015).

I now present the analysis of the first pair of lessons together with a discussion of the findings. In the analysis, I have used the theoretical framework of variation theory, as guided by Lo (2012) and Marton (2015) and summarised in Chapter 2 (section 2.7). The analysis looks at each of the three components of the object of learning, that is, the intended, the enacted and the lived object of learning. I conclude the chapter with an overall reflection on the two lessons.

4.2 Introduction to the Lessons

According to the Kenyan secondary schools' mathematics curriculum ([KIE], 2002), before teaching the topic of quadratic expressions and equations, teachers are supposed to teach the following topics as prerequisite knowledge. In Form 1 (14-16 years), factors, algebraic expressions, solutions of linear equations including simultaneous linear equations, and coordinates and graphs. In Form 2 (15-17 years), equations of a straight line. These topics are presented as Chapters 4, 10, 17, 19 and 27 respectively in the curriculum book ([KIE], 2002).

Under the topic of factors, students express composite numbers in their prime factor forms such as $10 = 2 \times 5$, $36 = 2^2 \times 3^2$ and so on. With regards to the topic of algebraic expressions, the students learn how to use letters to represent mathematical statements and how to simplify algebraic expressions such as $3(7x - 2) - 5(2 - 3x)$. For solutions of linear equations, the students learn to solve equations in one and two unknowns, which include the solution of simultaneous linear equations by the methods of elimination and substitution. Regarding coordinates and graphs, students learn to plot points in a Cartesian plane, sketch the graphs in their exercise books, and learn to solve simultaneous linear equations in two unknowns using the graphical method. In Form 2, they learn to find the gradients of straight lines and determine the equations of straight lines in the form $y = mx + c$. Teachers draw from the students' experiences in these topics to teach the topic of quadratic expressions and equations.

Before teaching factorisation of quadratic expressions, the topic of discussion in this chapter, the teachers had taught the following contents within the topic of quadratic expressions and equations. Expansion of algebraic expressions, such as $(x + 2)(x + 5)$, including the expansion of the three quadratic identities ([KIE], 2002, p. 22), $(p + q)^2$, $(p - q)^2$ and $(p + q)(p - q)$.

As is characteristic of a LS approach, before preparing a lesson, teachers identify the object of learning (Lo, 2012; Marton, 2015; Pang, 2008). The object of learning is identified after gathering information about the intended content/topic by consulting with the students on their prior knowledge, learning difficulties and their conceptions about the topic. Also taken into account are the syllabus, textbooks, research article(s), other related resources and teachers' past experience with the topic (Lo, 2012; Runesson, 2013).

In a LS approach, the gathering of prior information from the students is usually either through a diagnostic pre-lesson test or through short interview but in this research, I used short pre-lesson tests as explained in section 3.6.2.

In this first pair of lessons, after collecting responses from the students' diagnostic pre-lesson test, the teachers identified *factorisation of a quadratic expression with a unit coefficient of x^2 of the form $x^2 + bx + c$* as the object of learning. To discern the object of learning, the teachers focused on the relationship between the factors of the constant term and the addends of the coefficient of x as the critical features of the object of learning in an expression such as the one within this paragraph. The relationship narrows down to the identification of the factors of the constant term that sum to the coefficient of x. For example, to factorise an expression $x^2 + 7x + 10$, one would identify factors of 10 that sum to 7. The relationship is often referred to as "sum and product" in Kenyan mathematics textbooks ([KLB], 2003).

The lesson was going to be the first one to be taught in a learning environment different from the students' usual classroom setting. Teachers other than their mathematics teacher and the researcher were going to be present during the teaching and learning, and the students were expected to discuss in small groups and later report their work and discuss with the whole class. In Kenya, it is not a common practice to find other teachers observing their colleagues' lessons or even teaching a colleague's class in his/her absence, as already explained in Chapter 2 (section 2.6). Another cultural issue involved changing the usual classroom procedure – actively involving the students through small group discussion followed by group report. According to Stigler and Hiebert (1999), a classroom has a culture within which there are clear expectations for the teacher, the students, from the school administration and to some extent from the parents, who may have learned in the same way. Mulala (2015) describes Kenyan classroom teaching as "traditional instructional practices that centre on teacher dominated pedagogy (p. 20)."

The approach I applied in this research shifted teaching and learning from the usual traditional instructional practices where students are passive recipients of knowledge to a learner-centred approach where students are actively engaged during the lesson through small group and whole-class discussions. For this lesson, the teachers agreed to adapt the activity they had demonstrated during the orientation process as the lesson activity. It was related to the topic of discussion and they argued that since it was a practical activity (hands-on), which was new to the students, they would be curious about the activity and engage with it. The teachers had understood the activity and it was easy for them to supervise its implementation during the learning process.

The activity involved cutting up pieces of paper, as shown in Figure 11. The bigger piece is a square with x units; the five rectangular strips are of sides x units by one unit each and the six small pieces are each one-unit square. All these pieces together represented the expression $x^2 + 5x + 6$. There were two tasks in the activity. The first task required the students to form a rectangle from all the pieces and to determine its area. The formation of the rectangle was intended to help the students factorise the expression $x^2 + 5x + 6$, which is $(x + 2)(x + 3)$. The second task required the students to identify the relationship between the numerical terms of the factors $(x + 2)(x + 3)$, which are 2 and 3 and the coefficient of x and the constant term, in the expression $x^2 + 5x + 6$.

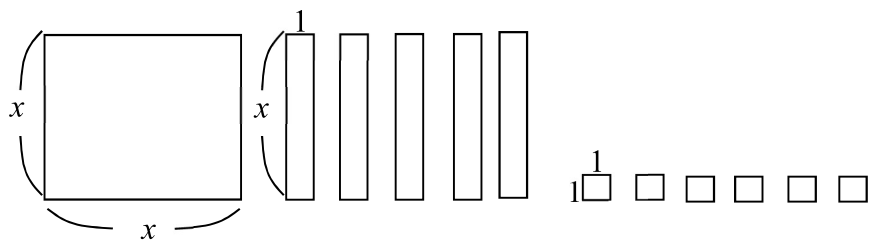


***Figure 11*: Paper cuttings for a hands-on activity aiming at the factorisation of $x^2 + 5x + 6$**

After discussing the first activity with the whole class, the students were instructed to use some of the pieces of paper from Figure 11 to do the second activity, which was the formation of a different rectangle leading to the factorisation of the expression

$x^2 + 3x + 2$. Although the second expression looks easier than the first one, it was meant to help the students observe the pattern leading to the *generalisation* of the condition for factorising a quadratic expression.

The activity was meant to help the students to appreciate the geometrical approach to the teaching and learning of algebra. In addition, the teachers felt that the activities were simple enough to motivate the students to discuss in their small groups. In the context of the LS approach to teaching and learning, the activities were designed in line with the generalisation pattern of variation and invariance (Marton, 2015) to help the students *generalise* the conditions for the factorisation of a quadratic expression with a unit coefficient of x^2.

Before planning the lesson, the teachers developed the following diagnostic pre-lesson questions to help them with the planning.

1. Why is this expression, $x^2 + 5x + 6$ called a quadratic expression?
2. What do we consider when attempting to factorise a quadratic expression such as the one given in (1)?
3. How many factors do we expect from a factorised quadratic expression?

As I reviewed earlier, Marton (2015) explains the aim of a diagnostic pre-lesson test as "the pre-lesson test is to find out whether or not the students discern certain aspects (dimension of variation) and thus the questions should not point out the aspects to be discerned (p. 261)." The teachers developed questions which were meant to help them find out the students' conceptions of certain aspects of the topic, which when addressed during the lessons help them discern the object of learning. In most cases, the same pre-lesson test is also the progressive post-lesson test administered after the lesson, which was the case with this study.

As I have already mentioned, the teachers had introduced the topic of quadratic expressions and equations, where they had expanded the linear factors to obtain quadratic expressions. The first question was intended to help them find out if the students understood the meaning of a quadratic expression. The second and the third questions were intended to help the teachers find out if the students would relate the expansion of the quadratic factors, which they had learnt in the previous lessons, with the factorisation of quadratic expressions to revert to the factors.

4.2.1 First Lesson

The expected answers to the diagnostic pre-lesson test stated in section 4.2 are: (1) the expression is called a quadratic expression because of the term x^2, (2) we consider factors of the constant term that sum to the coefficient of x, (3) we expect at most two factors. Although the teachers gave the above expected answers, there are cases like in question (2) when the given statement would not be authentic. A case where the coefficient of x^2 is not a unit. The teachers could have given a broader explanation to cater for such cases as well.

Table 12: ***The students' responses from the diagnostic pre-test (First lesson)***

Total number of students = 40					
Items	**Responses**				
Question 1	Correct (because of the term x^2)	It has unknowns	Because of x	blank	% of correct response
Frequency	1	29	2	8	3
Question 2	Correct (factors of the constant term that sum to the coefficient of x)		Like terms	blank	% of correct response
Frequency	1		35	4	3
Question 3	Correct (at most 2 factors)	Four factors	Three factors	blank	% of correct response
Frequency	28	7	3	2	70

Some of the frequent responses received from the students shown in Table 12 were: (1) the given expression is called a quadratic expression because it has unknowns. (2) We consider like terms in an attempt to factorise a quadratic expression and (3) there are two factors in a factorised quadratic expression. For the third question, this response was considered the correct answer. The responses were scored 1 for a correct answer and 0 otherwise.

Even though the students had been introduced to the topic of quadratic expression and equations, and although the students were partly right in that quadratic expressions and equations have unknowns, the responses from Question 1 suggest that most of the students might not have understood the meaning of a quadratic expression. Question 2 was the one addressing the topic of the lesson and its outcome confirms that almost all the students had no idea about the factorisation of a quadratic expression. Question 3 outcome shows that the majority of students could perhaps recall from the previous lesson that there are two factors arising from a factorised quadratic expression. The previous lesson was on the expansion of two linear factors leading to a quadratic expression.

In general, the nature of the questions was a deviation from the usual way the students were asked questions in mathematics, which was usually to solve a mathematical problem and not to explain definitions of a mathematical concepts or explain ways of solving mathematical problems.

The teachers considered the outcomes of the pre-lesson test when planning for the lesson. The outcomes helped them to understand the students' areas of weakness on which they needed to lay more emphasis during the lesson.

The Lesson

Peter introduced the lesson by explaining the rationale of including the topic in the mathematics curriculum. He informed the students that the topic is useful in some faculties such as Engineering in the Universities, and it is considered a prerequisite topic to other mathematics topics such as polynomials and binomial expansion.

The Kenyan mathematics curriculum requires teachers to explain the rationale of including the topics in the syllabus ([KIE], 2002). This is usually done, especially when introducing new topics, to help the students understand the importance of the topic and areas in which they expect to apply the topic. The curriculum developers found that students used to ask such questions as, where would they apply some mathematics concepts such as algebra ([KIE], 2002), which is supported by Nardi's & Steward's (2003) finding where some students questioned the rationale of teaching topics like algebra. The students said "Some of the topics are just so stupid they're … and … like algebra. […] Where are we going to use them for?" (p. 352).

As stated earlier, prior to this lesson, the teachers had taught the expansion of quadratic factors with a mixture of a variable and a numeral such as $(p + 2)\ (p + 5)$. Also, they had taught the expansion of factors such as $(p + q)^2$, $(p - q)^2$ and $(p + q)\ (p - q)$, the so-called quadratic identities. With this in mind, Peter wrote two expressions $x + 5$ and $x^2 - 6$ on the chalkboard and asked the students to identify which of the two expressions is quadratic. Many students were able to identify $x^2 - 6$ as the quadratic expression, explaining that it has the term x^2. This response was contrary to the pre-lesson test outcome, where only one student explained why $x^2 + 5x + 6$ is a quadratic expression. Perhaps Peter's approach, where he contrasted the two expressions, one linear and one quadratic, might have made the difference. Alternatively, some students might have discussed the questions amongst themselves ahead of the lesson, and that enabled them to identify a quadratic expression.

The teacher continued and asked the students to identify the coefficients of x^2 and x in the expression $x^2 + 4x + 3$. Whereas all the students could identify the coefficient of x as 4 correctly, the majority were unable to identify the coefficient of x^2 to be 1. Students gave various responses that included $x \times x$, x and 2 - presumably from the exponent 2. The teacher asked the students to discuss in pairs and seek a correct solution. After a while, a student correctly identified the coefficient, but could not explain why it is 1. Peter explained to the whole class why the coefficient is 1.

The pre-lesson test outcomes helped the teachers prepare the introductory remarks of the lesson and so clarified to the students the meaning of a quadratic expression that they had been taught in the previous lesson, but they could not remember. In addition, the introduction focused the students on the explanation of the term *coefficient*, which they would need in order to relate the factors of the constant term and the coefficient of x in discerning the object of learning.

After the introduction, Peter asked the students to factorise the expression $x^2 + 5x + 6$. After about five minutes, he asked them to form eight groups of five students each. The five minutes was for the students to think about the question individually and put down some attempts. The other teachers helped Peter distribute the pieces of paper shown in Figure 11 to each group. Peter explained the goal of the activity to the students, that is, to form a rectangle from all the pieces of paper provided and to determine the area of the rectangle formed. He allowed 15 minutes for the task. At the end of the 15 minutes, only two groups had formed the rectangle. Perhaps, Peter could have reversed the order of the activities to start with the one leading to the factorisation of $x^2 + 3x + 2$, which appears simpler than the activity leading to the factorisation of $x^2 + 5x + 6$.

He allowed a further 10 minutes for discussion, during which five more groups formed the rectangle with five out of eight groups calculating some areas as shown in Table 13. The students' responses to the formation of the rectangle and the determination of the area can be separated into four categories as shown in Table 13.

Table 13: ***Categories of the students' group work on the first activity from the first lesson***

Category	**Number of groups**	**Formation of the rectangle**	**Determination of the area**
One	4	Correct rectangle	Correct
Two	1	Correct rectangle	Not correct
Three	2	Correct rectangle	Not determined
Four	1	No rectangle formed	Incomplete

Peter asked three groups, one each from Categories One, Two and Four, to present their work for whole-class discussion. The Category One group presented the work shown in Figure 12. The group representative explained their working as follows:

Student 1[3]: Length = 1 + 1 + 1 + x = (3 + x) and width = 1 + 1 + x = (2 + x).
Area, A = (3 + x) (2 + x). Therefore, $x^2 + 5x + 6 = (3 + x)(2 + x)$

Although all the four groups in Category One determined the area in the same format as shown by student 1, they had different tessellations of the rectangles.

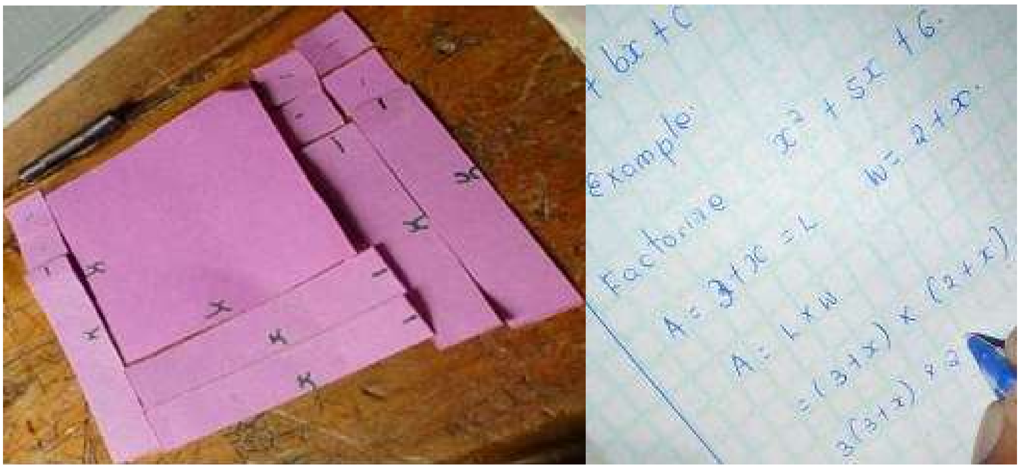


***Figure 12*: Representation of the Category One from the first lesson**

The Category Two group presented the work shown in Figure 13. The representative explained:

Student 2: The width has two pieces of x giving an area of $x \times x = x^2$. The length has a piece of x at the bottom and two pieces at the upper part plus the big piece whose length is x giving a total of $4x^2$. Total area,
A = $x^2 + 4x^2$.

[3] The numbering of students in Chapters 4, 5 and 6 was done in order of the students' responses during a lesson.

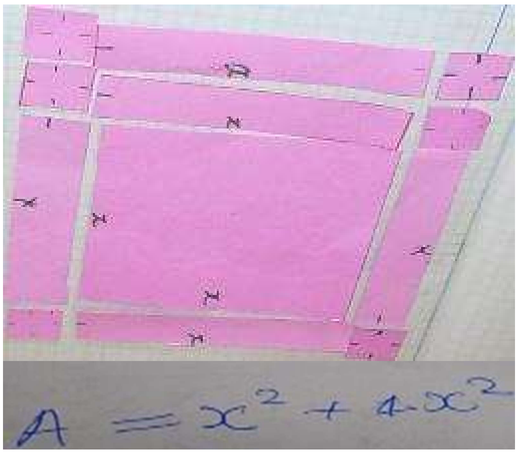


Figure 13: **Representation of the Category Two group from the first**

The group formed a correct rectangle but did not calculate the correct area, and hence failed to factorise the expression. From Student 2's explanation, it appears the group was trying to work out the area in parts, and then add them to obtain the area of the whole figure, which should be a correct procedure. However, the group made some mistakes in the process. For example, the group multiplied the two x unit by 1 unit pieces of paper on the extreme ends of the figure to obtain x^2. Similarly, they added the three x unit by 1 unit pieces of paper together with the square x unit by x unit, to obtain $4x^2$. The group did not consider the 6 pieces, each of which is a unit square, in their working of the area. It was difficult to comprehend this group's work as they were also not clear in their justification of the approach. However, their work showed that the students had difficulty in forming a correct algebraic expression from the diagram.

Category Four group presented their work represented by Figure 14. When the teacher asked them to explain their work, the group representative said:

Student 3: I have realised our mistake from the other groups that presented ahead of us, (*referring to Figures 12 and 13*).

Figure 14: **Representation of the Category Four group from the first lesson**

Student 3 realised that in attempting to form the rectangle they put all the x by 1 pieces on one side of the square x unit by x unit, leaving them with no way of using all the 6-unit squares to form a rectangle as required.

After the groups' presentations, Peter discussed Task Two with the whole class as he realised that he was running out of time, as by then it was the 40th minute, marking the end of the lesson period. The teacher referred to the area determined by Student 1, that is, $x^2 + 5x + 6 = (3 + x)(2 + x)$ and asked, "What do you notice between the values of 2 and 3, and the coefficient of x and the constant term of the expression $x^2 + 5x + 6$?" A student explained:

Student 4: Their sum is equal to 5, and their product is equal to 6.

The teacher acknowledged student 4's answer and explained that to factorise a quadratic expression such as the one given in the activity, one needs to identify the factors of the constant term that sum to the coefficient of x. Peter told the students, "For your homework factorise the expression $x^2 + 3x + 2$". He then gave the students the post-lesson test (same as the pre-lesson test). The students' responses are shown in Table 14.

Question 1 outcome suggests that although Peter explained the meaning of a quadratic expression at the introduction of the lesson, 28 out of 40 students seemed not to have comprehended the meaning. Similarly, most students, 36 out of 40, appeared not to have understood the conditions for factorising a quadratic expression as suggested by Question 2. Although 17 out of 40 students correctly answered Question 3, the majority of students, 21 out of 40, had varied responses, which might suggest that the students were not sure of the term *factor of a quadratic expression*. Many students, 14 out of 40, claimed that there are four factors arising from the factorisation of a quadratic expression. Perhaps these students considered each term in each of the two brackets $(3 + x)$ and $(2 + x)$ to be a factor. Other students said that there were three factors, as the question was left blank (see Table 14). The teachers discussed these outcomes during the reflection session.

Table 14: ***The post-test responses by students from the first lesson***

Total number of students = 40					
Items	**Responses**				
Question 1	Correct (because of the term x^2)	It has unknowns	Because of x	blank	% of correct response
Frequency	12	19	1	8	30
Question 2	Correct (factors of the constant term that sums to the coefficient of x)		Like terms	Consider coefficients	% of correct response
Frequency	4		3	33	10
Question 3	Correct (at most 2 factors)	Four factors	Three factors	blank	% of correct response
Frequency	17	14	5	4	43

First lesson's reflection session

As I explained in Chapter 3, the teachers who taught the lessons commented first. In this lesson, Peter was the first one to give his remarks (section 3.9).

Peter explained that the introduction took more time than he had expected and denied him the opportunity to teach the lesson as it had been planned.

Peter: I would have taught how to obtain the coefficient first before I taught this lesson. However, I think the students were able to come up with the rectangles, and some were able to work out the area of their rectangles. The only thing they could not identify is the relationship between the area they determined and the initial expression $x^2 + 5x + 6$ so that they could come up with the knowledge of "product and sum."

Peter caused some laughter when he evaluated the lesson and remarked,

Peter: So, I would say that the lesson was half good and half bad.

Peter said that the introduction took more time, which I confirmed to be true when I replayed the video. Peter planned to spend five minutes, but he spent seven minutes. Although Peter felt he should have explained the term coefficient before teaching the lesson, I think it may not have been his problem because this was not the first time the students were meeting the term. The students had been taught the topic of Algebraic Expressions in form one (14-16 years) where they used the term for the first time. In this lesson, Peter asked the students about the term at the introduction of the lesson to prepare them for its use in the lesson. The term was crucial in this lesson, as it was part of the explanation of the critical feature, which was the point of focus to discern the object of learning.

Dominic observed that the students were timid when they were working on the task. He explained that the way the students represented the determined area, that is $(3 + x)\ (2 + x)$ might have denied them the opportunity to see the relationship that was relevant in the second task. He explained:

Dominic: First, those students were nervous - I do not know what they feared. Secondly, they formed the rectangles very well, but if they could rearrange the factors from $(3 + x)\ (2 + x)$ to $(x + 3)\ (x + 2)$ they would have got the relationship because they could easily see the xs. Otherwise, the lesson was very much okay.

The students may have been justified in being timid given that this was the first lesson taught in an unfamiliar environment for them. However, Dominic's argument that the students would have obtained the intended relationship had they rearranged the factors, needs further investigation. It may or may not be true. The thinking is analogous to the thinking of teachers in Runesson's (2013) study of Exploring Teaching and Learning of Converting Sentences into Algebraic Expressions where the teachers conjectured that students find it difficult to write algebraic expression in terms of x and y when the variables in the statements are different. They made that argument their critical feature of the object of learning, and prepared the lesson using relevant letters such as b for a book and p for a pen. However, after the first lesson, they realised that students still had difficulties writing the expression representing the statement "3 pens and 1 book costs as much as 5 pens" (p. 178). The teachers had to change their critical feature to focus on in the second lesson. So, Dominic's proposal may require further investigation.

John agreed with Dominic's statement that Peter taught the lesson well. However, he pointed out that they had an oversight with their planning of the lesson. He commented:

John: To me the lesson was okay, but I think we should have done some peer teaching. We forgot how the students would present their work and this became a challenge that we did not expect.

From these observations and by referring to the lesson plan, the team realised that Peter ran out of time, which denied him the opportunity to implement the lesson fully as planned. They pointed out that part of the time management problem came from the preparation of the lesson. Due to that, the students did not discuss the second task of Activity One, which required them to relate the numerical values of the factors of the expression $x^2 + 5x + 6$, with the coefficient of x and the constant term, one of the critical features. In addition, the students did not discuss the second activity, which was meant to confirm the critical feature (sum and product) and to help the students *generalise* the condition for factorising a quadratic expression with a unit coefficient of x^2. Nevertheless, Peter explained the expected outcomes of the tasks in his conclusion to the lesson.

The team then agreed to modify the lesson, based on these observations, and for John to re-teach it to his class. The modification of the lesson included: (1) Explanation of both tasks of Activity One at the start of the group discussion so that any group that completed Task One could continue to Task Two without waiting for others to complete Task One, thus saving time. (2) The acquisition of drawing pins to help the groups pin their work during the groups' presentations. John re-taught the lesson to his class, hereafter referred to as the second lesson, one and half-hours after the end of the first lesson. The first lesson started at 11:50 and ended at 12:30, while the second lesson started at 14:00 and ended at 14:40.

A summary of the First Lesson

The students showed much enthusiasm working on the activity and the groups were fully involved in the discussion, although they were speaking softly, which made it difficult even for the group members to hear each other's explanations. Perhaps this could be attributed to the classroom culture the students are accustomed to, where they passively receive information from the teachers and only talk when answering a question from the teacher or when asking a question. The students might have thought that they would disturb other members of the class if they spoke loudly in their groups. Maybe this soft speaking together with the new classroom environment delayed completion of groupwork within the time the teachers had planned. As Rowland et al. (2011) point out, teachers usually respond to students' responses to ideas by taking contingency measures, so Peter responded to this delay and added more discussion time which helped many groups to complete the task.

In the end, the approach helped students discern the object of learning as was also suggested by the post-lesson test outcomes. Seven groups out of eight formed the required rectangle. Four groups out of the seven worked out the first activity correctly. Of the remaining three groups one did not determine the area of the rectangle correctly, while the remaining two did not determine the area at all. It was not clear whether they did not calculate the area due to time constraint, because the teacher stopped the discussion while some groups were still discussing, or because they just did not know how to find the area from the rectangles they formed.

The lesson revealed some difficulties that the students had with quadratic expressions. During the introduction, many students could not identify the 'invisible' 1 as the coefficient of x^2 in the expression $x^2 + 4x + 3$. In addition, out of the seven correct rectangles formed, three groups could not find correct algebraic expressions to represent the sides of the rectangles, which could have led them to the factorisation of the expression. These difficulties contributed to the time constraint experienced during the lesson. The students did not perform the second activity, neither did Peter demonstrate it for them. Instead, he assigned it as a homework.

During the reflection session, the teachers noted that part of the time constraint arose from their preparation of the lesson (the intended object of learning). They planned to explain the activities one at a time and this wasted time for groups that completed the first activity in time because they had to wait for the others. The teachers also observed that they did not plan how groups would report their work after discussion, which wasted time during presentations and subsequently wasted time of the lesson. These observations became part of the teachers' reasons for the modification of the second lesson.

4.2.2 Second Lesson

The students of the second lesson had done the diagnostic pre-lesson test at the same time as the students of the first lesson. The pre-lesson test was the same as on section 4.2.

Responses to the pre-lesson test from the students in the second lesson are shown in Table 15. This class had an enrollment of 33 students, but 27 students responded to the pre-lesson test. I was told that some students were unwell, while other absent students had issues of school fees to sort out with the administration. However, 28 students were present during the lesson and responded to the post-lesson test. The outcomes resembled those of the first lesson. The outcome to Question 1 suggests that 20 out of 27 students had not comprehended the term quadratic as was also observed in the first lesson. The students' responses to Question 2 clearly showed that they did not know the answer, which was understandable because the question was concerned with what was to be taught in the lesson.

Table 15: ***The diagnostic pre-test responses by students from the second lesson***

Total number of students = 27					
Items	**Responses**				
Question 1	Correct (because of the term x^2)	It has unknowns & knowns	Because of x	blank	% of correct response
Frequency	7	18	1	1	26
Question 2	Correct (factors of the constant term that sums to the coefficient of x)		Like terms	blank	% of correct response
Frequency	0		24	3	0
Question 3	Correct (at most 2 factors)		Four factors	Three factors	% of correct response
Frequency	13		8	6	48

The fact that many students, 24 out of 27, responded by saying that they would consider the like terms, alerted the teachers to the fact that they needed to prepare an explanation to correct the error. Students' responses to Question 3 showed that about a half of them might not have been sure with the term factors, as was also witnessed in the first lesson, although 13 out of 27 answered the question correctly. The teachers considered the responses in their preparation of the lesson.

The Lesson

At the introduction of the lesson, John explained the rationale of including the topic in the syllabus, as Peter did in the first lesson.

John then asked the students to expand the expression $(x + 2)(x + 1)$. Most students were able to expand the expression within two minutes. One student explained the answer:

Student 1 x multiplied by x gives x^2 plus x multiplied by 1 plus 2 multiplied by x plus 2 multiplied by 1, which gives $x^2 + 3x + 2$.

John wrote student 1's explanation on the board. He then asked the students to identify which of the expressions $x - 10$ and $x^2 + 3x + 2$ is quadratic. The students were able to identify $x^2 + 3x + 2$ as the quadratic expression and explained that it is so because of the term x^2, contrary to the answer by a majority of them in the pre-lesson test. Perhaps the approach John used by expanding the factors helped the students to remember the term *quadratic expression*.

John then asked the students to identify the coefficients of x^2 and x in both expressions $x^2 + 3x + 2$ and $x - 10$. The students identified the coefficients correctly. John explained further the terms 'coefficient' and 'constant term' by referring to the general form of a quadratic expression $ax^2 + bx + c$. The introduction took five minutes as planned.

John then informed the students that the intention of the lesson was to factorise the expression $x^2 + 5x + 6$. He asked each student to factorise the expression. After about five minutes, he asked the students to form seven groups of four students each to share their experience with the factorisation of the expression. Meanwhile, he distributed the pieces of paper shown in Figure 11 to each group. John gave the two instructions that addressed both tasks of Activity One before the start of the activity, as had been suggested during the reflection session after the first lesson. The instructions were:
(a) Form a rectangle with all the pieces of paper given and calculate the area of the rectangle formed.
(b) Find the relationships between the constant terms of the factors (sides of the rectangle) and i) the coefficient of x in the expression $x^2 + 5x + 6$, and ii) the constant term in the same expression $x^2 + 5x + 6$.

Note: the sides of the rectangle formed are represented by the factors of the expression $x^2 + 5x + 6$.

As the students worked on the activity, John and the observers walked around the class to observe their work. After 15 minutes, John asked all the groups to stop working. At that time, all the seven groups had formed the rectangles.

Three out of seven groups had determined the area while the others had not, as shown in Table 16.

Table 16: ***The responses of groups from the second lesson to the activity***

Category	**Number of group(s)**	**Formation of the rectangle**	**Determination of the area**
One	3	Correct rectangle	Correct
Two	1	Correct rectangle	Not correct
Three	3	Correct rectangle	Incomplete

John separated the students' responses into three categories as opposed to the four categories observed in the first lesson. The fourth category observed in the first lesson was for the group that did not form the rectangle; however, all the seven groups in this second lesson formed the rectangles. John asked two groups from Category One and the Category Two group to present their work.

Of the two groups from category one, only one explained how they had determined their area, as shown in Figure 15.

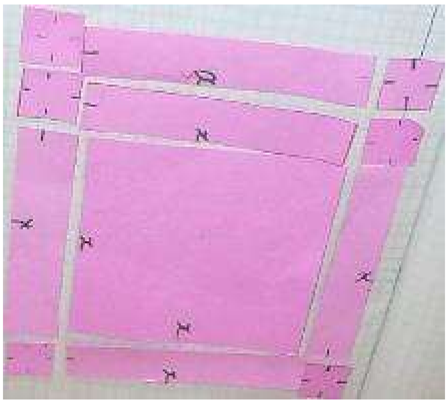


Figure 15: **Work by the first Category One group of the second lesson**

The representative of the first Category One group (represented by Figure 15) explained:

Student 2: Width = $1 + 1 + 1 + x = 3 + x$

Length $= 1 + 1 + x = 2 + x$

Area, $A = L \times W$

$= (2 + x)(3 + x) = x^2 + 5x + 6$

The other group represented by Figure 16, only presented the final factorised expression. The representative wrote their answer as:

Student 3: Area, $A = L \times W$

$= (3 + x)(2 + x) = x^2 + 5x + 6$

Figure 16: **Second Category One group in the second lesson**

The Category Two group explained the work as shown in Figure 17. The representative explained their work as follows:

Student 4: The two strips above (Figure 16) are multiplied to obtain x^2 and the four pieces on one side (pointing at the width with 1 unit by x units strip and the three 1 unit by 1unit square) are counted and multiplied by x to obtain $4x$.

Area, A = $x^2 + 4x$.

As in the first lesson (see Figure 13), the students of the Category Two group formed the correct rectangle but failed to determine the area correctly. It is difficult to comprehend why the students approached their work in the way Student 4 explained and the student could not explain why they had multiplied the xs from the two, I unit by x unit, strips on the upper side of the length to obtain x^2.

***Figure 17*: Category Two group in the second lesson**

After Student 4's presentation, John invited comments from the rest of the students. Three students gave comments; one of them explained:

Student 5: All the pieces were used to form one rectangle giving an area of $(3+x) \times (2+x)$. The three small pieces added to the one whose side is x cannot form $4x$ because x can take any value.

John did not comment further on the presentations but proceeded with the whole class discussion on task two. Referring to the first two presentations (from Student 2 and Student 3), John asked the students to identify any relationship between the constant terms of the factors ($x + 2$) and ($x + 3$) and: (1) the coefficient of x in $x^2 + 5x + 6$. (2) the constant term of the same expression $x^2 + 5x + 6$. Many students raised their hands; one of them stated that $3 + 2 = 5$ and $3 \times 2 = 6$. The teacher acknowledged the student's response, then proceeded further and asked the students to list all the factors of 6, which they listed as $\{(1 \times 6), (2 \times 3), (-1 \times -6), (-2 \times -3)\}$. The teacher explained that the addends of 5 can also be listed as $\{(0 + 5), (1 + 4), (2 + 3)\}$ and so on. Referring to the two sets, the teacher explained that, to factorise the expression $x^2 + 5x + 6$, one would look for the factors of 6, the constant term, which sum to 5, the coefficient of x. The whole class discussion from the groups' presentations to the discussion of task two, took 15 minutes.

John introduced the second activity, although there were only five minutes remaining to the end of the lesson. The second activity was to factorise the expression $x^2 + 3x + 2$. He asked the students to use some of the pieces of paper for activity 1, that is, the big square, two small squares (one unit by one unit) and three (x unit by one unit) strips, to form a rectangle and determine its area.

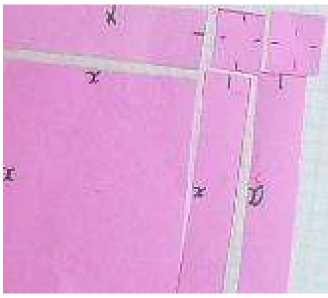


***Figure 18*: The rectangle for the factorisation of $x^2 + 3x + 2$**

John asked the first group to form the rectangle (shown in Figure 18) to present its work.

The representative from this group explained:

Student 6: Area, A = L × W

$= (1 + x) (2 + x)$

John summarised the lesson by explaining that one can use the relationship explained in Task Two of the first activity to factorise a given quadratic expression without necessarily using the pieces of paper, that is, look for factors of the constant term that sums to the coefficient of *x*. John invited the students to go to him, in their free time, for the pieces of paper and use them for the factorisation of other expressions.

Immediately after the lesson, John administered the post-lesson test, whose outcome is shown in Table 17. The Question 1 outcome showed that 24 out of 28 students answered the question correctly, which probably suggests that they understood the term quadratic after John had explained it.

***Table 17*: The post-test responses by students from the second lesson**

Total number of students = 28					
Items	**Responses**				
Question 1	Correct (because of the term x^2)	It has unknowns & knowns	Because of x	blank	% of correct response
Frequency	24	2	1	1	86
Question 2	Correct (factors of the constant term that sums to the coefficient of x)		Like terms	Coefficient	% of correct response
Frequency	20		3	5	71
Question 3	Correct (at most 2 factors)		Four factors	Three factors	% of correct response
Frequency	24		4	0	86

In addition, most students appeared to have discerned the conditions for the factorisation of a quadratic expression with a unit coefficient of x^2, the object of learning, after the lesson, as suggested by the outcome of Question 2. However, four students still felt that a quadratic expression has four factors, perhaps adopting the same thinking as students in the first lesson. Apparently, full implementation of the lesson seemed to have helped the students in this class, as suggested by their responses to the three questions of the post-lesson test.

Second lesson's reflection session

After the second lesson, the team again met for a reflection session. It was a moment for the team to find out if the modifications of the lesson suggested after the first lesson worked. Since John taught the lesson, he was accorded the first chance to express his views about the lesson as well as his experience with it.

John underscored the importance of reflection after every lesson, commenting that the reflection after the first lesson helped them to modify the second lesson, and he had enough time to discuss the second activity:

John: I realised that there was enough time: that is why I gave the other question. I realised that you could use the same cuttings to obtain factors of different quadratic expressions. Again, what I wanted to show is the fact that the number of factors in a quadratic expression are two. I realised that I did not stress much on the coefficients as I assumed that they had known that from the previous lesson, which is why I could not even ask them. They knew the coefficients of x^2 and x for the expression $x^2 + 3x + 2$.

Although John said that he had enough time, Peter noted that the lesson went beyond the 40 minutes, as he explained:

Peter: The lesson was good; it is only that it went beyond the time to about 45 minutes. The explanation at the beginning was okay, you tried to explain, and the students had prerequisite knowledge. Actually, you (John) said you did not emphasise so much but they already knew about the coefficients. You asked them, and they answered the question. In addition, when it came to the group work students came up with different arrangements of the rectangles. They were also able to identify that these rectangles do not just work with one quadratic expression because they formed another rectangle from a different expression. I am sure now they are aware that they can use the condition without the cuttings and factorise any quadratic expression.

Peter's finding about the term "coefficient" shows that members of this class were able to recall the term from their previous lesson. In addition, Peter's observation suggests that after the second activity students could probably factorise a quadratic expression without having to form a rectangle. This observation means that the students probably discerned the necessary condition for factorising a quadratic expression.

However, Peter noted some areas that required improvement in both lessons:

Peter: The only thing I realised that did not work well in the two lessons is the area of the key question. […] We did not focus the students on what we wanted to investigate beforehand, and they were making varied observations. […] At the beginning, we are supposed to tell them this is what you are going to do, and this is what we are going to find out. May be what we want to find out is more important at the beginning.

In a lesson study, the task for the class is referred to as "the key question". Corcoran (2011) explains key questions "as prompts to encourage student thinking" (p. 265). Peter noted that they (the teachers) delayed the explanation of the tasks, especially in the first lesson.

Peter also observed that some of the students were nervous when they were making presentations, and they did not fully explain what they were doing. In addition, he pointed out that the rectangle that was presented by the first group (Student 2) ought to have remained pinned up as other groups made their presentations, for comparison purposes.

Peter: I think only one point, when the students were at the board, they were nervous as they were not able to talk when they were pinning up their work. The students were able to come up with different rectangles and as such, the rectangle that was pinned on the board earlier by one group should have remained so that the students could realise that the rectangles were formed differently but gave the same area.

Peter's comment about students being nervous during the whole class presentations can be attributed to the change of classroom culture. Such nervousness was also observed during the first lesson as already explained in section 4.2.1. The students were not used to the approach adopted in this research where they were actively involved through small group and whole-class discussions.

Dominic noted that there was a great improvement on the implementation of the second lesson, as he explained:

Dominic: The students were able to make the rectangles fast. The fact that the whole class discussed the two activities helped the students to see the relationship and I am sure they can now factorise the quadratic expression without any problem. I noted that there were not enough pins, so all groups could not present their work at the same time.

Dominic concurred with Peter's observation that the second activity probably helped the students discern the object of learning. Both teachers were optimistic that the students could now factorise quadratic expressions without having to form a rectangle.

The teachers gave constructive comments during the reflective sessions. They focused their comments on the lessons and were able to point out areas of the lessons that did not work out well. The comments helped them improve the preparation, as well as the implementation of the subsequent lesson. Their comments after the first lesson saw them modify the second lesson, which in the end improved on the lesson's time management. This improvement seemed to have helped the students discern the object of learning after doing Activity Two.

They raised issues such as the right time to present the key question of the lesson, which in these lessons were tasks for the activities, and preparation for the students' presentation of their group findings. All these observations on the shortcomings of the lessons could be attributed to the change of classroom culture as has been explained. The students were used to the approach where they would listen to the teachers' explanations and only talk, one at a time, when asked by the teacher, what Mulala (2015) describes as "traditional instructional practices that centre on teacher dominated pedagogy (p. 20)."

The teachers complimented their colleagues on the areas in which they performed well, which helped them to settle emotionally since it was a new approach and they had to contend with many people observing their class during the lesson.

A summary of the Second Lesson

As in the first lesson, the students in the second lesson actively participated in the activities of the lesson and in the end discerned the object of learning as also suggested by the post-lesson test outcome. Although the groups still discussed softly, they worked quickly on the tasks and by the time the teacher stopped the group work for the first activity after 15 minutes, all seven groups had formed the rectangles. Perhaps this was due to John's explanation of both tasks at the beginning of the group work, so the groups proceeded to the second task as soon as they had completed the first one. Three of the seven groups correctly determined the areas of their rectangles. One group did not determine the area correctly, while the remaining three groups did not determine the areas of their rectangles. Maybe they did not have enough time or they did not know how to do it. As in the first lesson, (Figure 12) the Category Two group's presentation of their determination of the area, Figure 16, showed that the students had problems in forming algebraic expressions.

The approaches John adopted, both in the introduction of the lesson and at the enactment of the lesson, seemed to have worked well in this second lesson. This could be as a result of the reflection after the first lesson, which helped the teachers improve the lesson plan.

The students seemed to have understood the meaning of a quadratic expression and its factors, as was suggested by the outcomes of Question 1 and Question 3 in the post-lesson test. During the lesson, John explained both tasks for Activity One to the students before the start of the group discussion. The approaches also improved John's time management for the lesson as compared to the first one. The students had time to discuss the two tasks of Activity One and discussed Activity Two.

The discussion of both activities appeared to have helped the students discern the object of learning, as was suggested by the outcome of Question 2 and also observed in remarks by both Peter and Dominic. The modification of the lesson improved the teaching and learning of the second lesson.

4.3 Analysis based on Variation Theory as a Theoretical Framework

I considered each of the three components of the object of learning separately during my analysis, as was discussed in Chapter 2 (section 2.7). The three aspects are: the intended object of learning, the enacted object of learning and the lived object of learning.

4.3.1 Intended Object of Learning

The intended object of learning is the prepared lesson by the group of teachers. As explained in section 2.4, this is the teachers' view of the lesson. It is where the teachers plan on how to engage the students with the mathematics content in the classroom. As already explained in section 3.6.2, for this first pair of lessons, the teachers prepared the lesson on the factorisation of quadratic expressions with unit coefficients of x^2 since it was the introduction of factorisation.

The teachers prepared the material, shown in Figure 10, for the two practical (hands-on) activities they planned for the lesson. These activities were meant to help the students to factorise the two quadratic expressions, $x^2 + 5x + 6$ and $x^2 + 3x + 2$. Although the second expression was simpler than the first, the teachers explained that they started with the expression $x^2 + 5x + 6$ on purpose to allow each group to have all the pieces of paper shown in Figure 11 and to use all of them. In the second activity, the groups would use some of those pieces of paper to factorise the expression $x^2 + 3x + 2$. The students would obtain the factors by expressing the sides of their rectangles in algebraic forms, and then relate the factors with the expanded form of the expressions to determine the condition for factorising a quadratic expression.

The teachers planned to apply contrast and generalisation patterns of variation and invariance in this lesson. As explained in section 2.4, contrast is described as an awareness brought about by experiencing variation between two or more objects and generalisation is an awareness experienced by keeping the point of focus invariant as other forms of objects vary. The teachers planned to contrast the quadratic expressions $x^2 – 6$ and $x^2 + 4x + 3$, with the linear expressions $x + 5$ and x-10 respectively, to help the students discern a quadratic expression. They prepared two activities to help students factorise the expressions $x^2 + 5x + 6$ and $x^2 + 3x + 2$ by keeping the approach invariant, that is, making the rectangles. This would help the students generalise the conditions for factorising a quadratic expression with a unit coefficient of x^2.

The teachers prepared enough materials for all the groups to carry out the tasks. However, during the reflection sessions, the teachers noted some omissions in their preparation which might have contributed to the time management issues that occurred in the lessons. They realised that they ought to have planned for enough time to explain the activity tasks and how the students would report their group work. They considered some of these omissions in the modification of the lesson plan for the second lesson.

Drawing from the history of quadratic equations, which shows that quadratic equations developed from the process of finding the sides of a rectangle, the choice to introduce the topic through a hands-on activity was relevant to help the teachers connect the topic to the history and to help students to link algebra to basic geometrical objects such as the rectangle. The history of mathematics reveals that algebra, particularly quadratic equations, developed from geometry (Gandz, 1937; Katz, 2007) (section 2.2).

The teachers explained that in addition, they chose the activity to increase students' interest in mathematics by having a practical activity, especially in algebra, which both teachers and students usually refer to as a "dry topic," suggesting that it is regarded as an abstract topic without related - especially hands-on - activities, as Dominic pointed out during the orientation session. This is supported by Zahorik's (1996) finding that teachers use hands-on activities to arouse students' interest. In addition, the teachers argued that the practical activity would encourage more active participation in the group work in the new classroom procedure.

4.3.2 Enacted Object of Learning

As explained in section 2.4, Marton (2015) describes the enacted object of learning as the observers' views of the lesson. As has been discussed in section 4.3.1, the object of learning was the factorisation of a quadratic expression with a unit coefficient of x^2.

The teachers introduced their lessons by contrasting some quadratic expressions with linear expressions. The approach helped the students identify quadratic expressions and coefficients of various terms, which was helpful in discerning the critical feature of the object of learning – the relationship between the numerical values of the factors of a quadratic expression and the coefficient of x and the constant term. As explained in sections 4.2.1 and 4.2.2, Peter *contrasted* $x^2 - 6$ with $x + 5$ and later asked students to identify the coefficients of x^2 and x from the expression $x^2 + 4x + 3$, while John *contrasted* $x^2 + 3x + 2$ with $x - 10$ and asked the students to identify the coefficients of x^2 and x from the former. The teachers improved on the choice of a quadratic expression during John's lesson since the one in Peter's lesson was not ideal.

During the group discussion stage, the teachers varied the quadratic expressions while keeping the process of factorising the expressions invariant through the determination of areas of formed rectangles. In doing this, the teachers helped the students discern the necessary conditions for factorising a quadratic expression and to *generalise* these conditions since they applied them to more than one expression. Table 18 shows the summary of the generalisation pattern of variation and invariance as was applied in the second lesson that implemented the lesson plan in full. Although the teachers' plan for the first lesson comprised both expressions, Peter had not managed to give the students the second activity as already explained. However, he had explained the condition for factorisation of a quadratic expression in his summary of the lesson.

The teachers applied the contrast and generalisation patterns of variation and invariance as explained in the immediate paragraph above. However, had the teachers extended their contrast beyond the two expressions by including at least a third expression such as $x^3 + ax^2 + bx + c$, they would have given a better account of the principle of the contrast pattern of variation and invariance, which perhaps would have improved the students' understanding of the term "quadratic expression" more than it did. This approach would have helped the students to understand the existence of other expressions of higher orders and to explicitly discern that the term "quadratic" refers to the expressions of order 2. Even after the teachers explained the term after contrasting it with a linear expression, some students seem not to have understood, as was suggested by the post-lesson test outcomes.

Table 18: ***Summary of the generalisation pattern of variation and invariance***

Varied	**Invariant**	**Discernment**
Quadratic expressions (a) $x^2 + 5x + 6$ (b) $x^2 + 3x + 2$	Formation of the rectangle and the determination of its area. The areas of the two rectangles are $(x+2)\ (x+3)$ and $(x+2)\ (x+1)$ respectively.	Condition for factorising a quadratic expression with a unit coefficient of x^2, that is, factors of the constant term that sum to the coefficient of x

Being the first pair of lessons taught under LS approach, students appeared timid and took time discussing the first activity, especially in the first lesson. However, when

Peter realised that only two out of eight groups had completed the task within the stipulated time, he added more time, which enabled five more groups to complete the task. This was appropriate since it helped more students to discern the object of learning. This is supported by Rowland et al. (2011) and Thwaites, Jared and Rowland (2011) under contingency aspect in a Knowledge Quartet (KQ) analysis. Contingency concerns responses to unanticipated and unplanned events. In this case, it was Peter's flexibility with the planned timing of the activity, which allowed more students to complete the task. During the group presentations for whole-class discussion, it emerged that some groups had made mistakes in their calculation of the areas of formed rectangles. Had the teacher not responded to the contingent event, he might not have discovered the students' problems, which he now had time to correct. Otherwise, the teachers reorganised the lesson plan during the second lesson to accommodate the mistake that caused the delay in group discussion.

However, the group work revealed difficulties with forming an algebraic expression from the rectangles students had made. Eight out of fifteen groups were unable to form correct algebraic expressions that represented the sides of the rectangles to help them calculate the areas. This appears to be a common problem and is supported by findings from Clement's (1982), and Stacey and MacGregor's (2000) studies. In his study, Clement (1982) asked 150 first year university students to form algebraic expressions from word statements. About 40% of the students could not form correct algebraic expressions. He proceeded to represent the word statements diagrammatically, and still the students could not form the required algebraic expressions. In Stacey and MacGregor's case, high school students (fourth year) could not write an equation connecting the sides of a triangle with its perimeter. The triangle was drawn with its sides marked as x cm, $2x$ cm and 14 cm, and the perimeter stated as 44 cm. The students were to write an algebraic equation connecting the sides of the triangle with its perimeter, but 62% of the students did not write the correct equation.

In this research, the formation of the algebraic expression was not the main topic but a prerequisite knowledge to the topic of discussion. However, failing to form the correct algebraic expression, as happened with some students, might suggest that the students did not understand the topic or perhaps the earlier teaching approach might not have given then an opportunity to correct such errors. The Kenyan classroom teaching culture is dominated, as previously quoted from Mulala (2015), by traditional instructional practices that centre on teacher dominated pedagogy. It is an approach where a teacher works out the solution of the problem as he/she explains the procedure to the students, then asks the students to apply the procedure in the subsequent problems. Swan (2006) refers to such an approach as

> Traditional transmission method in which explanations, examples and exercises dominate but do not promote robust, transferrable learning that endures over time or that may be used in non-routine situations (p. 162).

Swan's statement could partly explain why some of the groups did not determine the area of their rectangles even though they had learnt algebraic formation from word statements and diagrams in Form 1 (14-16 years) according to the Kenyan curriculum. In this lesson, the students were given a chance through active collaborative learning, an approach supported by Swan (2006) in what he calls "Reverse traditional practices that allow students opportunities to tackle problems before teachers offer them guidance and support. This encourages students to apply pre-existing knowledge and allows teachers to assess and then help them build on that knowledge (p. 163)." So, the LS approach enabled the teachers to identify the students' difficulty in algebraic formation, which they corrected during the lesson, thereby helping the students to discern the object of learning.

4.3.3 Lived Object of Learning

As explained in section 2.4, the lived object of learning is the students' experience of the lesson (Marton, 2015). It is discussed through the students' participation in the lesson as well as their evaluation of the lessons, as usually suggested by the post-lesson tests (Lo, 2012).

The observations from the two lessons showed that the students discerned the object of learning and hence learned about factorisation of quadratic expressions. 14 out of 15 groups of students correctly performed Task 1 of the first activity, as summarised in Tables 13 and 16. Of these groups, seven were able to correctly discern the object of learning by factorising the quadratic expression $x^2 + 5x + 6$ as $(3 + x)(2 + x)$. The second class managed to perform the second activity and factorised the expression $x^2 + 3x + 2$ as $(2 + x)(1 + x)$. The groups that did not factorise the expressions benefited from whole-class discussions as groups presented their work, followed by the teachers' summary of the lessons, as suggested by the post-lesson test outcomes. The teachers' summaries and conclusions highlighted the condition for factorising a quadratic expression with a unit coefficient of x^2, that is, considering the factors of the constant term that sums to the coefficient of x.

The errors that the students made in the process of performing the activities became a learning moment for them. Such errors included failure to write algebraic expressions to represent the length and width of the formed rectangle and failure to identify 1 as the coefficient of x^2 in the expression $x^2 + 4x + 3$. The teachers did not anticipate these errors during preparation because the students had been taught these contents previously. Nevertheless, they got an opportunity to realise the students' difficulty with these contents and correct the mistakes, which helped the students discern the intended content.

In addition, the teachers' remarks during the reflection sessions suggested that the students discerned the object of learning set out at the beginning. Peter stated that:

Peter [...] I am sure that the students are aware that they can use the formula without the cuttings and factorise any quadratic expression because they were given the second activity.

Dominic echoed Peter's sentiment when he commented that:

Dominic: The students were able to come up with the rectangles very fast. The fact that the whole class discussed the two activities helped the

> students to see the relationship and I am sure they can now factorise the quadratic expression without any problem.

Both Peter and Dominic made these comments after the second lesson and they based their arguments on the fact that the students confirmed the condition for factorising a quadratic expression by performing the second activity.

4.5 Conclusion

The teachers' choice of the activities seemed very effective, as the students were fully engaged in group discussion, which in the end helped them discern the object of learning. In addition, the choice of the activities that linked algebra with geometry helped students to connect these two mathematics areas and corrected the belief that the topic of quadratic expressions and equations was a 'dry topic,' a topic without practical (hands-on) activities. As has been explained, most groups (14 out of 15) from both classes manipulated the hands-on activities well and formed the required rectangle, with half of the groups working out the area correctly. What became apparent was that the groups that did not correctly calculate the area of their rectangles had problems arising from previously taught contents, which were considered as prerequisite knowledge. Although the teachers corrected the mistakes, it was a reminder to them of the need to take care of prerequisite knowledge and skills in their planning, which is one of the LS requirements during the preparation stage, referred to as students' anticipated responses. The students' anticipated responses help teachers think of different approaches that the students might use in performing the lesson task or help teachers prepare for common errors usually made by students (Hiebert, 2003; Ryan & Williams, 2007 in Rowland et al. (2011); Wake, et al., 2015; Yoshida, 2012), some of which might be due to students' misinterpretations.

The LS approach helped the teachers realise the importance of teamwork among the teachers during the planning stage as they incorporated students views and the teachers' previous experience with the topic, which increased the originality of this study to a classroom that had all along followed a traditional approach to the teaching and learning of the topic.

During the enactment of the lessons, the teachers applied the planned patterns of variations and invariance. However, the teachers realised some shortcomings in implementing their planned lesson, especially in the first lesson, where students did not work on the second activity and Peter could not fully apply the generalisation pattern of variation and invariance. Peter was forced to take contingency measures to ensure that the groups completed the two tasks of the first activity. The teachers identified the problem and corrected it in the second lesson, which seems to have improved the students' learning of the content, as suggested by the post-lesson test outcomes. In addition, the teachers learned the importance of the reflection session, which helped them to modify the second lesson, and in the end, John was able to fully implement the lesson as had been planned. This helped the students to discern the critical features in steps, which resulted in them discerning the object of learning. This is an indication that full implementation of a LS approach lead to students' better understanding of the intended content.

Chapter 5 – Second Pair of Lessons: Solving Quadratic Equations by Completing the Square Method

5.1 Overview

This is the second of four chapters relating to the analysis of my study. In this chapter, I begin by briefly discussing the topic content, which is the solution of a quadratic equation by the method of completing the square. Thereafter, I present an account of each lesson as it took place, the reflection sessions after each lesson and the summary of each lesson.

Then, I will analyse both lessons in accordance with the theoretical framework of variation theory (Lo, 2012; Marton, 2015). My analysis will be grounded in the three aspects of the framework, namely: the intended object of learning – the preparation; the enacted object of learning – the actual teaching and learning of the content; and the lived object of learning – a review of the actual lesson. After the analysis, I will discuss the outcomes and link them with some existing literature. I conclude the chapter by reflecting on some key points about the lessons.

5.2 Introduction to the Lessons

Between the first pair of lessons discussed in Chapter 4 and the content of this second pair of lessons, the students were taught the following contents: solutions of quadratic equations by factorisation; formation of quadratic equations and their solutions by factorisation; perfect squares and completion of the square. There was a ten-day interval (19/01/2016 – 29/01/2016) between the first and the second pair of lessons, which was to enable the teachers to meet and prepare as was explained in section 3.3 about *Kyouzai Kenkyuu* in Japanese (Arani, 2017; Takahashi, 2009; Yoshida, 2012). The teachers are expected to meet together to identify and study the teaching materials.

In this pair of lessons, the teachers identified the object of learning as the solution of a non-factorisable quadratic equation with a unit coefficient of x^2, of the form

$x^2 + bx + c = 0$, by the method of completing the square. The teachers identified two critical features, which became the centres of focus to help discern the object of learning.

The first critical feature was the process of adding $(\frac{1}{2}$ coefficient of $x)^2$ to both sides of a quadratic equation to make its left-hand side (LHS) a perfect square. For example, to solve a quadratic equation such as $x^2 - 6x + 5 = 0$, one would rearrange the equation as $x^2 - 6x = -5$ then add $\left(\frac{-6}{2}\right)^2$ to both sides to make the LHS side of the equation a perfect square, although the equation $x^2 - 6x + 5 = 0$ can also be solved by the method of factorisation. This action would lead to the new equation, $x^2 - 6x + \left(\frac{-6}{2}\right)^2 = \left(\frac{-6}{2}\right)^2 - 5$ and to the factorisation of the LHS as shown in the equation $\left(x - \frac{6}{2}\right)^2 = \left(\frac{-6}{2}\right)^2 - 5$.

The second critical feature was the consideration of both positive and negative values when determining the square root of the RHS of an equation such as $\left(x - \frac{6}{2}\right)^2 = \left(\frac{-6}{2}\right)^2 - 5$, which simplifies to $x - 3 = \pm\sqrt{9 - 5}$, obtaining the two values of x as $x = 5$ or $x = 1$.

5.2.1 First Lesson

The day before the lesson, the pre-lesson test was given to the students of both lessons. The questions were:

1. How do you make a quadratic expression a perfect square?
2. How would you solve a quadratic equation that cannot be factorised? For example, $x^2 + 6x - 9 = 0$

The expected answers to the questions were: (1) by adding $(\frac{1}{2}$ coefficient of $x)^2$ to the expression, although in this question the answer could as well be by multiplying the expression by zero. The teachers decided on the chosen answer because it was the one that would lead to the process of solving a quadratic equation by the method of completing the square. (2) by the method of completing the square or by applying quadratic formula. The students had not been taught any of the two expected answers for Question 2.

The first question assessed the content that was to be taught prior to this lesson, and the second question could be addressed by the topic of the lesson or by a topic yet to be taught. Therefore, the students probably did not have any idea on how to respond to the second question. The following were frequent responses from the students: (1) Quadratic expressions are made perfect squares by squaring the expression. (2) By considering the "sum and product" approach, which is the factorisation method.

As in the first pair of lessons, the pre-lesson test responses were scored 1 for a correct answer and 0 otherwise. Table 19 shows the distribution of the students' responses from the pre-lesson test. The first question was meant to help the teachers see whether the students would recall the process of completion of the square, a process which is a necessary condition for solving a quadratic equation by completing the square. However, 44 students' responses suggest that they were not able to recall the process, despite the fact that they had discussed a perfect square in the previous lesson. Thirty-seven students suggested that one would make a quadratic expression a perfect square by squaring the expression. Although the students did not explain what they meant by the statement, but it showed that they were aware of some squaring involved when making an expression a perfect square. It could be related to the addition of $(\frac{1}{2}$ coefficient of $x)^2$, which they had learned earlier but it seems they could not recall what was being squared.

Table 19: ***The students' responses to the diagnostic pre-test – First lesson class***

Total number of students = 46						
Items	**Responses**					
Question 1	Correct response (by adding ($\frac{1}{2}$ coefficient of x)^2 to the expression)			By squaring the expression	Blank	% of correct response
Frequency	2			37	7	4
Question 2	Correct response (completing the square)	Correct answer (quadratic formula)	Applying "sum and product."	Taking square root	Blank	% of correct response
Frequency	0	0	36	4	6	0

The second question was concerned with the topic of the lesson, which the students had not been taught. However, the teachers were trying to assess the students' ideas about such equations. The students' responses from Table 19 show that most of them thought they could apply the same method of factorisation that they used to solve factorisable quadratic equations. Probably the students' thinking was informed by the fact that they had only learnt one method and that they thought it could be applicable in all cases. The students' responses to question 2, although it seems obvious, further confirmed to the teachers that the students were not aware of other methods of solving quadratic equations. The teachers considered the students' responses in planning for the lessons. They planned to introduce the lessons by solving a perfect-square quadratic equation by factorisation. This action was meant to remind the students about perfect square expressions, which would later help them when learning how to transform a non-perfect square expression into a perfect square one.

The Lesson

Peter taught the first lesson. He reminded the students of the process of solving a quadratic equation by factorisation.

Peter: In the previous lessons we solved quadratic equations by factorisation, where we considered factors of the product that sum to the coefficient of x. Which product?

Student 1: The product of the coefficient of x^2 and the constant term.

Peter: Correct. Today we want to learn how to solve quadratic equations by the method of completing the square, but before that, to solve the equation $x^2 + 6x + 9 = 0$.

The students correctly solved the equation as follows:

$x^2 + 6x + 9 = 0$

$x^2 + 3x + 3x + 9 = 0$

$x(x + 3) + 3(x + 3) = 0$

$(x + 3) (x +3) = 0$

$x = -3$ or $x = -3$

Peter asked a question,

Peter: How else would you write the LHS of the equation $(x + 3) (x + 3) = 0$?

Student 2: $(x + 3)^2 = 0$

Peter: This, "$(x + 3)^2$", is called a perfect square. Therefore, the LHS of the equation $x^2 + 6x + 9 = 0$ is a perfect square expression.

Peter then asked a further question:

Peter: How do we solve for x in the equation $(x + 3)^2 = 0$?

Many students could not solve the problem despite the fact that they had solved it when it was expressed as $(x + 3) (x + 3) = 0$. The teacher asked the students to discuss amongst themselves and work out the solution. After some time, one student explained,

Student 3: Work out the square root of $(x + 3)^2$ to obtain $x + 3 = 0$ and $x = -3$

Peter explained the process further.

The choice of the introduction question, which is the solution to the quadratic equation $x^2 + 6x + 9 = 0$, was meant to help the students revise the solution of a quadratic equation by the method of factorisation and in addition, to introduce the concept of a perfect square. It was envisaged that the perfect square concept would help the students when working on the activity of the lesson.

After the introduction session, Peter asked the students to solve the equation $x^2 + 6x - 9 = 0$, which was the activity for the lesson. The teachers' choice of the activity question was to *contrast* it with the introduction question, so that the students would realise that it could not be solved by the factorisation method. At the same time, it was meant to help the students be exploratory in their group discussions and rewrite the equation as $x^2 + 6x = 9$. This new equation could prompt the students to relate LHS with the introduction equation and so realise that they could add 9 to both sides and help them factorise the LHS.

After five minutes, the teacher asked the students to form groups of five members each. There were 41 students present as some students were unwell and had gone to the hospital. They formed eight groups: seven of them comprised five members each and one group had six members. The teacher allowed 15 minutes for group discussion, after which he asked some groups to present their work for whole class discussion. I separated the groups' approaches to the solution of the activity question into two groups as shown in Table 20.

Table 20: ***The distribution of the groups' approaches to the activity – First lesson***

Approaches	**Factorisation ("Sum and product")**	**Working out the square root of each term**	**No attempt**
Number of groups	4	2	2

What appears as the third approach is a representation of some two groups that had not done the activity by the time Peter called for whole class discussion. The first approach was represented by four groups who attempted to solve the activity question through the factorisation method, while the second approach was represented by two groups that attempted to solve the equation by distributing the square root over each term. Peter selected a group each from the two approaches to present their work. Figure 19 shows the presentation by the group that approached the solution from the factorisation method.

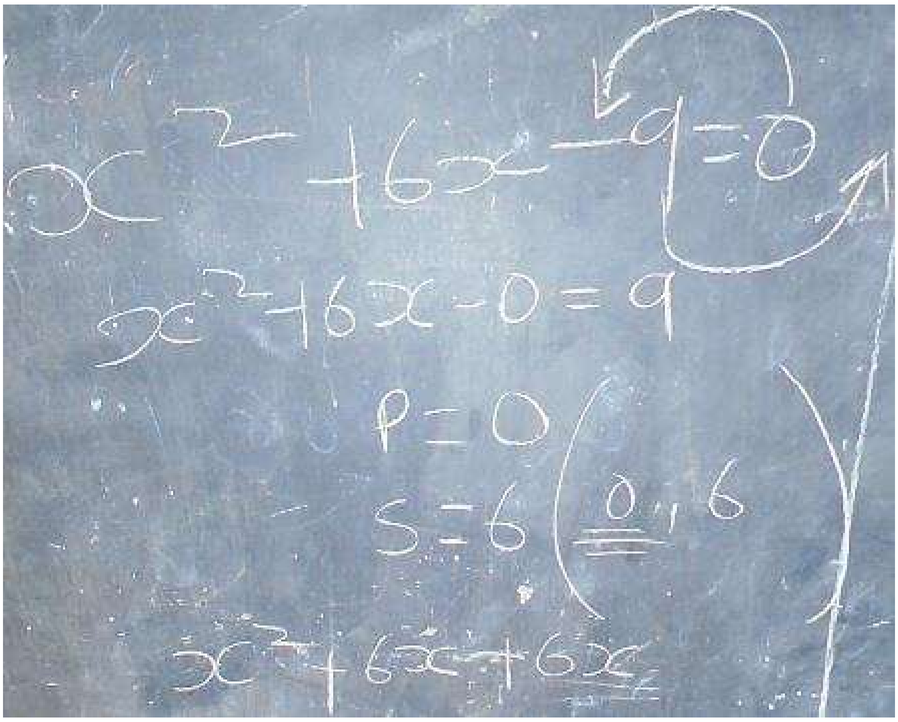


***Figure 19*: Attempted solution by the groups using "sum and product" approach – First lesson**

The student explained as follows:

Student 4: After realising that the LHS could not be factorised, we changed 0 and 9 then considered the product, p = 0 and sum, s = 6 and rearranged the LHS as $x^2 + 6x + 6x$ but realised it could not work.

From Student 4's statement, it seems that the students applied the method of factorisation to solve the equation, but they could not find the factors of -9 that sum to

6. They rearranged the equation, but still they could not factorise the LHS of the equation. Therefore, the application of the method of factorisation could not solve the equation.

Figure 20 shows the presentation by the group that approached the solution by determining the square root of each term. The photograph in Figure 20 was taken from the exercise book in which the group worked on the activity before working on the chalkboard. The group's representative erased what she had written on the chalkboard before I could take a photograph. She explained the work as follows:

Student 5: We added 9 to both sides of the equation and worked out the square root of x^2, $6x$ and 9, that is, $\sqrt{x^2} + \sqrt{6x} = \sqrt{9}$.

$$x + 2.45x = 3$$

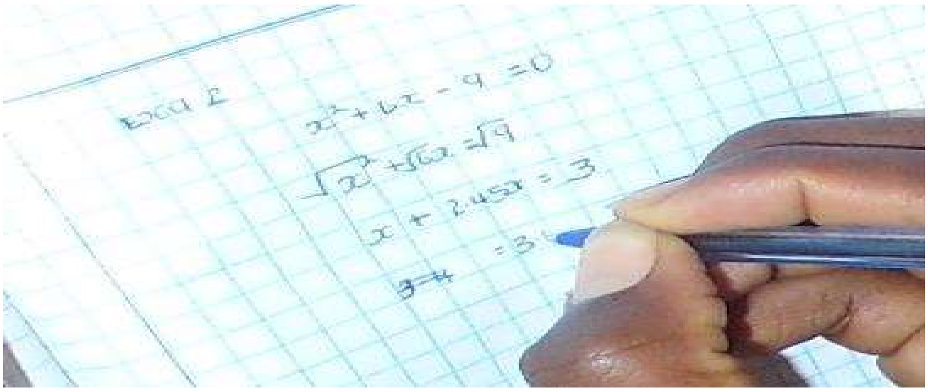


***Figure 20*: Attempted solution by the groups that used each term's square root approach**

When student 5 wrote the last equation, that is, $x + 2.45x = 3$, some students raised concern about the way the group determined the square root of $6x$ by only considering the square root of 6 but not x. One student asked, "Why have you only worked out the square root of 6 but not x?" Student 5 could not answer the question, and she did not continue with the calculation and cleaned the board.

Student 5's action of not continuing with the calculation after a fellow student asked her a question might suggest that she was not sure of their group's approach to the activity. Alternatively, she might have been conscious that avoiding determining the square root of x was not correct, but they did not have any other way of solving the problem.

Peter acknowledged the other students' concern and stated that the approach taken by the group, represented by Student 5, was not correct.

Peter engaged the students in a whole class discussion in solving the activity question.

Peter:	From the equation, $x^2 + 6x - 9 = 0$, we can add 9 to both sides and obtain, $x^2 + 6x = 9$, which most groups did. How can we make the LHS a perfect square?
Student 6:	By adding 9
Peter:	When you add 9 to the LHS you must also add it to the right-hand side (RHS) to obtain $x^2 + 6x + 9 = 18$. Factorise the LHS and work out the solution.

Almost all the students were able to work out the correct solutions, but only considered the positive square root of 18 and obtained one solution, that is, $(x + 3)^2 = 18$

$$x + 3 = \sqrt{18}$$
$$x = - 3 + 4.243$$
$$x = 1.243.$$

The Kenyan mathematics syllabus for secondary schools ([KIE], 2002) requires the students to give their answers correct to four significant figures unless stated otherwise. Although the students had been taught squares and square root in Form 1 (14-16 years) under Chapter 9 in the Kenyan secondary schools' mathematics curriculum ([KIE], 2002), the chapter seems to deal with natural numbers and does not stress the negative root of a number.

Peter proceeded and solved the equation by considering the two values of root 18,

$$(x + 3)^2 = 18$$

$(x + 3) = \pm\sqrt{18}$.

$x = -3 \pm\sqrt{18}$

$= 1.243$ or -7.243

He explained the need to include the negative value of the square root of the RHS term, for example, 18 in this case, to obtain two solutions of a quadratic equation. Peter continued with the whole class discussion and explained the relationship between the 9 that was added to make the LHS a perfect square and 6 the coefficient of the term with *x* as follows:

Peter: What do you notice between the added value 9 and the coefficient of *x*, which is 6?

As the students were still trying to establish the relationship between 9 and 6, Peter wrote $\left(\frac{6}{2}\right)^2$ and asked,

Peter: What is the value of this expression, $\left(\frac{6}{2}\right)^2$?

Student 7: 9

Peter: Which is ($\frac{1}{2}$ of the coefficient of x)^2, this is added to both sides of the equation to make the LHS expression a perfect square. This method of solving a quadratic equation by making the LHS of the equation a perfect square is called completing the square method of solving quadratic equations.

Peter: The equation $x^2 + 6x + 9 = 18$ can be written as $x^2 + 6x + (3)^2 = 18$. The LHS can then be factorised and the equation is written as $(x + 3)^2 = 18$. This is like taking the 3 in the bracket, add it to *x* then square the whole expression.

Peter then told the students, "For your homework, use the method of completing the square to solve the equation, $x^2 + 5x - 1 = 0$".

Peter appeared constrained by time during the summary of the lesson and introduced $\left(\frac{6}{2}\right)^2$ to the working of the solution and eventually explained the term $(\frac{1}{2}$ of the coefficient of $x)^2$, but this introduction appeared abrupt given that the students had only solved one equation. In addition, I think he could have chosen a homework question with an even number coefficient of x. This would have helped the students to solve the equation step by step and later related their working with the general form given by the teacher. However, Peter seemed to have noted the abrupt introduction of the general format, as he later commented on it during the reflection session.

Peter gave the students the same post-lesson test as the pre-lesson test with the same questions discussed in section 5.1. The distribution of the students' responses is shown in Table 21.

Table 21: ***The post-test responses by students from the first lesson***

Total number of students = 41					
Items	Responses				
Question 1	Correct response (by adding $(\frac{1}{2}$ coefficient of $x)^2$ to the expression)	Squaring expression	Adding 9	Blank	% of correct response
Frequency	3	21	13	4	7
Question 2	Correct response (Completing the square or quadratic formula)	"Sum and product."	Adding a square	Blank	% of correct response
Frequency	32	3	4	2	78

Had Peter given a different question in the post-lesson test from the one he used during the lesson, he could have evaluated the lesson better and could have given the students the opportunity to apply the taught method in a different question.

The students' responses to question 1 suggest that most of them had not comprehended the process of making an expression a perfect square, even after Peter's explanation. The activity question and the introduction question were similar except for the last operation sign, which as explained, was meant to *contrast* the two equations to help the students understand the method of completing the square to solve a quadratic equation. However, the process involved the addition of 9 in the activity question to make the LHS a perfect square, which made some students think that one needs to add 9 to make an expression a perfect square, as was suggested by the 13 out of 41 students in answering question 1. This is where a different question could perhaps have helped these 13 students to realise that it is not 9 that is added but $(\frac{1}{2}\text{ coefficient of } x)^2$.

Many students responded positively to question 2, which might mean that the majority of students remembered the name of the method that can be applied to solve a non-factorisable quadratic equation, but they might not have understood how to apply the method, as was suggested by question 1. Perhaps they answered the question correctly because it was the same question they used during the lesson and that is why a different question would probably have helped more in evaluating the students' understanding of the process.

First lesson's reflection session

After the lesson, the two teachers and I met for a reflection session. Although Dominic had attended the lesson, he could not attend the reflection session as he had a class immediately afterwards. During the reflection session, Peter stated that he realised the students were able to recall the factorisation approach to the solution of a quadratic equation when they were asked to solve the equation $x^2 + 6x + 9 = 0$.

Peter: We had a recap of what was taught previously, solution of a quadratic equation by factorisation method and students were able to do that well.

However, he noted that students were not able to link the introduction question with the activity question to determine a working method to solve the activity equation, even after discussing in groups for 15 minutes.

Peter: I gave them the quadratic equation $x^2 + 6x - 9 = 0$ in which they were not able to use the factorisation method. They struggled in the groups for a long time, and some came up with methods that could not work out. The concept is a bit technical and abstract; the students were not able to come up with completing the square method.

Peter commented that he had a short time after the discussion, but after his explanation of the method of completing the square, he felt students were able to learn the method and would be able to apply it to solve quadratic equations.

Peter: The time was short because the students took a lot of time in the group discussion, but in the end, they were able to see how they could handle quadratic equations that cannot be handled through factorisation method. Their responses regarding the questions we were handling were not bad, and I think the lesson went well.

However, he stated that although he felt the students understood the concept, he would explain the method further in the subsequent lesson.

Peter: […] I think the only thing I will do is to reinforce the method in the next lesson.

Although Peter said that the students understood the concept, his last statement appears to confirm the observation that the generalisation of the formula was introduced rather abruptly.

John noted that the students responded well to the introduction task.

John: The introduction part, the students were able to solve the equation $x^2 + 6x + 9 = 0$. They already knew how to solve the equation through "product and sum" approach.

John appreciated some groups' effort to try to solve the equation by taking the square root of each term, despite the fact that it was not a correct method. He also appreciated Peter's explanation to help the students learn the method of completing the square to solve quadratic equations.

John: During the discussion, some students were trying to use some methods to solve the equation, though not correct, but if one tries to get the values, somehow a value was coming up. […] I think Peter did well with his explanation, he really explained a lot until the students were able to verify the method. The students were able to see clearly how completing the square method came about after Peter's explanation.

It became apparent from John's assertion that Peter "really explained a lot until the students were able to verify the method" and that students had difficulty in understanding the process of solving a quadratic equation by the method of completing the square, as was suggested by students' responses to question 1. Based on these observations, the teachers decided to modify the lesson by stressing the condition of making a quadratic expression a perfect square during the introduction of the lesson before the students worked on the activity.

A summary of the First Lesson

The students appeared focused on working out the solution to the activity question as expressed in the groups' presentations. However, no group was able to obtain the expected solution. Many groups were able to reorganise the equation by adding 9 to both sides and obtain the equation $x^2 + 6x = 9$, a move that was to help them think of making the LHS a perfect square, which in turn would help them to work out the solution to the equation. However, many groups, four out of eight, still attempted to use the factorisation method to solve the equation $x^2 + 6x = 9$ despite the fact that they had confirmed that they could not work out the solution by this method. Being the only method, they had learned, some groups perhaps thought that by swapping the positions of 0 and 9 to have, $x^2 + 6x + 0 = 9$, they could still factorise the LHS.

Two groups chose to distribute the square root sign (radical sign) over the terms making the equation $x^2 + 6x = 9$ to obtain the equation $\sqrt{x^2} + \sqrt{6x} = \sqrt{9}$ as a first step to work out the solution. As a second step, the groups simplified the equation to obtain $x + 2.45x = 3$.

Although the students' steps appeared mathematical, the groups made two mistakes. The first mistake was to distribute the square root sign over the operation of addition, while the second mistake was determining the square root of 6 alone from a composite number $6x$. The nature of the mistakes suggests a naïve approach to the notation and algebraic structure of the expression. This mistake seems to be a result of linearity misuse in non-linear situations, especially in algebra, an observation supported by De Bock, Van Dooren, Janssens & Verschaffel (2002), who state that "In secondary education, 'linearity errors' are often reported in the fields of algebra and (pre)calculus… e.g. the square root of a sum is the sum of the square roots… (p. 313)." This is what the two groups representing the second approach, Table 20, did. They appeared to reason out that $\sqrt{x^2 + 6x} = \sqrt{9} \Leftrightarrow \sqrt{x^2} + \sqrt{6x} = \sqrt{9}$, which is not correct. De Bock et al. (2002) argue that these linearity errors result from students' over-generalisation of the distributive law, a situation observed in this lesson and in the second lesson, as we will see.

Some groups did not attempt to work out the solution to the activity question but appeared to have realised that the two approaches used by the presenting groups were not correct, as was observed from their comments on the second group's working.

The teachers had hoped that the introduction question, $x^2 + 6x + 9 = 0$, would help the students to see that they needed to add 9 on the LHS of the equation $x^2 + 6x = 9$ so as to have a perfect square on the LHS expression, but the students did not realise the link.

Instead, the students resorted to other methods that did not give the correct answer. However, the students' mistakes made the teachers realise the students' linearity misuse in a non-linear equation, which they commented on during the reflection sessions of each of the lessons. The teachers viewed the errors positively and addressed them during their explanation of completing the square method.

5.2.2 Second Lesson

The students in the second lesson had done the pre-lesson test, shown in section 5.2, at the same time as the students in the first lesson. All 33 students in this class responded to the pre-lesson test as well as the post-lesson test. The distribution of the students' responses to the pre-lesson test is shown in Table 22.

Table 22: ***The students' responses to the diagnostic pre-test – Second lesson class***

Total number of students = 33						
Items	**Responses**					
Question 1	Correct response (by adding ($\frac{1}{2}$ coefficient of x)^2 to the expression)			By squaring the expression	Blank	% of correct response
Frequency	2			29	2	6
Question 2	Correct response (completing the square)	Correct response (quadratic formula)	Applying "sum and product."	Taking square root	Blank	% of correct response
Frequency	1	0	27	4	1	3

As in the first lesson, many students, 29 out of 33, responded to question 1 that a quadratic expression would be made a perfect square by squaring the expression, which may suggest that the students did not recall the addition of $(\frac{1}{2}$ coefficient of $x)^2$ but remembered that some squaring exists because they had discussed perfect square in the previous lesson. Again, as in the first lesson, many students, 27 out of 33, responded to question 2 by saying that the solution to a non-factorisable quadratic equation would be by “sum and product,” which is a solution by factorisation method.

As I argued in the first lesson, question 2 was addressing the content of the topic of the lesson, which had not been taught. This scenario happened with all the lessons; perhaps it arose from the teachers’ interpretation of the diagnostic pre-test, which they seemed to think should try to find out what the students know about the topic to be discussed. Still, the students’ responses were considered by the teachers during their preparation of the lesson.

The similarity of the responses from the students in both lessons was to be expected since, as I explained in section 3.2.2, the streaming of the classes (Form 3 East and Form 3 West) was done randomly and not in any way according to students’ performance. The term ‘stream’ is used in Kenyan education policy documents to refer to schools with many classes per form, for example, a school with two classes in each of the Forms 1, 2, 3 and 4, would be referred to as a two-streamed school.

The Lesson

After the first lesson, the teachers modified the second lesson with the suggestion that the teacher of the second lesson help the students understand the condition for making a quadratic expression a perfect square before working on the activity question. The teacher was expected to identify an expression such as $x^2 + 8x$ and ask the students to find what to add to the expression to make it a perfect square, which was part of the previous lesson. The question would be worked out as,

$$\text{Let } x^2 + 8x + k = (x + a)^2$$
$$= x^2 + 2ax + a^2$$

By comparing the corresponding coefficients of the variable x and the constant term,

$2a = 8$ and $a = 4$ therefore, $k = a^2 = 16$.

The suggestion was meant to help the students work out the activity question. However, John introduced the lesson in the same way Peter did. I later learned during the interview that John was teaching Form 3 class for the first time. Perhaps this could explain why he did not take the students through the process suggested at the introduction of the lesson and could also explain the observation made in the next section when he explained the process of completing the square.

John introduced the lesson by asking the students to solve the quadratic equation $x^2 + 6x + 9 = 0$. The majority of students were able to solve the equation. One of the students explained:

Student 1: $x^2 + 6x + 9 = 0 \Leftrightarrow x^2 + 3x + 3x + 9 = 0$

$$(x + 3)(x + 3) = 0$$

$$x = -3 \text{ or } x = -3$$

John continued,

John: Is there a way in which we can further rewrite the LHS of the equation $(x+3)(x+3) = 0$?

Student 2: $(x + 3)^2 = 0$

John: How do we work out the solution?

Student 3: By taking the square root, $\sqrt{(x+3)^2} = \sqrt{0}$

$$x = -3$$

John: What does $(x + 3)^2$ remind you of?

The students remained silent. John then asked the students, "What happens when I replace x in the expression $(x + 3)^2$ with 1, 2 and 3?" As the students mentioned the values, he wrote the numbers on the chalkboard,

John: 16, 25 and 36, and asked, "What do you notice about these numbers?"

Student 4: Difference of two squares

Student 5: Perfect squares

John: Posed a question, “Who supports the first answer and who supports the second answer?”

Eleven students supported the first answer, while the rest of the students supported the second answer. Although both answers would be correct depending on how one argues out his/her point, it was clear from John’s follow-up statement that he expected the second answer.

John: Yes, they are perfect squares, hence $(x + 3)^2$ is also a perfect square.

John did not explain to the students why he did not accept the difference of two squares as a correct answer, nor enquired why the students thought the numbers were the difference between two squares. John seemed not to have anticipated a different answer such as Student 4’s answer, and as Huckstep et al. (2002) explain, “totally unexpected response can cause real difficulties to the teacher if they cannot follow the reasoning or the ideas the child is suggesting (p. 12).” John chose to ignore it and proceeded with the lesson to the activity stage, an observation noted by Rowland et al (2011) as one of a teacher’s responses to an unexpected idea or suggestion from students. They explain that “the teacher’s response to unexpected ideas and suggestions from students is one of the three kinds: to ignore, to acknowledge but put aside, and to acknowledge and incorporate” (p. 76).

However, the students had learnt about the term “the difference of two squares” and its application to the solutions of quadratic equations when they learned the expansion of quadratic factors such as $(p + q)$ $(p – q)$. This expression is one of the factors referred to as “quadratic identities” by the syllabus ([KIE], 2002). The expansion of the factors $(p + q)$ $(p - q)$ simplifies to $p^2 – q^2$, which is a difference of two squares. Perhaps this is what informed the students’ answers.

The introduction was meant to introduce the concept of the perfect square that was to be applied in the activity question to help the students solve the question. John asked the students to solve the quadratic equation $x^2 + 6x – 9 = 0$. Students worked individually for five minutes. John asked the students if any of them had obtained the solution, but none had. Some students responded that the expression could not be factorised. John then asked,

John: Is there an alternative method that you can use to solve this quadratic equation, given that it cannot be solved through factorisation?

While the students remained silent, John asked the students to form groups of five or six students each and discuss how to work out the solution to the equation. Students formed six groups; three groups of five members each and three groups of six members each. The groups worked on the activity through different approaches as shown in Table 23. As was the case with the first lesson, I separated the groups as per the approaches they used. In this second lesson, two groups tried to work out the activity through factorisation and two groups approached the activity by working out the square root of each term, as was observed in the first lesson. One group worked out the square roots on both sides of the equation as shown in Figure 21, while one group had not worked on the activity by the time all the groups were called upon to make presentations for whole class discussion.

Table 23: ***The distribution of the groups' work on the activity – Second lesson***

Approaches	**Factorisation ("Sum and product")**	**Working out the square roots of each term**	**Working out the square roots on both sides**	**No attempt**
Number of groups	2	2	1	1

John asked one of the two groups that considered the square root of each term, and the group that considered the square roots on both sides, to present their work for the whole class discussion. The work by the group that considered the square root of each term is shown in Figure 21.

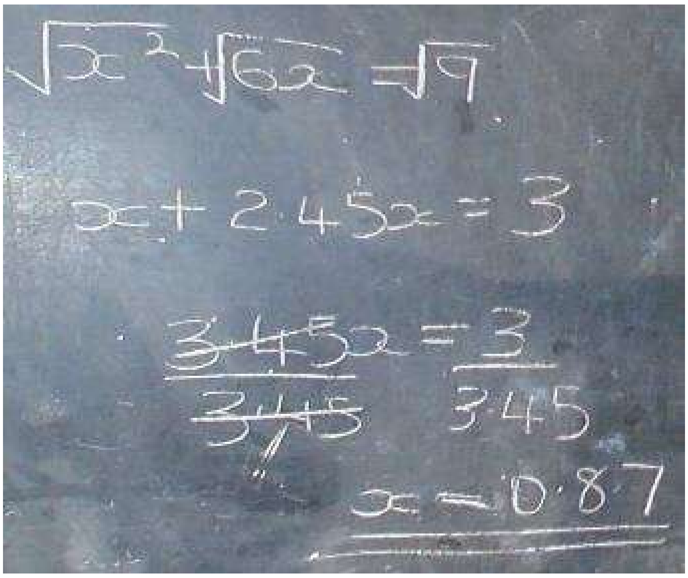


***Figure 21*: The work by one of the groups that considered the square root of each term**

As in the first lesson, the group distributed the square root sign (radical sign) over each term of the equation and only determined the square root of 6 but not x in the second term. The group proceeded and obtained a value of $x = 0.87$ by considering the positive square root of 9. As I explained in the first lesson, this is a linearity misuse on non-linear equations and it seems to be common among students in secondary schools as noted by De Bock et al. (2002).

John then asked the students to substitute the value of 0.87 in the original equation $x^2 + 6x - 9 = 0$ to confirm the answer. After working, one of the students responded,

Student 6: It does not work.

John: Therefore, it is not the answer.

John explained to the whole class that it is not correct to distribute the square root over each term, separated by either addition or subtraction operations. John then asked the second group to present their work, which is shown in Figure 22.

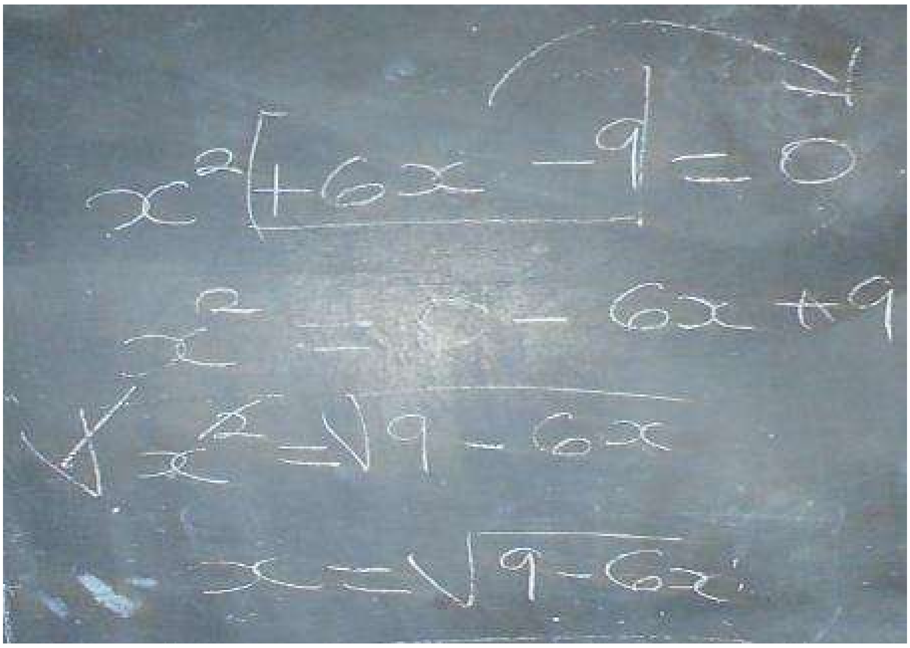


Figure 22: **The work by the group that considered the square root on both sides**

The group eliminated 6*x* and 9 from the LHS by subtracting 6*x* and adding -9 to both sides of the equation to obtain $x^2 = 9 - 6x$. The group then worked out the square roots of the expressions on both sides of the equation and obtained the solution as $x = \sqrt{9-6x}$. This solution is a correct mathematical argument; however, it did not lead to the expected solution, which was to obtain a value of *x*. They expressed *x* in terms of *x* – perhaps they considered the two *x*s from equation $x^2 = 9 - 6x$ as different unknowns. This consideration of the two *x*s in a quadratic equation as different variable appears to be a common student error, as is also reported by Vaiyavutjamai and Clement (2006), as will be seen later.

After the presentations, John guided the students, still in whole class discussion, to work out the solution by the method of completing the square. He guided the students to link the introduction equation, $x^2 + 6x + 9 = 0$, with the activity equation, $x^2 + 6x - 9 = 0$. He asked,

John: What do we add to the LHS of the equation $x^2 + 6x = 9$ to make it a perfect square?

Student 7: 9

John: Which is equal to $\left(\frac{6}{2}\right)^2$, where 6 is the coefficient of x. It is added to both sides of the equation, that is $x^2 + 6x + 9 = 9 + 9$.

As in the first lesson, the introduction of $\left(\frac{6}{2}\right)^2$ appeared abrupt at this stage and both teachers appeared determined to generalise the process of transforming a non-perfect square quadratic expression into a perfect square expression after only one example. John said, in general, what is added is $\left(\frac{b}{2}\right)^2$ to both sides of the equation, where b is the coefficient of x in a general quadratic equation such as $x^2 + bx + c = 0$.

John guided the students through a step-by-step process to work out the solution, as illustrated in Figure 23.

$x^2 + 6x + 9 = 9 + 9$ ………. Step 1
$x^2 + 6x + 3^2 = 18$ ………… Step 2
$x^2 + 3^2 = 18$ ……………… Step 3
$(x + 3)^2 = 18$ …………….. Step 4
$x + 3 = \pm\sqrt{18}$…………… Step 5
$x = \sqrt{18} - 3$ or $-\sqrt{18} - 3$ .. Step 6
$x = 4.243 - 3$ or $-4.243 - 3$ .Step 7
$x = 1.243$ or -7.243 ……. Step 8

***Figure 23*: Steps to solve the equation $x^2 + 6x - 9 = 0$ as guided by John**

As John moved from Step 1 to the Step 8, it appeared the students did not follow his progression from Step 2 to Step 4 through Step 3. This action prompted some questions from the students.

Student 8: Where did you take $6x$? ………………… Step 3

John: We drop $6x$ because we want a perfect square.

Student 9: Why don't you square 3? ……………….. Step 3

John: Because we want a perfect square.

Student 10: Do we need to take 3^2 and add it on the RHS? …………… Step 3

John: No.

Although the students did not ask further questions, their facial expressions appeared to suggest that they were not satisfied with the teacher's responses to the questions. John noted this and commented on it during the reflection session. He continued working through the steps up to the end as shown in Figure 23.

In his summary of the lesson, John asked,

John: What is the relationship between 3 and 6 in step 2?

Student 11: Three is a half of six.

John then explained, pointing at Step 2, that the 9 that was added is the square of 3 but 3 is a half of 6, which is the coefficient of x. Therefore, what was added is $\left(\frac{6}{2}\right)^2$. In general, what is added to both sides of an equation, such as the activity question to make the LHS a perfect square, is ($\frac{1}{2}$ the coefficient of x)^2. He explained that this process helps in the factorisation of the expression on the LHS, which culminates in the solution of the given quadratic equation. He went on to explain that "When working out the square root, we consider two values of the square root of the number on the RHS. This is done to have two values of the given quadratic equation."

From the episode illustrated in Figure 23, John appeared to be thinking of showing the students that after making the LHS of a quadratic equation a perfect square, the LHS simplifies to (x + $\frac{1}{2}$ the coefficient of x)^2, which is shown in Step 4. Perhaps his problem could be associated with part of the pedagogical content knowledge, which is "the ways of representing the subject which is comprehensible to others" (Huckstep, Rowland & Thwaites, 2002, p. 2), as they note in Shulman's (1986) work. John seemed to think that by introducing Step 3, the students would be able to understand the generalisation. However, the introduction of Step 3, whose equation was neither equivalent to the equation of Step 2 nor the equation of Step 4, could mislead the students. Had he asked the students to factorise the LHS of Step 2, they would have moved to Step 4 since they had done that in the introduction of the lesson. John could have used the students' solution to explain the general principle after that. Immediately after the lesson, the teacher gave the students the post-lesson test whose outcomes are shown in Table 24.

Table 24: ***Post-assessment responses of the students from the second class***

Total number of students = 22				
Items	Responses			
Question 1	Correct response (by adding ($\frac{1}{2}$ coefficient of x)^2 to the expression)	Squaring expression	Blank	% of correct responses
Frequency	7	11	4	32
Question 2	Correct response (Completing the square or quadratic formula)	"Sum and product."	Taking square root	% of correct response
Frequency	19	0	3	86

Although 33 students attended the lesson, John had brought only 22 copies of the post-lesson test, so 11 students were unable to complete it. As was the case in the first lesson, many students still seemed not to comprehend the process of transforming a non-perfect square quadratic expression into a perfect square as Question 1 responses suggest. This could be because they might not have internalised the general form from only one example. Had John introduced the lesson as was suggested for its modification, perhaps the students would have realised that the process works for different equations.

Although the groups did not work out the activity question correctly, responses to Question 2 suggest that all the students seemed to have been convinced that the factorisation method could not work as none of the students responded to the application of “sum and product.” As in the first lesson, many students responded that the non-factorisable quadratic equations might be solved by the method of completing the square, which was expected because that was the topic of the lesson and the same question was used for both lesson activity and post-lesson test. However, the students’ responses to Question 1 may suggest that the majority of students had not discerned the first critical feature of the lesson.

Second lesson’s reflection session

The reflection session after the second lesson in a (LS) approach is usually a moment for the teachers to assess how their suggested modifications after the first lesson were implemented in the second lesson (Runeson, 2013). However, it was noted during the lesson that John did not manage to include a different equation to explain the process of making a quadratic expression a perfect square, as had been suggested in the reflection session after the first lesson.

John appreciated their choices of the two equations for the introduction and the activity, that is, $x^2 + 6x + 9 = 0$ and $x^2 + 6x - 9 = 0$ respectively. The two equations *contrasted* well to help the students realise that one could be solved using factorisation method and the other could not.

John: The choice of the two equations was good. When I gave them the first question $x^2 + 6x + 9 = 0$, they did well by solving it through factorisation. This gave me confidence that they understood what I taught them. Later, I asked them to solve the equation $x^2 + 6x - 9 = 0$. They realised that the expression on the LHS could not be factorised.

He noted that different groups approached the solution to the activity question differently but that they were determined to obtain some solution.

John: Different groups had different approaches. Those who presented their work on the board had very good ideas on how to come up with the value of x, but they could not manage to get the expressions that could give the expected two values of x.

John also commented on the steps he laid out to explain the process of solving a quadratic equation by the method of completing the square. He realised that some students showed dissatisfaction with some of the steps (particularly Step 3) he applied. The students also seemed dissatisfied with his responses to some of the questions.

John: During my teaching to show the students how to use completing the square method, I realised some of my explanations were not understood, as "I saw from their eyes." Some were understanding, but some had doubts, and I think there is a lot that I am supposed to do. There are explanations that I am supposed to go back and give, that is, to redo the question that we answered.

Peter appreciated the way John introduced the lesson. He explained that the introduction seemed to focus the students on the topic of the lesson.

Peter: The introduction was well covered; the enquiry was made in the previous lesson. They were asked questions like "Is there an alternative method that you can use to solve this quadratic equation, now that we cannot use factorisation method?" I think that question made them alert; it made them realise that they were going to learn a new thing.

Peter noted that when John wrote the numbers 16, 25 and 36, some students responded that they noticed the difference of two squares; John should have asked them to explain why they thought so.

Peter: When John gave the values of 16, 25, 36, then asked, "What type of numbers are these?" There are those who answered, "a difference of two squares." The teacher should have asked them to explain why they responded so because they were square numbers but there was no difference.

However, Peter commended the determination that the students had shown to reach some solutions from the activity question even though their approaches were not correct. He pointed out the mistakes that the groups who presented their work for whole class discussion had made.

Peter: When they were working on the activity question, the most interesting approach was from the group that worked the square roots by distributing them over addition, that is, $\sqrt{x^2} + \sqrt{6x} = \sqrt{9}$. The square root sign can only be distributed over a division or multiplication. Then there was one group that wrote $x^2 = 9 - 6x$, then $x = \sqrt{9-6x}$, they were right. The only problem was that they worked out the value of x in terms of x. This approach would give some answer except that it would be x in terms of x, which may reach nowhere but they were active and alert.

In general, Peter felt that during the lesson, they (the teachers) should have allowed the students to factorise the LHS of the equation $x^2 + 6x + 9 = 18$ from first principles. That could have made the students realise how $6x$ is eliminated from the final equation $(x + 3)^2 = 18$.

Peter: When we were completing the square in both classes, most students wondered where the $6x$ had gone in the factorisation of the LHS, $(x + 3)^2$. I think that one did not come out well. I think had they factorised it from the expression they would have seen that when they open the expression, $6x$ will reappear.

Otherwise, Peter felt that the lesson went well, but there would be a need for improvement in the subsequent lesson.

The teachers' comments during the reflection session highlighted certain areas of concern in the lesson. Although both John and Peter appeared general in their observations, they noted that there was a need to take the students through the process of completing the square method in the subsequent lesson. John's observation, based on the students' facial expression, that they did not understand part of his explanation, arose from Step 3 shown in Figure 23. Peter alluded to the same, though in a general sense, when he commented that the students should have been allowed to work from first principles first before introducing the generalisation. Here, Peter meant that they, the teachers, could have given the students more problems to complete the square on the LHS and solve the equations by factorising the LHS and taking the square root on both sides. After a few solutions, they could have invited the students to observe the similarities in the completed squares and *generalise* that the final factor is ($x + \frac{1}{2}$ the coefficient of $x)^2$. It is after this point that they could have introduced the homework question that they gave the students, that is $x^2 + 5x - 1 = 0$.

Based on the classroom observations and the teachers' comments, I advised John to readdress the three questions raised by Student 8, Student 9 and Student 10 in the subsequent lesson. I alerted the teachers that the two equations that represented Step 3, $x^2 + 3^2 = 18$ and Step 4, $(x + 3)^2 = 18$ were not equivalent since the LHS expressions were not equal, that is, $x^2 + 3^2 \neq (x+3)^2$. Therefore, the way John moved from Step 3 to Step 4 could mislead the students into thinking that they could write $a^2 + b^2$ as $(a + b)^2$. I reiterated Peter's observation that they ought to have allowed students to solve more equations from the first principle before generalising the last step that factorises the LHS to $\left(x+\frac{b}{2}\right)^2$, which is Step 4 in Figure 23. By doing that, they would have allowed the students to comprehend the process by relating the similarities they would have observed from the different questions.

A summary of the Second Lesson

In this second lesson, the students formed six small groups that discussed the activity question. Three different approaches emerged from the attempts the groups made to work out a solution to the activity question. However, as was observed in the first lesson, none of the groups worked out the activity correctly. Two of the three attempted approaches were the same as the two approaches observed in the first lesson, that is the factorisation approach to the solutions of quadratic equations and the approach of distributing the square root sign over the terms of the activity equation to obtain $\sqrt{x^2} + \sqrt{6x} = \sqrt{9}$. The groups made a further mistake, as did the first class, of only determining the root of 6 and working out the value of x as $x = 0.87$. These groups fell victims to linearity misuse in non-linear quadratic equations, as has been explained in the first lesson.

In the third approach, the group obtained a solution of x in terms of another x thus, from $x^2 = 9 - 6x$ to obtain $x = \sqrt{9-6x}$. As has been noted in the lesson section, expressing x in terms of another x might suggest that these students considered the two xs to be different. Vaiyavutjamai and Clement (2006) interviewed some students in their study and found out that many students think that the two xs in a quadratic equation of the form $x^2 + bx + c = 0$ are different. The students solved a quadratic equation $x^2 - 8x + 15 = 0$ correctly by using the factorisation method, $(x - 3)(x - 5) = 0$ and obtained the x values as $x = 3$ and $x = 5$. When the students were asked to confirm their answers, they substituted 3 in the factor $(x - 3)$ and 5 in the factor $(x - 5)$.

After failing to obtain a correct solution to the activity question, the students appeared keen to understand the process of solving a quadratic equation through the method of completing the square. Both Peter and John made this observation when they commented that,

Peter	[...] most students wondered where the $6x$ had gone in the factorisation of the LHS, $(x + 3)^2$.
John	[...] I realised some of my explanations were not understood by them, as "I saw from their eyes."

Students were courageous enough to ask John to clarify some issues on the steps he used to solve the activity question. This action was observed from the three students 8, 9 and 10 who sought John's clarification on Step 3 of Figure 23.

Through the reflection session, teachers freely pointed out areas they felt they might not have done well during the lessons. They noted that they could have rushed the generalisation of the factorisation of the LHS after completing the square. They agreed to revisit the activity question, clarify the issues they felt the students might not have understood and give more practice questions in the subsequent lessons.

5.3 Analysis of the Lessons and of the Post-Lesson Teachers' Reflections in the Spirit of Variation Theory.

As was described in section 2.7 of the thesis, analysis through a variation theory framework focuses mainly on the three aspects of the object of learning. These are: the intended, the enacted, and the lived objects of learning. These are the same aspects considered in this analysis of the two lessons. In a LS approach, teachers collaborate at every stage of the three aspects (of the object of learning) either through discussions or observations.

5.3.1 Intended Object of Learning

The overall object of learning was the solution of the non-factorizable quadratic equation with a unit coefficient of x^2 such as $x^2 + bx + c = 0$. The teachers identified two critical features to help the students discern the object of learning. The decisions on the object of learning and the critical features were made according to the Kenyan secondary schools' mathematics curriculum that specifies the contents of the topic, and the teachers' experience of teaching that helped them to identify the students' areas of difficulty. The identified critical features were: (1) The process of adding $(\frac{1}{2}$ coefficient of $x)^2$ to both sides of a quadratic equation. This process was to help the students learn how to make the left-hand side (LHS) of the given quadratic equation a perfect square. (2) The process of obtaining two solutions of a quadratic equation by considering both positive and negative values of the square root of the RHS of a quadratic equation such as $x = -\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - c}$.

To help the students to discern these critical features, the teachers first planned to *contrast* two quadratic equations $x^2 + 6x + 9 = 0$ and $x^2 + 6x - 9 = 0$ with an intention to help the students: (1) realise that one equation could be solved through the factorisation method while the other one could not and (2) the LHS of the first equation is the square that would be needed in the second equation. The equations only differed in the sign of the last term on the LHS, which was the variation made in the second equation.

Secondly, the teachers intentionally identified a perfect square quadratic equation $x^2 + 6x + 9 = 0$ to be solved through the factorisation method. The intention was to help the students link the completion of the square in the second equation $x^2 + 6x - 9 = 0$ with the first equation. The teachers planned to apply part-whole separation patterns of variation and invariance (Marton, 2015) in discerning the first critical feature of the object of learning. Marton (2015) explains that:

> In order to develop a powerful way of seeing something, the learner must decompose the object of learning and bring it together again. Such decomposition happens in two ways: through delimiting parts and wholes and through the discernment of critical aspects (p. 145).

Both ways described by Marton (2015) were planned for in this pair of lessons. The teachers planned to separate the equation $x^2 + 6x - 9 = 0$ into $x^2 + 6x = 9$ to discern the first critical feature of the object of learning by completing the square on the LHS of the equation.

The teachers then planned to apply the fusion pattern of variation and invariance by bringing the parts together to help the students to discern the two critical features of the object of learning simultaneously, which are completing the square on the LHS and taking the square root on both sides to solve for the unknown. As Marton (2015) explains, "Putting together refers to the learner experiencing the different aspects simultaneously after having separated them (p. 51)." Finally, the teachers planned to summarise the lesson by harmonising the two critical features to discern the object of learning.

5.3.2 Enacted Object of Learning

Both Peter and John introduced their lessons by asking the students to solve the two quadratic equations $x^2 + 6x + 9 = 0$ and $x^2 + 6x - 9 = 0$. The teachers gave out the two problems one at a time. Whereas the majority of students were able to solve the first equation because they had been taught the solutions of a quadratic equation by factorisation, none of them could solve the second question. The two equations were contrasted to help the students to try and explore ways of linking up the square on the LHS of the first equation with the LHS of the second equation upon writing it out as $x^2 + 6x = 9$, so that they could complete the square in the LHS of the second equation.

Although none of the groups from both classes obtained the correct solution, many of the groups managed to rewrite the equation in the form $x^2 + 6x = 9$. The teachers built on this equation $x^2 + 6x = 9$ and applied the part-whole pattern of variation and invariance, which helped the students to discern the first critical feature of the object of learning, which was completion of the square on the LHS of the equation. Table 25 shows the summary of the application of part-whole separation pattern of variation and invariance.

The teachers then brought together the parts to work out the solution to the quadratic equations. This process completed the part-whole arrangement and helped the students to simultaneously discern both critical features of the object of learning, the second one being taking the square root on both sides of the equation with the consideration of both positive and negative values of the square root on the RHS

Table 25 - ***Summary of the part-whole separation pattern of variation and invariance applied***

Varied	Invariant	Discernment
Quadratic equations (a) $x^2 + 6x + 9 = 0$ (b) $x^2 + 6x - 9 = 0$	Left hand side (LHS) expressions of the two equations after completing the square of the second equation (a) $x^2 + 6x + 9 = 0$ (b) $x^2 + 6x + 9 = 18$	The addition of $(\frac{1}{2}$ coefficient of $x)^2$ on both sides of equation (b) to make its LHS expression a perfect square.

Table 26 shows the summary of the fusion pattern of variation and invariance applied.

Table 26 - ***Summary of the fusion pattern of variation and invariance applied to discern the two critical features of the object of learning simultaneously***

Varied	Invariant	Discernment
Quadratic equations (a) $x^2 + 6x + 9 = 0$ (b) $x^2 + 6x - 9 = 0$	The process of obtaining the solutions to the quadratic equations (a) $\sqrt{(x+3)^2} = \pm\sqrt{0}$ (b) $\sqrt{(x+3)^2} = \pm\sqrt{18}$	(1) The addition of $(\frac{1}{2}$ coefficient of $x)^2$ on both sides of equation (b) to make its LHS expression a perfect square. (2) Consideration of both positive and negative values of the square root on the RHS of the equations to obtain two values of the solution

The simultaneous discernment of the critical features of the object of learning, as shown in Table 26, helped the students discern it as identified during the preparation of the lessons, which was the solution of a non-factorisable quadratic equation by the method of completing the square. Both Peter and John summarised their lessons by explaining the process of working out a solution to any quadratic equation by the method of completing the square. However, the teachers noticed that the students needed more practice questions and further explanations of the process since they only used one question. Although they explained the general rule during the summary and conclusion of the lessons, they stated during the reflection sessions that they should have delayed the generalisation to give the students more practice questions and use the first principle approach to obtain the solutions.

Both teachers agreed to explain further the process of obtaining solutions to quadratic equations by the method of completing the square in the subsequent lessons. In addition, the teachers agreed that they would give students more practice questions involving equations with even coefficients of x. Such questions would in the end help the students to generalise the process of completing the square for non-factorizable quadratic equations. It is after such practice that the students would be able to solve quadratic equations such as the one Peter gave as homework, $x^2 + 5x - 1 = 0$, with the odd number coefficient of x.

The students' group work revealed some errors through some of the approaches which the teachers did not anticipate but appear to be common errors, as confirmed in other studies. Some groups approached the solution of $x^2 + 6x - 9 = 0$ by distributing the square root to each terms of the equation $\sqrt{x^2} + \sqrt{6x} = \sqrt{9}$. This suggests that: (1) they might have had a linearity problem with the application of distributive property to solve non-linear equations (De Bock et al., 2002) and (2) the understanding of the composite nature of algebraic expressions. Alternatively, the students might have been determined to find a solution to the equation and could manoeuvre their way to a solution without much consideration of the correct process. This interpretation is supported by Stacey and MacGregor (2000) who proposed that:

> Students' prior experiences with solving problems in arithmetic give them a compulsion to calculate which is manifested in (1) the meaning they give to "the

unknown"; (2) their interpretation of what an equation is; (3) the methods they choose to solve equations (p. 149).

The students in the groups that used this approach $\sqrt{x^2} + \sqrt{6x} = \sqrt{9}$ appeared determined to work out a solution by whatever means. Their interpretation of an equation might have been 'calculation.' They proceeded and solved a value of x in disregard of correct mathematical processes. This is supported by Tall et al.'s (2014) finding as I explained in Figure 3 of section 2.2. Tall et al. (2014) argue that such students attempt to move to a solution by a procedural approach that transforms a quadratic equation into a familiar linear problem, as the students did in both lessons in this study.

There was one group in the second class that approached the task by taking the square root on both sides, that is $\sqrt{x^2} = \sqrt{(9-6x)}$ and obtained a solution as $x = \sqrt{(9-6x)}$. The work of this group suggested that they might have considered the x values as different, that is x of the x^2 as different from x of $6x$. The assumption may be supported by the finding of Vaiyavutjamai and Clement (2006), which reveals that many students consider the two xs in a quadratic equation as different (see section 2.2).

Whereas the findings of Tall et al. (2014) and Vaiyavutjamai and Clement (2006) were from studies that tested the students after teaching the topic of quadratic expression and equations, which appears to be an evaluation study, in this study the errors were detected during the teaching process. The teaching and learning in this study actively involved the students in the lesson through small group discussion and whole class discussion, and in the process of discussions the students made mistakes which the teachers corrected during the summaries of the lessons. This detection of errors during the teaching and learning process might perhaps have a positive reflection on the students' future performance in the topic because it gave the teachers an opportunity to correct the errors in the process of teaching and learning.

5.3.3 Lived Object of Learning

The students learned the content at the end of the lessons as is suggested by the post-lesson test outcomes for Question 2 and later confirmed through the interviews as will be seen in Chapter 7. They also appeared to have enjoyed working on the activity of the lesson, despite the fact that none of the groups from either lesson obtained the correct answer. By introducing the lesson through a solution to a perfect square equation $x^2 + 6x + 9 = 0$ and giving out an activity question $x^2 + 6x - 9 = 0$, varied only at the last operation sign, the teachers might have thought that the students would try to link the first equation with the second equation and discover the method of completing the square. However, the groups came up with some workings, as shown in Table 20 and Table 23, which in themselves became learning moments for the students.

Gagatsis and Kyriakides (2010) report that "A reversal of the traditional view on errors is found in the work of Piaget… and for the first time, errors were viewed positively since they allow the tracing of a reasoning mechanism adopted by the student" (p. 25). This supports one of the components of a LS approach called Lived object of learning 1, which tries to determine the students' prior knowledge of the content to be addressed through diagnostic pre-lesson test and anticipated (contingency to) students' responses by the teachers. Through this, the students' conception and/or errors on the content become an important source of information for the preparation and enactment of the lesson. However, Rowland et al. (2011) notes "It is to be expected that the novice teacher is not very well-placed to anticipate contingent events: they lack the experience from which […] to learn about how students respond to certain pedagogical stimuli…" (p. 74). With respect to the LS approach, the teachers in this study could be considered novice.

This second pair of lessons became one such case where students learned through some of the errors they made during their group discussion. As already explained in section 5.3.2, some groups from both lessons tried to solve the activity question $x^2 + 6x - 9 = 0$ by distributing the square root sign over the terms of the equation and obtained an equation $\sqrt{x^2} + \sqrt{6x} = \sqrt{9}$. The students were made aware of their mistake after the discussion of their work in the group discussion. Had the teachers used their usual traditional approach, where they would have explained the procedure for solving a quadratic equation by completing the square, without involving the students, perhaps they (the teachers) might not have realised that the students had misused linearity in non-linear equations. Similarly, the students might not have learnt that it is a mistake to distribute the square root over addition and subtraction of terms.

However, the teachers got the opportunity to correct the students' mistakes. They asked them to substitute the value of x, which they obtained as their answer in the original equation and the students confirmed that it was not the correct answer. The teachers then explained to the students that it is incorrect to distribute the square root sign over addition or subtraction.

In addition, the students made a further mistake in working out the square root of the second term, that is $\sqrt{6x}$ in which they only worked out the square root of 6 but not x. Some students noted this mistake. Although in these two identified errors the teachers asked the rest of the class to substitute the answer back to the original equation to confirm that they were not correct, had they shown some specific examples such as $\sqrt{9+4} \neq \sqrt{9} + \sqrt{4}$ it might have perhaps convinced the students more. Even for $\sqrt{6x}$, had the teachers used a specific example such as $\sqrt{36}$ and shown that the number could be decomposed as $\sqrt{9} \times \sqrt{4}$ for them to see that $\sqrt{36} = \sqrt{9} \times \sqrt{4}$, perhaps they would have learned more.

However, through the LS component of group discussions and presentations, students learned by being made aware that it is a mistake to distribute square root over additions or subtractions and also to consider an algebraic number as a whole when taking the square root.

By learning through the mistakes, one group worked out the solution of the activity question by expressing x in terms of another x that is, $x = \sqrt{9-6x}$. As I have stated in section 5.2.2, the students approach was a correct mathematical process, but it could not lead to the expected solution, which was a solution for x. However, at the end of the lesson the students were able to learn how to solve such equations to obtain the expected values of x. After the group's presentation, John got the opportunity to explain to the students that it is a mistake to express a variable in terms of itself while trying to obtain a solution for the variable.

However, the teachers noted some areas that they felt did not work well and they could improve in the subsequent lesson to enhance the understanding of the method of completing the square to solve a quadratic expression. They spoke about those concerns during the reflection sessions, as already explained. They suggested that they would explain the concept further in the subsequent lessons.

5.4 Conclusion

Hiebert and Wearne (2003) suggest that teachers need to open up class activities to allow students to explore different methods of working on activities. In this Second pair of lessons, the teachers presented an activity that appeared to be a normal class exercise found in textbooks. However, the way they *contrasted* it with a similar equation in which the two equations only differed in the last operation sign, showed a good application of a variation theory approach to teaching and learning. As a result of teamwork during preparation, the teachers were able to come up with the two quadratic equations $x^2 + 6x + 9 = 0$ and $x^2 + 6x - 9$ that were relevant and engage the students in active discussion during the lesson. During the reflection session, John commented that "The choice of the two equations was good."

The students realised their mistakes during the whole class discussion and the teachers got the opportunity to correct the students' mistakes during the teaching and learning process. The students' active participation during whole class participation improved over the First pair of lessons. This was seen when students sought clarification from the teachers' explanations which seems to suggest that continued use of the LS approach to teaching and learning might help students change their classroom culture from the present state of being passive recipients of knowledge.

Although the teachers were implementing a LS approach to teaching and learning for the second time, they were still in the process of learning the design to understand it better. This was clear when the teachers tried to move fast to reach the generalisation of the conditions for completing the square when solving a quadratic equation by the method of completing square. However, the teachers were able to point out the areas they felt did not work well in the two lessons. As Pang (2008) argues, "LS offers potential gain…, which may open up certain possibilities to improve student learning, and over an individual teacher in the sense that teachers can learn from one another… (p. 19-20)." The teachers in this study also learned from one another during the reflection sessions after the lessons. They were able to state what they would do in the subsequent lessons to improve the students' learning of the method. For example, they noted that having rushed to generalise the method they should revisit it and approach it from first principles for two or three questions before introducing it. Also, the teachers were able to help John realise that the steps he followed in solving the activity problem could mislead the learners. This was a clear strength of the application of a LS approach to the teaching and learning of mathematics, which became a learning moment for the teachers that added to their professional development.

Chapter 6 – Third Pair of Lessons: Graphs of Quadratic Functions

6.1 Overview

This is the third analytic chapter, which describes the third pair of lessons and the last chapter that focuses on classroom observations. As with Chapters 4 and 5, the two lessons are described and then I provide some interpretations followed by discussion of the reflection sessions after each lesson. The lessons are then analysed together using the variation theory framework (Lo, 2012; Marton, 2015) and this analysis is related to the relevant literatures. The chapter concludes with an overall reflection on the analysis.

6.2 Introduction to the Lessons

The third pair of lessons was about the teaching and learning of graphs of quadratic functions, represented generally as $y = \pm ax^2 \pm bx \pm c$. In the Kenyan secondary school curriculum, the topic is referred to as "graphs of quadratic equations" ([KIE], 2002, p. 26). Between the second pair of lessons and this pair, the teachers had taught derivation of the quadratic equation solution formula, solutions of quadratic equations using the formula, forming and solving quadratic equations and tables of values for a given quadratic function. I did not observe the lessons on these stated subtopics.

In this pair of lessons, the teachers identified the object of learning as the shapes of graphs of quadratic functions and specifically, introduced the topic by drawing two types of graphs with a unit coefficient of x^2 of the forms $y = x^2 – bx – c$ and $y = -x^2 – bx – c,$ where one graph had a minimum turning point while the other had a maximum turning point. The teachers decided to maintain the same arithmetic operation signs, in this case a subtraction, between the terms of the functions, that is, $– bx – c$ in both. This was in order to help students with the critical feature of the object of learning, which they proposed as the relationship between the direction of a quadratic graph and the sign of the coefficient of x^2, that is, $\pm x^2$. A quadratic graph has a minimum turning point when the coefficient of x^2 is positive, while it has a maximum turning point when the coefficient is negative.

Before the lesson, the teachers gave the students diagnostic pre-lesson tests, which were:

1. Sketch the graphs of the following quadratic equations
   a) $y=x^2-x-6$
   b) $y=-x^2-5x-6$
2. Are quadratic graphs symmetrical or not?

The Kenyan secondary mathematics curriculum uses y to represent $f(x)$ for functions of polynomials. Students are introduced to the notation $f(x)$ when they join universities and other tertiary institutions such as national polytechnics that offer diploma courses. As mentioned in this introduction of the lessons, in the previous lessons the students had discussed tables of values of quadratic functions without drawing graphs. For Question 1, the teachers decided to see whether students would choose their own range of values of x to help them sketch the graphs.

For Question 2, the teachers seemed to mean the axis of symmetry, which is usually identifiable with a parabola. This became clear subsequently from a video replay in which I saw the teachers explaining that the axis of symmetry would be useful in learning the topic of volumes of revolution in universities. It appears that the teachers wanted the students to guess the symmetrical nature of parabolas, which would help them explain the axis of symmetry of a quadratic function with respect to the turning point. Usually, the coefficients a and b in a general quadratic function such as $y = \pm ax^2 \pm bx \pm c$ control the axis of symmetry of a parabola generated where the x-value at the turning point is calculated as $x = -\frac{b}{2a}$, which is the equation of the line of symmetry of the parabola.

Question 2 was another case where the teachers expected answer was not made clear from the question. The students could interpret the question differently.

The expected answers of the diagnostic pre-lesson tests were:

1 (a)

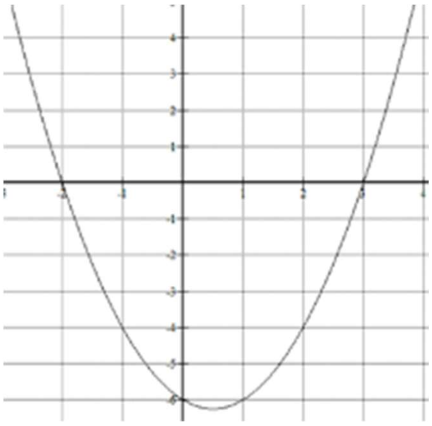


Figure 24: **Graph of $y = x^2 - x - 6$, for $-3 < x < 4$.**

1 (b)

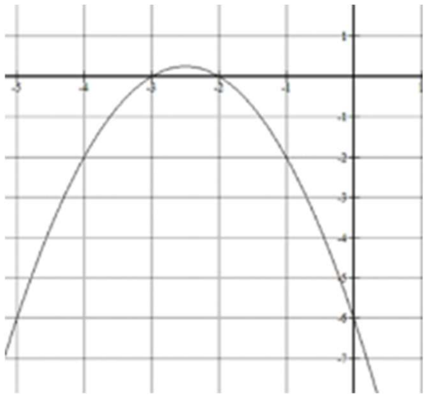


Figure 25: **Graph of $y = -x^2 - 5x - 6$, for $-5 \leq x \leq 0$**

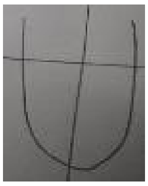
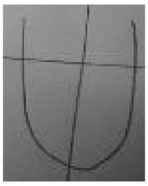
2. Graphs of quadratic functions are symmetrical.

6.2.1 First Lesson

The diagnostic pre-lesson test was given to the students a day before the lesson. In this first lesson, only 29 out of 49 students were present. The absent students had been sent home to fetch their school fees. In Kenya, there is free secondary education for public day schools, but parents pay school fees to maintain their children in boarding schools, such as the one where I conducted this research. Some parents delay paying the fees at the beginning of term, and in such cases the students are usually asked to go back home and collect the money.

The students' responses to the diagnostic pre-lesson test are shown in Table 27. The teachers thought that the students would use some tables of values of x and y to plot the graphs. Instead, the students simply sketched the graphs without showing the plotting. Nonetheless, a majority of students (19 out of 29) provided a correct sketch of the graph for Question 1 (a), which suggests that they might have had some trials for some values of x and y but did not indicate them. Few students (9 out of 29) sketched the graph for Question 1 (b) correctly. From the table, it appears as if the same students who drew the correct graph for (1a) drew the wrong graph for (1b). Similarly, it also appears as if the same students who drew the wrong graph for (1a) drew the correct graph for (1b) but this was not the case. Most students (22 out of 29) correctly responded to Question 2, that quadratic graphs are symmetrical.

Table 27: ***Distribution of students' responses for the diagnostic pre-test – First lesson***

Number of students present = 29			
Items	Responses		
Question (1a)	Correct answer 	Not correct 	% of correct response
Frequency	19	10	66
Question (1b)	Correct answer 	Not correct 	% of correct response
Frequency	9	20	31
Question 2	Correct response (Symmetrical)	Not symmetrical	% of correct response
Frequency	22	7	76

The Lesson

In the introduction, Peter asked the students to name the methods for solving a quadratic equation of the form $ax^2 + bx + c = 0$. The students named the methods taught earlier, that is, completing the square, using the quadratic formula and factorisation. Peter told the students that solutions to quadratic equations are also referred to as roots of the equations. This introduction was meant to help students recall the already taught methods of solving quadratic equation as the teachers prepared to introduce the graphical method of solving quadratic equations, whose solutions are popularly referred to as roots of quadratic equations.

The teachers had intended to include solutions to quadratic equations by graphical methods, but they changed their minds because they realised that they were dealing with many skills in this lesson, such as completion of a table of values of quadratic functions, plotting graphs in squared grids and drawing graphs, and all these were done manually by free hand drawing. The teachers decided that since the graphical solution to quadratic equations is a skill on its own, it should be deferred. This view is supported by Dreyfus and Eisenberg (1990 p. 2) quoted in Ainley, Nardi and Pratt (2000), that:

> Reading a diagram is a learned skill; it doesn't just happen by itself. To this point in time, graph reading and thinking visually have been taken to be serendipitous outcomes of the curriculum. But these skills are too important to be left to chance.

This quote supports the teachers' decision to defer that section of the topic.

Peter asked the students to identify the coefficients of x^2 in the functions $y=x^2-x-6$ and $y=-x^2-5x-6$. The students responded in chorus that the first one has a positive one while the second has a negative one. Peter then asked the students to form groups and draw graphs of the two functions, note and explain any differences in the graphs. Peter guided the students through the range of values of the independent variable x as he wrote the range $-5 \leq x \leq 5$, on the chalkboard. However, he did not explain why he chose that range since it was not contained in their (teachers') lesson plan, which did not have any range indicated. He drew the outline of the table for the function (a) $y=x^2-x-6$, as shown in Table 28 and asked the students to complete the table.

During this lesson, 10 of the students who were absent for the pre-lesson test had come back and 39 students attended the lesson. Peter asked the students to form eight groups, seven of them comprised five members and one group had four members. Working in groups, the students completed their tables and drew graphs. The students spent 10 minutes completing the table before drawing the graphs, which Peter considered a lot of time because according to the lesson plan, the teachers had expected filling in of the table to take five minutes as it had been taught in the previous lesson.

Table 28: ***The outline of the table for $y = x^2 - x - 6$ for $-5 \leq x \leq 5$***

x	**-5**	**-4**	**-3**	**-2**	**-1**	**0**	**1**	**2**	**3**	**4**	**5**
x^2											
$-x$											
- 6											
y											

There were two distinct graphs from the groups. Seven groups drew the graph shown in Figure 26 while one group drew the graph shown in Figure 27

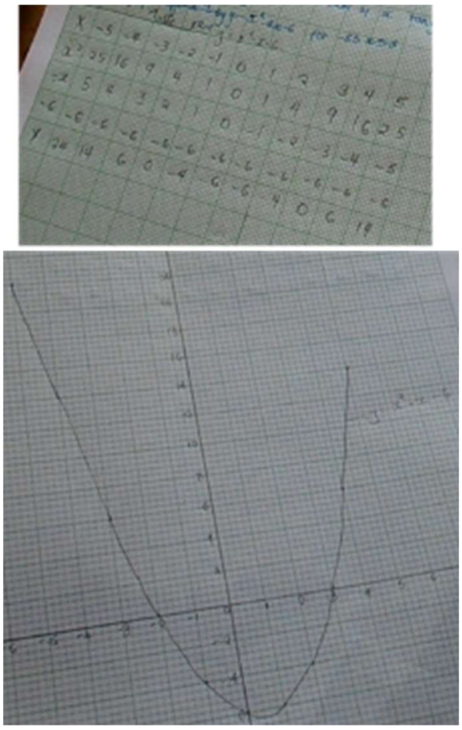


***Figure 26*: A representative graph of the seven groups for the function**

y = $x^2 - x - 6$, for $-5 \leq x \leq 5$

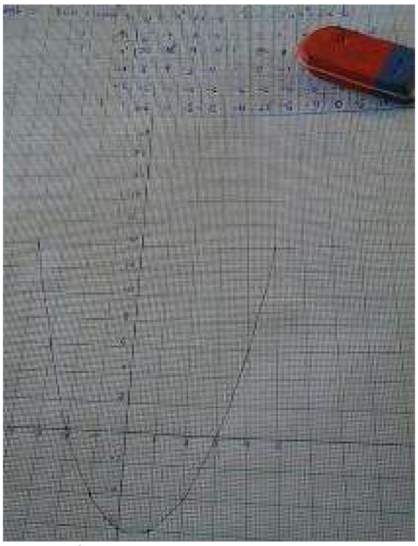


***Figure 27*: A graph of y = $x^2 - x - 6$, for $-4 \leq x \leq 5$ drawn by the group that modified the range of values of the independent variable.**

The graphs were similar, the only difference being that the Figure 27 group modified the range given by Peter of $-5 \leq x \leq 5$ to $-4 \leq x \leq 5$. They explained that they terminated the plotting of the graph at the coordinate (-4, 14) to cap the extending arms of the curve at the same level which would help them see the symmetrical nature of their graph. The students seemed to be considering the reflection of the curve along the axis of symmetry at the turning point.

Although the plotted points were from discrete integral values, the students were expected to use free-hand drawing to sketch the graph smoothly and continuously through all the points. In Kenya, high school teaching of mathematics has not integrated technologies such as graphical calculators that would help students sketch such graphs.

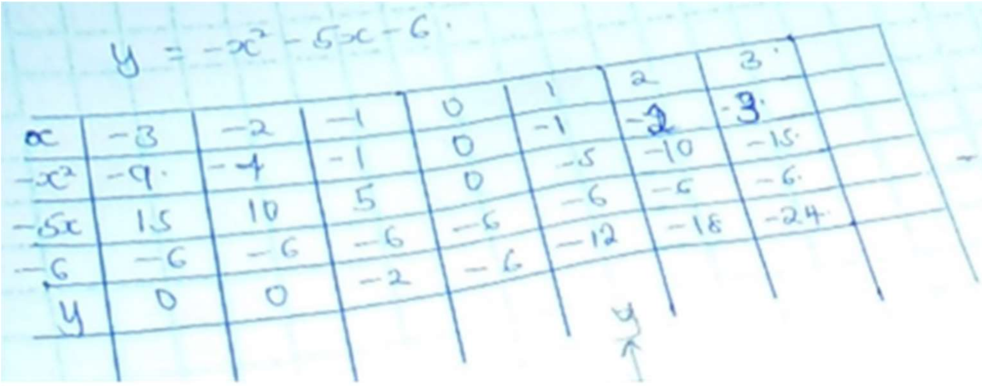
After the two presentations, Peter asked the groups to complete the table for part (b), $y = -x^2 - 5x - 6$ for a range of values for $-3 \leq x \leq 3$ as shown in Table 29. As in the first case, he did not explain why he chose that range.

As in part (a), the groups again spent more than five minutes completing the table in part (b). The teacher stopped the group work after 10 minutes so that the groups could present their solutions and by that time, only one group had drawn the graph, which is shown in Figure 28.

Table 29: ***The outline of the table for*** $y = -x^2 - 5x - 6$, ***for*** $-3 \leq x \leq 3$

x	-3	-2	-1	0	1	2	3
$-x^2$							
$-5x$							
- 6							
y							

Although Peter gave a range of x values for $-3 \leq x \leq 3$, the group drew a graph of x values ranging for $-5 \leq x \leq 2$, without physically adjusting the table, as shown in Figure 28.



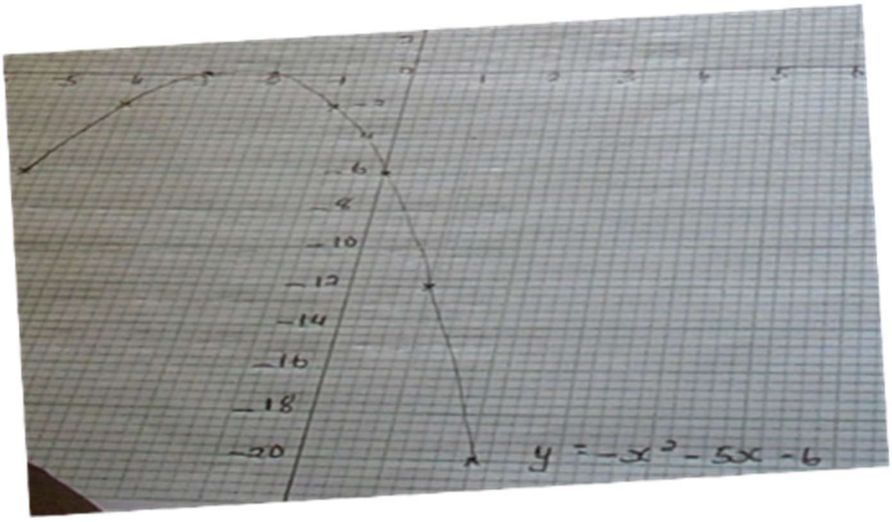


***Figure 28*: A graph of the quadratic function y = -x^2 – 5x – 6, for -5 ≤ x ≤ 2**

The group explained that they adjusted the range to help them see the turning point clearly. The range given by Peter stopped at the point which determines the turning point with the previous point, that the curve turned between (-3, 0) and (-2, 0). In addition, the students sketched the correct curve but the corresponding *y* values for the *x* values 2 and 3 in the table were incorrect. The mistakes for the *y* values arose from the substitution of *x* values (2 and 3) in the term $–x^2$, second row in the table. Instead of obtaining the values of *y* to be -4 and -9 respectively, the group wrote -2 and -3. Perhaps students were in a hurry since the group had correctly substituted -3 and -2 for *x* in the same term. The group's representative, Student 1, confirmed this hypothesis in his response to the teacher's request to explain the anomalies:

Student 1: The value of *y* for *x* = -2 and *x* = -3 was equal to 0 in both cases and we did not know how to draw the graph. We decided to extend the values of x to -4 and -5 to find the values of y, but because of time we did not correct them on the table. Also, when we tried to plot (2, -18) and (3, -24) the graph was not smooth. Members of the group worked out the values of y a fresh from $x = 0$, 1, 2 and 3, but again the person who was drawing the graph just corrected them on the graph because there was no time.

The group's behaviour is supported by what Ainley et al. (2000) call normalising of the data, "a behaviour […] in which the children are unhappy with the appearance of the graph and want to 'correct' it" (p. 17). In the current research, the students saw the trend of plotted points from the beginning and felt something was wrong with their data, so repeated the exercise of filling in the table and corrected the graph.

After the group's presentation, Peter summarized the lesson by referring to Figures 27 and 28. He asked the students, "What do you notice about the functions of the two graphs $y = x^2 - x - 6$ and $y = -x^2 - 5x - 6$?"

Student 2 The coefficient of x^2 in the second equation is negative and the coefficient of x in the second equation is -5 while the coefficient of x in the first equation is -1.

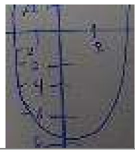
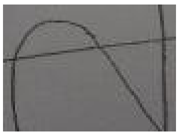
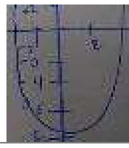
Peter explained that the differences in the shapes of the graphs were because of the signs of the coefficients of x^2.

Peter: A quadratic graph with a positive (+) coefficient of x^2 curves upwards while the one with a negative (–) coefficient of x^2 curves downwards. For your homework, draw the graph of $y = -5x^2 + 2x + 1$.

Peter then administered the post-lesson test, which was the same as the diagnostic pre-lesson test, whose outcome is shown in Table 30.

The table shows that all the students were able to sketch the correct curve for Question (1a) while 38 out of 39 students sketched the correct curve for Question (1b); 29 out of 39 students correctly answered Question 2.

Table 30: ***Post-lesson test responses by students from the first lesson***

Number of students present = 39				
Items	Responses			
Question (1a)	Correct answer 			% of correct response
Frequency	39			100
Question (1b)	Correct answer 	Not correct 		% of correct response
Frequency	38	1		97
Question 2	Correct (They are symmetrical)	Incorrect (Not symmetrical)	Blank	% of correct response
Frequency	29	6	4	74

Although Peter summarised the lesson by explaining the different shapes of the graphs, he did not address the axis of symmetry of quadratic graphs. Perhaps this is one of the reasons why 10 students could not answer the question correctly, especially the four students who left it blank. Also, the question asked whether the quadratic functions are symmetrical or not, but presumably the teachers meant the axis of symmetry of quadratic functions. I base this assumption on the video replay, in which I saw Peter mentioning the usefulness of the axis of symmetry of a parabola in calculating the volumes of revolutions generated by such curves, a concept the students would meet at university.

Reflections on the first lesson

During the reflection session, Peter explained that although the focus of the lesson was drawing graphs of quadratic functions, students spent more time than the teachers had expected in completing the tables of the dependent variable *y* for given values of the independent variable *x*.

Peter: The lesson was on drawing of the graphs but the students had to deal with the algebra and start with the completion of the table. Although all of them completed the table for part (a) and plotted the graph, they took a long time completing the table. For the second graph, I think there was limited time and I think we did not plan well.

Further alluding to the preparation of the lesson, Peter then reflected that he should have helped the students by completing some parts of the tables to allow them enough time to draw the graphs:

Peter: I should have made some rows to draw the first table and other rows to draw the second table then they should have made the comparison before time was over. They were able to see the nature of the two curves but we were not able to cover all concepts of quadratic curves. Next we will deal with the solutions from quadratic curves.

Dominic, one of the teacher-observers thought that the 40 minutes was not enough given the demands of the lesson:

Dominic: […] I think the content was too much for 40 minutes. Because when discussing the graphs, students must complete the two tables, they must draw the graphs then identify the one that curves upwards and the one that curves downwards. That is why I tend to believe that was too much for 40 minutes.

Dominic also felt that Peter should have engaged the students on other features of the graph, such as the turning points, during his summary of the lesson:

Dominic: Peter tried very much. The only key area, that is the turning point, you needed to have put more emphasis on that, especially graph two. For some students when same value of y in two points that follow one another such as (-2, 0) and (-3, 0), they are supposed to focus on the value in between to make that curve turn smoothly. During the subsequent lesson, lay more emphasis on the same.

John, the other teacher-observer, praised the students for their good work in plotting the points and drawing fairly smooth curves. However, he noted that students had difficulty in calculating the values of y to complete the tables:

John: As you look at the students, they were really good in plotting and drawing the graphs, but were fumbling in coming up with the tables. If Peter would have completed the first table with the students, I think he would have had enough time for all the students to draw the second graph and they could observe the differences clearly.

Concerning time management, John noted that the lesson commenced late due to a delay in the school assembly.

John: Just to emphasise on what Dominic has said, maybe we over planned and the content may have been too much for 40 minutes, but again the time spent might have been less than 40 minutes. Remember that the bell signalling the start of the lesson was rung when students were still in assembly, so again we may not say we planned too much for 40 minutes.

The teachers made three important observations that could improve future implementations of this lesson: firstly, they noted that students had difficulty calculating the values of y for the given values of x. Secondly, they noticed that students had difficulty drawing a smooth curve at the turning point and the need for the teacher to guide them on how to do it. Thirdly, they noted that 40 minutes was tight for delivering the lesson and that they needed to use it effectively and efficiently by helping students complete the tables during the lesson, so as to allow time to plot and draw the graphs and make comparisons.

As a participant observer, I reminded the teachers that completing the tables from quadratic expressions is a skill in its own right and that is why it was taught before embarking on the drawing of the graphs. Students would perfect it through homework practice. However, LS approach expects teachers to anticipate students' responses to the activities during preparation and plan how to address them. Had they drawn the graphs in advance themselves, they might have noticed the areas in which students would need guidance and would have chosen a more appropriate range.

Based on the discussions, the teachers decided to modify the lesson by helping the students to complete the tables. In the second lesson therefore, they incorporated the change of range values of x that was noticed by the group that presented on Figure 28. The teachers agreed that the tables would be completed in a whole-class discussion in the beginning, then groups would draw their graphs and discuss the shapes.

A Summary of the First Lesson

The students were able to complete the tables, plot and draw the graphs, especially the first function. In the end, they discerned the object of learning by identifying the condition for a quadratic function to have either a minimum or maximum turning point.

Group discussion helped some groups to normalise the graphs where there were errors in the table; it also helped them to realise some shortfalls with the range of values given by the teacher, which made them adjust the values accordingly to help them draw the correct graphs.

The teachers noted the students' difficulty in completing the table, especially substituting values in the second function that had negative coefficient of x^2. This difficulty resulted in a problem of time that affected most groups in drawing the graph of the second function. The teachers noted that there were many skills in the lesson and decided to modify the lesson by completing the table through a whole group discussion to allow ample time for drawing the graphs and discussing the condition of the nature of the curves.

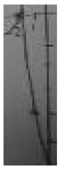
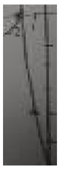
The teachers also noted students' difficulty in drawing a smooth curve, especially at the turning points. The teachers highlighted these shortfalls during the post-lesson reflection session, which later became the basis for modifying their plan for the second lesson.

6.2.2 Second Lesson

As stated earlier, the students of both lessons answered the same diagnostic pre-lesson test on the day before the lesson. The questions were the same as the ones presented in section 6.2.

Thirty-one out of the expected number of 33 responded to the diagnostic pre-lesson test. The two absent students were among those who went home to fetch the school fees as explained in section 6.2.1. The distribution of students' responses is shown in Table 31.

Table 31: ***Distribution of students' responses for the diagnostic pre-test – Second lesson***

Number of students present = 31					
Items	Responses				
Question (1a)	Correct 	Incorrect – curve 	Incorrect – lines  	Blank	% of correct response
Frequency	4	2	20	5	13
Question (1b)	Correct 	Incorrect-curve 	Incorrect – lines  	Blank	% of correct response
Frequency	3	5	20	3	10
Question 2	Symmetrical		Not symmetrical		% of correct response
Frequency	17		14		55

Compared with the pre-lesson test outcomes from the first lesson, the students in the second lesson had much greater difficulty sketching the required graphs. Most students

(20 out of 31) sketched some form of straight lines for both Questions (1a) and (1b). These students appeared to have calculated the intercepts of the axes and then drew the lines. Only four students correctly sketched the graph for Question (1a) and only three students correctly sketched the graph for Question (1b). The teachers were amazed by the big difference in sketching the graphs compared with the first lesson despite the fact that neither class had been taught the topic.

The Lesson

As in the first lesson, John asked students to name the methods for solving quadratic equations. The students listed the methods as completing the square, using the quadratic formula and factorisation. John then informed them about the topic of the lesson:

John: Today we want to draw graphs of quadratic functions and later on, may be in the next lesson I will show you how to use the graphs to solve quadratic equations.

John's promise to the students, which he said could be the next lesson, would be to draw graphs of quadratic functions such as $y = x^2 - 7x + 12$ and use the graph to solve the quadratic equation $x^2 - 7x + 12 = 0$; that is, obtaining the values of x when $y = 0$, which are the points of intercept between the curve and the line $y = 0$.

John wrote the two functions, $y = x^2 - x - 6$ and $y = -x^2 - 5x - 6$ on the chalkboard and asked the students, "What are the coefficients of x^2 in those two equations?" The students answered in chorus that the coefficient of the first is positive while the second is negative. John told them that we would like to draw these two graphs and see the nature of their curves and then asked them to form groups, and draw the two graphs: a) $y = x^2 - x - 6$ and b) $y = -x^2 - 5x - 6$, followed by discussing the shapes of the graphs.

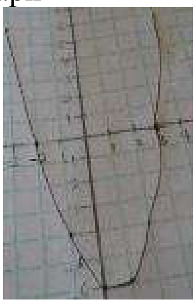
John asked the students to form groups of five; there were six groups, one of which had four members. John guided the students in a whole-class discussion in completing the tables for the stated quadratic functions, that is, $y = x^2 - x - 6$ and $y = -x^2 - 5x - 6$. The table for (a) $y = x^2 - x - 6$, was completed for integral values of x for -3 ≤ x ≤ 4 as shown in Table 32.

Table 32: ***The table for y = x^2 – x – 6, for -3 ≤ x ≤ 4***

x	**-3**	**-2**	**-1**	**0**	**1**	**2**	**3**	**4**
x^2	9	4	1	0	1	4	9	16
-x	3	2	1	0	-1	-2	-3	-4
-6	-6	-6	-6	-6	-6	-6	-6	-6
y	6	0	-4	-6	-6	-4	0	6

Groups plotted and drew their graphs as shown in Table 33.

Table 33: ***Categories of graphs by the six groups – Second lesson***

Categories	(a) Open-ended graph 	(b) Closed-ended with a straight line 	(c) Closed-ended with a curve 
Number of groups	3	1	2

I have categorised the graphs into three groups of: (a) open-ended graphs, (b) closed-ended graph with a straight line and (c) closed-ended graph with a curve.

All the three sketches are not satisfactorily smooth curves, but I decided to categorise these graphs as such because of the two anomalies of closing the graphs in (b) and (c). Usually, a graph of a quadratic function is left open-ended as shown in (a). However, three groups decided to close their graphs as shown in (b) and (c).

Although generally not smooth, all the three categories of the graphs were presented for discussion with the whole class. The groups in this lesson had difficulty drawing the graphs through the plotted points, especially the ones determining the turning points, as shown in Table 33. All of them joined the two adjacent points, signalling the turning point, by a straight line.

The students who presented the closed-ended graphs (b) and (c) explained that they thought they had to join all the points. One group represented by category (b) joined the open-ended with a straight line while two groups represented by (c) used a curve.

After the presentations, John explained to the students that the open ends of quadratic graphs do not need to be closed. He also told the students not to join the adjacent points with equal values of y such as (0, -6) and (1, -6) in the function $y = x^2 - x - 6$ with a straight line but to curve them slightly (demonstrating on the chalkboard) as shown in Figure 29, to show that the graph is turning between those points.

***Figure 29*: The arc representing the drawing John demonstrated on the chalkboard showing how to sketch a turning point between two adjacent points of a quadratic graph.**

John informed the students that they would learn how to accurately determine turning points the following year when they would be learning curve sketching under calculus.

After presenting the first graph, $y = x^2 - x - 6$, John again guided the students in a whole class discussion to complete the table for the second function, $y = -x^2 - 5x - 6$ as shown in Table 34.

Table 34: ***Completed table for the function*** $y = -x^2 - 5x - 6$, ***for*** $-4 \leq x \leq 3$.

x	-4	-3	-2	-1	0	1	2	3
$-x^2$	-16	-9	-4	-1	0	-1	-4	-9
$-5x$	20	15	10	5	0	-5	-10	-15
-6	-6	-6	-6	-6	-6	-6	-6	-6
y	-2	0	0	-2	-6	-12	-20	-30

John adopted the range of independent variable x, $-4 \leq x \leq 3$, which was suggested by a group in the first lesson, to clearly observe the turning point of the graph.

Four out of the six groups managed to draw the second graph within the discussion time. All the groups had almost similar graphs and the teacher selected two groups, whose graphs are shown in Figure 30 and Figure 31, to present their work for discussion with the whole class.

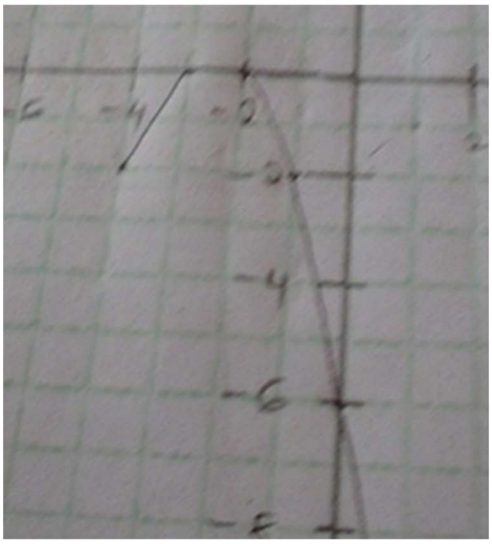


***Figure 30*: First group's presentation of $y = -x^2 - 5x - 6$, for $-4 \leq x \leq 0$.**

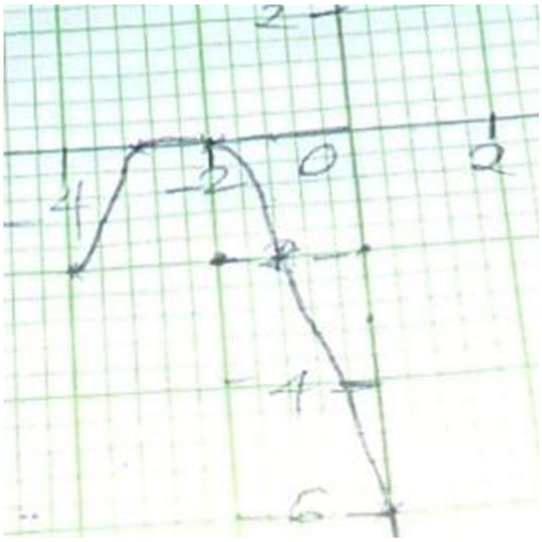


***Figure 31*: Second group's presentation of $y = -x^2 - 5x - 6$, for $-4 \leq x \leq 0$**

Despite John having demonstrated to the class how to draw graphs at the turning points, both graphs were still not very smooth, especially at the turning points. The groups represented by Figure 30 still did not make a curve at the turning point of the graph and instead used a straight line. However, none of the groups closed-ended the graphs as was the case in three groups of the first function.

John advised the whole class to practise drawing such graphs, as he noted that the ones they had drawn were inaccurate. He informed them that accurate graphs would help them in the next lesson to obtain accurate solutions of quadratic equations by graphical method.

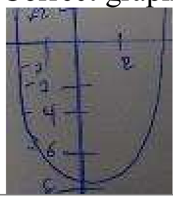
John summarised the lesson through interactions with the students.

John:	Explain the shapes of the graphs for functions (a) and (b)
Student 1:	The first graph faces upwards while the second graph faces downwards.
John:	Why do they curve differently? Do you notice any difference between the two equations?
Student 2:	The coefficients of x^2 are different and the coefficients of x are also different.

John: The sign of the coefficient of x^2 brings about the differences observed in the shapes. A positive coefficient like in the first function, $y = x^2 - x - 6$ produces a graph with a minimum turning point, while a negative coefficient like in the second function, $y = -x^2 - 5x - 6$, produces a curve with a maximum turning point.

As in the first lesson, John also did not discuss the symmetrical nature of the quadratic graphs. After the lesson, he gave the post-lesson test, the same as the pre-lesson test. The students' responses are shown in Table 35.

Table 35: ***Post-lesson test responses by the students of second lesson***

Number of students present = 29					
Items	Responses				
Question (1a)		Correct graph 			% of correct answer
Frequency		29			100
Question (1b)	Correct graph 		Blank		% of correct answer
Frequency	28		1		97
Question 2	Symmetrical	Not symmetrical	They are curves	Blank	% of correct answer
Frequency	19	5	3	2	66

All students sketched a correct graph for Question (1a), while all except one sketched a correct graph for Question (1b). Although 19 out of 29 students answered Question 2 as expected, there were still many students (10) who were not sure about the symmetrical nature of quadratic graphs, with three of them stating that they are curves. Perhaps the fact that John did not address the issue in his lesson summary, maybe due to time pressure, might have left the students uncertain of the answer.

Reflections on the second lesson

In reflecting on the first lesson, John pointed out that the lesson had started late due to the school assembly over-running. However, after teaching the second lesson he revised this view, and agreed with other observers that the content they had prepared was simply too much for a 40-minute lesson.

John: Looking at the lesson there were several values to fill in the tables. One, I think this is the first time the students were drawing graphs, two, I do not think they have ever seen such graphs before, three, some of them had problems in calculating the values to complete the table. I think the content was too much for the lesson.

One can infer that John realised that they might have chosen a bigger range of values of independent variable x than was necessary. In fact, the groups represented by Figures 30 and 31 only considered the values from the range $-4 \leq x \leq 0$ to draw the graph even though they spent time completing the table for the full range. In addition, John noticed that students had difficulty calculating the values of y, especially for the second function $y = -x^2 - 5x - 6$.

John: As the students completed the table in a whole class discussion, I realised that in the second table there were some wrong values, which we had to adjust before they could draw the graph. The time was little for the second graph and I did not conclude the lesson as I expected.

The difficulty arose from the substitution of x values in the first term of the function, that is, $-x^2$. Some students made the substitution then squared the whole term $(-x)^2$ instead of $-(x)^2$.

Peter, as one of the teacher-observers, appreciated John's guidance in completing the tables, explaining the tasks and correcting the students on the mistakes they had made.

Peter: At the beginning, John guided the students well to come up with the table. He posed questions at an appropriate time asking the students, "Why are the graphs different?" He also corrected the groups that drew wrong graphs by explaining to them what to consider when drawing quadratic graphs.

However, he noted that John delayed telling the students how to plot the graph after completing the table for the first graph. He argued that the delay affected John's conclusion of the lesson, because the students took some time before drawing the graphs after completing the table:

Peter: The students tried to plot the graphs, but John had not explained what was to be plotted against what. Some students were putting x and y in coordinate form and some students were plotting x against y. This affected John's conclusion of the lesson.

The other teacher-observer, Dominic, concurred that John had guided the students effectively in completing the table. However, he felt disappointed that almost all the groups drew inaccurate graphs:

Dominic: The teacher guided the students very well, but only some groups managed to draw the two graphs within the time. At the same time, many groups failed to draw good graphs.

Although Dominic expressed his disappointment with the kind of graphs students produced, it is this lesson that helped the teachers realise that drawing a graph is a skill that requires considerable practice.

A summary of the Second Lesson

All the groups plotted and drew graphs of the first function, while almost all the groups, except two, drew the graph of the second function. It seems that the change made to the lesson in terms of providing guidance in completing the tables in a whole-class discussion, improved time management. The students discerned the object of learning by identifying the condition that makes quadratic graphs have a minimum or a maximum turning point that is the sign of the coefficient of x^2. However, the students in this lesson had difficulty drawing smooth graphs, which was replicated in the second graph even after John had demonstrated how they could draw the graph at the turning points.

As in the first lesson, students in this class also adjusted the range of values of x for the second function for $-4 \leq x \leq 3$ given by the teacher to $-4 \leq x \leq 0$, even though they completed the table with the former range. The students realised that even after terminating the graph at (0, -6) they could observe the necessary features and draw conclusions.

Although there was improvement in time management, the teachers still realised that 40 minutes was not enough for the content they had prepared. Had they drawn the graphs in advance, they could have made more effective use of the time by choosing an appropriate range of values of x, which would help the students to fill the table faster.

6.3 Analysis based on Variation Theory as a Theoretical Framework

6.3.1 Intended Object of Learning

The teachers' choice of the quadratic functions $y = x^2 - x - 6$ and $y = -x^2 - 5x - 6$ was in line with variation theory, according to which students discern an object of learning when they observe some aspects of a lesson varying as others are kept invariant (Marton, 2015; Pang, 2008). In this pair of lessons, the teachers kept the signs between the terms of the two functions invariant (negative) while varying the signs of the coefficient of x^2 of the two functions. It is important to explain that although the absolute values of the coefficients of x are different, they do not have any effect on the shapes of the quadratic graphs, which is content of the lesson. The absolute value of a coefficient of x in a quadratic function, customarily b in $ax^2 + bx + c$, only causes the displacement of the vertex from the y-axis.

The choice of the object of learning, which was shapes of graphs of quadratic functions, suggested that the teachers planned to apply a separation pattern of variation and invariance, followed by a fusion pattern of variation and invariance (Lo, 2012; Marton, 2015; Marton, Runesson & Tsui, 2004; Pang, 2008). A separation pattern of variation and invariance requires teachers to isolate some feature of a dimension or some part of a lesson and concentrate on it, while other parts are kept in the background. In this lesson, the teachers planned to concentrate on one graph at a time, discussing its features with a focus on the nature of its turning point. After that, the two graphs would be compared and discussed together to note the differences and identify the cause of the differences. This is the fusion pattern of variation and invariance, explained as the observation of two or more features of a dimension simultaneously (Pang, 2008).

However, as has been observed in the last two analyses, Chapters 4 and 5, the teachers did not work on the lesson activities beforehand to identify students' possible actions and anticipated difficulties such as sketching the graph through the turning point. Some of the anticipated responses would have been the substitution of x values in the functions to calculate the values of y, especially in the second function where the coefficient of x^2 was negative. Also, they would have identified the appropriate range of values of the independent variable x, which would have saved time.

6.3.2 Enacted Object of Learning

Both Peter and John asked the students to draw the graphs of the two quadratic functions $y = x^2 - x - 6$ and $y = -x^2 - 5x - 6$, one at a time. In each case, the students presented their work and discussed the features of the graph, such as drawing a smooth curve in general and the turning point. They applied separation patterns of variation and invariance in what Chik and Lo (2004) call "an aspect-aspect relationship, which involves presentation and discussion of the first aspect followed by the presentation of the second aspect, after which both aspects are discussed together" (p. 93).

After drawing and discussing the graphs separately, the students were asked to explain what could be the reason for the difference in the shapes that they had observed. This was to help students observe any differences in the stated quadratic functions that could lead to the difference in shapes.

This action was an application of fusion pattern of variation and invariance where the two graphs were both brought into focus simultaneously to help the students discern the object of learning. Table 36 shows the summary of the applied patterns of variation and invariance. In both lessons, the teachers guided the students on the range of integral values of the independent variable x, which they used to calculate the values of the dependent variable y. The teachers chose the ranges that ensured that the drawn graphs included the turning points, which was the basis for comparison between the graphs leading to the discernment of the object of learning.

Table 36: ***Separation and Fusion patterns of variation and invariance applied in the lessons***

<table>
<tr><th>Varied</th><th>Varied</th><th>Invariant</th><th>Discernment</th></tr>
<tr><td>Quadratic functions</td><td rowspan="3">The signs of the coefficients of x^2 for the functions</td><td rowspan="3">The signs of the coefficients of x and the constant term</td><td>The shape of a graph of quadratic function depends on the sign of the coefficient of x^2</td></tr>
<tr><td>(a) $y = x^2 - x - 6$</td><td>When the sign is positive, the graph has a minimum turning point</td></tr>
<tr><td>(b) $y = -x^2 - 5x - 6$</td><td>When the sign is negative, the graph has a maximum turning point.</td></tr>
</table>

As the groups presented their graphs, the teachers realised that the students had difficulty in sketching accurate graphs, especially at the turning points as observed in Figures 28 and 30 and Table 33. The students joined the adjacent points, signalling the turning point in between them, with a straight line instead of an arc. In addition, the teachers noted that students had difficulty calculating the correct values of variable y when they substituted the values of x in the functions. During the reflection sessions, the teachers discussed these noted difficulties and suggested how they could be improved, which improved time management in the second lesson.

Perhaps the teachers could have realised some of these difficulties whilst preparing for the first lesson had they drawn the graphs in advance. In this regard, Wake et al. (2015) note that beginners implementing LS have difficulty in identifying students' anticipated responses during lesson preparation and the teachers in the current study were therefore no exception.

However, through discussion, the students adjusted some range of integral values of x that helped them observe the turning points clearly. They also *normalised* their points on the table (Ainley et al., 2000) to help them draw a correct graph.

6.3.3 Lived Object of Learning

The students learnt how to draw graphs of quadratic functions as was seen from the post-lesson tests outcomes shown in Tables 30 and 35. All eight groups in the first lesson were able to draw the correct graphs for the first function, $y = x^2 - x - 6$, while, all the groups in the second lesson had problems drawing correct graphs, even though they plotted the points correctly. All the groups from both classes had problems drawing correct graphs for the second function $y = -x^2 - 5x - 6$, especially drawing the graph at the turning point. However, they learnt how to draw the correct graphs after discussion with the whole class.

In addition, the students learnt about the condition of having either maximum or minimum turning points for quadratic graphs. However, many students still had difficulty with the axis of symmetry of a quadratic function as was suggested in post-lesson tests (Tables 30 and 35). The teachers realised this during the post-lesson reflection sessions and agreed to address the issue in the subsequent lesson.

During group discussions, groups from both lessons detected some problems with the range of x values, especially the second function where the range was terminating at a turning point of the graph. The students adjusted the ranges, which helped them accommodate the turning point clearly. The adjustment also saved them some time for drawing graphs after realising that continuing to plot other points beyond $x = 0$, in the second function, would not add anything new to the graph. All these changes were made without seeking help from the teachers. This may be an indication that the students are beginning to take responsibility for their learning through active involvement in the lesson as opposed to waiting for the teachers to make all decisions.

6.4 Conclusion

The students from both classes were able to discern the object of learning by identifying the conditions for maximum or minimum turning points for graphs of quadratic functions. Initially, they had difficulty calculating the values of y from the given values of x, which delayed the drawing of the graphs, especially in the first lesson. However, the situation improved in the second lesson after the teachers realised the problem, discussed it during the first reflection session and modified the lesson. The lessons also revealed the difficulty students had, especially those of the second lesson, in drawing smooth curves through the plotted points, mostly at the turning points.

However, this third pair of lessons has shown some good improvement on the effectiveness of group discussions in which students were able to identify some shortfall in the teachers' instructions, such as the range of values of variable x, and were able to correct them without involving the teachers. In addition, the students were able *normalise* the data (Ainley et al., 2000) to draw a correct graph when they realised that they had made some mistakes in completing the table. This had not been the practice earlier, where students waited for the teachers to show them how to proceed after making some mistakes. These students' actions showed significant strengths of the LS approach to the teaching and learning of the topic. Given the students classroom culture of passive recipients of knowledge, this action was a considerable change brought about by the LS approach and adds to the originality of this research.

The teachers' choice of tasks helped the students to discern the object of learning, as has been explained, because of the invariant components, which enabled students to clearly observe the conditions for difference in shapes from the varied component, the signs of the coefficient of x^2. In addition, the tasks were appropriate for the application of separation and fusion patterns of variations and invariances as the teachers planned. However, time constraints denied the teachers the opportunity to explain the axes of symmetry of quadratic graphs as was intended.

This third pair of lessons has reaffirmed the importance of solving class activities/tasks in advance during preparation to help teachers adjust their lesson plan accordingly and whenever necessary, performing an action called anticipated students' response in a LS lesson (Takahashi, 2009; Wake et al., 2015; Yoshida, 2012). The teachers realised, as John put it, that their ranges of values of x made them fill in more values than they needed to show the distinct shapes of the graphs. In addition, they realised from the students' adjustment that Peter's range for the second function terminated the left-hand limit prematurely at (-3, 0), while the curve had a turning point between (-2, 0) and (-3, 0). This did not allow the students to clearly observe the type of turning point they were expecting between the two points.

This chapter marks the end of the analytic chapters based on data collected from classroom observation and using Variation Theory theoretical framework. The next chapter, Chapter 7, analyses the data collected through interviews with teachers and students using Thematic Data Analysis.

Chapter 7 – Teachers' and Students' Experiences with Teaching and Learning the Topic of Quadratic Expressions and Equations in a Learning Study Approach

7.1 Introduction

This is the fourth and final chapter relating to the data analysis. This chapter mainly presents the analysis of data collected from the interviews with the teachers and students and follows a thematic data analysis approach using the themes summarised in Figure 10 and evidence included in Appendix 3. However, I have also included references to the data collected from classroom observations, as discussed in Chapters 4, 5 and 6, and any relevant literature that supports or contradicts the findings. In this chapter the eight interviewed students were assigned a number from 1 to 8 and will be referenced as Student 1, Student 2 ... Student 8.

As I explained in section 3.7.2, the global, organising and basic themes were refined before use by collating codes into sub-themes, basic themes, organising themes and eventually into a global theme. Part of the refining according to number 5 in Table 11 involved writing draft Chapter 7 and discussing it with the supervisors. After the discussion, I organised the data around two themes which I refer to as **Strengths** and **Challenges** (Figure 10). Each theme may have other sub-themes (Braun & Clarke, 2006 & 2013).

In **Strengths** I discuss the students' experiences in a learning study design and the teachers' professional development through learning study practice. These are presented in three stages - lesson preparation, actual lesson and post-lesson reflections.

In **Challenges** I discuss the changes introduced by the LS approach in classroom culture, regarding activities, and pre-lesson and post-lesson tests. Lesson duration, national examination pressure, syllabus coverage, shortage of teachers and workload are also discussed.

I conclude the chapter with a brief reflection on the gains made by the participants and how they navigated through various challenges. The final following chapter is then outlined.

7.2 Strengths

The students in this study said that the LS approach helped them in learning the topic of quadratic expressions and equations. They stated that the approach gave them an opportunity to interact with other students, not only during the lessons but also outside of class time. Through this interaction the students claimed that their communication improved with both the teachers and fellow students, improved their overall attitude towards mathematics and built confidence in their ability to solve mathematical problems.

The teachers echoed the students' assertions and claimed that the approach helped the students to learn the topic faster than students in previous years. They also supported the claim that the students' attitudes improved towards the subject of mathematics and their confidence was strengthened when solving mathematical problems. In addition, the teachers stated that the approach allowed them the opportunity to collaborate with each other and they were also able to learn from one another. Through team work they prepared lesson tasks that elicited student group discussions.

7.2.1 Student Learning Experiences in a LS Approach

During my interviews with the students they stated various ways in which the LS approach helped them in their learning. Many of them mentioned that the group work helped them in their learning of the topic.

Student 2 In the group discussion, you engage in one sum and many people came up with different ideas of calculating the sum and even a different method like in quadratic methods […] such as completing the square and factorisation. So, we collect ideas from different students and that makes students understand. The group agrees on the last answer.

Student 2's statement suggests that they learned through sharing ideas in small group discussions. One answer would be agreed upon as a group and this would then be presented for discussion with the whole class. Student 3 echoed student 2 and added that much was learned during the whole class discussion which followed the small group discussions. Student 3 explained that groups who could not obtain correct answers on their own could learn from the other groups.

Student 3 The class discussion after reporting helped us since […] some groups were not able to obtain correct answers in their groups so after reporting, these groups could correct their answers noting where they had gone wrong.

Student 3's statement is a confirmation of what I saw in my classroom observations. Some groups who did not obtain the correct answers at first were able to learn the topics after whole class discussions (as was also suggested by the post-lesson test outcomes). Indeed, in the second pair of classroom observations (Chapter 5) none of the groups were able to obtain the correct answer. However, after whole class discussion, the students understood how to solve quadratic equations by completing the square, which was the topic of discussion. Student 3 confirmed that students from their class then understood this method well enough to be able to use it to solve quadratic equations in an examination.

Student 3 The performance will be high because concerning the methods, students in our class understood completing the square method best and would apply it in solving quadratic equations.

The statement contrasts with the findings of Didis and Erbas (2015) which showed that solution of a quadratic equation through completing the square was unpopular with the students as "a few students who used completing the square method failed" (p. 1142). The same observation was made by Vaiyavutjamai and Clements (2006) who concluded by asking that “Are there realistically feasible forms of teaching that will result in students, and not just high-achieving students, learning quadratic equations, and other mathematics topics, in a relational way?” (p. 73). Student 3’s statement is encouraging and may suggest that, with more practice, perhaps the LS approach could be one method by which students could more easily understand the topic.

Student 4, while commenting on the approach, stated it helped them to extend group discussion beyond the classroom which then enabled further learning from their peers.

> Student 4 [...] group discussion helped some of us. We formed groups of about three members outside class. [...] in a case where one is good in mathematics and two members are not sure of the answer, they learned from the member who is good in mathematics instead of waiting to ask the teacher.

Student 4’s statement is a claim that the approach helped them to learn mathematics as a whole class. This, perhaps, helped to minimise the gap between the students perceived to be ‘weak’ and the ‘high’ achievers. The higher achievers were able to support the weaker achievers in their groups.

Student 2 confirmed Student 4’s statement when she said that her desk mate was not good at mathematics but, after discussions with her, she understood the topic.

> Student 2 [...] the approach was beneficial to many students because for example, my desk mate is not good in mathematics but when we discussed I see she understands that topic.

The statements from students 4 and 2 suggest that the approach helped them to learn from peers, and in the end, they understood the contents, that is, they were able to solve problems on quadratic equations from beginning to end and obtain correct solutions. This observation is supported by Nardi and Steward’s (2003, p. 354) finding that stated “...the students place emphasis on the significance of working with peers not for mere efficiency, not simply doing mathematics, but, also, for understanding it.”

These arguments are also supported by Elliot and Yu's (2008) evaluation of the Variation for the Improvement of Teaching and Learning (VITAL) Project. The students were able to recall what they learned two years after the project and said that interaction through class activities helped everybody to understand what they were taught.

These narratives by the students were supported both by the teachers in the current study and the principals in the VITAL project, as set out in the next paragraph.

John	[…] Most of the weak students understood solving quadratic equations by the method of factorisation by using the cuttings. It was so easy for them to factorise using the cuttings as I compared them to the previous group with whom I did not use this method.

John's statement about low achievers was supported by Elliot and Yu's (2008) VITAL evaluation report. The principals from the schools that participated in the project claimed that the approach enabled "teachers to reduce the gap between high and low achievers in a way that normal practice had not" (p. 157). One of the principals said, "The progress of those lower achievers was more apparent" (p. 157).

Dominic concurred with John's statement and added that the approach helped the students learn factorisation of quadratic expression faster.

Dominic	[…] the way we taught factorisation through the cutting of pieces of papers helped the students to comprehend it very fast. In contrast to the way we usually teach it where we force the students to learn that the value at the centre will always stand for the sum while the first one and the last one gives the product; making the students cram it in instead of allowing the students to know how they develop.

Dominic's statement suggests that even though the students were usually told that in factorising a quadratic expression such as $ax^2 + bx + c$, the coefficient of x represents the sum while the coefficient of x^2 and the constant term provide product, the students previously took more time to comprehend the answer than they did in this instance.

Apart from learning the topic of quadratic expressions and equations, which, according to Bloom et al.'s (1956) educational objectives, is a cognitive objective achievement, the students said that the approach helped to improve their attitude towards the topic and mathematics in general. They also said that it raised their confidence in their ability to solve mathematical problems and helped them improve their communication with fellow students as well as teachers, which is considered improvement in the affective domain (Bloom et al., 1956).

Student 1 […] I think group work was beneficial, because some students feared discussion at first but when people shared ideas, some got that confidence to do mathematics and discuss. When you look at the choice of the questions, some questions were not easy to answer as individual but after discussion people were confident to present.

Student 1's assertion was observed during the lessons where, in the beginning, some students were too shy to speak - this occurred especially in the first pair of lessons. However, during the second and third pairs of lessons there was an improvement in the students' participation in the classroom and in small group discussions. In the third pair of lessons, (Chapter 6), students confidently discussed the tasks in their small groups and even corrected what they considered an anomaly from the teacher's instruction (specifically the range of x values, which they were supposed to use to plot and draw graphs of quadratic functions, without consulting with the teachers).

As I mentioned earlier in Chapter 4, initially some students were too shy to speak. It appears that, not only did this shyness come from being in a new classroom practice, but the students may also have had problems with their communication in English, as Student 3 suggested.

Student 3 […] the approach improved the communication among the students. For example, one person would start to explain how to work on the Task and others might realise that the approach the person has used is wrong. Another person would come up with a new idea and everybody would discuss. […] at first, some students were only whispering because of fear but later everybody was talking loudly, and I realised that our English improved.

Student 3’s statement seems to suggest that the students corrected their colleagues’ spoken English during small group discussions and beyond.

The teachers concurred with the students’ opinions that there was some improvement in their behaviour after introducing the LS approach.

John [...] In the beginning, I thought the approach would be better for average students or students who are ready to speak their mind or who did not fear talking, but as we moved on some of the weak students could talk. [...] some of our students improved their communication, and at least they changed their attitude towards mathematics. [...] the relationship between some of us with some weak students has really improved. A group of students would come or an individual would come saying, please “mwalimu” (teacher) help me solve this problem.

John claimed that all categories of students improved their communication, which in turn improved the relationship between the students and teachers. He explained that, as result of this improved relationship, the students felt that they could approach the teachers during their free time to seek help with solving mathematical problems. This was not happening previously. Peter supported John’s observation of student/teacher consultations during their free time and added that the students were also now consulting teachers who were not necessarily their designated mathematics teachers.

Peter [...] earlier, learners only consulted their classroom teachers, but when they realised that the teachers were always preparing the lessons together and they teach the same thing, they now consult any of the teachers of mathematics in the school.

In view of these narrated behaviours, both the students and teachers claimed that the students’ performance in the topic, and in mathematics in general, would improve.

Student 1 [...] performance will improve since every student felt confident in herself because people shared ideas. Even some who "feared" mathematics gained confidence and were able to do maths and discuss.

Student 8 I think the performance will be good since after teaching we again used to meet in a group where everybody gave out her opinion. There we were able to understand somethings that we did not know. The practice will improve the performance.

Student 7 In the group discussion [...] we collect ideas from different students and that makes students understand. The performance will improve because we will be able to remember what we discussed in groups.

The students felt that the group participation, which extended beyond classroom time, helped them perform better in the topic. They predicted that their performance would improve in mathematics tests in general because they would remember what they had shared in the groups.

The teachers concurred with the students' suggestions that the performance would improve. A revelation came from Peter in an interview a month later. As I mentioned in section 3.6.2, Peter was attending a seminar away from the work station when I interviewed John and Dominic. Peter's interview happened after they had given the students a Continuous Assessment Test (CAT) on the topic of quadratic expressions and equations, and the end of term mathematics test.

Peter [...] in terms of performance, it is most likely going to improve. When we gave them a CAT after teaching quadratic expressions and equations, three-quarters of the students scored 10 out of 10.

According to Kenyan education policy, teachers are supposed to give students at least two CATs per subject per term and an end of term test (KIE, 2002). According to Peter they gave the CAT after teaching the complete topic of quadratic expressions and equations and three-quarters of the students managed to score 10 out of 10. For the end of term test, he claimed that almost all students selected a quadratic related question in Section II of the test.

In Kenya, the format of setting national mathematics examinations for secondary school students follows the format used by the Kenya National Examination Council (KNEC). In the KNEC format, the mathematics paper consists of two sections. Section I is compulsory for all students with 16 short answer questions giving a possible total of 50 marks. Section II is an elective section with eight questions. Students are expected to answer five questions, each of which carries a total of 10 marks. The quadratic question Peter referred to is a Section II question and is shown in Figure 32.

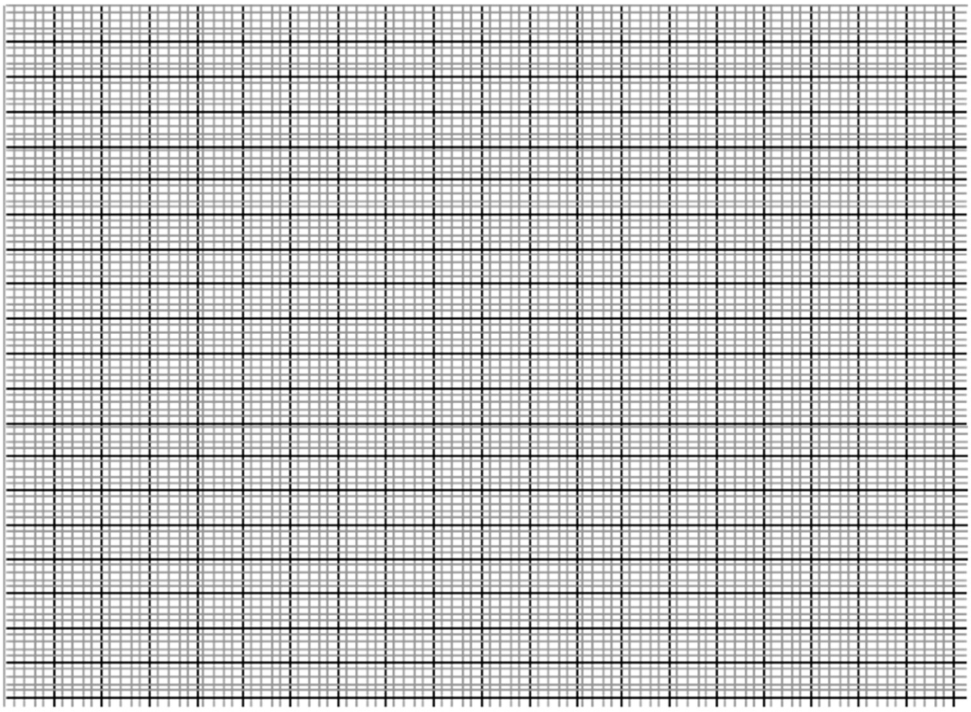
Peter	[…] when we gave them end of term exams, almost all the students answered the question on solution of quadratic equation by graphical method. […] those who attempted the question got at least five out of ten marks.

According to Peter, the fact that many students opted for this particular question would suggest that the students had understood the topic. However, he noted that a few students still had a problem drawing a smooth curve. This had also been observed during the third pair of lessons (section 6.2.2). Peter's observation indicates an improvement on what was observed during the lessons in which students had difficulty completing the table and almost all groups from the second lesson had problems with drawing smooth graphs.

20. Complete the table below for $y = 2x^2 - x - 3$ (2 marks)

x	-3	-2	-1	0	1	2	3
$2x^2$	18		2	0			18
$-x$	3		1	0	-3	-2	
-3	-3	-3	-3	-3	-3	-3	-3
y	18	7				3	12

(a) Draw the graph of $y = 2x^2 - x - 3$ for $-3 \leq x \leq 3$ (3 marks)



(b) Using a suitable straight line solve

(i) $2x^2 - x - 3 = 0$ (1 mark)

(ii) $2x^2 - 3x - 5 = 0$ (2 mark)

***Figure 32*: The Form 3 end of first term 2016 examination question that tested solutions of quadratic equations by graphical method.**

In addition, the Form 3 class in which I conducted the study in 2016 sat for their Kenya Certificate of Secondary Education (KCSE) examinations in October/November 2017. Upon the release of the examination results in January 2018 the Principal of the school called me (on 17th January 2018) and told me that their students' scores had improved in mathematics over scores from previous years and they had two students with a mean of A (plain), the highest grade in the Kenyan grading system of KCSE examinations. Apparently, those were the only A grades scored in the whole school. Later they sent me improvement mean scores of 3.97 in 2018 compared to 2.64 in 2017, an improvement index of 1.33 translating to a percentage improvement of 50.4. Although there are many factors that contribute to the students' examination results, but the Principal explained that, after introducing group work during this study's data collection, the students had continued with it and that could also be a contributing factor to their improvement.

In summary, the strengths observed in the LS approach to the students' learning of the topic included: improved learning of the topic including the subtopic which is internationally agreed as difficult method of solving quadratic equation (the completing square method) was accepted as preferred option of solving quadratic equations by the students. The teachers observed that the students learned the topic much faster than the previous students where they used the usual traditional method. Also, the students improved their attitude towards the topic and the subject in general and they improved their consultation with all the mathematics teachers in the school as well as fellow students outside of the class time. These actions led the teachers to predict that the students would do well in the final national mathematics examinations, which was eventually proved by the outcome of the KCSE mathematics examinations, a contribution to the originality of this study.

7.2.2 Teachers' Professional Development through Learning Study Practice

Garet, Porter, Desimone, Birman & Yoon (2001) say professional development occurs in various forms and format. They suggest that there are two main approaches to teachers' professional development: (1) the traditional approach, which is organised in the form of workshops and seminars; and (2) reform-based teachers' professional development such as study groups, mentoring, coaching and collective participation. However, Garet et al. (2001) argue that in any of the two main approaches:

> Core features of professional development activities that have significant, positive effects on teachers' self-reported increases in knowledge and skills and changes in classroom practice: (a) focus on content knowledge; (b) opportunities for active learning; and (c) coherence with other learning activities (p. 916).

Learning study (LS) gives teachers an opportunity to collaborate and work collectively in lesson preparation, teaching and post-lesson reflection. Therefore, LS falls under the reform-based teachers' professional development. The teachers in this current research cited some areas in which they felt they gained from the LS teaching and learning approach. These areas included: lesson preparation, classroom teaching and learning, and reflection sessions. These areas are supported by other studies that explain Learning study approach supports teachers' professional growth through collaborative activities at every stage of the teaching and learning process such as preparation, implementation and review of the lesson thereafter (Davies & Dunnill, 2008; Pang, 2006; Pang 2008; Runesson, 2013). This also has the advantage of ownership of the teaching and learning process since teachers decide on the object of learning by themselves (Pang, 2006; Pang, 2008; Runesson, 2013).

Lesson Preparation

The three teachers who participated in the study are trained teachers (section 3.2.2) and had been trained in lesson plan preparation. However, they realised LS teaching and learning required teachers to find out what students already knew prior to the lesson preparation. Students' prior knowledge is mostly sought through diagnostic pre-lesson testing or pre-lesson interviewing (Lo, 2012; Marton, 2015).

John	From the pre-lesson test, I would know what the students already know because that is what I want to use to get into what they do not know.

Previously they had planned lessons from what they thought the students needed to learn. The LS approach required them to incorporate the students' knowledge of the topic. The outcome of the pre-lesson testing reveals gaps that exist between what is expected of the students and what/how they understand the content. The teachers have an opportunity to plan for how to address and fill any gap that is realised following the testing.

The teachers felt that collaboration during preparation was an important learning achievement. Peter and John stated:

Peter	[…] It helped a lot especially for the part of the teacher preparation. You find that when teachers sit together to prepare a lesson, there are certain concepts that one or other teacher may understand or may find a better method of delivering the content.
John	[…] you know as a teacher there are certain things that you assume and concentrate on what textbooks offer, but when you prepare together you tap into other teachers' experiences. Preparing the lesson plan or discussing the lesson before we do the actual teaching made some of us think beyond what we usually think before going to class and add more on to what we usually do when teaching in class.

Peter's statement suggests that they shared their knowledge on each of the subtopics and were able to tap into an individual's relevant experience and expertise. This helped them to identify the tasks that elicited group discussions. The two teachers' statements show that they appreciated the support they obtained from one another during lesson preparation. John claimed that the discussions during the planning helped some of them think more deeply about the intended lesson, as opposed to what they used to do previously. The teachers' claims were evident in the activities they prepared for the lessons, as discussed in Chapters 4, 5 and 6. The students were engaged in the lesson activities and thereby learned the contents as suggested by the post-lesson tests and the students' statements as discussed in section 7.2.1.

The teachers' statements about learning from one another through collaborative teamwork are supported by Pang's (2006) finding in his study of the use of LS to enhance teacher professional learning in Hong Kong. Here, one of the teachers stated that: "I have really learnt from the collaboration. In the past, my focus was always on how to make students remember the key points… I now understand that we need to learn from the students as well" (p. 38). The teachers used students' prior knowledge about the topic during their preparation of the lessons.

During the Lesson

During the lessons, Peter and John had to occasionally adjust lesson plans to accommodate students' group work outcomes, which in some cases did not lead them to the correct solution. In the first lesson of the first pair of lessons (Chapter 4), Peter had to adjust the discussion time to allow the groups to conclude their discussions. In the second pair of lessons, both Peter and John had to prepare to explain why the groups' approaches to the solution of quadratic equations by completing the square were not correct. In the third pair of lessons, John had to adjust the lesson plan to accommodate the changes the students made concerning the range of x values for drawing graphs of quadratic functions. Peter summed up these observations by saying that:

Peter The approach requires that teachers to become flexible and understand the content very well [sic].

Peter's statement suggests that teachers should be prepared to adjust their planned lessons to accommodate students' views and to help them learn the intended content. If a teacher is not well versed in the content, he/she may not appropriately address students' concerns that emerge from a lesson. Peter's observation on content was witnessed in the second pair of lessons (Chapter 5), where students showed from their facial expressions that their questions had not been adequately addressed.

Peter's claim on flexibility is supported by the explanation of Lo (2012) and Marton (2015) that the object of learning is dynamic and can be adjusted depending on the students' responses during the learning process.

In addition, the teachers argued that teamwork in their preparation for the lessons appeared to have made their teaching ‘very easy’. John claimed:

John	After preparing together, where you share your colleagues’ ideas you find the teaching very easy. […] like the cuttings, I learnt how to use them with the help of my colleagues and that made my teaching very easy. Even weak students could discuss and present their work.

John’s comment seems to suggest that teamwork during preparation may translate into finding reasonable learning activities that are easily understood by the students. He claimed that even the students he previously perceived to be ‘weak’ participated in the discussions and presented their findings. Peter reiterated John’s observation that the teamwork was useful and claimed that it helped John who was teaching Form 3 for the first time.

Peter	[…] you see LS also encourages team teaching where the class is not owned by one teacher. It may help teachers who had not taught the topic before because teachers will be preparing together. […] that is what we did, John has never taught form three, so, such topics like perfect squares, completing the square method he had not taught, but because we planned together, he found it very easy.

Peter’s comment suggests that teamwork can help teachers develop confidence in preparing the lessons. In addition, teachers would develop confidence in each other so that they would be able to teach a colleague’s lesson in the spirit of team teaching.

The teachers appeared to acknowledge that the topic of quadratic expression and equation is a difficult topic to teach and to learn. However, they noted that the use of activities helped them to teach it more easily. Dominic stated that:

Dominic	[…] the topic of quadratic was a little bit complex, like when teaching […] the issue of the “sum and product” which is totally new. But the way we brought it through the cutting of the papers and the counting

technique meant the students could easily comprehend those things very fast.

Dominic was referring to the first lesson in which they used paper cuttings to help students factorise quadratic expressions. He claimed that the inclusion of practical activities enabled the students to learn factorisation of quadratic expressions quickly. John supported Dominic's claim that the introduction of the activity and the group discussions helped the students learn the topic faster than he had seen in previous years.

John [...] the topic is so challenging, [...] using the method that we introduced to the students' learning through activities, many students including "weak" students had to think and most of them understood the topic. It was so easy for the students to factorise quadratic expressions and solve equations as I compared them with the previous[4] group where we did not use that method.

Dominic and John's claims suggest that the use of activities, together with the students' involvement, helped the students learn the contents faster than previously when they were not involved as learners.

Post-Lesson Reflection Session

The teachers in the study appreciated the post-lesson reflection sessions, according to their comments during the interviews. One of the ways in which they felt that the sessions helped them was on the evaluation of the teaching and learning process, which culminated in the adjustment of subsequent lessons. Peter remarked:

Peter [...] after teaching, coming together to discuss the learning outcome is important to know whether the strategy the teacher used worked, if it

[4] Peter had said that John was teaching Form 3 for the first time. John's reference to the previous group in this excerpt refers to the aspects of quadratic expressions and equations that are usually taught in Form 2 according to the syllabus, which he had taught.

did not work then the other teacher can teach it in a new class after adjusting the lesson.

During the reflection sessions the teachers realised that some of the introductory activities, which they had considered to be pre-requisite knowledge to the topic of discussion, proved to be difficult. The students spent more time on them than they had anticipated. For example, in the first lesson of the first pair of lessons (as discussed in Chapter 4), the teachers realised that some students had difficulty recognising the 'invisible 1' as the coefficient of x^2 in the expression $x^2 + 4x + 3$. In the modified second lesson, the teachers adopted a new strategy to introduce the idea of a coefficient. Most of the adjustments the teachers made to the lessons during classroom observations were made to the introductory activities and group discussion schedules. These adjustments were made to reduce the time spent on the introductory activities.

In addition, Dominic observed that the reflection sessions were consultative, and the teachers used them to advise one another for future improvements on teaching and learning exercises. Dominic stated:

Dominic [...] In fact, LS is one of the best ways of teaching because you watch as teachers teach, and once you are through you sit down in something like a conference. There you consult with one another and tell the teacher, this is where the weakness was, and you are supposed to do this. As we do that, we are also learning and correcting one another.

Dominic's statement suggests that the reflection sessions were a learning time for them as they pointed out the areas for improvement. The sessions gave opportunities for teachers to review each other's lessons and helped the group to reach a consensus on the course of action to take to improve the teaching and learning of subsequent lessons. They also complimented each other on the areas they did well in. This was motivating for the teachers and helped them gain confidence in their teaching.

From the teachers' comments, it appears that they found the reflection sessions helpful in building teamwork spirit, which could be considered as one of the professional growths. The teachers' comments in this study are reinforced by similar one in Pang's (2008) study of using the learning study to improve students' mathematical understanding, stating, "After the study, the teachers gained precious experience of learning from one another and reflected on their own practice" (p. 20). Generally, the teachers in this study declared that they had gained professionally in these three aspects of preparation, enactment and evaluation of the lessons.

In summary, the LS approach brought teachers together to work collaboratively to prepare lessons, enact the lessons and evaluate the lessons in reflection sessions. As a result, the teachers embraced one another as they built teamwork spirit that elated the teachers' confidences. They also suggested improvement on the lesson plans that eventually improved effectiveness of lesson delivery as was observed in the modified lessons. All these activities were new to the teachers who were used to traditional approach to teaching where a teacher owns his/her class and decide what to teach by himself/herself. This adds to the strength and originality of this study in a culture where teachers are not used to be observed by others as they carry out their teaching.

7.3 Challenges

The LS approach posed some challenges to the teachers and the students in this study during the teaching and learning of quadratic expressions and equations. The main challenge appeared to be the changes in their usual classroom practices and culture, such as team preparation of activities used during the lessons, and pre-lesson and post-lesson tests. Other challenges included national examination pressure, syllabus coverage, and teacher shortages and workloads.

7.3.1 Change in Classroom Culture

As I have mentioned in section 7.3 above, the LS teaching and learning approach introduced new classroom cultures such as finding students' prior knowledge about the topic, team preparation among the teachers involved in the lesson, lesson observation by other teachers, and students' group discussions and reporting. This was different from the teachers' usual classroom practices. As I explained in section 2.6, having other people observe a teacher's lesson only happens in three scenarios in Kenya. As all these scenarios subject a teacher to some form of assessment or inspection, so a teacher would emotionally view his/her lesson observers as assessors of the lesson. In the initial classroom observations this was their impression of me, despite the fact that I informed the teachers that the observers would mainly be concerned with the students' learning activities during the lesson.

Apart from classroom observation, the LS approach also requires teachers to gather materials for the lessons and prepare them together, assess the students' conception of the content of the lesson before and after, and reflect on it after teaching. All these requirements were new to the teachers and would count as a change to normal classroom culture. First, the teachers had to arrange a time to meet as a team and prepare the lessons, which mostly happened outside of school hours as I explained in Chapter 3 (section 3.6.2). Secondly, the teachers were not used to student group discussions during the lesson, an arrangement they did not experience themselves when students and finding tasks/activities that would elicit group discussion was a challenge. Lastly, the teachers were not comfortable with the idea of testing students before the lesson as they pointed out in section 3.4; therefore, they found it a challenge to come up with relevant and useful pre-lesson tests.

Activities

Although the teachers appreciated the use of activities to teach quadratic expressions and equations, which supported the students' learning, they talked about the challenging task of identifying relevant activities. Dominic said that he was not sure whether they could apply the same approach to other topics.

Dominic [...] the only challenge, I do not know if it is applicable in all the topics in maths, because we have only tested it in quadratic and it is applicable. Especially the issue of making those things like the cuttings.

As I noted earlier in section 3.4, Dominic's doubt was mainly based on the practical activities when he said, "Especially the issue of making those things like the cuttings". He seemed to have been thinking about practical activities in other topics and, where he could not readily visualise a practical activity he doubted the implementation of the design. John expressed a similar concern when he suggested that there should be a pool of activities per topic that the teachers can easily access when teaching those topics.

John	[…] if this method is supposed to be implemented then I think there should be a lot of research done on the activities. It is not easy for a teacher to come up with activities in each topic. […] you know, at least there should be activities listed somewhere that you can use to make work easy.

The arguments by both John and Dominic point to the fact that it was an unusual practice which they had not experienced before, even as students. I observed their concern during the orientation sessions and I explained to them that an activity need not necessarily be practical in nature but could also be a discussion on usual textbook questions with a modification to elicit group discussion. In the second and third pairs of lessons, they selected non-practical questions, similar to those normally contained in textbook exercises, which were discussible. Peter was more categorical about the challenge on activities when he argued that teachers needed to be creative enough to develop good activities for teaching and learning. He noted that the heavy workload on teachers, as discussed in section 7.3.4, could pose a challenge to finding activities for each lesson, especially if the students enjoyed the teaching through the activities and would want it continued in every lesson.

Peter	[…] workload may limit the activities. […] however, the learners would wish that one continues to teach using activities once he/she introduces it. If he/she fails to use activities in some lessons […] students may feel the lesson is not enjoyable and they may doze off. I think […] the teacher needs to be creative and always look for things that will excite learners in every lesson and that is difficult.

The teachers claimed that their workload makes it difficult to have activities in every lesson. Also, as the use of activities was not their common classroom practice, it could very well be a source of worry. However, as Dominic suggested, with constant practice the teachers should overcome the challenge of finding suitable activities. Teachers would experience more of a challenge in the initial stages of the implementation because they would be finding new activities for all the topics. However, in subsequent years they would only need to be modifying those activities to improve on them.

For example, when teaching the topic of quadratic expressions and equations in subsequent years, the teachers involved in this study might just modify the current activities and use them again. During his interview Peter noted that there were only a few practice questions during the lessons, and the same questions in pre-lesson tests were given as activity questions, and this limited the students' learning. In such cases, the teachers would improve the situation by separating the activity questions from pre-lesson and post-lesson test questions. This could also improve on their evaluation of the lesson. The re-use of previous activities would help the teachers to create an activity bank, which could help resolve John's concern about having a pool of activities. The teachers would document the successful activities and share them with other teachers - a practice which is one of the aims of learning study (Pang, 2006).

Pre-lesson test and post-lesson test

Another challenge cited by both the teachers and the students were the diagnostic pre-tests and post-tests. This appears to be another classroom culture issue. During the orientation sessions, before the start of classroom observations, the teachers raised their concern about pre-testing the students. It was a genuine concern which I agreed with, especially for an examination-oriented country such as Kenya where students think of a test in terms of a competition. However, since pre- and post-lesson tests are part of the LS cycle, the team and I agreed to implement it with the teachers and the students giving their views at the end.

Some students found the pre- and post-lesson tests discouraging because they were tested on topics they had not already been taught, especially in the diagnostic pre-lesson tests. Some students claimed that they were boring, and others said that they were fun.

Student 1	The pre-lesson and post-lesson tests were discouraging because we were given questions we had not learnt. […] after the lesson, in some cases, still only some students could answer the questions.
Student 4	[…] before teaching you could not be happy about the pre-test questions, some students were complaining saying that they were boring. After the teaching, […] you could at least be encouraged to answer the questions.
Student 3	[…] they were fun. Before the lesson we were not using the right concepts to answer the questions but after the lesson, you know the concept. So, if you compare the first answer that you gave and the last answer that you have given they were just fun.

I felt that the students' complaints about the pre-lesson tests were justified as all the observed contents in this study were being introduced for the first time and therefore many students had no clue about them. The students' statements may suggest that they viewed the questions as irrelevant from their use of the terms such as 'boring and fun.' This is supported by Nardi and Steward (2003) study where a student in their study explained their use of the term 'boring' as "Students do not like irrelevant […] mathematical tasks" (p. 351).

Firstly, this had not been a common classroom practice, so it was new to the students. Secondly, as I have mentioned, these were students coming from a background of competition in tests and examinations, so testing them on what had not been taught could be discouraging. The students' concern in this research appeared supported by principals and teachers in Elliot and Yu's (2008) evaluation report of the VITAL project. Although the teachers, the principals and the academic consultants in the VITAL project underscored the importance of the diagnostic pre-test (this claim was also later stated by the teachers in this study), they cited some challenges with it and suggested amending its application. One of the challenges they mentioned was that it was time consuming. One of the principals commented that once they adopted the design they would simplify the pre- and post-lesson tests and said: *"The pre- and post-test for the subjects we will have will not be exactly like the VITAL […] will be something simplified"* (Elliot & Yu, 2008) (p. 232).

Although the principal did not suggest the format or nature of simplification, one of the academic consultants explained that: *"When teachers have reached consensus that Learning Study should be sustained for the continuing improvement on lesson teaching with collective efforts, pre- and post-tests could be replaced by interviews with students"* (p. 244).

Another challenge I noticed with the pre-post-tests in this study was the lack of clarity of some questions such as symmetry of graphs of quadratic functions asked in the last pair of lessons. This challenge could still be attributed to the fact that the teachers were trying to internalise the practice and appeared themselves to be rather unclear on the real objective of the tests.

7.3.2 Lesson Duration

In Kenya, the period for a single lesson in secondary schools is 40 minutes. There is a provision for a double lesson, but mathematics lessons are all taught as single lessons. The students raised concerns about the duration of the lessons in relation to the activities approach to teaching and learning of mathematics (as I have already explained under Strengths). They cited the limited time for the small group discussions as one of the challenges they experienced.

Student 2 […] the group discussion is good but during learning, the time is not enough. Take a case of 40 minutes for a lesson then the teacher explains a certain sum on the board then […] as you are discussing the bell is rung before you complete your group discussion.

Student 3 echoed Student 2’s observation and suggested that if the time is not enough then the teacher should shorten the small group discussions to allow more whole class discussion of the activity.

Student 3 […] may be the discussion can be short and… if some people have not understood, or the groups have not understood the activity then the teacher can elaborate it on the board for the whole class to discuss.

The concerns raised by the students about time constraint was observed during the lessons, especially the first lesson in all the pairs of lessons as presented in Chapters 4, 5 and 6. This was always one of the factors considered when modifying the lessons. This challenge of time was also experienced in Pang’s (2008) study in which he observed LS teaching and learning in a Hong Kong high school 40-minute mathematics class. The time constraint became one of their points of discussion during the post-lesson reflection session and they agreed to re-teach the lesson in a double lesson lasting 80 minutes. In Japan, where lesson study has been conducted for a long time, their single lesson lasts 50 minutes. However, since lesson duration is a policy/habit adopted by each country, teachers need to plan and manage the time as allocated. During the second lesson of each pair of the lessons, John improved the situation by modifying the lesson plans after hearing the reflection sessions. In some cases, this worked well. For example, during the second lesson of the first pair of lessons (Chapter 4), the teachers agreed to explain all the tasks at the beginning of the lesson to allow the groups that completed Task 1 ahead of the rest to proceed to Task 2. The modification worked, and the groups managed to do all the activities.

7.3.3 National Examination Pressure and Syllabus Coverage

Another issue that teachers cited as a challenge that would affect the implementation of the LS approach was national examination demands. Although the teachers taught the topic within the stipulated time according to the curriculum, they felt that it took a long time to cover the topic. The teaching was conducted in a Form 3 class. This is a senior class in a system where students spend four years in a secondary school and sit for a national examination at the end of the fourth year. School administration expects teachers to complete the syllabus ahead of the stipulated time and still have some time to revise the topics with the students before they sit for their end of secondary cycle examination. Peter explained:

Peter	[…] you see, what happens is that teachers would want to rush and finish the syllabus, so they do not pay attention to the stipulated time for syllabus coverage. […] I just strain to cover syllabus so that I finish early and have time for revision.

When I enquired about the rush that Peter talked about, the teachers said that they usually organise and teach extra lessons during the students' preparation times. However, because of this study they did not organise extra lessons for the topic of quadratic expressions and equations, but Dominic explained what they did to improve on the syllabus coverage.

Dominic	[…] this time round we were covering three different topics because of the teachers' strike and your programme. […] this was within the stipulated time, but we had to get time to cover others.

The concern of examination preparation in senior classes is also noted in other countries that practice lesson/learning studies. Bush (2003) notes that in Japan, "professional development through lesson study are more commonly found in elementary and middle schools than high schools because of the emphasis on national exam matriculation in the higher grades" (p. 89). Similar observation was made by Elliot and Yu (2008) in the VITAL project when one of the principals explained that they implemented the project in junior classes because of examination pressure in senior classes. He said, "yes, yes. We can bear the risks in junior classes" (p. 19). Perhaps the principal made the statement because it was a new arrangement which came with some uncertainties. These observations confirm the challenge of time for the implementation of a LS lesson.

Due to the demand for extra lessons, the teachers had less time to prepare for the LS lessons particularly because this required all three of them to be together. Perhaps this could explain some limitations noted in the observed lessons, such as having the same questions for both pre-and post-lesson test items, and these also being the same as lesson activities. Peter stated that:

Peter [...] this method may slow down the syllabus coverage because of the activities involved. [...] like when we were dealing with factorisation of quadratic expressions, [...] you see it took time and in one lesson we could only answer two questions.

Peter's concern was echoed by both John and Dominic:

John [...] I think there are some topics where you cannot develop practical (hands-on) activities or good activities and this can slow syllabus coverage.

Dominic [...] but the only problem, I do not know if it is applicable in almost all the topics in maths, [...] but maybe we have challenging topics where teachers frame words differently, which takes time to be comprehended by students

The teachers' comments showed that the identification of activities seemed to be a challenge and they felt that this would slow down the syllabus coverage. This would be tied to classroom culture issues as already discussed in section 7.3.1 above.

However, although the teachers stated that it appeared to slow down the syllabus, many students seemed to have understood the topics immediately after being taught and felt better about understanding the topic than they had with the traditional approach to teaching and learning.

Peter [...] however much it slows down the syllabus coverage I think what has been covered is understood better than if we cover the syllabus faster and learners do not understand well or only a few understands.

Peter's statement suggests that they may not need to spend so long revising the topic because many of the students understood it as they moved along. Dominic reiterated that, with continued practice, the syllabus coverage would improve.

Dominic [...] with continuous application of the strategy with proper preparation, I believe the syllabus can be covered fast.

Following Dominic's statement that the syllabus would be covered faster with continuous practice and preparation, I realised that during the lessons the teachers were already revising the topics that they considered pre-requisite to the topic of quadratic expressions and equations. This approach gave them an opportunity to identify problems that the students had with those pre-requisite topics. This action could reduce the revision time in the end.

7.3.4 Teachers' Shortage and Workload

As I explained in section 2.4, the LS approach requires teachers to prepare lessons as a group. One group member teaches the lesson while others observe. After that they meet to reflect on the lesson and modify it whenever necessary. It seems therefore, that a shortage of teachers in a school could be a challenge to the implementation of a LS approach in two ways. Firstly, in a situation where there is only one teacher teaching mathematics in the whole school, the teacher may not have anybody to prepare a lesson with or anybody to assist with the observation. Such a situation is common in single-streamed schools. Secondly, the shortage of teachers can lead to teachers having heavy workloads that may limit their free time to meet and prepare the lessons. For example, during the teaching of the second pair of lessons, (Chapter 5), Dominic could not attend the reflection sessions for both lessons because he was teaching other classes.

As I explained in section 1.4, teachers of secondary schools in Kenya teach two subjects and are expected to teach 27 lessons per week. However, those with responsibilities such as principals of schools, deputy principals and heads of departments have their workloads reduced in proportion to the number of streams in a school. For example, in a two-streamed school, such as the one where I did this research, a head of department such as Dominic should have 20 lessons per week. However, he was teaching 28 lessons per week, even higher than Peter and John who were teaching 27 lessons each.

Although Dominic was the head of mathematics, he was also the only teacher for physics in the school, so his workload surpassed that of a head of department.

The teachers' workloads show that they were teaching an average of five out of the eight lessons per day. The teachers' workloads restricted the days and times for observing the lessons because I had to select the days when all the three teachers were available to observe the lessons. As I explained in section 3.5.2, it is because of the stated restrictions that I observed Peter teaching all the first lessons and John teaching all the second lessons.

The teachers stated that the heavy workload could affect the preparation of the lessons, and especially the discussion of the activities. John stated that:

John [...] if workload is large, at times preparation may be a problem for the activities, especially hands-on activities. [...] in hands-on you need time to prepare and do it practically before you give it to the students, so it will take time.

Peter echoed John's argument on the workload when he said that:

Peter [...] time management, you will find that sometimes it is difficult to find that a teacher is free and the other teachers are also free to sit down to discuss or prepare.

Peter's argument may mean that they sometimes had inadequate time to prepare for the lessons including the deep discussion of the lesson materials (*Kyouzai kenkyuu*) as required by the LS approach. This could then lead to a hurriedly prepared lesson. As I have stated in section 7.3.1 under activities, the teachers' choices of pre- and post-lesson tests and the activity questions in the observed lessons might attest to the challenge of inadequate time of preparation. Also, in all pairs of lessons, the teachers did not seem to have fully prepared the lesson activities in advance as was noted during the lessons. Perhaps this was because of limited time to meet and prepare the lessons. This challenge of workload is supported by Lee's (2008) study where teachers stated that "They found it difficult to spare time for the scheduling of time, lesson planning and lesson observations" (p. 1122).

Dominic also commented on the issue of planning with respect to teacher shortages and the implementation of the LS approach to teaching and learning.

Dominic The only problem that might arise is the issue of teachers' shortage, if you do not have enough teachers, planning becomes very difficult. […] or may be if you have a school that has only one trained teacher and some teachers who just completed form four and have not gone for any further training, to some extent they might not bring out the concept the way it is expected.

Dominic's second sentence suggests that having untrained teachers may affect the implementation of a LS teaching approach and this could to some extent be true. However, the implementation of LS approach with such teachers could also be an advantage to them in the spirit of continuous professional development, as they will eventually gain from the group preparations, observations and discussions. `

7.4 Conclusion

In conclusion, as mentioned in various sections of Chapter 7, the main gains in the introduction of a LS approach to teaching and learning is the collaborative nature of the approach for both teachers and students. This action made the teaching and learning of the topic effective, which culminated in students discerning the intended objects of learning. Both the teachers and the students predicted that there would be improvement in performance of mathematics in the national examinations in Kenya.

Although there were challenges cited by both the students and the teachers, the main challenge was due to change in classroom culture, which if the approach is practised continuously will minimise those challenges or eliminate them. These awareness of the strengths and challenges of the LS approach gives this study an originality aspect in a culture of traditional approach to teaching and learning of mathematics where learners are usually passive recipients of knowledge.

The next chapter, which is the last chapter of the thesis, is the conclusion chapter. It focuses on the research questions as it collates the conclusions of the analyses chapters before making recommendations for the way forward.

Chapter 8 – Conclusion

8.1 Introduction

In Chapters 4, 5, 6 and 7, I reported the findings of this study, the subject of which is the teaching and learning of quadratic expressions and equations in Kenyan secondary schools utilising the LS approach. In this chapter, I draw conclusions from those findings and connect them to the theoretical framework underpinning this study, and I provide references to the relevant literature that informed these conclusions. The purpose of my study is: to explore the contributions made by LS approach to the teaching and learning of quadratic expressions and equations and, to document the teachers' and the students' perceptions on the application of LS approach when teaching and learning the aforementioned topic, with the possibility of applying the same approach to other mathematics topics in future. The research questions are as follows:

1. What is the outcome when a LS approach is applied to the teaching and learning of mathematics in a Kenyan cultural context?
2. What are the teachers' views on the application of a LS approach to the teaching and learning of the topic of quadratic expressions and equations, and the possibility of extending the same approach to other topics?
3. What are the students' perceptions and experiences of the application of a LS approach to the teaching and learning of the topic of quadratic expressions and equations?

In addressing these questions at this conclusion stage, I draw from Pang and Marton's (2003) assertion that:

> First, the students participating in the study are expected to learn about the object of learning and to learn better than they otherwise would have done. Second, the teachers participating in the study are expected to learn about handling the object of learning, […] Third, the researchers participating in the study are expected to learn about how the theory works […] and that theory is put to test (p. 180).

As I explained in section 2.4 and section 3.5, the choice of the objects of learning, in the lessons that I observed, depended upon the teachers' experience with the topic, the content of the lessons as stipulated in the curriculum, and the students' pre-lesson test outcomes. I utilised the LS approach as a teaching approach, within the stipulated time in the syllabus according to the Kenyan secondary schools' curriculum, because the Kenya National Examination Council ([KNEC], 2014) report had identified the topic as one of concern.

This study put the Variation Theory (Pang and Marton, 2003) into practice, taking into consideration that the culture is different from where it has previously been practised. I address the students' and teachers' reactions to the LS approach in the 'implications of the study' section; and I discuss the researcher's experience in the 'recommendation', 'reflection' and 'way forward' sections.

Firstly, I will address the contribution made by the application of the LS approach to teaching and learning the topic of quadratic expressions and equations. In this section I will refer to the aspects of the objects of learning, that is, the intended, the enacted, and the lived objects of learning.

Finally, I offer some recommendations on certain aspects, such as pre- and post-lesson tests, and the possibility of adopting LS as a teaching approach, before discussing some reflections and the way forward.

8.2 The Contribution of the LS Approach to the Teaching and Learning of Quadratic Expressions and Equations

As discussed in section 2.5, the LS approach encompasses teaching and learning in three broad areas – preparation, implementation and evaluation. The preparation stage begins with the participant teachers meeting to identify the object of learning, followed by the finding of relevant resource materials, (students' opinions gathered in pre-lesson tests could aid in the search for the learning materials). The team then decides on the critical feature(s) necessary for the discernment of the object of learning. They then make plans for the lesson activities, including planning for the necessary patterns of variation and invariance to be applied during the lesson. Marton (2015) notes, "This step is left entirely on the team's own ingenuity" (p. 198). The team is required to have a deep discussion of the teaching materials before the implementation of the lesson – this is called *Kyouzai Kenkyuu* in Japanese (Yoshida, 2012).

After teaching, the teachers converge for a session in which the team reflects on the lesson and the students' responses in the post-lesson assessment. This reflection leads to the modification of subsequent lessons whenever necessary. In the current study, the team devised various activities to align necessary patterns of variation and invariance in each pair of lessons, as discussed in section 8.2.1. Two of the three teachers taught one lesson each in each of the three pairs of lessons while the remaining two, and I, observed the lessons, and the team reflected on them later.

8.2.1 Intended object of learning

Marton (2015) explains that the intended object of learning is the teachers' view of what is to be learned in the lesson. I consider it to be the hub of the lesson where teachers plan how to make use of the information and materials gathered from various sources in order to offer an effective lesson. Teachers may have good materials, but if they do not plan well they may fail to enact the lesson in the intended manner. At this point, the teachers have some information about the students' prior knowledge of the content of the lesson through the outcome of the diagnostic pre-lesson test. The teachers plan the lesson so as to effectively address the students' responses from the pre-lesson tests, some of which may be misunderstandings of the content.

In addition, the teachers' preparation includes anticipation of the students' responses to the activities (Wake, Swan & Foster, 2016, Yoshida, 2012). This action helps teachers look at all possible approaches and methods that could lead to the solution of the activities. Sometimes, the teachers use their previous experience to anticipate frequent errors which the students usually make when working with such activities (Rowland et al., 2011). All these possible approaches and methods, plus others that would come from the students' work, help the teachers to respond to any errors during the whole class discussion. This is supported by Hiebert's (2003) suggestion that the teacher should allow the students to interact with the task and identify possible approaches, some of which may be wrong. The teacher would then harmonise the students' work and address any errors or add any approaches that the students did not identify. The identification of anticipated students' responses can be a challenge to teachers who are new to this requirement. Wake et al. (2016) state, in response to the teachers in their study, "[…] the teachers said that they found it difficult to anticipate student reasoning, and so they adopted a strategy to […] introduce the class to the initial problem in advance of the research lesson" (p. 251). It was also a challenge to the teachers in this study. They managed the challenge by allowing students to solve the tasks (with different approaches) in their groups, and address differences during the whole class discussion, as explained in the next section.

The planning includes decisions about how and when the teachers will give out the task(s) to the students during the lesson. Therefore, to manage the lesson in an effective way, as suggested by Marton, (2015) teachers should design the patterns of variation and invariance and apply them in their lessons. Marton (2015) notes that "creating a pattern of variation and invariance in line with the principles of variation may not in itself be enough to bring about learning. We might not even be able to create a particular pattern of variation" (p. 211). However, the activities should support the learning with necessary conditions, such as clear instructions, to guide the activities.

In the current research, the teachers prepared the lessons, which included various activities to support the learning of the intended content. In each pair of lessons there was evidence of the teachers incorporating the students' pre-lesson test responses and of teamwork during preparation. In the First pair of lessons, presented in Chapter 4, the teachers' plan was to ask the students to identify quadratic expressions from a list of quadratic and linear expressions which included: $x + 5$, $x – 10$, $x^2 – 6$, $x^2 + 4x +3$ and $x^2 + 3x + 2$. This action was taken in response to students' pre-lesson test outcome where many of them could not explain why $x^2 + 5x + 6$ is a quadratic expression.

The teachers then prepared a practical (hands-on) activity, which helped them introduce factorisation of a quadratic expression, differently from the usual approach of explaining the process. The teachers appreciated the teamwork among themselves as it helped them discuss and understand the hands-on activities, as they explained in section 7.2.2.

In the Second pair of lessons (Chapter 5), the teachers' plan was to have the students solve the equation $x^2 + 6x + 9 = 0$ by the factorisation method. The choice of the equation was to enable the teachers to explain the concept of a perfect square, which students were unable to explain during the pre-lesson test. In addition, the choice of the equation was meant to contrast with the activity question $x^2 + 6x - 9 = 0$. The choice of the two equations suggests that the teachers intended to help students explore the completion of the square on the LHS of the second equation after transforming it to $x^2 + 6x = 9$.

Similarly, the choice of the two quadratic functions $y = x^2 - x - 6$ and $y = -x^2 - 5x - 6$ for the Third pair of lessons (Chapter 6) shows that the teachers discussed their selection. The functions had their turning points in between two adjacent integral values of x for the ease of free-hand drawing of the graphs. In addition, the choice to have negative coefficients of x and the same constant term in both functions was to help students predict the cause of the difference in the nature of the two graphs, which was the object of learning.

During the interviews, the teachers confirmed that they planned the lessons together and stated that they improved in some areas that they might have missed in the previous lessons, (such as a key question to help the students work on the class activity). John commented:

John	[…] looking at the first lesson plan and the second lesson plan, the second lesson plan was like an improvement of the first because we realised that in the first lesson plan there was something missing like the key question.

What John calls the first lesson refers to the planning of the First pair of lessons while the second lesson refers to the planning of the Second pair of lessons.

Marton (2015) argues that:

> The theory outlined in this book does not suggest which particular examples should be used, but what follows from the theory is the pattern of variation and invariance that has to be created by means of the examples (p. 205).

This statement suggests that teachers should think of patterns of variation and invariance when choosing examples and activities. In this study, the teachers' arrangement of examples and activities showed planned applications of some patterns of variation and invariance. In Chapter 4, the teachers planned to introduce the lesson through the mixture of expressions, as explained in the fourth paragraph of 8.2.1 which implied the application of a contrasting pattern of variation and invariance. They also planned to use pieces of paper to help factorise the two expressions $x^2 + 5x + 6$ and $x^2 + 3x + 2$ one at a time, which was an application of the generalisation pattern of variation and invariance.

In Chapter 5, the choice of the quadratic equations $x^2 + 6x + 9 = 0$ and $x^2 + 6x - 9 = 0$ as the class activity was a plan designed to apply contrast, separation and fusion patterns of variation and invariance. They planned to *contrast* the two equations to help learners realise that the second equation could not be factorised. Then, after exploring ways of solving the second equation, they planned to rearrange the second equation as $x^2 + 6x = 9$, *separate* the left-hand side of the equation and explore ways of completing the square then *fuse* the two sides together to solve the equation.

In Chapter 6, the teachers planned to *separately* plot and draw each graph of the quadratic functions $y = x^2 - x - 6$ and $y = -x^2 - 5x - 6$, before discussing them together, which is an indication of a planned application of separation and fusion patterns of variations and invariances.

During the interviews, the teachers stated that the LS approach helped them to be imaginative, think of activities beyond textbooks ones, and prepare activities that elicited discussion among the students in small groups. The evidence of these statements is shown in section 7.2.2.

In view of all these narratives, the LS approach helped the teachers improve their lesson preparation, referred to here as intended object of learning, over their usual lesson preparation which would have been done by one teacher only. This research clearly showed that the approach helped teachers to embrace teamwork which resulted into effective lesson preparation with appropriate activities that supported learning, which was a contribution to the originality of the study in a culture where lesson preparation is a one-teacher affair.

8.2.2 Enacted object of learning

Marton (2015) writes:

> What the teacher has to do is not only to ensure that the necessary conditions are present, but she/he also has to try to make it possible that the necessary conditions are turned into sufficient conditions (p. 206).

As has been stated in the previous section, the necessary conditions availed by the teachers are the planned patterns of variation and invariance. During *enactment* of the lesson, teachers put the planned patterns of variation and invariance into action to ensure that students learn the content. The students should experience the variation in order to discern the object of learning. When that happens then the necessary conditions would become sufficient conditions, knowing that the sufficient conditions could be assumed to be the essence of lesson presentation, where one would expect the enacted object of learning to be the intended object of learning. As I explained in section 2.4, Lo (2012) and Marton (2015) categorise patterns of variation and invariance as contrast, separation, generalisation, and fusion. In any one lesson, the patterns may appear singly or in a combination of more than one pattern.

During the lessons observed in this research, the students were fully engaged with the activities in their small groups, before discussing the groups' findings in whole-class discussions. The teachers (Peter and John) involved various patterns of variation and invariance in the three pairs of lessons. In the First pair of lessons, they *contrasted* two algebraic expressions $x + 5$ and $x^2 - 6$; and $x–10$ and $x^2 + 3x + 2$ respectively, to help the students discern a quadratic expression. During the small group discussion session, the students formed rectangles from the pieces of paper representing the expressions $x^2 + 5x + 6$ and $x^2 + 3x + 2$. By calculating the areas of the rectangles formed, the students discerned and *generalised* the conditions for factorising a quadratic expression, which was the critical feature of the object of learning. In this way the teachers applied the contrast and generalisation patterns of variation and invariance putting the theory into practice.

In the Second pair of lessons, the teachers first *contrasted* the two equations $x^2 + 6x + 9 = 0$ and $x^2 + 6x - 9 = 0$, to create awareness in the students that factorisation cannot solve all quadratic equations. The teachers then applied the part-whole *separation* principle as explained by Marton, Runesson and Tsui (2004) to rewrite the equation $x^2 + 6x - 9 = 0$ as $x^2 + 6x = 9$. The separation helped the students to complete the square on the left-hand side (LHS) by linking it to the first equation $x^2 + 6x + 9 = 0$. After that, the students considered both parts together as they solved the equation $x^2 + 6x - 9 = 0$ by the method of completing the square. By bringing the parts together as they solved the equation, the teachers applied the fusion pattern of variation and invariance. The students discerned the conditions for solving a quadratic equation by the method of completing the square, which included the addition of $(\frac{1}{2}$ of the coefficient of $x)^2$ on both sides and also, considering both positive and negative values of the square root of the RHS expression. Therefore, in this Second pair of lessons, the teachers applied contrast, separation and fusion patterns of variation and invariance, which assisted learners in the discernment of the intended objects of learning.

In the Third pair of lessons, the students first drew graphs of the two quadratic functions, $y = x^2 - x - 6$ and $y = -x^2 - 5x - 6$ *separately before bringing them together* to discuss their shapes and the condition for the shapes. The students discerned that the shape of a quadratic graph depends on the sign of the coefficient of x^2 of a quadratic function written in the form $y = ax^2 + bx + c$. In this Third pair of lessons, the teachers applied separation and fusion patterns of variation and invariance.

Occasionally the teachers adjusted their plans due to unanticipated outcomes from group discussions, an action referred to as response to contingency in a Knowledge Quartet (KQ) analysis (Rowland et al., 2011; Thwaites et al., 2011) (section 4.2.1). For example, during the first lesson of the first pair of lessons, Peter extended the discussion time when he realised that many groups were still discussing the activity. This adjustment improved the outcome of the discussions with 7 out of 8 groups completing the first task and 4 out of 8 groups working out the task correctly.

Since the students were given an opportunity to explore ways of solving activity questions, some groups solved them in a way that showed certain weaknesses which teachers had not anticipated. They expected the students to know the correct workings, as these fell under what had been taught in earlier lessons in either Form 1 or Form 2, according to the Kenyan secondary schools' mathematics curriculum.

For example, in the Second pair of lessons, discussed in Chapter 5, some groups attempted to solve the quadratic equation in the activity by distributing the square root sign over the terms of the equation $x^2 + 6x = 9$ to obtain $\sqrt{x^2} + \sqrt{6x} = \sqrt{9}$. This approach was not a correct mathematical argument. They further proceeded to solve the equation as $x + 2.45x = 3$ and $x = 0.87$. Other groups expressed the value of x as $x = \sqrt{9 - 6x}$, which was a correct mathematical approach but could not give a correct solution required because x was expressed in terms of another x. In the Third pair of lessons, presented in Chapter 6, some students had difficulty in completing the table of values of y given the values of x for a quadratic function $y = -x^2 - 5x - 6$ (section 6.2).

All these errors were realised because the students were given an opportunity to recall their previous knowledge and to apply them in a new situation to solve the intended problem. This shows that the LS approach, when applied, helped in learning the correct way of solving such problems including revisions of previously learned topics, since the teachers corrected the errors during the enactment of the lessons. Therefore, the LS approach seems to be an effective way of learning mathematics as it helps learners connect mathematics topics during the teaching and learning process, which is a strength on this research over the previous approach to lesson delivery.

8.2.3 Lived object of learning

Marton (2015) explains that:

> The main question in a LS is the relation between the enacted and the lived object of learning (i.e. between the learning that is made possible on the one hand and what is learned on the other….). The idea is to determine what the students have learned and what it is that they were supposed to learn (p. 264).

Part of the evidence of the students' learning is their responses from the pre-lesson and post-lesson tests. Otherwise, part of the students' learning, which Lo (2012), Marton (2015) and Marton and Booth (1997) refer to as experienced awareness is observed from their actions during the lessons and from the post-lesson interviews with them and the teachers.

In this study, many students were able to discern the objects of learning after participating in the lessons, in their small group discussion and in the whole-class discussion, which included the summaries of the lessons. The students were able to learn factorisation of quadratic expressions, solutions of quadratic equation by completing the square, drawing graphs of quadratic functions and explaining the condition for the different shapes of the graphs. The post-lesson test outcomes suggested the same (sections 4.2, 5.2 and 6.2). John, one of the teachers in the study, commented that, because of the teaching approach they applied in which students solved lesson activities in the process of learning, the students learned factorisation of quadratic expressions faster than the previous years' students (section 7.2.1).

During the student group interviews, a student said that in their class they understood solving quadratic equations by the method of completing the square better than other methods and they were ready to apply it in the tests (section 7.2.1). The student's remark was contrary to Didis and Erbas's (2015) finding in which only 11 out of 217 students in their study attempted to solve the given quadratic equations by the method of completing the square, but they did not obtain the correct solutions. I hoped that the students in my study would confirm their claim later in a summative test given on the topic. Elliot and Yu (2008) in their evaluation report on VITAL project confirmed that students were able to recall what they had learnt through the LS approach two years after the project. I believe that the students in this study would also be able to recall and apply what they learned in any form of assessment in future. As was reported in section 7.2 that the students performed well in the 2017 KCSE mathematics examinations, this study shows that the LS approach maybe adding quality learning to the students.

However, as I mentioned in the previous section (section 8.2.2), students' group work revealed some knowledge gaps through errors which the teachers had not anticipated but which became learning moments for the students. In attempting to solve a quadratic equation by completing the square, some students distributed square root signs over each term of a quadratic equation $x^2 + 6x = 9$ and obtained $\sqrt{x^2} + \sqrt{6x} = \sqrt{9}$. The students made a further error of only working out the square root of 6 and ignored x and obtained a solution. This type of thinking and working, which is a misuse of linearity in non-linear equations, seems to be common to students, as was confirmed by the Tall et al. (2014) study (section 2.2). Students in both studies viewed the equations as calculations and tried to obtain a solution by any means (Stacey & MacGregor, 2000). In addition, some students attempted to solve the same quadratic equation $x^2 + 6x = 9$ by expressing x in terms of x, thus $x=\sqrt{9-6x}$. These students apparently considered the two x's as different, an error that appears to be common to students, as was confirmed by Vaiyavutjamai and Clements' (2006) study (section 2.2).

Whereas these other studies reported on students' performance in the tests after learning, in this current study, the difficulties and errors were identified during the teaching and learning process and the students benefitted from the teachers' explanations, which helped them learn the correct solutions.

Besides students learning the contents of the lessons, they also expressed their views during the interview concerning other ways the LS approach supported their learning. The students said that through learning activities which they discussed in groups, they were able to support each other in the learning of the topic, they improved their communications skills, and gathered courage to ask the teachers questions. They claimed that many students who initially appeared weak in mathematics, and had negative attitudes towards mathematics, changed their attitudes towards mathematics and formed groups to discuss mathematical problems outside the mathematics lesson. These sentiments were also supported by the teachers who felt that the students would improve in mathematics performance (section 7.2.1).

There are several factors that play out in the performance of the students in examination (such as the revision the students made before the test and the type of questions). However, when these students improved in their 2017 KCSE examination results over the 2016 result by about 50 %, the Principal of the school commented that one of the factors might have been the LS approach which I had introduced to the teachers and students. This action by the school administration is adding strength and originality of this study. Therefore, I would confidently encourage teachers, school administrators and other education stake holders in charge of secondary education to apply the approach in the teaching and learning of mathematics.

8.3 Implications of the Current Research

LS mainly focuses on the object of learning, which presents the contents of lessons, as well as opening up the students' capability to apply the learned contents in new situations. This is accomplished through lesson activities which, in the current research, was first discussed in small groups before being presented to the whole class.

8.3.1 Introducing LS in a Kenyan Cultural Context

LS is a relatively new approach to teaching and learning having been introduced as a theory of learning in 2001. It first came to the public's attention in a lecture at the Hong Kong Institute of Education in 2001 by the Swedish psychologist Ference Marton (Marton & Runesson, 2014). The origins of the LS lie in two separate projects, one conducted in Hong Kong in 2000 and later a different project conducted in Sweden in 2003. Since then, many studies have been conducted on LS, either to confirm its effectiveness in a classroom situation or as an application for teacher professional development (Davies & Dunnill, 2008; Pang, 2008). The published works on LS are mainly from the studies conducted in Hong Kong and Sweden, but it has since spread to other countries in Europe and Asia, such as the UK and Singapore (Davies & Dunnill, 2008, Wake, Swan & Foster, 2016; Runesson, 2013). Marton (2015) hoped that by the end of 2015 the completed studies on LS would be about one thousand. He stated thus:

> The LS model spread in Hong Kong and in some other parts of China, and subsequently in Sweden and quite a few other countries. I would guess that by the time this book is published, the number of completed learning studies will approach 1,000 altogether (p. 276).

However, the approach has not been taken up to any extent in Africa. I have obtained very little literature on its spread here, apart from a few papers from South Africa (Pillay, 2013; Pillay & Adler, 2015; Pillay, Ramaisa & Nyungu, 2014). This research will therefore be a contribution to the spread of LS in the African continent, and Kenya in particular, which makes it an original study in the region.

The culture in Kenya, regarding societal issues and classroom teaching and learning approaches, is different from the other cultures that have applied LS approach in their classrooms. Discussing Kenyan culture, I draw from Hofstede's five value dimensions of culture as discussed by Mai (2015) and discussed in Chapter 2 (section 2.6). Mai (2015) explains that universally people accept that, to a certain degree, they are not equal. He calls this Power Distance. In communities where there is high Power Distance, such as in Africa and some Asian communities, students are so submissive to their elders (including teachers), that students stand up when a teacher enters the classroom and may not answer any questions, unless asked to by the teacher.

Li (2017) supports Mai by comparing low power and high-power distance societies in education. Li (2017) writes that in low power distance society "Teachers expect initiative from students in class and quality of learning depends on two-way communication and excellence of students" while in a high-Power Distance society "Teachers take the initiative in class and quality of learning depends on the excellence of the teacher" (p. 3). However, Japan falls under high-Power Distance country, but teachers expect initiative from students in class.

The Kenyan culture, by Mai's description and classroom practice, reflects a high-Power Distance. Students are too shy to engage the teachers one-to-one, especially where the student's opinion may challenge the teacher's opinion on an issue.

I considered this factor when I introduced the group discussion element in the LS approach. From my experience as a teacher, I know that students feel more secure in a group and feel more able to give their opinions, even on challenging issues that may differ from a teacher's opinion. Seemingly this strategy worked, as discussed in the next section. The introduction of group discussion is another original piece that this study added as a contribution to the methodology of a LS approach to teaching and learning.

Students' Experience with the LS approach

The introduction of the LS approach to teaching and learning seemingly opened space for the students' participation during the lessons. Through group discussions, students managed to learn the intended contents, either in their small group discussions or during the whole class discussions. The students progressively improved in their classroom participation and by the time the Third pair of lessons were taught, the students made meaningful decisions by themselves. During the First pair of lessons, discussed in Chapter 4, the students appeared very timid in their group discussions and they spoke so softly that other group members could hardly hear them (section 4.2)

However, during the Second pair of lessons (Chapter 5), students improved in their group discussion and willingly reported their group findings. The students asked the teachers to clarify some issues they did not understand, especially in the second lesson when they did not understand John's explanation on how to complete the square on the left-hand side of the equation $x^2 + 6x = 9$ (section 5.2).

During the Third pair of lessons (Chapter 6), students modified the range of values for the independent variables, different from the ones given by the teachers, when they were completing the table to help them draw quadratic graphs (section 6.2). They explained that the range of values given by the teachers could not give them accurate graphs. Their arguments were, of course, valid and showed that the students were not only active participants but also critical, which was seen as an advantage of applying LS approach to teaching and learning f mathematics over a traditional approach.

The students' actions showed that the LS approach had allowed them to discuss lesson activities in groups during learning, helped them improve on the usual classroom culture, and allowed them to become active participants who could challenge the teachers' decision, thereby enabling better learning as Stigler and Hiebert (1999) report on a Japanese classroom. The students extended discussion groups outside class time in which they helped members perceived to be weak (section 7.2.1). The teachers reported that, because the students saw them working collaboratively during the lessons, they consulted any mathematics teacher for help outside lesson times. This was a positive move by the students which could lead to improved performance in examinations in this topic, which is internationally regarded as a difficult topic for students. From these observations, I can conclude that LS approach minimised the Power Distance between the teachers and the students for the benefit of the students' learning. This finding can be shared with other countries with high Power Distance with a view to improve students' learning of mathematics, especially on topics internationally viewed as topics of concern.

Experience of the Teachers with LS Approach to Teaching and Learning of Mathematics

I explained in Chapter 2 (section 2.6) about teachers being observed during classes in Kenya, and I here reiterate that they are rarely observed (apart from observation during training). Apparently, studies from many of the countries where lesson and learning studies have been conducted do not discuss the issue of observing a teacher in the classroom. Seemingly it is a non-issue in many of those countries. For example, in the UK teachers in high schools seem to frequently work collaboratively in their subjects, as suggested by Wake, Swan & Foster (2016). In their study, Professional Learning through the Collaborative Design of Problem-Solving Lessons, they stated that "Mathematics teachers in secondary schools are frequently organised to work collectively…" (p. 248). A recent study by Bussi, Bertolini and Ramploud (2017) on the cultural transportation of Chinese lesson study to Italy, has revealed that the Italian constitution gives teachers autonomy and freedom in their classrooms, hence it is not easy to observe a teacher. The research team had to convince the teacher, whose two lessons (initial and modified) were observed in two parallel classes, to agree to observation.

The challenges most often discussed in the reviewed literature concerning countries who are starting to practise learning studies include: misunderstanding/lack of understanding of learning study, insufficient content and pedagogical knowledge of teachers, lack of support and resources to conduct high-quality learning study, non-systematic approach to conducting effective learning study, short-sightedness in planning for improvement, and lack of time for professional development (Lewis, 2009; Yoshida, 2012). To a great extent these challenges were also experienced by the teachers in this study, and they will require continuous practice of the approach in order for teachers to address them appropriately.

However, at the end of the exercise, the teachers appeared happy and appreciated the application of LS approach to teach the topic of quadratic expressions and equations. Perhaps they were excited since it was the first time they were sharing with other teachers when planning a lesson. Some of the areas where they felt improvement included: (a) collaborative preparation of the lesson, (b) involvement of students in the learning process through group discussion and (c) the post-lesson reflection session.

The teachers stated that team preparation helped them tap into the others' experience as they discussed the resources to use in the lessons. They acknowledged that the demand of the LS approach requires teachers to be creative and develop or find activities that would engage learners in the discussions. They explained that they had to go beyond the textbook exercises to identify good activities for each of the three pairs of lessons. By doing this, the teachers are suggesting that the approach helped them improve their pedagogical content knowledge, an observation supported by teachers under the VITAL project, evaluated by Elliot and Yu (2008). They said: "Through Learning Study, teachers exchange and prepare lessons together. […] knowing teaching contents thoroughly and with teaching activities" (p. 176).

Also, a teacher in Pang's (2006) study stated that: "I find it very useful to discuss and share with other experienced teachers how to handle a topic, especially some difficult ones. From this, I can have the chance to reflect on my own teaching and learn from others" (p. 39). The study confirms that teachers need one another for effective preparation of the lesson to support students' learning.

The teachers underscored the importance of involving the students in the learning process right from the planning stage. Although, at the beginning, the teachers were sceptical about the inclusion of pre-lesson testing in the LS approach, in the end they said that it was an important aspect as it aided the preparation of the lesson. However, as I discussed in Chapter 7 (section 7.3.1), the issue of pre-lesson testing could be looked at again in order to develop a format that does not discourage students' learning. In addition, the teachers explained that the group discussions helped them understand the students' thinking about certain concepts, some of which were not correct mathematical arguments. This helped them correct the students' errors during the learning process.

As previously mentioned, the group discussions also helped the students improve their communication skills and develop a positive attitude towards mathematics. Due to this positive attitude, the students gained confidence in their ability to solve mathematics problems and improve their consultation with both teachers and other students outside the lesson time. The teachers noted that even the students perceived to be weak in mathematics improved and were active during group discussions; some of them represented their groups in presenting their work during the whole class discussion. Therefore, the LS approach helped the teachers to realise that students are capable of learning from each other when given a chance.

At the start of the study, the two teachers who taught appeared very anxious. Perhaps they were thinking in terms of assessment of their work, since that is what happened the last time they were observed during their training. They ignored students' wrong answers without any comment and they continuously paraphrased questions to help the students obtain the correct answers (section 4.2). Later, the teachers relaxed and made follow-ups to the students' wrong answers, asking students to explain their answers and in the end, they explained to the students why they were wrong (section 5.2). They attributed these observed improvements to the reflection sessions. This means that the approach helped them evaluate each other's teaching and improved the instruction of individual teachers. It also improved teamwork among the teachers, which was useful for the students' learning of the topic. All these teachers' experiences were considered strengths to the application of LS approach to teaching and learning of mathematics, especially to the countries such as Kenya where collaborative teaching is not practised. The good experiences by the teachers in this research can be shared with teachers in other countries where the approach has not been applied.

Adopting LS Approach as an approach to Teaching and Learning Mathematics in Secondary School

Most of the studies reporting research of LS confirm the improved functioning of the lesson or learning studies in classroom situations or the supporting of teachers' professional development or evaluation of a curriculum (Chen & Yang, 2013; Chokshi & Fernandez, 2004; Davies & Dunnill, 2008; Pang, 2006).

In this study I adopted the LS approach to teaching and learning mathematics on the topic of quadratic expressions and equations. I wanted to research the possibility of applying the LS as an approach to teaching mathematics, the same way the Japanese apply lesson study in teaching mathematics through problem-solving. To do this, I observed the lessons within the stipulated times, according to the school timetable, and within the 40-minutes duration per lesson. This was done in order to help me answer the Research Questions 2 and 3.

In many of these countries where LS has been tried before, such as Japan, Sweden, the UK and Italy, the duration of a lesson in secondary school is 60 minutes (Bussi et al., 2017; Runesson, 2013; Wake et al., 2016). However, in Hong Kong where the LS approach has been extensively trialled, a single lesson period is also 40 minutes, like Kenya (Pang, 2008).

In much of the reviewed literature on LS, a few studies were conducted in secondary schools, and most of them were in junior secondary schools. For example, Pang (2008) reports the teaching and learning in secondary three (15 years) in Hong Kong, which falls in junior secondary school. In the VITAL project, the schools comprised primary schools and junior secondary schools. When Elliot and Yu (2008) enquired about the reason for only involving the junior secondary school classes, one of the principals responded: "[…] yes, yes. We can bear the risks" (p. 19*).* The teachers and principals of the schools preferred using LS approach in junior secondary school classes because of the national examinations in the senior secondary school. Even in Japan, lesson study is applied mostly in elementary schools and junior high schools, due to the emphasis on higher grades in examinations (Bush, 2003; Stigler & Hiebert, 1999).

In this research I involved Form 3 students, a pre-candidate class in a system of four years secondary education, with a national examination at the end of the fourth year. As previously mentioned, the topic taught was a topic of concern according to the KNEC (2014) report as well as internationally published research. As it was a Form 3 topic, I specifically planned to address the issue by observing the relevant class. As discussed in section 8.2, the design helped the students to learn the topic of quadratic expressions and equations effectively. This shows that the LS approach can be used as a teaching approach to teach mathematics in secondary schools in all the classes, not necessarily only in junior secondary schools. This is an original report on upper class in secondary schools and a show case to the fact that the LS approach can be applied in all classes with some success.

The teachers were able to adjust their lesson plans to ensure that the planned activities were solved within the lesson duration, especially the second lessons in each pair (sections 4.2.2, 5.2.2 and 6.2.2). In the end, the teaching of the topic took the usual time as provided for in the syllabus. However, the teachers explained that they usually organise extra lessons in the senior classes (Forms 3 and 4) to help them complete the syllabus earlier than the stipulated time in order to allow them time for revision.

However, the approach seemed to have added some value to the students' learning of the topic as the teachers claimed that many students were able to understand the topic better than when they normally "rush" to complete the syllabus and revise the whole topic later. Even the students predicted that they would perform well in the topic, explaining that the discussions of the activities in groups helped them to understand it better. One of the students claimed that in their class they preferred the completing square method when solving quadratic equations because they understood it so well. This might mean that the students would perhaps be able to use the method to solve quadratic equations – this is in contrast to what other studies showed (see earlier discussion in the 'lived object of learning' section 8.2.3).

In conclusion, I would say that the LS approach can be adopted as a teaching approach in schools within the routine lesson durations of those countries. However, the 40-minute lesson duration did constrain the lesson activities, as was also observed by the Pang (2008) study. Perhaps these countries, such as Kenya, that are using the 40-minute lesson can adjust the time if they adopt the LS approach to teaching.

8.3.2 Teaching and Learning the Topic of Quadratic Expressions and Equations

International studies have shown that the topic of quadratic expressions and equations is a challenge to students (Didis & Erbas, 2015; Saglam & Alacaci, 2012; Stacey & MacGregor, 2000 Tall, Nogueira de Lima & Healy, 2014; Vaiyavutjamai & Clements, 2006). These studies highlight students' areas of weakness, such as transforming word problems into quadratic equations and solving quadratic equations – especially using methods other than the quadratic formula. They also cited cases such as students' misuse of linearity in non-linear equations as discussed in sections 2.2 and 5.2. In addition, the ([KNEC], 2014) report lists the topic as one of the topics of concern where students perform poorly in the national examination.

These studies, and the KNEC report, have shown students' weaknesses in performance in the solutions of quadratic equations with little reference to the teaching approach. Vaiyavutjamai and Clements (2006) claim that the few teaching approaches reported used traditional methods, and they state, "There is evidence that traditional […] teaching in mathematics classrooms isolates skills and fails to draw attention to connections" (p. 52). As I mentioned in section 2.2, they conclude their report by asking a question, "Are there realistically feasible forms of teaching that will result in students, and not just high-achieving students, learning quadratic equations, and other mathematics topics, in a relational way?" (p. 73). Although they did not specifically relate their statement to Skemp's relational approach to teaching, they could have implied it since they had referred to Skemp's (1976) study in their introduction.

In this study, I adopted the LS approach to teach quadratic expressions and equations. I noticed that some of the students' weaknesses such as failure to write correct algebraic expressions from statements, misuse of linearity in non-linear equations, considering the two *x*s in a quadratic equation as different, having difficulty in filling in a table of values for a quadratic function with a negative coefficient of x^2, were mostly from pre-requisite knowledge that they ought to have had before learning the current topic. These weaknesses were supported by earlier studies as I have explained (sections 4.2, 5.2, 6.2 and 8.2.3). In these earlier studies the problems were detected from the students' performance during the tests, while in this study the problems were detected during lessons because the students were given an opportunity to explore ways of solving quadratic equations. In the end, students claimed that they understood solutions of quadratic equations by completing the square better than other methods, contrary to other findings as already explained in section 8.2, which is a strength of this study that can be shared internationally to be adapted by other countries.

In conclusion, the approach adopted appears to have helped students to learn the topic of quadratic expressions and equations effectively, as was stated by both the teachers and the students. With continuous practice, the approach can be one of the relational teaching approaches that Vaiyavutjamai and Clements (2006) hoped could help salvage the teaching and learning of the topic.

8.4 Limitations of the Study

Having considered the contributions of the LS approach in my study, together with its implications, I also need to discuss a few limitations of my study. Firstly, the LS approach I adopted for my research was new to the teachers, the students and the school administration. This required ample time to discuss the approach with the parties concerned and it is better to observe the teachers actually in the classroom beforehand. However, an attempt to have that time was curtailed by the national teachers' strike, as was explained in Chapter 3 (section 3.4). Due to the resulting time constraint, the teachers mostly understood the approach during the process of applying it, which reduced the effectiveness of some aspects, such as diagnostic pre-lesson and post-lesson tests, that required insightful understanding before implementation.

Due to the limited time, the dataset I collected appeared relatively less exhaustive because the lessons I observed introduced new subtopics that would have required that I observe subsequent lessons to confirm students understanding of the introduced contents. The subtopics whose introductions were observed included: factorisation of quadratic expressions, solving quadratic equations by the method of completing the square and graphs of quadratic functions. Had I collected data as I planned, which is explained in section 3.4, I would have observed more lessons than I did. Also, the teachers learned more about LS in the process of data collection as opposed to the original plan where they would have had ample time to read more about LS before applying it in class. However, I bridged these gaps by interviewing the teachers and the students to inquire more about their teaching and learning of the subtopics beyond the observed lessons, which helped me link with the observed lessons. Also, the reflection sessions were intensive to help teachers express their feelings about the lessons and the approach in general, which helped them modify the second lessons as was observed in the post-lesson tests of the second lessons.

Also, the application of LS approach comes with some added undertakings such as collaborative planning, observation of other teachers' lessons and post-lesson meetings. These activities add on to the teachers' workload. This was considered in other international studies, such as VITAL project and lesson study in Japan, and the teacher's workload was reduced accordingly (Bush, 2003, Elliot & Yu, 2008). However, in this study the teachers retained their full workload and these added activities to the approach counted as extra load. This forced the teachers to meet after school to plan the lessons, which might have contributed to the choice of same activities for pre-post lesson tests and teaching activities. This is because the teachers might have been exhausted by the time they prepared the lesson.

8.5 Recommendations

Considering the findings and the challenges revealed by this research, my recommendations try to address areas for further research and areas that may require action on policy matters. The findings show that teachers and students embraced the approach, enumerating many successes, as I pointed out in the analyses chapters as well as in this concluding chapter. I conducted the research on one topic only, but the teachers and the students were optimistic that the approach would work with other topics, so long as a few challenges that they cited are addressed.

Some of these challenging areas in which I recommend some work could be done include the development of the diagnostic pre-test and post-test items, and civic awareness of the stakeholders involved with secondary education.

8.5.1 Diagnostic Pre-Lesson and Post-Lesson Assessments

Pre-lesson and post-lesson assessments form one of the components of a LS approach that contributes to an aspect of the lived object of learning. These assessments are administered as either tests or interviews to the students. In the studies that I reviewed, the assessments were given as tests, except in the study by Davies and Dunnill (2008) where both test and interviews were used (Lo, 2012; Marton & Pang, 2006: Pang, 2006; Pang, 2008; Runesson, 2013). In the current study, I used diagnostic pre- and post-lesson tests. Most of these studies did not seem to have interviewed the students, as their methods of data collection did not reflect it, and they did not report students' comments.

However, I gave the teachers and the students in this research the opportunity to express their views about the diagnostic pre-lesson test and post-lesson test and they raised some concerns. During the orientation session, teachers wondered why the students should be given a test on what has not been taught. I explained to them that it is one of the components of a LS approach and their opinion about the same would be useful in strengthening/improving the use of the approach.

The students were especially unhappy about the diagnostic pre-lesson tests, describing them as discouraging or boring, although some found them fun. As I explained in Chapter 7 (section 7.3), these are students from a culture of competitive examinations where tests are graded, and students are ranked each time they sit for a test, such as end of term tests. At the end of each cycle of education, students are assessed through a national examination, which determines their progress to the next cycle. Perhaps the students might have been thinking of the diagnostic pre-lesson tests in that context, hence they were conscious about failing these tests.

Also, the kind of questions asked required the students to give their opinion through explanation, which is in contrast with their usual way of solving mathematics questions. Some of the questions, as I have explained in Chapters 4, 5 and 6, appeared not very clear on the expected answers. This could contribute to the students' assertion that some questions were boring.

In Elliot and Yu's (2008) evaluation report of the VITAL project, some principals and teachers of the schools that participated also raised some concern about the pre- and post-assessment, mostly on time spent on the administration and marking of the tests. They proposed to change the format from a test to oral questions or interviews in their continued application of LS. However, they did not specify whether they would interview all the students or only a sample of them.

In addition, Erickson and Lindberg (2016) in their comparative studies on similarities and differences of the object of learning in two PhD theses from Sweden and Tanzania, raised the issues of validation of the pre- and post-test questionnaire items.

Although these issues have been raised concerning pre-post asssessment in a LS approach, all the studies conclude that it is an important component of the approach. Even the teachers in this study later stated that the pre- and post-tests were useful to them in the preparation and evaluation of the lessons. This is further supported by the teachers of the VITAL project (Elliot & Yu, 2008). Therefore, I recommend further deliberation on the appropriate application of the assessments, especially the diagnostic pre-test, to gather information for the lesson. Continued administration of the tests, without considering the students' plight, may lead to negative conditions for learning, such as a negative attitude towards the subject and a lack of self esteem by individual students.

I suggest that the application of the pre- and post-lesson assessments be adapted differently in different classroom/societal cultures in such a way that they support learning. For example, in Kenya where students are not used to testing before teaching and students are shy to be interviewed individually as I explained in section 3.6.2; I suggest an oral interview with the students in small groups for a diagnostic pre-lesson test. This could be done during students' preparation time prior to the lesson, the time the teachers gave the students the diagnostic pre-lesson tests during this study. The teachers would then collate the groups' responses and utilise them during lesson preparation. The approach can help shorten the time taken to interview all students individually as was suggested by the teachers of the VITAL project.

8.5.2 Cementing the Adoption of LS as a Teaching Approach through Increasing Stakeholder Awareness

The teachers and students in this study talked positively about the LS approach to teaching and learning of mathematics in general, and quadratic expressions and equations in particular. This has been supported by other studies, especially concerning the teachers' comments (Davies & Dunnil, 2008; Pang, 2006; Pang, 2008; Pang & Marton, 2008; Wake et al., 2015). Although many studies did not include students' comments, Elliot and Yu's (2008) report did include students' comments about the LS approach saying that it was very interactive between the students and teachers; and the students were active during the lessons as opposed to their earlier learning process where they remained passive participants.

The reported successes from these studies include students' learning of the content, teachers' professional development through teamwork, students' active involvement in the lessons and the use of students' ideas in planning. Pang (2008) reported students' improvement in learning about slope and the concept of equal share in fraction, as suggested by the performance in the post-test outcomes. One of the teachers from that study said, "After the study, the teachers gained precious experience of learning from one another and reflected on their own practice" (p. 20). Wake et al., (2015) paid tribute to post-lesson discussion and anticipated students' work during planning as they said "[…] the post-lesson discussion exploits the team's comparison of their jointly agreed intentions and actual enactment as a helpful way of promoting joint responsibility… […] and anticipated issues and progression tables are important in facilitating this collaboration and joint responsibility as well as stimulating professional learning" (p. 258). The teachers in Davies and Dunnill (2008) said that the LS approach helped them to change their focus from teaching to the learning of the students, as one of them stated "[…] like I said it made me switch the focus on the learning rather than teaching" (p. 14).

From these responses, and from the data of this research, I suggest that the LS approach can be adopted as a teaching and learning approach in order to promote students' active participation during the lesson and improve their learning. However, as I discussed in section 8.4, the application of the approach adds extra load to the existing teachers' workload. As such, its effective implementation requires the understanding of the school's administration to reduce the teacher's workload. For the Principal of the school to allow this means that another teacher should be employed to take care of the load left by the teacher. Also, adoption of the approach implies a change in the classroom culture, such as active involvement of students in classroom activities, which may require an increase in the lesson period. Therefore, to effectively implement the LS approach, the concerned stakeholders need to be made aware of the approach and its requirements. This can be implemented by policy makers, especially MoEST officers, after reading a report of this research, which will be deposited at the NACOSTI offices in both hard and soft copies as a requirement after every research is conducted.

8.6 Reflections and Way Forward

In this section my reflections for this study will focus on significant issues, situations and events that I experienced in the process of the study. This will be in terms of: the LS approach in teaching an internationally accepted difficult topic of quadratic expressions and equations, experiences of working with teachers and students, and my journey through this study.

This study set out to work with teachers towards overcoming the challenges of teaching a difficult topic of quadratic expressions and equations, which is acknowledged by international literature and Kenya Government as mentioned in Chapters 1 and 2. The study was conducted using a new approach, and in often adverse conditions in a Kenyan cultural context. Despite the challenges experienced during this study, some of which could be attributed to specific aspects of the Kenyan culture, by the end of the study students were able to solve problems involving the topic, an observation supported by the teachers as they compared the class with the previous ones.

Due to the success story of this study, I consider it as a first step of a preliminary study to conduct more LS researches in Kenya. The study has shown that LS has a potential to be built on in conducting more studies. Being the first original study on LS in Kenya, I would like to conduct more learning studies in Kenyan schools in other areas other than quadratic expressions and equations that are also internationally considered topics of concern. This would help answer the teachers' question that "Can it work with all topics?" (section 7.3.1 under *Activities*).

By conducting this study, I realise that the LS approach provides an opportunity for both the teachers and students to improve their teaching and learning respectively. The active involvement of teachers and students through collaboration from preparation to enactment of the lesson, helped the teachers embrace teamwork and helped students develop a positive attitude towards the subject. In addition, the students' interaction with one another as well as learning activities helped them build up their confidence in the ability to learn mathematics, which improves their reasoning, as explained earlier (section 8.3.1). That means, the research has showed that effective involvement of the students harmonises the affective domain objective with the cognitive objectives that improve students' learning achievement. This appears to be one of the original research in Kenya that has effectively harmonised the two domains as specified by Blooms et al. (1956). This means that through regular practice, the approach would be appropriate for upper secondary school classes as well and even in institutions of higher learning. This is because some components of it, such as discussing activities in groups and reporting the outcome, is similar to Problem Based Learning (PBL) practised in some courses, such as medicine, in institutions of higher learning (Hmelo-Silver, 2004; Hmelo-Silver & Barrows, 2006).

Therefore, institutions of higher learning charged with the responsibility of in-servicing already trained teachers such as CEMASTEA can be informed about the good classroom practices brought by LS approach through seminars and conferences with a view to incorporating them in their programmes. Similarly, pre-service teacher training institutions such as Universities and diploma training colleges can also be made aware of these practices and incorporate them in their training programmes so that they become classroom culture once the teachers are trained. In the process, these institutions (in-charge of INSET and pre-service training) can also be informed about the inadequacy to innovative practices noted from the teachers in this study in developing good learning activities, so that they can proactively help the teachers think beyond textbook examples and exercises. This can help teachers develop good learning activities that elicit discussions among learners.

However, as I mentioned in section 8.5.2, the implementation of the approach would require stakeholders' involvement. There are many stakeholders involved with the secondary education in Kenya, which include students, teachers, schools' administration, District Education Officers (DEO), County Directors of Education

(CDE), Ministry of Education Science and Technology (MoEST) officials and Teachers Service Commission (TSC) officials. I will be able to create awareness to these stakeholders starting from the top, that is TSC officials and MoEST officials, coming downwards to the teachers and students. I will start from the top to avoid the situation experienced with the SMASSE project discussed in section 1.3.1. In addition, I propose to introduce LS approach to teaching and learning through a University module/unit in my University. The stakeholders to be informed in this case appear to be fewer than secondary school. I would be able to present a proposal to the Dean of the school during school board meeting. In case it is accepted, the Dean would proceed and present it in the Deans' committee for consideration before the proposal is discussed in the Senate. If approved, then I will teach the module to the mathematics teacher trainees and they would implement the approach during their teaching practice in schools as I collect data to strengthen the study. After training, these teachers will be able to apply the same in schools as a teaching approach once they are employed.

During this study, which happened to be the first study in Kenya that brought the teachers together to collaborate at every stage of the lessons, it was clear that for teachers to come up with practical activities for students to be engaged in, they had to be creative. Teachers appreciated the creative aspect of cutting papers activity that enabled the students to enjoy the teaching and learning of the topic. However, they expressed reservations that it would be challenging for them to apply the same in the teaching and learning of other topics. This is an indication that teachers do not seem to go beyond the text book, which could be blamed on the teaching and learning culture of the country Kenya, which is still very traditional. The issue of creativity was also observed in the way teachers constructed pre-post-tests. One would therefore ask whether this is as a result of the teacher education programmes in our teacher training institutions. Educationists MaCleod (2007) and Akaempong (2003) also cast doubts on the quality of teacher education in Sub Saharan Africa. Although the Kenyan education policy and vision 2030 encourage child-centred approach to teaching and learning, my experience in this study indicates that teachers and by extension the teacher-training institutions may have been slow in implementing it.

In addition to the teachers' inadequate innovative practices to produce quality pre-post-tests and lesson activities, the diagnostic pre-test also posed other challenges to both the teachers and the students. This could be attributed to the fact that teachers in this context are not used to testing students before teaching, and students were also not used to being tested on what had not been taught. I may attribute part of these challenges to the classroom and the assessment culture of the Kenyan education system. Any test given to students count as continuous assessment test (CAT), which is included in the report form at the end of the term. At the same time, mathematics tests are usually set such that students solve the problems through calculation, drawing graphs or proving theorems, different from the way the pre-post-tests were constructed where students were expected to explain their answers in words. That could perhaps explain why students described them as discouraging and boring. The teachers were also uneasy to give such tests because they could make students develop negative attitudes towards the subject.

Regular use of pre-test with clear explanation to students on the objectives of the test, which I propose to be group interview, may improve the students' perception on it. This will also enable teachers to improve on the choice of the tests that will make them effective in the teaching and learning process. The explanation of the pre-test objectives can be well brought out by the lecturers in the teacher-training institutions once they embrace the LS approach.

The concept of collaboration was new to both teachers and students and they acknowledged the accrued benefits of it. Sentiments from both teachers and students indicate that they embraced collaborative work as a process of LS approach. Whilst the teachers argued that it helped them with the preparation and reflections of their lessons as they learnt from each other, the students argued that through collaboration with other students they learned the topic appropriately and assisted each other in the learning process. Continuous practice of such collaboration among teachers will help develop them professionally as effective classroom teachers. Whilst continuous collaboration among students in class may inculcate a child-centred classroom culture with improved learning of mathematics concepts as was observed in the topic of quadratic expressions and equations.

Though this journey of doing PhD has been challenging, there have been certain benefits from the whole process. The experience of this study has refined me as an upcoming international mathematics education researcher, especially in the interpretivism approach of research, though my background in research has been towards positivism. Therefore, this experience has been significant in influencing me into qualitative research which I have appreciated very much. This could be attributed to the interactions I had in the course of this study. Firstly, the personal and professional development (PPD) sessions organised by the faculty of social sciences exposed me to different approaches to qualitative research.

Secondly, the monthly meetings-cum-seminars under research in mathematics education (RME) group organised by lecturers, and incorporated post graduate research (PGR) students in mathematics education helped develop my skills in research considerably. This is as a result of the collaboration that members had as they read and commented on each other's research work. This considerably improved my analysis of data for this study.

Thirdly, during the course of my study, I attended three international conferences. Two of which (WALS conference held in Exeter in the UK in September 2016 and tenth CERME conference held in Dublin in February 2017) improved my study, especially the application of variation theory framework in analysing a lesson. During these conferences I also met with scholars/Professors who had conducted research in LS approach and they helped shape my analysis of the observed lessons using the variation theory framework. No doubt this journey has enabled me to join international scholars of mathematics education researchers.

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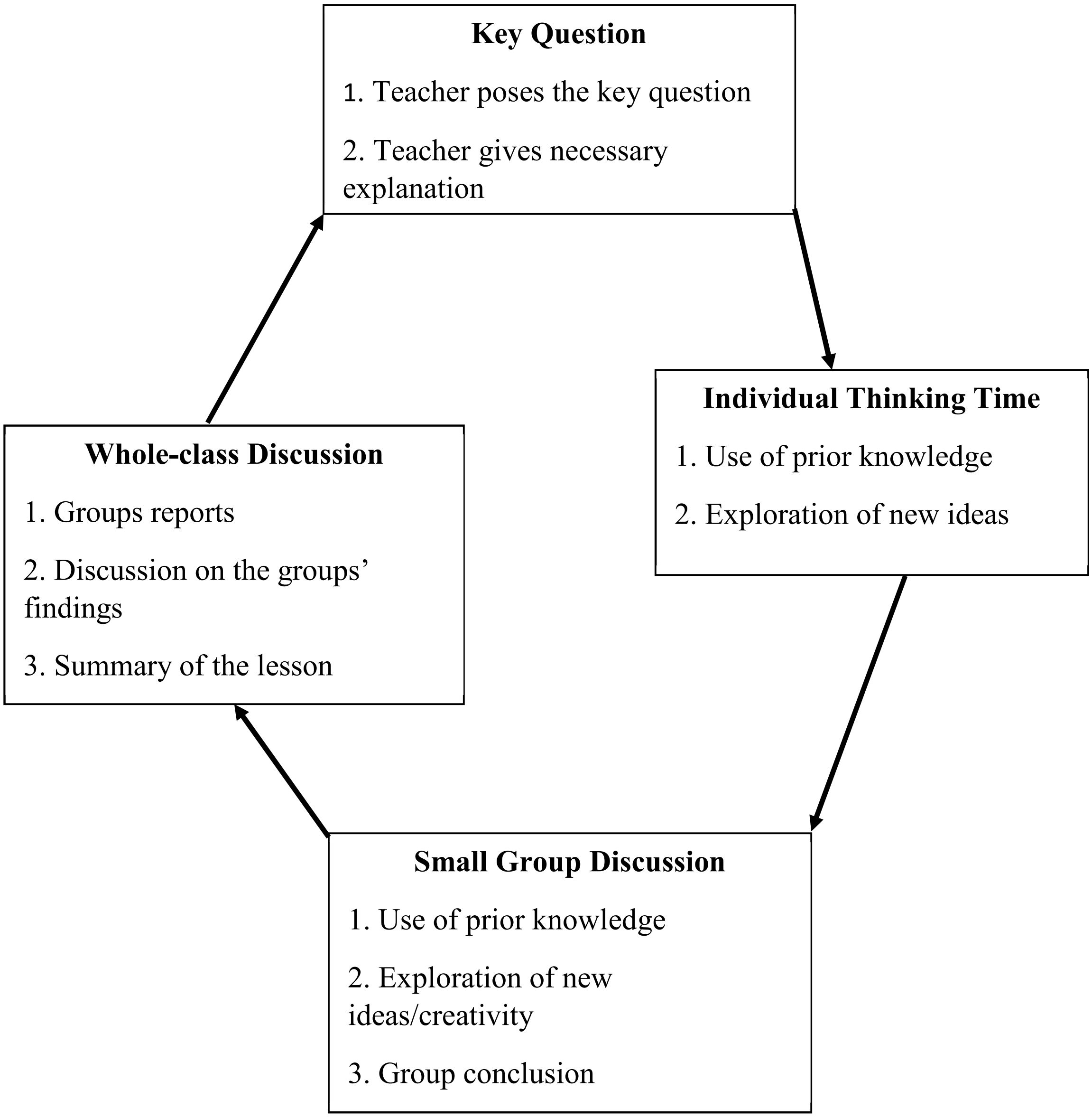
Appendices

Appendix 1 – Orientation Write-up

Conceptual Framework to be adapted.



Conceptual Framework adapted from Yackel and Cobb (1996, p. 460)

1. The teachers will be expected to lead the discussion in explaining the rationale of the topic and also in trying to know the students' entry behaviour in the topic, then pose the key question.
2. The students will be expected to reflect on the problem individually and make

remarks and or conclusions. This should take a short time, probably between 5-10 minutes.

3. The students then converge into small groups to explore possibilities of solving the problem, building on individual thinking and remarks from step two. The teacher moves round to assist groups that seek clarification. It is in this stage that the teacher will note different methods from different groups.
4. The teacher will select groups with different methods to report their findings and explain their method/approach. If there are several groups with the same method/approach, the teacher will ask one group to represent the others. This is done to avoid wastage of time.
5. In few occasions would the individual reflection be passed to small group discussion directly

Lesson Study

Lesson study is a research activity conducted within a classroom set-up through the collaboration of teachers on selected topics considered as topics of concern by education stakeholders, teachers and students (Fernandez, 2002). It is also adopted as a continuous professional development (CPD) of teachers, especially in Japan where newly recruited teachers and teachers who have been in the field for five years undergo in-service teacher education through lesson study approach.

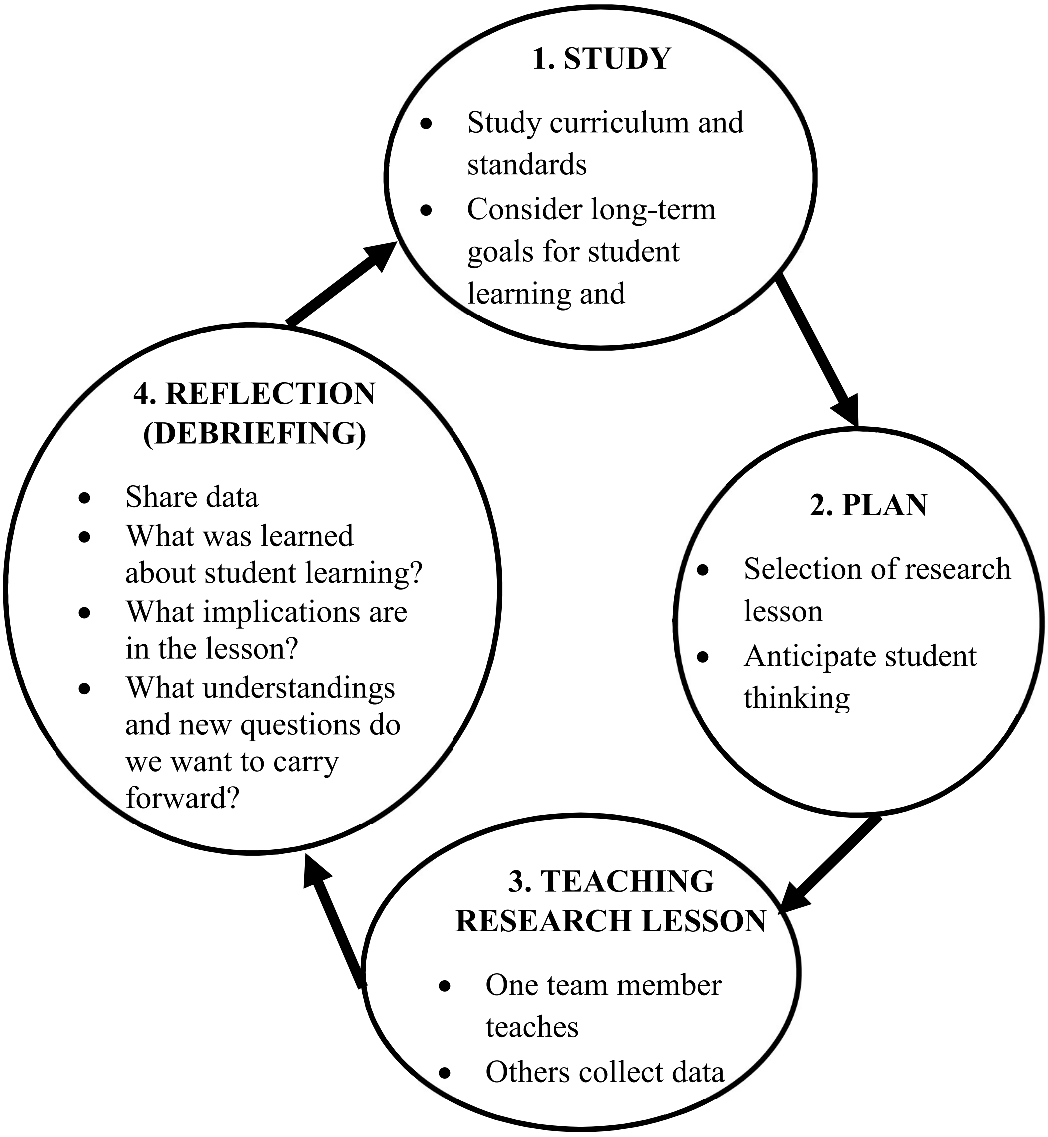
Lesson study has four main parts; Formulation of study goals, research-lesson planning, implementation, and reflection (debriefing) after the lesson as shown in the Figure below.

1. Teachers will gather information from the syllabus, textbooks and internet on the given topic. The information will include the critical feature that will be discerned during learning, the patterns of variation that will be used to discern the feature. I will explain and discuss with the teachers the technical terms used.

2. Teachers will collaboratively plan the lesson, determine the key question and identify the students' anticipated solutions and arguments.

3. One teacher will demonstrate the lesson explaining each step to the rest of the team members and myself.

4. After the demonstration, we will converge to discuss the demonstrated lesson and discuss areas of improvement.



Lesson study cycle, adapted from Lewis, 2009, p. 97.

Learning Study (LS)

LS draws its organization structure from lesson study, where a group of teachers prepares a lesson based on information gathered from the students, and resource materials such as textbooks, syllabus/curriculum, the internet and other research papers. One of the teachers teaches the lesson, others observe the lesson, and research data is collected before they all converge after the lesson for reflection.

In LS approach, the lesson is prepared on the premise of the object of learning, which is dynamic and can change in the process of learning as teachers and students interact (Lo, 2012). The object of learning is established by gathering information about the intended content/topic by consulting with the students on their prior knowledge (pre-test), learning difficulties and their conceptions about the content, and perusing through the syllabus, textbooks, research article(s) and other related resources, as has been mentioned in the second paragraph of lesson study. For the ease of monitoring the learning process, the object of learning is further categorised into: lived object of learning 1 and 2, intended object of learning, and enacted object of learning.

Variation Theory

According to Marton, Runesson and Tsui (2004) "Learning is the process of becoming capable of doing something ('doing' in the wide sense) as a result of having had certain experiences (of doing something or of something happening)" (p. 5). They explain that Variation Theory, which is a theory of learning, proposes that learning is always directed towards an object, which is the content, and could be a skill or a concept referred to as the object of learning (Lo, 2012). The object of learning differs from the educational learning objectives, which from their statements point to the end of the process of learning. They relate to what students can do at the end of the lesson. Learning objectives suggest that the result of learning is predetermined. However, "the object of learning refers to what the students need to learn to achieve the desired learning objectives. So, in a sense, it points to the starting point of the learning journey rather than to the end of the learning process" (Lo, 2012, p. 43).

Get more information from the handout provided (Pang, 2008).

Appendix 2 – Data Collection Instruments

2(a) Classroom Observation Checklist

Introduction	**Comments**	
	Teacher's Activities	Students' Activities
• Explanations on the rationale of the lesson • Enquiry on the pre-requisite knowledge • Posing of the Activity question		
Lesson development	**Comments**	

	Teacher's Activities	Students' Activities
• Individual work • Group work • Group report • Whole class discussion		
Summary and conclusion	**Comments**	
	Teacher's Activities	Students' Activities
• Summary of the concept from whole class discussion • Conclusion of the lesson		

General Remark.

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2(b) Teachers' Interview Schedule

1. What is your comment on the teaching and learning of quadratic expressions and equations using Learning Study design?
2. What part of the design did you like most? Explain your answer(s).
3. How did the use of the design impact on your teaching of the topic?
4. What would you comment in applying the design in teaching other mathematics topics?
5. What is your comment on the use of LS approach to teaching and learning of mathematics in relation to syllabus coverage?
6. What do you think about students' performance on this topic during examination after using this design?
7. What would you predict in performance in mathematics when the design is applied in all the topics? Explain your answer.

2(c) Students' Interview Schedule

1. What are your comments on the teaching and learning of quadratic expressions and equations through small group discussions?
2. What would you say on the use of the strategy in teaching and learning other mathematics topics in general?
3. Do you think there are benefits in learning mathematics using this style of learning? If so, what are the benefits? If not, why?
4. What are your comments on the pre-test and post-test questions by the teachers?
5. What are your comments on the tasks posed by the teacher?
6. What do you think will be your performance in quadratic expressions and equations during the tests?
7. What do you think about your performance in mathematics when group work is used in teaching mathematics?

Appendix 3 – Themes with the Participants Remarks

Global Theme	Organising Themes	Basic Themes	Excerpts
Teachers' and Students' Experiences in Teaching and Learning topic of Quadratic Expressions and Equations in LS approach	Strengths	Student Learning Experiences in a LS Approach	**Student 2**: In the group discussion, you engage in one sum and many people come up with different ideas of calculating the sum and even a different method like in quadratic methods […] such as completing the square and factorisation. So, we collect ideas from different students and that makes students understand. The group agrees on the last answer. **Student 3**: The class discussion after reporting helped us since […] some groups were not able to obtain correct answers in their groups so after reporting, these groups could correct their answers noting where they had gone wrong. **Student 3**: The performance will be high because concerning the methods, students in our class understood completing the square method best and would apply it in solving quadratic equations. **Student 4**: […] group discussion helped some of us. We formed groups of about three members outside class. […] in a case where one is good in mathematics and two members are not sure of the answer, they learned from the member who is good in mathematics instead of waiting to ask the teacher. **Student 2**: […] the approach was beneficial to many students because for

			example, my desk mate is not good in mathematics but when we discussed I see she understands that topic. **Student 2**: It helped everyone in the group to struggle and get the answer. **Student 1**: […] I think group work was beneficial, because some students feared discussion at first but when people shared ideas, some got that confidence to do mathematics and discuss. When you look at the choice of the questions, some questions were not easy to answer as individual but after discussion people were confident to present. **Student 3**: […] the approach improved the communication among the students. For example, one person would start to explain how to work on the Task and others might realise that the approach the person has used is wrong. Another person would come up with a new idea and everybody would discuss. […] at first, some students were only whispering because of fear but later everybody was talking loudly and I realised that our English improved. **Student 1**: I think it was beneficial, since every student felt confident in herself because some feared but when people share ideas some get that confident to do them and discussing. **Student 1**: […] performance will improve since every student felt confident in herself because people shared ideas. Even some who "feared" mathematics gained confidence and were able to do maths and discuss.

			Student 8: I think the performance will be good since after teaching we again used to meet in a group where everybody gave out her opinion. There we were able to understand somethings that we did not know. The practice will improve the performance. **Student 7**: In the group discussion […] we collect ideas from different students and that makes students understand. The performance will improve because we will be able to remember what we discussed in groups. **Student 1**: The lessons were good and we really participated. I can't say the lesson was bad, what I can say is that all the students cooperated and thank you for the organization. **Student 2**: The results will be high because the topic was understood well and since the formulas are also four you can't miss a formula that you understand between the four and… and in that case if you are given as many as more than five questions and even if you know only one formula you can't miss the five one formula or you can answer all. **Student 2**: When you look at the choice of the questions, since some questions were not easy to answer as individual so we go through them in the discussion and we accept how the sums were done. **Student 1**: I think the performance will be high since there are different methods, if you can't do the three, out of the four, you will have to find one that you think you can use.

			John: […] Most of the weak students understood solving quadratic equations by the method of factorisation by using the cuttings. It was so easy for them to factorise using the cuttings as I compared them to the previous group with whom I did not use this method. **John**: […] In the beginning, I thought the approach would be better for average students or students who are ready to speak their mind or who did not fear talking, but as we moved on some of the weak students could talk. […] some of our students improved their communication, and at least they changed their attitude towards mathematics. […] the relationship between some of us with some weak students has really improved. A group of students would come or an individual would come saying, please “mwalimu” (teacher) help me solve this problem. **Peter**: […] earlier, learners only consulted their classroom teachers, but when they realised that the teachers were always preparing the lessons together and they teach the same thing, they now consult any of the teachers of mathematics in the school. **Peter**: […] in terms of performance, it is most likely going to improve. When we gave them a CAT after teaching quadratic expressions and equations, three-quarters of the students scored 10 out of 10. **Peter**: […] when we gave them end of term exams, almost all the students answered the question on solution of

			quadratic equation by graphical method. […] those who attempted the question got at least five out of ten marks. **Dominic:** […] the way we taught factorisation through the cutting of pieces of papers helped the students to comprehend it very fast. In contrast to the way we usually teach it where we force the students to learn that the value at the centre will always stand for the sum while the first one and the last one gives the product; making the students cram it in instead of allowing the students to know how they develop.
		Teachers' Professional Development through LS practice	**John**: From the pre-lesson test, I would know what the students already know because that is what I want to use to get into what they do not know. **John**: […] you know as a teacher there are certain things that you assume and concentrate on what textbooks offer, but when you prepare together you tap into other teachers' experiences. Preparing the lesson plan or discussing the lesson before we do the actual teaching made some of us think beyond what we usually think before going to class and add more on to what we usually do when teaching in class. **John**: After preparing together, where you share your colleagues' ideas you find the teaching very easy. […] like the cuttings, I learnt how to use them with the help of my colleagues and that made my teaching very easy. Even weak students could discuss and present their work. **John**: […] the topic is so challenging, […] using the method that we introduced to the students' learning through activities, many students including

			"weak" students had to think and most of them understood the topic. It was so easy for the students to factorise quadratic expressions and solve equations as I compared them with the previous group where we did not use that method. **Peter**: […] It helped a lot especially for the part of the teacher preparation. You find that when teachers sit together to prepare a lesson, there are certain concepts that one or other teacher may understand or may find a better method of delivering the content. **Peter**: The approach requires that teachers to become flexible and understand the content very well. **Peter**: […] you see LS also encourages team teaching where the class is not owned by one teacher. It may help teachers who had not taught the topic before because teachers will be preparing together. […] that is what we did, John has never taught form three, so, such topics like perfect squares, completing the square method he had not taught, but because we planned together, he found it very easy. **Peter**: […] after teaching, coming together to discuss the learning outcome is important to know whether the strategy the teacher used worked, if it did not work then the other teacher can teach it in a new class after adjusting the lesson. **Dominic**: […] the topic of quadratic was a little bit complex, like when teaching […] the issue of the "sum and product" which is totally new. But the way we brought it through the cutting of the papers and the counting technique meant the students could easily comprehend those things very fast. **Dominic**: […] In fact, LS is one of the best ways of teaching because you watch as teachers teach, and once you are

			through you sit down in something like a conference. There you consult with one another and tell the teacher, this is where the weakness was and you are supposed to do this. As we do that, we are also learning and correcting one another.
	Challenges	Change in Classroom Culture	**Dominic**: […] the only challenge, I do not know if it is applicable in all the topics in maths, because we have only tested it in quadratic and it is applicable. Especially the issue of making those things like the cuttings. **John**: […] if this method is supposed to be implemented then I think there should be a lot of research done on the activities. It is not easy for a teacher to come up with activities in each topic. […] you know, at least there should be activities listed somewhere that you can use to make work easy. **Peter**: […] workload may limit the activities. […] however, the learners would wish that one continues to teach using activities once he/she introduces activities it. If he/she fails to use activities in some lessons […] students may feel the lesson is not enjoyable and they may dose off. I think […] the teacher needs to be creative and always look for things that will excite learners in every lesson and that is difficult. **Peter:** Eh… unless you if we look at… mostly if you look at form one topics, these topics are developed from primary schools most of the topics, so the strategy will work well, because then you will know what they have carried from primary. Ah, form two topics is now where we are introducing new mathematics to the learners and… if you look at a topic that specifically it may not work well is the use of logarithms where we are reading the table, the students have no idea how to read the table and all

			there is, I think the use of logarithms generally the topics that require the use of tables such as square roots, of course the students may start by getting the square roots but obtaining the square root from the table, obtaining the cube roots from the table, the use of tables that is where you may find the challenge when you use the strategy. But, the rest of the topics I think the strategy will work well on the rest of the topics. **John**: I would ah… maybe the challenge would be getting activities to students because at times some topics, most of them topics you would only use minds-on activities and getting hands-on activities at times is very difficult unless you do a deeper research on, extensive research for you to get an activity. You know as a teacher, as a young teacher you also need to get time to familiarise yourself to the activity and to see the challenges that you may have when you are teaching students using that activity. It is actually, I would prefer using same activity in teaching other topics because it will make my work easier with students. **Student 1**: The pre-lesson and post-lesson tests were discouraging because we were given questions we had not learnt. […] after the lesson, in some cases, still only some students could answer the questions. **Student 4**: […] before teaching you could not be happy about the pre-test questions, some students were complaining saying that they were boring. After the teaching, […] you could at least be encouraged to answer the questions. **Student 3**: […] they were fun. Before the lesson we were not using the right concepts to answer the questions but

			after the lesson, you know the concept. So, if you compare the first answer that you gave and the last answer that you have given they were just fun. **Student 1**: Students fear trigonometric ratios and I think if you apply it in that topic many students can benefit. **Student 4**: I think they were good since when we were given the questionnaires before we were taught it would help a student to predict what would be taught in the next lesson, may be to understand the concept very well when it is taught yes when it is taught. **Student 5**: I also think they were good since they help the students to do them by, you can use your own ability, you can try them and after the lesson you can come and look at what you had done before. **Student 5**: With me I think the performance will improve in some cases but take a case where the question is on application of quadratic expression, on that many students did not understood and we like to tell our teachers to repeat the application of quadratic equations because some students complain that they don't know how to handle such a question. **Student 8**: I think the performance will be good since may be you were taught and again went back to your groups and everybody gave out his or her opinion, there you may capture somethings that you did not know, but the performance will improve may be exam is brought and you are asked to solve a quadratic equation using completing square method and maybe you do not know that method. So it would be easy if it is a quadratic expression. In a quadratic equation we were told to use any method.

		Lesson Duration	**Student 2**: […] the group discussion is good but during learning, the time is not enough. Take a case of 40 minutes for a lesson then the teacher explains a certain sum on the board then […] as you are discussing the bell is rung before you complete your group discussion. **Student 3**: […] may be the discussion can be short and… if some people have not understood, or the groups have not understood the activity then the teacher can elaborate it on the board for the whole class to discuss.
		National Examination Pressure and Syllabus Coverage	**Peter**: […] you see, what happens is that teachers would want to rush and finish the syllabus so they do not pay attention to the stipulated time for syllabus coverage. […] I just strain to cover syllabus so that I finish early and have time for revision. **Peter**: […] this method may slow down the syllabus coverage because of the activities involved. […] like when we were dealing with factorisation of quadratic expressions, […] you see it took time and in one lesson we could only answer two questions. **Peter**: […] however much it slows down the syllabus coverage I think what has been covered is understood better than if we cover the syllabus faster and learners do not understand well or only a few understands **Dominic**: […] this time round we were covering three different topics because of the teachers' strike and your programme. […] this was within the stipulated time but we had to get time to cover others. **Dominic**: […] but the only problem, I do not know if it is applicable in almost all the topics in maths, […] but maybe we have challenging topics where teachers frame words differently, which takes time to be comprehended by students.

			John: […] I think there are some topics where you cannot develop practical (hands-on) activities or good activities and this can slow syllabus coverage. **Dominic**: […] with continuous application of the strategy with proper preparation, I believe the syllabus can be covered fast
		Teachers' Shortage and Workload	**John**: […] if workload is large, at times preparation may be a problem for the activities, especially hands-on activities. […] in hands-on you need time to prepare and do it practically before you give the students so it will take time. **Peter**: […] time management, you will find that sometimes it is difficult to find that a teacher is free and the other teachers are also free to sit down to discuss or prepare. **Peter:** […] time management, you will find that sometimes it is difficult to find that a teacher is free and the other teachers are also free to sit down to discuss or prepare. **Dominic:** The only problem that might arise is the issue of teachers' shortage, if you do not have enough teachers, planning becomes very difficult. […] or may be if you have a school that has only one trained teacher and some teachers who just completed form four and have not gone for any further training, to some extent they might not bring out the concept the way it is expected

Appendix 4 – Ethics Approval Documents

4(a) UEA Ethics Approval

EDU ETHICS APPLICATION FEEDBACK 2014-2015

APPLICANT DETAILS	
Name:	Fredrick Odindo
School:	EDU
Current Status:	**PG student**
UEA Email address:	F.Odindo@uea.ac.uk

EDU Recommendation	✓
Approved, data collection can begin	✓
Approved with minor revisions data collection can begin (see feedback below)	
Not Approved, resubmission required (see feedback below)	

EDU REC feedback to applicant: Chair review date …15.10.15…………………….

Comments: thank you for providing all of the relevant documentation required and you are free to start your data collection

Ethical approval has now been given:	Kate Russell

Signed: Course Ethics Coordinator (PGT courses)
Signed: EDU Chair, Research Ethics Committee

4(b) NACOSTI Research Approval

(i) *Research Authorization Letter*

NATIONAL COMMISSION FOR SCIENCE, TECHNOLOGY AND INNOVATION

Telephone: +254-20-2213471, 2241349, 310571, 2219420
Fax: +254-20-318245, 318249
Email: secretary@nacosti.go.ke
Website: www.nacosti.go.ke
When replying please quote

9th Floor, Utalii House
Uhuru Highway
P.O. Box 30623-00100
NAIROBI-KENYA

Ref. No.

Date:

21st August, 2015

NACOSTI/P/15/3606/7599

Fredrick Osena Odindo
University of East Anglia
UNITED KINGDOM.

RE: RESEARCH AUTHORIZATION

Following your application for authority to carry out research on ***"Perceptions of teachers and students on the problem solving strategy to teaching and learning of mathematics in Kenyan Secondary Schools: A case of lesson study approach,"*** I am pleased to inform you that you have been authorized to undertake research in **Siaya County** for a period ending **30th September, 2017.**

You are advised to report to **the County Commissioner and the County Director of Education, Siaya County** before embarking on the research project.

On completion of the research, you are expected to submit **two hard copies and one soft copy in pdf** of the research report/thesis to our office.

SAID HUSSEIN
FOR: DIRECTOR-GENERAL/CEO

Copy to:

The County Commissioner
Siaya County.

The County Director of Education
Siaya County.

NACOSTI

(ii) ***Research Clearance Permit***

REPUBLIC OF KENYA

NACOSTI

National Commission for Science, Technology and Innovation

RESEARCH CLEARANCE PERMIT

Serial No. A 6273

CONDITIONS: see back page

THIS IS TO CERTIFY THAT:
MR. FREDRICK OSENA ODINDO
of UNIVERSITY OF EAST ANGLIA, 0-40601 Bondo, has been permitted to conduct research in *Siaya County*

on the topic: *PERCEPTIONS OF TEACHERS AND STUDENTS ON THE PROBLEM SOLVING STRATEGY TO TEACHING AND LEARNING OF MATHEMATICS IN KENYAN SECONDARY SCHOOLS: A CASE OF LESSON STUDY APPROACH*

for the period ending:
30th September,2017

Applicant's Signature

Permit No : NACOSTI/P/15/3606/7599
Date Of Issue : 21st August,2015
Fee Recieved :Ksh. 2000

Director General
National Commission for Science, Technology & Innovation

4(c) Research Authorization by Siaya CDE

REPUBLIC OF KENYA
MINISTRY OF EDUCATION, SCIENCE & TECHNOLOGY
State Department of Education

Telephone:
Fax:

COUNTY DIRECTOR OF EDUCATION
SIAYA COUNTY
P.O. BOX 564
SIAYA

When replying please quote

Ref: SCA/10/VOL I

Thursday, August 27, 2015

TO WHOM IT MAY CONCERN

RE: RESEARCH AUTHORIZATION – FREDRICK OSENA ODINDO

The above mentioned has been mandated to carry out research in Siaya County vide the National Commission for Science, Technology and Innovation's authority Ref. No. NACOSTI/P/15/3606/7599 dated 21st August, 2015.

The research title is on *"**Perceptions of teachers and students on the problem solving strategy to teaching and learning of mathematics in Kenyan Secondary Schools.**"*

Kindly accord him the necessary assistance.

SIAYA COUNTY
27 AUG 2015
Box 199-40600

NERREAH OLICK
COUNTY DIRECTOR OF EDUCATION
SIAYA COUNTY