Properly stratified quotients of quiver Hecke algebras

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Abstract

Introduced in 2008 by Khovanov and Lauda, and independently by Rouquier, the quiver Hecke algebras are a family of infinite dimensional graded algebras which categorify the negative part of the quantum group associated to a graph. In finite types these algebras are known to have nice homological properties, in particular they are affine quasi-hereditary. In this thesis we utilise the affine quasi-hereditary structure to create finite dimensional quotients which preserve some of the homological structure of the original algebra.

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Contents

Introduction

Introduced in 2008 by Khovanov and Lauda [KL09], and separately Rouquier [Rou], the quiver Hecke algebras, or KLR algebras, are a family of graded algebras which categorify the negative part of the quantum group associated to a graph Γ. That is, for the KLR algebra $R_n(\Gamma)$ associated to Γ, there are canonical isomorphisms

$$(U_q^-(\mathfrak{g}))^* \cong \bigoplus_{n\geq 0} K_0(R_n(\Gamma)\text{-}\mathbf{gr.\,mod^{fd}}),$$

and, equivalently,

$$U_q^-(\mathfrak{g}) \cong \bigoplus_{n\geq 0} K_0(R_n(\Gamma)\text{-}\mathbf{p.\,mod}),$$

where $K_0(R_n(\Gamma)\text{-}\mathbf{gr.\,mod^{fd}})$ is the Grothendieck group of finite dimensional graded $R_n(\Gamma)$-modules, $K_0\left(R_n(\Gamma)\text{-}\mathbf{p.\,mod}\right)$ is the Grothendieck group of graded projective $R_n(\Gamma)$-modules, and $\mathfrak{g}$ is the Kac-Moody algebra associated to Γ. We have $U_q(\mathfrak{g})$ acting on the Grothendieck group as induction and restriction functors. Khovanov and Lauda also introduced certain cyclotomic finite dimensional graded quotients of the quiver Hecke algebra. Brundan and Kleshchev established an isomorphism between blocks of the cyclotomic Hecke algebra and blocks of the cyclotomic quiver Hecke algebra, which allowed them to introduce a grading on the cyclotomic Hecke algebra.

The affine cellularity of quiver Hecke algebras in finite type A was discovered by Kleshchev, Loubert and Miemietz [KLM13] and was later generalised by the first two authors to all finite types [KL15]. Establishing affine cellularity reproved finite global dimension for quiver Hecke algebras in finite type, a result that had already been shown by Kato [Kat]. An explicit value for the dimension was computed by McNamara [McN13].

In this thesis we construct an ideal $\mathcal{J}$ of the quiver Hecke algebra R_α and show that quotienting by this ideal produces a finite dimensional algebra which preserves much of the original algebra's homological structure. Our work concentrates on quiver Hecke algebras in type A as it uses foundations laid down in [KLM13]. Chapters 1 and 2 introduce the main players, bringing together definitions and theorems from the literature and establishing some technical results which are crucial to the construction of this ideal. In Chapter 3 we define the ideal $\mathcal{J}$ of the quiver Hecke algebra R_α, and define the quotient algebra $R_\alpha^{\mathcal{J}} := R_\alpha/\mathcal{J}$. We then provide some background on stratified algebras in Chapter 4 and establish a line of attack to prove that $R_\alpha^{\mathcal{J}}$ is properly stratified. Chapter 5 studies the homological structure of $R_\alpha^{\mathcal{J}}$, and highlights the similarities with R_α, in particular we have a quotient which preserves proper standard modules. We establish that $R_\alpha^{\mathcal{J}}$ is cellular and properly stratified. We then look at the case where every simple root has multiplicity at most one in the root α indexing the block R_α of $R_n(\Gamma)$. Here we provide a proof to a theorem of Brundan and Kleshchev, and use that to establish a special case in which the standard modules and proper standard modules of $R_\alpha^{\mathcal{J}}$ coincide, in particular this mean that $R_\alpha^{\mathcal{J}}$ is a quasi-hereditary quotient of the quiver Hecke algebra. Finally, Chapter 6 provides some worked examples and in particular highlights the example of $\alpha = 2\alpha_1 + \alpha_2$, for which one is unable to take a quasi-hereditary quotient of R_α while still preserving the proper standard modules.

Chapter 1

Background and definitions

We fix, once and for all, a field $\Bbbk$. Unless otherwise specified modules will be assumed to be left modules, when we need to distinguish that M is a left, resp. right, modules over an algebra A we write ${}_AM$, resp. M_A.

1.1 Quiver Hecke algebras

We begin with some Lie theoretic information, and fix notation that will be used throughout this report. We introduce the main objects here as well as some preliminary results. The content on graded algebras is taken from [HM10] and [Kle15], the rest of the chapter, unless otherwise indicated, can be found in [KLM13] and [Bru13].

Lie theoretic notation For a Dynkin quiver of type A_∞ with set of vertices $I = \mathbb{Z}$ we have the corresponding Cartan matrix with entries

$$a_{i,j} = \begin{cases} 2 & \text{if } i = j, \\ 0 & \text{if } |i-j| > 1, \\ -1 & \text{if } i = j \pm 1 \end{cases},$$

for $i, j \in I$. We also have a set of simple roots $\{\alpha_i \mid i \in I\}$ and the Cartan matrix defines a bilinear form such that $\alpha_i \cdot \alpha_j = a_{i,j}$ on the positive part of the root lattice $\mathcal{Q}_+ := \oplus_{i \in I} \mathbb{N}_0 \alpha_i$. The set of positive roots is given by

$$\Phi_+ := \{\alpha(m,n) := \alpha_m + \alpha_{m+1} + \cdots + \alpha_n \mid m, n \in I, m \leq n\}.$$

For $\alpha = \sum_{i \in I} c_i \alpha_i \in \mathcal{Q}_+$, we denote the *height* of α by $|\alpha| = \sum_{i \in I} c_i$.

The symmetric group $\mathfrak{S}_d$, generated by simple transpositions $s_1, \ldots, s_{d-1}$, acts on the set I^d by place permutation. The orbits under this action are the sets

$$\langle I\rangle_\alpha := \{\boldsymbol{i} = (i_1, \dots, i_d) \in I^d \mid \alpha_{i_1} + \cdots + \alpha_{i_d} = \alpha\}$$

for each $\alpha \in \mathcal{Q}_+$ with $|\alpha| = d$. We define a partial ordering $\leq$ based on the lexicographic order on $\langle I\rangle_\alpha$ which is determined by the natural order on $I = \mathbb{Z}$, by which we mean $(i_1, \cdots, i_d) < (i'_1, \cdots, i'_d)$ if and only if there is an integer k, with $1 \leq k \leq d$, such that $i_j = i'_j$ for $j < k$ and $i_k < i'_k$.

To a positive root $\beta = \alpha(m, n)$, we associate the word

$$\boldsymbol{i}_\beta := (m, m+1, \dots, n) \in \langle I\rangle_\beta.$$

We define a total order on Φ_+ by $\beta \leq \gamma$ if and only if $\boldsymbol{i}_\beta \leq \boldsymbol{i}_\gamma$, for $\beta, \gamma \in \Phi_+$.

Graded algebras An $\mathbb{I}$-*graded* $\mathbb{k}$-*module* is a $\mathbb{k}$-module M with a decomposition $M = \oplus_{i\in\mathbb{I}} M_i$, where $\mathbb{I}$ is some indexing set with a binary operation $+$. Elements $m \in M_i$ are called *homogeneous of degree* i. When we omit the grading set and just say graded module, etc, we shall mean $\mathbb{Z}$-graded.

A *graded* $\mathbb{k}$-*algebra* is a unital associative $\mathbb{k}$-algebra $A = \oplus_{i\in\mathbb{Z}} A_i$ which is a graded $\mathbb{k}$-module such that $A_i A_j \subseteq A_{i+j}$ for all $i, j \in \mathbb{Z}$. An A-module M is called a *graded (left) A-module* if it is a graded $\mathbb{k}$-module such that $A_i M_j \subseteq M_{i+j}$ for all $i, j \in \mathbb{Z}$. Graded submodules, graded right modules are all defined analogously. For a graded vector space $V = \oplus_{i\in\mathbb{Z}} V_i$ we say V is *locally finite* if each graded component V_i is finite, and we say it is *bounded below* if $V_i = 0$ for all $i << 0$. We define the *graded dimension* $\dim_q V := \sum_{i\in\mathbb{Z}} (\dim V_i) q^i$, where q is a formal variable. We also use q for the degree shift functor, so qV has $(qV)_i := V_{i-1}$. We call a graded vector space *Laurentian* if it is both locally finite and bounded below, in this case its graded dimension $\dim_q V$ is a formal Laurent series.

The KLR algebra Let $\alpha \in \mathcal{Q}_+$ be of height d and let $\mathbb{k}$ be a commutative unital ring. Then the *quiver Hecke algebra (of finite type A)* (also called the *Khovanov-Lauda-Rouquier (KLR) algebra*) $R_\alpha = R_\alpha(\mathbb{k})$ is the associative, unital $\mathbb{k}$-algebra generated by

$$\{e(\boldsymbol{i}) \mid \boldsymbol{i} \in \langle I\rangle_\alpha\} \cup \{y_1, \dots, y_d\} \cup \{\psi_1, \dots, \psi_{d-1}\}$$

subject to the following relations

$$e(\boldsymbol{i})e(\boldsymbol{j}) = \delta_{\boldsymbol{i},\boldsymbol{j}}e(\boldsymbol{i}); \qquad \sum_{\boldsymbol{i}\in\langle I\rangle_\alpha} e(\boldsymbol{i}) = 1;$$

$$y_r e(\boldsymbol{i}) = e(\boldsymbol{i})y_r; \qquad \psi_r e(\boldsymbol{i}) = e(s_r\cdot\boldsymbol{i})\psi_r; \quad y_r y_s = y_s y_r;$$

$$\psi_r y_s = y_s\psi_r \qquad \text{if } s \neq r, r+1;$$

$$\psi_r\psi_s = \psi_s\psi_r \qquad \text{if } |r-s| > 1;$$

$$\psi_r y_{r+1}e(\boldsymbol{i}) = (y_r\psi_r + \delta_{i_r,i_{r+1}})e(\boldsymbol{i}); \quad y_{r+1}\psi_r e(\boldsymbol{i}) = (\psi_r y_r + \delta_{i_r,i_{r+1}})e(\boldsymbol{i});$$

$$\psi_r^2 e(\boldsymbol{i}) = \begin{cases} 0 & \text{if } i_r = i_{r+1}, \\ e(\boldsymbol{i}) & \text{if } |i_r - i_{r+1}| > 1, \\ (y_{r+1}-y_r)e(\boldsymbol{i}) & \text{if } i_r = i_{r+1}-1, \\ (y_r - y_{r+1})e(\boldsymbol{i}) & \text{if } i_r = i_{r+1}+1; \end{cases}$$

$$\psi_r\psi_{r+1}\psi_r e(\boldsymbol{i}) = \begin{cases} (\psi_{r+1}\psi_r\psi_{r+1}+1)e(\boldsymbol{i}) & \text{if } i_{r+2} = i_r = i_{r+1}-1, \\ (\psi_{r+1}\psi_r\psi_{r+1}-1)e(\boldsymbol{i}) & \text{if } i_{r+2} = i_r = i_{r+1}+1, \\ \psi_{r+1}\psi_r\psi_{r+1}e(\boldsymbol{i}) & \text{otherwise.} \end{cases}$$

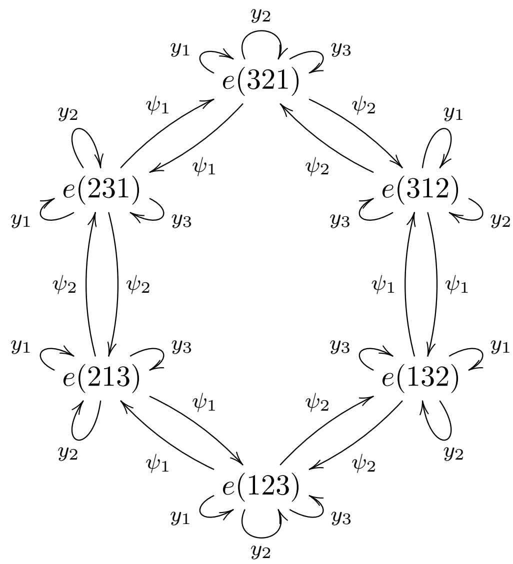
The algebra R_α possesses a unique $\mathbb{Z}$-grading such that all $e(\boldsymbol{i})$ are of degree 0, all y_r are of degree 2, and $\deg(\psi_r e(\boldsymbol{i})) = -a_{i_r,i_{r+1}}$, where $a_{i_r,i_{r+1}}$ is an entry in the Cartan matrix. For any reduced decomposition $w = s_{i_1}s_{i_2}\cdots s_{i_r} \in \mathfrak{S}_d$, define $\psi_w := \psi_{i_1}\psi_{i_2}\cdots\psi_{i_r}$.

Remark 1.1. Our ψ_w does depend on the choice of reduced expression for w, however, one deduces from the last relation that given two reduced expressions $\dot{w}$, $\ddot{w}$ of w, $\psi_{\dot{w}}$ and $\psi_{\ddot{w}}$ differ only by a sum of ψ_v for $l(v) < l(w)$. Henceforth we fix a reduced expression for every $w \in \mathfrak{S}_d$.

Example 1.2. Let us consider the root $\alpha = \alpha_1 + \alpha_2 + \alpha_3$, then R_α has generators

$$\{e(123), e(132), e(213), e(231), e(312), e(321), y_1, y_2, y_3, \psi_1, \psi_2\}$$

and we associate to R_α the following quiver.



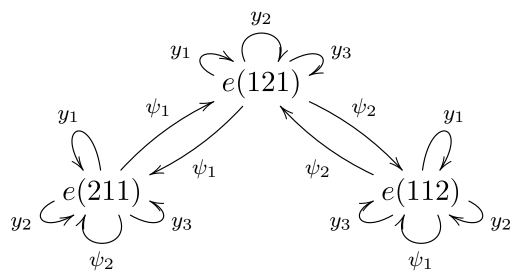
Relations give us, for example,

$$\psi_2^2 e(312) = (y_3 - y_2)e(312); \quad \psi_1^2 e(312) = e(312);$$
$$\psi_2 y_3 e(123) = y_2 \psi_2 e(123); \quad \psi_1\psi_2\psi_1 e(321) = \psi_2\psi_1\psi_2 e(321).$$

Example 1.3. If we consider the quiver Hecke algebra associated to the root $\alpha = \alpha_1 + \alpha_1 + \alpha_2$, then we have the generating set

$$\{e(112), e(121), e(211), y_1, y_2, y_3, \psi_1, \psi_2\}$$

and we associate to R_α the following quiver.



Relations give us, for example,

$$\psi_1^2 e(112) = 0; \quad \psi_1 y_2 e(112) = (y_1\psi_1 + 1)e(112);$$
$$\psi_1\psi_2\psi_1 e(121) = (\psi_2\psi_1\psi_2 + 1)e(121).$$

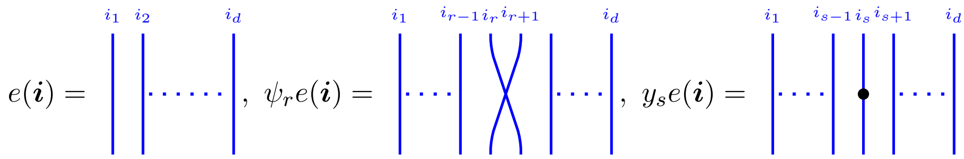
A theorem of Khovanov and Lauda provides a nice basis for this algebra.

Theorem 1.4. *[KL09, Theorem 2.5] For an arbitrary field $\mathbb{F}$, the elements*

$$\{\psi_w y_1^{r_1} \cdots y_d^{r_d} e(\boldsymbol{i}) \mid w \in \mathfrak{S}_d, r_1, \ldots, r_d \in \mathbb{Z}_{\geq 0}, \boldsymbol{i} \in \langle I \rangle_\alpha\}$$

form an $\mathbb{F}$-basis for $R_\alpha(\mathbb{F})$.

The quiver Hecke algebra can also be defined with diagrammatic notation, as introduced in [KL09]. For $\boldsymbol{i} = (i_1, \ldots, i_d) \in \langle I \rangle_\alpha$, we write



where $1 \leq r < d$ and $1 \leq s \leq d$. Multiplication of elements is concatenation of diagrams with matching labels, read from top to bottom and zero if the labels do not match.

The centre of R_α Let $\boldsymbol{i} \in \langle I \rangle_\alpha$ be such that $\mathfrak{S}_{\boldsymbol{i}} := \mathrm{Stab}_{\mathfrak{S}_d}(\boldsymbol{i})$ is a standard parabolic subgroup of $\mathfrak{S}_d$. It is easy to see that this is equivalent to all equal entries in $\boldsymbol{i}$ appearing consecutively. Let us denote by $\mathfrak{S}^{\boldsymbol{i}}$ the set of shortest length left coset representatives of $\mathfrak{S}_{\boldsymbol{i}}$ in $\mathfrak{S}_d$. Then for $j = 1, \ldots, d$ we define

$$z_j := \sum_{w \in \mathfrak{S}^{\boldsymbol{i}}} y_{w(j)} e(w(\boldsymbol{i})), \tag{1.1}$$

and we let $\mathfrak{S}_{\boldsymbol{i}}$ act on $\Bbbk[z_1, \ldots, z_d]$ by permuting the generators. For example, let $\alpha = 2\alpha_1 + \alpha_2$, and $\boldsymbol{i} = (112)$ then

$$z_1 = y_1 e(112) + y_1 e(121) + y_2 e(211), \tag{1.2}$$
$$z_2 = y_2 e(112) + y_3 e(121) + y_3 e(211), \tag{1.3}$$
$$z_3 = y_3 e(112) + y_2 e(121) + y_1 e(211). \tag{1.4}$$

Theorem 1.5 ([Bru13, Theorem 2.7])**.** *The centre of the algebra R_α is given by*

$$Z(R_\alpha) = \Bbbk[z_1, \ldots, z_d]^{\mathfrak{S}_{\boldsymbol{i}}}.$$

Root partitions and blocks Let $\alpha \in \mathcal{Q}_+$ with $|\alpha| = d$. A *root partition* of α is a way to write α as an ordered sum of positive roots

$$\alpha = p_1\beta_1 + \cdots + p_n\beta_n$$

so that $\beta_1 > \cdots > \beta_n$ and $p_1, \ldots, p_n > 0$. We denote such a root partition π as $\pi = \beta_1^{p_1} \ldots \beta_n^{p_n}$. Let $\Pi(\alpha)$ denote the set of root partitions of α. Within a root partition we call each β_i a *π-block of weight β_i*. Each root partition π has an associated idempotent $e(\boldsymbol{i}_\pi) \in R_\alpha$ with the word β_π given by the concatenation of $\boldsymbol{i}_{\beta_k}$ for $1 \leq k \leq n$

$$\boldsymbol{i}_\pi := \boldsymbol{i}_{\beta_1} \ldots \boldsymbol{i}_{\beta_1} \ldots \boldsymbol{i}_{\beta_n} \ldots \boldsymbol{i}_{\beta_n} \in \langle I \rangle_\alpha$$

where each $\boldsymbol{i}_{\beta_k}$ appears p_k times. Define the total order on $\Pi(\alpha)$ by $\pi \geq \sigma$ if and only if $\boldsymbol{i}_\pi \geq \boldsymbol{i}_\sigma$ for $\pi, \sigma \in \Pi(\alpha)$.

Lemma 1.6. *Let $\leq$ denote the lexicographic order on $\langle I \rangle_\alpha$. Assume that $\boldsymbol{i} \leq \boldsymbol{i}_\pi$ for all $\pi \in \Pi(\alpha)$, then $\boldsymbol{i} = \boldsymbol{i}_\pi$ if and only if $\pi = \alpha_1 + \cdots + \alpha_n$.*

Proof. Let $\pi = \alpha_1 + \cdots + \alpha_n$ then $\boldsymbol{i}_\pi \leq \boldsymbol{i}$ for all $\boldsymbol{i} \in \langle I \rangle_\alpha$, so $\boldsymbol{i} = \boldsymbol{i}_\pi$. Conversely, assume that $\pi \neq \alpha_1 + \cdots + \alpha_n$. Then either α contains repeated simple roots or there exists a $\sigma < \pi \in \Pi(\alpha)$ with $\sigma = \alpha_1 + \cdots + \alpha_n$ in the latter case, $\boldsymbol{i} \neq \boldsymbol{i}_\pi$. Without loss of generality let $\alpha = \alpha_1 + \cdots + 2\alpha_i + \cdots + \alpha_n$. Then

$$\pi = (\alpha_i + \cdots + \alpha_n)(\alpha_1 + \cdots + \alpha_i) \leq \sigma$$

for all $\sigma \in \Pi(\alpha)$, but $\boldsymbol{i} = 1 \cdots ii \cdots n <_{lex} \boldsymbol{i}_\pi \leq \boldsymbol{i}_\sigma$ for all $\sigma \in \Pi(\alpha)$. So the lowest root in $\Pi(\alpha)$ is α_i □

Example 1.7. For $\pi = (\alpha_3)^4(\alpha_2 + \alpha_3)^2(\alpha_2)^3(\alpha_1 + \alpha_2)$ we have

$$e(\boldsymbol{i}_\pi) = e(3333232322212)$$

and there are four (α_3) blocks, two $(\alpha_2 + \alpha_3)$ blocks, three (α_2) blocks and one $(\alpha_1 + \alpha_2)$ block.

To any π we associate the Young subgroup

$$\mathfrak{S}_\pi \cong \mathfrak{S}_{|\beta_1|}^{p_1} \times \cdots \times \mathfrak{S}_{|\beta_n|}^{p_n} \leq \mathfrak{S}_d,$$

and denote by $\mathfrak{S}^\pi$ the set of shortest left coset representatives for $\mathfrak{S}_\pi$ in $\mathfrak{S}_d$.

Lemma 1.8. *If $w \in \mathfrak{S}^\pi$ then $w(\boldsymbol{i}_\pi) \leq \boldsymbol{i}_\pi$.*

Proof. This follows directly from the definition of $\mathfrak{S}^\pi := \mathfrak{S}_d/\mathfrak{S}_\pi$. □

Example 1.9. Take the root partition $\pi = (\alpha_1 + \alpha_2)(\alpha_1)$. Then we label the generators of $\mathfrak{S}_3$ as s_1 and s_2, where the subscript tells us that they act on $\boldsymbol{i} = (121)$ by swapping the i^{th} and $(i+1)^{st}$ positions, we get $\mathfrak{S}_\pi = \langle e, s_1 \rangle \cong \mathfrak{S}_2$ and $\mathfrak{S}^\pi = \langle e, s_2, s_1 s_2 \rangle$.

1.2 Affine nil-Hecke algebras

A basic introduction to (affine) nil-Hecke algebras is detailed by Rouquier [Rou12]. In the case that $\alpha = a\alpha_n$, $a \in \mathbb{N}$, then R_α is isomorphic to the a^{th} affine nil-Hecke algebra, NH_a, where NH_a, is defined to be the associative unital ($\mathbb{Z}$-)algebra generated by $\{y_1, \ldots, y_a, \psi_1, \ldots, \psi_{a-1}\}$ subject to the relations

$$\begin{aligned} \psi_r^2 &= 0; \\ \psi_r\psi_s &= \psi_s\psi_r \qquad \text{if } |r-s|>1; \\ \psi_r\psi_{r+1}\psi_r &= \psi_{r+1}\psi_r\psi_{r+1}; \\ \psi_r y_s &= y_s\psi_r \qquad \text{if } s \neq r, r+1; \\ \psi_r y_{r+1} &= y_r\psi_r + 1; \\ y_{r+1}\psi_r &= \psi_r y_r + 1. \end{aligned}$$

Again we define $\psi_w := \psi_{i_1} \cdots \psi_{i_k}$ for a reduced decomposition of $w = s_{i_1} \cdots s_{i_k} \in \mathfrak{S}_a$, and the relations above show that ψ_w does not depend on the choice of reduced decomposition. It is noticed in [KL09, Section 2.2] that the element

$$\psi_{w_0} y_2 y_3^2 \cdots y_a^{a-1} \tag{1.5}$$

is an idempotent in NH_a, where w_0 denotes the longest element in $\mathfrak{S}_a$.

Schubert polynomials Schubert polynomials have been a powerful tool in both algebra and geometry. The set of Schubert polynomials forms a basis for the polynomial ring when viewed as a module over the ring of symmetric polynomials [Rou12, Theorem 2.11], and their connections to geometry are covered in [Ful99, Chapter 10]. Here we define a variant of the Schubert polynomial.

Given the polynomial ring $\mathbb{Z}[X_1, \ldots, X_m]$, define the *divided difference operator*, ∂_i by

$$\partial_i(P) := \frac{P - s_i(P)}{X_{i+1} - X_i}, \quad 1 \leq i \leq m-1,\ P \in \mathbb{Z}[X_1, \ldots, X_m],$$

where we use $s_i(P)$ to denote the result of interchanging X_i with X_{i+1} in P. The divided difference operator was first introduced by Bernstein, Gel'fand, and Gel'fand [BGG73] and Demazure [Dem74]. Given $w \in \mathfrak{S}_m$, write $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ a reduced expression. We define the *reverse Schubert polynomial* associated to w to be

$$f_w := \partial_{i_r} \circ \cdots \circ \partial_{i_2} \circ \partial_{i_1}(X_2 X_3^2 \cdots X_m^{m-1}).$$

Note that the total set of reverse Schubert polynomials $\{f_w \mid w \in \mathfrak{S}_m\}$ coincides with the total set of Schubert polynomials as defined in [Ful99, p.171]. Moreover, the reverse Schubert polynomial associated to w in variables $X_1, \dots, X_m$ is the same as the Schubert polynomial associated to $w_0 w$ in variables $X_m, \dots, X_1$, where w_0 is the longest reduced word in $\mathfrak{S}_m$. Henceforth we shall drop "reverse" when talking about these polynomials.

Example 1.10. In general for $\mathfrak{S}_n$ it follows from the definition that $f_{w_0} = 1$ and $f_{\mathrm{id}} = y_2 y_3^2 \cdots y_n^{n-1}$. Now, let $\mathfrak{S}_n = \mathfrak{S}_3$ and consider polynomials in $\Bbbk[y_1, y_2, y_3]$. If $w = s_1 s_2$ then

$$\begin{aligned}\partial_1 \partial_2 (y_2 y_3^2) &= \partial_1 \left(\frac{y_2 y_3^2 - y_2 y_2^2}{y_3 - y_2} \right) \\ &= \partial_1 (y_2 y_3) \\ &= \frac{y_2 y_3 - y_1 y_3}{y_2 - y_1} \\ &= y_3\end{aligned}$$

These polynomials appear naturally in the study of the affine nil-Hecke algebra since it is well known that NH_a is isomorphic to the ring of endomorphisms of $\mathbb{Z}[y_1, \dots, y_a]$ generated by the endomorphisms of multiplication and divided difference operators, see for instance [KL09], [Rou12].

Lemma 1.11. *[KL09, Section 2.2] [KLM13, Section 4.2] Let $w \in \mathfrak{S}_n$ be a reduced expression. Then in the affine nil-Hecke algebra of rank a,*

$$\psi_w y_2 y_3^2 \cdots y_a^{a-1} \psi_{w_0} = f_w \psi_{w_0},$$

where f_w denotes the corresponding Schubert polynomial in variables $y_1, \dots, y_a$.

Henceforth, let us use the notation

$$\begin{aligned}\psi_{\boldsymbol{a}} &:= \psi_{w_0} \in \mathrm{NH}_a; \\ y_{\boldsymbol{a}} &:= y_2 y_3^2 \cdots y_a^{a-1} \in \mathrm{NH}_a\end{aligned}$$

so that $\psi_{\boldsymbol{a}} y_{\boldsymbol{a}}$ is the idempotent (1.5). The following lemma is a well known property of NH_a.

Lemma 1.12. *We have $\psi_{\boldsymbol{a}} y_{\boldsymbol{a}} \psi_{\boldsymbol{a}} = \psi_{\boldsymbol{a}}$.*

Proof. This follows as a consequence of Lemma 1.11, since

$$\psi_{\boldsymbol{a}} y_{\boldsymbol{a}} \psi_{\boldsymbol{a}} = f_{w_0} \psi_{\boldsymbol{a}} = 1 \cdot \psi_{\boldsymbol{a}}.$$

□

Theorem 1.13. *[Rou12] The affine nil-Hecke algebra* NH_a *has a basis given by*

$$\{\psi_w y_1^{r_1} \cdots y_a^{r_a} \mid w \in \mathfrak{S}_a, r_i \geq 0 \ \forall i = 1, \dots, a\}.$$

Moreover, the action of NH_a *on* $\Bbbk[y_1, \dots, y_a]$ *induces a graded algebra isomorphism*

$$\mathrm{NH}_a \cong \mathrm{End}_{\Bbbk[y_1,\dots,y_a]^{\mathfrak{S}_a}}(\Bbbk[y_1, \dots, y_a]).$$

1.3 Motivation

Having introduced the quiver Hecke algebras and shown some of their first properties we now provide some motivating reasons behind their study. This chiefly falls into two sections, the famous categorification theorems which link the representation theory of R_α to half the quantized enveloping algebra associated to the Kac-Moody algebra $\mathfrak{g}$, and then the well studied cyclotomic quotients which have provided important advances in the representation theory of the symmetric group and related Hecke algebras. All of the information in this section can be found in the survey papers of Brundan [Bru13] and Kleshchev [Kle10], however we will highlight the origins of the main results.

For a loop free quiver with vertex set I we denote by $m_{i,j}$ the number of directed edges $i \to j$ for $i, j \in I$. The corresponding Cartan matrix $C = (c_{i,j})_{i,j \in I}$ is defined from $c_{i,i} = 2$, $c_{i,j} = -m_{i,j} - m_{j,i}$ for $i \neq j$. To C there is an associated Kac-Moody algebra $\mathfrak{g}$. We fix a choice of root datum for $\mathfrak{g}$. This gives a weight lattice P which is a finitely generated abelian group equipped with a symmetric bilinear from

$$\begin{aligned} P \times P &\to \mathbb{Q}; \\ (\lambda, \mu) &\mapsto \lambda \cdot \mu, \end{aligned}$$

containing simple roots $(\alpha_i)_{i \in I}$ and fundamental weights $(\Lambda_i)_{i \in I}$ such that, for $i, j \in I$, $\alpha_i \cdot \alpha_j = c_{i,j}$ and $\alpha_i \cdot \Lambda_i = \delta_{i,j}$. The root lattice is $Q := \oplus_{i \in I} \mathbb{Z}\alpha_i \subset P$ and the positive part is $Q_+ := \oplus_{i \in I} \mathbb{N}\alpha_i$.

Categorification The categorification theorems focus on the categories

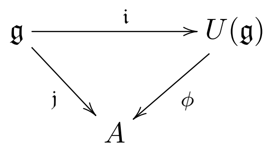
$$R\text{-}\mathbf{mod} = \oplus_{\alpha \in Q_+} R_\alpha\text{-}\mathbf{mod}, \qquad R\text{-}\mathbf{p.\,mod} = \oplus_{\alpha \in Q_+} R_\alpha\text{-}\mathbf{p.\,mod},$$

of finite dimensional R-modules and finitely generated projective R-modules, respectively.

Let $\mathfrak{g}$ be a semi-simple Lie algebra over some field $\mathbb{F}$. The *universal enveloping algebra* of $\mathfrak{g}$ is the associative unital algebra $U(\mathfrak{g})$ over $\mathbb{F}$ and a Lie algebra homomorphism

$$\mathfrak{i} : \mathfrak{g} \to U(\mathfrak{g})$$

satisfying the universal property that for every arbitrary associative unital algebra A over $\mathbb{F}$ and a Lie algebra homomorphism $\mathfrak{j} : \mathfrak{g} \to A$, there exists a unique homomorphism of associative algebras $\phi : U(\mathfrak{g}) \to A$ making the diagram commute.



Note that any associative algebra can be endowed with a Lie algebra structure using the commutator bracket $[x, y] = xy - yx$. The universal enveloping algebra of $\mathfrak{g}$ can be constructed explicitly as

$$U(\mathfrak{g}) := T(\mathfrak{g})/\langle x \otimes y - y \otimes x - [x, y] \mid x, y \in \mathfrak{g}\rangle,$$

where $T(\mathfrak{g})$ is the tensor algebra of $\mathfrak{g}$, i.e, $T(\mathfrak{g}) := \oplus_{i \geq 0} \mathfrak{g}^{\otimes i}$. There exists a deformation of this algebra known as the quantized universal enveloping algebra $U_q(\mathfrak{g})$ where $q \in \Bbbk^{\times}$, which decomposes into positive and negative parts, denoted $U_q^-(\mathfrak{g})$ and $U_q^+(\mathfrak{g})$, and a zero part $U_q^0(\mathfrak{g})$. It is often useful to utilise the existence of an algebra isomorphism between the algebra known as Lusztig's algebra $\mathbf{f}$ and $U_q^-(\mathfrak{g})$. Indeed, it is known that $\mathbf{f}$ is a Q_+-graded algebra so that $\mathbf{f} = \oplus_{\alpha \in Q_+} \mathbf{f}_\alpha$, and one can endow $\mathbf{f}$ with the structure of a twisted bialgebra. To avoid going beyond the scope of this brief motivational section we direct the reader to [Bru13] and [Kle10] for a detailed description of Lusztig's algebra.

The Grothendieck groups of the categories mentioned before can also be given twisted bialgebra structures in the following way. We have functors of induction and restriction between quiver Hecke algebras, for $\beta, \gamma \in Q_+$, there is natural embedding

$$R_\beta \otimes R_\gamma \hookrightarrow R_{\beta+\gamma}$$

where the tensor product acts as horizontal concatenation of diagrams. Denote the image of $1_\beta \otimes 1_\gamma \in R_\beta \otimes R_\gamma$ by $1_{\beta,\gamma} \in R_{\beta+\gamma}$. Then for $U \in R_{\beta+\gamma}$**-mod** and $V \in R_\beta \otimes R_\gamma$**-mod** we define functors

$$\mathrm{Res}_{\beta,\gamma}^{\beta+\gamma} : R_{\beta+\gamma}\textbf{-mod} \to R_\beta \otimes R_\gamma\textbf{-mod}$$

$$\mathrm{Ind}_{\beta,\gamma}^{\beta+\gamma} : R_\beta \otimes R_\gamma\textbf{-mod} \to R_{\beta+\gamma}\textbf{-mod}$$

by setting

$$\mathrm{Res}_{\beta,\gamma}^{\beta+\gamma} U = 1_{\beta,\gamma}U \qquad \mathrm{Ind}_{\beta,\gamma}^{\beta+\gamma} V := R_{\beta+\gamma}1_{\beta,\gamma} \otimes_{R_\beta \otimes R_\gamma} V.$$

Summing over all $\beta, \gamma \in Q_+$ gives functors Ind and Res, which act as multiplication and comultiplication (resp.) on the Grothendeick groups of the categories R-**mod** and R-**p. mod**, and endows them with the structure of a $\mathbb{Z}[q, q^{-1}]$-bialgebra. This result follows from the existence of an isomorphism between $K_0(R\text{-}\mathbf{p.\,mod})$ and a well known subalgebra ${}_{\mathbb{Z}[q,q^{-1}]}\mathbf{f}$ of $\mathbf{f}$, known as Lusztig's $\mathbb{Z}[q, q^{-1}]$-form. This is the first of the so-called categorification theorems.

Theorem 1.14. *[KL09, Theorem 1.1] There is a canonical twisted bialgebra isomorphisms*

$${}_{\mathbb{Z}[q,q^{-1}]}\mathbf{f} \to K_0(R\text{-}\mathbf{p.\,mod}).$$

Under this isomorphism ${}_{\mathbb{Z}[q,q^{-1}]}\mathbf{f}_\alpha$ corresponds to $K_0(R_\alpha\text{-}\mathbf{p.\,mod})$ for any $\alpha \in Q_+$, multiplication in ${}_{\mathbb{Z}[q,q^{-1}]}\mathbf{f}$ corresponds to induction in $K_0(R\text{-}\mathbf{p.\,mod})$, and comultiplication in ${}_{\mathbb{Z}[q,q^{-1}]}\mathbf{f}$ corresponds to restriction in $K_0(R\text{-}\mathbf{p.\,mod})$. The twisted multiplication on $K_0(R\text{-}\mathbf{mod}) \otimes K_0(R\text{-}\mathbf{mod})$ is defined by

$$(a \otimes b)(c \otimes d) = q^{-\beta\cdot\gamma} ac \otimes bd$$

for $a \in K_0(R_\alpha\text{-}\mathbf{mod})$, $b \in K_0(R_\beta\text{-}\mathbf{mod})$, $c \in K_0(R_\gamma\text{-}\mathbf{mod})$, and $d \in K_0(R_\delta\text{-}\mathbf{mod})$. For $\mathbb{F}$ with characteristic 0, the isomorphism also identifies a particularly nice basis, Lusztig's canonical basis, for $\mathbf{f}$ with the basis of the Grothendieck group $K_0(R\text{-}\mathbf{p.\,mod})$ consisting of isomorphism classes of projective indecomposable modules.

Theorem 1.15. *[Rou12, Corollary 5.8][VV11, Theorem 4.5] Assume $\mathbb{F}$ has characteristic 0. For every $\alpha \in Q_+$, the isomorphism*

$$\mathbf{f}_\alpha \to K_0(R_\alpha\text{-}\mathbf{p.\,mod})$$

maps Lusztig's canonical basis for $\mathbf{f}_\alpha$ to the basis of $K_0(R_\alpha\text{-}\mathbf{p.\,mod})$ consisting of isomorphism classes of indecomposable projective graded R_α-modules.

The above theorems describe what is meant in the vernacular of the subject when one says R categorifies $U_q^-(\mathfrak{g})$, and the indecomposable projectives categorify Lusztig's canonical basis.

Cyclotomic quotients The introduction of quiver Hecke algebras also allowed key developments in the representation theory of the symmetric group. To understand this one must introduce a special quotient of the quiver Hecke algebra in type

A. Recall that there is a bilinear form

$$(\cdot,\cdot) : P \times Q \to \mathbb{Z}$$

such that $(\Lambda_i, \alpha_j) = \delta_{ij}$, using this define, for a chosen $\Lambda \in P$, the ideal

$$I^{\Lambda} := \left\langle y_1^{(\Lambda,\alpha_{i_1})} e(\boldsymbol{i}) \mid \boldsymbol{i} \in \langle I \rangle_{\alpha} \right\rangle.$$

The quotient algebra $R_{\alpha}^{\Lambda} := R_{\alpha}/I^{\Lambda}$ is called the *cyclotomic quiver Hecke algebra.*

Proposition 1.16. *The elements $y_s e(\boldsymbol{i}) \in R_{\alpha}^{\Lambda}$ are nilpotent for all $1 \leq s \leq n$. Moreover, the algebra R_{α}^{Λ} is finite dimensional.*

Notice that once the nilpotence of the $y_s's$ is established the claim about finite dimensionality follows from Theorem 1.4.

For a fixed field $\mathbb{F}$ and $q \in \mathbb{F}^{\times}$ the affine Hecke algebra of type A, $H_d^{\text{aff}} = H_d^{\text{aff}}(\mathbb{F}, q)$, is the $\mathbb{F}$-algebra generated by

$$T_1, \cdots, T_{d-1}, X_1^{\pm 1}, \cdots, X_d^{\pm 1},$$

subject to the relations

$$\begin{gathered} X_r^{\pm 1} X_s^{\pm 1} = X_s^{\pm 1} X_r^{\pm 1}; \qquad X_r X_r^{-1} = 1; \\ T_r^2 = (q-1)T_r + q; \qquad T_r X_r T_r = q X_{r+1}; \qquad T_r T_{r+1} T_r = T_{r+1} T_r T_{r+1}; \\ T_r X_s = X_s T_r \qquad \text{if } s \neq r, r+1; \\ T_r T_s = T_s T_r \qquad \text{if } |r-s| > 1; \end{gathered}$$

There is a degenerate form $H_n^{\text{aff}}(\mathbb{F}, 1)$ for when $q = 1$, but we do not list the relations here. For a fixed $\Lambda \in P$, the cyclotomic Hecke algebra also known as the Ariki-Koike algebra is given by

$$H_n^{\Lambda} := H_n / \left\langle \prod_{i \in I} (X_1 - q^i)^{(\Lambda,\alpha_i)} \right\rangle.$$

These cyclotomic quotients give us the Hecke algebras $H_d \cong H_d^{\Lambda_i}$ and thus we recover the symmetric group from these by setting $q = 1$, ie, $\mathbb{F}\mathfrak{S}_d \cong H_d^{\Lambda_i}(\mathbb{F}, 1)$. By constructing an explicit basis, Brundan and Kleshchev established an isomorphism between blocks of H_d^{Λ} and the algebras R_{α}^{Λ}. This revealed a previously unknown grading on H_d^{Λ}, and thus on $\mathbb{F}\mathfrak{S}_d$.

Chapter 2

Cellular and affine cellular algebras

In this chapter we introduce the class of cellular algebras, these are finite dimensional algebras with particularly nice representation theory. We then introduce the more recent infinite dimensional analogue, the affine cellular algebras. We consider examples of both, and explain in detail the affine cellular structure of the quiver Hecke algebra of finite type A.

2.1 Definitions and examples

Cellular algebras Cellular algebras were introduced by Graham and Lehrer [GL66] as a class of algebras that have bases with nice multiplicative properties, inspired by those of the Kazhdan-Lusztig basis for Hecke algebras. Later Koenig and Xi [KX99] gave an abstract definition in terms of the existence of a particular ideal chain, called a cell chain. From this cell chain we are able to determine many aspects of the representation theory of these algebras, for instance, we get a complete classification of irreducible modules as well as a criterion for when the algebra is semi-simple.

Let A be an R algebra where R is a commutative Noetherian integral domain. Assume there is an involution τ on A, that is an automorphism such that $\tau(ab) = \tau(b)\tau(a)$ for all $a, b \in A$. A two sided ideal J in A is called a *cell ideal* if and only if $\tau(J) = J$ and there is a left ideal $\Delta \subset J$ such that Δ is finitely generated and free over R and there is an isomorphism of A-A-bimodule $\alpha : J \cong \Delta \otimes_R \tau(\Delta)$

making the following commute

$$\begin{array}{ccc} J & \xrightarrow{\alpha} & \Delta \otimes_R \tau(\Delta) \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle x\otimes y \mapsto \tau(y)\otimes\tau(x)} \\ J & \xrightarrow{\alpha} & \Delta \otimes_R \tau(\Delta). \end{array}$$

Then an algebra A (with involution τ) is called *cellular* if and only if there is an R-module decomposition $A = J'_1 \oplus \cdots \oplus J'_n$ with $\tau(J'_j) = J'_j$ for all $j = 1, \ldots, n$ and such that $J_j := \oplus_{l=1}^{j} J'_l$ gives a chain of two sided ideals

$$0 = J_0 \subset J_1 \subset \cdots \subset J_n = A$$

called a *cell chain*, such that for each $j = 1, \cdots, n$ the quotient J_j/J_{j-1} is a cell ideal of A/J_{j-1}. The Δ's are called *standard modules* as they coincide with the standard modules arising in the stratified algebras discussed in Chapter 4. Representatives for isomorphism classes of the irreducible modules of A can be taken as the heads of the standard modules.

Example 2.1. 1. The algebra $\mathbb{M}_{n\times n}(\Bbbk)$ is cellular with involution $\tau(A) = A^T$ and has cell chain of length 1. In this case

$$\Delta = \begin{pmatrix} * \\ * \\ \vdots \\ * \end{pmatrix}$$

and $\tau(\Delta) = \Delta^T$. It is clear that $\Delta \otimes \tau(\Delta) \cong \mathbb{M}_{n\times n}(\Bbbk)$.

2. The algebra $\Bbbk[x]/(x^n)$ is cellular with involution $\tau = \mathrm{id}$. The cell chain is given by
$$0 = (x^n) \subseteq (x^{n-1}) \subseteq \cdots \subseteq (x) \subseteq (1) = \Bbbk[x]/(x^n).$$

3. Let A be the path algebra of the quiver $e_1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} e_2 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} e_3$ modulo the ideal $(\alpha^2, \beta^2, \alpha\beta e_2 - \beta\alpha e_2)$. The Loewy structure of the left regular representation of A is given by

$$\begin{array}{c} 1 \\ 2 \\ 1 \end{array} \oplus \begin{array}{ccccc} & & {\scriptstyle 2} & & \\ & / & & \backslash & \\ {\scriptstyle 1} & & & & {\scriptstyle 3} \\ & \backslash & & / & \\ & & {\scriptstyle 2} & & \end{array} \oplus \begin{array}{c} 3 \\ 2 \\ 3 \end{array}.$$

The algebra A is cellular with respect to the involution τ defined by $\tau(e_i) = e_i$,

$\tau(\alpha) = \beta$, $\tau(\beta) = \alpha$. It has a cell chain given by

$$A(\alpha e_2 \beta)A \subseteq Ae_3A \subseteq A(e_2 + e_3)A \subseteq A.$$

Affine cellular algebras We define affine cellularity in the context of Koenig and Xi [KX12]. An *affine commutative algebra* is a commutative $\Bbbk$-algebra which is a quotient of a polynomial ring $\Bbbk[x_1, \cdots x_n]$ in finitely many variables. Let A be a unitary $\Bbbk$-algebra with a $\Bbbk$-anti-involution τ. A two-sided ideal J in A is called an *affine cell ideal* if the following conditions are satisfied:

1. the ideal J is fixed by τ, i.e., $\tau(J) = J$;
2. there exists a free $\Bbbk$-module V of finite rank and an affine commutative $\Bbbk$-algebra B with identity and with a $\Bbbk$-involution σ such that $\Delta := V \otimes_{\Bbbk} B$ can be given the structure of an A-B-bimodule, where the right B-module structure is induced by that of the regular right B-module B_B;
3. there is an A-A-bimodule isomorphism $\alpha : J \to \Delta \otimes_B \Delta'$, where $\Delta' := B \otimes_{\Bbbk} V$ is a B-A-bimodule with the left B-module structure induced by ${}_BB$ and with the right A-module structure via τ, that is,

   $$(b \otimes v)a := s(\tau(a)(v \otimes b)),$$

   for $a \in A$, $b \in B$, $v \in V$, and $s : V \otimes_{\Bbbk} B \to B \otimes_{\Bbbk} V$, $v \otimes b \mapsto b \otimes v$, such that the following diagram is commutative:

$$\begin{array}{ccc} J & \xrightarrow{\alpha} & \Delta \otimes_B \Delta' \\ \tau \downarrow & & \downarrow v_1 \otimes b_1 \otimes_B b_2 \otimes v_2 \mapsto v_2 \otimes \sigma(b_2) \otimes_B \sigma(b_1) \otimes v_1 \\ J & \xrightarrow{\alpha} & \Delta \otimes_B \Delta'. \end{array}$$

The algebra A (with involution τ) is called *affine cellular* if there is a $\Bbbk$-module decomposition $A = J_1' \oplus J_2' \oplus \cdots \oplus J_n'$ (for some n) with $\tau(J_j') = J_j'$ for each j and such that setting $J_j = \oplus_{l=1}^{j} J_l'$ gives a chain of two-sided ideals of A:

$$0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$$

(each of them fixed by τ) and for each $j = 1, \ldots, n$ the quotient J_j/J_{j-1} is an affine cell ideal of A/J_{j-1} (with respect to the involution induced by τ on the quotient). We call this chain a *cell chain* for the affine cellular algebra A. The module Δ is called a *cell module* for the affine cell ideal J.

Example 2.2. 1. The algebras $\mathbb{M}_{n\times n}(\Bbbk[x])$ are affine cellular with respect to the involution $\tau(A) = A^T$ with cell chains of length 1.

2. Moreover, the same is true of matrices over any affine algebra, in particular in light of the isomorphism

$$\mathrm{End}_{\Bbbk[y_1,\dots,y_a]^{\mathfrak{S}_a}}(\Bbbk[y_1,\cdots,y_a]) \cong \mathbb{M}_{a!\times a!}(\Bbbk[y_1,\dots,y_a])$$

and Theorem 1.13, the affine nil-Hecke algebra is affine cellular.

3. If $A := \Bbbk\mathcal{Q}/\mathcal{I} \otimes_{\Bbbk} \Bbbk[x]$ where $\mathcal{Q}:\ 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$ and $\mathcal{I} = \langle \alpha\beta \rangle$ then A is an affine cellular algebra with respect to the involution $\tau \otimes_{\Bbbk} \mathrm{id}$ where τ fixes idempotents and exchanges α and β. A has cell chain given by

$$0 \subseteq Ae_2A \otimes_{\Bbbk} \Bbbk[x] \subseteq A \otimes_{\Bbbk} \Bbbk[x].$$

4. More generally, if A is a cellular algebra and H is an affine algebra then $A \otimes_{\Bbbk} H$ is an affine cellular algebra with respect to the involution $i \otimes \mathrm{id}$ and has cell chain

$$0 \subseteq J_n \otimes_{\Bbbk} H \subseteq J_{n-1} \otimes_{\Bbbk} H \subseteq \cdots \subseteq J_1 \otimes_{\Bbbk} H = A \otimes_{\Bbbk} H$$

induced from the cell chain $0 \subseteq J_n \subseteq \cdots \subseteq J_1 = A$ of A.

2.2 Affine cellularity of $R_\alpha$the quiver Hecke algebra

The affine cellularity of quiver Hecke algebras in type A was established by Kleshchev, Loubert and Miemietz [KLM13]. To describe the affine cellular structure the authors make use of special elements y_π and ψ_π in R_α, which correspond to a root partition $\pi \in \Pi(\alpha)$, and are defined in the following way.

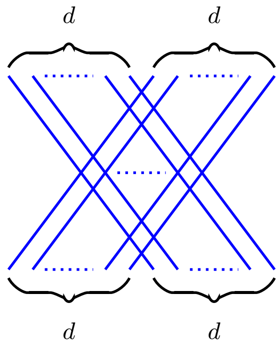
Elements y_π and ψ_π We fix a root $\alpha \in \mathcal{Q}_+$ of height d, and let $\alpha^1, \dots, \alpha^b \in \mathcal{Q}_+$ with $\alpha^1 + \cdots + \alpha^b = \alpha$. There is a natural embedding

$$\iota_{\alpha^1,\dots,\alpha^b} : R_{\alpha^1} \otimes \cdots \otimes R_{\alpha^b} \hookrightarrow R_\alpha$$

whose image $R_{\alpha^1,\dots,\alpha^b}$ is the parabolic subalgebra in R_α. Let us define $\psi_\alpha \in R_{2\alpha}$ to be the element

$$\psi_\alpha := (\psi_d \cdots \psi_{2d-1}) \cdots (\psi_2 \cdots \psi_{d+1})(\psi_1 \cdots \psi_d). \tag{2.1}$$

In explanation, ψ_α is the permutation of two α-blocks, as illustrated below.



Let $p \in \mathbb{N}$ then define

$$\psi_{\alpha,r} := \iota_{(r-1)\alpha,2\alpha,(p-r-1)\alpha}(1 \otimes \psi_\alpha \otimes 1) \in R_{p\alpha} \quad (1 \le r < p),$$

which is the element that permutes the r^{th} and $(r+1)^{th}$ α-blocks. Furthermore, for $w \in \mathfrak{S}_p$ and a reduced decomposition $w = s_{i_1} \cdots s_{i_m}$ define

$$\psi_{\alpha,w} := \psi_{\alpha,i_1} \cdots \psi_{\alpha,i_m} \in R_{p\alpha}.$$

Let us define

$$y_{\alpha,s} := \iota_{(s-1)\alpha,\alpha,(p-s)\alpha}(1 \otimes y_d \otimes 1) \in R_{p\alpha} \quad (1 \le s \le p).$$

In words, $y_{\alpha,s}$ is a dot on the last strand of the s^{th} block of size d.
We further define

$$y_{\alpha,\boldsymbol{p}} := y_{\alpha,2} y_{\alpha,3}^2 \cdots y_{\alpha,p}^{p-1} \in R_{p\alpha},$$

and denote the polynomial algebra and the symmetric polynomial algebra in these variables by

$$P_{\alpha,p} = \mathbb{Z}[y_{\alpha,1}, \dots, y_{\alpha,p}] \text{ and } \Lambda_{\alpha,p} = P_{\alpha,p}^{\mathfrak{S}_p}.$$

Now, let $\pi = \beta_1^{p_1} \cdots \beta_n^{p_n} \in \Pi(\alpha)$ be a root partition of α. For $1 \le k \le n$, and $x \in R_{p_k \beta_k}$ put

$$\iota^k(x) = \iota_{p_1\beta_1+\cdots+p_{k-1}\beta_{k-1}, p_k\beta_k, p_{k+1}\beta_{k+1}+\cdots+p_n\beta_n}(1 \otimes x \otimes 1) \in R_\alpha.$$

For all $1 \le k \le n$, $w \in \mathfrak{S}_{p_k}$, $1 \le r \le p_k$ and $1 \le s \le p_k$ define the elements of R_α

$$\psi_{k,w} := \iota^k(\psi_{\beta_k,w}),\ \psi_{k,r} := \iota^k(\psi_{\beta_k,r}),\ y_{k,s} := \iota^k(y_{\beta_k,s}).$$

In other words, $\psi_{k,r}$ is the permutation of the r, $r+1$ β_k-blocks and $y_{k,s}$ is a dot on final strand on s^{th} β_k-block. We define

$$y_\pi := \iota^1(y_{\beta_1,\boldsymbol{p}_1})\cdots\iota^n(y_{\beta_n,\boldsymbol{p}_n}),$$

$$\psi_\pi := \iota^1(\psi_{\beta_1,w_0^1})\cdots\iota^n(\psi_{\beta_n,w_0^n}),$$

where w_0^k is the longest element of $\mathfrak{S}_{p_k}$, for $k=1,\dots,n$. Also, let

$$\Lambda_\pi := \iota_{p_1\beta_1,\dots,p_n\beta_n}(\Lambda_{\beta_1,p_1}\otimes\cdots\otimes\Lambda_{\beta_n,p_n}) \cong \Lambda_{p_1}\otimes\cdots\otimes\Lambda_{p_n}, \tag{2.2}$$

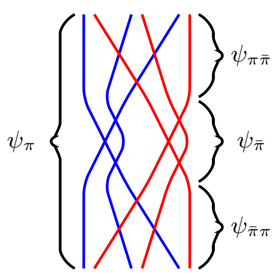
$$P_\pi := \iota_{p_1\beta_1,\dots,p_n\beta_n}(P_{\beta_1,p_1}\otimes\cdots\otimes P_{\beta_n,p_n}). \tag{2.3}$$

Let us consider some examples, as the elements y_π and ψ_π are clearer when illustrated.

Example 2.3.
1. When $\alpha=\alpha_i^a$, ie, $R_\alpha=\mathrm{NH}_a$, then $y_\pi=\boldsymbol{y_a}$, $\psi_\pi=\psi_{\boldsymbol{a}}$ and $\Lambda_\pi=\Bbbk[y_1,\dots,y_a]^{\mathfrak{S}_a}$.
2. For $\alpha=3\alpha_1+3\alpha_2$, let $\pi=(\alpha_1+\alpha_2)^3$. Then $y_\pi=y_4y_6^2$ and
$$\psi_\pi=\psi_2\psi_4\psi_3\psi_2\psi_1\psi_2\psi_5\psi_4\psi_5\psi_3\psi_4\psi_2.$$
3. For $\alpha=2\alpha_1+\alpha_2$, let $\pi=\alpha_2(\alpha_1)^2$, then $y_\pi=y_3$ and $\psi_\pi=\psi_2$, whereas for $\pi=(\alpha_1+\alpha_2)\alpha_1$ we have $y_\pi=e(\boldsymbol{i}_\pi)=\psi_\pi$.
4. Let $\alpha=2\alpha_1+2\alpha_2+2\alpha_3$, and $\pi=(\alpha_1+\alpha_2+\alpha_3)^2$. Then $y_\pi=y_6y_9^2$ and $\psi_\pi=\psi_3\psi_2\psi_4\psi_2\psi_4\psi_6\psi_4\psi_2\psi_3$.

Notice that we can split the element ψ_π into three distinct parts, namely, $\psi_\pi=\psi_{\pi\bar\pi}\psi_{\bar\pi}\psi_{\bar\pi\pi}$, where $\psi_{\bar\pi}$ consists of the part of ψ_π that contains only (i,i)-crossings of the same colour. Then $\psi_{\pi\bar\pi}$ contains only (i,j)-crossings of different colours, and $\psi_{\bar\pi\pi}$ is the reversal of $\psi_{\pi\bar\pi}$.

Example 2.4. For example, consider the root partition $\pi=(\alpha_1+\alpha_2)^3$. Then ψ_π can be written using diagrammatics as follows.



We now prove a generalised version of Lemma 1.12.

Lemma 2.5. *For $\pi \in \Pi(\alpha)$ and $\psi_\pi, y_\pi \in R_\alpha$ we have*

$$\psi_\pi y_\pi \psi_\pi e(\boldsymbol{i}_\pi) = \psi_\pi e(\boldsymbol{i}_\pi).$$

Proof. It suffices to prove this for a partition consisting of one block type since

$$\psi_\pi,\ y_\pi \in R_{p_1\beta_1} \otimes \cdots \otimes R_{p_n\beta_n} \subset R_\alpha.$$

So, let $\pi = (\alpha_1 + \cdots + \alpha_m)^a$. Then

$$\begin{aligned}
\psi_\pi y_\pi \psi_\pi e(\boldsymbol{i}_\pi) &= \psi_{\pi\bar{\pi}}\psi_{\bar{\pi}}\psi_{\bar{\pi}\pi} y_\pi \psi_{\pi\bar{\pi}}\psi_{\bar{\pi}}\psi_{\bar{\pi}\pi} e(\boldsymbol{i}_\pi) \\
&= \psi_{\pi\bar{\pi}}\psi_{\bar{\pi}}\psi_{\bar{\pi}\pi} \prod_{k=1}^{a-1} y^k_{(k+1)m} \psi_{\pi\bar{\pi}}\psi_{\bar{\pi}}\psi_{\bar{\pi}\pi} e(\boldsymbol{i}_\pi) \\
&= \psi_{\pi\bar{\pi}}\psi_{\bar{\pi}}\psi_{\bar{\pi}\pi}\psi_{\pi\bar{\pi}} \prod_{k=1}^{a-1} y^k_{a(m-1)+k+1} \psi_{\bar{\pi}}\psi_{\bar{\pi}\pi} e(\boldsymbol{i}_\pi).
\end{aligned}$$

Let us rename the polynomial part $y_{\bar{\pi}} e(\boldsymbol{i}) := \prod_{k=1}^{a-1} y^k_{a(m-1)+k+1}$. Direct computation shows that $\psi_{\bar{\pi}\pi}\psi_{\pi\bar{\pi}} e(\boldsymbol{i}) = pe(\boldsymbol{i})$, where p is a polynomial within a product of nil-Hecke algebras;

$$\mathrm{NH}_a^{(1)} \otimes \cdots \otimes \mathrm{NH}_a^{(m)},$$

and $\deg(\psi_{\pi\bar{\pi}}) = \sum_{a=1}^{m-1}(m-1)(a-k)$. We can write $p = p_1 + \cdots + p_r$, where each p_j is a monomial and $p_j = p_j^{(1)} \cdots p_j^{(m)}$ with $p_j^{(i)} \in \mathrm{NH}_a^{(i)}$. With the same convention of notation, write $\psi_{\bar{\pi}} = \psi_{\boldsymbol{a}}^{(1)} \cdots \psi_{\boldsymbol{a}}^{(m)}$. Note that $y_{\bar{\pi}} \in \mathrm{NH}_a^{(m)}$, this gives

$$\psi_{\bar{\pi}} p y_{\bar{\pi}} \psi_{\bar{\pi}} = \sum_j \psi_{\boldsymbol{a}}^{(1)} p_j^{(1)} \psi_{\boldsymbol{a}}^{(1)} \cdots \psi_{\boldsymbol{a}}^{(m)} p_j^{(m)} y_{\bar{\pi}} \psi_{\boldsymbol{a}}^{(m)}.$$

Let us denote by $\boldsymbol{p} := py_{\bar{\pi}}$, and carry this notation down so that $\boldsymbol{p}_j := p_j y_{\bar{\pi}}$ giving $bp_j^{(i)} := p_j^{(i)}$ and $\boldsymbol{p}_j^{(m)} := p_j^{(m)} y_{\bar{\pi}}$. Suppose $\psi_{\bar{\pi}} \boldsymbol{p}_j \psi_{\bar{\pi}} \neq 0$ for some $1 \le j \le r$, then we claim that $\deg(\boldsymbol{p}_j^{(i)}) = a(a-1)$ for each $1 \le i \le m$. If $\deg(\boldsymbol{p}_j^{(i)}) < a(a-1)$ then $\deg(\psi_{\boldsymbol{a}}^{(i)} \boldsymbol{p}_j^{(i)} \psi_{\boldsymbol{a}}^{(i)}) < \deg(\psi_{\boldsymbol{a}}^{(i)}) = -a(a-1)$ which contradicts $\psi_{\boldsymbol{a}}^{(i)}$ being the element of least degree in $\mathrm{NH}_a^{(i)}$. So $\deg(\boldsymbol{p}_j^{(i)}) \ge a(a-1)$, but if $\deg(\boldsymbol{p}_j^{(i)}) > a(a-1)$ for some i, then since

$$\deg(\boldsymbol{p}) = 2 \cdot \deg(\psi_{\pi\bar{\pi}}) + \deg(y_{\bar{\pi}}) = a(a-1) + 2\sum_{k=1}^{m-1}(m-1)(a-k) = a(a-1)m,$$

we would require $\deg(\boldsymbol{p}_j^{(i')}) < a(a-1)$ for some other i', which we already know cannot occur. So $\deg(\boldsymbol{p}_j^{(i)}) = a(a-1)$ for each $1 \le i \le m$. Since $\deg(y_{\bar{\pi}}) = a(a-1)$, we must have $p_j^{(m)} = 1$, so we can refine the polynomial $p_j = p_j^{(1)} \cdots p_j^{(m-1)}$.

We now claim that $p_j = \prod_{i=1}^{m-1} p_j^{(i)} = \prod_{i=1}^{m-1} y_{\boldsymbol{a}}^{(i)}$. The monomial $p_j^{(i)}$ has a variables, $y_{1+x_i}, \ldots, y_{a+x_i}$, where $x_i = a(i-1)$. Let us define

$$\deg_n(p_j^{(i)}) := \deg(p_j^{(i)}(y_{n+x_i})),$$

for $1 \leq n \leq a$. So $\deg_n(p_j^{(i)})$ is the degree of the nth variable of $p_j^{(i)}$, and is bounded above by twice the number of strands of $(i+1)$-colour that the n-strand crosses. Therefore, $\deg_n(p_j^{(i)}) \leq n-1$. So if $\psi_{\bar{\pi}} \boldsymbol{p}_j \psi_\pi \neq 0$ then $p_j = \prod_{i=1}^{m-1} y_{\boldsymbol{a}}^{(i)}$.

There is precisely one summand p_j with with this property. To show that this summand exists and is unique consider each $(i, i+1)$-crossing squared in $\psi_{\bar{\pi}\pi}\psi_{\pi\bar{\pi}}$, this produces a factor $(y_s - y_t)$ in p for some s and t, where y_s corresponds to a dot on the $(i-1)$-strand and y_t to a dot on the i-strand. When we multiply these out, picking the corresponding y_t term in each factor will produce $\prod_{i=1}^{m-1} y_{\boldsymbol{a}}^{(i)}$. It is easy to see that any other summand of p will not satisfy the above restrictions on degree.

So

$$\psi_\pi y_\pi \psi_\pi e(\boldsymbol{i}_\pi) = \psi_{\pi\bar{\pi}} \psi_{\bar{\pi}} \prod_{i=1}^{m-1} y_a^{(i)} y_{\bar{\pi}} \psi_{\bar{\pi}} \psi_{\bar{\pi}\pi} e(\boldsymbol{i}_\pi).$$

Notice that $y_{\bar{\pi}} = y_a^{(m)}$, now by Lemma 1.12 we get

$$\psi_{\pi\bar{\pi}} \psi_{\bar{\pi}} \prod_{i=1}^{m} y_a^{(i)} \psi_{\bar{\pi}} \psi_{\bar{\pi}\pi} e(\boldsymbol{i}_\pi) = \psi_{\pi\bar{\pi}} \psi_{\bar{\pi}} \psi_{\bar{\pi}\pi} e(\boldsymbol{i}_\pi) = \psi_\pi e(\boldsymbol{i}_\pi),$$

as required. □

Example 2.6. Let $p_j^{(k)} = y_{1+x}^3 y_{2+x}^2 y_{3+x}^7$. Then $\deg_1(p_j^{(k)}) = 3$, $\deg_2(p_j^{(k)}) = 2$ and $\deg_3(p_j^{(k)}) = 7$.

In particular, the previous lemma shows that $\psi_\pi y_\pi e(\boldsymbol{i}_\pi)$ are idempotents in R_α. This property is used when constructing an affine cellular basis for R_α.

Affine cell structure The authors of [KLM13] define

$$I'_\pi = \Bbbk - \operatorname{span}\{\psi_w y_\pi \Lambda_\pi \psi_\pi y_\pi e(\boldsymbol{i}_\pi) \psi_v^\tau \mid w, v \in \mathfrak{S}^\pi\},$$

$$I_\pi = \sum_{\sigma \geq \pi} I'_\sigma,$$

$$I_{>\pi} = \sum_{\sigma > \pi} I'_\sigma,$$

and conclude that the I_π form a cell chain for R_α, thus establishing affine cellularity for the quiver Hecke algebra.

Theorem 2.7. *[KLM13, Main Theorem] The algebra R_α is graded affine cellular with cell chain given by the ideals $\{I_\pi \mid \pi \in \Pi(\alpha)\}$. Moreover, setting $\bar{R}_\alpha := R_\alpha / I_{>\pi}$ for a fixed $\pi \in \Pi(\alpha)$, and $e_\pi := \psi_\pi y_\pi e(\boldsymbol{i}_\pi)$ we have:*

1. *the map $\Lambda_\pi \to \bar{e}_\pi \bar{R}_\alpha \bar{e}_\pi$, $b \mapsto \bar{b}\bar{y}_\pi \bar{\psi}_\pi \bar{e}(\boldsymbol{i}_\pi)$ is an isomorphism of graded algebras;*
2. *$\bar{R}_\alpha \bar{e}(\boldsymbol{i}_\pi)\bar{\psi}_\pi \bar{y}_\pi$ is a free right $\bar{e}_\pi \bar{R}_\alpha \bar{e}_\pi$-module with basis given by*
$$\{\bar{\psi}_w \bar{y}_\pi \bar{\psi}_\pi \bar{e}(\boldsymbol{i}_\pi)\bar{y}_\pi \mid w \in \mathfrak{S}^\pi\};$$
3. *$\bar{y}_\pi \bar{\psi}_\pi \bar{e}(\boldsymbol{i}_\pi)\bar{R}_\alpha$ is a free left $\bar{e}_\pi \bar{R}_\alpha \bar{e}_\pi$-module with basis given by*
$$\{\bar{\psi}_\pi \bar{e}(\boldsymbol{i}_\pi)\bar{y}_\pi \bar{\psi}_v^\tau \mid v \in \mathfrak{S}^\pi\};$$
4. *multiplication provides an isomorphism*
$$\bar{R}_\alpha \bar{e}(\boldsymbol{i}_\pi)\bar{\psi}_\pi \bar{y}_\pi \otimes_{\bar{e}_\pi \bar{R}_\alpha \bar{e}_\pi} \bar{y}_\pi \bar{\psi}_\pi \bar{e}(\boldsymbol{i}_\pi)\bar{R}_\alpha \to \bar{R}_\alpha \bar{\psi}_\pi \bar{e}(\boldsymbol{i}_\pi)\bar{y}_\pi \bar{R}_\alpha;$$
5. *$\bar{R}_\alpha \bar{\psi}_\pi \bar{e}(\boldsymbol{i}_\pi)\bar{y}_\pi \bar{R}_\alpha = I_\pi / I_{>\pi}$.*

In future examples it will become convenient to adopt the following notation. When referring to $\alpha = 2\alpha_1 + \alpha_2$ and $\pi = (\alpha_1 + \alpha_2)\alpha_1$ then we will often write $I_\pi = I_{121}$, and similarly for Λ_π and other such notation.

This gives rise to a basis for R_α which we call the *affine cellular basis* due to its combinatorial similarities with the bases of [GL66] for finite dimensional cellular algebras.

Corollary 2.8. *The algebra R_α has a basis given by*
$$\{\psi_w y_\pi \Lambda_\pi \psi_\pi y_\pi e(\boldsymbol{i}_\pi)\psi_v^\tau \mid \pi \in \Pi(\alpha); w, v \in \mathfrak{S}^\pi\}.$$

This work has since been generalised by Kleshchev and Loubert [KL15] to all finite types. Note that the affine cellular basis is not always the easiest basis to work with, as the next example illustrates.

Example 2.9. Let $\alpha = 2\alpha_1 + \alpha_2$ then $\Lambda_{121} = \Bbbk[y_2, y_3]$, so how is $e(121)y_1$ expressed as a linear combination of basis elements?

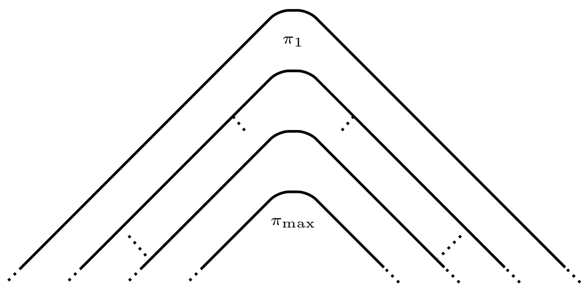
$$\begin{aligned}
e(121)y_1 &= (y_1 - y_2)e(121) + y_2 e(121) \\
&= -\psi_1 e(211)\psi_1 + y_2 e(121) \\
&= -\psi_1(\psi_2 y_3 - y_2\psi_2)e(211)\psi_1 + y_2 e(121) \\
&= \psi_1 y_2 \psi_2 y_3 \psi_2 e(211) y_3 \psi_2 \psi_1 - \psi_1 \psi_2 y_3 \psi_2 e(211) y_3 \psi_1 + y_2 e(121) \\
&= \psi_1 \psi_2 y_3 \psi_2 (y_2 + y_3) e(211) y_3 \psi_2 \psi_1 - \psi_1 y_3 \psi_2 e(211) y_3 \psi_2 \psi_1 \\
&\quad - \psi_1 \psi_2 y_3 \psi_2 e(211) y_3 \psi_1 + y_2 e(121).
\end{aligned}$$

This example is also illustrative of the property that $y_r e(\boldsymbol{i}_\pi) \equiv y_s e(\boldsymbol{i}_\pi) \mod I_{>\pi}$ when y_r and y_s are in the same $\pi-$block, see [KLM13, Corollary 5.10].

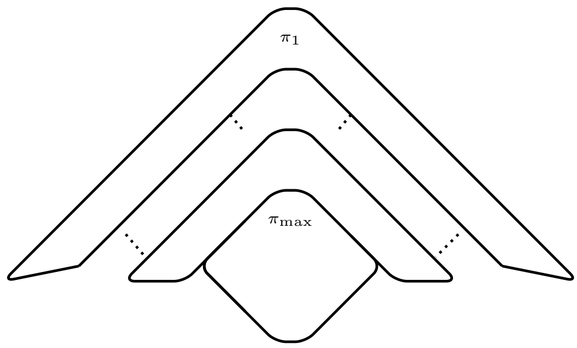
Chapter 3

An ideal of $R_\alpha$the quiver Hecke algebra

The affine cell chain structure of R_α described in the previous chapter can be thought of as follows



where each layer is a different affine cell ideal. The purpose of this chapter is to establish an ideal $\mathcal{J}$ such that the quotient $R_\alpha/\mathcal{J}$ is a truncation of the affine cell ideals to give a finite dimensional algebra.



In order to construct $\mathcal{J}$ we must first generalise Lemma 1.11 so that for any $w \in \mathfrak{S}^\pi$ such that $e(\boldsymbol{i}_\pi)\psi_w e(\boldsymbol{i}_\pi) \neq 0$ we may rewrite $\psi_w y_\pi \psi_\pi e(\boldsymbol{i}_\pi)$ as $f_w \psi_\pi e(\boldsymbol{i}_\pi)$ where f_w is a Schubert polynomial associated to w, this is done is Section 3.1. In Section 3.2 we construct $\mathcal{J}$, in doing so we make use of the fact that multiplying an element of the affine cell basis by any element of R_α either increases the degree of the polynomial from Λ_π at the centre of the basis element or yields a linear combination of basis elements from cells lower than the original (note that it is also an option that both of these eventualities occur). Crucially, the degree of the polynomial at the centre of our basis element is not decreased. Therefore, we can define an ideal by choosing basis elements from each cell ideal with central polynomial of sufficiently high degree to ensures that multiplication by elements of R_α yields linear combinations of basis elements above that degree in each cell ideal. It is worth noting that while we could define an ideal in the same way but containing all polynomials in Λ_π, the finite dimensional algebra obtained when R_α is quotiented by this ideal does not posses the homological properties R_α that we wish to preserve. A worked example of this is contained in Section 6.3. We start Section 3.2 by establishing a bound on the central polynomial and then go on to formally prove the properties of $\mathcal{J}$ that we describe here.

3.1 The group W_πW

Let β be a positive root of height h. Define the element $w_\beta \in \mathfrak{S}_{2h}$ to be $w_\beta := (s_h \dots s_{2h-1}) \dots (s_2 \dots s_{h+1})(s_1 \dots s_h)$. In other words, w_β permutes two β-blocks, and is the permutation in the symmetric group which yields $\psi_\beta = \psi_{w_\beta}$ in (2.1).

There is a natural embedding

$$\iota_{(r-1)h,2h,(p-r-1)h} : \mathfrak{S}_{(r-1)h} \times \mathfrak{S}_{2h} \times \mathfrak{S}_{(p-r-1)h} \hookrightarrow \mathfrak{S}_{ph}$$

We define

$$w_{\beta,r} := \iota_{(r-1)h,2h,(p-r-1)h}(1 \otimes w_\beta \otimes 1) \qquad (1 \leq r < p).$$

So $w_{\beta,r}$ is the element of the symmetric group that permutes the r^{th} and $(r+1)^{st}$ β-blocks. Now consider the root partition $\pi = \beta_1^{p_1} \cdots \beta_n^{p_n}$. For $1 \leq k \leq n$ and $x \in \mathfrak{S}_{p_k|\beta_k|}$, we define the embedding

$$\iota^k : \mathfrak{S}_{p_1|\beta_1|+\dots+p_{k-1}|\beta_{k-1}|} \times \mathfrak{S}_{p_k|\beta_k|} \times \mathfrak{S}_{p_{k+1}|\beta_{k+1}|+\dots+p_n|\beta_n|} \hookrightarrow \mathfrak{S}_d,$$

as

$$\iota^k(x) := \iota_{p_1|\beta_1|+\dots+p_{k-1}|\beta_{k-1}|,p_k|\beta_k|,p_{k+1}|\beta_{k+1}|+\dots+p_n|\beta_n|}(1 \otimes x \otimes 1).$$

Define, $w_{\beta_k,r} := \iota^k(w_{\beta,r})$ for all $1 \leq k \leq n$ and $1 \leq r < p_k$.

We now define the group W_π using the notation defined above,

$$W_\pi = \langle w_{\beta_k,r} \mid k = 1, \dots, n; r = 1, \dots, p_k - 1 \rangle.$$

In explanation, W_π is the group generated by permutations that swap π-blocks of weight β_k. The next collection of lemmas builds towards an alternative description of W_π.

Lemma 3.1. *If* $e(\boldsymbol{i}_\pi)\psi_w e(\boldsymbol{i}_\pi) \in \sum_{\sigma \leq \pi} I'_\sigma \subseteq R_\alpha$ *then* $w \in W_\pi$.

Proof. Assume that $w \notin W_\pi$, so ψ_w will "mix up" the blocks of π. Suppose we have a root $\beta = \alpha_t + \cdots + \alpha_{t+k}$ in the root partition π occupying the positions $i, \dots, i+k$. Additionally, suppose $i \leq j < j' \leq i+k$ such that $w(j) > w(j')$, without loss of generality we need only consider $j' = j+1$. Then $w = w's_j$ and,

$$\psi_w e(\boldsymbol{i}_\pi) = \psi_{w'}\psi_{s_j} e(\boldsymbol{i}_\pi) + \psi_v e(\boldsymbol{i}_\pi)$$

for v such that $l(\psi_v) < l(\psi_w)$. Clearly $\psi_{w'}\psi_{s_j} e(\boldsymbol{i}_\pi) = \psi_{w'} e(s_j \boldsymbol{i}_\pi)\psi_{s_j}$ and $s_j \boldsymbol{i}_\pi > \boldsymbol{i}_\pi$, which contradicts $e(\boldsymbol{i}_\pi)\psi_w e(\boldsymbol{i}_\pi) \in \sum_{\sigma \leq \pi} I'_\sigma$. □

Lemma 3.2. *[Mat99, Corollary 1.4] Suppose that* $w \in \mathfrak{S}_n$ *and that* s_i *is a simple transposition in* $\mathfrak{S}_n$. *Then*

$$l(ws_i) = \begin{cases} l(w) + 1; & \text{if } w(i) < w(i+1), \\ l(w) - 1; & \text{if } w(i) > w(i+1). \end{cases}$$

Lemma 3.3. *If* $w(i) < w(i+1)$ *for* $i < i+1$ *in the same* π*-block then* $w \in \mathfrak{S}^\pi$.

Proof. Let us consider ws_i for some transposition $s_i \in \mathfrak{S}_n$. Since $w(i) < w(i+1)$, $l(ws_i) = l(w) + 1$. Both w and ws_i are in the same $\mathfrak{S}_\pi$-coset, but $l(w) < l(ws_i)$ for all $s_i \in \mathfrak{S}^\pi$. Therefore, $l(w)$ is minimal, and $w \in \mathfrak{S}^\pi$. □

Lemma 3.4. *Diagrammatically a reduced expression is a diagram in which no two strands cross twice.*

Proof. Without loss of generality assume $\mathfrak{S}_n$ is acting on $(1 \cdots n)$ from the left. We proceed by induction on $l(w)$. If $l(w) = 0$ then we are done, so assume the claim is true for $l(w) = k$. Now let $\tilde{w} = ws_i$, by Lemma 3.2 either $l(\tilde{w}) = k+1$ or $l(\tilde{w}) = k-1$. If it is the latter, then our expression of $\tilde{w}$ is not reduced, and $w(i+1) < w(i)$, which means we have had a crossing of the i and $i+1$ strands, therefore adding s_i corresponds to a diagram in which the two strands cross twice. So, if $l(\tilde{w}) = k+1$, then our expression is still reduced, and since $w(i) < w(i+1)$, diagrammatically, we have not already had a crossing of the i and $i+1$ strands, and any other crossing of two strands only occurs once. □

Lemma 3.5. *For a root partition* $\pi \in \Pi(\alpha)$ *we have*

$$W_\pi = \mathfrak{S}^\pi \cap \{w \in \mathfrak{S}_n \mid e(\boldsymbol{i}_\pi)\psi_w e(\boldsymbol{i}_\pi) \neq 0\}.$$

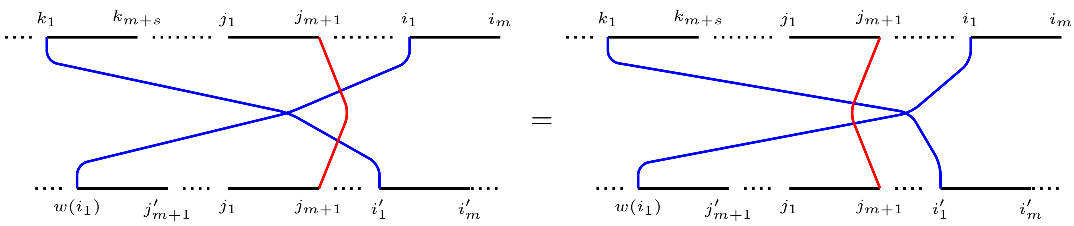
Proof. Let $\pi = \beta_1^{p_1} \cdots \beta_n^{p_n}$. We start with the $(\subseteq)$ inclusion. It follows from the definition of W_π that $W_\pi \subset \{w \in \mathfrak{S}_n \mid e(\boldsymbol{i}_\pi)\psi_w e(\boldsymbol{i}_\pi) \neq 0\}$. To see that $W_\pi \subset \mathfrak{S}^\pi$ take $w \in W_\pi$. Again by definition $w(i) < w(j)$ if i, j are in the same block, this implies $w \in \mathfrak{S}^\pi$.

Now for the $(\supseteq)$ inclusion. Take the element

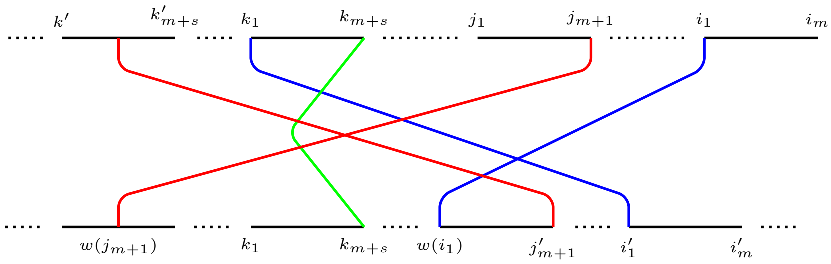
$$w \in \mathfrak{S}^\pi \cap \{w \in \mathfrak{S}_n \mid e(\boldsymbol{i}_\pi)\psi_w e(\boldsymbol{i}_\pi) \neq 0\}$$

and first consider the π-blocks of weight β_n. Without loss of generality, assume $\beta_n = \alpha_1 + \cdots + \alpha_m$. Pick the rightmost strand of colour 1, say this appears in the i_1^{th} position, then $w(i_1) \leq i_1$. We claim that $w(i_1)$ is also in a π-block of weight β_n. If $w(i_1) = i_1$ then the claim is satisfied. Assume $w(i_1) < i_1$ then if $w(i_1)$ is not in a π-block of weight β_n then it is in one of higher weight. Assume that $w(i_1)$ is not in a π-block of weight β_n, but is in a π-block of weight β_{n-1}. Since $\beta_{n-1} > \beta_n$ in the ordering on $\Pi(\alpha)$, β_{n-1} contains a strand of higher colour, without loss of generality say $m+1$. Let $w(i_1)$ be in the rightmost β_{n-1} block. Label the position of the last appearing strand of colour $m+1$ by j_{m+1}, then $w(j_{m+1}) \leq j_{m+1}$ since $e(\boldsymbol{i}_\pi)\psi_w e(\boldsymbol{i}_\pi) \neq 0$.

Assume that $w(j_{m+1}) = j_{m+1}$. By considering the braid diagram in the symmetric group, we see that for there to be a bijection between the top and bottom of the diagram we must have a strand of colour 1 going into the π-blocks of weight β_n from some π-block of weight $\beta_s > \beta_n$. This contradicts Lemma 1.8 as, using [GP00, Lemma 2.1.4], we can now find $w' \in \mathfrak{S}^\pi$ such that $w = \bar{w}w'$ and $w'(\boldsymbol{i}_\pi) > \boldsymbol{i}_\pi$ as illustrated below.



So we consider $w(j_{m+1}) < j_{m+1}$. Again, for a bijection of the diagram we need a strand of colour 1 from the left of j_{m+1} going to the π-blocks of weight β_n. If $w(j_{m+1})$ is in a π-block of weight β_{n-1} then we get the same situation as above so let $w(j_{m+1})$ fall in some π-block of weight β_s. Now, assuming $w(k_{m+s}) = k_{m+s}$ we reach a similar contradiction as illustrated below.

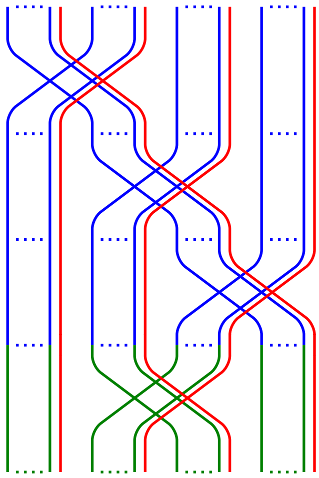


If $w(k_{m+s}) < k_{m+s}$ then recursive repetition of the argument means we run out of places to send a strand. So $w(i_1)$ is not in a π-block of weight β_{n-1}. Instead, if we assume $w(i_1)$ is in a π-block of weight $\beta_t < \beta_{n-1}$ then we reapply the previous arguments to that block and again get a contradiction. Inductively, we get that $w(i_1)$ must be in a π-block of weight β_n.

The same argument above can be applied to the next rightmost strand of colour 1 and so on giving us that all strands of colour 1 in block β_n have their image, under w, in a π-block of weight β_n. Thus, all strands of a block β_n have their images under w in a β_n block. Applying the above arguments recursively to β_{n-1} through β_1 gives us that $w \in \mathfrak{S}_{p_1|\beta_1|} \times \cdots \times \mathfrak{S}_{p_n|\beta_n|}$.

We now reduce our attention to $\pi_n = p_n(\alpha_1 + \cdots + \alpha_m) = p_n\beta_n$. For $i < i'$ in β_n we have $w(i) < w(i')$. So, consider neighbouring strands of colours i and $i+1$ in $p_n\beta_n$ and choose the q such that $w(i+qm)$ is maximal among all strands of colour i. Then $w(i+qm) = i+(p_n-1)m$ and $w(i+qm)+1 \le w(i+1+qm)$, since we are in the maximal block there is only one option and $w(i+1+qm) = i+1+(p_n-1)m$. Now proceed with downward induction on the images of $i+qm$ under w where q varies. To help keep track we introduce some quantifier κ, so that for q with $w(i+qm) > \kappa$ assume $w(i+1+qm) = w(i+qm)+1$. We now need to show the hypothesis for q such that $w(i+qm) = \kappa$. We know that $w(i+1+qm) \le w(i+qm)+1$, but by the inductive hypothesis the strictly greater options are already accounted for, so $w(i+1+qm) = w(i+qm)+1$. We have shown that for $i, i+1$ in β_n, $w(i+1) = w(i)+1$. Repeating this argument for each $p_i\beta_i$ gives us $w(i+1) = w(i)+1$ for all $i, i+1$ in the same π-block. Thus, $w \in W_\pi$. $\square$

We have one final lemma on reduced expressions in W_π before we generalize Lemma 1.11. When thinking about the proof of the lemma below, one should keep in mind a picture of the following sort.



Lemma 3.6. *If $\tilde{w} = s_{r_1} \cdots s_{r_n}$ is a reduced expression, then $w := w_{\beta,r_1} \cdots w_{\beta,r_n}$ is a reduced expression. We then define $\psi_w := \psi_{w_{\beta,r_1}} \cdots \psi_{w_{\beta,r_n}}$, moreover, $l(w) = \sum_{i=1}^{n} l(w_{\beta,r_i})$.*

Proof. We begin by induction on the length of $\tilde{w}$. For $l(\tilde{w}) = 0$ the hypothesis is clear, so assume it is also true for $l(\tilde{w}) = n-1$. Now for $\tilde{w}$ of length n we induct on the height of the root β. If $|\beta| = 1$, then $\tilde{w} = w$ and therefore is a reduced expression. Now assume that the claim is true for $|\beta| = m-1$, without loss of generality $\beta = \alpha_1 + \ldots + \alpha_{m-1}$. Then for $|\beta| = m$, assume w is not a reduced expression. So, the m^{th} strand in some copy must cross the same strand twice by Lemma 3.4. But since $w \in W_\pi$, we have no crossings within the root β by Lemma 3.5. Therefore, there must also be double crossings in each of the other strands, for instance the 1^{st} strand. This contradicts $\tilde{w}$ being a reduced expression. So w must be a reduced expression. □

Recall the polynomial ring P_π from (2.3), this is the polynomial ring in variables corresponding to the ends of roots. The polynomial ring Λ_π is a subset of P_π.

Example 3.7. Let $\pi = (\alpha_1 + \alpha_2 + \alpha_3)(\alpha_1 + \alpha_2)^2$. Then $P_\pi = \Bbbk[y_3, y_5, y_7]$.

Proposition 3.8. *Let $\pi \in \Pi(\alpha)$ be a root partition for α. Then*

$$\{e(i_\pi)\psi_w y_\pi \psi_\pi e(i_\pi) \mid w \in \mathfrak{S}^\pi\} = \{f_w \psi_\pi e(i_\pi) \mid w \in W_\pi\} \subseteq R_\alpha,$$

where f_w is the Schubert polynomial with variables in P_π associated to $w \in W_\pi$. Moreover, this is a term-wise equality.

Proof. Lemma 3.5 allows us to reduce our attention to the case of one repeated root, ie $\pi = \beta^n$. We prove this by induction on the length of ψ_w. For $l(w) = 0$ the equality is clear, so assume it is also true for $l(w') = l-1$ and let $w = w_{\beta,r_1}w'$. Using Lemma 3.6 we write $\psi_w = \psi_{w_{\beta,r_1}} \cdots \psi_{w_{\beta,r_n}}$, then $w_{\beta,r_2} \cdots w_{\beta,r_n} = \psi_{w'}$ for $w' \in W_\pi$. By length, we know the claim holds for $\psi_{w'}$, so

$$\begin{aligned}\psi_w y_\pi \psi_\pi e(\boldsymbol{i}_\pi) &= \psi_{w_{\beta,r_1}} \psi_{w'} y_\pi \psi_\pi e(\boldsymbol{i}_\pi), \\ &= \psi_{w_{\beta,r_1}} f_{w'} \psi_\pi e(\boldsymbol{i}_\pi), \\ &= \psi_{w_{\beta,r_1}} \psi_{\pi\bar{\pi}} \overline{f_{w'}} \psi_{\bar{\pi}} \psi_{\bar{\pi}\pi} e(\boldsymbol{i}_\pi).\end{aligned}$$

Our claim now reduces to showing that, for $w = w_{\beta,r_1}$,

$$\psi_w \psi_{\pi\bar{\pi}} \overline{f_{w'}} \psi_{\bar{\pi}} \psi_{\bar{\pi}\pi} e(\boldsymbol{i}_\pi) = \psi_{\pi\bar{\pi}} \overline{f_w} \psi_{\bar{\pi}} \psi_{\bar{\pi}\pi} e(\boldsymbol{i}_\pi).$$

Using the same convention as Example 2.4 we write $\psi_w = \psi_{w\bar{w}} \psi_{\bar{w}} \psi_{\bar{w}w}$, then

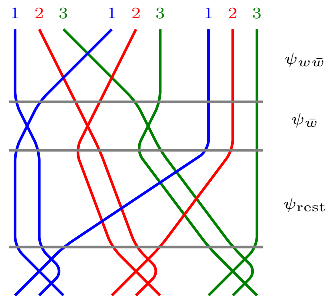
$$\psi_w \psi_{\pi\bar{\pi}} \overline{f_{w'}} \psi_{\bar{\pi}} \psi_{\bar{\pi}\pi} e(\boldsymbol{i}_\pi) = \psi_{w\bar{w}} \psi_{\bar{w}} \psi_{\bar{w}w} \psi_{\pi\bar{\pi}} \overline{f_{w'}} \psi_{\bar{\pi}} \psi_{\pi\bar{\pi}} e(\boldsymbol{i}_\pi).$$

Notice that $\psi_{\pi\bar{\pi}}$ can be written in two ways. We can either collect all the 1s, then all the 2s and so on. Or, we can order two adjacent blocks, then order a third adjacent block into that and so on. (The two options are illustrated in Example 3.9.)

Choosing the second option, and first ordering the r_1 and $(r_1 + 1)$ π-blocks of weight β then $\psi_{\pi\bar{\pi}}$ ends with the expression $\psi_{w\bar{w}}$, ie $\psi_{\pi\bar{\pi}} = \psi_{w\bar{w}} \psi_{\text{rest}}$ where ψ_{rest} is just the remaining part of $\psi_{\pi\bar{\pi}}$. So,

$$\begin{aligned}\psi_{w\bar{w}} \psi_{\bar{w}} \psi_{\bar{w}w} \psi_{w\bar{w}} \psi_{\text{rest}} \overline{f_{w'}} \psi_{\bar{\pi}} \psi_{\bar{\pi}\pi} &= \psi_{w\bar{w}} \psi_{\bar{w}} p \psi_{\text{rest}} \overline{f_{w'}} \psi_{\bar{\pi}} \psi_{\bar{\pi}\pi} e(\boldsymbol{i}_\pi), \\ &= \psi_{w\bar{w}} \psi_{\bar{w}} \psi_{\text{rest}} \bar{p} \overline{f_{w'}} \psi_{\bar{\pi}} \psi_{\bar{\pi}\pi} e(\boldsymbol{i}_\pi).\end{aligned}$$

We now claim that it is possible to "braid" $\psi_{\bar{w}}$ through ψ_{rest} to give $\psi_{\text{rest}} \overline{\psi_{\bar{w}}}$. This follows from the fact that $\psi_{\bar{w}}$ contains only (i,i)-crossings and ψ_{rest} contains only (i,j)-crossings. Thus eliminating any non-trivial braid relations as $\psi_{\bar{w}}$ passes through. It is also worth noting that for each (i,i)-crossing, if one of these i's crosses a j, then this implies that the other i will also cross that j-strand. See the following picture in the case of $\pi = 3(\alpha_1 + \alpha_2 + \alpha_3)$, the $\psi_{w\bar{w}}$ is at the top of the braid diagram with $\psi_{\bar{w}}$ in the section below followed by ψ_{rest}.



So we have

$$\psi_{\bar{w}w}\psi_{\bar{w}}\psi_{\text{rest}}\bar{p}\overline{f_{w'}}\psi_{\bar{\pi}}\psi_{\bar{\pi}\pi} = \psi_{w\bar{w}}\psi_{\text{rest}}\overline{\psi_{\bar{w}}}\bar{p}\overline{f_{w'}}\psi_{\bar{\pi}}\psi_{\bar{\pi}\pi}e(\boldsymbol{i}_\pi).$$

Part of the equation between ψ_{rest} and $\psi_{\bar{\pi}\pi}$ takes place in the product of nil-Hecke algebras, if we write that part explicitly we get

$$\psi_{w\bar{w}}\psi_{\text{rest}}\prod_{i=0}^{m-1}\psi_{r_1+ia}\prod_{i=0}^{m-2}(y_{(r_1+1)+ia} - y_{r_1+ia})\overline{f_{w'}}\prod_{i=0}^{m-1}\prod_{j=1}^{a-1}\prod_{k=1}^{a-j}\psi_{k+ia}\psi_{\bar{\pi}\pi}e(\boldsymbol{i}_\pi).$$

When we expand the polynomial product we get a series of summands, all bar one of which equate to zero. The non-zero summand is the one which results from choosing the y corresponding to a dot on the $(r+1)^{st}$ strand of each nil-Hecke algebra, so

$$\prod_{i=0}^{m-2}(y_{(r_1+1)+ia} - y_{r_1+ia}) = \prod_{i=0}^{m-2}(y_{(r_1+1)+ia}).$$

If we focus on just the part in the nil-Hecke algebras $\text{NH}_a^{(1)} \otimes \cdots \otimes \text{NH}_a^{(m)}$ we get

$$\prod_{i=1}^{m-1}\left(\psi_{r_1}^{(i)}y_{r_1+1}^{(i)}\prod_{j=1}^{a-1}\prod_{k=1}^{a-j}\psi_k^{(i)}\right)\cdot\psi_{r_1}^{(m)}\overline{f_{w'}}^{(m)}\prod_{j=1}^{a-1}\prod_{k=1}^{a-j}\psi_k^{(m)}.$$

Since $\psi_{\boldsymbol{a}}^{(i)} = \prod_{j=1}^{a-1}\prod_{k=1}^{a-j}\psi_k^{(i)}$ and we can chose a reduced expression for $\boldsymbol{a}$ starting with r_1, we obtain

$$\prod_{i=1}^{m-1}\left(\prod_{j=1}^{a-1}\prod_{k=1}^{a-j}\psi_k^{(i)}\right)\cdot\psi_{r_1}^{(m)}\overline{f_{w'}}^{(m)}\prod_{j=1}^{a-1}\prod_{k=1}^{a-j}\psi_k^{(m)}.$$

When we consider our Schubert polynomial f_w, we have

$$\deg(f_{\text{id}}) = \deg(y_\pi) = 2\cdot a(a-1).$$

The length of w_0, the longest possible reduced word, is $a(a-1)/2$, and $f_{w_0} = 1$. So each time we increase the length of ψ_w by 1, whilst still being a reduced expression, we reduce the degree of f_w by 2. This corresponds to losing a y_i from the polynomial expression of f_w. There is precisely one y_i for each transposition in the reduced expression of w_0, taking this into account we get

$$\prod_{i=1}^{m-1}\left(\prod_{j=1}^{a-1}\prod_{k=1}^{a-j}\psi_k^{(i)}\right)\cdot\overline{f_w}^{(m)}\prod_{j=1}^{a-1}\prod_{k=1}^{a-j}\psi_k^{(m)}=\overline{f_w}\psi_{\bar{\pi}},$$

with $\deg(\overline{f_w})=\deg(\overline{f_{w'}})-2$. Having simplified the part in the nil-Hecke algebra we can return to our full picture where we have

$$\psi_{w\bar{w}}\psi_{\text{rest}}\overline{\psi_{\bar{w}}}\bar{p}\overline{f_{w'}}\psi_{\bar{\pi}}\psi_{\bar{\pi}\pi}e(\boldsymbol{i}_\pi)=\psi_{w\bar{w}}\psi_{\text{rest}}\overline{f_w}\psi_{\bar{\pi}}\psi_{\bar{\pi}\pi}e(\boldsymbol{i}_\pi).$$

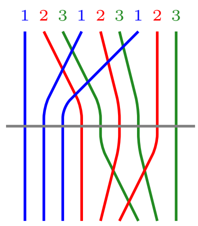
We can now move $\overline{f_w}$ back through to the front, and rewrite $\psi_{\pi\bar{\pi}}=\psi_{w\bar{w}}\psi_{\text{rest}}$ to get

$$f_w\psi_{\pi\bar{\pi}}\psi_{\bar{\pi}}\psi_{\bar{\pi}\pi}e(\boldsymbol{i}_\pi)=f_w\psi_\pi e(\boldsymbol{i}_\pi).$$

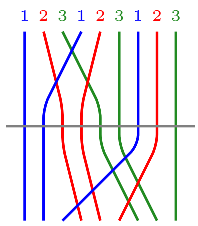
Hence, $\{e(i_\pi)\psi_w y_\pi\psi_\pi e(i_\pi)\mid w\in\mathfrak{S}^\pi\}=\{f_w\psi_\pi e(i_\pi)\mid w\in W_\pi\}$.

∎

Example 3.9. We illustrate the two ways that $\psi_{\pi\bar{\pi}}$ can be written for $\pi=(\alpha_1+\alpha_2+\alpha_3)^3$. Here we first collect all the 1s, then all the 2s and that gives us all the 3s together.



Here we order the first two blocks, then order the third block into that.



Corollary 3.10. *If $e(i_\pi)\psi_w e(i_\pi)$ then $\deg(\psi_w)<0$, unless $w=\text{id}$, for $w\in W_\pi$.*

Proof. We know $\deg(e(\boldsymbol{i}_\pi)\psi_w y_\pi e(\boldsymbol{i}_\pi))=\deg(f_w)<\deg(y_\pi)$ unless $w=\text{id}$. So $\deg(\psi_w)<0$ unless $w=\text{id}$. ∎

3.2 The ideal $\mathcal{J}$J

We now set about constructing an ideal $\mathcal{J}$, with which we intend to define a quotient of the quiver Hecke algebra with nice homological properties. To do this we need to introduce a function that takes a root partition and gives us out a number. This number is then used as a bound on the degree of a polynomial in the definition of our ideal $\mathcal{J}$.

Proposition 3.11. *For $\alpha \in Q_+$ with $\mid \alpha \mid= n$ and $\nu \geq \sigma > \pi \in \Pi(\alpha)$ there exists a function,*

$$\boldsymbol{d} : \Pi(\alpha) \to \mathbb{N};$$

$$\pi \mapsto d_\pi,$$

iteratively constructed on

$$d_\pi = \max_{\nu,\sigma,\pi}\{d_\nu + \deg(y_\nu) - \deg(y_\sigma) - y_\pi + 4n(n-1)\}$$

such that for reduced expressions $w', v' \in \mathfrak{S}^\sigma$, $v \in \mathfrak{S}^\pi$ and polynomial $p \in \Lambda_\pi$ with $\deg(p) \geq d_\pi$*, we have,*

$$\psi_{w'} y_\sigma e(\boldsymbol{i}_\sigma)\psi_\sigma y_\sigma \psi^\tau_{v'} y_\pi e(\boldsymbol{i}_\pi)\psi_\pi y_\pi p \psi^\tau_v = \sum_{\substack{\nu \geq \sigma;\\ \tilde{u},\tilde{v}\in\mathfrak{S}^\nu;\\ q\in\mathfrak{B}_\nu}} c_{\nu,p,\tilde{u},\tilde{v}} \psi_{\tilde{u}} y_\nu e(\boldsymbol{i}_\nu)\psi_\nu y_\nu q \psi^\tau_{\tilde{v}}, \tag{3.1}$$

for all ν where $\mathfrak{B}_\nu$ is a basis for Λ_ν and if $c_{\nu,q,\tilde{u},\tilde{v}} \neq 0$ then $\deg(q) \geq d_\nu$*.*

Proof. We prove this by downward induction on root partitions. For $\pi_{max} \in \Pi(\alpha)$ we set $d_\pi = 1$. Assume there exists a d_σ for all $\sigma > \pi \in \Pi(\alpha)$. Now take the element

$$\psi_{w'} y_\sigma e(\boldsymbol{i}_\sigma)\psi_\sigma y_\sigma \psi^\tau_{v'} y_\pi e(\boldsymbol{i}_\pi)\psi_\pi y_\pi p \psi^\tau_v \in I_{\geq\sigma},$$

then by [KLM13, Theorem 5.6] we can rewrite this as

$$\sum_{\substack{\nu \geq \sigma;\\ \tilde{u},\tilde{v}\in\mathfrak{S}^\nu;\\ q\in\mathfrak{B}_\nu}} \alpha_{\nu,p,\tilde{u},\tilde{v}} \psi_{\tilde{u}} y_\nu e(\boldsymbol{i}_\nu)\psi_\nu y_\nu q \psi^\tau_{\tilde{v}}.$$

We proceed by arguing that if we choose p with a sufficiently high degree then $\alpha_{\nu,p,\tilde{u},\tilde{v}} \neq 0$ will imply $\deg(q) \geq d_\nu$. If we compare degrees on either side of the equality (3.11) we have

$$\begin{aligned}\deg(\psi_{w'} e(\boldsymbol{i}_\sigma)) + \deg(y_\sigma) + \deg(e(\boldsymbol{i}_\sigma)\psi^\tau_{v'} e(\boldsymbol{i}_\pi)) + \deg(y_\pi) + \deg(p) + \deg(\psi^\tau_v e(\boldsymbol{i}_\pi))\\ = \deg(\psi_{\tilde{u}} e(\boldsymbol{i}_\nu)) + \deg(y_\nu) + \deg(q) + \deg(e(\boldsymbol{i}_\nu)\psi^\tau_{\tilde{v}}),\end{aligned}$$

bearing in mind that we need $\deg(q) \geq d_\nu$ we want

$$\begin{aligned}\deg(\psi_{w'}e(\boldsymbol{i}_\sigma)) + \deg(y_\sigma) + \deg(e(\boldsymbol{i}_\sigma)\psi^\tau_{v'}e(\boldsymbol{i}_\pi)) + \deg(y_\pi) + \deg(p)\\ + \deg(\psi^\tau_v e(\boldsymbol{i}_\pi)) - \deg(\psi_{\tilde{u}}e(\boldsymbol{i}_\nu)) - \deg(y_\nu) - \deg(e(\boldsymbol{i}_\nu)\psi^\tau_{\tilde{v}}) \geq d_\nu.\end{aligned}$$

So we require

$$\begin{aligned}\deg(p) \geq d_\nu + \deg(\psi_{\tilde{u}}e(\boldsymbol{i}_\nu)) + \deg(y_\nu) + \deg(e(\boldsymbol{i}_\nu)\psi^\tau_{\tilde{v}}) - \deg(\psi_{w'}e(\boldsymbol{i}_\sigma))\\ - \deg(y_\sigma) - \deg(e(\boldsymbol{i}_\sigma)\psi^\tau_{v'}e(\boldsymbol{i}_\pi)) - \deg(y_\pi) - \deg(\psi^\tau_v e(\boldsymbol{i}_\pi))\end{aligned}$$

Since the longest word in $\mathfrak{S}_n$ has length $n(n-1)/2$ we determine an upper bound on the degrees

$$\deg(\psi_{\tilde{u}}e(\boldsymbol{i}_\nu)),\ \deg(e(\boldsymbol{i}_\nu)\psi^\tau_{\tilde{v}}) \leq \frac{n(n-1)}{2}.$$

Also,

$$\deg(\psi_{w'}e(\boldsymbol{i}_\sigma)), \deg(e(\boldsymbol{i}_\sigma)\psi^\tau_{v'}e(\boldsymbol{i}_\pi)), \deg(\psi^\tau_v e(\boldsymbol{i}_\pi)) \geq -n(n-1).$$

So take

$$\begin{aligned}\deg(p) &\geq d_\nu + n(n-1) + \deg(y_\pi) - \deg(y_\sigma) - \deg(y_\pi) + 3n(n-1)\\ &\geq d_\nu + \deg(y_\nu) - \deg(y_\sigma) - \deg(y_\pi) + 4n(n-1),\end{aligned}$$

therefore we set $d_\pi = \max_{\nu,\sigma,\pi}\{d_\nu + \deg(y_\nu) - \deg(y_\sigma) - \deg(y_\pi) + 4n(n-1)\}$. □

Throughout the remainder of this chapter we fix a $\boldsymbol{d}$ satisfying the conditions of Proposition 3.11, and for a root partition $\pi \in \Pi(\alpha)$ we define

$$\mathcal{J}'_\pi = \Bbbk - \operatorname{span}\{\psi_w y_\pi \psi_\pi e(\boldsymbol{i}_\pi) y_\pi p \psi^\tau_v | w, v \in \mathfrak{S}^\pi, p \in \Lambda_\pi, \deg(p) \geq d_\pi\},$$

$$\mathcal{J}_\pi = \sum_{\sigma \geq \pi} \mathcal{J}'_\pi,$$

$$\mathcal{J}_{>\pi} = \sum_{\sigma > \pi} \mathcal{J}'_\pi.$$

Now define

$$\mathcal{J} = \sum_{\pi \in \Pi(\alpha)} \mathcal{J}'_\pi.$$

We are about to show that $\mathcal{J}$ is an ideal for R_α, but first we need a technical lemma and a generalization of [Rou12, Theorem 2.11].

Lemma 3.12. *[KLM13, Corollary 5.10] If y_r and y_s are in the same π-block, then*

$$y_r e(\boldsymbol{i}_\pi) \equiv y_s e(\boldsymbol{i}_\pi) \mod I_{>\pi}.$$

For an example of this in the case of $\alpha = 2\alpha_1 + \alpha_2$ see Example 2.9 in which it is shown that $y_1 e(121) \equiv y_2 e(121) \mod I_{211}$.

We need to use some classic results on Schubert polynomials, but adapted to our particular setting.

Theorem 3.13. *[Rou, Theorem 2.11] Schubert polynomials in $y_1, \cdots, y_d$ form a basis for the polynomial ring $\Bbbk[y_1, \cdots, y_d]$ as a free module over the ring $\Bbbk[y_1, \cdots, y_d]^{\mathfrak{S}}$ of symmetric polynomials.*

Recall that P_π is the polynomial ring in the same variables as Λ_π but without any symmetry. Let us consider the set of Schubert polynomials in P_π with respect to W_π, by which we mean the subring of P_π generated by Schubert polynomials in P_{β_i, p_i} for each $i = 1, \cdots, n$.

Corollary 3.14. *Schubert polynomials in P_π with respect to W_π form a basis for P_π as a free module over Λ_π.*

Theorem 3.15. *$\mathcal{J}$ is an ideal in R_α.*

Proof. For $a \in R_\alpha$ we have

$$a\psi_w y_\pi \psi_\pi e(\boldsymbol{i}_\pi) y_\pi p \psi_v^\tau = a' y_\pi \psi_\pi e(\boldsymbol{i}_\pi) y_\pi p \psi_v^\tau$$

for some $a' \in R_\alpha$. So setting $b = y_\pi \psi_\pi e(\boldsymbol{i}_\pi) y_\pi p \psi_v^\tau$ it suffices to check that $hb \in \mathcal{J}$ for all $h \in R_\alpha$. Recalling the basis in Theorem 1.4 we shall take

$$h \in \{\psi_u y_1^{r_1} \cdots y_d^{r_d} e(\boldsymbol{i}) \mid r_i \geq 0;\ u \in \mathfrak{S}_d;\ \boldsymbol{i} \in \langle I \rangle_\alpha\}.$$

We proceed by induction on $\pi \in \Pi(\alpha)$. Let $\pi = \pi_{\max}$, then each β_i in π has $|\beta_i| = 1$ so $P_\pi = \Bbbk[y_1, \ldots, y_d]$. First consider $h = y_1^{r_1} \cdots y_d^{r_d} e(\boldsymbol{i}_\pi)$ then using Corollary 3.14, we get

$$hy_\pi e(\boldsymbol{i}_\pi) = \sum_{w \in \mathfrak{S}_d} f_w p_w e(\boldsymbol{i}_\pi)$$

where f_w is the Schubert polynomial associated to w and p_w is a symmetric polynomial. Therefore,

$$hb = \sum_{w \in \mathfrak{S}_d} f_w p_w \psi_\pi e(\boldsymbol{i}_\pi) y_\pi p \psi_v^\tau = \sum_{w \in \mathfrak{S}_d} f_w \psi_\pi e(\boldsymbol{i}_\pi) y_\pi p \bar{p}_w \psi_v^\tau.$$

Notice that $\mathfrak{S}^{\pi_{\max}} = \mathfrak{S}_d$ so Proposition 3.8 gives

$$hb = \sum_{w \in W_\pi} \psi_w y_\pi \psi_\pi e(\boldsymbol{i}_\pi) y_\pi p' \psi_v^\tau \in \mathcal{J}.$$

Now, it remains to check $h = \psi_u \psi_w$ for $u \in \mathfrak{S}_d$. When $\psi_w = 1$, since $\mathfrak{S}_d = \mathfrak{S}^\pi$ we have that hb is a basis element in $\mathcal{J}_\pi$. Now consider $\psi_w \neq 1$, Corollary 3.10 gives $\deg(\psi_w e(\boldsymbol{i}_\pi)) < 0$ therefore

$$\deg(\psi_u e(\boldsymbol{i}_\pi)\psi_w e(\boldsymbol{i}_\pi)) < \deg(\psi_u e(\boldsymbol{i}_\pi)).$$

Proceed by induction on the degree of $\psi_u e(\boldsymbol{i}_\pi)$. For the base case let $\psi_u e(\boldsymbol{i}_\pi)$ be of minimal degree then $\psi_u \psi_w e(\boldsymbol{i}_\pi) = 0 \in \mathcal{J}$. Now assume $hb \in \mathcal{J}$ for all $\psi_u e(\boldsymbol{i}_\pi)$ of degree less than $m \in \mathbb{Z}$ and consider $u \in \mathfrak{S}_d$ with $\deg(\psi_u e(\boldsymbol{i}_\pi)) = m$. Untwisting double crossings give

$$\psi_u e(\boldsymbol{i}_\pi)\psi_w e(\boldsymbol{i}_\pi) = \sum_{\tilde{u} \in \mathfrak{S}_d} \psi_{\tilde{u}} q_{\tilde{u}} e(\boldsymbol{i}_\pi)$$

where $q_{\tilde{u}} \in \Bbbk[y_1, \dots, y_d]$ and $\deg(\psi_{\tilde{u}}) \leq \deg(\psi_u)$. If $\deg(q_{\tilde{u}}) = 0$ then

$$\psi_{\tilde{u}} q_{\tilde{u}} y_\pi \psi_\pi e(\boldsymbol{i}_\pi) y_\pi p \psi_v^\tau = \psi_{\tilde{u}} y_\pi \psi_\pi e(\boldsymbol{i}_\pi) y_\pi p \psi_v^\tau$$

is a basis element of $\mathcal{J}_\pi$. If $\deg(q_{\tilde{u}}) > 0$ then

$$\begin{aligned} q_{\tilde{u}} y_\pi \psi_\pi e(\boldsymbol{i}_\pi) y_\pi p \psi_v^\tau &= \sum_{w' \in W_\pi} f_{w'} \psi_\pi e(\boldsymbol{i}_\pi) y_\pi p_{w'} p \psi_v^\tau \\ &= \sum_{w' \in W_\pi} \psi_{w'} y_\pi \psi_\pi e(\boldsymbol{i}_\pi) y_\pi p' \psi_v^\tau. \end{aligned}$$

Now $\psi_{\tilde{u}} \psi_{w'} y_\pi \psi_\pi e(\boldsymbol{i}_\pi) y_\pi p' \psi_v^\tau$ as the same shape as $\psi_u \psi_w b$ but with

$$\deg(\psi_{\tilde{u}} e(\boldsymbol{i}_\pi)) < \deg(\psi_u e(\boldsymbol{i}_\pi))$$

so is in $\mathcal{J}$ by induction.

The arguments are symmetric, as multiplication on the right works in the same way, so $\mathcal{J}_{\pi_{\max}}$ is a two-sided ideal.

Now, for an arbitrary $\pi \in \Pi(\alpha)$ assume that $\mathcal{J}_{>\pi}$ is an ideal and use this to show $\mathcal{J}_\pi$ is an ideal. Using Lemma 3.12 we rewrite $h = y_1^{r_1} \cdots y_d^{r_d} e(\boldsymbol{i}_\pi)$ as $\bar{h} e(\boldsymbol{i}_\pi) + B$ for $\bar{h} \in P_\pi$ and $B \in I_{>\pi}$. Then $hb = \bar{h}b + Bb$ for $B \in I_{>\pi}$. By Proposition 3.11 we can rewrite B so that

$$Bb = \sum_{a_\sigma \in I'_\sigma} a_\sigma b = \sum_{\substack{a_\nu \in \mathcal{J}'_\nu \\ \nu \geq \sigma > \pi}} a_\nu$$

thus $Bb \in \mathcal{J}_{>\pi}$. Now consider $h \in P_\pi$, as before $he(\boldsymbol{i}_\pi) = \sum_{w \in W_\pi} f_w p_w e(\boldsymbol{i}_\pi)$ and by

Corollary 3.14

$$hb = \sum_{w \in W_\pi} f_w p_w b = \sum_{w \in W_\pi} \psi_w y \pi \psi_\pi e(\boldsymbol{i}_\pi) y_\pi p' \psi_v^\tau \in \mathcal{J}.$$

Now consider $h = \psi_u e(\boldsymbol{i}_\pi)\psi_w$ for $u \in \mathfrak{S}_d$. If $u \notin \mathfrak{S}^\pi$ then ψ_u factors over some I_σ where $\sigma > \pi$ in which case Proposition 3.11 puts this into $\mathcal{J}_{>\pi}$ which is covered by the inductive assumption. It is therefore sufficient to consider $u \in \mathfrak{S}^\pi$. If $\psi_w = 1$ then $\psi_u b$ is a basis element for $\mathcal{J}_\pi$. If $\psi_w \neq 1$ then $\deg(\psi_w e(\boldsymbol{i}_\pi)) < 0$ and $\deg(\psi_u \psi_w e(\boldsymbol{i}_\pi)) < \deg(\psi_u)$. Proceed by induction on the degree of $\psi_u e(\boldsymbol{i}_\pi)$. For $\psi_u e(\boldsymbol{i}_\pi)$ of minimal degree for $u \in \mathfrak{S}^\pi$ then $\psi_u \psi_w e(\boldsymbol{i}_\pi) = 0 \in \mathcal{J}$. Assume $\psi_u \psi_w b \in \mathcal{J}$ for all $\psi_u e(\boldsymbol{i}_\pi)$ such that $\deg(\psi_u e(\boldsymbol{i}_\pi)) < m \in \mathbb{Z}$. Consider $u \in \mathfrak{S}^\pi$ such that $\deg(\psi_u e(\boldsymbol{i}_\pi)) = m$ and we write

$$\psi_u \psi_w e(\boldsymbol{i}_\pi) = \sum_{\tilde{u} \in \mathfrak{S}^\pi} \psi_{\tilde{u}} q_{\tilde{u}} e(\boldsymbol{i}_\pi).$$

Since $\deg(\psi_u \psi_w e(\boldsymbol{i}_\pi)) < \deg(\psi_u e(\boldsymbol{i}_\pi))$ and $\deg(q_{\tilde{u}}) \geq 0$ we have $\deg(\psi_{\tilde{u}}) \leq \deg(\psi_u)$. Now consider

$$hb = \sum_{\tilde{u} \in \mathfrak{S}^\pi} \psi_{\tilde{u}} q_{\tilde{u}} y_\pi \psi_\pi e(\boldsymbol{i}_\pi) y_\pi p \psi_v^\tau.$$

If $\deg(q_{\tilde{u}}) = 0$ then hb is a basis element for $\mathcal{J}_\pi$. If $\deg(q_{\tilde{u}}) > 0$ then using Corollary 3.14 we rewrite as $q_{\tilde{u}} y_\pi e(\boldsymbol{i}_\pi) = \sum_{w' \in \mathfrak{S}^\pi} f_{w'} p_{w'} e(\boldsymbol{i}_\pi)$ which together with Proposition 3.8 gives

$$q_{\tilde{u}} y_\pi \psi_\pi e(\boldsymbol{i}_\pi) y_\pi p \psi_v^\tau = \sum_{w' \in \mathfrak{S}^\pi} \psi_{w'} y_\pi \psi_\pi e(\boldsymbol{i}_\pi) y_\pi p_{w'} p \psi_v^\tau.$$

Now $\psi_{\tilde{u}} \psi_{w'} y_\pi \psi_\pi e(\boldsymbol{i}_\pi) y_\pi p' \psi_v^\tau$ has the same shape as $\psi_u \psi_w b$ but with

$$\deg(\psi_{\tilde{u}} e(\boldsymbol{i}_\pi)) < \deg(\psi_u e(\boldsymbol{i}_\pi))$$

so by induction $hb \in \mathcal{J}$ and $\mathcal{J}$ is a two sided ideal.

□

3.3 An improvement on $d_\pi$our bound d

When constructing the ideal $\mathcal{J}$ polynomials in Λ_π are chosen to have degree greater than some d_π, for each $\pi \in \Pi(\alpha)$. The bound on d_π is far from optimal, and is currently given by

$$d_\pi \geq d_\nu + \deg(y_\nu) - \deg(y_\sigma) - \deg(y_\pi) + 4n(n-1).$$

The $4n(n-1)$ aspect is obtained by crudely taking the following upper bounds on the degrees of ψ_w type elements where w is a coset representative of some parabolic subgroup of $\mathfrak{S}_n$, hence its length is bound by the length of the longest element of $\mathfrak{S}_n$ which is $n(n-1)/2$. Each ψ_i has a degree 0, 1, or -2 so

$$\begin{aligned}
\deg(e(\boldsymbol{i}_\sigma)\psi^\tau_{v'}e(\boldsymbol{i}_\pi)) &\geq -n(n-1)\\
\deg(\psi_{w'}e(\boldsymbol{i}_\sigma)) &\geq -n(n-1)\\
\deg(\psi^\tau_v e(\boldsymbol{i}_\pi)) &\geq -n(n-1)\\
\deg(\psi_{\bar{u}}e(\boldsymbol{i}_\nu)) &\leq n(n-1)/2\\
\deg(e(\boldsymbol{i}_\nu)\psi^\tau_{\bar{v}}) &\leq n(n-1)/2
\end{aligned}$$

where $\pi < \sigma \leq \nu \in \Pi(\alpha)$, $\bar{u}, \bar{v} \in \mathfrak{S}^\nu$, $w', v' \in \mathfrak{S}^\sigma$ and $v \in \mathfrak{S}^\pi$. Recall that $\alpha = \sum_{i\in I} c_i\alpha_i$, where α_i are simple roots and for all $w \in \mathfrak{S}^\sigma$ such that $e(\boldsymbol{i})\psi_w e(\boldsymbol{i}_\sigma) \neq 0$, we have $\boldsymbol{i} \leq_{lex} e(\boldsymbol{i}_\sigma)$.

Lemma 3.16. *Let $\alpha = \sum_i c_i\alpha_i$ and let d_π be the positive integer defined in Proposition 3.11. It is sufficient to take*

$$d_\pi \geq d_\nu + \deg(y_\nu) - \deg(y_\sigma) - \deg(y_\pi) + 2\sum_i c_i c_{i+1} + 3\sum_i c_i(c_i - 1).$$

Proof. The bounds above can be greatly reduced by observing that, at the lower end, the most negative degree for $\psi_w e(\boldsymbol{i}_\pi)$ occurs when π is maximal among $\Pi(\alpha)$, and when ψ_w is the longest permutation of like-coloured strands. The longest word on strands of colour i has length $c_i(c_i-1)/2$, and the quiver Hecke algebra element corresponding to that has degree $-c_i(c_i-1)$ so

$$\sum_i c_i(1-c_i) \leq \deg(\psi_w e(\boldsymbol{i}_\pi))$$

which is clearly greater that $-n(n-1)$. At the upper end, the greatest degree for $\psi_w e(\boldsymbol{i}_\pi)$ again occurs when $e(\boldsymbol{i}_\pi)$ is maximal, as this allows us to have longer words. The element ψ_j is of positive degree whenever $i_j = i_{j+1} + 1$, for each collection of strands of neighbouring index there can be only $c_j c_{j+1}$ crossings that are not subject to relations. So $\deg(\psi_w e(\boldsymbol{i}_\sigma)) \leq \sum_i c_i c_{i+1}$. We can also place an upper bound on the degree of the element y_π. This is again of maximal degree when π is maximal. It is

$$\deg(y_\pi) \leq \sum_{c_i \neq 0} (c_i - 1)!.$$

Hence we get our bound. $\square$

Chapter 4

Stratified algebras

Quasi-hereditary algebras are a class of finite dimensional algebras introduced by Cline, Parshall and Scott [CPS88] that have particularly nice representation theory. They arise naturally in Lie theory and also overlap with the class of cellular algebras. There are several natural generalizations of quasi-hereditary algebras, these include the so-called standardly stratified algebras introduced in [CPS96], and the so-called properly stratified algebras introduced in [Dla00] which form a proper subclass of the class of standardly stratified algebras.

Definitions Let A be a finite dimensional $\Bbbk$-algebra, and let Λ be an indexing set for isomorphism classes of simple A-modules $L(\lambda)$, $\lambda \in \Lambda$. Let us denote by $P(\lambda)$ and $I(\lambda)$ the projective cover and injective hull, respectively, of the simple module $L(\lambda)$. For a subclass $\mathcal{C}$ of objects from A-**mod** we define $\mathcal{F}(\mathcal{C})$ to be the full subcategory of A-**mod** consisting of all modules M having a filtration whose subquotients are isomorphic to modules from $\mathcal{C}$, ie, a chain of submodules

$$0 \subseteq M_n \subseteq \cdots \subseteq M_1 \subseteq M$$

such that $M_i/M_{i+1} \in \mathcal{C}$. Define $\mathrm{add}(M)$ to be the full subcategory of A-**mod** consisting of modules N isomorphic to a direct summand of M^k for some $k \geq 0$. For A-modules M and N we define the *trace* $\mathrm{Tr}_M(N)$ of M in N as the sum of images of all A-homomorphisms from M to N.

Fix a partial pre-order $\leq$, by which we mean $\leq$ is reflexive and transitive, on Λ. For $\lambda, \mu \in \Lambda$ we write $\lambda < \mu$ if $\lambda \leq \mu$ and $\mu \not\leq \lambda$; and $\lambda \sim \mu$ if $\lambda \leq \mu$ and $\mu \leq \lambda$. For $\lambda \in \Lambda$ define $P^{>\lambda} = \oplus_{\mu>\lambda} P(\mu)$ and $I^{>\lambda} = \oplus_{\mu>\lambda} I(\mu)$. For each $\lambda \in \Lambda$ we define

- the *standard module* $\Delta(\lambda)$ to be the maximal quotient of $P(\lambda)$ such that $[\Delta(\lambda) : L(\mu)] = 0$ for $\mu > \lambda$,
- the *proper standard module* $\bar{\Delta}(\lambda)$ to be the maximal quotient of $\Delta(\lambda)$ satisfying

$[\bar{\Delta}(\lambda) : L(\lambda)] = 1$,

- the *costandard module* $\nabla(\lambda)$ to be the maximal submodule of $I(\lambda)$ such that $[\nabla(\lambda) : L(\mu)] = 0$ for all $\mu > \lambda$,
- the *proper costandard module* $\bar{\nabla}(\lambda)$ to be the maximal submodule of $\nabla(\lambda)$ satisfying $[\nabla(\lambda) : L(\lambda)] = 1$.

These definitions yield the following equations

$$\Delta(\lambda) = P(\lambda)/\operatorname{Tr}_{P^{>\lambda}}(P(\lambda)), \tag{4.1}$$

$$\bar{\Delta}(\lambda) = P(\lambda)/\operatorname{Tr}_{P^{\geq\lambda}}(\operatorname{rad}(P(\lambda))), \tag{4.2}$$

$$\nabla(\lambda) = \bigcap_{f:I(\lambda)\to I^{>\lambda}} \operatorname{Ker} f, \tag{4.3}$$

and $\bar{\nabla}(\lambda)$ is the pre-image under the canonical epimorphism $I(\lambda) \to I(\lambda)/\operatorname{soc}(I(\lambda))$ of

$$\bar{\nabla}(\lambda) = \bigcap_{f:I(\lambda)/\operatorname{soc}(I(\lambda))\to I^{\geq\lambda}} \operatorname{Ker} f. \tag{4.4}$$

We now define three types of stratified algebra. We follow the definitions in [FM06] and will refer back to this as the FM definition. The pair $(A, \leq)$ is called a *standardly stratified algebra* if

(SS1) the kernel of the canonical epimorphism $P(\lambda) \twoheadrightarrow \Delta(\lambda)$ has a filtration whose subquotients are isomorphic to $\Delta(\mu)$ with $\mu > \lambda$.

(SS2) the kernel of the canonical epimorphism $\Delta(\lambda) \twoheadrightarrow L(\lambda)$ has a filtration whose subquotients are isomorphic to $L(\mu)$ with $\mu \leq \lambda$.

If $\leq$ is a partial (or equivalently, linear) order and the above conditions are satisfied then we call $(A, \leq)$ a *strongly standardly stratified algebra* or, for brevity an *SSS-algebra*. The next class of algebras form a proper subclass of the class of standardly stratified algebras. We say that $(A, \leq)$ is a *properly stratified algebra* if it satisfies (SS1), (SS2) and the following condition:

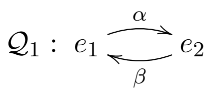
(PS1) for each $\lambda \in \Lambda$ the module $\Delta(\lambda)$ has a filtration with subquotients isomorphic to $\bar{\Delta}(\lambda)$.

An SSS-algebra is properly stratified if and only if A^{op} is an SSS-algebra. In particular, an algebra A is properly stratified if and only if A^{op} is also properly stratified, see [Fri06]. Finally, assume that $\leq$ is a partial order, then $(A, \leq)$ is a *quasi-hereditary algebra* if it satisfies (SS1), (SS2) and the following condition

(QH) for each $\lambda \in \Lambda$ we have

$$\Delta(\lambda) = \bar{\Delta}(\lambda).$$

Example 4.1. 1. Consider the path algebra $A_1 = \Bbbk\mathcal{Q}_1/I_1$ of the quiver

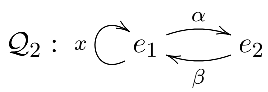


modulo the ideal $I_1 = \langle \alpha\beta \rangle$. The left regular module of A has Loewy structure

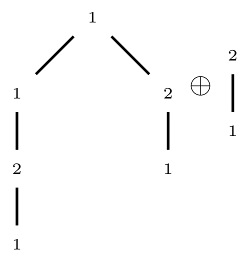
$$\begin{matrix} 1 \\ 2 \\ 1 \end{matrix} \oplus \begin{matrix} 2 \\ 1 \end{matrix}.$$

Hence A_1 is quasi-hereditary with $\Delta(2) = P(2) = \bar{\Delta}(2)$ and $\Delta(1) = L(1) = \bar{\Delta}(1)$.

2. Consider the path algebra $A_2 = \Bbbk\mathcal{Q}_2/I_2$ of the quiver



modulo the ideal $I_2 = \langle \alpha\beta, x\beta, x^2 \rangle$. The left regular module of A_2 has Loewy structure

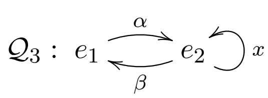


Hence A_2 is properly stratified with $\Delta(2) = P(2) = \bar{\Delta}(2)$ and

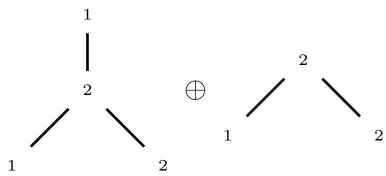
$$\Delta(1) = \begin{matrix} 1 \\ 1 \end{matrix}, \quad \bar{\Delta}(1) = L(1),$$

since $\Delta(1) \neq \bar{\Delta}(1)$ the algebra A_2 is not quasi-hereditary.

3. Consider the path algebra $A_3 = \Bbbk\mathcal{Q}_3/I_3$ of the quiver



modulo the ideal $I_3 = \langle \alpha\beta, \beta x, x^2 \rangle$. The left regular module of A_3 has Loewy structure



Hence A_3 is standardly stratified with $\Delta(2) = P(2)$, $\Delta(1) = L(1) = \bar{\Delta}(1)$ and

$$\bar{\Delta}(2) = \begin{matrix} 2 \\ 1 \end{matrix} .$$

The algebra A_3 is not properly stratified as $\Delta(2)$ does not possess a filtration by $\bar{\Delta}(2)$.

4. A whole class of examples of properly stratified algebras can be obtained from quasi-hereditary algebras in the following way. If A is quasi-hereditary then the algebra obtained from the tensor product $A \otimes_{\mathbb{k}} \mathbb{k}[x_1, \cdots, x_n]/(x_1^{t_1}, \cdots, x_n^{t_n})$ is properly stratified.

4.1 The category $\mathcal{F}(\Delta)$F(D) and tilting

If A is a stratified algebra then $\mathcal{F}(\Delta)$ denotes the category $\mathcal{F}(\mathcal{C})$ where $\mathcal{C} = \{\Delta(\lambda) \mid \lambda \in \Lambda\}$, in a similar way define $\mathcal{F}(\bar{\Delta})$, $\mathcal{F}(\nabla)$, $\mathcal{F}(\bar{\nabla})$. We also define $\mathcal{C}_{\leq\lambda}$ the subclass of $\mathcal{C}$ consisting of modules in $\mathcal{C}$ with index less or equal to $\lambda \in \Lambda$ (equivalently define $\mathcal{C}_{\geq\lambda}$, $\mathcal{C}_{<\lambda}$ and $\mathcal{C}_{>\lambda}$), using this notation we can define the respective categories $\mathcal{F}(\Delta_{\leq\lambda})$, $\mathcal{F}(\Delta_{\geq\lambda})$, $\mathcal{F}(\Delta_{<\lambda})$ and $\mathcal{F}(\Delta_{>\lambda})$. We define *tilting modules* to be the objects in the category $\mathcal{F}(\Delta) \cap \mathcal{F}(\bar{\nabla})$. For $\lambda \in \Lambda$ there exists a unique (up to isomorphism) indecomposable tilting module $T(\lambda)$ with the property that its standard filtration starts with $\Delta(\lambda)$ when reading from the bottom. It is shown in [AHLU00b, Theorem 2.1 & Proposition 2.3] that there exists a multiplicity free tilting module $T = \oplus_{\lambda\in\Lambda} T(\lambda)$ such that $\mathcal{F}(\Delta) \cap \mathcal{F}(\bar{\nabla}) = \mathrm{add}(T)$. We call this T the *characteristic tilting module*. Dually, the objects of $\mathcal{F}(\bar{\Delta}) \cap \mathcal{F}(\nabla)$ are called the *cotilting modules*, and for $\lambda \in \Lambda$ we denote by $C(\lambda)$ the cotilting module whose costandard filtration ends with $\nabla(\lambda)$. Define the *characteristic cotilting module* $C = \oplus_{\lambda\in\Lambda} C(\lambda)$, we have $\mathcal{F}(\bar{\Delta}) \cap \mathcal{F}(\nabla) = \mathrm{add}(C)$. For more details on tilting theory we refer the reader to [HHK07] and for the particular case of standardly stratified algebras [AHLU00b].

The following theorem is well known see [DR92, Lemma 1.5] and [Rin91, Theorem 2].

Theorem 4.2. *The category $\mathcal{F}(\Delta)$ is*

1. closed under kernels of epimorphisms;

2. closed under extensions;

3. closed under direct summands of direct sums.

4.2 A strategy for proving standardly stratified

We remind the reader that we are referring to the previous definition of standardly stratified as the FM definition. We now take inspiration from an earlier definition of standardly stratified which we refer to as the ADL definition [ADL98]. Let A be a basic connected finite dimensional $\mathbb{k}$-algebra and $A_A = \oplus_{i=1}^{n} P^{\mathrm{op}}(i) = \oplus_{i=1}^{n} e_i A$. Denote by $\boldsymbol{e} = (e_1, \cdots, e_n)$ the complete sequence of its indecomposable orthogonal idempotents and set $\epsilon_i = \sum_{j=i}^{n} e_j$. For $(A, \boldsymbol{e})$ we define right standard and proper standard A-modules by

$$\Delta_A(i) = e_i A / e_i \operatorname{rad} A \epsilon_{i+1} A, \; 1 \leq i \leq n, \text{ and,}$$
$$\bar{\Delta}_A(i) = e_i A / e_i \operatorname{rad} A \epsilon_i A, \; 1 \leq i \leq n,$$

respectively. Then, according to the ADL definition, the algebra $(A, \boldsymbol{e})$ is standardly stratified if each factor $A\epsilon_i A / A\epsilon_{i+1} A$ of the trace filtration of A_A belongs to $\mathcal{F}(\bar{\Delta}_A)$. This is equivalent to (see [Dla96] or [Lak00]) each factor of the trace filtration of ${}_A A$ belonging to $\mathcal{F}(\Delta)$.

We now give an alternative characterisation of standardly stratified which does not require the algebra A to be basic, but does require the existence of a set of idempotents with properties inspired by the properties of $\{e(\boldsymbol{i}_\pi) \mid \pi \in \Pi(\alpha)\} \subset R_\alpha$.

Theorem 4.3. *Let A be an algebra with idempotents $e_1, \ldots, e_n$ such that*

(a) $A(e_1 + \cdots + e_n)A = A$;

(b) and each idempotent e_i has a decomposition $e_i = f_i + f_i'$ where;

(i) $f_1, \ldots, f_n$ are indecomposable pairwise orthogonal idempotents with

$$A(f_1 + \cdots + f_n)A = A;$$

(ii) and $f_i' \in A\varepsilon_{i+1}A$ where $\varepsilon_i = \sum_{j=i}^{n} e_j$.

Then ${}_A(A\varepsilon_i A / A\varepsilon_{i+1} A) \in \mathcal{F}(\Delta)$ if and only if A is (strongly) standardly stratified.

Before proving this theorem we need a few other results. Note that if an algebra is standardly stratified in the sense of the FM definition then for each class of projective module there exists a primitive idempotent e_λ such that $Ae_\lambda \cong P(\lambda)$.

Lemma 4.4. *Let A be a standardly stratified $\Bbbk$-algebra (in the sense of the FM definition) and let e_n be the highest idempotent in the associated order, then there is an isomorphism*

$$\phi : Ae_n \otimes_{e_nAe_n} e_nA \to Ae_nA$$

where $a \otimes b \mapsto ab$.

Proof. Since A is standardly stratified we have a filtration of Ae_nAe_i by $\Delta(n)$ and since $\Delta(n)$ is projective we choose, for each $i = 1, \cdots, n$, a decomposition of Ae_nAe_i into s direct summands isomorphic to Ae_n so

$$Ae_nAe_i \cong Ae_n^{\oplus s}.$$

Let $e_nb_je_i$ be a generator for the j^{th} summand, for $1 \leq j \leq s$. We now claim that e_nAe_i is free as a left e_nAe_n-module with basis

$$\{b_j = e_nb_je_i \mid j = 1, \dots, s\}.$$

Let $x \in e_nAe_i$ then $x = 1 \cdot x \in Ae_nAe_i$ and can be written uniquely as a sum

$$\sum_j a_je_nb_je_i, \qquad a_j \in A;$$

and since $e_nx = x$ we have $e_na_j = a_j$. So $a_j \in e_nAe_n$ and the claim holds. Returning to the map ϕ, since multiplication is surjective

$$\phi : Ae_n \otimes_{e_nAe_n} e_nA \twoheadrightarrow Ae_nA.$$

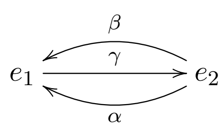
It follows from e_nAe_i being a free left e_nAe_n-module of rank s that for each fixed $1 \leq i \leq n$ we have $Ae_n \otimes_{e_nAe_n} e_nAe_i \cong Ae_n^{\oplus s}$ (where s depends on i). On the other hand we have Ae_nAe_i isomorphic to $Ae_n^{\oplus s}$ from the start. So the map

$$Ae_n \otimes_{e_nAe_n} e_nAe_i \twoheadrightarrow Ae_nAe_i$$

is an isomorphism and hence ϕ is an isomorphism. □

The following is an example of why the lemma above only applies to the idempotent that is highest in the associated order.

Example 4.5. Let A be the path algebra of the quiver



modulo the ideal $(\gamma\beta, \alpha\gamma\alpha)$, we set $\Lambda = \{1 < 2\}$. This 11 dimensional algebra is standardly stratified and $Ae_2A \cong Ae_2 \otimes_{e_2Ae_2} e_2A$ but

$$\begin{array}{rcl} Ae_1A & = & \{e_1, \alpha, \beta, \gamma, \gamma\alpha, \alpha\gamma, \beta\gamma, \gamma\alpha\gamma, \beta\gamma\alpha, \beta\gamma\alpha\gamma\} \\ Ae_1 \otimes_{e_1Ae_1} e_1A & = & \left\{ \begin{array}{cccc} e_1 \otimes e_1, & e_1 \otimes \alpha, & e_1 \otimes \beta, & \gamma \otimes e_1, \\ \gamma \otimes \alpha, & \gamma \otimes \beta, & \beta\gamma \otimes e_1, & \alpha\gamma \otimes e_1, \\ \beta\gamma \otimes \alpha, & \gamma\alpha\gamma \otimes e_1, & \beta\gamma\alpha\gamma \otimes e_1 & \end{array} \right\} \end{array}$$

which are clearly not isomorphic since the dimensions are not equal.

Before we continue we will need the following well known lemma which can be found in [Wei95, Exercise 1.3.3].

Lemma 4.6 (The Five Lemma)**.** *In any commutative diagram*

$$\begin{array}{ccccccccc} A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \\ \cong\downarrow a & & \cong\downarrow b & & \downarrow c & & \cong\downarrow d & & \cong\downarrow e \\ A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \end{array}$$

with exact rows in any abelian category, if a, b, d, and e are isomorphism, the c is also an isomorphism. More precisely, this lemma comes in two halves. If b and d are monomorphisms and a is an epimorphism then c is a monomorphism. If b and d are epimorphisms and e is a mononomorphism then c is an epimorphism.

Next we show that certain subcategories of $\mathcal{F}(\Delta)$ satisfy the conditions $1-3$ of Theorem 4.2.

Proposition 4.7. *The categories $\mathcal{F}(\Delta_{\geq i})$ and $\mathcal{F}(\Delta_{\leq i})$ also satisfy*

1. *closed under kernels of epimorphisms;*

2. *closed under extensions;*

3. *closed under direct summands of direct sums.*

Proof. Let $B := \varepsilon_i A \varepsilon_i$. Define a functor

$$\begin{array}{rl} \varepsilon_i \cdot : A\textbf{-mod} \to B\textbf{-mod} \\ M \mapsto \varepsilon_i M. \end{array}$$

It follows from the definition that

$$\Delta(j) \mapsto \left\{ \begin{array}{cc} \Delta^B(j) & \text{if } j \geq i \\ 0 & \text{otherwise.} \end{array} \right.$$

So $\varepsilon_i\cdot$ restricts to a functor $\bar{\varepsilon}_i\cdot : \mathcal{F}(\Delta_{\geq i}) \to \mathcal{F}(\Delta^B)$. We claim that the above functor is mutually inverse to

$$A\varepsilon_i \otimes_{\varepsilon_i A \varepsilon_i} - : \mathcal{F}(\Delta^B) \to \mathcal{F}(\Delta_{\geq i}),$$

and provides an isomorphism of categories

$$\mathcal{F}(\Delta_{\geq i}) \cong \mathcal{F}(\Delta^B).$$

In one direction the composition is clearly isomorphic to the identity

$$\varepsilon_i \cdot \circ A\varepsilon_i \otimes_{\varepsilon_i A\varepsilon_i} - \cong \mathrm{Id}_{\varepsilon_i A \varepsilon_i}$$

hence restricts to $\mathrm{Id}_{\mathcal{F}(\Delta^B)}$. So, now consider $A\varepsilon_i \otimes_{\varepsilon_i A\varepsilon_i} - \circ\, \varepsilon_i$. Under this functor $M \in \mathcal{F}(\Delta_{\geq i})$ maps to $A\varepsilon_i \otimes_{\varepsilon_i A\varepsilon_i} \varepsilon_i M$. If $M = \Delta(j)$ with $j \geq i$ then

$$\Delta(j) = Ae_j/A\varepsilon_{j+1}Ae_j \mapsto \varepsilon_i Ae_j/\varepsilon_i A\varepsilon_{j+1}Ae_j \mapsto A\varepsilon_i Ae_j/A\varepsilon_i A\varepsilon_{j+1}Ae_j = \Delta(j).$$

The final equality holds since we claim that $A\varepsilon_i Ae_j = Ae_j$. In one direction ($\subseteq$) the inclusion is clear, and for the other ($\supseteq$) notice that $1 \cdot \varepsilon_i \cdot 1 \cdot e_j = e_j$, thus equality follows. Now we apply induction and need to show the claim for M filtered by $\Delta(j)$. Let

$$N \hookrightarrow M \twoheadrightarrow \Delta(j)$$

be a short exact sequence. Then we have the commutative diagram

$$\begin{array}{ccccc} N & \hookrightarrow & M & \twoheadrightarrow & \Delta(j) \\ \| \wr & & \uparrow & & \| \wr \\ N & \hookrightarrow & A\varepsilon_i M & \twoheadrightarrow & \Delta(j). \end{array}$$

The Five Lemma 4.6 gives us an isomorphism taking $A\varepsilon_i \otimes_{\varepsilon_i A\varepsilon_i} \varepsilon_i M \mapsto M$, hence

$$A\varepsilon_i \otimes_{\varepsilon_i A\varepsilon_i} - \circ\, \varepsilon_i \cdot \cong \mathrm{Id}_{\mathcal{F}(\Delta_{\geq i})}\,.$$

Now, the two categories are equivalent. Since, $\mathcal{F}(\Delta^B)$ satisfied the properties of Theorem 4.2 we may deduce that $\mathcal{F}(\Delta_{\geq i})$ also satisfied these properties.

First notice that $\mathcal{F}(\Delta_{\leq i})$ is a full subcategory of $\mathcal{F}(\Delta)$. We know that $[\Delta(i) : L(j)] = 0$ if $j > i$. So for $M \in \mathcal{F}(\Delta_{\leq i})$ we have $[M : L(j)] = 0$ for $j > i$. If we have the epimorphism $f : M \twoheadrightarrow N$ where both $M, N \in \mathcal{F}(\Delta_{\leq i})$, then we know that $\ker f \in \mathcal{F}(\Delta)$, but since neither M nor N contain simples with index greater than i we may deduce that neither does $\ker f$, so $\ker f \in \mathcal{F}(\Delta_{\leq i})$. Similarly, if

$M, N \in \mathcal{F}(\Delta_{\leq i})$ fit into the short exact sequence

$$M \hookrightarrow X \twoheadrightarrow N,$$

$X \in A$-**mod**, then $X \in \mathcal{F}(\Delta)$ and $[X : L(j)] = 0$ for $j > i$ so $X \in \mathcal{F}(\Delta_{\leq i})$. Closure under direct summands is clear. $\square$

Let us return to proving Theorem 4.3.

Proof. The task is to prove (under the conditions given in the theorem) that $A\varepsilon_i A/A\varepsilon_{i+1}A$ is in $\mathcal{F}(\Delta)$ if and only if $A(\sum_{j>i} f_j)Af_i$ is in $\mathcal{F}(\Delta_{\geq i+1})$ for all $i = 1, \cdots, n$. Notice that $A(\sum_{j>i} f_j)Af_i = A\varepsilon_{i+1}Af_i$. For the forward direction we proceed by downward induction on i. For $i = n$, $A\varepsilon_{n+1}A = 0$, so by assumption $Ae_nA \in \mathcal{F}(\Delta)$. We prove that $Ae_nA \in \mathcal{F}(\Delta_{\geq n})$, from which it then follows that $Ae_nAf_{n-1} \in \mathcal{F}(\Delta_{\geq n})$ since $\mathcal{F}(\Delta_{\geq n})$ is closed under direct summands by Proposition 4.7. Indeed, there exists a $k > 0$ such that $\operatorname{top}(Ae_nA) = L(n)^{\oplus k}$, and hence we have the surjection

$$\phi : Ae_nA \twoheadrightarrow \Delta(n)^{\oplus k}.$$

Since $\Delta(n)$ is projective we have $Ae_nA = \Delta(n)^{\oplus k} \oplus \ker\phi$. However, $\operatorname{top}(\ker\phi)$ is made up of some copies of $L(n)$, and thus we must have $\ker\phi = 0$. Hence $Ae_nA \in \mathcal{F}(\Delta_{\geq n})$.

Now inductively assume that $A\varepsilon_{i+1}A \in \mathcal{F}(\Delta_{\geq i+1})$, then $A\varepsilon_iA/A\varepsilon_{i+1}A$ is a sum of $\Delta(i)$ by the base step for the algebra $A/A\varepsilon_{i+1}A$. We construct the short exact sequence

$$A\varepsilon_{i+1}A \hookrightarrow A\varepsilon_iA \twoheadrightarrow A\varepsilon_iA/A\varepsilon_{i+1}A,$$

and observe that $A\varepsilon_{i+1}A \in \mathcal{F}(\Delta_{\geq i+1}) \subset \mathcal{F}(\Delta_{\geq i})$ and $A\varepsilon_iA/A\varepsilon_{i+1}A \in \mathcal{F}(\Delta_{\geq i})$, since $\mathcal{F}(\Delta_{\geq i})$ is closed under extensions $A\varepsilon_iA \in \mathcal{F}(\Delta_{\geq i})$. Now, consider $A\varepsilon_{i+1}A$, we can write this as

$$A\varepsilon_{i+1}A = \bigoplus_{j=1}^{n} A\varepsilon_{i+1}Af_j.$$

Now $A\varepsilon_{i+1}Af_i$ appears as a direct summand of $A\varepsilon_{i+1}A$, and $\mathcal{F}(\Delta)$ is closed under direct summands, so $A\varepsilon_{i+1}Af_i \in \mathcal{F}(\Delta_{\geq i+1})$.

For the converse, assume that the kernel of $P(i) \twoheadrightarrow \Delta(i)$ has a filtration with subquotients $\Delta(j)$, for $j > i$. The lowest cell is given by

$$A\varepsilon_nA = Ae_nA = Af_nA$$

which gives us $Ae_nA \cong Ae_n^{\oplus l} \cong \Delta(n)^{\oplus l}$, where l is the rank of e_nA as a left e_nAe_n-module by Lemma 4.4, and hence $Ae_nA \in \mathcal{F}(\Delta)$. We proceed by downward induction on the index of cells, so assume that all factors down to

$$A\varepsilon_{i+1}A/A\varepsilon_{i+2}A \in \mathcal{F}(\Delta).$$

Then considering $A\varepsilon_i A/A\varepsilon_{i+1}A$, we rewrite this as

$$A\varepsilon_i A/A\varepsilon_{i+1}A = (Ae_iA + A\varepsilon_{i+1}A)/A\varepsilon_{i+1}A = (Af_iA + A\varepsilon_{i+1}A)/A\varepsilon_{i+1}A,$$

and then apply the second isomorphism theorem

$$(Af_iA + A\varepsilon_{i+1}A)/A\varepsilon_{i+1} \cong Af_iA/(Af_iA \cap A\varepsilon_{i+1}A).$$

Now, if we view $Af_iA/Af_iA \cap A\varepsilon_{i+1}A$ as an ideal of $A/A\varepsilon_{i+1}A$ then f_i is the highest indexed idempotent. Since A is standardly stratified

$$Ae_iA/(Ae_iA \cap A\varepsilon_{i+1}A) \cong (A/A\varepsilon_{i+1}A)e_i^{\oplus m_i} = \Delta(i)^{\oplus m_i}.$$

where m_i is the rank of $e_i(A/A\varepsilon_{i+1}A)$ as a left $e_i(A/AA\varepsilon_{i+1}A)e_i$-module. So $A\varepsilon_{i+1}A/A\varepsilon_iA \in \mathcal{F}(\Delta)$.

□

4.3 Properties of stratified algebras

These stratifications have reasonably nice homological properties which have been studied by [Rin91], [AHLU00b], [FM06]. If one knows an algebra is quasi-hereditary then one knows that it has finite global dimension, unfortunately this does not carry over to properly or standardly stratified algebras, which can have infinite global dimension.

Theorem 4.8. *[AHLU00b, Theorem 2.4] Let $(A, \leq)$ be a standardly stratified algebra. Then A is quasi-hereditary if and only if* $\mathrm{gl.dim}(A) < \infty$.

For properly stratified algebras another invariant is well understood, namely the finitistic dimension. The *(projectively defined) finitistic dimension* of an algebra A is the number

$$\mathrm{fin.dim}(A) := \sup\{\mathrm{p.dim}(M) | M \in A\text{-}\mathbf{mod}, \mathrm{p.dim}(M) < \infty\}.$$

This homological property is the subject of a still open conjecture since 1960.

Conjecture 4.9. *Let A be a finite dimensional algebra, then* $\mathrm{fin.dim}(A) < \infty$.

The conjecture has been shown to hold for many classes of algebras, and for more information on its history we refer the reader to [ZH95]. For our purposes we need only note that the conjecture has been shown to hold for the class of stratified algebras [AHLU00a, Theorem 2.1]. Obtaining optimum bounds on the finitistic dimension of standardly and properly stratified algebras is studied in [AHLU00a], [MO04], [Maz04]. Another property, originally studied for quasi-hereditary algebras by Ringel, is the endomorphism ring of the characteristic tilting module. For an SSS-algebra $(A, \leq)$ the *Ringel Dual* R of A is defined to be

$$R := \mathrm{End}_A(T).$$

For quasi-hereditary algebras the Ringel dual is a well behaved object.

Theorem 4.10. *[Rin91] If $(A, \leq)$ be a quasi-hereditary algebra, then the Ringel dual R of A is quasi-hereditary with respect to the opposite order on the poset. Moreover, the Ringel dual of R is Morita equivalent to A.*

However, the Ringel dual of a properly stratified algebra need not be properly stratified. Indeed, we will see examples in Chapter 6 that illustrate this fact.

The class of cellular algebras, described in Chapter 2, overlaps with the class of stratified algebras. The following result illustrates part of that overlap.

Proposition 4.11. *[KX99] Let A be a cellular algebra with involution τ then the following are equivalent:*

- *A is quasi-hereditary*
- *A has finite global dimension*
- *there is a cell chain of A whose length equals the number of isomorphism classes of simple A-modules.*

4.4 Affine stratified algebras

The stratified notions in this chapter have been extended to infinite dimensional algebras by Kleshchev [Kle15]. A graded algebra whose graded dimension is a Laurent series is called a *Laurentian algebra*. Kleshchev shows that Laurentian algebras are graded semiperfect (i.e. every finitely generated graded module has a graded projective cover) have finite dimensional irreducible modules, and have only finitely many irreducible modules up to isomorphism and degree shift. Let R be a left Noetherian Laurentian algebra with simple indexing set Π. For every $\pi \in \Pi$ we have an indecomposable projective $P(\pi)$. A two sided ideal $J \subseteq R$ is called *affine stratifying* if it satisfies:

(ASI1) $\mathrm{Hom}_R(J, R/J) = 0$;

(ASI2) As a left module $J \cong \oplus_{\pi\in\Upsilon} m_\pi(q)P(\pi)$ for some graded multiplicities $m_\pi(q)$ and some subset $\Upsilon \subseteq \Pi$ such that for $P_\Upsilon := \oplus_{\pi\in\Upsilon} P(\pi)$ we have $B_\Upsilon := \mathrm{End}_r(P_\Upsilon)^{\mathrm{op}}$ is an affine algebra.

An affine stratifying ideal is called affine standardly stratifying if

(ASS1) it is finitely generated as a right $B_\Upsilon-$module.

An affine standardly stratifying ideal is called affine properly stratifying if

(APS1) it is flat as a right $B_\Upsilon-$module.

An affine stratifying ideal is called an affine hereditary ideal if it is affine properly stratifying with $|\Upsilon| = 1$. The algebra R is called affine stratifying (resp. affine standardly stratifying, affine properly stratifying, affine quasihereditary) if there exists a finite chain of ideals

$$(0) = J_n \subset \cdots \subset J_1 \subset J_0 = R$$

with J_i/J_{i+1} an affine stratifying (resp. affine standardly stratifying, affine properly stratifying, affine hereditary) ideal in R/J_{i+1} for all $0 \leq i < n$. Such a chain of ideals is called an affine stratifying (resp. affine standardly stratifying, affine properly stratifying, affine hereditary) chain.

Lemma 4.12. *[Kle15] If J is an ideal in R such that ${}_RJ$ is projective, then the following are equivalent*

1 (ASI1) $\mathrm{Hom}_R(J, R/J) = 0$*;*

2 $J^2 = J$*;*

3 $J = ReR$ *for an idempotent* $e \in R$*.*

Example 4.13. If $(A, \leq)$ is a quasi-hereditary $\Bbbk$-algebra with indexing set Π and $\mathbb{A}$ is a polynomial $\Bbbk$-algebra then $H := A \otimes_\Bbbk \mathbb{A}$ is affine quasi-hereditary. Since A is quasi-hereditary it comes with a set of idempotents $\{e_i\}_{i\in\Pi}$ that give rise to a chain of hereditary ideals $A\varepsilon_i A$ where $\varepsilon_i = \sum_{j\geq i} e_j$. The ideals $J_i := H(\varepsilon_i \otimes_\Bbbk 1_\mathbb{A})H$ are affine properly stratifying in H, and $|\Upsilon| = 1$.

Example 4.14. [KLM13, KL15] The quiver Hecke algebras (of finite type) are affine quasi-hereditary.

Chapter 5

Homological structure of $R_\alpha^{\mathcal{J}}$our quotient

In this chapter we describe a cellular structure for $R_\alpha^{\mathcal{J}}$ induced from the affine cellular structure of $R_\alpha^{\mathcal{J}}$, from this we are able to obtain a parametrisation of cell modules, standard modules and simple modules. We then give a way to obtain the standard and proper standard modules of $R_\alpha^{\mathcal{J}}$ from the standard and proper standard modules of R_α. We use this to prove that $R_\alpha^{\mathcal{J}}$ is properly stratified.

5.1 Cellular structure

Before describing the cellular structure of $R_\alpha^{\mathcal{J}}$ we prove the following useful result from homological algebra.

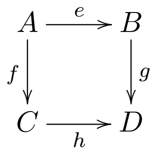
Lemma 5.1. *For R-modules A, B, C and D and R-module morphisms e, f, g and h, the following diagram*

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow[h]{} & D \end{array}$$

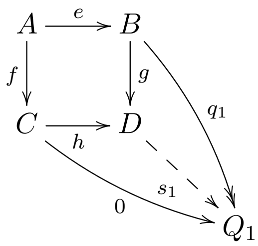
1. *is a pushout if and only if there is an isomorphism on the cokernels of e and h and an epimorphism on the kernels of e and h.*

2. *is a pullback if and only if there is an isomorphism on the kernels of e and h and a monomorphism on the cokernels of e and h.*

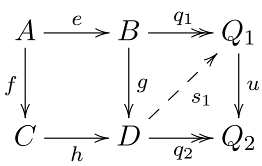
Proof. 1. (⇒) If the following diagram is a pushout



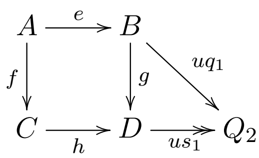
and $q_1 : B \to Q_1$ is the cokernel of e, then there is a unique map $s_1 : D \to Q_1$ such that, $s_1 g = q_1$ and s_1 is an epimorphism. The existence follows from $q_1 e = 0$, since we can consider the zero map from $C \to Q_1$, and we get s_1 and its uniqueness from the universal property of pushouts.



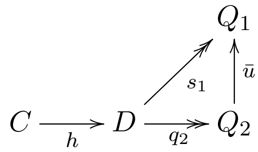
Now, let $q_2 : D \to Q_2$ be the cokernel of $h : C \to D$ and $u : Q_1 \to Q_2$ be the morphism induced from g.



We get also $us_1 : D \to Q_2$ and the following diagram commutes



since $us_1 = q_2$ we get that u is an epimorphism. Now we get a map $\bar{u} : Q_2 \to Q_1$ since $s_1 h = 0$ and so factors over Q_2. The diagram



commutes so $s_1 = \bar{u} q_2 = \bar{u} u s_1$, since s_1 is an epimorphism we get $\bar{u} u = \mathrm{id}_{Q_1}$. We also have $u \bar{u} q_2 = u s_1 = q_2$ and q_2 is an epimorphism so $u \bar{u} = \mathrm{id}_{Q_2}$.

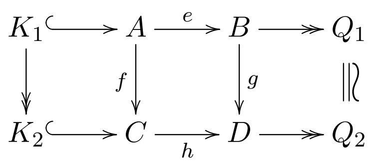
Therefore, $Q_1 \cong Q_2$.

Let K_1 be the kernel of e and K_2 the kernel of h. If $y \in K_2 \subset C$, then $h(y) = 0$, but

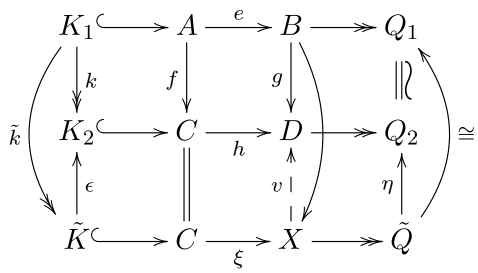
$$D \cong B \oplus C / \langle (e(x), 0) - (0, f(x)) \mid x \in A \rangle.$$

So $h(y) = (0, y) = 0$, we can write this as $h(y) = (0, y) - (0, 0)$, so $y = f(x)$ for some $x \in A$, and $e(x) = 0$. So $x \in K_1$, and $k : K_1 \to K_2$ is an epimorphism.

($\Leftarrow$) Assume we have the following diagram



If X is the pushout of e and f then the first half of the proof gives $\tilde{Q} \cong Q_1$ and there exists a unique $v : X \to D$ and induced maps ϵ, η making everything commute



We clearly get that η is an isomorphism, and $\epsilon\tilde{k} = k$. Since k is an epimorphism we get that ϵ is an epimorphism. The relevant half of the Five Lemma 4.6 implies that v is a monomorphism. Let $x \in D$, and label $u : Q_1 \leftrightarrow Q_2$ then there is a $y \in B$ such that $q_2(x) = uq_1(y) = q_2(g(y))$. We get that

$$x - g(y) \in \ker q_2 = \operatorname{Im} g,$$

so $x = g(y) + h(z)$ for some $z \in C$. So $h \oplus g$ is onto. Now, label $\zeta : B \to X$ so $h = v\xi$ and $g = v\zeta$, giving $h \oplus g = v(\xi \oplus \zeta)$. Hence v is an epimorphism, and therefore an isomorphism.

The result on pullbacks is proved dually.

$\square$

Recall the definition of the polynomial ring Λ_π from (2.2), and the cell ideals $I_\pi = \sum_{\sigma \geq \pi} I'_\sigma$ where

$$I'_\pi = \Bbbk\text{- span}\{\psi_w y_\pi \Lambda_\pi \psi_\pi y_\pi e(\boldsymbol{i}_\pi) \psi_v^\tau \mid w, v \in \mathfrak{S}^\pi\}.$$

Now let us define

$$A_\pi = \Lambda_\pi / \langle p \in \Lambda_\pi \mid \deg(p) \geq d_\pi \rangle; \tag{5.1}$$

$$\mathcal{I}'_\pi := \Bbbk - \operatorname{span}\{\psi_w y_\pi \psi_\pi e(\boldsymbol{i}_\pi) p y_\pi \psi_v^\tau + \mathcal{J} \mid w, v \in \mathfrak{S}^\pi, \pi \in \Pi, p \in \mathfrak{B}(A_\pi)\} \subset R_\alpha^{\mathcal{J}}; \tag{5.2}$$

$$\mathcal{I}_\pi = \sum_{\sigma \geq \pi} \mathcal{I}'_\sigma; \qquad \mathcal{I}_{>\pi} = \sum_{\sigma > \pi} \mathcal{I}'_\sigma.$$

Proposition 5.2. *$\mathcal{I}_\pi$ is the image of I_π in $R_\alpha^{\mathcal{J}}$. Moreover, $\mathcal{I}_\pi$ is the two sided ideal $\sum_{\sigma \geq \pi} R_\alpha^{\mathcal{J}} e(\boldsymbol{i}_\sigma) R_\alpha^{\mathcal{J}}$.*

Proof. Both I_π and $\mathcal{J}$ are ideals of R_α and thus embed into R_α under the inclusions ι_1 and ι_2, respectively. If we take the pullback, that is

$$X := \{(a, b) \in I_\pi \times \mathcal{J} \mid \iota_1(a) = \iota_2(b)\}$$

then since $I_\pi, \mathcal{J} \in R_\alpha$**-mod** we have ([Rot09, Example 5.2]) that

$$X = I_\pi \cap \mathcal{J}$$

adding cokernels we get

$$\begin{array}{ccccc} I_\pi \cap \mathcal{J} & \hookrightarrow & I_\pi & \twoheadrightarrow & I_\pi/(I_\pi \cap \mathcal{J}) \\ \downarrow & & \downarrow & & \downarrow f \\ \mathcal{J} & \hookrightarrow & R_\alpha & \twoheadrightarrow & R_\alpha^{\mathcal{J}} \end{array}$$

Here the map f is a monomorphism since pullbacks induce monomorphisms on cokernels by Lemma 5.1. So, we can choose a vector space splitting of $R_\alpha^{\mathcal{J}}$ such that $\mathcal{I}_\pi$ is the image of I_π in the quotient. Since the quotient map is an algebra homomorphism and $I_\pi = \sum_{\sigma \geq \pi} R_\alpha e(\boldsymbol{i}_\sigma) R_\alpha$ we get $\mathcal{I}_\pi = \sum_{\sigma \geq \pi} R_\alpha^{\mathcal{J}} e(\boldsymbol{i}_\pi) R_\alpha^{\mathcal{J}}$. □

Theorem 5.3. *The algebra $R_\alpha^{\mathcal{J}}$ is a cellular $\Bbbk$-algebra with respect to the involution τ.*

Proof. We obtain a chain of ideals $\{\mathcal{I}_\pi \mid \pi \in \Pi(\alpha)\}$ in $R_\alpha^{\mathcal{J}}$ from the affine cell chain $\{I_\pi \mid \pi \in \Pi(\alpha)\}$ of R_α. To simplify notation let us set $d_\pi = r$, we take a chain of ideals in A_π, filtered by degree

$$0 = M_r \subset M_{r-1} \subset \cdots \subset M_1 \subset M_0 = A_\pi \tag{5.3}$$

where $M_i = \langle p \in A_\pi \mid \deg(p) \geq i \rangle$, denote subquotients $\mathcal{M}_i := M_i / M_{i+1}$. Recall that $\mathfrak{B}(M)$ denotes a basis for M, we now define

$$\mathcal{I}'_{\pi,i} := \langle \psi_w y_\pi \psi_\pi e(\boldsymbol{i}_\pi) p y_\pi \psi_v^\tau \mid w, v \in \mathfrak{S}^\pi, \pi \in \Pi(\alpha), p \in \mathfrak{B}(\mathcal{M}_i) \rangle,$$

and thus define a refinement of the ideal chain $\{\mathcal{I}_\pi \mid \pi \in \Pi(\alpha)\}$ to a chain of ideals given by

$$\mathcal{I}_{\pi,i} := \sum_{\sigma>\pi} \mathcal{I}_\sigma + \sum_{j\geq i} \mathcal{I}'_{\pi,j}.$$

We choose a total order on $\mathfrak{B}(A_\pi)$ that refines the partial order on degrees using this we refine (5.3) to the Jordon-Hölder series

$$0 = M_{r,m_r} \subset M_{r,m_r-1} \subset \cdots \subset M_{r,1} \subset M_{r-1,m_{r-1}} \subset \cdots \subset M_{1,1} \subset M_0 = A_\pi, \quad (5.4)$$

where $M_{i,k}$ denotes the submodule generated by elements of degree i less than k in the total order and elements of degree greater than i. Let $\mathcal{M}_{i,k}$ denote the subquotient $M_{i,k}/M_{i,k+1}$ and $\mathcal{M}_{i,m_i}$ denote $M_{i,m_i}/M_{i+1,1}$. Let us define

$$\mathcal{I}'_{\pi,i,k} := \langle \psi_w y_\pi \psi_\pi e(\boldsymbol{i}_\pi) p y_\pi \psi_v^\tau \mid w, v \in \mathfrak{S}^\pi, \pi \in \Pi(\alpha), p \in \mathfrak{B}(\mathcal{M}_{i,k})\},$$

and refine the ideal chain $\{\mathcal{I}_{\pi,i} \mid \pi \in \Pi\}$ to a chain

$$\mathcal{I}_{\pi,i,k} := \sum_{\sigma>\pi} \mathcal{I}_\sigma + \sum_{j>i} \mathcal{I}_{\pi,j} + \sum_{l\geq k} \mathcal{I}'_{\pi,i,l}.$$

Let us further define

$$I_{>(\pi,i,k)} = \begin{cases} \sum_{\sigma>\pi} \mathcal{I}'_\sigma & \text{if } k = m_i \text{ and } i+1 = r; \\ \sum_{\sigma>\pi} \mathcal{I}'_\sigma + \sum_{j>i} \mathcal{I}'_{\pi,j} & \text{if } k = m_i; \\ \sum_{\sigma>\pi} \mathcal{I}'_\sigma + \sum_{j>i} \mathcal{I}'_{\pi,j} + \sum_{l>k} \mathcal{I}'_{\pi,i,l} & \text{otherwise.} \end{cases}$$

Note that the bases of the $\mathcal{I}'_{\pi,i,k}$ partition the basis of $\mathcal{I}'_\pi$, hence

$$\bigoplus_{i,k} \mathcal{I}'_{\pi,i,k} = \mathcal{I}'_\pi,$$

and thus $\oplus_{\pi,i,k}\mathcal{I}'_{\pi,i,k} = R^{\mathcal{J}}_\alpha$. We now claim that $\mathcal{I}_{\pi,i,k}/\mathcal{I}_{>(\pi,i,k)}$ is a cell ideal in $R^{\mathcal{J}}_\alpha/\mathcal{I}_{>(\pi,i,k)}$. Let us write $\bar{\mathcal{I}}_{\pi,i,k} := \mathcal{I}_{\pi,i,k}/\mathcal{I}_{>(\pi,i,k)}$ and $\bar{R}^{\mathcal{J}}_\alpha := R^{\mathcal{J}}_\alpha/\mathcal{I}_{\pi,i,k}$. By construction $\bar{\mathcal{I}}_{\pi,i,k}$ is a two sided ideal in $\bar{R}^{\mathcal{J}}_\alpha$. It follows directly from the basis and [KLM13, Lemma 5.5] that $\tau(\bar{\mathcal{I}}_{\pi,i,k}) = \bar{\mathcal{I}}_{\pi,i,k}$.

We define a left ideal $\Delta \subset \bar{\mathcal{I}}_{\pi,i,k}$ with $\Bbbk$-basis

$$\{\bar{\psi}_w \bar{y}_\pi \bar{\psi}_\pi \bar{b}_{i,k} \bar{e}(\boldsymbol{i}_\pi) \mid \pi \in \Pi(\alpha), w \in \mathfrak{S}^\pi, b_{i,k} \in \mathfrak{B}(\mathcal{M}_{i,k})\}.$$

Clearly Δ is finitely generated and free over $\Bbbk$. We also have a $\Bbbk$-basis for $\tau(\Delta)$ given by

$$\{\bar{e}(\boldsymbol{i}_\pi) \bar{b}_{i,k} \bar{\psi}_\pi \bar{y}_\pi \bar{\psi}_v^\tau \mid \pi \in \Pi(\alpha), v \in \mathfrak{S}^\pi, b_{i,k} \in \mathfrak{B}(\mathcal{M}_{i,k})\}.$$

The map

$$\alpha : \Delta \otimes \tau(\Delta) \to \bar{\mathcal{I}}_{\pi,i,k}$$
$$\bar{\psi}_w \bar{y}_\pi \bar{\psi}_\pi \bar{y}_\pi \bar{b}_{i,k} \bar{e}(\boldsymbol{i}_\pi) \otimes_{\Bbbk} \bar{e}(\boldsymbol{i}_\pi) \bar{b}_{i,k} \bar{y}_\pi \bar{\psi}_\pi \bar{y}_\pi \bar{\psi}_w^\tau \mapsto \bar{\psi}_w \bar{y}_\pi \bar{\psi}_\pi \bar{b}_{i,k} \bar{e}(\boldsymbol{i}_\pi) \bar{y}_\pi \bar{\psi}_w^\tau$$

defines a $\bar{R}_\alpha^{\mathcal{J}}$-$\bar{R}_\alpha^{\mathcal{J}}$-bimodule isomorphism $\mathcal{I}_{\pi,i,k}/\mathcal{I}_{>(\pi,i,k)} \cong \Delta \otimes_{\Bbbk} \tau(\Delta)$ which satisfies

$$\begin{array}{ccc} \bar{\mathcal{I}}_{\pi,i,k} & \xrightarrow{\alpha} & \Delta \otimes_{\Bbbk} \tau(\Delta) \\ \downarrow \tau & & \downarrow x\otimes y \mapsto \tau(y)\otimes\tau(x) \\ \bar{\mathcal{I}}_{\pi,i,k} & \xrightarrow{\alpha} & \Delta \otimes_{\Bbbk} \tau(\Delta) \end{array}$$

so $\bar{\mathcal{I}}_{\pi,i,k}$ is a cell ideal as claimed. □

5.2 Projective, standard and proper standard modules

In this section we prove that $R_\alpha^{\mathcal{J}} := R_\alpha/\mathcal{J}$ is a properly stratified algebra.

First we describe the projective, standard and proper standard modules for $R_\alpha^{\mathcal{J}}$. We shall keep notation clear by saying $\Delta(\lambda)$ is a standard module over the algebra R_α, similarly for $P(\lambda)$, whereas $\Delta^{\mathcal{J}}(\lambda)$ and $P^{\mathcal{J}}(\lambda)$ are standard and projective (resp.) modules in $R_\alpha^{\mathcal{J}}$ **-mod**.

Lemma 5.4. *For $\lambda \in \Pi(\alpha)$, the modules $P^{\mathcal{J}}(\lambda) := R_\alpha^{\mathcal{J}} \otimes_{R_\alpha} P(\lambda)$ are indecomposable projective modules for $R_\alpha^{\mathcal{J}}$.*

Proof. Since $P(\lambda)$ is a projective module for R_α, there is an idempotent e_λ such that $P(\lambda) = R_\alpha e_\lambda$. Now,

$$P^{\mathcal{J}}(\lambda) = R_\alpha^{\mathcal{J}} \otimes_{R_\alpha} P(\lambda) = R_\alpha^{\mathcal{J}} \otimes_{R_\alpha} R_\alpha e_\lambda = R_\alpha^{\mathcal{J}} \bar{e}_\lambda.$$

Thus $P^{\mathcal{J}}(\lambda)$ is a projective module for $R_\alpha^{\mathcal{J}}$. The indecomposability follows from the fact that $\bar{e}_\lambda$ lifts to e_λ and [Lam99, 21.22]. □

Before classifying the standard modules we include a well known result from homological algebra [Wei95, Snake Lemma 1.3.2]

Lemma 5.5 (The Snake Lemma)**.** *Consider a commutative diagram of R-modules of the form*

$$\begin{array}{ccccccc} & & A' & \longrightarrow & B' & \xrightarrow{g} & C' & \longrightarrow & 0 \\ & & \downarrow a & & \downarrow b & & \downarrow c & & \\ 0 & \longrightarrow & A & \xrightarrow{f} & B & \longrightarrow & C & & \end{array}$$

If the rows are exact, there is an exact sequence

$$\ker(a) \to \ker(b) \to \ker(c) \to \operatorname{coker}(a) \to \operatorname{coker}(b) \to \operatorname{coker}(c)$$

with $\partial : \ker(c) \to \operatorname{coker}(a)$ *defined by the formula*

$$\partial(x) = f^{-1}bg^{-1}(x), \quad x \in \ker(c).$$

Proposition 5.6. *The modules* $R_\alpha^{\mathcal{J}} \otimes_{R_\alpha} \Delta(\lambda)$ *form a set of standard modules for* $R_\alpha^{\mathcal{J}}$.

Proof. Let $\Delta^{\mathcal{J}}(\lambda)$ be the standard module obtained from $P^{\mathcal{J}}(\lambda)$ in $R_\alpha^{\mathcal{J}}$. By definition, these modules fit into the short exact sequence

$$\operatorname{Tr}_{P^{\mathcal{J}}_{>\lambda}}(P^{\mathcal{J}}(\lambda)) \hookrightarrow P^{\mathcal{J}}(\lambda) \twoheadrightarrow \Delta^{\mathcal{J}}(\lambda).$$

Since the functor $R_\alpha^{\mathcal{J}} \otimes_{R_\alpha} -$ is right exact we also have a surjection from $P^{\mathcal{J}}(\lambda)$ onto $R_\alpha^{\mathcal{J}} \otimes_{R_\alpha} \Delta(\lambda)$, let the kernel of this surjection be K so that there is the short exact sequence

$$K \hookrightarrow P^{\mathcal{J}}(\lambda) \twoheadrightarrow R_\alpha^{\mathcal{J}} \otimes_{R_\alpha} \Delta(\lambda).$$

The module $\Delta^{\mathcal{J}}(\lambda)$ is the largest quotient of $P^{\mathcal{J}}(\lambda)$ with $\left[\Delta^{\mathcal{J}}(\lambda) : L^{\mathcal{J}}(\mu)\right] = 0$ for $\mu > \lambda$. So there is a surjection $f : \Delta^{\mathcal{J}}(\lambda) \to R_\alpha^{\mathcal{J}} \otimes_{R_\alpha} \Delta(\lambda)$. Combining these facts we get the following diagram

$$\begin{array}{ccccc}
\operatorname{Tr}_{P^{\mathcal{J}}_{>\lambda}}(P^{\mathcal{J}}(\lambda)) & \hookrightarrow & P^{\mathcal{J}}(\lambda) & \twoheadrightarrow & \Delta^{\mathcal{J}}(\lambda) \\
{\scriptstyle g}\downarrow & & \| & & \downarrow{\scriptstyle f} \\
K & \hookrightarrow & P^{\mathcal{J}}(\lambda) & \twoheadrightarrow & R_\alpha^{\mathcal{J}} \otimes_{R_\alpha} \Delta(\lambda).
\end{array}$$

Applying the Snake Lemma 5.5 gives the diagram

$$\begin{array}{ccccc}
0 & \longrightarrow & 0 & \longrightarrow & \ker(f) \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Tr}_{P^{\mathcal{J}}_{>\lambda}}(P^{\mathcal{J}}(\lambda)) & \hookrightarrow & P^{\mathcal{J}}(\lambda) & \twoheadrightarrow & \Delta^{\mathcal{J}}(\lambda) \\
{\scriptstyle g}\downarrow & & \| & & \downarrow{\scriptstyle f} \\
K & \hookrightarrow & P^{\mathcal{J}}(\lambda) & \twoheadrightarrow & R_\alpha^{\mathcal{J}} \otimes_{R_\alpha} \Delta(\lambda) \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{coker}(g) & \longrightarrow & 0 & \longrightarrow & 0.
\end{array}$$

from which we get that g is a monomorphism and $\ker f \cong \operatorname{coker} g$. Importantly, since $\ker f \subset \Delta^{\mathcal{J}}(\lambda)$ it too must have composition factors $L(\mu)$ with $\mu \leq \lambda$ and so must $\operatorname{coker} g$. In R_α we have the short exact sequence

$$\mathrm{Tr}_{P_{>\lambda}}(P(\lambda)) \hookrightarrow P(\lambda) \twoheadrightarrow \Delta(\lambda),$$

and if we apply $R_\alpha^{\mathcal{J}} \otimes_{R_\alpha} -$ we can induce the long exact sequence

$$\begin{array}{c} \cdots \longrightarrow \mathrm{Tor}(R_\alpha^{\mathcal{J}}, \Delta(\lambda)) \\ \swarrow \\ R_\alpha^{\mathcal{J}} \otimes_{R_\alpha} \mathrm{Tr}_{P_{>\lambda}}(P(\lambda)) \xrightarrow{h} P^{\mathcal{J}}(\lambda) \twoheadrightarrow R_\alpha^{\mathcal{J}} \otimes_{R_\alpha} \Delta(\lambda). \end{array}$$

The map h factors through K. Since everything in $\mathrm{Tr}_{P_{>\lambda}}(P(\lambda))$ is the sum of some images of maps from $P_{>\lambda}$, we have $P_{>\lambda} \twoheadrightarrow \mathrm{Tr}_{P_{>\lambda}}(P(\lambda))$ and so

$$R_\alpha^{\mathcal{J}} \otimes_{R_\alpha} P_{>\lambda} = P_{>\lambda}^{\mathcal{J}} \twoheadrightarrow R_\alpha^{\mathcal{J}} \otimes_{R_\alpha} \mathrm{Tr}_{P_{>\lambda}}(P(\lambda)).$$

Therefore, $\mathrm{top}(R_\alpha^{\mathcal{J}} \otimes \mathrm{Tr}_{P_{>\lambda}}(P(\lambda))) \in \mathrm{add}(\{L(\mu) \mid \mu > \lambda\})$. The long exact sequence above gives us $R_\alpha^{\mathcal{J}} \otimes_{R_\alpha} \mathrm{Tr}_{P_{>\lambda}}(P(\lambda)) \twoheadrightarrow K$. This implies that

$$\mathrm{top}(K) \in \mathcal{F}(\{L(\mu) \mid \mu > \lambda\}).$$

We know, however, that $\operatorname{coker} g \cong \ker f \in \mathrm{add}(\{L(\mu) \mid \mu \leq \lambda\})$, so since K surjects onto $\operatorname{coker} g$, we must have $\operatorname{coker} g = 0$. Thus we deduce that f and g are isomorphisms. □

Let us first include a characterisation of proper standard modules for affine quasi-hereditary algebras.

Proposition 5.7. *[Kle15, Proposition 5.6] If A is affine quasi-hereditary with simple indexing set Π. Then*

$$\bar{\Delta}(\pi) \cong \Delta(\pi)/\Delta(\pi)N_\pi,$$

where N_π is the Jacobson radical of the affine algebra B_π, $\pi \in \Pi$ and the notation $\Delta(\pi)N_\pi$ means $\sum_{f \in N_\pi} \mathrm{Im}\, f \subseteq \Delta(\pi)$.

Proposition 5.8. *The proper standard modules in $R_\alpha^{\mathcal{J}}$ are of the form $R_\alpha^{\mathcal{J}} \otimes_{R_\alpha} \bar{\Delta}(\lambda)$, where $\bar{\Delta}(\lambda)$ is a proper standard module for R_α. Moreover, if*

$$j : R_\alpha^{\mathcal{J}}\textbf{-mod} \to R_\alpha\textbf{-mod}$$

is the inclusion functor, $j(\bar{\Delta}^{\mathcal{J}}(\lambda)) \cong \bar{\Delta}(\lambda)$.

Proof. Let us assume that $\bar{\Delta}^{\mathcal{J}}(\lambda)$ is the proper standard module coming from $P^{\mathcal{J}}(\lambda)$

in $R_\alpha^{\mathcal{J}}$. In a similar way to the proof above we get the diagram

$$\begin{array}{ccccc}
 & & & & \ker f \\
 & & & & \big\downarrow \\
\mathrm{Tr}_{P\geq\lambda}(\operatorname{rad} P(\lambda)) & \hookrightarrow & P^{\mathcal{J}}(\lambda) & \twoheadrightarrow & \bar{\Delta}^{\mathcal{J}}(\lambda) \\
\big\downarrow & & \big\| & & \big\downarrow \\
K & \hookrightarrow & P^{\mathcal{J}}(\lambda) & \twoheadrightarrow & R_\alpha^{\mathcal{J}} \otimes_{R_\alpha} \bar{\Delta}(\lambda) \\
\big\downarrow & & & & \\
\operatorname{coker} g & & & & .
\end{array}$$

By the snake lemma $\ker f \cong \operatorname{coker} g$, and since $\ker f$ is strictly contained in $\bar{\Delta}^{\mathcal{J}}(\lambda)$ it has composition factors $L(\mu)$ with $\mu < \lambda$. We induce the long exact sequence

$$\begin{array}{c}
\cdots \longrightarrow \mathrm{Tor}(R_\alpha^{\mathcal{J}}, \bar{\Delta}(\lambda)) \\
\swarrow \\
R_\alpha^{\mathcal{J}} \otimes_{R_\alpha} \mathrm{Tr}_{P_{\geq\lambda}}(\operatorname{rad} P(\lambda)) \xrightarrow{h} P^{\mathcal{J}}(\lambda) \twoheadrightarrow R_\alpha^{\mathcal{J}} \otimes_{R_\alpha} \bar{\Delta}(\lambda).
\end{array}$$

Again, the map h must factor through K. We have that $P_{\geq\lambda}^{\mathcal{J}}$ surjects onto

$$R_\alpha^{\mathcal{J}} \otimes \mathrm{Tr}_{P_{\geq\lambda}}(\operatorname{rad} P(\lambda)),$$

so

$$\operatorname{top} R_\alpha^{\mathcal{J}} \otimes \mathrm{Tr}_{P_{\geq\lambda}}(\operatorname{rad} P(\lambda)) \in \operatorname{add}(\{L(\mu) \mid \mu \geq \lambda\}).$$

Since $R_\alpha^{\mathcal{J}} \otimes \mathrm{Tr}_{P_{\geq\lambda}}(\operatorname{rad} P(\lambda))$ surjects onto K we get that $\operatorname{top} K \in \operatorname{add}(\{L(\mu) \mid \mu \leq \lambda\})$, but $\operatorname{coker} g \cong \ker f \in \mathcal{F}(\{L(\mu) \mid \mu < \lambda\})$, so $\operatorname{coker} g = 0$ and $K \cong \mathrm{Tr}_{P_{\geq\lambda}}(\operatorname{rad} P(\lambda))$.

For the moreover statement, we have a chain of isomorphisms

$$\bar{\Delta}(\lambda) \cong \Delta(\lambda)/\Delta(\lambda) \operatorname{rad} \Lambda_\lambda \cong \Delta^{\mathcal{J}}(\lambda)/\Delta^{\mathcal{J}}(\lambda) \operatorname{rad} A_\lambda \cong \bar{\Delta}^{\mathcal{J}}(\lambda)$$

recalling the definitions of Λ_π and A_π from (2.2) and (5.1) respectively, the middle isomorphism follows from writing down bases for either side as given in [KL15, Lemma 3.10]. □

Theorem 5.9. *The functor $R_\alpha^{\mathcal{J}} \otimes_{R_\alpha} - : R_\alpha$*-**mod** $\to R_\alpha^{\mathcal{J}}$-**mod** *is exact on $\mathcal{F}(\Delta)$.*

Proof. Let $A := R_\alpha^{\mathcal{J}}$ and $R := R_\alpha$. Also, for a $\Bbbk$-module M let M^* denote the vector space dual of M achieved by applying the functor $\mathrm{Hom}_\Bbbk(-, \Bbbk)$, then

$$A \otimes_R M \cong \mathrm{Hom}_\Bbbk(A \otimes_R M, \Bbbk)^*.$$

Utilising the tensor-hom adjunction

$$\mathrm{Hom}_{\Bbbk}(A \otimes_R M, \Bbbk)^* \cong \mathrm{Hom}_R(M, \mathrm{Hom}_{\Bbbk}(A, \Bbbk))^*$$

and then $\mathrm{Hom}_R(M, \mathrm{Hom}_{\Bbbk}(A, \Bbbk))^* \cong \mathrm{Hom}_R(M, A^*)^*$ by definition.

Since A is filtered by proper standard modules, and we have a simple preserving duality it follows that A^* is filtered by proper costandard modules. From [AHLU00b, Theorem 1.6]

$$\mathcal{F}(\Delta) = \{X \mid \mathrm{Ext}^1_A(X, \mathcal{F}(\bar{\nabla})) = 0\}$$

hence $\mathrm{Hom}(-, A^*)$ is exact on $\mathcal{F}(\Delta)$. Since * is exact we get that $A \otimes_R -$ is exact on $\mathcal{F}(\Delta)$. □

$R_\alpha^{\mathcal{J}}$ is properly stratified

In this section we show that $R_\alpha^{\mathcal{J}}$ satisfies the conditions of Theorem 4.3, i.e. that $R_\alpha^{\mathcal{J}}$ has a full set of idempotents each of which decompose as $e_i = f_i + f_i'$ where the set of f_i form a full set of pairwise orthogonal idempotents and the $f_i' \in A(\sum_{j \geq i+1} e_j)A$, and hence that $R_\alpha^{\mathcal{J}}$ is standardly stratified.

Lemma 5.10. *The idempotents $e_\pi := \psi_\pi y_\pi e(\boldsymbol{i}_\pi) \in R_\alpha$ satisfy*

$$\sum_{\sigma \geq \pi} R_\alpha e_\sigma R_\alpha = \sum_{\sigma \geq \pi} R_\alpha e(\boldsymbol{i}_\sigma) R_\alpha.$$

Proof. The inclusion $\sum_{\sigma \geq \pi} R_\alpha e_\sigma R_\alpha \subseteq \sum_{\sigma \geq \pi} R_\alpha e(\boldsymbol{i}_\sigma) R_\alpha$ is clear. For equality, recall that $I_\pi = \sum_{\sigma \geq \pi} R_\alpha e(\boldsymbol{i}_\sigma) R_\alpha$ and has a basis given by elements of the form

$$\psi_w y_\sigma e(\boldsymbol{i}_\sigma) \psi_\sigma b y_\sigma \psi_v^\tau = \psi_w y_\sigma e_\sigma b \psi_v^\tau$$

with $\sigma \geq \pi \in \Pi(\alpha)$, $w, v \in \mathfrak{S}^\sigma$ and $b \in \Lambda_\sigma$. In particular, for a $\nu > \pi$, we have

$$e(\boldsymbol{i}_\nu) = \sum_{\sigma \geq \pi} a_{\nu,\sigma} \psi_w y_\sigma \psi_\sigma e(\boldsymbol{i}_\sigma) b_\sigma y_\sigma \psi_v^\tau = \sum_{\sigma \geq \pi} a_{\nu,\sigma} \psi_w y_\sigma e_\sigma b_\sigma \psi_v^\tau.$$

Therefore, $e(\boldsymbol{i}_\nu) \in I_\pi \subseteq \sum_{\sigma \geq \pi} R_\alpha e_\sigma R_\alpha$ and the claim follows. □

Proposition 5.11. *The algebra $R_\alpha^{\mathcal{J}}$ is standardly stratified.*

Proof. Firstly, we claim that the idempotents $\{e_\pi := y_\pi \psi_\pi e(\boldsymbol{i}_\pi) \mid \pi \in \Pi(\alpha)\}$ in R_α satisfy the conditions (a) and (b) in Theorem 4.3. Namely, by [KLM13, Main Theorem] we have

$$\sum_{\pi \in \Pi(\alpha)} R_\alpha e_\pi R_\alpha = R_\alpha$$

and since $\bar{e}_\pi \bar{R}_\alpha \cong \Delta(\pi)$ we get that $\bar{e}_\pi$ is primitive. Let $e_\pi = \epsilon_{\pi,1} + \epsilon_{\pi,2} + \cdots + \epsilon_{\pi,r}$ be a decomposition into primitive idempotents, then $(\epsilon_{\pi,1} + \cdots + \epsilon_{\pi,r}) + I_{>\pi}$ is primitive. Without loss of generality $\epsilon_{\pi,1} \notin I_{>\pi}$, and $e_\pi + I_{>\pi} = \epsilon_{\pi,1} + I_{>\pi}$. This gives

$$\bar{R}_\alpha \bar{e}_\pi \bar{R}_\alpha = I_\pi / I_{>\pi} = \bar{R}_\alpha \bar{\epsilon}_{\pi,1} \bar{R}_\alpha$$

so $\sum_{\pi \in \Pi(\alpha)} R_\alpha \epsilon_{\pi,1} R_\alpha = R_\alpha$. Now, $e_\pi + \mathcal{J}$ is non-zero in $R_\alpha^{\mathcal{J}}$ and we have a chain of ideals given by

$$\{\mathcal{I}_\pi \mid \pi \in \Pi(\alpha)\}.$$

We have seen that the ideal $\mathcal{I}_\pi \cong \sum_{\sigma \geq \pi} R_\alpha^{\mathcal{J}} e(\boldsymbol{i}_\sigma) R_\alpha^{\mathcal{J}}$. Now as a left $R_\alpha^{\mathcal{J}}$-module

$$\mathcal{I}_\pi / \mathcal{I}_{>\pi} \cong \Delta^{\mathcal{J}}(\pi) \otimes_{\Bbbk} V_\pi.$$

So $\mathcal{I}_\pi / \mathcal{I}_{>\pi} \in \mathcal{F}(\Delta^{\mathcal{J}})$ and hence we obtain the result. □

Proposition 5.12. *For all $\pi \in \Pi(\alpha)$, $\Delta^{\mathcal{J}}(\pi) \in \mathcal{F}(\bar{\Delta}^{\mathcal{J}})$.*

Proof. We have $\Delta^{\mathcal{J}}(\pi) \cong V_\pi \otimes_{\Bbbk} A_\pi$ as vector spaces. We obtain a filtration of $\Delta^{\mathcal{J}}(\pi)$ by taking

$$V_\pi \otimes M_n \subseteq V_\pi \otimes M_{n-1} \subseteq \cdots \subseteq V_\pi \otimes A_\pi,$$

each subquotient is isomorphic, as a $R_\alpha^{\mathcal{J}}$ module, to $\bar{\Delta}^{\mathcal{J}}(\pi)$. □

Corollary 5.13. *The algebra $R_\alpha^{\mathcal{J}}$ is properly stratified.*

5.3 Finitistic dimension

We now provide a bound for the finitistic dimension of $R_\alpha^{\mathcal{J}}$. First note that the standard module in R_α with largest projective dimension is the standard module corresponding to the root lowest in the order.

Lemma 5.14. *[BKM14, Corollary 4.11] For $\alpha \in Q^+$ of height n and*

$$\pi = p_1\beta_1 + \cdots + p_n\beta_n \in \Pi(\alpha),$$

the projective dimension of $\Delta(\pi)$ satisfies $\operatorname{p.dim} \Delta(\pi) \leq n - l$ where $l = \sum_{i=1}^n p_i$.

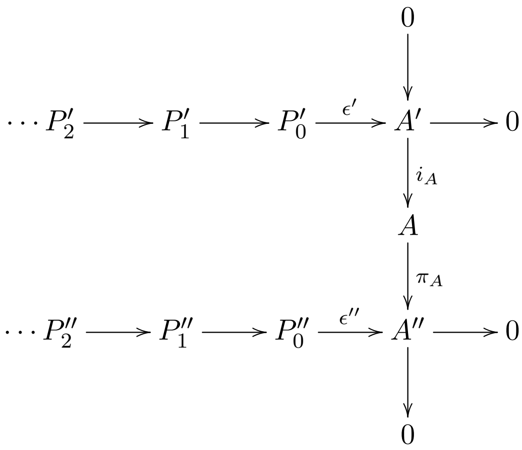
Recall the definition of the characteristic tilting module T from Section 4.1.

Theorem 5.15. *[Maz04] Let A be a properly stratified algebra with a simple preserving duality, then we have the following bound on $\operatorname{fin.dim}(A)$:*

$$\operatorname{fin.dim}(A) \leq 2 \cdot \operatorname{p.dim}(T).$$

The following lemma is well known in homological algebra [Wei95, Horseshoe Lemma 2.2.8].

Lemma 5.16 (Horseshoe Lemma)**.** *Suppose given a commutative diagram*



where the column is exact and the rows are projective resolutions. Set $P_n = P'_n \oplus P''_n$*. Then the* P_n *form a projective resolution* P *of* A*, and the right-hand column lifts to an exact sequence of complexes*

$$0 \longrightarrow P' \xrightarrow{i} P \xrightarrow{\pi} P'' \longrightarrow 0,$$

where $i_n : P'_n \to P_n$ *and* $\pi_n : P_n \to P''_n$ *are the natural inclusion and projection respectively.*

Proposition 5.17. *Let* $|\alpha| = d$*, and* $\pi_1 \in \Pi(\alpha)$ *be such that* $\pi_1 \leq \pi$ *for all* $\pi \in \Pi(\alpha)$ *and let* T *be the characteristic tilting module for* $R^{\mathcal{J}}_{\alpha}$*. We have the following bound on its projective dimension:*

$$\mathrm{p.dim}(T) \leq \mathrm{p.dim}(\Delta(1)) = d - l.$$

Proof. The module T fits into a short exact sequence

$$0 \to K \to T \to \Delta(\pi_1) \to 0.$$

The result follows from the Horseshoe Lemma 5.16 and Lemma 5.14. □

Corollary 5.18. *We get the following bound on the finitistic dimension of* $R^{\mathcal{J}}_{\alpha}$*,*

$$\mathrm{fin.dim}(R^{\mathcal{J}}_{\alpha}) \leq 2(d - l).$$

5.4 The multiplicity one case

Throughout this section let the underlying quiver of R_α be a Dynkin diagram A_n and let $\alpha = \alpha_1 + \cdots + \alpha_n$ be the highest root. By multiplicity one we mean that the root α_i appears only once for each $1 \le i \le n$. In this case, it is worth noting that the relations of the quiver Hecke algebra reduce to the following.

$$\psi_r y_s = y_s \psi_r \qquad \text{if } s \neq r, r+1; \tag{5.5}$$

$$\psi_r \psi_s = \psi_s \psi_r \qquad \text{if } |r-s| > 1; \tag{5.6}$$

$$\psi_r y_{r+1} e(\boldsymbol{i}) = (y_r \psi_r) e(\boldsymbol{i}); \quad y_{r+1} \psi_r e(\boldsymbol{i}) = (\psi_r y_r) e(\boldsymbol{i}); \tag{5.7}$$

$$\psi_r^2 e(\boldsymbol{i}) = \begin{cases} e(\boldsymbol{i}) & \text{if } |i_r - i_{r+1}| > 1, \\ (y_{r+1} - y_r) e(\boldsymbol{i}) & \text{if } i_r = i_{r+1} - 1, \\ (y_r - y_{r+1}) e(\boldsymbol{i}) & \text{if } i_r = i_{r+1} + 1; \end{cases} \tag{5.8}$$

$$\psi_r \psi_{r+1} \psi_r e(\boldsymbol{i}) = \psi_{r+1} \psi_r \psi_{r+1} e(\boldsymbol{i}). \tag{5.9}$$

In this chapter we show that when α is the highest root, the module category of the quiver Hecke algebra is equivalent to that of the tensor products of path algebras of a particular quiver and a polynomial ring. More generally, this notion is known as Morita Equivalence.

Morita equivalence Morita equivalence is an important tool in the study of rings and algebras. A full introduction to Morita theory can be found in Chapter 7 of Lam [Lam99]. We say that a ring T is *Morita equivalent* to a ring S if there exists a category equivalence between their categories of modules T-**mod** and S-**mod**. The following theorem is useful when it comes to showing Morita equivalence.

Theorem 5.19. *[Lam99, Theorem 17.25] The ring T is Morita equivalent to S if and only if $T \cong \mathrm{End}_S(P)$, where P is a projective generator in S-**mod**.*

For the left S-module P to be a *projective generator* in S mod , we require that P is a finitely generated projective module, and $\mathrm{Tr}_S(P) =_S S$.

5.4.1 A theorem of Brundan and Kleshchev

First, a comment on root partitions.

Lemma 5.20. *If $\alpha = \alpha_1 + \cdots + \alpha_n$ then there are 2^{n-1} root partitions of α, determined by*

$$\Pi(n) := \{(a_1, a_2, \ldots, a_{n-1}) | a_i \in \{1, 2\}\}$$

Proof. The set of root partitions $\Pi(\alpha)$ is in bijection with $\Pi(n)$. The bijection is given by

$$\Theta : \Pi(\alpha) \longleftrightarrow \Pi(n),$$

$$\pi \leftrightarrow (a_1, \ldots, a_{n-1})$$

such that

$$a_i = \begin{cases} 1 & \text{if } \alpha_i \text{ appears before } \alpha_{i+1}, \\ 2 & \text{if } \alpha_i \text{ appears after } \alpha_{i+1}. \end{cases}$$

□

Example 5.21. Let $n = 3$ so that $\alpha = \alpha_1 + \alpha_2 + \alpha_3$, then the bijection $\Theta : \Pi(\alpha) \leftrightarrow \Pi(n)$ in the previous lemma is:

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &\leftrightarrow (1, 1) \\ (\alpha_2 + \alpha_3)\alpha_1 &\leftrightarrow (2, 1) \\ \alpha_3(\alpha_1 + \alpha_2) &\leftrightarrow (1, 2) \\ (\alpha_3)(\alpha_2)(\alpha_1) &\leftrightarrow (2, 2) \end{aligned}$$

Now, let $\mathcal{A}$ be the path algebra of the following quiver,

$$e_1 \underset{\tau}{\overset{\tau}{\rightleftarrows}} e_2$$

It was noticed by Brundan and Kleshchev that when α is of multiplicity one R_α is Morita equivalent to tensor products of this path algebra with a polynomial ring. There is no published proof of their theorem so we include one here.

Theorem 5.22. *[Bru13, Theorem 3.13] Suppose the graph underlying the quiver is a Dynkin diagram A_n and that $\alpha = \alpha_1 + \cdots + \alpha_n$ is the highest root. Then, R_α is graded Morita equivalent to $\mathcal{A}^{\otimes(n-1)} \otimes \Bbbk[x]$, which is of global dimension n.*

Proof. Let $\pi_1, \ldots, \pi_r \in \Pi(\alpha)$, and let $P_1, \ldots, P_r$ be the left ideals generated by the idempotents $e(\boldsymbol{i}_{\pi_1}), \ldots, e(\boldsymbol{i}_{\pi_r})$, respectively. Let $\mathfrak{B}$ be a basis for $\Bbbk[y_1, \ldots, y_n]$, we can compute the endomorphism algebra of the minimal projective generator $\tilde{P} = P_1 \oplus \cdots \oplus P_r$, which consists of matrices

$$\left\{ \left(\begin{array}{ccc} e(\boldsymbol{i}_{\pi_1})be(\boldsymbol{i}_{\pi_1}) & \cdots & e(\boldsymbol{i}_{\pi_1})\psi_w be(\boldsymbol{i}_{\pi_r}) \\ \vdots & \ddots & \vdots \\ e(\boldsymbol{i}_{\pi_r})\psi_w be(\boldsymbol{i}_{\pi_1}) & \cdots & e(\boldsymbol{i}_{\pi_r})be(\boldsymbol{i}_{\pi_r}) \end{array} \right) \middle| \begin{array}{l} b \in \mathfrak{B}, \\ w \in \mathfrak{S}_n, \\ \text{a min. length red. expr.} \end{array} \right\}$$

Let us define a map

$$\phi : A^{\otimes(n-1)} \otimes \Bbbk[x] \to \mathrm{End}_{R_\alpha}(\tilde{P})$$

$$\begin{array}{rcl} e_{j_{n-1}} \otimes \cdots \otimes e_{j_1} \otimes 1 & \mapsto & \Theta^{-1}\left((j_{n-1}, \ldots, j_1)\right) \\ 1 \otimes \cdots \otimes 1 \otimes x & \mapsto & z \\ e_{j_{n-1}} \otimes \cdots \otimes e_{j_{k'}} \tau e_{j_k} \otimes \cdots \otimes e_{j_1} \otimes 1 & \mapsto & e(\boldsymbol{i}_\sigma)\psi_w e(\boldsymbol{i}_\pi) \end{array}$$

where $z \in Z(R_\alpha)$ is the element $z = z_1 := \sum_{w \in \mathfrak{S}^{\boldsymbol{i}}} y_{w(1)} e(w(\boldsymbol{i}))$ from (1.1), π and σ are neighbouring root partitions with respect to the partial ordering on $\Pi(\alpha)$ and $\pi = \Theta^{-1}(j_{n-1} \cdots j_k \cdots j_1)$, $\sigma = \Theta^{-1}(j_{n-1} \cdots j_{k'} \cdots j_1)$, and w is the unique element in $\mathfrak{S}_n$ such that $w(\boldsymbol{i}_\pi) = (\boldsymbol{i}_\sigma)$.

We claim that the map ϕ is surjective, and since ψ_w is unique we are only required to show that $y_j e(\boldsymbol{i}_\pi)$ is in the image of ϕ. For this we use the following algorithm. Associated to y_j we have a number i_j, which is the number occupying the j^{th} position in $\boldsymbol{i}_\pi$. Write $y_j e(\boldsymbol{i}_\pi) = (y_j - y_k + y_k)e(\boldsymbol{i}_\pi)$ where $i_k = i_j - 1$, we then write y_k in a similar fashion and continue recursively until we have

$$y_j e(\boldsymbol{i}_\pi) = (y_j - y_k + y_k - \cdots - y_l + y_l)e(\boldsymbol{i}_\pi),$$

where $i_l = 1$. Then $y_l e(\boldsymbol{i}_\pi)$ is one of summands of z. Then

$$(y_j - y_k)e(\boldsymbol{i}_\pi) = (\psi_{w_1}^2 + \cdots + \psi_{w_{r+1}}^2)e(\boldsymbol{i}_\pi),$$

therefore

$$y_j e(\boldsymbol{i}_\pi) = (\psi_{w_1}^2 + \cdots + \psi_{w_{r+1}}^2 + y_k)e(\boldsymbol{i}_\pi) = \phi(\psi_{w_1}^2 + \cdots + \psi_{w_{r+1}}^2 e(\boldsymbol{i}_\pi)) + \phi(y_k e(\boldsymbol{i}_\pi)),$$

and each ψ_{w_k} is one of the $\phi(\cdots \otimes \tau \otimes \cdots)$.

For injectivity we introduce a dimension formula for the algebra $\mathcal{A}^{\otimes n} \otimes \Bbbk[x]$,

$$\dim_q \mathcal{A}^{\otimes(n-1)} \otimes \Bbbk[x] = \frac{2^{n-1}}{(1-q)^{n-1}(1-q^2)}.$$

We verify this by noticing $\dim_q \mathcal{A} = 2/(1-q)$, since we have a choice of τe_1 or τe_2, each of which are in degree one. There are therefore, two options for each power of τ, giving the degree determining polynomial $2 + 2q + 2q^2 + 2q^3 + \cdots$, which is the Laurent expansion of $2/(1-q)$. For $\Bbbk[x]$, each x has degree 2, so the dimension formula for the polynomial ring is $1 + q^2 + q^4 + \cdots$ which is the Laurent expansion of $1/(1-q^2)$. Bringing this information together gives the dimension formula above.

We now claim that the dimension formula for $\mathrm{End}_{R_\alpha}(\tilde{P})$ is

$$\dim_q \mathrm{End}_{R_\alpha}(\tilde{P}) = \frac{2^{n-1}}{(1-q)^{n-1}(1-q^2)} = \dim_q \mathcal{A}^{\otimes(n-1)} \otimes \Bbbk[x].$$

To see this, first notice that there are 2^{n-1} root partitions in $\Pi(\alpha)$. Therefore, we have 2^{n-1} elements in degree zero. Each $y_1, \ldots, y_n$ is in degree 2, so we count their contribution to the degree with $1/(1-q^2)^n$. We then need to account for the ψ_w. The map ϕ is a degree preserving map, clearly idempotents and polynomial elements have their degree preserved by ϕ. If we consider the unique $w \in \mathfrak{S}_n$ that takes the partition π to π', then, $\deg(e(\boldsymbol{i}_{\pi'})\psi_w e(\boldsymbol{i}_\pi))$ is equal to the number of $(i, i+1)$ such that i appears before $i+1$ in one of $e(\boldsymbol{i}_\pi)$ or $e(\boldsymbol{i}_{\pi'})$, and then i appears after $i+1$ in the other. This equates to the number of positions in which the representatives $(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n) \in \Pi(n)$ of $\pi, \pi' \in \Pi(\alpha)$ (resp.) differ. Therefore, the degree of ψ_w is equal to the number of τ that appear in $\mathcal{A}^{\otimes(n-1)}$. Since ϕ is degree preserving, we have a bijection between

$$\left\{ e(\boldsymbol{i}_{\pi_j})\psi_w e(\boldsymbol{i}_{\pi_i}) \in \mathrm{End}_{R_\alpha}(\tilde{P}) \,\middle|\, 1 \le i, j \le n-1 \right\}$$

$$\updownarrow$$

$$\left\{ \gamma \in \mathcal{A}^{\otimes(n-1)} \,\middle|\, \gamma = \gamma_{n-1} \otimes \cdots \otimes \gamma_1,\ \deg(\gamma_i) \le 1,\ \forall\ 1 \le i \le n-1 \right\},$$

and each of these sets has cardinality 2^{n-1}. Let us denote by $\mathcal{A}^{\otimes(n-1)}_{loc\le 1}$ the vector space spanned by $\langle \gamma_{n-1} \otimes \cdots \otimes \gamma_1 \,|\, \deg(\gamma_i) \le 1 \rangle$. Then $\dim_q \mathcal{A}^{\otimes(n-1)}_{loc\le 1} = 2^{n-1}(1+q)^{n-1}$. Therefore,

$$\sum_{\pi,\pi'} q^{\deg(e(\boldsymbol{i}_\pi)\psi_w e(\boldsymbol{i}_{\pi'}))} = \dim_q \mathcal{A}^{\otimes(n-1)}_{loc\le 1} = 2^{n-1}(1+q)^{n-1}$$

and

$$\begin{aligned}
\dim_q \mathrm{End}_{R_\alpha}(\tilde{P}) &= \left(\sum_{\pi,\pi'} q^{\deg(e(\boldsymbol{i}_\pi)\psi_w e(\boldsymbol{i}_{\pi'}))} \right) \frac{1}{(1-q^2)^n} \\
&= 2^{n-1} \frac{(1+q)^{n-1}}{(1-q^2)^n} \\
&= \frac{2^{n-1}}{(1-q)^{n-1}(1-q^2)} = \dim_q \mathcal{A}^{\otimes(n-1)} \otimes \Bbbk[x].
\end{aligned}$$

Since the dimensions in each graded part match up, and are finite, surjectivity gives us injectivity. Therefore, ϕ is an isomorphism, and $\mathcal{A}^{\otimes(n-1)} \otimes \Bbbk[x]$ is Morita equivalent to R_α for highest root $\alpha = \alpha_1 + \cdots + \alpha_n$. □

Proposition 5.23. *There exists a quotient of the algebra $\mathcal{A}^{\otimes(n-1)} \otimes \Bbbk[x]$ that is quasi-hereditary.*

Proof. Let $\mathcal{I} = \langle x, \tau_i^2 e_2 \rangle$, then $\mathcal{A}^{\otimes(n-1)} \otimes \Bbbk[x]/\mathcal{I}$ is isomorphic to a tensor product of algebras $A = \Bbbk\mathcal{A}/\langle \tau^2 e_2 \rangle$. This algebra is quasi-hereditary with standard modules $\Delta_1 = Ae_2$, $\Delta_2 = Ae_1/Ae_2$. □

Corollary 5.24. *There is a quotient of the algebra R_α that is quasi-hereditary.*

This question corresponds with taking $d_\pi = 1$ for all π.

Chapter 6

Worked examples

6.1 Multiplicity free - $\alpha = \sum_{i=1}^{n} \alpha_i$

Here we consider some worked examples in the case where there are no repeated root.

Example 6.1. Let $\alpha = \alpha_1 + \alpha_2$. Let $\pi_1 = \alpha_1 + \alpha_2$ and $\pi_2 = \alpha_2\alpha_1$, the set of root partitions $\Pi(\alpha) = \{\pi_1, \pi_2\}$ is ordered such that $\pi_1 < \pi_2$.

$$\mathcal{J}_{\pi_1} = {}_{\Bbbk}\langle \psi_w e(12) p \psi_v^\tau \mid w, v \in \mathfrak{S}^{\pi_1}, p \in \mathfrak{B}(\Bbbk[y_2]), \deg(p) \geq 1 \rangle$$
$$\mathcal{J}_{\pi_2} = {}_{\Bbbk}\langle \psi_w e(21) p \psi_v^\tau \mid w, v \in \mathfrak{S}^{\pi_2}, p \in \mathfrak{B}(\Bbbk[y_1, y_2]), \deg(p) \geq 1 \rangle$$

The quotient $R_\alpha/\mathcal{J}$ is a five dimensional algebra with basis

$$\{e(12), e(21), \psi_1 e(12), \psi_1 e(21), y_1 e(12)\}$$

note that $y_1^2 e(12) = 0 \in R_\alpha^{\mathcal{J}}$ since $y_1^2 = \psi_1^2 e(12) = \psi_1(y_1 - y_2)e(21)\psi_1 = 0$. The left regular representation of the algebra decomposes into the sum of left projective modules as follows

$$_{R_\alpha^{\mathcal{J}}} R_\alpha^{\mathcal{J}} = \begin{array}{c} 1 \\ 2 \\ 1 \end{array} \oplus \begin{array}{c} 2 \\ 1 \end{array}$$

Where 1 and 2 denote the simple modules indexed by π_1 and π_2 respectively. This is clearly quasi-hereditary with standard modules $\Delta(\pi_1) = L(\pi_1)$ and $\Delta(\pi_2) = P(\pi_2)$. The costandard modules are $\nabla(\pi_1) = L(\pi_1)$, $\nabla(\pi_2) = I(\pi_2)$. The tilting modules are $T(\pi_1) = L(\pi_1)$ and $T(\pi_2) = P(\pi_1)$.

We have the following linear tilting coresolutions of $\Delta(\pi_1)$ and $\Delta(\pi_2)$;

$$0 \longrightarrow \Delta(\pi_1) \longrightarrow L(\pi_1) \longrightarrow 0 \longrightarrow 0,$$

$$0 \longrightarrow \Delta(\pi_2) \longrightarrow P(\pi_1) \longrightarrow L(\pi_1) \longrightarrow 0,$$

and the following linear tilting resolutions of $\nabla(\pi_1)$ and $\nabla(\pi_2)$;

$$0 \longrightarrow 0 \longrightarrow L(\pi_1) \longrightarrow \nabla(\pi_1) \longrightarrow 0,$$

$$0 \longrightarrow L(\pi_1) \longrightarrow P(\pi_1) \longrightarrow \nabla(\pi_2) \longrightarrow 0.$$

The generalised tilting module

$$T = \oplus_{\pi_i \in \Pi(\alpha)} T(\pi_i) = T(\pi_1) \oplus T(\pi_2) = L(\pi_1) \oplus P(\pi_1)$$

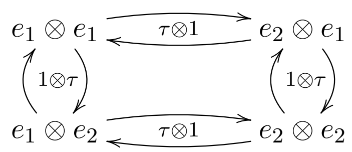
The Ringel dual is $\mathrm{End}_R(T) \cong R_\alpha/\mathcal{J}$, hence Ringel self-dual. Let $L(\pi_1) = A$ and $P(\pi_1) = B$, then $\mathrm{Hom}_R(A \oplus B, A \oplus B) = {}_{R'}R'_{R'}$, we have

$$\begin{aligned} R'_{e_A} &= \mathrm{Hom}_R(A \oplus B, A) \cong P(\pi_2) \\ R'_{e_B} &= \mathrm{Hom}_R(A \oplus B, B) \cong P(\pi_1). \end{aligned}$$

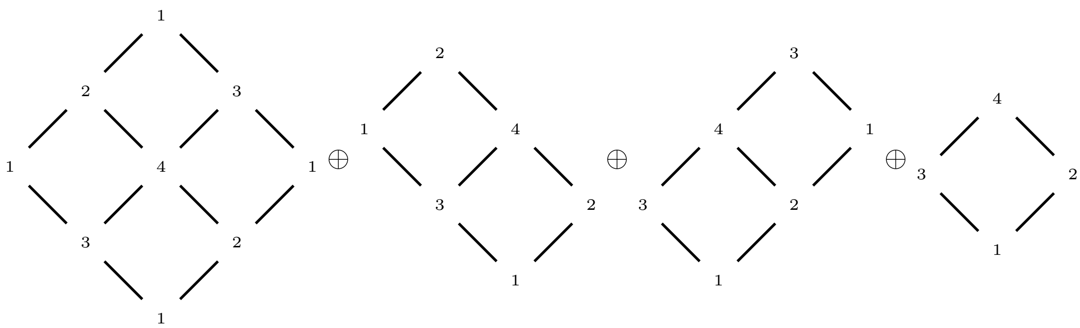
Example 6.2. Let $\alpha = \alpha_1 + \alpha_2 + \alpha_3$. Label the root partitions in the following way $\pi_1 := \alpha_1 + \alpha_2 + \alpha_3$, $\pi_2 := (\alpha_2 + \alpha_3)\alpha_1$, $\pi_3 := \alpha_3(\alpha_1 + \alpha_2)$, $\pi_4 := \alpha_3\alpha_2\alpha_1$, the ordering is $\pi_1 \leq \pi_2 \leq \pi_4$ and $\pi_1 \leq \pi_3 \leq \pi_4$.

$$\begin{aligned} \mathcal{J}_{\pi_1} &= {}_{\Bbbk}\langle \psi_w e(123) p \psi_v^\tau \mid w, v \in \mathfrak{S}^{\pi_1}, p \in \mathfrak{B}(\Bbbk[y_3]), \deg(p) \geq 1 \rangle \\ \mathcal{J}_{\pi_2} &= {}_{\Bbbk}\langle \psi_w e(231) p \psi_v^\tau \mid w, v \in \mathfrak{S}^{\pi_2}, p \in \mathfrak{B}(\Bbbk[y_2, y_3]), \deg(p) \geq 1 \rangle \\ \mathcal{J}_{\pi_3} &= {}_{\Bbbk}\langle \psi_w e(312) p \psi_v^\tau \mid w, v \in \mathfrak{S}^{\pi_3}, p \in \mathfrak{B}(\Bbbk[y_1, y_3]), \deg(p) \geq 1 \rangle \\ \mathcal{J}_{\pi_4} &= {}_{\Bbbk}\langle \psi_w e(321) p \psi_v^\tau \mid w, v \in \mathfrak{S}^{\pi_4}, p \in \mathfrak{B}(\Bbbk[y_1, y_2, y_3]), \deg(p) \geq 1 \rangle \end{aligned}$$

The quotient $R_\alpha/\mathcal{J}$ is a 25-dimensional algebra which, by Section 5.4.1, is Morita equivalent to the path algebra of



modulo the relation $\tau^2 e_2 = 0$. The left regular representation decomposes into a direct sum of left projective modules in the following way



The quasi-hereditary structure has standard modules $\Delta(\pi_1) = L(\pi_1)$, $\Delta(\pi_4) = P(\pi_4)$, and

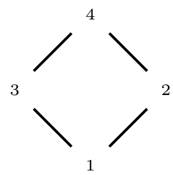
$$\Delta(\pi_2) = \begin{matrix} 2 \\ 1 \end{matrix} \qquad \Delta(\pi_3) = \begin{matrix} 3 \\ 1 \end{matrix}$$

The costandard modules are $\nabla(\pi_1) = L(\pi_1)$, $\nabla(\pi_4) = I(\pi_4)$, and

$$\nabla(\pi_2) = \begin{matrix} 1 \\ 2 \end{matrix} \qquad \nabla(\pi_3) : \begin{matrix} 1 \\ 3 \end{matrix} .$$

The characteristic tilting module is given by

$$T = 1 \oplus \begin{matrix} 1 \\ 2 \\ 1 \end{matrix} \oplus \begin{matrix} 1 \\ 3 \\ 1 \end{matrix} \oplus$$



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6.2 Affine nil-Hecke algebra

In this section we look at the opposite extreme, that where we have only one repeated simple root.

Let $\alpha = 2\alpha_1$, then the affine cellular basis for NH_2 is given by

$$\left\{ \psi_w y_2 e(11) \psi_1 \mathfrak{B}(\Bbbk[y_1, y_2]^{\mathfrak{S}}) y_2 \psi_v^{\tau} \mid w, v \in \mathfrak{S}_2 \right\}.$$

Now, let $e = \psi_1 y_2$.

We know from [Bru13, Theorem 2.3] that for $e_a := x_2 x_3^2 \cdots x_n^{n-1} \tau_{w_0}$ we have $P_a := q^{\frac{1}{2}a(a-1)} \mathrm{NH}_a \, e_a \cong q^{-\frac{1}{2}a(a-1)} \Bbbk[y_1, \cdots, y_a]$ and from [KLM13, Theorem 4.3] that P_a is free as a Λ_a module with basis $\{\psi_w y_2 y_3 \cdots \psi_{w_0} \mid w \in \mathfrak{S}_a\}$. But P_a is only free as a NH_a-module if NH_a is local, which it is not. As an $e_a \, \mathrm{NH}_a \, e_a$-module, for $a = 2$, we have

$$P_a = \langle y_2 \psi_1 e(11) y_2 b \psi_1 \mid b \in \mathfrak{B}(\Bbbk[y_1, y_2]^{\mathfrak{S}}) \rangle.$$

For

$$\mathcal{J} := \langle \psi_w y_2 e(11)\psi_1 p y_2 \psi_v^\tau \mid w, v \in \mathfrak{S}_2, p \in \mathfrak{B}(\Bbbk[y_1, y_2]^{\mathfrak{S}}), \deg(p) \geq 1 \rangle,$$

the algebra $(e\,\mathrm{NH}_2\,e)^{\mathcal{J}}$ is one dimensional as

$$\psi_1 y_2^2 e(11)\psi_1 y_2 \psi_1 y_2 = \psi_1 y_2 e(11)(y_1 + y_2)\psi_1 y_2 = 0 \tag{6.1}$$

$$\psi_1 y_2 \psi_1 y_2 e(11)\psi_1 y_2 \psi_1 y_2 = \psi_1 y_2 e(11)\psi_1 y_2 \tag{6.2}$$

$$\psi_1 y_2^2 e(11)\psi_1 y_2 \psi_1^2 y_2 = 0 \tag{6.3}$$

$$\psi_1 y_2 \psi_1 y_2 e(11)\psi_1 y_2 \psi_1^2 y_2 = 0 \tag{6.4}$$

and $\mathrm{NH}_2^{\mathcal{J}}$ is semi-simple.

6.3 $\alpha = 2\alpha_1 + \alpha_2$ **112**

We devote this section to the example of $\alpha = 2\alpha_1 + \alpha_2$. We relabel the root partitions of α as $1 = (\alpha_1 + \alpha_2)\alpha_1$ and $2 = \alpha_2\alpha_1^2$. This is the smallest case in which we have a repeated simple root, but are not isomorphic to a nil-Hecke algebra. Whilst our bound on d_π would give a much larger quotient, this example is sufficiently small to determine that we are able to take a quotient ideal given by the sum of

$$\mathcal{J}_{121} := {}_{\Bbbk}\langle \psi_w e(121) p \psi_v^\tau \mid w, v \in \mathfrak{S}^\pi, p \in \mathfrak{B}(\Bbbk[y_2, y_3]), \deg(p) \geq 2 \rangle$$

$$\mathcal{J}_{211} := {}_{\Bbbk}\langle \psi_w y_3 \psi_2 e(211) p y_3 \psi_v^\tau \mid w, v \in \mathfrak{S}^\pi, p \in \mathfrak{B}(\Bbbk[y_1] \otimes \Bbbk[y_2, y_3]^{\mathfrak{S}}), \deg(p) \geq 1 \rangle.$$

Let us recall why we cannot just kill all positive degree polynomials in the higher cell.

Remark 6.3. Consider $h = \psi_1\psi_2 e(121) y_3 \in \mathcal{J}_{121}$, then

$$\begin{aligned} h\psi_1 &= \psi_1\psi_2\psi_1 y_3 e(211) \\ &= \psi_2\psi_1\psi_2 y_3 e(211) \\ &= \psi_2\psi_1\psi_2 y_3 e(211)\psi_2 y_3 \in \mathcal{J}_{211}. \end{aligned}$$

Note that in this instance the algebra R_α is not basic. The idempotents $e(\boldsymbol{i})$ decompose into primitive orthogonal idempotents in the following way:

$$e(112) = (\psi_1 y_2 - y_1\psi_1)e(112) \tag{6.5}$$

$$e(121) = (\psi_1\psi_2\psi_1 - \psi_2\psi_1\psi_2)e(121) \tag{6.6}$$

$$e(211) = (\psi_2 y_3 - y_2\psi_2)e(211). \tag{6.7}$$

Lemma 6.4. *The idempotent* $f = \psi_2 y_3 e(211) - \psi_2\psi_1\psi_2 e(121)$ *is a full idempotent in* R_α.

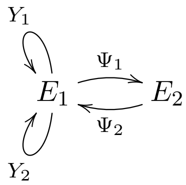
Proof. The inclusion $R_\alpha f R_\alpha \subseteq R_\alpha$ is clear. For the other direction notice that

$$(\psi_1 e(211) + \psi_2\psi_1\psi_2 e(121)) f (e(211)\psi_2\psi_1) = e(121) \tag{6.8}$$
$$(e(211)) f (e(211) - \psi_2 y_2 e(211)) = e(211). \tag{6.9}$$

By [KLM13, Lemma 5.13], if a two sided ideal J contains all idempotents $e(\boldsymbol{i}_\pi)$ such that $\pi \in \Pi(\alpha)$ then $J = R_\alpha$. Hence $R_\alpha f R_\alpha = R_\alpha$ and f is a full idempotent. $\square$

We now compute the basic algebra $fR_\alpha^{\mathcal{J}} f = fR_\alpha f/\mathcal{J}$ associated to $R_\alpha^{\mathcal{J}}$.

Proposition 6.5. *The algebra* $fR_\alpha^{\mathcal{J}} f$ *is a seven dimensional properly stratified algebra isomorphic to the path algebra* $\Bbbk\mathcal{Q}/I$ *where* $\mathcal{Q}$ *is*



and $I = \langle Y_1^2, Y_iY_j, Y_i\Psi_j, \Psi_jY_i, \Psi_1\Psi_2, \Psi_2\Psi_1 - Y_2^2\rangle$.

Proof. For all basis elements x of R_α we compute $fxf + \mathcal{J}$, the only surviving elements are:

$\Bbbk\mathcal{Q}/I$	fxf	$fxf + \mathcal{J}$	degree
E_1	$fe(121)f$	$-\psi_2\psi_1\psi_2 e(121)$	0
E_2	$f\psi_2y_3\psi_2e(211)y_3f$	$\psi_2y_3e(211)$	0
Ψ_1	$f\psi_2y_3\psi_2e(211)y_3\psi_1f$	$\psi_2y_3\psi_2e(211)y_3\psi_1$	1
Y_1	$fy_2e(121)f$	$-\psi_2\psi_1\psi_2e(121)y_2$	2
Y_2	$fy_3e(121)f$	$-\psi_2\psi_1\psi_2e(121)(y_1+y_3-y_2)$	2
Ψ_2	$f\psi_1y_3\psi_2e(211)y_3f$	$\psi_1y_3\psi_2e(211)y_3$	3
$\Psi_2\Psi_1 = Y_2^2$	$f\psi_1y_3\psi_2e(211)y_3\psi_1f$	$\psi_1y_3\psi_2e(211)y_3\psi_1$	4

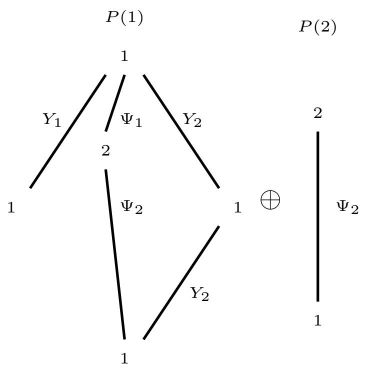
When lifted to R_α the elements above corresponding to E_1, E_2, Y_1, Y_2 are not written in terms of the affine cellular basis, but can be written as:

$$\begin{aligned}
E_1 &= e(121) - \psi_1\psi_2y_3e(211)\psi_2y_3\psi_2\psi_1 \\
E_2 &= \psi_2y_3e(211)\psi_2y_3 \\
Y_1 &= -e(121)y_2 + \mathcal{J} \\
Y_2 &= \psi_1y_3e(211)\psi_2y_3\psi_2\psi_1 + \psi_1\psi_2y_3e(211)\psi_2y_3\psi_1 - e(121)y_3 + \mathcal{J}
\end{aligned}$$

We show that Y_1 is annihilated by all non-idempotent elements and Y_2 is annihilated by all elements except itself.

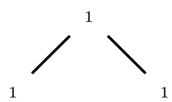
$$
\begin{aligned}
\Psi_1 Y_1 &= -\psi_2 y_3 \psi_2 e(211) y_3 y_1^2 \psi_2 \psi_1 + \psi_2 y_3 \psi_2 y_3 e(211) y_1 \psi_1 = 0 + \mathcal{J} \\
\Psi_1 Y_2 &= -\psi_2 y_3 \psi_2 e(211) y_1 (y_2 + y_3 - y_1) y_3 \psi_2 \psi_1 + \psi_2 y_3 \psi_2 y_3 e(211)(y_3 + y_2 - y_1)\psi_1 \\
&= 0 + \mathcal{J} \\
Y_1 Y_2 &= \psi_1 \psi_2 y_3 (y_1(y_2 + y_3) - y_1^2) \psi_2 y_3 \psi_2 \psi_1 - e(121(y_3 y_3) + \psi_1 \psi_2 y_3 \psi_2 y_2 y_3 \psi_1 \\
&\quad + \psi_1 y_3 \psi_2 y_3 y_1 \psi_2 \psi_1 = 0 + \mathcal{J} \\
Y_2 Y_1 &= Y_1 Y_2 = 0 + \mathcal{J} \\
Y_1^2 &= \psi_1 \psi_2 y_3 \psi_2 e(211) y_1^2 y_3 \psi_2 \psi_1 - e(121) y_2^2 = 0 + \mathcal{J} \\
Y_1 \Psi_2 &= \psi_1 y_3 \psi_2 e(211) y_1 y_3 - \psi_1 \psi_2 y_3 \psi_2 e(211) y_1^2 y_3 = 0 + \mathcal{J} \\
Y_2 \Psi_2 &= \psi_1 y_3 \psi_2 e(211)(y_2 + y_3 - y_1) y_3 - \psi_1 \psi_2 y_3 \psi_2 e(211) y_1 (\psi_2 + \psi_3 - \psi_1) y_3 = 0 + \mathcal{J} \\
Y_2^2 &= \psi_1 y_3 \psi_2 y_3 \psi_1 + \mathcal{J}.
\end{aligned}
$$

Hence the left regular representation of $fR_\alpha^{\mathcal{J}}f$ decomposes into a sum of indecomposable projectives with Loewy structure

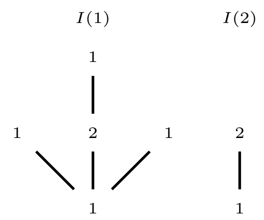


□

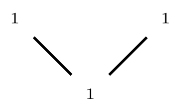
Clearly the quotient above is not the most optimal properly stratified quotient of R_α as we could also quotient by y_2 to remove the element Y_1. The standard modules are $\Delta(2) = P(2)$ and $\Delta(1)$ has Leowy structure



The proper standard modules are $\bar{\Delta}(1) = L(1)$ and $\bar{\Delta}(2) = \Delta(2) = P(2)$. The socle filtrations of the injectives are



The costandard module $\nabla(1)$ has Loewy structure



and $\bar{\nabla}(1) = L(1)$, $\nabla(2) = I(2) = \bar{\nabla}(2)$.

From which we get tilting modules $T(1) = \Delta(1)$ and $T(2) = P(1)$, so the characteristic tilting module is $T = \Delta(1) \oplus P(1)$.

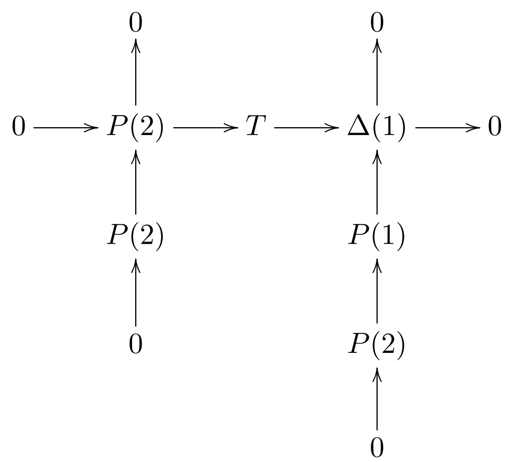
We define modules $S(\lambda) := \mathrm{Tr}_{T^{>\lambda}}(T(\lambda))$ and $N(\lambda) := T(\lambda)/S(\lambda)$ that fit into the following short exact sequence

$$0 \longrightarrow S(\lambda) \longrightarrow T(\lambda) \longrightarrow N(\lambda) \longrightarrow 0$$

For this example we get $S(1) = 0$ and $S(2) = L(1) \oplus L(1) \oplus L(1)$ and thus $N(1) = T(1) = \Delta(1)$ and $N(2) = I(2)$.

Since $S(2) \notin \mathcal{F}(N)$we use [FM06, Theorem 3] to deduce that the Ringel dual is not properly stratified.

We now compute the projective dimension of $T = \oplus_\lambda T(\lambda)$. Notice that T fits into the split exact sequence, to which we've added projective resolutions.



Applying [Wei95, Horseshoe Lemma 2.2.8] we deduce that

$$\mathrm{p.dim}(T) \leq \mathrm{p.dim}(\Delta(1)) = 1.$$

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