Quasi-hereditary Covers and Derived Equivalences of Higher Zigzag Algebras

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Abstract

In this thesis we look at higher zigzag algebras Z_s^d of type A as a generalization of Brauer tree algebras whose tree is a line. We recall the presentation of these algebras as path algebras with relations and their relation with higher preprojective algebras of d-representation finite and Koszul algebras. The algebras Z_s^d are not Koszul, since simple modules do not have linear projective resolutions. To overcome this lack of regularity we give an explicit construction of a quasi-hereditary cover for Z_s^d as a quotient algebra of Z_{s+1}^d and we study different Koszul properties of these quasi-hereditary algebras. We prove that they are Koszul in the classical sense, standard Koszul and, endowed with an appropriate grading, Koszul with respect to the standard module Δ. This more general Koszul property leads to a well-defined notion of duality, generalizing the classical Koszul duality. We will show that the Δ-Koszul dual of our quasi-hereditary cover is again a Koszul algebra in the classical sense. Using the fact that Koszul algebras are quadratic we will be able to give a presentation of the Δ-Koszul dual algebras as path algebras with relations.

The last chapter of this thesis will be about derived Morita theory and silting objects for higher zigzag algebras. Since in the case of Brauer tree algebras the Okuyama–Rickard method to obtain two-term tilting complexes leads to the complete classification of the derived equivalence class, we focus our attention on two-term tilting complexes in the derived category $\mathcal{D}^b(Z_s^d)$. For Z_s^d we give a more explicit description of irreducible Okuyama–Rickard mutations of Z_s^d. To conclude we describe the derived equivalence class of Z_3^2 by showing all the algebras derived equivalent to it.

Contents

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Introduction

Zigzag algebras and, in particular, type A zigzag algebras have made their appearance in many areas of representation theory of algebras and finite groups. First examples of these algebras appeared for instance in modular representation theory of finite groups, since some zigzag algebras are Morita equivalent to blocks of finite groups with cyclic defect ([Alp86]). They have also been studied in relation with the action of braid groups on derived categories ([ST01], [RZ03] and [HK01]). In [HK01] the authors give an explicit construction for the zigzag algebra $A(G)$ of a connected graph G without loops or multiple edges and they show that this algebra is a graded, quadratic, trivial extension algebra, hence symmetric. Moreover Brenner, Butler and King in [BBK02] also show that, when the graph G is bipartite, then the quadratic dual of $A(G)$ is isomorphic to the *preprojective algebra* of G.

For an acyclic quiver Q, the preprojective algebra of Q was introduced by Gelfand and Ponomarev ([GP79]), who gave an explicit construction as path algebra with relations. Then Baer, Geigle and Lenzing gave a more abstract construction for these algebras as direct sums of spaces $\mathrm{Hom}_{kQ}(kQ, \tau^{-l}kQ)$, where τ and τ^{-} denote the Auslander–Reiten translate and its inverse respectively. Hence these algebras decompose as the direct sum of preprojective modules, from which their name.

More recently, Iyama developed a higher-dimensional version of Auslander–Reiten theory ([Iya07]), with higher analogues of translates τ_d and τ_d^{-} as well as higher versions of preprojective algebras ([HI11b], [IO11], [IO13], [HIO14]). Trying to find a more explicit presentation for higher preprojective algebras, Grant and Iyama prove the following result (see Definition 2.1.1 for the definition of Triv_d):

Theorem 1 ([GI])**.** Let Λ be a Koszul algebra of global dimension d and Π its higher preprojective algebra.

1. There exists a morphism of graded k-algebras $\phi : \Pi^! \to \mathrm{Triv}_d(\Lambda^!)$.

2. If Λ is d-hereditary then ϕ is surjective; in this case ϕ is an isomorphism if and only if $\mathrm{Triv}_d(\Lambda^!)$ is quadratic.

d-hereditary algebras were defined by Herschend, Iyama and Oppermann in [HIO14]. A particularly nice set of examples of d-hereditary and d-representation finite algebras is given by type A d-Auslander algebras $T_s^{(d)}(k)$ defined by Iyama in [Iya11]. Applying the previous result to this family of algebras Grant and Iyama showed that, in this case, ϕ gives an example of *almost Koszul duality*.

Having in mind the duality between preprojective algebras and zigzag algebras, Grant defined *higher zigzag algebras* of Koszul algebras of finite global dimension d:

Definition 2 ([Gra17], Definition 2.5)**.** Let Λ be a Koszul algebra of finite global dimension d. The $(d+1)$-*zigzag algebra* of Λ is $Z_{d+1} = \mathrm{Triv}_{d+1}(\Lambda^!)$.

These algebras are the main objects of study of this thesis. Historically higher zigzag algebras were first defined in [Guo16] and [GL16] under the name of *d-cubic pyramid algebras*, in relation with translation quivers appearing in higher representation theory. Independently they appeared in [Gra17] and [GI] where the authors focused on their connection with higher preprojective algebras. In this thesis a special role will be played by higher zigzag algebras of d-Auslander algebras of type A, $T_s^{(d)}(k)$: Iyama's "cone" construction is recursive, so we have examples of higher zigzag algebras of algebras $T_s^{(d)}(k)$ with any given global dimension. Since they come from higher Auslander algebras of type A quivers, these algebras have been called *type A* higher zigzag algebras. In [Gra17] the author proved that they are symmetric and have a nice presentation as path algebras of some quivers modulo quadratic relations, so it will be easier to explain some particular constructions in this case.

Some particular quotients of type A classical zigzag algebras Z_s^1, often denoted in the literature by $\mathcal{A}_{s+1}$, appeared also in the work of many authors about *quasi-hereditary algebras* (Definition 1.2.5; see also [Erd93] and [KS02]). Working with fields of positive characteristic, Erdmann showed that the algebras $\mathcal{A}_s$ describe certain blocks of polynomial representations of general linear groups (see [Erd93], Section 3.1 and Proposition 4.1). More recently, in the case $k = \mathbb{C}$, Brundan and Stroppel in [BS11] described the category $\mathcal{O}^{\mathfrak{p}}$ in the case $\mathfrak{g} = \mathfrak{gl}(l+s)$ and when $\mathfrak{p}$ is the parabolic subalgebra with Levi component $\mathfrak{gl}(l+s) \oplus \mathfrak{gl}(m)$. When $l = 1$ the algebra associated to the principal block of $\mathcal{O}^{\mathfrak{p}}$ is isomorphic to $\mathcal{A}_{s+1}$ (see [BJ77], [Irv85]). The algebra $\mathcal{A}_{s+1}$ is a *quasi-hereditary cover* for the classical zigzag algebra Z_s^1 which can be defined as a quotient of the algebra Z_{s+1}^1. The definition of quasi-hereditary cover that we use is due to Rouquier:

Definition 3 ([Rou08])**.** Let Λ be a quasi-hereditary algebra and P a finitely generated projective Λ-module. The pair (Λ, P) is a *quasi-hereditary cover* for $\Lambda' = \mathrm{End}_\Lambda(P)$ if the restriction of the functor

$$F = \mathrm{Hom}_\Lambda(P, -) : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Lambda'$$

to the subcategory $\mathrm{proj}\,\Lambda$ of projective Λ-modules is fully faithful.

An interesting feature of the quasi-hereditary algebras $\mathcal{A}_{s+1}$ is that they are Koszul and, when endowed with different gradings, provide an interesting example of Δ-Koszul algebras (or *Koszul with respect to the standard module* Δ, see Definition 1.2.2). This is a special instance of the following generalized Koszul property:

Definition 4 ([Mad11]). Let $\Lambda = \bigoplus_{n\geq 0} \Lambda_{[n]}$ be a graded algebra such that $\operatorname{gldim} \Lambda_{[0]} < \infty$ and let T be a graded Λ-module concentrated in degree zero. Denote by $T\langle j \rangle$ the graded shift in the category of finitely generated graded Λ-modules. Then we say that Λ is *Koszul with respect to* T, or T*-Koszul*, if:

1. T is a tilting $\Lambda_{[0]}$-module.

2. T is graded self-orthogonal as a Λ-module, that is

$$\operatorname{Ext}^i_{\mathsf{gr}\Lambda}(T, T\langle j \rangle) = 0, \text{ whenever } i \neq j.$$

The theory of such a generalization of Koszul properties was first introduced in [GRS02], then developed by Madsen in [Mad11] and, focusing on the quasi-hereditary case, in [Mad13]. The above notion of T-Koszul algebras is of particular interest since it leads to a well defined T-Koszul duality that specializes to the classical one when the algebra Λ is Koszul. The quasi-hereditary covers of Z^1_s provide a nice example for which the Δ-Koszul dual algebras admit a nice presentation as path algebras with relations. The Δ-Koszul dual is precisely the extension algebra of the standard module $\operatorname{Ext}^*(\Delta, \Delta)$ and these algebras are again Koszul in the classical sense. Moreover, a presentation as path algebras of quivers with relations has been computed for instance in [KS12] and [MT13].

The first part of this thesis is devoted to generalizing this construction of quasi-hereditary covers to the case of higher zigzag algebras and to showing that they satisfy different Koszul properties. In particular we will show that the results proved in [Mad13] about generalized Koszul duality apply for such algebras and we will compute explicitly their Δ-Koszul dual. The main results of Chapter 3 can be summarized as follows:

Theorem 5. Let Γ be the quasi-hereditary cover of the higher zigzag algebra Z^d_s as defined in Section 3.1.

- If we put all arrows of the quiver of Γ in degree one, the graded algebra Γ is Koszul in the classical sense. Moreover the standard module Δ (see Definition 1.2.2) has a linear projective resolution (hence Γ is *standard Koszul*).

- If we put $\deg \alpha_i = 1$ for $i \neq 0$ and $\deg \alpha_0 = 0$, then the corresponding graded algebra Γ is Koszul with respect to Δ.

- The Δ-Koszul dual of Γ is the bound path algebra described in Theorem 3.3.7. Moreover it is a Koszul algebra in the classical sense.

In the last chapter we study silting (equivalently, tilting) mutations in the bounded derived category $\mathcal{D}^b(Z_s^d)$ focusing on two-term tilting complexes. Our interest in this topic is motivated by existing results about *Brauer tree algebras*. As their name suggests, Brauer tree algebras can be represented by a tree T endowed with a cyclic ordering on the edges with a common vertex. The simple modules of a Brauer tree algebra are in one to one correspondence with the edges of the tree T and each indecomposable projective module $P(e)$, for some edge e of T, has simple top and socle both isomorphic to $S(e)$ (the simple module corresponding to e). Moreover $\operatorname{rad} P/\operatorname{soc} P$ is the direct sum of two uniserial modules whose composition factors are given by the two cyclic orderings in which the edge e appears. A lot of results have been proved about these algebras which are, for instance, a special case of Brauer graph algebras. We are particular interested in them since, when the tree is a line with, e.g., s edges and without exceptional vertex, then the associated Brauer tree algebra coincides with the zigzag algebra Z_s^1, so it is natural to try to generalize some of the already known results to higher zigzag algebras Z_s^d. In particular, the derived equivalence class of Brauer tree algebras has been completely classified using results by Rickard, Gabriel and Riedtmann ([Ric89a], [GR79]).

Theorem 6 ([Ric89a], Theorem 4.2)**.** Let $B = B(T, s, m)$ be a Brauer tree algebra over a Brauer tree T with s edges and exceptional vertex with multiplicity $m \in \mathbb{N}$. Then B is derived equivalent to the Brauer tree algebra $B(s, m)$ whose tree is a star with exceptional vertex in the center.

Moreover, by Theorem 2 in [GR79], any algebra derived equivalent to a Brauer tree algebra is Morita equivalent to a Brauer tree algebra, so the derived equivalence class consists precisely of Brauer tree algebras with s edges and exceptional vertex with multiplicity m. Rickard showed that this derived equivalence can be decomposed in a sequence of equivalences given by two-term tilting objects, called *Okuyama–Rickard complexes*. In the more general setting of silting mutation introduced by Iyama and Aihara ([AI12]) these complexes correspond to irreducible silting mutations of the algebra viewed as a silting object. Hoping to generalize some of these results to higher zigzag algebras, we will look at iterated derived equivalences given by Okuyama–Rickard complexes with respect to irreducible idempotents (*irreducible Okuyama–Rickard mutations*) and we will prove that, with some conditions on the algebra Λ, the operation $\mu_x(-)$ of mutation at the simple S_x "commutes" with taking the trivial extension of Λ:

$$\mu_x(\operatorname{Triv}(\Lambda)) \cong \operatorname{Triv}(\mu_x(\Lambda)).$$

As a corollary we will have the following result:

Corollary 7. Every algebra that is a tilting mutation of a type A higher zigzag algebra Z_s^d is a trivial extension algebra.

This thesis is organized as follows. In Chapter 1 we fix the notation that will be used in the thesis and we recall known classical results. We start by giving the definition of triangulated categories and derived categories of modules, in order to introduce Rickard's theorem characterizing derived Morita equivalences. Then we recall the definition and basic properties of quasi-hereditary algebras, as well as Koszul algebras, T-Koszul algebras and the respective dualities.

In Chapter 2 we recall the definition of higher zigzag algebras paying particular attention to the type A case. In this case we give the presentations of the algebras Z_s^d as path algebras with relations. All the results of this chapter are taken from [Gra17] and [GI].

Chapter 3 is about our results on quasi-hereditary covers for Z_s^d. We present the construction of our quasi-hereditary covers and we prove that they are Koszul with respect to the radical grading, standard Koszul (the standard module has a linear projective resolution) and, with a different grading, Koszul with respect to the standard module Δ. Then we compute the Δ-Koszul dual as the path algebra of a quiver with relations and we show that it is again a Koszul algebra.

To conclude, in Chapter 4, we present some result about irreducible tilting mutations in the bounded derived category of finitely generated Z_s^d-modules. In particular we give a description of Okuyama–Rickard mutation via the mutation of some particular quotient of trivial extension algebras. At the end of this last chapter we also give a full description of the algebras in the derived equivalence class of Z_3^2. This result has been achieved by brute force, computing iterated Okuyama–Rickard mutations, after realizing that Z_3^2 is of finite representation type, every derived equivalence can be decomposed into irreducible mutations.

1

Background results

In this chapter we set the notation that will be used through this thesis and we recall classical results from the literature. We start with an overview on triangulated and derived categories, derived functors and derived Morita equivalences. Then we recall basic properties of quasi-hereditary algebras, Koszul algebras and T-Koszul algebras.

Throughout this thesis k will denote an algebraically closed field and we will only consider algebras of finite dimension over k. All modules will be finitely generated right modules and the composition of morphisms fg means that g is applied first and then f. For a k-algebra Λ, we will denote by $\operatorname{mod}\Lambda$ the category of finitely generated Λ-modules and by $\operatorname{Hom}_\Lambda(M,N)$ the vector space of Λ-module morphisms between M and N. If $Q = (Q_0, Q_1)$ is a quiver and $\alpha \in Q_1$ is an arrow of Q, $s(\alpha)$ denotes the source of α and $t(\alpha)$ the target: $s(\alpha) \xrightarrow{\alpha} t(\alpha)$. For two consecutive arrows $\xrightarrow{\alpha}\xrightarrow{\beta}$ their concatenation is denoted by $\alpha\beta$ so that, in the path algebra kQ, $\alpha\beta \neq 0$ and $\beta\alpha = 0$.

1.1 Derived categories and equivalences

In this first section we will recall the definitions of triangulated category and derived category necessary to state and understand the theorem of Rickard about derived equivalences between categories of modules over k-algebras. An example will be given at the end of the section. All the arguments are taken from [Kel07] and [RS07].

1.1.1 Basic notions

Let $\mathcal{A}$ be an abelian category and $\mathcal{C}(\mathcal{A})$ be the category of *chain complexes* in $\mathcal{A}$. For any chain complex $C = (C_n, d_n)$ in $\mathcal{C}(\mathcal{A})$, the *shift* of complexes $[-]$ acts as $C[1]_n = C_{n-1}$, hence shifting the complex to the left. The *homology groups* of C are defined as

$$H_n(C) = \operatorname{Ker} d_n / \operatorname{Im} d_{n+1}, \quad \forall n \in \mathbb{Z}.$$

Similarly, if (C^n, d^n) is a cochain complex, the cohomology groups are defined as

$$H^n(C) = \operatorname{Ker} d^n / \operatorname{Im} d^{n-1}, \quad \forall n \in \mathbb{Z}.$$

Any morphism of chain (cochain respectively) complexes $f : C \to D$ induces a morphism of groups $H_n(f) : H_n(C) \to H_n(D)$ ($H^n(f) : H^n(C) \to H^n(D)$ respectively) in each degree; we say that f is a *quasi-isomorphsm* if $H_n(f)$ is an isomorphism (respectively if $H^n(f)$ is an isomorphism) for every n.

Note that we can think of objects of $\mathcal{A}$ as complexes concentrated in degree zero: this correspondence gives a well defined additive fully faithful functor:

$$\mathcal{A} \to \mathcal{C}(\mathcal{A}).$$

The *derived category* of $\mathcal{A}$, $\mathcal{D}(\mathcal{A})$, is obtained from $\mathcal{C}(\mathcal{A})$ by formally inverting all quasi-isomorphisms. This construction is motivated by the fact that in homological algebra we deal usually with *projective resolutions* of objects, which are unique up to quasi-isomorphisms.

Example 1.1.1. Let Λ be a finite dimensional algebra over a field k. Let us recall the notions of *projective cover* and *injective envelope*.

Definition 1.1.2.

1. Let M be a Λ module and L a submodule of M. The module L is called *superfluous* if for any submodule X of M, the equality $L + X = M$ implies $X = M$. The module M is called an *essential extension* of L if for any submodule X of M, the equality $X \cap L = \{0\}$ implies $X = \{0\}$.
2. An epimorphism $h : M \to N$ in $\operatorname{mod}\Lambda$ is called *minimal* if $\operatorname{Ker} h$ is a superfluous submodule of M.
3. An epimorphism $p : P \to M$ in $\operatorname{mod}\Lambda$ is called a *projective cover* of M if P is projective and $\operatorname{Ker} h$ is a superfluous submodule of P. Dually, an inclusion of M into an injective module $0 \to M \to I$ is called an *injective envelope* of M if I is an essential extension of M.

The category $\operatorname{mod}\Lambda$ has enough projectives since every module admits a projective cover.

Assume that $\mathcal{A}$ has enough projectives and let $F : \mathcal{A} \to Ab$ be a covariant right exact functor from $\mathcal{A}$ to the category of abelian groups. Then any M in $\mathcal{A}$ has a projective resolution, i.e. a complex of projective objects $P_\bullet = \ldots \to P_1 \to P_0 \to 0 \to \ldots$ with a morphism $P_\bullet \xrightarrow{\pi} M$ such that the following complex is exact:

$$\ldots \to P_2 \to P_1 \to P_0 \xrightarrow{\pi} M \to 0.$$

If we think of M as a complex concentrated in degree zero, this is equivalent to requiring that the morphism of complexes $P_\bullet \to M$ is a quasi-isomorphism.

For $n \geq 0$, the *n-th left derived functor* $L_nF(-) : \mathcal{A} \to Ab$ is defined on the objects of $\mathcal{A}$ as $L_nF(M) = H_n(F(P_\bullet))$ that is the n-th homology group of the complex

$$\ldots \to F(P_2) \to F(P_1) \to F(P_0) \to 0 \to \ldots$$

Let M' be another object of $\mathcal{A}$ with a projective resolution $P'_\bullet \to M'$ and let $f : M \to M'$ be a morphism in $\mathcal{A}$. By the *Comparison Theorem* ([Wei94, Theorem 2.2.6]) f lifts to a morphism of complexes $\tilde{f} : P_\bullet \to P'_\bullet$ that is unique up to homotopy equivalence. Since F is an additive functor, it can be extended to functors $F : \mathcal{C}(\mathcal{A}) \to \mathcal{C}(Ab)$ and $F : \mathcal{H}(\mathcal{A}) \to \mathcal{H}(Ab)$. Hence the image $F(\tilde{f})$ induces well defined maps between the homology groups $F(\tilde{f})_n : H_n(F(P_\bullet)) \to H_n(F(P'_\bullet))$ and we set $L_nF(f) := F(\tilde{f})_n$.

Note that since F is right exact we always have $L_0F(M) \simeq F(M)$; moreover L_nF is a well-defined functor since projective resolutions are unique up to quasi-isomorphisms.

For a left exact covariant functor, the *n-th right derived functor* $R^nF(M)$ is defined in a similar way.

From the previous construction we can see that it can be useful to work in the derived category instead of $\mathcal{C}(\mathcal{A})$. However, in order to give an explicit definition of $\mathcal{D}(\mathcal{A})$, it is better to proceed by steps and invert all the quasi-isomorphisms in the *homotopy category* $\mathcal{H}(\mathcal{A})$. This category has by definition the same objects of $\mathcal{C}(\mathcal{A})$ and as morphisms homotopy equivalence classes of morphisms in $\mathcal{C}(\mathcal{A})$,

$$\mathrm{Hom}_{\mathcal{H}(\mathcal{A})}(A, B) = \mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(A, B)/\sim$$

where $f \sim g$ if and only if for every $n \in \mathbb{Z}$ there exists $s_n : A_n \to B_{n+1}$ such that $f_n - g_n = d^B_{n+1}s_n + s_{n-1}d^A_n$. In this case we say that f is *homotopic* to g.

The homotopy category is a well-defined additive category and the quotient functor $\mathcal{C}(\mathcal{A}) \to \mathcal{H}(\mathcal{A})$ is an additive functor.

1.1.2 Triangulated categories

The main reason for defining the derived category starting from the homotopy category lies in the fact that, even though it is not abelian, $\mathcal{H}(\mathcal{A})$ has another important structure, that will allow us to construct the derived category more explicitly.

Let $\mathcal{T}$ be an additive category with an additive auto-equivalence $T : \mathcal{T} \to \mathcal{T}$. Motivated by the *shift* functor on complexes, we will write $X[n]$ and $f[n]$ instead of $T^n(X)$ and $T^n(f)$ respectively. A *triangle* in $\mathcal{T}$ is a diagram of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

denoted by (X, Y, Z, f, g, h) (or $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{+}$); a morphism of triangles is the data of three morphisms

$$(\alpha, \beta, \gamma) : (X, Y, Z, f, g, h) \to (X', Y', Z', f', g', h')$$

such that the following diagram is commutative in $\mathcal{T}$:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[r, "g"] \arrow[d, "\beta"'] & Z \arrow[r, "h"] \arrow[d, "\gamma"'] & X[1] \arrow[d, "\alpha[1]"'] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & X'[1] \end{tikzcd}$$

A morphism of triangles is said to be an isomorphism if α, β and γ are isomorphisms.

Definition 1.1.3. A structure of *triangulated category* on $\mathcal{T}$ is given by a translation functor T as above and a class of triangles, called *distinguished triangles*, satisfying the following axioms:

TR1 Every triangle isomorphic to a distinguished one is distinguished;

TR2 For any object X in $\mathcal{T}$, $(X, X, 0, id_X, 0, 0)$ is a distinguished triangle;

TR3 Every morphism $X \xrightarrow{f} Y$ can be embedded in a distinguished triangle (X, Y, Z, f, g, h);

TR4 (Rotation) A triangle (X, Y, Z, f, g, h) is distinguished if and only if $(Y, Z, X[1], g, h, -f[1])$ is distinguished;

TR5 (Morphisms) Every commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[r, "g"] \arrow[d, "\beta"'] & Z \arrow[r, "h"] & X[1] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & X'[1] \end{tikzcd}$$

whose rows are distinguished triangles can be completed to a morphism of triangles by a morphism $Z \to Z'$;

TR6 (Octahedral axiom) Given $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ morphisms in $\mathcal{T}$, and distinguished triangles

$$(X, Y, X', f, f', s), \quad (X, Z, Y', gf, h, r), \quad (Y, Z, Z', g, g', t)$$

there exist morphisms $X' \xrightarrow{u} Y'$,$Y' \xrightarrow{v} Z'$ such that

$$(X', Y', Z', u, v, f'[1]t)$$

is a distinguished triangle and

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "id_X"'] & Y \arrow[r, "f'"] \arrow[d, "g"'] & X' \arrow[r, "s"] \arrow[d, "u"'] & X[1] \arrow[d, "id_X[1]"'] \\
X \arrow[r, "gf"] \arrow[d, "f"'] & Z \arrow[r, "h"] \arrow[d, "id_Z"'] & Y' \arrow[r, "r"] \arrow[d, "v"'] & X[1] \arrow[d, "f[1]"'] \\
Y \arrow[r, "g"] \arrow[d, "f'"'] & Z \arrow[r, "g'"] \arrow[d, "h"'] & Z' \arrow[r, "t"] \arrow[d, "id_{Z'}"'] & Y[1] \arrow[d, "f'[1]"'] \\
X' \arrow[r, "u"] & Y' \arrow[r, "v"] & Z' \arrow[r, "w"] & X[1]
\end{tikzcd}$$

is a commutative diagram.

The notion of homological δ-functor between abelian categories generalizes to triangulated categories in a natural way:

Definition 1.1.4. Let $\mathcal{T}$ be a triangulated category and $\mathcal{A}$ an abelian category. An additive functor $H : \mathcal{T} \to \mathcal{A}$ is called a *homological functor* if, for any distinguished triangle (X, Y, Z, f, g, h) in $\mathcal{T}$, we get an exact sequence in $\mathcal{A}$:

$$H_i(X) \xrightarrow{H_i(f)} H_i(Y) \xrightarrow{H_i(g)} H_i(Z)$$

for every $i \in \mathbb{Z}$, where $H_i := H \circ T^i = H(-[i])$.

Note that by the Rotation axiom, if (X, Y, Z, f, g, h) is a distinguished triangle then also $(Y, Z, X[1], g, h, -f[1])$ is distinguished. So, if H is a homological functor, the sequence

$$H_i(Y) \xrightarrow{H_i(g)} H_i(Z) \xrightarrow{H_i(h)} H_{i-1}(X)$$

is exact and we get a long exact sequence in $\mathcal{A}$:

$$\ldots \to H_i(X) \xrightarrow{H_i(f)} H_i(Y) \xrightarrow{H_i(g)} H_i(Z) \xrightarrow{H_i(h)} H_{i-1}(X) \to \ldots$$

Remark 1.1.5. *Cohomological functors* are defined in a similar way, the only difference being that the indices in the long exact sequence are increasing:

$$\ldots \to H^i(X) \xrightarrow{H^i(f)} H^i(Y) \xrightarrow{H^i(g)} H^i(Z) \xrightarrow{H^i(h)} H^{i+1}(X) \to \ldots$$

Corollary 1.1.6.

1. *If (X, Y, Z, f, g, h) is a distinguished triangle, then $gf = 0$;*

2. *For any object $U \in \mathcal{T}$, the functors $\mathrm{Hom}_{\mathcal{T}}(U, -) : \mathcal{T} \to Ab$, and $\mathrm{Hom}_{\mathcal{T}}(-, U) : \mathcal{T}^{op} \to Ab$ are cohomological functors;*

3. *Any distinguished triangle is determined up to isomorphism by one of its*

morphisms.

Proof. Let (X, Y, Z, f, g, h) be any distinguished triangle.

1. From TR2, $(X, X, 0, id_X, 0, 0)$ is a distinguished triangle and by TR5 we know that there exists a morphism of triangles

$$\begin{tikzcd} X \arrow[r, "id"] \arrow[d, "id"'] & X \arrow[r] \arrow[d, "f"'] & 0 \arrow[r] \arrow[d] & X[1] \arrow[d, "id[1]"'] \\ X \arrow[r, "f'"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & X[1] \end{tikzcd}$$

then $gf = 0$.

2. We have to show that the sequence

$$\mathrm{Hom}_{\mathcal{T}}(U, X) \to \mathrm{Hom}_{\mathcal{T}}(U, Y) \to \mathrm{Hom}_{\mathcal{T}}(U, Z)$$

is exact, i.e. we can complete the following diagram to a morphism of triangles:

$$\begin{tikzcd} U \arrow[r, "id"] & U \arrow[r] \arrow[d, "f"'] & 0 \arrow[r] \arrow[d] & U[1] \\ X \arrow[r, "f'"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & X[1] \end{tikzcd}$$

and this can be done using axioms TR2-4-5. The statement can be proved similarly for $\mathrm{Hom}_{\mathcal{T}}(-, U)$.

3. From TR4 it suffices to prove that the distinguished triangles (X, Y, Z, f, g, h) and (X, Y, Z', f, g', h') are isomorphic. By TR5 there exists a morphism of triangles

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "id"'] & Y \arrow[r, "g"] \arrow[d, "id"'] & Z \arrow[r, "h"] \arrow[d, "t"'] & X[1] \arrow[d, "id[1]"'] \\ X \arrow[r, "f"] & Y \arrow[r, "g'"] & Z' \arrow[r, "h'"] & X[1] \end{tikzcd}$$

If we apply the cohomological functors $\mathrm{Hom}_{\mathcal{T}}(-, Z)$ and $\mathrm{Hom}_{\mathcal{T}}(Z', -)$ we get two maps

$$t^* : \mathrm{Hom}_{\mathcal{T}}(Z', Z) \to \mathrm{Hom}_{\mathcal{T}}(Z, Z), \quad t_* : \mathrm{Hom}_{\mathcal{T}}(Z', Z) \to \mathrm{Hom}_{\mathcal{T}}(Z', Z')$$

that are isomorphisms by the 5-lemma. It follows that t has a right and left inverse and thus it is an isomorphism.

□

Now let $\mathcal{C}(\mathcal{A})$ be the category of complexes over $\mathcal{A}$ and $[1] : \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{A})$ the shift functor on complexes. [1] is an additive automorphism; moreover, since $f[1]$

is homotopic to zero if and only if f is homotopic to zero, it induces an additive automorphism $[1] : \mathcal{H}(\mathcal{A}) \to \mathcal{H}(\mathcal{A})$ on the homotopy category of $\mathcal{A}$.

In order to define the triangulated structure of $\mathcal{H}(\mathcal{A})$ we have to show what the distinguished triangles are. Recall that for every morphism $X \xrightarrow{f} Y$ in $\mathcal{C}(\mathcal{A})$, the *mapping cone* of f is the complex cone(f) defined by $\text{cone}(f)_n = X_{n-1} \oplus Y_n$ and with differential

$$d_n^f = \begin{bmatrix} -d_{n-1}^X & 0 \\ f_{n-1} & d_n^Y \end{bmatrix} : X_{n-1} \oplus Y_n \to X_{n-2} \oplus Y_{n-1}.$$

Definition 1.1.7. A triangle in $\mathcal{H}(\mathcal{A})$ is a distinguished triangle if and only if it is isomorphic in $\mathcal{H}(\mathcal{A})$ to one of the form

$$X \xrightarrow{f} Y \xrightarrow{\alpha(f)} \text{cone}(f) \xrightarrow{\beta(f)} X[1]$$

where, in each degree, $\alpha(f)$ and $\beta(f)$ are the canonical immersion and projection respectively.

Theorem 1.1.8. *The category $\mathcal{H}(\mathcal{A})$ with translation functor $[-]$ and the class of distinguished triangles as above is a triangulated category.*

A complete proof can be found for instance in the book by C. Weibel "An introduction to homological algebra", Proposition 10.2.4 [Wei94]. We can point out some remarks:

Remarks 1.1.9. (a) TR1 and TR3 are obvious.

(b) TR4 follows from the following lemma:

Lemma 1.1.10. *For every morphism $f : X \to Y$ in $\mathcal{C}(\mathcal{A})$ there exists $\phi : X[1] \to \text{cone}(\alpha(f))$ such that*

- *ϕ is iso in $\mathcal{H}(\mathcal{A})$ and*
- *the following diagram is commutative in $\mathcal{H}(\mathcal{A})$:*

$$\begin{tikzcd}
Y \arrow[r, "\alpha(f)"] \arrow[d, "id"'] & \text{cone}(f) \arrow[r, "\beta(f)"] \arrow[d, "id"'] & X[1] \arrow[r, "-f[1]"] \arrow[d, "\phi"'] & Y[1] \arrow[d, "id[1]"'] \\
Y \arrow[r, "\alpha(f)"] & \text{cone}(f) \arrow[r, "\alpha\alpha(f)"] & \text{cone}(\alpha(f)) \arrow[r, "\beta\alpha(f)"] & Y[1]
\end{tikzcd}$$

(For a proof see the book of Kashiwara-Schapira "Sheaves on Manifolds" [KS94]).

This result doesn't hold in $\mathcal{C}(\mathcal{A})$, thus TR3 holds only in the homotopy category.

(c) TR2 holds by TR4 since the mapping cone of the zero map $0 \to X$ gives the distinguished triangle $0 \to X \to X \to 0$ and we can conclude applying TR3.

(d) Also TR5 holds only in $\mathcal{H}(\mathcal{A})$.

1.1.3 Localization and Derived category

The formal process by which we "add" inverses of quasi-isomorphisms to the category $\mathcal{C}(\mathcal{A})$ is called *localization*. This construction can be defined for any category $\mathcal{C}$.

Definition 1.1.11. Let $\mathcal{C}$ be any category and $\mathcal{S}$ a family of morphisms of $\mathcal{C}$. The *localization* of $\mathcal{C}$ by $\mathcal{S}$ is the data $(\mathcal{C}_{\mathcal{S}}, Q)$ where $\mathcal{C}_{\mathcal{S}}$ is a category and $Q : \mathcal{C} \to \mathcal{C}_{\mathcal{S}}$ is a functor, satisfying the following universal property:

- for every $s \in \mathcal{S}$, $Q(s)$ is an isomorphism in $\mathcal{C}_{\mathcal{S}}$;
- for every functor $F : \mathcal{C} \to \mathcal{D}$ such that $F(s)$ is an isomorphism for every $s \in \mathcal{S}$, there exists a unique functor $F_{\mathcal{S}}$ such that $F = F_{\mathcal{S}} \circ Q$.

In this case Q is called the *localization functor*.

Definition 1.1.12. The *derived category* $\mathcal{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$ is the localization of $\mathcal{H}(\mathcal{A})$ by the class Σ of all the quasi-isomorphisms.

Since the previous definition is too abstract to work with, we will give an equivalent, more explicit construction of $\mathcal{C}_{\mathcal{S}}$.

Definition 1.1.13. A family of morphism $\mathcal{S}$ of $\mathcal{C}$ is said to be a *multiplicative system* if it satisfies the following axioms:

S1 For every object X of $\mathcal{C}$, $id_X \in \mathcal{S}$;

S2 For every $f, g \in \mathcal{S}$ then $fg \in \mathcal{S}$, if the composition exists;

S3 Any diagram $\begin{tikzcd} & C \arrow[d, "s"'] \\ A \arrow[r, "f"] & B \end{tikzcd}$ with $s \in \mathcal{S}$, fits into a commutative diagram $\begin{tikzcd} D \arrow[r, "g"] \arrow[d, "t"'] & C \arrow[d, "s"] \\ A \arrow[r, "f"] & B \end{tikzcd}$ with $t \in \mathcal{S}$ (Same with arrow reversed for the right version).

S4 For every $f, g : A \rightrightarrows B$, if there exists $t : B \to C$ in $\mathcal{S}$ such that $tf = tg$, then there exists $s : D \to A$ in $\mathcal{S}$ such that $fs = gs$.

If $\mathcal{S}$ is a multiplicative system, we can define $\mathcal{C}_{\mathcal{S}}$, the localization of $\mathcal{C}$ by $\mathcal{S}$, by setting

$$\mathrm{Ob}\, \mathcal{C}_{\mathcal{S}} = \mathrm{Ob}\, \mathcal{C}, \quad \mathrm{Hom}_{\mathcal{C}_{\mathcal{S}}}(X, Y) = \{(f, s) : x \xrightarrow{f} Y' \xleftarrow{s} Y, s \in \mathcal{S}\}/\simeq$$

where the equivalence relation is given by: $(f, s) \simeq (f', s')$ if and only if there exists a commutative diagram

$$\begin{tikzcd} & Y' \arrow[d] & \\ X \arrow[ur, "f"] \arrow[r, "g"] \arrow[dr, "f'"'] & Y''' & Y \arrow[ul, "s"'] \arrow[l, "t"] \arrow[dl, "s'"] \\ & Y'' \arrow[u] & \end{tikzcd}$$

with $t \in \mathcal{S}$. The composition law is given by using S3: $(f, s) \circ (g, t) = (hf, rt)$:

$$\begin{tikzcd}
 & & W & & \\
 & Y' \arrow[ur, dashed, "h"] & & Z' \arrow[ul, dashed, "r\in\mathcal{S}"'] & \\
X \arrow[ur, "f"] & & Y \arrow[ul, "s"'] \arrow[ur, "g"] & & Z \arrow[ul, "t"']
\end{tikzcd}$$

Using this construction, the *localization functor* $Q : \mathcal{C} \to \mathcal{C}_{\mathcal{S}}$ is defined by $Q(X) = X$ on any object X and $Q(X \to Y) = [X \to Y \xleftarrow{id} Y]$.

If $\mathcal{C}$ is moreover a triangulated category and $H : \mathcal{C} \to \mathcal{D}$ is a homological functor, we say that the family of morphisms $\mathcal{S}$ *arises from the homological functor* H if $\mathcal{S}$ consists precisely of those morphisms s of $\mathcal{C}$ such that $H_n(s)$ is an isomorphism for every n.

Proposition 1.1.14. *If $\mathcal{S}$ arises from a homological functor then:*

1. *$\mathcal{S}$ is a multiplicative system;*
2. *the localization $\mathcal{C}_{\mathcal{S}}$ is a triangulated category and the localization functor Q is a triangulated functor in the sense that Q is additive, commutes with the translation functor $T = (-)[1]$ and sends distinguished triangles to distinguished triangles.*

Proof. See for instance the book by Weibel, "An introduction to homological algebra", Proposition 10.4.1. [Wei94] □

Remark 1.1.15. In the proof of Proposition 1.1.14 the distinguished triangles of $\mathcal{C}_{\mathcal{S}}$ are defined to be the images under Q of the distinguished triangles in $\mathcal{C}$.

In the case of the homotopy category $\mathcal{H}(\mathcal{A})$, the family of all the quasi-isomorphisms Σ arises precisely from the homology functor $H_0 = \operatorname{Ker} d_0 / \operatorname{Im} d_1$, so it is a multiplicative system and we can define the derived category of $\mathcal{A}$ as $\mathcal{D}(\mathcal{A}) = \mathcal{H}(\mathcal{A})_\Sigma$. Moreover we know that $\mathcal{D}(\mathcal{A})$ is a triangulated category with translation functor induced by the shift functor (note that it sends quasi-isomorphisms to quasi-isomorphisms, thus factorizes uniquely through the localization functor) and distinguished triangles given by images of distinguished triangles in $\mathcal{H}(\mathcal{A})$.

Remark 1.1.16. The category of complexes $\mathcal{C}(\mathcal{A})$ is not a triangulated category and the family of all the quasi-isomorphism between complexes is not a multiplicative system. Then starting from $\mathcal{C}(\mathcal{A})$ we could not describe the triangulated structure of $\mathcal{D}(\mathcal{A})$ or use "calculus of fractions" dealing with morphisms.

Proposition 1.1.17. *The composition $\mathcal{A} \to \mathcal{C}(\mathcal{A}) \to \mathcal{H}(\mathcal{A}) \xrightarrow{Q} \mathcal{D}(\mathcal{A})$ is a fully faithful functor.*

Proof. See for instance [RS07], Proposition 4.6. □

Then we can identify the category $\mathcal{A}$ as complexes concentrated in degree zero in $\mathcal{D}(\mathcal{A})$. If X is an object of $\mathcal{A}$ we will denote always by X the corresponding complex in the derived category. Note that $\mathcal{D}(\mathcal{A})$ is no longer abelian, as the following example shows:

Example 1.1.18. Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a short exact sequence in $\mathcal{A}$, so that the same sequence can be viewed as a short exact sequence in $\mathcal{C}(\mathcal{A})$. Since

$$\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] \arrow[d] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d, "g"'] & 0 \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & Z \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

is a quasi isomorphism between complexes, in $\mathcal{D}(\mathcal{A})$ we can replace the complex $Z[0]$ with the complex $W := \cdots 0 \to X \xrightarrow{f} Y \to 0 \cdots$ appearing in the first line of the previous diagram. Thus we obtain the following sequence in $\mathcal{D}(\mathcal{A})$:

$$0 \to X \xrightarrow{f} Y \to W \to 0$$

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "f"'] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r, equal] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

which is not an exact sequence of complexes.

Nevertheless, from the short exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in $\mathcal{A}$ we obtain a triangle in $\mathcal{D}(\mathcal{A})$:

$$X \xrightarrow{f} Y \to W \to X[1]$$

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[d] \\
X \arrow[r, "f"] \arrow[d] & Y \arrow[r, equal] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

with complexes in the columns and morphisms of complexes from left to right.

Since the functor $\mathcal{A} \to \mathcal{D}(\mathcal{A})$ is fully faithful we know that if X and Y are complexes concentrated in degree zero, then $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X, Y) \cong \mathrm{Hom}_{\mathcal{A}}(X, Y)$. Moreover $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X[n], Y[m]) = \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X, Y[m-n])$.

Now let $(X \xrightarrow{f} Z \xleftarrow{s} Y[n])$ be a representative of a map from X to $Y[n]$. Since s is a quasi-isomorphism, the complex Z has homology zero except in degree n, corresponding to the non-trivial degree of the complex $Y[n]$. Consider the following quasi-isomorphism:

$$\begin{tikzcd}
Z \arrow[d, "u"'] & \dots \arrow[r] & Z_{n+1} \arrow[r, "d_{n+1}"] \arrow[d] & Z_n \arrow[r] \arrow[d] & Z_{n-1} \arrow[r] \arrow[d, equal] & \dots \\
U & \dots \arrow[r] & 0 \arrow[r] & Z_n/\operatorname{Im} d_{n+1} \arrow[r] & Z_{n-1} \arrow[r] & \dots
\end{tikzcd}$$

If $n < 0$ then $uf = 0$ and $s^{-1}f \sim (us)^{-1}(uf) \sim 0$. Thus $\operatorname{Hom}_{\mathcal{D}(\mathcal{A})}(X, Y[n]) = 0$ for every $n < 0$.

Proposition 1.1.19. *(a) If I is a left bounded complex with injective components, then for every X in $\mathcal{H}(\mathcal{A})$ we have* $\operatorname{Hom}_{\mathcal{D}(\mathcal{A})}(A, I) \cong \operatorname{Hom}_{\mathcal{H}(\mathcal{A})}(A, I)$.

(b) If P is a right bounded complex with projective components, then for every X in $\mathcal{H}(\mathcal{A})$, $\operatorname{Hom}_{\mathcal{D}(\mathcal{A})}(P, A) \cong \operatorname{Hom}_{\mathcal{H}(\mathcal{A})}(P, A)$.

(c) Assume that $\mathcal{A}$ has enough injectives, then for every X, Y in $\mathcal{A}$, $\operatorname{Hom}_{\mathcal{D}(\mathcal{A})}(A, B[n]) \cong \operatorname{Ext}^n_{\mathcal{A}}(A, B)$, *for every $n \geq 0$.*

Proof. (a) Consider a morphism from A to I in the derived category, i.e. equivalence classes of diagrams $A \to Y \xleftarrow{s} I$ with $s \in \Sigma$. Our claim is that every quasi isomorphisms $I \xrightarrow{s} Y$ admits a quasi-inverse, that is, it is invertible in $\mathcal{H}(\mathcal{A})$. So every equivalence class as above determines a well defined morphism $A \to I$ in the homotopy category and this gives a bijection between the spaces of morphisms $\operatorname{Hom}_{\mathcal{D}(\mathcal{A})}(A, I) \cong \operatorname{Hom}_{\mathcal{H}(\mathcal{A})}(A, I)$ sending the class $[A \to Y \xleftarrow{s} I]$ to $A \to Y \xrightarrow{s^{-1}} I$.

To prove the claim let $Z = \operatorname{cone}(s)$ be the mapping cone of the quasi-isomorphism s:

$$Z = I[1] \oplus Y, \quad d^Z = \begin{bmatrix} -d^I & 0 \\ s & d^Y \end{bmatrix}.$$

Z is acyclic since it is the mapping cone of a quasi-isomorphism, so the projection $v = [id_{I[1]}, 0] : Z \to I[1]$ is null-homotopic. Thus there exists a map $[k, t] : Z \to I$ such that $v = [k, t]d^Z + d^{I[1]}[k, t]$, i.e.

$$\begin{bmatrix} k & t \end{bmatrix} \begin{bmatrix} -d^I & 0 \\ s & d^Y \end{bmatrix} - \begin{bmatrix} d^I k & d^I t \end{bmatrix} = \begin{bmatrix} -kd^I + ts & td^Y \end{bmatrix} - \begin{bmatrix} d^I k & d^I t \end{bmatrix}$$

that gives

$$\begin{cases} id_{I[1]} = -kd^I + ts - d^I k \\ td^Y = d^I t. \end{cases}$$

But this means exactly that t is a morphism of complexes and is the quasi-inverse of s.

(b) The proof is similar to the first case.

(c) If I is any injective resolution of B, then we have a quasi-isomorphism $B \xrightarrow{s} I$.

Hence we have isomorphisms

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(A, B[n]) \cong \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(A, I[n]) \cong \mathrm{Hom}_{\mathcal{H}(\mathcal{A})}(A, I[n]) = \mathrm{Ext}^n_{\mathcal{A}}(A, B)$$

where the first isomorphism is given by the fact that s is invertible in $\mathcal{D}(\mathcal{A})$ and the second is given by point (a). □

1.1.4 Total derived functors

Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories and $F : \mathcal{A} \to \mathcal{B}$ an additive functor between them. Then F induces a well defined functor, always denoted by F, between the corresponding homotopy categories $F : \mathcal{H}(\mathcal{A}) \to \mathcal{H}(\mathcal{B})$ but, if F is not exact, it doesn't preserve in general the quasi-isomorphisms. So it doesn't induce a functor between the derived categories. Before we go on we need the notion of *triangulated functor* that is a functor that preserves the structure between triangulated categories.

Definition 1.1.20. Let $(\mathcal{T}, T)$ and $(\mathcal{T}', T')$ be two triangulated categories. A *triangulated functor* is given by a pair (F, η) where $F : \mathcal{T} \to \mathcal{T}'$ is a functor which sends distinguished triangle to distinguished triangles and $\eta : F \circ T \simeq_{nat} T' \circ F$ is a natural isomorphism. If moreover F is an equivalence, the it is called a *triangle equivalence.*

Since for technical reasons it is better to avoid dealing with unbounded complexes and all the results of this thesis concern bounded complexes, we will give the construction of left and right *total derived functors* for the categories $\mathcal{C}^b(\mathcal{A})$, $\mathcal{H}^b(\mathcal{A})$ and $\mathcal{D}^b(\mathcal{A})$ of bounded complexes.

We can look at the following example as a motivation for the formal definition of total derived functor.

Example 1.1.21. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. Assume that the category $\mathcal{A}$ has enough injectives and let $\mathcal{H}^b(\mathcal{I})$ be the homotopy category of bounded complexes of injective objects. In the proof of Proposition 1.1.19 we proved that every quasi-isomorphism in $\mathcal{H}^b(\mathcal{I})$ is invertible, so $\mathcal{H}^b(\mathcal{I})$ is isomorphic to its derived category $\mathcal{D}^b(\mathcal{I})$. Remember that we denoted by $Q : \mathcal{H}^b(-) \to \mathcal{D}^b()$ the localization functor from the homotopy category to the derived category of any abelian category. The functor $QFQ^{-1} : \mathcal{D}^b(\mathcal{I}) \xrightarrow{\cong} \mathcal{H}^b(\mathcal{I}) \to \mathcal{H}^b(\mathcal{B}) \to \mathcal{D}^b(\mathcal{B})$ satisfies $QF \cong (QFQ^{-1})Q$, so it makes the following diagram commutative (up to a natural isomorphism of functors):

$$\begin{tikzcd} \mathcal{H}^b(\mathcal{I}) \arrow[r, "F"] \arrow[d, "Q"'] & \mathcal{H}^b(\mathcal{B}) \arrow[d, "Q"] \\ \mathcal{D}^b(\mathcal{I}) \arrow[r, "QFQ^{-1}"] & \mathcal{D}(\mathcal{B}) \end{tikzcd}$$

Since it not always possible to find a commutative diagram as in Example 1.1.21, the definition of total derived functor has to be more general.

Definition 1.1.22. Let $F : \mathcal{H}^b(\mathcal{A}) \to \mathcal{H}^b(\mathcal{B})$ be a triangulated functor. A *total right derived functor* of F is a triangulated functor $\mathbb{R}F : \mathcal{D}^b(\mathcal{A}) \to \mathcal{D}^b(\mathcal{B})$ together with a natural transformation $\eta : QF \to (\mathbb{R}F)Q$ satisfying the following universal property:

- For any triangulated functor $G : \mathcal{D}^b(\mathcal{A}) \to \mathcal{D}^b(\mathcal{B})$ equipped with a natural transformation $\zeta : QF \to GQ$, there exists a unique natural transformation $\zeta' : \mathbb{R}F \to G$ such that $\zeta_A = \zeta'_{Q(A)} \circ \eta_A$ for every $A \in \mathcal{D}^b(\mathcal{A})$.

Dually, a *total left derived functor* of F is a triangulated functor $\mathbb{L}F : \mathcal{D}^b(\mathcal{A}) \to \mathcal{D}^b(\mathcal{B})$ together with a natural transformation $\eta : (\mathbb{L}F)Q \to QF$ satisfying the dual universal property.

Remark 1.1.23. The universal property of Definition 1.1.22 assures that if right/left total derived functors exist, then they are unique up to natural isomorphism.

Example 1.1.24. In Example 1.1.21, QFQ^{-1} is both the right and left total derived functor of F.

Remark 1.1.25. If $\mathcal{A}$ has enough projectives/injectives then Theorem 10.5.6 of [Wei94] assures that the left/right derived functor of F exists (at least for bounded derived categories).

Now we will focus on the example of the tensor product functor, since the corresponding total derived functor will be used to state the generalization of the Morita theorem for derived categories.

If $C = (C_{p,q}, d_p, d_q)$ is a bounded double complex with elements in an abelian category $\mathcal{A}$, the *total complex* $\mathrm{Tot}^{\oplus}(C)$ is a complex in $\mathcal{C}^b(\mathcal{A})$ defined by

$$\mathrm{Tot}^{\oplus}(C)_n = \bigoplus_{p+q=n} C_{p,q}$$

with differential

$$d = d_p + d_q : \mathrm{Tot}^{\oplus}(C)_n \to \mathrm{Tot}^{\oplus}(C)_{n-1}$$

Example 1.1.26. If R is any ring and (A, d^A) and (B, d^B) are two complexes of right and left R-modules respectively, then we can construct the double complex $A \otimes_R B$ by setting

$$(A \otimes_R B)_{p,q} = A_p \otimes_R B_q$$

and with differential defined using the sign rule, i.e., the horizontal arrows are given by $d^A \otimes 1$ and the vertical by $(-1)^p \otimes d^B$. $\mathrm{Tot}^{\oplus}(A \otimes_R B)$ is called the *total tensor product complex* of A and B.

If A is a bounded complex in $\mathcal{C}^b(\mathcal{A})$ we can construct explicitly a *projective resolution* for A by giving a double complex $P = (P_{pq}, d_p, d_q)$ whose entries are projective modules together with a map $P \xrightarrow{\pi} A$ such that:

- P is an upper half-plane complex, i.e. $P_{\bullet q} = 0$ if $q < 0$,

- $P_{p\bullet} \xrightarrow{\pi_p} A_p$ with differential given by $(-1)^p d_v$ is a projective resolution of A_p.

If $\mathcal{A}$ has enough projectives such a complex always exists, since we can find a standard projective resolution for every term of A, $P_{p\bullet} \to A_p$, and we can construct horizontal differentials using the comparison theorem. Moreover the induced map $\mathrm{Tot}^{\oplus}(P) \to A$ is a quasi-isomorphism in $\mathcal{H}^b(\mathcal{A})$ and it is also possible to define chain homotopy between double complexes, such that every two such resolutions of the same complex A are homotopy equivalent.

Definition 1.1.27. Let $\mathcal{A} = \mathrm{mod}\, R$, $\mathcal{B} = R\,\mathrm{mod}$ and A, B two complexes of modules in $\mathcal{A}$ and $\mathcal{B}$ as above. Consider two projective resolutions $P \to A$ and $Q \to B$ of A and B and $\widetilde{P} = \mathrm{Tot}^{\oplus}(P)$, $\widetilde{Q} = \mathrm{Tot}^{\oplus}(Q)$. Then the *total (left) derived tensor product* $A \otimes_R^{\mathbb{L}} B$ is defined as

$$A \otimes_R^{\mathbb{L}} B = \mathrm{Tot}^{\oplus}(\widetilde{P} \otimes_R \widetilde{Q})$$

This defines a bifunctor

$$- \otimes_R^{\mathbb{L}} - : \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{B}) \to \mathcal{D}(Ab)$$

called the *total derived tensor product.*

1.1.5 Derived Morita equivalences

Here we state the theorem of Rickard that generalizes the classical theorem on Morita equivalences in the context of derived categories of modules over k-algebras. We will also give some examples of these equivalences from representation theory of quivers.

Let k be a field and Λ, Γ two k-algebras. The classical result on Morita theory characterizes the equivalences between the corresponding categories of modules $\mathrm{mod}\,\Lambda$ and $\mathrm{mod}\,\Gamma$:

Theorem 1.1.28. *The following statements are equivalent:*

(a) there is a k-linear equivalence $F : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Gamma$;

(b) there is a Λ-Γ-bimodule X such that the tensor product $- \otimes_\Lambda X : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Gamma$ is an equivalence and any k-linear equivalence is naturally equivalent to $- \otimes_\Lambda X$;

(c) there is a finitely generated projective Γ-module P which generates $\mathrm{mod}\,\Gamma$ *and whose endomorphism ring is isomorphic to Λ.*

The following generalization of Morita equivalence in the derived setting is due to Rickard.

Theorem 1.1.29 ([Ric89a],[Ric89b])**.** *Let Λ and Γ be k-algebras, $\mathcal{D}^b(\Lambda), \mathcal{D}^b(\Gamma)$ the derived categories of the corresponding categories of modules. Then the following are equivalent:*

1. *there is a k-linear triangle equivalence* $F : \mathcal{D}^b(\Lambda) \to \mathcal{D}^b(\Gamma)$*;*

2. *there is a complex of* Λ-Γ*-modules* X *such that the total left derived functor*

$$(- \otimes^{\mathbb{L}}_{\Lambda} X) : \mathcal{D}^b(\Lambda) \to \mathcal{D}^b(\Gamma)$$

   is a triangle equivalence;

3. *there is a complex* T *of* Γ*-modules that satisfies the following conditions:*

   (a) T *is quasi-isomorphic to a bounded complex of finitely generated projective modules,*

   (b) T *generates* $\mathcal{D}^b(\Gamma)$ *as a triangulated category,*

   (c) *we have*

$$\mathrm{Hom}_{\mathcal{D}^b(\Gamma)}(T, T[n]) = 0 \textit{ for } n \neq 0 \textit{ and } \mathrm{Hom}_{\mathcal{D}^b(\Gamma)}(T, T) \cong \Lambda.$$

By definition, if the conditions of the theorem hold, Λ is said to be *derived equivalent* to Γ, T is called a *tilting complex*, X a *two-sided tilting complex* and the functor $- \otimes^{\mathbb{L}}_{\Lambda} X$ a *standard equivalence*.

We know that dealing with categories of modules, every equivalence is naturally isomorphic to one given by the tensor product by a bimodule. However, in the more general setting of derived categories, it is not known whether or not every k-linear triangle equivalence is isomorphic to a standard equivalence.

A useful special case of the previous theorem is when $F(\Lambda_\Lambda)$ is a complex concentrated in degree zero. Then, following the proof of the classical Morita theorem, one can put $X = T = F(\Lambda)$ and T becomes a Λ-Γ-bimodule under the natural actions. Moreover in this case T is a (generalized) tilting module of finite projective dimension, i.e.

1. as a Γ-module, T admits a finite resolution

$$0 \to P_n \to P_{n-1} \to \ldots \to P_0 \to T \to 0$$

   of finitely generated projective modules;

2. $\mathrm{Ext}^i_\Gamma(T, T) = 0$ for every $i > 0$ and $\mathrm{End}_\Gamma(T) \cong \Lambda$, and

3. there is a long exact sequence

$$0 \to \Gamma \to T^0 \to \ldots \to T^{m-1} \to T^m \to 0$$

   in $\mathrm{mod}\,\Gamma$, where T^i is in $\mathrm{add}\,T$ for any $i = 0, \ldots, m$.

Recall that by $\mathrm{add}\,T$ we mean the smallest full subcategory of $\mathrm{mod}\,\Lambda$ containing T and closed under direct sums and direct summands.

Example 1.1.30. Our example consists of some algebras derived equivalent to the path algebra $\Lambda = kA_3$ where k is any field and A_3 is the quiver

$$\begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[r, "\beta"] & 3 \end{tikzcd}$$

There are three simple modules S_1, S_2 and S_3, generated by the trivial paths at each vertex, e_1, e_2 and e_3 respectively. If we represent every indecomposable Λ-module by its Loewy-structure, we can easily see that the indecomposable projective modules in $\operatorname{mod}\Lambda$ are the following:

$$P_1 = \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}, \quad P_2 = \begin{smallmatrix} 2 \\ 3 \end{smallmatrix}, \quad P_3 = S_3 = 3.$$

The indecomposable injective modules correspond, via the duality $D(-) = \operatorname{Hom}_k(-, k)$, to indecomposable projective of the dual $D\Lambda = \operatorname{Hom}_k(\Lambda, k)$ considered as a right Λ-module. Then they can be computed dualizing the projective indecomposable modules of kA_3^{op} where A_3^{op} is the quiver obtained reversing the arrows of A_3:

$$\begin{tikzcd} 1 & 2 \arrow[l, "\alpha"'] & 3 \arrow[l, "\beta"'] \end{tikzcd}$$

Then we have

$$I_3 = P_1 = \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}, \quad I_2 = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}, \quad I_1 = S_1 = 1.$$

and we can represent (isomorphism classes of) indecomposable modules in the following quiver:

$$\begin{tikzcd}
 & & P_1 = I_3 \arrow[dr] & & \\
 & P_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[dr] & \\
P_3 = S_3 \arrow[ur] & & S_2 \arrow[ur] & & I_1 = S_1
\end{tikzcd}$$

where the arrows are the obvious epimorphisms/monomorphisms between the modules.

Consider first of all the module $T = P_1 \oplus P_2 \oplus S_2$ viewed as a complex concentrated in degree zero and denote it always by T. Since $S_2 \cong P_2/P_3$, T is quasi-isomorphic to the complex

$$\cdots \to P_3 \to P_1 \oplus P_2 \oplus P_2 \to \dots$$

so T is a bounded complex of finitely generated projective modules. Moreover it has zero homology in each degree $i \neq 0$, then we have that $\operatorname{Ext}^i_\Lambda(T, T) = 0$ for every $i \geq 1$ and it is easily seen by direct computations that also $\operatorname{Ext}^1_\Lambda(T, T) = 0$. Finally, since we have an exact sequence

$$0 \to \Lambda \to P_1 \oplus P_2 \oplus P_2 \to S_2 \to 0$$

we can conclude that T is a tilting complex and we have a derived equivalence $\Lambda \simeq_d \mathrm{End}(T)$. Straightforward computations show that $\mathrm{End}(T) \simeq \Gamma = k(1 \xleftarrow{\gamma} 2 \xrightarrow{\delta} 3)$.

If we repeat the previous calculations with $T' = P_1 \oplus I_2 \oplus S_2$ where $I_2 \cong P_1/P_3$ we obtain that $\mathrm{End}(T') \simeq \Gamma' = k(1 \xrightarrow{\eta} 2 \xleftarrow{\lambda} 3)$.

In this case we are not able to find an exact sequence

$$0 \to \Lambda \to T^1 \to \ldots \to T^n \to 0 \to \ldots$$

with $T^i \in \mathrm{add}\, T$ because, differently from the first example, T' is not a generalized tilting module of finite projective dimension. Nevertheless it is a tilting complex since it generates $\mathcal{D}(\Lambda)$ as a triangulated category and thanks to the Rickard's Theorem we have the derived equivalence $\Lambda \simeq_d \Gamma'$.

These examples show that the $k-$algebra of the quiver A_3 is derived equivalent to any $k-$algebra of a quiver obtained from A_3 changing the orientation of an arrow.

As a last example we point out that the complex given by $T'' = P_3 \oplus P_1 \oplus I_1$ concentrated in degree zero gives us a derived equivalence between Λ and $\Gamma'' = \mathrm{End}(T'') = k(1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3)/\langle \alpha\beta \rangle$.

Convention. In this thesis we will always work with chain complexes, hence every complex will have decreasing indices:

$$X_\bullet : \cdots \xrightarrow{d_{n+2}} X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \xrightarrow{d_{n-1}} \cdots$$

However, since when working with path algebras it is a standard notation to denote by P_x the indecomposable projective cover of the simple module S_x, we will use upper indices for projective resolutions. We apologize for this little incongruence in the notation and we hope that this will not cause too much confusion.

1.2 Quasi-hereditary algebras

In this section we recall some results from the theory of *quasi-hereditary algebras* that will be used in Chapter 3. These algebras were defined originally by Cline, Parshall and Scott in [Sco87] and [CPS88] and they can be seen as algebras whose category of modules satisfies some "directedness" property. As before let Λ be a finitely generated k-algebra and suppose that we can fix a partial order on a complete set of non-isomorphic simple Λ-modules $S_1, \ldots, S_r$. To simplify the notation let $I = \{1, 2, \ldots, r\}$ be a labelling set for the simple modules we have chosen and $(I, \leq)$ the fixed partial order; this is often called a *set of weights* for Λ. Moreover, for $x \in I$, let P_x (resp. I_x) be the indecomposable projective cover (resp. injective envelope) of S_x.

Recall also the definition of *syzygy* and *cosyzygy* modules:

Definition 1.2.1. For any Λ-module M, the *syzygy module* $\Omega(M)$ is the kernel of the projective cover of M:

$$0 \to \Omega(M) \to P \to M \to 0.$$

Dually, the *cosyzygy module* $\Omega^{-1}(M)$ is the cokernel of the injective envelope of M:

$$0 \to M \to I \to \Omega^{-1}(M) \to 0.$$

Definition 1.2.2. For every $x \in I$ the *standard module* Δ_x is the largest quotient of P_x having no simple composition factor S_y for $x < y$. Here by *largest quotient* we mean that if S_z is a composition factor of $\operatorname{top}\Omega(\Delta_x)$, then $x < z$. Dually the *costandard module* ∇_x is the largest submodule of I_x having no composition factor S_y for $x < y$ (i.e. $\operatorname{soc}\Omega^{-1}(\nabla_x)$ has composition factors S_w such that $y < w$). Let $\Delta = \bigoplus_{x\in I}\Delta_x$ and $\nabla = \bigoplus_{x\in I}\nabla_x$.

Remark 1.2.3 ([DR92], Lemma 1.1)**.** For any $x \in I$, the standard module Δ_x can also be defined as the quotient $P_x/\operatorname{Im}\left(\sum_{y>x} P_y \to P_x\right)$, where the module $\operatorname{Im}\left(\sum_{y>x} P_y \to P_x\right)$ is the maximal submodule of P_x generated by the set of projective modules $\{P_y \mid y > x\}$.

Note that the duality $D(-) = \operatorname{Hom}_k(-, k)$ sends standard modules of $\operatorname{mod}\Lambda$ to costandard modules of $\operatorname{mod}\Lambda^{op}$ so any statement about standard modules holds also for the costandard ones.

From the definition we can deduce the following well known result (see e.g. [CPS88], [DR89b]). The proof is taken from [Mad17].

Lemma 1.2.4. *Let Λ be a k-algebra as above. Then* $\operatorname{Ext}^1_\Lambda(\Delta, \nabla) = 0$.

Proof. Suppose $\operatorname{Ext}^1_\Lambda(\Delta_x, \nabla_y) \neq 0$ for some $x, y \in X$. Applying the functor $\operatorname{Hom}_\Lambda(-, \nabla_y)$ to the short exact sequence

$$0 \to \Omega(\Delta_x) \to P_x \to \Delta_x \to 0$$

we obtain the following long exact sequence:

$$0 \to \operatorname{Hom}_\Lambda(\Delta_x, \nabla_y) \to \operatorname{Hom}_\Lambda(P_x, \nabla_y) \to \operatorname{Hom}_\Lambda(\Omega(\Delta_x), \nabla_y) \to \operatorname{Ext}^1_\Lambda(\Delta_x, \nabla_y) \to 0.$$

If $\operatorname{Ext}^1_\Lambda(\Delta_x, \nabla_y) \neq 0$ then $\operatorname{Hom}_\Lambda(\Omega(\Delta_x), \nabla_y) \neq 0$, so there must be a composition factor S_z of $\operatorname{top}\Omega(\Delta_x)$ such that $z \leq y$. But by definition of Δ_x, the composition factors of $\operatorname{top}\Omega(\Delta_x)$ must have index greater than x, hence $x < z \leq y$. Dually, by applying $\operatorname{Hom}_\Lambda(\Delta_x, -)$ to

$$0 \to \nabla_y \to I_y \to \Omega^{-1}(\nabla_y) \to 0$$

we obtain that $\operatorname{soc}\Omega^{-1}(\nabla_y)$ must have a composition factor S_w such that $y < w \leq x$. Therefore we have $x < y$ and $y < x$ that is a contradiction. $\square$

Definition 1.2.5. The algebra Λ is said to be *quasi-hereditary* if the following hold:

(i) $\mathrm{End}_\Lambda(\Delta_x) \cong k$ for any $x \in I$.

(ii) Λ_Λ admits a Δ-filtration, that is there exists a filtration

$$0 = M_0 \subset M_1 \subset \ldots \subset M_{n-1} \subset M_n = \Lambda$$

such that M_k/M_{k-1} is a standard module for every $k = 1, \ldots, n$.

Example 1.2.6 ([DR89b], Theorem 2)**.** Any semiprimary ring of global dimension 2 is quasi-hereditary. Hence any finite-dimensional algebra of global dimension 2 is quasi-hereditary.

Lemma 1.2.7. *Let Λ be any finite-dimensional k-algebra. If M is a Λ-module that admits a Δ-filtration, then* $\mathrm{Ext}^1_\Lambda(M, \nabla) = 0$.

Proof. This follows from Lemma 1.2.4 and by induction on the length of a Δ-filtration of M. (See also [Mad17], Section 1 for a precise reference.) □

The following result gives an equivalent self-dual definition of quasi-hereditary algebra.

Proposition 1.2.8. *The algebra Λ is quasi-hereditary if and only if the following conditions hold:*

(i) $\mathrm{Ext}^2_\Lambda(\Delta, \nabla) = 0$.

(ii) $\mathrm{End}_\Lambda(\Delta_x) \cong k$ *for any $x \in I$.*

Proof. See, for example, [Mad17], Theorem 1.6. □

Corollary 1.2.9. *The algebra Λ is quasi-hereditary if and only if Λ^{op} is quasi-hereditary. In particular, Λ_Λ is Δ-filtered if and only if the module $D(\Lambda)_\Lambda$ is ∇-filtered.*

Proof. See, for example, [Mad17] Corollary 1.7. □

From now to the end of this section we will assume that Λ is a quasi-hereditary algebra. The following proposition gives two homological properties of quasi-hereditary algebras that will be useful in the sequel: namely that there are no extensions in any degree between the standard module Δ and the costandard module ∇ and that any quasi-hereditary algebra has finite global dimension.

Proposition 1.2.10. *1. For any $n > 0$,* $\mathrm{Ext}^n_\Lambda(\Delta, \nabla) = 0$.

2. Λ has finite global dimension.

Proof. See [Mad17] Theorem 2.1 and Theorem 2.5. □

Another important notion is given by the *distance* between indices of simple Λ-modules.

Definition 1.2.11. For any $x, y \in I$ such that $x \leq y$ the *distance* between them is defined as

$$d(x,y) = max\{n \in \mathbb{N} : \exists\, (x = x_0 < \ldots < x_n = y)\}.$$

If x and y are not comparable with respect with the partial order on I we set $d(x,y) = \infty$.

Lemma 1.2.12. *If Λ is a quasi-hereditary algebra, then the following hold:*

1. *If $x > y$ then* $\mathrm{Hom}_\Lambda(\Delta_x, \Delta_y) = 0$.
2. *If $l > 0$ and $x \not< y$ then* $\mathrm{Ext}^l_\Lambda(\Delta_x, S_y) \cong \mathrm{Ext}^l_\Lambda(\Delta_x, \Delta_y) = 0$.
3. *If $x \leq y$ and $l > d(x,y)$ then* $\mathrm{Ext}^l_\Lambda(\Delta_x, S_y) \cong \mathrm{Ext}^l_\Lambda(\Delta_x, \Delta_y) = 0$.

Proof. See [Far08], Lemma 3. □

The definition of *quasi-hereditary cover* was first introduced by Rouquier in [Rou08, Section 4.2]. For instance, such covers play a fundamental role in the proof that the category $\mathcal{O}$ for a rational Cherednik algebra of type A is equivalent to the category of modules over a q-Schur algebra when $q \notin \frac{1}{2} + \mathbb{Z}$.

Let P be a finitely generated projective Λ-module and $\Lambda' = \mathrm{End}_\Lambda(P)$. Consider the functors $F = \mathrm{Hom}_\Lambda(P, -) : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Lambda'$ and $G = \mathrm{Hom}_{\Lambda'}(F\Lambda, -) : \mathrm{mod}\,\Lambda' \to \mathrm{mod}\,\Lambda$; the canonical natural isomorphism $\mathrm{Hom}_\Lambda(P, \Lambda) \otimes_\Lambda - \xrightarrow{\cong} \mathrm{Hom}_\Lambda(P, -)$ makes (F, G) an adjoint pair. Denote by $\varepsilon : FG \to \mathrm{Id}$ (resp. $\eta : \mathrm{Id} \to GF$) the unit (resp. the counit) of the adjunction; note that ε is an isomorphism.

Definition 1.2.13. Let Λ be quasi-hereditary and P a finitely generated projective Λ-module. The pair (Λ, P) is a *quasi-hereditary cover* for $\Lambda' = \mathrm{End}_\Lambda(P)$ if the restriction of the functor

$$F = \mathrm{Hom}_\Lambda(P, -) : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Lambda'$$

to the subcategory $\mathrm{proj}\,\Lambda$ of projective Λ-module is fully faithful.

Here we have two statements equivalent to the fact that the restriction of the functor F is fully faithful.

Lemma 1.2.14. *The following are equivalent:*

- *The canonical map* $\Lambda \to \mathrm{End}_{\Lambda'}(F\Lambda)$ *is an isomorphism of algebras.*
- *For all $M \in \mathrm{proj}\,\Lambda$, the map $\eta(M) : M \to GF(M)$ is an isomorphism.*
- *F restricted to* $\mathrm{proj}\,\Lambda$ *is fully faithful.*

Proof. For a proof see for example [Rou08], Lemma 4.33. □

Remark 1.2.15. Without further assumptions the uniqueness of quasi-hereditary covers is not guaranteed. However, the existence of quasi-hereditary covers for any finite-dimensional algebra (and more generally for any semiprimary ring) was proved by Dlab and Ringel in [DR89a].

1.2.1 Strong exact Borel subalgebras

An important notion related to quasi-hereditary algebras are *exact Borel subalgebras*, first defined by König in [Koe95]. The aim was, given a quasi-hereditary algebra Λ, to find a subalgebra of Λ that somehow encodes the information about the standard filtration of projective Λ-modules.

Definition 1.2.16. [Koe95] Let Λ be a quasi-hereditary algebra, I an index set for the simple Λ-modules and $\leq$ the partial order on I. A subalgebra B of Λ is called an *exact Borel subalgebra* if and only if the following three conditions are satisfied:

- The algebra B has the same partially ordered set of indices of simple modules as Λ and B is directed, i.e. it is quasi-hereditary with simple standard modules;
- The functor $- \otimes_B \Lambda$ is exact;
- The functor $- \otimes_B \Lambda$ sends simple B-modules to standard Λ-modules and this correspondence preserves the indices.

An exact Borel subalgebra B of Λ is called *strong* if Λ has a maximal semisimple subalgebra that is also a maximal semisimple subalgebra of B.

In general it is not true that every quasi-hereditary algebra has an exact Borel subalgebra. Nevertheless it has been proved in [KKO14] that every quasi-hereditary algebra is Morita equivalent to a quasi-hereditary algebra that has an exact Borel subalgebra.

A useful tool to determine the existence of an exact Borel subalgebra is the following:

Theorem 1.2.17 ([Koe95], Theorem A)**.** *Let Λ be a quasi-hereditary algebra and B a subalgebra of Λ. Suppose that the index set of simple Λ-modules I is in bijection with the index set of simple B-modules so that we have an induced partial order on simple B-modules. Then B is an exact Borel subalgebra of Λ if and only if B with the partial order defined above is directed and satisfies the following condition:*

- *For each $x \in I$, the restriction from Λ-modules to B-modules gives an isomorphism of costandard modules: $(\nabla_x)_\Lambda \cong (\nabla_x)_B$.*

1.3 Koszul properties for algebras

In this section we will recall the definition of (classical) Koszul algebras and the main results on Koszul duality. Then we will focus on a more general form of Koszul property that leads to an equivalence between certain derived categories of graded modules. Since we will be interested in the study of algebras that are quasi-hereditary and Koszul, we will also recall from [Mad13] some properties about quasi-hereditary algebras that are Koszul with respect to the standard module Δ.

We will work with non-negatively graded algebras: if $\Lambda = \bigoplus_{n\geq 0} \Lambda_n$ is a graded k-algebra, then we denote by $\mathsf{gr}\Lambda$ the category of finitely generated graded Λ-modules with graded shift $\langle\cdot\rangle$ acting as $(M\langle j\rangle)_i = M_{i-j}$ for any graded Λ-module M. $\mathrm{Hom}_{\mathsf{gr}\Lambda}(M,N)$ is the vector space of graded morphisms of degree zero between M and N and we have $\mathrm{Hom}_\Lambda(M,N) = \bigoplus_{i\geq 0} \mathrm{Hom}_{\mathsf{gr}\Lambda}(M, N\langle i\rangle)$.

1.3.1 Koszul algebras and Koszul duality

We refer to [BGS96] for the definition of (classical) Koszul algebras and their basic properties.

Definition 1.3.1 ([BGS96], Definition 1.2.1)**.** A *Koszul algebra* is a non-negatively graded algebra $\Lambda = \bigoplus_{n\geq 0} \Lambda_n$ such that:

1. Λ_0 is semisimple and
2. Λ_0, considered as a right Λ-module, admits a projective resolution of graded modules
$$\cdots \to P^2 \to P^1 \to \Lambda_0 \to 0$$
such that each P^i is generated in degree i, that is $P^i = P^i_i\Lambda$.

Definition 1.3.2. A finitely generated graded Λ-module M is called *linear* if M admits a graded projective resolution

$$\cdots \to P^2 \to P^1 \to P^0 \to M \to 0$$

such that P^i is generated by its component of degree i, for any $i \geq 0$.

Remark 1.3.3. By the definition of linear module, condition (2) of Definition 1.3.1 is equivalent to: (2') Λ_0 is a linear Λ-module.

Recall that, following our notation, a *graded morphism* between graded Λ-modules is a module homomorphism of degree zero. Moreover, for each graded morphism $f : M \to N$, $\mathrm{Ker}\, f$ and $\mathrm{Coker}\, f$ are graded modules and the morphisms $\mathrm{Ker}\, f \to M$ and $N \to \mathrm{Coker}\, f$ are graded as well. Therefore, if we define standard modules Δ_x as in Definition 1.2.2, each Δ_x is a graded module generated in degree zero.

Definition 1.3.4. [ÁDL03] A graded quasi-hereditary algebra is called *standard Koszul* if the standard module is linear.

Theorem 1.3.5 ([ÁDL03], Theorem 1.4)**.** *If Λ is quasi-hereditary and standard Koszul then it is Koszul.*

Proposition 1.3.6. *Let M be a finitely generated graded Λ-module. The following are equivalent:*

1. *M is linear.*

2. $\mathrm{Ext}^i_{\mathsf{gr}\Lambda}(M, \Lambda_0\langle j\rangle) = 0$ *unless* $i = j$.

Proof. See [BGS96], Proposition 1.14.2. □

The following result gives an alternative definition of Koszul algebra based on extension groups:

Proposition 1.3.7. *Let $\Lambda = \bigoplus_{n\geq 0} \Lambda_n$ be a graded algebra such that $\Lambda_0 \simeq k^s$ is semisimple. The following conditions are equivalent:*

1. *Λ is Koszul.*

2. *For any two graded Λ-modules M, N concentrated in degree m and n respectively, we have* $\mathrm{Ext}^i_{\mathsf{gr}\Lambda}(M, N) = 0$ *unless* $i = n - m$.

3. $\mathrm{Ext}^i_{\mathsf{gr}\Lambda}(\Lambda_0, \Lambda_0\langle n\rangle) = 0$ *unless* $i = n$.

Proof. See [BGS96], Proposition 2.1.3. □

Recall that a graded algebra Λ is *quadratic* if Λ_0 is semisimple, Λ is generated by Λ_1 as a tensor algebra over Λ_0 and the ideal of relations is generated by elements in degree 2. More explicitly this means that Λ is isomorphic to a quotient of the tensor algebra

$$T_{\Lambda_0}\Lambda_1 = \Lambda_0 \oplus \Lambda_1 \oplus (\Lambda_1 \otimes_{\Lambda_0} \Lambda_1) \oplus \cdots = \bigoplus_{i\geq 0} \Lambda_1^{\otimes i}$$

by an ideal $I := (R)$ such that $R \subset \Lambda_1 \otimes_{\Lambda_0} \Lambda_1$.

Theorem 1.3.8. *Let Λ be graded with Λ_0 semisimple. The following conditions are equivalent:*

1. $\mathrm{Ext}^1_{\mathsf{gr}\Lambda}(\Lambda_0, \Lambda_0\langle n\rangle) \neq 0$ *only if* $n = 1$.

2. *Λ is generated by Λ_1 over Λ_0.*

Under the above equivalent assumptions, if $\mathrm{Ext}^2_{\mathsf{gr}\Lambda}(\Lambda_0, \Lambda_0\langle n\rangle) \neq 0$ *implies* $n = 2$ *then Λ is quadratic.*

Proof. The equivalence between the two statements is Proposition 2.3.1 in [BGS96]. The last statement is Theorem 2.3.2 in [BGS96]. □

Corollary 1.3.9 ([BGS96], Corollary 2.3.3)**.** *Any Koszul algebra is quadratic.*

Let $\Lambda = kQ/I$ be the path algebra of the quiver Q with relations given by a homogeneous admissible ideal $I = (R) \subseteq kQ_2$, where Q_i denotes the set of paths of length i. Let $\mathcal{B} = \bigcup_{i\geq 0} \mathcal{B}_i$ be a basis of Λ consisting of paths such that $\mathcal{B}_0 = Q_0$, $\mathcal{B}_1 = Q_1$ and $\mathcal{B}_i \subseteq Q_i$. Suppose moreover that there is a total order $<$ on Q_1 that we extend to each Q_i lexicographically, then to each $\mathcal{B}_i \subset Q_i$ and finally to the union $\mathcal{B}_+ = \bigcup_{i>0} \mathcal{B}_i$, by refining the degree order.

Definition 1.3.10. The pair $(\mathcal{B}, <)$ is a *PBW basis* for Λ if the following hold:

- if p and q are paths in $\mathcal{B}$ then either pq is in $\mathcal{B}$ or it is a linear combination of elements $r \in \mathcal{B}$ such that $r < pq$ as paths in Q_i, for some $i > 0$.
- a path $\pi = \alpha_1\alpha_2 \cdots \alpha_i$ of length $i \geq 3$ is in $\mathcal{B}$ if and only if for each $1 \leq j \leq i-1$ the paths $\alpha_1 \cdots \alpha_j$ and $\alpha_{j+1} \cdots \alpha_i$ are in $\mathcal{B}$.

The following fact is a generalization of [Pri70, Theorem 5.2]:

Theorem 1.3.11 ([Gra17], Theorem 2.18)**.** *If Λ has a PBW basis then it is Koszul.*

For any k-vector space V, let $V^* = \operatorname{Hom}_k(V, k)$ be the dual vector space via the standard duality $D(-) = \operatorname{Hom}_k(-, k)$. Recall that, for any quadratic algebra Λ, the *quadratic dual* is the (quadratic) algebra $V^! = T_{\Lambda_0}\Lambda_1^*/(R^\perp)$. Moreover let $E(\Lambda) = \operatorname{Ext}^*_\Lambda(\Lambda_0, \Lambda_0)$ be the graded algebra of self-extensions of Λ_0.

Theorem 1.3.12 ([BGS96], Theorems 2.10.1, 2.10.2)**.** *Let Λ be a Koszul algebra, then $E(\Lambda) \cong (\Lambda^!)$ and $E(E(\Lambda)) \cong \Lambda$ canonically.*

This "duality" between Λ and $E(\Lambda)$ gives rise to an equivalence of triangulated categories as explained in the following Theorem:

Theorem 1.3.13 ([BGS96], Theorems 2.12.5, 2.12.6)**.** *Let $\Lambda = kQ/I$ be a finite dimensional Koszul k-algebra with a presentation as path algebra with relations. There exists an equivalence of triangulated categories*

$$\mathcal{K} : \mathcal{D}^b(\Lambda) \to \mathcal{D}^b(\Lambda^!)$$

between the (graded) bounded derived category of Λ and that of $\Lambda^!$ such that:

(a) $\mathcal{K}(M\langle n\rangle) = (\mathcal{K}M)[-n]\langle -n\rangle$ canonically for any $M \in \mathcal{D}^b(\Lambda)$.

(b) Let $S_x = e_x\Lambda_0$ be the simple Λ-module associated to the vertex x, I_x its injective envelope and $P_x = e_x\Lambda^!$ the projective cover of the simple $\Lambda^!$-module $e_x\Lambda^!_0 = T_x$, then $\mathcal{K}(S_x) = P_x$ and $\mathcal{K}(I_x) = T_x$.

The functor $\mathcal{K}$ is called the "Koszul duality functor".

In Sections 2.13 and 2.14 of [BGS96] the authors characterized the class of linear modules of $\Lambda^!$ (see also the Remark following Theorem 2.12.5 in the same paper). Let

$$\mathsf{gr}\Lambda^{\uparrow} = \{M \in \mathsf{gr}\Lambda \mid M_j = 0 \text{ for } j \ll 0\}$$

and

$$\mathsf{gr}\Lambda^{\downarrow} = \{M \in \mathsf{gr}\Lambda \mid M_j = 0 \text{ for } j \gg 0\}$$

be the subcategories of $\mathsf{gr}\Lambda$ consisting of modules whose degree is bounded below and above respectively.

Proposition 1.3.14. *The class of linear modules of* $\Lambda^!$ *consists precisely of* $\mathsf{gr}(\Lambda^!)^{\uparrow} \cap \mathcal{K}(\mathsf{gr}\Lambda^{\downarrow})$.

Proof. See [BGS96], Corollary 2.13.3. □

1.3.2 T-Koszul algebras and generalized Koszul duality

In what follows we will consider a grading that is different from the radical grading. To avoid confusion in the notation, given a graded algebra Λ, we will denote its graded subspaces by $\Lambda_{[i]}$ whenever the grading is not the radical grading. Later on (Subsection 1.3.3) this grading will coincide with the $\langle\cdot\rangle^{\flat}$-grading. The definition *T-Koszul* algebra is taken from [Mad13], [Mad11].

Definition 1.3.15. [Mad11] Let Λ be a graded algebra such that $\operatorname{gldim}\Lambda_{[0]} < \infty$ and let T be a graded Λ-module concentrated in degree zero. Then we say that Λ is *Koszul with respect to* T or T*-Koszul* if:

1. T is a tilting $\Lambda_{[0]}$-module.

2. T is graded self-orthogonal as a Λ-module, that is

$$\operatorname{Ext}^i_{\mathsf{gr}\Lambda}(T, T\langle j\rangle) = 0, \text{ whenever } i \neq j.$$

We recall the following results about graded self-orthogonal modules:

Lemma 1.3.16. *Let* $\Lambda = \bigoplus_{i\geq 0}\Lambda_{[i]}$ *be a graded algebra (with* $\Lambda_{[0]}$ *not necessarily semisimple) and* T *a finitely generated* Λ*-module concentrated in degree zero.*

1. *If* T *is linear then it is graded self-orthogonal.*

2. *If* $\operatorname{Ext}^i_{\mathsf{gr}\Lambda}(T, \Lambda_{[0]}\langle j\rangle) = 0$ *unless* $i = j$*, then* T *is linear.*

Proof. To prove the first statement, let P^i be a projective module in a minimal graded projective resolution of T; then P^i is generated in degree i and, since $T\langle j\rangle$ is concentrated in degree j, we have that $\operatorname{Hom}_{\mathsf{gr}\Gamma}(P^i, T\langle j\rangle) = 0$ if $i \neq j$. Hence $\operatorname{Ext}^i_{\mathsf{gr}\Gamma}(T, T\langle j\rangle) = 0$ whenever $i \neq j$.

The proof of the second statement can be found in [BGS96], Proposition 2.14.2, in the case when $\Lambda_{[0]}$ is semisimple and the part of the proof we are interested in is still true without any assumption on $\Lambda_{[0]}$. We include the proof here for the convenience of the reader.

Since T is concentrated in degree zero over Λ, its projective cover consists of a projective module P^0 generated in degree zero. We want to find a linear graded projective resolution for T by induction, so let us assume that we have a projective resolution

$$P^i \to P^{i-1} \to \cdots \to P^0 \to T \to 0$$

such that P^i is generated in degree i over Λ and the differential is injective on the degree i part of P^i, $P^i_{[i]}$. Then if we put $K = \mathrm{Ker}(P^i \to P^{i-1})$, we have that $K_{[j]} = 0$ for $j < i+1$. If N is any Λ-module that is concentrated in one single degree then $\mathrm{Ext}^{i+1}_{\mathsf{gr}\Lambda}(T, N) = \mathrm{Hom}_{\mathsf{gr}\Lambda}(K, N)$. But then our assumption means that $\mathrm{Hom}_{\mathsf{gr}\Lambda}(K, \Lambda_{[0]}\langle j\rangle) = 0$ unless $i+1 = j$, that is, K is generated in degree $i+1$ over Λ. Then we can find a projective cover P^{i+1} of K that is generated in degree $i+1$ and we can conclude by induction. □

It is important to underline that in general the two conditions in part (1) of Lemma 1.3.16 are not equivalent as the following example shows.

Example 1.3.17. Let Λ be the path algebra of the following quiver:

$$\begin{tikzcd} 1 & 2 \arrow[l, bend right, "c"'] \arrow[l, bend left, "a"] & 3 \arrow[l, bend right, "d"'] \arrow[l, bend left, "b"] \end{tikzcd}$$

with relations $ba = 0$, $da = bc$. Define a grading $|\cdot|$ on Λ by setting $|a| = |b| = 0$ and $|d| = |c| = 1$ and let $\Lambda_{[0]}$ be the subalgebra of Λ concentrated in degree zero. Then Λ is Koszul with respect to $D\Lambda_{[0]}$ but the simple module S_3 is a direct summand of $D\Lambda_{[0]}$ and its (graded) projective resolution is:

$$0 \to P_1 \oplus P_1\langle 1\rangle \to P_2 \oplus P_2\langle 1\rangle \to P_3 \to S_3 \to 0$$

hence it is not linear.

Lemma 1.3.18. *Let Λ be a graded k-algebra with $\Lambda_{[0]}$ not necessarily semisimple and T a graded self-orthogonal module. Then*

$$\mathrm{Ext}^i_\Lambda(T, T) \cong \mathrm{Ext}^i_{\mathsf{gr}\Lambda}(T, T\langle i\rangle)$$

for each $i \geq 0$. Moreover there is an isomorphism of graded algebras

$$\bigoplus_{i\geq 0} \mathrm{Ext}^i_\Lambda(T, T) \cong \bigoplus_{i \geq 0} \mathrm{Ext}^i_{\mathsf{gr}\Lambda}(T, T\langle i\rangle).$$

Proof. See [Mad11], Proposition 3.1.2 and Corollary 3.1.3. □

An analogous of Koszul duality holds for T-Koszul algebras:

Theorem 1.3.19 ([Mad11], Theorem 4.2.1)**.** *Let Λ be a graded k-algebra such that* $\operatorname{gldim}\Lambda_{[0]} < \infty$ *and suppose that Λ is T-Koszul for a module T. Let $\Lambda^\dagger = \operatorname{Ext}^*_\Lambda(T,T)$ endowed with the* Ext*-grading, then:*

1. $\operatorname{gldim}\Lambda^\dagger_{[0]} < \infty$ *and $\Lambda^\dagger$ is Koszul with respect to $DT_{\Lambda^\dagger}$.*
2. *There is an isomorphism of graded algebras $\Lambda \simeq \operatorname{Ext}^*_{\Lambda^\dagger}(DT, DT)$*

If this is the case we say that the pair $(\Lambda^\dagger, DT)$ is the *Koszul dual* of (Λ, T).

When Λ is T-Koszul there exists a complex of bigraded Λ-$\Lambda^\dagger$-modules X that defines two functors

$$F_T = - \otimes^{\mathbb{L}}_{\mathsf{gr}\Lambda^\dagger} X : \mathcal{D}(\mathsf{gr}\Lambda^\dagger) \rightleftharpoons \mathcal{D}(\mathsf{gr}\Lambda) : G_T = \mathbb{R}\operatorname{Hom}_{\mathsf{gr}\Lambda}(X, -)$$

such that (F_T, G_T) is an adjoint pair ([Mad11], Section 3 and [Kel94]).

In general the two functors above are not quasi-inverses of each other but they induce an equivalence on certain subcategories. Let $\mathcal{F}_{\mathsf{gr}\Lambda}$ be the full subcategory of $\mathsf{gr}\Lambda$ of modules M having a finite filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_t = M$ with factors that are graded shifts of direct summands of T. Let $\mathcal{L}^b(\Lambda^\dagger)$ be the category of bounded linear complexes of graded projective $\Lambda^\dagger$-modules.

Theorem 1.3.20 ([Mad11], Theorem 4.3.2 and Theorem 4.3.4)**.** *The functor $G_T : \mathcal{D}(\mathsf{gr}\Lambda) \to \mathcal{D}(\mathsf{gr}\Lambda^\dagger)$ restricts to an equivalence $G_T : \mathcal{F}_{\mathsf{gr}\Lambda} \to \mathcal{L}^b(\Lambda^\dagger)$. If moreover Λ is Artinian, $\Lambda^\dagger$ is Noetherian and* $\operatorname{gldim}\Lambda^\dagger < \infty$*, then there is an equivalence of triangulated categories $G^b_T : \mathcal{D}^b(\mathsf{gr}\Lambda) \to \mathcal{D}^b(\mathsf{gr}\Lambda^\dagger)$ between the bounded derived categories.*

The following Proposition gives some useful properties of the adjoint pair (F_T, G_T):

Proposition 1.3.21. *Let T be a graded self-orthogonal Λ-module, M a finitely cogenerated Λ-module and N an object in $\mathcal{D}(\mathsf{gr}\Lambda)$. Then, for every $i, j \in \mathbb{Z}$, we have:*

(a) $G_T(T) \cong \Lambda^\dagger$.

(b) If $\phi : F_T G_T \to id_{\mathcal{D}(\mathsf{gr}\Lambda)}$ is the counit of the adjunction, then $\phi_T : F_T G_T(T) \to T$ is an isomorphism.

(c) There is a functorial isomorphism $G_T(N\langle j\rangle) \cong G_T(N)\langle -j\rangle[-j]$.

(d) There is a functorial isomorphism $F_T G_T(N\langle j\rangle) \cong F_T G_T(N)\langle j\rangle$.

(e) $(H^i G_T(M))_j \cong \operatorname{Ext}^{i+j}_{\mathsf{gr}\Lambda}(T, M\langle j\rangle)$.

(f) $G_T(D\Lambda) \cong DT_{\Lambda^\dagger}$.

Proof. See Proposition 3.2.1 in [Mad11]. □

1.3.3 Bigraded Δ-Koszul algebras

We are particularly interested in the case when a quasi-hereditary algebra is Koszul with respect to the standard module Δ. Example 2.4 and 4.7 in [Mad13] show that if Z is the Brauer algebra associated to the Brauer line, then there exists a quasi-hereditary cover Γ that is standard Koszul with grading given by path-length but it is also possible to define another grading in order to make it Δ-Koszul.

Example 1.3.22. [Mad11] [Mad13] Consider the case of the Brauer tree algebra when the tree is a line with s edges. It can be presented as the path algebra of the following quiver:

$$\begin{tikzcd} 1 \arrow[r, shift left, bend left, "\alpha"] & 2 \arrow[l, shift left, bend left, "\beta"] \arrow[r, shift left, bend left, "\alpha"] & \cdots \arrow[l, shift left, bend left, "\beta"] \arrow[r, shift left, bend left, "\alpha"] & s-1 \arrow[l, shift left, bend left, "\beta"] \arrow[r, shift left, bend left, "\alpha"] & s \arrow[l, shift left, bend left, "\beta"] \end{tikzcd}$$

with relations $\alpha^2 = \beta^2 = 0$ and $e_x\alpha\beta = e_x\beta\alpha$ for any $x = 2, \dots s-1$. Then a quasi-hereditary cover Γ is the path algebra of the following quiver:

$$\begin{tikzcd} 1 \arrow[r, shift left, bend left, "\alpha"] & 2 \arrow[l, shift left, bend left, "\beta"] \arrow[r, shift left, bend left, "\alpha"] & \cdots \arrow[l, shift left, bend left, "\beta"] \arrow[r, shift left, bend left, "\alpha"] & s \arrow[l, shift left, bend left, "\beta"] \arrow[r, shift left, bend left, "\alpha"] & s+1 \arrow[l, shift left, bend left, "\beta"] \end{tikzcd}$$

bound by the ideal of relations $I = \big(\alpha\beta - \beta\alpha, \alpha^2, \beta^2, e_{s+1}\beta\alpha\big)$. We can see that Γ is Koszul (and standard Koszul) or Koszul with respect to Δ depending on the grading that we put on the algebra:

1. If all the arrows are given degree 1 then Γ is Koszul in the classical sense and standard Koszul.

2. If we put $deg\alpha = 1$ and $deg\beta = 0$, this define an algebra grading on Γ and the conditions of Definition 1.3.15 are satisfied with $T = \Gamma_{[0]} = \Delta$. Hence Γ is Koszul with respect to Δ and the Koszul dual algebra $\Gamma^\dagger = \mathrm{Ext}^*_\Gamma(\Delta, \Delta)$ is the path algebra of the following quiver:

$$\begin{tikzcd} 1 & 2 \arrow[l, shift right, bend right, "\alpha^*"'] \arrow[l, shift left, bend left, "\beta"] & \cdots \arrow[l, shift right, bend right, "\alpha^*"'] \arrow[l, shift left, bend left, "\beta"] & s \arrow[l, shift right, bend right, "\alpha^*"'] \arrow[l, shift left, bend left, "\beta"] & s+1 \arrow[l, shift right, bend right, "\alpha^*"'] \arrow[l, shift left, bend left, "\beta"] \end{tikzcd}$$

with relations $I' = \big(\beta^2, \alpha^*\beta - \beta\alpha^*\big)$. Moreover it is shown in [Mad13] that $\Gamma^\dagger$ is Koszul in the classical sense.

We will need to consider a bigraded structure on Λ. Suppose we can define two gradings $|\cdot|^\flat$ and $|\cdot|^\sharp$ on Λ, with shifts $\langle\cdot\rangle^\flat$ and $\langle\cdot\rangle^\sharp$, and corresponding categories of graded modules $\mathsf{gr}^\flat\Lambda$ and $\mathsf{gr}^\sharp\Lambda$ respectively. Let $|\cdot|^{tot}$ be the total grading on Λ obtained by adding the $|\cdot|^\flat$-degree and the $|\cdot|^\sharp$-degree. For $i \geq 0$, denote by $\Lambda_{[i]}$ the degree-i subspace of Λ with respect to $|\cdot|^\flat$, and by Λ_i the degree-i subspace of Λ with respect to $|\cdot|^{tot}$; then $\Lambda_0 \subseteq \Lambda_{[0]}$. Suppose moreover that $(\Lambda, |\cdot|^\flat)$ is $\Lambda_{[0]}$-Koszul and let $(\Lambda^\dagger, D\Lambda_{[0]})$ be the Koszul dual of $(\Lambda, \Lambda_{[0]})$.

The grading $|\cdot|^\sharp$ on Λ induces a grading on $\Lambda^\dagger$ in the following way. Since $\Lambda_{[0]}$ is concentrated in $|\cdot|^\flat$-degree zero, the $|\cdot|^\sharp$-degree on $\Lambda_{[0]}$ coincides with $|\cdot|^{tot}$, so $\Lambda_{[0]}$ inherits a graded structure from $|\cdot|^\sharp$ by defining the graded parts $(\Lambda_{[0]})_n = \Lambda_n \cap \Lambda_{[0]}$. Put $V_{n,j} = \mathrm{Ext}^n_{\mathsf{gr}^\sharp\Lambda}(\Lambda_{[0]}, \Lambda_{[0]}\langle j\rangle^\sharp)$; the Yoneda extension groups of $\Lambda_{[0]}$ are graded k-vector spaces:

$$\mathrm{Ext}^n_\Lambda(\Lambda_{[0]}, \Lambda_{[0]}) = \bigoplus_{j\geq 0} \mathrm{Ext}^n_{\mathsf{gr}^\sharp\Lambda}(\Lambda_{[0]}, \Lambda_{[0]}\langle j\rangle^\sharp) = \bigoplus_{j\geq 0} V_{n,j}.$$

Setting $V_{\bullet,j} = \bigoplus_{n\geq 0} V_{n,j}$ gives a grading on $\Lambda^\dagger = \bigoplus_{j\geq 0} V_{\bullet,j}$ that we will denote again by $|\cdot|^\sharp$. The algebra $\Lambda^\dagger = \mathrm{Ext}^*_\Lambda(\Lambda_{[0]}, \Lambda_{[0]})$ is also a graded algebra with respect to the Ext-grading since, for any $n, m \geq 0$ we have

$$\mathrm{Ext}^n_\Lambda(\Lambda_{[0]}, \Lambda_{[0]})\,\mathrm{Ext}^m_\Lambda(\Lambda_{[0]}, \Lambda_{[0]}) \subseteq \mathrm{Ext}^{n+m}_\Lambda(\Lambda_{[0]}, \Lambda_{[0]}).$$

Note that, since $\Lambda_{[0]}$ is graded self-orthogonal with respect to $|\cdot|^\flat$, the Ext-grading on $\Lambda^\dagger$ is precisely the one induced by $|\cdot|^\flat$; hence we will denote the Ext-grading on $\Lambda^\dagger$ again by $|\cdot|^\flat$. The decomposition of $\Lambda^\dagger$ in bigraded subspaces is $\Lambda^\dagger = \bigoplus_{n,j\geq 0} V_{n,j} = \bigoplus_{n\geq 0}\left(\bigoplus_{j\geq 0} V_{n,j}\right)$. We can define $V_n = \bigoplus_{i+j=n} V_{i,j}$ so that

$$\begin{aligned} V_{0,0} &= \mathrm{Hom}_{\mathsf{gr}^\sharp\Lambda}(\Lambda_{[0]}, \Lambda_{[0]})(\cong \Lambda_0), \\ V_{0,1} &= \mathrm{Hom}_{\mathsf{gr}^\sharp\Lambda}(\Lambda_{[0]}, \Lambda_{[0]}\langle 1\rangle^\sharp), \\ V_{1,0} &= \mathrm{Ext}^1_{\mathsf{gr}^\sharp\Lambda}(\Lambda_{[0]}, \Lambda_{[0]}), \\ &\dots \end{aligned} .$$

Then we can write $\Lambda^\dagger = \bigoplus_{n\geq 0} V_n$ and this defines a new graded structure on $\Lambda^\dagger$ as a k-vector space. From the above we have that

$$\begin{aligned} V_n V_m &= \left(\bigoplus_{i+j=n} \mathrm{Ext}^i_{\mathsf{gr}^\sharp\Lambda}(\Lambda_{[0]}, \Lambda_{[0]}\langle j\rangle^\sharp)\right)\left(\bigoplus_{h+l=m} \mathrm{Ext}^h_{\mathsf{gr}^\sharp\Lambda}(\Lambda_{[0]}, \Lambda_{[0]}\langle l\rangle^\sharp)\right) \\ &\subseteq \bigoplus_{i+j+h+l=n+m} \mathrm{Ext}^{i+h}_{\mathsf{gr}^\sharp\Lambda}(\Lambda_{[0]}, \Lambda_{[0]}\langle j+l\rangle^\sharp) \end{aligned}$$

hence this gives us a graded structure on $\Lambda^\dagger$ as a k-algebra. Finally we will denote this grading on $\Lambda^\dagger$ by $|\cdot|^{tot}$ and the category of (finitely generated) graded modules by $\mathsf{tgr}\Lambda$.

Example 1.3.23. Consider the algebra from Example 1.3.22:

$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] & 2 \arrow[l, bend left, "\beta"] \arrow[r, bend left, "\alpha"] & \cdots \arrow[l, bend left, "\beta"] \arrow[r, bend left, "\alpha"] & s \arrow[l, bend left, "\beta"] \arrow[r, bend left, "\alpha"] & s+1 \arrow[l, bend left, "\beta"] \end{tikzcd}$$

We can take as $\flat$-grading the one considered in part (2) of the example, so $deg_\flat\alpha = 1$ and $deg_\flat\beta = 0$ and as $\sharp$-grading the grading such that $deg_\sharp\alpha = 0$ and $deg_\sharp\beta = 1$. Then the total grading coincides with the radical grading (by path length).

The first result of the following is essentially Proposition 4.2 of [Mad13] when Λ is quasi-hereditary and Δ-Koszul, with $\flat$-grading given by $|\cdot|^\flat$ so that $\Lambda_{[0]} = \Delta$. Recall that $G_\Delta = \mathrm{Hom}_{\mathcal{D}(\mathsf{gr}^\flat\Lambda)}(\Delta, -)$.

Proposition 1.3.24 ([Mad13])**.** *Let Λ be a bigraded quasi-hereditary algebra, with gradings $|\cdot|^\flat$ and $|\cdot|^\sharp$ as before, that is also Δ-Koszul with respect to $|\cdot|^\flat$. Then*

1. $G_\Delta(\nabla_x) \cong S_x$
2. $S_x\langle j\rangle^\sharp \cong G_\Delta(\nabla_x\langle j\rangle^\sharp)$

where ∇_x denotes the costandard Λ-module of weight x and S_x is the simple $\Lambda^\dagger$-module whose projective cover is $G_\Delta(\Delta_x)$.

The original statement in [Mad13] is about standard Koszul algebras admitting a particular height function but the proof is still valid in the case of Δ-Koszul algebras. We include the original argument here for the convenience of the reader.

Proof. 1. By Proposition 1.3.21(e), we have

$$(H^k G_\Delta(\nabla_x))_j \cong \mathrm{Ext}^{k+j}_{\mathsf{gr}^\flat\Lambda}(\Delta, \nabla_x\langle j\rangle^\flat) = 0$$

whenever $k \neq 0$ or $j \neq 0$. Then

$$\begin{aligned} G_\Delta(\nabla_x) &\cong (H^0(G_\Delta(\nabla_x))_0 \\ &\cong \mathrm{Hom}_{\mathsf{gr}^\flat\Lambda}(\Delta, \nabla_x) \\ &\cong \mathrm{Hom}_{\mathsf{gr}^\flat\Lambda}(\Delta_x, \nabla_x), \end{aligned}$$

that is a one-dimensional k-vector space. Moreover, if $y \neq x$,

$$\mathrm{Hom}_{\mathcal{D}(\mathsf{gr}^\flat\Lambda^\dagger)}(G_\Delta(\Delta_y), G_\Delta(\nabla_x)) \cong \mathrm{Hom}_{\mathcal{D}(\mathsf{gr}^\flat\Lambda)}(\Delta_y, \nabla_x) = 0$$

so we must have $G_\Delta(\nabla_x) \cong \operatorname{top} G_\Delta(\Delta_x)$.

2. We have

$$\begin{aligned} S_x\langle j\rangle^\sharp &\cong G_\Delta(\nabla_x)\langle j\rangle^\sharp \\ &\cong \mathrm{Hom}_{\mathcal{D}(\mathsf{gr}^\flat\Lambda)}(\Delta, \nabla_x)\langle j\rangle^\sharp \\ &\cong \bigoplus_{k\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}(\mathsf{tgr}\Lambda)}(\Delta, \nabla_x\langle 0, k+j\rangle) \cong G_\Delta(\nabla_x\langle j\rangle^\sharp) \end{aligned}$$

∎

2

Higher zigzag algebras

In this chapter we recall the definition of *higher zigzag algebras* as described in [Gra17]. After giving the general construction, we will focus on the case of *type A* higher zigzag-algebras, since they are the central object of this thesis.

2.1 General construction

We start by recalling some basic facts. Given any algebra Λ and a Λ-Λ-bimodule M, the *extension* of Λ by M, denoted by $\Lambda \ltimes M$, is the algebra whose vector space is $\Lambda \oplus M$ with multiplication given by $(a,m)(b,n) = (ab, mb + an)$. When $M = \Lambda^*$, this is called the *trivial extension* algebra of Λ, $\mathrm{Triv}(\Lambda) = \Lambda \ltimes \Lambda^*$.

If $\phi \in \mathrm{Aut}(\Lambda)$ is an automorphism of Λ and M a (left/right/bi-) Λ-module, we denote by M_ϕ the (resp. left/right/bi-) Λ-module given by M with action twisted by ϕ: $m \cdot \lambda = m\phi(\lambda)$.

Now let $\Lambda = \bigoplus_{i \geq 0} \Lambda_i$ be graded, with Λ_0 semisimple and generated in degree 1 as a tensor algebra over Λ_0. We have an automorphism $\zeta \in \mathrm{Aut}(\Lambda)$ defined by $\zeta(\lambda) = (-1)^i \lambda$, for $\lambda \in \Lambda_i$ an homogeneous element of degree i.

Definition 2.1.1 ([GI], Section 5)**.** The $(d+1)$-*trivial extension* of Λ is the trivial extension of Λ by the twisted bimodule $\Lambda^*_{\zeta^d}\langle d+1 \rangle$ and it is denoted by $\mathrm{Triv}_{d+1}(\Lambda)$.

Hence, as a graded vector space, $\mathrm{Triv}_{d+1}(\Lambda) = \Lambda \oplus \Lambda^*\langle d+1 \rangle$ and the multiplication is given by

$$(a, f)(b, g) = (ab, fb + (-1)^{di} ag)$$

for $a \in \Lambda_i$.

We are now ready to give our main definition.

Definition 2.1.2 ([Gra17], Definition 2.5)**.** Let Λ be a Koszul algebra such that $\operatorname{gldim} \Lambda \leq d < \infty$. The $(d+1)$-*zigzag algebra* of Λ is $Z_{d+1}(\Lambda) = \mathrm{Triv}_{d+1}(\Lambda^!)$.

When $\operatorname{gldim} \Lambda = d$ (hence d is a parameter determined by Λ) we will refer to $Z_{d+1}(\Lambda)$ as the *higher zigzag algebra* of Λ, denoting it by $Z(\Lambda)$.

Remark 2.1.3. Given any graph G without loops or multiple edges, Huerfano and Khovanov in [HK01] described the construction of the *zigzag algebra* $A(G)$ of G and they proved that it is isomorphic to $\mathrm{Triv}(kQ^!)$, where Q is the quiver obtained by taking any orientation on the edges of G [HK01, Proposition 9]. Hence we can talk about the zigzag algebra of any hereditary algebra kQ. In general though Definition 2.1.2 is different from the classical one since $A(kQ)$ is not always isomorphic to $Z(kQ)$, as the following example shows.

Example 2.1.4 ([Gra17], Example 2.11)**.** Let Q be the following quiver:

$$
\begin{tikzcd}
 & 2 \arrow[dr, "\alpha_2"] & \\
1 \arrow[ur, "\alpha_1"] & & 3 \arrow[ll, "\alpha_3"]
\end{tikzcd}
$$

Then the algebras $A = Z_2(kQ)$ and $B = \mathrm{Triv}(kQ^!)$, graded by path length, both have basis given by $e_i, \alpha_i, \alpha_i^*$ and e_i^* for $i = 1, 2, 3$. Any graded isomorphism $A \xrightarrow{\cong} B$ would permute the idempotents e_i and this permutation would determine the images of the arrows up to scalars. Without loss of generality we can suppose that e_i is sent to e_i; suppose moreover that α_i is sent to $\lambda_i \alpha_i$ and that α_i^* to $\mu_i \alpha_i^*$. Then the relation $\alpha_i \alpha_i^* = -\alpha_{i-1}^* \alpha_{i-1}$ implies that $\lambda_i \mu_i = -\mu_{i-1}\lambda_{i-1}$, so $\lambda_1\mu_1 = -\lambda_2\mu_2 = \lambda_3\mu_3 = -\lambda_1\mu_1$ and this is possible only if $\operatorname{char} k = 2$.

Lemma 2.1.5. *Let $\Lambda = kQ/I$ be graded by path length and Koszul. If the underlying graph of Q is bipartite then $Z_{d+1}(\Lambda) \simeq \mathrm{Triv}(\Lambda^!)$.*

Proof. See [Gra17], Lemma 2.6. □

2.2 Type A higher zigzag algebras

In this section we will focus on the construction of *type A* higher zigzag algebras and their presentation as path algebras with relations [Gra17, Section 3]. This family of algebras is the main object of this thesis because in the case $d = 1$ they coincide with *Brauer tree algebras* whose underlying tree is a line. Brauer tree algebras have been widely studied and Brauer line algebras appear in many different areas of Representation Theory of algebras.

In order to motivate the construction of these higher zigzag algebras we start with recalling some definitions from *higher Auslander–Reiten theory* [Iya11].

Definition 2.2.1 ([Iya11], Definition 1.1)**.** Let $d > 0$. A module $M \in \operatorname{mod}\Lambda$ is called *d-cluster tilting* if

$$
\begin{aligned}
\operatorname{add} M &= \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(M, X) = 0 \text{ for any } 0 < i < d\} \\
&= \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(X, M) = 0 \text{ for any } 0 < i < d\}.
\end{aligned}
$$

We say that the algebra Λ is *d-representation finite* if $\operatorname{gldim}\Lambda \leq d$ and there exists a d-cluster tilting module $M \in \operatorname{mod}\Lambda$.

Note that 1-cluster tilting module are additive generators of $\operatorname{mod}\Lambda$. So Λ is 1-representation finite if and only if it is representation finite and hereditary.

Definition 2.2.2 ([Iya11], Theorem 1.10)**.** Let Λ be a d-representation finite algebra with d-cluster tilting module M. We call the algebra $\Gamma = \operatorname{End}_\Lambda(M)$ the *d-Auslander algebra* of Λ.

In [Iya11] Iyama defined recursively a family of algebras Λ_s^d, for $d, s \geq 1$, such that Λ_s^d is d-representation finite, with d-Auslander algebra Λ_s^{d+1}, and Λ_s^{d+1} is $d+1$-representation finite. Type A higher zigzag algebras are defined as $Z_s^d = Z_{d+1}(\Lambda_s^d)$ so let us recall Iyama's construction.

Let Λ_s^1 be the path algebra of the linearly oriented quiver A_s:

$$1 \to 2 \to \cdots \to s$$

This algebra is hereditary and representation finite so Λ_s^1 is 1-representation finite; let M_s^1 be the cluster tilting module in $\operatorname{mod}\Lambda_s^1$. By [Iya11, Corollary 1.16], $\Lambda_s^2 = \operatorname{End}_{\Lambda_s^1}(M_s^1)$ is 2-representation finite so we can inductively define Λ_s^d by:

$$\Lambda_s^d = \operatorname{End}_{\Lambda_s^{d-1}}(M_s^{d-1})$$

where M_s^{d-1} is a $(d-1)$-cluster tilting module for Λ_s^{d-1}.

The following result is the last step we need to define higher zigzag algebras of type A.

Proposition 2.2.3. *For $s, d \geq 1$ the algebra Λ_s^d is Koszul.*

Proof. See [Gra17], Proposition 3.4. □

Definition 2.2.4 ([Gra17], Definition 3.5)**.** The $(d+1)$-zigzag algebra of type A_s is $Z_s^d = Z_{d+1}(\Lambda_s^d)$.

The presentation of Z_s^d as path algebra of a quiver with relations is the main result of [Gra17, Section 3] and it is achieved using the fact that type A higher zigzag algebras are quadratic duals of type A *higher preprojective algebras.*

Definition 2.2.5 ([IO11])**.** Assume that Λ has finite global dimension d. The $(d+1)$-*preprojective algebra* of Λ, denoted by $\Pi = \Pi_{d+1}(\Lambda)$, is the tensor algebra, over Λ, of the Λ-Λ-bimodule $E = \operatorname{Ext}_\Lambda^d(\Lambda^*, \Lambda)$:

$$\Pi := T_\Lambda \operatorname{Ext}_\Lambda^d(\Lambda^*, \Lambda).$$

In [GI] the authors established a connection between higher zigzag algebras and higher preprojective algebras.

Theorem 2.2.6 ([GI], Section 5). *Let Λ be Koszul and of global dimension d and let Π be its $(d+1)$-preprojective algebra. Then Π is a quadratic algebra and there is a morphism $\phi : \Pi^! \to Z(\Lambda)$ that is an isomorphism in degree 0 and 1.*

In [GI] it is conjectured that, for type A_s higher zigzag algebra with $s \geq 3$, the morphism ϕ is an isomorphism and this has been proved in [Gra17]. As a consequence it is possible to adapt the presentation of higher preprojective algebras as path algebras with relations given by Iyama, Oppermann and Grant ([IO11], [GI]) to describe higher zigzag algebras.

Theorem 2.2.7 ([GI], Theorem 3.1). *The algebra Z_s^d has a presentation*

$$Z_s^d \cong kQ_s^d / I_s^d$$

where Q_s^d is a quiver with set of vertices:

$$Q_0 = \left\{ x = (x_0, x_1, \ldots, x_d) \in \mathbb{Z}_{\geq 0}^{d+1} \mid \sum_{i=0}^{d} x_i = s - 1 \right\}.$$

If $s = 1$ then Q_0 consists of a single vertex and Q_s^d has a single loop α in degree $d+1$. In this case we put $I_1^d = (\alpha^2)$.

If $s \geq 2$ the arrows of Q_s^d are all in degree one and they are given by the following set:

$$Q_1 = \left\{ x \xrightarrow{\alpha_i} x + f_i \mid i \in \{0, \ldots, d\}, x, x + f_i \in Q_0 \right\},$$

where f_i denotes the vector

$$f_i = (0, \ldots, 0, \overset{i-1}{-1}, \overset{i}{1}, 0, \ldots, 0) \in \mathbb{Z}^{d+1}$$

and we put $f_0 = (1, 0, \ldots, 0, -1)$.

If $s = 2$ then Q_2^d consists of an oriented cycle of $d+1$ vertices and I_2^d is the ideal of paths of length $d+2$.

If $s \geq 3$ then I_s^d is generated by the following relations:

for any $x \in Q_0$ and $i, j \in \{0, \ldots, d\}$ satisfying $x + f_i$, $x + f_i + f_j \in Q_0$,

$$(x \xrightarrow{\alpha_i} x + f_i \xrightarrow{\alpha_j} x + f_i + f_j) = \begin{cases} (x \xrightarrow{\alpha_j} x + f_j \xrightarrow{\alpha_i} x + f_i + f_j) & \textit{if } x + f_j \in Q_0, \\ 0 & \textit{if } i = j. \end{cases}$$

Remark 2.2.8. To simplify the notation we will usually write the relations as

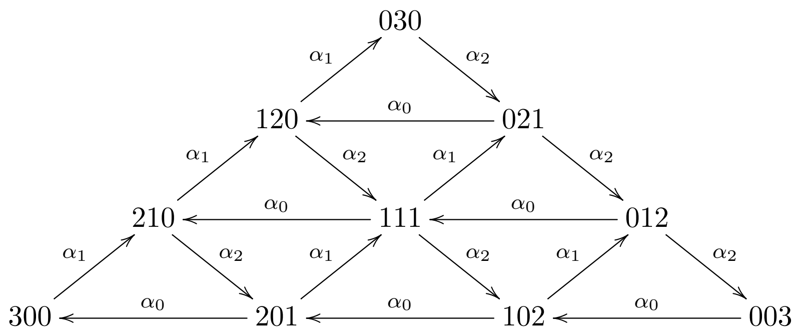
$$\alpha_i \alpha_j = \begin{cases} \alpha_j \alpha_i & \text{if } \alpha_j \alpha_i \neq 0, \\ 0 & \text{if } i = j. \end{cases}$$

Any vertex of the quiver is contained in a $(d+1)$-cycle of the form:

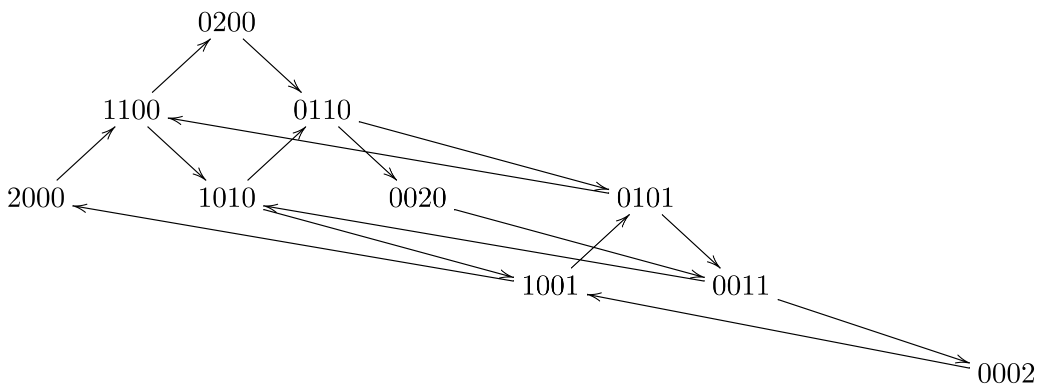
$$x \to x + f_{\sigma(1)} \to x + f_{\sigma(1)} + f_{\sigma(2)} \to \cdots \to x + f_{\sigma(1)} + \cdots + f_{\sigma(d)} \to x$$

for some permutation $\sigma \in \mathfrak{S}_{d+1}$.

Example 2.2.9. As an example we show the quiver of the higher zigzag algebra Z_4^2:



and the quiver of Z_3^3:



The following are important results that will be frequently used in the sequel:

Proposition 2.2.10. *$Z_s^d \cong \mathrm{Triv}((\Lambda_s^d)^!)$, thus Z_s^d is a symmetric algebra.*

Proof. See [Gra17], Proposition 3.11. □

Note that Proposition 2.2.10 tells us that for type A higher zigzag algebras we can forget about the change of sign introduced by the twisted trivial extension in the general definition. This very particular property is also reflected by the fact that we can express the relations of these algebras as commutativity relations instead of anti-commutativity relations, as the quadratic duality between Z_s^d and $\Pi_{d+1}(\Lambda_s^d)$ suggests.

Proposition 2.2.11.
- *Let $\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_l}$ be a path in the quiver of Z_s^d starting at the vertex x. If σ is a permutation in $\mathfrak{S}_l$ and $\alpha_{i_{\sigma(1)}}\alpha_{i_{\sigma(2)}}\cdots\alpha_{i_{\sigma(l)}}$ is another path starting at x, then the two paths are equal.*

- *Let $s \geq 3$. Then any path in the quiver of Z_s^d that contains α_i more than once is zero.*

Proof. The proofs of the two facts can be found in [Gra17], Lemma 3.17 and Lemma 3.19 respectively. □

Proposition 2.2.12. *Let $Z = Z_s^d$ be a higher zigzag algebra of type A and P_x, P_y two indecomposable projective Z-modules. Then*

- *If $x = y$ then $\dim_k \operatorname{Hom}_Z(P_x, P_x) = 2$ and the* Hom*-space is generated by the identity and the map $\varepsilon_x : P_x \to \operatorname{soc}(P_x)$.*
- *If $x \neq y$ then $\dim_k \operatorname{Hom}_Z(P_x, P_y) \leq 1$.*

Proof. By Proposition 2.2.11 each minimal cycle as in Remark 2.2.8 is non-zero and it is also a maximal path starting at the vertex x. Hence it corresponds in Z to the generator of the socle of the indecomposable projective module P_x and this shows the existence of $\varepsilon_x : P_x \to \operatorname{soc}(P_x)$. Since the vertices of each $d+1$-cycle are distinct, for any indecomposable projective Z-module P, the composition factors of $P/\operatorname{soc} P$ are all distinct and the first claim is proved.

The second claim follows from the fact that the simple module S_x can appear as a composition factor of P_y at most once. □

We also recall the following fact about the endomorphism ring of projective modules.

Proposition 2.2.13. *Fix $s, d \geq 1$, $1 \leq n \leq s$, $0 \leq m \leq d$ and let $Z = Z_s^d$. Let P be the direct sum of the indecomposable projective modules $e_x Z$ such that $x_m \geq n$. Then $\operatorname{End}_Z(P) \simeq Z_{s-n}^d$.*

Proof. See [Gra17], Proposition 4.4. □

3

Quasi-hereditary covers

In this chapter we define some quasi-hereditary covers for higher zigzag algebras as quotients of higher zigzag algebras of bigger size. We prove that these algebras are quasi-hereditary and that they are Koszul in the classical sense, standard Koszul and Koszul with respect to the standard module Δ. To conclude we compute the Δ-Koszul dual as a path algebra with relations.

3.1 Construction of quasi-hereditary covers

Let $Z^+ = Z^d_{s+1}$ be the $(d+1)$-zigzag-algebra of type A_{s+1} and consider the presentation of Z^+ as the path algebra of the quiver $Q^d_{s+1} = (Q_0, Q_1)$ with relations as given in Section 2.2 (Theorem 2.2.7). Denote by I the set of vertices. In the quiver we have arrows labelled by

$$\alpha_k : (x_0, \ldots, x_{k-1}, x_k, \ldots, x_d) \to (x_0, \ldots, x_{k-1} - 1, x_k + 1, \ldots, x_d)$$

for $k = 1, \ldots, d$ and α_0 is defined cyclically. Denote by $Z = Z^d_s$ the $(d+1)$-zigzag-algebra of type A_s; by Proposition 2.2.13 we can select a subset $J \subset I$ such that $Z \cong \operatorname{End}_{Z^+}(\bigoplus_{y \in J} P^+_y)$ where P^+_y is the projective cover of the simple Z^+-module associated to the vertex y. To be precise $J = \{(x_0, x_1, \ldots, x_d) \in I \mid x_0 \neq 0\}$. Put $P^+_J = \bigoplus_{y \in J} P^+_y$. For $K = I \setminus J$, let

$$\Gamma(Z) = Z^+ / \left(e_z \alpha_0 \alpha_1 | z \in K \right)$$

so that we have a surjective morphism of k-algebras: $Z^+ \to \Gamma(Z)$ that induces a fully faithful embedding: $\operatorname{mod} \Gamma(Z) \to \operatorname{mod} Z^+$. Since $\Gamma(Z)$ is defined as a quotient of Z^+ by an admissible ideal, the underlying quivers of Z^+ and of Γ coincide. Note that the vertices in K are precisely the ones of the kind $(0, x_1, \ldots, x_d)$, so they are not the target of any arrow α_0. Viceversa, vertices in J are always the target of some arrow α_0.

Example 3.1.1. Consider the 2-zigzag algebra $Z = Z^2_3$, with labels on the arrows accordingly to Theorem 2.2.7:

$$
\begin{tikzcd}
 & & 020 \arrow[dr, "\alpha_2"] & & \\
 & 110 \arrow[ur, "\alpha_1"] \arrow[dr, "\alpha_2"] & & 011 \arrow[ll, "\alpha_0"'] \arrow[dr, "\alpha_2"] & \\
200 \arrow[ur, "\alpha_1"] & & 101 \arrow[ll, "\alpha_0"] \arrow[ur, "\alpha_1"] & & 002 \arrow[ll, "\alpha_0"]
\end{tikzcd}
$$

Then the quiver of $\Gamma(Z)$ is

$$
\begin{tikzcd}
 & & & 030 \arrow[dr, "\alpha_2"] & & & \\
 & & 120 \arrow[ur, "\alpha_1"] \arrow[dr, "\alpha_2"] & & 021 \arrow[ll, "\alpha_0"'] \arrow[dr, "\alpha_2"] & & \\
 & 210 \arrow[ur, "\alpha_1"] \arrow[dr, "\alpha_2"] & & 111 \arrow[ll, "\alpha_0"'] \arrow[ur, "\alpha_1"] \arrow[dr, "\alpha_2"] & & 012 \arrow[ll, "\alpha_0"'] \arrow[dr, "\alpha_2"] & \\
300 \arrow[ur, "\alpha_1"] & & 201 \arrow[ll, "\alpha_0"'] \arrow[ur, "\alpha_1"] & & 102 \arrow[ll, "\alpha_0"'] \arrow[ur, "\alpha_1"] & & 003 \arrow[ll, "\alpha_0"']
\end{tikzcd}
$$

In this example we have $J = \{300, 210, 201, \dots, 102\}$ and $K = \{030, 021, 012, 003\}$.

In the following we put $\Gamma = \Gamma(Z)$.

Lemma 3.1.2. *The action of Z^+ on the module P_J^+ factors over Γ.*

Proof. It is enough to show that, for any $y \in J$ in the quiver of Z^+, any element $e_z^+\alpha_0\alpha_1$ annihilates $e_y^+Z^+ = P_y^+$, where e_z^+ (resp. e_y^+) is the primitive idempotent in Z^+ associated to the vertex $z \in K$ (resp. $y \in J$). Let $e_y^+\alpha_i \cdots \alpha_j e_z^+$ be a path from y to z corresponding to a generator of P_y^+, such that we can non-trivially multiply it on the right by $e_z^+\alpha_0\alpha_1$. By the relations $\alpha_i\alpha_j = \alpha_j\alpha_i$ of Z^+, such a path is equivalent to $e_y^+\alpha_l \cdots \alpha_1 e_z^+$. Then, again using the commutativity relations, we have $e_y^+\alpha_l \cdots \alpha_1 e_z^+\alpha_0\alpha_1 \sim e_y^+\alpha_l \cdots \alpha_0\alpha_1\alpha_1 = 0$ □

As a corollary we have that:

$$\mathrm{End}_\Gamma(P_J^+) \cong \mathrm{End}_{Z^+}(P_J^+) \cong Z$$

where the first isomorphism comes from the fully faithful embedding $\mathrm{mod}\,\Gamma \to \mathrm{mod}\,Z^+$. So P_J^+ is also a left Z-module and there is an adjunction

$$G = - \otimes_Z P_J^+ : \mathrm{mod}\,Z \rightleftharpoons \mathrm{mod}\,\Gamma : F = \mathrm{Hom}_\Gamma(P_J^+, -)$$

such that $GF \cong id$ when restricted to $\mathrm{add}\,P_J^+$.

Lemma 3.1.3. *Let P_y be an indecomposable projective Γ-module such that $y \in J$. Then $P_y = I_y$ is also injective.*

Proof. Note that for any $y \in J$, every path in the quiver of Γ is contained in a minimal cycle at the vertex y and this cycle is non-zero in Γ since P_y is also a module over Z^+. Denote by ε_y the non-zero element in Γ corresponding to such a cycle (note that this is well defined since by commutativity relations all the minimal cycles at y are equivalent). Let $\pi \in e_y\Gamma = P_y$ be a path starting at y and π' be its complementary path in a minimal cycle: $\varepsilon_y = e_y\pi\pi' e_y$. Then the map $\pi \mapsto D(\pi')$ extends to a morphism of Γ-modules and it gives an isomorphism $P_y = e_y\Gamma \xrightarrow{\cong} I_y = D(\Gamma e_y)$. $\square$

We can define a partial order on I by putting $x < y$ if and only if there is a path $x \overset{\pi}{\dashrightarrow} y$ in the quiver of Z^+, such that π does not involve any arrow α_0 (recall the presentation of Z^+ as in Theorem 2.2.7).

Lemma 3.1.4. *The indecomposable standard modules of Γ are either simple or uniserial with radical length two. Whenever we have an arrow $\alpha_0 : x \to y$ in the quiver of Γ then the standard module Δ_x has simple top S_x and simple socle S_y. Therefore $\mathrm{End}_\Gamma(\Delta_x) \cong k$ for any $x \in I$.*

Dually indecomposable costandard modules are either $\nabla_x = e_x\alpha_0\Gamma \subseteq I_x$, if $x \in J$, or $\nabla_x = I_x$, if $x \in K$.

Proof. By the construction of the quiver of Z^+, there exists an arrow $\alpha_l : y \to x$ for $x < y$ only if $l = 0$. Hence Δ_x is non simple if and only if there is an arrow $x \xrightarrow{\alpha_0} y$; S_y is the only composition factor in a radical filtration of P_x such that $y < x$, since $\alpha_0\alpha_0 = 0$. Then standard modules are uniserial with radical length at most 2.

Dually we show that, for any $y \in J$, the costandard module ∇_y is given by $e_y\alpha_0\Gamma \subseteq I_y \cong P_y$. First $e_y\alpha_0\Gamma \subseteq \nabla_y$ because α_0 appears only once in any path starting at y, so any composition factor S_z of $e_y\alpha_0\Gamma$ is such that $z \leq y$. Also $\nabla_y \subseteq e_y\alpha_0\Gamma$: if not we could find an element $e_y\alpha_j \cdots \alpha_k e_z$ in ∇_y with $j, \ldots, k \neq 0$ and S_z would be a composition factor of ∇_y such that $z > y$, a contradiction.

To conclude, if $z \in K$ then $\nabla_z = I_z$. Indeed, using commutativity relations and $e_z\alpha_0\alpha_1 = 0$, any path ending in z involving an arrow α_0 is zero in Γ. So any composition factor S_z of I_y satisfies $z \leq y$. $\square$

Example 3.1.5. Consider the 2-zigzag algebra $Z = Z_3^2$ as in Example 3.1.1:

$$\begin{tikzcd}
 & & 020 \arrow[dr, "\alpha_2"] & & \\
 & 110 \arrow[ur, "\alpha_1"] \arrow[dr, "\alpha_2"'] & & 011 \arrow[ll, "\alpha_0"'] \arrow[dr, "\alpha_2"] & \\
200 \arrow[ur, "\alpha_1"] & & 101 \arrow[ll, "\alpha_0"] \arrow[ur, "\alpha_1"'] & & 002 \arrow[ll, "\alpha_0"]
\end{tikzcd}$$

Then the quiver of Γ is

$$\begin{tikzcd}
& & & 030 \arrow[dr, "\alpha_2"] & & & \\
& & 120 \arrow[ur, "\alpha_1"] \arrow[dr, "\alpha_2"] & & 021 \arrow[ll, "\alpha_0"'] \arrow[dr, "\alpha_2"] & & \\
& 210 \arrow[ur, "\alpha_1"] \arrow[dr, "\alpha_2"] & & 111 \arrow[ll, "\alpha_0"'] \arrow[ur, "\alpha_1"] \arrow[dr, "\alpha_2"] & & 012 \arrow[ll, "\alpha_0"'] \arrow[dr, "\alpha_2"] & \\
300 \arrow[ur, "\alpha_1"] & & 201 \arrow[ll, "\alpha_0"'] \arrow[ur, "\alpha_1"] & & 102 \arrow[ll, "\alpha_0"'] \arrow[ur, "\alpha_1"] & & 003 \arrow[ll, "\alpha_0"']
\end{tikzcd}$$

and the ideal of relations is generated by the usual relations of Z_4^2 with moreover $e_{021}\alpha_0\alpha_1 = e_{012}\alpha_0\alpha_1 = e_{003}\alpha_0\alpha_1 = 0$. Note that in this example we have $J = \{300, 210, 201, \ldots, 102\}$ and $K = \{030, 021, 012, 003\}$.

The quiver of the partial order on I is the following:

$$\begin{tikzcd}
& & & 030 \arrow[dr] & & & \\
& & 120 \arrow[ur] \arrow[dr] & & 021 \arrow[dr] & & \\
& 210 \arrow[ur] \arrow[dr] & & 111 \arrow[ur] \arrow[dr] & & 012 \arrow[dr] & \\
300 \arrow[ur] & & 201 \arrow[ur] & & 102 \arrow[ur] & & 003
\end{tikzcd}$$

so that $300 < 210 < 201 < 111\ldots$, $300 < 210 < 120 < 111\ldots$, $300 < 210 < 120 < 030\ldots$, etc.

Proposition 3.1.6. *The algebra $\Gamma = \Gamma(Z)$ is a quasi-hereditary cover of Z.*

Proof. First we show that Γ is quasi-hereditary. By Lemma 3.1.4, $\mathrm{End}_\Gamma(\Delta_x) \cong k$ for every $x \in I$ so we only need to show that Γ is Δ-filtered.

We will prove that every indecomposable injective module is ∇-filtered. Let $x, y, z \in I$ such that there are arrows $z \xrightarrow{\alpha_0} y \xrightarrow{\alpha_0} x$ and consider the corresponding morphisms between indecomposable injective modules $I_x \xrightarrow{\alpha_0\cdot} I_y \xrightarrow{\alpha_0\cdot} I_z$. Since the vertices x and y must belong to J we have that $I_x = P_x = e_x\Gamma$ and $I_y = P_y = e_y\Gamma$. Hence $\mathrm{Im}(I_x \xrightarrow{\alpha_0\cdot} I_y) = e_y\alpha_0\Gamma = \nabla_y$ and, since by the relations we have $\mathrm{Ker}(I_y \xrightarrow{\alpha_0\cdot} I_z) = e_y\alpha_0\Gamma$, the sequence $I_x \xrightarrow{\alpha_0\cdot} I_y \xrightarrow{\alpha_0\cdot} I_z$ is exact. If $z \in K$, then $\nabla_z = I_z$ and $I_y \xrightarrow{\alpha_0\cdot} I_z$ is an epimorphism since the image is the costandard module ∇_z. Then for every $x \in I$ such that $z \xrightarrow{\alpha_0} y' \xrightarrow{\alpha_0} \cdots \xrightarrow{\alpha_0} y \xrightarrow{\alpha_0} x$ is a subquiver of the quiver of Γ with $z \in K$, an injective resolution of the costandard module ∇_x is given by

$$0 \to \nabla_x \to I_x \xrightarrow{\alpha_0\cdot} I_y \xrightarrow{\alpha_0\cdot} \cdots \xrightarrow{\alpha_0\cdot} I_{y'} \xrightarrow{\alpha_0\cdot} I_z \to 0.$$

This means that for any $x \in I$, either x is in K and $I_x = \nabla_x$ or there exists a short exact sequence

$$0 \to \nabla_x \to I_x \to \nabla_y \to 0$$

so that I_x is ∇-filtered.

The functor

$$F = \mathrm{Hom}_{\Gamma(Z)}(P_J^+, -) : \mathrm{mod}\,\Gamma(Z) \to \mathrm{mod}\,Z$$

is clearly full. To prove that it is faithful let P_x and P_z be two indecomposable projective Γ-modules and $\pi : P_x \to P_z$ a morphism between them; hence π is given by an equivalence class in Γ of a path from z to x. The image of π through F is $F\pi : \mathrm{Hom}_\Gamma(P_J^+, P_x) \to \mathrm{Hom}_\Gamma(P_J^+, P_z)$ and it is given by composing with π on the left. We consider two cases:

- If $x \in J$ then $id_{P_x} \in \mathrm{Hom}_\Gamma(P_J^+, P_x)$. Suppose $F\pi = 0$, then $F\pi(id_{P_x}) = \pi \cdot id_{P_x} = 0$ implies $\pi = 0$.

- If $x \in K$, suppose $\pi \neq 0$ and let π' be such that $\pi\pi'$ is a maximal path contained in a minimal cycle based at z ending in a vertex $y \in J$. Obviously in $\pi\pi'$ any arrow α_i can appear at most once because it is part of a minimal cycle. Moreover it does not contain any subpath $\alpha_0\alpha_1$: if this was the case, another α_0 would have to appear in $\pi\pi'$ for this path to end in J. Then $F\pi(\pi') = \pi\pi'$ is a non-zero morphism.

We have proved that $\pi \neq 0$ implies $F\pi \neq 0$ so the functor F is faithful and the proof is complete. $\square$

Example 3.1.7. Let $\Gamma = \Gamma(Z_3^2)$ be the quasi-hereditary cover of Z_3^2. We give here a slightly different presentation of Γ as path algebra with relations, in order to give a cleaner description of the structure of indecomposable standard modules, projective modules and their filtrations. To do that we label the vertices of the quiver with integers and we keep the same notation for the arrows:

$$\begin{tikzcd}
 & & & 7 \arrow[dr, "\alpha_2"] & & & \\
 & & 4 \arrow[ur, "\alpha_1"] \arrow[dr, "\alpha_2"] & & 8 \arrow[ll, "\alpha_0"'] \arrow[dr, "\alpha_2"] & & \\
 & 2 \arrow[ur, "\alpha_1"] \arrow[dr, "\alpha_2"] & & 5 \arrow[ll, "\alpha_0"'] \arrow[ur, "\alpha_1"] \arrow[dr, "\alpha_2"] & & 9 \arrow[ll, "\alpha_0"'] \arrow[dr, "\alpha_2"] & \\
1 \arrow[ur, "\alpha_1"] & & 3 \arrow[ll, "\alpha_0"'] \arrow[ur, "\alpha_1"] & & 6 \arrow[ll, "\alpha_0"'] \arrow[ur, "\alpha_1"] & & 0 \arrow[ll, "\alpha_0"']
\end{tikzcd}$$

The ideal of relations is generated by $\alpha_i\alpha_j - \alpha_j\alpha_i$ whenever the two compositions exist in the quiver, $\alpha_i\alpha_i$ and $e_z\alpha_0\alpha_1$ for any $i, j \in \{0, \ldots, d\}$ and $z \in K$ where $K = \{7, 8, 9, 0\}$.

Indecomposable projective modules have the following structure:

$$P_1 = \begin{smallmatrix} 1 \\ 2 \\ 3 \\ 1 \end{smallmatrix}, \quad P_2 = \begin{smallmatrix} & & 2 & \\ & 3 & & 4 \\ 1 & & 5 & \\ & 2 & & \end{smallmatrix}, \quad P_3 = \begin{smallmatrix} & 3 & & \\ 1 & & 5 & \\ & 2 & & 6 \\ & & 3 & \end{smallmatrix}, \quad P_4 = \begin{smallmatrix} & & 4 & \\ & 5 & & 7 \\ 2 & & 8 & \\ & 4 & & \end{smallmatrix}, \quad P_5 = \begin{smallmatrix} & 5 & \\ 2 & 8 & 6 \\ 4 & 9 & 3 \\ & 5 & \end{smallmatrix},$$

$$P_6 = \begin{smallmatrix} & 6 & & \\ 3 & & 9 & \\ & 5 & & 0 \\ & & 6 & \end{smallmatrix}, \quad P_7 = \begin{smallmatrix} 7 \\ 8 \\ 4 \end{smallmatrix}, \quad P_8 = \begin{smallmatrix} & 8 & \\ 4 & & 9 \\ & 5 & \end{smallmatrix}, \quad P_9 = \begin{smallmatrix} & 9 & \\ 5 & & 0 \\ & 6 & \end{smallmatrix}, \quad P_0 = \begin{smallmatrix} 0 \\ 6 \end{smallmatrix}.$$

Indecomposable standard modules are as follows:

$$\Delta_1 = S_1, \quad \Delta_2 = S_2, \quad \Delta_3 = \begin{smallmatrix} 3 \\ 1 \end{smallmatrix}, \quad \Delta_4 = S_4, \quad \Delta_5 = \begin{smallmatrix} 5 \\ 2 \end{smallmatrix},$$

$$\Delta_6 = \begin{smallmatrix} 6 \\ 3 \end{smallmatrix}, \quad \Delta_7 = S_7, \quad \Delta_8 = \begin{smallmatrix} 8 \\ 4 \end{smallmatrix}, \quad \Delta_9 = \begin{smallmatrix} 9 \\ 5 \end{smallmatrix}, \quad \Delta_0 = \begin{smallmatrix} 0 \\ 6 \end{smallmatrix}.$$

To conclude we show the filtration of some indecomposable projective modules by standard modules. We will show the filtrations of P_1, P_2, P_5 and P_8 since, by the symmetry of the structure of the projective modules, the other filtrations are very similar.

$$P_1 : 0 \subset \begin{smallmatrix} 3 \\ 1 \end{smallmatrix} \subset \begin{smallmatrix} 2 \\ 3 \\ 1 \end{smallmatrix} \subset \begin{smallmatrix} 1 \\ 2 \\ 3 \\ 1 \end{smallmatrix}, \quad P_2 : 0 \subset \begin{smallmatrix} 5 \\ 2 \end{smallmatrix} \subset \begin{smallmatrix} & 3 & \\ 1 & & 5 \\ & 2 & \end{smallmatrix} \subset \begin{smallmatrix} & 3 & & 4 \\ 1 & & 5 & \\ & 2 & & \end{smallmatrix} \subset \begin{smallmatrix} & & 2 & \\ & 3 & & 4 \\ 1 & & 5 & \\ & 2 & & \end{smallmatrix},$$

$$P_5 : 0 \subset \begin{smallmatrix} 9 \\ 5 \end{smallmatrix} \subset \begin{smallmatrix} & 8 & \\ 4 & & 9 \\ & 5 & \end{smallmatrix} \subset \begin{smallmatrix} & 8 & 6 \\ 4 & 9 & 3 \\ & 5 & \end{smallmatrix} \subset \begin{smallmatrix} & 5 & \\ 2 & 8 & 6 \\ 4 & 9 & 3 \\ & 5 & \end{smallmatrix}, \quad P_8 : 0 \subset \begin{smallmatrix} 9 \\ 5 \end{smallmatrix} \subset \begin{smallmatrix} & 8 & \\ 4 & & 9 \\ & 5 & \end{smallmatrix}.$$

Notation 3.1.8. Despite the non-uniqueness of quasi-hereditary covers, we will from now on refer to $\Gamma = \Gamma(Z)$ as the quasi-hereditary cover defined in this section.

Proposition 3.1.9. *Quasi-hereditary covers of higher zigzag-algebras have strong exact Borel subalgebras.*

Proof. Let Q be the quiver of Γ and Q' the subquiver of Q with the same set of vertices $Q'_0 = Q_0$ and arrows $Q'_1 = \{\alpha_i \in Q_1 \mid i \neq 0\}$. The path algebra of Q' bound by the ideal of relations $R' = \{\alpha_i\alpha_i \mid i \in \{0, \dots, d\}\} \cup \{\alpha_i\alpha_j = \alpha_j\alpha_i \mid i, j \in \{0, \dots, d\}$ and $\alpha_i\alpha_j \neq 0\}$ is a subalgebra of Γ and we will denote it by B. Since to obtain Q' from Q we removed exactly the arrows α_0, we can identify the vertices of the two quivers and label them with the same set I. Note that, by comparing their presentations, the algebras B and the quadratic dual of the higher Auslander algebra Λ^d_{s+1} of type A are isomorphic. The partial order on I that we gave before is still well defined in the quiver Q' since we have not removed any arrow α_i for $i \neq 0$. Then B has simple standard modules and, for every projective indecomposable B-module P, any composition series of P gives a Δ-filtration. This means that B is a quasi-hereditary algebra with simple standard modules. The restriction functor $\iota : \operatorname{mod}\Gamma \to \operatorname{mod} B$ sends costandard modules to costandard modules, since they are generated by paths not involving arrows α_0, and the indices are preserved. Then the condition of Theorem 1.2.17 is satisfied and B is a strong exact Borel subalgebra of Γ. □

3.1.1 Koszulity of quasi-hereditary covers

Note that quasi-hereditary covers of higher zigzag-algebras are quadratic and we can define an order on the arrows of the quiver of Γ such that $e_x\alpha_i < e_x\alpha_{i+1}$ for $i = 1, \dots, d-1$. Moreover we set $e_x\alpha_i < e_x\alpha_0$ for every $i \neq 0$.

Proposition 3.1.10. *If Z is a higher zigzag-algebra, then its quasi-hereditary cover Γ is a Koszul algebra.*

Proof. We want to show that Γ has a PBW basis, in order to use Theorem 1.3.11. Since we already have an order on the arrows, we need to show that we can extend this order lexicographically to paths of any length. Remember that if $Z = Z^d_s$, then Γ is a quotient of Z^d_{s+1}, so we can label the vertices of the underlying quiver by $x = (x_0, x_1, \dots, x_d)$ where $\sum_i x_i = s$. If we want to extend our order we have to prove that, for every $i < j$, if $e_x\alpha_j\alpha_i \neq 0$ then $e_x\alpha_j\alpha_i = e_x\alpha_i\alpha_j$. If this is the case then

$$\mathcal{B} = \{e_x\alpha_{i_1} \cdots \alpha_{i_s} \mid x \in Q_0, i_1 < i_2, \dots < i_s\}$$

is a PBW basis for Γ.

Now suppose we have $i < j$ and $e_x\alpha_j\alpha_i \neq 0$:

$$\begin{array}{c} x = (\dots, x_{i-1}, x_i, \dots, x_{j-1}, x_j, \dots) \\ \big\downarrow{\scriptstyle \alpha_j} \\ (\dots, x_{i-1}, x_i, \dots, x_{j-1} - 1, x_j + 1, \dots) \\ \big\downarrow{\scriptstyle \alpha_i} \\ (\dots, x_{i-1} - 1, x_i + 1, \dots, x_{j-1} - 1, x_j + 1, \dots) \end{array}$$

It is clear that the composition $e_x\alpha_i\alpha_j$ always exists unless $j = d$, $i = 0$ and $x_i = x_0 = 0$. But in this last situation we have that $x \in K$ and $e_x\alpha_j\alpha_i = e_x\alpha_0\alpha_1 = 0$ in Γ. □

It is known that standard Koszul algebras are Koszul in the classical sense; this follows from a characterization of standard Koszul algebras, that is Theorem 1.4 in [ÁDL03]. However, in the case of our quasi-hereditary covers, we have to prove that they are standard Koszul explicitly.

Theorem 3.1.11. *The quasi-hereditary algebra Γ is standard Koszul.*

Proof. Every standard module Δ_x is either simple or the extension of two simple modules

$$0 \to S_y\langle 1\rangle \to \Delta_x \to S_x \to 0$$

such that there exists an arrow $x \xrightarrow{\alpha_0} y$ in the quiver of Γ. Let $P^\bullet(x)$ and $P^\bullet(y)$ be linear projective resolutions of S_x and S_y respectively (their existence is provided by the Koszulity of Γ). Then in $\mathcal{D}^b(\Gamma)$ there is a triangle:

$$\Delta_x \to P^\bullet(x) \to P^\bullet(S_y\langle 1\rangle)[1] \xrightarrow{+}$$

Now consider the Koszul duality functor

$$\mathcal{K} : \mathcal{D}^b(\Gamma^!) \to \mathcal{D}^b(\Gamma)$$

and denote its quasi-inverse by $\mathcal{K}^{-1}$. Applying $\mathcal{K}^{-1}$ to the previous triangle we obtain a triangle in $\mathcal{D}^b(\Gamma^!)$:

$$C \to \mathcal{K}^{-1}(P^\bullet(x)) \to \mathcal{K}^{-1}(P^\bullet(S_y\langle 1\rangle)[1]) \xrightarrow{+}$$

where $\mathcal{K}(C) \cong \Delta_x$, $\mathcal{K}^{-1}(P^\bullet(x)) \cong \mathcal{K}^{-1}(S_x) \cong I_x$ and

$$\mathcal{K}^{-1}(P^\bullet(S_y\langle 1\rangle)[1]) \cong \mathcal{K}^{-1}(S_y\langle 1\rangle[1]) \cong \mathcal{K}^{-1}(\mathcal{K}(I_y)\langle 1\rangle[1]) \cong \mathcal{K}^{-1}\mathcal{K}(I_y\langle -1\rangle) \cong I_y\langle -1\rangle$$

where I_x and I_y are the injective envelopes of the simple $\Gamma^!$-modules T_x and T_y respectively.

We claim that the map $I_x \xrightarrow{f} I_y\langle -1\rangle$ is surjective. From this follows that C is quasi-isomorphic to $\operatorname{Ker} f$ and then $\Delta_x \in \mathsf{gr}\Lambda^\uparrow \cap \mathcal{K}(\mathsf{gr}(\Lambda^!)^\downarrow)$ is a linear module.

The map $I_x \xrightarrow{f} I_y\langle -1\rangle$ is surjective if and only if the corresponding dual map between left $\Gamma^!$-modules

$$\Gamma^! e_y\langle -1\rangle \to \Gamma^! e_x$$

is injective and this map is given by right multiplication by the arrow $x \xrightarrow{\alpha_0} y$. If we put $B = (\Gamma^!)^{op}$, we can equivalently show that the map given by left multiplication by α_0 between right projective B-modules is injective:

$$e_y B \xrightarrow{\alpha_0 \cdot} e_x B.$$

To prove our claim we need the following really useful construction that we recall from the proof of Theorem 3.5 in [HI11a]. Note that we will modify slightly the ideals of relations to adapt to our quasi-hereditary setting.

First of all note that, since $B = (\Gamma^!)^{op}$, with a little abuse of notation we can describe the quiver Q of B using the same notation as in Definition 2.2.4

$$Q_0 = \left\{ x = (x_0, x_1, \cdots, x_d) \in \mathbb{Z}_{\geq 0}^{d+1} \mid \sum_{i=0}^{d} x_i = s \right\}$$

and

$$Q_1 = \left\{ x \xrightarrow{\alpha_i} x + f_i \mid i \in \{0, \dots, d\}, x, x + f_i \in Q_0 \right\}$$

where $f_i = (0, \dots, \overset{i-1}{-1}, \overset{i}{1}, \dots, 0)$ and $f_0 = (1, \dots, -1)$. Remember moreover that we called K the subset of Q_0 consisting of vertices x such that $x_0 = 0$. Then $B = kQ/I$ where I is the ideal generated by the elements

$$e_x \alpha_i \alpha_j = \begin{cases} e_x \alpha_j \alpha_i & \text{if } x + f_j \in Q_0 \\ 0 & \text{if } x + f_j \notin Q_0 \text{ and } (i, j) \neq (0, 1) \\ e_x \alpha_i \alpha_j & \text{if } x + f_j \notin Q_0 \text{ and } (i, j) = (0, 1) \end{cases}$$

for any $x \in Q_0$ such that $x + f_i, x + f_i + f_j \in Q_0$. The elements $e_z \alpha_0 \alpha_1$, for $z \in K$ are non-zero in B (in contrast with the higher preprojective algebras described in [HI11a]) since in Γ we have $e_z \alpha_0 \alpha_1 = 0$ and $B = (\Gamma^!)^{op}$.

Define the quiver $\widehat{Q}$ by

$$\widehat{Q}_0 = \left\{ x = (x_0, \cdots, x_d) \in \mathbb{Z}^{d+1} \mid \sum_{i=0}^{d} x_i = s, x_0 \geq 0 \right\}$$

$$\widehat{Q}_1 = \left\{ x \xrightarrow{\alpha_i} x + f_i \mid i \in \{0, \dots, d\}, x + f_i \in \widehat{Q}_0 \right\}$$

and let $\widehat{I}$ be the ideal of $k\widehat{Q}$ defined by all the possible commutativity relations $\alpha_i \alpha_j = \alpha_j \alpha_i$. If we set $\widehat{B} = k\widehat{Q}/\widehat{I}$ then we have a surjective morphism of k-algebras $\pi : \widehat{B} \to B$ with kernel

$$R = \sum_{z \notin Q_0} \widehat{B} e_z \widehat{B}$$

and the residue classes of paths that are not in R are mapped bijectively to residue classes of paths in Q. For two paths in $\widehat{Q}$ p, p' from x to y, we will write $p \equiv p'$ if $p - p' \in \widehat{I}$.

We define a $\mathbb{Z}^{d+1}$-grading g on $\widehat{B}$ by $(g(\alpha_i))_j = \delta_{ij}$. This is a well-defined algebra grading on $\widehat{B}$ since $\widehat{I}$ is generated by homogeneous relations. Let p be a path from x to y, then $y - x = \sum_i b_i f_i$, where $b = (b_0, \cdots, b_d) = g(p)$ is the degree of p. In fact we can always write $p \equiv p_{x,b,y}$ where

$$p_{x,b,y} = x \xrightarrow{\alpha_0} \cdots \xrightarrow{\alpha_0} x + d_0 f_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{d-1}} y - b_d f_d \xrightarrow{\alpha_d} \cdots \xrightarrow{\alpha_d} y.$$

hence, in $\widehat{B}$, $p + \widehat{I}$ is determined by its degree b and either x or y. Moreover, for each path p' from x to z, we have $p \equiv p'q$ if and only if $g(p')_i \leq b_i$ for all $0 \leq i \leq d$ (take for example $q = p_{z,b-g(p'),y}$). Hence the residue class $p + \widehat{I}$ is in the ideal R if and only if $p \equiv p'q$ where p' is a path from x to z and $z \notin Q_0$ and this is equivalent to say that there exist an index $j \neq 1$ such that $x_j < d_{j+1}$ where we work modulo $d+1$ on the indices (equivalently $p + \widehat{I} \notin R$ if and only if $x_j \geq d_{j+1}$ for all $j \neq 1$).

Lemma 3.1.12. *Let $P_x = e_x B$ and $\pi \in \mathrm{soc}(P_x)$. Then π corresponds to a maximal path starting at x and ending at a vertex in K.*

Proof. Note that since $B \cong \widehat{B}/R$ is finite dimensional over k, for any $x \in Q_0$ there are (a finite number of) maximal paths in Q starting at x, up to relations. We will show that any non-zero path p from x to z with $z \notin K$ can be prolonged to a path ending in $z' \in K$. Let $x = (x_0, \cdots, x_d) \in I$, $z = (z_0, \cdots, z_d) \in J$ and p a path in $\widehat{Q}$ from x to z such that $p + \widehat{I} \notin R$. Then $z' = z + z_0 f_1 \in K$ so, if $q = e_z \alpha_1^{z_0}$ is the path from z to z' given by the arrows α_1 and $g(p) = b$, we have $b' = g(pq) = b + (0, z_0, \cdots, 0)$. Since $p + \widehat{I} \notin R$ we have $x_j \geq b_{j+1}$ for every $j \neq 1$ and this implies $x_j \geq b'_{j+1}$. So $pq + \widehat{I} \notin R$ is a non-zero path from x to $z' \in K$. ☐

Lemma 3.1.13. $\mathrm{soc}(P_x)$ *is simple for every $x \in I$.*

Proof. Let p and p' be two paths in $\widehat{Q}$ from x to z and z' respectively, such that $p + \widehat{I}, p' + \widehat{I} \notin R$. By the Lemma 3.1.12 we can suppose $z, z' \in K$. Let $g(p) = c$ and $g(p') = b$ so that we can write $z = x + \sum_i c_i f_i$ and $z' = x + \sum_i b_i f_i$. Since the full subquiver of Q that has K as set of vertices is directed, there exists a vertex $z'' \in K$ and two paths q, q' in K (hence not involving arrows α_0 and α_1) from z and z' respectively to z''. Since q and q' are paths in K we have that $pq + \widehat{I}, p'q' + \widehat{I} \notin R$ because $z_i \geq g(q)_{i+1}$ and $z'_i \geq g(q')_{i+1}$ for any $i \neq 0$. Hence they must coincide (up to equivalence) since $z'' - x = \sum_i g(pq)_i f_i = \sum_i g(p'q')_i f_i$. Hence $dim_k \, \mathrm{soc}(P_x) = 1$ and $\mathrm{soc}(P_x)$ is simple. ☐

Lemma 3.1.14. *Let $P_{x'} \xrightarrow{\alpha_0 \cdot} P_x$ be the irreducible morphism between indecomposable projective B-modules given by left multiplication by $x \xrightarrow{\alpha_0} x'$. Then $\alpha_0 \cdot (\mathrm{soc}(P_{x'})) \neq 0$ and $P_{x'} \xrightarrow{\alpha_0 \cdot} P_x$ is injective.*

Proof. Let p' be a path in $\widehat{Q}$ from x' to z such that $\pi(p' + \widehat{I}) = p' + I$ generates $\mathrm{soc}(P_{x'})$. Lemma 3.1.13 guarantees the existence and the uniqueness of such path. We want to show that $\alpha_0 p' + \widehat{I} \notin R$. If $g(p') = b'$, then $g(\alpha_0 p') = b' + (1, 0, \cdots, 0) = (b'_0 + 1, b'_1, \cdots, b'_d)$. Since $x' = x + f_0$, we have $x_i = x'_i$ for every $i \neq 0, d$; moreover $x'_d = x_d - 1$ therefore $x_d = x'_d + 1 \geq b'_0 + 1$ and we conclude that $\alpha_0 p' + \widehat{I} \notin R$. The morphism $P_{x'} \xrightarrow{\alpha_0 \cdot} P_x$ is injective since it doesn't annihilate the socle of $P_{x'}$. ☐

As a consequence of Lemma 3.1.14 we have that any irreducible morphism between indecomposable projective B-modules is a monomorphism and the proof of Theorem 3.1.11 is complete. ☐

3.2 Δ-Koszulity of quasi-hereditary covers

Motivated by Example 1.3.22, we want to show that similar results are true for our quasi-hereditary covers of higher zigzag-algebras (see also Theorem 4.1 and 4.4 of [Mad13]). We already know that for a higher zigzag-algebra Z, its quasi-hereditary cover Γ is Koszul and standard Koszul. Now we want to prove that Γ is Koszul with respect to Δ.

We define a new grading on Γ, that we will denote by $|\cdot|^\flat$, by setting

$$deg_\flat(e_x) = 0 \quad \forall x \in I, \quad deg_\flat(\alpha_k) = \begin{cases} 1 \text{ if } k \neq 0 \\ 0 \text{ if } k = 0 \end{cases}$$

The fact that this is a well defined algebra grading is assured by the fact that every non-monomial relation is of the form $\alpha_i\alpha_j = \alpha_j\alpha_i$. We will call this grading the $\flat$-grading (according to Subsection 1.3.3); we denote by $\Gamma_{[i]}$ the $\flat$-degree i part of Γ and by $\mathsf{gr}^\flat\Gamma$ the category of finitely generated graded Γ-modules. We have the following:

Proposition 3.2.1. *Let Γ be our quasi-hereditary cover of the higher zigzag-algebra Z.*

1. *Consider Γ with the ordinary grading. If $\mathrm{Ext}^u_{\mathsf{gr}\Gamma}(\Delta_y, S_x\langle v\rangle) \neq 0$ then $u = v = d(x, y)$.*
2. *According to the $\flat$-grading, $\Gamma_{[0]} \cong \Delta$ as graded Γ-modules.*
3. *If Γ is given the $\flat$-grading, then minimal resolutions of standard modules are linear with respect to the $\flat$-grading.*

Proof. 1. Suppose that $\mathrm{Ext}^u_{\mathsf{gr}\Gamma}(\Delta_y, S_x\langle v\rangle) \neq 0$. Since Γ is standard Koszul we have $u = v$. Consider a linear projective resolution of Δ_y:

$$\ldots \to P^u \to \ldots \to P^1 \to P^0 = P_y \to \Delta_y \to 0$$

where the indecomposable projective module P_x appears as a direct summand in P^u. By part (2) of Lemma 1.2.12 we have that $y < x$ and so, by the definition of the partial order on the set of weights of Γ, there exists a path π from y to x that involves only arrows α_k for $k \neq 0$. The length of this path π is equal to $d(x, y)$ and so we have $d(x, y) \leq u$. On the other hand by part (3) of Lemma 1.2.12 we deduce that $u \leq d(x, y)$ and this proves the claim.

2. This follows from the definition of $\flat$-grading.

3. By what has been proved in part (1), if $P^u \to P^v$ is a map in a linear projective resolution of a standard module Δ_x, then the image of generators of P^u in P^v are linear combinations of elements of the form $e_a\alpha_k e_b$ with $k \neq 0$, since $d(a, b) = 1$. Then the resolution is also linear with respect to the $\flat$-grading.

$\square$

From the above results we have the following theorem:

Theorem 3.2.2. *Consider Γ as a graded algebra according to the $\flat$-grading. Then Γ is Koszul with respect to Δ.*

Proof. The algebra $\Gamma_{[0]}$ can be decomposed in subalgebras each of which is isomorphic to a type A algebra with underlying quiver:

$$x_1 \xrightarrow{\alpha_0} x_2 \xrightarrow{\alpha_0} \cdots \xrightarrow{\alpha_0} x_k$$

bound by relations $\alpha_0\alpha_0 = 0$, for $1 \leq k \leq s+1$. These algebras have all finite global dimension, hence $\Gamma_{[0]}$ has finite global dimension as well. Δ is a tilting $\Gamma_{[0]}$-module by part (2) of Proposition 3.2.1. Now let P^i be a projective module in a minimal graded projective resolution of Δ; then P^i is generated in degree i and since $\Delta\langle j\rangle$ is concentrated in degree j we have that $\mathrm{Hom}_{\mathsf{gr}^\flat\Gamma}(P^i, \Delta\langle j\rangle) = 0$ if $i \neq j$. Hence $\mathrm{Ext}^i_{\mathsf{gr}^\flat\Gamma}(\Delta, \Delta\langle j\rangle) = 0$ whenever $i \neq j$. □

3.3 Δ-Koszul duality

In this last section we want to study the Δ-Koszul dual $\Gamma^\dagger$ when Γ is the quasi-hereditary cover that we defined for a higher zigzag-algebra. First we want to show that $\Gamma^\dagger$ is a bigraded algebra and that it is Koszul when endowed with the total grading $|\cdot|^{tot}$. This is true when we consider the quasi-hereditary cover of the Brauer line by [Mad13, Theorem 4.4] and the proof is based on the existence of a particular *height function* on the set of vertices of the quiver of Λ (see [Mad13]). It is then reasonable to try to generalize this result for higher zigzag-algebras (of type A). To conclude we compute the quiver of the Δ-dual algebra and, using the fact that Koszulity implies quadraticity, we determine its ideal of relations.

3.3.1 Δ-Koszul dual of quasi-hereditary covers

Let us describe the bigraded structure that we will consider on Γ and on its Δ-dual $\Gamma^\dagger$. Recall that we have already defined the $\flat$-grading on Γ and we denoted by $\Gamma_{[0]}$ the degree zero part with respect to this grading. We can define another grading $|\cdot|^\sharp$ on Γ such that the total grading corresponds to the radical grading:

$$|e_x|^\sharp = 0 \quad \forall x \in I, \quad |\alpha_k|^\sharp = \begin{cases} 0 \text{ if } k \neq 0 \\ 1 \text{ if } k = 0. \end{cases}$$

When considering the dual algebra $\Gamma^\dagger$, we will denote the Ext-grading by $|\cdot|^\flat$ (since it is induced by the $\flat$-grading) and the grading induced by $|\cdot|^\sharp$ always by $|\cdot|^\sharp$. For every bigraded Γ-module (or $\Gamma^\dagger$ in the same way) M, we will denote by $M\langle i, j\rangle$ the bigraded module obtained by shifting M of i with respect to $|\cdot|^\flat$ and of j with respect to $|\cdot|^\sharp$. Then we will denote by $|\cdot|^{tot}$ the total grading on Γ (and on $\Gamma^\dagger$ similarly).

Let $\Gamma^\dagger = \mathrm{Ext}^*_\Gamma(\Delta, \Delta)$ so, by Theorem 1.3.19, $\Gamma^\dagger$ is Koszul with respect to $D\Delta$ and $(\Gamma^\dagger, D\Delta)$ is the Koszul dual of (Γ, Δ). The $|\cdot|^\flat$-degree zero part of $\Gamma^\dagger$ is $\mathrm{End}_\Gamma(\Delta) \simeq \mathrm{End}_{\Gamma_{[0]}}(\Delta) \simeq \Delta_{\Gamma^\dagger}$ considered as a right $\Gamma^\dagger$-module. We have the following corollary of Theorem 3.2.2, Theorem 1.3.19 and Theorem 1.3.20:

Corollary 3.3.1. *There is an isomorphism*

$$\Gamma \cong \mathrm{Ext}^*_{\Gamma^\dagger}(D\Delta, D\Delta)$$

as ungraded algebras. Moreover, if Γ is given the $\flat$-grading and $\Gamma^\dagger$ the Ext*-grading, then there is an equivalence of triangulated categories $G_\Delta = \mathbb{R}\,\mathrm{Hom}_{\mathsf{gr}\Lambda}(\Delta, -) : \mathcal{D}^b(\mathsf{gr}^\flat\Gamma) \to \mathcal{D}^b(\mathsf{gr}^\flat\Gamma^\dagger)$ which restricts to an equivalence $G_\Delta : \mathcal{F}_{\mathsf{gr}^\flat\Gamma}(\Delta) \to \mathcal{L}^b(\Gamma^\dagger)$.*

Proof. Since, by Theorem 3.2.2, Γ with the $\flat$-grading is Koszul with respect to Δ, the isomorphism follows from Theorem 1.3.19. By Theorem 1.3.20 the functor $G_\Delta : \mathcal{D}(\mathsf{gr}^\flat\Gamma) \to \mathcal{D}(\mathsf{gr}^\flat\Gamma^\dagger)$ restricts to an equivalence $G_\Delta : \mathcal{F}_{\mathsf{gr}^\flat\Gamma}(\Delta) \to \mathcal{L}^b(\Gamma^\dagger)$. Moreover, since Γ has finite global dimension, $\Gamma^\dagger$ is finite dimensional and it is directed since the extension algebra of standard modules is always directed [Par98, Theorem 1.8(b)]. Then $\Gamma^\dagger$, being directed, has finite global dimension and, again by Theorem 1.3.20, we also have an equivalence between the bounded derived categories $G_\Delta : \mathcal{D}^b(\mathsf{gr}^\flat\Gamma) \to \mathcal{D}^b(\mathsf{gr}^\flat\Gamma^\dagger)$. Since obviously $\Delta \in \mathcal{D}^b(\mathsf{gr}^\flat\Gamma)$, the category $\mathcal{F}_{\mathsf{gr}^\flat\Gamma}(\Delta)$ is a subcategory of $\mathcal{D}^b(\mathsf{gr}^\flat\Gamma)$. Therefore the restriction of $G_\Delta : \mathcal{D}^b(\mathsf{gr}^\flat\Gamma) \to \mathcal{D}^b(\mathsf{gr}^\flat\Gamma^\dagger)$ to $\mathcal{F}_{\mathsf{gr}^\flat\Gamma}(\Delta)$ gives the desired equivalence $\mathcal{F}_{\mathsf{gr}^\flat\Gamma}(\Delta) \simeq \mathcal{L}^b(\Gamma^\dagger)$. □

The following lemma gives a useful description of the graded parts of $\Gamma^\dagger = \mathrm{Ext}^*_\Gamma(\Delta, \Delta)$ when Γ is the quasi-hereditary cover of a higher zigzag algebra.

Lemma 3.3.2. *Let x, y be two vertices in the quiver of Γ and $b = d(x, y)$ their distance in the quiver. If $\mathrm{Ext}^i_{\mathsf{gr}^\sharp\Gamma}(\Delta_x, \Delta_y\langle j\rangle^\sharp) \neq 0$ for some $i, j \geq 0$, then $i = b - dj$. As a consequence we have that*

$$|\,\mathrm{Ext}^i_{\mathsf{gr}^\sharp\Gamma}(\Delta_x, \Delta_y\langle j\rangle^\sharp)|^{tot} = b - j(d-1).$$

Proof. We proceed by induction on $b = d(x, y)$. Recall that $d(x, y)$ is the length of a path from x to y not involving any arrow α_0, if such a path exists, and it is ∞ otherwise. The distance between two vertices is infinite precisely when they are not comparable in the partial order on the set of vertices. But this can not happen under our assumptions since, by Lemma 1.2.12(2), $\mathrm{Ext}^i_\Gamma(\Delta_x, \Delta_y) \neq 0$ implies $x < y$, for any $i > 0$.

Note first that if $b = 0$ then $x = y$ and $\mathrm{Ext}^*_\Gamma(\Delta_x, \Delta_x) \cong \mathrm{Hom}_\Gamma(\Delta_x, \Delta_x)$ by quasi-heredity.

Suppose $b = 1$: by Lemma 1.2.12, if $x > y$ then $\mathrm{Ext}^i_\Gamma(\Delta_x, \Delta_y) = 0$, so we can assume $x < y$. In this case $\mathrm{Ext}^i_\Gamma(\Delta_x, \Delta_y) \neq 0$ only if $i \leq b = 1$. We must have $i \neq 0$ since $b = 1$

implies that there exists an arrow $x \xrightarrow{\alpha_k} y$ with $k \neq 0$. So, since $d > 1$, there can not be an arrow $y \xrightarrow{\alpha_0} x$ and $\mathrm{Hom}_\Gamma(\Delta_x, \Delta_y) = 0$. Therefore $j = 0$ and $i = b = 1$.

Assume now $b > 1$ and, for any vertex y' such that $b' = d(x, y') < b$, if $\mathrm{Ext}^i_{\mathsf{gr}^\sharp\Gamma}(\Delta_x, \Delta_{y'}\langle j\rangle^\sharp) \neq 0$ then $i = b' - dj$. Let $P^\bullet(x)$ be a projective resolution of Δ_x, linear with respect to the $\flat$-grading on Γ, and consider a morphism $f : P^i(x) \to \Delta_y$ that gives a non-zero homogeneous element in $\mathrm{Ext}^i_{\mathsf{gr}^\sharp\Gamma}(\Delta_x, \Delta_y\langle j\rangle^\sharp)$. If $(S_y, S_{y'})$ are the composition factors of Δ_y then $b' = d(x, y') = b - d$. We distinguish two cases:

- If f is epi, then P_y is a direct summand of $P^i(x)$ and so $i = b$ and $j = 0$.
- If f is not epi then it factors through the morphism $\Delta_{y'} \to \Delta_y$ and we have the following commutative diagram:

$$\begin{tikzcd} P^i(x) \arrow[d, "g"'] \arrow[dr, "f"] & \\ \Delta_{y'} \arrow[r] & \Delta_y \end{tikzcd}$$

where g is non-zero and epi, hence it belongs to $\mathrm{Ext}^i_{\mathsf{gr}^\sharp\Gamma}(\Delta_x, \Delta_{y'})$. Since $b' = b - d < b$, by induction we have that $g \in \mathrm{Ext}^i_{\mathsf{gr}^\sharp\Gamma}(\Delta_x, \Delta_{y'}\langle k\rangle^\sharp)$ for $k \geq 0$ such that $i = b' - dk$. Therefore we have

$$f \in \mathrm{Hom}_{\mathsf{gr}^\sharp\Gamma}(\Delta_{y'}, \Delta_y\langle 1\rangle^\sharp) \cdot \mathrm{Ext}^{b'-dk}_{\mathsf{gr}^\sharp\Gamma}(\Delta_x, \Delta_{y'}\langle k\rangle^\sharp) \subseteq \mathrm{Ext}^{b'-dk}_{\mathsf{gr}^\sharp\Gamma}(\Delta_x, \Delta_y\langle k+1\rangle^\sharp)$$

and we know that

$$\mathrm{Ext}^{b'-dk}_{\mathsf{gr}^\sharp\Gamma}(\Delta_x, \Delta_y\langle k+1\rangle^\sharp) = \mathrm{Ext}^i_{\mathsf{gr}^\sharp\Gamma}(\Delta_x, \Delta_y\langle j\rangle^\sharp),$$

so $i = b' - dk = b - d - dk = b - d(k+1)$.

$\square$

Denote by $\mathsf{tgr}\Gamma^\dagger$ the category of graded $\Gamma^\dagger$-modules with respect to the total grading.

Lemma 3.3.3. *If $Q_x\langle s\rangle \to Q_y$ is a non-zero morphism between indecomposable projective modules in $\mathsf{tgr}\Gamma^\dagger$, then $s = b - j(d-1)$ for some $j \geq 0$ and $b = d(x, y)$.*

Proof. Any non-zero morphism $Q_x\langle s\rangle \to Q_y$ is given by left multiplication by an element in $\mathrm{Ext}^i_{\mathsf{gr}^\sharp\Gamma}(\Delta_x, \Delta_y\langle j\rangle^\sharp) \subseteq \mathrm{Ext}^*_\Gamma(\Delta_x, \Delta_y)$ whose total grading is $b - j(d-1)$ by Lemma 3.3.2. Hence $s = b - j(d-1)$. $\square$

We can now prove the following:

Theorem 3.3.4. *Let Γ be our quasi-hereditary cover of an n-zigzag algebra with $n > 1$, and let $\Gamma^\dagger = \mathrm{Ext}^*_\Gamma(\Delta, \Delta)$. Then $\Gamma^\dagger$ endowed with the total grading $|\cdot|^{tot}$ is Koszul in the classical sense.*

Proof. We want to show that, for any two simple $\Gamma^\dagger$-modules S_x, S_y, if $\mathrm{Ext}^s_{\mathsf{tgr}\Gamma^\dagger}(S_x, S_y\langle i,j\rangle) \neq 0$ then $i+j=s$. First recall that $S_x \cong G_\Delta(\nabla_x)$ by Proposition 1.3.24. Moreover

$$S_x\langle i,j\rangle \cong G_\Delta(\nabla_x)\langle i,j\rangle \cong G_\Delta(\nabla_x\langle -i,j\rangle[-i])$$

by Proposition 1.3.24 and Proposition 1.3.21, (c). Then we have:

$$\begin{aligned}\mathrm{Ext}^s_{\mathsf{tgr}\Gamma^\dagger}(S_x, S_y\langle i,j\rangle) &\cong \mathrm{Hom}_{\mathcal{D}(\mathsf{tgr}\Gamma^\dagger)}(S_x, S_y\langle i,j\rangle[s])\\ &\cong \mathrm{Hom}_{\mathcal{D}(\mathsf{tgr}\Gamma)}(\nabla_x, \nabla_y\langle -i,j\rangle[s-i])\\ &\cong \mathrm{Ext}^{s-i}_{\mathsf{tgr}\Gamma}(\nabla_x, \nabla_y\langle -i,j\rangle).\end{aligned}$$

Recall also that from Proposition 3.1.6, an (ungraded) injective coresolution of ∇_y is:

$$0 \to \nabla_y \to I_y \to I_{y_1} \to \cdots \to I_{y_k} = \nabla_z \to 0$$

such that

$$z \xrightarrow{\alpha_0} \cdots \xrightarrow{\alpha_0} y_1 \xrightarrow{\alpha_0} y$$

is a subquiver of the quiver of Γ.

Each (indecomposable) injective module in such a coresolution is cogenerated in $|\cdot|^\flat$-degree zero since applying the duality $D = \mathrm{Hom}_k(-,k)$ the corresponding maps between projective Γ^{op}-modules are given by right multiplication by α_0^* that are in degree zero. Moreover, for the same reason, we see that the coresolution is also linear with respect to $|\cdot|^\sharp$:

$$0 \to \nabla_y\langle -i,j\rangle \to I_y\langle -i,j\rangle \to I_{y_1}\langle -i,j-1\rangle \to \cdots \to I_{y_k}\langle -i,j-k\rangle = \nabla_z\langle -i,j-k\rangle \to 0.$$

Hence if $\nabla_x \to I_{y_{s-i}}\langle -i, j-s+i\rangle$ is a non-zero map that gives a non-trivial element in the Ext-group, we must have $j = s-i$ since ∇_x is concentrated in $|\cdot|^\sharp$-degree zero. □

3.3.2 Presentation of $\Gamma^\dagger$ as bound quiver algebra

Let $(Q_0^{\Gamma^\dagger}, Q_1^{\Gamma^\dagger})$ be the quiver of $\Gamma^\dagger$. The set of vertices $Q_0^{\Gamma^\dagger}$ is in bijection with the set of vertices of the quiver of Γ since standard Γ-modules correspond bijectively to simple $\Gamma^\dagger$-modules via the functor G_Δ; we will index this set always by I. By Theorem 3.3.4 $\Gamma^\dagger$ is Koszul with respect to the total grading so $\Gamma^\dagger$ is generated by elements of degree one over its semisimple subalgebra in degree zero. Then the arrows of the quiver of $\Gamma^\dagger$ can be divided in two spaces:

1. Arrows in Ext-degree zero: they correspond to the generators of $\mathrm{Hom}_\Gamma(\Delta_x, \Delta_y)$ whenever we have an arrow $y \xrightarrow{\alpha_0} x$ in the quiver of Γ. This Hom-space is clearly one dimensional and gives us an arrow $a_0 : y \to x$.

2. Arrows in Ext-degree one: they correspond to the generators of $\operatorname{Ext}^1_\Gamma(\Delta_x, \Delta_y)$ that do not factor through a morphism as in point (1). Suppose that $\operatorname{Ext}^1_\Gamma(\Delta_x, \Delta_y) \neq 0$ and let $P^\bullet \to \Delta_x$ be a (graded linear) projective resolution of Δ_x (note that $P^0 = P_x$). The first Ext-space is the first cohomology group of the complex $\operatorname{Hom}_\Gamma(P^\bullet, \Delta_y)$ so its elements are equivalence classes of morphisms $P^1 \to \Delta_y$. But since standard modules are concentrated in only one Δ-degree and the differentials of $P^\bullet$ are in Δ-degree one, the cohomology classes correspond to the Hom-spaces. By Lemma 3.1.4 the standard module Δ_y is either the simple module S_y or, in case there exists an arrow $y \xrightarrow{\alpha_0} z$, it is uniserial with radical length two and composition factors $S_y (= \operatorname{top} \Delta_y)$ and $S_z (= \operatorname{soc} \Delta_y)$. If $\operatorname{Im}(P^1 \to \Delta_y) = S_z$ then the morphism factors through Δ_z hence is not irreducible; we have a (unique up to scalar multiplication) irreducible morphism $f : P^1 \to \Delta_y$ if and only if $P^1 = P' \oplus P_y$ and $f = 0 \oplus (P_y \twoheadrightarrow \Delta_y)$. Therefore we have an arrow in Ext-degree one $a_i : y \to x \in Q_1^{\Gamma^\dagger}$ whenever there is an arrow $\alpha_i : x \to y \in Q_1^\Gamma$ for $i \neq 0$, since this happens if and only if the linear projective resolution of Δ_x is the following:

$$\cdots \to P^2 \to P^1 = P' \oplus P_y \xrightarrow{[g, \alpha_i \cdot]} P_x \twoheadrightarrow \Delta_x$$

for some g in degree one.

This means that $\operatorname{Ext}^1_\Gamma(\Delta_x, \Delta_y)$ decomposes, as a k-vector space, in the direct sum of two at most one-dimensional vector spaces $\mathcal{U} \oplus \mathcal{V}$ where $\mathcal{U} = \operatorname{Ext}^1_\Gamma(\Delta_x, \Delta_y)_0$ is generated by an irreducible morphism and $\mathcal{V} = \operatorname{Ext}^1_\Gamma(\Delta_x, \Delta_y)_1$ is generated by a morphism that factors through a morphism between standard modules.

Example 3.3.5. Let $Z = Z_3^2$ and remember the quiver of Γ from Example 3.1.5. Then the quiver of $\Gamma^\dagger = \operatorname{Ext}^*_\Gamma(\Delta, \Delta)$ is obtained by the quiver of Γ by keeping the same arrows in Δ-degree zero and reversing the arrows in Δ-degree one:

$$\begin{tikzcd}
 & & & 030 \arrow[dl, "a_1"'] & & & \\
 & & 120 \arrow[dl, "a_1"'] & & 021 \arrow[ll, "a_0"'] \arrow[ul, "a_2"'] \arrow[dl, "a_1"'] & & \\
 & 210 \arrow[dl, "a_1"'] & & 111 \arrow[ll, "a_0"'] \arrow[ul, "a_2"'] \arrow[dl, "a_1"'] & & 012 \arrow[ll, "a_0"'] \arrow[ul, "a_2"'] \arrow[dl, "a_1"'] & \\
300 & & 201 \arrow[ll, "a_0"'] \arrow[ul, "a_2"'] & & 102 \arrow[ll, "a_0"'] \arrow[ul, "a_2"'] & & 003 \arrow[ll, "a_0"'] \arrow[ul, "a_2"']
\end{tikzcd}$$

By Theorem 3.3.4, the ideal of relations of $\Gamma^\dagger$ is generated by homogeneous elements of degree two (with respect to the total grading). Then we need to find all the degree two relations of $\Gamma^\dagger$.

Proposition 3.3.6. *Let $(Q_0^{\Gamma^\dagger}, Q_1^{\Gamma^\dagger})$ be the quiver of $\Gamma^\dagger$.*

(I) There are quadratic commutativity relations given by:

(a) *For any relation* $\alpha_i\alpha_j = \alpha_j\alpha_i$ *with* $i, j \neq 0$ *in* Γ: $\begin{tikzcd}[row sep=small] & y \arrow[dr, "\alpha_j"] & \\ x \arrow[ur, "\alpha_i"] \arrow[dr, "\alpha_j"'] \arrow[rr, dash, dotted] & & z \\ & w \arrow[ur, "\alpha_i"'] & \end{tikzcd}$, *in* $\Gamma^\dagger$ *we have the relation* $a_ia_j = a_ja_i$: $\begin{tikzcd}[row sep=small] & y \arrow[dl, "a_i"'] & \\ x \arrow[rr, dash, dotted] & & z \arrow[ul, "a_j"'] \arrow[dl, "a_i"] \\ & w \arrow[ul, "a_j"] & \end{tikzcd}$.

(b) *For any relation* $\alpha_0\alpha_i = \alpha_i\alpha_0$ *in* Γ: $\begin{tikzcd}[row sep=small] & y \arrow[dr, "\alpha_i"] & \\ x \arrow[ur, "\alpha_0"] \arrow[dr, "\alpha_i"'] \arrow[rr, dash, dotted] & & z \\ & w \arrow[ur, "\alpha_0"'] & \end{tikzcd}$, *in* $\Gamma^\dagger$ *we have the relation* $a_ia_0 = a_0a_i$: $\begin{tikzcd}[row sep=small] & y \arrow[dd, dash, dotted] & \\ x \arrow[ur, "a_0"] & & z \arrow[ul, "a_i"'] \\ & w \arrow[ul, "a_i"] \arrow[ur, "a_0"'] & \end{tikzcd}$.

(II) *There are quadratic monomial relations given by:*

(a) $a_0a_0 = 0$

(b) $a_ia_j = 0$ *for any* $i, j \neq 0$ *such that* $\alpha_i\alpha_j$ *is not defined in the quiver of* Γ.

Proof. For any $v \in I$ denote by $P^\bullet(v)$ a projective resolution of the standard module Δ_v.

(I) (a) Consider the first three terms of a linear projective resolution of Δ_x

$$\cdots \to P^2(x) \to \bigoplus_{x \xrightarrow{\alpha_i} x', i\neq 0} P_{x'} = P^1(x) \xrightarrow{f} P_x$$

In particular P_y and P_w are direct summands of $P^1(x)$ and the restriction of f on these modules is $P_y \oplus P_w \xrightarrow{[\alpha_i\cdot, \alpha_j\cdot]} P_x$. The element $e_y\alpha_je_z - e_w\alpha_ie_z$ is such that $f(e_y\alpha_je_z - e_w\alpha_ie_z) = e_x\alpha_i\alpha_je_z - e_x\alpha_j\alpha_ie_z = 0$ hence it lies in the image of $P^2(x) \to P_y \oplus P_w$. But this is a linear map between projective modules, so we must have that $e_y\alpha_je_z - e_w\alpha_ie_z$ is in the image of $P_z \xrightarrow{[\alpha_j\cdot, \alpha_i\cdot]} P_y \oplus P_w$ and then P_z must be a direct summand of $P^2(x)$. In particular $a_ia_j \in \mathrm{Ext}^2_\Gamma(\Delta_x, \Delta_z)$ is non-zero. Moreover a_ia_j and a_ja_i are respectively represented by the following diagrams:

$$\begin{tikzcd}
\cdots \arrow[r] & P_z \oplus Q_2 \arrow[r, "{\begin{bmatrix} \alpha_j\cdot & * \\ \alpha_i\cdot & * \\ * & * \end{bmatrix}}"] \arrow[d, "\mathrm{id}"] & P_y \oplus P_w \oplus Q_1 \arrow[r, "{[\alpha_i\cdot \ \alpha_j\cdot \ *]}"] \arrow[d, "{[1\ 0\ 0]}"] & P_x \arrow[r, "0"] \arrow[d, "0"] & 0 \\
\cdots \arrow[r] & P_z \oplus Q_2' \arrow[r, "{[\alpha_j\cdot \ *]}"] \arrow[d, "{[\mathrm{id}\ 0]}"] & P_y \arrow[r, "0"] \arrow[d, "0"] & 0 & \\
\cdots \arrow[r] & P_z \arrow[r, "0"] & 0 & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\cdots \arrow[r] & P_z \oplus Q_2 \arrow[r, "{\left[\begin{smallmatrix} \alpha_j\cdot & * \\ \alpha_i\cdot & * \\ * & * \end{smallmatrix}\right]}"] \arrow[d, "\mathrm{id}"] & P_y \oplus P_w \oplus Q_1 \arrow[r, "{[\alpha_i\cdot \ \alpha_j\cdot \ *]}"] \arrow[d, "{[0\ 1\ 0]}"] & P_x \arrow[r, "0"] \arrow[d, "0"] & 0 \\
\cdots \arrow[r] & P_z \oplus Q_2'' \arrow[r, "{[\alpha_i\cdot \ *]}"] \arrow[d, "{[\mathrm{id}\ 0]}"] & P_w \arrow[r, "0"] \arrow[d, "0"] & 0 & \\
\cdots \arrow[r] & P_z \arrow[r, "0"] & 0 & &
\end{tikzcd}$$

where Q_1, Q_2, Q_2' and Q_2'' are projective Γ-modules such that $P_y \oplus P_w \oplus Q_1 = P^1(x), P_z \oplus Q_2 = P^2(x), P_z \oplus Q_2' = P^1(y)$ and $P_z \oplus Q_2'' = P^1(w)$. Therefore we see that $a_i a_j = a_j a_i \in \operatorname{Ext}^2_\Gamma(\Delta_x, \Delta_z)$.

(b) An element $a_i a_0 \in \operatorname{Ext}^1_\Gamma(\Delta_x, \Delta_w) \operatorname{Hom}_\Gamma(\Delta_y, \Delta_x)$, $i \neq 0$, is given by a diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & P^2(y) \arrow[r] \arrow[d] & P_z \oplus Q \arrow[r, "{[\alpha_i\cdot \ *]}"] \arrow[d, "{\left[\begin{smallmatrix} \alpha_0\cdot & 0 \\ * & * \end{smallmatrix}\right]}"] & P_y \arrow[r, "0"] \arrow[d, "\alpha_0\cdot"] & 0 \\
\cdots \arrow[r] & P^2(x) \arrow[r] \arrow[d] & P_w \oplus Q' \arrow[r, "{[\alpha_i\cdot \ *]}"] \arrow[d, "{[\mathrm{id}\ 0]}"] & P_x \arrow[r, "0"] \arrow[d, "0"] & 0 \\
\cdots \arrow[r] & P^2(w) \arrow[r] & P_w \arrow[r, "0"] & 0 &
\end{tikzcd}$$

where Q and Q' are projective modules such that $P_z \oplus Q = P^1(y)$ and $P_w \oplus Q' = P^1(x)$. Note that the map $P_z \oplus Q \xrightarrow{\left[\begin{smallmatrix} \alpha_0\cdot & 0 \\ * & * \end{smallmatrix}\right]} P_w \oplus Q'$ has to be in Δ-degree zero. Therefore the top-right entry of the corresponding matrix is zero since $P_z \to P_w$ is the unique Δ-degree zero map with codomain P_w.

On the other hand the element $a_0 a_i \in \operatorname{Hom}_\Gamma(\Delta_z, \Delta_w) \operatorname{Ext}^1_\Gamma(\Delta_y, \Delta_z)$ is given by:

$$\begin{tikzcd}
\cdots \arrow[r] & P_z \oplus Q \arrow[r, "{[\alpha_i\cdot \ *]}"] \arrow[d, "{[\mathrm{id}\ 0]}"] & P_y \arrow[r, "0"] \arrow[d, "0"] & 0 \\
\cdots \arrow[r] & P_z \arrow[r, "0"] \arrow[d, "\alpha_0\cdot"] & 0 \arrow[d, "0"] & \\
\cdots \arrow[r] & P_w \arrow[r, "0"] & 0 &
\end{tikzcd}$$

Hence we can conclude that $a_i a_0 = a_0 a_i$ since $[\mathrm{id}\ 0]\left[\begin{smallmatrix} \alpha_0\cdot & 0 \\ * & * \end{smallmatrix}\right] = [\alpha_0\cdot\ 0] = \alpha_0 [\mathrm{id}\ 0]$.

(II) (a) Obvious, since by the structure of standard modules we have $\operatorname{Hom}_\Gamma(\Delta_y, \Delta_z) \operatorname{Hom}_\Gamma(\Delta_x, \Delta_y) = 0$ for any $x, y, z \in I$.

(b) Assume $i, j \neq 0$, so that $a_i a_j$ comes from some non-zero composition $x \xrightarrow{\alpha_j} y \xrightarrow{\alpha_i} z$ in the quiver of Γ and we have no paths $x \overset{\alpha_i\alpha_j}{\dashrightarrow} z$. Let $\cdots \to P^2(z) \to P^1(z) \to P_z \twoheadrightarrow \Delta_z$ be a graded linear projective resolution of Δ_z; under our assumptions P_x can not appear as a direct summand of $P^2(z)$ since the composition $P^2(z) \to P^1(z) \to P_z$ restricted to P_x must coincide with $\alpha_j\alpha_i \cdot$ and so it would be non-zero. This means that $\mathrm{Ext}^2_\Gamma(\Delta_z, \Delta_x) \cong \mathrm{Hom}_\Gamma(P^2(z), \Delta_x) = 0$.

□

Theorem 3.3.7. *The algebra $\Gamma^\dagger$ is isomorphic to the path algebra of the following quiver:*

$$Q_0^{\Gamma^\dagger} = Q_0^\Gamma$$

$$Q_1^{\Gamma^\dagger} = \left\{ x \xrightarrow{a_0} y : \text{ there exists } x \xrightarrow{\alpha_0} y \in Q_1^\Gamma \right\} \cup \left\{ w \xrightarrow{a_i} z : \text{ there exists } z \xrightarrow{\alpha_i} w \in Q_1^\Gamma, i \neq 0 \right\}$$

bound by the ideal of relations $\mathcal{R}$ generated by elements as in Proposition 3.3.6.

Proof. We are left to prove that the relations of Proposition 3.3.6 are the only quadratic relations of $\Gamma^\dagger$. First of all note that, by the description of the quiver Q^Γ given in Definition 2.2.4 and after reversing the arrows α_i with $i \neq 0$, given any two vertices $x, z \in Q_0^{\Gamma^\dagger}$ the k-basis of the vector space $z(kQ^{\Gamma^\dagger})_2 x$ is either (1) the element $a_i a_i$ with $i \in \{0, \ldots, n\}$, or (2) the set $\{a_i a_j, a_j a_i\}$ or (3) the element $a_i a_j$ with $i, j \in \{0, \ldots, n\}$, $i \neq j$, such that the composition $a_j a_i$ is not defined in the quiver. Case (2) occurs precisely when the path $a_i a_j$ is part of a mesh
$$\begin{tikzcd} & y \arrow[dl, "a_i"'] & \\ x \arrow[rr, dotted, no head] & & z \arrow[ul, "a_j"'] \arrow[dl, "a_i"] \\ & w \arrow[ul, "a_j"] & \end{tikzcd}$$
for $i, j \in \{0, 1, \cdots, n\}$, as described in Proposition 3.3.6. Let us discuss the possible relations in these three cases:

(1) The elements $a_0 a_0$ are zero in $\Gamma^\dagger$ by Proposition 3.3.6 (II)(a). The elements $a_i a_i$ for $i \neq 0$ are non-zero since the same argument used in Proposition 3.3.6 (I)(a) for $i = j$ shows an explicit extension.

(2) Any element $a_i a_j$ that is part of a square is subject to the commutativity relation $a_i a_j = a_j a_i$ by Proposition 3.3.6 (I)(a,b). Moreover such an element is non-zero since, as before, an explicit extension is given in the proof of Proposition 3.3.6.

(3) Let $a_i a_j$ be a path of length two and suppose that $i = 0$ and $j \neq 0$. Then any composition $a_0 a_j$ in the quiver of $\Gamma^\dagger$ comes from two arrows
$$\begin{tikzcd} y \arrow[dr, "\alpha_j"] & \\ & z \\ w \arrow[ur, "\alpha_0"'] & \end{tikzcd}$$
in the

quiver of Γ. But any such couple of arrows is part of a square

$$\begin{tikzcd} & y \arrow[dr, "\alpha_j"] & \\ x \arrow[ur, "\alpha_0"] \arrow[dr, "\alpha_j"'] \arrow[rr, dotted, no head] & & z \\ & w \arrow[ur, "\alpha_0"'] & \end{tikzcd}$$

where $\alpha_0\alpha_j \neq 0$ is defined. Moreover the corresponding square in $Q^{\Gamma^\dagger}$ is commutative by Proposition 3.3.6 (I)(b) . The case for $j = 0$ and $i \neq 0$ is similar. Hence, for elements $a_i a_j$ such that $a_j a_i$ is not part of the quiver, we can always assume $i, j \neq 0$ and these elements are zero by Proposition 3.3.6 (II)(b).

We have shown all the possible quadratic relations so the proof is complete. □

4

Irreducible mutations

In this chapter we describe some derived equivalences for the category of finitely generated modules over higher zigzag algebras of type A. We start by recalling the notions of silting objects, silting mutation and Okuyama–Rickard complexes. Then we focus on two-term tilting complexes, characterizing two-term partial tilting complexes for higher zigzag algebras of type A. To conclude we study irreducible mutations of tilting objects in the derived category $\mathcal{D}^b(Z_{d+1}(\Lambda))$, showing that the operation of mutation "commutes" with taking the trivial extension.

4.1 Silting objects and mutation

In this section we introduce the notion of *silting objects* and *silting mutation* in derived categories, as presented in [AI12]. Let $\mathcal{T}$ be a triangulated category; we will always assume that $\mathcal{T}$ is k-linear and Hom-finite. Silting objects are a generalization of tilting objects (recall Theorem 1.1.29).

Definition 4.1.1. Let $T \in \mathcal{T}$.

(a) T is called *pre-silting* (respectively, *pre-tilting*) if $\mathrm{Hom}_{\mathcal{T}}(T, T[i]) = 0$ for any $i < 0$ (respectively, $i \neq 0$).

(b) T is called *silting* (respectively, *tilting*) if it is pre-silting (respectively, pre-tilting) and it satisfies $\mathcal{T} = \mathsf{thick}T$, where $\mathsf{thick}T$ is the smallest triangulated subcategory of $\mathcal{T}$ containing T and closed under direct summands. We denote by $\mathrm{silt}\,\mathcal{T}$ the set of isomorphism classes of basic silting objects in $\mathcal{T}$ and by $\mathrm{tilt}\,\mathcal{T}$ the set of isomorphism classes of basic tilting objects.

(c) T is called *partial silting* if it is a direct summand of a silting object.

Working with derived categories, or more in general with triangulated categories of complexes, we say that a silting (resp. tilting) complex is a *stalk complex* if it is non-zero in only one position and we say that it is a *two-term* silting (resp. tilting) complex if it is non-zero in only two neighbouring positions.

The procedure of *mutation* of silting objects is often very useful in the study of $\mathrm{silt}\,\mathcal{T}$ since it gives a method to produce new silting objects from a given one. The definition of mutation we recall here is taken from [Aih13] but it is important to notice that this notion is the result of the work of many other authors (for example [AI12]).

Definition 4.1.2 ([Aih13], Definition 2.3)**.** Let $T \in \operatorname{silt} \mathcal{T}$ be a silting object and X a direct summand of T such that $T = X \oplus M$ and $\operatorname{add} X \cap \operatorname{add} M = 0$. Take a left $\operatorname{add} M$-approximation of X, $f : X \to M'$ and a triangle

$$X \xrightarrow{f} M' \to Y \to X[1].$$

Then we put

$$\mu_X^+(T) := Y \oplus M$$

and we call it a *left mutation* of T with respect to X. Dually we can define a *right mutation* $\mu_X^-(T)$ of T with respect to X. A *silting mutation* is a right or left mutation. We say that a mutation is *tilting* if both T and its mutation are tilting. Moreover a mutation is called *irreducible* if X is indecomposable.

The following theorem tells us that mutation is well defined only between silting objects.

Theorem 4.1.3 ([AI12], Theorem 2.30)**.** *Any mutation of a silting object is again a silting object.*

As pointed out in [AI12], Section 2.7, silting mutation specializes to other similar techniques in the study of silting and tilting objects (e.g. APR-tilting modules, BB-tilting modules and Okuyama–Rickard complexes). We will focus in particular on *Okuyama–Rickard complexes* and their iterated mutations because this method provides a solution to the problem of classifying Brauer tree algebras up to derived equivalence.

As before let Λ be a finite dimensional (basic) k-algebra. For any Λ-module X we denote by $P(X)$ its projective cover (Definition 1.1.2) and let $\nu = D\operatorname{Hom}_\Lambda(-, \Lambda) : \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda$ be the Nakayama functor.

Definition 4.1.4. Let $e \in \Lambda$ be an idempotent. The *Okuyama–Rickard complex* with respect to e is defined as follows:

$$T := \begin{cases} \overset{0}{P(e\Lambda(1-e)\Lambda)} & \xrightarrow{p_e} \quad \overset{-1}{e\Lambda} \\ \oplus & \\ (1-e)\Lambda & \end{cases}$$

where p_e gives the projective cover of the submodule $e\Lambda(1-e)\Lambda$ of $e\Lambda$.

Okuyama–Rickard complexes were first introduced by Rickard in his PhD thesis in some particular cases, then defined by Okuyama in [Oku97]. In the same document the author proved that they are tilting complexes when Λ is a symmetric algebra ([Oku97], Proposition 1.1). We summarise the key properties of Okuyama–Rickard complexes in the following theorem which also describes these complexes as irreducible (silting) mutation of Λ.

Theorem 4.1.5 ([AI12], Theorem 2.50)**.** *Let $e \in \Lambda$ be an idempotent and T the Okuyama–Rickard complex with respect to e.*

(a) T is isomorphic to right mutation $\mu^{-}_{e\Lambda}(\Lambda)$ of Λ with respect to $e\Lambda$.

(b) T is a silting object in $\mathcal{H}^b(\operatorname{proj}\Lambda)$.

(c) The following conditions are equivalent.

(i) T is a tilting object in $\mathcal{H}^b(\operatorname{proj}\Lambda)$.

(ii) $\operatorname{Hom}_\Lambda(e\Lambda/e\Lambda(1-e)\Lambda, (1-e)\Lambda) = 0$.

(d) If Λ is a self-injective algebra and $e\Lambda \simeq \nu(e\Lambda)$, then T is a tilting object in $\mathcal{H}^b(\operatorname{proj}\Lambda)$.

4.2 Derived equivalences for higher zigzag-algebras

The purpose of this section is to study derived equivalences for derived categories of higher zigzag algebras of type A (recall Definition 2.2.4). In particular we will look at Okuyama–Rickard tilting complexes in the derived category of Z_s^d and we will try to describe the induced derived equivalence. Since for $d = 1$ the zigzag algebra of the path algebra of a type A quiver is Morita equivalent to a Brauer tree algebra, we will look also at two-term (partial) tilting complexes, generalizing some existing results from [AZ13].

Consider the type A higher zigzag algebra $Z_s^1 = Z_2(kA_s)$ for $d = 1$ (Definition 2.2.4), where A_s is the linearly oriented type A Dynkin quiver with s vertices. By Lemma 2.1.5 this coincides with the classical zigzag algebra (see for example [HK01]). It is easy to see that Z_s^1 is Morita equivalent to the *Brauer tree algebra* associated to the linear tree with s edges and without exceptional vertex. Denote by $B(s, m)$ the Brauer tree algebra associated to the tree with s edges and with exactly one common vertex, that coincides with the exceptional vertex and has multiplicity m. Rickard, in his paper [Ric89a], proved that up to derived equivalence any Brauer tree algebra is determined by the number of the edges of the tree and by the multiplicity of the exceptional vertex:

Theorem 4.2.1 ([Ric89a], Theorem 4.2)**.** *Let $B = B(T, s, m)$ be the Brauer tree algebra over a Brauer tree T with s edges and exceptional vertex with multiplicity m. Then B is derived equivalent to $B(s, m)$.*

It is also important to notice that such a derived equivalence can be decomposed into equivalences induced by Okuyama–Rickard complexes, hence iterated irreducible silting mutations.

Moreover Brauer tree algebras completely classify the derived equivalence class of Z_s^1. Indeed from Theorem 2.1 and Corollary 2.2 of [Ric89a], if B is a k-algebra derived equivalent to Z_s^1, and hence to $B(s,1)$, then B is symmetric (see [Ric91], Section 5) and thus it is stably equivalent to $B(s,1)$. Therefore we can deduce that B is given by a Brauer tree algebra by Theorem 2 of [GR79] (see also [ARS95], Theorem X.3.14).

Thus derived equivalence classes of zigzag algebras of type A are completely characterized: every algebra derived equivalent to Z_s^1 is Morita equivalent to a Brauer tree algebra over a tree with s edges and no exceptional vertex.

4.2.1 Okuyama–Rickard mutations

Motivated by the classical results about Brauer tree algebras we focus our attention on Okuyama–Rickard complexes in $\mathcal{H}^b(\operatorname{proj} Z_s^d)$. For the rest of this section we will always assume that any quiver Q has no loops, no multiple arrows and every arrow in Q is contained in an oriented cycle.

Definition 4.2.2. Let $\Lambda = kQ/I$, e_x be a primitive idempotent of Λ and T the Okuyama–Rickard complex for Λ associated to e_x. We say that the algebra $\mu_x(\Lambda) = \operatorname{End}_{\mathcal{D}^b(\Lambda)}(T)$ is the *tilting mutation* of Λ at the vertex x (equivalently, at the idempotent e_x).

Recall that, by Proposition 2.2.10, type A higher zigzag algebras are trivial extension algebras, so it is useful to have a better description of the quiver of a trivial extension algebra.

Proposition 4.2.3. *Let Λ be the path algebra of the quiver Q_Λ with relations and $\Lambda^e = \Lambda \otimes \Lambda^{op}$ its enveloping algebra. The quiver of the trivial extension* $\operatorname{Triv}(\Lambda)$ *is given by:*

1. $(Q_{\operatorname{Triv}(\Lambda)})^0 = (Q_\Lambda)^0$
2. $(Q_{\operatorname{Triv}(\Lambda)})^1 = (Q_\Lambda)^1 \cup \{\beta_{p_1}, \dots, \beta_{p_t}\}$ *where* $\{p_1, \dots, p_t\}$ *is a k-basis of* $\operatorname{soc}_{\Lambda^e} \Lambda$ *consisting of linear combinations p_i of paths with the same origin $s(p_i)$ and the same endpoint $t(p_i)$ and β_{p_i} is an arrow from $t(p_i)$ to $s(p_i)$.*

Proof. See [Pla10], Proposition 2.2. □

We also recall the following result about algebras with isomorphic trivial extensions:

Theorem 4.2.4 ([FP06], Theorem 3.6)**.** *Let Λ and Λ' two finite dimensional algebras such that* $\operatorname{Triv}(\Lambda) \cong \operatorname{Triv}(\Lambda')$. *Assume moreover that any oriented cycle in Q_Λ is zero in Λ. Then $\Lambda' \cong \operatorname{Triv}(\Lambda)/J$ where J is an ideal generated by exactly one arrow from any non-zero oriented cycle in* $\operatorname{Triv}(\Lambda)$.

The set of arrows generating the ideal of relations J are called a *cut* of the quiver Q.

Definition 4.2.5 ([IO11], Definition 5.8)**.**

1. A *cutting set* for Q is a set of arrows $C \subset Q_1$ such that C contains exactly one arrow from every oriented cycle of Q. Denote by Q_C the quiver obtained by removing from Q all the arrows in C.
2. If x is a source of Q_C, we define a subset $\mu_x^+(C)$ of Q^1 by removing from C all the arrows ending at x and adding all the arrows starting at x.
3. Dually, if x is a sink of Q_C, we define another subset $\mu_x^-(C)$ of Q^1 by removing from C all the arrows starting at x and adding the ones ending at x.
4. When x is a source or a sink, we call the procedure of replacing C by $\mu_x^+(C)$ or $\mu_x^-(C)$ *mutation of cuts.*

To make full sense of the definition of cut mutation we need to prove that mutations of cuts are cuts.

Proposition 4.2.6. *Let Q be a quiver, x a vertex of Q and C a cut such that x is a source in Q_C. Then we have the following:*

1. *Any arrow in Q ending at x belongs to C and any arrow in Q starting at x does not belong to C.*
2. *$\mu_x^+(C)$ (equivalently $\mu_x^-(C)$) is again a cut.*
3. *x is a sink of the quiver $Q_{\mu_x^+(C)}$.*

Proof. The proof of these facts can be found in [IO11, Proposition 5.14] when Q is the quiver of higher zigzag algebras of type A. However the results are true more generally under our assumptions on the quiver Q. We include here the proof for the convenience of the reader.

1. Clearly any arrow ending at x belongs to C since x is a source in Q_C. Let α be an arrow starting at x and assume α belongs to C. By our assumptions on Q, α belongs to an oriented cycle c passing through x, which then contains two arrows in C, contradiction.
2. Let c be an oriented cycle. We need to check that exactly one arrow of c belongs to $\mu_x^+(C)$. This is clear if c does not pass through x. Assume then that x is in c and let α and β be the two arrows in c ending and starting in x respectively. Then α is the unique arrow in c contained in C and β is the unique arrow in c contained in $\mu_x^+(C)$.
3. This is obvious from (1).

□

Lemma 4.2.7. *Assume that Q admits a cut C such that Q_C has at least a source (or a sink). Then for any vertex x of Q there exists a cut C' such that x is a source in $Q_{C'}$.*

Proof. We can obtain the cut C' by iterated mutations of C. By the definition of cut and by our assumptions on Q, the quiver Q_C has no oriented cycles so it is the Hasse quiver of a partial order on the set of vertices. Denote by S the set of sources of Q_C. These are the greatest elements with respect to the partial order defined by Q_C and after mutating at every vertex in S we obtain a new quiver $Q_{\mu_S(C)}$ where the sources are those vertices that were the immediate successors of vertices in S. Since Q is finite and connected, by repeating this argument we will eventually obtain a quiver $Q_{C'}$ such that the chosen vertex x is a source. □

The following result relates the operations of taking trivial extension and mutation:

Proposition 4.2.8. *Let $\Gamma = kQ/I$ be a finite dimensional algebra with quiver Q without loops and multiple arrows, such that any arrow of Q is contained in an oriented cycle. Let x be a vertex of Q and suppose that*

- *there exists a cut C for Q such that x is a source in Q_C;*
- *any oriented cycle is zero in the quotient $\Lambda = \Gamma/C$;*
- $\Gamma = \mathrm{Triv}(\Lambda)$.

Then

- $\mu_x(\Gamma) \cong \mathrm{Triv}(\mu_x(\Lambda))$*;*
- *denote by T_Λ (resp. T_Γ) the Okuyama–Rickard complex at x in $\mathcal{D}^b(\Lambda)$ (resp. $\mathcal{D}^b(\Gamma)$). Then $T_\Gamma \cong T_\Lambda \otimes \Gamma$.*

Proof. It is known that if $\Lambda \simeq_d \Lambda'$ via an equivalence given by the tilting complex T_Λ, then $T = T_\Lambda \otimes_\Lambda \Gamma$ is again a tilting complex and it gives a derived equivalence $\Gamma \simeq_d T(\Lambda')$ between the corresponding trivial extensions (see [Ric89a, Theorem 3.1]).

In order to prove the claims we have to show that T_Λ is actually a tilting complex and that its extension $T_\Lambda \otimes_\Lambda \Gamma$ is exactly the Okuyama–Rickard complex for Γ associated to the same vertex.

The fact that T_Λ is tilting is provided by Theorem 4.1.5, since in our case we have:

$$\mathrm{Hom}_\Lambda(e_x\Lambda/e_x\Lambda(1-e_x)\Lambda, (1-e_x)\Lambda) = \mathrm{Hom}_\Lambda\left(S_x, \bigoplus_{y\neq x} P_y\right) = 0$$

where the last equality holds because, by the choice of the cut C, the simple S_x is not in the socle of any projective module.

Let $A = \{\alpha \in Q_\Lambda^1 \mid s(\alpha) = x\}$ be the set of all arrows in the quiver of Λ with source x. Then since

$$T_\Lambda = \left(\bigoplus_{y\neq x} P_y \to 0\right) \oplus \left(\bigoplus_{\alpha\in A} P_{t(\alpha)} \xrightarrow{\varphi} P_x\right)$$

$T_\Lambda \otimes_\Lambda \Gamma$ is the direct sum of the following two complexes:

$$\bigoplus_{y \neq x} P_y \otimes \Gamma \to 0 \quad \text{and} \quad \bigoplus_{\alpha \in A} P_{t(\alpha)} \otimes \Gamma \xrightarrow{\varphi \otimes id} P_x \otimes \Gamma.$$

Now, if P_y is an indecomposable projective module over Λ, then $Q_y = P_y \otimes_\Lambda \Gamma$ is the indecomposable projective Γ-module still corresponding to the vertex y. So T is the Okuyama–Rickard complex for Γ associated to the vertex x if $\mathrm{Coker}(\varphi \otimes id)$ has composition factors isomorphic to S_x only. This is true if $\mathrm{Im}(\varphi \otimes id) = \mathrm{rad}\, Q_x$ since in this case $\mathrm{Coker}(\varphi \otimes id) \cong \mathrm{top}\, Q_x = S_x$.

Since $Q_y = P_y \otimes_\Lambda \Gamma$ for any y and $\varphi \otimes id : \bigoplus_{\alpha \in A} P_{t(\alpha)} \otimes \Gamma \to P_x \otimes \Gamma$, we have that

$$\varphi \otimes id : \bigoplus_{\alpha \in A} Q_{t(\alpha)} \to Q_x$$

so our claim holds if in the quiver of trivial extension Γ, the set of arrows with source x coincides with A. But this is true because, by Proposition 4.2.3, since x a source in the quiver of Λ, we do not need to add any new arrow starting at x to obtain the quiver of Γ. □

We can apply the results of this section to the case of higher zigzag algebras of type A to obtain the following corollary.

Corollary 4.2.9. *Every algebra that is a tilting mutation of a type A higher zigzag algebra Z_s^d is a trivial extension algebra.*

Proof. To apply Proposition 4.2.8 we only need to show that for any given vertex x of the quiver Q of Z_s^d there exists a cut C such that x is a source of Q_C. Obviously there are cuts for Q such that the quiver obtained after removing the corresponding arrows have sources or sinks: having in mind the presentation of Z_s^d given in Section 2.2 we can choose for example $C = \{\alpha_0 \in Q^1\}$ the set of all the arrows α_0, so that $x = (s+1, 0, \ldots, 0)$ is a source. Then we can use Lemma 4.2.7 to find the desired cut. □

Example 4.2.10. Consider the 2-zigzag algebra Z_3^2:

$$\begin{tikzcd}
 & & 6 \arrow[dr] & & \\
 & 2 \arrow[ur] \arrow[dr] & & 4 \arrow[ll, "\alpha_1"'] \arrow[dr] & \\
1 \arrow[ur] & & 3 \arrow[ll, "\alpha_2"] \arrow[ur] & & 5 \arrow[ll, "\alpha_3"]
\end{tikzcd}$$

The set $C_\alpha = \{\alpha_1, \alpha_2, \alpha_3\}$ is a cutting set such that in the quiver of $\Lambda = Z_3^2/C_\alpha$ the vertex 1 is a source:

$$\begin{tikzcd}
 & & 6 \arrow[dr] & & \\
 & 2 \arrow[ur] \arrow[dr] & & 4 \arrow[dr] & \\
1 \arrow[ur] & & 3 \arrow[ur] & & 5
\end{tikzcd}$$

By Proposition 4.2.8, the algebra $\mu_1(Z_3^2)$ is isomorphic to $\mathrm{Triv}(\mu_1(\Lambda))$. It is easy to see that the quivers of the two algebras coincides: the quiver of $\mu_1(\Lambda)$ is the following

$$\begin{tikzcd}
 & 6 \arrow[rr, "\gamma"] & & 4 \arrow[rr, "\delta"] & & 5 \\
1 \arrow[ur, "\alpha"] & & 2 \arrow[ll, "\epsilon"'] \arrow[rr, "\nu"] & & 3 \arrow[ul, "\eta"'] &
\end{tikzcd}$$

and a k-basis of $\mathrm{soc}_{\Lambda^e} \Lambda$ is $\{p_1 = \epsilon\alpha\gamma = \nu\eta, p_2 = \eta\delta\}$. So the quiver of $\mathrm{Triv}(\mu_1(\Lambda))$ is

$$\begin{tikzcd}
 & 6 \arrow[rr, "\gamma"] & & 4 \arrow[rr, "\delta"] \arrow[dl, "\beta_1"'] & & 5 \arrow[dl, "\beta_2"] \\
1 \arrow[ur, "\alpha"] & & 2 \arrow[ll, "\epsilon"'] \arrow[rr, "\nu"] & & 3 \arrow[ul, "\eta"'] &
\end{tikzcd}$$

and we can see that the set $C_\beta = \{\beta_1, \beta_2\}$ is a cutting set for $\mu_1(Z_3^2)$ so that $\mu_1(Z_3^2) \cong \mathrm{Triv}(\mu_1(Z_3^2/C_\beta)$.

4.3 Derived class of Z_3^2

Here we describe the derived equivalence class of the 2-zigzag algebra Z_3^2:

$$\begin{tikzcd}
 & & 6 \arrow[dr, "\alpha_2"] & & \\
 & 2 \arrow[ur, "\alpha_1"] \arrow[dr, "\alpha_2"] & & 4 \arrow[ll, "\alpha_0"'] \arrow[dr, "\alpha_2"] & \\
1 \arrow[ur, "\alpha_1"] & & 3 \arrow[ll, "\alpha_0"] \arrow[ur, "\alpha_1"] & & 5 \arrow[ll, "\alpha_0"]
\end{tikzcd}$$

by showing explicitly all the algebras in the class. This classification has been achieved by brute force, computing iterated irreducible tilting mutations at every vertex of any algebra found during the process. These computations produce twenty-four non-isomorphic symmetric algebras, hence we only need to show that any algebra derived equivalent to Z_3^2 can be obtained by iterated tilting mutations. This can be made more precise using the notion of *tilting-connected* symmetric algebras.

Definition 4.3.1. A symmetric algebra is called *tilting-connected* if the action of iterated irreducible tilting mutation on the set of basic tilting complexes is transitive.

First of all it relatively easy to show that Z_3^2 is derived equivalent to the (classical) zigzag algebra of kD_6:

$$\begin{tikzcd}
 & & & & 2 \arrow[dl, bend left, "\beta"] \\
1 \arrow[r, bend left, "\alpha"] & 3 \arrow[l, bend left, "\beta"] \arrow[r, bend left, "\alpha"] & 5 \arrow[l, bend left, "\beta"] \arrow[r, bend left, "\alpha"] & 4 \arrow[l, bend left, "\beta"] \arrow[ur, bend left, "\alpha"] \arrow[dr, bend left, "\alpha"] & \\
 & & & & 6 \arrow[ul, bend left, "\beta"]
\end{tikzcd}$$

with relations: $\alpha\beta\alpha = \beta\alpha\beta = 0$, $\alpha\beta = \beta\alpha$. This can be seen for example mutating the algebra Z_3^2 twice at vertex 2, then once at vertex 4 in the above presentation. Since the zigzag algebra of kD_6 is the trivial extension of an algebra of finite Dynkin type, it is again representation finite (of the same Cartan class) by the following result.

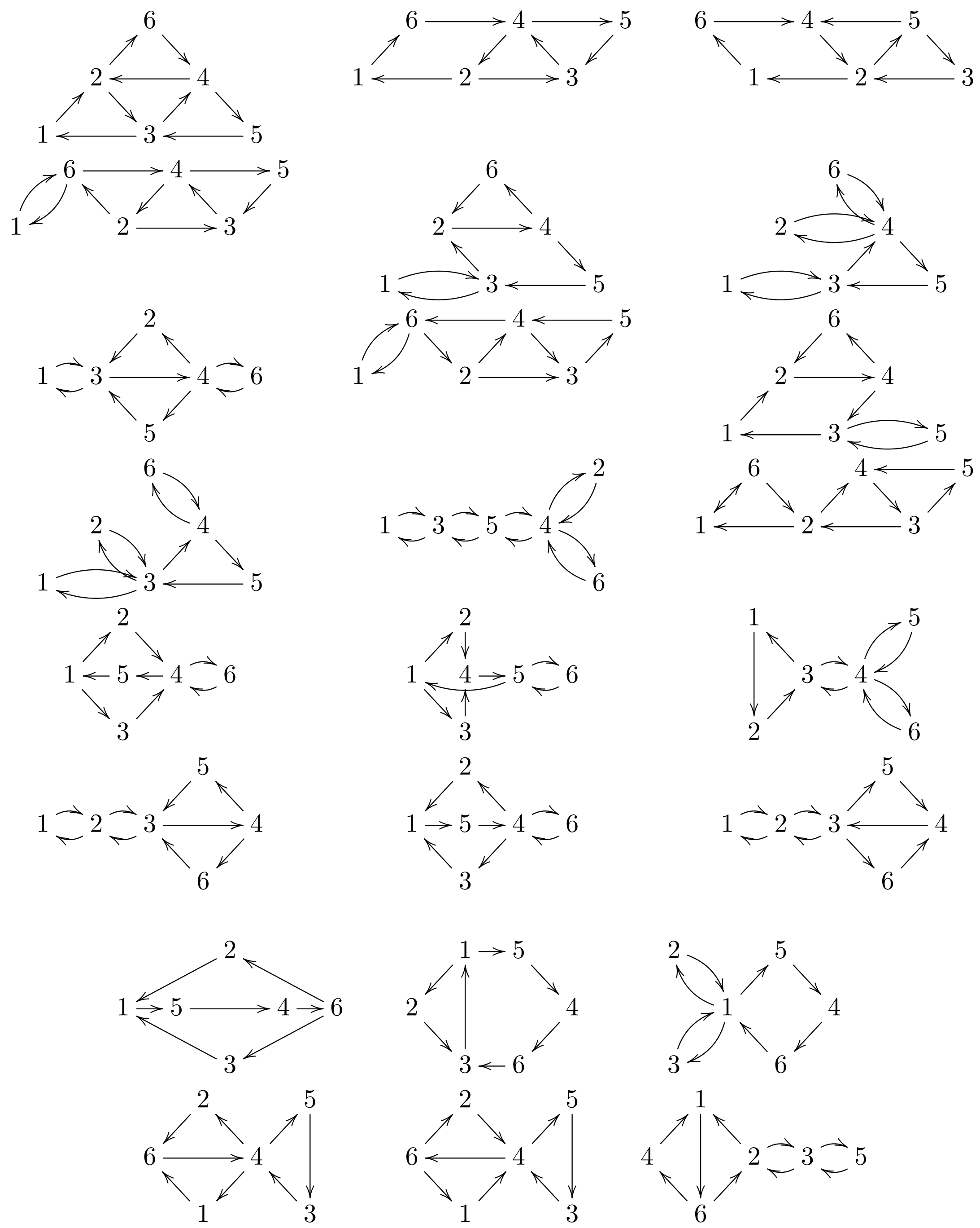
Theorem 4.3.2. *Let* Λ *be a (basic, connected) finite dimensional* k*-algebra and* $\operatorname{Triv}\Lambda$ *its trivial extension algebra. Then* $\operatorname{Triv}\Lambda$ *is representation finite of Cartan class* G *if and only if* Λ *is an iterated tilted algebra of Dynkin type* G*.*

Proof. For a proof see [AHR84], Theorem 3.1. □

The result we need was proved by Aihara in [Aih13]:

Theorem 4.3.3 ([Aih13], Theorem 1.2)**.** *Any representation-finite symmetric algebra is tilting-connected.*

Therefore any algebra derived equivalent to Z_3^2 can be obtained via a finite number of irreducible tilting mutations. We can then conclude by showing the presentations of all the algebras derived equivalent to Z_3^2, up to isomorphism. Recall that the relations are always such that any path passing through the same vertex twice is zero, any path that is contained in more than one cycle but is not contained in the intersection of such cycles is zero and any two non-zero paths with same starting and ending vertices are equal.



Note moreover that, as a direct consequence of Theorem 4.3.2, we can deduce that

Z_3^2 can not be derived equivalent to the path algebra of the following quiver

$$\begin{tikzcd}
 & 2 \arrow[dr] & & 5 \arrow[ll] \\
1 \arrow[ur] \arrow[dr] & & 4 \arrow[ll] \arrow[ur] \arrow[dr] & \\
 & 3 \arrow[ur] & & 6 \arrow[ll]
\end{tikzcd}$$

with relations as before. This is because this algebra is an iterated tilting mutation of the zigzag algebra of kE_6, hence of a different Cartan class.

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