Singular Graphs

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Abstract

Let Γ be a simple graph on a finite vertex set V and let A be its adjacency matrix. Then Γ is said to be singular if and only if 0 is an eigenvalue of A. The nullity (singularity) of Γ, denoted by $\text{null}(\Gamma)$, is the algebraic multiplicity of the eigenvalue 0 in the spectrum of Γ. In 1957, Collatz and Sinogowitz [57] posed the problem of characterizing singular graphs. Singular graphs have important applications in mathematics and science. In chemistry the importance of singular graphs lies in the fact that a singular molecular graph, with vertices formed by atoms, edges corresponding to bonds between the atoms in the molecule, often is associated to compounds that are more reactive or unstable. By this reason, the chemists have a great interest in this problem. The general problem of characterising singular graphs is easy to state but it seems too difficult at this time. In this work, we investigate this problem for graphs in general and graphs with a vertex transitive group G of automorphisms. In some cases we determine the nullity of such graphs. We characterize singular Cayley graphs over cyclic groups. We show that vertex transitive graphs where $|V|$ is prime are non-singular. The relationship between the irreducible representations of G and the eigenspaces of Γ is studied.

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Introduction

Let Γ be a graph on the finite vertex set V of size n, and let A be its adjacency matrix. Then Γ is *singular* if A is singular. The *spectrum* of Γ consists of all eigenvalues $\lambda_1, ..., \lambda_n$ of A and so Γ is singular if and only if 0 belongs to the spectrum of Γ. The *nullity* (*singularity*) of Γ, denoted by null(Γ), is the algebraic multiplicity of the eigenvalue 0 in the spectrum of Γ. Through this thesis all graphs are undirected, simple and finite.

There are applications of graph spectra and singularity in the representation theory of permutation groups. In physics and chemistry, nullity is important for the study of a molecular graph stability, see Section 5 of Chapter 3. The nullity of a graph is also important in mathematics generally since it is relevant for the rank of the adjacency matrix.

In this thesis, we investigate singular graphs. Collatz and Sinogowitz [57] posed the problem of characterizing all graphs with zero nullity. These are the *non-singular graphs*. In particular, this research could begin with examining the rich literature on graph spectra. We refer for instance to [11], [16] and [37]. Graph theorists started to investigate this problem in special classes of graphs which include trees, cycles, paths, line graphs of a tree, bipartite graphs, circulant graphs, graphs with cut-points, directed graphs, graphs with one cycle, graphs with exactly two cycles and others. We give a short survey on graph singularity at the end of this introduction.

Our aim is to develop some general theory on graph nullity. We develop an algebraic language for this problem by representing the graph and the adjacency relation by a vector space $\mathbb{C}V$ and an adjacency map

$$\alpha : \mathbb{C}V \to \mathbb{C}V.$$

The nullity of Γ essentially concerns the nullity of α.

Some of our main results are of the general type. For instance, in Theorem 3.5.9 we show that singular graphs can be characterized by a Balance Condition on the vertex set. We also have some general comments on the nullity of α for bipartite graphs. We investigate the nullity of α for $L(\Gamma)$ (line graph) and $\overline{\Gamma}$ (graph complement), see Corollary 3.5.5 and Corollary 3.5.7 respectively. We study the nullity of a sub-graph of Γ in the terms of nullity of Γ, see Proposition 3.5.8.

As before Γ is a graph with finite vertex set V. A permutation g of V is an *automorphism* of Γ if the pair of vertices (u^g, v^g) forms an edge in Γ if and only if the pair of vertices (u, v) forms an edge in Γ. Here u^g is the image of u under the action of g. The set of all automorphisms of Γ forms a subgroup of the symmetric group on V, called the *automorphism group* of the graph Γ. It is denoted by $Aut(\Gamma)$, for more details see Section 3 of Chapter 2. A graph Γ is *vertex transitive* if $Aut(\Gamma)$ acts transitively on V.

A particular class of vertex transitive graphs is the so-called Cayley graphs. These are denoted by $Cay(G, H)$ where G is an arbitrary group and H is a connecting set in G; see the definition in Section 1 of Chapter 4. Cayley graphs are vertex transitive by construction. But the converse is not true, see [8] as a reference. For example, the Petersen graph is vertex transitive but not a Cayley graph. In fact, the Petersen graph is the smallest vertex transitive graph which is not a Cayley graph, see [45] as a reference. The complete graph K_n is an example of a vertex transitive graph with automorphism group $Sym(n)$, the symmetric group of degree n. Another example of a vertex transitive graph is the cycle graph of order n. Its automorphism group is the dihedral group D_n of order $2n$.

The main body of our work concerns graphs which have a transitive group G of automorphisms where G is a subgroup of the automorphism group of a graph. In this situation the singularity question can be discussed in terms of the representation theory of G. We first concentrate on the Cayley graphs and the results for Cayley graphs include the fact that $Cay(G, H)$ is singular if H is a union of right cosets of a subgroup $K \neq \{1_G\}$, see Corollary 4.2.6.

The next important case concerns the situation where H is a normal connecting set, that is $Hg = gH$ for all $g \in G$. Here the singularity problem can be discussed in terms of the irreducible characters of G. In fact, we have several eigenvalue formula in terms of characters, see Theorem 4.1.6. We also have such an eigenvalue formula when Γ is a vertex transitive graph, see Theorem 4.4.1.

The material in the thesis is organised as follows: In Chapter 2 we give a brief introduction to the methods and concepts from linear algebra that are relevant for us. Here we also discuss the basic ideas from group and graph theory which are used. Indeed these topics are considered as a regular part of the standard literature on Graph Spectra.

In Chapter 3 we study graphs and their linear maps. We also discuss the representation theory of finite groups that is related to our problem. We introduce the projection maps onto the eigenspaces of α and discuss their properties. We use these properties to study the spectral decomposition of the elements of $\mathbb{C}V$. We determine conditions for a graph in general to be singular. The main goal of this Chapter is to go through the ideas and notations of the graph spectra and we also include some results in this area to be used later, mostly in Chapter 4.

In Chapter 4 we investigate the spectrum of Cayley graphs and that of vertex transitive graphs. Our aims in this Chapter is to extend the results of Lovász [43], Zieschang [59], M.Ram [47], Diaconis [19] and Babai [6] to find out sufficient conditions for a vertex transitive graph to be singular. We list several results on the singularity of such graphs, see Theorem 4.2.2, Theorem 4.2.3 and Theorem 4.3.1. We also specialize to Cayley graphs on abelian groups. We investigate the singularity of such graphs. We show that vertex transitive graphs with $|V|$ is a prime number are non-singular, see Theorem 4.2.16.

All results from the literature are fully referenced. Where no reference is given the result is new to the best our knowledge. Occasionally we include new proofs that are based on our techniques.

We conclude this introduction with survey about the existing literature on singular graphs. Sookyang, Arworn and Wojtylak [55] characterize non-singular cycles, paths and trees. They proved that a cycle on n vertices is non-singular if and only if n is not divided by 4. Similarly, a path on n vertices is non-singular if and only if n is even and a tree on n vertices is non-singular if and only if n is even and contains a sesquivalent spanning subgraph (a sesquivalent graph is a simple graph whose components are single edges or cycles). Fiorini, Gutman and Sciriha [21] discuss the maximum nullity of trees. They proved that the maximum nullity of a tree on n vertices with maximum degree Δ is $((n-2)\cdot\lceil\frac{n-1}{\Delta}\rceil)$ where $\lceil x\rceil$ denote the smallest integer $a \geq x$. Furthermore, they showed how trees with such maximum nullity can be constructed. Also Li and Chang [42] characterize trees with maximum nullity. Cvetković, Dragoš and Gutman [17] first found that the nullity of a tree can be given in explicit form in the terms of the matching number of the tree. The nullity of the line graph of a tree is studied in [50] and [44]. In both papers they proved that the multiplicity of the eigenvalue 0 in such graphs is at most 1 and showed that every tree whose line graph is singular has an even order (here order means the number of the vertices of a graph). Ashraf and Bamdad [3] determine the possible order for a graph to be non-singular.

Fan and Qian [20] characterize bipartite graphs on n vertices with nullity $n-4$ and regular bipartite graphs on n vertices with nullity $n-6$. Moreover, they showed that the nullity set of bipartite graphs on n vertices is $\{n-2l : l = 0, 1, ..., \lceil \frac{n}{2} \rceil\}$. Bapat [7] showed that the nullity of the line graph of a bipartite graph is at most 1 when the bipartite graph has an odd number of spanning trees and also proved that the bipartite graph with this property has an even number of vertices. Leonor [2] determine necessary and sufficient conditions for two classes of circulant graphs which are C_n^r (r^{th}-power graph on n vertices) and $C(2n, r)$ (the r^{th}-power graph of the cycle graph on $2n$ vertices) to be non-singular. Lal and Reddy [38] give sufficient conditions for a few classes of known circulant graphs and/or digraphs to be singular.

Gong and Xu [25] investigate the nullity of a graph with cut-points. Moreover, they proved that the nullity of the line graph of a connected graphs is at most $l+1$ when the graph has l induced cycles. Cheng and Liu [14] characterize graphs on n vertices with nullity $n-2$ and $n-3$. Sciriha [51] determine necessary and sufficient conditions for a graph to be singular in terms of admissible induced sub-graphs. Chang, Huang and Yeh [12] and [13] characterize graphs of order n with nullity $n-4$ and $n-5$ respectively. Siemons and Zalesski [54] discuss singular Cayley graphs over finite simple groups and alternating groups in particular. They suggested some approaches for constructing singular Cayley graphs for finite simple groups.

Hu, Xuezhong and Liu [31] give the nullity set of bi-cyclic graphs on n vertices for $n \geq 6$, that is $\{0, 1, 2, ..., n-4\}$ and characterise such graphs with maximum nullity. Li, Chang and Shiu [41] also give the nullity set of two kinds of bi-cyclic graphs and characterize such graphs with maximum nullity. Xuezhong and Liu [58] show that the nullity set $\{0, 1, 2, ..., n-4\}$ for uni-cyclic graphs on n vertices for $n \geq 5$. Moreover, they characterize the uni-cyclic graphs with maximum nullity. Guo, Yan and Yeh [28] compute the nullity of uni-cyclic graphs in terms the matching number. Moreover, they determine conditions for uni-cyclic graphs to be non-singular and characterize uni-cyclic graphs with maximum nullity. Gong, Fan and Yin [24] express the nullity of graphs with pendant trees in terms of its sub-graphs. Furthermore, they characterize uni-cyclic graphs with a given nullity. Nath and Sarma [48] determine sufficient and necessary conditions for acyclic and uni-cyclic graphs to be singular and they showed that the characterization of such graphs can be used to construct a basis of the null space.

Some graph theorists have a great interest in calculating the determinant of the adjacency matrix of a graph. Clearly, the determinant of this matrix is 0 if and only if the graph is singular. For instance, Abdollahi [1] discuss the set of all determinants of the adjacency matrix of a graph on at most 11 vertices. Moreover, he evaluate the determinants of the adjacency matrix of a graph with exactly two cycles. Shengbiao [53] compute the determinants of the adjacency matrix of a connected graph with exactly one cycle. Harary [29] introduce a procedure for computing the determinant of the adjacency matrix of a graph in terms of spanning sub-graphs.

2

Preliminaries

In this chapter, we introduce the basic notations and definitions that are needed in this thesis. We deal with vector space, representation theory of a group and graphs. Note all graphs and groups in this work are finite.

2.1 Vector Space and Linear Maps

In this section we give the basic ideas of linear algebra. All these definitions have been taken from [5]. Let $\mathbb{F}$ be a field of characteristic 0 such as $\mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$. Let W and U be two finite-dimensional vector spaces over $\mathbb{F}$ with non-degenerate inner products $\langle , \rangle$. The set of all *linear maps* from W to U is denoted by $Hom(W, U)$. Suppose $\vartheta \in Hom(W, U)$. If there is some $\vartheta^* \in Hom(U, W)$ for which

$$\langle \vartheta(w), u \rangle = \langle w, \vartheta^*(u) \rangle \qquad (2.1.1)$$

for all $w \in W$ and $u \in U$ then ϑ^* is said to be the unique *adjoint* of ϑ. A linear map ϑ is *symmetric* if and only if $W = U$ and $\vartheta = \vartheta^*$.

Let $\varphi \in Hom(W, W)$. Let $\overline{W}$ be a subset of W. If $\overline{W}$ is itself a vector space then $\overline{W}$ is said to be a *subspace* of W. We shall say that $\overline{W}$ is an *invariant subspace* of φ if and only if $\varphi(\overline{w}) \in \overline{W}$ for all $\overline{w} \in \overline{W}$. In this case $\varphi\mid_{\overline{W}}: \overline{W} \longrightarrow \overline{W}$ is the *restriction* of φ to the subspace $\overline{W}$. The *kernel* of φ is defined as $ker(\varphi) = \{w \in W | \varphi(w) = 0\}$. The dimension of $\varphi(W)$ is called the *rank* of φ.

A scalar $\lambda \in \mathbb{F}$ is called an *eigenvalue* of φ if there exists a non-zero vector $w \in W$ such that

$$\varphi(w) = \lambda w.$$

This is equivalent to $(\varphi - \lambda I)w = 0$ and the vector w is called an *eigenvector* of φ corresponding to the eigenvalue λ. The *characteristic polynomial* of φ is the polynomial

$$C_\varphi(x) = det(M_\varphi - xI).$$

Here I is the identity matrix and M_φ is the matrix associated to φ for some basis. It is clear that the roots of the characteristic polynomial of φ equal the eigenvalues of φ. The *algebraic multiplicity* of the eigenvalue λ is the largest positive integer n for which $(x - \lambda)^n$ is a factor of the characteristic polynomial. The *eigenspace* of φ corresponding to eigenvalue λ is the vector space $E_\lambda = ker((M_\varphi - \lambda I)$. Hence, the *eigenspace* E_λ is the span of all eigenvectors corresponding to the eigenvalue λ. Thus from the above each eigenspace E_λ is a subspace of W. The *geometric multiplicity* of eigenvalue λ is the dimension of its eigenspace. The *spectrum* of φ is the set of its eigenvalues together with their algebraic multiplicities. In general, over $\mathbb{C}$ the algebraic multiplicity is bigger or equal to the geometric multiplicity, and they are the same if and only if the matrix can be diagonalized. Furthermore, this property hold for all matrices in this thesis, see Lemma 2.1.4.

Lemma 2.1.1. *[33] The determinant of a direct sum matrix is the product of the determinant of the constituent matrices.*

Theorem 2.1.2. *[30] Let A be an $m \times n$ matrix. Let B is a matrix created by deleting rows and / or columns of A, then $rank(B) \leq rank(A)$.*

Theorem 2.1.3. *[30] Let A be an $m \times n$ matrix. Let P, Q be invertible matrices of size $m \times m$ and $n \times n$ respectively. Then* *(i)* $rank(AQ) = rank(A)$

(ii) $rank(PA) = rank(A)$

(iii) $rank(A) = rank(A^t)$.

Lemma 2.1.4. *[23] Let M be a real symmetric $n \times n$ matrix. Then*

(i) All eigenvalues are real.

(ii) The eigenvectors of distinct eigenvalues are orthogonal.

(iii) There are matrices L and D such that $LL^T = L^T L = I_n$ and $LML^T = D$, where D is the diagonal matrix of eigenvalues of M. In particular, we have that the algebraic multiplicity and the geometric multiplicity of an eigenvalue of M are equal.

2.2 Representation Theory

We give some basic definitions and theorems in representation theory and group theory. All groups considered are finite. Our notations and definitions of groups and their representation have been taken from [32]. Let G be a finite group. We use 1_G to denote the *identity* element of G. We use $\{1_G\}$ to denote the *trivial* subgroup.

For all $x, y \in G$ we say that x is *conjugate* to y in G if $y = g^{-1}xg$ for some $g \in G$. The set of all elements conjugate to x in G is $Cl(x) = \{g^{-1}xg : g \in G\}$ and is called the *conjugacy class* of x in G.

A *generating set* of G is a subset of G so that every element of the group can be expressed as the combination (under the group operation) of finitely many elements of the subset. If S is a subset of G thus the *subgroup* $\langle S \rangle$ *generated* by S is the smallest subgroup of G containing every element of S; equivalently $\langle S \rangle$ is the subgroup of all elements of G that can be expressed as a finite product of the elements of S. If $G = \langle S \rangle$ then we say S *generates* G and the elements in S are called *group generators*. Let $g \in G$ and r be the least positive integer such that $g^r = 1$. Then r is the number of the elements in $\langle g \rangle$, here $\langle g \rangle = \{1, g, g^2, \cdots, g^{r-1}\}$. We denote to the *order* of the element g by $ord(g)$. If $G = \langle g \rangle$ for some $g \in G$ then we call G a *cyclic group*.

Let X be a finite set. Then the map $\phi : X \times G \to X$ is an *action,* and we say G *acts* on X, if $\phi(x, g) \in X$ for all $g \in G, x \in X$ and the following conditions hold for all $x \in X$:

(1) $\phi(x, 1_G) = x$

(2) $\phi(x, gh) = \phi(\phi(x, g), h)$

for all $g, h \in G$. Sometimes, when it is clear what action we have we write $x^g = xg = \phi(x, g)$ for all $x \in X$ and $g \in G$. A one-to-one mapping from a finite set onto itself is called a *permutation*. A *permutation group* is a group whose elements are certain permutations acting on the same finite set called the *object set*. Note the group operation is the composition of mappings. Let X be the object set and G be the permutation group. Then $|G|$ is the *order* of the group and $|X|$ is the *degree* of the group. The set of all permutations of X is denoted by $Sym(X)$ or $Sym(n)$ if $|X| = n$. Here $Sym(X)$ is the *symmetric group* of degree n where its elements are the set of all permutations on n symbols. Therefore a permutation group of the object set X is a subgroup of $Sym(X)$.

The *orbit* of an element $x \in X$ is defined as

$$x^G = \{x^g \mid g \in G\}.$$

Then $G_x = \{g \in G \mid x^g = x\}$ is called the *stabilizer* of x. A group action $X \times G \to X$ is *transitive* if it is possesses only a single orbit. In other words, for every $x, y \in X$ there is $g \in G$ such that $x^g = y$. A group G acts *semi-regularly* on X if $G_x = 1_G$ for all $x \in X$. A group G is *regular* if it is semiregular and transitive. An *automorphism* of a group G is a bijective map σ from G to itself that satisfies the following condition: $\sigma(gh) = \sigma(g)\sigma(h)$ for all $g, h \in G$. We mean by this σ is a group homomorphism.

Theorem 2.2.1. *(Orbit-Stabilizer Theorem)[23, Lemma 2.2.2] Let G be a permutation group acting on a set V and let u be a point in V. Then $|G| = |u^G||G_u|$.*

Let $\mathbb{C}$ be the field of complex numbers. Let G be a finite group and W be a finite dimensional vector space over $\mathbb{C}$. A *representation* of G over $\mathbb{C}$ is a group homomorphism ρ from G to $GL(W)$. Here $GL(W)$ is the group of all bijective linear maps $\beta : W \to W$. The *degree* of the representation ρ is the dimension of the vector space W. We also say that ρ is a representation of degree n over $\mathbb{C}$. So if ρ is a map from G to $GL(W)$ then ρ is a representation if and only if

$$\rho(gh) = \rho(g)\rho(h)$$

for all $g, h \in G$. A representation ρ of a group G is called *faithful* if ρ is a one to one function on G.

We denote the *group algebra* of G over $\mathbb{C}$ by $\mathbb{C}G$. Then $\mathbb{C}G$ is the vector space over $\mathbb{C}$ with basis G and multiplication defined by extending the group multiplication linearly. Thus $\mathbb{C}G$ is the set of all formal sums $f = \sum_{g \in G} c_g g$ where $c_g \in \mathbb{C}$. Identifying $\sum_{g \in G} c_g g$ with the function g maps to c_g we view $\mathbb{C}G$ as the space of all $\mathbb{C}$-valued function on G. If we put $g = 1.g$ for all $g \in G$ so $G \subseteq \mathbb{C}G$. For $x, y \in G$; we define an inner product as follows: $\langle x, y \rangle = 1$ if $x = y$ and $\langle x, y \rangle = 0$ otherwise, only holds for the basis elements. Thus G becomes an *orthonormal basis* of $\mathbb{C}G$. This turns $\mathbb{C}G$ into a $\mathbb{C}$-*algebra* of dimension $|G|$.

Let $W = \mathbb{C}G$. The *right regular representation* is a map (in fact homomorphism) $\rho_r : G \to GL(W)$ of G given by $\rho_r(h)(g) = gh$ for each $h \in G$ and all $g \in G$. The *left regular representation* $\rho_l : G \to GL(W)$ of G is given by $\rho_l(h)(g) = h^{-1}g$ for each $h \in G$ and all $g \in G$.

Suppose $\rho : G \to GL(W_1)$ and $\sigma : G \to GL(W_2)$ are two representations of G over $\mathbb{C}$. Then we say that ρ is *equivalent* to σ if and only if there is a linear isomorphism T from W_1 to W_2 such that $T\sigma(g) = \rho(g)T$ for all $g \in G$.

A $\mathbb{C}G$-*module* is a vector space W over $\mathbb{C}$ if an action $(w, g) \to w^g \in W(w \in W, g \in G)$ is defined satisfying the following conditions:

(1) $w^g \in W$

(2) $(w^g)^h = w^{gh}$

(3) $(\lambda w)^g = \lambda w^g$

(4) $(u + w)^g = u^g + w^g$

for all $u, w \in W, \lambda \in \mathbb{C}$ and $g, h \in G$. We use the letters $\mathbb{C}$ and G in the name $\mathbb{C}G$-module to indicate that W is a vector space over the field $\mathbb{C}$ and that G is the group from which we are taking the elements g to form the products $w^g(w \in W)$. A subset $\overline{W}$ of W is said to be an $\mathbb{C}G$-*sub-module* of W if $\overline{W}$ is a subspace and $\overline{w}^g \in \overline{W}$ for all $g \in G$ and for all $\overline{w} \in \overline{W}$. An $\mathbb{C}G$-module W is said to be *irreducible* if it is non-zero and it has no $\mathbb{C}G$-sub-modules other than $\{0\}$ and W. If W is an $\mathbb{C}G$-module and U is an irreducible $\mathbb{C}G$-module then we say that U is a *composition factor* of W if W has an $\mathbb{C}G$-sub-module which is isomorphic to U. Two $\mathbb{C}G$-modules W and U are said to have a *common composition factor* if there is an irreducible $\mathbb{C}G$-module which is a composition factor of both W and U.

A representation $\rho : G \to GL(W)$ is *irreducible* if the corresponding $\mathbb{C}G$-module W given by $w^g = w^{\rho(g)}$ where $w \in W, g \in G$ is irreducible. If we restrict ρ to an irreducible sub-module $\overline{W}$ we will get

$$\rho_{\overline{W}} : G \to GL(\overline{W})$$

which is a representation of G on $\overline{W}$ called the *restricted representation* or *sub-representation* of ρ on $\overline{W}$. An $\mathbb{C}G$-module W is said to be *completely reducible* if $W = W_1 \oplus ... \oplus W_s$ where each W_i is an irreducible $\mathbb{C}G$-sub-module of W. Then the representation $\rho : G \to GL(W)$ is completely reducible and it is a direct sum of irreducible representations. Then we write ρ as $\rho = \rho_{W_1} \oplus ... \oplus \rho_{W_s}$. If $B_i, i = 1, ..., s$ is an ordered basis for $W_i, i = 1, ..., s$ then a basis of W is $B = B_1 \cup ... \cup B_s$ and the relation between the corresponding matrix representation of ρ and $\rho_{W_i}, i = 1, ..., s$ is as follows:

$$\rho(g) = \begin{pmatrix} \rho_{W_1}(g) & 0 & \dots & 0 \\ 0 & \rho_{W_2}(g) & \dots & 0 \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ 0 & 0 & \dots & \rho_{W_s}(g) \end{pmatrix}.$$

Theorem 2.2.2. *(Maschke's Theorem)[32, Theorem 8.1] If G is a finite group and ρ is a representation of G over $\mathbb{C}$ then ρ is completely reducible.*

The *character* associated with ρ is the function $\chi_\rho : G \to \mathbb{C}$ denoted by $\chi_\rho(g) = \text{tra}(\rho(g))$ for all $g \in G$. Here $\text{tra}(\rho(g))$ is the trace of the representation matrix. *The degree* of the character is the degree of the representation and it is equal to $\chi_\rho(1_G)$. It is clear that characters are *class functions* (functions of G which are constant on all conjugacy classes,) see [32, Proposition 13.5] and it is a deep result of representation theory that the set of all irreducible characters is a basis of the vector space of all class functions on G; see Theorem 2.2.12. Let θ and ϑ be two class functions of G. Then the inner product of θ and ϑ is

$$\langle \theta, \vartheta \rangle = \langle \vartheta, \theta \rangle = \frac{1}{|G|} \sum_{g \in G} \theta(g)\vartheta(g^{-1}).$$

A character of degree one is called a *linear character*. We say that the character is *irreducible* if the corresponding representation is irreducible. If N is a non-trivial normal subgroup of G and $\tilde{\chi}$ is a character of G/N, then the character of G which is given by

$$\chi(g) = \tilde{\chi}(gN)$$

for all $g \in G$ is called the *lift* of $\tilde{\chi}$ to G. The *character table* of G is a square matrix whose rows are indexed by the irreducible characters of G and whose columns are indexed by the conjugacy classes in G. The entries of the matrix are the characters evaluated for each conjugacy class.

A character is *faithful* if $Ker(\chi) = 1_G$ where

$$Ker(\chi) = \{g \in G \,|\, \chi(g) = \chi(1_G)\}.$$

Let $X = \{x_1, x_2, ..., x_n\}$ and let G be a subgroup of $Sym(X)$. The $\mathbb{C}G$-module W with basis $\{x_1, ..., x_n\}$ and the G action $g : x_i \mapsto x_i^g$ for $g \in G$. is a *permutation module* for G over $\mathbb{C}$. The character of the permutation module is the number of fixed points of X under the action of g. We denote this by ψ, so $\psi(g)$ is the number of x_i such that $x_i^g = x_i$.

We list some theorems which we will use later.

Theorem 2.2.3. *(Sylow's Theorem)[32, Theorem 30.9] Let p be a prime number, and let G be a finite group of order $p^a b$, where a, b are positive integers and p does not divide b then*

(1) G contains a subgroup of order p^a; such a subgroup is called a Sylow p-subgroup of G.

(2) All Sylow's p-subgroups are conjugate in G.

(3) The number of Sylow p-subgroups is congruent to 1 modulo p.

Theorem 2.2.4. *[22, 49] Every cyclic group G is isomorphic either to the additive group $\mathbb{Z}$ or to the additive group $\mathbb{Z}/n\mathbb{Z}$ for some positive integer n.*

Theorem 2.2.5. *[32, Theorem 22.11] If χ is an irreducible character of G over $\mathbb{C}$ then $\chi(1_G)$ divides $|G|$.*

Theorem 2.2.6. *(Schur's Lemma in the terms of representations)[32, Lemma 9.1] Let V and W be irreducible $\mathbb{C}G$-modules.*

(1) If $\varphi : V \longrightarrow W$ is a $\mathbb{C}G$-homomorphism then either φ is a $\mathbb{C}G$-isomorphism or $\varphi(v) = 0$ for all $v \in V$.

(2) If $\varphi : V \longrightarrow V$ is a $\mathbb{C}G$-isomorphism then $\varphi = \lambda id_V$ where $\lambda \in \mathbb{C}$.

Theorem 2.2.7. *[32, Proposition 9.5] If G is a finite abelian group then every irreducible $\mathbb{C}G$-module has dimension 1.*

Proposition 2.2.8. *[32, Proposition 11.3] Let W and U be $\mathbb{C}G$-modules and suppose that $Hom_{\mathbb{C}G}(W, U) \neq \{0\}$. Then W and U have a common composition factor.*

Theorem 2.2.9. *[32, Corollary 11.6] Let U a $\mathbb{C}G$-module with $U = U_1 \oplus ... \oplus U_s$, where each U_i is an irreducible $\mathbb{C}G$-module. Let W be any irreducible $\mathbb{C}G$-module. Then the dimension of $Hom_{\mathbb{C}G}(U, W)$ and $Hom_{\mathbb{C}G}(W, U)$ are both equal to the number of $\mathbb{C}G$-module U_i such that $U_i \cong W$.*

Theorem 2.2.10. *[32, Theorem 14.24] Let U and W be $\mathbb{C}G$-modules with character χ and ψ, respectively. Then $dim(Hom_{\mathbb{C}G}(U, W)) = \langle \chi, \psi \rangle$.*

Theorem 2.2.11. *[32, Theorem 11.9] Suppose that*

$$\mathbb{C}G = U_1 \oplus ... \oplus U_r$$

is a direct sum of irreducible $\mathbb{C}G$-modules. If U is any irreducible $\mathbb{C}G$-module then the number of $\mathbb{C}G$-modules U_i with $U_i \cong U$ is equal to $dim(U)$.

Theorem 2.2.12. *[32, Corollary 15.4] The irreducible characters $\chi_1, \chi_2, ..., \chi_s$ of the group G form a basis for the vector space of all class functions on G. Indeed, if φ is a class function, then*

$$\varphi = \sum_{i=1}^{s} a_i \chi_i$$

where $a_i = \langle \varphi, \chi_i \rangle$ *for* $1 \leq i \leq s$.

2.3 Graph Theory

In this section we give the basic definitions and basic ideas of graph theory. These can be found in any book or lecture notes on graph theory, see as a reference [4] and [27]. An *undirected graph* $\Gamma = (V, E)$ consists of a set V of vertices and a set E of unordered pairs of vertices. Two vertices v and w are said to be *adjacent* if and only if $\{v, w\} \in E$. The *endpoints* of the edge $\{v, w\}$ are v and w. We use $v \sim w$ to say that there is an edge between v and w. A *loop* is an edge from a vertex to itself. A graph with no loops is called *simple*. Note, in this thesis Γ is a simple graph. The *adjacency matrix* of Γ is the integer matrix with rows and columns indexed by the vertices of Γ, such that the A_{vw}-entry is equal to 1 if and only if $v \sim w$ and 0 otherwise and it is denoted by A. The *spectrum* of Γ consists of all eigenvalues $\lambda_1, ..., \lambda_n$ of A where $|V| = n$.

The *complement graph* $\bar{\Gamma}$ of Γ is a simple graph with vertex set V in which two vertices are adjacent if and only if they are not adjacent in Γ. The *order* of Γ is the number of vertices of Γ and the *size* of Γ is the number of its edges.

Let $v \in V$. The *degree* of v denoted by $d(v)$ is the number of vertices which are adjacent to v. A graph Γ is said to be *k-regular* if and only if every vertex of Γ has the same degree k. A graph Γ of order n is said to be a *complete graph*, and is denoted by K_n, if and only any two distinct vertices are adjacent. The complement of the complete graph is the *null graph*. A graph Γ is said to be *bipartite* if we can partition the vertex set into two parts, say V_1 and V_2, so that each edge has exactly one end point in V_1 and one end point in V_2. A *complete bipartite graph* is a bipartite graph in which each vertex in V_1 is joined to each vertex in V_2 by just one edge and is denoted by $K_{n,m}$ where $|V_1| = n$ and $|V_2| = m$.

A *walk* in Γ is a finite sequence of edges of the form $v_0v_1, v_1v_2, ..., v_{m-1}v_m$ in which any two consecutive edges are adjacent or identical. We call v_0 is the *initial vertex* and v_m is the *final vertex* of the walk. A *path* is a walk in which all vertices and edges are distinct. A *cycle* is a closed path with at least three edges such that the initial vertex and the final vertex are the same. The *length* of a cycle or path is the number of vertices in this cycle or path. A *connected graph* is a graph in which any two vertices are connected by a path otherwise it is a disconnected graph. The *distance* between two vertices is the length of the shortest path between these vertices.

The *line graph* of Γ denoted by $L(\Gamma)$, is the graph whose vertex set is the edge set of Γ. Two vertices are adjacent in $L(\Gamma)$ if and only if these edges are incident in Γ (that is, the two edges have a same endpoint). A *strongly regular* graph with parameters (n, k, λ, μ) is a graph on n vertices which is regular with degree k and has the following properties:

(1) any two adjacent vertices have exactly λ common neighbours;

(2) any two non adjacent vertices have exactly μ common neighbours.

Lemma 2.3.1. *[23, Lemma 10.2.1] A connected regular graph with exactly three distinct eigenvalues is a strongly regular.*

A graph X is *sub-graph* of a graph Γ if each of its vertices belong to $V(\Gamma)$ and each of its edges belongs to $E(\Gamma)$. If X is a sub-graph of Γ and we have $V(X) = V(\Gamma)$ then we call X *spanning subgraph* and we say X spans Γ.

In this research we deal with finite connected undirected and simple graphs. In this work a graph means a finite simple connected undirected graph.

In the remainder of this section we will give the definition and properties of the automorphism group of a graph. We will show that for each graph there is an associated group called the *automorphism group* of the graph.

This concept established the link between group theory and graph theory.

Given a graph Γ, a permutation g of V is an *automorphism* if $u^g \sim v^g$ if and only if $u \sim v$ for all $u, v \in V$. The set of all automorphisms of Γ under the operation of composition of mappings forms a subgroup of the symmetric group on V, called the *automorphism group* of Γ. It is denoted by $Aut(\Gamma)$. Thus each automorphism of Γ is a one-to-one and onto relations of the vertices of Γ which preserve the adjacency and non adjacency. This implies that an automorphism maps any vertex onto a vertex of the same degree. The identity of the automorphism group of Γ is denoted by $1_{Aut(\Gamma)} = 1$.

Lemma 2.3.2. *[23, Lemma 1.3.3] If Γ is a graph then $Aut(\Gamma) = Aut(\overline{\Gamma})$.*

A graph is *rigid* if it admits only the trivial automorphism. The automorphism group of the complete graph K_n on n vertices is $Sym(n)$. Note, any permutation of its n vertices is in fact an automorphism for adjacency is never lost. The automorphism group of the complete bipartite graph $K_{n,m}$ where $n \neq m$ is $Sym(n) \times Sym(m)$ since the n vertices in the first class can be permuted by $n!$ ways and similarly $m!$ for the second class. On the other hand, there is no automorphism that can be obtained from swapping a vertex from the first class and a vertex from the second class because $n \neq m$. Therefore the automorphism group of complete bipartite graph where $n \neq m$ is $Sym(n) \times Sym(m)$. However the automorphism group of the complete bipartite graph $K_{n,n}$ is $(Sym(n) \times Sym(n)) \rtimes Z_2$.

We say that Γ is a *vertex transitive graph* if $Aut(\Gamma)$ acts transitively on V. In other words, a graph is a *vertex transitive* graph if the action of its automorphism group on the vertex set has only one orbit. This means that for any two vertices u and v of Γ there is an automorphism $g \in Aut(\Gamma)$ such that $u^g = v$. We can conclude from the above that a vertex transitive graph is a regular graph. Examples of a vertex transitive graphs include the complete graph K_n and its automorphism group $Sym(n)$. Another example of a vertex transitive graph is the cycle. Its automorphism group is the dihedral group D_n of $2n$ elements when the cycle is of length n. Many other examples of vertex transitive graphs arise from the Cayley graphs which we will define in Chapter 4.

Let G be a transitive group of automorphisms of Γ. A non-emptsubset S of $V(\Gamma)$ is a block of *imprimitivity* for G if for any $g \in G$, either $S^g = S$ or $S^g \cap S = \phi$. Because G is transitive, it is clear that the translates of S form a partition of V. This is called the partition associated to S. This set of distinct translates is called a system of imprimitivity for G. Then the group G is called imprimitive if there is system of imprimitivity with some S in the system such that $S \neq \{v\}, S \neq \{V\}$ for some $v \in V$. Otherwise, G is primitive.

3

Graphs and their Maps

Let Γ be a finite graph with vertex set V and let G a group of automorphisms of Γ. In this chapter we introduce some basic concepts and notations that allow us to discuss the singularity problem in Γ. In particular we discuss a *vector space* $\mathbb{C}V$ associated to the vertices of Γ and show how the adjacency relation gives raise to a linear map

$$\alpha : \mathbb{C}V \to \mathbb{C}V.$$

This allows us to apply many techniques from linear algebra. We want to develop an algebraic language to help us study the relationship between the eigenspaces of Γ and the irreducible characters of G.

In particular, we are interested in the eigenspaces of α and the projection maps for $\mathbb{C}V$ onto these eigenspaces. We use the projection maps to derive information about the eigenvalues of the graph and we give an example for graphs with three distinct eigenvalues. These include strongly regular graphs. In the last section we discuss conditions for a graph to be singular.

3.1 Graphs and their Adjacency Map

Let Γ be a finite graph with vertex set V. Two distinct vertices u, v are adjacent, denoted by $u \sim v$ if and only if $\{u, v\}$ is an edge of Γ. It is convenient to introduce a vector space and an adjacency map that is represents this graph structure.

Let $\mathbb{C}$ be the field of complex numbers. Let $\mathbb{C}V$ denote to the vector space over $\mathbb{C}$ with basis V, we call this the *vertex space* of Γ. Its elements are the formal sums

$$f = \sum_{v \in V} c_v v$$

where $v \in V$ and $c_v \in \mathbb{C}$. If we have $f = \sum_{v \in V} c_v v$ and $h = \sum_{v \in V} \bar{c}_v v$ then $f = h$ if and only if $c_v = \bar{c}_v$ for all v. We define

$$f + h = \sum_{v \in V} (c_v + \bar{c}_v) v$$

for all $f, h \in \mathbb{C}V$ and $sf = \sum_{v \in V} s c_v v$ for all $s \in \mathbb{C}$. So indeed, with these operations $\mathbb{C}V$ is a vector space over $\mathbb{C}$.

We can describe this vector space in another way as the set of all functions $f : V \to \mathbb{C}$ where we think of $f = \sum_{v \in V} c_v v$ as the function $f : v \to c_v$.

We define a *natural inner product* on $\mathbb{C}V$ by $\langle u, v \rangle = 1$ if $u = v$ and $\langle u, v \rangle = 0$ if $u \neq v$, for all $u, v \in V$. In particular, we identify $v = 1v$ so that V is a subset of $\mathbb{C}V$. Therefore V is an orthonormal basis of $\mathbb{C}V$. We put $||f||^2 = \langle f, f \rangle$ and call $||f||$ the *length* of f.

Proposition 3.1.1. *[5, Theorem 6.17] Let Γ be a finite graph with vertex set V and let $f = \sum_{v \in V} c_v v$ be an element of $\mathbb{C}V$. Then $c_v = \langle f, v \rangle$ for all $v \in V$.*

Proof: Let $f \in \mathbb{C}V$ with

$$f = c_{v_1} v_1 + ... + c_{v_n} v_n. \tag{3.1.1}$$

Take the inner product with v_j on both sides of Equation 3.1.1. We will get $\langle f, v_j \rangle = c_{v_j}$ since $\langle v_i, v_j \rangle = 1$ if and only if $i = j$ and 0 otherwise. □

We define the *adjacency map* of Γ as the linear map

$$\alpha : \mathbb{C}V \longrightarrow \mathbb{C}V$$

given on the basis V by $\alpha(v) = \sum_{u \sim v} u$ for all $v \in V$. If $u, v \in V$ then $\langle \alpha(u), v \rangle = \langle u, \alpha(v) \rangle = 1$ if $u \sim v$ and $\langle \alpha(u), v \rangle = \langle u, \alpha(v) \rangle = 0$ otherwise. Hence, the adjacency map is symmetric for the given inner product. The matrix of α with respect to the basis V is the *adjacency matrix* $A = A(\Gamma)$ of Γ.

Since A is symmetric all eigenvalues of A are real by Lemma 2.1.4. We denote the *distinct eigenvalue* by $\lambda_1 > \lambda_2 > ... > \lambda_t$ and let μ_1, μ_2, ..., μ_t be their *multiplicity*. The *spectrum* of Γ consists of all eigenvalues of A,

$$Spec(\Gamma) = \lambda_1^{\mu_1}, \lambda_2^{\mu_2}, ..., \lambda_t^{\mu_t}$$

where $\lambda_1^{\mu_1}$ indicates that λ_1 has multiplicity μ_1, and so on. We denote by E_1, E_2, ... E_t the corresponding eigenspaces. Throughout we denote the kernel of Γ by E_*. Thus $E_* = E_i$ for some i where E_i is the eigenspace corresponding to the eigenvalue 0 of Γ and $E_* = 0$ otherwise.

There are important connections between eigenvalues of α and the structure of the graph.

Theorem 3.1.2. *[37, Proposition 1.48] Let Γ be a k-regular graph with n vertices then the following hold.*

(i) We have that $\lambda_1 = k$, *and* $\mu_1 = 1$ *if and only if* Γ *is connected.*

(ii) For each eigenvalue λ_i *of* Γ *we have* $|\lambda_i| \leq k$.

(iii) If Γ *is bipartite then the spectrum of* Γ *is symmetric about* 0.

(iv) We have that $-k$ *is an eigenvalue of* Γ *if and only if* Γ *is bipartite.*

In addition we can apply the Spectral Theorem to graphs as in the following theorem:

Theorem 3.1.3. *[23, Theorem 8.4.5](Decomposition Theorem) Let* $\Gamma = (V, E)$ *be a graph. Suppose that* $\lambda_1 > \lambda_2 > ... > \lambda_t$ *are the distinct eigenvalues of* Γ *and that* $E_1, ..., E_t$ *are the corresponding eigenspaces. Then*

$$\mathbb{C}V = E_1 \oplus ... \oplus E_t$$

is the orthogonal decomposition of the vertex space of Γ. *Furthermore,* $dim(E_i)$ *is the (algebraic) multiplicity* μ_i *of* λ_i.

This means that every vector $f \in \mathbb{C}V$ can be written as

$$f = f_1 + f_2 + ... + f_t \tag{3.1.2}$$

with uniquely determined $f_i \in E_i$. We call the f_i the *orthonormal components* of f and we call

$$f = f_1 + f_2 + ... + f_t$$

the *spectral decomposition* of f. Note by Lemma 2.1.4 we have that the components are vectors with f_i perpendicular to f_j if $i \neq j$ and

$$||f||^2 = ||f_1||^2 + ... + ||f_t||^2.$$

3.2 The Projection Maps onto Eigenspaces

Results in this section on linear algebra have been taken from the lectures notes on Linear Algebra by Peter Cameron[1]. For each $i = 1, 2, ..., t$ we define the *projection maps* $\pi_i : \mathbb{C}V \to \mathbb{C}V$ so that $\pi_i(f) = f_i$ where f_i is as above in Equation 3.1.2. In particular $\pi_i(\mathbb{C}V) \subseteq E_i$. It is clear that π_i is a linear map. Formally the projection maps satisfy $\pi_i^2 = \pi_i$, $\pi_i\pi_j = 0$ if $i \neq j$ and $\sum_{i=1}^{t} \pi_i = id$.

Note, as the eigenvalues of π_i are 0 and 1 it is clear that

$$\mathrm{tra}(\pi_i) = rank(\pi_i) \tag{3.2.1}$$

for each i. Hence, by Equation 3.2.1 and the definition of the rank we have $dim(E_i) = \mathrm{tra}(\pi_i)$. This is an instance of the following more general result.

Theorem 3.2.1. *Let W be a finite dimensional vector space over the field of complex numbers with inner product and let $\varphi : W \to W$ be a symmetric linear map. Let $\lambda_1, ..., \lambda_t$ be the distinct eigenvalues of φ and let $E_1, ..., E_t$ be the corresponding eigenspaces. Suppose that π_i are the projection maps onto E_i where $1 \leq i \leq t$. Then $\mathrm{tra}(\pi_i) = dim(E_i)$.*

The π_i are the *minimal idempotents* associated to α. We now determine these idempotents in terms of the eigenvalues. The following is well-known, see for instance [2].

Proposition 3.2.2. *Let $\lambda_1 > \lambda_2 > ... > \lambda_t$ be the distinct eigenvalues of the adjacency map α with spectral decomposition $\mathbb{C}V = E_1 \oplus ... \oplus E_t$. Let $1 \leq i \leq t$ and let $\psi_i : \mathbb{C}V \to \mathbb{C}V$ be the map given by*

$$\psi_i = \prod_{j \neq i} \frac{1}{(\lambda_i - \lambda_j)}(\alpha - \lambda_j).$$

[1]https://cameroncounts.files.wordpress.com/2013/11/linalg.pdf
[2]http://infohost.nmt.edu/~ iavramid/notes/mp2-5.pdf

Then $\psi_i = \pi_i$ is the projection map $\pi_i : \mathbb{C}V \to \mathbb{C}V$ associated to α.

Proof: Let $f \in \mathbb{C}V$. By the Decomposition Theorem 3.1.3 we have $f = f_1 + f_2 + ... + f_t$ where $f_i \in E_i$. We will compute

$$\psi_i(f) = \psi_i(f_1 + f_2 + ... + f_t) = \Big[\prod_{i \neq j} \frac{1}{(\lambda_i - \lambda_j)}(\alpha - \lambda_j)\Big](f_1 + f_2 + ... + f_t).$$

Note,

$$\prod_{i \neq j} \frac{1}{(\lambda_i - \lambda_j)}(\alpha - \lambda_j)(f_k) = \begin{cases} 0 & k \neq i \\ f_i & k = i. \end{cases}$$

Hence $\psi_i(f) = f_i$. □

Corollary 3.2.3. *Let $\lambda_1 > \lambda_2 > ... > \lambda_t$ be the distinct eigenvalues of the symmetric map α with spectral decomposition $\mathbb{C}V = E_1 \oplus ... \oplus E_t$. Let $1 \leq i \leq t$ and let $\psi_i : \mathbb{C}V \to \mathbb{C}V$ be the map given by*

$$\psi_i = \prod_{i \neq j} \frac{1}{(\lambda_i - \lambda_j)}(\alpha - \lambda_j).$$

Then we have that $dim(E_i) = \mathrm{tra}(\psi_i)$ for each i.

Corollary 3.2.4. *Let $\lambda_1 > \lambda_2 > ... > \lambda_t$ be the distinct eigenvalues of the symmetric map α with spectral decomposition $\mathbb{C}V = E_1 \oplus ... \oplus E_t$ into eigenspaces E_i. If $\pi_i : \mathbb{C}V \to \mathbb{C}V$ is the projection onto the eigenspaces E_i then $\alpha = \lambda_1 \pi_1 + ... + \lambda_t \pi_t$.*

Proof: Let $f \in \mathbb{C}V$. By the Decomposition Theorem 3.1.3 we have $f = f_1 + f_2 + ... + f_t$ where $f_i \in E_i$. We will compute $\alpha(f) = \alpha(f_1) + ... + \alpha(f_t)$. Thus $\alpha(f) = \lambda_1 f_1 + ... + \lambda_t f_t$. Note, $\pi_i(f) = f_i$ by the previous Proposition. Hence, $\alpha(f) = \lambda_1 \pi_1(f) + ... + \lambda_t \pi_t(f)$. □

3.3 Eigenvalue Inequalities

As above let $\pi_i : \mathbb{C}V \longrightarrow \mathbb{C}V$ with $\pi_i(\mathbb{C}V) \subseteq E_i$ be the projection maps onto the eigenspace E_i of Γ for $i = 1, 2, ..., t$. Given $f \in \mathbb{C}V$ we have that $f = f_1 + ... + f_t$ with $\pi_i(f) = f_i$ by the Decomposition Theorem. We can derive interesting inequalities from this decomposition. Note the E_i are orthogonal to each other. Hence,

$$\langle f, f_i \rangle = \langle f_1 + ... + f_t, f_i \rangle = \langle f_i, f_i \rangle \geq 0$$

for $i = 1, ..., t$. We formulate this as a theorem.

Theorem 3.3.1. *[Delsarte's Linear Programming Bound][18] For all* $i = 1, 2, ..., t$ *we have* $\langle f, f_i \rangle \geq 0$.

Delsarte's Bound appears in the context of association Schemas. Here however we see that the same principle applies more generally. While the proof of this theorem is extremely simple this bound has many important application in the theory of association schemes, see [18] as a reference. In the rest of this section we will give a method that allows us to derive some inequalities for the spectrum of graphs with three distinct eigenvalues. The strengths of this method lies in the fact that we have an explicit formula for the projective maps.

Example: Let $\Gamma = (V, E)$ be a k-regular graph with three distinct eigenvalues $k = \lambda_1 > \lambda_2 > \lambda_3$. Let A be its adjacency matrix. By the Decomposition Theorem 3.1.3 we have

$$\mathbb{C}V = E_1 \oplus E_2 \oplus E_3$$

where $E_i = \pi_i(\mathbb{C}V)$. We develop an inequalities for these eigenvalues, using Delsarte's Bound above. Let $v \in V$. Since α is a symmetric map so is π_i for

$i = 1, 2, 3$. Hence

$$\begin{aligned}\langle \pi_2(v), \pi_2(v)\rangle &= \langle \pi_2^2(v), v\rangle \\ &= \langle \pi_2(v), v\rangle \\ &= R_2\langle(\alpha - k)(\alpha - \lambda_3)v, v\rangle\end{aligned}$$

where $R_2 = (\lambda_2 - k)^{-1}(\lambda_2 - \lambda_3)^{-1}$. Note $\langle \pi_2(v), \pi_2(v)\rangle \geq 0$. Since $k > \lambda_2 > \lambda_3$ we have that $R_2 < 0$. Hence

$$\begin{aligned}0 \geq \langle(\alpha - k)(\alpha - \lambda_3)v, v\rangle &= \langle(\alpha - k)v, (\alpha - \lambda_3)v\rangle \\ &= k - 0 - 0 + k\lambda_3.\end{aligned}$$

Thus $0 \geq k + k\lambda_3 = k(1 + \lambda_3)$. So that

$$\lambda_3 \leq -1. \tag{3.3.1}$$

Moreover

$$\begin{aligned}\langle \pi_3(v), \pi_3(v)\rangle &= \langle \pi_3^2(v), v\rangle \\ &= \langle \pi_3(v), v\rangle \\ &= R_3\langle(\alpha - k)(\alpha - \lambda_2)v, v\rangle\end{aligned}$$

where $R_3 = (\lambda_3 - k)^{-1}(\lambda_3 - \lambda_2)^{-1}$. Note $\langle \pi_3(v), \pi_3(v)\rangle \geq 0$. Since $k > \lambda_2 > \lambda_3$ we have that $R_3 > 0$. So that we have

$$\begin{aligned}0 \leq \langle(\alpha - k)(\alpha - \lambda_2)v, v\rangle &= \langle(\alpha - k)v, (\alpha - \lambda_2)v\rangle \\ &= k - 0 - 0 + k\lambda_2.\end{aligned}$$

Thus $0 \leq k + k\lambda_2 = k(1 + \lambda_2)$. Hence we conclude that

$$\lambda_2 \geq -1. \tag{3.3.2}$$

In addition

$$\langle \pi_1(v), \pi_1(v)\rangle = \langle \pi_1^2(v), v\rangle.$$

Note by the projection map properties we have that

$$\begin{aligned}\langle \pi_1^2(v), v\rangle &= \langle \pi_1(v), v\rangle \\ &= R_1\langle(\alpha - \lambda_2)(\alpha - \lambda_3)(v), v\rangle\end{aligned}$$

where $R_1 = (k-\lambda_2)^{-1}(k-\lambda_3)^{-1}$. Note, $\langle \pi_1(v), \pi_1(v)\rangle \geq 0$. Since $k > \lambda_2 > \lambda_3$ we have that $R_1 > 0$. Therefore

$$\begin{aligned} 0 < R_1\langle(\alpha-\lambda_2)(\alpha-\lambda_3)(v), v\rangle &\leq \langle(\alpha-\lambda_2)(\alpha-\lambda_3)(v), v\rangle \\ &= \langle(\alpha-\lambda_2)v, (\alpha-\lambda_3)v\rangle \\ &= k - 0 + \lambda_2\lambda_3. \end{aligned}$$

Note we have that $k + \lambda_2 + \lambda_3 = \mathrm{tra}(A) = 0$ and by Equation 3.3.1 and Equation 3.3.2 we have that

$$k > |\lambda_2\lambda_3|. \tag{3.3.3}$$

Proposition 3.3.2. *Let* Γ *be a* k*-regular graph with exactly three distinct eigenvalues* $k = \lambda_1 > \lambda_2 > \lambda_3$*. Then we have*

(i) $k > |\lambda_2\lambda_2|$,

(ii) $\lambda_2 \geq -1$ *and*

(ii) $\lambda_3 \leq -1$.

These properties can be obtained in many other ways. According to Lemma 2.3.1 regular graphs with 3 distinct eigenvalues are strongly regular and the inequalities can be obtained directly from the matrix equation satisfied for the adjacency matrix of the graph.

Our method here is based on Delsarte's bound in Theorem 3.3.1 and this method works more generally for graphs with more than three distinct eigenvalues. This could be shown by looking at a few examples but we omit these discussions.

3.4 Groups of Automorphisms and Eigenspaces

Throughout this section let Γ be a finite graph with vertex set V. In this section we study the relationship between the irreducible representations of a group of automorphisms of Γ and its eigenspaces. As before $\mathbb{C}V$ is the vector space with basis V and $\alpha : \mathbb{C}V \longrightarrow \mathbb{C}V$ is the adjacency map of Γ. Suppose that $E_1, ..., E_t$ are the eigenspaces of α corresponding to the distinct eigenvalues $\lambda_1, ..., \lambda_t$.

Let G be a group of automorphisms of Γ. We denote the image of v under g by v^g or vg for all $v \in V$ and $g \in G$. Then every element $g \in G$ acts linearly on $\mathbb{C}V$ by $g : c_v v \mapsto c_v v^g$. In this way $\mathbb{C}V$ becomes a $\mathbb{C}G$-module. Note, g preserves the inner product, in the sense that $\langle v, u\rangle = \langle v^g, u^g\rangle$.

Proposition 3.4.1. *[16, p. 134] A permutation g of V is an automorphism of Γ if and only if $\alpha(f^g) = (\alpha(f))^g$ where α is the adjacency map of Γ for all $f \in \mathbb{C}V$.*

Proof: It suffices to show this property when $f = v$ for some $v \in V$. Assume that $\alpha(v^g) = (\alpha(v))^g$. Suppose that $u \sim v$ where $u, v \in V$. Therefore we want to prove that g is an automorphism of Γ. So by the definition of the automorphism this is enough to prove that $u^g \sim v^g$. Since $u \sim v$ so $\langle \alpha(u), v\rangle = 1$. Then

$$\begin{aligned} \langle \alpha(u^g), v^g\rangle &= \langle (\alpha(u))^g, v^g\rangle \\ &= \langle \alpha(u), v^{gg^{-1}}\rangle \\ &= \langle \alpha(u), v\rangle \\ &= 1. \end{aligned}$$

Hence $u^g \sim v^g$ which means that g is an automorphism of Γ.

Now suppose that g is an automorphism of Γ and we want to prove that

$\alpha(v^g) = (\alpha(v))^g$ for all $v \in V$. Then

$$\alpha(v^g) = \sum_w \langle \alpha(v^g), w \rangle w \tag{3.4.1}$$

$$(\alpha(v))^g = \sum_w \langle (\alpha(v))^g, w \rangle w. \tag{3.4.2}$$

Hence $\langle \alpha(v^g), w \rangle = \langle v^g, \alpha(w) \rangle = \begin{cases} 1 & v^g \sim w \\ 0 & v^g \nsim w \end{cases}$

and $\langle (\alpha(v))^g, w \rangle = \langle \alpha(v), w^{g^{-1}} \rangle = \begin{cases} 1 & v \sim w^{g^{-1}} \\ 0 & v \nsim w^{g^{-1}}. \end{cases}$

Now, by the definition of automorphisms, $v^g \sim w$ if and only if $v^{gg^{-1}} = v \sim w^{g^{-1}}$. Therefore by Equation 3.4.1 and Equation 3.4.2 we have that $\alpha(f^g) = (\alpha(f))^g$ for all $f \in \mathbb{C}V$. □

Let $f \in E_i$ be an eigenvector of α corresponding to the eigenvalue λ_i. Then we can show that $\alpha(f^g) = \lambda_i f^g$ since

$$\alpha(f^g) = (\alpha(f))^g = \lambda_i f^g$$

by Proposition 3.4.1. Therefore we have proved the following important theorem, see for instance [8] as a reference.

Theorem 3.4.2. *Let Γ be a finite graph with adjacency map α and eigenspaces $E_1, E_2, ..., E_t$ corresponding to the distinct eigenvalues of α. Let G be a group of automorphisms of Γ. Then each E_i is a $\mathbb{C}G$-module.*

According to the general representation theory of finite groups our group G has irreducible $\mathbb{C}$-modules

$$U_1, U_2, ..., U_s.$$

This means that there are representations $\rho_1, \rho_2, ..., \rho_s$ which are homomorphisms $\rho_i : G \longrightarrow GL(U_i)$ such that only the spaces 0 and U_i are invariant under $\rho_i(G)$. The function

$$\chi_i(g) = \text{tra}(\rho_i(g))$$

is the *character* associated to ρ_i. Here $\text{tra}(\rho_i(g))$ is the matrix trace. As we are working over $\mathbb{C}$ every G-module decomposes into a direct sum of irreducible modules. In this case we can write

$$\mathbb{C}V = \underbrace{U_1 + U_1 + ... + U_1}_{m_1} \oplus \underbrace{U_2 + U_2 + ... + U_2}_{m_2} \oplus ... \oplus \underbrace{U_s + U_s + ... + U_s}_{m_s}.$$

The m_i are the *multiplicities* of U_i in $\mathbb{C}V$.

Using the same principle again each eigenspace E_i of α can also be decomposed into irreducible modules

$$E_i = \underbrace{(U_1 + U_1 + ... + U_1)}_{m_{i1}} \oplus \underbrace{(U_2 + U_2 + ... + U_2)}_{m_{i2}} \oplus ... \oplus \underbrace{(U_s + U_s + ... + U_s)}_{m_{is}}.$$

Now we can use this theory and the properties of projection maps to determine the multiplicity of $\mathbb{C}$-modules in each eigenspace of Γ.

Proposition 3.4.3. *We have $\sum_{i=1}^{t} m_{ij} = m_j$ for each $j = 1, 2, ..., s$. Furthermore, if G is transitive on V, then there exist a G-embedding of $\mathbb{C}V$ into $\mathbb{C}G$. In particular, $m_j \leq dim(U_j)$ and $m_j = dim(U_j)$ for all j if and only if G acts regularly on V.*

Proof: It is clear from the above that $\sum_{i=1}^{t} m_{ij} = m_j$ for each $j = 1, 2, ..., s$. Now let G be transitive on V. It follows that Γ is a regular graph, say of degree k. Then fix some $v \in V$ and let L be the stabilizer group of v. Now we prove that there is a map

$$\varphi : \mathbb{C}V \to \mathbb{C}G$$

which is injective and commutes with G. So, φ is an embedding of G-

modules. We put

$$\varphi(v) = \sum_{l \in L} l \in \mathbb{C}G.$$

Then, given any other $v' \in V$, by the transitivity of G we have that $v' = v^{g'}$ for some $g' \in G$. Hence we define

$$\varphi(v') = \sum_{l \in L} lg'$$

and extended linearly. If g'' is any other element with $v^{g'} = v^{g''}$ then $g''{g'}^{-1}$ fixes v, so $g''{g'}^{-1} \in L$. So

$$\varphi(v) = \sum_{l \in L} lg' = \sum_{l \in L} lg''.$$

Now suppose that $\sum t_i v^{g_i} \in \mathbb{C}V$ with $\varphi(\sum_i t_i v^{g_i}) = 0$ for some i. So we have that $0 = \sum_i t_i \varphi(v^{g_i}) = \sum_i t_i \sum_{l \in L} lg_i$ so that $t_i = 0$ for all i, because cosets do not intersect. So φ is injective.

Now we prove that φ is a G-homomorphism. Let $\tilde{v} \in V$ where $\tilde{v} = v^{\tilde{g}}$ for some $\tilde{g} \in G$. Let $g \in G$ we have that

$$\begin{aligned}
\varphi(\tilde{v}^g) &= \varphi(v^{\tilde{g}g}) \\
&= \sum_{l \in L} l(\tilde{g}g) \\
&= \sum_{l \in L} (l\tilde{g})g \\
&= \varphi(\tilde{v})g \\
&= (\varphi(\tilde{v}))^g.
\end{aligned}$$

Therefore we conclude from the above that φ is a G-homomorphism hence it is an embedding of $\mathbb{C}G$-modules. □

Let ρ_1 be the trivial representation of G and let U_1 be the trivial sub-module of dimension 1. In general it is not clear how the trivial representation is distributed among the $E_1, ..., E_t$. But if G is transitive then the trivial representation appears exactly once in $\mathbb{C}V$ and hence in exactly one E_i. In this case this eigenspace is E_k (the eigenspace of the degree of Γ). This is clear, the vector $v_1 + v_2 + ... + v_n$ spans the 1-dimensional eigenspace for the eigenvalue k. Often we do not differentiate between character and representation, in particular for 1-dimensional representation.

We will try to determine which of the irreducible representations of G are a part of some given eigenspace of α. This is a difficult problem in general. Note, for each i we have that $\pi_i : \mathbb{C}V \longrightarrow \mathbb{C}V$ is a projection map onto the eigenspace E_i of α by Proposition 3.2.2. Note E_i is a G-invariant submodule of $\mathbb{C}V$ for each $i = 1, 2, ..., t$ the restriction of g to E_i is $\pi_i g = g\pi_i$, noting that π_i is a polynomial of α, and hence commutes with g. In this way we have a representation of

$$G \to GL(E_i)$$

with $g \mapsto \pi_i g = g\pi_i$. The character of $\pi_i g$ is $\beta_i(g) = \text{tra}(\pi_i g)$ and it is a class function on G since for $g, h \in G$ we have

$$\begin{aligned} \beta_i(h^{-1}gh) &= \text{tra}(\pi_i h^{-1}gh) \\ &= \text{tra}(h^{-1}\pi g h) \\ &= \text{tra}(\pi_i g h h^{-1}) \\ &= \text{tra}(\pi_i g). \end{aligned}$$

Therefore the multiplicity m_{ij} of the irreducible G-module U_j in E_i is $\langle \text{tra}(\pi_i g), \chi_j \rangle$. Therefore by Theorem 2.2.10 we have the following theorem.

Theorem 3.4.4. *The multiplicity of the irreducible module U_j in the eigenspace E_i is $m_{ij} = \langle \chi_j, \beta_i \rangle$ where $\beta_i(g) = \text{tra}(\pi_i g)$ for all $g \in G$ and where χ_j is the irreducible character corresponding to U_j.*

Note, we have explicit formula for π_i in terms of α and $Spec(\Gamma)$. We illustrate this method in the following example in great details.

EXAMPLE: Let Γ be the cycle graph of length 4. In this graph v_i is adjacent to v_{i+1} modulo 4. The full automorphism group of Γ is D_4, with transitive subgroup C_4. We illustrate the theorem by taking $G = C_4$ and also by considering $G = D_4$. The computations differ significantly. The adjacency map α has matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

It is easy to see that the spectrum of Γ is $2^1, 0^2$ and -2^1. Now the projection maps are

$$\begin{aligned} \pi_1 &= \prod_{j\neq 1} \frac{1}{\lambda_1 - \lambda_j}(A - \lambda_j) \\ &= \frac{1}{(2-0)(2-(-2))}(A-0)(A-(-2)) \end{aligned}$$

with matrix

$$P_1 = \frac{1}{8}\begin{pmatrix} 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{pmatrix}.$$

Similarly,

$$\begin{aligned} \pi_2 &= \prod_{j\neq 2} \frac{1}{\lambda_2 - \lambda_j}(A - \lambda_j) \\ &= \frac{1}{(0-2)(0+2)}(A-2)(A+2) \end{aligned}$$

with matrix

$$P_2 = \frac{1}{-4}\begin{pmatrix} -2 & 0 & 2 & 0 \\ 0 & -2 & 0 & 2 \\ 2 & 0 & -2 & 0 \\ 0 & 2 & 0 & -2 \end{pmatrix}$$

and

$$\begin{aligned}\pi_3 &= \prod_{j\neq 3} \frac{1}{\lambda_3 - \lambda_j}(A - \lambda_j) \\ &= \frac{1}{(-2-0)(-2-2)}(A-0)(A-2)\end{aligned}$$

with matrix

$$P_3 = \frac{1}{8}\begin{pmatrix} 2 & -2 & 2 & -2 \\ -2 & 2 & -2 & 2 \\ 2 & -2 & 2 & -2 \\ -2 & 2 & -2 & 2 \end{pmatrix}.$$

It is clear that

$$P_1 + P_2 + P_3 = I$$

where I is the identity matrix of dimension 4×4.

First we consider the cyclic subgroup $G = C_4$. The character table of this group is shown in Table 3.1.

	1_G	a	a^2	a^3
χ_1	1	1	1	1
χ_2	1	-1	1	-1
χ_3	1	i	-1	$-i$
χ_4	1	$-i$	-1	i

Table 3.1: The character table of C_4

We evaluate $\beta_1(g), \beta_2(g)$ and $\beta_3(g)$ for each $g \in G$ in Table 3.2.

	1_G	a	a^2	a^3
β_1	1	1	1	1
β_2	2	0	-2	0
β_3	1	-1	1	-1

Table 3.2: $\beta_i(g)$ for $i = 1, 2, 3$.

Thus $\langle \beta_1, \chi_1 \rangle = 1$, $\langle \beta_1, \chi_2 \rangle = 0$, $\langle \beta_1, \chi_3 \rangle = 0$ and $\langle \beta_1, \chi_4 \rangle = 0$. Therefore $\beta_1 = \chi_1$. It follows that χ_1 is a part of the eigenspace E_1, giving that

$$E_1 = 1 \cdot U_1 + 0 \cdot U_2 + 0 \cdot U_3 + 0 \cdot U_4.$$

Thus $\langle \beta_2, \chi_1 \rangle = 0$, $\langle \beta_2, \chi_2 \rangle = 0$, $\langle \beta_2, \chi_3 \rangle = 1$ and $\langle \beta_2, \chi_4 \rangle = 1$. Therefore $\beta_2 = \chi_3 + \chi_4$. It follows that χ_3 and χ_4 are part of the the eigenspace E_2, giving that

$$E_2 = 0 \cdot U_1 + 0 \cdot U_2 + 1 \cdot U_3 + 1 \cdot U_4.$$

Thus $\langle \beta_3, \chi_1 \rangle = 0$, $\langle \beta_3, \chi_2 \rangle = 1$, $\langle \beta_3, \chi_3 \rangle = 0$ and $\langle \beta_3, \chi_4 \rangle = 0$. Therefore $\beta_3 = \chi_2$. It follows that χ_2 is a part of the eigenspace E_3, giving that

$$E_3 = 0 \cdot U_1 + 1 \cdot U_2 + 0 \cdot U_3 + 0 \cdot U_4.$$

Hence we have

$$\mathbb{C}V = 1 \cdot U_1 + 1 \cdot U_2 + 1 \cdot U_3 + 1 \cdot U_4.$$

Observe that here we have the regular (transitive) representation of $G = C_4$ of degree 4.

Next let $G = D_4 = \langle a, b : a^4 = b^2 = 1, b^{-1}ab = a^{-1} \rangle$. The character table of this group is shown in Table 3.3.

	1_G	a^2	a	b	ab
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

Table 3.3: The character table of D_4

Note here one irreducible representation has degree 2, as D_4 is not abelian. We evaluate $\beta_1(g), \beta_2(g)$ and $\beta_3(g)$ for each $g \in G$ in Table 3.4.

	1_G	a^2	a	b	ab
β_1	1	1	1	1	1
β_2	2	-2	0	0	0
β_3	1	1	-1	1	-1

Table 3.4: $\beta_i(g)$ for $i = 1, 2, 3$.

Thus $\langle \beta_1, \chi_1 \rangle = 1$, $\langle \beta_1, \chi_2 \rangle = 0$, $\langle \beta_1, \chi_3 \rangle = 0$, $\langle \beta_1, \chi_4 \rangle = 0$ and $\langle \beta_1, \chi_5 \rangle = 0$. Therefore $\beta_1 = \chi_1$. It follows that χ_1 is a part of the eigenspace E_1, giving that

$$E_1 = 1 \cdot U_1 + 0 \cdot U_2 + 0 \cdot U_3 + 0 \cdot U_4 + 0 \cdot U_5.$$

Thus $\langle \beta_2, \chi_1 \rangle = 0$, $\langle \beta_2, \chi_2 \rangle = 0$, $\langle \beta_2, \chi_3 \rangle = 0$, $\langle \beta_2, \chi_4 \rangle = 0$ and $\langle \beta_2, \chi_5 \rangle = 1$. Therefore $\beta_2 = \chi_5$. It follows that χ_5 is a part of the eigenspace E_2, giving that

$$E_2 = 0 \cdot U_1 + 0 \cdot U_2 + 0 \cdot U_3 + 0 \cdot U_4 + 1 \cdot U_5.$$

Thus $\langle \beta_3, \chi_1 \rangle = 0$, $\langle \beta_3, \chi_2 \rangle = 0$, $\langle \beta_3, \chi_3 \rangle = 1$, $\langle \beta_3, \chi_4 \rangle = 0$ and $\langle \beta_3, \chi_5 \rangle = 0$. Therefore $\beta_3 = \chi_3$. It follows that χ_3 is a part of the eigenspace E_3 giving that

$$E_3 = 0 \cdot U_1 + 0 \cdot U_2 + 1 \cdot U_3 + 0 \cdot U_4 + 0 \cdot U_5.$$

Hence we have

$$\mathbb{C}V = 1 \cdot U_1 + 0 \cdot U_2 + 1 \cdot U_3 + 0 \cdot U_4 + 1 \cdot U_5.$$

3.5 Singular Graphs in General and Applications

Let $\Gamma = (V, E)$ be a finite graph with vertex set V. let A be its adjacency matrix. Then Γ is *singular* if A is singular. In this section, we discuss general properties of singular graphs and provide some sufficient conditions for a graph to be singular. Furthermore, the *nullity* of Γ is the dimension of the null space of Γ and we denote this by $\mathrm{null}(\Gamma)$. Note $|V| = \mathrm{null}(\Gamma) + r(\Gamma)$ where $r(\Gamma)$ is the rank of A. Hence singular graphs have a non-trivial null space.

Therefore we are interested in conditions for a graph to have a non-trivial null space. Let $Spec(\Gamma)$ be the set of all eigenvalues of Γ, with their multiplicities.

Singular graphs have important applications in chemistry. The eigenvalue problem has the same structure as the time-independent Schrödinger Equation

$$H\psi = E\psi.$$

Its solutions are the eigenvalues and eigenfunctions (eigenspaces) of the system. Here ψ is the *wave function*, E is the *energy* and H is the Hamiltonian operator of the system considered. When applied to a particular molecule, the Schrödinger Equation enables us one to describe the behaviour of the electrons in this molecule and to establish their energy. The approximation of the π-electron *energy* in chemistry was given by, Erich Hückel in 1930, and the name of this method is the Hückel Molecular Orbital. From this formulation we can write H in terms of the adjacency matrix A of the molecular graph as

$$H = aI + bA$$

where I is the identity matrix and a, b are constants. From this we conclude that finding the spectrum of A is equivalent to finding the spectrum of the Hamiltonian operator H. In chemistry the importance of singular graphs lies in the fact that a singular molecular graph, with vertices formed by atoms, edges corresponding to bonds between the atoms in the molecule, often is associated to compounds that are more reactive or unstable. The problems that we are discussing therefore relate to the stability of a class of molecules. Chemists have significant important applications of Spectral Graph Theory, see [26] and [52] for a reference.

Singular graphs have also important applications for the representation theory of finite groups. For instance, the famous Foulkes's conjecture on the representations of $Sym(a) \wr Sym(b)$ will hold if certain graphs related to symmetric groups are non-singular, see [15], [9] and [46] for a reference.

Let $X \subseteq V$. Then the *induced sub-graph* $\Gamma' = \Gamma[X]$ is the graph (X, E') where E' consists of all $\{v, v'\} \in E$ with both v and v' in X where E is the edge set of Γ. The *incident matrix* M of Γ is the integer matrix with rows and columns indexed by the vertices and edges of Γ, respectively such that the M_{ij}-entry of M is equal to 1 if and only if the vertex v_i is an end vertex of the edge e_j. Note M has dimension $n \times m$ where $|V| = n$ and $|E| = m$.

Let Γ_1 and Γ_2 be two simple graphs. We define the *union* of Γ_1 and Γ_2 to be a graph with vertex set $V(\Gamma_1) \cup V(\Gamma_2)$ and edge set $E(\Gamma_1) \cup E(\Gamma_2)$ and it is denoted by $\Gamma_1 \cup \Gamma_2$. If Γ_1 and Γ_2 are *disjoint* we denote their union by $\Gamma_1 + \Gamma_2$. The *tensor product* of Γ_1 and Γ_2 is a graph has vertex set $V(\Gamma_1) \times V(\Gamma_2)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if $u_1 \sim u_2$ in Γ_1 and $v_1 \sim v_2$ in Γ_2. It is denoted by $\Gamma_1 \otimes \Gamma_2$. Note

$$Spec(\Gamma_1 \otimes \Gamma_2) = \{\lambda_i \mu_j : \lambda_i \in Spec(\Gamma_1), \mu_j \in Spec(\Gamma_2)\}, \tag{3.5.1}$$

see [11] as a reference.

The following properties are obvious.

Theorem 3.5.1. *1.* $\Gamma_1 \otimes \Gamma_2$ *is singular if and only if at least one of* Γ_1 *and* Γ_2 *is singular.*

2. $\Gamma_1 + \Gamma_2$ *(the disjoint union) is singular if and only if at least one of* Γ_1 *and* Γ_2 *is singular.*

Proposition 3.5.2. *Suppose that* $\Gamma = (V, E)$ *is a bipartite graph with parts* $V_1 \dot{\cup} V_2 = V$ *where* $|V_1| \geq |V_2|$*. Then* $\text{null}(\Gamma) \geq |V_1| - |V_2|$.

Proof: Let A be the adjacency matrix of Γ. Hence A has the following shape

$$A = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$$

where B is an $|V_1| \times |V_2|$ matrix. Note by Theorem 2.1.3 we have that $r(A) = 2r(B)$ as $r(B) = r(B^T)$. Therefore we have the following $\text{null}(\Gamma) = |V_1| + |V_2| - 2r(B)$ since $r(B) \leq |V_2|$. So in this case we have that

$$\begin{aligned} \text{null}(\Gamma) &\geq |V_1| + |V_2| - 2|V_2| \\ &\geq |V_1| - |V_2|. \quad \square \end{aligned}$$

Lemma 3.5.3. *[23, Lemma 8.2.3] Let* W *and* U *be vector spaces with linear maps*

$$\varphi : W \to U \text{ and } \varsigma : U \to W.$$

Then $\varphi\varsigma : U \to U$ *and* $\varsigma\varphi : W \to W$ *have the same non-zero eigenvalues. Furthermore, if* λ *is a non-zero eigenvalue with eigenspace* $W_\lambda \subseteq W$ *and* $U_\lambda \subseteq U$ *for* $\varsigma\varphi$ *and* $\varphi\varsigma$ *respectively then* φ *and* ς *restrict to isomorphisms* $\varphi : W_\lambda \to U_\lambda$ *and* $\varsigma : U_\lambda \to W_\lambda$.

Theorem 3.5.4. *[11] Let* $\Gamma = (V, E)$ *be a* k*-regular graph with vertex set* V *of size* n *and edge set* E *of size* m*. Let* $\lambda_1, ..., \lambda_n$ *be the eigenvalues of* Γ.

Then the line graph $L(\Gamma)$ of Γ is a $(2k-2)$-regular graph with eigenvalues $\lambda_i + k - 2$ for $1 \leq i \leq n$ and -2 with multiplicity of $|E| - |V|$. Furthermore, if Γ is bipartite, then the multiplicity of -2 in $L(\Gamma)$ is $|E| - |V| + 1$.

Proof: Let M be the incident matrix of Γ. Then we show that

$$M^T M = 2I_m + A^*$$

where A^* is the adjacency matrix of $L(\Gamma)$. Note $M^T M$ is the $m \times m$ matrix with entries $(M^T M)_{ii} = 2$ as each edge of Γ is incident with two vertices, and $(M^T M)_{ij} = 1$ if and only if e_i, e_j have an end vertex in common; with $(i \neq j)$ and $(M^T M)_{ij} = 0$ otherwise. From this we deduce that $M^T M = 2I_m + A^*$.

Now we prove that

$$MM^T = kI_n + A$$

where A is the adjacency matrix of Γ. Note MM^T is the $n \times n$ matrix with entries $(MM^T)_{ii} = k$ as Γ is k-regular, and $(MM^T)_{ij} = 1$ if and only if $v_i \sim v_j$, and 0 otherwise. From this we conclude that $MM^T = kI_n + A$. By Lemma 3.5.3 we have that $M^T M$ and MM^T have the same non-zero eigenvalues, hence the spectrum of $L(\Gamma)$ is $\lambda_i + k - 2$ for $1 \leq i \leq n$ where λ_i is an eigenvalue of Γ and -2 with multiplicity of $|E| - |V|$. Note if Γ is a bipartite graph then by Theorem 3.1.2 we have that $-k$ is an eigenvalue of Γ, and so the multiplicity of -2 is $|E| - |V| + 1$. □

Corollary 3.5.5. *Let Γ be a k-regular graph. Then Γ is singular if and only if $k-2$ is an eigenvalue of $L(\Gamma)$.*

Proof: Suppose that Γ is singular. So 0 is an eigenvalue of Γ. Hence by Theorem 3.5.4 we have that $k-2$ is an eigenvalue of $L(\Gamma)$. Conversely, if $k-2$ is an eigenvalue of $L(\Gamma)$, then by Theorem 3.5.4 we have that $k - 2 = \lambda_i + k - 2$ where λ_i is an eigenvalue of Γ for some i. From this we conclude that $\lambda_i = 0$ so that Γ is singular. Other possibility we have that

$k - 2 = -2$ hence $k = 0$ and this gives us a contradiction. □

Proposition 3.5.6. *[34] Let Γ be a k-regular graph with n vertices. Then Γ and $\overline{\Gamma}$ have the same eigenvectors and the eigenvalues of $\overline{\Gamma}$ are $n - k - 1$ and $-1 - \lambda_i$ where k and λ_i for $1 \leq i \leq n - 1$ are the eigenvalues of Γ.*

Corollary 3.5.7. *Let Γ be k-regular graph. Then Γ is singular if and only if -1 is an eigenvalue of $\overline{\Gamma}$.*

Proposition 3.5.8. *Let Γ be a graph with nullity $\mathrm{null}(\Gamma) = l \geq 1$ and let $0 \leq i \leq l$. Suppose that V' is a subset of V with $|V'| = |V| - i$. Then $\Gamma[V']$ has nullity $\mathrm{null}(\Gamma[V']) \geq l - i$.*

Proof: Let $V' = V - i$ and $r(\Gamma) = r(A)$ where A is the adjacency matrix of Γ. Note we have that

$$l = |V| - r(\Gamma) \tag{3.5.2}$$

and so

$$\begin{aligned}\mathrm{null}(\Gamma[V']) &= |V'| - r(\Gamma[V']) \\ &= |V| - i - r(\Gamma[V']).\end{aligned}$$

Hence by Equation 3.5.2 we have that

$$\begin{aligned}\mathrm{null}(\Gamma[V']) &= l + r(\Gamma) - i - r(\Gamma[V']) \\ &= l - i - (r(\Gamma[V']) - r(\Gamma)).\end{aligned}$$

Note by Theorem 2.1.2 we have that $r(\Gamma) \geq r(\Gamma[V'])$. From the above we conclude that $\mathrm{null}(\Gamma[V']) \geq l - i$. □

Next we list a further general properties of singular graphs. Our first criterion for singular graphs is a balance condition.

Theorem 3.5.9 (Balance Condition)**.** *Let Γ be a graph with vertex set V. Then Γ is singular if and only if there are two disjoint non-empty subsets*

X, $Y \subseteq V$ and a function $f : X \cup Y \to \mathbb{N}$ with $f(u) \neq 0$ for all $u \in X \cup Y$ such that the following holds:

If v is a vertex in V, then

$$\sum_{v \sim u \in X} f(u) = \sum_{v \sim u \in Y} f(u).$$

In particular, if $X, Y \subset Z$, then $\Gamma[Z]$ is singular.

Proof: Suppose that Γ is singular on n vertices and let $h \in E_*$ with $h \neq 0$ be an element in the kernel of A. In particular, A is singular over $\mathbb{Q}$ as all entries of A are 0 and 1. So we may assume that h_v is rational where $h = \sum_{v \in V} h_v v$ and after multiplying by the least common multiple of all denominators, that h_v is an integer for all v. Let X be the set of all v such that $h_v \geq 1$ and Y the set of all v such that $h_v \leq -1$. Define f^X and f^Y in $\mathbb{C}V$ by $f^X_v = h_v$ for $v \in X$ and $f^X_v = 0$ otherwise, while $f^Y_v = -h_v$ for $v \in Y$ and $f^Y_v = 0$ otherwise. Thus

$$Af^X = Af^Y. \tag{3.5.3}$$

For any $v \in V$ we have that $\langle v, Af^X \rangle = \langle Av, f^X \rangle = \sum_{v \sim u \in X} f^X(u)$. Here we use that A is self-adjoint, that is

$$\langle h, Ak \rangle = \langle Ah, k \rangle$$

for all $h, k \in \mathbb{C}V$. Similarly, $\langle v, Af^Y \rangle = \sum_{v \sim u \in X} f^Y(u)$. Hence by Equation 3.5.3 we have that $\sum_{v \sim u \in X} f(u) = \sum_{v \sim u \in Y} f(u)$ for all $v \in V$.

Suppose that the above condition holds. This means that

$$\sum_{v \sim u \in X} f(u) = \sum_{v \sim u \in Y} f(u)$$

for all $v \in V$. Now we prove that Γ is singular. As before A is the adjacency matrix of Γ. Note A is non-singular if and only if its rows are linearly independent. Suppose that A_v be the row of A labelled by a vertex $v \in V(\Gamma)$. Note we have that

$$\sum_{x \in X} f(x)A_x - \sum_{y \in Y} f(y)A_y = 0$$

where $f(v) \neq 0$ for all $v \in X \cup Y$. From this we conclude that the rows of A are linearly dependent and so A is singular. □

Example 1: Let $\Gamma = (V, E)$ be a graph. Suppose that $w, u \in V$ such that $w \nsim u$, and w and u have the same neighbour set. In this case put $X = \{w\}, Y = \{u\}$ and $f(u) = f(w) = 1$ while $f(v) = 0$ for $u \neq v \neq w$. Then f has the property of the theorem. More directly of course, $\alpha(w) = \alpha(u)$ implies that $0 \neq w - u \in E_*$.

Example 2: Let $\Gamma = C^n$ be an n-cycle on $V = \{1, 2, ..., n\}$. Then it is easy to compute the eigenvalues of Γ. These are the numbers $\lambda_r = 2\cos(\frac{2\pi r}{n})$ where $r = 0, 1, 2, ..., n-1$, see [10] as a reference. In particular, C^n is singular if and only if n is divisible by 4. If 4 does divide n we may take $X = \{a \in V \ : \ a \equiv 0 \text{ or } 1 \pmod 4\}$, $Y = \{b \in V \ : \ b \equiv 2 \text{ or } 3 \pmod 4\}$ and $f(v) = 1$ for all $v \in X \cup Y$ while $f(v) = 0$ for all $v \notin X \cup Y$.

Example 3: Let $\Gamma = P^n$ be a path on n vertices, $V = \{1, 2, ..., n\}$. Then it is easy to compute the eigenvalues of Γ. These are the numbers $\lambda_r = 2cos(\frac{\pi r}{n+1})$ where $r = 1, 2, ..., n$. In particular, P^n is singular if and only if n is odd see [10] as a reference. If n is odd then we have that $X \cup Y$ contains the odd numbers $\leq n$, and we choose $X = \{1, 5, 9, ...\}$, $Y = \{3, 7, 11, ...\}$ and $f(v) = 1$ for all $v \in X \cup Y$ while $f(v) = 0$ for all $v \notin X \cup Y$.

This concludes our comments about singular graphs in general. In the next chapter we turn to singular vertex transitive graphs.

4

Vertex Transitive Graphs

A graph Γ is said to be *vertex transitive* if its automorphism group acts transitively on its vertex set. In other words, for any two vertices u, v of Γ there is $g \in Aut(\Gamma)$ such that $v^g = u$. It is clear that vertex transitive graphs are regular. If the degree of Γ is k then by Theorem 3.1.2 we have that k is an eigenvalue of Γ with multiplicity of $\geqslant 1$ and all other eigenvalues will be less than k and greater than or equal to $-k$. The multiplicity of k in fact is the number of components of Γ. In this chapter we use a transitive group of automorphisms to find the spectrum of Γ and determine sufficient conditions for Γ to have 0 as an eigenvalue.

This chapter is divided into four main sections. In the first section, we study properties of Cayley graphs and their spectrum. In the second section, we investigate the singularity of Cayley graphs. In the third section, we reduce the problem of finding the spectrum of a vertex transitive graph to finding the spectrum of an associated Cayley graph. This method is due to Lovász [43]. In the last section, we compute the spectrum of a vertex transitive graph in terms of irreducible characters of a transitive group of automorphisms. To our knowledge this method is new. In each section of this chapter we provide conditions that distinguish singular graphs.

4.1 Cayley Graphs

We now discuss the definition of Cayley graph associated to a finite group and a certain set that generates the group. We introduce basic properties of the automorphism group of a Cayley graph and compute its spectrum in terms of the irreducible representations and the irreducible characters of the group. Moreover we investigate the relationship between the representations of the group and the eigenspaces of the Cayley graph.

Let G be a finite group with identity element $1 = 1_G$. A subset H of G is called a *connecting set* if

(1) $H^{-1} = \{h^{-1} \,|\, h \in H\} = H$

(2) $1_G \notin H$

(3) H generates G.

In this case we can define a graph Γ with vertex set $V(\Gamma) = G$. Two vertices v, w are adjacent, $v \sim w$, if and only if $wv^{-1} \in H$ if and only if $w \in Hv$ if and only if $w = h^{-1}v$ for some $h \in H$. Note that $wv^{-1} \in H$ implies $(wv^{-1})^{-1} = vw^{-1} \in H$ and therefore $w \sim v$. In addition, $v \not\sim v$ since $v^{-1}v = 1_G \notin H$. This means that $\sim$ defines a simple connected undirected graph on G.

This graph is called the *Cayley graph* on G with connecting set H. It is denoted by $Cay(G, H)$. It follows that the adjacency map $\alpha : \mathbb{C}G \to \mathbb{C}G$ has the form

$$\alpha(v) = \sum_{h \in H} h^{-1}v \tag{4.1.1}$$

for all $v \in G$.

An arbitrary graph X is said to be a Cayley graph if there exists a group G and a connecting set H such that X is isomorphic to $Cay(G, H)$. Note a graph Γ can be a Cayley graph for several different groups and connecting set. For instance, K_n is the Cayley graph for any group G of order n and connecting set $H = G \backslash 1_G$.

Note also that if H is a subset of G which satisfies the first two requirements above but not the last one then we still have the Cayley graph $Cay(G^*, H)$ where G^* is the group generated by H.

We collect a few properties of Cayley graphs. Let G be a group and H a connecting set. Denote the Cayley graph of G for connecting set H by $\Gamma = Cay(G, H)$. Let v be a vertex of Γ. Then the set of all neighbours of v is Hv. It follows that Γ is k-regular with $k = |H|$. Applying the same argument again the set of all neighbours of vertices in Hv is HHv. Hence the connected component containing v consists of the vertices in $HH...Hv$. Since H generates G we have $G = HH...H$ and therefore Γ is connected. Therefore we have the following well-known standard result.

Theorem 4.1.1. *[37, Proposition 1.29] Let G be a group and H a connecting set for the graph $\Gamma = Cay(G, H)$. Then Γ is a connected k-regular graph with $k = |H|$.*

Now we give some examples of Cayley graphs. The Cayley graphs over cyclic groups have played a special role in the study of Cayley graphs. These graphs are widely known as *circulant graphs.* The adjacency matrix of a circulant graph is a *circulant matrix.*

The *complete bipartite graphs* $K_{n,n} = Cay(G, H)$ where $|G| = 2n = 2|H|$, $H = G \backslash K$ and K is a subgroup of G, are Cayley graphs. Similarly, the *k-dimensional cube graph* Q_k is the Cayley graph defined on the elementary abelian group $(\mathbb{Z}_2)^k$ where the connecting set is the standard generating set for $(\mathbb{Z}_2)^k$.

The graph formed on the finite field $\mathbb{F}_q$ (addition group) as vertex set where $q \equiv 1 \bmod 4$ and where the connecting set is $H = \{x^2 : x \in \mathbb{F}_q, x \neq$

$0\}$ is called the *Paley graph* for q. Note, the condition that $q \equiv 1 \bmod 4$ guarantees that $H = -H$.

Let n be even and $G = \mathbb{Z}/nZ$ and let $H = \{\mp 1, \frac{n}{2}\} \subset G$. Then the Cayley graph $\Gamma = Cay(G, H)$ is known as the *Mőbius ladder* graph of order n.

In the rest of this section we discuss the automorphisms of a Cayley graph. There are special properties for the automorphism group of $\Gamma = Cay(G, H)$ related to the group G and the connecting set H. The problem of determining the full automorphism group of Γ is difficult in general. The full automorphism group $Aut(\Gamma)$ of Γ is the set of all permutations of the set $V = G$ preserving the edge structure, see Section 3 of Chapter 2. We describe some aspects of the automorphisms of Γ.

The multiplication on the right by the element g in G, that is $v \mapsto vg$ for $v \in V$, induces an automorphism on Γ. To prove this let $v, v' \in V$. If $v \sim v'$ then $v' = h^{-1}v$ for some $h \in H$ and so $v'g = h^{-1}(vg)$ giving that $vg \sim v'g$. Conversely, if $v \nsim v'$ then $vg \nsim v'g$. We can understand this automorphism in terms of the right regular representations $\rho_r : G \to GL(\mathbb{C}G)$ of G given by $\rho_r(g)(v) = vg$.

The left regular action $v \mapsto g^{-1}v$ however is in general not an automorphism of Γ. In fact, it is easy to see that $v \mapsto g^{-1}v$ for $v \in V$ is an automorphism if and only if $gH = Hg$. We say that H is *normal* if $gH = Hg$ for all $g \in G$.

The right regular action is transitive on vertices and only the identity element fixes any vertex. This is therefore the regular action of G on itself. This property characterises Cayley graphs.

Theorem 4.1.2. *(Sabidussi's Theorem)[23, Lemma 3.7.1] Let $\Gamma = (V, E)$ be a graph. Then Γ is a Cayley graph if and only if $Aut(\Gamma)$ contains a subgroup G which is regular on V.*

We now describe other automorphisms of Γ. Let ϕ be a group automorphism of G fixing H as a set. Such ϕ induces an automorphism of Γ. To prove this let $v, v' \in V$. If $v \sim v'$ then $v' = h^{-1}v$ for some $h \in H$ and so

$$\begin{aligned} \phi(v') &= \phi(h^{-1}v) \\ &= \phi(h^{-1})\phi(v) \\ &= h'^{-1}\phi(v) \end{aligned}$$

for some $h' \in H$ as $\phi(H) = H$. Hence $\phi(v) \sim \phi(v')$. Conversely, if $v \nsim v'$ then $\phi(v) \nsim \phi(v')$. The following is due to this result.

Theorem 4.1.3. *[40] Suppose that ϕ is an automorphism of the group G that fixes H set-wise. Then ϕ is an automorphism of $Cay(G, H)$ fixing the identity element of G.*

4.1.1 Representations and Spectrum

We fix some group G and a connecting set H. Let $\Gamma = Cay(G, H)$. Now we study the relationship between the irreducible representations of G and the eigenspaces of Γ. We know that G acts on $V(\Gamma)$ regularly by multiplication on the right and so G is isomorphic to a regular subgroup of $Aut(\Gamma)$. We come back to use the general representation theory of finite groups to study the relationship between the irreducible characters of G and the eigenspaces of Γ. According to this theory our group G has irreducible $\mathbb{C}$-modules $U_1, ..., U_s$. This means that there are representations $\rho_1, \rho_2, ..., \rho_s$ which are homomorphisms $\rho_i : G \longrightarrow GL(U_i)$ such that only the trivial spaces 0 and U_i are invariant under $\rho_i(G)$. The function

$$\chi_i(g) = \mathrm{tra}(\rho_i(g))$$

is the *character* associated to ρ_i. Note we define $\chi_i(H) = \sum_{h \in H} \chi_i(h)$.

As we are working over $\mathbb{C}$ every G-module decomposes into a direct sum of irreducible modules. We apply this in particular to $\mathbb{C}V$, the vertex space of Γ. In our case we can therefore write

$$\mathbb{C}V = \underbrace{U_1 + U_1 + ... + U_1}_{m_1} \oplus \underbrace{U_2 + U_2 + ... + U_2}_{m_2} \oplus ... \oplus \underbrace{U_s + U_s + ... + U_s}_{m_s}.$$

In particular, $m_i = dim(U_i)$ since G acts regularly on $V(\Gamma)$.

Recall that each eigenspace E_i for $i = 1, ..., t$ is invariant under the right multiplication by G, by Theorem 3.4.2. Using the same principle again, each eigenspace E_i of Γ can also be decomposed into irreducible modules

$$E_i = \underbrace{(U_1 + ... + U_1)}_{m_{i_1}} \oplus \underbrace{(U_2 + ... + U_2)}_{m_{i_2}} \oplus ... \oplus \underbrace{(U_s + ... + U_s)}_{m_{i_s}}.$$

It is clear that

$$\sum_{i=1}^{t} m_{i_1} = m_1, \ \sum_{j=1}^{t} m_{j_2} = m_2, \ ...$$

and so on. Let $\pi_i : \mathbb{C}V \longrightarrow \mathbb{C}V$ with $\pi_i(\mathbb{C}V) = E_i$ be the projection onto the eigenspace E_i of Γ. Hence by the discussions in the last chapter we can apply Theorem 3.4.4 to determine which of the irreducible representations of G are a part of each eigenspace of Γ.

Next consider the left-regular representation of G : For each $h \in G$ we have the linear map $\rho_l(h) : \mathbb{C}V \to \mathbb{C}V$ defined on the standard basis of $\mathbb{C}V$ by

$$\rho_l(h)(v) = h^{-1}v, \quad \text{for } v \in V.$$

Recall that $\alpha(v) = \sum_{h \in H} h^{-1}v$ for all $v \in G$ by Equation 4.1.1 and therefore

$$\alpha = \sum_{h \in H} \rho_l(h) \ \text{ as a map } \mathbb{C}V \to \mathbb{C}V. \tag{4.1.2}$$

Therefore Equation 4.1.2 establishes the link between graph theory and representation theory of finite groups. Hence by Theorem 2.2.11 we can decompose

$$\rho_l = m_1\rho_1 \oplus ... \oplus m_s\rho_s$$

into a direct sum of irreducible representations of G. Thus

$$\alpha = \sum_{h\in H} \rho_l(h) = \sum_{h\in H}(m_1\rho_1(h) \oplus ... \oplus m_s\rho_s(h))$$

where $\rho_i(h) : G \to GL(U_i)$ and m_i is the degree of ρ_i, and at the same time m_i is the dimension of the irreducible G-module U_i of $\mathbb{C}V$ for each i. The following is an important general fact.

Theorem 4.1.4. *Let G be a finite group and let $\rho_1, ..., \rho_s$ be the set of all inequivalent irreducible representations of G. Then λ is an eigenvalue of $Cay(G, H)$ if and only if there is some ρ_i such that $\sum_{h\in H} \rho_i(h) - \lambda$ is singular.*

Proof: Let $U_1, ..., U_s$ be the irreducible G-modules. Let $E_1, ..., E_t$ be the eigenspaces of α. By Theorem 3.4.2 $E_1, ..., E_t$ are G-invariant (under multiplication on the right) and so each E_j can be decomposed into

$$E_j = m_{j1}U_1 \oplus ... \oplus m_{js}U_s,$$

as before. Now E_j is the eigenspace of α for the eigenvalue λ if and only if $\alpha - \lambda$ is singular on E_j. This in turn implies that $\sum_{h\in H} \rho_l(h) - \lambda$ is singular on E_j. Let U_i be an irreducible G-module that appears in E_j. Then $\sum_{h\in H} \rho_i(h) - \lambda$ is singular.

Conversely, if $\sum_{h\in H} \rho_i(h) - \lambda$ is singular on U_i then U_i appears in the decomposition of

$$\mathbb{C}G = E_1 \oplus ... \oplus E_t$$

as this the regular G-module, and so $\alpha - \lambda$ is singular. □

Theorem 4.1.5. *[6] Let G be a finite group and let H be a connecting set. Let $\rho_1, ..., \rho_s$ be the complete set of all inequivalent irreducible* *representations of* G *and let* $m_1, ..., m_s$ *be their degrees. Then the spectrum of* $\Gamma = Cay(G, H)$ *is given by*

$$Spec(\Gamma) = \{\lambda_{1,1}^{m_1}, ..., \lambda_{s,1}^{m_s}, ..., \lambda_{s,m_s}^{m_s}\}$$

where $\lambda_{i,j}^{m_i}$ *is the* j^{th} *eigenvalue of* $\sum_{h\in H} \rho_i(h)$ *with the multiplicity* m_i*. More generally we have that*

$$\lambda_{i,1}^r + \lambda_{i,2}^r + ... + \lambda_{i,m_i}^r = \mathrm{tra}((\pi_i\alpha)^r) = \mathrm{tra}((\sum_{h\in H} \rho_i(h))^r) = \sum_{h_1,h_2,...,h_r\in H} \chi_i(h_1h_2...h_r)$$

for any natural number $r \leq m_i$*. Note,* $\lambda_{i,j}^r$ *is the* r^{th} *power of* $\lambda_{i,j}$ *(not a multiplicity) and* $m_1^2 + ... + m_s^2 = n$*.*

Proof: Note we have that

$$\alpha = \sum_{h\in H} \rho_l(h) = m_1 \sum_{h\in H} \rho_1(h) \oplus ... \oplus m_s \sum_{h\in H} \rho_s(h).$$

Here $\sum_{h\in H} \rho_i(h)$ is an m_i by m_i matrix and by Theorem 4.1.4 we have that each eigenvalue of $\sum_{h\in H} \rho_i(h)$ is an eigenvalue of α for $i = 1, ..., s$. Hence $\sum_{h\in H} \rho_i(h)$ has m_i eigenvalues and in the same time these eigenvalues of α. Note for each i we have that $\{\lambda_{i,1}, ..., \lambda_{i,m_i}\}$ are the eigenvalues of $\sum_{h\in H} \rho_i(h)$. Hence by the trace properties we have that

$$\begin{aligned} \lambda_{i,1} + \lambda_{i,2} + ... + \lambda_{i,m_i} &= \mathrm{tra}(\sum_{h\in H} \rho_i(h)) \\ &= \sum_{h\in H} \mathrm{tra}(\rho_i(h)) \\ &= \sum_{h\in H} \chi_i(h). \end{aligned}$$

Similarly, let $r \in \mathbb{N}$ where $r \leq m_i$. Then we have that

$$\lambda_{i,1}^r + \lambda_{i,2}^r + ... + \lambda_{i,m_i}^r = \mathrm{tra}((\sum_{h\in H} \rho_i(h))^r).$$

Note ρ_i is a group homomorphism. Therefore we have that

$$\begin{aligned}\lambda_{i,1}^r + \lambda_{i,2}^r + ... + \lambda_{i,m_i}^r &= \text{tra}(\sum_{h1,h_2,...,h_r \in H} \rho_i(h_1 h_2 ... h_r)) \\ &= \sum_{h_1,h_2,...,h_r \in H} \chi_i(h_1 h_2 ... h_r).\end{aligned}$$

Note, about the multiplicity we have that

$$\alpha = \sum_{h \in H} \rho_l(h) = m_1 \sum_{h \in H} \rho_1(h) \oplus ... \oplus m_s \sum_{h \in H} \rho_s(h).$$

From the above we conclude that each $\rho_i(h)$ appears m_i times in α so that the multiplicity of $\lambda_{i,j}$ is m_i for each j. □

Example: Let $D_4 = \langle a, b : a^4 = b^2 = 1_{D_4}, bab = a^3 \rangle$. Let $\Gamma = Cay(D_4, H)$ where $H = \{a, a^3, b\}$. So Γ has 8 vertices of degree 3. In this example we apply Theorem 4.1.5 to compute the spectrum of Γ. Note the character table of D_4 is shown in the Table 4.1.

	1_{D_4}	a^2	a	b	ab
$\lvert g_i^{D_4} \rvert$	1	1	2	2	2
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

Table 4.1: The character table of D_4.

Note χ_1, χ_2, χ_3 and χ_4 are of degree 1 so that $\lambda_{i,1} = \sum_{h \in H} \chi_i(h)$ for $i = 1, 2, 3, 4$. In this case we have that

$$\lambda_{1,1} = \sum_{h \in H} \chi_1(h) = 3, \;\; \lambda_{2,1} = \sum_{h \in H} \chi_2(h) = 1, \;\; \lambda_{3,1} = \sum_{h \in H} \chi_3(h) = -1$$

and $\lambda_{4,1} = \sum_{h \in H} \chi_4(h) = -3$. However χ_5 is of degree 2 in this case we have that

$$\begin{aligned} \lambda_{5,1} + \lambda_{5,2} &= \sum_{h \in H} \chi_5(h) = 0 \\ \lambda_{5,1}^2 + \lambda_{5,2}^2 &= \sum_{h_i, h_j \in H} \chi_5(h_i h_j) = 2. \end{aligned}$$

By solving these equations we have that $\lambda_{5,1} = 1$ and $\lambda_{5,2} = -1$. Hence we have that $Spec(\Gamma) = \{3^1, -3^1, 1^3, -1^3\}$.

From the above, we conclude that we can determine the spectrum of the Cayley graph $Cay(G, H)$ by Theorem 4.1.4 and Theorem 4.1.5 at least in principle.

In the next results we can compute the spectrum of the Cayley graph $Cay(G, H)$ in terms of the irreducible characters of G when H is a normal connecting set.

Theorem 4.1.6. *Let Γ be the Cayley graph $\Gamma = Cay(G, H)$ where H is a normal connecting set of G. Let U be an irreducible sub-module of $\mathbb{C}V = \mathbb{C}G$, (by right multiplication). Then $\alpha(U) = U$ and furthermore U is contained in the eigenspace of α for*

$$\lambda = \frac{1}{\chi(1_G)} \sum_{h \in H} \chi(h)$$

where χ is the irreducible character corresponding to U.

Proof: Let $\mathbb{C}V$ be the vertex G-module of Γ. Let $\rho = \rho_i : G \to GL(U)$. Then by using Equation 4.1.2 we have that $\alpha = \sum_{h \in H} \rho_l(h)$ where $\rho_l(h)$ is the left regular representation of G. Let $\rho_1, ..., \rho_s$ be the irreducible representations of G. Hence we have that

$$\sum_{h \in H} \rho_l(h) = \oplus_{j=1}^{s} \sum_{h \in H} m_j \rho_j(h)$$

where m_j is the degree of ρ_j. Thus we compute

$$\begin{aligned}\alpha \circ \rho(g) &= \sum_{h \in H} \rho_l(h) \circ \rho(g) \\ &= \oplus_{j=1}^{s} \sum_{h \in H} m_j \rho_j(h) \circ \rho(g).\end{aligned}$$

Since, $\rho_j(h)\rho(g) = \rho_i(hg)$ if $j = i$ and 0 otherwise. We continue

$$\begin{aligned}\alpha \circ \rho(g) &= \sum_{h \in H} \rho_i(hg) \\ &= \sum_{h \in H} \rho_i(ghg^{-1}g) \\ &= \sum_{h \in H} \rho_i(gh).\end{aligned}$$

Since H is a normal connecting set. Continuing, we get

$$\begin{aligned}\alpha \circ \rho(g) &= \rho_i(g) \sum_{h \in H} \rho_i(h) \\ &= \rho_i(g) \oplus_{j=1}^{s} \sum_{h \in H} m_j \rho_j(h) \\ &= \rho(g) \circ \alpha\end{aligned}$$

for all $g \in G$.

Therefore by using Theorem 2.2.6 we have that $\alpha(u) = \lambda u$ for some $\lambda \in \mathbb{C}$ and all $u \in U$. So we have that $U \subseteq E_i$ for some i where E_i is the eigenspace corresponding to the eigenvalue λ. Note we have that

$$m\lambda = \text{tra}(\lambda u) = \text{tra}(\alpha(u)) = \text{tra}(\sum_{h \in H} \rho(h)(u)) = \sum_{h \in H} \chi(h)$$

where $m = \chi(1_G)$. So we have that

$$\lambda = \frac{1}{\chi(1_G)} \sum_{h \in H} \chi(h).$$

For the multiplicity each irreducible character χ there are $\chi(1_G)$ copies of

U in $\mathbb{C}V$ and on each copy α acts as λid_U. Therefore λ has multiplicity $(\chi(1))^2$. □

Comment: There are several papers in which a formula for eigenvalues is given, for instance [43], [6], [37], [59], [11] and [19]. The precise characterization of an arbitrary G-invariant irreducible module as an eigenspace we believe is new.

Note: In the first example we have $\Gamma = Cay(D_4, H)$ where $H = \{a, a^3, b\}$. We have that

$$\mathbb{C}D_4 = E_1 \oplus E_2 \oplus E_3 \oplus E_4$$

where E_1, E_2, E_3, E_4 are the eigenspaces for the eigenvalue $3, -3, 1, -1$, respectively. We noted that U_5 appears in both E_3 and E_4. Note this is possibly as H is not normal in G.

We note the following special cases of the theorem.

Corollary 4.1.7. *Let $\Gamma = Cay(G, H)$ where H is a normal connecting set. If U, U^* are G-isomorphic irreducible sub-modules of $\mathbb{C}V$ then U, U^* are contained in the same eigenspace E_i for some i where E_i is the eigenspace corresponding to the eigenvalue $\lambda = \frac{1}{\chi(1_G)} \sum_{h \in H} \chi(h)$, where χ is the irreducible character corresponding to U, U^*. In particular, each eigenspace E_j of Γ has dimension $\sum_{i=i_1,\ldots,i_l} (\chi_i(1_G))^2$ where the sum over all irreducible modules $U_i \subseteq E_j$.*

Corollary 4.1.8. *Let $\Gamma = Cay(G, H)$ where H is a normal connecting set. If U, U^* are not G-isomorphic irreducible sub-modules of $\mathbb{C}V$ then U, U^* are in the same eigenspace of α if and only if*

$$\frac{1}{\chi(1_G)} \sum_{h \in H} \chi(h) = \frac{1}{\chi^*(1_G)} \sum_{h \in H} \chi^*(h)$$

where χ and χ^* are the irreducible character of U and U^* respectively.

Theorem 4.1.9. *[35] Let G be a group and H a connecting set of G. Let Γ*

be the graph $\Gamma = Cay(G, H)$ *and let* χ *be any* 1*-dimensional character of* G*. Then* $f = \sum_{v \in G} \chi(v)v$ *is an eigenvector of* Γ *with eigenvalue* $\sum_{h \in H} \chi(h)$.

The proof was given in [35] for an additive abelian groups. However, the following is my version of proof for general groups.

Proof: Let α be the adjacency map of Γ and let $f \in \mathbb{C}G$ where $f = \sum_{v \in G} \chi(v)v$. Then

$$\begin{aligned} \alpha(f) &= \sum_{v \in G} \chi(v)\alpha(v) \\ &= \sum_{v \in G} \chi(v) \sum_{h \in H} (h^{-1}v) \\ &= \sum_{v \in G} \chi(v) \sum_{h \in H} u \end{aligned}$$

where $u = h^{-1}v$. Hence

$$\begin{aligned} \alpha(f) &= \sum_{u \in G} \sum_{h \in H} \chi(hu)u \\ &= \sum_{u \in G} \sum_{h \in H} \chi(h)\chi(u)u \\ &= \sum_{h \in H} \chi(h) \sum_{u \in G} \chi(u)u \\ &= (\sum_{h \in H} \chi(h))f. \end{aligned}$$

□

This theorem determines the spectrum and eigenspaces of Cayley graphs over abelian groups. The following is an example for a non-commutative group.

Example: Let $\Gamma = (Sym(3), H)$ and

$$H = \{(12), (13), (23)\}.$$

In this example we compute the the spectrum of Γ and we apply Theorem 3.4.4 to determine which of the irreducible representations of $Sym(3)$ are part of the kernel $(\Gamma) = E_*$ where $*$ denotes the number of the zero-eigenspace and β_2 is the character of that space. Note, here H is a normal connecting set.

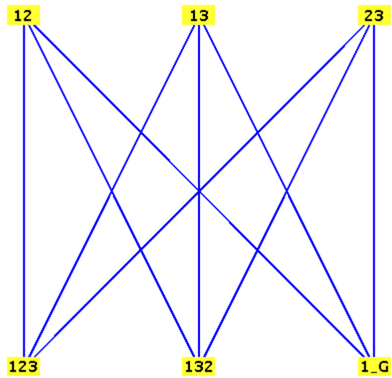

Figure 4.1.1: $\Gamma = Cay(Sym(3), H)$ and $H = \{(12), (13), (23)\}$

The adjacency map α has the adjacency matrix (on the standard basis)

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then we apply Theorem 4.1.6 to compute the spectrum. The character table of $Sym(3)$ is shown in Table 4.2.

	$1_{Sym(X)}$	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Table 4.2: The character table of $Sym(3)$

Therefore for $i = 1, 2, 3$ we have that $\lambda_i = \frac{1}{m_i} \sum_{h \in H} \chi_i(h)$. Hence $\lambda_1 = 3$

with multiplicity of 1, $\lambda_2 = 0$ with multiplicity of 4 and $\lambda_3 = -3$ with multiplicity 1. Note in this example we have that $* = 2$. Now the projection map is

$$\begin{aligned}\pi_2 &= \prod_{2\neq j} \frac{1}{\lambda_2 - \lambda_j}(A - \lambda_j) \\ &= \frac{1}{-9}(A+3)(A-3)\end{aligned}$$

and has matrix

$$P_2 = \frac{1}{-9}\begin{pmatrix} -6 & 3 & 3 & 0 & 0 & 0 \\ 3 & -6 & 3 & 0 & 0 & 0 \\ 3 & 3 & -6 & 0 & 0 & 0 \\ 0 & 0 & 0 & -6 & 3 & 3 \\ 0 & 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 0 & 3 & 3 & -6 \end{pmatrix}.$$

We evaluate $\beta_2(g)$ for each $g \in Sym(3)$ in the following table

	$1_{Sym(X)}$	(12)	(123)
β_2	4	0	-2

Table 4.3: The class function $\beta_*(g)$

Thus we have that $\langle\beta_2, \chi_1\rangle = 0$, $\langle\beta_2, \chi_2\rangle = 0$ and $\langle\beta_2, \chi_3\rangle = 2$. We see that $\beta_2 = 2\chi_3$. Therefore χ_3 is part of the kernel E_2. In fact, the kernel E_2 has dimension 4 and so $E_2 = 2U_3$.

4.2 Singular Cayley Graphs

In this section we determine conditions for the singularity of a Cayley graph. Let G be a finite group and H a connecting set of G. Let Γ be the graph $\Gamma = Cay(G, H)$. In this section we denote by K a subgroup of G.

Theorem 4.2.1. *[35] Let H be a connecting set of a group G and let H be a union of left cosets of a non-trivial subgroup K in G. Suppose there is some element $k \in K$ and 1-dimensional character χ of G such that $\chi(k) \neq 1$. Then we have that $\sum_{h\in H}\chi(h) = 0$. In particular, $\Gamma = Cay(G,H)$ is singular.*

This theorem is mentioned in [35] for a subgroup of an additive abelain group. However, the following proof is my version for a union of cosets of a non-trivial subgroup of the group in general.

Proof: Suppose that $H = a_1K \cup a_2K \cup ...$ for $a_1, a_2, ... \in G$ and χ is an 1-dimensional character of G with $\chi(k) \neq 1$ for some $k \in K$. Then we have that

$$\begin{aligned}\sum_{h\in H}\chi(h) &= \sum_{a_iK\cap H\neq\phi}\chi(a_iK)\\ &= \sum_{a_iK\cap H\neq\phi}\chi(a_iKk)\\ &= \sum_{a_iK\cap H\neq\phi}\chi(a_iK)\chi(k)\\ &= \sum_{h\in H}\chi(h)\chi(k).\end{aligned}$$

Hence we have that $\sum_{h\in H}\chi(h)(1-\chi(k)) = 0$. Since $\chi(k) \neq 1$ so that $\sum_{h\in H}\chi(h) = 0$. □

As a consequence to Theorem 4.1.4 and Theorem 4.1.6 we have the following results:

Theorem 4.2.2. *Let G be a finite group and let H be a connecting set of G. Let $\rho_1, .., \rho_s$ denote the irreducible representations of G. Then $\Gamma = Cay(G,H)$ is singular if and only if there exists some i such that $\sum_{h\in H}\rho_i(h)$ is singular.*

Theorem 4.2.3. *Let G be a finite group and let H be a connecting set and normal subset of G. Then $Cay(G,H)$ is singular if and only if there is an* *irreducible character* χ *of* G *such that* $\sum_{h\in H}\chi(h) = 0$. *In particular, we have that* $\text{null}(\Gamma) \geq (\chi(1_G))^2$.

Proof: Suppose that Γ is singular. Then we have that 0 is an eigenvalue of Γ. Note H is normal subset of G and so by Theorem 4.1.6 each eigenvalue of $Cay(G, H)$ is given by

$$\lambda = \frac{1}{\chi(1_G)} \sum_{h\in H} \chi(h)$$

where χ is an irreducible character of G. Hence we have that $\sum_{h\in H}\chi(h) = 0$ for some irreducible character of G.

Suppose that $\sum_{h\in H}\chi(h) = 0$ for some irreducible character of G. Then by Theorem 4.1.6 we have that $\lambda = 0$ for some eigenvalues of $Cay(G, H)$ so $Cay(G, H)$ is singular. Furthermore, by Corollary 4.1.7 we have that E_* contains the module of χ with multiplicity of $(\chi(1_G))^2$. □

Corollary 4.2.4. *Suppose* G *is non-abelian simple group. Suppose* H *is any subset of* G *with* $1_G \notin G, H = H^{-1}$ *and* H *is normal. Then nullity of* $\Gamma = Cay(G, H)$ *is either* 0 *or* $\geq m^2$ *where* $m \neq 1$ *is the least degree of an irreducible character of* G.

Example: The possible value of m in the following simple groups: $m = m(A_6) = 5$, $m = m(A_{11}) = 10$ and $m = m(PSL(2, 23)) = 11$. So, for instance, if $G = PSL(2, 23)$ and if H is any normal subset with $1 \notin H$ and $H = H^{-1}$, then $\Gamma = Cay(G, H)$ is either non-singular or has nullity $\geq 11^2$, see Theorem 4.2.3.

Example: Let $\Gamma = Cay(Sym(3), H)$ where

$$H = \{(12), (23), (123), (132)\}.$$

In this example we apply Theorem 4.2.2 to decide the singularity of Γ.

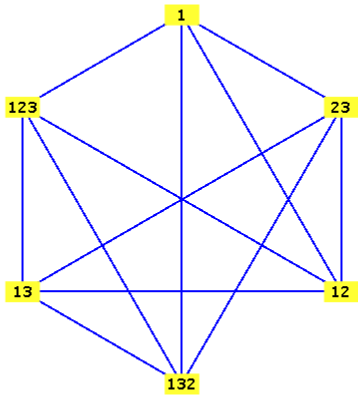

Figure 4.2.1: $\Gamma = Cay(Sym(3), H)$ where $H = \{(12), (23), (123), (132)\}$

Let α be the adjacency map of Γ and let A be the matrix represents α on the basis $Sym(3)$. So we have that A is equivalent to

$$\sum_{h \in H} \rho_l(h) = \sum_{h \in H} (m_1 \rho_1(h) \oplus ... \oplus m_s \rho_s(h))$$

where $\rho_i(h)$ is an irreducible representation of $Sym(3)$. Let $\mu_i = \sum_{h \in H} \rho_i(h)$ for $i = 1, 2, 3$. The irreducible representations of $Sym(3)$ for H are shown in Table 4.4.

H	(12)	(23)	(123)	(132)	μ_i
ρ_1	1	1	1	1	4
ρ_2	-1	-1	1	1	0
ρ_3	$\begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}$	$\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$	$\begin{pmatrix} -1 & -\omega \\ -\omega^2 & -1 \end{pmatrix}$

Table 4.4: $\rho_i(h)$ and $\mu_i = \sum_{h \in H} \rho_i(h)$ for $i = 1, 2, 3$.

The adjacency matrix of Γ is equivalent to the following matrix

$$\begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -\omega & 0 & 0 \\ 0 & 0 & -\omega^2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -\omega \\ 0 & 0 & 0 & 0 & -\omega^2 & -1 \end{pmatrix}.$$

It is clear that μ_2 and μ_3 are singular matrices and the eigenvalues of μ_3 are $\{-2, 0\}$. So Γ is singular by Theorem 4.2.2 and its nullity is in fact 3.

Theorem 4.2.5. *Let H be a connecting set in the group G and suppose that H is a union of right cosets of the subgroup K of G with $|K| \neq 1$. Then $A(\Gamma)$ is of the form $A(\Gamma^*) \otimes J$ where $\Gamma = Cay(G, H)$, Γ^* is some graph defined on the right cosets of K in G and J is the $|K| \times |K|$ matrix with all entries equal to* 1.

COMMENTS:(1) If H is a union of left cosets of K then it is a union of right cosets since $H = H^{-1}$. However, $aK \cup Ka^{-1}$ may not be a union of left or right cosets.

(2) Note Γ^* is a graph with vertex set as the right cosets of K in G and $Kg_i \sim Kg_j$ in Γ^* if and only there is an element in Kg_i adjacent to an element in Kg_j in Γ. In general may not be a Cayley graph and in some cases Γ^* is a Cayley graph, for instance if K is normal.

Proof of Theorem 4.2.5: Suppose that $H = Ka_1 \cup Ka_2 \cup Ka_3 \cup ...$ for some $a_1, a_2, ... \in G$. We want to prove that any two elements in the same right coset of K are not adjacent and if an element in Kg_i is adjacent to an element in Kg_j then all elements in Kg_i are adjacent to all elements in Kg_j.

Let $x, \tilde{x} \in Kg_i$. Suppose that $x \sim \tilde{x}$. Hence by the Cayley graph definition we have $\tilde{x} = h^{-1}x$ for some $h \in H$. So we have that $\tilde{x} = \tilde{k_1}g_i = a'k_2k_1g_i$

for some $a' \in G$ where $x = k_1 g_i$ and $h^{-1} = a' k_2$ so that $\tilde{k_1} = a' k_3$ for some $\tilde{k_1}, k_1, k_2, k_3 \in K$. This give us a contradiction as the right cosets of K are disjoint. From the above we conclude that the elements of the same right coset are non-adjacent.

Now let $x, \tilde{x} \in K g_i$ and $y, \tilde{y} \in K g_j$. Suppose that $x \sim y$ in Γ and we want to show that $\tilde{x} \sim \tilde{y}$ in Γ. Note by the Cayley graph definition we have that $y = h^{-1} x$ for some $h \in H$. In this case we have that $k_2 g_j = a' k_3 k_1 g_i$ so $g_j = k_2^{-1} a' k_4 g_i$. So we have that

$$\begin{aligned} \tilde{y} &= \tilde{k_2} g_j \\ &= \tilde{k_2} k_2^{-1} a' k_4 g_i \\ &= \tilde{k_3} a' k_4 g_i \end{aligned} \qquad (4.2.1)$$

for $k_1, k_2, k_3, k_4, \tilde{k_2}, \tilde{k_3} \in K$. Now assume that $\tilde{x} = k_4 g_i$ hence $\tilde{x} \sim \tilde{y}$. From the above we conclude that all elements in $K g_i$ are adjacent to all elements in $K g_j$. From the above we deduce

$$A(\Gamma) = A(\Gamma^*) \otimes J.$$

□

Corollary 4.2.6. *If H is a connecting set in the group G and if H is a union of right cosets of the subgroup $K \subseteq G$ with $|K| \neq 1$, then $\Gamma = Cay(G, H)$ is singular and* $\text{null}(\Gamma) \geq \frac{|G|}{|K|} \cdot (|K| - 1)$.

Proof: Note J is singular and the eigenvalues of J are $|K|$ with multiplicity of 1 and 0 with multiplicity of $|K| - 1$. Hence we have that

$$Spec(\Gamma) = Spec(\Gamma^*) \otimes (|K|, \underbrace{0, 0, ..., 0}_{|K|-1}).$$

From this we conclude that $\text{null}(\Gamma) \geq \frac{|G|}{|K|} \cdot (|K| - 1)$. □

EXAMPLE: Let $G = D_5$ be the dihedral group of order 10 where $D_5 = \langle a, b : a^5 = b^2 = 1_G,\ bab = a^{-1}\rangle$ and let $K = \{1_G, b\}$. Let $H = aK \cup a^{-1}K$ be the connecting set for the graph $\Gamma = Cay(D_5, H)$. So $H = \{a, a^4, ab, a^4b\}$. Therefore according to Corollary 4.2.6 we have that Γ is singular with $\text{null}(\Gamma) \geq 5$. The spectrum of Γ by using a GAP program is displayed in Table 4.5.

	Eigenvalues of Γ	Multiplicities
λ_1	4	1
λ_2	$\sqrt{5}-1$	2
λ_3	$-\sqrt{5}-1$	2
λ_4	0	5

Table 4.5: The eigenvalues of the graph $\Gamma = Cay(D_5, H)$.

The above table verifies that Γ is singular with $null(\Gamma) = 5$.

4.2.1 Cayley Graphs over Cyclic Groups

In this section we derive simple conditions which characterise singular Cayley graphs over a cyclic group. Note that a Cayley graph over a cyclic group is also called a *circulant graph.* Let $C_n = \langle a\rangle$ be a cyclic group of order n and let H be a connecting set of C_n. Denote the Cayley graph $Cay(C_n, H)$ by Γ. It is clear that H is a normal subset of C_n.

Let l be a positive integer and let Ω_l be the group of l^{th}*roots of unity*, that is $\Omega_l = \{z \in \mathbb{C}\backslash\{0\} : z^l = 1\}$. Then Ω_l is a cyclic group of order l with generator $e^{\frac{2\pi i}{l}}$. Note this is not the only generator of Ω_l, indeed any power $e^{\frac{2\pi i m}{l}}$ where $gcd(l, m) = 1$ is a generator too. A generator of Ω_l is called *a primitive l^{th} root of unity. Euler's totient function* of l is defined as the number of positive integer $\leq l$ that are relatively prime to l and it is denoted by $\phi(l)$.

Let n be a positive integer and let $\Phi_n(x)$ denote the n^{th}*cyclotomic polynomial.* Then $\Phi_n(x)$ is the unique irreducible integer polynomial with leading coefficient 1 so that $\Phi_n(x)$ divides $x^n - 1$ but does not divide of $x^k - 1$ for any $k < n$. Its roots are all primitive n^{th} roots of unity. So

$$\Phi_n(x) = \prod_{1 \le m < n} (x - e^{\frac{2\pi i m}{n}})$$

where $gcd(m, n) = 1$.

Lemma 4.2.7. *[56, Lemma 3.1.1] If n is a prime power, $n = p^m$, if ω is a primitive n^{th} root of unity and if $a(1), ..., a(k)$ are integers with*

$$\omega^{a(1)} + ... + \omega^{a(k)} = 0 \tag{4.2.2}$$

then k is a multiple of p.

Proof: Assume that $0 \le a(i) \le p^m - 1$ for all i. Construct the polynomial

$$P(x) = x^{a(1)} + ... + x^{a(k)}.$$

Note $P(\omega) = 0$ and keep in minding that $a(1), ..., a(k)$ are not necessary distinct numbers. It is clear that $degP(x) \le p^m - 1$ and $P(x)$ is not the zero polynomial, since $P(1) = k$. Hence $degP(x) \ge 0$. The n^{th} cyclotomic polynomial

$$\Phi_{p^m}(x) = 1 + x^{p^{m-1}} + x^{2p^{m-1}} + ... + x^{(p-1)p^{m-1}}$$

is irreducible over the field of rational numbers $\mathbb{Q}$ and it consists of p monomials. Therefore, as ω is a common root of $P(x)$ and $\Phi_{p^m}(x)$, it follows that $\Phi_{p^m}(x)$ divides $P(x)$. Hence we have that

$$P(x) = \Phi_{p^m}(x)Q(x),$$

for some polynomial $Q(x) \in \mathbb{Q}[x]$. In particular,

$$\begin{aligned} 0 &\leq deg(Q(x)) \\ &= degP(x) - deg\Phi_{p^m}(x) \\ &\leq (p^m - 1) - (p-1)p^{m-1} \\ &= p^{m-1} - 1. \end{aligned}$$

Suppose $x^{a+tp^{m-1}} = x^{b+t^*p^{m-1}}$ are two equal terms in $P(x)$ when $a > b$. Then

$$\begin{aligned} a + tp^{m-1} &= b + t^*p^{m-1} \\ a - b &= (t - t^*)p^{m-1} \end{aligned}$$

Note, x^a and x^b are monomials of $Q(x)$ and so $a, b \leq deg(Q(x)$. Hence this gives us a contradiction as $a - b = (t - t^*)p^{m-1}$ but $degQ(x) \leq p^{m-1} - 1$. Therefore, when multiplying monomials $x^{ip^{m-1}}$ for $i = 0, 1, ..., p-1$ of $\Phi_{p^m}(x)$ by a monomial x^a of $Q(x)$ no two products have the same exponent. It follows that $Q(x)\Phi_{p^m}(x)$ consists of a multiple of the p monomials of $\Phi_{p^m}(x)$. □

By similar techniques one can prove.

Lemma 4.2.8. *[56, Lemma 3.1.3] Let ω be a primitive n^{th} root of unity and let $a(1), ..., a(k)$ be integers. If n is a product of two prime powers, say $n = p^e q^f$, and if*

$$\omega^{a(1)} + ... + \omega^{a(k)} = 0,$$

then

$$\omega^{a(1)} + ... + \omega^{a(k)} = l(1 + \delta + ... + \delta^{p-1}) + r(1 + \varepsilon + ... + \varepsilon^{q-1}),$$

where l, r are sums of powers of ω, and δ, ε are primitive p^{th} and q^{th} roots of unity respectively.

Before stating the theorems and results about the singularity of Cayley graphs over cyclic groups we give a brief introduction to the irreducible representations and the irreducible characters of finite cyclic groups.

As before C_n is a finite cyclic group of order n. Note that the conjugacy class of any element a of C_n consists of that element only. Thus there are exactly n irreducible representations of C_n. According to Theorem 2.2.7 each irreducible representation of C_n has degree 1. Let $\rho : C_n \to GL(\mathbb{C}C_n)$ be a representation of the group C_n on $\mathbb{C}C_n$. The *character* associated with ρ is the function $\chi_\rho : C_n \to \mathbb{C}$ denoted by $\chi_\rho(a) = \text{tra}(\rho(a))$ for all $a \in C_n$. Here $\text{tra}(\rho(a))$ is the trace of the representation matrix. Note that an irreducible representation of degree one and the associated irreducible character are the same thing. Thus every irreducible representation or irreducible character of C_n is uniquely determined by its value on any generator set of C_n. Then $\rho_i(a) = \omega^{i-1}$ for $i = 1, 2, ..., n$ are the complete list of the irreducible representations of C_n and the same time these are the irreducible characters of C_n. This fact provides an easy construction of all irreducible representations or irreducible characters of a cyclic group. They are simply the homomorphisms $C_n \to \mathbb{C}$. Two representations of a finite group are equivalent if and only if their characters are equal. Thus the two irreducible representations of degree one are inequivalent if and only if they are unequal.

As a consequence of Theorem 4.1.9 we have the general theorem on abelian groups:

Theorem 4.2.9. *Let $\Gamma = Cay(G, H)$ be a Cayley graph for the abelian group G, and denote the irreducible characters of G by $\chi_1, \chi_2, ..., \chi_n$. Then $f_1, f_2, ..., f_n$ with $f_i = \sum_{v \in G} \chi_i(v)v$ span the eigenspaces of Γ with eigenvalue $\lambda_i = \sum_{h \in H} \chi_i(h)$. In particular, Γ is singular if and only if there is some character $\chi = \chi_i$ for which $\sum_{h \in H} \chi(h) = 0$. Furthermore, if the number of distinct characters with this property is c then $null(\Gamma) = c$.*

Therefore by Theorem 4.2.9 we deduce that each eigenvalue of Γ is a certain sum of n^{th} roots of unity. Note that in this work the irreducible character χ_i for $1 \leq i \leq n$ of C_n is generated by ω^{i-1} where ω is a fixed primitive n^{th} root of unity and the corresponding eigenvalue of Γ will be λ_i.

We consider the case where $G = C_{n_1} \times C_{n_2}$. Suppose that $\chi_1, ..., \chi_{n_1}$ and $\psi_1, ..., \psi_{n_2}$ are the irreducible characters of C_{n_1} and C_{n_2} respectively. Hence

$$\chi_j \times \psi_k(a, b) = \chi_j(a).\psi_k(b)$$

for $(a, b) \in G$ and $j = 1, ..., n_1, k = 1, ..., n_2$ are the distinct characters of G.

Now let $H \subset G$ be a connecting set and consider the Cayley graph $\Gamma = Cay(G, H)$. By Theorem 4.1.6 the eigenvalues of Γ have the shape

$$\lambda = \sum_{(h_1,h_2)\in H} \chi_j \times \psi_k(h_1, h_2). \tag{4.2.3}$$

In some special cases it is possible to determine these eigenvalues in terms of eigenvalues of certain graph $\Gamma_i = Cay(C_{n_i}, H_i)$ for $i = 1, 2$ depending on the shape of H. We consider two cases .

We say that H is of the *box shape* if $H = H_1 \times H_2$ with $H_i \subset C_{n_i}$ and where H_i does not contain the identity element of C_{n_i}, for $i = 1, 2$, as is shown in the Figure 4.2.2.

Here it is easy to verify that H_i is a connecting set for C_{n_i}, and so we have Cayley graphs $\Gamma_i = Cay(C_{n_i}, H_i)$. Here the formula in Equation 4.2.3 takes the shape

$$\begin{aligned} \lambda &= \sum_{(h_1,h_2)\in H} \chi_j \times \psi_k(h_1, h_2) \\ &= \sum_{h_1\in H_1} \chi_j(h_1) \cdot \sum_{h_2\in H_2} \psi_k(h_2) \\ &= \lambda' \cdot \lambda'' \end{aligned}$$

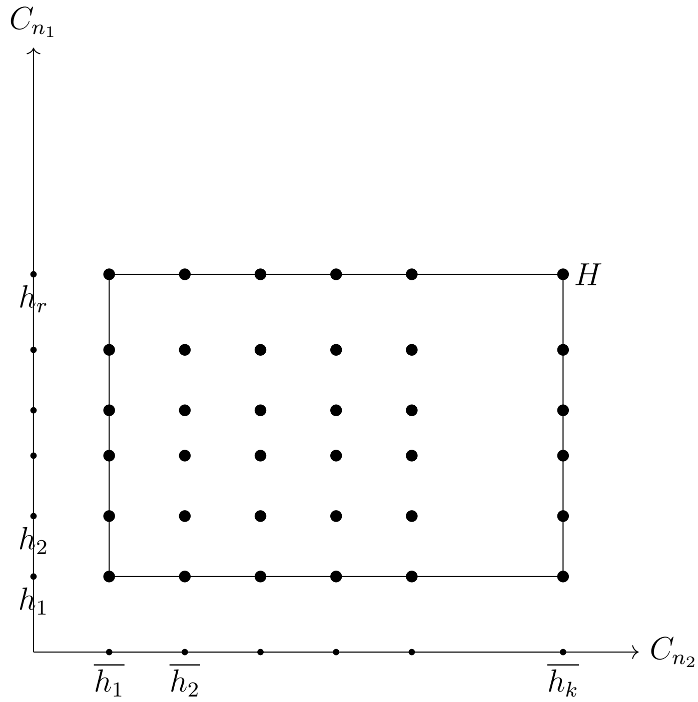

Figure 4.2.2: H has Box shape

where λ', λ'' are eigenvalues of Γ_1, Γ_2 respectively. Thus,

$$Spec(\Gamma) = Spec(\Gamma_1) \otimes Spec(\Gamma_2)$$

where $Spec(\Gamma_1) \otimes Spec(\Gamma_2)$ is the multi-set $\{\lambda_t \cdot \lambda_l : \lambda_t \in Spec(\Gamma_1), \lambda_l \in Spec(\Gamma_2)\}$ for $1 \leq t \leq n_1$ and $1 \leq l \leq n_2$. In particular, since $|H_i|$ is the degree of Γ_i, then $|H_i|$ is the largest eigenvalue of Γ_i. Thus $|H_1| \cdot Spec(\Gamma_2) \subset Spec(\Gamma)$ and $|H_2| \cdot Spec(\Gamma_1) \subset Spec(\Gamma)$. So we have proved the following result.

Theorem 4.2.10. *Let $G = C_{n_1} \times C_{n_2}$ and let $H \subset G$ be a connecting set of box shape. Let $\Gamma = Cay(G, H)$ and $\Gamma_i = Cay(C_{n_i}, H_i)$. Then $Spec(\Gamma) = Spec(\Gamma_1) \otimes Spec(\Gamma_2)$. In particular, $|H_1| \cdot Spec(\Gamma_2) \subset Spec(\Gamma)$ and $|H_2| \cdot Spec(\Gamma_1) \subset Spec(\Gamma)$. Furthermore, Γ is singular if and only if at least one of Γ_1, Γ_2 are singular.*

We next consider a generalization of this idea. Let $G = C_{n_1} \times C_{n_2}$ as above. We say that H is of *brick shape* if there are elements $a_1, ..., a_r \in C_{n_1}$ with

$a_i \neq 1_{C_{n_1}}$, and subsets $A_1, ..., A_r \subseteq C_{n_2}$ with $|A_i| = l$, for some $r, l \in \mathbb{N}$, so that $H = a_1 A_1 \cup ... \cup a_r A_r$. We call l the *brick length*, as is shown in Figure 4.2.3.

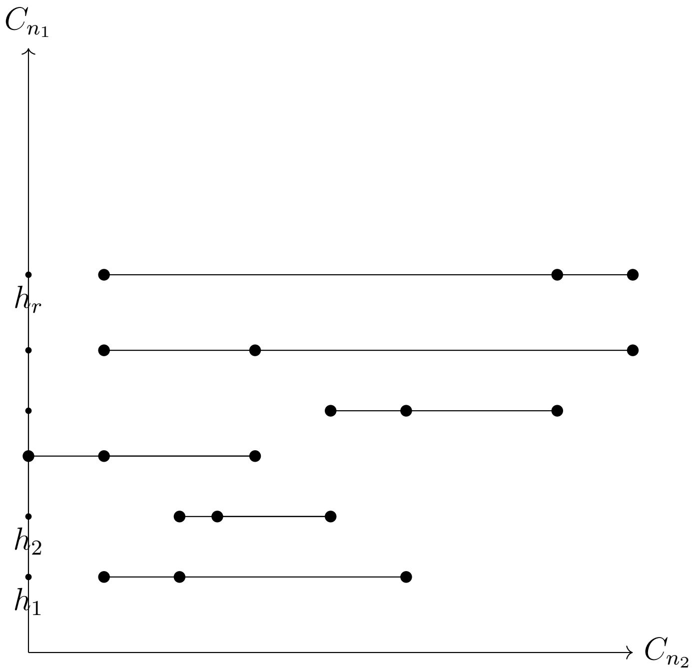

Figure 4.2.3: H has Brick shape of length 3 for H_1

Here it is easy to show that $H_1 = \{a_1, .., a_r\}$ is a connecting set and so we have Cayley graph $\Gamma_1 = Cay(C_{n_1}, H_1)$. Here we can evaluate the expression in Equation 4.2.3 to get

$$\begin{aligned} \lambda &= \sum_{(h_1,h_2)\in H} \chi_j \times \psi_k(h_1, h_2) \\ &= \chi_j(a_1)(\sum_{b\in A_1} \psi_k(b)) + ... + \chi_j(a_r)(\sum_{b\in A_r} \psi_k(b)). \end{aligned} \tag{4.2.4}$$

We can evaluate this formula where ψ_k is the trivial character. In this case Equation 4.2.4 becomes $\lambda = l \cdot \sum_{i=1,...,r} \chi_j(a_i)$. Hence we have the following:

Theorem 4.2.11. *Suppose that $G = C_{n_1} \times C_{n_2}$ and that H is a connecting set of G has brick shape of length l, as above. Let $\Gamma = Cay(G, H)$ and let Γ_1 be defines as above. Then $l \cdot Spec(\Gamma_1) \subset Spec(\Gamma)$. In particular, Γ is* *singular if* Γ_1 *is singular.*

EXAMPLE: Let $\Gamma = Cay(G,H)$ where $G = Z_6 \times Z_3$ and $H = \{(\underline{1},\underline{1}),(\underline{2},\underline{5})\}$. Let $\Gamma_1 = Cay(Z_3,H_1)$ where $H_1 = \{\underline{1},\underline{2}\}$ and let $\Gamma_2 = Cay(Z_6,H_2)$ where $H_2 = \{\underline{1},\underline{5}\}$. Note $H \neq H_1 \times H_2$ so we can not conclude that $Spec(\Gamma) = Spec(\Gamma_1) \otimes Spec(\Gamma_2)$. However we have that H has the brick shape twice, this means that for H_1 brick of length 1 and for H_2 brick of length 1. Hence we can apply Theorem 4.2.11 for H_1 and H_2 so by this we have that $Spec(\Gamma_1) \subset Spec(\Gamma)$ and $Spec(\Gamma_2) \subset Spec(\Gamma)$. By using a GAP program we find the spectrum of Γ as well as the spectrum of Γ_1 and Γ_2 as are shown in the following tables:

	Eigenvalues of Γ	Multiplicities
λ_1	2	3
λ_2	-2	3
λ_3	1	6
λ_4	-1	6

Table 4.6: The eigenvalues of the graph $\Gamma = Cay(Z_3 \times Z_6, H)$

	Eigenvalues of Γ_1	Multiplicity
λ_1	2	1
λ_2	-1	2

Table 4.7: The eigenvalues of the graph $\Gamma_1 = Cay(Z_3, H_1)$

	Eigenvalues of Γ_2	Multiplicities
λ_1	2	1
λ_2	-2	1
λ_3	1	2
λ_4	-1	2

Table 4.8: The eigenvalues of the graph $\Gamma_2 = Cay(Z_6, H_2)$.

The above tables verify that $Spec(\Gamma_i) \subset Spec(\Gamma)$ for $i = 1, 2$.

Character sum of this type are a well-established topic. More generally, if G is a group and H a connecting set of G then we say that H is *vanishing* on the irreducible character χ if $\sum_{h \in H} \chi(h) = 0$.

In the following we are interested in conditions for a subset H to be vanishing. Our first example comes from subgroups of G.

We conclude from Lemma 4.2.7 and Lemma 4.2.8 that the vanishing of certain sums of n^{th} roots of unity can occur when the sum is over union of cosets of some non-trivial subgroup of Ω_n, the group of the n^{th} roots of unity. Therefore we have the following important result:

Theorem 4.2.12. *Let $C_n = \langle a \rangle$ and H a connecting set of C_n. Let Γ be the graph $\Gamma = Cay(C_n, H)$ and let Ω_n be the group of n^{th} roots of unity. For $i = 1, 2, 3, ..., n$ consider the homomorphism*

$$\varphi_i : C_n \to \Omega_n$$

given by $\varphi_i(a^m) = \omega^{(i-1)m}$ where ω is a primitive n^{th} root of unity and $0 \leq m \leq n-1$. Then Γ is a singular graph if the multi-set

$$\varphi_i(H) = \{\varphi_i(h_1), ..., \varphi_i(h_k)\},$$

with $|H| = k$, is a union of cosets of some non-trivial subgroup $\Upsilon \subseteq \Omega_n$ for some i.

Proof: It is clear by Lemma 4.2.7 and Lemma 4.2.8. □

Example: Let $\Gamma = Cay(C_8, H)$ where $H = \{a, a^3, a^5, a^7\}$. It is clear that H is a coset of $K = \langle a^2 \rangle$ so by Corollary 4.2.6 we have that Γ is singular.

Example: Let $\Gamma = Cay(C_8, H)$ where $H = \{a, a^2, a^6, a^7\}$. Clearly

$$\Omega_8 = \{1, \omega, \omega^2, \omega^3, \omega^4, \omega^5, \omega^6, \omega^7\}$$

and it has two non-trivial subgroups $\Upsilon_1 = \{1, \omega^4\}$ and $\Upsilon_2 = \{1, \omega^2, \omega^4, \omega^6\}$. Note H is not a union of cosets however its image under χ_i for some i is a coset of a non-trivial subgroup of Ω_8. So to decide the singularity of Γ, according to Theorem 4.2.12 we need to look at the irreducible characters of C_8 which are generated by the elements of Υ_1 and Υ_2 are shown in the following table:

	1_{C_8}	a	a^2	a^3	a^4	a^5	a^6	a^7
χ_3	1	ω^2	ω^4	ω^6	1	ω^2	ω^4	ω^6
χ_5	1	ω^4	1	ω^4	1	ω^4	1	ω^4
χ_7	1	ω^6	ω^4	ω^2	1	ω^6	ω^4	ω^2

Table 4.9: The irreducible characters of C_8 that are generated by the elements of Υ_1 and Υ_2

Now we find the elements of theses characters correspond to the elements of H as is shown in the following table:

	a	a^2	a^6	a^7
χ_3	ω^2	ω^4	ω^4	ω^6
χ_5	ω^4	1	1	ω^4
χ_7	ω^6	ω^4	ω^4	ω^2

Table 4.10: The χ_3, χ_5 and χ_7 values which are corresponding to elements of H

Note we have that $\chi_5(H) = 2 \times \Upsilon_1$ so Γ is singular and $\lambda_5 = 0$.

Now we generalise this result for groups, which may not be abelian as shown in the following:

Proposition 4.2.13. *Let G be a group with normal subgroup K and a homomorphism*

$$\varphi : G \to G/K.$$

Suppose that H is a subset of G such that:

(1) $\varphi(H)$ is vanishing in G/K for some character χ of G/K.

(2) There is a constant c such that every coset of K in G meets H in 0 or c elements.

Then H is vanishing in G.

Proof: Let χ be the irreducible character of G/K on which χ is vanishing. Then we have that

$$0 = \sum_{h \in H} \chi(\varphi(h)) = \sum_{g_i K \cap H \neq \phi} \chi(g_i K) \tag{4.2.5}$$

where $g_1 K \uplus g_2 K \uplus ... \uplus g_m K = G$. Hence by Equation 4.2.5 we have that

$$\sum_{h \in H} \tilde{\chi}(h) = c \sum_{g_i K \cap H \neq \phi} \chi(g_i K) = 0$$

where $\tilde{\chi}$ is the lift character corresponding to χ. □

Corollary 4.2.14. *Let G be a group with normal subgroup K such that G/K is abelian. Let H be a connecting set of G and let $\Gamma = Cay(G, H)$. Suppose that every coset of K in G meets H in exactly c elements for some c. Then Γ is singular with nullity $\geq |G/K| - 1$.*

Proof: We need to show that $A \cong \sum_{h \in H} \rho_l(h)$ is singular and of nullity $\geq |G/K| - 1$, where A is the adjacency matrix of Γ and ρ_l is the left regular representation of G. Then we can decompose

$$\sum_{h \in H} \rho_l(h) = m_1 \sum_{h \in H} \rho_1(h) \oplus ... \oplus m_s \sum_{h \in H} \rho_s(h)$$

where $\rho_1, ..., \rho_s$ are the irreducible representations of G and $m_1, ..., m_s$ are their degrees respectively. For this it is sufficient to show that $\sum_{h \in H} \rho_i(h)$ is singular for some i. Now let $\chi_1, ..., \chi_{|G/K|}$ be the irreducible characters of G/K. Then we have that $\sum \chi_j(gK) = 0$ for all non-trivial irreducible characters of G/K where the sum over all the cosets of K in G. Hence we have that $\sum_{h \in H} \tilde{\chi}_j(h) = c \sum \chi_j(gK) = 0$ where $\tilde{\chi}_j$ is the lift character corresponding to χ_j. From this we conclude that $\sum_{h \in H} \rho_j(h)$ is singular where ρ_j is the irreducible representation of G which is corresponding to $\tilde{\chi}_j$. So by Theorem 4.1.4 we have that A is singular with nullity $\geq |G/K| - 1$ as there are $|G/K| - 1$ non-trivial character for G/K. □

The problem of determining vanishing set of elements in a group is very difficult. Here we have the following consequence of the Corollary 4.2.6 and Proposition 4.2.13 for singular graphs so far.

Example: Let $\Gamma = Cay(C_9, H)$ where $H = \{a, a^2, a^3, a^6, a^7, a^8\}$. Let $K = \langle a^3 \rangle$ be a non-trivial subgroup of C_9. Note we have that each coset of K meets H in exactly 2 elements. Hence we have that

$$\varphi(H) = \sum_{h \in H} \varphi(h) = 2 \sum \tilde{\varphi}(Kg)$$

where the sum over all cosets of K, and φ is the left character corresponding to the irreducible character $\tilde{\varphi}$ of C_9/K. In this case we have that $\varphi(H) = 0$ if and only if $\tilde{\varphi}$ is a non-trivial character of C_9/K. Therefore by Corollary 4.2.14 we have that Γ is singular.

In the next example we consider an instance where $|H|$ divides $n = |C_n|$. In this case however H does not satisfy the criteria in Theorem 4.2.12, and Proposition 4.2.13 and indeed $\Gamma = Cay(C_n, H)$ is non-singular.

Example: Let $\Gamma = Cay(C_{20}, H)$ where $H = \{a, a^5, a^{10}, a^{15}, a^{19}\}$. Note Ω_{20} has four non-trivial subgroups which are $\Upsilon_1, \Upsilon_2, \Upsilon_3$ and Υ_4 and these generate by $\omega^5, \omega^4, \omega^2, \omega^{10}$ respectively. We have $|H| = 5$ which divides $|C_{20}|$. Note $\lambda_i = \sum_{a^j \in H} \omega^{(i-1) \times j}$ where $1 \leq i \leq 20$. Therefore we can use a GAP program to find the spectrum of Γ as is shown in the Table 4.11.

It is clear that Γ is non-singular graph.

	Eigenvalues of Γ	Multiplicities
λ_1	5	1
λ_2	-3	1
λ_3	-1	2
λ_4	$5/2-(1/2)\times\sqrt{5}$	2
λ_5	$5/2+(1/2)\times\sqrt{5}$	2
λ_6	$(1/2)\times\sqrt{5}-1/2$	2
λ_7	$-1/2-(1/2)\times\sqrt{5}$	2
λ_8	$-1-(1/2)\times\sqrt{(10-(2\times\sqrt{5}))}$	2
λ_9	$-1-(1/2)\times\sqrt{(10+(2\times\sqrt{5}))}$	2
λ_{10}	$-1+(1/2)\times\sqrt{(10-(2\times\sqrt{5}))}$	2
λ_{11}	$-1+(1/2)\times\sqrt{(10+(2\times\sqrt{5}))}$	2

Table 4.11: The eigenvalues of the graph $\Gamma = Cay(Z_{20}, H)$

As before C_n is a cyclic group of order n and H a connecting set of C_n. We have the problem of understanding vanishing character sums. According to Lemma 4.2.7 and Lemma 4.2.8 we have that if n is a prime power or the multiple of two distinct primes power then $\sum_{h\in H}\chi_i(h) = 0$ if and only if the image of H under χ_i is a union of cosets of some non-trivial subgroup of Ω_n for some irreducible character χ_i of C_n. However this does not hold if n is a product of three distinct primes power as shown in the following example:

Example: Let C_{30} be a group of order 30 and let

$$C_2 = \{1, a\}, C_3 = \{1, b_2, b_3\} \text{ and } C_5 = \{1, c_2, c_3, c_4, c_5\}$$

be subgroups of order $2, 3$ and 5 respectively. Suppose that χ is a 1-dimensional character with $\chi(a), \chi(b_2)$ and $\chi(c_2)$ all $\neq 1$. Then by the

Theorem 4.2.1 we have that

$$\begin{aligned}\chi(a)(\chi(b_2)+\chi(b_3))+(\chi(c_2)+\chi(c_3)+\chi(c_4)+\chi(c_5)) &= (-1)(-1)+(-1)\\ &= 0.\end{aligned}$$

Therefore $H = (C_2\backslash\{1_G\}).(C_3\backslash\{1_G\}\cup(C_5\backslash\{1_G\}$ is a set of 6 elements which vanishes for χ. It clearly is not a union of cosets of a non-trivial subgroup.

Thus we conclude from the above example that the vanishing of the sums of roots of unity can also occur when no union of cosets of Ω_n is involved, see [39] as a reference.

This leads us to study the singularity of Cayley graphs over a cyclic group according to the cyclotomic polynomial. As before $C_n = \langle a\rangle$ is a cyclic group of order n and H a connecting set of C_n. Let Γ be the graph $\Gamma = Cay(C_n, H)$ and let H^* be the set of all $0 < m \leq n-1$ such that $H = \{a^m : m \in H^*\}$. Now consider the polynomial

$$\Psi_\Gamma(x) = \sum_{m\in H^*} x^m$$

associated to Γ. Note that Ψ_Γ depends on the choice of the generator a. If a' is some other generator, then $a = (a')^r$ for some r with $gcd(r,n) = 1$. Therefore $(H')^* \subseteq \{1,2,...,n-1\}$ given by $(H')^* \equiv rH^* \mod n$ and so

$$\Psi'_\Gamma(x) = x^r \sum_{m\in H^*} x^m \equiv x^r\Psi_\Gamma(x) \mod x^n.$$

Examples: Suppose $G = \langle a\rangle$ has order 7 and $H = \{a, a^6\}$. Thus $H^* = \{1,6\}$ and so

$$\Psi_\Gamma(x) = x + x^6.$$

Now a^2 is also a generator, $a = (a^2)^4$ giving $(H')^* = \{4, 3\}$ and so

$$\Psi'_\Gamma(x) = x^4 + x^3.$$

NOTE:

(1) Isomorphic graphs may have different $\Psi_\Gamma(x)$ polynomials.

(2) If we change the generator of C_n then $\Psi_\Gamma(x)$ will change. If we go back to the example and we change the generator of C_n as $C_n = \langle a^2 \rangle$ with the same connecting set H we have $\Psi_\Gamma(x) = x^4 + x^3 \neq x + x^6$.

Theorem 4.2.15. *Let C_n be a cyclic group of order n and let $\Gamma = Cay(C_n, H)$ be the Cayley graph for the connecting set $H \subset C_n$. Let $\Psi_\Gamma(x)$ be the polynomial associated to Γ for some generator of C_n. Then Γ is singular if and only if $\Phi_d(x)$ divides $\Psi_\Gamma(x)$ for some divisor d of n with $1 < d \leq n$ where $\Phi_d(x)$ is the d^{th} cyclotomic polynomial. Furthermore, let $d_1, d_2, ..., d_l$ be the divisors of n. Then we have that $\text{null}(\Gamma) = \sum \phi(d_j)$ where the sum is over all d_j such that $\Phi_{d_j}(x)$ divides $\Psi_\Gamma(x)$ and $\phi(d_j)$ is Euler's totient function of d_j.*

Proof: Let $\lambda = \lambda_i$ be any eigenvalue of Γ. Thus by Theorem 4.1.9 we have that

$$\begin{aligned} \lambda_i &= \sum_{h \in H} \chi_i(h) \\ &= \sum_{m \in H^*} \chi_i(a^m) \\ &= \sum_{m \in H^*} \chi_i(a)^m \\ &= \sum_{m \in H^*} (\omega^{i-1})^m \\ &= \Psi_\Gamma(\omega^{i-1}) \end{aligned}$$

where ω is a primitive n^{th} root of unity. Now if $\lambda_i = 0$, then $\Psi_\Gamma(\omega^{i-1}) = 0$ and so; if ω^{i-1} is a primitive n^{th} root of unity then we have that $\Phi_n(x) | \Psi_\Gamma(x)$

as $\Psi_\Gamma(x)$ and $\Phi_n(x)$ have a common root and if ω^{i-1} is not a primitive n^{th} root of unity. In this case ω^{i-1} is a primitive r^{th} root of unity for some divisor r of n where $1 < r < n$. Hence we have that $\Phi_r(x)|\Psi_\Gamma(x)$ as $\Psi_\Gamma(x)$ and $\Phi_r(x)$ have a common root. Therefore for both cases we have that $\Phi_d(x)$ divides $\Psi_\Gamma(x)$ for some divisor d of n with $1 < d \leq n$.

Conversely, suppose $\Phi_d(x)$ divides $\Psi_\Gamma(x)$ for some divisor d of n. Then we have that $\Phi_d(\omega^*) = 0$ where ω^* is a primitive d^{th} root of unity. So we have that $\Psi_\Gamma(\omega^*) = 0$ then $\lambda_i = 0$ for some i. By this we deduce that Γ is singular.

By the second part of the proof we have that $\lambda_i = \Psi_\Gamma(\omega^*) = 0$ if and only if ω^* is a primitive d^{th} root of unity for some divisor d of n. In this case we have that $\phi(d)$ of primitives d^{th} root of unity. Hence we deduce that $\text{null}(\Gamma) = \sum \phi(d_j)$ where the sum is over all divisors of n such that $\Phi_{d_j}(x)$ divides $\Psi_\Gamma(x)$. □

EXAMPLE: Let $\Gamma = Cay(C_8, H)$ where $H = \{a, a^3, a^5, a^7\}$. Then we have that $\Psi_\Gamma(x) = x + x^3 + x^5 + x^7$. We know that $\Phi_8(x) = x^4 + 1$ and $\Phi_4(x) = x^2 + 1$. It is clear that $\Psi_\Gamma(x) = \Phi_8(x)(x^3 + x)$ and $\Psi_\Gamma(x) = \Phi_4(x)(x^5 + x)$. Hence by Theorem 4.2.15 we have that $\text{null}(\Gamma) = \phi(8) + \phi(4) = 6$. By using a GAP program the spectrum of Γ is shown in the Table 4.12.

	Eigenvalues of Γ	Multiplicities
λ_1	4	1
λ_2	-4	1
λ_3	0	6

Table 4.12: The eigenvalues of the graph $\Gamma = Cay(C_8, H)$

Theorem 4.2.16. *Let Γ be a vertex transitive graph on p vertices with at least one edge where p is a prime number. Then Γ is non-singular.*

Proof: Let V be the vertex set of Γ, with $|V| = p$ and p is a prime number.

Let G be a vertex transitive group on Γ. By Theorem 2.2.1 we have that

$$|G| = |V| \cdot |G_v|$$

for some $v \in V$. So by Sylow's Theorem there exist a subgroup K of G with $|K| = p$. Note K is cyclic. Now apply Theorem 2.2.1 again we have that $|K| = |v^K|\cdot|K_v|$ for $k \in K$. Note we have that $|K_v| = 1_G$, so K acts regularly on V. Hence by Sabidussi's Theorem 4.1.2 we have that Γ is a Cayley graph $Cay(K, H)$ for some connecting set H of K. Suppose for contradiction that Γ is singular. As p is a prime number, according to Theorem 4.2.15 we have that $\Phi_p(x)$ divides $\Psi_\Gamma(x)$. So there is $Q(x) \in \mathbb{Q}[x]$ such that

$$\Psi_\Gamma(x) = \Phi_p(x) \times Q(x).$$

This gives us a contradiction as $\Phi_p(x)$ has degree $p - 1$ and $Q(x) \neq 0$ but the maximum degree of $\Psi_\Gamma(x)$ is less than p. $\square$

Recently we have found related results for circulant matrices in [38] and [36] which are similar to Theorem 4.2.15.

Now we will look at some examples:

Example: Let $\Gamma = Cay(C_{30}, H)$ where $H = \{a^5, a^6, a^{12}, a^{18}, a^{24}, a^{25}\}$. Note in this example H is not union of cosets of a non-trivial subgroup of C_{30}. However, we have that

$$\Phi_{30} = x^8 + x^7 - x^5 - x^4 - x^3 + x + 1$$

divides

$$\Psi_\Gamma(x) = x^{25} + x^{24} + x^{18} + x^{12} + x^6 + x^5$$

as is shown in the following

$$x^{25}+x^{24}+x^{18}+x^{12}+x^6+x^5 = (x^{17}+x^{14}+x^{12}+x^{10}x^8+x^5)(x^8+x^7-x^5-x^4-x^3+x+1).$$

Hence by Theorem 4.2.15 we have that Γ is singular with nullity 8. Then by using a GAP program the spectrum of Γ is shown as in the Table 4.13.

	Eigenvalues of Γ	Multiplicities
λ_1	6	1
λ_2	2	1
λ_3	5	2
λ_4	3	2
λ_5	1	4
λ_6	-3	4
λ_7	-2	8
λ_8	0	8

Table 4.13: The eigenvalues of the graph $\Gamma = Cay(C_{30}, H)$

EXAMPLE: Let $\Gamma = Cay(C_{12}, H)$ where $H = \{a^1, a^{11}\}$. Note $\Psi_\Gamma(x) = x + x^{11}$ so we need to check all d^{th} Cyclotomoc polynomials $\Phi_d(x)$ where d is a divisor of n. Hence we find that $\Phi_4(x) = x^2 + 1$ divides $\Psi_\Gamma(x)$ so by Theorem 4.2.15 we have that Γ is singular with nullity 2. By using a GAP program the spectrum of Γ is shown in the Table 4.14.

	Eigenvalues of Γ	Multiplicities
λ_1	2	1
λ_2	-2	1
λ_3	0	2
λ_4	1	2
λ_5	-1	2
λ_6	$\sqrt{3}$	2
λ_7	$-\sqrt{3}$	2

Table 4.14: The eigenvalues of the graph $\Gamma = Cay(C_{12}, H)$

Final comments: Similar techniques allow us to determine the singularity of Cayley graphs over dihedral groups by using the character formula of

Babai.

4.3 The First Method

In this section we determine the spectrum of a vertex transitive graph in terms of an associated Cayley graph. This method can be found in [43]. We generalise and introduce new points of view and we determine the condition for a vertex transitive graph to be singular.

Through out this section Γ is a simple connected graph with vertex set V and $|V| = n$. Let G be a group of automorphisms of Γ which acts transitively on V. Note, if G is regular then Γ is a Cayley graph by Sabidussi's Theorem 4.1.2. Therefore we assume it is not regular.

We now determine the spectrum of Γ in terms of some Cayley graphs. Fix a vertex $v \in V(\Gamma)$. Let $H = \{g \in G : v \sim v^g\}$. We now prove that H is a connecting set of G satisfying the three conditions for a Cayley graphs. Clearly (i) $v \nsim v^{1_G}$ as Γ is simple so that H is free-identity (that means $1_G \notin H$), and (ii) for each $g \in H$ we have that $g^{-1} \in H$ as $g \in H$ *if and only if* $v \sim v^g$ since Γ is undirected and by the automorphism definition we have that $v^{g^{-1}} \sim v$ so that $H = H^{-1}$.

For (iii) let $g \in G$. As Γ is connected there is a path $v \sim v_1 \sim v_2 \sim ... \sim v_l = v^g$ and by vertex transitivity, there are $g_1, g_2, ...$ so that $v_i = v^{g_i}$ with $g_i \in G$. Since $v_{i-1} \sim v_i$ we have that $v^{g_{i-1}} \sim v^{g_i}$. This implies that $v \sim v^{g_i g_{i-1}^{-1}}$ so that $h_i = g_i g_{i-1}^{-1}$ belongs to H for $i = 2, ..., l$. and $h_1 = g_1 \in H$. Therefore we have that

$$g = g_l = h_l g_{l-1}$$

where $g_{l-1} = h_{l-1} g_{l-2}$ and so on. By the above we conclude that $g = g_l = h_l h_{l-1} .....h_2 h_1$ where $h_l, h_{l-1}, ..., h_2, h_1 \in H$ and so also the third condition is satisfied.

Now consider the Cayley graph $\Gamma^* = Cay(G, H)$. Let K be the stabilizer group of v. In particular, $|G| = |K| \cdot |V|$. Then we have that $KH = \{g \in G : v \sim v^{Kg} = v^g\}$ hence from this we conclude that $KH = H$ is a union of (right) coests. Let Kg_i and Kg_j be two distinct cosets of K. Let $x = k_1 g_i$ and $y = k_2 g_j$ be in Kg_i and Kg_j respectively where $k_1, k_2 \in K$. Then $x \sim y$ in Γ^* if and only if $v^{k_1 g_i} \sim v^{k_2 g_j}$ in Γ. Note by the stabilizer group properties we have that $v^{k_1 g_i} = v^{g_i}$ and $v^{k_2 g_j} = v^{g_j}$ so that $x \sim y$ if and only if $v^{g_i} \sim v^{g_j}$ in Γ. From this we conclude that all vertices in Kg_i are adjacent to all vertices in Kg_j. While the elements of any coset of K are not adjacent to each other as Γ is simple.

It follows that the adjacency matrix of Γ^* is of the form $A(\Gamma) \otimes J$ where J is the $|K| \times |K|$ matrix with all entries equal to 1. Note J is singular and the eigenvalues of J are $|K|$ with multiplicity of 1 and 0 with multiplicity of $|K| - 1$. In particular, we have that

$$Spec(\Gamma^*) = \{|K|\lambda_1, |K|\lambda_2, ..., |K|\lambda_n, 0, 0, ..., 0\}$$

where $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of Γ. From the above we conclude the following result that determines vertex transitive singular graphs:

Theorem 4.3.1. *Let $\Gamma = (V, E)$ be a connected graph which admit a group G of automorphisms that is transitive on V. Let v be a vertex of Γ and assume that the stabilizer of v has order $c > 1$. Let $H = \{ g \in G : v \sim v^g \}$. Then H is a connecting set and $\Gamma^* = \mathrm{Cay}(G, H)$ has nullity $\mathrm{null}(\Gamma^*) \geq (|V|) \cdot (c-1)$. Furthermore, Γ is singular if and only if $\mathrm{null}(\Gamma^*) > |V| \cdot (c-1)$.*

Example: We apply this method to find the spectrum of $K_3 = (V, E)$ where $V = \{1, 2, 3\}$. It is clear that $G = Aut(K_3) = Sym(V)$. Note we have that

$$G = \{1_{Sym(V)}, (12), (13), (23), (123), (132)\}$$

without loss of generality. The graph $K_3^* = (V^*, E^*)$ is defined as $V^* = G$

and its edge are defined by

$$(\sigma, \gamma) \in E^* \text{ if and only if } (v^\sigma, v^\gamma) \in E$$

where $\sigma, \gamma \in G$ and $v \in V$. Therefore we chose the vertex 1 of K_3 to construct K_3^*. Note we have that

$$1_G(1) = 1,\ (12)(1) = 2,\ (13)(1) = 3,\ (23)(1) = 1,\ (123)(1) = 2$$

and $(132)(1) = 3$. From this we have that

$$H = \{(12), (13), (123), (132)\}$$

and

$$K = \{1_G, (23)\}$$

the stabilizer of the vertex 1. Thus we can compute the spectrum of K_3^* by applying Theorem 4.1.4. Therefore we have that

$$\mu_i = \sum_{h \in H} \rho_i(h)$$

where ρ_i is the irreducible representation of $sym(V)$ for $i = 1, 2, 3$, see Table 4.4. Hence we have that $\mu_1(H) = 4$ where $\rho_1(\sigma) = 1$ for all $\sigma \in H$, $\mu_2(H) = 0$ where $\rho_2(\sigma) = sign(\sigma)$ for all $\sigma \in H$ and

$$\mu_3 = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}.$$

Thus the spectrum of K_3^* is $\{4, -2, -2, 0, 0, 0\}$ and by the first method we have the spectrum of K_3 is $\{2, -1, -1\}$, as is well-known.

Example: Let $\Gamma = (V, E)$ be Petersen graph and let V be the collection of all 2-sets from $\Omega = \{1, 2, 3, 4, 5\}$. Then two vertices are adjacent in Γ if and only if their sets are disjoint sets as shown in the following

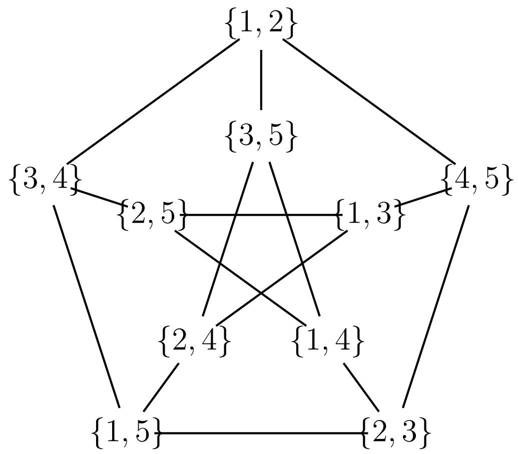

Figure 4.3.1: Petersen Graph

We demonstrate the method by computing its spectrum. Let

$$G = AGL(1,5) = \langle (12345), (2354) \rangle.$$

Note G acts transitively on V. Fix a vertex $v = \{1, 2\}$ of $V(\Gamma)$, and put $H = \{g \in G : v \sim v^g\}$ and $K = \{g \in G : v^g = v\}$. Hence we have that

$$H = \{(13524), (1325), (14)(23), (14253), (1523), (15)(24)\}$$

and

$$K = \{1_G, (12)(35)\}.$$

Now consider $\Gamma^* = Cay(G, H)$. This is a graph on 20 vertices of degree 6.

In this figure every box consists two vertices and these are adjacent to the vertices in the adjacent boxes.

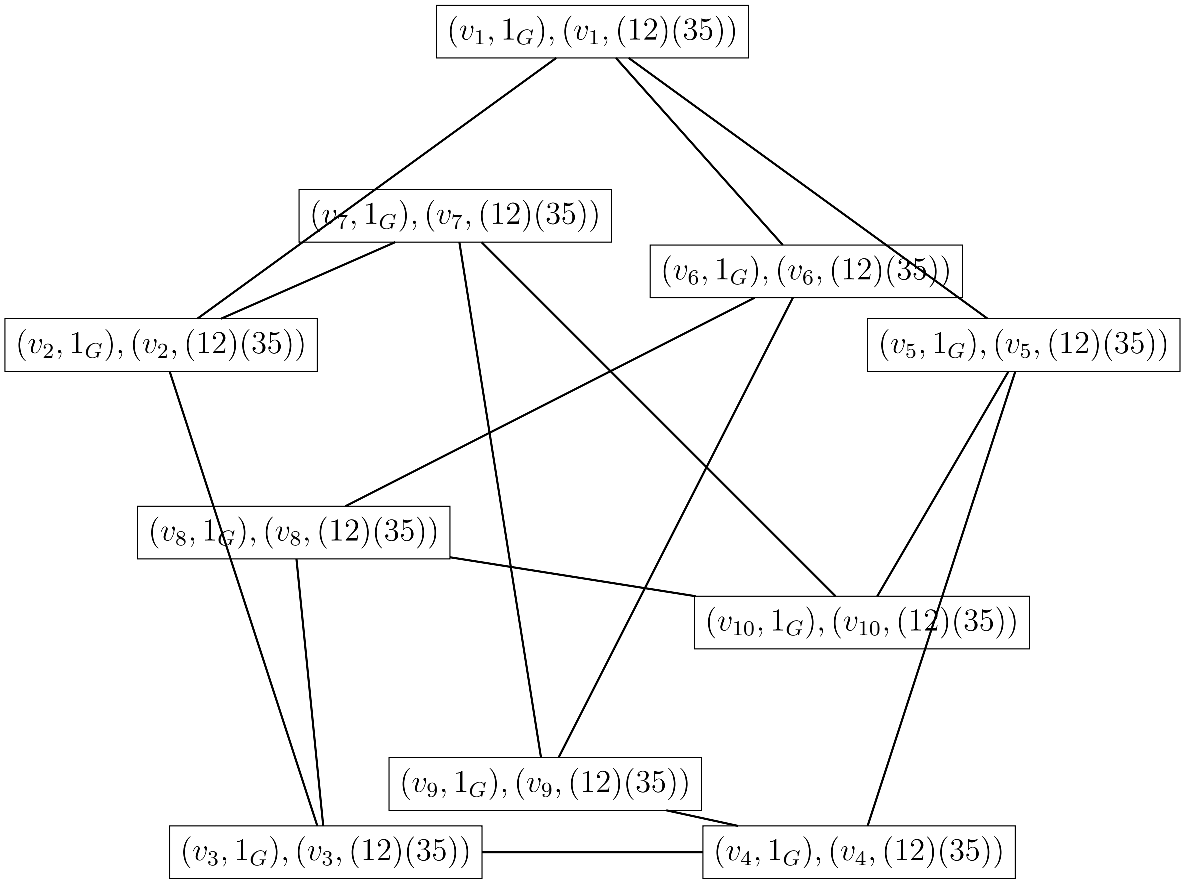

Figure 4.3.2: $\Gamma^* = Cay(AGL(1,5), H)$

By a GAP program we have that the spectrum of Γ^* is $\{6^1, -4^4, 2^5, 0^{10}\}$. It is clear that the spectrum of Γ^* is divided into two sets of size $|V| = 10$ which are $\{6^1, -4^4, 2^5\}$ and $\{0^{10}\}$. According to this method the spectrum of Γ is $\{3^1, -2^4, 1^5\}$ by dividing $\{6^1, -4^4, 2^5\}$ by $|K|$.

4.4 The Second Method

In this method we compute the spectrum of a vertex transitive graph in terms of the irreducible characters of a transitive group of automorphisms. This method is new as far as we know. We apply this method to some examples.

As before Γ is a simple connected graph with vertex set V. Let G be a vertex transitive group of automorphisms of Γ. Let $U_1, ..., U_s$ be the irreducible modules of G with corresponding characters $\chi_1, ..., \chi_s$. Let $E_1, ..., E_t$ be the eigenspaces of α with corresponding eigenvalues $\lambda_1, ..., \lambda_t$. Let $m_{j,i}$ be the multiplicity of U_i in E_j.

Theorem 4.4.1. *Let G be a group of automorphisms of the graph Γ which acts transitively on $V = V(\Gamma)$. Consider $\tau(g) := \text{tra}(g\alpha)$ for $g \in G$ where α is the adjacency map of Γ. Then $\tau(g)$ is a class function and $\langle \tau, \chi_i \rangle = \sum_{j=1}^{t} m_{j,i}\lambda_j$. If the permutation action of G on V is multiplicity-free then the following hold*

(i) Every eigenvalue of Γ is equal to $\langle \tau, \chi_i \rangle$ for some i with multiplicity of $\chi_i(1_G)$.

(ii) Γ is singular if and only if $\sum \chi_i(1) < |V|$ where the sum runs over all characters χ_i with $\langle \tau, \chi_i \rangle \neq 0$.

Comments: 1. It is clear that $\tau(g) = \text{tra}(g\alpha)$ is the number of times a vertex $v \in V$ is adjacent to its image v^g under g. In particular, $\tau(1_G) = 0$.

(2) A permutation character ψ of G is *multiplicity-free* if and only if each irreducible character of G appears with multiplicity ≤ 1 in ψ. In particular, if ψ is multiplicity-free then G is transitive.

(3) As before G, is transitive on the vertex set V of Γ. Therefore the permutation representation of G on V is a sub-representation of the regular representation. Therefore $\sum_{j=1...t} m_{j,i} \leq \dim(U_i)$ for all i. For instance, if G is abelian then all non-zero eigenvalues are of the form $\lambda_j = \langle \tau, \chi_i \rangle$ for some i.

(4) If $\Gamma = Cay(G, H)$ where H is a normal connecting set of G then $\langle \tau, \chi_i \rangle = m_i\lambda_i$ where λ_i is an eigenvalue of Γ and m_i is the dimension of U_i. Here the multiplicity of λ_i is m_i^2. Since by Theorem 4.1.6 we have that U_i appears in $\mathbb{C}V$ with multiplicity m_i and so on m_iU_i appears in one eigenspace.

(5) We consider the case where the permutation character ψ of G is not multiplicity free. Let r be the multiplicity of χ_i in ψ. Define $\tau^l(g) = \mathrm{tra}(\alpha^l g)$ for some $l \in \mathbb{N}$. Then, using the same ideas as in the proof of the theorem, we have

$$\begin{aligned}
\sum_{j=1}^{t} m_{j,i}\lambda_j &= \langle \tau, \chi_i \rangle \\
\sum_{j=1}^{t} m_{j,i}\lambda_j^2 &= \langle \tau^2, \chi_i \rangle \\
&\vdots \\
\sum_{j=1}^{t} m_{j,i}\lambda_j^r &= \langle \tau^r, \chi_i \rangle
\end{aligned} \tag{4.4.1}$$

where $r \leq m_i$. These are additional equations to determine the spectrum of Γ. Please see the example of the Petersen graph with the General Affine Group.

Proof of Theorem 4.4.1 : As before let $\mathbb{C}V$ be the vertex module of Γ and α be the adjacency map of Γ.

First we show that $\tau(g)$ is a class function. Note by Proposition 3.4.1 we have that $\alpha h^{-1}gh = h^{-1}\alpha gh$ for $h \in G$. Since

$$\begin{aligned}
\mathrm{tra}(\alpha h^{-1}gh) &= \mathrm{tra}(h^{-1}\alpha gh) \\
&= \mathrm{tra}(\alpha ghh^{-1}) \\
&= \mathrm{tra}(\alpha g)
\end{aligned}$$

we have that $\tau(g) = \mathrm{tra}(\alpha g)$ is a class function. So we can write τ in the following shape

$$\tau(g) = \langle \tau(g), \chi_1(g) \rangle \chi_1(g) + ... + \langle \tau(g), \chi_s(g) \rangle \chi_s(g) \tag{4.4.2}$$

as $\chi_1, ..., \chi_s$ is an orthonormal basis of the vector space of all class functions.

Let $\pi_1, ..., \pi_t$ be the projections $\pi_j \colon \mathbb{C}V \to \mathbb{C}V$ with $\pi_j(\mathbb{C}V) \subseteq E_j$. Since G

preserves eigenspaces and commutes with the π_j (in both cases as G commutes with α) we have $g\alpha\pi_j = g\lambda_j\pi_j = \lambda_j g\pi_j$. Since $\pi_1 + ... + \pi_t = id$ we have

$$\begin{aligned}\langle \tau, \chi_i \rangle &= \langle \text{tra}(\alpha g), \chi_i \rangle \\ &= \langle \sum_j \text{tra}(g\alpha\pi_j), \chi_i(g) \rangle \\ &= \sum_j \lambda_j \langle \text{tra}\, g\pi_j, \chi_i(g) \rangle \\ &= \sum_j \lambda_j\, m_{ji}.\end{aligned}$$

Note, if the permutation character of G on vertices is multiplicity-free then $0 \leq m_{ji} \leq 1$ for all j, i and for every i there is at most one j with $m_{ji} = 1$. Hence $\lambda_j = \langle \tau, \chi_i \rangle$ for such a pair. By the same argument, $\lambda_j = 0$ is an eigenvalue if and only if $\sum \chi_i(1) < |V|$ where the sum is over all characters with $\langle \tau, \chi_i \rangle \neq 0$. □

Note, if $\Gamma = Cay(G, H)$ and H is a normal connecting set of G, then by Theorem 4.1.6 we have that each irreducible G-module say U_i appears in exactly one eigenspace of Γ with multiplicity of m_i. Hence we conclude that $\langle \tau, \chi_i \rangle = \lambda_i m_i$ where λ_i is an eigenvalue of Γ with multiplicity of m_i^2.

In the remainder of this section we apply Theorem 4.4.1 to some examples.

Example 1: Once again, let Γ be Petersen graph. We determine the spectrum of Γ by the second method.

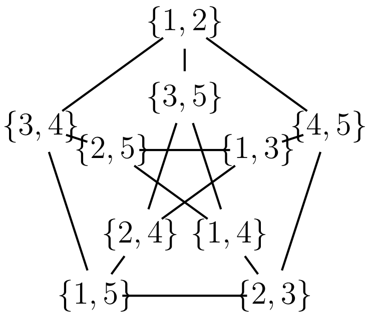

Figure 4.4.1: Petersen Graph

Note, A_5 acts transversely on $V(\Gamma)$. Consider $\tau = \text{tra}(\alpha g)$ where $g \in A_5$. Therefore the character table of A_5 with the function ψ and τ are shown in the Table 4.15.

	1_{A_5}	(123)	(12)(34)	(12345)	(13425)
$\lvert g_i^{A_5} \rvert$	1	20	15	12	12
χ_1	1	1	1	1	1
χ_2	4	1	0	-1	-1
χ_3	5	-1	1	0	0
χ_4	3	0	-1	ε	ϵ
χ_5	3	0	-1	ϵ	ε
ψ	10	1	2	0	0
τ	0	0	4	5	5

Table 4.15: The character table of A_5, ψ and τ

where $\varepsilon = \frac{1}{2}(1+\sqrt{5})$ and $\epsilon = \frac{1}{2}(1-\sqrt{5})$.

Next, represents ψ and τ as a linear combination of the χ_i. We see $\psi = \chi_1 + \chi_2 + \chi_3$ and $\tau = 3\chi_1 - 2\chi_2 + \chi_3$. We note that ψ is multiplicity-free and so that the spectrum of Γ is $3^1, -2^4$ and 1^5.

In this example $G = A_5$ acted multiplicity freely on V. In the next example we replace G by $AGL(1,5)$ which is not multiplicity free.

Example 2: We apply Theorem 4.4.1 to compute the spectrum of Γ using $G = AGL(1,5) = \langle (12345), (2354) \rangle$. As we have seen that G acts transitively on $V(\Gamma)$. The character table of G with the functions ψ, τ and τ^2 are shown in the Table 4.16.

	$1_{GA(1,5)}$	(1342)	(1243)	(25)(34)	(12345)
$\lvert g^G\rvert$	1	5	5	5	4
χ_1	1	1	1	1	1
χ_2	1	-1	-1	1	1
χ_3	1	A	$-A$	-1	1
χ_4	1	$-A$	A	-1	1
χ_5	4	0	0	0	-1
ψ	10	0	0	2	0
τ	0	2	2	4	5
τ^2	30	$*$	$*$	$*$	5

Table 4.16: The character table of G, ψ, τ and τ^2

where $A = -E(4) = -\sqrt{-1} = -i$. Note $* \in \mathbb{N}$.

Next represents ψ as a linear combination of the χ_i for $i = 1, 2, 3, 4, 5$. We see $\psi = \chi_1 + \chi_2 + 2\chi_5$. It is clear that ψ is not multiplicity-free. Now we represent τ as a linear combination of χ_i. We have that $\tau = 3\chi_1 + \chi_2 - \chi_5$. Note χ_1 and χ_2 have multiplicity of 1 in ψ so that the coefficients of these characters in τ are eigenvalues of Γ. However the multiplicity of χ_5 in ψ is 2. In this case Theorem 4.4.1 by itself is not sufficient to solve. However using Comment 5 above we have the additional equations are shown in the following:

$$\begin{aligned} \lambda_{5,1} + \lambda_{5,2} &= -1 \\ \lambda_{5,1}^2 + \lambda_{5,2}^2 &= 5. \end{aligned}$$

Thus by solve these equations we have that $\lambda_{5,1} = -2$ and $\lambda_{5,2} = 1$. From the above we conclude that the $Spec(\Gamma) = \{3^1, -2^4, 1^5\}$.

Example3: Let $D_6 = \langle a, b : a^6 = b^2 = 1_{D_6}, bab = a^{-1}\rangle$. Let Γ be the graph $\Gamma = Cay(D_6, H)$ where $H = \{a, a^5, b, a^2b, a^4b\}$.

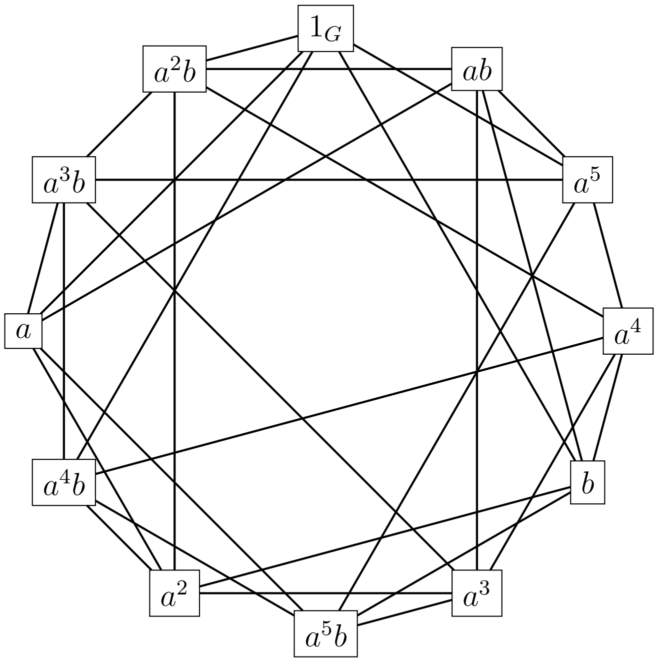

Figure 4.4.2: $\Gamma = Cay(D_6, H)$ where $H = \{a, a^5, b, a^2b, a^4b\}$

In this example we apply Theorem 4.4.1 to compute the spectrum of Γ. The character table of D_6 with function ψ and τ are shown in the Table 4.17.

	1_{D_6}	a^3	a	a^2	b	ab
$g_i^{D_6}$	1	1	2	2	3	3
χ_1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	-1
χ_3	1	-1	-1	1	1	-1
χ_4	1	-1	-1	1	-1	1
χ_5	2	-2	1	-1	0	0
χ_6	2	2	-1	-1	0	0
ψ	12	0	0	0	0	0
τ	0	0	12	0	12	0

Table 4.17: The character table of D_6, ψ and τ

Next represents ψ as a linear combination of the χ_i for $i = 1, 2, 3, 4, 5, 6$. We see that $\psi = \chi_1 + \chi_2 + \chi_3 + \chi_4 + 2\chi_5 + 2\chi_6$. It is clear that ψ is the regular permutation character and since H is the normal connecting set hence according to Theorem 4.4.1 we have that

$$\langle \tau, \chi_i \rangle = m_i \lambda_i$$

for $i = 1, 2, 3, 4, 5, 6$. In this case we have that $\tau = 5\chi_1 - \chi_2 + \chi_3 - 5\chi_4 + 2\chi_5 - 2\chi_6$. Thus we have that $Spec(\Gamma) = \{5^1, -5^1, -1^5, 1^5\}$.

Bibliography

[1] A. Abdollahi. Determinants of adjacency matrices of graphs. *arXiv preprint arXiv:0908.3324*, 2009.

[2] L. Aquino-Ruivivar. Singular and nonsingular circulant graphs. *Journal of Research in Science, Computing, and Engineering*, 3(3), 2010.

[3] F. Ashraf and H. Bamdad. A note on graphs with zero nullity. *MATCH Commun. Math. Comput. Chem*, 60(1):15–19, 2008.

[4] M. Axenovich. Lecture Notes Graph Theory. Online, 2014.

[5] S. J. Axler. *Linear algebra done right*, volume 2. Springer, 1997.

[6] L. Babai. Spectra of Cayley graphs. *Journal of Combinatorial Theory, Series B*, 27(2):180–189, 1979.

[7] R. Bapat. A note on singular line graphs. *Bull. Kerala Math. Assoc*, 8(2), 2011.

[8] N. Biggs. *Algebraic graph theory*. Cambridge University Press, 1993.

[9] S. Black and R. List. A note on plethysm. *European Journal of Combinatorics*, 10(1):111–112, 1989.

[10] B. Borovicanin and I. Gutman. Nullity of graphs. *Applications of Graph Spectra*, page 107, 2009.

[11] A. E. Brouwer and W. H. Haemers. *Spectra of Graphs*. Springer Science & Business Media, 2011.

[12] G. J. Chang, L.-H. Huang, and H.-G. Yeh. A characterization of graphs with rank 4. *Linear Algebra and its Applications*, 434(8):1793–1798, 2011.

[13] G. J. Chang, L.-H. Huang, and H.-G. Yeh. A characterization of graphs with rank 5. *Linear Algebra and its Applications*, 436(11):4241–4250, 2012.

[14] B. Cheng and B. Liu. On the nullity of graphs. *Electronic Journal of Linear Algebra*, 16(1):5, 2007.

[15] C. Coker. A problem related to Foulkes's Conjecture. *Graphs and Combinatorics*, 9(2-4):117–134, 1993.

[16] D. Cvetković, M. Doob, and H. Sachs. Spectra of Graphs–Theory and Application, Johann Ambrosius Barth Verlag. *Heidelberg, Leipzig*, 1995.

[17] D. M. Cvetković and I. M. Gutman. The algebraic multiplicity of the number zero in the spectrum of a bipartite graph. *Matematički Vesnik*, 9(56):141–150, 1972.

[18] P. Delsarte. *An Algebraic Approach to the Association Schemes of Coding Theory*. Philips journal of research / Supplement. N.V. Philips' Gloeilampenfabrieken, 1973.

[19] P. Diaconis and M. Shahshahani. Generating a random permutation with random transpositions. *Probability Theory and Related Fields*, 57(2):159–179, 1981.

[20] Y.-Z. Fan and K.-S. Qian. On the nullity of bipartite graphs. *Linear Algebra and its Applications*, 430(11-12):2943–2949, 2009.

[21] S. Fiorini, I. Gutman, and I. Sciriha. Trees with maximum nullity. *Linear Algebra and its Applications*, 397:245–251, 2005.

[22] J. B. Fraleigh. *A first course in abstract algebra*. Pearson Education India, 2003.

[23] C. Godsil and G. F. Royle. *Algebraic Graph Theory*, volume 207. Springer Science & Business Media, 2013.

[24] S.-C. Gong, Y.-Z. Fan, and Z.-X. Yin. On the nullity of graphs with pendant trees. *Linear Algebra and its Applications*, 433(7):1374–1380, 2010.

[25] S.-C. Gong and G.-H. Xu. On the nullity of a graph with cut-points. *Linear Algebra Appl*, 436(1):135–142, 2012.

[26] A. Graovac, I. Gutman, N. Trinajstić, and T. Živković. Graph theory and molecular orbitals. *Theoretical Chemistry Accounts: Theory, Computation, and Modeling (Theoretica Chimica Acta)*, 26(1):67–78, 1972.

[27] J. L. Gross and J. Yellen. *Handbook of Graph Theory*. CRC press, 2004.

[28] J.-M. Guo, W. Yan, and Y.-N. Yeh. On the nullity and the matching number of unicyclic graphs. *Linear Algebra and its Applications*, 431(8):1293–1301, 2009.

[29] F. Harary. The determinant of the adjacency matrix of a graph. *Siam Review*, 4(3):202–210, 1962.

[30] L. Hogben. *Handbook of Linear Algebra*. CRC Press, 2006.

[31] S. Hu, T. Xuezhong, and B. Liu. On the nullity of bicyclic graphs. *Linear Algebra and its Applications*, 429(7):1387–1391, 2008.

[32] G. James and M. W. Liebeck. *Representations and Characters of Groups*. Cambridge University Press, 2001.

[33] A. W. Joshi. *Matrices and Tensors in Physics*. New Age International, 1995.

[34] D. Kiani, M. M. H. Aghaei, Y. Meemark, and B. Suntornpoch. Energy of unitary Cayley graphs and gcd-graphs. *Linear Algebra and its Applications*, 435(6):1336–1343, 2011.

[35] W. Klotz and T. Sander. Integral Cayley graphs over abelian groups. *Electron. J. Combin*, 17(1):R81, 2010.

[36] I. Kra and S. R. Simanca. On circulant matrices. *Notices of the AMS*, 59(3):368–377, 2012.

[37] M. Krebs and A. Shaheen. *Expander Families and Cayley Graphs: A Beginner's Guide*. Oxford University Press, 2011.

[38] A. Lal and A. S. Reddy. Non-singular circulant graphs and digraphs. *arXiv preprint arXiv:1106.0809*, 2011.

[39] T. Y. Lam and K. H. Leung. On vanishing sums of roots of unity. *Journal of Algebra*, 224(1):91–109, 2000.

[40] V. I. Levenshtein and J. Siemons. Error graphs and the reconstruction of elements in groups. *Journal of Combinatorial Theory, Series A*, 116(4):795–815, 2009.

[41] J. Li, A. Chang, and W. C. Shiu. On the nullity of bicyclic graphs. *MATCH Commun. Math. Comput. Chem*, 60(1):21–36, 2008.

[42] W. Li and A. Chang. On the trees with maximum nullity. *MATCH Commun. Math. Comput. Chem*, 56(3):501–508, 2006.

[43] L. Lovász. Spectra of graphs with transitive groups. *Periodica Mathematica Hungarica*, 6(2):191–195, 1975.

[44] M. Marino, I. Sciriha, S. Simić, and D. Tošić. More about singular line graphs of trees. *Publications de l'Institut Mathematique*, 79(93):1–12, 2006.

[45] B. D. McKay and C. E. Praeger. Vertex-transitive graphs which are not Cayley graphs, I. *Journal of the Australian Mathematical Society*, 56(1):53–63, 1994.

[46] J. Müller and M. Neunhöffer. Some computations regarding Foulkes' conjecture. *Experimental Mathematics*, 14(3):277–283, 2005.

[47] M. R. Murty. Ramanujan graphs. *Journal-Ramanujan Mathematical Society*, 18(1):33–52, 2003.

[48] M. Nath and B. K. Sarma. On the null-spaces of acyclic and unicyclic singular graphs. *Linear Algebra and its Applications*, 427(1):42–54, 2007.

[49] W. Paulsen. *Abstract Algebra: An interactive approach*. CRC Press, 2009.

[50] I. Sciriha. On singular line graphs of trees. *Congressus Numerantium*, pages 73–92, 1998.

[51] I. Sciriha. A characterization of singular graphs. *Electronic Journal of Linear Algebra*, 16(1):38, 2007.

[52] I. Sciriha and P. W. Fowler. On nut and core singular fullerenes. *Discrete Mathematics*, 308(2):267–276, 2008.

[53] H. Shengbiao. The classification and maximum determinants of the adjacency matrix of graphs with one cycle [j]. *Journal of Mathematical Study*, 1:016, 2003.

[54] J. Siemons and A. Zalesski. Remarks on singular cayley graphs and vanishing elements of simple groups. *arXiv preprint arXiv:1804.01204*, 2018.

[55] S. Sookyang, S. Arworn, and P. Wojtylak. Characterizations of non-singular cycles, path and trees. *Thai Journal of Mathematics*, 6(2):331–336, 2012.

[56] S. Szabó. *Topics in Factorization of Abelian Groups*. Springer Science & Business Media, 2004.

[57] L. Von Collatz and U. Sinogowitz. Spektren endlicher Grafen. In *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, volume 21, pages 63–77. Springer, 1957.

[58] T. Xuezhong and B. Liu. On the nullity of unicyclic graphs. *Linear Algebra and its Applications*, 408:212–220, 2005.

[59] P.-H. Zieschang. Cayley graphs of finite groups. *Journal of Algebra*, 118(2):447–454, 1988.

A

Thesis Programming

In this appendix, we provide all necessary code regarding Cayley graphs spectra and spectral decompositions. We apply these code in GAP or Maple.

A.1 Spectrum Computation with GAP

In this section, we introduce a code for computing the spectrum of Cayley graph over a finite group. Note we apply this code in version 4.8.9 of GAP and use the package "grape". We list the commands to generate a finite group. For instance, the following steps generate the dihedral group of order 24.

```
 f:=FreeGroup("a","b");
<free group on the generators [ a, b ]>
 G:=f/[f.1^12,f.2^2,f.2*f.1*f.2*f.1];
 <fp group on the generators [ a, b ]>
a:=g.1;; b:=g.2;;
```

We use the following function to return a Cayley graph over the group G and connecting set H.

```
C:=CayleyGraph(G,[H]);
```

Now we apply the following code to compute the spectrum of C.

```
# Generate empty list
 M:=[];
 # n is the number of vertices of C
  n:=Length(Vertices(C));
 i:=1;
while i <= n do
# L generates the vector which represents the set of the vertices of C
L:=Vertices(C);
# L is the column vector representing the set of vertices that are adjacent
to vertex i
for j in L do
 if j in Adjacency(C,i) then
L[j]:=1;
else
L[j]:=0;
fi;
od;
# Add this column to the list M
Add(M,L);
i:=i+1;
od;
# M is the adjacency matrix of the graph C
M;
# Computing the spectrum of C over n^{th} Cyclotomic field
Eigenvalues(CyclotomicField(n),M);
```

A.2 Spectrum computation with Maple

In this section, we provide a code for Maple to compute the spectrum of a Cayley graph over a cyclic group C_n. In this code we choose H arbitrary by the choosing function. For each choice we compute the spectrum of $Cay(G, H)$. For this code we need to load the packages combinat and linear algebra.

```
Loading combinat
Loading LinearAlgebra
# calculate eigenvalue of the adjacency matrix of a cayley graph
CayleyEigenvalue := proc (s, H, p)
 local k, t, n1, i;
  n1 := Dimension(H);
   t := 0;
  for i to n1 do
   k := H[i];
   # t is the eigenvalue Cayley graph
  t := t+exp((2*I)*k*s*Pi/p)
  end do;
  return t
  end proc;
  n:=|C_n|;
f := j-> j:
# Create a vector with respect to f
w := Vector(n-1, f);
m := Dimension(w);
# ceil((m/2)) is the smallest integer greater than or equal to (m/2)
k := ceil((m/2));
# Creating a vector of dimension k with respect to f
Z := Vector(k, f);
```

```
# create the matrix M where its rows are the combinations of size |H|/2
M := Matrix(choose(k, |H|/2));
l := RowDimension(M);
A := Matrix(n, l);
for i to l do
h := M[i]-~ n/2;
# remove zeros from h
r := remove[flatten](t -> t = 0 , h);
R := r+~n/2;
# H1 is closed under the inverses
H1 := <M[i]|n-~ R>;
 H := convert(H1, Vector);
 for j from 0 to n-1 do
  A(j+1, i) := CayleyEigenvalue(j, H, n):
   end do
    end do:
# A is a matrix where its columns  are the spectrum of Cay(C_n,H_1)
A;
```