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Abstract

We prove the existence of S-integral solutions of simultaneous diophantine inequalities for pairs (Q, L) involving one quadratic form and one linear form satisfying some arithmetico-geometric conditions. This result generalises previous results of Gorodnik and Borel-Prasad. The proof uses Ratner's theorem for unipotent actions on homogeneous spaces combined with an argument of strong approximation.

Keywords: Quadratic forms, Diophantine approximation, Algebraic groups, Strong approximation, Ratner's Orbit Closure Theorem.

1. Introduction

A famous conjecture made by Oppenheim in 1929 and proved by G.A. Margulis in the mid-eighties states that given a nondegenerate indefinite real quadratic form Q in $n \geqslant 3$ variables which is not proportional to a form with rational coefficients then $Q(\mathbb{Z}^n)$ is dense in $\mathbb{R}^n$. It is not difficult to see that the latter statement is equivalent to the following assertion,

$$\forall \varepsilon > 0, \exists\, x \in \mathbb{Z}^n - \{0\}, \quad 0 < |Q(x)| < \varepsilon\,. \tag{1}$$

A natural generalization of the Oppenheim conjecture concerns the existence of integral solutions of system of inequalities involving several quadratic forms. More precisely given a family $(Q_j)_{1 \leq j \leq r}$ of real nondegenerate quadratic forms in n variables we may ask whether there exist solutions to the diophantine inequalities

$$\forall \varepsilon > 0, \exists\, x \in \mathbb{Z}^n - \{0\}, \ |Q_j(x)| < \varepsilon \ \text{ for } j = 1, \dots, r. \tag{2}$$

Such kind of problems have been intensively studied and a general solution is still an open problem when $r > 1$. Some partial results have been obtained and almost all of them assume two fundamental necessary conditions for (2) to hold. The first condition is the existence of nonzero real solution $x \in \mathbb{R}^n$ ($n \geq 3$):

$$Q_1(x) = \ldots = Q_r(x) = 0.$$

The second condition is of arithmetical nature: it asks that for any $(\alpha_1, \dots, \alpha_r) \in \mathbb{R}^r - \{0\}$, the pencil forms $\sum_{i=1}^{r} \alpha_i Q_i(x)$ are not proportional to a rational form. Notice that these two conditions are natural generalisation of the assumptions of the Oppenheim conjecture to the general case of systems of quadratic inequalities.

To the best of our knowledge, the most general result proving that (2) has a solution is due to W. Müller [14]. The method used by W. Müller is a variant of the circle method which applies to families for which the rank of each pencil form is greater than $8r$. One of the main inconvenient of those analytic methods is that it fails to work when n is small. In view of this, it is natural to try to generalise Margulis's proof to systems of quadratic forms with the help of Ratner's rigidity theorems. A first obstruction regarding this, is that the use of Ratner's theory requires that the intersection $\bigcap_{j=1}^{r} SO(Q_j)$ is large enough, in the sense that it should at least contain a unipotent one-parameter subgroup. If it is the case then one can take advantage in working in low dimension in order to elucidate the structure of intermediate subgroups arising from the possiblities for the orbit closure. Unfortunately solving (2) using rigidity theorems leads to untractable situations in the general case. In [10], Gorodnik gave necessary and sufficient conditions for such systems (2) to have a solution. One of these conditions is the existence of a nonzero vector $v \in \mathbb{R}^n$ lying in the intersection $\bigcap_{1\leq j\leq r} \{Q_j = 0\}$ such that for some $v_i \neq 0$ the vector $(\dfrac{v_k}{v_i} : 1 \leq k \leq nd, k \neq i)$ is well-approximable of order one. In general, the latter condition appears to be very difficult to check even for a system involving only two quadratic forms. However if we consider a pair of two quadratic forms Q_1 and Q_2 with $Q_2 = L^2$ where L is a linear form, a solution for (2) has been obtained using ergodic theory. The first result in that direction is due to S.G Dani and G.A. Margulis [7] and concerns the dimension 3 for a pair (Q, L^2) consisting of one nondegenerate indefinite quadratic form and a nonzero linear form in dimension 3 such that the cone $\{Q = 0\}$ intersects tangentially the plane $\{L = 0\}$ and no linear combination of Q and L^2 is rational. Under those conditions they proved using the original method used to prove the Oppenheim conjecture that the set $\{(Q(x), L(x)) : x \in \mathbb{Z}^3\}$ is dense in $\mathbb{R}^2$. In higher dimension, the density for pairs holds if one replaces the previous transversality condition by the assumption that $Q_{|L=0}$ is indefinite, this result is due to A.Gorodnik [9]:

Theorem 1.1 (Gorodnik). *Let $F = (Q, L)$ be a pair consisting of a quadratic form Q and L a nonzero linear form in dimension $n \geq 4$ satisfying the the following conditions*

1. *Q is nondegenerate.*
2. *$Q_{|L=0}$ is indefinite.*
3. *No linear combination of Q and L^2 is rational.*

Then the set $F(\mathcal{P}(\mathbb{Z}^n))$ is dense in $\mathbb{R}^2$ where $\mathcal{P}(\mathbb{Z}^n)$ is the set of primitive integer vectors.

The conclusion of the theorem implies immediately that the set $F(\mathbb{Z}^n)$ is dense in $\mathbb{R}^2$. The proof of this theorem reduces to the case of the dimension 4. The condition (2) is a sufficient condition to ensure that we have $F(\mathbb{R}^n) = \mathbb{R}^2$. The most important obstruction to prove density for pairs is that the identity component of the stabilizer of a pair (Q, L) is no longer maximal among the connected Lie subgroups of $G = \mathrm{SL}(4, \mathbb{R})$ in contrast with the case of the isotropy groups $\mathrm{SO}(3, 1)^\circ$ or $\mathrm{SO}(2, 2)^\circ$.
The stabilizer of the pair (Q, L) is defined by the following subgroup of G,

$$\mathrm{Stab}(Q, L) = \{h \in SO(Q) \mid L(hx) = L(x)\}.$$

The pairs such that $Q_{|L=0}$ is nondegenerate (resp. degenerate) are said to be of type (I) (resp. II). The proof of Theorem 1.1 is divided in two parts following each type and consists to apply Ratner's orbit closure theorem, and to study the action of the stabiliser on the dual space of $\mathbb{C}^4$. A remarkable fact is that the density is proved without showing the density of the orbit closure of the stabilizer in the homogeneous space G/Γ. Indeed the intermediate subgroups which possess non-trivial irreducible components have closed orbits in G/Γ, in particular they are not maximal. However, one is able to classify all the complex semisimple Lie algebras in $\mathfrak{sl}(4,\mathbb{C})$, and Gorodnik used this classification to check density case by case using the constrain on rationality given by the condition (3). The situation for pairs of type (II) is more complicated compared with the pairs of type (I) since the dual action of the stabilizer has three irreducible components for the pairs of type (II), instead of two for the pairs of type (I).

2. Main results

2.1. S-arithmetic setting

Let us recall what we mean by S-arithmetic setting by fixing some notations. Let k be a number field, that is a finite extension of $\mathbb{Q}$ and let $\mathcal{O}$ be the ring of integers of k. For every normalised absolute value $|.|_s$ on k, let k_s be the completion of k at s. We identify s with the specific absolute value $|.|_s$ on k_s defined by the formula $\mu(a\Omega) = |a|_s\mu(\Omega)$, where μ is any Haar measure on the additive group k_s, $a \in k_s$ and Ω is a measurable subset of k_s of finite measure. We denote by Σ_k the set of places of k. In the sequel S is a finite subset of Σ_k which contains the set S_∞ of archimedean places, k_S the direct sum of the fields $k_s (s \in S)$ and $\mathcal{O}_S$ the ring of S-integers of k (i.e. the ring of elements $x \in k$ such that $|x|_s \leq 1$ for $s \notin S$). We denote by S_f the set of non-arichimedean places of S. For $s \in S_f$, the valuation ring of the local field k_s is defined to be $\mathcal{O}_s = \{x \in k \mid |x|_s \leq 1\}$. We denote by K an algebraic closure of k and for each $s \in S$, we denote K_s the algebraic closure of the completion k_s. Note that K_s is not necessarily the completion of K at s, indeed K_s is not complete at least at finite places.

In view of a generalisation of the previous results to the S-adic setting, let us consider a number field k of degree d with ring of integers $\mathcal{O}$ and a finite set of places S containing the archimedean ones. Suppose we are given a familiy of r quadratic forms $Q_1, \dots, Q_r$ with coefficients in the product of the completions k_s $(s \in S)$. The S-arithmetic version of the system (2) is given by the following diophantine inequalities,

$$\forall \varepsilon > 0, \exists\, x \in \mathcal{O}_S^n - \{0\},\ \forall s \in S, \quad |Q_{j,s}(x)|_s < \varepsilon \ \text{ for } j = 1, \dots, r. \tag{3}$$

In the case when the set S only contains archimedean places, $\mathcal{O}$ is a free $\mathbb{Z}$-module of rank d and let us choose a basis for $\mathcal{O}$ given by $\omega = (\omega_1, \dots, \omega_d)$. if we introduce the quadratic form

$$Q_j^\omega(x_{1,1}, \dots, x_{d,n}) := Q_j\left(\sum_{k=1}^d x_{k,1}\omega_k, \dots, \sum_{k=1}^d x_{k,n}\omega_k\right).$$

Then (3) reduces to the system (2)

$$\forall \varepsilon > 0, \exists\, x \in \mathbb{Z}^{nd} - \{0\},\ |Q_j^\omega(x)| < \varepsilon \ \text{ for } j = 1, \dots, r. \tag{4}$$

Using such kind of restriction of scalar does not simply the problem. Indeed this operation increases the number of variables and the hope to apply Ratner's orbit closure together with the classification of intermediate subgroups seems compromised. For example, even in the simplest case of a quadratic number field and $n \geq 3$, we already have to deal with quadratic forms in at least six variables for which the classification of intermediate subgroups is out of reach. The only case for which system (3) has been solved is when $r = 1$ and $n \geq 3$ which corresponds to the S-arithmetic version of the Oppenheim conjecture proved by Borel and Prasad [3]. Their proof appeals to the same method used in the original proof of Margulis, namely, it shows that the orbit closure of the lattice $\mathcal{O}_S^n$ under $\prod_{s\in S} SO(Q_s)^+$ is dense in the homogeneous space $\Omega_S = SL(n, k_S)/SL(n, \mathcal{O}_S)$ provided the form Q_S is an isotropic form not proportional to a form with coeffcients in k. The aim of this paper is to give a S-adic version of the Theorem 1.1 corresponding to the system (3) for pairs given by $(Q_s, L_s^2)_{s\in S}$ using Ratner's orbit closure theorem.

2.2. Main results

Let (Q, L) be a pair consisting of one quadratic form and one nonzero linear form on k_S^n. Equivalently, (Q, L) can be viewed as a family $(Q_s, L_s)(s \in S)$, where Q_s is a quadratic form on k_s^n and L_s a nonzero linear form on k_s^n. The form Q is nondegenerate if and only each Q_s is nondegenerate. We say that Q is isotropic if each Q_s is so, i.e. if there exists for every $s \in S$ an element $x_s \in k_s^n - \{0\}$ such that $Q_s(x_s) = 0$, in particular if s is a real place an isotropic form is also said to be indefinite. For any quadratic form Q, we denote by rad(Q) (resp. $c(Q)$) the radical (resp. the isotropy cone) of Q. By definition Q is nondegenerate (resp. isotropic) if and only if rad$(Q) \neq 0$ (resp. c$(Q) \neq 0$). The form Q is said to be rational (over k) if there exists a quadratic form Q_o on k^n and a unit c of k_S such that $Q = c.Q_0$, and irrational otherwise. If G is a locally compact group, G° denotes the connected component of the identity in G and G^+ is the subgroup of G generated by its one parameter unipotent subgroups.

Given a pair $F = (Q_s, L_s)_{s\in S}$ on k_S^n, we say that the set $F(\mathcal{O}_S^n)$ is dense in k_S^2 for the S-adic topology if for any $(a, b) \in k_S^2$ and any $\varepsilon > 0$, there exists an S-integral vector $x \in \mathcal{O}_S^n$

$$|Q_s(x) - a_s|_s < \varepsilon \text{ and } |L_s(x) - b_s|_s < \varepsilon \text{ for each } s \in S.$$

Our main result gives the required conditions for density to hold when $S = S_\infty$. This may be seen as an S-arithmetic version of Theorem 1.1 for archimedean places.

Theorem 2.1. *Assume $S = S_\infty$ and let $Q = (Q_s)_{s\in S}$ be a quadratic form on k_S^n and $L = (L_s)_{s\in S}$ be a linear form on k_S^n with $n \geq 4$ and $L_s \neq 0$ for all $s \in S$. Suppose that the pair $F = (Q, L)$ satisfies the following conditions,*

1. *Q is nondegenerate.*
2. *$Q_{|L=0}$ is nondegenerate and isotropic.*
3. *For each $s \in S$ the forms $\alpha_s Q_s + \beta_s L_s^2$ are irrational given any α_s, β_s in k_s with $(\alpha_s, \beta_s) \neq (0, 0)$.*

Then the set $F(\mathcal{O}^n)$ is dense in k_S^2.

Whenever S contains in addition non-archimedean places, one can easily deduce from Gorodnik's theorem a weaker conclusion than the one appearing in Theorem 2.1.

Corollary 2.2. *Assume S be a finite set of places of k such that $S \supsetneq S_\infty$. Let (Q, L) be a pair satisfying conditions of Theorem 2.1 for S. Then for any $\varepsilon > 0$, there exists $x \in \mathcal{O}_S^n - \{0\}$ such that*

$$|Q_s(x)|_s < \varepsilon \text{ and } |L_s(x)|_s < \varepsilon \quad \text{for each } s \in S.$$

2.3. Remarks.

(1) The proof of Theorem 2.1 reduces to dimension 4, (see § 3) this reduction is necessary since the proof of Theorem 2.1 relies essentially on classification of intermediate subgroups and requires low dimension[1]. The reduction process is made possible by the weak approximation property applied to Grassmanian varieties.

(2) The conclusion of Corollary 2.2 follows immediately from Theorem 1.1 when $k = \mathbb{Q}$ and $S = \{p_1, \dots, p_r\} \cup \{\infty\}$ where the p_i are distinct primes. In this case we have $\mathbb{Q}_S = \prod_{p_i} \mathbb{Q}_{p_i} \times \mathbb{R}$ and $\mathcal{O}_S = \mathbb{Z}_S = \mathbb{Z}[1/(p_1 \dots p_r)]$. Given any $\varepsilon > 0$, there exists an integer N sufficiently divisible by the primes p in S such that for any $y \in \mathbb{Z}^n$ we have

$$N^2|Q(y)|_p < \varepsilon \text{ and } N|L(y)|_p < \varepsilon \text{ for any } p \in S.$$

From Theorem 1.1 there exists a non-zero integer $x \in \mathbb{Z}^n$ such that

$$|Q(x)| < \varepsilon/N^2 \text{ and } |L(x)| < \varepsilon/N.$$

Thus $Nx \in \mathcal{O}_S^n$ satisfies the conclusion of Corollary 2.2.

(3) The proof of Theorem 2.1 relies on Ratner's Theorem which gives a precise description of the closure orbits of lattices under the action of a Lie group generated by its unipotent one parameter subgroups. We need to apply an S-adic version of Ratner's theorem in order to find an integral solution simultaneously at all places. We treat first the case when $S = S_\infty$, by Weil's restriction of scalars we can use results of [9] to elucidate the structure of the intermediate subgroups. This is exactly where we need to work in dimension 4, indeed the proof relies on the classifications of semisimple Lie algebras in $\mathfrak{sl}(4, \mathbb{C})$ which contain the Lie algebra of the stabilizer. For a general finite set of places S containing both archimedean and nonarchimedean places, we need to use strong approximation for number fields in order to prove Corollary 2.2.

(4) For Theorem 2.1 even if we assume that $\alpha Q + \beta L^2$ is irrational, it can be possible that the pencil form $\alpha_s Q_s + \beta_s L_s^2$ is rational for some place s, in this situation it is not possible to apply Ratner's theorem. It can be possible that the result is still true in this situation but there are serious obstacles to (see § 8).

(5) Unfortunately we are not able to show the density of $F(\mathcal{O}_S^n)$ under the conditions of theorem 2.1 when S contains both archimedean and non-archimedean places with our method.

(6) It should be noticed that it can be possible that $|Q_s(x)|_s$ and $|L_s(x)|_s$ are both zero for any $s \in S$ and $x \in \mathcal{O}_S^n$ as in the conclusion of Corollary 2.2.

(7) In the real case, one can hope to relax condition (2) by only asking $\alpha Q + \beta L^2$ to be isotropic as it is conjectured by Gorodnik (see § 8, Conjecture 8.1). The major issue is that reduction to lower dimension fails to hold.

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[1]The classification of intermediate Lie subgroups becomes rapidly unfeasible when the dimension exceeds four.

3. Weak approximation, rationality and reduction to dimension 4

3.1. Weak approximation in number fields and Grassmannian varieties

Number fields satisfy a nice *local-global principle* called the weak approximation which can be seen as a refinement of the Chinese remainder theorem.

Theorem 3.1 (Weak approximation in number fields)**.** *Let S be a finite set of Σ_k. Let us give $\alpha_s \in k_s$ for each $s \in S$. Then there exists an $\alpha \in k$ which is arbitrarily close to α_s for all $s \in S$ with respect to the s-adic topology.*

Proof. (See e.g. [12], Theorem1, p.35)

One can reformulate this theorem as follows: the diagonal embedding $k \hookrightarrow \prod_{s\in S} k_s$ is dense, the product being equipped with the product of the s-adic topologies.

Definition 3.2 (Weak approximation in algebraic varieties)**.** *Let X be an algebraic variety defined over k, then X is said to satisfy the weak approximation property with respect to a finite set of places S if the diagonal embedding $X(k) \hookrightarrow \prod_{s\in S} X(k_s)$ is dense for the S-adic topology.*

To prove reduction we need to introduce a useful class of algebraic varieties which satisfies weak approximation,

Definition 3.3. *Let V be a k-vector space of dimension $n \geq 1$ and for each $1 \leq m \leq n$ let us define the set*

$$\mathcal{G}_m(V) = \Big\{ k\text{-vector subspaces } W \subset V \text{ with } \dim W = m \Big\}.$$

This is an algebraic variety defined over K called the Grassmannian variety, if $V = k^n$ the set of k-rational points of $\mathcal{G}_m(V)$ is simply denoted $\mathcal{G}_{n,m}(k)$.

Proposition 3.4. *Let be given two integers $1 \leq m \leq n$, then the Grassmannian variety $\mathcal{G}_{n,m}$ satisfies the weak approximation property with respect to S, that is,*

$$\mathcal{G}_{n,m}(k) \hookrightarrow \prod_{s\in S} \mathcal{G}_{n,m}(k_s) \text{ is dense.}$$

Proof. Let us give a family $(V_s)_{s\in S}$ of k_s-vector subspaces of dimension m in k_s^n for each $s \in S$. Each of these subspaces V_s are determined by m linearly independent vectors in k_s^n. For each of the V_s, the coefficients of these vectors in the standard basis of k_s^n give rise to a $m \times n$-matrix A_s with coefficients in k_s. By weak approximation property in k_S^{nm} we obtain a matrix $B \in \mathcal{M}_{m,n}(k)$ such that for any $s \in S$, B_s is arbitrarily close to A_s. Since the rank is locally constant and B_s is in a sufficiently small open neighbourhood of A_s, we get that rank A_s = rank B_s. Let V' be the vector subspace generated by the n columns of B, obviously $V \in \mathcal{G}_{n,m}(k)$ and V_s' is arbitrary close to V_s for all $s \in S$.

3.2. Rationality criterion for polynomial maps

Let us recall that K denotes an algebraic closure of k and K_s an algebraic closure of k_s for each $s \in S$. The following result gives a nice criterion in order to prove that a polynomial map is defined over k (see also Proposition 3.1.10, [22]).

Proposition 3.5. *Let be given $s \in S$ and $f : K_s^n \longrightarrow K_s$ be a regular map. Suppose there exists a k-rational subspace V in k^n which is Zariski dense in K_s^n and such that $f(V) \subset k$. Then f is defined over k.*

Proof. Since f is regular, there exists a $\alpha_1, \ldots, \alpha_r \in K$ such that $\{1, \alpha_1, \ldots, \alpha_r\}$ are linearly independent over k so that we can write $f = f_0 + \sum_{i=1}^{r} \alpha_i f_i$ where $f_0, f_1, \ldots, f_r$ are polynomials with coefficients in k. By assumption, $f(V) \subset k$ and the linear independence of the family $\{1, \alpha_1, \alpha_2, \ldots, \alpha_r\}$, it implies that $f(x) = f_0(x)$ for all $x \in V$. Since the Zariski density of V in K_s^n is equivalent to V having dimension n, we obtain $f = f_0$ on all K_s^n and f is defined over k.

3.3. Reduction of Theorem 2.1 to the dimension 4

Proposition 3.6. *Let $F = (Q, L)$ be a pair consisting of a quadratic form Q and a nonzero linear form L in k_S^n $(n \geqslant 5)$ such that*

(1) *Q is nondegenerate*
(2) *$Q_{|L=0}$ is isotropic*
(3) *Any quadratic form $\alpha_s Q_s + \beta_s L_s^2$ with α_s, β_s in k_s such that $(\alpha_s, \beta_s) \neq (0,0)$ for all $s \in S$ is irrational.*

Then there exists a k-rational subspace V of k^n of codimension 1 such that $F_{|V_S}$ satisfies the conditions $(1)(2)(3)$, moreover V can be chosen such that $Q_{|\{L=0\}\cap V_S}$ is nondegenerate.

Proof. When s is an archimedean real place, it is proved in ([9], Proposition 4) that there exists a subspace V_s of k^n of codimension 1 such that $F_{s|V_s}$ verifies conditions $(1)(2)(3)$. (therein the condition that $Q_{s|L_s=0}$ is nondegenerate for $s \in S$ refers to the condition $Q_{s|L_s=0}$ of type (I) in [9]). The proof non-archimedean case of ([9], Proposition 4) is analog to the real case. Therefore there exists a subspace V_s of k^n of codimension 1 such that $F_{s|V_s}$ verifies conditions $(1)(2)(3)$. Hence for any $s \in S$ we may find V_s a subspace of k^n of codimension 1 so that the conditions $(1)(2)(3)$ are satisfied by $F_{s|V_s}$ and one can choose V_s to be such that $Q_{s|\{L_s=0\}\cap V_s}$ is nondegenerate.
Assume that $n \geqslant 5$. Let us give $s \in S$ and V_s a k-subspace of codimension 1 in k_s^n such that the restriction of Q_s on V_s is non-degenerate and isotropic. Let us define $\mathcal{H}_s := SO(Q_s)$ the k_s-algebraic subgroup of the orthogonal group, the set of k_s-points $H_s = \mathcal{H}_s(K_s)$ is a Lie group over the algebraic closure K_s of k_s. For our needs, we introduce the following lemma which is valid for any field of characteristic zero (see e.g. [18], §3.1, Corollary 2).

Lemma 3.7. *Let $G \times X \longrightarrow X$ a K-defined action of a K-algebraic group on K-algebraic variety. If $x \in X(K)$ and Y is the closure of the orbit Gx then for any open $F \subset G(K)$, the set Fx is open on $Y(K)$. In particular Gx is open in $Y(K)$.*

Following Borel and Prasad ([3] Proposition 1.3), let us consider the k_s-action of the group H_s on the Grassmanian variety $\mathcal{G}_{n-1,n}$ of the hyperplanes over k_s. Since $Q_{|V_s}$ is non-degenerate we can use the previous lemma above in order to infer that the orbit H_sV is open in $\mathcal{G}_{n-1,n}(k_s)$ for the S-adic topology. Moreover by weak approximation in k_S we can find a rational subspace in V' of codimension 1 in k^n such that $V' \otimes_k k_s$ is arbitrarily close to V_s for all $s \in S$, in particular they belong to the same open orbit under H_s. We have established that $F_{s|V_s}$ satisfies conditions (1) and (2), it is equivalent to say that

$$\text{rad}(Q_s) \cap V_s = \{0\} \text{ and } c(Q_{s|L_s=0}) \cap V_s \neq \{0\}. \ (*)$$

The condition (2) remains true if we replace V_s by any subspace sufficiently close to V_s. Since the subspace rad(Q_s) is invariant under the action of the orthogonal group $\mathrm{SO}(Q_s)$, the condition (1) above is verified by any element of $\mathcal{G}_{n-1,n}(k_s)$ which lies in the orbit of V_s under $\mathrm{SO}(Q_s)$. In particular, $V' \otimes_k k_s$ satisfies $(*)$ for each $s \in S$. Hence we obtain a k-rational subspace V' of k^n such that $F_{|V'_S}$ satisfies the conditions $(1)(2)$. It remains to find such V such that in addition $F_{|V_S}$ satisfies condition (3). Let us put

$$\mathcal{V} = \Big\{ V \in \mathcal{G}_{n-1,n}(k) \,|\; F_{|V_S} \text{ satisfies conditions}\,(1)\,(2) \Big\}.$$

It is nonempty because it contains V'. Suppose there exists no V in $\mathcal{V}$ for which $F_{|V_S}$ satisfies condition (3), that is to say that for any $V \in \mathcal{V}$, it should exists some $s \in S$ and some $(\alpha_s, \beta_s) \in k_s^2 - \{(0,0)\}$, such that the quadratic form $\alpha_s Q_s(x) + \beta_s L_s(x)^2_{|V_s}$ is rational. Let us consider the map $f : k_s^n \longrightarrow k_s$ given by

$$f : x \mapsto \alpha_s Q_s(x) + \beta_s L_s(x)^2.$$

Clearly f is a regular function on K_s^n and for each $V \in \mathcal{V}$ we have $f(V(k)) \subset k$.The Zariski density of $\bigcup_{V \in \mathcal{V}} V(k)$ in K_s^n implies by Proposition 3.5 that f is defined over k. In other words, $\alpha_s Q_s(x) + \beta_s L_s(x)^2$ is rational over k, contradiction. Hence there exists $V \in \mathcal{V}$ such that $F_{|V_S}$ satisfies condition (3).

Corollary 3.8. *It suffices to prove Theorem 2.1 for $n = 4$.*

Proof. It follows from the proposition by descending induction on n.

3.4. Adeles and strong approximation for number fields

The set of adeles $\mathbb{A}$ of k is the subset of the direct product $\prod_{s \in \Sigma_k} k_s$ consisting of those $x = (x_s)$ such that $x \in \mathcal{O}_s$ for almost all $s \in \Sigma_k$. The set of adeles $\mathbb{A}$ is a locally topological ring with respect to the adele topology given by the base of open sets of the form $\prod_{s \in S} U_s \times \prod_{s \notin S} \mathcal{O}_s$ where $S \subset \Sigma_k$ is finite with $S \supset S_\infty$ and U_s are open subsets of k_s for each $s \in S$. reversemarginpar For any finite subset $S \subset \Sigma_k$ with $S \supset S_\infty$, the ring of S-integral adeles is defined by:

$$\mathbb{A}(S) = \textstyle\prod_{s \in S} k_s \times \prod_{s \notin S} \mathcal{O}_s, \text{ thus we can see that } \mathbb{A} = \displaystyle\bigcup_{S \supset S_\infty} \mathbb{A}(S).$$

We define also $\mathbb{A}_S$ to be the image of $\mathbb{A}$ onto $\prod_{s \notin S} k_s$, clearly $\mathbb{A} = k_S \times \mathbb{A}_S$.

Theorem 3.9 (Strong approximation)**.** *If $S \neq \emptyset$ the image of k under the diagonal embedding is dense in $\mathbb{A}_S$.*

4. Stabilizers of pairs (Q, L)

For each $s \in S$ let us define $G_s = \mathrm{SL}_4(k_s)$, $G_S = \prod_{s\in S} \mathrm{SL}_4(k_s) = \mathrm{SL}_4(k_S)$. Let $F = (Q, L)$ be a pair on k_S^4 satisfying the conditions $(1)(2)(3)$ of Theorem 2.1.
For every $s \in S$ we realize Q_s on a four-dimensional quadratic vector space (k_s^4, Q_s) over k_s equipped with the standard basis $\mathcal{B} = \{e_1, \cdots, e_4\}$. For each $s \in S$, let us define H_s to be the stabilizer of the pair F_s under the action of G_s, in other words

$$H_s = \Big\{ g \in G_s \mid Q_s \circ g = Q_s \,,\, L_s \circ g = L_s \Big\}.$$

Equivalently one can write $H_s = \Big\{ g \in \mathrm{SO}(Q_s) \mid L_s \circ g = L_s \Big\}$, clearly it is a linear algebraic group defined over k_s. Also let us define $V_s = \{L_s = 0\}$, it is an hyperplane of k_s^4 which induces a quadratic isotropic subspace $(V_s, Q_{s|V_s})$ of dimension 3 in k_s^4. We have two cases following $(V_s, Q_{s|V_s})$ is nondegenerate or not. If s is a real place the first case corresponds to pairs of type (I) in the terminology of [9].

Lemma 4.1. *Let be given a pair (Q, L) satisfying the conditions of Theorem 2.1 in dimension 4. Then the stabilizer of (Q, L) under the action of G is of the form (up to conjugation)*

$$H = \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array} \right) \mid A \in \mathrm{SO}(Q_{|L=0}) \right\} \subseteq \ \mathrm{SL}_4(\overline{k_s}).$$

In particular, H is semisimple. Moreover, any quadratic form $\widetilde{Q}$ which is H-invariant is of the form $\alpha Q + \beta L^2$ for some $\alpha, \beta \in k_S$ not both zero.

Proof. Since $(V_s, Q_{s|V_s})$ is nondegenerate, one can write the the following decomposition $k_s^4 = V_s \oplus V_s^{\perp}$ where $V_s^{\perp}$ the orthogonal complement w.r.t. Q. Since $\dim V_s^{\perp} = 1$ there exists some nonzero u vector of k_s^4 such that $V_s^{\perp} =< u >$ with $L_s(u) \neq 0$. Moreover by definition L_s is H_s-invariant so V_s is H_s-invariant. Moreover any element of $h \in H_s$ is in particular an element of $SO(Q_s)$, that is, $h^T = h$ hence $V_s^{\perp} =< u >$ is also H_s-invariant. Then for any $h \in H_s$, the restriction $h_{|V_s}$ induces an automorphism of V_s and $hu = u$. Let us put $w_4 = u$, and complete with a basis $\{w_1, w_2, w_3\}$ of V_s to obtain the following matrix representation of H_s up to a k_s-isomorphism of k_s^4,

$$H_s \simeq \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array} \right) \mid A \in \mathrm{SO}(Q_{s|V_s}) \right\} \subseteq \ \mathrm{SL}_4(K_s).$$

It is well-known that the orthogonal group of a nondegenerate quadratic form is a semisimple group. The last statement is the Lemma 9 in [9].

S-adic products

Now let $F = (Q_s, L_s)_{s\in S}$ be a pair satisfying the conditions of the main theorem. Let $\mathcal{H}_s$ be the algebraic group defined over k_s such that $\mathcal{H}_s(k_s) = H_s$. Given any subgroup H, the notation H^+ denotes the subgroup of H generated by its one-dimensional unipotent subgroups. Let us put

$$H_S = \prod_{s\in S} H_s \text{ and } H_S^+ = \prod_{s\in S} H_s^+.$$

Therefore H_S is an algebraic subgroup of $\mathrm{SL}_4(k_S)$ which leaves invariant the pair $F = (Q, L)$ with respect to the S-basis $\mathcal{B}' = \{w_1, w_2, w_3, w_4\}$ as in the previous lemma. In other words, we have

$$H_S \simeq \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array} \right) \mid A \in \mathrm{SO}(Q_{|V_S}) \right\} \subseteq \prod_{s\in S} \mathrm{SL}_4(K_s)$$

and

$$H_S^+ \simeq \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array} \right) \mid A \in \mathrm{SO}(Q_{|V_S})^+ \right\} \subseteq \prod_{s\in S} \mathrm{SL}_4(K_s).$$

5. Topological rigidity in S-adic homogeneous spaces

5.1. Ratner's topological rigidity theorem for unipotent groups actions

In this section, we are assuming that S is the set of archimedean places in Σ_k i.e. $S = S_\infty$. However, we allow ourselves to use the explicit notation S_∞ if necessary.

Let us consider the S-adic linear group defined over k_S, $G_S = \mathrm{SL}_n(k_S)$. It is not difficult to see that G_S is a Lie group which has a discrete subgroup a finite covolume $\Gamma_S = \mathrm{SL}_n(\mathcal{O}_S)$. Let us define Ω_S to be the quotient space given by G_S/Γ_S where $G_S = \mathrm{SL}_n(k_S)$. A well-known easy fact says that the homogeneous space Ω_S is one-to-one with the space of unimodular free $\mathcal{O}_S$-modules of maximal rank in k_S^n. For every $s \in S$, let $\mathcal{U}_s$ be a k_s-subgroup of G_s generated by its one-parameter unipotent subgroups. We denote by $\mathcal{U} = \prod_{s\in S} \mathcal{U}_s(k_s)$ the product subgroup of G_S. We are interested in the left action of $\mathcal{U}$ on the homogeneous space Ω_S and more particularly with the closure of such orbits taken with respect to the topology on Ω_S induced by the projection $\pi : G_S \to \Omega_S$. If $x \in \Omega_S$ it turns out that the closure of the orbit $\mathcal{U}x$ is also an orbit of x. The following result is the generalisation of Ratner's orbit closure theorem for S-products proven independently by Margulis-Tomanov and Ratner (see [15], [20]).

Theorem 5.1 (Ratner's Theorem for S-adic groups)**.** *For each $x \in \Omega_S$, there exists a closed subgroup $M = M(x) \subset G_S$ containing $\mathcal{U}$ such that the closure of the orbit $\mathcal{U}x$ coincides with Mx and Mx admits M-invariant probability measure.*

In the real case, this result was conjectured by Raghunathan who stated it with $G = SL(3, \mathbb{R})$, $U = SO(2, 1)^\circ$ and $\Gamma = SL(3, \mathbb{Z})$. He noticed that the proof of this Conjecture should lead to a solution of the Oppenheim conjecture. Despite appearences, the proof of this theorem is measure theoretic and consists to classify ergodic measures under the action of the unipotent flow on the homogeneous space Ω. In both proofs made by Margulis-Tomanov ([15]) and Ratner ([20]) the notion of entropy plays a central role for the measure classification.

5.2. The structure of intermediate subgroups in the archimedean case.

The application of Ratner's Theorem 5.1 above gives a nice description of the orbit closure under the action of a subgroup H^+ generated by unipotent one parameter subgroups in G_S. Considering a unimodular lattice $x \in \Omega_S$, we get that

$$\overline{H^+x} = L.x$$

for some closed connected subgroup L (depending on x) such that $H^+ \subset L \subset G$.
The following theorem due to N. Shah gives extra information about the structure of the intermediate subgroups arising from Ratner's orbit closure theorem above,

Comment (1)

Theorem 5.2 ([21], Prop. 3.2)**.** *Let $G = \mathcal{G}(\mathbb{R})^\circ$ with $\mathcal{G}$ an algebraic subgroup of $SL(n, \mathbb{C})$ defined over $\mathbb{Q}$ and $\Gamma = G \cap SL(n, \mathbb{Z})$. Let H be a subgroup of G generated by its unipotent one parameter subgroups and assume that*

$$\overline{H\Gamma} = P\Gamma \text{ where } P \text{ is a closed connected subgroup of } G$$

such that $P \cap \Gamma$ has finite covolume in P. Then $P = \widetilde{P}(\mathbb{R})^\circ$ where $\widetilde{P}$ is the smallest $\mathbb{Q}$-subgroup of $\mathcal{G}$ whose group of real points contains H.

By combining Ratner's theorem and previous proposition, we obtain immediately the following corollary stated in the set-up of the previous proposition.

Corollary 5.3 ([3], Corollary 7.2)**.** *Let $g \in G$ such that $x = g.0$ where 0 is the coset of Γ in G/Γ. Then $g^{-1}Lg = \widetilde{P}(\mathbb{R})$ where $\widetilde{P}$ is the smallest subgroup of whose group of real points contains $g^{-1}Hg$.*

Comment (2)

To use the results of this section for our purposes, we need to reduces to the case when $k = \mathbb{Q}$. This can be done by using the functor of restriction of scalars for algebraic groups.

Proposition 5.4. *Let k be a number field. Given any algebraic group $G \subset GL_n(\overline{k})$ defined over k, there exists an algebraic group G' defined over $\mathbb{Q}$ such that $G'(\mathbb{Q}) \simeq G(k)$.*

Proof. See e.g. [18], §2.1.2 for a general construction for finite separable extensions.

Definition 5.5 (Restriction of scalars)**.** *We denote G' by $\mathcal{R}_{k/\mathbb{Q}}(G)$ and it is called the algebraic group associated to G obtained by restriction of scalars from k to $\mathbb{Q}$.*

The operation $\mathcal{R}_{k/\mathbb{Q}}$ defines a functor from the category of k-groups to the category of $\mathbb{Q}$-groups. This functor has a nice arithmetic property regarding the set of integral points, given any algebraic group defined over k, we have[2]

$$\mathcal{R}_{k/\mathbb{Q}}(G)(\mathbb{Z}) \simeq G(\mathcal{O}_k)$$

Assume that $G = SL_{n|k}$ viewed as a algebraic group defined over k and all the places are archimedean i.e. $S = S_\infty$. The fact that $k_S = k \otimes_\mathbb{Q} \mathbb{R}$ implies $G_S = \mathcal{R}_{k/\mathbb{Q}}(G)(\mathbb{R})$ and the intermediate subgroups have still an interesting structure by means of restriction of scalars

Proposition 5.6 ([3], Prop. 7.3)**.** *Let H_s be a closed subgroup of $G_s = SL_n(k_s)$ for each $s \in S_\infty$ and H the product of the H_s. Then the smallest $\mathbb{Q}$-algebraic subgroup $\mathcal{L}$ of $\mathcal{G}$ whose group of real points contains H is of the form $\mathcal{L} = \mathcal{R}_{k/\mathbb{Q}}\mathcal{L}'$, where $\mathcal{L}'$ is a connected k-subgroup of $\mathcal{G}$.*

[2] This result will not be used in the sequel.

6. Proof of the Theorem 2.1

Let $F = (Q, L)$ be a pair in k_S^n which satisfies the conditions of Theorem 2.1. After § 3, we know that it suffices to show it for $n = 4$. By condition (3) all the forms $\alpha_s Q_s + \beta_s L_s^2$ are irrational for each α_s, β_s in k_s such that $(\alpha_s, \beta_s) \neq (0, 0)$ for any $s \in S$. Let $g \in G_S$ be the matrix of the basis $\mathcal{B}'_S$ with respect to the standard basis of k_S^4. By definition $g^{-1} H_S g$ leaves invariant the pair $F = (Q_s, L_s)_{s\in S}$, and H_S^+ is generated by one-dimensional unipotent subgroups. We consider Γ_S as an element of the homogeneous space Ω_S. By applying Ratner's Theorem 5.1, one obtains

$$\overline{g^{-1} H_S^+ g \Gamma_S} = P \Gamma_S \tag{5}$$

where P is a closed subgroup of G_S which contains $g^{-1} H_S^+ g$.

Since we assume that $S = S_\infty$, we have $\mathcal{O}_S^4 = \mathcal{O}^4$. Let us write k_∞, H_∞ and G_∞ respectively for k_{S_∞}, H_{S_∞} and G_{S_∞}. One can also note that H_∞^+ is nothing else than the component of the identity H_∞°. Using equality (5) one deduces that the set $F(\mathcal{O}^4)$ is dense in k_∞^2. Indeed, we are going to adapt the proof of ([9], Proposition 10) to the S_∞-products, as follows[3]. We first reduce the ground field from k to the field of rational numbers. To achieve this we realise G_∞ as the group of real points of an algebraic group $\mathcal{G}$ defined over $\mathbb{Q}$. In view of Proposition 5.6 this is given explicitly by taking $\mathcal{G} = R_{k/\mathbb{Q}} \mathrm{SL}_4$ where $R_{k/\mathbb{Q}}$ is the functor restriction of scalars of the field extension $k/\mathbb{Q}$ and where SL_4 is regarded as the usual algebraic group over k. In other words, $G_\infty = \mathcal{G}(\mathbb{R})$ with $\mathcal{G}$ an algebraic group defined over $\mathbb{Q}$. Now let us precise the structure of P. From Corollary 5.3 above, we infer that there exists an algebraic group $\widetilde{P}$ defined over $\mathbb{Q}$ which is the smallest $\mathbb{Q}$-algebraic group whose group of real points contains $g^{-1} H_\infty^\circ g$. In the other hand, Proposition 5.2 implies that $P = \widetilde{P}(\mathbb{R})^\circ$ and the unipotent radical $\mathcal{U}$ of $\widetilde{P}$ is also defined over $\mathbb{Q}$. Thus equality (5) may be read as

Comment (3)

$$\overline{g^{-1} H_\infty^\circ g \Gamma} = \widetilde{P}(\mathbb{R})^\circ \Gamma. \tag{6}$$

Let us set the group $U = \mathcal{U}(\mathbb{R})$, then for each $s \in S_\infty$, we define P_s (resp. U_s) to be the intersection of P (resp. U) with G_s , where G_s is identified with the subgroup of G consisting of elements $(g_t)_{t\in S}$ with $g_t = 1$ for all $s \neq t$.

Lemma 6.1. *If P_s acts irreducibly on $\mathbb{C}^4$, then $P_s = G_s$. Otherwise, $P_s = M_s U_s$ where*

$$M_s = u g_s^{-1} \left(\begin{array}{c|c} \mathrm{SL}_3 & 0 \\ \hline 0 & 1 \end{array} \right) g_s u^{-1} \text{ for some } u \in U_s.$$

Proof the Lemma. This result is the core of the proof of Proposition 10 in [9] for which we recall the outlines. If P_s acts irreducibly on $\mathbb{C}^4$, then P_s is semisimple and the classification of irreducible semisimple Lie groups in SL_4 implies that P_s is equal either to G_s or $SO(B_s)$ for some nondegenerate form B_s (Proposition 7 and Lemma 8, [9]). Such form B_s being H_s-invariant is necessarily of the form $\alpha_s Q_s + \beta_s L_s^2$ for some $(\alpha_s, \beta_s) \neq (0, 0)$ (Lemma 4.1). As seen before P_s is defined over $\mathbb{Q}$, so that B_s is forced to be rational which is a contradiction. Hence $P_s = G_s$. For the second assertion, we consider the induced action of P_s on the space $\mathcal{L}$ of linear forms in $\mathbb{C}^4$, it is reducible by hypothesis. There only two reducible components in $\mathcal{L}$ under H, namely

[3] For more precisions, the reader is invited to read the original proof which is similar.

$\langle x_1, x_2, x_3\rangle$ and $\langle x_4\rangle$. Since P_s contains $g^{-1}H_s g$, there are only two P_s-invariant subspaces in $\mathcal{L}$, which are given by $\mathcal{L}_1 = \langle L_1, L_2, L_3\rangle$ and $\mathcal{L}_2 = \langle L_4\rangle$ where $L_i(x) = (gx)_i$ for $i = 1, \ldots, 4$, note that $L_4 = L$. Since P_s is defined over $\mathbb{Q}$, one infers that M is semisimple thus admitting a Levi decomposition

$$P_s = M_s U_s$$

where M_s and U_s are respectively a Levi subgroup and the unipotent radical of P_s. The Levi subgroup M_s is defined over $\mathbb{Q}$ since P_s is. Also as seen above, $\mathcal{U}$ is defined over $\mathbb{Q}$ and Malcev's theorem ensures that the Levi subgroups are unique up to conjugacy (e.g. see §4.3 [16]), in particular

$$g_s^{-1} H_s^\circ g_s \subseteq u^{-1} M_s u \qquad \text{for some } u \in U_s. \tag{7}$$

Moreover this inclusion is strict, indeed H_s leaves invariant any linear combination of Q_s and L_s^2 which by assumption are always irrational thus H_s is not defined over $\mathbb{Q}$. The latter fact and the maximality of $\mathrm{SO}(Q_{|L=0})$ in SL_3 ($Q_{|L=0}$ is isotropic) give the equality

$$M_s = u g_s^{-1} \left(\begin{array}{c|c} \mathrm{SL}_3 & 0 \\ \hline 0 & 1 \end{array}\right) g_s u^{-1}.$$

This achieves the proof of the Lemma.

Let us define the subgroup

$$M_s' := u^{-1} M_s u = g_s^{-1} \left(\begin{array}{c|c} \mathrm{SL}_3 & 0 \\ \hline 0 & 1 \end{array}\right) g_s.$$

By the previous Lemma 6.1, one can rephrase equality (6) in the following way

$$\overline{g^{-1} H_\infty^\circ g\Gamma} = M'(\mathbb{R})^\circ \mathcal{U}(\mathbb{R})\Gamma. \tag{8}$$

Now let be given $(a, b) \in k_\infty^2$ and let us choose $x \in \mathcal{O}^4 - \langle g^{-1} e_4\rangle$. It is not difficult to see that there exists $m \in M'(\mathbb{R})^\circ$ and $u \in \mathcal{U}(\mathbb{R})$ such that

$$F(mux) = (Q(mux), L(mux)) = (a, b).$$

Using density in (8), we infer that there exists $h_n \in g^{-1} H_\infty^\circ g$ and $\gamma_n \in \Gamma$ such that

$$h_n \gamma_n \to mu \text{ as } n \to \infty.$$

We conclude that

$$F(\gamma_n x) = F(h_n \gamma_n x) \to F(umx) = (a, b) \text{ as } n \to \infty.$$

Hence $F(\mathcal{O}^4)$ is dense in k_∞^2 and Theorem 2.1 is proved.

7. Proof of Corollary 2.2

Now let us assume $S \neq S_\infty$ and let be given $s \in S_f$. The set $\mathcal{O}^4$ is bounded in k_s^4, thus for any neighbourhood U of the origin in k_s^4 one can find an integer $a_s \in \mathcal{O}_s$ such that $a_s.\mathcal{O}^4 \subset U$. In other words, given any $\varepsilon > 0$ one can find $a_s \in \mathcal{O}_s$ such that :

$$|Q_s(a_s x)|_s \leqslant \varepsilon \quad \text{and} \quad |L_s(a_s x)|_s \leqslant \varepsilon \quad \text{for} \quad \text{all} \quad x \in \mathcal{O}^4. \tag{9}$$

Thus for each $s \in S_f$, we can associate an integer $a_s \in \mathcal{O}_s$ satisfying the previous inequalities. By strong approximation one can find $a \in \mathcal{O}$ such that $|a|_s = |a_s|_s$ for all $s \in S_f$. Put $\|a\|_\infty = \max_{s \in S_\infty} |a|_s$, by Theorem 2.1 we can find $x \in \mathcal{O}^4 - \{0\}$ such that:

$$|Q_s(x)|_s \leqslant \varepsilon/\|a\|_\infty^2 \text{ and } |L_s(x)|_s \leqslant \varepsilon/\|a\|_\infty \text{ for all } s \in S_\infty.$$

We immediately obtain for all $s \in S_\infty$

$$|Q_s(a_s x)|_s = |a_s|_s^2 |Q_s(x)|_s \leqslant \varepsilon \quad \text{and} \quad |L_s(a_s x)|_s = |a_s|_s |L_s(x)|_s \leqslant \varepsilon. \tag{10}$$

Let us consider the initial choice of $\epsilon > 0$. By combining (10) with the choice of a satisfying (9), we obtain a nonzero element $y = a.x \in \mathcal{O}_S^4$ satisfiying the conclusion of Corollary 2.2, i.e.

$$|Q_s(y)|_s \leqslant \varepsilon \text{ and } |L_s(y)|_s \leqslant \varepsilon \text{ for all } s \in S.$$

8. Comments and open problems

Irrationality of the pencils forms

The rationality condition in Theorem 2.1, namely asking irrationality of all the pencils of Q and L^2 at all places of S is more restrictive than assuming irrationality of the pencils over k_S. Indeed the latter condition leaves the possibility that some pencil could be rational at some place(s) of S. In this case, using Ratner's theorem and therefore the classification of intermediate subgroups cannot be achieved by our methods. By analogy with the work of Borel and Prasad in the case of a family of quadratic forms $(Q_s)_{s \in S}$, it may be possible to apply strong approximation and avoiding reduction of dimension. The problem is that this method does not give integral solutions $x \in \mathcal{O}_S^n$ of inequalities $|Q(x)| \leq \varepsilon$ and $|L(x)| \leq \varepsilon$ but only nonzero integral solutions of the pencil forms $|\alpha Q(x) + \beta L^2(x)| \leq \varepsilon$ which may depend on the coefficients α and β. The most serious issue is to eliminate the dependance on the coefficients, that is, to replace the valid assertion

$$\forall \varepsilon > 0, \forall P \in \mathbb{P}^1(k_S), \exists x \in \mathcal{O}_S^n - \{0\}, \quad |\widetilde{Q}_P(x)|_S \leq \varepsilon$$

by the one we would like

$$\forall \varepsilon > 0, \exists x \in \mathcal{O}_S^n - \{0\}, \forall P \in \mathbb{P}^1(k_S), \quad |\widetilde{Q}_P(x)|_S \leq \varepsilon.$$

Indeed if one is able to do so, such x will satisfies those inequalities for both $P_1 = [0 : 1]$ and $P_2 = [0 : 1]$ and by homogeneity it would give the solution of our problem.

Towards density

It should be possible to obtain the density of $F(\mathcal{O}_S^n)$ for a pair $F = (Q, L)$ over k_S under the same assumptions generalising those of Theorem 1.1. For this we need a analog of Lemma 6 of [9] for nonarchimedean completions which has no clear reason to fail in characteristic zero. A significant difference with the classical Oppenheim conjecture is that the stabilizer of such pairs is no more maximal, and the classification of intermediate subgroups is much more involved. Unfortunately we are not able to prove Lemma 6.1 for non archimedean completions and to avoid the use of strong approximation.

Some Open problems

We conclude by mentioning two conjectures of Gorodnik (see [9], conjecture 15) and (see [10], conjecture 13) which concerns the assumption (2) of Theorem 1.1 in the real case. It is conjectured that the condition $Q_{|L=0}$ is isotropic can be replaced by the weaker assumption that the pencil $\alpha Q + \beta L^2$ is isotropic for any real numbers α, β such that $(\alpha, \beta) \neq (0, 0)$.

Conjecture 8.1 (Gorodnik)**.** *Let $F = (Q, L)$ be a pair consisting of one nondegenerate quadratic and one nonzero linear form in dimension $n \geq 4$. Suppose that*

1. *For every $\beta \in \mathbb{R}$, $Q + \beta L^2$ is indefinite.*
2. *For every $(\alpha, \beta) \neq (0, 0)$, with $\alpha, \beta \in \mathbb{R}$, $\alpha Q + \beta L^2$ is irrational.*

Then $F(\mathcal{P}(\mathbb{Z}^n))$ is dense in $\mathbb{R}^2$.

The first condition is necessary for density to hold. The main issue is that this condition (contrarily to the condition that $Q_{|L=0}$ is indefinite) does not allow us to reduce to the four dimensional case. Hence all the strategy of the proof of Theorem 1.1 becomes needless regarding the impossibility to classify all the intermediate subgroups in higher dimension.

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