On the A_∞-Structure and Quiver for the Weyl Extension Algebra of $GL_2(\overline{\mathbb{F}_p})$

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Robin Cussol
School of Mathematics, UEA, Norwich, NR4 7TJ England

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Abstract

This research belongs to the field of Representation Theory and tries to solve questions through homological algebraic methods. This project deals with the study of symmetries of the plane and aims at measuring how much a mathematical object of importance for that study fails to satisfy the property of not needing bracketing when multiplying three elements together, which is called associativity. More precisely, we study the rational representations of $GL_2(\overline{\mathbb{F}_p})$, the general linear group of order 2 over an algebraically closed field of prime characteristic p. Representations are a means to understand group or algebra elements as linear transformations on a vector space of a given dimension, and it is possible to "build" representations from smaller ones, e.g. the set of so-called *standard representations*. The way to glue these building blocks together is governed by the algebra of extensions between standard representations. In a series of papers culminating with [MT13], Miemietz and Turner described precisely the algebra structure of that extension algebra. It is the homology of a differential-graded algebra and this project aims at estimating how non-associative it is by computing its A_∞-algebra structure. For any p, we give the quiver of that extension algebra, and for $p = 2$, we show that there exists a subalgebra of the extension algebra which admits a trivial A_∞-algebra structure, and what's more, in a somewhat peculiar way. We also give its quiver and discuss some of its properties.

Contents

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Introduction

Foreword

This PhD project is based on the work of Miemietz and Turner, more precisely on their paper *The Weyl extension algebra of* $GL_2(\overline{\mathbb{F}_p})$ (2013) [MT13]. They provide an alternative description of the extension algebra of standard modules belonging to the principal block of rational representations of $GL_2(\overline{\mathbb{F}_p})$ and that description gives the algebra structure: a basis is parametrised by some polytopes in $\mathbb{Z}^7$ and the multiplication is given explicitly in terms of those polytopes. Let us introduce the setup for this project.

Rational Representations of $GL_n(\overline{\mathbb{F}_p})$

Let F be an algebraically closed field of positive characteristic p. We consider the polynomial representations of $GL_n(F)$, namely those morphisms of algebraic groups

$$\rho : GL_n(F) \to GL(V),$$

for some m-dimensional vector space V over F, such that, after choosing a basis for $GL(V)$, all the entries of $\rho(g)$ are polynomials in the coordinate functions of $GL_n(F)$.

Denote by $R_n = F[G]$ the ring of coordinates of $G = GL_n(F)$. As a polynomial ring, it has a coalgebra structure. In addition, it contains the subcoalgebra $A(n,r)$ of polynomials of degree r. Dualising this coalgebra with respect to F, we obtain the Schur algebra:

$$S(n,r) = A(n,r)^*.$$

Theorem. *[Gre81] Denote by Rep G the category of polynomial representations of G, and by* $Rep_r G$ *the category of polynomial representations of G of degree r. Then we have:*

$$Rep\ G = \bigoplus_{r \geq 0} Rep_r G,$$

namely, if $M \in Rep\ G$*, then* M *splits as*

$$M = \bigoplus_{r \geq 0} M_r,$$

where $M_r \in Rep_r G$ *for all* $r \geq 0$.

In addition,

Theorem. *[Gre81] There is an equivalence of categories*

$$Rep_r G \simeq S(n,r) - mod.$$

The simple modules are labelled by partitions of r with at most n parts.

The Schur algebra $S(n,r)$ decomposes into blocks:

$$S(n,r) \cong A_1 \times \ldots \times A_s,$$

i.e. as a direct product of indecomposable algebras A_i.

We now restrict to the case $n = 2$, so that $G = GL_2(F)$. Suppose A_1 and A_2 are blocks of $S(2,r_1)$, $S(2,r_2)$ resp. but A_1 and A_2 have the same number of simple modules, then $A_1 - \text{mod} \cong A_2 - \text{mod}$ ([EH02, Theorem 13]). Note that in that case, it is possible to label simple modules by an integer a: a partition (λ_1, λ_2) of r_i with two parts is uniquely identified by $a := \lambda_1 - \lambda_2$. Given a degree $r \in \mathbb{N}$, there exists a combinatorial description of which such a's are in the same block (cf. [Par07]).

Finally, we have

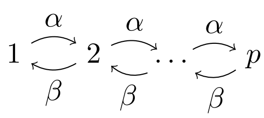
Theorem. *[MT10] If a block A has p^r simple modules, where $p = char\, F$, then A is Morita equivalent to the algebra $\mathbf{c}_p{}^r(F,F)$.*

We define $\mathbf{c}_p{}^r(F,F)$ in the next section.

Inductive Construction

As this project relies on the paper [MT13], we need to explain their notation and results. The category G-mod of rational representations of $G = GL_2(\overline{\mathbb{F}_p})$ is a highest weight category and the standard modules are called *Weyl modules.* They give an explicit description of the algebra structure of the Yoneda extension algebra $\mathbf{w}$ of the Weyl modules (belonging to the principal block) of the category G-mod. This description relies on an inductive construction of some algebra $\boldsymbol{\mu}$ using some algebraic operators which turn out to be well-behaved with respect to homology.

The starting point is to consider the very small quasi-hereditary algebra $\mathbf{c}_p$ which is the path algebra of the following quiver:



modulo the relations $(\alpha^2, \beta^2, \alpha\beta + \beta\alpha, \alpha\beta e_p)$. Our convention to write paths is the same as that to write the composition of maps, namely ab corresponds to the path

$$\bullet \xrightarrow{b} \bullet \xrightarrow{a} \bullet.$$

The algebra $\mathbf{c}_p$ is a trigraded algebra, with j-grading the path length, k-grading being identically zero and d-grading the grading with respect to the quasi-hereditary structure of $\mathbf{c}_p$ (the filtration of $\mathbf{c}_p$ by standard modules is unique). We can turn it into a differential trigraded algebra by adding a differential on it which we choose to be the zero map.

Computing the endomorphism ring of its tilting module $\mathbf{t}_p$ as a left $\mathbf{c}_p$-module, we see that $\mathbf{c}_p$ is Ringel self-dual. This yields an isomorphism of left $\mathbf{c}_p$-modules $\mathbf{t}_p \otimes_{\mathbf{c}_p} \mathbf{t}_p \cong \mathbf{c}_p{}^*$. Because $(-)^*$ is a simple preserving duality, $\mathbf{c}_p{}^*$ looks like $\mathbf{c}_p$ upside down. Note that $\mathbf{t}_p$ is *not* trigraded as a $\mathbf{c}_p$-module: the d-grading on $\mathbf{t}_p$ is not a module grading over the d-graded algebra $\mathbf{c}_p$ but is a vector space grading (cf. [MT13, Corrigendum]).

We now want to make the construction of $\mathbf{c}_p{}^r(F,F)$ explicit. Let us first fix some notation: let $a = \oplus a^{jk}$ be a differential bigraded algebra, $m = \oplus m^{jk}$ be a differential bigraded a-a-bimodule, $A = \oplus_{k\in\mathbb{Z}} A^k$ be a differential graded algebra and $M = \oplus_{k\in\mathbb{Z}} M^k$ be a graded bimodule. For simplicity, we assume a and m are non negatively j-graded. We write:

$$\mathbb{P}_{a,m}(A,M) := (a(A,M), m(A,M)),$$

where

$$a^k(A,M) := \bigoplus_j a^{jk} \otimes_F M^{\otimes_A j},$$
$$m^k(A,M) := \bigoplus_j m^{jk} \otimes_F M^{\otimes_A j}.$$

Informally, what we do is glue a, resp. m, with a tensor product of copies of M of length the j-degree of a, resp. m. The first coordinate $a(A,M)$ of $\mathbb{P}_{a,m}(A,M)$ is a differential graded algebra with multiplication

$$\begin{array}{ccc} a(A,M)\otimes a(A,M) & & \left(x^{\,jk} \otimes_F y_1 \otimes_A \ldots \otimes_A y_j\right) \otimes \left(x'^{j'k'} \otimes_F y'_1 \otimes_A \ldots \otimes_A y'_{j'}\right) \\ \downarrow & & \downarrow \\ a(A,M) & & \left(xx'^{(j+j')(k+k')} \otimes_F y_1 \otimes_A \ldots \otimes_A y_j \otimes_A y'_1 \otimes_A \ldots \otimes_A y'_{j'}\right) \end{array}$$

with k-grading and differential the total k-grading and total differential on the tensor products of complexes. The second coordinate $m(A,M)$ is a differential graded $a(A,M)$-$a(A,M)$-bimodule, with left action

$$\begin{array}{ccc} a(A,M)\otimes m(A,M) & & \left(x^{\,jk} \otimes_F y_1 \otimes_A \ldots \otimes_A y_j\right) \otimes \left(m^{j'k'} \otimes_F y'_1 \otimes_A \ldots \otimes_A y'_{j'}\right) \\ \downarrow & & \downarrow \\ m(A,M) & & \left(xm^{(j+j')(k+k')} \otimes_F y_1 \otimes_A \ldots \otimes_A y_j \otimes_A y'_1 \otimes_A \ldots \otimes_A y'_{j'}\right) \end{array}$$

and the right action is defined likewise. The k-grading and differential are defined similarly as for $a(A,M)$.

We can now define ${\mathbf{c}_p}^r(F,F)$: it is the algebra part of $\mathbb{P}^r_{\mathbf{c}_p,\mathbf{t}_p}(F,F)$.

Example. To illustrate this construction, consider the case $p=2$ and $r=2$. There are two simple modules denoted by 1 and 2, with corresponding idempotents e_1 and e_2. The tilting module $\mathbf{t}_2$ of $\mathbf{c}_2$ admits the following decomposition as a left module:

$$\mathbf{t}_2 e_1 \oplus \mathbf{t}_2 e_2 = 1^0_1 \oplus \begin{array}{c} 1^0_0 \\ 2^1_1 \\ 1^1_2 \end{array},$$

where the superscript corresponds to the d-grading and the subscript to the j-grading. Note that the way the right $\mathbf{c}_2$-action is defined ([MT13][Section 6.]) - so that $\mathbf{t}_2$ is a $\mathbf{c}_2$-$\mathbf{c}_2$-bimodule - imposes that the first tilting module only basis element has j-degree 1.

We can now compute $\mathbf{c}_2^2(F,F)$. Recall that $\mathbf{t}_2 \otimes_{\mathbf{c}_2} \mathbf{t}_2 \cong \mathbf{c}_2^*$. Pictorially, we have:

$$\begin{array}{c} 1^0_0 \otimes \mathbf{c}_2 \\ 2^1_1 \otimes \mathbf{t}_2 \\ 1^1_2 \otimes {\mathbf{c}_2}^* \end{array} \oplus \begin{array}{c} 2^0_0 \otimes \mathbf{c}_2 \\ 1^0_1 \otimes \mathbf{t}_2 \\ \end{array}$$

This algebra has four simple modules (i,j) for $1 \le i,j \le 2$, which we write 2-adically. We denote the corresponding idempotents by e_s where $s \in \{1,2,3,4\}$. To sum things up, we see in Figure 1 the decomposition of $\mathbf{c}_2^2(F,F)$ into indecomposable left projective modules.

The algebra describing the principal block of rational representations is isomorphic to the homology of the algebra part of (the inverse limit of) $\mathbb{P}^r_{\mathbf{c}_p,\mathbf{t}_p}(F,F)$. Let $\mathbf{d}_p$ be the extension algebra of the standard modules of $\mathbf{c}_p$, and $\underline{\mathbf{u}} = (\mathbf{u},\mathbf{u}^{-1})$ be the image of

$$\mathbf{c}_2^2(F,F)e_1 \oplus \mathbf{c}_2^2(F,F)e_2 \oplus \mathbf{c}_2^2(F,F)e_3 \oplus \mathbf{c}_2^2(F,F)e_4 =$$

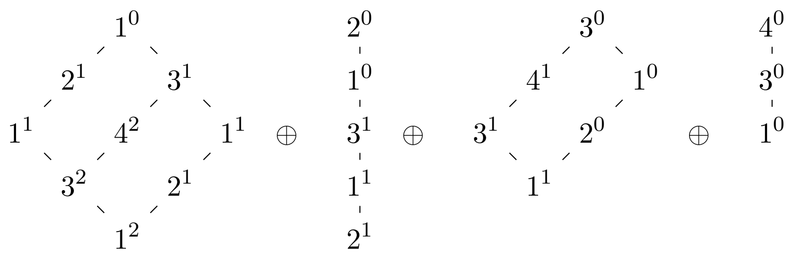


Figure 1: Decomposition into indecomposable left projective modules of $\mathbf{c}_2^2(F,F)$.

$\underline{\mathbf{t}}_p = (\mathbf{t}_p, \mathbf{t}_p^{-1})$, where $\mathbf{t}_p$ is the tilting module of $\mathbf{c}_p$ and $\mathbf{t}_p^{-1} := \mathrm{Hom}_{\mathbf{c}_p}(\mathbf{t}_p, \mathbf{c}_p)$, under the dg derived equivalence given in Proposition 25 of [MT13]. The algebra $\boldsymbol{\mu}$ mentioned above is the result of a similar iteration using $\mathbf{d}_p$ and the homology of $\underline{\mathbf{u}}_p$, instead of $\mathbf{c}_p$ and $\mathbf{t}_p$, and using an operator $\mathfrak{O}$ instead of $\mathbb{P}$. The operator $\mathfrak{O}$ has a similar definition as operator $\mathbb{P}$:

$$\mathfrak{O}_\Gamma(\Sigma)^{ik} = \bigoplus_{j,k_1+k_2=k} \Gamma^{ijk_1} \otimes_F \Sigma^{jk_2},$$

where $\Gamma = \oplus_{i,j,k\in\mathbb{Z}}\Gamma^{ijk}$ is a $\mathbb{Z}$-trigraded algebra and $\Sigma = \oplus_{j,k\in\mathbb{Z}}\Sigma^{jk}$ is a $\mathbb{Z}$-bigraded algebra. Informally, we glue the two algebras along the j-degree.

Denoting by $\mathbf{w}_q$ an idempotent truncation of $\mathbf{w}$ with p^q simple modules, Miemietz and Turner prove the following in their paper:

Proposition. *[MT13, Proposition 28.] We have*

$$\mathbf{w}_q \cong \boldsymbol{\mu}_q := \mathfrak{O}_F \mathfrak{O}^q_{\mathbb{H}\mathbb{T}_{\mathbf{d}}(\underline{\mathbf{u}})}(F[z, z^{-1}]),$$

where

- $\mathbb{H}$ means take homology;
- $\mathbb{T}_{\mathbf{d}}(\underline{\mathbf{u}}) := \bigoplus_{l>0} \mathbf{u}^{-1 \otimes_{\mathbf{d}} l} \oplus \mathbf{d} \oplus \bigoplus_{l>0} \mathbf{u}^{\otimes_{\mathbf{d}} l}$ is a sum of tensor products of $\mathbf{u}^{-1}$ when the index is negative and of $\mathbf{u}$ when it is positive. Note that it is not an algebra as the multiplication is not well-defined; however, its homology is an algebra.

In particular, $\mathbf{w}_q$ can be identified with a subalgebra of $\mathbf{d} \otimes_F \mathbb{H}\mathbb{T}_{\mathbf{d}}(\underline{\mathbf{u}})^{\otimes q-1}$. After closer analysis ([MT13, Lemma 29]), it turns out we only need a truncation of $\mathbb{H}\mathbb{T}_{\mathbf{d}}(\underline{\mathbf{u}})$; we can keep the non-positive powers of $\mathbf{u}$ and $\mathbf{u}$ itself, which we denote $\mathbb{H}\mathbb{T}_{\mathbf{d}}(\underline{\mathbf{u}})^{\leq 1}$.

We are interested in $\boldsymbol{\mu}$ because it admits a much more explicit algebra structure. Identifying $\mathbf{w}_q$ and $\boldsymbol{\mu}_q$ through that isomorphism of algebras, and since $\mathbf{w}_q$ appears as a subalgebra of $\mathbf{d} \otimes \mathbb{H}\mathbb{T}_{\mathbf{d}}(\underline{\mathbf{u}})^{\otimes q-1}$, it is possible to express the basis elements of a basis of $\mathbf{w}_q$ in terms of basis elements of a basis of $\mathbb{H}\mathbb{T}_{\mathbf{d}}(\underline{\mathbf{u}})^{\leq 1}$ which is indexed by a polytope in $\mathbb{Z}^7$.

Overview

In this section, we wish to give the reader an overview of how all the different objects introduced so far come into play and relate to each other. Keller's duality is a homological duality inducing a dg-derived equivalence.

In Figure 2, we can see that, starting from $\mathbf{c}_p$ and the pair $(\mathbf{t}_p, \mathbf{t}_p^{-1})$, we can either apply the iterative construction, then Keller's duality and take homology to obtain the

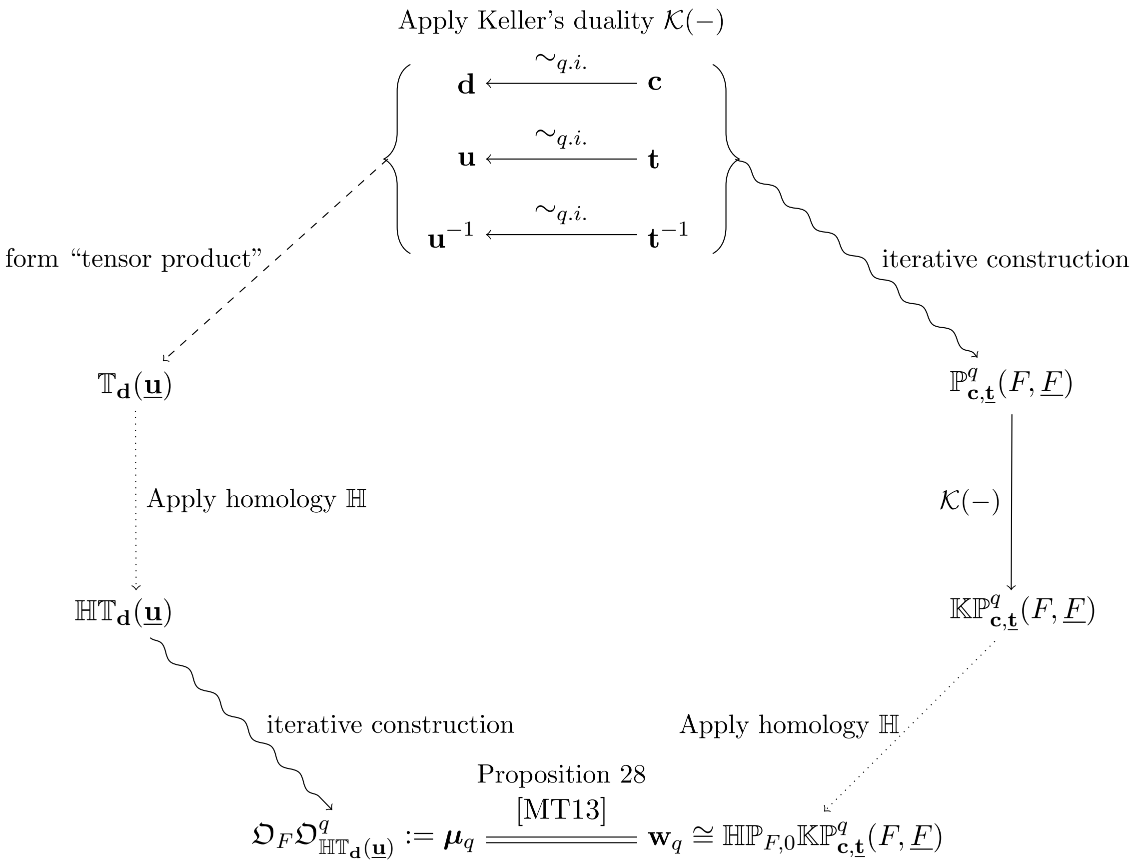


Figure 2: Overview of the constructions in [MT13].

extension algebra $\mathbf{w}_q$ we are interested in, or we can first apply Keller's duality, then take homology and finally apply another iterative construction to obtain $\boldsymbol{\mu}_q$, which is isomorphic as algebras to $\mathbf{w}_q$ by Proposition 28 in [MT13]. The fact that we can compute homology once and for all and then do the iterative construction is the crucial contribution from Miemietz and Turner.

Project and results

Our project aims at computing the A_∞-algebra structure on the Yoneda extension algebra of Weyl modules of the principal block of rational representations of $G = GL_2(\overline{\mathbb{F}_p})$; it appears as a subalgebra of a tensor product of the form $\mathbf{d} \otimes \mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})^{q-1}$.

First, we give a description of the quiver of $\mathbf{w}_q$ for any p. Since the previous iteration $\mathbf{w}_{q-1}$ appears as a subalgebra of $\mathbf{w}_q$ in the form $e_{s_1} \otimes \mathbf{w}_{q-1}$ for $1 \leq s_1 \leq p$, where e_s is an idempotent of $\mathbf{d}$, we only give the arrows for which the first constituent of the basis element is not an idempotent of $\mathbf{d}$. We call them the *new* arrows. Let $p = 2$. We have:

Theorem (Theorem 3.3.10)**.** *The new arrows for the quiver of* $\mathbf{w}_q$ *are of the form*

- $\xi \otimes (e_{s_2} \otimes e^*_{p+1-s_2}) \otimes \ldots \otimes (e_{s_q} \otimes e^*_{p+1-s_q})$;
- $x \otimes (e_2 \otimes e_1) \otimes \ldots \otimes (e_2 \otimes e_1)$;
- $x \otimes (e_2 \otimes e_1) \otimes \ldots \otimes (e_2 \otimes e_1) \otimes (\xi \otimes e_1) \otimes e_{l_1} \otimes e_{l_2} \otimes \ldots e_{l_s}$ *if there exists* $1 \leq i \leq s$ *such that* $l_i = 2$ $(s > 1)$;
- $x \otimes (e_2 \otimes e_1) \otimes \ldots \otimes (e_2 \otimes e_1) \otimes (e_2 \otimes \xi) \otimes e_{l_1} \otimes e_{l_2} \otimes \ldots e_{l_r}$ *if there exists* $1 \leq i \leq r$ *such that* $l_i = 1$ $(r > 1)$.

For $p > 2$, we have:

Theorem (Theorem 4.6.1)**.** *The new arrows for the quiver of* $\mathbf{w}_q$ *are of the form*

$$e_{s_1}\xi e_{s_1+1} \otimes \bigotimes_{l=2}^{q} \left(e_{s_l} \otimes e^*_{p+1-s_l}\right) \qquad 1 \le s_1 \le p-1;$$

$$e_{s_1}xe_{s_1+1} \otimes (e_p \otimes e_1)^{\otimes n} \otimes \left(e_{s_{n+2}}\xi \otimes \xi e_{p+1-s_{n+2}}\right) \otimes \bigotimes_{l=n+3}^{q} \left(e_{s_l} \otimes e^*_{p+1-s_l}\right) \qquad 1 \le s_1 \le p-1,\ 0 \le n \le q-2,\ 1 \le s_{n+2} \le p-2;$$

$$e_{s_1}xe_{s_1+1} \otimes (e_p \otimes e_1)^{\otimes q-1} \qquad 1 \le s_1 \le p-1;$$

$$e_{s_1}xe_{s_1+1} \otimes (e_p \otimes e_1)^{\otimes n} \otimes (\xi \otimes e_1) \otimes \bigotimes_{l=3+n}^{q} e_{s_l} \qquad 1 \le s_1 \le p-1,\ 0 \le n \le q-3,\ \exists l \ge 3+n \text{ s.t. } s_l \ne 1;$$

$$e_{s_1}xe_{s_1+1} \otimes (e_p \otimes e_1)^{\otimes n} \otimes (e_p \otimes \xi) \otimes \bigotimes_{l=3+n}^{q} e_{s_l} \qquad 1 \le s_1 \le p-1,\ 0 \le n \le q-3,\ \exists l \ge 3+n \text{ s.t. } s_l \ne p;$$

$$e_{s_1}xe_{s_1+1} \otimes (e_p \otimes e_1)^{\otimes n} \otimes (x \otimes e_1) \otimes e_{s_{3+n}}we_{s_{3+n}} \otimes \bigotimes_{l=4+n}^{q} e_{s_l} \qquad 1 \le s_1 \le p-1,\ 0 \le n \le q-3,\ s_{3+n} \ne 1, p;$$

$$e_{s_1}xe_{s_1+1} \otimes (e_p \otimes e_1)^{\otimes n} \otimes (x \otimes e_1) \otimes e_1we_1 \otimes \bigotimes_{l=4+n}^{q} e_{s_l} \qquad 1 \le s_1 \le p-1,\ 0 \le n \le q-4,\ \exists l \ge 4+n \text{ s.t. } s_l \ne 1;$$

$$e_{s_1}xe_{s_1+1} \otimes (e_p \otimes e_1)^{\otimes n} \otimes (x \otimes e_1) \otimes e_pwe_p \otimes \bigotimes_{l=4+n}^{q} e_{s_l} \qquad 1 \le s_1 \le p-1,\ 0 \le n \le q-4,\ \exists l \ge 4+n \text{ s.t. } s_l \ne p;$$

$$e_{s_1}xe_{s_1+1} \otimes (e_p \otimes e_1)^{\otimes n} \otimes (e_p \otimes x) \otimes e_{s_{3+n}}we_{s_{3+n}} \otimes \bigotimes_{l=4+n}^{q} e_{s_l} \qquad 1 \le s_1 \le p-1,\ 0 \le n \le q-3,\ s_{3+n} \ne 1, p;$$

$$e_{s_1}xe_{s_1+1} \otimes (e_p \otimes e_1)^{\otimes n} \otimes (e_p \otimes x) \otimes e_1we_1 \otimes \bigotimes_{l=4+n}^{q} e_{s_l} \qquad 1 \le s_1 \le p-1,\ 0 \le n \le q-4,\ \exists l \ge 4+n \text{ s.t. } s_l \ne 1;$$

$$e_{s_1}xe_{s_1+1} \otimes (e_p \otimes e_1)^{\otimes n} \otimes (e_p \otimes x) \otimes e_pwe_p \otimes \bigotimes_{l=4+n}^{q} e_{s_l} \qquad 1 \le s_1 \le p-1,\ 0 \le n \le q-4,\ \exists l \ge 4+n \text{ s.t. } s_l \ne p;$$

$$e_{s_1}x\xi e_{s_1+2} \otimes \bigotimes_{l=2}^{q} e_{s_l} \qquad 1 \le s_1 \le p-2,\ (s_2, \dots, s_q) \notin S;$$

$$e_{s_1}x^2e_{s_1+2} \otimes e_{s_2}we_{s_2} \otimes \bigotimes_{l=3}^{q} e_{s_l} \qquad 1 \le s_1 \le p-2,\ 2 \le s_2 \le p-1;$$

$$e_{s_1}x^2e_{s_1+2} \otimes e_1we_1 \otimes \bigotimes_{l=3}^{q} e_{s_l} \qquad 1 \le s_1 \le p-2,\ \exists l \ge 3 \text{ s.t. } s_l \ne 1;$$

$$e_{s_1}x^2e_{s_1+2} \otimes e_pwe_p \otimes \bigotimes_{l=3}^{q} e_{s_l} \qquad 1 \le s_1 \le p-2,\ \exists l \ge 3 \text{ s.t. } s_l \ne p;$$

where $1 \leq s_l \leq p$ *(unless otherwise stated) and* $S := \{(1, \ldots, 1), (p, \ldots, p)\}$.

Second, we exhibit a peculiar A_∞-structure on $\mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})$ (note that we get rid of the positive part $\mathbf{u}$ of $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})$):

Proposition (Proposition 6.2.6). *Let* $p = 2$. *Under the assumption that for all* $2 < r < n$, *the higher multiplication* $m_r : \mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})^{\otimes r} \to \mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})$ *can be chosen to be identically zero, there is a map* $f_n : \mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})^{\otimes n} \to \mathbb{T}_{\mathbf{d}}(\mathbf{u}^{-1})$ *defined by:*

$$
\begin{aligned}
&f_n\left(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1} \otimes \ldots \otimes e_2^{\epsilon_n}\xi^{m_n}x^{l_n}e_1^{\eta_n}\right) := \\
&\sum_{i_1=1}^{m_n}\sum_{i_2=1}^{l_{n-1}}\cdots\sum_{i_{2(n-j)-1}=1}^{m_{j+1}+i_{2(n-(j+1))-1}-1}\;\sum_{i_{2(n-j)}=1}^{l_j}\;\sum_{i_{2(n-(j-1))-1}=1}^{m_j+i_{2(n-j)-1}-1}\;\sum_{i_{2(n-(j-1))}=1}^{l_{j-1}}\cdots\sum_{i_{2n-3}=1}^{m_2+i_{2n-5}-1}\;\sum_{i_{2n-2}=1}^{l_1} \\
&(-1)^{\sum_{k=1}^{n-1}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}}e_2^{\epsilon_1}\xi^{m_1+i_{2n-3}-1}x^{l_1-i_{2n-2}}e_1e_2x^{i_{2n-2}-1}\xi^{m_2+i_{2n-5}-1-i_{2n-3}}\ldots \\
&\qquad\ldots x^{l_2-i_{2n-4}}e_1e_2x^{i_{2n-4}-1}\xi^{m_3+i_{2n-7}-1-i_{2n-5}}\ldots \\
&\qquad\ldots \\
&\qquad\ldots x^{l_j-i_{2(n-j)}}e_1e_2x^{i_{2(n-j)}-1}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}\ldots \\
&\qquad\ldots \\
&\qquad\ldots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n},
\end{aligned}
$$

if $\eta_i \neq \epsilon_{i+1}$ *for all* $1 \leq i \leq n-1$, *and zero otherwise. Moreover, it is a graded map of k-degree* $1-n$ *such that* $-m_1 f_n = \Phi_n$. *In particular, the higher multiplication* $m_n : \mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})^{\otimes n} \to \mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})$ *can be chosen as identically zero.*

The peculiarity of that A_∞-structure is that the components of the A_∞-morphism are non-identically zero for any n, even though we can choose all the higher multiplication maps ($n \geq 3$) to be zero. That result leads us to finding a particular subalgebra of $\mathbf{w}_q$.

Proposition (Proposition 6.3.1). *Let* $p = 2$. *There is a large subalgebra of* $\mathbf{w}_q$ *which is formal. We denote it by* $\boldsymbol{\omega}_q$.

However, we know from some examples that there is a non-trivial A_∞-structure on $\mathbf{w}_q$, therefore that A_∞-structure must "come from $\mathbf{u}$", in a sense that needs to be made precise. The combinatorics to obtain Proposition 6.2.6 were not easy to grasp and we doubt that the same approach could be used to reach the same result for greater p - even trying to compute f_3 in the case $p = 3$ is difficult.

Finally, we give a description of that formal subalgebra.

Proposition (Proposition 6.3.4). *Let* $p = 2$. *Let* $v = v_1 \otimes \ldots \otimes v_q$ *be an irreducible monomial of the formal subalgebra* $\boldsymbol{\omega}_q$. *Then it is of the form*

- $e_{s_1} \otimes \ldots \otimes e_{s_q}$;
- $e_{s_1} \otimes \ldots \otimes e_{s_{q-1}} \otimes \xi$;
- $e_{s_1} \otimes \ldots \otimes e_{s_n} \otimes x \otimes (e_2 \otimes e_1) \otimes \ldots \otimes (e_2 \otimes e_1)$;
- $e_{s_1} \otimes \ldots \otimes e_{s_n} \otimes x \otimes (e_2 \otimes e_1) \otimes \ldots \otimes (e_2 \otimes e_1) \otimes (\xi \otimes e_1) \otimes e_{l_1} \otimes e_{l_2} \otimes \ldots e_{l_s}$, $s > 1$;
- $e_{s_1} \otimes \ldots \otimes e_{s_n} \otimes x \otimes (e_2 \otimes e_1) \otimes \ldots \otimes (e_2 \otimes e_1) \otimes (e_2 \otimes \xi) \otimes e_{l_1} \otimes e_{l_2} \otimes \ldots e_{l_r}$, $r > 1$.

Proposition (Proposition 6.3.7). *Let* $p = 2$. *The algebra* $\boldsymbol{\omega}_q$ *is not d-Koszul for any* $d \geq 2$.

Description of the Chapters

We now wish to give the reader an overview of the organisation of the present document. In Chapter 1, we recall important theories underlying the work of Miemietz and Turner in [MT13] such as quasi-hereditary algebras, tilting theory and Ringel duality. We introduce some key objects as examples. In Chapter 2, we give a more detailed account of [MT13] to introduce the setup and notation for the project. In Chapter 3, we give the quiver of the algebra $\mathbf{w}_q$ for $p = 2$ and we do the same for $p > 2$ in Chapter 4. The methods used for $p > 2$ may work for $p = 2$, but the treatment of case $p = 2$ exhibits some nice properties of the combinatorics underlying the iterative construction of $\mathbf{w}_q$. In Chapter 5, we recall the necessary definitions and concepts for A_∞-algebras and we extend some well-known results to a multi-graded setting. Finally, we show in Chapter 6 the existence of a formal subalgebra of $\mathbf{w}_q$ in case $p = 2$ and we characterize it. An Appendix completes the chapters by providing one lengthy computation needed for a proof.

Chapter 1

Quasi-hereditary algebras and related theories

1.1 Quasi-hereditary algebras

Since the algebra $\mathbf{c}_p$ mentioned in the introduction is quasi-hereditary, and that we consider the extension algebra of some standard modules (the Weyl modules), it appears important to introduce those concepts and theories. This part relies on [DR92], and missing proofs were added whenever possible.

1.1.1 First definitions

Let F be a field, and A be a finite-dimensional F-algebra. We denote by A-mod the category of all (finite-dimensional left) A-modules. If Θ is a class of A-modules (closed under isomorphisms), $\mathcal{F}(\Theta)$ denotes the class of all A-modules M which have a Θ-filtration, i.e. a filtration $M = M_0 \supset M_1 \supset \ldots \supset M_{m-1} \supset M_m = 0$ such that all factors M_{t-1}/M_t, $1 \leq t \leq m$, belong to Θ.

Let $E(\lambda)$, $\lambda \in \Lambda$, be the simple A-modules (one from each isomorphism class) and we assume that the index set Λ is endowed with a partial ordering.

If $M \in A$-mod, we denote the Jordan-Hölder multiplicity of $E(\lambda)$ in M by $[M : E(\lambda)]$.

For each $\lambda \in \Lambda$, let $P(\lambda)$ be the projective cover of $E(\lambda)$ and $Q(\lambda)$ be the injective hull of $E(\lambda)$.

Definition 1.1.1. Denote by $\Delta(\lambda)$ (or $\Delta_A(\lambda)$, $\Delta_\Lambda(\lambda)$) the maximal factor module of $P(\lambda)$ with composition factors of the form $E(\mu)$ where $\mu \leq \lambda$; these modules $\Delta(\lambda)$ are called *standard* modules, and we obtain the set of standard modules $\Delta := \{\Delta(\lambda) | \lambda \in \Lambda\}$.

Similarly,

Definition 1.1.2. Denote by $\nabla(\lambda)$ (or $\nabla_A(\lambda)$, $\nabla_\Lambda(\lambda)$) the maximal submodule of $Q(\lambda)$ with composition factors of the form $E(\mu)$ where $\mu \leq \lambda$; these modules $\nabla(\lambda)$ are called *costandard* modules, and we obtain the set of costandard modules $\nabla := \{\nabla(\lambda) | \lambda \in \Lambda\}$.

Let us point out that $\nabla(\lambda)$ is the dual of a corresponding standard module. Let $D := \mathrm{Hom}_F(-, F)$ be the duality with respect to the base field F. Let A^o be the opposite algebra of A. We have:

$$D : A - \mathrm{mod} \to \mathrm{mod} - A \cong A^o - \mathrm{mod},$$

so we choose simple A^o-modules $E_{A^o}(\lambda)$ to be the image of the simple A-modules $E_A(\lambda)$ (note the use of the same index).

Lemma 1.1.3. *We have:* $\nabla_A(\lambda) = D\Delta_{A^o}(\lambda)$.

In particular, any statement on standard modules yields a corresponding statement for costandard modules.

Proof. This choice of simple A^o-modules is made so that the A-action on the left coincides with that on the left of $E_A(\lambda)$. Let us compute $\nabla_A(\lambda)$. By definition, it fits in a short exact sequence of the form:

$$0 \to \nabla_A(\lambda) \to Q_A(\lambda) \to C \to 0,$$

where C is the cokernel of the natural injection map. Applying the exact functor D to this short exact sequence, we get:

$$0 \to D(C) \to D(Q_A(\lambda)) \to D(\nabla_A(\lambda)) \to 0,$$

and $D(Q_A(\lambda)) = P_{A^o}(\lambda)$ since D is an exact contravariant functor. The A-module $\nabla_A(\lambda)$ satisfies that its composition factors $E_A(\mu)$ are such that $\mu \leq \lambda$ and it is the maximal submodule of $Q_A(\lambda)$ with this property. Therefore, $D(\nabla_A(\lambda))$ satisfies the same properties about its composition factors $D(E_A(\mu)) = E_{A^o}(\mu)$ and it is the maximal quotient of $P_{A^o}(\lambda)$ with this property. By definition, $D(\nabla_A(\lambda)) = \Delta_{A^o}(\lambda)$. Since we consider finitely generated modules over a finite dimensional F-algebra, these are in particular finite-dimensional, hence applying D twice amounts to applying the identity functor, so that we finally have:

$$\nabla_A(\lambda) = D\Delta_{A^o}(\lambda).$$

□

Note that the only module M with $\mathrm{Hom}(\Delta(\lambda), M) = 0$ for all $\lambda \in \Lambda$ is the zero module: for $M \neq 0$, let $E(\lambda)$ be a submodule, then $\mathrm{Hom}(\Delta(\lambda), M) \neq 0$. Dually, the only module M with $\mathrm{Hom}(M, \nabla(\lambda)) = 0$ for all $\lambda \in \Lambda$ is the zero module.

Given a set χ of A-modules, then for any A-module M, we denote by $\eta_\chi M$ the *trace* of χ in M; it is the maximal submodule of M generated by χ.

The standard modules may be characterized as follows:

Lemma 1.1.4. *For any A-module M, and $\lambda \in \Lambda$ the following assertions are equivalent:*

(i) $M \cong \Delta(\lambda)$;

(ii) $\mathrm{top}\, M \cong E(\lambda)$, *all composition factors of M are of the form $E(\mu)$, with $\mu \leq \lambda$, and* $\mathrm{Ext}^1(M, E(\mu)) = 0$ *for all $\mu \leq \lambda$;*

(iii) $M \cong P(\lambda)/\eta_{\{P(\mu)|\mu \not\leq \lambda\}} P(\lambda)$.

Proof.

(i) ⇔ (iii) Let $M \cong P(\lambda)/\eta_{\{P(\mu)|\mu \not\leq \lambda\}} P(\lambda)$, and consider one of its composition factors $E(\mu)$. Then the composition $P(\mu) \to P(\lambda) \twoheadrightarrow M$ is non-zero, so that $\mu \leq \lambda$. Now let $M \cong \Delta(\lambda)$, and consider one of its composition factors $E(\mu)$. Since $\Delta(\lambda)$ is a quotient of $P(\lambda)$, there is a short exact sequence:

$$0 \to K(\lambda) \to P(\lambda) \to \Delta(\lambda) \to 0.$$

We want to show that $K(\lambda) = \eta_{\{P(\nu)|\nu \not\leq \lambda\}} P(\lambda)$. Let $\nu \not\leq \lambda$. Then the composition $P(\nu) \to P(\lambda) \to \Delta(\lambda)$ is zero since $E(\nu)$ is not a composition factor of $\Delta(\lambda)$. So this map factors through $K(\lambda)$. This means $\eta_{\{P(\nu)|\nu \not\leq \lambda\}} P(\lambda)$ is a submodule of $K(\lambda)$, which is equivalent to saying that $P(\lambda)/\eta_{\{P(\nu)|\nu \not\leq \lambda\}} P(\lambda)$ surjects onto $\Delta(\lambda)$. By the first part of the proof, we know that all composition factors $E(\mu)$ of $P(\lambda)/\eta_{\{P(\nu)|\nu \not\leq \lambda\}} P(\lambda)$ satisfy $\mu \leq \lambda$. By maximality of $\Delta(\lambda)$, we obtain $\Delta(\lambda) \cong P(\lambda)/\eta_{\{P(\nu)|\nu \not\leq \lambda\}} P(\lambda)$.

(i),(iii) $\Rightarrow$ (ii) Assume $M \cong \Delta(\lambda)$. From the definition of $\Delta(\lambda)$, we know there is an epimorphism $P(\lambda) \twoheadrightarrow \Delta(\lambda)$. This yields an epimorphism $\operatorname{top} P(\lambda) \twoheadrightarrow \operatorname{top} \Delta(\lambda)$, i.e. $E(\lambda) \twoheadrightarrow \operatorname{top} \Delta(\lambda)$, so $\operatorname{top} \Delta(\lambda)$ is either 0 or $E(\lambda)$. Since $\Delta(\lambda) \cong P(\lambda)/\eta_{\{P(\mu)|\mu \nleq \lambda\}} P(\lambda)$, $\operatorname{top} \Delta(\lambda) \cong E(\lambda)$. The fact that all composition factors of M are of the form $E(\mu)$ with $\mu \leq \lambda$ comes from the definition of $\Delta(\lambda)$. We have the following short exact sequence:

$$0 \to K(\lambda) \to P(\lambda) \to \Delta(\lambda) \to 0,$$

where $K(\lambda) = \eta_{\{P(\mu)|\mu \nleq \lambda\}} P(\lambda)$, and since $\operatorname{Ext}(-, E(\mu))$ ($\mu \in \Lambda$) is a cohomological functor, we get the following long exact sequence:

$$0 \to \operatorname{Hom}(\Delta(\lambda), E(\mu)) \to \operatorname{Hom}(P(\lambda), E(\mu)) \to \operatorname{Hom}(K(\lambda), E(\mu)) \to \operatorname{Ext}^1(\Delta(\lambda), E(\mu)) \to \operatorname{Ext}^1(P(\lambda), E(\mu)) \longrightarrow \dots$$

Since $P(\lambda)$ is projective, $\operatorname{Ext}^1(P(\lambda), E(\mu)) = 0$. We have two cases to consider; assume first that $\mu = \lambda$. Then, both $\operatorname{Hom}(\Delta(\lambda), E(\mu))$ and $\operatorname{Hom}(P(\lambda), E(\mu))$ are isomorphic to F, so that $\operatorname{Hom}(K(\lambda), E(\mu))$ is isomorphic to $\operatorname{Ext}^1(\Delta(\lambda), E(\mu))$. Assume now that $\mu < \lambda$. Then $\operatorname{Hom}(P(\lambda), E(\mu)) = 0$ since a homomorphism from $P(\lambda)$ is only determined by the image of its top $E(\lambda)$, hence must be zero according to Schur's Lemma. This means that $\operatorname{Ext}^1(\Delta(\lambda), E(\mu))$ and $\operatorname{Hom}(K(\lambda), E(\mu))$ are isomorphic. Hence, in both cases, we have that $\operatorname{Ext}^1(\Delta(\lambda), E(\mu))$ and $\operatorname{Hom}(K(\lambda), E(\mu))$ are isomorphic. Recall that $K(\lambda) = \eta_{\{P(\mu)|\mu \nleq \lambda\}} P(\lambda)$ is generated by the images in $P(\lambda)$ of all maps $P(\epsilon) \to P(\lambda)$ for $\epsilon \nleq \lambda$ if $E(\epsilon)$ is a composition factor of $P(\lambda)$. So $E(\mu)$ cannot be a composition factor of $\operatorname{top} K(\lambda)$, hence $\operatorname{Hom}(K(\lambda), E(\mu))$ must be zero.

(ii) $\Rightarrow$ (i),(iii) Let M be an A-module such that $\operatorname{top} M \cong E(\lambda)$, all composition factors of M are of the form $E(\mu)$, with $\mu \leq \lambda$, and $\operatorname{Ext}^1(M, E(\mu)) = 0$ for all $\mu \leq \lambda$. Since $E(\lambda)$ is a composition factor of M, there is a map $P(\lambda) \to M$, which is necessarily an epimorphism. So M is isomorphic to a quotient of $P(\lambda)$, say $P(\lambda)/K(\lambda)$. $K(\lambda)$ must contain all composition factors $E(\mu)$ of $P(\lambda)$ such that $\mu \nleq \lambda$, so $K(\lambda) \supset \eta_{\{P(\mu)|\mu \nleq \lambda\}} P(\lambda)$. As a consequence, $\Delta(\lambda) \twoheadrightarrow M$. We then have the following short exact sequence:

$$0 \to K' \to \Delta(\lambda) \xrightarrow{\pi} M \to 0.$$

Let $\mu \leq \lambda$. Apply $\operatorname{Ext}(-, E(\mu))$ to this short exact sequence to obtain a long exact sequence:

$$0 \to \operatorname{Hom}(M, E(\mu)) \to \operatorname{Hom}(\Delta(\lambda), E(\mu)) \to \operatorname{Hom}(K', E(\mu)) \to \operatorname{Ext}^1(M, E(\mu)) \to \operatorname{Ext}^1(\Delta(\lambda), E(\mu)) \longrightarrow \dots$$

which yields the following short exact sequence since $\operatorname{Ext}^1(M, E(\mu)) = 0$ by hypothesis:

$$0 \to \operatorname{Hom}(M, E(\mu)) \to \operatorname{Hom}(\Delta(\lambda), E(\mu)) \to \operatorname{Hom}(K', E(\mu)) \to 0.$$

We know that $\operatorname{top} \Delta(\lambda) \cong E(\lambda) \cong \operatorname{top} M$, so that

$$\operatorname{Hom}(\Delta(\lambda), E(\mu)) \cong \operatorname{Hom}(M, E(\mu)) = \begin{cases} 0 & \text{if } \mu \neq \lambda \\ F & \text{if } \mu = \lambda \end{cases}.$$

In either case, we obtain from the previous short exact sequence that for all $\mu \leq \lambda$, $\mathrm{Hom}(K', E(\mu)) = 0$. Since K' is a submodule of $\Delta(\lambda)$ and by definition of $\Delta(\lambda)$, we also have $\mathrm{Hom}(K', E(\mu)) = 0$ for all $\mu > \lambda$. Therefore $\mathrm{Hom}(K', E(\mu)) = 0$ for all $\mu \in \Lambda$, namely $K' = 0$.

□

Lemma 1.1.5. *Let M be an A-module, and $\lambda, \mu \in \Lambda$. Then*

(a) $\mathrm{Hom}(\Delta(\lambda), M) \neq 0 \Rightarrow [M : E(\lambda)] \neq 0$;

(b) $\mathrm{Hom}(\Delta(\lambda), \Delta(\mu)) \neq 0 \Rightarrow \lambda \leq \mu$;

(c) $\mathrm{Hom}(\Delta(\lambda), \nabla(\mu)) \neq 0 \Rightarrow \lambda = \mu$.

Proof. (a) Suppose $[M : E(\lambda)] = 0$, then $\mathrm{Hom}(P(\lambda), M) = 0$. Let $f : \Delta(\lambda) \to M$. Then the composition $P(\lambda) \overset{p}{\twoheadrightarrow} \Delta(\lambda) \overset{f}{\to} M$ is zero, which yields $f = 0$, and as a result $\mathrm{Hom}(\Delta(\lambda), M) = 0$.

(b) Applying *(a)*, we get that $[\Delta(\mu) : E(\lambda)] \neq 0$, which means $E(\lambda)$ is a composition factor of $\Delta(\mu)$, so that by definition, $\lambda \leq \mu$.

(c) Applying *(a)* again, $\mathrm{Hom}(\Delta(\lambda), \nabla(\mu)) \neq 0$ gives $[\nabla(\mu) : E(\lambda)] \neq 0$, and so $\lambda \leq \mu$. Applying the dual statement of *(a)*, namely $\mathrm{Hom}(M, \nabla(\lambda)) \neq 0 \Rightarrow [M : E(\lambda)] \neq 0$, we obtain that $\mu \leq \lambda$.

□

The sets Δ and ∇ depend in an essential way on the given partial ordering Λ. This gives rise to the notion of equivalence of two posets Λ, Λ' used as index sets for the simple module: Λ and Λ' are equivalent if the sets of standard modules indexed by Λ, resp. Λ' coincide, and the sets of costandard modules indexed by Λ, resp. Λ' coincide.

In general, the standard and costandard modules will change when refining the ordering. We then need to consider *adapted* orderings in order to avoid this situation.

Definition 1.1.6. A partial ordering Λ of the sets of simple A-modules $\{E(\lambda) | \lambda \in \Lambda\}$ is said to be *adapted* provided that the following condition holds:

For every A-module M with $\mathrm{top}\, M \cong E(\lambda_1)$ and $\mathrm{soc}\, M \cong E(\lambda_2)$, with $\lambda_1, \lambda_2 \in \Lambda$ incomparable, there is some $\mu > \lambda_1, \lambda_2$ such that $[M : E(\mu)] \neq 0$.

Remark 1.1.7. This definition is just an ad-hoc definition in the sense that we choose μ to be greater than both λ_1 and λ_2, when we could possibly choose it to be smaller than both of them. It just makes more sense like this in the view of A being quasi-hereditary (i.e. filtered by standard modules). Besides, suppose we could find a μ such that $\mu < \lambda_1$ and $\lambda_2 < \mu$, then combining both yields $\lambda_2 < \lambda_1$, which is a contradiction since they are incomparable by assumption.

For Λ adapted, we may always assume that we deal with a total ordering; in such a case, we may replace Λ by the equivalent index set $\{1, \ldots, n\}$ with its natural ordering.

Then, Lemma 1.1.4 rewrites:

Lemma 1.1.8. *For any A-module M, and $\lambda \in \Lambda$, where Λ is adapted, the following assertions are equivalent:*

(i) $M \cong \Delta(\lambda)$;

(ii) $\mathrm{top}\, M \cong E(\lambda)$, *all composition factors of M are of the form $E(\mu)$, with $\mu \leq \lambda$, and* $\mathrm{Ext}^1(M, E(\mu)) \neq 0$ *implies $\mu > \lambda$;*

(iii) $M \cong P(\lambda)/\eta_{\{P(\mu)|\mu>\lambda\}}P(\lambda)$.

As a consequence, we see:

Lemma 1.1.9. *Assume Λ is adapted. Let M be an A-module, and $\lambda, \mu \in \Lambda$. Then*

(a) $\mathrm{Ext}^1(\Delta(\lambda), M) \neq 0 \Rightarrow [M : E(\mu)] \neq 0$ *for some* $\mu > \lambda$*;*

(b) $\mathrm{Ext}^1(\Delta(\lambda), \Delta(\mu)) \neq 0 \Rightarrow \lambda < \mu$*;*

(c) $\mathrm{Ext}^1(\Delta(\lambda), \nabla(\mu)) = 0$.

Proof. See [DR92][Lemma 1.3]. □

1.1.2 Schurian modules

Definition 1.1.10. A module is called *Schurian* if its endormorphism ring is a division ring.

Example 1.1.11. By Schur's Lemma, simple modules are Schurian.

Lemma 1.1.12. *The following statements are equivalent, for any $\lambda \in \Lambda$:*

(i) $\Delta(\lambda)$ *is a Schurian module;*

(ii) $[\Delta(\lambda) : E(\lambda)] = 1$*;*

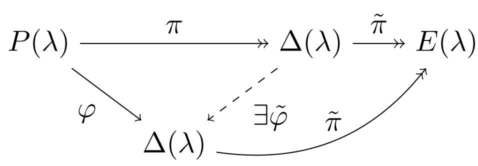
(iii) If M is an A-module with top and socle isomorphic to $E(\lambda)$, and $[M : E(\mu)] \neq 0$ only for $\mu \leq \lambda$, then $M \cong E(\lambda)$;

*(ii)** $[\nabla(\lambda) : E(\lambda)] = 1$*;*

*(i)** $\nabla(\lambda)$ *is a Schurian module.*

Proof.

(i) $\Rightarrow$ (ii) Suppose $\Delta(\lambda)$ is a Schurian module, namely $\mathrm{End}_{A\text{-mod}}(\Delta(\lambda))$ is a division ring and suppose by contradiction that $[\Delta(\lambda) : E(\lambda)] > 1$ (it cannot be zero since $\mathrm{top}\,\Delta(\lambda) = E(\lambda)$). This means there is at least another composition factor of the form $E(\lambda)$ in $\Delta(\lambda)$. We then have the following diagram:



where the first line is the usual sequence of epimorphisms, and $\varphi : P(\lambda) \to \Delta(\lambda)$ is the map obtained from sending the top of $P(\lambda)$ to the other composition factor $E(\lambda)$ of $\Delta(\lambda)$. Since the kernel of π only contains composition factors $E(\mu)$ with $\mu > \lambda$, $\varphi(\ker \pi) = 0$, hence φ factors through π: there exists $\tilde{\varphi} : \Delta(\lambda) \to \Delta(\lambda)$ such that $\tilde{\varphi} \circ \pi = \varphi$. Composing $\tilde{\varphi}$ with $\tilde{\pi}$ yields the zero map since $\tilde{\pi} \circ \varphi$ is zero. This amounts to saying that $\tilde{\varphi}$ is not invertible, which is a contradiction.

(ii) $\Rightarrow$ (iii) Suppose $[\Delta(\lambda) : E(\lambda)] = 1$ and let M be an A-module with top and socle isomorphic to $E(\lambda)$, and such that $[M : E(\mu)] \neq 0$ only for $\mu \leq \lambda$. Since $M \twoheadrightarrow E(\lambda)$, we have $P(\lambda) \twoheadrightarrow M$. This means that M is isomorphic to a quotient of $P(\lambda)$ satisfying $\mathrm{top}\, M \cong E(\lambda)$ and $[M : E(\mu)] \neq 0$ only for $\mu \leq \lambda$, but by definition, $\Delta(\lambda)$ is the largest such quotient of $P(\lambda)$. As a consequence, there is an epimorphism $\Delta(\lambda) \twoheadrightarrow M$. Hence, $[M : E(\lambda)]$ is necessarily 1, and as $\mathrm{soc}\, M \cong E(\lambda)$, this shows that $M \cong E(\lambda)$.

(iii) ⇒ (i) Suppose by contradiction that $\dim_F \operatorname{End}_{A\text{-mod}}(\Delta(\lambda)) > 1$. $\Delta(\lambda)$ has simple top $E(\lambda)$, so our hypothesis yields that there is another composition factor $E(\lambda)$ of $\Delta(\lambda)$. We have the following composition:

$$\operatorname{rad} P(\lambda) \xrightarrow{r} P(\lambda) \xrightarrow{\varphi} \Delta(\lambda),$$

where r is the inclusion, and φ is the map from $P(\lambda)$ to $\Delta(\lambda)$ obtained by sending the top of $P(\lambda)$ to the composition factor $E(\lambda)$ of $\Delta(\lambda)$ which is not in the top of $\Delta(\lambda)$. In particular, $\operatorname{Im}\varphi$ is a submodule of $\Delta(\lambda)$ such that its top is $E(\lambda)$ and $\operatorname{Im}\varphi \circ r$ is a submodule of $\operatorname{Im}\varphi$ which corresponds to $\operatorname{rad}\operatorname{Im}\varphi$. We set $M := \Delta(\lambda)/\operatorname{Im}\varphi \circ r$. Note that all composition factors $E(\mu)$ of M satisfy $\mu \leq \lambda$, and that $\operatorname{top} M \cong E(\lambda)$. In addition, $\operatorname{soc} M \cong E(\lambda)$. According to *(iii)*, $M \cong E(\lambda)$, which is a contradiction.

□

1.1.3 Definition of a quasi-hereditary algebra

It is now possible to formulate the definition of a quasi-hereditary algebra.

Theorem 1.1.13. *Assume that Λ is adapted, and that all standard modules are Schurian. Then the following conditions are equivalent:*

(i) $\mathcal{F}(\Delta)$ *contains* ${}_AA$*;*

(ii) $\mathcal{F}(\Delta) = \{X \mid \operatorname{Ext}^1(X, \nabla) = 0\}$*;*

(iii) $\mathcal{F}(\Delta) = \{X \mid \operatorname{Ext}^i(X, \nabla) = 0$ *for all* $i \geq 1\}$*;*

(iv) $\operatorname{Ext}^2(\Delta, \nabla) = 0$*;*

*(iii)** $\mathcal{F}(\nabla) = \{Y \mid \operatorname{Ext}^i(\Delta, Y) = 0$ *for all* $i \geq 1\}$*;*

*(ii)** $\mathcal{F}(\nabla) = \{Y \mid \operatorname{Ext}^1(\Delta, Y) = 0\}$*;*

*(i)** $\mathcal{F}(\nabla)$ *contains* $D(A_A)$*.*

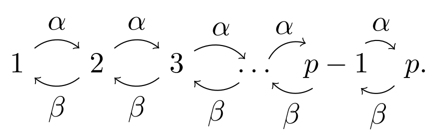
Proof. See [DR92][Theorem 1]. □

We can now give the definition of a quasi-hereditary algebra:

Definition 1.1.14. An algebra A with an adapted partial ordering Λ, whose standard modules are Schurian and such that the equivalent conditions of Theorem 1.1.13 are satisfied is said to be *quasi-hereditary*.

1.1.4 Example: the algebra $\mathbf{c}_p$

Before going any further, let us consider $\mathbf{c}_p$ again, which we will describe completely. Consider the following quiver Q:



Let F be a field. We will consider the quotient of the path algebra over the quiver Q modulo the relations $(\alpha^2, \beta^2, \alpha\beta + \beta\alpha, \alpha\beta e_p)$:

$$\mathbf{c}_p := FQ/(\alpha^2, \beta^2, \alpha\beta + \beta\alpha, \alpha\beta e_p),$$

where e_i is the constant path at the vertex indexed by i. Those constant paths form a family of orthogonal primitive idempotents, and $\mathbf{c}_p$ admits the unit $1 := \sum_{\lambda=1}^{p} e_\lambda$,.

Considering $\mathbf{c}_p$ as a left module over itself, we see that it decomposes as

$${}_{\mathbf{c}_p}\mathbf{c}_p = \mathbf{c}_p e_1 \oplus \mathbf{c}_p e_2 \oplus \ldots \oplus \mathbf{c}_p e_p,$$

and $\mathbf{c}_p e_\lambda$ consists in all paths in Q starting at λ. Being direct summands of a free module, the $\mathbf{c}_p e_\lambda$'s are projective $\mathbf{c}_p$-modules, which in addition are indecomposable as the e_λ's are primitive. We can display a basis for $\mathbf{c}_p$, with respect to the decomposition into projectives:

$${}_{\mathbf{c}_p}\mathbf{c}_p = \begin{array}{c} e_1 \\ e_2\alpha e_1 \\ e_1\beta\alpha e_1 \end{array} \oplus \begin{array}{c} e_2 \\ e_1\beta e_2 \quad e_3\alpha e_2 \\ e_2\alpha\beta e_2 \end{array} \oplus \ldots \oplus \begin{array}{c} e_{p-1} \\ e_{p-2}\beta e_{p-1} \quad e_p\alpha e_{p-1} \\ e_{p-1}\alpha\beta e_{p-1} \end{array} \oplus \begin{array}{c} e_p \\ e_{p-1}\beta e_p \\ \end{array}. \tag{1.1}$$

We set $\Lambda = \{1, 2, \ldots, p-1, p\}$ with its natural order. We label the simple modules as $E(\lambda) =< e_\lambda >$, for all $\lambda \in \Lambda$ and order them using the order on Λ. We will then very often simplify notations (1.1) as follows:

$${}_{\mathbf{c}_p}\mathbf{c}_p = \begin{array}{c} 1 \\ 2 \\ 1 \end{array} \oplus \begin{array}{ccc} & 2 & \\ 1 & & 3 \\ & 2 & \end{array} \oplus \ldots \oplus \begin{array}{ccc} & p-1 & \\ p-2 & & p \\ & p-1 & \end{array} \oplus \begin{array}{c} p \\ p-1 \\ \end{array}.$$

As noticed earlier, the $\mathbf{c}_p e_\lambda$'s are indecomposable projective modules, and their tops are $E(\lambda)$. We can check that $P(\lambda) := \mathbf{c}_p e_\lambda$ is the projective cover of $E(\lambda)$, for $\lambda \in \Lambda$. So we have, for $2 \leq \lambda \leq p-1$:

$$P(1) = \begin{array}{c} 1 \\ 2 \\ 1 \end{array}, \ P(\lambda) = \begin{array}{ccc} & \lambda & \\ \lambda-1 & & \lambda+1 \\ & \lambda & \end{array}, \ P(p) = \begin{array}{c} p \\ p-1 \\ \end{array}.$$

According to Lemma 1.1.8, we may obtain the set Δ of the standard modules of $\mathbf{c}_p$:

$$\Delta(1) = 1, \ \Delta(\lambda) = \begin{array}{c} \lambda \\ \lambda-1 \end{array},$$

where $2 \leq \lambda \leq p$.

Using the duality introduced earlier, and considering ${\mathbf{c}_p}^* := \mathrm{Hom}_F(\mathbf{c}_p, F)$, the dual of $\mathbf{c}_p$, we know that ${\mathbf{c}_p}^*$ can be seen as a left $\mathbf{c}_p$-module: $\forall x \in \mathbf{c}_p, \forall f \in {\mathbf{c}_p}^*, x \cdot f := (m \mapsto f(mx))$ (which amounts to seeing $\mathbf{c}_p$ as a right $\mathbf{c}_p$-module, i.e. as a left ${\mathbf{c}_p}^o$-module). We get the following decomposition into injective indecomposable right $\mathbf{c}_p$-modules:

$$\begin{array}{c} (e_1\beta\alpha e_1)^* \\ (e_1\beta e_2)^* \\ e_1^* \end{array} \oplus \begin{array}{c} (e_2\alpha\beta e_2)^* \\ (e_2\beta e_3)^* \quad (e_2\alpha e_1)^* \\ e_2^* \end{array} \oplus \ldots \oplus \begin{array}{c} (e_{p-1}\alpha\beta e_{p-1})^* \\ (e_{p-1}\beta e_p)^* \quad (e_{p-1}\alpha e_{p-2})^* \\ e_{p-1}^* \end{array} \oplus \begin{array}{c} (e_p\alpha e_{p-1})^* \\ e_p^* \\ \end{array}. \tag{1.2}$$

As a consequence, we get the injective hulls of the simple modules since direct summands of the dual of $\mathbf{c}_p$ correspond to injective modules:

$$Q(1) = \begin{array}{c} 1 \\ 2 \\ 1 \end{array}, \ Q(\lambda) = \begin{array}{ccc} & \lambda & \\ \lambda+1 & & \lambda-1 \\ & \lambda & \end{array}, \ Q(p) = \begin{array}{c} p-1 \\ p \\ \end{array},$$

where $2 \leq \lambda \leq p-1$. And following the definition of a costandard module, we obtain the set ∇ of costandard modules:

$$\nabla(1) = 1,\ \nabla(\lambda) = \begin{matrix} \lambda - 1 \\ \lambda \end{matrix}\ ,$$

where $2 \leq \lambda \leq p$.

1.2 Tilting theory and Ringel duality

1.2.1 Tilting theory

Let A be a finite-dimensional F-algebra with a labelling of the simple A-modules by some adapted poset $(\Lambda, \leq)$. We defined what standard and costandard A-modules are, and A is quasi-hereditary if and only if standard modules are Schurian and ${}_AA$ is filtered by standard modules.

Tilting modules are A-modules filtered both by standard and costandard modules. We have the following theorem which makes this statement more precise:

Theorem 1.2.1 (Theorem 2, [Koe02])**.** *Let $(A, \leq)$ be a quasi-hereditary algebra with set Λ of isomorphism classes of simple modules. Then, for each $\lambda \in \Lambda$, there is a unique (up to isomorphism) indecomposable module $T(\lambda)$ which has both a filtration with subquotients of the form $\Delta(\mu)$ (for $\mu \leq \lambda$ and $\Delta(\lambda)$ itself occurring with multiplicity one) and another filtration with subquotients of the form $\nabla(\mu)$ (for $\mu \leq \lambda$ and $\nabla(\lambda)$ itself occurring with multiplicity one).*

Definition 1.2.2. We define the characteristic tilting module T of $(A, \leq)$:

$$T = \bigoplus_{\lambda \in \Lambda} T(\lambda).$$

It can be characterised as the minimal A-module T such that

$$\operatorname{add}(T) = \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla),$$

where $\operatorname{add}(M)$ is the full subcategory of A-mod consisting of direct summands of M^n for all $n \geq 1$.

This characteristic tilting module completely determines the full subcategories $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$ as shown by the next result.

Proposition 1.2.3 (Proposition 3.2, [DR92])**.** *Let T be the characteristic tilting module of a quasi-hereditary algebra A. Then*

$$\mathcal{F}(\Delta) = \left\{ X \in A\text{-}mod \,|\, \operatorname{Ext}^i(X, T) = 0 \text{ for all } i \geq 1 \right\},$$

and

$$\mathcal{F}(\nabla) = \left\{ Y \in A\text{-}mod \,|\, \operatorname{Ext}^i(T, Y) = 0 \text{ for all } i \geq 1 \right\}.$$

1.2.2 Ringel duality

We define the Ringel dual of a quasi-hereditary algebra $(A, \leq)$ as follows:

Definition 1.2.4. Let $(A, \leq)$ be a quasi-hereditary algebra. Then denote by T its characteristic tilting module. We define the Ringel dual $(A', \geq)$ of $(A, \leq)$ by

$$A' = \operatorname{End}_A({}_AT).$$

In [KKO13], Koenig, Külshammer and Ovsienko remark that Ringel duality accounts for a central symmetry in the class of quasi-hereditary algebras. Indeed, we have the following result

Theorem 1.2.5 (Theorem 6, [Rin91])**.** *Consider the functor* $F := \mathrm{Hom}_A(T, -) : A-$ *mod* $\to A' -$ *mod. Then* F *sends costandard* A*-modules to standard* A'*-modules. In particular,* A' *is quasi-hereditary with standard modules* $F\nabla(\lambda)$ *but with the order on the simples being reversed.*

Remark 1.2.6 (Theorem 7, [Rin91]). Assuming A is basic, Ringel duality is a duality in the class of quasi-hereditary algebras: $(A')' \cong A$, with the same ordering of the simple modules.

Chapter 2

More about object of study

2.1 Preliminaries

Let F be an algebraically closed field of characteristic p. Recall from the introduction the following definition. The quasi-hereditary algebra $\mathbf{c}_p$ is defined as the path algebra of the following quiver:

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \cdots \underset{\beta}{\overset{\alpha}{\rightleftarrows}} p$$

modulo the relations $(\alpha^2, \beta^2, \alpha\beta + \beta\alpha, \alpha\beta e_p)$.

We denote by $\mathbf{d}$ its extension algebra; it is the path algebra of the following quiver:

$$1 \underset{\xi}{\overset{x}{\leftleftarrows}} 2 \underset{\xi}{\overset{x}{\leftleftarrows}} \cdots \underset{\xi}{\overset{x}{\leftleftarrows}} p$$

modulo the relations $(x\xi - \xi x, \xi^2)$, where x is given djk-grading $(-1, -1, 1)$ and ξ is given djk-degree $(0, 1, 0)$.

We write $\mathrm{Hom}_{A-}(-,-)$, resp. $\mathrm{Hom}_{-A}(-,-)$, to mean morphisms of left A-modules, resp. of right A-modules.

2.2 Two important bimodules

We define the two important $\mathbf{d}$-$\mathbf{d}$-bimodules $\mathbf{u}$ and $\mathbf{u}^{-1}$ in this section. Let us introduce some notation first. Let $\mathbf{d}^0$ be the semisimple quotient of $\mathbf{d}$ modulo its radical and write it $\mathbf{d}^0 = Fe_1 \oplus Fe_2 \oplus \ldots \oplus Fe_p$. We denote by $\sigma \in \mathrm{Aut}(\mathbf{d}^0)$ the automorphism of $\mathbf{d}^0$ which maps e_l to e_{p+1-l}. Finally, for a jk-bigraded $\mathbf{d}$-$\mathbf{d}$-bimodule M, we denote by $M\langle n\rangle$ the jk-bigraded $\mathbf{d}$-$\mathbf{d}$-bimodule M shifted by n in the j-degree.

Definition 2.2.1. Let $\mathbf{u}$ and $\mathbf{u}^{-1}$ be the $\mathbf{d}$-$\mathbf{d}$-bimodules given by

$$\begin{aligned} \mathbf{u} &:= \mathbf{d}^\sigma \otimes_{\mathbf{d}^0} \mathbf{d}^*\langle 1\rangle; \\ \mathbf{u}^{-1} &:= \mathbf{d}^\sigma \otimes_{\mathbf{d}^0} \mathbf{d}\langle -1\rangle. \end{aligned}$$

Note that they inherit a jk-grading from $\mathbf{d}$ by taking the total j-grading and total k-grading and then applying the shift in the j-degree.

More generally, for $i \geq 1$, we can define

$$\mathbf{u}^{-i} := \underbrace{\mathbf{d}^{\sigma} \otimes_{\mathbf{d}^0} \ldots \otimes_{\mathbf{d}^0} \mathbf{d}^{\sigma}}_{i} \otimes_{\mathbf{d}^0} \mathbf{d}\langle -i \rangle.$$

An alternative description of $\mathbf{u}^{-i}$, using canonical isomorphisms, is as follows:

$$\mathbf{u}^{-i} = \underbrace{\mathbf{u}^{-1} \otimes_{\mathbf{d}} \ldots \otimes_{\mathbf{d}} \mathbf{u}^{-1}}_{i}.$$

Remark that if $i = 0$, we obtain $\mathbf{d}$. Similarly, for $i \geq 1$, we can define

$$\mathbf{u}^{i} = \underbrace{\mathbf{u} \otimes_{\mathbf{d}} \otimes \mathbf{u} \otimes_{\mathbf{d}} \ldots \otimes_{\mathbf{d}} \mathbf{u}}_{i},$$

and, again, if $i = 0$, $\mathbf{u}^0 = \mathbf{d}$. Thus, it makes sense to define the following sum of tensor products of $\mathbf{u}^{-1}$ and $\mathbf{u}$ over $\mathbf{d}$:

$$\mathbb{T}_{\mathbf{d}}(\underline{\mathbf{u}}) := \bigoplus_{i \in \mathbb{Z}} \mathbf{u}^{-i}.$$

However, we need some multiplication map to turn this vector space into an algebra: it is not necessarily obvious how to multiply $\mathbf{u}$ with $\mathbf{u}^{-1}$, or the other way round. We have the following:

Claim: $\mathbf{u}^{-1} \cong \mathrm{Hom}_{\mathbf{d}-}(\mathbf{u}, \mathbf{d})$ as $\mathbf{d}$-$\mathbf{d}$-bimodules.

Proof. We have:

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{d}}(\mathbf{u}, \mathbf{d}) &\overset{\text{def}}{=} \mathrm{Hom}_{\mathbf{d}}(\mathbf{d}^{\sigma} \otimes_{\mathbf{d}^0} \mathbf{d}^*\langle 1 \rangle, \mathbf{d}) \\
&= \mathrm{Hom}_{\mathbf{d}}(\mathbf{d}^{\sigma} \otimes_{\mathbf{d}^0} \mathbf{d}^*, \mathbf{d})\langle -1 \rangle \\
&\cong \mathrm{Hom}_{\mathbf{d}^0}(\mathbf{d}^*, \mathrm{Hom}_{\mathbf{d}}(\mathbf{d}^{\sigma}, \mathbf{d}))\langle -1 \rangle \\
&\cong \mathrm{Hom}_{\mathbf{d}^0}(\mathbf{d}^*, {}^{\sigma}\mathbf{d})\langle -1 \rangle \\
&\cong \mathrm{Hom}_{\mathbf{d}^0}(\mathbf{d}^*, \mathbf{d}^0) \otimes_{\mathbf{d}^0} {}^{\sigma}\mathbf{d}\langle -1 \rangle \qquad \text{(cf. Remark 2.2.2.1 below)} \\
&\cong \mathrm{Hom}_{F}(\mathbf{d}^*, F) \otimes_{\mathbf{d}^0} {}^{\sigma}\mathbf{d}\langle -1 \rangle \qquad \text{(cf. Remark 2.2.2.2 below)} \\
&\cong \mathbf{d} \otimes_{\mathbf{d}^0} {}^{\sigma}\mathbf{d}\langle -1 \rangle \\
&\overset{\text{def}}{=} \mathbf{u}^{-1}.
\end{aligned}$$

□

Remark 2.2.2. 1. This is due to $\mathbf{d}^*$ being a projective $\mathbf{d}^0$-module ($\mathbf{d}^0$ is semisimple) and the following result taken from [AF92]:

Proposition (Proposition 20.10.). *Given modules ${}_SP$, ${}_SU_T$ and ${}_TN$ there is a homomorphism, natural in P, U and N:*

$$\eta : \mathrm{Hom}_S(P, U) \otimes_T N \to \mathrm{Hom}_S(P, (U \otimes_T N)),$$

defined via

$$\eta(\gamma \otimes_T n) : p \longmapsto \gamma(p) \otimes_T n.$$

If ${}_SP$ is finitely generated and projective, then η is an isomorphism.

2. Since $\mathbf{d}^0$ is semisimple, there is an isomorphism: $\mathbf{d}^0 \cong \mathrm{Hom}_F(\mathbf{d}^0, F)$. We then have:

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{d}^0}(\mathbf{d}^*, \mathbf{d}^0) &\cong \mathrm{Hom}_{\mathbf{d}^0}(\mathbf{d}^*, \mathrm{Hom}_F(\mathbf{d}^0, F)) \\
&\cong \mathrm{Hom}_F(\mathbf{d}^0 \otimes_{\mathbf{d}^0} \mathbf{d}^*, F) \\
&\cong \mathrm{Hom}_F(\mathbf{d}^*, F) \\
&= \mathbf{d}.
\end{aligned}$$

In particular, there is a map $\mathbf{u} \otimes \mathbf{u}^{-1} \to \mathbf{d}$ which corresponds to the evaluation map

$$\mathbf{u} \otimes \mathbf{u}^{-1} \cong \mathbf{u} \otimes \mathrm{Hom}_{\mathbf{d}-}(\mathbf{u}, \mathbf{d}) \to \mathbf{d}.$$

However, $\mathbf{u} \ncong \mathrm{Hom}_{\mathbf{d}-}(\mathbf{u}^{-1}, \mathbf{d})$. Indeed, we have

$$\begin{array}{rcl} \mathrm{Hom}_{\mathbf{d}}(\mathbf{u}^{-1}, \mathbf{d}) & \stackrel{\text{def}}{=} & \mathrm{Hom}_{\mathbf{d}}(\mathbf{d}^{\sigma} \otimes_{\mathbf{d}^0} \mathbf{d}\langle -1 \rangle, \mathbf{d}) \\ & \cong & \mathrm{Hom}_{\mathbf{d}^0}(\mathbf{d}, \mathrm{Hom}_{\mathbf{d}}(\mathbf{d}^{\sigma}, \mathbf{d}))\langle 1 \rangle \\ & \cong & \mathrm{Hom}_{\mathbf{d}^0}(\mathbf{d}, {}^{\sigma}\mathbf{d})\langle 1 \rangle \\ & \cong & \mathrm{Hom}_{\mathbf{d}^0}(\mathbf{d}, \mathbf{d}^0) \otimes_{\mathbf{d}^0} {}^{\sigma}\mathbf{d}\langle 1 \rangle \\ & \cong & \mathrm{Hom}_F(\mathbf{d}, F) \otimes_{\mathbf{d}^0} {}^{\sigma}\mathbf{d}\langle 1 \rangle \\ & \cong & \mathbf{d}^* \otimes_{\mathbf{d}^0} {}^{\sigma}\mathbf{d}\langle 1 \rangle \\ & =: & \tilde{\mathbf{u}}. \end{array}$$

That means we cannot find a similar map for $\mathbf{u}^{-1} \otimes \mathbf{u} \to \mathbf{d}$. In particular, $\mathbb{T}_{\mathbf{d}}(\underline{\mathbf{u}})$ is not an algebra.

Nonetheless, $\mathbf{u}$, $\tilde{\mathbf{u}}$ and $\mathbf{u}^{-1}$ are differential graded bimodules with differential given by

$$\delta(a \otimes b) = (-1)^{|a|_k}(ax \otimes \xi b + a\xi \otimes xb),$$

where $|a|_k$ is the k-degree of a, and they are jk-bigraded bimodules; they inherit their gradings from that on $\mathbf{d}$. In particular, the differential is of k-degree 1. We can extend that differential to $\mathbf{u}^i$ for $i \in \mathbb{Z}$ using the standard way of defining a differential on a tensor product.

We can then take the homology of $\mathbb{T}_{\mathbf{d}}(\underline{\mathbf{u}})$, and we denote it by $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})$; we have

$$\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}}) = \oplus_{i \in \mathbb{Z}} \mathbb{H}(\mathbf{u}^i),$$

where $\mathbb{H}(\mathbf{u}^i)$ is the homology of $\mathbf{u}^i$.

It turns out that $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})$ is an algebra in its own right as $\mathbb{H}(\mathbf{u}) \cong (\mathbf{d}^0)^{\sigma}$ as $\mathbf{d}^0$-$\mathbf{d}^0$-bimodules by Lemma 30 in [MT13], and thanks to the following lemma:

Lemma 2.2.3. *As a $\mathbf{d}^0$-$\mathbf{d}^0$-bimodule,* $\mathbb{H}(\tilde{\mathbf{u}}) \cong (\mathbf{d}^0)^{\sigma}$.

Proof. This follows from direct computation. The basis elements spanning the homology of $\tilde{\mathbf{u}}$ are $e_s^* \otimes e_{p+1-s}$ for $s = 1, \ldots, p$. □

That means that $\mathbf{u}$ and $\tilde{\mathbf{u}}$ have the same homology. In particular, we now have a well-defined multiplication in homology via the induced maps in homology:

$$\mathbb{H}(\mathbf{u}) \otimes \mathbb{H}(\mathbf{u}^{-1}) \cong \mathbb{H}(\mathbf{u}) \otimes \mathbb{H}(\mathrm{Hom}_{\mathbf{d}-}(\mathbf{u}, \mathbf{d})) = \mathbb{H}(\mathbf{u}) \otimes \mathrm{Hom}_{\mathbf{d}-}(\mathbb{H}(\mathbf{u}), \mathbf{d}) \to \mathbf{d},$$

and

$$\begin{array}{rcl} \mathbb{H}(\mathbf{u}^{-1}) \otimes \mathbb{H}(\mathbf{u}) & \cong & \mathbb{H}(\mathbf{u}^{-1}) \otimes \mathbb{H}(\tilde{\mathbf{u}}) \\ & \cong & \mathbb{H}(\mathbf{u}^{-1}) \otimes \mathbb{H}(\mathrm{Hom}_{\mathbf{d}-}(\mathbf{u}^{-1}, \mathbf{d})) \\ & = & \mathbb{H}(\mathbf{u}^{-1}) \otimes \mathrm{Hom}_{\mathbf{d}-}(\mathbb{H}(\mathbf{u}^{-1}), \mathbf{d}) \to \mathbf{d}. \end{array}$$

Despite this problem, all the main results in [MT13] remain true (cf. [MT13, Corrigendum])

$$
\begin{array}{ccccccc}
 & & & & & \mathbf{d}^{0\sigma} & \mathbb{H}(\mathbf{u}) \\
 & & & & \mathbf{d} & & \mathbb{H}(\mathbf{d}) = \mathbf{d} \\
 & & & \overline{M}^{\tau} & \to & \overline{\mathbf{d}^{0\sigma}} & \mathbb{H}(\mathbf{u}^{-1}) \\
 & & M & \oplus & \mathbf{d} & & \mathbb{H}(\mathbf{u}^{-2}) \\
 & M^{\tau} & \oplus & \overline{M}^{\tau} & \to & \overline{\mathbf{d}^{0\sigma}} & \mathbb{H}(\mathbf{u}^{-3}) \\
M & \oplus & M & \oplus & \mathbf{d} & & \mathbb{H}(\mathbf{u}^{-4}) \\
 & & & & \cdots\cdots & &
\end{array}
$$

Figure 2.1: Decomposition of $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})^{\leq 1}$ $(p > 2)$

2.3 Case $p > 2$

2.3.1 Description of $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})^{\leq 1}$ using polytopes

In this section, we collect a number of facts about the truncation $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})^{\leq 1}$ of $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})$ which will be useful to prove later results. Let $q > 1$. We denote $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})$ by Υ and we will use both notations in the following.

The space $\Upsilon^{\leq 1}$ admits a polytopal basis, and the different polytopes involved in that basis correspond to different parts of the homology of $\mathbb{T}_{\mathbf{d}}(\underline{\mathbf{u}})^{\leq 1}$ as shown on Figure 2.1 (cf. [MT13]).

The superscripts τ and σ indicate that there is a twist in the $\mathbf{d}$-action on the right on some components, and we refer to [MT13] for their precise definitions as well as those of M and $\overline{M}$. We note that there is a non-trivial extension of $\overline{M}$ by $\overline{\mathbf{d}^{0\sigma}}$, which we have indicated using an arrow in the diagram; we refer to Remark 2.3.3 below for more details.

Definition 2.3.1 (Lemma 51, [MT13])**.** Define

$$\mathcal{P}_{\mathbf{d}} = \left\{ (s, j_0, k_0, t) \in \mathbb{Z}^4 \quad \middle| \quad \begin{array}{ll} 1 \leq s \leq t \leq p, & 0 \leq j_0 + k_0 \leq 1, \\ t - s = j_0 + 2k_0, & j_0 = 0 = k_0 \text{ if } s = t \end{array} \right\};$$

$$\mathcal{P}_{\overline{\mathbf{d}^0}} = \left\{ (s, j_0, k_0, t) \in \mathbb{Z}^4 \quad \middle| \quad \begin{array}{ll} 1 \leq s, t \leq p, & s + t = p + 1, \\ j_0 = 0 = k_0 & \end{array} \right\} \setminus \{(p, 0, 0, 1)\};$$

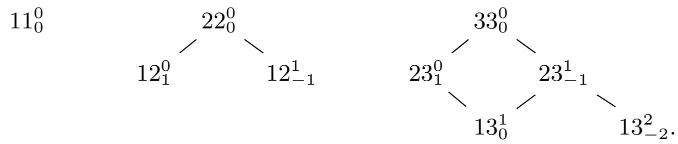
$$\mathcal{P}_{M} = \left\{ (s, j_0, k_0, t) \in \mathbb{Z}^4 \quad \middle| \quad \begin{array}{ll} 1 \leq s, t \leq p, & 0 \leq j_0 + k_0 + 2 \leq 1, \\ t - s - 1 + p = j_0 + 2k_0 + 2 & \end{array} \right\};$$

$$\mathcal{P}_{\overline{M}} = \mathcal{P}_M \setminus \{(p, 0, -1, 1)\}.$$

Example 2.3.2 (Example 52, [MT13])*.* The following is a diagram of the polytope for M in case $p = 3$ (we depict its structure as a left module):

$$
\begin{array}{ccccccccc}
31_0^{-1} & & & & 31^0_{-2} & & & & \\
 & \diagdown & & \diagup & & \diagdown & & & \\
 & & 21^0_{-1} & & & & 21^1_{-3} & & \\
 & & & \diagdown & & \diagup & & \diagdown & \\
 & & & & 11^1_{-2} & & & & 11^2_{-4}
\end{array}
\quad
\begin{array}{ccccccccc}
32^0_{-1} & & & & 32^1_{-3} & & & & \\
 & \diagdown & & \diagup & & \diagdown & & & \\
 & & 22^1_{-2} & & & & 22^2_{-4} & & \\
 & & & \diagdown & & \diagup & & \diagdown & \\
 & & & & 12^2_{-3} & & & & 12^3_{-5}
\end{array}
\quad
\begin{array}{ccccccccc}
33^1_{-2} & & & & 33^2_{-4} & & & & \\
 & \diagdown & & \diagup & & \diagdown & & & \\
 & & 23^2_{-3} & & & & 23^3_{-5} & & \\
 & & & \diagdown & & \diagup & & \diagdown & \\
 & & & & 13^3_{-4} & & & & 13^4_{-6}.
\end{array}
$$

In the diagram an element (s, j, k, t) is written st_j^k. Similarly a diagram of the polytope for $\mathbf{d}$ in case $p = 3$ is given by



Now to obtain the polytopal basis of $\mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})$ (regardless of the polytope encoding $\mathbb{H}(\mathbf{u})$ which we will define later on), we use two integers $a, b \geq 0$ such that the i-degree satisfies $i = -a - b$; they determine the coordinates of the component in Figure 2.1, where a indicates the position on the northeast to southwest axis, and b that on the northwest to southeast axis, with the origin placed at $\mathbf{d}$ on row $\mathbb{H}(\mathbf{d}) = \mathbf{d}$ ($i = 0$). There exists a basis for $\Upsilon^{\leq 0}$ indexed by the subset

$$\begin{aligned}
\mathcal{P}_{\leq 0} \quad := \quad & \{(s, j_0, k_0, a, b, t) \in \mathbb{Z}^6 \quad | \quad (s, j_0, k_0, t) \in \mathcal{P}_{\mathbf{d}}, a, b \geq 0, a = b\} \\
\cup \quad & \{(s, j_0, k_0, a, b, t) \in \mathbb{Z}^6 \quad | \quad (s, j_0, k_0, t) \in \mathcal{P}_{\overline{\mathbf{d}^0}}, a, b \geq 0, a = b - 1 \\
& (s, j_0, k_0, t) \neq (p-1, 0, 0, 2)\} \\
\cup \quad & \{(s, j_0, k_0, a, b, t) \in \mathbb{Z}^6 \quad | \quad (s, j_0, k_0, t) \in \mathcal{P}_{\overline{M}}, a, b \geq 0, a = b + 1\} \\
\cup \quad & \{(s, j_0, k_0, a, b, t) \in \mathbb{Z}^6 \quad | \quad (s, j_0, k_0, t) = (p-1, 0, 0, 2), a, b \geq 0, a = b + 1\} \\
\cup \quad & \{(s, j_0, k_0, a, b, t) \in \mathbb{Z}^6 \quad | \quad (s, j_0, k_0, t) \in \mathcal{P}_M, a, b \geq 0, a > b + 1\}.
\end{aligned}$$

The ijk-degree of such an element is given by the formulas

$$\begin{aligned}
i &= -a - b; \\
j &= \begin{cases} j_0 - (a - b - 1)p + 1 & \text{for } a \geq b + 1, \\ j_0 & \text{for } a = b, \\ j_0 + 1 & \text{for } a = b - 1; \end{cases} \\
k &= \begin{cases} k_0 + (a - b - 1)(p - 1) & \text{for } a \geq b + 1, \\ k_0 & \text{for } a \leq b. \end{cases}
\end{aligned}$$

Remark 2.3.3. In Corollary 39 in [MT13], the generator $e_p \otimes e_1 = (p, -1, -1, 0, 1, 0, 1)$ generates more than what is claimed. The element

$$e_{p-1}\xi e_p \otimes e_1 \xi e_2$$

is generated by $e_p \otimes e_1$, hence its (a, b)-degree should be $(1, 0)$ and not $(0, 1)$. More generally, the element

$$e_{p-1}\xi \otimes (\xi \otimes \xi)^{\otimes b} \otimes \xi e_2$$

is generated by $e_p \otimes (\xi \otimes \xi)^{\otimes b} \otimes e_1$, hence its (a, b)-degree should be $(b + 1, b)$.

We define $\mathcal{P}_{\Upsilon^{\leq 0}}$ to be the corresponding set of elements (s, i, j, k, a, b, t) in $\mathbb{Z}^7$. We define $\mathcal{P}_{\Upsilon^{\leq 1}}$ to be $\mathcal{P}_{\Upsilon^{\leq 0}} \cup \mathcal{P}_{\Upsilon^1}$, where $\mathcal{P}_{\Upsilon^1} := \{(s, 1, 1, 0, -1, 0, p + 1 - s) \in \mathbb{Z}^7 | 1 \leq s \leq p\}$ corresponds to $\mathbb{H}(\mathbf{u})$.

Note that the description of a polytopal basis for $\mathbb{H}(\mathbf{u})$ has been corrected because of the following remark.

Remark 2.3.4. Due to the position of $\mathbb{H}(\mathbf{u})$ in the diagrammatic description of the pieces of homology (cf. Figure 2.1), we see that the elements of $\mathbb{H}(\mathbf{u})$ have (a, b)-degree $(-1, 0)$.

That change makes sense as we would want the following multiplications to exist and to be non-identically zero (for $i \leq 0$):

$$\mathbb{H}(\mathbf{u}) \otimes \mathbb{H}(\mathbf{u}^i) \to \mathbb{H}(\mathbf{u}^{i+1}),$$

and

$$\mathbb{H}(\mathbf{u}^i) \otimes \mathbb{H}(\mathbf{u}) \to \mathbb{H}(\mathbf{u}^{i+1}).$$

Indeed, let us consider the i-degree of the resulting element:

$$\begin{cases} i &= i_1 + i_2 \\ i &= -(a_1 + a_2) - (b_1 + b_2) \end{cases} \Leftrightarrow i_1 + i_2 = -(a_1 + a_2) - (b_1 + b_2).$$

We assume $i_1 = 1$ (the case $i_2 = 1$ is similar). With the (a, b) degree stated in [MT13], we would obtain:

$$i_2 + 1 = -a_2 - b_2 = i_2,$$

which is a contradiction. Hence the need to have either $(a, b) = (-1, 0)$ or $(a, b) = (0, -1)$.

Since we want $(e_p \otimes e_1) \cdot (e_1 \otimes e_p^*)$ to be equal to e_p and $(e_1 \otimes e_p^*) \cdot (e_p \otimes e_1)$ to be equal to e_1, it seems that $(a, b) = (-1, 0)$ is the most reasonable choice. In addition, it is coherent with the position of $\mathbf{d}^{0\sigma}$ in the diagram representing the homology of $\mathbb{T}_{\mathbf{d}}(\underline{\mathbf{u}})^{\leq 1}$.

This allows to describe explicitly the multiplication in $\Upsilon^{\leq 1}$.

Theorem (Theorem 53, [MT13])**.** $\Upsilon^{\leq 1}$ *has basis* $\{m_v\}_{v \in \mathcal{P}_{\Upsilon^{\leq 1}}}$ *with product given by*

$$m_u m_{u'} = \begin{cases} (-1)^{aj_0' + bj_0' + ba'} m_v & \text{if } v_1 = u_1, u_7 = u_1', u_7' = v_7, v_l = u_l + u_l' \\ & \text{for } 2 \leq l \leq 6 \text{ and } v \in \mathcal{P}_{\Upsilon^{\leq 1}}, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.3.5. We call this product of $\Upsilon^{\leq 1}$ the *concatenation* product.

We now want to explain how $\mathbf{w}_q$ is constructed from $\Upsilon^{\leq 1}$.

Definition 2.3.6. We call *chained elements* the elements of $(\Upsilon^{\leq 1})^{\otimes q}$ which are of the form $(s_1, i_1, j_1, k_1, a_1, b_1, t_1) \otimes (s_2, j_1, j_2, k_2, a_2, b_2, t_2) \otimes \ldots \otimes (s_q, j_{q-1}, j_q, k_q, a_q, b_q, t_q)$, i.e. the j-degree of the n-th component is the i-degree of the $(n+1)$-th.

Recall from [MT13, Proposition 28.] that $\mathbf{w}_q$ can be identified to a subalgebra of $\mathbf{d} \otimes \Upsilon^{\otimes q-1}$, and after identification, every basis element of $\mathbf{w}_q$ is a chained element of the form

$$(s_1, 0, j_1, k_1, 0, 0, t_1) \otimes (s_2, j_1, j_2, k_2, a_2, b_2, t_2) \otimes \ldots \otimes (s_q, j_{q-1}, j_q, k_q, a_q, b_q, t_q),$$

where each $(s_n, i_n, j_n, k_n, a_n, b_n, t_n)$ is an element of $\mathbb{H}(\mathbf{u}^{i_n})$ with $i_n \leq 1$ for all n. Hence, to multiply two basis elements of $\mathbf{w}_q$, we just need to apply the concatenation product component wise.

2.3.2 An alternative description of the polytopes

In a following paper ([BLM13]), the authors produce a more uniform combinatorial description of $\mathcal{P}_{\Upsilon^{\leq 1}}$, which will prove useful later on. Let us introduce the following sets.

Definition 2.3.7 (Definition 8. [BLM13])**.** We define sets $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ by

$$\mathcal{S}_1 := \left\{ (s,i,j,k,a,b,t) \in \mathbb{Z}^7 \ : \ \begin{array}{l} 1 \le s \le p, \ a \ge b \ge 0, \ 1 \le t \le p, \\ t-s \ge 0 \text{ if } a-b=0, \ i=-a-b, \\ j = -p(a-b)-(t-s)+2u \text{ and} \\ k=(p-1)(a-b)+(t-s)-u \\ \text{with } u \in \{0,1\}, \\ u=0 \text{ if } t-s=0 \text{ and } a-b=0, \\ t-s \ge 2-p \text{ if } u=1 \text{ and } a-b=1 \end{array} \right\}$$
$$\bigcup \{(p-1, -2b-1, 1, 0, b+1, b, 2) \ : \ b \ge 0\};$$
$$\mathcal{S}_2 := \left\{ (s,i,j,k,a,b,t) \in \mathbb{Z}^7 \ : \ \begin{array}{l} 1 \le s \le p-2, \ t=p+1-s, \ a \ge 0, \\ b=a+1, \ i=-2a-1, \ j=1, \ k=0 \end{array} \right\};$$
$$\mathcal{S}_3 := \left\{ (s,i,j,k,a,b,t) \in \mathbb{Z}^7 \ : \ \begin{array}{l} 1 \le s \le p, \ i=1, \ j=1, \ k=0, \\ a=-1, \ b=0, \ t=p+1-s \end{array} \right\}.$$

These three sets are disjoint ([BLM13][Proposition 9.]). Note that we implemented some corrections to the original definition because of Remark 2.3.3 and Remark 2.3.4 above. In addition, the added elements to $\mathcal{S}_1$ fit the combinatorial description for $u = 2$; this is because the value of u counts the number of multiplications by the elements $e_s \xi e_{s+1}$ and the element

$$(p-1, -2b-1, 1, 0, b+1, b, 2) = e_{p-1}\xi \otimes (\xi \otimes \xi)^{\otimes b} \otimes \xi e_2$$

is obtained from $e_p \otimes (\xi \otimes \xi)^{\otimes b} \otimes e_1$ by multiplying twice by ξ, once on the left, and once on the right.

That alternative description is very useful to characterize the kind of chained basis elements we can have in $\mathbf{w}_q$. We define the type of a basis element below.

Definition 2.3.8 (Definition 10. [BLM13])**.** We say that the q-tuple of vectors $(\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_q)$ belongs to case $(x_1, x_2, ..., x_q)$ if $\mathbf{v}_g \in \mathcal{S}_{x_g}$ with $x_g \in \{1,2,3\}$ for all $g \in \{1, ..., q\}$. If several adjacent x_g take the same value, we will also say that $(\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_q)$ belongs to case $(x_1^{h_1} x_{h_1+1}^{h_2}, \dots)$ to mean that $x_{h_1} = x_{h_1-1} = \cdots = x_2 = x_1$, $x_{h_1+h_2} = x_{h_1+h_2-1} = \cdots = x_{h_1+2} = x_{h_1+1}$, etc.

Denoting by $\mathcal{B}^k(m,\ell)$ a basis of the subspace $\mathrm{Ext}^k(\Delta_m, \Delta_\ell)$ (see [BLM13] for an explicit definition), we can formulate the following lemma.

Lemma 2.3.9 (Lemma 11. [BLM13])**.** *A q-tuple of vectors $(\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_q) \in \mathcal{B}^k(m,\ell)$ belongs either to case (1^q) or to case $(1^h, 2, 3^{q-h-1})$ or to case $(1^h 3^{q-h})$ for $1 \le h \le q-1$.*

We see that there is a restricted number of cases to consider for basis elements of $\mathbf{w}_q$. Building upon Lemma 11 from [BLM13] we will call elements belonging to set S_l *elements of type l*.

Remark 2.3.10. Note that for $v \in \mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})^{\le 1}$, if $v \in S_1$, then its a- and b-degree a_v and b_v satisfy $a_v - b_v \ge 0$; if $v \in S_2$, then $a_v - b_v = -1$; finally, if $v \in S_3$, then $a_v - b_v = -1$. In particular, for all $v \in \mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})^{\le 1}$, we have $a_v - b_v \ge -1$.

2.3.3 Polytopal and x, ξ form

We would like to explain how to go from one description to the other as we will rely on it later on; both forms are useful in different contexts. In Figure 2.1, we see that $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})^{\le 1}$ is made up of different pieces. We recall the following results from [MT13], which give the generators of the different parts in homology:

Theorem (Theorem 44, Lemma 45). *1. The generators of the* $\mathbf{d}$-$\mathbf{d}$*-bimodule* M *or* $\overline{M}$ *are*

$$\begin{aligned}\mathbf{x}_{f,-i} &= e_p \otimes (\xi\otimes\xi)^{\otimes f} \otimes (x^{p-1})^{\otimes -i-1-2f} \otimes e_1 \\ &\cong \pm e_p \otimes (x^{p-1})^{\otimes -i-1-2f} \otimes (\xi\otimes\xi)^{\otimes f} \otimes e_1 \\ &\in \mathbb{H}(\mathbf{u}^i)\end{aligned}$$

and

$$\begin{aligned}\mathbf{y}_{f,-i} &= e_p \otimes (\xi\otimes\xi)^{\otimes f} \otimes (x^{p-1})^{\otimes -i-2-2f} \otimes x^{p-2}\xi \otimes e_1 \\ &\cong \pm e_p \otimes x^{p-2}\xi \otimes (x^{p-1})^{\otimes -i-1-2f} \otimes (\xi\otimes\xi)^{\otimes f} \otimes e_1 \quad . \\ &\in \mathbb{H}(\mathbf{u}^i)\end{aligned}$$

If i is even, $0 \leq f \leq \dfrac{-i-2}{2}$*, and if i is odd, the parameter f satisfies* $0 \leq f \leq \dfrac{-i-1}{2}$ *for* $\mathbf{x}_{f,-i}$ *and it satisfies* $0 \leq f \leq \dfrac{-i-3}{2}$ *for* $\mathbf{y}_{f,-i}$*.*

2. *For i even, the elements* $e_l w^{\frac{-i}{2}} e_l$ *(for* $1 \leq l \leq p$*) generate a factor isomorphic to* $\mathbf{d}$ *in* $\mathbb{H}(\mathbf{u}^i)$*, where*
$$e_l w e_l := e_l\,(\xi\otimes\xi\otimes 1 - 1\otimes\xi\otimes\xi)\,e_l.$$

3. *For i odd, the elements* $e_l\,(\xi\otimes\xi)^{\frac{1-i}{2}}\,e_{p+1-l}$ *(for* $1 \leq l \leq p-1$*) generate the factor isomorphic to* $\overline{\mathbf{d}^{0\sigma}}$ *in* $\mathbb{H}(\mathbf{u}^i)$*.*

Note that, for $i \leq 0$, since $\mathbf{u}^i = \mathbf{d}^\sigma \otimes_{\mathbf{d}^0} \mathbf{d}^\sigma \otimes_{\mathbf{d}^0} \ldots \otimes_{\mathbf{d}^0} \mathbf{d}\langle i\rangle$, we need to apply a shift in the j-degree to obtain the right degree.

Since $x \in \mathbf{d}$ is the only element given non-zero k-degree, the number of x's in the element will give the right k-degree. Besides, since $x \in \mathbf{d}$ is given j-degree -1 and $\xi \in \mathbf{d}$ is given j-degree 1, we see that the j-degree is given by $|\xi| - |x| + i$ if the element lives in $\mathbb{H}(\mathbf{u}^i)$. For instance, for $\mathbf{x}_{f,-i}$, we have:

$$\begin{aligned}k-\deg &= (-i-1-2f)(p-1) \\ j-\deg &= (p-1)(i+1+2f)+2f+i\end{aligned}$$

We also need to find a and b. We know that they should satisfy $-a-b = i$. Let us rearrange the previous system:

$$\begin{aligned}k-\deg &= (p-1)(-i-2f)+(1-p) \\ j-\deg &= p(i+2f)-(1-p)-i-2f+2f+i.\end{aligned}$$

For the k-degree, we almost recognise the expression giving the k-degree in the set $\mathcal{S}_1$, namely $k-\deg = (p-1)(a-b)+(t-s)-u$. Indeed, we have:

$$-i-2f = a-b \Leftrightarrow a+b-2f = a-b \Leftrightarrow f = b.$$

We can finally write:

$$\begin{aligned}s &= p \\ i &= -a-b \\ j &= -p(a-b)-(t-s) \\ k &= (p-1)(a-b)+(t-s) \\ a &= -i-f \\ b &= f \\ t &= 1.\end{aligned}$$

Let (S, I, J, K, A, B, T) be an element of $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})^{\leq 1}$. Then, we have the following dictionary.

- If $A = B$, then

$$(s, -2a, -(t-s) + 2u, t-s-u, a, a, t) = e_s w^a e_s x^{t-s-u} \xi^u e_t;$$

- If $A = B + 1$, then

$$\begin{aligned}(s, -2b-1, -p-(t-s)+2u, (p-1)+(t-s)-u, b+1, b, t)\\ = e_s x^{p-s-u} \xi^u e_p \otimes (\xi \otimes \xi)^{\otimes b} \otimes e_1 x^{t-1} e_t\\ \cong e_s x^{p-s} e_p \otimes (\xi \otimes \xi)^{\otimes b} \otimes e_1 x^{t-1-u} \xi^u e_t\end{aligned}$$

if $u \in \{0, 1\}$, and if $u = 2$, then we have

$$(p-1, -2b-1, 1, 0, b+1, b, 2) = e_{p-1} \xi e_p \otimes (\xi \otimes \xi)^{\otimes b} \otimes e_1 \xi e_2;$$

- If $A > B + 1$, then

$$\begin{aligned}(s, -a-b, -p(a-b)-(t-s), (p-1)(a-b)+(t-s), a, b, t)\\ = e_s x^{p-s} e_p \otimes (x^{p-1})^{\otimes a-b-1} \otimes (\xi \otimes \xi)^{\otimes b} \otimes e_1 x^{t-1} e_t\end{aligned}$$

if $u = 0$, and

$$\begin{aligned}(s, -a-b, -p(a-b)-(t-s)+2, (p-1)(a-b)+(t-s)-1, a, b, t)\\ = e_s x^{p-s-1} \xi e_p \otimes (x^{p-1})^{\otimes a-b-1} \otimes (\xi \otimes \xi)^{\otimes b} \otimes e_1 x^{t-1} e_t\\ \cong e_s x^{p-s} e_p \otimes (x^{p-1})^{\otimes a-b-1} \otimes (\xi \otimes \xi)^{\otimes b} \otimes e_1 x^{t-2} \xi e_t\\ \cong e_s x^{p-s} e_p \otimes x^{p-2} \xi \otimes (x^{p-1})^{\otimes a-b-2} \otimes (\xi \otimes \xi)^{\otimes b} \otimes e_1 x^{t-1} e_t\end{aligned}$$

if $u = 1$;

- If $B = A + 1$, then

$$(s, -2a-1, 1, 0, a, a+1, p+1-s) = e_s (\xi \otimes \xi)^{a+1} e_{p+1-s}$$

if $a \geq 0$, and

$$(s, 1, 1, 0, -1, 0, p+1-s) = e_s \otimes e^*_{p+1-s}$$

if $a = -1$.

2.3.4 Multiplication in $\Upsilon^{\leq 1}$

In this subsection, we want to understand the type of the product of two elements of $\Upsilon^{\leq 1}$. Note that we do *not* consider the sign obtained from multiplying two monomial basis elements as seen in Theorem 53 in [MT13]. We have nine cases to consider. We denote the product of an element of type $\mathbf{n}$ by and element of type $\mathbf{m}$ by $\mathbf{n} \cdot \mathbf{m}$, and we will write $\mathbf{n} \cdot \mathbf{m} = \mathbf{l}$ to mean that the product $\mathbf{n} \cdot \mathbf{m}$ is of type $\mathbf{l}$. We will write the different degrees and parameters of the product obtained with capital letters, e.g. (S, I, J, K, A, B, T). We assume that *idempotents match*, i.e. $t_1 = s_2$, as otherwise we obtain zero.
The structure of the nine cases is pretty straightforward:

1. we describe what the product looks like,
2. we try to discriminate some types,
3. for each remaining type, we assume the product is of that type and we see what that implies for the factors.

1 · 1 – We have:

$$\begin{aligned}
& (s_1, -a_1 - b_1, -p(a_1 - b_1) - (t_1 - s_1) + 2u_1, (p-1)(a_1 - b_1) + (t_1 - s_1) - u_1, a_1, b_1, t_1) \\
\cdot \; & (s_2, -a_2 - b_2, -p(a_2 - b_2) - (t_2 - s_2) + 2u_2, (p-1)(a_2 - b_2) + (t_2 - s_2) - u_2, a_2, b_2, t_2) \\
= \; & (s_1, -(a_1 + a_2) - (b_1 + b_2), -p((a_1 + a_2) - (b_1 + b_2)) - (t_2 - s_1) + 2(u_1 + u_2), \\
& \quad (p-1)((a_1 + a_2) - (b_1 + b_2)) + (t_2 - s_1) - (u_1 + u_2), a_1 + a_2, b_1 + b_2, t_2).
\end{aligned}$$

Since $a_l \geq b_l \geq 0$ for $l = 1, 2$, we know $a_1 + a_2 \geq b_1 + b_2 \geq 0$. In particular, that element cannot be of type **2** nor **3**. So it must be of type **1** or it is zero otherwise. For it to be of type **1**, we see that we only need to ensure $u_1 + u_2 \in \{0, 1\}$. This is equivalent to $u_1 \cdot u_2 = 0$.

1 · 2 – We have:

$$\begin{aligned}
& (s_1, -a_1 - b_1, -p(a_1 - b_1) - (t_1 - s_1) + 2u_1, (p-1)(a_1 - b_1) + (t_1 - s_1) - u_1, a_1, b_1, t_1) \\
\cdot \; & (s_2, -2a_2 - 1, 1, 0, a_2, a_2 + 1, p + 1 - s_2) \\
= \; & (s_1, -a_1 - b_1 - 2a_2 - 1, -p(a_1 - b_1) - (t_1 - s_1) + 2u_1 + 1, \\
& \quad (p-1)(a_1 - b_1) + (t_1 - s_1) - u_1, a_1 + a_2, b_1 + a_2 + 1, p + 1 - t_1),
\end{aligned}$$

where $1 \leq t_1 \leq p - 1$ since $1 \leq s_2 \leq p - 1$ and $t_1 = s_2$. We have $a_1 + a_2 \geq 0$ since $a_1, a_2 \geq 0$, therefore the product cannot be of type **3**.

Let us assume that the product is of type **2**. We need:

$$\begin{cases}
A \geq 0 & \Leftrightarrow & a_1 + a_2 \geq 0 & & \checkmark \\
B = A + 1 & \Leftrightarrow & b_1 + a_2 + 1 = a_1 + a_2 + 1 & \Leftrightarrow & a_1 = b_1 \\
T = p + 1 - S & \Leftrightarrow & p + 1 - t_1 = p + 1 - s_1 & \Leftrightarrow & t_1 = s_1 \\
J = 1 & \Leftrightarrow & -p(a_1 - b_1) - (t_1 - s_1) + 2u_1 + 1 = 1 & \Leftrightarrow & u_1 = 0 \\
K = 0 & \Leftrightarrow & (p-1)(a_1 - b_1) + (t_1 - s_1) - u_1 = 0 & & \checkmark .
\end{cases}$$

So the type **1** factor must be of the form $(s_1, -2a_1, 0, 0, a_1, a_1, s_1)$ with $1 \leq s_1 \leq p - 1$ for the product to be of type **2**.

Let us now assume that the product is of type **1**. We need:

$$\begin{cases}
A \geq B \geq 0 & \Leftrightarrow & a_1 + a_2 \geq b_1 + a_2 + 1 \geq 0 \\
J = -p(A - B) - (T - S) + 2U & & \\
K = (p-1)(A - B) + (T - S) - U & &
\end{cases}$$

The first condition is equivalent to $a_1 + a_2 \geq b_1 + a_2 + 1 \geq 0$, i.e. to $a_1 - b_1 \geq 1$ since $a_1 \geq b_1 \geq 0$.
The two remaining conditions are equivalent to the following system:

$$\begin{cases}
-p(a_1 - b_1) - (t_1 - s_1) + 2u_1 + 1 & = & -p(A - B) - (T - S) + 2U \\
(p-1)(a_1 - b_1) + (t_1 - s_1) - u_1 & = & (p-1)(A - B) + (T - S) - U
\end{cases}$$

$\Leftrightarrow$

$$\begin{cases}
-p(a_1 - b_1) - (t_1 - s_1) + 2u_1 + 1 & = & -p(a_1 - b_1 - 1) - (p + 1 - t_1 - s_1) + 2U \\
(p-1)(a_1 - b_1) + (t_1 - s_1) - u_1 & = & (p-1)(a_1 - b_1 - 1) + (p + 1 - t_1 - s_1) - U
\end{cases}$$

$\Leftrightarrow$

$$\begin{cases}
-t_1 + 2u_1 + 1 & = & -(1 - t_1) + 2U \\
t_1 - u_1 & = & 2 - t_1 - U
\end{cases}$$

$\Leftrightarrow$

$U = u_1$ and $t_1 = 1$.

So the type **1** factor must be of the form

$$(s_1, -a_1 - b_1, -p(a_1 - b_1) - (1 - s_1) - 2u_1, (p-1)(a_1 - b_1) + (1 - s_1) - u_1, a_1, b_1, 1)$$

with $a_1 - b_1 \geq 1$ for the product to be of type **1**.

1 · **3** – We have:

$$\begin{aligned} & (s_1, -a_1 - b_1, -p(a_1 - b_1) - (t_1 - s_1) + 2u_1, (p-1)(a_1 - b_1) + (t_1 - s_1) - u_1, a_1, b_1, t_1) \\ \cdot \quad & (s_2, 1, 1, 0, -1, 0, p + 1 - s_2) \\ = \quad & (s_1, -a_1 - b_1 + 1, -p(a_1 - b_1) - (t_1 - s_1) + 2u_1 + 1, \\ & \qquad (p-1)(a_1 - b_1) + (t_1 - s_1) - u_1, a_1 - 1, b_1, p + 1 - t_1). \end{aligned}$$

Let us assume that the product is of type **3**. We need:

$$\begin{cases} A = -1 & \Leftrightarrow & a_1 - 1 = -1 & \Leftrightarrow & a_1 = 0 \\ B = 0 & \Leftrightarrow & b_1 = 0 & & \\ T = p + 1 - S & \Leftrightarrow & p + 1 - t_1 = p + 1 - s_1 & \Leftrightarrow & t_1 = s_1 \\ J = 1 & \Leftrightarrow & -p(a_1 - b_1) - (t_1 - s_1) + 2u_1 + 1 = 1 & \Leftrightarrow & u_1 = 0 \\ K = 0 & \Leftrightarrow & (p-1)(a_1 - b_1) + (t_1 - s_1) - u_1 = 0 & & \checkmark. \end{cases}$$

So the type **1** factor must be of the form $(s_1, 0, 0, 0, 0, 0, s_1)$ for the product to be of type **3**.

Let us now assume that the product is of type **2**. We need:

$$\begin{cases} A \geq 0 & \Leftrightarrow & a_1 - 1 \geq 0 & \Leftrightarrow & a_1 \geq 1 \\ B = A + 1 & \Leftrightarrow & b_1 = (a_1 - 1) + 1 & \Leftrightarrow & a_1 = b_1 \\ T = p + 1 - S & \Leftrightarrow & p + 1 - t_1 = p + 1 - s_1 & \Leftrightarrow & t_1 = s_1 \\ J = 1 & \Leftrightarrow & -p(a_1 - b_1) - (t_1 - s_1) + 2u_1 + 1 = 1 & \Leftrightarrow & u_1 = 0 \\ K = 0 & \Leftrightarrow & (p-1)(a_1 - b_1) + (t_1 - s_1) - u_1 = 0 & & \checkmark \end{cases}$$

So the type **1** factor must be of the form $(s_1, -2a_1, 0, 0, a_1, a_1, s_1)$ with $1 \leq s_1 \leq p - 1$ and $a_1 \geq 1$ for the product to be of type **2**.

Let us finally assume that the product is of type **1**. We need:

$$\begin{cases} A \geq B \geq 0 \quad \Leftrightarrow \quad a_1 - 1 \geq b_1 \geq 0 \\ J = -p(A - B) - (T - S) + 2U \\ K = (p-1)(A - B) + (T - S) - U \end{cases}$$

The first condition is equivalent to $a_1 - b_1 \geq 1$, since $a_1 \geq b_1 \geq 0$.
The two remaining conditions are equivalent to the following system:

$$\begin{cases} -p(a_1 - b_1) - (t_1 - s_1) + 2u_1 + 1 & = & -p(A - B) - (T - S) + 2U \\ (p-1)(a_1 - b_1) + (t_1 - s_1) - u_1 & = & (p-1)(A - B) + (T - S) - U, \end{cases}$$

which is the same system as for the case **1** · **2** when assuming the product is of type **1**. It is therefore equivalent to

$$U = u_1 \text{ and } t_1 = 1.$$

So the type **1** factor must be of the form

$$(s_1, -a_1 - b_1, -p(a_1 - b_1) - (1 - s_1) - 2u_1, (p-1)(a_1 - b_1) + (1 - s_1) - u_1, a_1, b_1, 1)$$

with $a_1 - b_1 \geq 1$ for the product to be of type **1**.

2 · 1 – We have:

$$\begin{aligned} & (s_1, -2a_1-1, 1, 0, a_1, a_1+1, p+1-s_1) \\ \cdot\ & (s_2, -a_2-b_2, -p(a_2-b_2)-(t_2-s_2)+2u_2, (p-1)(a_2-b_2)+(t_2-s_2)-u_2, a_2, b_2, t_2) \\ =\ & (s_1, -2a_1-1-a_2-b_2, -p(a_2-b_2)-(t_2-s_2)+2u_2+1, \\ & \qquad (p-1)(a_2-b_2)+(t_2-s_2)-u_2, a_1+a_2, a_1+1+b_2, t_2) \end{aligned}$$

where $1 \leq s_1 \leq p-1$ and $s_2 = p+1-s_1$ (so $2 \leq s_2 \leq p$). We have $a_1 + a_2 \geq 0$ since $a_1, a_2 \geq 0$, therefore the product cannot be of type **3**.

Let us assume that the product is of type **2**. We need:

$$\begin{cases} A \geq 0 & \Leftrightarrow\ a_1 + a_2 \geq 0 & & \checkmark \\ B = A+1 & \Leftrightarrow\ a_1+1+b_2 = a_1+a_2+1 & \Leftrightarrow & a_2 = b_2 \\ T = p+1-S & \Leftrightarrow\ t_2 = p+1-s_1 & \Leftrightarrow & t_2 = s_2 \\ J = 1 & \Leftrightarrow\ -p(a_2-b_2)-(t_2-s_2)+2u_2+1 = 1 & \Leftrightarrow & u_2 = 0 \\ K = 0 & \Leftrightarrow\ (p-1)(a_2-b_2)+(t_2-s_2)-u_2 = 0 & & \checkmark. \end{cases}$$

So the type **1** factor must be of the form $(s_2, -2a_2, 0, 0, a_2, a_2, s_2)$ with $2 \leq s_2 \leq p$ for the product to be of type **2**.

Let us now assume that the product is of type **1**. We need:

$$\begin{cases} A \geq B \geq 0 & \Leftrightarrow\ a_1 + a_2 \geq a_1 + 1 + b_2 \geq 0 \\ J = -p(A-B)-(T-S)+2U \\ K = (p-1)(A-B)+(T-S)-U \end{cases}$$

The first condition is equivalent to $a_2 - b_2 \geq 1$, since $a_1 \geq 0$ and $a_2 \geq b_2 \geq 0$. The two remaining conditions are equivalent to the following system:

$$\begin{cases} -p(a_2-b_2)-(t_2-s_2)+2u_2+1 &= -p(A-B)-(T-S)+2U \\ (p-1)(a_2-b_2)+(t_2-s_2)-u_2 &= (p-1)(A-B)+(T-S)-U \end{cases}$$

$$\Leftrightarrow$$

$$\begin{cases} -p(a_2-b_2)-(t_2-s_2)+2u_2+1 &= -p(a_2-b_2-1)-(t_2-s_1)+2U \\ (p-1)(a_2-b_2)+(t_2-s_2)-u_2 &= (p-1)(a_2-b_2-1)+(t_2-s_1)-U \end{cases}$$

$$\Leftrightarrow$$

$$\begin{cases} s_2+2u_1+1 &= p+s_1+2U \\ -s_2-u_1 &= -p+1-s_1-U \end{cases}$$

$$\Leftrightarrow$$

$$\begin{cases} p+1-s_1+2u_1+1 &= p+s_1+2U \\ -p-1+s_1-u_1 &= -p+1-s_1-U \end{cases}$$

$$\Leftrightarrow$$

$U = u_1$ and $s_1 = 1 (\Rightarrow s_2 = p)$.

So the type **1** factor must be of the form

$$(p, -a_2-b_2, -p(a_2-b_2)-(t_2-p)-2u_1, (p-1)(a_2-b_2)+(t_2-p)-u_1, a_2, b_2, t_2)$$

with $a_2 - b_2 \geq 1$ for the product to be of type **1**.

2 · 2 – We have:

$$\begin{aligned} & (s_1, -2a_1 - 1, 1, 0, a_1, a_1 + 1, p + 1 - s_1) \\ \cdot \quad & (s_2, -2a_2 - 1, 1, 0, a_2, a_2 + 1, p + 1 - s_2) \\ = \quad & (s_1, -2a_1 - 1 - 2a_2 - 1, 2, 0, a_1 + a_2, a_1 + 1 + a_2 + 1, s_1) \end{aligned}$$

as $s_2 = p + 1 - s_1 \Rightarrow p + 1 - s_2 = s_1$. We see that $J = 2$ in the product and there is no such element in $\Upsilon^{\leq 1}$. Hence multiplying two type **2** elements together yields zero.

2 · 3 – We have:

$$\begin{aligned} & (s_1, -2a_1 - 1, 1, 0, a_1, a_1 + 1, p + 1 - s_1) \\ \cdot \quad & (s_2, 1, 1, 0, -1, 0, p + 1 - s_2) \\ = \quad & (s_1, -2a_1, 2, 0, a_1 - 1, a_1 + 1, s_1) \end{aligned}$$

as $s_2 = p + 1 - s_1 \Rightarrow p + 1 - s_2 = s_1$. We see that $J = 2$ in the product and there is no such element in $\Upsilon^{\leq 1}$. Hence this product is zero.

3 · 1 – We have:

$$\begin{aligned} & (s_1, 1, 1, 0, -1, 0, p + 1 - s_1) \\ \cdot \quad & (s_2, -a_2 - b_2, -p(a_2 - b_2) - (t_2 - s_2) + 2u_2, (p - 1)(a_2 - b_2) + (t_2 - s_2) - u_2, a_2, b_2, t_2) \\ = \quad & (s_1, 1 - a_2 - b_2, 1 - p(a_2 - b_2) - (t_2 - s_2) + 2u_2, \\ & \qquad (p - 1)(a_2 - b_2) + (t_2 - s_2) - u_2, -1 + a_2, b_2, t_2) \end{aligned}$$

so $s_2 = p + 1 - s_1$

Let us assume that the product is of type **3**. We need:

$$\begin{cases} A = -1 & \Leftrightarrow & a_2 - 1 = -1 & \Leftrightarrow & a_2 = 0 \\ B = 0 & \Leftrightarrow & b_2 = 0 & & \\ T = p + 1 - S & \Leftrightarrow & t_2 = p + 1 - s_1 & \Leftrightarrow & t_2 = s_2 \\ J = 1 & \Leftrightarrow & -p(a_2 - b_2) - (t_2 - s_2) + 2u_2 + 1 = 1 & \Leftrightarrow & u_2 = 0 \\ K = 0 & \Leftrightarrow & (p - 1)(a_2 - b_2) + (t_2 - s_2) - u_2 = 0 & & \checkmark. \end{cases}$$

So the type **1** factor must be of the form $(s_2, 0, 0, 0, 0, 0, s_2)$ for the product to be of type **3**.

Let us now assume that the product is of type **2**. We need:

$$\begin{cases} A \geq 0 & \Leftrightarrow & a_2 - 1 \geq 0 & \Leftrightarrow & a_2 \geq 1 \\ B = A + 1 & \Leftrightarrow & b_2 = (a_2 - 1) + 1 & \Leftrightarrow & a_2 = b_2 \\ T = p + 1 - S & \Leftrightarrow & t_2 = p + 1 - s_1 & \Leftrightarrow & t_2 = s_2 \\ 1 \leq S \leq p - 1 & \Leftrightarrow & 1 \leq s_1 \leq p - 1 & \Leftrightarrow & 2 \leq s_2 \leq p \\ J = 1 & \Leftrightarrow & -p(a_2 - b_2) - (t_2 - s_2) + 2u_2 + 1 = 1 & \Leftrightarrow & u_2 = 0 \\ K = 0 & \Leftrightarrow & (p - 1)(a_2 - b_2) + (t_2 - s_2) - u_2 = 0 & & \checkmark. \end{cases}$$

So the type **1** factor must be of the form $(s_2, -2a_2, 0, 0, a_2, a_2, s_2)$ with $2 \leq s_2 \leq p$ and $a_2 \geq 1$ for the product to be of type **2**.

Let us finally assume that the product is of type **1**. We need:

$$\begin{cases} A \geq B \geq 0 & \Leftrightarrow & a_2 - 1 \geq b_2 \geq 0 \\ J = -p(A - B) - (T - S) + 2U & & \\ K = (p - 1)(A - B) + (T - S) - U & & \end{cases}$$

The first condition is equivalent to $a_2 - b_2 \geq 1$ since $a_2 \geq b_2 \geq 0$. The two remaining conditions are equivalent to the following system:

$$\begin{cases} -p(a_2-b_2)-(t_2-s_2)+2u_2+1 &= -p(A-B)-(T-S)+2U \\ (p-1)(a_2-b_2)+(t_2-s_2)-u_2 &= (p-1)(A-B)+(T-S)-U \end{cases}$$
$$\Leftrightarrow$$
$$\begin{cases} -p(a_2-b_2)-(t_2-s_2)+2u_2+1 &= -p(a_2-b_2-1)-(t_2-s_1)+2U \\ (p-1)(a_2-b_2)+(t_2-s_2)-u_2 &= (p-1)(a_2-b_2-1)+(t_2-s_1)-U \end{cases}$$
$$\Leftrightarrow$$
$$\begin{cases} s_2+2u_2+1 &= p+s_1+2U \\ -s_2-u_2 &= -(p-1)-s_1-U \end{cases}$$
$$\Leftrightarrow$$
$$\begin{cases} p+1-s_1+2u_2+1 &= p+s_1+2U \\ -p-1+s_1-u_2 &= -(p-1)-s_1-U \end{cases}$$
$$\Leftrightarrow$$
$$U = u_2 \text{ and } s_1 = 1 (\Rightarrow s_2 = p).$$

So the type **1** factor must be of the form

$$(p, -a_2-b_2, -p(a_2-b_2)-(t_2-p)-2u_2, (p-1)(a_2-b_2)+(t_2-p)-u_2, a_2, b_2, t_2)$$

with $a_2 - b_2 \geq 1$ for the product to be of type **1**.

$\mathbf{3}\cdot\mathbf{2}$ – We have:

$$\begin{aligned} & (s_1, 1, 1, 0, -1, 0, p+1-s_1) \\ \cdot\ & (s_2, -2a_2-1, 1, 0, a_2, a_2+1, p+1-s_2) \\ =\ & (s_1, -2a_2, 2, 0, a_2-1, a_2+1, s_1) \end{aligned}$$

as $s_2 = p+1-s_1 \Rightarrow p+1-s_2 = s_1$. We see that $J = 2$ in the product and there is no such element in $\Upsilon^{\leq 1}$. Hence this product is zero.

$\mathbf{3}\cdot\mathbf{3}$ – We have:

$$\begin{aligned} & (s_1, 1, 1, 0, -1, 0, p+1-s_1) \\ \cdot\ & (s_2, 1, 1, 0, -1, 0, p+1-s_2) \\ =\ & (s_1, 2, 2, 0, -2, 0, s_1) \end{aligned}$$

as $s_2 = p+1-s_1 \Rightarrow p+1-s_2 = s_1$. We see that $I = J = 2$ in the product and there is no such element in $\Upsilon^{\leq 1}$. Hence this product is zero.

	1	**2**	**3**
1	$\mathbf{1} \Leftrightarrow u_1 \cdot u_2 = 0$	$\mathbf{1} \Leftrightarrow \begin{cases} a_1 - b_1 \geq 1 \\ t_1 = 1 \end{cases}$ $\mathbf{2} \Leftrightarrow \begin{cases} a_1 = b_1 \\ t_1 = s_1 \\ 1 \leq s_1 \leq p-1 \end{cases}$	$\mathbf{1} \Leftrightarrow \begin{cases} a_1 - b_1 \geq 1 \\ t_1 = 1 \end{cases}$ $\mathbf{2} \Leftrightarrow \begin{cases} a_1 = b_1 \\ a_1 \geq 1 \\ t_1 = s_1 \\ 1 \leq s_1 \leq p-1 \end{cases}$ $\mathbf{3} \Leftrightarrow \begin{cases} a_1 = b_1 = 0 \\ t_1 = s_1 \end{cases}$
2	$\mathbf{1} \Leftrightarrow \begin{cases} a_2 - b_2 \geq 1 \\ s_2 = p \end{cases}$ $\mathbf{2} \Leftrightarrow \begin{cases} a_2 = b_2 \\ t_2 = s_2 \\ 2 \leq s_2 \leq p \end{cases}$	0	0
3	$\mathbf{1} \Leftrightarrow \begin{cases} a_2 - b_2 \geq 1 \\ s_2 = p \end{cases}$ $\mathbf{2} \Leftrightarrow \begin{cases} a_2 = b_2 \\ a_2 \geq 1 \\ t_2 = s_2 \\ 2 \leq s_2 \leq p \end{cases}$ $\mathbf{3} \Leftrightarrow \begin{cases} a_2 = b_2 = 0 \\ t_2 = s_2 \end{cases}$	0	0

Figure 2.2: Summary of type multiplication for $p > 2$

2.4 Case $p = 2$

2.4.1 The polytopes for $p = 2$

In [MT13], the authors explain that it is possible to use the same combinatorics for the basis elements: the polytopal description is still valid, even though the module structure of the homology of $\mathbf{u}^{-i}$ $(i \geq 1)$ is different. Indeed, Lemma 49 in [MT13] shows that the homology of those bimodules is indecomposable, which means Figure 2.1 is not really relevant. However, in this subsection, we explain, through examples, how to recover a polytopal description of the basis from Figure 2.1. In particular, we will see how to get rid of the integers a and b and write explicitly the extension of $\overline{M}$ by $\overline{\mathbf{d}^0}$.

Definition 2.4.1. Define

$$\mathcal{P}_{\mathbf{d}} = \left\{ (s, j_0, k_0, t) \in \mathbb{Z}^4 \quad \middle| \quad \begin{array}{ll} 1 \leq s \leq t \leq 2, & 0 \leq j_0 + k_0 \leq 1, \\ t - s = j_0 + 2k_0, & j_0 = 0 = k_0 \text{ if } s = t \end{array} \right\};$$

$$\mathcal{P}_{\overline{\mathbf{d}^0}} = \left\{ (s, j_0, k_0, t) \in \mathbb{Z}^4 \quad \middle| \quad \begin{array}{ll} 1 \leq s, t \leq 2, & s + t = 3, \\ j_0 = 0 = k_0 & \end{array} \right\} \setminus \{(2,0,0,1)\};$$

$$\mathcal{P}_M = \left\{ (s, j_0, k_0, t) \in \mathbb{Z}^4 \quad \middle| \quad \begin{array}{ll} 1 \leq s, t \leq 2, & 0 \leq j_0 + k_0 + 2 \leq 1, \\ t - s + 1 = j_0 + 2k_0 + 2 & \end{array} \right\};$$

$$\mathcal{P}_{\overline{M}} = \mathcal{P}_M \setminus \{(2, 0, -1, 1)\}.$$

From the definition, we easily see that the polytopes are the following sets of tuples.

Lemma 2.4.2. $\mathcal{P}_{\mathbf{d}} = \{(1,0,0,1), (2,0,0,2), (1,1,0,2), (1,-1,1,2)\}$*;*

$$\mathcal{P}_{\overline{\mathbf{d}^0}} = \{(1,0,0,2)\};$$

$$\mathcal{P}_M = \left\{ \begin{array}{l} (1,-1,0,1), (1,-3,1,1), (2,-1,0,2), (2,-3,1,2), (1,-2,1,2), \\ (1,-4,2,2), (2,-2,0,1), (2,0,-1,1) \end{array} \right\};$$

$$\mathcal{P}_{\overline{M}} = \left\{ \begin{array}{l} (1,-1,0,1), (1,-3,1,1), (2,-1,0,2), (2,-3,1,2), (1,-2,1,2), \\ (1,-4,2,2), (2,-2,0,1) \end{array} \right\}.$$

Example 2.4.3. The following is a diagram of the polytope for M in case $p = 2$ (we depict its structure as a left module):

$$\begin{array}{ccccccccccccc} 21_0^{-1} & & & & 21_{-2}^0 & & & & 22_{-1}^0 & & & & 22_{-3}^1 & & \\ & \diagdown & & \diagup & & \diagdown & & & & \diagdown & & \diagup & & \diagdown & \\ & & 11_{-1}^0 & & & & 11_{-3}^1 & & & & 12_{-2}^1 & & & & 12_{-4}^2. \end{array}$$

Similarly a diagram of the polytope for $\mathbf{d}$ in case $p = 2$ is given by

$$\begin{array}{ccccc} 11_0^0 & & & 22_0^0 & & \\ & & \diagup & & \diagdown & \\ & 12_1^0 & & & & 12_{-1}^1. \end{array}$$

We see that it is as if we had cut the lowest row in the picture for $p = 3$ (cf. Example 2.3.2) and replaced $p = 3$ by $p = 2$ (hence only two components).

We consider the following example to describe explicitly the extension of $\overline{M}$ by $\overline{\mathbf{d}^0}$. Note that $\overline{\mathbf{d}^0}$ is one-dimensional.

Example 2.4.4. $\mathbb{H}(\mathbf{u}^{-1})$ is given by a copy $\overline{M}$ in position $(a, b) = (1, 0)$ and a copy $\overline{\mathbf{d}^{0\sigma}}$ in position $(a, b) = (0, 1)$ for $p > 2$. However, for $p = 2$, $\overline{\mathbf{d}^{0\sigma}}$ is reduced to the element $e_1\xi \otimes \xi e_2$ and it is generated by $e_2 \otimes e_1$, which means it corresponds to the non-trivial extension of $\overline{M}$ by $\overline{\mathbf{d}^{0\sigma}}$. Therefore, as left modules, the homology of $\mathbf{u}^{-1}$ is given by

$$\begin{array}{cccccccccccc} & & 21_{-2}^0 & & & & & 22_{-1}^0 & & & & 22_{-3}^1 & & \\ & \diagup & & \diagdown & & & \diagup & & \diagdown & & \diagup & & \diagdown & \\ 11_{-1}^0 & & & & 11_{-3}^1 & 12_0^0 & & & & 12_{-2}^1 & & & & 12_{-4}^2. \end{array}$$

The extension corresponds to multiplying $e_2 \otimes \xi e_2 = 22_{-1}^0$ on the left by $e_1\xi e_2 = 12_1^0 \in \mathcal{P}_{\mathbf{d}}$ to obtain $e_1\xi \otimes \xi e_2 = 12_0^0$.

In the next diagrams, an element (s, j, k, a, b, t) is written ${}_b^a st_j^k$.

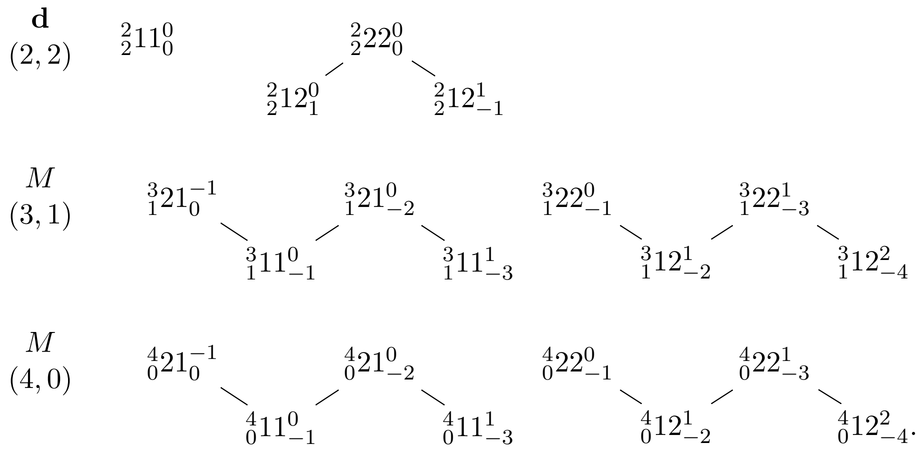
Example 2.4.5. In Figure 2.1, we see that $\mathbb{H}(\mathbf{u}^{-4})$ decomposes in case $p > 2$ as

$$\begin{array}{ccccc} M & \oplus & M & \oplus & \mathbf{d} \\ (4,0) & & (3,1) & & (2,2) \end{array}$$

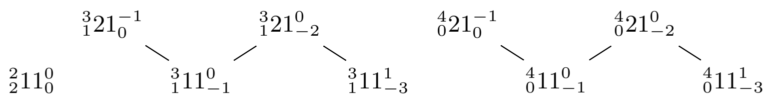
with the corresponding positions (a, b) indicated underneath. We reverse the order in which it is presented to obtain

$$\begin{array}{ccccc} \mathbf{d} & \oplus & M & \oplus & M \\ (2,2) & & (3,1) & & (4,0) \end{array}$$

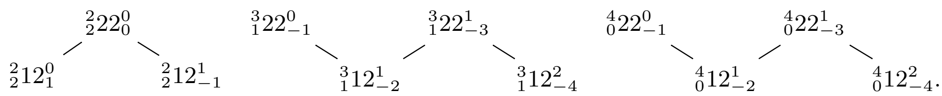
and writing the structure as left modules, we see



That means that, at this stage of the explanation, $\mathbb{H}(\mathbf{u}^{-4})e_1$ is given by



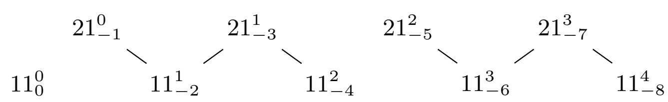
and $\mathbb{H}(\mathbf{u}^{-4})e_2$ by



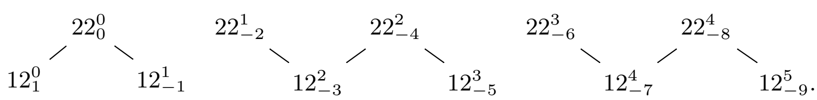
Applying the corrections to the j- and k-degree as prescribed in the case $p > 2$, we see for instance that if $(a, b) = (4, 0)$, since $a \geq b + 1$, we have

$$\begin{aligned} j &= j_0 - 5, \\ k &= k_0 + 3. \end{aligned}$$

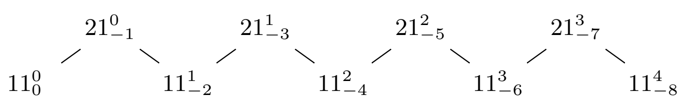
Note that we should have applied corrections on the j- and k-degrees in the previous example, but for the clarity and progression of the argument, we omitted it. Removing the indices a and b since we used them to correct the degrees appropriately, we see that $\mathbb{H}(\mathbf{u}^{-4})e_1$ is thus given by



and $\mathbb{H}(\mathbf{u}^{-4})e_2$ by



It is now clear that we can glue the different pieces together: multiplying 21^0_{-1}, 21^2_{-5}, 22^1_{-2}, 22^3_{-6} on the left by the element $12^0_1 = e_1\xi e_2$, we see that we have the following structures for $\mathbb{H}(\mathbf{u}^{-4})e_1$ and $\mathbb{H}(\mathbf{u}^{-4})e_2$



$$\begin{array}{ccccccccccc} & 22^0_0 & & 22^1_{-2} & & 22^2_{-4} & & 22^3_{-6} & & 22^4_{-8} & \\ \nearrow & & \searrow \nearrow & & \searrow \nearrow & & \searrow \nearrow & & \searrow \nearrow & & \searrow \\ 12^0_1 & & 12^1_{-1} & & 12^2_{-3} & & 12^3_{-5} & & 12^4_{-7} & & 12^5_{-9}. \end{array}$$

Indeed, we have

$$\begin{aligned} 12^0_1 \cdot 21^0_{-1} &= e_1\xi e_2 \cdot e_2 \otimes \xi \otimes \xi \otimes \xi \otimes e_1 \\ &= e_1\xi \otimes \xi \otimes \xi \otimes \xi \otimes e_1 \\ &= 11^0_0; \end{aligned}$$

$$\begin{aligned} 12^0_1 \cdot 21^2_{-5} &= e_1\xi e_2 \cdot e_2 \otimes x \otimes x \otimes \xi \otimes e_1 \\ &= e_1\xi \otimes x \otimes x \otimes \xi \otimes e_1 \\ &= 11^2_{-4}; \end{aligned}$$

$$\begin{aligned} 12^0_1 \cdot 22^1_{-2} &= e_1\xi e_2 \cdot e_2 \otimes x \otimes \xi \otimes \xi \otimes \xi e_2 \\ &= e_1\xi \otimes x \otimes \xi \otimes \xi \otimes \xi e_2 \\ &= 12^1_{-1}; \end{aligned}$$

$$\begin{aligned} 12^0_1 \cdot 22^3_{-6} &= e_1\xi e_2 \cdot e_2 \otimes x \otimes x \otimes x \otimes \xi e_2 \\ &= e_1\xi \otimes x \otimes x \otimes x \otimes \xi e_2 \\ &= 12^3_{-5}. \end{aligned}$$

From this, we see there exists a basis for $\Upsilon^{\leq 0}$ indexed by the subset

$$\begin{aligned} \mathcal{P}_{\leq 0} \;:=\; & \{(s,i,j,k,t) \in \mathbb{Z}^5 \;\mid\; (s,j,k,t) \in \mathcal{P}_{\mathbf{d}}, i \in -2\mathbb{N}\} \\ \cup\; & \{(s,i,j,k,t) \in \mathbb{Z}^5 \;\mid\; (s,j-1,k,t) \in \mathcal{P}_{\overline{\mathbf{d}^0}}, i \in -2\mathbb{N}_{>0}+1\} \\ \cup\; & \{(s,i,j,k,t) \in \mathbb{Z}^5 \;\mid\; (s,j-1,k,t) \in \mathcal{P}_{\overline{M}}, i \in -2\mathbb{N}-1\} \\ \cup\; & \{(s,i,j,k,t) \in \mathbb{Z}^5 \;\mid\; i \leq -2, a > \frac{1-i}{2}, \\ & \quad (s, j+2(2a+i-1)-1, k-(2a+i-1), t) \in \mathcal{P}_M\}. \end{aligned}$$

Recalling the explicit description of the different polytopes given in Lemma 2.4.2, we can write

$$\begin{aligned} \mathcal{P}_{\leq 0} \;:=\; & \{(1,-2n,0,0,1),(2,-2n,0,0,2),(1,-2n,1,0,2),(1,-2n,-1,1,2) \;\mid\; n \in \mathbb{N}\} \\ \cup\; & \{(1,-2n+1,1,0,2) \;\mid\; n \in \mathbb{N}_{>0}\} \\ \cup\; & \{(1,-2n-1,0,0,1),(1,-2n-1,-2,1,1),(2,-2n-1,0,0,2), \\ & (2,-2n-1,-2,1,2),(1,-2n-1,-1,1,2),(1,-2n-1,-3,2,2), \\ & (2,-2n-1,-1,0,1) \;\mid\; n \in \mathbb{N}\} \\ \cup\; & \{(1,i,2-2(2a+i),2a+i-1,1),(1,i,-2(2a+i),2a+i,1), \\ & (2,i,2-2(2a+i),2a+i-1,2),(2,i,-2(2a+i),2a+i,2), \\ & (1,i,1-2(2a+i),2a+i,2),(1,i,-1-2(2a+i),2a+i+1,2), \\ & (2,i,1-2(2a+i),2a+i-1,1),(2,i,3-2(2a+i),2a+i-2,1) \\ & \quad \left. \mid \; i \leq -2, \frac{1-i}{2} < a \leq -i \right\}. \end{aligned}$$

We define $\mathcal{P}_{\Upsilon^{\leq 1}}$ to be $\mathcal{P}_{\leq 0} \cup \mathcal{P}_{\Upsilon^1}$, where $\mathcal{P}_{\Upsilon^1} = \{(s,1,1,0,3-s) \in \mathbb{Z}^5 | 1 \leq s \leq 2\}$ corresponds to the homology of $\mathbf{u}$.

This enables us to write the multiplication map on $\Upsilon^{\leq 1}$ explicitly.

Theorem (Theorem 53, [MT13]). *$\Upsilon^{\leq 1}$ has basis $\{m_v\}_{v \in \mathcal{P}_{\Upsilon^{\leq 1}}}$ with product given by*

$$m_u m_{u'} = \begin{cases} m_v & \text{if } v_1 = u_1, u_5 = u_1', u_5' = v_5, v_l = u_l + u_l' \\ & \text{for } 2 \leq l \leq 4 \text{ and } v \in \mathcal{P}_{\Upsilon^{\leq 1}}, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.4.6. We call this product of $\Upsilon^{\leq 1}$ *concatenation* product.

We now want to explain how $\mathbf{w}_q$ is constructed from $\Upsilon^{\leq 1}$.

Definition 2.4.7. We call *chained elements* of $(\Upsilon^{\leq 1})^{\otimes q}$ elements of the form $(s_1, i_1, j_1, k_1, t_1) \otimes (s_2, j_1, j_2, k_2, t_2) \otimes \ldots \otimes (s_q, j_{q-1}, j_q, k_q, t_q)$, i.e. the j-degree of the n-th component is the i-degree of the $(n+1)$-th.

Recall from [MT13, Proposition 28.] that $\mathbf{w}_q$ is a subalgebra of $\mathbf{d} \otimes \Upsilon^{\otimes q-1}$, and every basis element of $\mathbf{w}_q$ is a chained element of the form

$$(s_1, 0, j_1, k_1, t_1) \otimes (s_2, j_1, j_2, k_2, t_2) \otimes \ldots \otimes (s_q, j_{q-1}, j_q, k_q, t_q),$$

where each $(s_n, i_n, j_n, k_n, t_n)$ is an element of $\mathbb{H}(\mathbf{u}^{i_n})$. Thus, to multiply two basis elements of $\mathbf{w}_q$, we just need to apply the concatenation product component wise.

2.4.2 Polytopal and x, ξ form

Similarly as for the case $p > 2$, we explain how to translate elements in polytopal form to a more explicit x and ξ form. It is easier in this case since we do not need to consider the a- and b-degree any longer.

Since $x \in \mathbf{d}$ is the only element with non-zero k-degree, that degree corresponds to the number of x's. To obtain the j-degree, recall that $x \in \mathbf{d}$ is given in j-degree -1 and $\xi \in \mathbf{d}$ is given in j-degree 1. By definition of $\mathbf{u}^{-i}$, we need to apply a shift in the j-degree as well, so that the j-degree of an element of i-degree i is $|\xi| - |x| + i$.

For instance, we have:

$$\begin{aligned} (1,0,0,0,1) &= e_1 \\ (2,0,0,0,2) &= e_2 \\ (1,0,1,0,2) &= \xi \\ (1,0,-1,1,2) &= x. \end{aligned}$$

More generally, we have:

$$\begin{aligned} (1,i,j,k,1) &= x^{\otimes k} \otimes \xi^{\otimes j+k-i} \otimes e_1 & (j+2k=0) \\ (2,i,j,k,1) &= e_2 \otimes x^{\otimes k} \otimes \xi^{\otimes j+k-i} \otimes e_1 & (j+2k=-1) \\ (1,i,j,k,2) &= x^{\otimes k} \otimes \xi^{\otimes j+k-i} & (j+2k=1) \\ (2,i,j,k,2) &= e_2 \otimes x^{\otimes k} \otimes \xi^{\otimes j+k-i} & (j+2k=0). \end{aligned}$$

2.4.3 Multiplication in $\Upsilon^{\leq 1}$

Using the description of basis elements given by polytopes, we see that $\mathbb{H}(\mathbf{u}^{-i})$ decomposes as follows:

For both pictures, going left from the semi-simple top to the socle corresponds to multiplying on the left by $\xi = (1,0,1,0,2)$ when going right corresponds to multiplying on the left by $x = (1,0,-1,1,2)$.

Elements of $\mathbb{H}(\mathbf{u}^{-i})$ can be of three types:

Type I. $(s, -i, -2l, l, s)$, where $s \in \{1,2\}$, $0 \leq l \leq i$ (circled in blue);

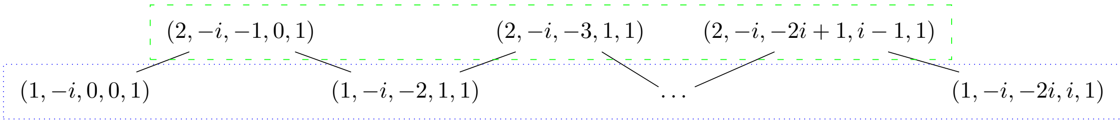


Figure 2.3: Structure of $\mathbb{H}(\mathbf{u}^{-i})e_1$ as a $\mathbf{d}$-module

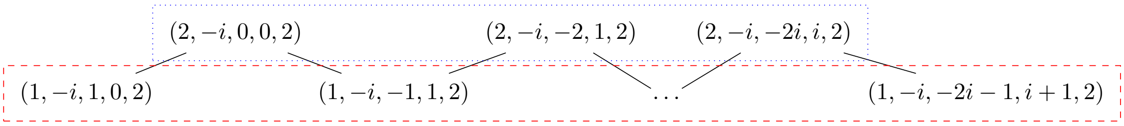


Figure 2.4: Structure of $\mathbb{H}(\mathbf{u}^{-i})e_2$ as a $\mathbf{d}$-module

Type II. $(2, -i, -2l + 1, l - 1, 1)$, where $1 \leq l \leq i$ (circled in green);

Type III. $(1, -i, -2l - 1, l + 1, 2)$, where $-1 \leq l \leq i$ (circled in red).

Remark 2.4.8. We can deduce the three types from the explicit description of $\mathcal{P}_{\leq 0}$ as

$$\frac{1-i}{2} \leq a \leq -i \quad \Leftrightarrow \quad 1 \leq 2a + i \leq -i.$$

Set $l = 2a + i$ and we are almost done: the range for l must be corrected since elements of the same type with different shifts of l appear in the description, e.g. $(2, i, 1 - 2(2a + i), 2a + i - 1, 1)$ and $(2, i, 3 - 2(2a + i), 2a + i - 2, 1)$, which we may rewrite

$$(2, i, -2l + 1, l - 1, 1), (2, i, -2(l - 1) + 1, (l - 1) - 1, 1).$$

Let us multiply elements of different types together:

I × I: $(s_1, -i_1, -2l_1, l_1, s_1) \times (s_2, -i_2, -2l_2, l_2, s_2) = (s_1, -(i_1 + i_2), -2(l_1 + l_2), l_1 + l_2, s_1)$ if $s_1 = s_2$ (zero otherwise), where

$$\left.\begin{array}{l} 0 \leq l_1 \leq i_1 \\ 0 \leq l_2 \leq i_2 \end{array}\right\} \Rightarrow 0 \leq l_1 + l_2 \leq i_1 + i_2,$$

which means type I elements multiplied with type I elements give type I elements.

I × II: $(s_1, -i_1, -2l_1, l_1, s_1) \times (2, -i_2, -2l_2 + 1, l_2 - 1, 1) = (2, -(i_1 + i_2), -2(l_1 + l_2) + 1, (l_1 + l_2) - 1, 1)$ if $s_1 = 2$ (zero otherwise), where

$$\left.\begin{array}{l} 0 \leq l_1 \leq i_1 \\ 1 \leq l_2 \leq i_2 \end{array}\right\} \Rightarrow 1 \leq l_1 + l_2 \leq i_1 + i_2,$$

which means type I elements multiplied with type II elements give type II elements.

I × III: $(s_1, -i_1, -2l_1, l_1, s_1) \times (1, -i_2, -2l_2 - 1, l_2 + 1, 2) = (1, -(i_1 + i_2), -2(l_1 + l_2) - 1, (l_1 + l_2) + 1, 2)$ if $s_1 = 1$ (zero otherwise), where

$$\left.\begin{array}{l} 0 \leq l_1 \leq i_1 \\ -1 \leq l_2 \leq i_2 \end{array}\right\} \Rightarrow -1 \leq l_1 + l_2 \leq i_1 + i_2,$$

which means type I elements multiplied with type III elements give type III elements.

II × I: $(2, -i_1, -2l_1 + 1, l_1 - 1, 1) \times (s_2, -i_2, -2l_2, l_2, s_2) = (2, -(i_1 + i_2), -2(l_1 + l_2) + 1, (l_1 + l_2) - 1, 1)$ if $s_2 = 1$ (zero otherwise), where

$$\left.\begin{array}{l} 1 \leq l_1 \leq i_1 \\ 0 \leq l_2 \leq i_2 \end{array}\right\} \Rightarrow 1 \leq l_1 + l_2 \leq i_1 + i_2,$$

which means type II elements multiplied with type I elements give type II elements.

$II \times II$: Multiplying type II elements together gives zero as idempotents do not match.

$II \times III$: $(2, -i_1, -2l_1+1, l_1-1, 1) \times (1, -i_2, -2l_2-1, l_2+1, 2) = (2, -(i_1+i_2), -2(l_1+l_2), (l_1+l_2), 2)$, where

$$\left.\begin{array}{l} 1 \leq l_1 \leq i_1 \\ -1 \leq l_2 \leq i_2 \end{array}\right\} \Rightarrow 0 \leq l_1 + l_2 \leq i_1 + i_2,$$

which means type II elements multiplied with type III elements give type I elements with idempotent $s_1 = 2$.

$III \times I$: $(1, -i_1, -2l_1-1, l_1+1, 2) \times (s_2, -i_2, -2l_2, l_2, s_2) = (1, -(i_1+i_2), -2(l_1+l_2)-1, (l_1+l_2)+1, 2)$ if $s_2 = 2$ (zero otherwise), where

$$\left.\begin{array}{l} -1 \leq l_1 \leq i_1 \\ 0 \leq l_2 \leq i_2 \end{array}\right\} \Rightarrow -1 \leq l_1 + l_2 \leq i_1 + i_2,$$

which means type III elements multiplied with type I elements give type III elements.

$III \times II$: $(1, -i_1, -2l_1-1, l_1+1, 2) \times (2, -i_2, -2l_2+1, l_2-1, 1) = (1, -(i_1+i_2), -2(l_1+l_2), (l_1+l_2), 1)$, where

$$\left.\begin{array}{l} -1 \leq l_1 \leq i_1 \\ 1 \leq l_2 \leq i_2 \end{array}\right\} \Rightarrow 0 \leq l_1 + l_2 \leq i_1 + i_2,$$

which means type III elements multiplied with type II elements give type I elements with idempotent $s_1 = 1$.

$III \times III$: Multiplying type III elements together gives zero as idempotents do not match.

These calculations are summarised in Table 2.1.

×	I	II	III
I	I	II	III
II	II	0	I
III	III	I	0

Table 2.1: Multiplication table of elements (by type) of $\mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})$

Remark 2.4.9. We see from the computations that $m_2 : \mathbb{H}(\mathbf{u}^{-i_1}) \otimes \mathbb{H}(\mathbf{u}^{-i_2}) \to \mathbb{H}(\mathbf{u}^{-(i_1+i_2)})$ is surjective for any $i_1, i_2 \geq 0$, and thus $m_2 : \coprod_{i_1+i_2=i_3} \mathbb{H}(\mathbf{u}^{-i_1}) \otimes \mathbb{H}(\mathbf{u}^{-i_2}) \to \mathbb{H}(\mathbf{u}^{-i_3})$ is also surjective.

Chapter 3

Quiver of $\mathbf{w}_q$ for $p = 2$

In this chapter, we give a description of the quiver of $\mathbf{w}_q$ in the case $p = 2$. Let us first understand how to decompose chained elements.

3.1 Decomposition of chained elements of $\mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})$

Since the multiplication m_2 is surjective (cf. Remark 2.4.9), we can decompose any element from $\mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})$ as follows.

Lemma 3.1.1. *Let $(s,i,j,k,t) \in \mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})$. Let $-1 \geq \hat{i} \geq i$. There exists an integer $\tilde{i} \leq 0$ such that (s,i,j,k,t) decomposes as the product*

$$(s,i,j,k,t) = (s,\tilde{i},-2\tilde{l},\tilde{l},s)(s,\hat{i},j+2\tilde{l},k-\tilde{l},t), \tag{*}$$

such that

1. *$(s,\tilde{i},-2\tilde{l},\tilde{l},s)$ is a type I element;*
2. *$(s,\hat{i},j+2\tilde{l},k-\tilde{l},t) \in \mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})$;*
3. *$i = \tilde{i}+\hat{i}$, $0 \leq \tilde{l} \leq -\tilde{i}$.*

Proof. If we let $\hat{i} = i$, then $\tilde{i} = 0$, and there is a trivial solution for the type I element, namely $(s,0,0,0,s)$, and $(s,i,j,k,t) = (s,0,0,0,s)(s,i,j,k,t)$.

Let $i < \hat{i} \leq 0$ (so that $\tilde{i} = i - \hat{i}$). We want to show there exists a type I element such that (*) holds, i.e. we need to show there exists $\tilde{i} \leq 0$ and $0 \leq \tilde{l} \leq -i$. Note that the type of $(s,\hat{i},j+2\tilde{l},k-\tilde{l},t)$ is the same as that of the element we start with, (s,i,j,k,t) (cf. Table 2.1).

Type I. Assume that (s,i,j,k,t) is of type I, i.e. $(s,i,j,k,t) = (s,i,-2l,l,t)$ with $0 \leq l \leq -i$. Then $(s,\hat{i},j+2\tilde{l},k-\tilde{l},t) = (s,\hat{i},-2(l-\tilde{l}),l-\tilde{l},t)$ is of type I too, therefore we have $0 \leq l-\tilde{l} \leq -\hat{i}$. Rearranging this inequality yields

$$l+\hat{i} \leq \tilde{l} \leq l.$$

For $(s,\tilde{i},-2\tilde{l},\tilde{l},s)$ to be a type I element, $\tilde{l}$ must satisfy $0 \leq \tilde{l} \leq -\tilde{i}$ with $\tilde{i} \leq 0$. Thus, we obtain the following inequality

$$\max\{l+\hat{i},0\} \leq \tilde{l} \leq \min\{l,-\tilde{i}\},$$

and we must show the corresponding interval in the integers is not empty.

(a) If $\max\{l+\hat{i},0\} = l+\hat{i}$ and $\min\{l,-\tilde{i}\} = -\tilde{i}$, then

$$-\tilde{i} - l - \hat{i} = -l - i \geq 0$$

since $0 \leq l \leq -i$;

(b) If $\max\{l+\hat{i},0\} = 0$ and $\min\{l,-\tilde{i}\} = -\tilde{i}$, then

$$-\tilde{i} - 0 = -\tilde{i} \geq 0$$

since such a type I element would satisfy $\tilde{i} \leq 0$;

(c) If $\max\{l+\hat{i},0\} = l+\hat{i}$ and $\min\{l,-\tilde{i}\} = l$, then

$$l - l - \hat{i} = -\hat{i} \geq 0$$

since $\hat{i} \leq 0$;

(d) If $\max\{l+\hat{i},0\} = 0$ and $\min\{l,-\tilde{i}\} = l$, then

$$l - 0 = l \geq 0$$

since $0 \leq l \leq -i$ by assumption.

Type II. Assume that (s,i,j,k,t) is of type II, i.e. $(s,i,j,k,t) = (2,i,-2l+1,l-1,1)$ with $1 \leq l \leq -i$. In particular, we need to assume that $i \leq -1$ as type II elements only occur then. Then $(s,\hat{i},j+2\tilde{l},k-\tilde{l},t) = (2,\hat{i},-2(l-\tilde{l})+1,l-\tilde{l}-1,1)$ is of type II too, therefore we have $1 \leq l-\tilde{l} \leq -\hat{i}$ and $\hat{i} \leq -1$. Rearranging this inequality yields

$$l+\hat{i} \leq \tilde{l} \leq l-1.$$

For $(s,\tilde{i},-2\tilde{l},\tilde{l},s)$ to be a type I element, $\tilde{l}$ must satisfy $0 \leq \tilde{l} \leq -\tilde{i}$ with $\tilde{i} \leq 0$. Thus, we obtain the following inequality

$$\max\{l+\hat{i},0\} \leq \tilde{l} \leq \min\{l-1,-\tilde{i}\},$$

and we must show the corresponding interval in the integers is not empty.

(a) If $\max\{l+\hat{i},0\} = l+\hat{i}$ and $\min\{l-1,-\tilde{i}\} = -\tilde{i}$, then

$$-\tilde{i} - l - \hat{i} = -l - i \geq 0$$

since $1 \leq l \leq -i$;

(b) If $\max\{l+\hat{i},0\} = 0$ and $\min\{l-1,-\tilde{i}\} = -\tilde{i}$, then

$$-\tilde{i} - 0 = -\tilde{i} \geq 0$$

since such a type I element would satisfy $\tilde{i} \leq 0$;

(c) If $\max\{l+\hat{i},0\} = l+\hat{i}$ and $\min\{l-1,-\tilde{i}\} = l-1$, then

$$l - 1 - l - \hat{i} = -1 - \hat{i} \geq 0$$

since $\hat{i} \leq -1$;

(d) If $\max\{l+\hat{i},0\} = 0$ and $\min\{l-1,-\tilde{i}\} = l-1$, then

$$l - 1 - 0 = l - 1 \geq 0$$

since $1 \leq l \leq -i$ by assumption.

Type III. Assume that (s,i,j,k,t) is of type III, i.e. $(s,i,j,k,t) = (1,i,-2l-1,l+1,2)$ with $-1 \leq l \leq -i$. Then $(s,\hat{\imath},j+2\tilde{l},k-\tilde{l},t) = (1,\hat{\imath},-2(l-\tilde{l})-1,l-\tilde{l}+1,2)$ is of type III too, therefore we have $-1 \leq l-\tilde{l} \leq -\hat{\imath}$. Rearranging this inequality yields

$$l+\hat{\imath} \leq \tilde{l} \leq l+1.$$

For $(s,\tilde{\imath},-2\tilde{l},\tilde{l},s)$ to be a type I element, $\tilde{l}$ must satisfy $0 \leq \tilde{l} \leq -\tilde{\imath}$ with $\tilde{\imath} \leq 0$. Thus, we obtain the following inequality

$$\max\{l+\hat{\imath},0\} \leq \tilde{l} \leq \min\{l+1,-\tilde{\imath}\},$$

and we must show the corresponding interval in the integers is not empty.

(a) If $\max\{l+\hat{\imath},0\} = l+\hat{\imath}$ and $\min\{l+1,-\tilde{\imath}\} = -\tilde{\imath}$, then

$$-\tilde{\imath}-l-\hat{\imath} = -l-i \geq 0$$

since $0 \leq l \leq -i$;

(b) If $\max\{l+\hat{\imath},0\} = 0$ and $\min\{l+1,-\tilde{\imath}\} = -\tilde{\imath}$, then

$$-\tilde{\imath}-0 = -\tilde{\imath} \geq 0$$

since such a type I element would satisfy $\tilde{\imath} \leq 0$;

(c) If $\max\{l+\hat{\imath},0\} = l+\hat{\imath}$ and $\min\{l+1,-\tilde{\imath}\} = l+1$, then

$$l+1-l-\hat{\imath} = 1-\hat{\imath} \geq 2 \geq 0$$

since $\hat{\imath} \leq -1$;

(d) If $\max\{l+\hat{\imath},0\} = 0$ and $\min\{l+1,-\tilde{\imath}\} = l+1$, then

$$l+1-0 = l+1 \geq 0$$

since $-1 \leq l \leq -i$ by assumption.

□

Remark 3.1.2. This means in particular for any element (s,i_0,j,k,t) of $\mathbb{H}(\mathbf{u}^{i_0})$, and for any integer $-i_0 \geq n \geq 0$, we can choose an element of the same type $(s,i_0+n,j+2l,k-l,t)$ in $\mathbb{H}(\mathbf{u}^{i_0+n})$ such that (s,i_0,j,k,t) decomposes as the product of a type I element $(s,-n,-2l,l,s) \in \mathbb{H}(\mathbf{u}^{-n})$ by that same type element we chose. We have one degree of freedom in the choice of the i-degree for the decomposition.

Corollary 3.1.3. *Let* $(s_1,i_1,j_1,k_1,t_1) \otimes (s_2,j_1,j_2,k_2,t_2) \otimes \ldots \otimes (s_q,j_{q-1},j_q,k_q,t_q) \in \left(\mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})\right)^{\otimes q}$ *be a chained element with components in* $\mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})$*. Then there exists a decomposition*

$$\begin{aligned}
& (s_1,i_1,j_1,k_1,t_1) \otimes (s_2,j_1,j_2,k_2,t_2) \otimes \ldots \otimes (s_q,j_{q-1},j_q,k_q,t_q) \\
= \; & (s_1,\tilde{\imath}_1,-2l_1,l_1,s_1) \otimes (s_2,-2l_1,-2l_2,l_2,s_2) \otimes \ldots \otimes (s_q,-2l_{q-1},-2l_q,l_q,s_q) \\
& \cdot (s_1,\hat{\imath}_1,j_1+2l_1,k_1-l_1,t_1) \otimes (s_2,j_1+2l_1,j_2+2l_2,k_2-l_2,t_2) \otimes \ldots \\
& \qquad \otimes (s_q,j_{q-1}+2l_{q-1},j_q+2l_q,k_q-l_q,t_q)
\end{aligned}$$

such that, for all $1 \leq n \leq q$,

1. $(s_n,\tilde{\imath}_n,-2\tilde{l}_n,\tilde{l}_n,s_n)$ *is a type I element;*

2. $(s_n,\hat{\imath}_n,j_n+2\tilde{l}_n,k_n-\tilde{l}_n,t_n) \in \mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})$*;*

3. $i_n = \tilde{i}_n + \hat{i}_n$, $0 \leq \tilde{l}_n \leq -\tilde{i}_n$.

In addition, both $(s_1, \tilde{i}_1, -2l_1, l_1, s_1) \otimes \ldots \otimes (s_q, -2l_{q-1}, -2l_q, l_q, s_q)$ *and* $(s_1, \hat{i}_1, j_1 + 2l_1, k_1 - l_1, t_1) \otimes \ldots \otimes (s_q, j_{q-1} + 2l_{q-1}, j_q + 2l_q, k_q - l_q, t_q)$ *are chained elements with components in* $\mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})$.

Proof. From Lemma 3.1.1, we know that for every $(s_n, \hat{i}_n, j_n + 2l_n, k_n - l_n, t_n)$, there is a type I element $(s_n, \tilde{i}_n, -2l_n, l_n, s_n)$ such that their product (with the type I element on the left to respect idempotents) yields $(s_n, i_n, j_n, k_n, t_n)$. In particular, we can choose $\hat{i}_n = j_{n-1} + 2l_{n-1}$ for $n = 2, \ldots, q$, and as a result $\tilde{i} = i_n - \hat{i} = -2l_{n-1}$. That means the chaining rule is preserved and both elements obtained are chained elements. □

Remark 3.1.4. The decomposition might be "trivial" in the sense that all the type I elements on the left could be idempotents of $\mathbf{d}$. We shall use that decomposition result later on and the only thing we will need, to make sure the decomposition is not trivial, is if at least one term in the left component is not an idempotent.

3.2 Action of $\mathbb{H}(\mathbf{u})$ on $\mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})$

We now want to understand how the elements of $\mathbf{u}$ act on $\mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})$. Let $(s_1, -i_1, j_1, l_1, t_1) \in \mathbb{H}(\mathbf{u}^{-i_1})$, $i_1 \geq 1$ and let $(s_2, 1, 1, 0, 3 - s_2) \in \mathbb{H}(\mathbf{u})$. We have three cases:

I. $(s_1, -i_1, j_1, l_1, t_1) = (s_1, -i_1, -2l_1, l_1, s_1)$ is of type I. Then we have:

$$\begin{aligned}(s_1, -i_1, -2l_1, l_1, s_1) \times (s_2, 1, 1, 0, 3 - s_2) &= (s_1, -(i_1 - 1), -2l_1 + 1, l_1, 3 - s_2) \\ &\qquad \text{if } s_1 = s_2 (\text{zero otherwise}) \\ (s_2, 1, 1, 0, 3 - s_2) \times (s_1, -i_1, -2l_1, l_1, s_1) &= (s_2, -(i_1 - 1), -2l_1 + 1, l_1, 3 - s_2) \\ &\qquad \text{if } s_1 = 3 - s_2 (\text{zero otherwise})\end{aligned}$$

In both cases, we obtain the expression $(s_2, -(i_1 - 1), -2l_1 + 1, l_1, 3 - s_2)$, which we can rewrite: $(s_2, -(i_1 - 1), -2(l_1 - 1) - 1, (l_1 - 1) + 1, 3 - s_2)$. For that to be an element of $\mathbb{H}(\mathbf{u}^{-(i_1-1)})$, since idempotents on the left and on the right are different, it must be of type II or III. We see that $0 \leq l_1 \leq i_1$, so that $-1 \leq l_1 - 1 \leq i_1 - 1$. Therefore, it is an element of type III if $s_2 = 1$ and zero otherwise.

II. $(s_1, -i_1, j_1, l_1, t_1) = (2, -i_1, -2l_1 + 1, l_1 - 1, 1)$ is of type II. Then we have:

$$\begin{aligned}(2, -i_1, -2l_1 + 1, l_1 - 1, 1) \times (s_2, 1, 1, 0, 3 - s_2) &= (2, -(i_1 - 1), -2l_1 + 2, l_1 - 1, 2) \\ &\qquad \text{if } s_2 = 1 (\text{zero otherwise}); \\ (s_2, 1, 1, 0, 3 - s_2) \times (2, -i_1, -2l_1 + 1, l_1 - 1, 1) &= (1, -(i_1 - 1), -2l_1 + 2, l_1 - 1, 1) \\ &\qquad \text{if } s_2 = 1 (\text{zero otherwise}).\end{aligned}$$

In both cases, we obtain an expression of the form $(s_1, -(i_1 - 1), -2(l_1 - 1), l_1 - 1, s_1)$ and this clearly is an element of type I of $\mathbb{H}(\mathbf{u}^{-(i_1-1)})$. We note that $1 \leq l_1 \leq i_1$, and so $0 \leq l_1 - 1 \leq i_1 - 1$, which means that

$$m_2 : e_2 \mathbb{H}(\mathbf{u}^{-i}) e_1 \otimes e_1 \mathbb{H}(\mathbf{u}) e_2 \twoheadrightarrow e_2 \mathbb{H}(\mathbf{u}^{-i+1}) e_2,$$

and

$$m_2 : e_1 \mathbb{H}(\mathbf{u}) e_2 \otimes e_2 \mathbb{H}(\mathbf{u}^{-i}) e_1 \twoheadrightarrow e_1 \mathbb{H}(\mathbf{u}^{-i+1}) e_1$$

are surjective maps.

III. $(s_1, -i_1, j_1, l_1, t_1) = (1, -i_1, -2l_1 - 1, l_1 + 1, 2)$ is of type III. Then we have:

$$(1, -i_1, -2l_1 - 1, l_1 + 1, 2) \times (s_2, 1, 1, 0, 3 - s_2) = (1, -(i_1 - 1), -2l_1, l_1 + 1, 1) \quad \text{if } s_2 = 2 \text{(zero otherwise)};$$
$$(s_2, 1, 1, 0, 3 - s_2) \times (1, -i_1, -2l_1 - 1, l_1 + 1, 2) = (2, -(i_1 - 1), -2l_1, l_1 + 1, 2) \quad \text{if } s_2 = 2 \text{(zero otherwise)}.$$

In both cases, we obtain an expression of the form $(s_1, -(i_1 - 1), -2l_1, l_1 + 1, s_1)$ and this clearly should be an element of type I of $\mathbb{H}(\mathbf{u}^{-(i_1-1)})$ since idempotents on the left and on the right agree. We note however that the j-degree and the k-degree of that expression are not related appropriately, which means that this expression does not correspond to any element of $\mathbb{H}(\mathbf{u}^{-(i_1-1)})$ and the product is zero.

To summarise, we see that $\mathbb{H}(\mathbf{u})$ has a non trivial action on $\mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})$ only through the element $(1, 1, 1, 0, 2) = e_1 \otimes e_2^*$, and it sends type I elements to type III elements, and it sends type II elements on type I elements. We can see it pictorially in Figure 3.1.

3.3 Irreducible monomials

Definition 3.3.1. A basis element of $\mathbf{w}_q$ is called an irreducible monomial if it cannot be written as a non-trivial product of two other basis elements of $\mathbf{w}_q$.

We denote by V_q the set of irreducible monomials of $\mathbf{w}_q$. It is non-empty as it contains the idempotents of $\mathbf{w}_q$, namely all the basis elements of the form

$$(s_1, 0, 0, 0, s_1) \otimes \ldots \otimes (s_q, 0, 0, 0, s_q) = e_{s_1} \otimes \ldots \otimes e_{s_q}.$$

Since $\mathbf{w}_q$ is a subalgebra of $\mathbf{d} \otimes (\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}}))^{\otimes q-1}$, we know that the first component of any basis element of $\mathbf{w}_q$ is an element of $\mathbf{d}$, i.e. it is either e_1, e_2, ξ or x.

3.3.1 Irreducible monomials starting with an idempotent of d

Lemma 3.3.2. *Let $a_1 \otimes \ldots \otimes a_q \in V_q$. Then $e_1 \otimes a_1 \otimes \ldots \otimes a_q$ and $e_2 \otimes a_1 \otimes \ldots \otimes a_q$ are irreducible monomials of $\mathbf{w}_{q+1}$.*

Proof. Suppose $e_i \otimes a_1 \otimes \ldots \otimes a_q \in \mathbf{w}_{q+1}$ is reducible, i.e. $e_i \otimes a_1 \otimes \ldots \otimes a_q = b_0 \otimes \ldots \otimes b_q \cdot c_0 \otimes \ldots \otimes c_q$, with $b_0 \otimes \ldots \otimes b_q, c_0 \otimes \ldots \otimes c_q \in \mathbf{w}_{q+1}$. Since $b_0, c_0 \in \mathbf{d}$ and $e_i = b_0 c_0$, we must have $b_0 = c_0 = e_i$. This shows $b_1 \otimes \ldots \otimes b_q$ and $c_1 \otimes \ldots \otimes c_q$ are elements of $\mathbf{w}_q$ as both b_1 and c_1 are in $\mathbf{d}$, and their product is $a_1 \otimes \ldots \otimes a_q$. ☐

Corollary 3.3.3. *The set V_{q+1} contains two copies of V_q: $\{e_1 \otimes A, e_2 \otimes A | A \in V_q\} \subset V_{q+1}$.*

3.3.2 Irreducible monomials starting with ξ

Lemma 3.3.4. *Let $a_1 \otimes \ldots \otimes a_q \in \mathbf{w}_q$ such that $a_1 = \xi$. Then $a_1 \otimes \ldots \otimes a_q \in V_q$.*

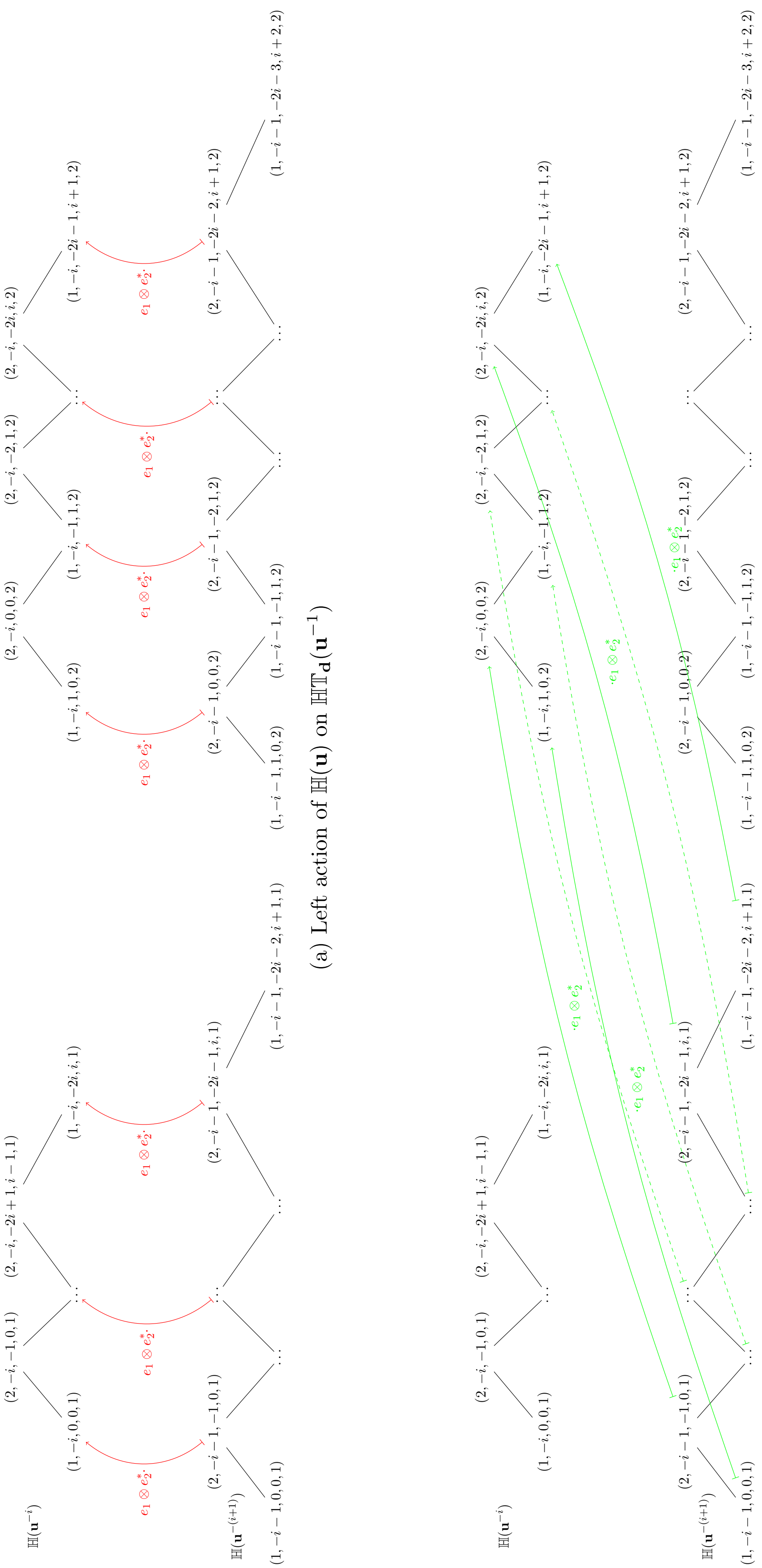
Proof. Since $a_1 = \xi = (1, 0, 1, 0, 2)$ has j-degree 1, a_2 must be an element of $\mathbb{H}(\mathbf{u}) = \{(1, 1, 1, 0, 2), (2, 1, 1, 0, 1)\}$. In either case, a_2 has j-degree 1. More generally, we see that $a_l \in \mathbb{H}(\mathbf{u})$ for $2 \leq l \leq q$. No a_l can be decomposed for $1 \leq l \leq q$, hence $a_1 \otimes \ldots \otimes a_q \in V_q$. ☐

Corollary 3.3.5. *For $1 \leq l \leq q$, any basis element of the form*

$$e_{s_1} \otimes \ldots \otimes e_{s_{l-1}} \otimes \xi \otimes (e_{s_{l+1}} \otimes e^*_{3-s_{l+1}}) \otimes \ldots \otimes (e_{s_q} \otimes e^*_{3-s_q})$$

is in V_q.

Proof. Use Lemma 3.3.2 together with Lemma 3.3.4. ☐



(a) Left action of $\mathbb{H}(\mathbf{u})$ on $\mathbb{HT}_\mathbf{d}(\mathbf{u}^{-1})$

(b) Right action of $\mathbb{H}(\mathbf{u})$ on $\mathbb{HT}_\mathbf{d}(\mathbf{u}^{-1})$

Figure 3.1: Action of $\mathbb{H}(\mathbf{u})$ on $\mathbb{HT}_\mathbf{d}(\mathbf{u}^{-1})$

3.3.3 Irreducible monomials starting with x

Lemma 3.3.6. *Let $a_1 \otimes \ldots \otimes a_q \in V_q$ such that $a_1 = x$. Suppose there is an index $i_0 > 1$ such that $a_{i_0} \in \mathbf{d}$. Then $a_{i_0} \in \{e_1, e_2\}$.*

Proof. Suppose that $a_{i_0} \in \{\xi, x\}$. Then we can write

$$a_1 \otimes \ldots \otimes a_q = a_1 \otimes \ldots \otimes a_{i_0-1} \otimes e_1 \otimes e_{s_{i_0+1}} \otimes \ldots \otimes e_{s_q} \cdot e_{t_1} \otimes \ldots \otimes e_{t_{i_0-1}} \otimes a_{i_0} \otimes a_{i_0+1} \otimes \ldots \otimes a_q,$$

where e_{s_l}, resp. e_{t_l}, is the left, resp. right, idempotent of $a_l = (s_l, i_l, j_l, k_l, t_l)$. This decomposition is not trivial since at least a_{i_0} is not an idempotent of $\mathbf{d}$. □

Lemma 3.3.7. *Let $a_1 \otimes \ldots \otimes a_q \in \mathbf{w}_q$ such that $a_1 = x$. Suppose there is an index $l > 1$ such that $\deg_k a_l \geq 1$. Then $a_1 \otimes \ldots \otimes a_q \notin V_q$. In particular, irreducible monomials of $\mathbf{w}_q$ have total k-degree at most 1.*

Proof. Writing $a_1 \otimes a_2 \otimes \ldots \otimes a_q = (1, 0, -1, 1, 2) \otimes (s_2, -1, j_2, k_2, t_2) \otimes \ldots \otimes (s_q, j_{q-1}, j_q, k_q, t_q)$, we let $l_0 = \min\{2 \leq l \leq q \;\mid\; k_l \geq 1\}$.

Case $l_0 = 2$. The element $(s_2, -1, j_2, k_2, t_2)$ is such that $k_2 \geq 1$. We can choose it among

$$\begin{array}{rcl} (s, -1, -2, 1, s) & = & x \otimes e_1 \text{ or } e_2 \otimes x \\ (1, -1, -1, 1, 2) & = & x \otimes \xi \\ (1, -1, -3, 2, 2) & = & x \otimes x \end{array}$$

We can write those elements as follows

$$\begin{array}{rcl} x \otimes e_1 & = & x \cdot e_2 \otimes e_1 \\ e_2 \otimes x & = & e_2 \otimes e_1 \cdot x \\ x \otimes \xi & = & x \cdot e_2 \otimes \xi \\ x \otimes x & = & x \cdot e_2 \otimes x \end{array}$$

and note that $e_2 \otimes e_1 = (2, -1, -1, 0, 1)$ and $e_2 \otimes x = (2, -1, -2, 1, 2)$ have negative j-degree, i.e. the element a_3 following them has negative i-degree; $e_2 \otimes \xi = (2, -1, 0, 0, 2)$ has j-degree 0 so the element a_3 following must be chosen in $\mathbf{d}$.

Let $n_0 = \min\{l_0 < n \leq q \;\mid\; i_n = j_{n-1} \geq 0\}$. Then $a_n \in \mathbb{H}(\mathbf{u}^{-i_n})$ ($i_n \geq 1$) for all $l_0 = 2 \leq n \leq n_0 - 1$. Applying Corollary 3.1.3, there exists a decomposition

$$a_3 \otimes \ldots \otimes a_{n_0-1} = b_3 \otimes \ldots \otimes b_{n_0-1} \cdot c_3 \otimes \ldots \otimes c_{n_0-1},$$

such that we can choose the i-degree of b_3 and c_3 thanks to the decomposition of a_2 so that the chaining rule is respected. For a_{n_0} to be in $\mathbf{d}$, resp. $\mathbb{H}(\mathbf{u})$, a_{n_0-1} must have j-degree 0, resp. 1. So it is one of

$$\begin{array}{rcl} (s_{n_0-1}, i_{n_0-1}, 0, 0, s_{n_0-1}) & = & \xi^{\otimes -i_{n_0-1}} \otimes e_1 \text{ or } e_2 \otimes \xi^{\otimes -i_{n_0-1}} \\ (1, i_{n_0-1}, 1, 0, 2) & = & \xi^{\otimes -i_{n_0-1}+1} \end{array}$$

and we see that we can choose $(s_{n_0-1}, i_{n_0-1}+j_{n_0-1}, 0, 0, s_{n_0-1}) = \xi^{\otimes -i_{n_0-1}-j_{n_0-1}} \otimes e_1$ or $e_2 \otimes \xi^{\otimes -i_{n_0-1}-j_{n_0-1}}$ for the type I element b_{n_0-1} of the decomposition, and, depending on the j-degree, $(s_{n_0-1}, -1, 0, 0, s_{n_0-1}) = \xi \otimes e_1$ or $e_2 \otimes \xi$ ($j_{n_0-1} = 1$) or $(1, 0, 1, 0, 2) = \xi$ ($j_{n_0-1} = 0$) for the element c_{n_0-1}. Finally, we can decompose $a_1 \otimes a_2 \otimes \ldots \otimes a_q$ as

$$\begin{array}{rl} & (1, 0, -1, 1, 2) \otimes (s_2, -1, j_2, k_2, t_2) \otimes \ldots \otimes (s_q, j_{q-1}, j_q, k_q, t_q) \\ = & (1, 0, 0, 0, 1) \otimes (1, 0, -1, 1, 2) \otimes b_3 \otimes \ldots \otimes b_{n_0-2} \otimes \\ & (s_{n_0-1}, i_{n_0-1} + j_{n_0-1}, 0, 0, s_{n_0-1}) \otimes (s_{n_0}, 0, 0, 0, s_{n_0}) \otimes \ldots \otimes (s_q, 0, 0, 0, s_q) \\ & \cdot (1, 0, -1, 1, 2) \otimes \left\{ \begin{array}{c} (2, -1, -1, 0, 1) \\ (2, -1, -2, 1, 1) \\ (2, -1, 0, 0, 1) \end{array} \right. \otimes c_3 \otimes \ldots \otimes c_{n_0-2} \otimes \\ & \qquad \left\{ \begin{array}{r} (s_{n_0-1}, -1, 0, 0, s_{n_0-1}) \\ (1, 0, 1, 0, 2) \end{array} \right. \otimes a_{n_0} \ldots \otimes a_q \end{array}$$

if a_2 can be decomposed as $x \cdot \tilde{a}_2$, and as

$$\begin{array}{rl} & (1,0,-1,1,2) \otimes (s_2,-1,j_2,k_2,t_2) \otimes \ldots \otimes (s_q,j_{q-1},j_q,k_q,t_q) \\ = & (1,0,-1,1,2) \otimes (2,-1,-1,0,1) \otimes b_3 \otimes \ldots \otimes b_{n_0-2} \otimes \\ & (s_{n_0-1}, i_{n_0-1}+j_{n_0-1},0,0,s_{n_0-1}) \otimes (s_{n_0},0,0,0,s_{n_0}) \otimes \ldots \otimes (s_q,0,0,0,s_q) \\ & \cdot (2,0,0,0,2) \otimes (1,0,-1,1,2) \otimes c_3 \otimes \ldots \otimes c_{n_0-2} \otimes \\ & \qquad \left\{ \begin{array}{r} (s_{n_0-1},-1,0,0,s_{n_0-1}) \\ (1,0,1,0,2) \end{array} \right. \otimes a_{n_0} \ldots \otimes a_q \end{array}$$

if $a_2 = e_2 \otimes x = e_2 \otimes e_1 \cdot x$.

Case $l_0 > 2$**.** This means that $k_l = 0$ for all $1 < l < l_0$. The elements a_2 up to a_{l_0-1} can be chosen among

$$\begin{array}{rcl} (s,-i,0,0,s) & = & \xi^{\otimes i} \otimes e_1 \text{ or } e_2 \otimes \xi^{\otimes i} \\ (2,-i,-1,0,1) & = & e_2 \otimes \xi^{\otimes i-1} \otimes e_1 \\ (1,-i,1,0,2) & = & \xi^{\otimes i+1} \\ (s,0,0,0,s) & = & e_1 \text{ or } e_2 \\ (1,0,1,0,2) & = & \xi \\ (s,1,1,0,3-s) & = & e_1 \otimes e_2^* \text{ or } e_2 \otimes e_1^* \end{array}$$

and this is an exhaustive list of k-degree 0 elements of $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})^{\leq 1}$. One must obviously choose a_α before choosing $a_{\alpha+1}$ to respect the construction of $\mathbf{w}_q$ from $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})^{\leq 1}$.

If at any point an element of j-degree 1 is chosen for an a_α, i.e. $(1,-i,1,0,2)$, $(1,0,1,0,2)$ or $(s,1,1,0,3-s)$, the following elements must all be in $\mathbb{H}(\mathbf{u})$ as seen in the proof of Lemma 3.3.4, including a_{l_0}. This is a contradiction as elements of $\mathbb{H}(\mathbf{u})$ have k-degree 0.

Our choice is thus narrowed to

$$\begin{array}{rcl} (s,-i,0,0,s) & = & \xi^{\otimes i} \otimes e_1 \text{ or } e_2 \otimes \xi^{\otimes i} \\ (2,-i,-1,0,1) & = & e_2 \otimes \xi^{\otimes i-1} \otimes e_1 \\ (s,0,0,0,s) & = & e_1 \text{ or } e_2 \end{array}$$

In particular, a_2 is chosen among $\xi \otimes e_1$, $e_2 \otimes \xi$, and $e_2 \otimes e_1$. If it is $\xi \otimes e_1$ or $e_2 \otimes \xi$, since they both have j-degree 0, a_3 is an element of $\mathbf{d}$ and thus either e_1 or e_2 by Lemma 3.3.6. Inductively, since a_3 has j-degree 0, we see that $a_n = (s_n,0,0,0,s_n) = e_{s_n}$ for all $3 \leq n < l_0$ and $a_{l_0} = x$. We can then decompose $a_1 \otimes \ldots \otimes a_q$ as

$$\begin{array}{rl} & (1,0,-1,1,2) \otimes (s_2,-1,j_2,k_2,t_2) \otimes \ldots \otimes (s_q,j_{q-1},j_q,k_q,t_q) \\ = & (1,0,0,0,1) \otimes (s_2,0,0,0,s_2) \otimes \ldots \otimes (s_{l_0-1},0,0,0,s_{l_0-1}) \otimes (1,0,-1,1,2) \otimes \\ & \qquad a_{l_0+1} \otimes \ldots \otimes a_q \\ & \cdot (1,0,-1,1,2) \otimes a_2 \otimes \ldots \otimes a_{l_0-1} \otimes (2,0,0,0,2) \otimes (t_{l_0+1},0,0,0,t_{l_0+1}) \otimes \\ & \qquad \ldots \otimes (t_q,0,0,0,t_q) \end{array}$$

If $a_2 = e_2 \otimes e_1$, then $a_3 \in \mathbb{H}(\mathbf{u}^{-1})$ and it has k-degree 0: a_3 is either $\xi \otimes e_1$, $e_2 \otimes \xi$, and $e_2 \otimes e_1$. Choosing $\xi \otimes e_1$ or $e_2 \otimes \xi$, we see that we can apply the same argument as for $a_2 = \xi \otimes e_1$ or $e_2 \otimes \xi$. This extends to $a_n = \xi \otimes e_1$ or $e_2 \otimes \xi$ for $2 \leq n < l_0$. The only remaining case to study is that when $a_n = e_2 \otimes e_1$ for $2 \leq n < l_0$. That means a_{l_0} is an element of $\mathbb{H}(\mathbf{u}^{-1})$ with k-degree greater than or equal to one (by definition of l_0). So a_{l_0} is one of

$$\begin{array}{rcl} (1,-1,-2,1,1) & = & x \otimes e_1 \\ (2,-1,-2,1,2) & = & e_2 \otimes x \\ (1,-1,-1,1,2) & = & x \otimes \xi \\ (1,-1,-3,2,2) & = & x \otimes x \end{array}$$

We see that using the same reasoning as for $l_0 = 2$ yields a decomposition of $a_1 \otimes \ldots \otimes a_q$. □

Remark 3.3.8. This means that $\mathrm{Ext}^*(\Delta,\Delta)$ is generated in $\mathrm{Ext}^1(\Delta,\Delta)$ and $\mathrm{Ext}^0(\Delta,\Delta)$ elements. This is false for $p>2$; in Section 3 of [MT13], the Ext^1-quiver has a degree 2 arrow from the 8th simple to the 2nd.

Corollary 3.3.9. *Let $a_1\otimes\ldots\otimes a_q$ be an irreducible monomial of $\mathbf{w}_q$ such that $a_1=x$. Then $a_1\otimes\ldots\otimes a_q$ has one of the following forms:*

- $x\otimes(e_2\otimes e_1)\otimes\ldots\otimes(e_2\otimes e_1)$;
- $x\otimes(e_2\otimes e_1)\otimes\ldots\otimes(e_2\otimes e_1)\otimes(\xi\otimes e_1)\otimes e_{l_1}\otimes e_{l_2}\otimes\ldots e_{l_s}$ *if there exists* $1\le i\le s$ *such that* $l_i=2$ $(s>1)$;
- $x\otimes(e_2\otimes e_1)\otimes\ldots\otimes(e_2\otimes e_1)\otimes(e_2\otimes\xi)\otimes e_{l_1}\otimes e_{l_2}\otimes\ldots e_{l_r}$ *if there exists* $1\le i\le r$ *such that* $l_i=1$ $(r>1)$.

Proof. Since $a_1=x$ has j-degree -1, a_2 is in $\mathbb{H}(\mathbf{u}^{-1})$. Besides by the previous Lemma, it has k-degree 0, so it could be chosen among $\{e_2\otimes e_1,\xi\otimes e_1,e_2\otimes\xi,\xi\otimes\xi\}$. They have j-degrees comprised between -1 and 1.

(i) If $e_2\otimes e_1$ is chosen for a_2, then we have the same choices for a_3 as $e_2\otimes e_1$ has j-degree -1. Inductively, we see that we can choose $a_n=e_2\otimes e_1$ for $2\le n\le q$ and this element is indecomposable.

(ii) If $\xi\otimes e_1$ or $e_2\otimes\xi$ is chosen for a_2; the element a_3 has j-degree 0, so is in $\mathbf{d}$. By Lemma 3.3.6, a_3 is an idempotent and so must be $a_4,\ldots,a_q$ inductively.

In that case, the element obtained is irreducible if the "right" idempotents from $\mathbf{d}$ follow. Consider

$$x\otimes(\xi\otimes e_1)\otimes e_1\otimes e_1\otimes\ldots e_1.$$

This element is not irreducible because we have

$$\begin{aligned} & e_1\otimes\xi\otimes(e_1\otimes e_2^*)\otimes\ldots\otimes(e_1\otimes e_2^*)\\ \cdot\ & x\otimes(e_2\otimes e_1)\otimes\ldots\otimes(e_2\otimes e_1)\\ =\ & x\otimes(\xi\otimes e_1)\otimes e_1\otimes e_1\otimes\ldots e_1.\end{aligned}$$

The only way for this instance not to happen is if we have at least some e_2 replacing an e_1 in the tail of the element; this is because e_2 can only be obtained from the right multiplication of $e_2\otimes e_1$ by $e_1\otimes e_2^*$ and because of the position of ξ in $\xi\otimes e_1$, we are forced to have $\mathbb{H}(\mathbf{u})$ act on the left (cf. Section 3.2).

For the same reason, we see that

$$x\otimes(e_2\otimes\xi)\otimes e_2\otimes e_2\otimes\ldots e_2$$

is not irreducible; we need at least some e_1 replacing an e_2.

Note that we could have chosen a_2 to be $e_2\otimes e_1$, and then have a_3 be either $\xi\otimes e_1$ or $e_2\otimes\xi$, and the same argument applies. Inductively, we see that we can choose $e_2\otimes e_1$ for $a_2,\ldots,a_{l_0}$ for $2\le l_0<q$, and we choose $\xi\otimes e_1$ or $e_2\otimes\xi$ for a_{l_0+1}. This element is irreducible if and only if the appropriate idempotent of $\mathbf{d}$ appears at least once. Therefore, we must have $l_0+1<q$.

(iii) Finally, if $\xi\otimes\xi$ is chosen for a_2 (or for any a_l, $2\le l\le q$ with all the preceding a_n's equal to $e_2\otimes e_1$), then we obtain a reducible element:

$$\begin{aligned} & x\otimes(e_2\otimes e_1)^{\otimes ?}\otimes(\xi\otimes\xi)\otimes(e_{s_1}\otimes e^*_{3-s_1})\otimes\ldots\otimes(e_{s_l}\otimes e^*_{3-s_l})\\ =\ & e_1\otimes e_2^{\otimes ?}\otimes\xi\otimes(e_{s_1}\otimes e^*_{3-s_1})\otimes\ldots\otimes(e_{s_l}\otimes e^*_{3-s_l})\\ \cdot\ & x\otimes(e_2\otimes e_1)^{\otimes ?}\otimes(e_2\otimes\xi)\otimes e_{3-s_1}\otimes\ldots\otimes e_{3-s_l}.\end{aligned}$$

□

Theorem 3.3.10. *The new arrows for the quiver of* $\mathbf{w}_q$ *are of the form*

- $\xi \otimes (e_{s_2} \otimes e^*_{p+1-s_2}) \otimes \ldots \otimes (e_{s_q} \otimes e^*_{p+1-s_q})$*;*
- $x \otimes (e_2 \otimes e_1) \otimes \ldots \otimes (e_2 \otimes e_1)$*;*
- $x \otimes (e_2 \otimes e_1) \otimes \ldots \otimes (e_2 \otimes e_1) \otimes (\xi \otimes e_1) \otimes e_{l_1} \otimes e_{l_2} \otimes \ldots e_{l_s}$ *if there exists* $1 \leq i \leq s$ *such that* $l_i = 2$ $(s > 1)$*;*
- $x \otimes (e_2 \otimes e_1) \otimes \ldots \otimes (e_2 \otimes e_1) \otimes (e_2 \otimes \xi) \otimes e_{l_1} \otimes e_{l_2} \otimes \ldots e_{l_r}$ *if there exists* $1 \leq i \leq r$ *such that* $l_i = 1$ $(r > 1)$*.*

Proof. Use Corollary 3.3.5 and Corollary 3.3.9, and the fact that the first component is either x or ξ. □

Lemma 3.3.11. *The set* V_q *of irreducible monomials for* $\mathbf{w}_q$ *has* N_q *elements, where*

$$N_q := 2^{q-1}(3q-4) + 2q + 3.$$

Proof. From Corollary 3.3.3 and the fact that irreducible monomials of $\mathbf{w}_q$ start by either e_1, e_2, ξ or x, we know that

$$|V_q| = 2|V_{q-1}| + |\boldsymbol{\xi}_q| + |\mathbf{x}_q|,$$

where $\boldsymbol{\xi}_q$ is the set of irreducible monomials of $\mathbf{w}_q$ starting with ξ and $\mathbf{x}_q$ is the set of irreducible monomials of $\mathbf{w}_q$ starting with x.

By Corollary 3.3.5, we know that $v \in \boldsymbol{\xi}_q$ if and only if

$$v = \xi \otimes (e_{s_2} \otimes e^*_{3-s_2}) \otimes \ldots \otimes (e_{s_q} \otimes e^*_{3-s_q})$$

with $s_l \in \{1, 2\}$ for all $2 \leq l \leq q$. In particular,

$$|\boldsymbol{\xi}_q| = 2^{q-1}.$$

By Corollary 3.3.9, we know that $v \in \mathbf{x}_q$ if and only if

1. $v = x \otimes (e_2 \otimes e_1) \otimes \ldots \otimes (e_2 \otimes e_1)$;
2. $v = x \otimes (e_2 \otimes e_1) \otimes \ldots \otimes (e_2 \otimes e_1) \otimes (\xi \otimes e_1) \otimes e_{l_1} \otimes e_{l_2} \otimes \ldots e_{l_s}$ if there exists $1 \leq i \leq s$ such that $l_i = 2$ $(s > 1)$;
3. $v = x \otimes (e_2 \otimes e_1) \otimes \ldots \otimes (e_2 \otimes e_1) \otimes (e_2 \otimes \xi) \otimes e_{l_1} \otimes e_{l_2} \otimes \ldots e_{l_r}$ if there exists $1 \leq i \leq r$ such that $l_i = 1$ $(r > 1)$.

We could rewrite the last two possibilities in the form

$$v = x \otimes (e_2 \otimes e_1)^{\otimes n} \otimes v_{n+2} \otimes \bigotimes_{l=n+3}^{q} e_{s_l}$$

where $0 \leq n \leq q-3$, and the condition $\exists n+3 \leq l \leq q$ such that $s_l = i$ ($i = 1$ or $i = 2$ depending on v_{n+2}) is equivalent to $\exists n+3 \leq l \leq q$ such that $s_l \neq 3-i$, and finally we see that it is equivalent to $(s_{n+3}, \ldots, s_q) \neq (3-i, \ldots, 3-i)$.

In particular, the first possibility provides one element and the last two possibilities each provide the following number of elements:

$$\sum_{n=0}^{q-3}(2^{q-2-n}-1) = 2^{q-1} - q.$$

Hence

$$\begin{aligned} |\mathbf{x}_q| &= 1 + 2(2^{q-1} - q) \\ &= 2^q - 2q + 1. \end{aligned}$$

We thus obtain

$$\begin{aligned} |V_q| &= 2|V_{q-1}| + |\boldsymbol{\xi}_q| + |\mathbf{x}_q| \\ &= 2(2|V_{q-2}| + |\boldsymbol{\xi}_{q-1}| + |\mathbf{x}_{q-1}|) + |\boldsymbol{\xi}_q| + |\mathbf{x}_q| \\ &\vdots \\ &= 2^{q-1}|V_1| + 2^{q-2}(|\boldsymbol{\xi}_2| + |\mathbf{x}_2|) + \ldots + 2(|\boldsymbol{\xi}_{q-1}| + |\mathbf{x}_{q-1}|) + |\boldsymbol{\xi}_q| + |\mathbf{x}_q| \\ &= 2^{q-1}|V_1| + \sum_{n=0}^{q-2} 2^n(|\boldsymbol{\xi}_{q-n}| + |\mathbf{x}_{q-n}|) \end{aligned}$$

and, substituting the expressions for $|\boldsymbol{\xi}_{q-n}|$ and $|\mathbf{x}_{q-n}|$, we see

$$\begin{aligned} |V_q| &= 2^{q-1}|V_1| + \sum_{n=0}^{q-2} 2^n(2^{q-n-1} + 2^{q-n} - 2(q-n) + 1) \\ &= 2^{q-1}|V_1| + \sum_{n=0}^{q-2}(2^{q-1} + 2^q - q2^{n+1} + n2^{n+1} + 2^n) \\ &= 2^{q-1}|V_1| + (q-1)(2^{q-1} + 2^q) - 2q\frac{2^{q-1}-1}{2-1} + \sum_{n=1}^{q-2} n2^{n+1} + \frac{2^{q-1}-1}{2-1} \\ &= 2^{q-1}|V_1| + (q-1)(2^{q-1} + 2^q) - q2^q + 2q + \sum_{n=1}^{q-2} n2^{n+1} + 2^{q-1} - 1. \end{aligned}$$

Now, to simplify this expression, we need a formula for the term $\sum_{n=1}^{q-2} n2^{n+1}$. Let a be a formal variable. We know

$$\sum_{k=0}^{n} a^k = \frac{a^{n+1}-1}{a-1}$$

and deriving both sides of the equality with respect to a gives

$$\sum_{k=1}^{n} ka^{k-1} = \frac{na^n}{a-1} + \frac{1-a^n}{(a-1)^2}.$$

Setting $a = 2$, we obtain:

$$\begin{aligned} \sum_{n=1}^{q-2} n2^{n+1} &= 2^2 \sum_{n=1}^{q-2} n2^{n-1} \\ &= 2^2\left((q-2)2^{q-2} + 1 - 2^{q-2}\right) \\ &= 2^q(q-3) + 4. \end{aligned}$$

Finally, since $|V_1| = 4$, we get

$$\begin{aligned} |V_q| &= 2^{q+1} + (q-1)(2^{q-1} + 2^q) - q2^q + 2q + 2^q(q-3) + 4 + 2^{q-1} - 1 \\ &= 2^{q-1}\left(2^2 + 2(q-1-q+q-3) + (q-1+1)\right) + 2q + 3 \end{aligned}$$

so that

$$|V_q| = 2^{q-1}(3q-4) + 2q + 3.$$

□

3.4 The quiver of $\mathbf{w}_q$

For any index $q \in \mathbb{N}_{>0}$, we have an explicit description of the basis elements of $\mathbf{w}_q$ as q-tuples made up from elements of $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})^{\leq 1}$. These constituting elements are (i, j, k)-graded, and together with their idempotents on the left and on the right, that grading completely determines them. Therefore, basis elements of $\mathbf{w}_q$ are $(i, j, k)^q$-graded and are fully determined by that grading. We denote the simples of $\mathbf{w}_q$ by their 2-adic expansion, namely

$$(s_1, 0, 0, 0, s_1) \otimes \ldots \otimes (s_q, 0, 0, 0, s_q) \leftrightarrow \sum_{l=2}^{q} (s_l - 1)2^{l-1} + s_1.$$

Since $\mathbf{w}_q$ is the extension algebra of the standard modules of an algebra of finite global dimension, it is finite-dimensional. By definition of V_q, we can write $\mathbf{w}_q$ as the quotient of a tensor algebra by some ideal, namely

$$\mathbf{w}_q \cong T_B V_q / \mathcal{I},$$

where the tensor product is taken over the semi-simple algebra B made up by the idempotents of $\mathbf{w}_q$.

Now, V_q is a (finite) subset of monomial basis elements of $\mathbf{w}_q$ which is a multiplicative basis for $\mathbf{w}_q$. Since $\mathbf{w}_q$ is multigraded, $\mathcal{I}$ must be homogeneous with respect to that $(i, j, k)^q$-grading. Since an element $z \in V_q$ is uniquely determined by its $(i, j, k)^q$-degree (together with idempotents on the left and on the right), we obtain that $\mathcal{I}$ cannot contain any element of V_q: let $v_1 + \ldots + v_s \in \mathcal{I}$, with v_j's words in elements of V_q; then all v_j's have the same $(i, j, k)^q$-degree since $\mathcal{I}$ is homogeneous. In particular, at most one v_j is in V_q. This is a contradiction as we would obtain a linear relation between basis elements of $\mathbf{w}_q$. Therefore, all v_j's are words in at least two elements of V_q, i.e. $\mathcal{I} \subset V_q \otimes_B V_q$. In addition, since $\mathbf{w}_q$ is finite-dimensional, there cannot be words in V_q of infinite length. Thus, there exists $N > 2$ such that

$$V_q^{\otimes_B N} \subset \mathcal{I} \subset V_q \otimes_B V_q,$$

so that $\mathcal{I}$ is admissible.

We can therefore interpret V_q as the quiver of $\mathbf{w}_q$. We see that the vertices are given by the simples of $\mathbf{w}_q$ and the set of arrows of the quiver corresponds to V_q.

Example 3.4.1. To illustrate that section, we give the quiver of $\mathbf{w}_q$ for $q = 1, 2, 3$. We denote arrows of degree 1 with a decorated tail. From the proof of Lemma 3.3.11, we know the number of new arrows starting with ξ or x:

$$\begin{array}{rclrcl} |\boldsymbol{\xi}_q| & = & 2^{q-1} & |\boldsymbol{x}_q| & = & 2^q - 2q + 1 \\ |\boldsymbol{\xi}_1| & = & 1 & |\boldsymbol{x}_1| & = & 1 \\ |\boldsymbol{\xi}_2| & = & 2 & |\boldsymbol{x}_2| & = & 1 \\ |\boldsymbol{\xi}_3| & = & 4 & |\boldsymbol{x}_3| & = & 3 \end{array}$$

1. The quiver of $\mathbf{w}_1 = \mathbf{d}$ is given in Figure 3.2.

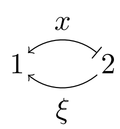


Figure 3.2: Quiver of $\mathbf{w}_1$

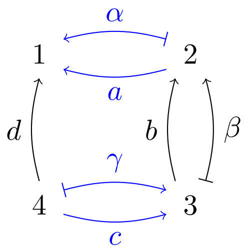


Figure 3.3: Quiver of $\mathbf{w}_2$

2. The quiver of $\mathbf{w}_2$ is given in Figure 3.3.

   The two copies of $\mathbf{w}_1$ have been coloured in blue. The label of the arrows correspond to the following elements of V_2:

$$\begin{array}{rcl} a & = & e_1 \otimes \xi \\ \alpha & = & e_1 \otimes x \\ b & = & \xi \otimes (e_2 \otimes e_1^*) \\ \beta & = & x \otimes (e_2 \otimes e_1) \\ c & = & e_2 \otimes \xi \\ \gamma & = & e_2 \otimes x \\ d & = & \xi \otimes (e_1 \otimes e_2^*). \end{array}$$

3. The quiver of $\mathbf{w}_3$ is given in Figure 3.4.

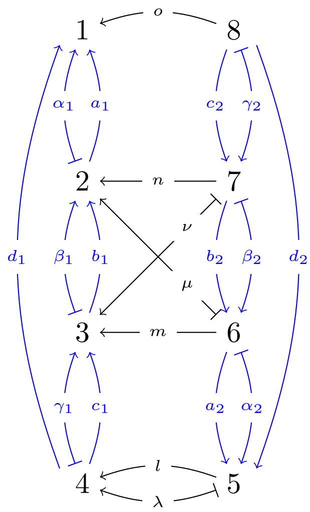


Figure 3.4: Quiver of $\mathbf{w}_3$

The two copies of $\mathbf{w}_2$ have been coloured in blue. The label of the arrows correspond to the following elements of V_3:

$$\begin{aligned}
a_i &= e_i \otimes e_1 \otimes \xi \\
\alpha_i &= e_i \otimes e_1 \otimes x \\
b_i &= e_i \otimes \xi \otimes (e_2 \otimes e_1^*) \\
\beta_i &= e_i \otimes x \otimes (e_2 \otimes e_1) \\
c_i &= e_i \otimes e_2 \otimes \xi \\
\gamma_i &= e_i \otimes e_2 \otimes x \\
d_i &= e_i \otimes \xi \otimes (e_1 \otimes e_2^*) \\
l &= \xi \otimes (e_2 \otimes e_1^*) \otimes (e_2 \otimes e_1^*) \\
\lambda &= x \otimes (e_2 \otimes e_1) \otimes (e_2 \otimes e_1) \\
m &= \xi \otimes (e_2 \otimes e_1^*) \otimes (e_1 \otimes e_2^*) \\
\mu &= x \otimes (\xi \otimes e_1) \otimes e_2 \\
\nu &= x \otimes (e_2 \otimes \xi) \otimes e_1 \\
n &= \xi \otimes (e_1 \otimes e_2^*) \otimes (e_2 \otimes e_1^*) \\
o &= \xi \otimes (e_1 \otimes e_2^*) \otimes (e_1 \otimes e_2^*)
\end{aligned}$$

where $i \in \{1, 2\}$.

Chapter 4

Quiver of $\mathbf{w}_q$ for $p > 2$

4.1 Introduction

We consider the basis elements of $\mathbf{w}_q$ and would like to find the set V_q of algebra generators of $\mathbf{w}_q$. Those basis elements come under the form

$$v_1 \otimes \ldots \otimes v_q.$$

We know from [BLM13, Lemma 11] that these elements come in three types, namely $\mathbf{1}^q$, $\mathbf{1}^h\ \mathbf{3}^{q-h}$ or $\mathbf{1}^h\ \mathbf{2}\ \mathbf{3}^{q-h-1}$. In the following section, we define the notion of irreducibility for an element of $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})$ and we prove that type $\mathbf{3}$ elements are irreducible. That means in particular that the q-tensor products we are interested in can only be split in the type $\mathbf{1}$ or type $\mathbf{2}$ part of the tensor product. Therefore we need to study tensor products of type $\mathbf{1}^{q-\epsilon}\ \mathbf{2}^{\epsilon}$ in more detail.

Due to the chaining rule, if v_1 is an idempotent of $\mathbf{d}$, then v_2 is an element of $\mathbf{d}$ and hence

$$v_2 \otimes \ldots \otimes v_q$$

is a basis element of $\mathbf{w}_{q-1}$. That means in particular that V_q contains p copies of V_{q-1}, namely

$$e_i \otimes V_{q-1} \subset V_q$$

for all $1 \leq i \leq p$.

So by induction, we just need to determine the tensor products such that v_1 is not an idempotent, i.e.

$$v_1 = e_s x^{n_1} \xi^{\epsilon_1} e_{s+n_1+\epsilon_1},$$

with $1 \leq s \leq p$, $1 \leq n_1 + \epsilon_1 \leq p - s$ and $\epsilon_1 \in \{0, 1\}$.

4.2 Irreducibility

In this section, we define what we mean for a monomial basis element of an algebra to be irreducible. We then provide all the irreducible monomial basis elements of $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})^{\leq 1}$.

Definition 4.2.1. A monomial basis element of an algebra A is said to be *irreducible* if it cannot be written as a non-trivial product. It is otherwise called *reducible*.

By non-trivial product we mean a product $a \cdot a'$ where neither a nor a' are idempotents of A.

Remark 4.2.2. The irreducible monomial basis elements of an algebra are precisely a minimal set of generators for that algebra.

Definition 4.2.3. Let $B \subseteq A$ be a subspace of an algebra A. We say that a monomial basis element is *irreducible in* B if it cannot be written as a non-trivial product of elements in B.

In the rest of the section, we take $A = \mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})$ and $B = \mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})^{\leq 1}$.

Lemma 4.2.4. *The elements of* $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})^{\leq 1}$ *of type* **3** *are irreducible.*

Proof. According to Table 2.2, we can write a type **3** element as a product of the form $\mathbf{1} \cdot \mathbf{3}$ or $\mathbf{3} \cdot \mathbf{1}$. However, the conditions on the type **1** element involved show that it must be an idempotent. Therefore, there aren't any non-trivial products yielding a type **3** element. □

Let us now determine the irreducible elements of types **1** and **2**, if they exist. Let us consider type **2** elements first. Note that for any element of that type, the i-degree is less or equal to -1 ($i = -2a + 1$ with $a \geq 0$). Let

$$v := (s, -2a - 1, 1, 0, a, a + 1, p + 1 - s)$$

with $1 \leq s \leq p - 2$ be a type **2** element. Assume that its i-degree is less than -1, i.e. $a \geq 1$. Then we can write

$$v = (s, -2a, 0, 0, a, a, s) \cdot (s, -1, 1, 0, 0, 1, p + 1 - s),$$

and the element $(s, -2a, 0, 0, a, a, s)$ a valid type **1** element of $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})^{\leq 1}$ which is not an idempotent since $a \geq 1$, so that v is not irreducible. Therefore, we know that if v is irreducible, then $a = 0$ and its i-degree is -1. Let us then assume $a = 0$. From Table 2.2, we see that a type **2** element can be obtained as the products

$(\mathbf{1} \cdot \mathbf{2})$

$$\begin{aligned} & (s_1, -2a_1, 0, 0, a_1, a_1, s_1) \\ \cdot\ & (s_2, -2a_2 - 1, 1, 0, a_2, a_2 + 1, p + 1 - s_2) \\ =\ & (s_2, -2(a_1 + a_2) - 1, 1, 0, (a_1 + a_2), (a_1 + a_2) + 1, p + 1 - s_2) \end{aligned}$$

with $a_1 \geq 0$ and $a_2 \geq 0$ so that $a_1 + a_2 \geq 0$. Now, writing v in such a way is possible if $a_1 = a_2 = 0$, and that decomposition is then trivial.

$(\mathbf{2} \cdot \mathbf{1})$

$$\begin{aligned} & (s_1, -2a_1 - 1, 1, 0, a_1, a_1 + 1, p + 1 - s_1) \\ \cdot\ & (s_2, -2a_2, 0, 0, a_2, a_2, s_2) \\ =\ & (s_1, -2(a_1 + a_2) - 1, 1, 0, (a_1 + a_2), (a_1 + a_2) + 1, p + 1 - s_1) \end{aligned}$$

with $a_1 \geq 0$ and $a_2 \geq 0$. Similar to the previous case, we see $a_1 = a_2 = 0$, and the decomposition is trivial.

$(\mathbf{1} \cdot \mathbf{3})$

$$\begin{aligned} & (s_1, -2a_1, 0, 0, a_1, a_1, s_1) \\ \cdot\ & (s_2, 1, 1, 0, -1, 0, p + 1 - s_2) \\ =\ & (s_2, -2a_1 + 1, 1, 0, a_1 - 1, a_1, p + 1 - s_2) \end{aligned}$$

exists if $a_1 \geq 1$. Writing v in such a way means we must choose $a_1 = 1$, and this product is non-trivial. Hence v is not irreducible.

$(\mathbf{3} \cdot \mathbf{1})$

$$\begin{aligned} & (s_1, 1, 1, 0, -1, 0, p + 1 - s_1) \\ \cdot\ & (s_2, -2a_2, 0, 0, a_2, a_2, s_2) \\ =\ & (s_1, -2a_2 + 1, 1, 0, a_2 - 1, a_2, p + 1 - s_1) \end{aligned}$$

exists if $a_2 \geq 1$. Writing v in such a way means we must choose $a_2 = 1$, and this product is non-trivial. Hence v is not irreducible.

Remark 4.2.5. Since type **3** elements correspond to the $\mathbb{H}(\mathbf{u})$ part of homology, we see from these considerations that type **2** elements of i-degree -1 are irreducible in $\mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})$, but not in $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})^{\leq 1}$.

Let us consider type **1** elements now. Let

$$v := (s, -a-b, -p(a-b)-(t-s)+2u, (p-1)(a-b)+(t-s)-u, a, b, t)$$

be a type **1** element. Then we have the following non-trivial decomposition

$$\begin{aligned} & (s, -2, 0, 0, 1, 1, s) \\ \cdot \quad & (s, -a-b+2, -p(a-b)-(t-s)+2u, (p-1)(a-b)+(t-s)-u, a-1, b-1, t) \\ = \quad & (s, -a-b, -p(a-b)-(t-s)+2u, (p-1)(a-b)+(t-s)-u, a, b, t) \end{aligned}$$

into two type **1** elements, unless $a = b = 0$, or $a = 1$ and $b = 0$, or $a = b = 1$ and $s = t$. Let us analyse those three cases.

1. If $a = b = 0$, then v is of the form

$$(s, 0, -(t-s)+2u, (t-s)-u, 0, 0, t),$$

and by definition we must have $t - s \geq 0$, and if $t - s = 0$, then u must be equal to 0. We can write it as the non-trivial product

$$\begin{aligned} & (s, 0, -1+2u, 1-u, 0, 0, s+1) \\ \cdot \quad & (s+1, 0, -(t-(s+1)), (t-(s+1)), 0, 0, t) \\ = \quad & (s, 0, -(t-s)+2u, (t-s)-u, 0, 0, t) \end{aligned}$$

unless $t-s \leq 1$. Therefore, $v = (s, 0, -1+2u, 1-u, 0, 0, s+1)$ and $v = (s, 0, 0, 0, 0, 0, s)$ are candidates for irreducibility.

2. If $a = 1$ and $b = 0$, then v is of the form

$$(s, -1, -p-(t-s)+2u, (p-1)+(t-s)-u, 1, 0, t),$$

and, assuming $u \in \{0, 1\}$, we can write it as the non-trivial product

$$\begin{aligned} & (s, 0, -1+2u, 1-u, 0, 0, s+1) \\ \cdot \quad & (s+1, -1, -p-(t-(s+1)), (p-1)+(t-(s+1)), 1, 0, t) \\ = \quad & (s, -1, -p-(t-s)+2u, (p-1)+(t-s)-u, 1, 0, t) \end{aligned}$$

unless $s = p$. Similarly, assuming $u \in \{0, 1\}$, we can write it as the non-trivial product

$$\begin{aligned} & (s, -1, -p-((t-1)-s), (p-1)+((t-1)-s), 1, 0, t-1) \\ \cdot \quad & (t-1, 0, -1+2u, 1-u, 0, 0, t) \\ = \quad & (s, -1, -p-(t-s)+2u, (p-1)+(t-s)-u, 1, 0, t) \end{aligned}$$

unless $t = 1$. Hence, $v = (p, -1, -1, 0, 1, 0, 1)$ is a candidate for irreducibility. In addition, if $u = 2$, so that v is necessarily of the form

$$(p-1, -1, 1, 0, 1, 0, 2),$$

we see that we can write it as the non-trivial product

$$\begin{aligned} & (p-1, 0, 1, 0, 0, 0, p) \\ \cdot \quad & (p, -1, 0, 0, 1, 0, 2) \\ = \quad & (p-1, -1, 1, 0, 1, 0, 2) \end{aligned}$$

which means it is not irreducible.

3. If $a = b = 1$ and $s = t$, then v is of the form
$$(s, -2, 0, 0, 1, 1, s),$$
and it is another candidate for irreducibility.

So far, the possible irreducible elements of type **1** are

1. $(s, 0, 0, 0, 0, 0, s)$, for $1 \leq s \leq p$;
2. $(s, 0, 1, 0, 0, 0, s+1)$, for $1 \leq s \leq p-1$;
3. $(s, 0, -1, 1, 0, 0, s+1)$, for $1 \leq s \leq p-1$;
4. $(p, -1, -1, 0, 1, 0, 1)$;
5. $(s, -2, 0, 0, 1, 1, s)$, for $1 \leq s \leq p$;

since these are the only ones that don't arise as products of type $\mathbf{1} \cdot \mathbf{1}$. Now we can analyse which ones of those can be obtained by multiplying different types. According to Table 2.2, we see that type **1** elements can be obtained in five ways. We cover the four remaining ones now.

Let us write down a product $\mathbf{1} \cdot \mathbf{2}$ of type **1**.

$$\begin{array}{rl} & (s_1, -a_1 - b_1, -p(a_1 - b_1) - (1 - s_1) + 2u_1, (p-1)(a_1 - b_1) + (1 - s_1) - u_1, a_1, b_1, 1) \\ \cdot & (1, -2a_2 - 1, 1, 0, a_2, a_2 + 1, p) \\ = & (s_1, -(a_1 + a_2) - (b_1 + a_2 + 1), -p(a_1 - b_1) - (1 - s_1) + 2u_1 + 1, \\ & \quad (p-1)(a_1 - b_1) + (1 - s_1) - u_1, a_1 + a_2, b_1 + a_2 + 1, p) \end{array}$$

with $a_1 - b_1 \geq 1$. Since the b-degree is $b_1 + a_2 + 1 \geq 1$, only the last candidate could fit. But since $t = p$, only one element could be decomposed like so, namely $(p, -2, 0, 0, 1, 1, p)$. Indeed, we have

$$\begin{array}{rl} & (p, -1, -1, 0, 1, 0, 1) \\ \cdot & (1, -1, 1, 0, 0, 1, p) \\ = & (p, -2, 0, 0, 1, 1, p). \end{array}$$

Let us write down a product $\mathbf{2} \cdot \mathbf{1}$ of type **1**.

$$\begin{array}{rl} & (1, -2a_1 - 1, 1, 0, a_1, a_1 + 1, p) \\ \cdot & (p, -a_2 - b_2, -p(a_2 - b_2) - (t_2 - p) + 2u_2, (p-1)(a_2 - b_2) + (t_2 - p) - u_2, a_2, b_2, t_2) \\ = & (1, -(a_1 + a_2) - (a_1 + b_2 + 1), -p(a_2 - b_2) - (t_2 - p) + 2u_2 + 1, \\ & \quad (p-1)(a_2 - b_2) + (t_2 - p) - u_2, a_1 + a_2, a_1 + b_2 + 1, t_2) \end{array}$$

with $a_2 - b_2 \geq 1$. Since the b-degree is $a_1 + b_2 + 1 \geq 1$, only the last candidate could fit. But since $s = 1$, only one element could be decomposed like so, namely $(1, -2, 0, 0, 1, 1, 1)$. Indeed, we have

$$\begin{array}{rl} & (1, -1, 1, 0, 0, 1, p) \\ \cdot & (p, -1, -1, 0, 1, 0, 1) \\ = & (1, -2, 0, 0, 1, 1, 1). \end{array}$$

Let us write down a product $\mathbf{1} \cdot \mathbf{3}$ of type **1**.

$$\begin{array}{rl} & (s_1, -a_1 - b_1, -p(a_1 - b_1) - (1 - s_1) + 2u_1, (p-1)(a_1 - b_1) + (1 - s_1) - u_1, a_1, b_1, 1) \\ \cdot & (1, 1, 1, 0, -1, 0, p) \\ = & (s_1, -a_1 - b_1 + 1, -p(a_1 - b_1) - (1 - s_1) + 2u_1 + 1, \\ & \quad (p-1)(a_1 - b_1) + (1 - s_1) - u_1, a_1 - 1, b_1, p) \end{array}$$

with $a_1 - b_1 \geq 1$. Since $t = p$, the following elements could be decomposed like so: $(p, 0, 0, 0, 0, 0, p)$, $(p-1, 0, 1, 0, 0, 0, p)$, $(p-1, 0, -1, 1, 0, 0, p)$ and $(p, -2, 0, 0, 1, 1, p)$. Indeed, we have

$$\begin{aligned} & (p, -1, -1, 0, 1, 0, 1) \\ \cdot\ & (1, 1, 1, 0, -1, 0, p) \\ =\ & (p, 0, 0, 0, 0, 0, p); \end{aligned}$$

$$\begin{aligned} & (p-1, -1, 0, 0, 1, 0, 1) \\ \cdot\ & (1, 1, 1, 0, -1, 0, p) \\ =\ & (p-1, 0, -1, 0, 0, 0, p); \end{aligned}$$

$$\begin{aligned} & (p-1, -1, -2, 1, 1, 0, 1) \\ \cdot\ & (1, 1, 1, 0, -1, 0, p) \\ =\ & (p-1, 0, -1, 1, 0, 0, p); \end{aligned}$$

$$\begin{aligned} & (p, -3, -1, 0, 2, 1, 1) \\ \cdot\ & (1, 1, 1, 0, -1, 0, p) \\ =\ & (p, -2, 0, 0, 1, 1, p). \end{aligned}$$

Let us write down a product $\mathbf{3} \cdot \mathbf{1}$ of type $\mathbf{1}$.

$$\begin{aligned} & (1, 1, 1, 0, -1, 0, p) \\ \cdot\ & (p, -a_2 - b_2, -p(a_2 - b_2) - (t_2 - p) + 2u_2, (p-1)(a_2 - b_2) + (t_2 - p) - u_2, a_2, b_2, t_2) \\ =\ & (1, -a_2 - b_2 + 1, -p(a_2 - b_2) - (t_2 - p) + 2u_2 + 1, \\ & \quad (p-1)(a_2 - b_2) + (t_2 - p) - u_2, a_2 - 1, b_2, t_2) \end{aligned}$$

with $a_2 - b_2 \geq 1$. Since $s = 1$, the following elements could be decomposed like so: $(1, 0, 0, 0, 0, 0, 1)$, $(1, 0, 1, 0, 0, 0, 2)$, $(1, 0, -1, 1, 0, 0, 2)$ and $(1, -2, 0, 0, 1, 1, 1)$. Indeed, we have

$$\begin{aligned} & (1, 1, 1, 0, -1, 0, p) \\ \cdot\ & (p, -1, -1, 0, 1, 0, 1) \\ =\ & (1, 0, 0, 0, 0, 0, 1); \end{aligned}$$

$$\begin{aligned} & (1, 1, 1, 0, -1, 0, p) \\ \cdot\ & (p, -1, 0, 0, 1, 0, 2) \\ =\ & (1, 0, 1, 0, 0, 0, 2); \end{aligned}$$

$$\begin{aligned} & (1, 1, 1, 0, -1, 0, p) \\ \cdot\ & (p, -1, -2, 1, 1, 0, 2) \\ =\ & (1, 0, -1, 1, 0, 0, 2); \end{aligned}$$

$$\begin{aligned} & (1, 1, 1, 0, -1, 0, p) \\ \cdot\ & (p, -3, -1, 0, 2, 1, 1) \\ =\ & (1, -2, 0, 0, 1, 1, 1). \end{aligned}$$

Remark 4.2.6. As for type **2** elements, we see that some elements of type **1** are irreducible in $\mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})$ but not in $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})^{\leq 1}$.

We have proved the following:

Proposition 4.2.7. *Let v be an irreducible monomial basis element for $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})^{\leq 1}$. Then v is one of the following elements:*

Type **1**
- $(s, 0, 0, 0, 0, 0, s) = e_s$, *for* $2 \leq s \leq p-1$*;*
- $(s, 0, 1, 0, 0, 0, s+1) = e_s \xi e_{s+1}$, *for* $2 \leq s \leq p-2$*;*
- $(s, 0, -1, 1, 0, 0, s+1) = e_s x e_{s+1}$, *for* $2 \leq s \leq p-2$*;*
- $(p, -1, -1, 0, 1, 0, 1) = e_p \otimes e_1$*;*

– $(s, -2, 0, 0, 1, 1, s) = e_s w e_s$, *for* $2 \leq s \leq p-1$;

Type **3** – $(s, 1, 1, 0, -1, 0, p+1-s) = e_s \otimes e^*_{p+1-s}$, *for* $1 \leq s \leq p$.

and

Proposition 4.2.8. *Let v be an irreducible monomial basis element for $\mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})$. Then v is one of the following elements:*

Type **1** – $(s, 0, 0, 0, 0, 0, s) = e_s$, *for* $1 \leq s \leq p$;
– $(s, 0, 1, 0, 0, 0, s+1) = e_s \xi e_{s+1}$, *for* $1 \leq s \leq p-1$;
– $(s, 0, -1, 1, 0, 0, s+1) = e_s x e_{s+1}$, *for* $1 \leq s \leq p-1$;
– $(p, -1, -1, 0, 1, 0, 1) = e_p \otimes e_1$;
– $(s, -2, 0, 0, 1, 1, s) = e_s w e_s$, *for* $2 \leq s \leq p-1$;

Type **2** – $(s, -1, 1, 0, 0, 1, p+1-s) = e_s \xi \otimes \xi e_{p+1-s}$, *for* $1 \leq s \leq p-2$.

The following Corollary is merely an observation.

Corollary 4.2.9. *Let v be an irreducible monomial of $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})^{\leq 1}$. Then its i-degree is in the set $\{-2, -1, 0, 1\}$.*

Remark 4.2.10. Comparing the two previous propositions, we see that the following elements are irreducible in $\mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})$ but not in $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})^{\leq 1}$:

$$(1, 0, 0, 0, 0, 0, 1) = e_1$$
$$(p, 0, 0, 0, 0, 0, p) = e_p$$

$$(1, 0, 1, 0, 0, 0, 2) = e_1 \xi e_2$$
$$(p-1, 0, 1, 0, 0, 0, p) = e_{p-1} \xi e_p$$

$$(1, 0, -1, 1, 0, 0, 2) = e_1 x e_2$$
$$(p-1, 0, -1, 1, 0, 0, p) = e_{p-1} x e_p$$

In addition, the elements

$$(1, -2, 0, 0, 1, 1, 1) = e_1 w e_1$$
$$(p, -2, 0, 0, 1, 1, p) = e_p w e_p$$

are not irreducible in either $\mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})$ or $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})^{\leq 1}$.

4.3 Study of the j-degree for type 1 elements

Because of the chaining rule making the j-degree of one element correspond to the i-degree of the next element in the tensor product, it is necessary to understand what values the j-degree of a given element of type **1** can be. Note that type **2** and type **3** elements have j-degree 1, hence we just need to study the j-degree of type **1** elements.

The j-degree of a type **1** element is of the form $-p(a-b)-(t-s)+2u$, so let us solve the following equation for some parameter N,

$$-p(a-b)-(t-s)+2u = N.$$

Since $1 \leq s \leq p$, $1 \leq t \leq p$, we have that

$$1-p \leq -(t-s) \leq p-1,$$

and so

$$1 - p - N + 2u \leq -(t - s) - N + 2u \leq p - 1 - N + 2u,$$

i.e.

$$1 - p - N + 2u \leq p(a - b) \leq p - 1 - N + 2u,$$

which we can further write

$$\frac{1 - N + 2u}{p} - 1 \leq a - b \leq 1 + \frac{-1 - N + 2u}{p}.$$

Since $a \geq b \geq 0$, we have $a - b \geq 0$ and is an integer; we denote it by n. We want to determine the interval

$$\llbracket \frac{1 - N + 2u}{p} - 1\,; 1 + \frac{-1 - N + 2u}{p} \rrbracket \cap \mathbb{N}_{\geq 0}$$

and since it has length $1 + \frac{-1 - N + 2u}{p} - \left(\frac{1 - N + 2u}{p} - 1\right) = 2 - \frac{2}{p}$, it contains at most 2 integers, namely

$$\left\lceil \frac{1 - N + 2u}{p} - 1 \right\rceil \text{ and } \left\lfloor 1 + \frac{-1 - N + 2u}{p} \right\rfloor.$$

Going back to our original equation, we see:

$$t = s - pn + 2u - N, \tag{4.1}$$

where $n \in \left\{ \left\lceil \frac{1 - N + 2u}{p} - 1 \right\rceil \,,\, \left\lfloor 1 + \frac{-1 - N + 2u}{p} \right\rfloor \right\} \cap \mathbb{N}_{\geq 0}$.

Lemma 4.3.1. *The monomial basis elements of* $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})^{\leq 1}$ *of* j*-degree 1 are*

$$\begin{aligned}
(s_1, -2a, -1, 1, b, b, s_1 + 1) &= e_{s_1} w^b \xi e_{s_1+1} \\
(p - 1, -2b - 1, 1, 0, b + 1, b, 2) &= e_{p-1} \xi \otimes (\xi \otimes \xi)^{\otimes b} \otimes \xi e_2 \\
(s_2, -2a - 1, 1, 0, a, a + 1, p + 1 - s_2) &= e_{s_2} (\xi \otimes \xi)^{\otimes a+1} e_{p+1-s_2} \\
(s_3, 1, 1, 0, -1, 0, p + 1 - s_3) &= e_{s_3} \otimes e^*_{p+1-s_3}
\end{aligned}$$

with $1 \leq s_1 \leq p - 1$, $1 \leq s_2 \leq p - 2$, $1 \leq s_3 \leq p - 1$, $a, b \geq 0$.

Proof. As said in the introduction, we know that type **2** and type **3** elements have j-degree 1, and they correspond to the last two possibilities in the list above.

Let us now set $N = 1$ in equation (4.1). We see that

$$n \in \left\{ \left\lceil \frac{2u}{p} - 1 \right\rceil \,,\, \left\lfloor 1 + \frac{-2 + 2u}{p} \right\rfloor \right\} \cap \mathbb{N}_{\geq 0},$$

and

$$0 > \frac{2u}{p} - 1,$$

or equivalently $p > 2u$, which is always true since $p \geq 3 > 2 \geq 2u$;

$$\frac{2u}{p} - 1 > -1$$

is equivalent to $u > 0$. Hence, if $u = 1$, $\left\lceil \frac{2u}{p} - 1 \right\rceil = 0$, and if $u = 0$, $\left\lceil \frac{2u}{p} - 1 \right\rceil = \lceil -1 \rceil = -1$.

In addition, we have
$$1+\frac{-2+2u}{p}>0$$
if and only if we have
$$2u>-p+2,$$
which is always true since $p\geq 3$, $-p+2\leq -1<0\leq 2u$, and
$$1>1+\frac{-2+2u}{p}$$
is equivalent to
$$0>-2+2u$$
and this is if and only if
$$2>2u$$
which means that $u=0$. Hence, if $u=0$, $\left\lfloor 1+\frac{-2+2u}{p}\right\rfloor=0$, and if $u=1$, $\left\lfloor 1+\frac{-2+2u}{p}\right\rfloor=\lfloor 1\rfloor=1$.

Therefore, we have $n\in\{0,1\}$, and $n=1$ only if $u=1$.

The corresponding elements of type **1** can be written
$$(s,-2b-n,1,n(p-1)+(-pn+2u-1)-u,b+n,b,s-pn+2u-1),$$
where $u\in\{0,1\}$ and $n\in\{0,1\}$. Re-arranging the k-degree, we have
$$(s,-2b-n,1,-n-1+u,b+n,b,s-pn+2u-1).$$

If $n=0$, the element becomes
$$(s,-2b,1,-1+u,b,b,s+2u-1).$$

- If $u=0$, we have
$$(s,-2b,1,-1,b,b,s-1),$$
which is not a valid element since by definition, if the a- and b-degree coincide, then then the s- and t-degree must satisfy $t-s\geq 0$.
- If $u=1$, we have
$$(s,-2b,1,0,b,b,s+1).$$

If $n=1$, in particular $u=1$, the element becomes
$$(s,-2b-1,1,-1,b+1,b,s-p+1),$$
but since $s-p+1-s=1-p<2-p$, this is not a valid **1** element.

Finally, we need to recall that there is an additional type **1** element of j-degree 1, for which the parameter u is equal to 2, and $s=p-1$, $t=2$; this is due to the fact that $e_p\otimes(\xi\otimes\xi)^{\otimes b}\otimes e_1$ generates more than what is announced in [MT13] (e.g. see Corollary 39). That element is
$$(p-1,-2b-1,1,0,b+1,b,2).$$

□

Lemma 4.3.2. *The monomial basis elements of* $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})^{\leq 1}$ *of j-degree 0 are*

$$\begin{array}{rcl}(s,-2a,0,u,a,a,s+2u) & = & e_s w^a (x\xi)^u e_{s+2u} \\ (p-1,-2a-1,0,0,a+1,a,1) & = & e_{p-1}\xi e_p \otimes (\xi\otimes\xi)^{\otimes a} \otimes e_1 \\ (p,-2a-1,0,0,a+1,a,2) & = & e_p \otimes (\xi\otimes\xi)^{\otimes a} \otimes e_1 \xi e_2\end{array}$$

with $u \in \{0,1\}$, $1 \leq s \leq p-2u$, $a \geq 0$.

Proof. Let us set $N = 0$ in equation (4.1). We see that

$$\left\{ \left\lceil \frac{1+2u}{p} - 1 \right\rceil , \left\lfloor 1 + \frac{-1+2u}{p} \right\rfloor \right\} \cap \mathbb{N}_{\geq 0} = \{0, u\}.$$

The corresponding elements of type **1** can be written

$$(s, -2a-n, 0, n(p-1)+2u-pn-u, a+n, a, s+2u-pn),$$

where $u \in \{0,1\}$, and $n \in \{0,u\}$. Re-arranging the k-degree, we have

$$(s, -2a-n, 0, u-n, a+n, a, s+2u-pn).$$

If $n = 0$, the element becomes

$$(s, -2a, 0, u, a, a, s+2u).$$

If $n = 1$ (so necessarily $u = 1$), we have

$$(s, -2a-1, 0, 0, a+1, a, s+2-p),$$

and since $1 \leq s+2-p \leq p$, we see that $s \in \{p-1, p\}$, which gives the other two possibilities. ☐

Lemma 4.3.3. *The monomial basis elements of* $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})^{\leq 1}$ *of j-degree -1 are*

$$\begin{array}{rcl}(s_1,-2a,-1,1,a,a,s_1+1) & = & e_{s_1} w^a x e_{s_1+1} \\ (s_2,-2a,-1,2,a,a,s_2+3) & = & e_{s_2} w^a x^2 \xi e_{s_2+3} \\ (p,-2a-1,-1,0,a+1,a,1) & = & e_p \otimes (\xi\otimes\xi)^{\otimes a} \otimes e_1 \\ (p,-2a-1,-1,1,a+1,a,3) & = & e_p \otimes (\xi\otimes\xi)^{\otimes a} \otimes x\xi e_3 \\ (p-1,-2a-1,-1,1,a+1,a,2) & = & e_{p-1}\xi \otimes (\xi\otimes\xi)^{\otimes a} \otimes x e_2 \\ & (\cong & e_{p-1}x \otimes (\xi\otimes\xi)^{\otimes a} \otimes \xi e_2) \\ (p-2,-2a-1,-1,1,a+1,a,1) & = & e_{p-2}x\xi \otimes (\xi\otimes\xi)^{\otimes a} \otimes e_1\end{array}$$

with $1 \leq s_1 \leq p-1$, $1 \leq s_2 \leq p-3$, $a \geq 0$.

Proof. Let us set $N = -1$ in equation (4.1). We see that

$$n \in \left\{ \left\lceil \frac{2+2u}{p} - 1 \right\rceil , \left\lfloor 1 + \frac{2u}{p} \right\rfloor \right\} \cap \mathbb{N}_{\geq 0},$$

and

$$1 > \frac{2+2u}{p} - 1$$

which is equivalent to

$$p - 1 > u.$$

That is always true since $p \geq 3$;

$$\frac{2+2u}{p} - 1 > 0$$

is equivalent to

$$2u > p - 2$$

which means in particular that $u = 1$ and $p = 3$. Furthermore, we have

$$\frac{2+2u}{p} - 1 > -1$$

or equivalently

$$2u > -2,$$

which is always true since $u \in \{0, 1\}$.

That means that $\left\lceil \frac{2+2u}{p} - 1 \right\rceil = 0$ for all $p \geq 3$ and $u \in \{0, 1\}$ unless $p = 3$ and $u = 1$, in which case $\left\lceil \frac{2+2u}{p} - 1 \right\rceil = 1$.

Also,

$$2 > 1 + \frac{2u}{p} \geq 1$$

is equivalent to

$$p > 2u \geq 0$$

and this is true for all $p \geq 3$ and $u \in \{0, 1\}$, hence $\left\lfloor 1 + \frac{2u}{p} \right\rfloor = 1$ for all $p \geq 3$ and $u \in \{0, 1\}$.

Therefore, we have

$$n \in \{0, 1\}$$

for all $p \geq 3$ and $u \in \{0, 1\}$. In addition, $n \neq 0$ if $p = 3$ and $u = 1$.

The corresponding elements of type **1** can be written

$$(s, -2a - n, -1, n(p-1) + 2u - pn + 1 - u, a + n, a, s + 2u - pn + 1),$$

where $u \in \{0, 1\}$, and $n \in \{0, 1\}$. Re-arranging the k-degree, we have

$$(s, -2a - n, -1, 1 + u - n, a + n, a, s + 2u - pn + 1).$$

If $n = 0$, the element becomes

$$(s, -2a, -1, 1 + u, a, a, s + 2u + 1).$$

- If $u = 0$, we have
$$(s, -2a, -1, 1, a, a, s + 1).$$
- If $u = 1$, we have
$$(s, -2a, -1, 2, a, a, s + 3).$$

If $n = 1$, we have

$$(s, -2a - 1, -1, u, a + 1, a, s + 2u - p + 1).$$

- If $u = 0$, we have
$$(s, -2a - 1, -1, 0, a + 1, a, s - p + 1),$$
and since $1 \leq s - p + 1 \leq p$, we see that $s = p$ so that the element is
$$(p, -2a - 1, -1, 0, a + 1, a, 1);$$

- If $u = 1$, we have
$$(s, -2a-1, -1, 1, a+1, a, s-p+3),$$
and since $1 \leq s-p+3 \leq p$, we see that $s \in \{p, p-1, p-2\}$ so that this case gives the three elements
$$\begin{aligned}&(p, -2a-1, -1, 1, a+1, a, 3);\\&(p-1, -2a-1, -1, 1, a+1, a, 2);\\&(p-2, -2a-1, -1, 1, a+1, a, 1).\end{aligned}$$

□

Remark 4.3.4. 1. Comparing the elements in Lemma 4.3.2 and those in Lemma 4.3.3, we see that the elements of the second lemma are obtained from the elements of the first lemma by multiplication with $x = (s, 0, -1, 1, 0, 0, s+1)$;

2. We see that the conditions in the previous lemma do not make sense for all values of p: some elements do not exist if $p = 3$, and this corresponds to the fact that $n \neq 0$ if $p = 3$ and $u = 1$ in the proof.

Lemma 4.3.5. *The monomial basis elements of* $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})^{\leq 1}$ *of j-degree -2 are*

$$\begin{array}{rcll}(s_1, -2a, -2, 2, a, a, s_1+2) &=& e_{s_1} w^a x^2 e_{s_1+2} & \\ (s_2, -2a, -2, 3, a, a, s_2+4) &=& e_{s_2} w^a x^3 \xi e_{s_2+4} & \\ (p, -2a-1, -2, 1, a+1, a, 2) &=& e_p \otimes (\xi \otimes \xi)^{\otimes a} \otimes x e_2 & \\ (p-1, -2a-1, -2, 1, a+1, a, 1) &=& e_{p-1} x \otimes (\xi \otimes \xi)^{\otimes a} \otimes e_1 & \\ (p, -2a-1, -2, 2, a+1, a, 4) &=& e_p \otimes (\xi \otimes \xi)^{\otimes a} \otimes x^2 \xi e_4 & \\ (p-1, -2a-1, -2, 2, a+1, a, 3) &=& e_{p-1} \xi \otimes (\xi \otimes \xi)^{\otimes a} \otimes x^2 e_3 & \\ & (\cong & e_{p-1} x \otimes (\xi \otimes \xi)^{\otimes a} \otimes x \xi e_3) & \\ (p-2, -2a-1, -2, 2, a+1, a, 2) &=& e_{p-2} x \xi \otimes (\xi \otimes \xi)^{\otimes a} \otimes x e_2 & \\ & (\cong & e_{p-2} x^2 \otimes (\xi \otimes \xi)^{\otimes a} \otimes \xi e_2) & \\ (p-3, -2a-1, -2, 2, a+1, a, 1) &=& e_{p-3} x^2 \xi \otimes (\xi \otimes \xi)^{\otimes a} \otimes e_1 & \\ (3, -2a-2, -2, 1, a+2, a, 1) &=& e_3 \otimes x\xi \otimes (\xi \otimes \xi)^{\otimes a} \otimes e_1 & (\text{ if } p = 3)\end{array}$$

with $1 \leq s_1 \leq p-2$, $1 \leq s_2 \leq p-4$, $a \geq 0$.

Proof. Let us set $N = -2$ in equation (4.1). We see that

$$n \in \left\{ \left\lceil \frac{3+2u}{p} - 1 \right\rceil \ , \ \left\lfloor 1 + \frac{1+2u}{p} \right\rfloor \right\} \cap \mathbb{N}_{\geq 0},$$

and

$$1 > \frac{3+2u}{p} - 1$$

which is equivalent to

$$2p - 3 > 2u,$$

and that is always true since $p \geq 3$;

$$\frac{3+2u}{p} - 1 > 0$$

or equivalently

$$p < 3 + 2u$$

which means in particular that $p = 3$ and $u = 1$. In addition, we have

$$\frac{3+2u}{p} - 1 > -1$$

which may be equivalently written

$$2u > -3.$$

That is always true since $u \in \{0,1\}$.

That means that $\left\lceil \dfrac{3+2u}{p} - 1 \right\rceil = 0$ for all $p \geq 3$ and $u \in \{0,1\}$ unless $p = 3$ and $u = 1$, in which case $\left\lceil \dfrac{3+2u}{p} - 1 \right\rceil = 1$.

Also,

$$2 > 1 + \frac{1+2u}{p} \geq 1$$

is equivalent to

$$p > 1 + 2u \geq 0$$

and this is true for all $p \geq 5$ and $u \in \{0,1\}$, and for $p = 3$ and $u = 0$, hence in that case, $\left\lfloor 1 + \dfrac{1+2u}{p} \right\rfloor = 1$; if $p = 3$ and $u = 1$, we have

$$\left\lfloor 1 + \frac{1+2u}{p} \right\rfloor = \left\lfloor 1 + \frac{3}{3} \right\rfloor = 2.$$

Therefore, we have

$$n \in \{0,1\}$$

for all $p \geq 5$ and $u \in \{0,1\}$. In addition,

$$n \in \{0,1\}$$

if $p = 3$ and $u = 0$, and

$$n \in \{1,2\}$$

if $p = 3$ and $u = 1$.

The corresponding elements of type **1** can be written

$$(s, -2a-n, -2, n(p-1) + 2u - pn + 2 - u, a+n, a, s + 2u - pn + 2),$$

where $u \in \{0,1\}$, and $n \in \{0,1,2\}$. Re-arranging the k-degree, we have

$$(s, -2a-n, -2, 2+u-n, a+n, a, s+2u-pn+2).$$

If $n = 0$, the element becomes

$$(s, -2a, -2, 2+u, a, a, s+2u+2).$$

- If $u = 0$, we have
$$(s, -2a, -2, 2, a, a, s+2).$$
- If $u = 1$, we have
$$(s, -2a, -2, 3, a, a, s+4).$$

If $n = 1$, we have

$$(s, -2a-1, -2, 1+u, a+1, a, s+2u-p+2).$$

- If $u = 0$, we have
$$(s, -2a-1, -2, 1, a+1, a, s-p+2),$$
and since $1 \leq s - p + 2 \leq p$, we see that $s \in \{p, p-1\}$ so that the corresponding elements are
$$(p, -2a-1, -2, 1, a+1, a, 2);$$
$$(p-1, -2a-1, -2, 1, a+1, a, 1).$$

- If $u = 1$, we have
$$(s, -2a-1, -2, 2, a+1, a, s-p+4),$$
and since $1 \leq s-p+4 \leq p$, we see that $s \in \{p, p-1, p-2, p-3\}$ so that this case gives the four elements
$$\begin{array}{l}(p, -2a-1, -2, 2, a+1, a, 4);\\(p-1, -2a-1, -2, 2, a+1, a, 3);\\(p-2, -2a-1, -2, 2, a+1, a, 2);\\(p-3, -2a-1, -2, 2, a+1, a, 1).\end{array}$$

If $n = 2$ (for $p = 3$ and $u = 1$), we have the element
$$(s, -2a-2, -2, 1, a+2, a, s-2),$$
which implies $s = 3$, namely
$$(3, -2a-2, -2, 1, a+2, a, 1).$$

□

4.4 Decomposition of chained elements of $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})^{\leq 1}$

In this section, we will exhibit a few key decompositions of chained elements of $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})^{\leq 1}$. They will enable us to find a criterion to decide when a chained element is reducible and it will be critical to describe the irreducible monomials of $\mathbf{w}_q$.

4.4.1 Decompositions using $e_1 \otimes \ldots \otimes e_{l-1} \otimes x\xi \otimes e_{l+1} \otimes \ldots \otimes e_q$

Proposition 4.4.1. *Let $v_1 \otimes \ldots \otimes v_q$ be an element of $\mathbf{w}_q$. Let $2 \leq l \leq q-1$ be such that v_l is a type **1** element such that $u_l = 1$ and such that its k-degree is greater or equal to 1. If $s_l \leq p-2$ or $t_l \geq 3$, then $v_1 \otimes \ldots \otimes v_q$ is reducible.*

Proof. Note that $e_s x\xi e_{s+2}$ corresponds to $(s, 0, 0, 1, 0, 0, s+2)$. We have the following decompositions

$$\begin{aligned}
& v_1 \otimes \ldots \otimes v_{l-1} \otimes v_l \otimes v_{l+1} \otimes \ldots \otimes v_q \\
= \; & e_{s_1} \otimes \ldots \otimes e_{s_{l-1}} \otimes e_{s_l} x\xi e_{s_l+2} \otimes e_{s_{l+1}} \otimes \ldots \otimes e_{s_q} \\
\cdot \; & v_1 \otimes \ldots \otimes v_{l-1} \otimes \\
& (s_l+2, -a_l-b_l, -p(a_l-b_l)-(t_l-s_l)+2u_l, \\
& \quad (p-1)(a_l-b_l)+(t_l-s_l)-u_l-1, a_l, b_l, t_l) \\
& \qquad \otimes v_{l+1} \otimes \ldots \otimes v_q \\
= \; & v_1 \otimes \ldots \otimes v_{l-1} \otimes \\
& (s_l, -a_l-b_l, -p(a_l-b_l)-(t_l-s_l)+2u_l, \\
& \quad (p-1)(a_l-b_l)+(t_l-s_l)-u_l-1, a_l, b_l, t_l-2) \\
& \qquad \otimes v_{l+1} \otimes \ldots \otimes v_q \\
\cdot \; & e_{t_1} \otimes \ldots \otimes e_{t_{l-1}} \otimes e_{t_l-2} x\xi e_{t_l} \otimes e_{t_{l+1}} \otimes \ldots \otimes e_{t_q},
\end{aligned}$$

if and only if the elements

$$(s_l+2, -a_l-b_l, -p(a_l-b_l)-(t_l-s_l)+2u_l, (p-1)(a_l-b_l)+(t_l-s_l)-u_l-1, a_l, b_l, t_l),$$

resp.

$$(s_l, -a_l-b_l, -p(a_l-b_l)-(t_l-s_l)+2u_l, (p-1)(a_l-b_l)+(t_l-s_l)-u_l-1, a_l, b_l, t_l-2),$$

are valid type **1** elements. The latter is true if and only if

1. $(p-1)(a_l-b_l)+(t_l-s_l)-u_l-1 \geq 0$, which is equivalent to $(p-1)(a_l-b_l)+(t_l-s_l)-u_l \geq 1$, and $(p-1)(a_l-b_l)+(t_l-s_l)-u_l$ is the k-degree of v_l which is greater of equal to 1 by assumption;

2. and

$$\begin{cases} -p(a_l-b_l)-(t_l-s_l)+2u_l &= -p(a_l-b_l)-(t_l-(s_l+2))+2u'_l \\ (p-1)(a_l-b_l)+(t_l-s_l)-u_l-1 &= (p-1)(a_l-b_l)+(t_l-(s_l+2))-u'_l \end{cases}$$

resp.

$$\begin{cases} -p(a_l-b_l)-(t_l-s_l)+2u_l &= -p(a_l-b_l)-((t_l-2)-s_l)+2u'_l \\ (p-1)(a_l-b_l)+(t_l-s_l)-u_l-1 &= (p-1)(a_l-b_l)+((t_l-2)-s_l)-u'_l \end{cases}$$

and both are equivalent to

$$\begin{cases} 2u_l &= 2+2u'_l \\ -u_l-1 &= -2-u'_l, \end{cases}$$

and, simplifying further, to

$$\begin{cases} u_l &= 1 \\ u'_l &= 0; \end{cases}$$

3. and

$$s_l+2 \leq p$$

which is if and only if

$$s_l \leq p-2,$$

resp.

$$t_l-2 \geq 1$$

which is equivalent to

$$t_l \geq 3.$$

Since $l \geq 2$, v_1 is left untouched and it is not an idempotent by assumption. Hence those decompositions are non-trivial and $v_1 \otimes \ldots \otimes v_q$ is reducible. □

Proposition 4.4.2. *Let $v_1 \otimes \ldots \otimes v_q$ be an element of $\mathbf{w}_q$. Let v_1 be such that $u_1 = 1$ and such that its k-degree is greater or equal to 2. Then $v_1 \otimes \ldots \otimes v_q$ is reducible.*

Proof. The proof uses the same decompositions as in the proof of the previous proposition. We just need to show that these decompositions are non-trivial. Let us consider the factors

$$(s_1+2, 0, -(t_1-s_1)+2u_1, (t_1-s_1)-u_1-1, 0, 0, t_1),$$

and

$$(s_1, 0, -(t_1-s_1)+2u_1, (t_1-s_1)-u_1-1, 0, 0, t_1-2).$$

Since the k-degree of v_1 is greater or equal to 2, i.e.

$$(t_1-s_1)-u_1 \geq 2,$$

we see that the k-degree of the factors satisfies

$$(t_1-s_1)-u_1-1 \geq 1 > 0,$$

hence they are not idempotents and the decompositions are non-trivial. Therefore, $v_1 \otimes \ldots \otimes v_q$ is reducible. □

Remark 4.4.3. A consequence of these two propositions is that $v_1 \otimes \ldots \otimes v_q$ is reducible if we can factor $x\xi$ from any of its components v_l, where $2 \leq l \leq q$, and if we can factor $x\xi$ from v_1 where v_1 has k-degree at least 2; hence **we assume from here onwards that it is not possible to factor $x\xi$ from v_l**, i.e. for all $2 \leq l \leq q$, v_l is of the form

$$\begin{cases} (s, -a-b, -p(a-b)-(t-s), (p-1)(a-b)+(t-s), a, b, t) \\ \qquad = x^{p-s} \otimes (x^{p-1})^{\otimes a-b-1} \otimes (\xi \otimes \xi)^{\otimes b} \otimes x^{t-1} \\ (p-1, -a-b, -p(a-b)-(t-p+1)+2, (p-1)(a-b)+(t-p+1)-1, a, b, t) \\ \qquad = \xi \otimes (x^{p-1})^{\otimes a-b-1} \otimes (\xi \otimes \xi)^{\otimes b} \otimes x^{t-1} \\ (s, -a-b, -p(a-b)-(2-s)+2, (p-1)(a-b)+(2-s)-1, a, b, 2) \\ \qquad = x^{p-s} \otimes (x^{p-1})^{\otimes a-b-1} \otimes (\xi \otimes \xi)^{\otimes b} \otimes \xi \\ (p, -a-b, -p(a-b)-(1-p)+2, (p-1)(a-b)+(1-p)-1, a, b, 1) \\ \qquad = e_p \otimes x^{p-2}\xi \otimes (x^{p-1})^{\otimes a-b-2} \otimes (\xi \otimes \xi)^{\otimes b} \otimes e_1 \end{cases}$$

if $a-b \geq 2$, or of the form

$$\begin{cases} (s, -2b-1, -p-(t-s), (p-1)+(t-s), b+1, b, t) \\ \qquad = x^{p-s} \otimes (\xi \otimes \xi)^{\otimes b} \otimes x^{t-1} \\ (p-1, -2b-1, -p-(t-p+1)+2, (p-1)+(t-p+1)-1, b+1, b, t) \\ \qquad = \xi \otimes (\xi \otimes \xi)^{\otimes b} \otimes x^{t-1} \\ (s, -2b-1, -p-(2-s)+2, (p-1)+(2-s)-1, b+1, b, 2) \\ \qquad = x^{p-s} \otimes (\xi \otimes \xi)^{\otimes b} \otimes \xi \end{cases}$$

if $a-b=1$, or of the form

$$\begin{cases} (s, -2a, -(t-s), (t-s), a, a, t) & = & e_s w x^{t-s} e_t \\ (s, -2a, 1, 0, a, a, s+1) & = & e_s w \xi e_{s+1} \end{cases}$$

if $a-b=0$, or of the form

$$(s, -2a-1, 1, 0, a, a+1, p+1-s) = e_s(\xi \otimes \xi)^{\otimes a} e_{p+1-s}$$

if $a-b=-1, a \geq 0$, or of the form

$$(s, 1, 1, 0, -1, 0, p+1-s) = e_s \otimes e^*_{p+1-s}$$

if $a-b=-1$, with $a=-1, b=0$.

In addition, v_1 is of the form

$$\begin{cases} (s, 0, -(t-s), t-s, 0, 0, t) & = & e_s x^{t-s} e_t \\ (s, 0, 0, 1, 0, 0, s+2) & = & e_s x\xi e_{s+2} \\ (s, 0, 1, 0, 0, 0, s+1) & = & e_s \xi e_{s+1} \end{cases}$$

where $t-s \geq 0$.

We can therefore rewrite Lemmas 4.3.2, 4.3.3 and 4.3.5 to only include elements which do not yield these easy decompositions.

Lemma 4.4.4. *The monomial basis elements of $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})^{\leq 1}$ of j-degree 0 from which $x\xi$ cannot be non-trivially factored are*

$$\begin{array}{rcl} (s_1, 0, 0, 1, 0, 0, s_1+2) & = & e_{s_1} x\xi e_{s_1} \\ (s_2, -2a, 0, 0, a, a, s_2) & = & e_{s_2} w^a e_{s_2} \\ (p-1, -2a-1, 0, 0, a+1, a, 1) & = & e_{p-1}\xi e_p \otimes (\xi \otimes \xi)^{\otimes a} \otimes e_1 \\ (p, -2a-1, 0, 0, a+1, a, 2) & = & e_p \otimes (\xi \otimes \xi)^{\otimes a} \otimes e_1 \xi e_2 \end{array}$$

with $1 \leq s_1 \leq p-2$, $1 \leq s_2 \leq p$, $a \geq 0$.

Lemma 4.4.5. *The monomial basis elements of $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})^{\leq 1}$ of j-degree -1 from which $x\xi$ cannot be non-trivially factored are*

$$\begin{array}{rcl} (s,-2a,-1,1,a,a,s+1) & = & e_s w^a x e_{s+1} \\ (p,-2a-1,-1,0,a+1,a,1) & = & e_p \otimes (\xi\otimes\xi)^{\otimes a} \otimes e_1 \\ (p-1,-2a-1,-1,1,a+1,a,2) & = & e_{p-1}\xi \otimes (\xi\otimes\xi)^{\otimes a} \otimes xe_2 \\ & (\cong & e_{p-1}x \otimes (\xi\otimes\xi)^{\otimes a} \otimes \xi e_2) \end{array}$$

with $1 \leq s \leq p-1$, $a \geq 0$.

Lemma 4.4.6. *The monomial basis elements of $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})^{\leq 1}$ of j-degree -2 from which $x\xi$ cannot be non-trivially factored are*

$$\begin{array}{rcl} (s,-2a,-2,2,a,a,s+2) & = & e_s w^a x^2 e_{s+2} \\ (p,-2a-1,-2,1,a+1,a,2) & = & e_p \otimes (\xi\otimes\xi)^{\otimes a} \otimes xe_2 \\ (p-1,-2a-1,-2,1,a+1,a,1) & = & e_{p-1}x \otimes (\xi\otimes\xi)^{\otimes a} \otimes e_1 \\ (3,-2a-2,-2,1,a+2,a,1) & = & e_3 \otimes x\xi \otimes (\xi\otimes\xi)^{\otimes a} \otimes e_1 \quad (\text{ if } p=3) \end{array}$$

with $1 \leq s \leq p-2$, $a \geq 0$.

4.4.2 Decompositions using $v_1 \otimes \ldots \otimes v_l \otimes (\xi^u e_p \otimes e_1 \xi^{1-u}) \otimes e_{l+1} \otimes \ldots \otimes e_q$

Lemma 4.4.7. *Let $v = v_1 \otimes \ldots \otimes v_q$ be a monomial of $\mathbf{w}_q$. Let $1 \leq l \leq q-1$. Assume that there are non-trivial decompositions*

$$v_1 \otimes \ldots \otimes v_l = \hat{v}_1 \otimes \ldots \otimes \hat{v}_l \cdot \tilde{v}_1 \otimes \ldots \otimes \tilde{v}_l,$$

with the j-degree of $\hat{v}_l$ being equal to -1, resp. the j-degree of $\tilde{v}_l$, and

$$v_{l+1} = (\xi \otimes e_1) \cdot \tilde{v}_{l+1}, \quad (\textit{resp. } \hat{v}_{l+1} \cdot (e_p \otimes \xi)),$$

then v is reducible.

Proof. Since $\hat{v}_l$ has j-degree -1, resp. $\tilde{v}_l$, we see that $\hat{v}_l \otimes (\xi \otimes e_1)$, resp. $\tilde{v}_l \otimes (e_p \otimes \xi)$ are chained. Besides, $\xi \otimes e_1$, resp. $e_p \otimes \xi$ has j-degree 0, hence can be followed by idempotents. We can therefore write the following non-trivial decomposition

$$\begin{array}{rl} & v_1 \otimes \ldots \otimes v_q \\ = & \hat{v}_1 \otimes \ldots \otimes \hat{v}_l \otimes (\xi \otimes e_1) \otimes e_{s_{l+2}} \ldots \otimes e_{s_q} \\ \cdot & \tilde{v}_1 \otimes \ldots \otimes \tilde{v}_l \otimes \tilde{v}_{l+1} \otimes v_{l+2} \otimes \ldots \otimes v_q \end{array}$$

resp.

$$\begin{array}{rl} & v_1 \otimes \ldots \otimes v_q \\ = & \hat{v}_1 \otimes \ldots \otimes \hat{v}_l \otimes \hat{v}_{l+1} \otimes v_{l+2} \otimes \ldots \otimes v_q \\ \cdot & \tilde{v}_1 \otimes \ldots \otimes \tilde{v}_l \otimes (e_p \otimes \xi) \otimes e_{t_{l+2}} \ldots \otimes e_{t_q}. \end{array}$$

Thus v is reducible. □

4.4.3 Decomposition of elements such that $a - b \geq 2$

Lemma 4.4.8. *Let $v \in \mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})^{\leq 1}$ such that its a- and b-degree satisfy $a - b \geq 2$. Then for all $1 \leq m \leq p$, there exists a decomposition*

$$\begin{array}{rl} v = & (s,-a-b,-p(a-b)-(t-s)+2u,(p-1)(a-b)+(t-s)-u,a,b,t) \\ = & (s,-\alpha-\beta,-p(\alpha-\beta)-(m-s)+2\nu,(p-1)(\alpha-\beta)+(m-s)-\nu,\alpha,\beta,m) \\ \cdot & (m,-(a-\alpha)-(b-\beta),-p((a-\alpha)-(b-\beta))-(t-m)+2(u-\nu), \\ & \quad (p-1)((a-\alpha)-(b-\beta))+(t-m)-(u-\nu),a-\alpha,b-\beta,t), \end{array}$$

with $\alpha \leq a$, $\beta \leq b$ such that $\alpha \geq \beta \geq 0$.

Proof. Since $a - b \geq 2$, v is of the form

$$L \otimes (x^{p-1})^{\otimes a-b-1} \otimes (\xi \otimes \xi)^{\otimes b} \otimes R,$$

where $(L, R) \in \{(x^{p-s}, x^{t-1}), (\xi, x^{t-1}), (x^{p-s}, \xi)\}$, and we know there is at least a x^{p-1} component, or v is of the form

$$e_p \otimes x^{p-2}\xi \otimes (x^{p-1})^{\otimes a-b-2} \otimes (\xi \otimes \xi)^{\otimes b} \otimes e_1,$$

and we know there is a $x^{p-2}\xi$ component.

We decompose v along one x^{p-1} component

$$\begin{array}{rl} & L \otimes (x^{p-1})^{\otimes a-b-1} \otimes (\xi \otimes \xi)^{\otimes b} \otimes R \\ = & L \otimes (x^{p-1})^{\otimes \alpha-\beta-1} \otimes (\xi \otimes \xi)^{\otimes \beta} \otimes e_1 x^{m-1} \\ \cdot & x^{p-m} \otimes (x^{p-1})^{\otimes (a-\alpha)-(b-\beta)-1} \otimes (\xi \otimes \xi)^{\otimes b-\beta} \otimes R \end{array}$$

in the first case, or along the $x^{p-2}\xi$ component

$$\begin{array}{rl} & e_p \otimes x^{p-2}\xi \otimes (x^{p-1})^{\otimes a-b-2} \otimes (\xi \otimes \xi)^{\otimes b} \otimes e_1 \\ = & e_p \otimes (x^{p-1})^{\otimes \alpha-\beta-1} \otimes (\xi \otimes \xi)^{\otimes \beta} \otimes e_1 x^{m-1-\nu}\xi^{\nu} \\ \cdot & x^{p-m-(1-\nu)}\xi^{1-\nu} \otimes (x^{p-1})^{\otimes (a-\alpha)-(b-\beta)-1} \otimes (\xi \otimes \xi)^{\otimes b-\beta} \otimes e_1 \end{array}$$

in the second case. Writing these decompositions in the form (s, t, i, j, k, a, b, t) yields the result. □

Proposition 4.4.9. *Let $v = v_1 \otimes \ldots \otimes v_q$ be a monomial of $\mathbf{w}_q$ such that there is an index $2 \leq l \leq q$ such that for all $l \leq n \leq q$ the a- and b-degree of v_n satisfy $a_n - b_n \geq 2$. We let l be minimal with that property. Then there exists a decomposition*

$$\begin{array}{rl} & v_1 \otimes \ldots \otimes v_{l-1} \otimes v_l \otimes v_{l+1} \otimes \ldots \otimes v_q \\ = & e_{s_1} \otimes \ldots \otimes e_{s_{l-2}} \otimes x \otimes \hat{v}_l \otimes \hat{v}_{l+1} \otimes \ldots \otimes \hat{v}_q \\ \cdot & v_1 \otimes \ldots \otimes v_{l-2} \otimes \tilde{v}_{l-1} \otimes \tilde{v}_l \otimes \tilde{v}_{l+1} \otimes \ldots \otimes \tilde{v}_q \end{array}$$

or

$$\begin{array}{rl} & v_1 \otimes \ldots \otimes v_{l-1} \otimes v_l \otimes v_{l+1} \otimes \ldots \otimes v_q \\ = & v_1 \otimes \ldots \otimes v_{l-2} \otimes \hat{v}_{l-1} \otimes \hat{v}_l \otimes \hat{v}_{l+1} \otimes \ldots \otimes \hat{v}_q \ . \\ \cdot & e_{t_1} \otimes \ldots \otimes e_{t_{l-2}} \otimes x \otimes \tilde{v}_l \otimes \tilde{v}_{l+1} \otimes \ldots \otimes \tilde{v}_q \end{array}$$

In particular, v is reducible.

Proof. Let us note that since $a_n - b_n \geq 2$ for all $l \leq n \leq q$, there exists a decomposition

$$v_n = \hat{v}_n \cdot \tilde{v}_n$$

for all $l \leq n \leq q$ by Lemma 4.4.8, and more precisely there exists $1 \leq m_n \leq p$ for all $l \leq n \leq q$ such that the idempotent on the right of $\hat{v}_n$ is m_n ($\hat{t}_n = m_n$) and such that the idempotent on the left of $\tilde{v}_n$ is m_n ($\tilde{s}_n = m_n$).

Consider the following identities:

$$i = -a - b \Leftrightarrow a = -i - b \Leftrightarrow a - b = -i - 2b,$$

and for all $1 \leq n \leq q$, we have

$$j_n = -p(a_n - b_n) - (t_n - s_n) + 2u_n,$$

which we may rewrite

$$j_n = -p(-i_n - 2b_n) - (t_n - s_n) + 2u_n.$$

Now, if $n \geq 1$, we can replace i_n by j_{n-1} and we see

$$j_n = -p(-j_{n-1} - 2b_n) - (t_n - s_n) + 2u_n,$$

hence, if $n \geq 2$,

$$j_n = -p(-(-p(-j_{n-2} - 2b_{n-1}) - (t_{n-1} - s_{n-1}) + 2u_{n-1}) - 2b_n) - (t_n - s_n) + 2u_n,$$

which we may rewrite

$$\begin{array}{rcl} j_n & = & -p(p(-j_{n-2} - 2b_{n-1}) + (t_{n-1} - s_{n-1}) - 2u_{n-1} - 2b_n) - (t_n - s_n) + 2u_n \\ & = & -p^2(-j_{n-2} - 2b_{n-1}) - p(t_{n-1} - s_{n-1}) + 2pu_{n-1} + 2pb_n - (t_n - s_n) + 2u_n \\ & = & p^2 j_{n-2} + 2p^2 b_{n-1} + 2pb_n - p(t_{n-1} - s_{n-1} - 2u_{n-1}) - (t_n - s_n - 2u_n). \end{array}$$

In particular, for $1 \leq n' < n$, we have

$$j_n \;=\; p^{n-n'} j_{n'} - \sum_{r=n'+1}^{n} p^{n-r}(t_r - s_r - 2u_r) + 2 \sum_{r=n'+1}^{n} p^{n-r+1} b_r \;.$$

If we let $n' = l - 1$, then, since there exists $1 \leq m_z \leq p$ for all $l \leq z \leq q$ such that $\hat{t}_z = m_z$ and $\tilde{s}_z = m_z$, we have

$$\begin{array}{rcl} j_n & = & p^{n-l+1} j_{l-1} - \sum_{r=l}^{n} p^{n-r}(t_r - m_r + m_r - s_r - 2u_r + 2\nu_r - 2\nu_r) \\ & & +2 \sum_{r=l}^{n} p^{n-r+1}(b_r - \beta_r + \beta_r) \\ & = & p^{n-l+1} j_{l-1} - \sum_{r=l}^{n} p^{n-r}(t_r - m_r - 2\nu_r) - \sum_{r=l}^{n} p^{n-r}(m_r - s_r - 2(u_r - \nu_r)) \\ & & +2 \sum_{r=l}^{n} p^{n-r+1}\beta_r + 2 \sum_{r=l}^{n} p^{n-r+1}(b_r - \beta_r) \end{array}$$

for all $l - 1 < n \leq q$.

That means that the decomposition of each individual $v_n = \hat{v}_n \cdot \tilde{v}_n$ is compatible with the chaining rule after index $l - 1$, namely we have

$$v_l \otimes \ldots \otimes v_q = \hat{v}_l \otimes \ldots \otimes \hat{v}_q \cdot \tilde{v}_l \otimes \ldots \otimes \tilde{v}_q,$$

where $\hat{v}_n$ has j-degree $-\sum_{r=l}^{n} p^{n-r}(t_r - m_r) + 2\sum_{r=l}^{n} p^{n-r+1}\beta_r$ and $\tilde{v}_n$ has j-degree

$$-\sum_{r=l}^{n} p^{n-r}(m_r - s_r - 2u_r) + 2\sum_{r=l}^{n} p^{n-r+1}(b_r - \beta_r).$$

Obviously, for this decomposition to propagate to the whole element, we at least need to split v_{l-1} too so that it is compatible with that decomposition. By minimality of l, we know that $1 \geq a_{l-1} - b_{l-1} \geq 0$, i.e. it is of the form

$$\begin{cases} (s, -2b-1, -p-(t-s), (p-1)+(t-s), b+1, b, t) \\ \qquad\qquad = x^{p-s} \otimes (\xi \otimes \xi)^{\otimes b} \otimes x^{t-1} \\ (p-1, -2b-1, 1-t, t-1, b+1, b, t) \\ \qquad\qquad = \xi \otimes (\xi \otimes \xi)^{\otimes b} \otimes x^{t-1} \\ (s, -2b-1, s-p, p-s, b+1, b, t) \\ \qquad\qquad = x^{p-s} \otimes (\xi \otimes \xi)^{\otimes b} \otimes \xi \end{cases}$$

if $a_{l-1} - b_{l-1} = 1$, or of the form

$$\begin{cases} (s, -2a, -(t-s), (t-s), a, a, t) & = & e_s w x^{t-s} e_t \\ (s, -2a, 1, 0, a, a, s+1) & = & e_s w \xi e_{s+1} \end{cases}$$

if $a_{l-1} - b_{l-1} = 0$. Since $a_l - b_l \geq 2$ is equivalent to $a_l \geq b_l + 2$, we have in particular

$$j_{l-1} = i_l = -a_l - b_l \leq -2b_l - 2 \leq -2,$$

so v_{l-1} could be

$$\begin{cases} \begin{array}{r} (s, -2b-1, -p-(t-s), (p-1)+(t-s), b+1, b, t) \\ = x^{p-s} \otimes (\xi \otimes \xi)^{\otimes b} \otimes x^{t-1}, \end{array} & \text{with } t - s \geq 2 - p \\ \begin{array}{r} (p-1, -2b-1, 1-t, t-1, b+1, b, t) \\ = \xi \otimes (\xi \otimes \xi)^{\otimes b} \otimes x^{t-1}, \end{array} & \text{with } t \geq 3 \\ \begin{array}{r} (s, -2b-1, s-p, p-s, b+1, b, t) \\ = x^{p-s} \otimes (\xi \otimes \xi)^{\otimes b} \otimes \xi, \end{array} & \text{with } s \leq p-2 \end{cases}$$

if $a_{l-1} - b_{l-1} = 1$, or

$$(s, -2a, -(t-s), (t-s), a, a, t) = e_s w x^{t-s} e_t, \qquad \text{with } t - s \geq 2$$

if $a_{l-1} - b_{l-1} = 0$. In particular, it is possible to factor one x from v_{l-1}, either on the left or on the right, which means we can write

$$\begin{aligned} j_n &= p^{n-l+1}(-1+1+j_{l-1}) - \sum_{r=l}^{n} p^{n-r}(t_r - m_r - 2\nu_r) - \sum_{r=l}^{n} p^{n-r}(m_r - s_r - 2(u_r - \nu_r)) \\ &\quad + 2\sum_{r=l}^{n} p^{n-r+1}\beta_r + 2\sum_{r=l}^{n} p^{n-r+1}(b_r - \beta_r) \\ &= \hat{j}_n + \tilde{j}_n \end{aligned}$$

for all $l - 1 < n \leq q$, where

$$\begin{aligned} \hat{j}_n &= -p^{n-l+1} - \sum_{r=l}^{n} p^{n-r}(t_r - m_r - 2\nu_r) + 2\sum_{r=l}^{n} p^{n-r+1}\beta_r \\ \tilde{j}_n &= p^{n-l+1}(1 + j_{l-1}) - \sum_{r=l}^{n} p^{n-r}(m_r - s_r - 2(u_r - \nu_r)) + 2\sum_{r=l}^{n} p^{n-r+1}(b_r - \beta_r) \end{aligned}$$

or

$$\begin{aligned} \hat{j}_n &= p^{n-l+1}(1 + j_{l-1}) - \sum_{r=l}^{n} p^{n-r}(t_r - m_r - 2\nu_r) + 2\sum_{r=l}^{n} p^{n-r+1}\beta_r \\ \tilde{j}_n &= -p^{n-l+1} - \sum_{r=l}^{n} p^{n-r}(m_r - s_r - 2(u_r - \nu_r)) + 2\sum_{r=l}^{n} p^{n-r+1}(b_r - \beta_r). \end{aligned}$$

Hence the result. □

4.4.4 Decomposition using $v_1 \otimes \ldots \otimes v_{l-1} \otimes w \otimes e_{l+1} \otimes \ldots \otimes e_q$

Lemma 4.4.10. *Let v be an element of $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})^{\leq 1}$ such that its i-degree is greater or equal to -2 and its j-degree is less or equal to -3. Then v is one of*

$$\begin{cases} (s_{l-1}, 0, -n, n, 0, 0, s+n) & = x^n & n \geq 3 \\ (s, -1, -p-(t-s), \\ \quad p-1+(t-s), 1, 0, t) & = x^{p-s} \otimes x^{t-1} & t-s \geq 3-p \\ (p-1, -1, -t+1, \\ \quad t-1, 1, 0, t) & = \xi \otimes x^{t-1} & t-1 \geq 3 \\ (s, -1, -p+s, \\ \quad p-s, 1, 0, 2) & = x^{p-s} \otimes \xi & p-s \geq 3 \\ (s, -2, -n, n, 1, 1, s+n) & = e_s w x^n & n \geq 3 \\ (s, -2, -2p-(t-s), \\ \quad 2(p-1)+(t-s), 2, 0, t) & = x^{p-s} \otimes x^{p-1} \otimes x^{t-1} & \\ (p-1, -2, -p-t+1, \\ \quad p+t-2, 2, 0, t) & = \xi \otimes x^{p-1} \otimes x^{t-1} & \\ (s, -2, -2p+s, \\ \quad 2p-s-1, 2, 0, 2) & = x^{p-s} \otimes x^{p-1} \otimes \xi & \\ (p, -2, -p+1, p-2, 2, 0, 1) & = e_p \otimes x^{p-2}\xi \otimes e_1 & p > 3 \end{cases}$$

Proof. Let $v \in \mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})^{\leq 1}$ such that its i-degree i_v is greater or equal to -2.

- If $i_v = 1$, then $j_v = 1$ and no element of i-degree less or equal to -3 could follow.
- If $i_v = 0$, then
$$v = (s, 0, -(t-s) + 2u, t-s-u, 0, 0, t)$$
and by Remark 4.4.3, u must be equal to 0. It has j-degree less or equal to -3 if and only if $-(t-s) \leq -3$, i.e. if and only if $t \geq s+3$, hence the description.
- If $i_v = -1$ and v is of type **2**, then $j_v = 1$ and no element of i-degree less or equal to -3 could follow.
- If $i_v = -1$ and v is of type **1**, then
$$v = (s, -1, -p-(t-s) + 2u, p-1+t-s-u, 1, 0, t)$$
and by Remark 4.4.3, we have three subcases:
  - No ξ at all, in which case $u = 0$, and the element has j-degree less or equal to -3 if and only if $-p-(t-s) \leq -3$ which is equivalent to $t-s \geq 3-p$;
  - A ξ on the left, and all its x's on the right, and the element has j-degree less or equal to -3 if and only if $-p-(t-p+1)+2 = -t+1 \leq -3$, or equivalently $t-1 \geq 3$;
  - A ξ on the right, and all its x's on the left, and the element has j-degree less or equal to -3 if and only if $-p-(2-s)+2 = -(p-s) \leq -3$, i.e. if and only if $p-s \geq 3$.

  Hence the description.

- If $i_v = -2$, then v is of type **1** and
  - if $a = b = 1$, then
  $$v = (s, -2, -(t-s) + 2u, t - s - u, 1, 1, t)$$
  and by Remark 4.4.3, u must be equal to 0; it has j-degree less or equal to -3 if and only if $-(t-s) \leq -3$, or equivalently $t \geq s + 3$;
  - if $a = 2, b = 0$, then
  $$v = (s, -2, -2p - (t-s) + 2u, 2(p-1) + t - s - u, 2, 0, t),$$
  and, similarly to case $i_v = -1$, we have four subcases by Remark 4.4.3. In all but one case, the j-degree is greater or equal to -3:
    * No ξ at all, in which case $u = 0$, and the element has j-degree less or equal to -3 if and only if $-2p - (t-s) \leq -3$ which is equivalent to $2p + t - s \geq 3$, and since $t - s \geq 1 - p$, we have $2p + t - s \geq 2p + 1 - p = p + 1 \geq 3$ as $p > 2$;
    * A ξ on the left, and all its x's on the right, and the element has j-degree less or equal to -3 if and only if $-2p - (t - p + 1) + 2 = -p - t + 1 \leq -3$, or equivalently $p + t - 1 \geq 3$ and since $t \geq 1$, $p + t - 1 \geq p > 2$;
    * A ξ on the right, and all its x's on the left, and the element has j-degree less or equal to -3 if and only if $-2p - (2 - s) + 2 = -2p + s \leq -3$, i.e if and only if $2p - s \geq 3$ and since $s \leq p$, we have $2p - s \geq p > 2$;
    * We have a ξ in "the middle" which correspond to the $\mathbf{y}_{f,-i}$ in [MT13], namely
    $$(p, -2, -p + 1, p - 2, 2, 0, 1) = e_p \otimes x^{p-2}\xi \otimes e_1,$$
    and $-p + 1 \leq -3$ is equivalent to $p \geq 4$.

Hence the description. □

Lemma 4.4.11. *Let $v = v_1 \otimes \ldots \otimes v_q$ be a monomial of $\mathbf{w}_q$. Suppose there exists $2 \leq l \leq q$ such that the b-degree of v_l satisfies $b_l \geq 1$, and*

(a) the i-degree of v_l is less or equal to -3;

or

(b) the a- and b-degree of v_{l-1} satisfy $a_{l-1} - b_{l-1} \geq 2$.

If $s_{l-1} \leq p - 2$ or $t_{l-1} \geq 3$, v is reducible.

Proof. Assume $b_l \geq 1$. We have the following non-trivial decomposition

$$\begin{aligned}
& v_1 \otimes v_2 \otimes \ldots \otimes v_l \otimes \ldots \otimes v_q \\
= \quad & e_{s_1} \otimes e_{s_2} \otimes \ldots \otimes e_{s_{l-2}} \\
& \quad \otimes e_{s_{l-1}} x^2 e_{s_{l-1}+2} \otimes e_{s_l} w e_{s_l} \\
& \quad \otimes e_{s_{l+1}} \otimes \ldots \otimes e_{s_q} \\
\cdot \quad & v_1 \otimes v_2 \otimes \ldots \otimes v_{l-2} \\
& \quad \otimes (s_{l-1} + 2, i_{l-1}, j_{l-1} + 2, k_{l-1} - 2, a_{l-1}, b_{l-1}, t_{l-1}) \\
& \quad \otimes (s_l, i_l + 2, j_l, k_l, a_l - 1, b_l - 1, t_l) \\
& \quad \otimes v_{l+1} \otimes \ldots \otimes v_q
\end{aligned}$$

or

$$\begin{aligned}
& v_1 \otimes v_2 \otimes \ldots \otimes v_l \otimes \ldots \otimes v_q \\
= \; & v_1 \otimes v_2 \otimes \ldots \otimes v_{l-2} \\
& \otimes (s_{l-1}, i_{l-1}, j_{l-1}+2, k_{l-1}-2, a_{l-1}, b_{l-1}, t_{l-1}-2) \\
& \otimes (s_l, i_l+2, j_l, k_l, a_l-1, b_l-1, t_l) \\
& \otimes v_{l+1} \otimes \ldots \otimes v_q \\
\cdot \; & e_{t_1} \otimes e_{t_2} \otimes \ldots \otimes e_{t_{l-2}} \\
& \otimes e_{t_{l-1}-2} x^2 e_{t_{l-1}} \otimes e_{t_l} w e_{s_l} \\
& \otimes e_{t_{l+1}} \otimes \ldots \otimes e_{t_q}
\end{aligned}$$

unless $a_l - 1 < 0$ or $(s_{l-1}+2 > p$ and $t_{l-1}-2 < 1)$. Equivalently, this decomposition exists unless $a_l < 1$ or $(s_{l-1} > p-2$ and $t_{l-1} < 3)$.

Let us analyse these conditions.

- If v_l is of type **1**, then $a_l \geq b_l \geq 1$ by assumption.
- If v_l is of type **2**, then
  - if the i-degree of v_l is less or equal to -3, then $-3 \geq i_l = -2a_l - 1$ and $a_l \geq 1$;
  - if the a- and b-degree of v_{l-1} satisfy $a_{l-1} - b_{l-1} \geq 2$, then the obstruction $a_l < 1$ can only occur if $a_l = 0$ and $b_l = 1$, namely v_l has i-degree -1, which means v_{l-1} must have j-degree -1, so from Lemma 4.4.5, it must be one of

$$\begin{aligned}
(s_1, -2a, -1, 1, a, a, s_1+1) &= e_{s_1} w^a x e_{s_1+1} \\
(p, -2a-1, -1, 0, a+1, a, 1) &= e_p \otimes (\xi \otimes \xi)^{\otimes a} \otimes e_1, \\
(p-1, -2a-1, -1, 1, a+1, a, 2) &= e_{p-1} \xi \otimes (\xi \otimes \xi)^{\otimes a} \otimes x e_2 \\
&(\cong e_{p-1} x \otimes (\xi \otimes \xi)^{\otimes a} \otimes \xi e_2).
\end{aligned}$$

    with $1 \leq s_1 \leq p-1$, $1 \leq s_2 \leq p-3$, $a \geq 0$. But by assumption, the a- and b-degree of v_{l-1} must satisfy $a_{l-1} - b_{l-1} \geq 2$. So v_l cannot be of type **2** with $a_l = 0$ and $b_l = 1$ in that case, and there is no obstruction to the decomposition.
- If v_l is of type **3**, in particular $b_l = 0$, which is a contradiction. Hence v_l cannot be of type **3**.

That means the previous decomposition fails if and only if $s_{l-1} > p-2$ and $t_{l-1} < 3$. □

Lemma 4.4.12. *Let $v = v_1 \otimes \ldots \otimes v_q$ be a monomial of $\mathbf{w}_q$ such that v_1 is not an idempotent of $\mathbf{d}$. If there exists $2 \leq l \leq q$ such that the a- and b-degree of v_l are equal and the idempotents on either side of v_l are different, then v is reducible.*

Proof. Assume there exists $2 \leq l \leq q$ such that the $a_l - b_l = 0$ and $s_l \neq t_l$. Then we have the following decomposition

$$\begin{aligned}
& v_1 \otimes \ldots \otimes v_q \\
= \; & v_1 \otimes \ldots \otimes v_{l_1-1} \otimes (s_{l_1}, -2a_{l_1}, 0, 0, a_{l_1}, a_{l_1}, s_{l_1}) \\
& \otimes e_{s_{l_1+1}} \otimes \ldots \otimes e_{s_q} \\
\cdot \; & e_{t_1} \otimes \ldots \otimes e_{t_{l_1-1}} \otimes (s_{l_1}, 0, -(t_{l_1} - s_{l_1}) + 2u_{l_1}, (t_{l_1} - s_{l_1}) - u_{l_1}, 0, 0, t_{l_1}) \\
& \otimes v_{l_1+1} \otimes \ldots \otimes v_q
\end{aligned}$$

and it is not trivial since v_1 is not an idempotent and $s_l \neq t_l$. Thus v is reducible. □

Corollary 4.4.13. *Let $v = v_1 \otimes \ldots \otimes v_q$ be a monomial of $\mathbf{w}_q$ such that v_1 is not an idempotent of $\mathbf{d}$. Assume v is irreducible. If there exists $2 \leq l \leq q$ such that the a- and b-degree of v_l are equal and the idempotents on either side of v_l are equal, then $v_n = e_{s_n}$ for all $l < n \leq q$.*

Proof. Assume there exists $2 \leq l \leq q$ such that the $a_l - b_l = 0$ and $s_l = t_l$. Then in particular, v_{l+1} has i-degree 0. By the previous lemma, if $s_{l+1} \neq t_{l+1}$, then v is reducible. That is a contradiction. Hence $s_{l+1} = t_{l+1}$ and $v_{l+1} = e_{s_{l+1}}$. That shows that v_{l+2} has i-degree 0 too, and repeating the argument yields the proof. □

Proposition 4.4.14. *Let $v = v_1 \otimes \ldots \otimes v_q$ be a monomial of $\mathbf{w}_q$ such that v_1 is not an idempotent of $\mathbf{d}$. Suppose there exists $2 \leq l \leq q$ such that the i-degree i_l of v_l is less or equal to -3, and that there exists $l < l' \leq q$ such that the a- and b- degree of $v_{l'}$ are equal. Then v is reducible.*

Proof. We choose l and l' to be minimal with that property. If $a_{l'} - b_{l'} = 0$, then by Lemma 4.4.12, v is reducible as long as $s_{l'} \neq t_{l'}$. Furthermore, by Corollary 4.4.13, it is reducible unless $v_n = e_{s_n}$ for all $l' < n \leq q$. Hence, we assume that the idempotents on either side of $v_{l'}$ are equal and that $v_n = e_{s_n}$ for all $l' < n \leq q$, i.e. we are in the situation

$$v = v_1 \otimes \ldots \otimes v_l \otimes \ldots \otimes v_{l'-1} \otimes e_{s_{l'}} w^{a_{l'}} e_{s_{l'}} \otimes e_{s_{l'+1}} \otimes \ldots \otimes e_q.$$

By minimality of l, for all $1 \leq m < l$, v_m has at least i-degree -2, and for all $l \leq n < l'$, v_n is of type **1** and its a- and b-degree satisfy $a_n - b_n \geq 1$. Using Remark 4.4.3, we see that v_n is of the form

$$(s, -a-b, -p(a-b)-(t-s), (p-1)(a-b)+(t-s), a, b, t) \\ = x^{p-s} \otimes (x^{p-1})^{\otimes a-b-1} \otimes (\xi \otimes \xi)^{\otimes b} \otimes x^{t-1}$$

$$(p-1, -a-b, -p(a-b)-(t-p+1)+2, (p-1)(a-b)+(t-p+1)-1, a, b, t) \\ = \xi \otimes (x^{p-1})^{\otimes a-b-1} \otimes (\xi \otimes \xi)^{\otimes b} \otimes x^{t-1}$$

$$(s, -a-b, -p(a-b)-(2-s)+2, (p-1)(a-b)+(2-s)-1, a, b, 2) \\ = x^{p-s} \otimes (x^{p-1})^{\otimes a-b-1} \otimes (\xi \otimes \xi)^{\otimes b} \otimes \xi$$

$$(p, -a-b, -p(a-b)-(1-p)+2, (p-1)(a-b)+(1-p)-1, a, b, 1) \\ = e_p \otimes x^{p-2}\xi \otimes (x^{p-1})^{\otimes a-b-2} \otimes (\xi \otimes \xi)^{\otimes b} \otimes e_1$$

if $a_n - b_n \geq 2$, or of the form

$$(s, -2b-1, -p-(t-s), (p-1)+(t-s), b+1, b, t) \\ = x^{p-s} \otimes (\xi \otimes \xi)^{\otimes b} \otimes x^{t-1}$$

$$(p-1, -2b-1, -p-(t-p+1)+2, (p-1)+(t-p+1)-1, b+1, b, t) \\ = \xi \otimes (\xi \otimes \xi)^{\otimes b} \otimes x^{t-1}$$

$$(s, -2b-1, -p-(2-s)+2, (p-1)+(2-s)-1, b+1, b, 2) \\ = x^{p-s} \otimes (\xi \otimes \xi)^{\otimes b} \otimes \xi$$

if $a_n - b_n = 1$.

In addition, since $a_{l'} = b_{l'}$, $v_{l'}$ has an even i-degree, and therefore $v_{l'-1}$ must have an even j-degree.

- Assume $a_{l'} \geq 1$. Then we have the following non-trivial decomposition

$$
\begin{aligned}
& v_1 \otimes \ldots \otimes v_q \\
= \; & e_{s_1} \otimes \ldots \otimes e_{s_{l'-2}} \otimes e_{s_{l'-1}} x^2 e_{s_{l'-1}+2} \otimes e_{s_{l'}} w e_{s_{l'}} \otimes e_{s_{l'+1}} \otimes \ldots \otimes e_{s_q} \\
\cdot \; & v_1 \otimes \ldots \otimes v_{l'-2} \\
& \quad \otimes (s_{l'-1}+2, i_{l'-1}, -p(a_{l'-1}-b_{l'-1}) - (t_{l'-1}-(s_{l'-1}+2)) + 2u_{l'-1}, \\
& \quad (p-1)(a_{l'-1}-b_{l'-1}) + (t_{l'-1}-(s_{l'-1}+2)) - u_{l'-1}, a_{l'-1}, b_{l'-1}, t_{l'-1}) \\
& \quad \otimes e_{s_{l'}} w^{a_{l'}-1} e_{s_{l'}} \otimes e s_{l'+1} \otimes \ldots \otimes e_{s_q}
\end{aligned}
$$

or

$$
\begin{aligned}
& v_1 \otimes \ldots \otimes v_q \\
= \; & v_1 \otimes \ldots \otimes v_{l'-2} \\
& \quad \otimes (s_{l'-1}, i_{l'-1}, -p(a_{l'-1}-b_{l'-1}) - ((t_{l'-1}-2) - s_{l'-1}) + 2u_{l'-1}, \\
& \quad (p-1)(a_{l'-1}-b_{l'-1}) + ((t_{l'-1}-2) - s_{l'-1}) - u_{l'-1}, a_{l'-1}, b_{l'-1}, t_{l'-1}-2) \\
& \quad \otimes e_{s_{l'}} w^{a_{l'}-1} e_{s_{l'}} \otimes e s_{l'+1} \otimes \ldots \otimes e_{s_q} \\
\cdot \; & e_{t_1} \otimes \ldots \otimes e_{t_{l'-2}} \otimes e_{t_{l'-1}-2} x^2 e_{t_{l'-1}} \otimes e_{s_{l'}} w e_{s_{l'}} \otimes e_{s_{l'+1}} \otimes \ldots \otimes e_{s_q}
\end{aligned}
$$

unless $s_{l'-1}+2 > p$ and $t_{l'-1}-2 < 1$, i.e. unless $s_{l'-1} > p-2$ and $t_{l'-1} < 3$. We need to investigate that case further. Assume $s_{l'-1} > p-2$ and $t_{l'-1} < 3$. Recall that $v_{l'-1}$ must have even j-degree and since $a_{l'} \geq 1$, $-2 \geq i_{l'} = j_{l'-1}$. That means the element in position $l'-1$ is one of the following elements

$$
\begin{cases}
(p-1, -2b-1, -p-(2-p), (p-1)+(2-p), b+1, b, 1) \\
\qquad\qquad = x \otimes (\xi \otimes \xi)^{\otimes b} \otimes e_1 \\
(p, -2b-1, -p-(2-p), (p-1)+(2-p), b+1, b, 2) \\
\qquad\qquad = e_p \otimes (\xi \otimes \xi)^{\otimes b} \otimes x \\
(p-1, -2b-1, -p-(2-p), (p-1)+(2-p), b+1, b, 2) \\
\qquad\qquad = e_p \otimes (\xi \otimes \xi)^{\otimes b} \otimes x
\end{cases}
$$

if $a_{l'-1} - b_{l'-1} = 1$, and

$$\begin{cases} (p-1,-a-b,-p(a-b)-(2-p),(p-1)(a-b)+(2-p),a,b,1) \\ \qquad = x\otimes(x^{p-1})^{\otimes a-b-1}\otimes(\xi\otimes\xi)^{\otimes b}\otimes e_1 \\ \text{if } a-b-1 \text{ is even,} \\ (p-1,-a-b,-p(a-b)-(3-p),(p-1)(a-b)+(3-p),a,b,2) \\ \qquad = x\otimes(x^{p-1})^{\otimes a-b-1}\otimes(\xi\otimes\xi)^{\otimes b}\otimes x \\ \text{if } a-b-1 \text{ is odd,} \\ (p,-a-b,-p(a-b)-(1-p),(p-1)(a-b)+(1-p),a,b,1) \\ \qquad = e_p\otimes(x^{p-1})^{\otimes a-b-1}\otimes(\xi\otimes\xi)^{\otimes b}\otimes e_1 \\ \text{if } a-b-1 \text{ is odd,} \\ (p,-a-b,-p(a-b)-(2-p),(p-1)(a-b)+(2-p),a,b,2) \\ \qquad = e_p\otimes(x^{p-1})^{\otimes a-b-1}\otimes(\xi\otimes\xi)^{\otimes b}\otimes x \\ \text{if } a-b-1 \text{ is even,} \\ (p-1,-a-b,-p(a-b)-(1-p+1)+2,(p-1)(a-b)+(1-p+1)-1,a,b,1) \\ \qquad = \xi\otimes(x^{p-1})^{\otimes a-b-1}\otimes(\xi\otimes\xi)^{\otimes b}\otimes e_1 \\ \text{if } a-b-1 \text{ is even,} \\ (p-1,-a-b,-p(a-b)-(2-p+1)+2,(p-1)(a-b)+(2-p+1)-1,a,b,2) \\ \qquad = \xi\otimes(x^{p-1})^{\otimes a-b-1}\otimes(\xi\otimes\xi)^{\otimes b}\otimes x \\ \text{if } a-b-1 \text{ is odd,} \\ (p-1,-a-b,-p(a-b)-(2-p+1)+2,(p-1)(a-b)+(2-p+1)-1,a,b,2) \\ \qquad = x\otimes(x^{p-1})^{\otimes a-b-1}\otimes(\xi\otimes\xi)^{\otimes b}\otimes \xi \\ \text{if } a-b-1 \text{ is odd,} \\ (p,-a-b,-p(a-b)-(2-p)+2,(p-1)(a-b)+(2-p)-1,a,b,2) \\ \qquad = e_p\otimes(x^{p-1})^{\otimes a-b-1}\otimes(\xi\otimes\xi)^{\otimes b}\otimes \xi \\ \text{if } a-b-1 \text{ is even,} \\ (p,-a-b,-p(a-b)-(1-p)+2,(p-1)(a-b)+(1-p)-1,a,b,1) \\ \qquad = e_p\otimes x^{p-2}\xi\otimes(x^{p-1})^{\otimes a-b-2}\otimes(\xi\otimes\xi)^{\otimes b}\otimes e_1 \\ \text{if } a-b-1 \text{ is odd,} \end{cases}$$

if $a_{l'-1} - b_{l'-1} \geq 2$.

We notice that in all cases, we can factor from $v_{l'-1}$, both on the left and on the right, an element of $\mathbb{H}(\mathbf{u}^{-1})$ of j-degree -2, namely one of

$$\begin{array}{rcl} (p-1,-1,-2,1,1,0,1) & = & x\otimes e_1 \\ (p,-1,-2,1,1,0,2) & = & e_p\otimes x \\ (p-1,-1,-2,2,1,0,3) & = & \xi\otimes x^2 \\ (p-2,-1,-2,2,1,0,2) & = & x^2\otimes\xi \end{array}$$

so that we could still have a similar non-trivial decomposition as before. However, for that decomposition to exist, we must be able to find a decomposition of $v_{l'-2}$ of the form $\hat{v}_{l'-2}\cdot\tilde{v}_{l'-2}$ with one of the factor being of j-degree -1. If we can factor x from either side of $v_{l'-2}$, then we are done, as we could write the following decomposition

$$\begin{array}{cl} & v_1\otimes\ldots\otimes v_l\otimes\ldots\otimes v_{l'-2}\otimes v_{l'-1}\otimes v_{l'}\otimes\ldots\otimes v_q \\ = & es_1\otimes\ldots\otimes es_l\otimes\ldots\otimes x\otimes L\otimes es_{l'}wes_{l'}\otimes e_{s_{l'+1}}\otimes\ldots\otimes es_q \\ \cdot & v_1\otimes\ldots\otimes v_l\otimes\ldots\otimes\tilde{v}_{l'-2}\otimes\tilde{v}_{l'-1}\otimes\tilde{v}_l\otimes v_{l'+1}\otimes\ldots\otimes v_q \end{array}$$

if we can factor x on the left of $v_{l'-2}$, or

$$\begin{array}{rl} & v_1 \otimes \ldots \otimes v_l \otimes \ldots \otimes v_{l'-2} \otimes v_{l'-1} \otimes v_{l'} \otimes \ldots \otimes v_q \\ = & v_1 \otimes \ldots \otimes v_l \otimes \ldots \otimes \tilde{v}_{l'-2} \otimes \tilde{v}_{l'-1} \otimes \tilde{v}_l \otimes v_{l'+1} \otimes \ldots \otimes v_q \\ \cdot & et_1 \otimes \ldots \otimes et_l \otimes \ldots \otimes x \otimes R \otimes et_{l'} w et_{l'} \otimes e_{t_{l'+1}} \otimes \ldots \otimes et_q \end{array}$$

if we can factor x on the right of $v_{l'-2}$, where $\tilde{v}_c$ is the remaining part of v_c after factorisation and $L, R \in \{x \otimes e_1, e_p \otimes x, \xi \otimes x^2, x^2 \otimes \xi\}$ depending on the element $v_{l'-1}$.

If not, that means that $v_{l'-2}$ is of the form

$$\begin{cases} (p, -a-b, -p(a-b)-(1-p), (p-1)(a-b)+(1-p), a, b, 1) \\ \qquad = e_p \otimes (x^{p-1})^{\otimes a-b-1} \otimes (\xi \otimes \xi)^{\otimes b} \otimes e_1 \\ \text{if } a-b-1 \text{ is odd,} \\ \\ (p-1, -a-b, -p(a-b)-(1-p+1)+2, (p-1)(a-b)+(1-p+1)-1, a, b, 1) \\ \qquad = \xi \otimes (x^{p-1})^{\otimes a-b-1} \otimes (\xi \otimes \xi)^{\otimes b} \otimes e_1 \\ \text{if } a-b-1 \text{ is even,} \\ \\ (p, -a-b, -p(a-b)-(2-p)+2, (p-1)(a-b)+(2-p)-1, a, b, 2) \\ \qquad = e_p \otimes (x^{p-1})^{\otimes a-b-1} \otimes (\xi \otimes \xi)^{\otimes b} \otimes \xi \\ \text{if } a-b-1 \text{ is even,} \\ \\ (p, -a-b, -p(a-b)-(1-p)+2, (p-1)(a-b)+(1-p)-1, a, b, 1) \\ \qquad = e_p \otimes x^{p-2}\xi \otimes (x^{p-1})^{\otimes a-b-2} \otimes (\xi \otimes \xi)^{\otimes b} \otimes e_1 \\ \text{if } a-b-1 \text{ is odd,} \end{cases}$$

and in particular, $a_{l'-2} - b_{l'-2} \geq 2$ in that case.

We note that it is possible to factor an element of $\mathbb{H}(\mathbf{u}^{-1})$ of j-degree -1 for all of these possibilities, both on the left and on the right, namely we can factor one of

$$\begin{array}{rcl} (p, -1, -1, 0, 1, 0, 1) & = & e_p \otimes e_1 \\ (p-1, -1, -1, 1, 1, 0, 2) & = & x \otimes \xi \\ & \cong & \xi \otimes x \end{array}$$

Again, for the decomposition to make sense, we need to be able to factor an element of j-degree -1 from $v_{l'-3}$, and again if we can factor x then we are done. If not, we apply the same reasoning as for $v_{l'-2}$. Eventually, in the worst case, we reach $l' - (l' - l) = l$ and v_l has i-degree less or equal to -3. Since l is chosen minimal with that property, we know that v_{l-1} has at least i-degree -2 with j-degree at most -3.

By Lemma 4.4.10, v_{l-1} is one of

$$\begin{cases} (s_{l-1}, 0, -n, n, 0, 0, s_{l-1}+n) & = x^n & n \geq 3 \\ (s_{l-1}, -1, -p-(t_{l-1}-s_{l-1}), \\ \quad p-1+(t_{l-1}-s_{l-1}), 1, 0, t_{l-1}) & = x^{p-s_{l-1}} \otimes x^{t_{l-1}-1} & t_{l-1}-s_{l-1} \geq 3-p \\ (p-1, -1, -t_{l-1}+1, \\ \quad t_{l-1}-1, 1, 0, t_{l-1}) & = \xi \otimes x^{t_{l-1}-1} & t_{l-1}-1 \geq 3 \\ (s_{l-1}, -1, -p+s_{l-1}, \\ \quad p-s_{l-1}, 1, 0, 2) & = x^{p-s_{l-1}} \otimes \xi & p-s_{l-1} \geq 3 \\ (s_{l-1}, -2, -n, n, 1, 1, s_{l_0-1}+n) & = e_{s_{l-1}} w x^n & n \geq 3 \\ (s_{l-1}, -2, -2p-(t_{l-1}-s_{l-1}), \\ \quad 2(p-1)+(t_{l-1}-s_{l-1}), 2, 0, t_{l-1}) & = x^{p-s_{l-1}} \otimes x^{p-1} \otimes x^{t_{l-1}-1} \\ (p-1, -2, -p-t_{l-1}+1, \\ \quad p+t_{l-1}-2, 2, 0, t_{l-1}) & = \xi \otimes x^{p-1} \otimes x^{t_{l-1}-1} \\ (s_{l-1}, -2, -2p+s_{l-1}, \\ \quad 2p-s_{l-1}-1, 2, 0, 2) & = x^{p-s_{l-1}} \otimes x^{p-1} \otimes \xi \\ (p, -2, -p+1, p-2, 2, 0, 1) & = e_p \otimes x^{p-2}\xi \otimes e_1 & p > 3 \end{cases}$$

and if it is possible to factor an x, then by the same reasoning, we are done. If not, then v_{l-1} is one of the following elements

$$\begin{cases} (p, -2, -p-1, p-1, 2, 0, 1) & = e_p \otimes x^{p-1} \otimes e_1 \\ (p-1, -2, -p, p-1, 2, 0, 1) & = \xi \otimes x^{p-1} \otimes e_1 \\ (p, -2, -p, p-1, 2, 0, 2) & = e_p \otimes x^{p-1} \otimes \xi \\ (p, -2, -p+1, p-2, 2, 0, 1) & = e_p \otimes x^{p-2}\xi \otimes e_1 \quad p > 3 \end{cases}$$

by Lemma 4.4.6, noticing that higher values of a give rise to smaller i-degrees. We note that it is possible to factor an element of $\mathbb{H}(\mathbf{u}^{-1})$ of j-degree -1 for all of these possibilities, both on the left and on the right, namely we can factor one of

$$\begin{array}{rcl} (p, -1, -1, 0, 1, 0, 1) & = & e_p \otimes e_1 \\ (p-1, -1, -1, 1, 1, 0, 2) & = & x \otimes \xi \\ & \cong & \xi \otimes x \end{array}$$

In particular v_{l-1} has i-degree -2, hence v_{l-2} has i-degree at most -2 with j-degree -2; it must be one of

$$\begin{array}{rcl} (s_1, -2a, -2, 2, a, a, s_1+2) & = & e_{s_1} w^a x^2 e_{s_1+1}, \quad 0 \leq a \leq 1 \\ (p, -1, -2, 1, 1, 0, 2) & = & e_p \otimes x e_2, \\ (p-1, -1, -2, 1, 1, 0, 1) & = & e_{p-1} x \otimes e_1, \\ (3, -2, -2, 1, 2, 0, 1) & = & e_3 \otimes x\xi \otimes e_1 \qquad (\text{ if } p = 3) \end{array}$$

Again, we can factor an x in all cases but $e_3 \otimes x\xi \otimes e_1$ if $p = 3$. That means we are done, unless $p = 3$ and $v_{l-2} = e_3 \otimes x\xi \otimes e_1$. Since that element has j-degree -2, the same reasoning as for v_{l-2} can be applied to v_{l-3}. Hence, in the worst case, and if $p = 3$, we have a sequence of $e_3 \otimes x\xi \otimes e_1$ for all v_f where $2 \leq f \leq l-1$, and we can factor $e_p \otimes e_1 = e_3 \otimes e_1$ both on the left and on the right of all of them. But then v_1 is an element of $\mathbf{d}$ (its i-degree is 0) with j-degree -2, thus $v_1 = x^2$ and we can factor an x, and v is reducible.

- Assume $a_{l'} = 0$. That means in particular that $v_{l'-1}$ has j-degree 0 with its a- and b-degree satisfying $a_{l'-1} - b_{l'-1} \geq 1$ by minimality of l' with respect to $a_{l'} = b_{l'}$. By Lemma 4.4.4, it is one of

$$\begin{cases} (p-1, -2a-1, 0, 0, a+1, a, 1) &= e_{p-1}\xi e_p \otimes (\xi \otimes \xi)^{\otimes a} \otimes e_1 \\ (p, -2a-1, 0, 0, a+1, a, 2) &= e_p \otimes (\xi \otimes \xi)^{\otimes a} \otimes e_1 \xi e_2 \end{cases}$$

We notice that we can factor an element of $\mathbb{H}(\mathbf{u}^{-1})$ of j-degree 0, namely $\xi \otimes e_1$ on the left for the first possibility or $e_p \otimes \xi$ on the right for the second possibility. We will treat the two cases in parallel, namely factoring on the left for the first possibility, or factoring on the right for the second possibility (compare to the previous case when we could factor both on the left and on the right). Similar to the previous case, we want to factor on the left, resp. right, an element of j-degree -1 from element $v_{l'-2}$. If we can factor x on the left, resp. right, then we are done. If we cannot, that means $v_{l'-2}$ is one of

$$\begin{aligned} &(p-u, -2b-1, -p-(t-p+u)+2u, (p-1)+(t-p+u)-u, b+1, b, t) \\ &\qquad = \xi^u e_p \otimes (\xi \otimes \xi)^{\otimes b} \otimes e_1 x^{t-1} \end{aligned}$$

resp.

$$\begin{aligned} &(s, -2b-1, -p-(1+u-s)+2u, (p-1)+(1+u-s)-u, b+1, b, 1+u) \\ &\qquad = x^{p-s} e_p \otimes (\xi \otimes \xi)^{\otimes b} \otimes e_1 \xi^u \end{aligned}$$

if $a_{l'-2} - b_{l'-2} = 1$, and

$$\begin{aligned} &(p, -a-b, -p(a-b)-(t-p)+2u, (p-1)(a-b)+(t-p)-u, a, b, t) \\ &\qquad = e_p \otimes (x^{p-1-u}\xi^u) \otimes (x^{p-1})^{\otimes a-b-2} \otimes (\xi \otimes \xi)^{\otimes b} \otimes e_1 x^{t-1} \end{aligned}$$

resp.

$$\begin{aligned} &(s, -a-b, -p(a-b)-(1-s)+2u, (p-1)(a-b)+(1-s)-u, a, b, 1) \\ &\qquad = x^{p-s} e_p \otimes (x^{p-1-u}\xi^u) \otimes (x^{p-1})^{\otimes a-b-2} \otimes (\xi \otimes \xi)^{\otimes b} \otimes e_1 \end{aligned}$$

if $a_{l'-2} - b_{l'-2} \geq 2$.

In all cases, we can factor the elements of $\mathbb{H}(\mathbf{u}^{-1})$ $\xi^u e_p \otimes e_1$, resp. $e_p \otimes e_1 \xi^u$ with $u \in \{0, 1\}$ from the left, resp. right of $v_{l'-2}$. Suppose we can decompose v with that factorisation at $v_{l'-2}$, then without loss of generality, we can assume $u = 0$; indeed, if $u = 1$ instead, that means we would factor $\xi \otimes e_1$, resp. $e_p \otimes \xi$, which has j-degree 0, and by Lemma 4.4.7 we then obtain a non-trivial decomposition. Hence, we can assume we can factor $e_p \otimes e_1$ on the left, resp. right, from $v_{l'-2}$, which has the right j-degree -1 to be compatible with the decomposition of $v_{l'-1}$ we discussed earlier.

Repeating the same argument as before, if we can factor an x from $v_{l'-3}$ on the left, resp. right, then we have our decomposition. If not, we must be able to factor $e_p \otimes e_1$ from it on the left, resp. right. In the worst case, we reach $l' - (l' - l) = l$ and v_l has i-degree less or equal to -3. Since l is chosen minimal with that property, we know that v_{l-1} has at least i-degree -2 with j-degree at most -3. By Lemma 4.4.10, v_{l-1}

is one of

$$\begin{cases}
(s_{l-1},0,-n,n,0,0,s_{l-1}+n) & = x^n & n \geq 3 \\
(s_{l-1},-1,-p-(t_{l-1}-s_{l-1}), \\ \quad p-1+(t_{l-1}-s_{l-1}),1,0,t_{l-1}) & = x^{p-s_{l-1}} \otimes x^{t_{l-1}-1} & t_{l-1}-s_{l-1} \geq 3-p \\
(p-1,-1,-t_{l-1}+1, \\ \quad t_{l-1}-1,1,0,t_{l-1}) & = \xi \otimes x^{t_{l-1}-1} & t_{l-1}-1 \geq 3 \\
(s_{l-1},-1,-p+s_{l-1}, \\ \quad p-s_{l-1},1,0,2) & = x^{p-s_{l-1}} \otimes \xi & p-s_{l-1} \geq 3 \\
(s_{l-1},-2,-n,n,1,1,s_{l_0-1}+n) & = e_{s_{l-1}} w x^n & n \geq 3 \\
(s_{l-1},-2,-2p-(t_{l-1}-s_{l-1}), \\ \quad 2(p-1)+(t_{l-1}-s_{l-1}),2,0,t_{l-1}) & = x^{p-s_{l-1}} \otimes x^{p-1} \otimes x^{t_{l-1}-1} & \\
(p-1,-2,-p-t_{l-1}+1, \\ \quad p+t_{l-1}-2,2,0,t_{l-1}) & = \xi \otimes x^{p-1} \otimes x^{t_{l-1}-1} & \\
(s_{l-1},-2,-2p+s_{l-1}, \\ \quad 2p-s_{l-1}-1,2,0,2) & = x^{p-s_{l-1}} \otimes x^{p-1} \otimes \xi & \\
(p,-2,-p+1,p-2,2,0,1) & = e_p \otimes x^{p-2}\xi \otimes e_1 & p > 3
\end{cases}$$

Again, if we can factor x from the left, resp. right, then we are done. If not, v_{l-1} must be one of

$$\begin{cases}
(p,-1,-p-(t_{l-1}-p), \\ \quad p-1+(t_{l-1}-p),1,0,t_{l-1}) & = e_p \otimes x^{t_{l-1}-1} & \text{if } t_{l-1} \geq 3 \\
(p-1,-1,-p-(t_{l-1}-p+1)+2, \\ \quad p-1+(t_{l-1}-p+1)-1,1,0,t_{l-1}) & = \xi \otimes x^{t_{l-1}-1} & \text{if } t_{l-1}-1 \geq 3 \\
(p,-2,-2p-(t_{l-1}-p), \\ \quad 2(p-1)+(t_{l-1}-p),2,0,t_{l-1}) & = e_p \otimes x^{p-1} \otimes x^{t_{l-1}-1} & \\
(p-1,-2,-p-t_{l-1}+1, \\ \quad p+t_{l-1}-2,2,0,t_{l-1}) & = \xi \otimes x^{p-1} \otimes x^{t_{l-1}-1} & \\
(p,-2,-p,p-1,2,0,2) & = e_p \otimes x^{p-1} \otimes \xi & \\
(p,-2,-p+1,p-2,2,0,1) & = e_p \otimes x^{p-2}\xi \otimes e_1 & p > 3
\end{cases}$$

resp. one of

$$\begin{cases} (s_{l-1}, -1, -p-(1-s_{l-1}), \\ \quad p-1+(1-s_{l-1}), 1, 0, 1) & = x^{p-s_{l-1}} \otimes e_1 & \text{if } p-s_{l-1} \geq 2 \\ (s_{l-1}, -1, -p-(2-s_{l-1})+2, \\ \quad p-1+(2-s_{l-1})-1, 1, 0, 2) & = x^{p-s_{l-1}} \otimes \xi & \text{if } p-s_{l-1} \geq 3 \\ (s_{l-1}, -2, -2p-(1-s_{l-1}), \\ \quad 2(p-1)+(1-s_{l-1}), 2, 0, 1) & = x^{p-s_{l-1}} \otimes x^{p-1} \otimes e_1 \\ (p-1, -2, -p, p-1, 2, 0, 1) & = \xi \otimes x^{p-1} \otimes e_1 \\ (s_{l-1}, -2, -2p+s_{l-1}, \\ \quad 2p-s_{l-1}-1, 2, 0, 2) & = x^{p-s_{l-1}} \otimes x^{p-1} \otimes \xi \\ (p, -2, -p+1, p-2, 2, 0, 1) & = e_p \otimes x^{p-2}\xi \otimes e_1 & p > 3 \end{cases}$$

It is possible to factor either an $e_p \otimes e_1$ on the left, resp. right, or $\xi \otimes x$ on the left, resp. $x \otimes \xi$ on the right. Now, if v_{l-2} has j-degree -1, we see that we are done as we have the following decomposition

$$\begin{aligned} v = \; & v_1 \otimes \ldots \otimes v_l \otimes \ldots \otimes v_{l'-1} \otimes e_{s_{l'}} \otimes e_{s_{l'+1}} \otimes \ldots \otimes e_q \\ = \; & v_1 \otimes \ldots \otimes v_{l-2} \otimes (e_p \otimes e_1) \otimes \ldots \otimes (e_p \otimes e_1) \otimes (\xi \otimes e_1) \otimes e_{s_{l'}} \otimes e_{s_{l'+1}} \otimes \ldots \otimes e_q \\ \cdot \; & e_{t_1} \otimes \ldots \otimes e_{t_{l-2}} \otimes e_1 x^{t_{l-1}-1} \otimes \tilde{v}_l \otimes \ldots \otimes \tilde{v}_{l'-2} \otimes e_1 w^{a_{l'-1}} e_1 \otimes e_{s_{l'}} \otimes \ldots \otimes e_{s_q} \end{aligned}$$

resp.

$$\begin{aligned} v = \; & v_1 \otimes \ldots \otimes v_l \otimes \ldots \otimes v_{l'-1} \otimes e_{s_{l'}} \otimes e_{s_{l'+1}} \otimes \ldots \otimes e_q \\ = \; & e_{s_1} \otimes \ldots \otimes e_{s_{l-2}} \otimes e_1 x^{p-s_{l-1}} \otimes \hat{v}_l \otimes \ldots \otimes \hat{v}_{l'-2} \otimes e_1 w^{a_{l'-1}} e_1 \otimes e_{s_{l'}} \otimes \ldots \otimes e_{s_q} \\ \cdot \; & v_1 \otimes \ldots \otimes v_{l-2} \otimes (e_p \otimes e_1) \otimes \ldots \otimes (e_p \otimes e_1) \otimes (e_p \otimes \xi) \otimes e_{s_{l'}} \otimes e_{s_{l'+1}} \otimes \ldots \otimes e_q \end{aligned}$$

or

$$\begin{aligned} v = \; & v_1 \otimes \ldots \otimes v_l \otimes \ldots \otimes v_{l'-1} \otimes e_{s_{l'}} \otimes e_{s_{l'+1}} \otimes \ldots \otimes e_q \\ = \; & v_1 \otimes \ldots \otimes v_{l-2} \otimes (\xi \otimes x) \otimes \ldots \otimes (e_p \otimes e_1) \otimes (\xi \otimes e_1) \otimes e_{s_{l'}} \otimes e_{s_{l'+1}} \otimes \ldots \otimes e_q \\ \cdot \; & e_{t_1} \otimes \ldots \otimes e_{t_{l-2}} \otimes e_2 x^{t_{l-1}-2} \otimes \tilde{v}_l \otimes \ldots \otimes \tilde{v}_{l'-2} \otimes e_1 w^{a_{l'-1}} e_1 \otimes e_{s_{l'}} \otimes \ldots \otimes e_{s_q} \end{aligned}$$

resp.

$$\begin{aligned} v = \; & v_1 \otimes \ldots \otimes v_l \otimes \ldots \otimes v_{l'-1} \otimes e_{s_{l'}} \otimes e_{s_{l'+1}} \otimes \ldots \otimes e_q \\ = \; & e_{s_1} \otimes \ldots \otimes e_{s_{l-2}} \otimes e_1 x^{p-s_{l-1}-1} \otimes \hat{v}_l \otimes \ldots \otimes \hat{v}_{l'-2} \otimes e_1 w^{a_{l'-1}} e_1 \otimes e_{s_{l'}} \otimes \ldots \otimes e_{s_q} \\ \cdot \; & v_1 \otimes \ldots \otimes v_{l-2} \otimes (x \otimes \xi) \otimes \ldots \otimes (e_p \otimes e_1) \otimes (e_p \otimes \xi) \otimes e_{s_{l'}} \otimes e_{s_{l'+1}} \otimes \ldots \otimes e_q \end{aligned}$$

and v is reducible. If v_{l-2} has j-degree -2, then the same analysis yields that v is reducible unless v_{l-3} has j-degree -2. In the worst case, v_1 has j-degree -2 and it is an element of $\mathbf{d}$. Hence $v_1 = x^2$ and it is possible to factor an x on the left, resp. right. Hence v is reducible.

□

4.4.5 Criterion for reducibility

Proposition 4.4.15. *Let $v = v_1 \otimes \ldots \otimes v_q$ be a monomial of $\mathbf{w}_q$ such that v_1 is not an idempotent of $\mathbf{d}$. If there exists $2 \leq l \leq q$ such that the i-degree i_l of v_l is less or equal to -3, then v is reducible.*

Proof. Assume there exists $2 \leq l \leq q$ such that $i_l \leq -3$, and let l_0 be the minimal l with that property. Then $i_l \geq -2$ for all $l < l_0$. In particular, by Lemma 4.4.10, v_{l_0-1} is one of

$$\begin{cases} (s_{l_0-1}, 0, -n, n, 0, 0, s_{l_0-1}+n) & = x^n & n \geq 3 \\ (s_{l_0-1}, -1, -p-(t_{l_0-1}-s_{l_0-1}), \\ \quad p-1+(t_{l_0-1}-s_{l_0-1}), 1, 0, t_{l_0-1}) & = x^{p-s_{l_0-1}} \otimes x^{t_{l_0-1}-1} & t_{l_0-1}-s_{l_0-1} \geq 3-p \\ (p-1, -1, -t_{l_0-1}+1, \\ \quad t_{l_0-1}-1, 1, 0, t_{l_0-1}) & = \xi \otimes x^{t_{l_0-1}-1} & t_{l_0-1}-1 \geq 3 \\ (s_{l_0-1}, -1, -p+s_{l_0-1}, \\ \quad p-s_{l_0-1}, 1, 0, 2) & = x^{p-s_{l_0-1}} \otimes \xi & p-s_{l_0-1} \geq 3 \\ (s_{l_0-1}, -2, -n, n, 1, 1, s_{l_0-1}+n) & = e_{s_{l_0-1}} w x^n & n \geq 3 \\ (s_{l_0-1}, -2, -2p-(t_{l_0-1}-s_{l_0-1}), \\ \quad 2(p-1)+(t_{l_0-1}-s_{l-1}), 2, 0, t_{l_0-1}) & = x^{p-s_{l_0-1}} \otimes x^{p-1} \otimes x^{t_{l_0-1}-1} \\ (p-1, -2, -p-t_{l_0-1}+1, \\ \quad p+t_{l_0-1}-2, 2, 0, t_{l_0-1}) & = \xi \otimes x^{p-1} \otimes x^{t_{l_0-1}-1} \\ (s_{l_0-1}, -2, -2p+s_{l_0-1}, \\ \quad 2p-s_{l_0-1}-1, 2, 0, 2) & = x^{p-s_{l_0-1}} \otimes x^{p-1} \otimes \xi \\ (p, -2, -p+1, p-2, 2, 0, 1) & = e_p \otimes x^{p-2}\xi \otimes e_1 & p > 3 \end{cases}$$

Note that the a- and b-degree in these elements satisfy $a-b \in \{0,1,2\}$. Let us rewrite that list with respect to the value of $a-b$:

1. If $a_{l_0-1} - b_{l_0-1} = 0$, v_{l_0-1} is one of

$$\begin{cases} (s_{l_0-1}, 0, -n, n, 0, 0, s_{l_0-1}+n) & = x^n & n \geq 3 \\ (s_{l_0-1}, -2, -n, n, 1, 1, s_{l_0-1}+n) & = e_{s_{l_0-1}} w x^n & n \geq 3 \end{cases} \tag{4.2}$$

2. If $a_{l_0-1} - b_{l_0-1} = 1$, v_{l_0-1} is one of

$$\begin{cases} (s_{l_0-1}, -1, -p-(t_{l_0-1}-s_{l_0-1}), \\ \quad p-1+(t_{l_0-1}-s_{l_0-1}), 1, 0, t_{l_0-1}) & = x^{p-s_{l_0-1}} \otimes x^{t_{l_0-1}-1} & t_{l_0-1}-s_{l_0-1} \geq 3-p \\ (p-1, -1, -t_{l_0-1}+1, \\ \quad t_{l_0-1}-1, 1, 0, t_{l_0-1}) & = \xi \otimes x^{t_{l_0-1}-1} & t_{l_0-1}-1 \geq 3 \\ (s_{l_0-1}, -1, -p+s_{l_0-1}, \\ \quad p-s_{l_0-1}, 1, 0, 2) & = x^{p-s_{l_0-1}} \otimes \xi & p-s_{l_0-1} \geq 3 \end{cases} \tag{4.3}$$

3. If $a_{l_0-1} - b_{l_0-1} = 2$, v_{l_0-1} is one of

$$\begin{cases} (s_{l_0-1}, -2, -2p-(t_{l_0-1}-s_{l_0-1}), \\ \quad 2(p-1)+(t_{l_0-1}-s_{l-1}), 2, 0, t_{l_0-1}) & = x^{p-s_{l_0-1}} \otimes x^{p-1} \otimes x^{t_{l_0-1}-1} \\ (p-1, -2, -p-t_{l_0-1}+1, \\ \quad p+t_{l_0-1}-2, 2, 0, t_{l_0-1}) & = \xi \otimes x^{p-1} \otimes x^{t_{l_0-1}-1} \\ (s_{l_0-1}, -2, -2p+s_{l_0-1}, \\ \quad 2p-s_{l_0-1}-1, 2, 0, 2) & = x^{p-s_{l_0-1}} \otimes x^{p-1} \otimes \xi \\ (p, -2, -p+1, p-2, 2, 0, 1) & = e_p \otimes x^{p-2}\xi \otimes e_1 & p > 3 \end{cases} \tag{4.4}$$

We need to study four cases, namely

- $l_0 = 2$;
- $l_0 > 2$ and
  1. $a_{l_0-1} - b_{l_0-1} = 0$;
  2. $a_{l_0-1} - b_{l_0-1} = 1$;
  3. $a_{l_0-1} - b_{l_0-1} = 2$.

The strategy is similar in all cases:

1. if $b_{l_0} \geq 1$, we factor a w from v_{l_0} and v is reducible;
2. otherwise, $b_{l_0} = 0$. This divides into two subcases:
   (i) either the difference between the a- and b-degree of all the subsequent elements is greater or equal to 2, and Proposition 4.4.9 provides a non-trivial decomposition and v is reducible;
   (ii) or there exists an index l such that the difference between the a- and b-degree of v_l is at most 1 and we take l to be minimal with that property. In that case, we have three additional subcases:
   (a) if $a_l - b_l = 0$, then by Proposition 4.4.14 v is reducible;
   (b) if $b_l \geq 1$, we factor a w from v_l, and v is reducible;
   (c) if $b_l = 0$ and $a_l - b_l \neq 0$, then $a_l = 1$ or $a_l = -1$ as $-1 \leq a_l - b_l$ by Remark 2.3.10, and $a_l - b_l \leq 1$ by definition of l. We see that these two situations lead to a contradiction.

All the decompositions we try to construct in the following paragraphs are of the form

$$\begin{array}{cl} & v_1 \otimes \ldots \otimes v_q \\ = & e_{s_1} \otimes \ldots \otimes e_{s_{l-2}} \otimes x \otimes \tilde{v}_l \otimes \ldots \otimes \tilde{v}_q \\ \cdot & v_1 \otimes \ldots \otimes v_{l-2} \otimes \hat{v}_{l-1} \otimes \hat{v}_l \otimes \ldots \otimes \hat{v}_q \end{array}$$

if we can factor x on the right of v_{l-1}, or

$$\begin{array}{cl} & v_1 \otimes \ldots \otimes v_q \\ = & v_1 \otimes \ldots \otimes v_{l-2} \otimes \tilde{v}_{l-1} \otimes \tilde{v}_l \otimes \ldots \otimes \tilde{v}_q \\ \cdot & e_{t_1} \otimes \ldots \otimes e_{t_{l-2}} \otimes x \otimes \hat{v}_l \otimes \ldots \otimes \hat{v}_q \end{array}$$

if we can factor x on the left of v_{l-1}, where $\tilde{v}_c \cdot \hat{v}_c = v_c$. In particular, when we say that we factor w, say from v_{m+1}, the decompositions will look like

$$\begin{array}{cl} & v_1 \otimes \ldots \otimes v_q \\ = & e_{s_1} \otimes \ldots \otimes e_{s_{l-2}} \otimes x \otimes \tilde{v}_l \otimes \ldots \otimes \ldots \otimes \tilde{v}_m \otimes w \otimes e_{s_{m+2}} \otimes \ldots \otimes e_{s_q} \\ \cdot & v_1 \otimes \ldots \otimes v_{l-2} \otimes \hat{v}_{l-1} \otimes \hat{v}_l \otimes \ldots \otimes \hat{v}_m \otimes \hat{v}_{m+1} \otimes v_{m+2} \otimes \ldots \otimes v_q \end{array}$$

if we can factor x on the right of v_{l-1}, or

$$\begin{array}{cl} & v_1 \otimes \ldots \otimes v_q \\ = & v_1 \otimes \ldots \otimes v_{l-2} \otimes \tilde{v}_{l-1} \otimes \tilde{v}_l \otimes \ldots \otimes \tilde{v}_m \otimes \tilde{v}_{m+1} \otimes v_{m+2} \otimes \ldots \otimes v_q \\ \cdot & e_{t_1} \otimes \ldots \otimes e_{t_{l-2}} \otimes x \otimes \hat{v}_l \otimes \ldots \otimes \ldots \otimes \hat{v}_m \otimes w \otimes e_{t_{m+2}} \otimes \ldots \otimes e_{t_q} \end{array}$$

if we can factor x on the left of v_{l-1}.

The aim is to determine those elements $\hat{v}_c$ that we can factor from v_c to make the decomposition work.

- Assume $l_0 = 2$. We have in particular that $a_{l_0-1} - b_{l_0-1} = 0$, as $a_{l_0-1} = a_1 = 0$ and $b_{l_0-1} = b_1 = 0$. This means that $v_{l_0-1} = v_1 = e_{s_1}x^n e_{s_1+n}$ for some $n \geq 3$ by Expression 4.2.
  - If $b_{l_0} = b_2 \geq 1$, since $i_2 \leq -3$, we can apply Lemma 4.4.11: if $s_1 \leq p-2$ or $t_1 \geq 3$, v is reducible. Assume now that $s_1 > p-2$ and $t_1 < 3$. We have $-3 \geq i_2 = j_1 = -t_1 + s_1 + 2u_1$ if and only if
$$-3 - 2u_1 \geq -t_1 + s_1 > -3 + p - 2 = p - 5,$$
    and $p - 5 \geq -3$ for all $p \geq 2$. Hence we have a contradiction and that situation cannot arise. (Note that this means that we have at least three powers of x in first position, which is why it is possible to factor x^2).
    Hence v is reducible.
  - If $b_{l_0} = b_2 = 0$, then $a_2 \geq 3$ since $-a_2 - b_2 = i_2 \leq -3$.
    * Assume $a_n - b_n \geq 2$ for all $2 \leq n \leq q$. Then by Proposition 4.4.9, there exists a non-trivial decomposition and v is reducible.
    * Otherwise, let l be the minimal index, $2 < l \leq q$, such that $a_l - b_l \leq 1$. In particular, $a_n - b_n \geq 2$ for all $2 \leq n < l$. We are in the following situation
$$v_1 \otimes \overbrace{v_2 \otimes \ldots \otimes v_{l-1}}^{a_n - b_n \geq 2} \otimes v_l \otimes \ldots \otimes v_q$$
      We need to examine three cases.
      · Assume $a_l - b_l = 0$. Then by Proposition 4.4.14, v is reducible.
      · Assume $b_l = 0$ and $a_l - b_l \neq 0$. Since $a_l - b_l \leq 1$, we see that $a_l = 1$ or $a_l = -1$ ($a_l < -1$ does not correspond to any element in $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})^{\leq 1}$).
      Assume $a_l = -1$, i.e. v_l is of type **3** and has i-degree 1. Then v_{l-1} has j-degree 1 and by Lemma 4.3.1 must be of the form
$$\begin{array}{rcl} (s_1, -2a, -1, 1, b, b, s_1+1) & = & e_{s_1} w^b \xi e_{s_1+1} \\ (p-1, -2b-1, 1, 0, b+1, b, 2) & = & e_{p-1}\xi \otimes (\xi \otimes \xi)^{\otimes b} \otimes \xi e_2 \\ (s_2, -2a-1, 1, 0, a, a+1, p+1-s_2) & = & e_{s_2}(\xi \otimes \xi)^{\otimes a+1} e_{p+1-s_2} \\ (s_3, 1, 1, 0, -1, 0, p+1-s_3) & = & e_{s_3} \otimes e^*_{p+1-s_3} \end{array}$$
      with $1 \leq s_1 \leq p-1$, $1 \leq s_2 \leq p-2$, $1 \leq s_3 \leq p-1$, $a, b \geq 0$. By minimality of l, the a- and b-degree of v_{l-1} must satisfy $a_{l-1} - b_{l-1} \geq 2$. Hence a_l cannot be equal to -1.
      Assume $a_l = 1$. That means v_{l-1} has j-degree -1 and by Lemma 4.4.5 is of the form
$$\begin{array}{rcl} (s, -2a, -1, 1, a, a, s+1) & = & e_s w^a x e_{s+1} \\ (p, -2a-1, -1, 0, a+1, a, 1) & = & e_p \otimes (\xi \otimes \xi)^{\otimes a} \otimes e_1 \\ (p-1, -2a-1, -1, 1, a+1, a, 2) & = & e_{p-1}\xi \otimes (\xi \otimes \xi)^{\otimes a} \otimes x e_2 \\ & (\cong & e_{p-1} x \otimes (\xi \otimes \xi)^{\otimes a} \otimes \xi e_2) \end{array}$$
      with $1 \leq s \leq p-1$, $a \geq 0$. However, again by minimality of l, the a- and b-degree of v_{l-1} must satisfy $a_{l-1} - b_{l-1} \geq 2$. Hence a_l cannot be equal to 1 either.
      · Assume $b_l \geq 1$, and $a_l - b_l \neq 0$; in particular v_l is not of type **3**, and thus v_{l-1} must be of type **1**. Since $a_{l-1} - b_{l-1} \geq 2$ by minimality of l, we can apply Lemma 4.4.11: if $s_{l-1} \leq p-2$ or $t_{l-1} \geq 3$, v is reducible. Assume now that $s_{l-1} > p-2$ and $t_{l-1} < 3$.

That means that v_{l-1} is of the form

$$\begin{aligned}&(p, -a-b, -p(a-b)-(1-p)+2u,\\ &\quad (p-1)(a-b)+(1-p)-u, a, b, 1)\\ &= e_p \otimes (x^{p-2}\xi)^u \otimes (x^{p-1})^{\otimes a-b-1-u} \otimes (\xi \otimes \xi)^{\otimes b} \otimes e_1\end{aligned}$$

$$\begin{aligned}&(p, -a-b, -p(a-b)-(2-p)+2u,\\ &\quad (p-1)(a-b)+(2-p)-u, a, b, 2)\\ &= e_p \otimes (x^{p-2}\xi)^u \otimes (x^{p-1})^{\otimes a-b-1-u} \otimes (\xi \otimes \xi)^{\otimes b} \otimes xe_2\end{aligned}$$

$$\begin{aligned}&(p-1, -a-b, -p(a-b)-(1-(p-1))+2u,\\ &\quad (p-1)(a-b)+(1-(p-1))-u, a, b, 1)\\ &= e_{p-1}x \otimes (x^{p-2}\xi)^u \otimes (x^{p-1})^{\otimes a-b-1-u} \otimes (\xi \otimes \xi)^{\otimes b} \otimes e_1\end{aligned}$$

$$\begin{aligned}&(p-1, -a-b, -p(a-b)-(2-(p-1))+2u,\\ &\quad (p-1)(a-b)+(2-(p-1))-u, a, b, 2)\\ &= e_{p-1}x \otimes (x^{p-2}\xi)^u \otimes (x^{p-1})^{\otimes a-b-1-u} \otimes (\xi \otimes \xi)^{\otimes b} \otimes xe_2\end{aligned}$$

with $a_{l-1} - b_{l-1} \geq 2$ by minimality of l. Recall that

$$e_p \otimes (x^{p-2}\xi)^u \otimes (x^{p-1})^{\otimes a-b-1-u} \otimes (\xi \otimes \xi)^{\otimes b} \otimes e_1$$

is the same element as

$$e_p \otimes (\xi \otimes \xi)^{\otimes b} \otimes (x^{p-1})^{\otimes a-b-1-u} \otimes (x^{p-2}\xi)^u \otimes e_1$$

in homology. Hence in all cases, we can factor j-degree -2 elements $x \otimes e_1$ or $e_p \otimes x$ from v_{l-1} both from the left side and the right side. Let $\lambda \leq l-2$ be the largest index for which x can be factored from v_λ, either from the left or from the right. Note that such an index λ exists: $v_1 = x^n$ with $n \geq 3$ by assumption, so $\lambda \geq 1$. In particular, for all $\lambda < \lambda' \leq l-2$, it is impossible to factor x both from the left and from the right of $v_{\lambda'}$. In addition, recall that for $l_0 = 2 \leq \lambda' \leq l$, the a- and b-degree of $v_{\lambda'}$ satisfy $a_{\lambda'} - b_{\lambda'} \geq 2$ by minimality of l. Hence for all $\lambda < \lambda' \leq l-2$, $v_{\lambda'}$ must be of the form

$$\begin{aligned}&(p, -a-b, -p(a-b)-(1-p)+2u, (p-1)(a-b)+(1-p)-u, a, b, 1)\\ &\quad = e_p \otimes (x^{p-2}\xi)^u \otimes (x^{p-1})^{\otimes a-b-1-u} \otimes (\xi \otimes \xi)^{\otimes b} \otimes e_1\\ &\quad \cong e_p \otimes (\xi \otimes \xi)^{\otimes b} \otimes (x^{p-1})^{\otimes a-b-1-u} \otimes (x^{p-2}\xi)^u \otimes e_1\end{aligned}$$

and we can factor $e_p \otimes e_1$ on both sides. Therefore, we can write the following decomposition

$$\begin{aligned} & v_1 \otimes \ldots \otimes v_\lambda \otimes \ldots \otimes v_l \otimes \ldots \otimes v_q\\ = \; & e_{s_1} \otimes \ldots \otimes e_{s_{\lambda-1}} \otimes x \otimes (e_p \otimes e_1) \otimes \ldots \otimes (e_p \otimes e_1) \otimes L \otimes e_{s_l} w e_{s_l}\\ & \quad \otimes e_{s_{l+1}} \otimes \ldots \otimes e_{s_q}\\ \cdot \; & v_1 \otimes \ldots \otimes v_{\lambda-1} \otimes \tilde{v}_\lambda \otimes \tilde{v}_{\lambda+1} \otimes \ldots \otimes \tilde{v}_{l-1} \otimes \tilde{v}_l \otimes v_{l+1} \otimes \ldots \otimes v_q\end{aligned}$$

if we can factor x from the left of v_λ, or

$$\begin{aligned} & v_1 \otimes \ldots \otimes v_\lambda \otimes \ldots \otimes v_l \otimes \ldots \otimes v_q\\ = \; & v_1 \otimes \ldots \otimes v_{\lambda-1} \otimes \tilde{v}_\lambda \otimes \tilde{v}_{\lambda+1} \otimes \ldots \otimes \tilde{v}_{l-1} \otimes \tilde{v}_l \otimes v_{l+1} \otimes \ldots \otimes v_q\\ \cdot \; & e_{t_1} \otimes \ldots \otimes e_{t_{\lambda-1}} \otimes x \otimes (e_p \otimes e_1) \otimes \ldots \otimes (e_p \otimes e_1) \otimes R \otimes e_{t_l} w e_{t_l}\\ & \quad \otimes e_{t_{l+1}} \otimes \ldots \otimes e_{t_q}\end{aligned}$$

if we can factor x from the right of v_λ, where $\tilde{v}_c$ is the remaining part of v_c after factorisation, and $L, R \in \{x \otimes e_1, e_p \otimes x\}$ depending on the element v_{l-1}. Hence v is reducible.

- If $l_0 > 2$, then we are in the following situation

$$v_1 \otimes \overbrace{v_2 \otimes \ldots \otimes v_{l_0-1}}^{i\geq -2} \otimes v_{l_0} \otimes \ldots \otimes v_q.$$

1. If $a_{l_0-1} - b_{l_0-1} = 0$, then by Lemma 4.4.12, v is reducible;
2. If $a_{l_0-1} - b_{l_0-1} = 1$, then in particular, $a_{l_0-1} = 1$ and $b_{l_0-1} = 0$ since $-a_{l_0-1} - b_{l_0-1} = i_{l_0-1} \geq -2$. By Expression 4.3, we see that v_{l_0-1} is one of

$$\begin{cases} \begin{array}{l} (s_{l_0-1}, -1, -p-(t_{l_0-1}-s_{l_0-1}), \\ \quad p-1+(t_{l_0-1}-s_{l_0-1}), 1, 0, t_{l_0-1}) \end{array} & = x^{p-s_{l_0-1}} \otimes x^{t_{l_0-1}-1} \quad t_{l_0-1}-s_{l_0-1} \geq 3-p \\ \begin{array}{l} (p-1, -1, -t_{l_0-1}+1, \\ \quad t_{l_0-1}-1, 1, 0, t_{l_0-1}) \end{array} & = \xi \otimes x^{t_{l_0-1}-1} \quad t_{l_0-1}-1 \geq 3 \\ \begin{array}{l} (s_{l_0-1}, -1, -p+s_{l_0-1}, \\ \quad p-s_{l_0-1}, 1, 0, 2) \end{array} & = x^{p-s_{l_0-1}} \otimes \xi \quad p-s_{l_0-1} \geq 3 \end{cases}$$

   - If $b_{l_0} \geq 1$, since $i_{l_0} \leq -3$, we can apply Lemma 4.4.11: if $s_{l_0-1} \leq p-2$ or $t_{l_0-1} \geq 3$, v is reducible. Assume now that $s_{l_0-1} > p-2$ and $t_{l_0-1} < 3$. We have $-3 \geq i_{l_0} = j_{l_0-1} = -p - t_{l_0-1} + s_{l_0-1} + 2u_{l_0-1}$, i.e.

$$p - 3 - 2u_{l_0-1} \geq -t_{l_0-1} + s_{l_0-1} > -3 + p - 2 = p - 5$$

     with $u_{l_0-1} \in \{0,1\}$. We get $u_{l_0-1} = 0$, $s_{l_0-1} = p-1$ and $t_{l_0-1} = 2$. In particular, $i_{l_0} = -3$.
     Hence we are in the situation where $v_{l_0} \in \mathbb{H}(\mathbf{u}^{-3})$ and v writes

$$v_1 \otimes \ldots \otimes v_{l_0-2} \otimes (x \otimes x) \otimes v_{l_0} \otimes \ldots \otimes v_q,$$

     and we clearly cannot factor any x^2 from v_{l_0-1}. However, we can always write

$$\begin{aligned} & v_1 \otimes \ldots \otimes v_q \\ = \; & v_1 \otimes \ldots \otimes v_{l_0-2} \otimes \overbrace{(p-1,-1,-2,1,1,0,1)}^{e_{p-1}x \otimes e_1} \otimes \overbrace{(s_{l_0},-2,0,0,1,1,s_{l_0})}^{e_{s_{l_0}} w e_{s_{l_0}}} \\ & \quad \otimes e_{s_{l_0+1}} \otimes \ldots \otimes e_{s_q} \\ \cdot \; & e_{t_1} \otimes \ldots \otimes e_{t_{l_0-2}} \\ & \otimes \overbrace{(1,0,-1,1,0,0,2)}^{e_1 x e_2} \otimes (s_{l_0}, i_{l_0}+2, j_{l_0}, k_{l_0}, a_{l_0}-1, b_{l_0}-1, t_{l_0}) \\ & \quad \otimes v_{l_0+1} \otimes \ldots \otimes v_q. \end{aligned}$$

     So v is reducible in that case too.
   - The remaining case to cover is if $b_{l_0} = 0$. Since $i_{l_0} = -a_{l_0} - b_{l_0}$, we have $a_{l_0} \geq 3$, and in particular, $a_{l_0} - b_{l_0} \geq 3 \geq 2$.
     * Assume that $a_n - b_n \geq 2$ for all $l_0 \leq n \leq q$. Then, by Proposition 4.4.9, v is reducible.
     * Otherwise, we see that there exists an index $l_0 < l \leq q$ such that $a_l - b_l \leq 1$, and we assume it is minimal with that property.

- If $a_l - b_l = 0$, then by Proposition 4.4.14 v is reducible.
- If $b_l \geq 1$ and $a_l - b_l \neq 0$, then in particular v_l is not of type **3**, and thus v_{l-1} must be of type **1**. Since $a_{l-1} - b_{l-1} \geq 2$ by minimality of l, we can apply Lemma 4.4.11: if $s_{l-1} \leq p-2$ or $t_{l-1} \geq 3$, v is reducible. Assume now that $s_{l-1} > p-2$ and $t_{l-1} < 3$.
  That means that v_{l-1} is of the form

$$\begin{aligned}&(p, -a-b, -p(a-b) - (1-p) + 2u,\\ &\quad (p-1)(a-b) + (1-p) - u, a, b, 1)\\ &= e_p \otimes (x^{p-2}\xi)^u \otimes (x^{p-1})^{\otimes a-b-1-u} \otimes (\xi \otimes \xi)^{\otimes b} \otimes e_1\end{aligned}$$

$$\begin{aligned}&(p, -a-b, -p(a-b) - (2-p) + 2u,\\ &\quad (p-1)(a-b) + (2-p) - u, a, b, 2)\\ &= e_p \otimes (x^{p-2}\xi)^u \otimes (x^{p-1})^{\otimes a-b-1-u} \otimes (\xi \otimes \xi)^{\otimes b} \otimes xe_2\end{aligned}$$

$$\begin{aligned}&(p-1, -a-b, -p(a-b) - (1-(p-1)) + 2u,\\ &\quad (p-1)(a-b) + (1-(p-1)) - u, a, b, 1)\\ &= e_{p-1}x \otimes (x^{p-2}\xi)^u \otimes (x^{p-1})^{\otimes a-b-1-u} \otimes (\xi \otimes \xi)^{\otimes b} \otimes e_1\end{aligned}$$

$$\begin{aligned}&(p-1, -a-b, -p(a-b) - (2-(p-1)) + 2u,\\ &\quad (p-1)(a-b) + (2-(p-1)) - u, a, b, 2)\\ &= e_{p-1}x \otimes (x^{p-2}\xi)^u \otimes (x^{p-1})^{\otimes a-b-1-u} \otimes (\xi \otimes \xi)^{\otimes b} \otimes xe_2\end{aligned}$$

  with $a_{l-1} - b_{l-1} \geq 2$ by minimality of l. Recall that

$$e_p \otimes (x^{p-2}\xi)^u \otimes (x^{p-1})^{\otimes a-b-1-u} \otimes (\xi \otimes \xi)^{\otimes b} \otimes e_1$$

  is the same element as

$$e_p \otimes (\xi \otimes \xi)^{\otimes b} \otimes (x^{p-1})^{\otimes a-b-1-u} \otimes (x^{p-2}\xi)^u \otimes e_1$$

  in homology. Hence in all cases, we can factor j-degree -2 elements $x \otimes e_1$ or $e_p \otimes x$ from v_{l-1} both from the left side and the right side. Let $1 \leq \lambda \leq l-2$ be the largest index for which x can be factored from v_λ, from the left or from the right. In particular, for all $\lambda < \lambda' \leq l-2$, it is both impossible to factor x from the left and from the right of $v_{\lambda'}$. Hence if $\max \lambda, l_0 - 1 < \lambda' \leq l-2$, $v_{\lambda'}$ must be of the form

$$\begin{aligned}&(p, -a-b, -p(a-b) - (1-p) + 2u, (p-1)(a-b) + (1-p) - u, a, b, 1)\\ &\quad = e_p \otimes (x^{p-2}\xi)^u \otimes (x^{p-1})^{\otimes a-b-1-u} \otimes (\xi \otimes \xi)^{\otimes b} \otimes e_1\\ &\quad \cong e_p \otimes (\xi \otimes \xi)^{\otimes b} \otimes (x^{p-1})^{\otimes a-b-1-u} \otimes (x^{p-2}\xi)^u \otimes e_1\end{aligned}$$

  as $a_{\lambda'} - b_{\lambda'} \geq 2$ for all $l_0 \leq \lambda' \leq l-2$, and we can factor $e_p \otimes e_1$ on both sides of $v_{\lambda'}$. Thus, if $\lambda \geq l_0 - 1$, we have a decomposition as announced in the introduction, namely we have

$$\begin{aligned}&v_1 \otimes \ldots \otimes v_{l_0} \otimes \ldots \otimes v_\lambda \otimes \ldots \otimes v_q\\ = \;&e_{s_1} \otimes \ldots \otimes e_{s_{\lambda-1}} \otimes x \otimes (e_p \otimes e_1) \ldots \otimes (e_p \otimes e_1) \otimes L \otimes w\\ &\quad \otimes e_{s_{l+1}} \otimes \ldots \otimes e_{s_q}\\ \cdot \;&v_1 \otimes \ldots \otimes v_{l_0} \otimes \ldots \otimes v_{\lambda-1} \otimes \tilde{v}_\lambda \otimes \ldots \otimes \tilde{v}_l \otimes v_{l+1} \otimes \ldots \otimes v_q\end{aligned}$$

  or

$$\begin{aligned}&v_1 \otimes \ldots \otimes v_{l_0} \otimes \ldots \otimes v_\lambda \otimes \ldots \otimes v_q\\ = \;&v_1 \otimes \ldots \otimes v_{l_0} \otimes \ldots \otimes v_{\lambda-1} \otimes \tilde{v}_\lambda \otimes \ldots \otimes \tilde{v}_l \otimes v_{l+1} \otimes \ldots \otimes v_q\\ \cdot \;&e_{t_1} \otimes \ldots \otimes e_{t_{\lambda-1}} \otimes x \otimes (e_p \otimes e_1) \ldots \otimes (e_p \otimes e_1) \otimes R \otimes w\\ &\quad \otimes e_{t_{l+1}} \otimes \ldots \otimes e_{t_q}\end{aligned}$$

where $R, L \in \{x \otimes e_1, e_p \otimes x\}$, and v is reducible.
Going back to the introduction of Case $a_{l_0-1} - b_{l_0-1} = 1$, we see that by Expression 4.3, v_{l_0-1} is one of

$$\begin{cases} (s,-1,-p-(t-s), \\ \quad p-1+(t-s),1,0,t) & = x^{p-s} \otimes x^{t-1} \quad t-s \geq 3-p \\ (p-1,-1,-t+1, \\ \quad t-1,1,0,t_{l_0-1}) & = \xi \otimes x^{t-1} \quad t-1 \geq 3 \\ (s,-1,-p+s, \\ \quad p-s,1,0,2) & = x^{p-s} \otimes \xi \quad p-s \geq 3 \end{cases}$$

and all the elements above have at least one factor x on the left or on the right; in particular, $\lambda \geq l_0 - 1$, so v is reducible.

- The only outstanding cases are if $a_l - b_l \leq 1$, $a_l - b_l \neq 0$, and $b_l = 0$, namely if $a_l = 1$ or if $a_l = -1$.
  Assume $a_l = -1$, i.e. v_l is of type **3**. Then by Lemma 4.3.1, the element v_{l-1} must then be of the form

$$\begin{aligned} (s_1,-2a,-1,1,b,b,s_1+1) &= e_{s_1} w^b \xi e_{s_1+1} \\ (p-1,-2b-1,1,0,b+1,b,2) &= e_{p-1}\xi \otimes (\xi\otimes\xi)^{\otimes b} \otimes \xi e_2 \\ (s_2,-2a-1,1,0,a,a+1,p+1-s_2) &= e_{s_2}(\xi\otimes\xi)^{\otimes a+1} e_{p+1-s_2} \\ (s_3,1,1,0,-1,0,p+1-s_3) &= e_{s_3} \otimes e^*_{p+1-s_3} \end{aligned}$$

  with $1 \leq s_1 \leq p-1$, $1 \leq s_2 \leq p-2$, $1 \leq s_3 \leq p-1$, $a, b \geq 0$. By minimality of l, the a- and b-degree of v_{l-1} must satisfy $a_{l-1} - b_{l-1} \geq 2$. Hence a_l cannot be equal to -1.
  Assume $a_l = 1$. That means v_{l-1} has j-degree -1 and by Lemma 4.4.5 is of the form

$$\begin{aligned} (s_1,-2a,-1,1,a,a,s_1+1) &= e_{s_1} w^a x e_{s_1+1} \\ (s_2,-2a,-1,2,a,a,s_2+3) &= e_{s_2} w^a x^2 \xi e_{s_2+3} \\ (p,-2a-1,-1,0,a+1,a,1) &= e_p \otimes (\xi\otimes\xi)^{\otimes a} \otimes e_1, \\ (p,-2a-1,-1,1,a+1,a,3) &= e_p \otimes (\xi\otimes\xi)^{\otimes a} \otimes x\xi e_3, \\ (p-1,-2a-1,-1,1,a+1,a,2) &= e_{p-1}\xi \otimes (\xi\otimes\xi)^{\otimes a} \otimes x e_2 \\ &(\cong e_{p-1}x \otimes (\xi\otimes\xi)^{\otimes a} \otimes \xi e_2), \\ (p-2,-2a-1,-1,1,a+1,a,1) &= e_{p-2}x\xi \otimes (\xi\otimes\xi)^{\otimes a} \otimes e_1. \end{aligned}$$

  with $1 \leq s_1 \leq p-1$, $1 \leq s_2 \leq p-3$, $a \geq 0$. However, again by minimality of l, the a- and b-degree of v_{l-1} must satisfy $a_{l-1} - b_{l-1} \geq 2$. Hence a_l cannot be equal to 1 either.

3. If $a_{l_0-1} - b_{l_0-1} = 2$, then in particular, $a_{l_0-1} = 2$ and $b_{l_0-1} = 0$ since $-a_{l_0-1} - b_{l_0-1} = i_{l_0-1} \geq -2$. This means v_{l_0-1} is one of

$$\begin{cases} (s_{l_0-1},-2,-2p-(t_{l_0-1}-s_{l_0-1}), \\ \quad 2(p-1)+(t_{l_0-1}-s_{l-1}),2,0,t_{l_0-1}) & = x^{p-s_{l_0-1}} \otimes x^{p-1} \otimes x^{t_{l_0-1}-1} \\ (p-1,-2,-p-t_{l_0-1}+1, \\ \quad p+t_{l_0-1}-2,2,0,t_{l_0-1}) & = \xi \otimes x^{p-1} \otimes x^{t_{l_0-1}-1} \\ (s_{l_0-1},-2,-2p+s_{l_0-1}, \\ \quad 2p-s_{l_0-1}-1,2,0,2) & = x^{p-s_{l_0-1}} \otimes x^{p-1} \otimes \xi \\ (p,-2,-p+1,p-2,2,0,1) & = e_p \otimes x^{p-2}\xi \otimes e_1 \quad p > 3 \end{cases}$$

– If $b_{l_0} \geq 1$, since $i_{l_0} \leq -3$, we can apply Lemma 4.4.11: if $s_{l_0-1} \leq p-2$ or $t_{l_0-1} \geq 3$, v is reducible. Assume now that $s_{l_0-1} > p-2$ and $t_{l_0-1} < 3$. Thus v_{l_0-1} is one of

$$\begin{cases} (p,-2,-2p-(1-p), \\ \quad 2(p-1)+(1-p),2,0,1) & = e_p \otimes x^{p-1} \otimes e_1 \\ (p-1,-2,-2p-(1-p+1), \\ \quad 2(p-1)+(1-p+1),2,0,1) & = x \otimes x^{p-1} \otimes e_1 \\ (p,-2,-2p-(2-p), \\ \quad 2(p-1)+(2-p),2,0,2) & = e_p \otimes x^{p-1} \otimes x \\ (p-1,-2,-2p-(2-p+1), \\ \quad 2(p-1)+(2-p+1),2,0,2) & = x \otimes x^{p-1} \otimes x \\ (p-1,-2,-p-1+1, \\ \quad p+1-2,2,0,1) & = \xi \otimes x^{p-1} \otimes e_1 \cong x \otimes x^{p-2}\xi \otimes e_1 \\ (p-1,-2,-p-2+1, \\ \quad p+2-2,2,0,2) & = \xi \otimes x^{p-1} \otimes x \cong x \otimes x^{p-2}\xi \otimes x \\ (p,-2,-2p+p, \\ \quad 2p-p-1,2,0,2) & = e_p \otimes x^{p-1} \otimes \xi \cong e_p \otimes x^{p-2}\xi \otimes x \\ (p-1,-2,-2p+p-1, \\ \quad 2p-p-1-1,2,0,2) & = x \otimes x^{p-1} \otimes \xi \cong x \otimes x^{p-2}\xi \otimes x \\ (p,-2,-p+1,p-2,2,0,1) & = e_p \otimes x^{p-2}\xi \otimes e_1 \qquad p > 3 \end{cases}$$

In all cases, we can factor an element of $\mathbb{H}(\mathbf{u}^{-1})$ of j-degree -2, namely one of

$$\begin{cases} (p,-1,-2,1,1,0,2) & = \quad e_p \otimes x \\ (p-1,-1,-2,1,1,0,1) & = \quad x \otimes e_1 \end{cases}$$

both from the left and from the right. Let $1 \leq \lambda \leq l_0 - 2$ be the largest index such that x can be factored from v_λ from the left or from the right. In particular, if $\lambda < l_0 - 2$, for all $\lambda < \lambda' \leq l_0 - 2$, it is impossible to factor x both from the left and from the right of $v_{\lambda'}$. Since v_{l_0-1} has i-degree -2 by assumption, v_{l_0-2} has j-degree -2 and by Lemma 4.4.6, it implies that

* either $\lambda = l_0 - 2$, i.e. we can factor x from the left or from the right of v_{l_0-2}, and we can write the following non-trivial decomposition

$$\begin{array}{rl} & v_1 \otimes \ldots \otimes v_{\lambda-1} \otimes v_\lambda \otimes \ldots \otimes v_{l_0-1} \otimes v_{l_0} \otimes v_{l_0+1} \otimes \ldots \otimes v_q \\ = & e_{s_1} \otimes \ldots \otimes e_{s_{l_0-3}} \otimes x \otimes L \otimes e_{s_{l_0}} w e_{s_{l_0}} \otimes e_{s_{l_0+1}} \otimes \ldots \otimes e_{s_q} \\ \cdot & v_1 \otimes \ldots \otimes v_{l_0-3} \otimes \tilde{v}_{l_0-2} \otimes \tilde{v}_{l_0-1} \otimes \tilde{v}_{l_0} \otimes v_{l_0+1} \otimes \ldots \otimes v_q \end{array}$$

  if we can factor x from the left of v_{l_0-2}, or

$$\begin{array}{rl} & v_1 \otimes \ldots \otimes v_{\lambda-1} \otimes v_\lambda \otimes \ldots \otimes v_{l_0-1} \otimes v_{l_0} \otimes v_{l_0+1} \otimes \ldots \otimes v_q \\ = & v_1 \otimes \ldots \otimes v_{l_0-3} \otimes \tilde{v}_{l_0-2} \otimes \tilde{v}_{l_0-1} \otimes \tilde{v}_{l_0} \otimes v_{l_0+1} \otimes \ldots \otimes v_q \\ \cdot & e_{t_1} \otimes \ldots \otimes e_{t_{l_0-3}} \otimes x \otimes R \otimes e_{t_{l_0}} w e_{t_{l_0}} \otimes e_{t_{l_0+1}} \otimes \ldots \otimes e_{t_q} \end{array}$$

  if we can factor x from the right of v_{l_0-2}, where $\tilde{v}_c$ is the remaining part of v_c after factorisation, and $L, R \in \{e_p \otimes x, x \otimes e_1\}$ depending on the element v_{l_0-1};

∗ or we cannot factor x from either side of v_{l_0-2}, and hence by Lemma 4.4.6 $p = 3$ and v_{l_0-2} must be

$$(3, -2a-2, -2, 1, a+2, a, 1) \quad = \quad e_3 \otimes x\xi \otimes (\xi \otimes \xi)^{\otimes a} \otimes e_1$$

By minimality of l_0, it has i-degree at least -2, hence $a = 0$. Thus it has i-degree -2 and as a result, we see that $v_{\lambda'}$ must be the same element

$$(3, -2, -2, 1, 2, 0, 1) \quad = \quad e_3 \otimes x\xi \otimes e_1$$

for all $\lambda < \lambda' \leq l_0 - 2$. We see that we can factor

$$(3, -1, -1, 0, 1, 0, 1) = e_3 \otimes e_1$$

both on the left and on the right from all $v_{\lambda'}$, $\lambda < \lambda' \leq l_0 - 2$, and hence, if we factor x from the left of v_λ, we can write the following decomposition

$$\begin{aligned} & v_1 \otimes \ldots \otimes v_{\lambda-1} \otimes v_\lambda \otimes \ldots \otimes v_{l_0-1} \otimes v_{l_0} \otimes v_{l_0+1} \otimes \ldots \otimes v_q \\ = \quad & e_{s_1} \otimes \ldots \otimes e_{s_{\lambda-1}} \otimes x \otimes (e_3 \otimes e_1) \otimes \ldots \otimes (e_3 \otimes e_1) \otimes L \\ & \qquad \otimes e_{s_{l_0}} w e_{s_{l_0}} \otimes e_{s_{l_0+1}} \otimes \ldots \otimes e_{s_q} \\ \cdot \quad & v_1 \otimes \ldots \otimes v_{\lambda-1} \otimes \tilde{v}_\lambda \otimes \tilde{v}_{\lambda+1} \otimes \ldots \otimes \tilde{v}_{l_0} \otimes v_{l_0+1} \otimes \ldots \otimes v_q \end{aligned}$$

or

$$\begin{aligned} & v_1 \otimes \ldots \otimes v_{\lambda-1} \otimes v_\lambda \otimes \ldots \otimes v_{l_0-1} \otimes v_{l_0} \otimes v_{l_0+1} \otimes \ldots \otimes v_q \\ = \quad & v_1 \otimes \ldots \otimes v_{\lambda-1} \otimes \tilde{v}_\lambda \otimes \tilde{v}_{\lambda+1} \otimes \ldots \otimes \tilde{v}_{l_0} \otimes v_{l_0+1} \otimes \ldots \otimes v_q \\ \cdot \quad & e_{t_1} \otimes \ldots \otimes e_{t_{\lambda-1}} \otimes x \otimes (e_3 \otimes e_1) \otimes \ldots \otimes (e_3 \otimes e_1) \otimes R \\ & \qquad \otimes e_{t_{l_0}} w e_{t_{l_0}} \otimes e_{t_{l_0+1}} \otimes \ldots \otimes e_{t_q} \end{aligned}$$

if we factor x from the right of v_λ, where $\tilde{v}_c$ is the remaining part of v_c after factorisation, and $L, R \in \{e_p \otimes x, x \otimes e_1, \xi \otimes x^2, x^2 \otimes \xi\}$ depending on the element v_{l_0-1}. Note that in both cases, if $\lambda = 1$, then v_1 must be x^2 and it is possible to factor x both from the left and from the right, hence v is reducible.

– If $b_{l_0} = 0$, then $a_{l_0} \geq 3$ since $-a_{l_0} - b_{l_0} = i_{l_0} \leq -3$.

∗ Assume $a_n - b_n \geq 2$ for all $l_0 \leq n \leq q$. Then by Proposition 4.4.9, there exists a non-trivial decomposition and v is reducible.

∗ Otherwise, let l be the minimal index, $l_0 < l \leq q$, such that $a_l - b_l \leq 1$. In particular, $a_n - b_n \geq 2$ for all $l_0 \leq n < l$. We need to examine three cases.

· Assume $a_l - b_l = 0$. Then by Proposition 4.4.14, v is reducible.

· Assume $b_l \geq 1$, and $a_l - b_l \neq 0$; in particular v_l is not of type **3**, and thus v_{l-1} must be of type **1**. Since $a_{l-1} - b_{l-1} \geq 2$ by minimality of l, we can apply Lemma 4.4.11: if $s_{l-1} \leq p-2$ or $t_{l-1} \geq 3$, v is reducible. Assume now that $s_{l-1} > p-2$ and $t_{l-1} < 3$.

That means that v_{l-1} is of the form

$$\begin{aligned}&(p, -a-b, -p(a-b)-(1-p)+2u,\\ &\quad (p-1)(a-b)+(1-p)-u, a, b, 1)\\ &= e_p \otimes (x^{p-2}\xi)^u \otimes (x^{p-1})^{\otimes a-b-1-u} \otimes (\xi \otimes \xi)^{\otimes b} \otimes e_1\end{aligned}$$

$$\begin{aligned}&(p, -a-b, -p(a-b)-(2-p)+2u,\\ &\quad (p-1)(a-b)+(2-p)-u, a, b, 2)\\ &= e_p \otimes (x^{p-2}\xi)^u \otimes (x^{p-1})^{\otimes a-b-1-u} \otimes (\xi \otimes \xi)^{\otimes b} \otimes xe_2\end{aligned}$$

$$\begin{aligned}&(p-1, -a-b, -p(a-b)-(1-(p-1))+2u,\\ &\quad (p-1)(a-b)+(1-(p-1))-u, a, b, 1)\\ &= e_{p-1}x \otimes (x^{p-2}\xi)^u \otimes (x^{p-1})^{\otimes a-b-1-u} \otimes (\xi \otimes \xi)^{\otimes b} \otimes e_1\end{aligned}$$

$$\begin{aligned}&(p-1, -a-b, -p(a-b)-(2-(p-1))+2u,\\ &\quad (p-1)(a-b)+(2-(p-1))-u, a, b, 2)\\ &= e_{p-1}x \otimes (x^{p-2}\xi)^u \otimes (x^{p-1})^{\otimes a-b-1-u} \otimes (\xi \otimes \xi)^{\otimes b} \otimes xe_2\end{aligned}$$

with $a_{l-1} - b_{l-1} \geq 2$ by minimality of l. Recall that

$$e_p \otimes (x^{p-2}\xi)^u \otimes (x^{p-1})^{\otimes a-b-1-u} \otimes (\xi \otimes \xi)^{\otimes b} \otimes e_1$$

is the same element as

$$e_p \otimes (\xi \otimes \xi)^{\otimes b} \otimes (x^{p-1})^{\otimes a-b-1-u} \otimes (x^{p-2}\xi)^u \otimes e_1$$

in homology. Hence in all cases, we can factor j-degree -2 elements $x \otimes e_1$ or $e_p \otimes x$ from v_{l-1} both from the left side and the right side. Let $1 \leq \lambda \leq l-2$ be the largest index for which x can be factored from v_λ, from the left or from the right. In particular, for all $\lambda < \lambda' \leq l-2$, it is both impossible to factor x from the left and from the right of $v_{\lambda'}$. Hence if $\max l_0 - 1, \lambda < \lambda' \leq l-2$, $v_{\lambda'}$ must be of the form

$$\begin{aligned}&(p, -a-b, -p(a-b)-(1-p)+2u, (p-1)(a-b)+(1-p)-u, a, b, 1)\\ &\quad = e_p \otimes (x^{p-2}\xi)^u \otimes (x^{p-1})^{\otimes a-b-1-u} \otimes (\xi \otimes \xi)^{\otimes b} \otimes e_1\\ &\quad \cong e_p \otimes (\xi \otimes \xi)^{\otimes b} \otimes (x^{p-1})^{\otimes a-b-1-u} \otimes (x^{p-2}\xi)^u \otimes e_1\end{aligned}$$

as $a_{\lambda'} - b_{\lambda'} \geq 2$.
And for all $l_0 \leq \lambda' \leq l-2$, and we can factor $e_p \otimes e_1$ on both sides of $v_{\lambda'}$.
Thus, if $\lambda \geq l_0 - 1$, we have a decomposition as announced in the introduction, namely we have

$$\begin{aligned}& v_1 \otimes \ldots \otimes v_{l_0} \otimes \ldots \otimes v_\lambda \otimes \ldots \otimes v_q\\ = \;& e_{s_1} \otimes \ldots \otimes e_{s_{\lambda-1}} \otimes x \otimes (e_p \otimes e_1) \ldots \otimes (e_p \otimes e_1) \otimes L \otimes w\\ & \quad \otimes e_{s_{l+1}} \otimes \ldots \otimes e_{s_q}\\ \cdot \;& v_1 \otimes \ldots \otimes v_{l_0} \otimes \ldots \otimes v_{\lambda-1} \otimes \tilde{v}_\lambda \otimes \ldots \otimes \tilde{v}_l \otimes v_{l+1} \otimes \ldots \otimes v_q\end{aligned}$$

or

$$\begin{aligned}& v_1 \otimes \ldots \otimes v_{l_0} \otimes \ldots \otimes v_\lambda \otimes \ldots \otimes v_q\\ = \;& v_1 \otimes \ldots \otimes v_{l_0} \otimes \ldots \otimes v_{\lambda-1} \otimes \tilde{v}_\lambda \otimes \ldots \otimes \tilde{v}_l \otimes v_{l+1} \otimes \ldots \otimes v_q\\ \cdot \;& e_{t_1} \otimes \ldots \otimes e_{t_{\lambda-1}} \otimes x \otimes (e_p \otimes e_1) \ldots \otimes (e_p \otimes e_1) \otimes R \otimes w\\ & \quad \otimes e_{t_{l+1}} \otimes \ldots \otimes e_{t_q}\end{aligned}$$

where $R, L \in \{x \otimes e_1, e_p \otimes x\}$, and v is reducible.

Now, if $1 \leq \lambda \leq l_0 - 2$, we use the same factorisations found so far for $v_{\lambda'}$ where $l_0 \leq \lambda' \leq l$. For all $1 \leq \lambda' \leq l_0 - 1$ $v_{\lambda'}$ has at least i-degree -2 by assumption. Thus the element v_{l_0-1} has i-degree at least -2 and j-degree at most -3 and it is not possible to factor any x from it. Recall that by assumption $a_{l_0-1} - b_{l_0-1} = 2$, so by Expression (4.4), v_{l_0-1} is one of

$$\begin{cases} (s, -2, -2p - (t-s), \\ \quad 2(p-1) + (t-s), 2, 0, t) & = x^{p-s} \otimes x^{p-1} \otimes x^{t-1} \\ (p-1, -2, -p-t+1, \\ \quad p+t-2, 2, 0, t) & = \xi \otimes x^{p-1} \otimes x^{t-1} \\ (s, -2, -2p+s, \\ \quad 2p-s-1, 2, 0, 2) & = x^{p-s} \otimes x^{p-1} \otimes \xi \\ (p, -2, -p+1, p-2, 2, 0, 1) & = e_p \otimes x^{p-2}\xi \otimes e_1 \qquad p > 3 \end{cases}$$

Since it is impossible to factor x from it (as $\lambda \leq l_0 - 2$), v_{l_0-1} is then one of

$$\begin{cases} (p, -2, -p-1, p-1, 2, 0, 1) & = e_p \otimes x^{p-1} \otimes e_1 \\ (p-1, -2, -p, p-1, 2, 0, 1) & = \xi \otimes x^{p-1} \otimes e_1 \\ (p, -2, -p, p-1, 2, 0, 2) & = e_p \otimes x^{p-1} \otimes \xi \\ (p, -2, -p+1, p-2, 2, 0, 1) & = e_p \otimes x^{p-2}\xi \otimes e_1 \quad p > 3 \end{cases}$$

Due to the following equalities in homology,

$$\xi \otimes x^{p-1} \otimes e_1 = x \otimes x^{p-2}\xi \otimes e_1$$

and

$$e_p \otimes x^{p-1} \otimes \xi \cong e_p \otimes x^{p-2}\xi \otimes x,$$

we see that x can actually be factored from the middle two possibilities and v is reducible. Again, we can factor $e_p \otimes e_1$ from both sides for the first and last possibility.
Now, for all $\lambda < \lambda' < l_0 - 1$, $v_{\lambda'}$ is an element with i-degree at least -2 such that x cannot be factored from it. Considering the i-degree, $v_{\lambda'}$ is one of

$$\begin{cases} (s, 1, 1, 0, -1, 0, p+1-s) & = e_s \otimes e^*_{p+1-s} \\ (s, 0, -(t-s)+2u, t-s-u, 0, 0, t) & = e_s x^{t-s-u}\xi^u e_t \\ (s, -1, 1, 0, 0, 1, p+1-s) & = e_s\xi \otimes \xi e_{p+1-s} \\ (p-1, -1, 1, 0, 1, 0, 2) & = e_{p-1}\xi \otimes \xi e_2 \\ (s, -1, -p-(t-s)+2u, \\ \quad (p-1)+(t-s)-u, 1, 0, t) & = e_s x^{p-s-u}\xi^u \otimes x^{t-1} e_t \\ & \cong e_s x^{p-s} \otimes x^{t-1-u}\xi^u e_t \\ (s, -2, -(t-s)+2u, t-s-u, 1, 1, t) & = e_s w x^{t-s-u}\xi^u e_t \\ (s, -2, -2p-(t-s)+2u, \\ \quad 2(p-1)+(t-s)-u, 2, 0, t) & = e_s x^{p-s} \otimes x^{p-1-u}\xi^u \otimes x^{t-1} e_t \end{cases}$$

and removing the elements from which x can be factored, we are left with

$$\begin{cases} (s,1,1,0,-1,0,p+1-s) & = e_s \otimes e^*_{p+1-s} \\ (s,0,u,0,0,0,s+u) & = e_s\xi^u e_{s+u} \\ (s,-1,1,0,0,1,p+1-s) & = e_s\xi \otimes \xi e_{p+1-s} \\ (p-1,-1,1,0,1,0,2) & = e_{p-1}\xi \otimes \xi e_2 \\ (p-u,-1,u-1,0,1,0,1) & = e_{p-u}\xi^u \otimes e_1 \\ (p,-1,u-1,0,1,0,1+u) & = e_p \otimes \xi^u e_{1+u} \\ (s,-2,u,0,1,1,s+u) & = e_s w\xi^u e_{s+u} \\ (p,-2,-p-1+2u, & \\ \quad p-1-u,2,0,1) & = e_p \otimes x^{p-1-u}\xi^u \otimes e_1 \end{cases}$$

and finally, we can remove the elements with j-degree 1 as type **3** elements cannot follow: v_{l_0-1} is a type **1** element and can only be preceded by type **1** elements. That means the possibilities reduce to

$$\begin{cases} (s,0,0,0,0,0,s) & = e_s \\ (p-u,-1,u-1,0,1,0,1) & = e_{p-u}\xi^u \otimes e_1 \\ (p,-1,u-1,0,1,0,1+u) & = e_p \otimes \xi^u e_{1+u} \\ (s,-2,0,0,1,1,s) & = e_s w e_s \\ (p,-2,-p-1+2u, & \\ \quad p-1-u,2,0,1) & = e_p \otimes x^{p-1-u}\xi^u \otimes e_1 \end{cases}$$

Now, if $v_{\lambda'} \in \{e_{s_{\lambda'}}, \xi \otimes e_1, e_p \otimes \xi, e_{s_{\lambda'}} w e_{s_{\lambda'}}\}$ for some $\lambda < \lambda' < l_0 - 1$, since it has j-degree 0, $v_{\lambda'+1}$ is an element of $\mathbf{d}$. In particular, its a- and b-degree are equal. By Lemma 4.4.12, we see that v is reducible unless $v_{\lambda'+1}$ is an idempotent. By Corollary 4.4.13, if v is irreducible, then $v_\eta = e_{s_\eta}$ for all $\lambda' < \eta < l_0 - 1$. In particular, v_{l_0-1} has i-degree 0, which is a contradiction since we assumed v_{l_0-1} is one of the two following i-degree -2 elements

$$\begin{aligned} (p,-2,-p-1,p-1,2,0,1) &= e_p \otimes x^{p-1} \otimes e_1 \\ (p,-2,-p+1,p-2,2,0,1) &= e_p \otimes x^{p-2}\xi \otimes e_1 \end{aligned}$$

Hence, if $v_{\lambda'} \in \{e_{s_{\lambda'}}, \xi \otimes e_1, e_p \otimes \xi, e_{s_{\lambda'}} w e_{s_{\lambda'}}\}$ for some $\lambda < \lambda' < l_0 - 1$, then v is reducible.

Furthermore, if $v_{\lambda'} = e_p \otimes e_1$ for some $\lambda < \lambda' < l_0 - 1$, then $v_{\lambda'+1} \in \{\xi \otimes e_1, e_p \otimes \xi, e_p \otimes e_1\}$. By the same analysis as before, we see that v is reducible if $v_{\lambda'+1}$ is one of the two j-degree 0 elements $\xi \otimes e_1$ or $e_p \otimes \xi$. Thus, it remains to consider the case $v_\eta = e_p \otimes e_1$ for all $\lambda' \leq \eta < l_0 - 1$. We reach another contradiction as $v_{l_0-2} = e_p \otimes e_1$ has j-degree -1 when v_{l_0-1} has i-degree -2 by assumption.

Finally, if $v_{\lambda'} = e_p \otimes x^{p-1-u}\xi^u \otimes e_1$ for some $\lambda < \lambda' < l_0 - 1$, since v_η has i-degree at least -2 for all $2 \leq \eta \leq l_0 - 1$, the j-degree of $v_{\lambda'}$

must satisfy

$$-p-1+2u \geq -2$$

or equivalently

$$1+2u \geq p$$

which is if and only if $u=1$ and $p=3$. That means that if $p>3$, then v is reducible and $\lambda \geq l_0-2$. If $p=3$, we see that in particular, $v_{\lambda'} = e_3 \otimes x\xi \otimes e_1$ has j-degree -2 and $v_\eta \in \{e_s w e_s, e_3 \otimes x\xi \otimes e_1\}$ for all $\lambda' < \eta < l_0 - 1$. Similarly to previous considerations, if v_η is equal to the j-degree 0 element $e_s w e_s$, then v is reducible. Thus we need to consider the case when v_η is equal to $e_3 \otimes x\xi \otimes e_1$ for all $\lambda' \leq \eta < l_0 - 1$, and v_{l_0-1} equals $e_3 \otimes x^2 \otimes e_1$. In particular, for the chaining rule to be respected, if $v_{\lambda'} = e_3 \otimes x\xi \otimes e_1$ (an (i,j)-degree (-2,-2) element) for some $\lambda < \lambda' < l_0 - 1$, then $v_\eta = e_3 \otimes x\xi \otimes e_1$ for all $\lambda < \eta < l_0 - 1$. In addition, v_λ must have j-degree -2. Hence, we are in the following situation

$$v_1 \otimes \ldots \otimes v_\lambda \otimes (e_3 \otimes x\xi \otimes e_1)^{\otimes l_0-2-\lambda} \otimes \left(e_3 \otimes x^2 \otimes e_1\right) \otimes v_{l_0} \otimes \ldots \otimes v_q$$

and we can again factor $e_p \otimes e_1 = e_3 \otimes e_1$ both from the left and from the right of v_η for all $\lambda < \eta \leq l_0 - 1$. By assumption, we can factor x from v_λ and we obtain the following decomposition

$$\begin{aligned} & v_1 \otimes \ldots \otimes v_\lambda \otimes \ldots \otimes v_l \otimes \ldots \otimes v_q \\ = \; & e_{s_1} \otimes \ldots \otimes e_{s_{\lambda-1}} \otimes x \otimes (e_p \otimes e_1) \otimes \ldots \otimes (e_p \otimes e_1) \otimes L \otimes e_{s_l} w e_{s_l} \\ & \quad \otimes e_{s_{l+1}} \otimes \ldots \otimes e_{s_q} \\ \cdot \; & v_1 \otimes \ldots \otimes v_{\lambda-1} \otimes \tilde{v}_\lambda \otimes \tilde{v}_{\lambda+1} \otimes \ldots \otimes \tilde{v}_{l-1} \otimes \tilde{v}_l \otimes v_{l+1} \otimes \ldots \otimes v_q \end{aligned}$$

if we can factor x from the left of v_λ, or

$$\begin{aligned} & v_1 \otimes \ldots \otimes v_\lambda \otimes \ldots \otimes v_l \otimes \ldots \otimes v_q \\ = \; & v_1 \otimes \ldots \otimes v_{\lambda-1} \otimes \tilde{v}_\lambda \otimes \tilde{v}_{\lambda+1} \otimes \ldots \otimes \tilde{v}_{l-1} \otimes \tilde{v}_l \otimes v_{l+1} \otimes \ldots \otimes v_q \\ \cdot \; & e_{t_1} \otimes \ldots \otimes e_{t_{\lambda-1}} \otimes x \otimes (e_p \otimes e_1) \otimes \ldots \otimes (e_p \otimes e_1) \otimes R \otimes e_{t_l} w e_{t_l} \\ & \quad \otimes e_{t_{l+1}} \otimes \ldots \otimes e_{t_q} \end{aligned}$$

if we can factor x from the right of v_λ, where $\tilde{v}_c$ is the remaining part of v_c after factorisation, and $L, R \in \{x \otimes e_1, e_p \otimes x\}$ depending on the element v_{l-1}.
Note that if $\lambda = 1$, $v_\lambda = v_1$ is an element of $\mathbf{d}$ with j-degree -2, hence $v_1 = x^2$ and we can factor x. In particular, λ exists.

· Assume $b_l = 0$ and $a_l - b_l \neq 0$. Since $a_l - b_l \leq 1$, we see that $a_l = 1$ or $a_l = -1$ ($a_l < -1$ does not correspond to any element in $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})^{\leq 1}$). Assume $a_l = -1$, i.e. v_l is of type **3**. Then by Lemma 4.3.1, the element v_{l-1} must then be of the form

$$\begin{aligned} (s_1, -2a, -1, 1, b, b, s_1+1) &= e_{s_1} w^b \xi e_{s_1+1} \\ (p-1, -2b-1, 1, 0, b+1, b, 2) &= e_{p-1}\xi \otimes (\xi \otimes \xi)^{\otimes b} \otimes \xi e_2 \\ (s_2, -2a-1, 1, 0, a, a+1, p+1-s_2) &= e_{s_2} (\xi \otimes \xi)^{\otimes a+1} e_{p+1-s_2} \\ (s_3, 1, 1, 0, -1, 0, p+1-s_3) &= e_{s_3} \otimes e^*_{p+1-s_3} \end{aligned}$$

with $1 \leq s_1 \leq p-1$, $1 \leq s_2 \leq p-2$, $1 \leq s_3 \leq p-1$, $a, b \geq 0$. By minimality of l, the a- and b-degree of v_{l-1} must satisfy $a_{l-1} - b_{l-1} \geq 2$. Hence a_l cannot be equal to -1.

Assume $a_l = 1$. That means v_{l-1} has j-degree -1 and by Lemma 4.4.5 is of the form

$$\begin{array}{rcl}(s_1, -2a, -1, 1, a, a, s_1+1) & = & e_{s_1} w^a x e_{s_1+1} \\ (p, -2a-1, -1, 0, a+1, a, 1) & = & e_p \otimes (\xi \otimes \xi)^{\otimes a} \otimes e_1 \\ (p-1, -2a-1, -1, 1, a+1, a, 2) & = & e_{p-1}\xi \otimes (\xi \otimes \xi)^{\otimes a} \otimes x e_2 \\ & (\cong & e_{p-1} x \otimes (\xi \otimes \xi)^{\otimes a} \otimes \xi e_2)\end{array}$$

with $1 \leq s_1 \leq p-1$, $1 \leq s_2 \leq p-3$, $a \geq 0$. However, again by minimality of l, the a- and b-degree of v_{l-1} must satisfy $a_{l-1} - b_{l-1} \geq 2$. Hence a_l cannot be equal to 1 either.

Hence v is reducible.

□

4.5 New irreducible monomials of $\mathbf{w}_q$

A consequence of the previous result is that we can only build irreducible monomials from elements of i-degree at least -2, and they must have j-degree at least -2 as well.

Lemma 4.5.1. *The monomial basis elements of $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})^{\leq 1}$ of i-degree at least -2 and j-degree at least -2 from which $x\xi$ cannot be non-trivially factored are given below*

$$\begin{array}{rrrrrrll}
(s, & 1, & 1, & 0, & -1, & 0, & p+1-s) & = e_s \otimes e^*_{p+1-s} \\
\\
(s, & 0, & 1, & 0, & 0, & 0, & s+1) & = e_s \xi e_{s+1} \\
(s, & 0, & 0, & 0, & 0, & 0, & s) & = e_s \\
(s, & 0, & 0, & 1, & 0, & 0, & s+2) & = e_s x\xi e_{s+2} \\
(s, & 0, & -1, & 1, & 0, & 0, & s+1) & = e_s x e_{s+1} \\
(s, & 0, & -2, & 2, & 0, & 0, & s+2) & = e_s x^2 e_{s+2} \\
\\
(p-1, & -1, & 1, & 0, & 1, & 0, & 2) & = e_{p-1}\xi \otimes \xi e_2 \\
(s, & -1, & 1, & 0, & 0, & 1, & p+1-s) & = e_s \xi \otimes \xi e_{p+1-s} \\
(p, & -1, & 0, & 0, & 1, & 0, & 2) & = e_p \otimes \xi e_2 \\
(p-1, & -1, & 0, & 0, & 1, & 0, & 1) & = e_{p-1}\xi \otimes e_1 \\
(p, & -1, & -1, & 0, & 1, & 0, & 1) & = e_p \otimes e_1 \\
(p-1, & -1, & -1, & 1, & 1, & 0, & 2) & = e_{p-1}\xi \otimes x e_2 \cong e_{p-1} x \otimes \xi e_2 \\
(p, & -1, & -2, & 1, & 1, & 0, & 2) & = e_p \otimes x e_2 \\
(p-1, & -1, & -2, & 1, & 1, & 0, & 1) & = e_{p-1} x \otimes e_1 \\
\\
(s, & -2, & 0, & 0, & 1, & 1, & s) & = e_s w e_s \\
(3, & -2, & -2, & 1, & 2, & 0, & 1) & = e_3 \otimes x\xi \otimes e_1 (\text{ if } p = 3).
\end{array}$$

Proof. Let v be such a monomial basis element of $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})^{\leq 1}$. Then its i-degree i_v satisfies $-2 \leq i_v \leq 1$. Let us study these four different cases.

($i_v = 1$) If $i_v = 1$, then v is an element of $\mathbb{H}(\mathbf{u})$ and so is of the form $v = (s, 1, 1, 0, -1, 0, p+1-s) = e_s \otimes e^*_{p+1-s}$. Note that it is irreducible.

($i_v = 0$) If $i_v = 0$, then v is an element of $\mathbf{d}$, so it is of the form

$$(s, 0, -(t-s) + 2u, (t-s) - u, 0, 0, t),$$

with $t-s \geq 0$. The element v has at least j-degree -2 if and only if

$$-(t-s)+2u \geq -2$$

i.e. if and only if

$$t-s \leq 2+2u.$$

If $u=0$, we then have

- ◇ $t-s=0$, and $v=(s,0,0,0,0,0,s)=e_s$;
- ◇ $t-s=1$, and $v=(s,0,-1,1,0,0,s+1)=e_s x e_{s+1}$;
- ◇ $t-s=2$, and $v=(s,0,-2,2,0,0,s+2)=e_s x^2 e_{s+2}$.

If $u=1$, in particular $t-s>0$ by definition of type **1** elements, and we have

- ◇ $t-s=1$, and $v=(s,0,1,0,0,0,s+1)=e_s \xi e_{s+1}$;
- ◇ $t-s=2$, and $v=(s,0,0,1,0,0,s+2)=e_s x \xi e_{s+2}$;
- ◇ $t-s=3$, and $v=(s,0,-1,2,0,0,s+3)=e_s x^2 \xi e_{s+3}$;
- ◇ $t-s=4$, and $v=(s,0,-2,3,0,0,s+4)=e_s x^3 \xi e_{s+4}$.

By Proposition 4.4.2, we know that we must eliminate $e_s x^2 \xi e_{s+3}$ and $e_s x^3 \xi e_{s+4}$ from that list of possibilities. The elements of i-degree 0 satisfying the required conditions are

$$\begin{array}{l}(s,0,0,0,0,0,s)=e_s\\(s,0,-1,1,0,0,s+1)=e_s x e_{s+1}\\(s,0,-2,2,0,0,s+2)=e_s x^2 e_{s+2}\\(s,0,1,0,0,0,s+1)=e_s \xi e_{s+1}\\(s,0,0,1,0,0,s+2)=e_s x \xi e_{s+2}.\end{array}$$

($i_v=-1$) If $i_v=-1$, then v is an element of $\mathbb{H}(\mathbf{u}^{-1})$ and so is of the form

$$(s,-1,-p-(t-s)+2u,p-1+(t-s)-u,1,0,t),$$

or

$$(p-1,-1,1,0,1,0,2),$$

or

$$(s,-1,1,0,0,1,p+1-s),$$

with $1 \leq s \leq p-2$ in the last case.

The last two possibilities satisfy the conditions required and so are on the list. We need to study the first possibility more carefully. Recall that $1 \leq s,t \leq p$, so that $1-p \leq t-s \leq p-1$. The element v has at least j-degree -2 if and only if

$$-p-(t-s)+2u \geq -2$$

which writes equivalently

$$t-s \leq 2+2u-p.$$

If $u=0$, then $1-p \leq t-s \leq 2-p$, so we have

- ◇ $t-s=1-p$, and $v=(p,-1,-1,0,1,0,1)=e_p \otimes e_1$;
- ◇ $t-s=2-p$, and
  - ∗ $v=(p,-1,-2,1,1,0,2)=e_p \otimes x e_2$;
  - ∗ or $v=(p-1,-1,-2,1,1,0,1)=e_{p-1} x \otimes e_1$.

If $u = 1$, then $t - s \geq 2 - p$ by definition of type **1** elements as $u = 1$ and $a - b = 1$, hence $2 - p \leq t - s \leq 4 - p$, and we have

- ⋄ $t - s = 2 - p$, and
  - ∗ $v = (p, -1, 0, 0, 1, 0, 2) = e_p \otimes \xi e_2$;
  - ∗ or $v = (p - 1, -1, 0, 0, 1, 0, 1) = e_{p-1}\xi \otimes e_1$;
- ⋄ $t - s = 3 - p$, and
  - ∗ $v = (p, -1, -1, 1, 1, 0, 3) = e_p \otimes x\xi e_3$;
  - ∗ or $v = (p - 1, -1, -1, 1, 1, 0, 2) = e_{p-1}\xi \otimes xe_2 \cong e_{p-1}x \otimes \xi e_2$;
  - ∗ or $v = (p - 2, -1, -1, 1, 1, 0, 1) = e_{p-2}x\xi \otimes e_1$;
- ⋄ $t - s = 4 - p$, and
  - ∗ $v = (p, -1, -2, 2, 1, 0, 4) = e_p \otimes x^2\xi e_4$;
  - ∗ or $v = (p - 1, -1, -2, 2, 1, 0, 3) = e_{p-1}\xi \otimes x^2 e_3 \cong e_{p-1}x \otimes x\xi e_3$;
  - ∗ or $v = (p - 2, -1, -2, 2, 1, 0, 2) = e_{p-2}x\xi \otimes xe_2 \cong e_{p-2}x^2 \otimes \xi e_2$;
  - ∗ or $v = (p - 3, -1, -2, 2, 1, 0, 1) = e_{p-3}x^2\xi \otimes e_1$.

By Proposition 4.4.1, s and t must satisfy $s > p - 2$ and $t < 3$, so we must eliminate all the elements of case $u = 1$ and $t - s = 4 - p$, and elements $e_p \otimes x\xi e_3$ and $e_{p-2}x\xi \otimes e_1$ of case $u = 1$ and $t - s = 3 - p$.

The elements of i-degree -1 satisfying the required conditions are

$$\begin{aligned}
&(p, -1, -1, 0, 1, 0, 1) = e_p \otimes e_1\\
&(p, -1, -2, 1, 1, 0, 2) = e_p \otimes xe_2\\
&(p - 1, -1, -2, 1, 1, 0, 1) = e_{p-1}x \otimes e_1\\
&(p, -1, 0, 0, 1, 0, 2) = e_p \otimes \xi e_2\\
&(p - 1, -1, 0, 0, 1, 0, 1) = e_{p-1}\xi \otimes e_1\\
&(p - 1, -1, -1, 1, 1, 0, 2) = e_{p-1}\xi \otimes xe_2 \cong e_{p-1}x \otimes \xi e_2\\
&(p - 1, -1, 1, 0, 1, 0, 2) = e_{p-1}\xi \otimes \xi e_2\\
&(s, -1, 1, 0, 0, 1, p + 1 - s) = e_s\xi \otimes \xi e_{p+1-s}.
\end{aligned}$$

$(i_v = -2)$ If $i_v = -2$, then v is an element of $\mathbb{H}(\mathbf{u}^{-2})$ and so is of the form

$$(s, -2, -(t - s) + 2u, (t - s) - u, 1, 1, t),$$

and $t - s \geq 0$, or

$$(s, -2, -2p - (t - s) + 2u, 2(p - 1) + (t - s) - u, 2, 0, t).$$

Consider the j-degree in the latter case:

$$-2p - (t - s) + 2u \geq -2$$

is equivalent to

$$t - s \leq 2(1 + u - p),$$

but since $1 \leq s, t \leq p$, $1 - p \leq t - s$, and we have

$$1 - p \leq 2(1 + u - p)$$

or equivalently

$$p \leq 1 + 2u.$$

If $u = 0$, then we obtain a contradiction, so $u = 1$ and $p = 3$. We get $1 - p = -2 \leq t - s \leq -2 = 2(1+1-3)$, so $s = t+2$ and it satisfies $1 \leq t+2 \leq 3$, i.e. $s = 3$ and $t = 1$. The corresponding element is $v = (3, -2, -2, 1, 2, 0, 1) = e_3 \otimes x\xi \otimes e_1$.

For the first possibility, by Lemma 4.4.12, we must have $s = t$ in order not to get any splitting. The corresponding element is $v = (s, -2, 0, 0, 1, 1, s) = e_s w e_s$.

The elements of i-degree -2 satisfying the required conditions are

$$\begin{aligned}&(s, -2, 0, 0, 1, 1, s) = e_s w e_s\\&(3, -2, -2, 1, 2, 0, 1) = e_3 \otimes x\xi \otimes e_1 (\text{ if } p = 3).\end{aligned}$$

□

Corollary 4.5.2. *The element in first position of an irreducible monomial v_1 can be one of*

$$\begin{array}{lllllllll}(s, & 0, & 1, & 0, & 0, & 0, & s+1) & = e_s\xi e_{s+1}\\(s, & 0, & 0, & 0, & 0, & 0, & s) & = e_s\\(s, & 0, & 0, & 1, & 0, & 0, & s+2) & = e_s x\xi e_{s+2}\\(s, & 0, & -1, & 1, & 0, & 0, & s+1) & = e_s x e_{s+1}\\(s, & 0, & -2, & 2, & 0, & 0, & s+2) & = e_s x^2 e_{s+2}\end{array}$$

Since we want to understand the new arrows of $\mathbf{w}_q$, we can assume v_1 is not an idempotent of $\mathbf{d}$.

4.5.1 Irreducible monomials starting with ξ, $x\xi$ or x^2

Proposition 4.5.3 (Irreducible monomials starting with ξ)**.** *Let $v = v_1 \otimes \ldots \otimes v_q$ be an irreducible monomial of $\mathbf{w}_q$ such that $v_1 = e_{s_1}\xi e_{s_1+1}$ for some $1 \leq s_1 \leq p-1$. Then v is of the form*

$$e_{s_1}\xi e_{s_1+1} \otimes \left(e_{s_2} \otimes e^*_{p+1-s_2}\right) \otimes \ldots \otimes \left(e_{s_q} \otimes e^*_{p+1-s_q}\right).$$

Proof. If $v_1 = e_{s_1}\xi e_{s_1+1}$, it has j-degree 1, hence only elements of type **3** can follow and these elements are irreducible. We saw that only $e_1\xi e_2$ and $e_{p-1}\xi e_p$ are not always irreducible (cf Remark 4.2.10). However, being in first position, they could only be obtained as a product of two elements of $\mathbf{d}$ and we saw they are irreducible in $\mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})$, hence the result. □

Proposition 4.5.4 (Irreducible monomials starting with $x\xi$)**.** *Let $v = v_1 \otimes \ldots \otimes v_q$ be an irreducible monomial of $\mathbf{w}_q$ such that $v_1 = e_{s_1}x\xi e_{s_1+2}$ for some $1 \leq s_1 \leq p-2$. Then v is of the form*

$$e_{s_1}x\xi e_{s_1+2} \otimes e_{s_2} \otimes \ldots \otimes e_{s_q},$$

such that $(s_2, \ldots, s_q) \notin \{(1, \ldots, 1), (p, \ldots, p)\}$.

Proof. If $v_1 = e_{s_1}x\xi e_{s_1+2}$, it has j-degree 0, hence an element of $\mathbf{d}$ must follow. By Corollary 4.4.13 and since v is irreducible, $v_l = e_{s_l}$ for all $2 \leq l \leq q$. Note that v_1 is not irreducible: $v_1 = e_{s_1}xe_{s_1+1} \cdot e_{s_1+1}\xi e_{s_1+2} = e_{s_1}\xi e_{s_1+1} \cdot e_{s_1+1}xe_{s_1+2}$. Suppose we can write

$$\begin{aligned}& e_{s_1}x\xi e_{s_1+2} \otimes e_{s_2} \otimes \ldots \otimes e_{s_q}\\= \;& e_{s_1}xe_{s_1+1} \otimes \tilde{v}_2 \otimes \ldots \otimes \tilde{v}_q\\\cdot \;& e_{s_1+1}\xi e_{s_1+2} \otimes \left(e_{t_2} \otimes e^*_{p+1-t_2}\right) \otimes \ldots \otimes \left(e_{t_q} \otimes e^*_{p+1-t_q}\right)\end{aligned}$$

or

$$\begin{aligned}& e_{s_1}x\xi e_{s_1+2} \otimes e_{s_2} \otimes \ldots \otimes e_{s_q}\\= \;& e_{s_1}\xi e_{s_1+1} \otimes \left(e_{p+1-s_2} \otimes e^*_{s_2}\right) \otimes \ldots \otimes \left(e_{p+1-s_q} \otimes e^*_{s_q}\right)\\\cdot \;& e_{s_1+1}xe_{s_1+2} \otimes \tilde{v}_2 \otimes \ldots \otimes \tilde{v}_q\end{aligned}$$

then in both cases we see that $\tilde{v}_l$ must have j-degree -1 for all $2 \leq l \leq q$. By Table 2.2, the only way to obtain a type **1** element of the form $\mathbf{1} \cdot \mathbf{3}$ or $\mathbf{3} \cdot \mathbf{1}$ is if the idempotents of the type **1** element satisfy $t = 1$ or $s = p$ (in particular, the element of type **3** is $e_1 \otimes e_p^*$). Hence, the first decomposition is possible if and only if $e_{s_l} = e_p$ for all $2 \leq l \leq q$, and the second is possible if and only if $e_{s_l} = e_1$ for all $2 \leq l \leq q$. □

Proposition 4.5.5 (Irreducible monomials starting with x^2)**.** *Let $v = v_1 \otimes \ldots \otimes v_q$ be an irreducible monomial of $\mathbf{w}_q$ such that $v_1 = e_{s_1}x^2e_{s_1+2}$ for some $1 \leq s_1 \leq p-2$. Then v is of the form*

$$e_{s_1}x^2e_{s_1+2} \otimes e_{s_2}we_{s_2} \otimes e_{s_3} \otimes \ldots \otimes e_{s_q},$$

such that

- *if $s_2 = 1$, there exists $3 \leq l \leq q$ such that $s_l \neq 1$;*
- *if $s_2 = p$, there exists $3 \leq l \leq q$ such that $s_l \neq p$.*

Proof. If $v_1 = e_{s_1}x^2e_{s_1+2}$, it has j-degree -2, hence by Lemma 4.5.1, v_2 is one of

$$\begin{aligned} (s,-2,0,0,0,0,s) &= e_swe_s \\ (3,-2,-2,1,2,0,1) &= e_3 \otimes x\xi \otimes e_1 \quad (\text{if } p=3). \end{aligned}$$

Assume $p = 3$ and $v_2 = e_3 \otimes x\xi \otimes e_1$. Since it has j-degree -2, v_3 has i-degree -2. We can write the following decomposition:

$$\begin{aligned} & e_{s_1}x^2e_{s_1+2} \otimes (e_3 \otimes x\xi \otimes e_1) \otimes v_3 \otimes \ldots \otimes v_q \\ = \; & e_{s_1}xe_{s_1+1} \otimes (e_3 \otimes x) \otimes v_3 \otimes \ldots \otimes v_q \\ \cdot \; & e_{s_1+1}xe_{s_1+2} \otimes (\xi \otimes e_1) \otimes e_{t_3} \otimes \ldots \otimes e_{t_q} \end{aligned}$$

since $e_3 \otimes x\xi \otimes e_1$ is the product of j-degree -2 element $e_3 \otimes x$ with j-degree 0 element $\xi \otimes e_1$.

Therefore v_2 must be of the form $e_{s_2}we_{s_2}$ for v to be irreducible. By Corollary 4.4.13, $v_n = e_{s_n}$ for all $3 \leq n \leq q$ and v is of the form

$$e_{s_1}x^2e_{s_1+2} \otimes e_{s_2}we_{s_2} \otimes e_{s_3} \otimes \ldots \otimes e_{s_q}.$$

By Remark 4.2.10, we know that e_swe_s is not irreducible if and only if $s = 1$ or $s = p$, in which case it can be more conveniently written

$$e_1we_1 = e_1\xi \otimes \xi \otimes e_1,$$

and

$$e_pwe_p = -e_p \otimes \xi \otimes \xi e_p.$$

We can write the following decompositions if $s_2 = 1$

$$\begin{aligned} & e_{s_1}x^2e_{s_1+2} \otimes e_{s_2}we_{s_2} \otimes e_{s_3} \otimes \ldots \otimes e_{s_q} \\ = \; & e_{s_1}xe_{s_1+1} \otimes (e_1\xi \otimes \xi e_p) \otimes \left(e_{s_3} \otimes e^*_{p+1-s_3}\right) \otimes \ldots \otimes \left(e_{s_q} \otimes e^*_{p+1-s_q}\right) \\ \cdot \; & e_{s_1+1}xe_{s_1+2} \otimes (e_p \otimes e_1) \otimes \tilde{v}_3 \otimes \ldots \otimes \tilde{v}_q \end{aligned}$$

and we can write the following decomposition if $s_2 = p$

$$\begin{aligned} & e_{s_1}x^2e_{s_1+2} \otimes e_{s_2}we_{s_2} \otimes e_{s_3} \otimes \ldots \otimes e_{s_q} \\ = \; & e_{s_1}xe_{s_1+1} \otimes (e_p \otimes e_1) \otimes \tilde{v}_3 \otimes \ldots \otimes \tilde{v}_q \\ \cdot \; & e_{s_1+1}xe_{s_1+2} \otimes (e_1\xi \otimes \xi e_p) \otimes \left(e_{p+1-s_3} \otimes e^*_{s_3}\right) \otimes \ldots \otimes \left(e_{p+1-s_q} \otimes e^*_{s_q}\right) \end{aligned}$$

where, for all $3 \leq n \leq q$, $\left(e_{s_n} \otimes e^*_{p+1-s_n}\right) \cdot \tilde{v}_n = e_{s_n}$ if $s_2 = 1$ or $\tilde{v}_n \cdot \left(e_{p+1-s_n} \otimes e^*_{s_n}\right) = e_{s_n}$ if $s_2 = p$. By Table 2.2, we know that type **1** elements of the form $\mathbf{3} \cdot \mathbf{1}$ or $\mathbf{1} \cdot \mathbf{3}$ must satisfy $s_n = 1$ if $s_2 = 1$, or $s_n = p$ if $s_2 = p$. Thus, if there exists $3 \leq n \leq q$ such that $s_n \neq 1$ if $s_2 = 1$, that decomposition fails and v is irreducible. Similarly, if there exists $3 \leq l \leq q$ such $s_n \neq p$ if $s_2 = p$, v is irreducible. □

4.5.2 Irreducible monomials starting with x

We will now give a few results concerning the irreducible monomials starting with x. By Lemma 4.5.1, we know v_2 is one of

$$\begin{array}{rrrrrrll}
(p-1, & -1, & 1, & 0, & 1, & 0, & 2) & = e_{p-1}\xi \otimes \xi e_2 \\
(s, & -1, & 1, & 0, & 0, & 1, & p+1-s) & = e_s\xi \otimes \xi e_{p+1-s} \\
(p, & -1, & 0, & 0, & 1, & 0, & 2) & = e_p \otimes \xi e_2 \\
(p-1, & -1, & 0, & 0, & 1, & 0, & 1) & = e_{p-1}\xi \otimes e_1 \\
(p, & -1, & -1, & 0, & 1, & 0, & 1) & = e_p \otimes e_1 \\
(p-1, & -1, & -1, & 1, & 1, & 0, & 2) & = e_{p-1}\xi \otimes xe_2 \cong e_{p-1}x \otimes \xi e_2 \\
(p, & -1, & -2, & 1, & 1, & 0, & 2) & = e_p \otimes xe_2 \\
(p-1, & -1, & -2, & 1, & 1, & 0, & 1) & = e_{p-1}x \otimes e_1.
\end{array}$$

We study the different possibilities below.

Lemma 4.5.6 (Elements starting with $x\otimes({}_{p-1}\xi\otimes\xi_2)$)**.** *Let $v = v_1\otimes\ldots\otimes v_q$ be a monomial of $\mathbf{w}_q$ such that $v_1 = e_{s_1}xe_{s_1+1}$ for some $1 \le s_1 \le p-1$, and $v_2 = e_{p-1}\xi \otimes \xi e_2$. Then v is reducible.*

Proof. If $v_1 = e_{s_1}xe_{s_1+1}$ for some $1 \le s_1 \le p-2$, and $v_2 = e_{p-1}\xi \otimes \xi e_2$, then v is of the form

$$e_{s_1}xe_{s_1+1} \otimes (e_{p-1}\xi \otimes \xi e_2) \otimes \left(e_{s_3} \otimes e^*_{p+1-s_3}\right) \otimes \ldots \left(e_{s_q} \otimes e^*_{p+1-s_q}\right)$$

since v_2 has j-degree 1. Note that we can write v_2 as the product of j-degree 1 element $e_{p-1}\xi e_p$ by j-degree 0 element $e_p \otimes \xi e_2$. That gives us the following decomposition

$$\begin{aligned}
& e_{s_1}xe_{s_1+1} \otimes (e_{p-1}\xi \otimes \xi e_2) \otimes \left(e_{s_3} \otimes e^*_{p+1-s_3}\right) \otimes \ldots \left(e_{s_q} \otimes e^*_{p+1-s_q}\right) \\
= \; & e_{s_1} \otimes e_{p-1}\xi e_p \otimes \left(e_{s_3} \otimes e^*_{p+1-s_3}\right) \otimes \ldots \left(e_{s_q} \otimes e^*_{p+1-s_q}\right) \\
\cdot \; & e_{s_1}xe_{s_1+1} \otimes (e_p \otimes \xi e_2) \otimes e_{p+1-s_3} \otimes \ldots e_{p+1-s_q}.
\end{aligned}$$

Hence v is reducible. □

Lemma 4.5.7 (Elements starting with $x \otimes (\xi \otimes \xi)$)**.** *Let $v = v_1 \otimes \ldots \otimes v_q$ be a monomial of $\mathbf{w}_q$ such that $v_1 = e_{s_1}xe_{s_1+1}$ for some $1 \le s_1 \le p-1$, and $v_2 = e_{s_2}\xi \otimes \xi e_{p+1-s_2}$ for some $1 \le s_2 \le p-2$. Then v is irreducible.*

Proof. If $v_1 = e_{s_1}xe_{s_1+1}$ for some $1 \le s_1 \le p-1$, and $v_2 = e_{s_2}\xi \otimes \xi e_{p+1-s_2}$, then v is of the form

$$e_{s_1}xe_{s_1+1} \otimes (e_{s_2}\xi \otimes \xi e_{p+1-s_2}) \otimes \left(e_{s_3} \otimes e^*_{p+1-s_3}\right) \otimes \ldots \left(e_{s_q} \otimes e^*_{p+1-s_q}\right).$$

By Proposition 4.2.7, Proposition 4.2.8 and Remark 4.2.10, the only possibly non irreducible component is v_1 if $s_1 = 1$ or $s_1 = p-1$. However, since v_1 is in first position, it can only be obtained as a product of elements of $\mathbf{d}$ and it is irreducible in $\mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})$ by Proposition 4.2.8. Hence v_n is irreducible for all $1 \le n \le q$ and v is irreducible. □

Lemma 4.5.8 (Elements starting with $x\otimes(e_p\otimes\xi)$)**.** *Let $v = v_1\otimes\ldots\otimes v_q$ be an irreducible monomial of $\mathbf{w}_q$ such that $v_1 = e_{s_1}xe_{s_1+1}$ for some $1 \le s_1 \le p-1$, and $v_2 = e_p \otimes \xi e_2$. Then v is of the form*

$$e_{s_1}xe_{s_1+1} \otimes (e_p \otimes \xi e_2) \otimes e_{s_3} \otimes \ldots \otimes e_{s_q},$$

such that there exists $3 \le n \le q$ satisfying $s_n \ne p$.

Proof. Since $e_p \otimes \xi e_2$ has j-degree 0, by Corollary 4.4.13, we know that $v_n = e_{s_n}$ for all $3 \leq n \leq q$. Hence, v is of the form

$$e_{s_1} x e_{s_1+1} \otimes (e_p \otimes \xi e_2) \otimes e_{s_3} \otimes \ldots \otimes e_{s_q}.$$

However, $e_p \otimes \xi e_2$ can be decomposed as the product of j-degree -1 element $e_p \otimes e_1$ by j-degree 1 element $e_1 \xi e_2$. So we could write the following decomposition

$$\begin{aligned} & e_{s_1} x e_{s_1+1} \otimes (e_p \otimes \xi e_2) \otimes e_{s_3} \otimes \ldots \otimes e_{s_q} \\ = \ & e_{s_1} x e_{s_1+1} \otimes (e_p \otimes e_1) \otimes \tilde{v}_3 \otimes \ldots \otimes \tilde{v}_q \\ \cdot \ & e_{s_1+1} \otimes e_1 \xi e_2 \otimes \left(e_{p+1-s_3} \otimes e^*_{s_3}\right) \otimes \ldots \otimes \left(e_{p+1-s_q} \otimes e^*_{s_q}\right) \end{aligned}$$

if and only if $e_{s_n} = \tilde{v}_n \cdot \left(e_{p+1-s_n} \otimes e^*_{s_n}\right)$ for all $3 \leq n \leq q$. From Table 2.2, we see that the only idempotent which can be written as a product of type $\mathbf{1} \cdot \mathbf{3}$ is e_p. Thus v is irreducible if there exists $3 \leq n \leq q$ such that $s_n \neq p$. □

Lemma 4.5.9 (Elements starting with $x \otimes (\xi \otimes e_1)$)**.** *Let $v = v_1 \otimes \ldots \otimes v_q$ be an irreducible monomial of $\mathbf{w}_q$ such that $v_1 = e_{s_1} x e_{s_1+1}$ for some $1 \leq s_1 \leq p-1$, and $v_2 = e_{p-1}\xi \otimes e_1$. Then v is of the form*

$$e_{s_1} x e_{s_1+1} \otimes (e_{p-1}\xi \otimes e_1) \otimes e_{s_3} \otimes \ldots \otimes e_{s_q},$$

such that there exists $3 \leq n \leq q$ satisfying $s_n \neq 1$.

Proof. The proof is very similar to that of Lemma 4.5.8. Since $e_{p-1}\xi \otimes e_1$ has j-degree 0, by Corollary 4.4.13, we know that $v_n = e_{s_n}$ for all $3 \leq n \leq q$. Hence, v is of the form

$$e_{s_1} x e_{s_1+1} \otimes (e_{p-1}\xi \otimes e_1) \otimes e_{s_3} \otimes \ldots \otimes e_{s_q}.$$

However, $e_{p-1}\xi \otimes e_1$ can be decomposed as the product of j-degree 1 element $e_{p-1}\xi e_p$ by j-degree -1 element $e_p \otimes e_1$. So we could write the following decomposition

$$\begin{aligned} & e_{s_1} x e_{s_1+1} \otimes (e_p \otimes \xi e_2) \otimes e_{s_3} \otimes \ldots \otimes e_{s_q} \\ = \ & e_{s_1} \otimes e_{p-1}\xi e_p \otimes \left(e_{s_3} \otimes e^*_{p+1-s_3}\right) \otimes \ldots \otimes \left(e_{s_q} \otimes e^*_{p+1-s_q}\right) \\ \cdot \ & e_{s_1} x e_{s_1+1} \otimes (e_p \otimes e_1) \otimes \tilde{v}_3 \otimes \ldots \otimes \tilde{v}_q \end{aligned}$$

if and only if $e_{s_n} = \left(e_{s_n} \otimes e^*_{p+1-s_n}\right) \cdot \tilde{v}_n$ for all $3 \leq n \leq q$. From Table 2.2, we see that the only idempotent which can be written as a product of type $\mathbf{3} \cdot \mathbf{1}$ is e_1. Thus v is irreducible if there exists $3 \leq n \leq q$ such that $s_n \neq 1$. □

Lemma 4.5.10 (Elements starting with $x \otimes (\xi \otimes x)$)**.** *Let $v = v_1 \otimes \ldots \otimes v_q$ be a monomial of $\mathbf{w}_q$ such that $v_1 = e_{s_1} x e_{s_1+1}$ for some $1 \leq s_1 \leq p-1$, and $v_2 = e_{p-1}\xi \otimes x e_2$. Then v is reducible.*

Proof. The element $v_2 = e_{p-1}\xi \otimes x e_2$ has j-degree -1, hence v_3 has i-degree -1. Note that v_2 can be decomposed as the product of j-degree 0 element $e_{p-1}\xi \otimes e_p$ by j-degree -1 element $e_1 x e_2$. We then have the following decomposition

$$\begin{aligned} & e_{s_1} x e_{s_1+1} \otimes (e_{p-1}\xi \otimes x e_2) \otimes v_3 \otimes \ldots \otimes v_q \\ = \ & e_{s_1} x e_{s_1+1} \otimes (e_{p-1}\xi \otimes e_1) \otimes e_{s_3} \otimes \ldots \otimes e_{s_q} \\ \cdot \ & e_{s_1+1} \otimes e_1 x e_2 \otimes v_3 \otimes \ldots \otimes v_q. \end{aligned}$$

Therefore, v is reducible. □

Lemma 4.5.11 (Elements starting with $x \otimes (e_p \otimes x)$ or $x \otimes (x \otimes e_1)$)**.** *Let $v = v_1 \otimes \ldots \otimes v_q$ be an irreducible monomial of $\mathbf{w}_q$ such that $v_1 = e_{s_1} x e_{s_1+1}$ for some $1 \leq s_1 \leq p-1$, and $v_2 = e_{p-1}x \otimes e_1$ or $v_2 = e_p \otimes xe_2$. Then v is of the form*

$$e_{s_1} x e_{s_1+1} \otimes v_2 \otimes e_{s_3} w e_{s_3} \otimes e_{s_4} \otimes \ldots \otimes e_{s_q},$$

such that

- *if $s_3 = 1$, there exists $4 \leq l \leq q$ such that $s_l \neq 1$;*
- *if $s_3 = p$, there exists $4 \leq l \leq q$ such that $s_l \neq p$.*

Proof. If $v_2 = e_{p-1}x \otimes e_1$ or $v_2 = e_p \otimes xe_2$, we see that it can be decomposed into two parts of j-degree -1:

$$e_{p-1}x \otimes e_1 = e_{p-1}xe_p \cdot (e_p \otimes e_1)$$

or

$$e_p \otimes xe_2 = (e_p \otimes e_1) \cdot e_1 x e_2.$$

Note also that v_2 has j-degree -2, hence by Lemma 4.5.1, v_3 is one of

$$\begin{array}{rcl} (s, -2, 0, 0, 0, 0, s) & = & e_s w e_s \\ (3, -2, -2, 1, 2, 0, 1) & = & e_3 \otimes x\xi \otimes e_1 \quad (\text{if } p = 3). \end{array}$$

Assume $p = 3$ and $v_3 = e_3 \otimes x\xi \otimes e_1$. Since it has j-degree -2, v_4 has i-degree -2. If $v_2 = e_{p-1}x \otimes e_1$, we can write the following decomposition

$$\begin{array}{rl} & e_{s_1} x e_{s_1+1} \otimes e_{p-1}x \otimes e_1 \otimes (e_3 \otimes x\xi \otimes e_1) \otimes v_4 \otimes \ldots \otimes v_q \\ = & e_{s_1} \otimes e_{p-1} x e_p \otimes (e_3 \otimes x) \otimes v_4 \otimes \ldots \otimes v_q \\ \cdot & e_{s_1} x e_{s_1+1} \otimes (e_p \otimes e_1) \otimes (\xi \otimes e_1) \otimes e_{t_4} \otimes \ldots \otimes e_{t_q} \end{array}$$

since $e_3 \otimes x\xi \otimes e_1$ is the product of j-degree -2 element $e_3 \otimes x$ with j-degree 0 element $\xi \otimes e_1$.

If $v_2 = e_p \otimes xe_2$, we can write the following decomposition

$$\begin{array}{rl} & e_{s_1} x e_{s_1+1} \otimes e_{p-1}x \otimes e_1 \otimes (e_3 \otimes x\xi \otimes e_1) \otimes v_4 \otimes \ldots \otimes v_q \\ = & e_{s_1} x e_{s_1+1} \otimes (e_p \otimes e_1) \otimes (e_3 \otimes \xi) \otimes e_{s_4} \otimes \ldots \otimes e_{s_q} \\ \cdot & e_{s_1+1} \otimes e_1 x e_2 \otimes (x \otimes e_1) \otimes v_4 \otimes \ldots \otimes v_q \end{array}$$

since $e_3 \otimes x\xi \otimes e_1$ is the product of j-degree 0 element $e_3 \otimes \xi$ with j-degree -2 element $x \otimes e_1$.

Therefore v_3 must be of the form $e_{s_3} w e_{s_3}$ for v to be irreducible. By Corollary 4.4.13, $v_n = e_{s_n}$ for all $4 \leq n \leq q$ and v is of the form

$$e_{s_1} x e_{s_1+1} \otimes v_2 \otimes e_{s_3} w e_{s_3} \otimes e_{s_4} \otimes \ldots \otimes e_{s_q}.$$

By Remark 4.2.10, we know that $e_s w e_s$ is not irreducible if and only if $s = 1$ or $s = p$, in which case it can be more conveniently written

$$e_1 w e_1 = e_1 \xi \otimes \xi \otimes e_1,$$

and

$$e_p w e_p = -e_p \otimes \xi \otimes \xi e_p.$$

We will ignore the sign of $e_p w e_p$ for convenience. Assume that $s_3 = 1$. Then, if $v_2 = e_{p-1}x \otimes e_1$, we can write the following decomposition

$$\begin{array}{rl} & e_{s_1} x e_{s_1+1} \otimes (e_{p-1}x \otimes e_1) \otimes e_{s_3} w e_{s_3} \otimes e_{s_4} \otimes \ldots \otimes e_{s_q} \\ = & e_{s_1} \otimes e_{p-1} x e_p \otimes (e_1 \xi \otimes \xi e_p) \otimes \left(e_{s_4} \otimes e^*_{p+1-s_4}\right) \otimes \ldots \otimes \left(e_{s_q} \otimes e^*_{p+1-s_q}\right) \\ \cdot & e_{s_1} x e_{s_1+1} \otimes (e_p \otimes e_1) \otimes (e_p \otimes e_1) \otimes \tilde{v}_4 \otimes \ldots \otimes \tilde{v}_q \end{array}$$

and if $v_2 = e_p \otimes xe_2$, we can write the following decomposition

$$\begin{aligned} & e_{s_1}xe_{s_1+1} \otimes (e_p \otimes xe_2) \otimes e_{s_3}we_{s_3} \otimes e_{s_4} \otimes \ldots \otimes e_{s_q} \\ = \; & e_{s_1}xe_{s_1+1} \otimes (e_p \otimes e_1) \otimes (e_1\xi \otimes \xi e_p) \otimes \left(e_{s_4} \otimes e^*_{p+1-s_4}\right) \otimes \ldots \otimes \left(e_{s_q} \otimes e^*_{p+1-s_q}\right) \\ \cdot \; & e_{s_1+1} \otimes e_1xe_2 \otimes (e_p \otimes e_1) \otimes \tilde{v}_4 \otimes \ldots \otimes \tilde{v}_q. \end{aligned}$$

Assume now that $s_3 = p$. If $v_2 = e_{p-1}x \otimes e_1$, we can write the following decomposition

$$\begin{aligned} & e_{s_1}xe_{s_1+1} \otimes (e_{p-1}x \otimes e_1) \otimes e_{s_3}we_{s_3} \otimes e_{s_4} \otimes \ldots \otimes e_{s_q} \\ = \; & e_{s_1} \otimes e_{p-1}xe_p \otimes (e_p \otimes e_1) \otimes \tilde{v}_4 \otimes \ldots \otimes \tilde{v}_q \\ \cdot \; & e_{s_1}xe_{s_1+1} \otimes (e_p \otimes e_1) \otimes (e_1\xi \otimes \xi e_p) \otimes \left(e_{p+1-s_4} \otimes e^*_{s_4}\right) \otimes \ldots \otimes \left(e_{p+1-s_q} \otimes e^*_{s_q}\right) \end{aligned}$$

and if $v_2 = e_p \otimes xe_2$, we can write the following decomposition

$$\begin{aligned} & e_{s_1}xe_{s_1+1} \otimes (e_p \otimes xe_2) \otimes e_{s_3}we_{s_3} \otimes e_{s_4} \otimes \ldots \otimes e_{s_q} \\ = \; & e_{s_1}xe_{s_1+1} \otimes (e_p \otimes e_1) \otimes (e_p \otimes e_1) \otimes \tilde{v}_4 \otimes \ldots \otimes \tilde{v}_q \\ \cdot \; & e_{s_1+1} \otimes e_1xe_2 \otimes (e_1\xi \otimes \xi e_p) \otimes \left(e_{p+1-s_4} \otimes e^*_{s_4}\right) \otimes \ldots \otimes \left(e_{p+1-s_q} \otimes e^*_{s_q}\right). \end{aligned}$$

These decompositions are possible if and only if for all $4 \leq n \leq q$, $\left(e_{s_n} \otimes e^*_{p+1-s_n}\right) \cdot \tilde{v}_n = e_{s_n}$ if $s_3 = 1$ or $\tilde{v}_n \cdot \left(e_{p+1-s_n} \otimes e^*_{s_n}\right) = e_{s_n}$ if $s_3 = p$. By Table 2.2, we know that type **1** elements of the form $\mathbf{3} \cdot \mathbf{1}$ or $\mathbf{1} \cdot \mathbf{3}$ must satisfy $s_n = 1$ if $s_3 = 1$, or $s_n = p$ if $s_3 = p$. Thus, if there exists $4 \leq n \leq q$ such that $s_n \neq 1$ if $s_3 = 1$, that decomposition fails and v is irreducible. Similarly, if there exists $4 \leq l \leq q$ such $s_n \neq p$ if $s_3 = p$, v is irreducible. □

All the previous results can be summarised in the proposition below.

Proposition 4.5.12 (Irreducible monomials starting with x)**.** *Let $v = v_1 \otimes \ldots \otimes v_q$ be an irreducible monomial of $\mathbf{w}_q$ such that $v_1 = e_{s_1}xe_{s_1+1}$ for some $1 \leq s_1 \leq p-1$. Then v is* *of the form*

$$x \otimes (e_p \otimes e_1)^{\otimes n} \otimes \left(e_{s_{n+2}}\xi \otimes \xi e_{p+1-s_{n+2}}\right) \otimes \bigotimes_{l=n+3}^{q} (e_{s_l} \otimes e^*_{p+1-s_l}) \qquad 0 \le n \le q-2,\ 1 \le s_{n+2} \le p-2;$$

$$x \otimes (e_p \otimes e_1)^{\otimes q-1}$$

$$x \otimes (e_p \otimes e_1)^{\otimes n} \otimes (\xi \otimes e_1) \otimes \bigotimes_{l=3+n}^{q} e_{s_l} \qquad 0 \le n \le q-3,\ \exists l \ge 3+n\ s.t.\ s_l \ne 1;$$

$$x \otimes (e_p \otimes e_1)^{\otimes n} \otimes (e_p \otimes \xi) \otimes \bigotimes_{l=3+n}^{q} e_{s_l} \qquad 0 \le n \le q-3,\ \exists l \ge 3+n\ s.t.\ s_l \ne p;$$

$$x \otimes (e_p \otimes e_1)^{\otimes n} \otimes (x \otimes e_1) \otimes e_{s_{3+n}} w e_{s_{3+n}} \otimes \bigotimes_{l=4+n}^{q} e_{s_l} \qquad 0 \le n \le q-3,\ s_{3+n} \ne 1, p;$$

$$x \otimes (e_p \otimes e_1)^{\otimes n} \otimes (x \otimes e_1) \otimes e_1 w e_1 \otimes \bigotimes_{l=4+n}^{q} e_{s_l} \qquad 0 \le n \le q-4,\ \exists l \ge 4+n\ s.t.\ s_l \ne 1;$$

$$x \otimes (e_p \otimes e_1)^{\otimes n} \otimes (x \otimes e_1) \otimes e_p w e_p \otimes \bigotimes_{l=4+n}^{q} e_{s_l} \qquad 0 \le n \le q-4,\ \exists l \ge 4+n\ s.t.\ s_l \ne p;$$

$$x \otimes (e_p \otimes e_1)^{\otimes n} \otimes (e_p \otimes x) \otimes e_{s_{3+n}} w e_{s_{3+n}} \otimes \bigotimes_{l=4+n}^{q} e_{s_l} \qquad 0 \le n \le q-3,\ s_{3+n} \ne 1, p;$$

$$x \otimes (e_p \otimes e_1)^{\otimes n} \otimes (e_p \otimes x) \otimes e_{s_1} w e_1 \otimes \bigotimes_{l=4+n}^{q} e_{s_l} \qquad 0 \le n \le q-4,\ \exists l \ge 4+n\ s.t.\ s_l \ne 1;$$

$$x \otimes (e_p \otimes e_1)^{\otimes n} \otimes (e_p \otimes x) \otimes e_{s_p} w e_p \otimes \bigotimes_{l=4+n}^{q} e_{s_l} \qquad 0 \le n \le q-4,\ \exists l \ge 4+n\ s.t.\ s_l \ne p.$$

Proof. For $v_1 = e_{s_1} x e_{s_1+1}$ with $1 \le s_1 \le p-1$, an element of i-degree -1 must follow. So by Lemma 4.5.1, we know v_2 is one of

$$\begin{aligned}
(p-1,\ -1,\ 1,\ 0,\ 1,\ 0,\ 2) &= e_{p-1}\xi \otimes \xi e_2\\
(s,\ -1,\ 1,\ 0,\ 0,\ 1,\ p+1-s) &= e_s\xi \otimes \xi e_{p+1-s}\\
(p,\ -1,\ 0,\ 0,\ 1,\ 0,\ 2) &= e_p \otimes \xi e_2\\
(p-1,\ -1,\ 0,\ 0,\ 1,\ 0,\ 1) &= e_{p-1}\xi \otimes e_1\\
(p,\ -1,\ -1,\ 0,\ 1,\ 0,\ 1) &= e_p \otimes e_1\\
(p-1,\ -1,\ -1,\ 1,\ 1,\ 0,\ 2) &= e_{p-1}\xi \otimes x e_2 \cong e_{p-1}x \otimes \xi e_2\\
(p,\ -1,\ -2,\ 1,\ 1,\ 0,\ 2) &= e_p \otimes x e_2\\
(p-1,\ -1,\ -2,\ 1,\ 1,\ 0,\ 1) &= e_{p-1}x \otimes e_1.
\end{aligned}$$

By Lemma 4.5.6, v_2 cannot be $e_{p-1}\xi \otimes \xi e_2$ as v would then be reducible.

By Lemma 4.5.7, v_2 can be $e_{s_2}\xi \otimes e_{p+1-s_2}$ for $1 \le s_2 \le p-2$, and v is of the form

$$e_{s_1} x e_{s_1+1} \otimes (e_{s_2}\xi \otimes \xi e_{p+1-s_2}) \otimes (e_{s_3} \otimes e^*_{p+1-s_3}) \otimes \dots (e_{s_q} \otimes e^*_{p+1-s_q}).$$

By Lemma 4.5.8, v_2 can be $e_p \otimes \xi e_2$ as long as there exists $3 \leq l \leq q$ such that $v_l \neq e_p$; v is then of the form

$$e_{s_1} x e_{s_1+1} \otimes (e_p \otimes \xi e_2) \otimes e_{s_3} \otimes \ldots \otimes e_{s_q}.$$

By Lemma 4.5.9, v_2 can be $e_{p-1}\xi \otimes e_1$ as long as there exists $3 \leq l \leq q$ such that $v_l \neq e_1$; v is then of the form

$$e_{s_1} x e_{s_1+1} \otimes (e_{p-1}\xi \otimes e_1) \otimes e_{s_3} \otimes \ldots \otimes e_{s_q}.$$

By Lemma 4.5.8, v_2 cannot be $e_{p-1}\xi \otimes x e_2 \cong e_{p-1} x \otimes \xi e_2$ as v would then be reducible.

By Lemma 4.5.11, v_2 can be $e_p \otimes x e_2$ or $e_{p-1} x \otimes e_1$ as long as there exists $4 \leq l \leq q$ such that $v_l \neq e_1$ if $s_3 = 1$, and such that $v_l \neq e_p$ if $s_3 = p$; v is then of the form

$$e_{s_1} x e_{s_1+1} \otimes v_2 \otimes e_{s_3} w e_{s_3} \otimes e_{s_4} \otimes \ldots \otimes e_{s_q}.$$

Finally, we need to address the case when $v_2 = e_p \otimes e_1$. Assume it is the case. Since that element has j-degree -1, by Lemma 4.5.1 again, we know that v_3 must be one of

$$\begin{array}{rrrrrrll}
(p-1, & -1, & 1, & 0, & 1, & 0, & 2) & = e_{p-1}\xi \otimes \xi e_2 \\
(s, & -1, & 1, & 0, & 0, & 1, & p+1-s) & = e_s\xi \otimes \xi e_{p+1-s} \\
(p, & -1, & 0, & 0, & 1, & 0, & 2) & = e_p \otimes \xi e_2 \\
(p-1, & -1, & 0, & 0, & 1, & 0, & 1) & = e_{p-1}\xi \otimes e_1 \\
(p, & -1, & -1, & 0, & 1, & 0, & 1) & = e_p \otimes e_1 \\
(p-1, & -1, & -1, & 1, & 1, & 0, & 2) & = e_{p-1}\xi \otimes x e_2 \cong e_{p-1} x \otimes \xi e_2 \\
(p, & -1, & -2, & 1, & 1, & 0, & 2) & = e_p \otimes x e_2 \\
(p-1, & -1, & -2, & 1, & 1, & 0, & 1) & = e_{p-1} x \otimes e_1.
\end{array}$$

Considering all the decompositions provided in the previous proofs of the results concerning irreducible monomials starting with x, we see that they all are of the same form, namely

$$\begin{array}{rl}
 & e_{s_1} x e_{s_1+1} \otimes v_2 \otimes v_3 \otimes \ldots \otimes v_q \\
= & e_{s_1} x e_{s_1+1} \otimes \tilde{v}_2 \otimes \tilde{v}_3 \otimes \ldots \otimes \tilde{v}_q \\
\cdot & e_{s_1+1} \otimes \hat{v}_2 \otimes \hat{v}_3 \otimes \ldots \otimes \hat{v}_q
\end{array}$$

or

$$\begin{array}{rl}
 & e_{s_1} x e_{s_1+1} \otimes v_2 \otimes v_3 \otimes \ldots \otimes v_q \\
= & e_{s_1} \otimes \hat{v}_2 \otimes \hat{v}_3 \otimes \ldots \otimes \hat{v}_q \\
\cdot & e_{s_1} x e_{s_1+1} \otimes \tilde{v}_2 \otimes \tilde{v}_3 \otimes \ldots \otimes \tilde{v}_q
\end{array}$$

such that $\hat{v}_2$ is an element of $\mathbf{d}$, and $\tilde{v}_2$ is an element of $\mathbb{H}(\mathbf{u}^{-1})$. Informally, we could append the element $e_p \otimes e_1$ in position 2 and still obtain a valid decomposition, namely

$$\begin{array}{rl}
 & e_{s_1} x e_{s_1+1} \otimes (\mathbf{e_p} \otimes \mathbf{e_1}) \otimes v_2 \otimes v_3 \otimes \ldots \otimes v_q \\
= & e_{s_1} x e_{s_1+1} \otimes (\mathbf{e_p} \otimes \mathbf{e_1}) \otimes \tilde{v}_2 \otimes \tilde{v}_3 \otimes \ldots \otimes \tilde{v}_q \\
\cdot & e_{s_1+1} \otimes \mathbf{e_1} \otimes \hat{v}_2 \otimes \hat{v}_3 \otimes \ldots \otimes \hat{v}_q
\end{array}$$

or

$$\begin{array}{rl}
 & e_{s_1} x e_{s_1+1} \otimes (\mathbf{e_p} \otimes \mathbf{e_1}) \otimes v_2 \otimes v_3 \otimes \ldots \otimes v_q \\
= & e_{s_1} \otimes \mathbf{e_p} \otimes \hat{v}_2 \otimes \hat{v}_3 \otimes \ldots \otimes \hat{v}_q \\
\cdot & e_{s_1} x e_{s_1+1} \otimes (\mathbf{e_p} \otimes \mathbf{e_1}) \otimes \tilde{v}_2 \otimes \tilde{v}_3 \otimes \ldots \otimes \tilde{v}_q
\end{array}$$

as $e_p \otimes e_1$ has j-degree -1.

Furthermore, we see that we can append as many $e_p \otimes e_1$'s as we want (as long as we are still in the bounds regarding the length of the element), and the decompositions and results will go through. One just needs to adjust the indices upon which the results rely. Hence, if $v_l = e_p \otimes e_1$ for all $2 \leq l \leq n+1$, we have

- by Lemma 4.5.6, v_{n+2} cannot be $e_{p-1}\xi \otimes \xi e_2$ as v would then be reducible;
- by Lemma 4.5.7, v_{n+2} can be $e_{s_{n+2}}\xi \otimes e_{p+1-s_{n+2}}$ for $1 \leq s_{n+2} \leq p-2$, and v is of the form
$$e_{s_1}xe_{s_1+1}\otimes(e_p \otimes e_1)^{\otimes n}\otimes\big(e_{s_{n+2}}\xi \otimes \xi e_{p+1-s_{n+2}}\big)\otimes(e_{s_{n+3}}\otimes e^*_{p+1-s_{n+3}})\otimes\dots(e_{s_q}\otimes e^*_{p+1-s_q});$$
- by Lemma 4.5.8, v_{n+2} can be $e_p \otimes \xi e_2$ as long as there exists $n+3 \leq l \leq q$ such that $v_l \neq e_p$; v is then of the form
$$e_{s_1}xe_{s_1+1} \otimes (e_p \otimes e_1)^{\otimes n} \otimes (e_p \otimes \xi e_2) \otimes e_{s_{n+3}} \otimes \dots \otimes e_{s_q};$$
- by Lemma 4.5.9, v_{n+2} can be $e_{p-1}\xi \otimes e_1$ as long as there exists $n+3 \leq l \leq q$ such that $v_l \neq e_1$; v is then of the form
$$e_{s_1}xe_{s_1+1} \otimes (e_p \otimes e_1)^{\otimes n} \otimes (e_{p-1}\xi \otimes e_1) \otimes e_{s_{n+3}} \otimes \dots \otimes e_{s_q};$$
- by Lemma 4.5.8, v_{n+2} cannot be $e_{p-1}\xi \otimes xe_2 \cong e_{p-1}x \otimes \xi e_2$ as v would then be reducible;
- by Lemma 4.5.11, v_{n+2} can be $e_p \otimes xe_2$ or $e_{p-1}x \otimes e_1$ as long as there exists $n+4 \leq l \leq q$ such that $v_l \neq e_1$ if $s_{n+3} = 1$, and such that $v_l \neq e_p$ if $s_{n+3} = p$; v is then of the form
$$e_{s_1}xe_{s_1+1} \otimes (e_p \otimes e_1)^{\otimes n} \otimes v_{n+2} \otimes e_{s_{n+3}}we_{s_{n+3}} \otimes e_{s_{n+4}} \otimes \dots \otimes e_{s_q}.$$

Finally, since $e_p \otimes e_1$ is irreducible, we see that
$$e_{s_1}xe_{s_1+1} \otimes (e_p \otimes e_1) \otimes \dots \otimes (e_p \otimes e_1)$$
is irreducible. The description is now complete. □

4.6 The quiver of $\mathbf{w}_q$

4.6.1 Description of V_q

We now gather all the results obtained so far.

Theorem 4.6.1. *The new arrows for the quiver of $\mathbf{w}_q$ are of the form*

$$e_{s_1}\xi e_{s_1+1} \otimes \bigotimes_{l=2}^{q} \left(e_{s_l} \otimes e^*_{p+1-s_l}\right) \qquad 1 \leq s_1 \leq p-1;$$

$$e_{s_1}xe_{s_1+1} \otimes (e_p \otimes e_1)^{\otimes n} \otimes \left(e_{s_{n+2}}\xi \otimes \xi e_{p+1-s_{n+2}}\right) \otimes \bigotimes_{l=n+3}^{q} \left(e_{s_l} \otimes e^*_{p+1-s_l}\right) \qquad 1 \leq s_1 \leq p-1,\ 0 \leq n \leq q-2,\ 1 \leq s_{n+2} \leq p-2;$$

$$e_{s_1}xe_{s_1+1} \otimes (e_p \otimes e_1)^{\otimes q-1} \qquad 1 \leq s_1 \leq p-1;$$

$$e_{s_1}xe_{s_1+1} \otimes (e_p \otimes e_1)^{\otimes n} \otimes (\xi \otimes e_1) \otimes \bigotimes_{l=3+n}^{q} e_{s_l} \qquad 1 \leq s_1 \leq p-1,\ 0 \leq n \leq q-3,\ \exists l \geq 3+n \ s.t.\ s_l \neq 1;$$

$$e_{s_1}xe_{s_1+1} \otimes (e_p \otimes e_1)^{\otimes n} \otimes (e_p \otimes \xi) \otimes \bigotimes_{l=3+n}^{q} e_{s_l} \qquad 1 \leq s_1 \leq p-1,\ 0 \leq n \leq q-3,\ \exists l \geq 3+n \ s.t.\ s_l \neq p;$$

$$e_{s_1}xe_{s_1+1} \otimes (e_p \otimes e_1)^{\otimes n} \otimes (x \otimes e_1) \otimes e_{s_{3+n}}we_{s_{3+n}} \otimes \bigotimes_{l=4+n}^{q} e_{s_l} \qquad 1 \leq s_1 \leq p-1,\ 0 \leq n \leq q-3,\ s_{3+n} \neq 1, p;$$

$$e_{s_1}xe_{s_1+1} \otimes (e_p \otimes e_1)^{\otimes n} \otimes (x \otimes e_1) \otimes e_1we_1 \otimes \bigotimes_{l=4+n}^{q} e_{s_l} \qquad 1 \leq s_1 \leq p-1,\ 0 \leq n \leq q-4,\ \exists l \geq 4+n \ s.t.\ s_l \neq 1;$$

$$e_{s_1}xe_{s_1+1} \otimes (e_p \otimes e_1)^{\otimes n} \otimes (x \otimes e_1) \otimes e_pwe_p \otimes \bigotimes_{l=4+n}^{q} e_{s_l} \qquad 1 \leq s_1 \leq p-1,\ 0 \leq n \leq q-4,\ \exists l \geq 4+n \ s.t.\ s_l \neq p;$$

$$e_{s_1}xe_{s_1+1} \otimes (e_p \otimes e_1)^{\otimes n} \otimes (e_p \otimes x) \otimes e_{s_{3+n}}we_{s_{3+n}} \otimes \bigotimes_{l=4+n}^{q} e_{s_l} \qquad 1 \leq s_1 \leq p-1,\ 0 \leq n \leq q-3,\ s_{3+n} \neq 1, p;$$

$$e_{s_1}xe_{s_1+1} \otimes (e_p \otimes e_1)^{\otimes n} \otimes (e_p \otimes x) \otimes e_1we_1 \otimes \bigotimes_{l=4+n}^{q} e_{s_l} \qquad 1 \leq s_1 \leq p-1,\ 0 \leq n \leq q-4,\ \exists l \geq 4+n \ s.t.\ s_l \neq 1;$$

$$e_{s_1}xe_{s_1+1} \otimes (e_p \otimes e_1)^{\otimes n} \otimes (e_p \otimes x) \otimes e_pwe_p \otimes \bigotimes_{l=4+n}^{q} e_{s_l} \qquad 1 \leq s_1 \leq p-1,\ 0 \leq n \leq q-4,\ \exists l \geq 4+n \ s.t.\ s_l \neq p;$$

$$e_{s_1}x\xi e_{s_1+2} \otimes \bigotimes_{l=2}^{q} e_{s_l} \qquad 1 \leq s_1 \leq p-2,\ (s_2, \dots, s_q) \notin S;$$

$$e_{s_1}x^2e_{s_1+2} \otimes e_{s_2}we_{s_2} \otimes \bigotimes_{l=3}^{q} e_{s_l} \qquad 1 \leq s_1 \leq p-2,\ 2 \leq s_2 \leq p-1;$$

$$e_{s_1}x^2e_{s_1+2} \otimes e_1we_1 \otimes \bigotimes_{l=3}^{q} e_{s_l} \qquad 1 \leq s_1 \leq p-2,\ \exists l \geq 3 \ s.t.\ s_l \neq 1;$$

$$e_{s_1}x^2e_{s_1+2} \otimes e_pwe_p \otimes \bigotimes_{l=3}^{q} e_{s_l} \qquad 1 \leq s_1 \leq p-2,\ \exists l \geq 3 \ s.t.\ s_l \neq p;$$

where $1 \leq s_l \leq p$ (unless otherwise stated) and $S := \{(1, \dots, 1), (p, \dots, p)\}$.

Remark 4.6.2 (On the value of q and irreducibility). Suppose $q = 2$. Note that the element

$$x^2 \otimes e_1 w e_1 = x \otimes (e_1\xi \otimes \xi e_p) \cdot x \otimes (e_p \otimes e_1)$$

is not irreducible, but the element

$$x^2 \otimes e_s w e_s$$

is irreducible for all $2 \leq s \leq p-1$. We see that going to $q = 3$ and appending idempotent e_2 to the element $x^2 \otimes e_1 w e_1$ makes the resulting element

$$x^2 \otimes e_1 w e_1 \otimes e_2$$

irreducible.

Corollary 4.6.3. *The algebra $\mathbf{w}_q$ is generated in degrees 0, 1 and 2.*

4.6.2 Quiver of $\mathbf{w}_q$

Similarly to the case $p = 2$, we can view V_q as the quiver of $\mathbf{w}_q$. Since $\mathbf{w}_q$ is the extension algebra of the standard modules of an algebra of finite global dimension, it is finite-dimensional. By definition of V_q, we can write $\mathbf{w}_q$ as the quotient of a tensor algebra by some ideal, namely

$$\mathbf{w}_q \cong T_B V_q / \mathcal{I},$$

where the tensor product is taken over the semi-simple algebra B made up by the idempotents of $\mathbf{w}_q$.

Now, V_q is a (finite) subset of monomial basis elements of $\mathbf{w}_q$ which is a multiplicative basis for $\mathbf{w}_q$. Since $\mathbf{w}_q$ is multigraded, $\mathcal{I}$ must be homogeneous with respect to that $(i,j,k,a,b)^q$-grading. Since an element $z \in V_q$ is uniquely determined by its $(i,j,k,a,b)^q$-degree (together with idempotents on the left and on the right), we obtain that $\mathcal{I}$ cannot contain any element of V_q: let $v_1 + \ldots + v_s \in \mathcal{I}$, with v_j's words in elements of V_q; then all v_j's have the same $(i,j,k,a,b)^q$-degree since $\mathcal{I}$ is homogeneous. In particular, at most one v_j is in V_q. This is a contradiction as we would obtain a linear relation between basis elements of $\mathbf{w}_q$. Therefore, all v_j's are words in at least two elements of V_q, i.e. $\mathcal{I} \subset V_q \otimes_B V_q$. In addition, since $\mathbf{w}_q$ is finite-dimensional, there cannot be words in V_q of infinite length. Thus, there exists $N > 2$ such that

$$V_q^{\otimes_B N} \subset \mathcal{I} \subset V_q \otimes_B V_q,$$

so that $\mathcal{I}$ is admissible.

We can therefore interpret V_q as the quiver of $\mathbf{w}_q$. We see that the vertices are given by the simples of $\mathbf{w}_q$ and the set of arrows of the quiver corresponds to V_q.

Example 4.6.4. To illustrate that section, we give the quiver of $\mathbf{w}_2$ for $p = 3$ in Figure 4.1.

The labels correspond to the following irreducible monomials of $\mathbf{w}_2$:

$$\begin{aligned}
a_i &= e_i \otimes e_1\xi e_2 \\
b_i &= e_i \otimes e_2\xi e_3 \\
c_i &= e_1\xi e_2 \otimes (e_{4-i} \otimes e_i^*) \\
d_i &= e_2\xi e_3 \otimes (e_{4-i} \otimes e_i^*) \\
\alpha_i &= e_i \otimes e_1 x e_2 \\
\beta_i &= e_i \otimes e_2 x e_3 \\
\gamma_1 &= e_1 x e_2 \otimes (e_1\xi \otimes \xi e_3) \\
\gamma_3 &= e_1 x e_2 \otimes (e_3 \otimes e_1) \\
\delta_1 &= e_2 x e_3 \otimes (e_3 \otimes e_1) \\
\delta_3 &= e_2 x e_3 \otimes (e_1\xi \otimes \xi e_3) \\
\mu &= e_1 x\xi e_3 \otimes e_2 \\
\nu &= e_1 x^2 e_3 \otimes e_2 w e_2.
\end{aligned}$$

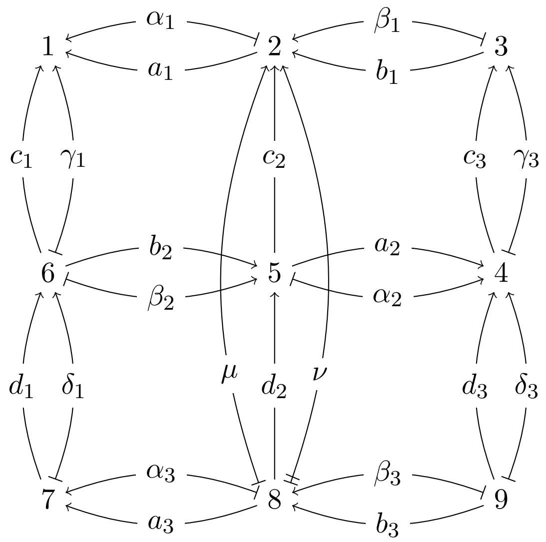


Figure 4.1: Quiver of $\mathbf{w}_2$ for $p = 3$

Chapter 5

A_∞-algebras

This chapter relies on [Kel99], [Lef03] and [Amo12].

5.1 First definitions

5.1.1 Motivations

Let F be a field. The following problem constitute a motivation for the study of the A_∞-algebra structure of the extension algebra of Weyl modules of the principal block of rational representations of $GL_2(\overline{\mathbb{F}_p})$.

Let A be an associative unital F-algebra and let $\Delta_1, \Delta_2, \ldots, \Delta_n$ be A-modules. Denote by $\mathcal{F}(\Delta)$ the full subcategory of the category of left A-modules whose objects admit finite filtrations with subquotients among the Δ_i. We can describe that category as the closure under extensions of the Δ_i's. One can ask if $\mathcal{F}(\Delta)$ is determined by the extension algebra

$$\mathrm{Ext}^*_A(\Delta, \Delta), \text{where } \Delta = \oplus_{1 \le i \le n} \Delta_i.$$

In particular, how can one reconstruct the category of iterated extensions of Δ_i's from $\mathrm{Ext}^*_A(\Delta, \Delta)$?

In [Kel99], Keller shows that one can do so if the A_∞-structure on the extension algebra is known.

5.1.2 Definition of an A_∞-algebra

Definition 5.1.1. [Kel99] Let F be a field. An *A_∞-algebra over F* is the datum of a $\mathbb{Z}$-graded vector space

$$A = \bigoplus_{p \in \mathbb{Z}} A^p$$

together with graded maps (i.e. homogeneous F-linear maps)

$$m_n : A^{\otimes n} \to A, n \ge 1,$$

of degree $2 - n$ satisfying the following relation for all $n \ge 1$

$$\sum_{s=1}^{n} \sum_{r=0}^{n-s} (-1)^{r+s(n-r-s)} m_{n-s+1}(\mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{\otimes n-r-s}) = 0.$$

We call these relations *Stasheff identities* of order n.

Remark 5.1.2 (About signs). In the definition, the sign convention adopted by Getzler-Jones [GJ+90] is used. In addition, note that when formulae are applied to elements, additional signs appear because of the Koszul sign rule:

$$(f \otimes g)(x \otimes y) = (-1)^{|x||g|} f(x) \otimes g(y),$$

and

$$(f \otimes g)(h \otimes k) = (-1)^{|h||g|} f \circ h \otimes g \circ k,$$

if f, g, h, k are homogeneous F-linear maps and x, y homogeneous elements whose degree is denoted between vertical bars, e.g. $|f|$ for the degree of f.

For small values of $n \geq 1$, the Stasheff identities give:

- For $n = 1$, we have $m_1 m_1 = 0$, which means that (A, m_1) is a differential complex;
- For $n = 2$, we have
$$m_1 m_2 = m_2(m_1 \otimes \mathbf{1} + \mathbf{1} \otimes m_1)$$
so m_1 is a graded derivation with respect to the multiplication m_2. We have indeed:
$$\begin{aligned} m_1 m_2(x \otimes y) &= m_2(m_1 \otimes \mathbf{1} + \mathbf{1} \otimes m_1)(x \otimes y) \\ &= m_1(x) \otimes y + (-1)^{|m_1||x|} x \otimes m_1(y) \end{aligned}$$
and since $|m_1| = 1$, the map m_1 is then the usual differential on the tensor product;
- For $n = 3$, we have
$$m_2(\mathbf{1} \otimes m_2 - m_2 \otimes \mathbf{1}) = m_1 m_3 + m_3(m_1 \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes m_1 \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes m_1),$$
so that m_2 is associative up to a homotopy given by m_3.

The definition yields three immediate consequences:

(i) In general, an A_∞-algebra is not associative, but its homology with respect to m_1, $H^*(A)$, is an associative graded algebra for the multiplication induced by m_2.

(ii) If $A^l = 0$ for all $l \neq 0$, then A is concentrated in degree 0 and is an ordinary associative algebra. Indeed, since m_n is of degree $2 - n$, all m_n other than m_2 have to vanish.

(iii) If m_n vanishes for all $n \geq 3$, then A is an associative differential $\mathbb{Z}$-graded algebra (or dg-algebra) and conversely, each dg-algebra yields an A_∞-algebra with $m_n = 0$ for all $n \geq 3$.

5.1.3 Morphisms of A_∞-algebras

Definition 5.1.3. [Kel99] *A morphism of* A_∞*-algebras* $f : A \to B$ is a family

$$f_n : A^{\otimes n} \to B$$

of graded maps of degree $n - 1$ such that for all $n \geq 1$

$$\sum_{s=1}^{n} \sum_{r=0}^{n-s} (-1)^{\#} f_{n-s+1}(\mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{\otimes n-r-s}) = \sum_{r=1}^{n} \sum_{i_1+\ldots+i_r=n} (-1)^{*} m_r(f_{i_1} \otimes \cdots \otimes f_{i_r}),$$

where $\# = r + s(n - r - s)$ and $* = \sum_{l=1}^{r-1} (r - l)(i_l - 1)$.

As for the multiplication maps in the definition of an A$_\infty$-algebra, we write down the relations defining morphisms of A$_\infty$-algebras for small values of $n \geq 1$:

- For $n = 1$, we have $f_1 m_1 = m_1 f_1$, namely f_1 is morphism of complexes;
- For $n = 2$, we have

$$f_1 m_2 = m_2(f_1 \otimes f_1) + m_1 f_2 + f_2(m_1 \otimes \mathbf{1} + \mathbf{1} \otimes m_1)$$

  so f_1 commutes with the multiplication m_2 up to a homotopy given by f_2.

Definition 5.1.4. We define *the composition of two morphisms* $f : B \to C$ *and* $g : A \to B$ by

$$(f \circ g)_n = \sum_{r=1}^{n} \sum_{i_1+\ldots+i_r=n} (-1)^* f_r \circ (g_{i_1} \otimes \cdots \otimes g_{i_r}),$$

where $* = \sum_{l=1}^{r-1}(r-l)(i_l - 1)$.

Definition 5.1.5. We say that a morphism $f = (f_n)_{n\geq 1}$ is:

1. a *quasi-isomorphism* if f_1 is a quasi-isomorphism;
2. *strict* if $f_i = 0$ for all $i \neq 1$.

5.2 Minimal models and formality

We now give an important result in the field of A$_\infty$-algebras which is important for our task of computing an A$_\infty$-algebra structure.

Theorem 5.2.1 (Kadeishvili [Kad80],[Kel99],[Lef03])**.** *Let A be an A_∞-algebra. Then the homology H^*A has an A_∞-algebra structure such that*

1. *we have $m_1 = 0$ and m_2 is induced by m_2^A;*
2. *there is a quasi-isomorphism of A_∞-algebras $H^*A \to A$ lifting the identity of H^*A.*

Moreover, this structure is unique up to (non unique) isomorphism of A_∞-algebras.

Proof. See in [Kad80] for instance. □

Definition 5.2.2. An A$_\infty$-algebra such that m_1 is identically zero is called *minimal.*

In the context of Theorem 5.2.1, we say that H^*A is a *minimal model* for the A$_\infty$-algebra A.

Definition 5.2.3. An A$_\infty$-algebra whose minimal model can be chosen such that all higher multiplications vanish, namely $m_n^{H^*} = 0$ for all $n \geq 3$ is called *formal.*

5.3 Multi-graded $\mathbf{A}_\infty$-algebras

Our task at hand relies on several multi-graded algebras and we see that we can easily extend the definitions of A_∞-algebras and morphims to a multi-graded setting. Furthermore, we also have an equivalent of Kadeishvili's Theorem in that setting.

Let F be a field. Let A be an $(d_1, d_2, \dots, d_N, k)$-graded dg-algebra, whose differential graded structure is taken with respect to the k-grading. We want to prove that its homology H^*A carries an A_∞-structure with respect to the k-grading and which is also $(d_1, d_2, \dots, d_N, k)$-graded. Let us make this statement more precise:

Definition 5.3.1. A $(d_1, d_2, \dots, d_N, k)$*-graded* A_∞*-algebra* A is the datum of an $(i_1, i_2, \dots, i_n, k)$-graded F-vector space together with graded maps

$$m_n : A^{\otimes n} \to A$$

of degree $(0, 0, \dots, 0, 2-n)$ satisfying Stasheff multiplication identities:

$$\textbf{SI(n)} \qquad \sum (-1)^{r+st} m_u(\mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{\otimes t}) = 0,$$

for all $n \geq 1$, where the sum runs over decompositions $n = r+s+t$ and we let $u = r+1+t$, where $r, t \geq 0$ and $s \geq 1$.

Note that the identity $\mathbf{SI(n)}$ has homogeneous degree $(0, \dots, 0, 3-n)$.

Similarly, we can extend the definition of A_∞-morphisms:

Definition 5.3.2. *A morphism of* $(d_1, d_2, \dots, d_N, k)$*-graded* A_∞*-algebras* $f : A \to B$ is a family

$$f_n : A^{\otimes n} \to B$$

of $(i_1, i_2, \dots, i_N, k)$-graded graded maps of degree $(0, 0, \dots, 0, 1-n)$ such that for all $n \geq 1$

$$\sum_{s=1}^{n} \sum_{r=0}^{n-s} (-1)^{\#} f_{n-s+1}(\mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{\otimes n-r-s}) = \sum_{r=1}^{n} \sum_{i_1+\ldots+i_r=n} (-1)^{*} m_r(f_{i_1} \otimes \cdots \otimes f_{i_r}),$$

where $\# = r + s(n-r-s)$ and $* = \sum_{l=1}^{r-1} (r-l)(i_l - 1)$.

Theorem 5.3.3 (Kadeishvili's Theorem analogue)**.** *Let A be an $(d_1, d_2, \dots, d_N, k)$-graded dg-algebra, whose differential graded structure is taken with respect to the k-grading. There is an A_∞-algebra structure on its homology H^*A with $m_1 = 0$ and m_2 induced by the multiplication on A, such that there is a quasi-isomorphism of A_∞-algebras $H^*A \to A$ lifting the identity of H^*A.*

Proof. We follow in essence the proof provided by Kadeishvili in [Kad80] and show it is compatible with the additional gradings.

We denote by δ the differential on A; it is in particular a graded F-linear map of degree $(0, \dots, 0, 1)$. We let m_2^A denote the multiplication map in A; it is a graded bilinear map of degree $(0, \dots, 0, 0)$.

We need to construct graded higher multiplications m_n on H^*A satisfying the Stasheff identities and a graded A_∞-morphism $f = (f_n)_n$ between H^*A and A which lifts the identity of H^*A such that the gradings $(d_1, \dots, d_N)$ are preserved by both the higher multiplications and the A_∞-morphism, by which we mean that they are of $(d_1, \dots, d_N)$-degree $(0, \dots, 0)$.

We will do this inductively. Let $i = 1$. We take $m_1 = 0$ of k-degree 1 and, since we are working over a field, we can define $f_1 : H^*A \to A$ to be an embedding of the homology into A, i.e. $\delta f_1 = 0$ and it has k-degree 0. In particular,

$$m_1 m_1 = 0,$$

and

$$f_1 m_1 = 0 = \delta f_1,$$

hence the A_∞-relations for multiplications and for morphisms are satisfied, and by definition, they preserve the gradings $(d_1, \ldots, d_N)$ and are of the right k-degree. In addition, f_1 lifts the identity of H^*A.

Suppose that m_n and f_n are constructed for $n \leq i-1$ in such a way that they satisfy the A_∞-relations. Let

$$\begin{aligned} U_n = & \sum_{i_1=1}^{n-1} (-1)^{i_1-1} m_2^A (f_{i_1} \otimes f_{n-i_1}) \\ + & \sum_{s=2}^{n-1} \sum_{r=0}^{n-s} (-1)^{\#+1} f_{n-s+1} (\mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{n-r-s}) \end{aligned}$$

where $\# = r + s(n-r-s)$. This is well defined as U_n only involves maps m_l and f_l such that $l \leq n \leq i-1$, and they exist by assumption. In particular, U_n preserves the $(d_1, \ldots, d_N)$ grading.

The defining relations for morphisms of multi-graded A_∞-algebras in Definition 5.3.2 write

$$\delta f_n = f_1 m_n - U_n,$$

or equivalently

$$U_n = f_1 m_n - \delta f_n.$$

Applying δ, we see

$$\delta U_n = \delta f_1 m_n - \delta\delta f_n,$$

and by definition of a differential and by definition of f_1, we know that $\delta^2 = 0$ and $\delta f_1 = 0$, so that $\delta U_n = 0$. This means that $U_n(a_1 \otimes \ldots \otimes a_n)$ is a cycle in A, so we can define $m_n(a_1 \otimes \ldots \otimes a_n)$ to be the class of $U_n(a_1 \otimes \ldots \otimes a_n)$; we have

$$f_1 m_n (a_1 \otimes \ldots \otimes a_n) = U_n (a_1 \otimes \ldots \otimes a_n)$$

as f_1 is an embedding of the homology into A. In addition, we see that m_n preserves the $(d_1, \ldots, d_N)$ grading and is of k-degree $2-n$.

Finally, we can define $f_n(a_1 \otimes \ldots \otimes a_n)$ to be an element of A which is a boundary for the difference $f_1 m_n - U_n$, and assuming $a_1, \ldots, a_n$ are generators of H^*A, we can extend f_n linearly for it to be a well-defined graded map. Again, defining f_n in such a way means that it is a $(d_1, \ldots, d_N, k)$-degree $(0, \ldots, 0, 1-n)$ map, and the maps m_n and f_n satisfy the relations for f to be a morphism of multi-graded A_∞-algebras.

We still need to check that the higher multiplications satisfy the Stasheff identity $\mathbf{SI(n+1)}$, i.e. the relation

$$\sum_{r=0}^{n} \sum_{t=0}^{n-r} (-1)^{r+(n+1-r-t)t} m_{r+1+t} (\mathbf{1}^{\otimes r} \otimes m_{n+1-r-t} \otimes \mathbf{1}^{\otimes t}) = 0.$$

Note that this is the first relation in which m_n appears; in $\mathbf{SI(n)}$, the terms containing m_n are $m_1 m_n$ and $\sum_{r=1}^{n-1} m_n (\mathbf{1}^{\otimes r} \otimes m_1 \otimes \mathbf{1}^{\otimes n-1-r})$, and since we chose m_1 to be identically zero, these terms vanish.

Due to the complexity of the formulas, we will show that m_3 satisfies **SI(4)**; the general case follows in the same manner after very lengthy computations using the different Stasheff identities for higher multiplications and morphisms of A_∞-algebras. Note that these relations preserve the $(d_1, \dots, d_N)$-grading.

Assume $n = 3$, the higher multiplication m_3 writes

$$m_3 = m_2(f_1 \otimes f_2 - f_2 \otimes f_1) + f_2(\mathbf{1} \otimes m_2 - m_2 \otimes \mathbf{1}),$$

and **SI(4)** is the relation

$$\begin{aligned} & m_3(m_2 \otimes \mathbf{1}^{\otimes 2} - \mathbf{1} \otimes m_2 \otimes \mathbf{1} + \mathbf{1}^{\otimes 2} \otimes m_2) \\ & -m_2(m_3 \otimes \mathbf{1} + \mathbf{1} \otimes m_3) \\ & = 0. \end{aligned}$$

We will call the first line of **SI(4)** expression A, and the second expression B. We have

$$\begin{array}{rllll} A = & m_2(f_1 m_2 \otimes f_2 & -f_2(m_2 \otimes \mathbf{1}) \otimes f_1) & +f_2(m_2 \otimes m_2 & -m_2(m_2 \otimes \mathbf{1}) \otimes \mathbf{1}) \\ & -m_2(f_1 \otimes f_2(m_2 \otimes \mathbf{1}) & -f_2(\mathbf{1} \otimes m_2) \otimes f_1) & -f_2(\mathbf{1} \otimes m_2(m_2 \otimes \mathbf{1}) & -m_2(\mathbf{1} \otimes m_2) \otimes \mathbf{1}) \\ & +m_2(f_1 \otimes f_2(\mathbf{1} \otimes m_2) & -f_2 \otimes f_1 m_2) & +f_2(\mathbf{1} \otimes m_2(\mathbf{1} \otimes m_2) & -m_2 \otimes m_2) \\ = & m_2(f_2(\mathbf{1} \otimes m_2 & -m_2 \otimes \mathbf{1}) \otimes f_1) & +m_2(f_1 \otimes f_2(\mathbf{1} \otimes m_2 & -m_2 \otimes \mathbf{1})) \\ & +m_2(f_1 m_2 \otimes f_2) & -m_2(f_2 \otimes f_1 m_2) & & \end{array}$$

and

$$\begin{array}{rllll} B = & -m_2(m_2(f_1 \otimes f_2 & -f_2 \otimes f_1) \otimes \mathbf{1}) & +f_2(1 \otimes m_2 & -m_2 \otimes \mathbf{1}) \otimes \mathbf{1} \\ & -m_2(\mathbf{1} \otimes m_2(f_1 \otimes f_2 & -f_2 \otimes f_1) & +\mathbf{1} \otimes f_2(\mathbf{1} \otimes m_2 & -m_2 \otimes \mathbf{1})) \end{array}$$

so that, cancelling elements with opposite signs, $A + B$ gives

$$\begin{aligned} A + B = \ & m_2(f_1 m_2 \otimes f_2 - f_2 \otimes f_1 m_2 \\ & -m_2(m_2(f_1 \otimes f_2 - f_2 \otimes f_1) \otimes \mathbf{1} + \mathbf{1} \otimes m_2(f_1 \otimes f_2 - f_2 \otimes f_1)) \end{aligned}$$

Recalling the A_∞-morphism relations for $n = 2$, which, since $m_1 = 0$, is

$$f_1 m_2 = m_2(f_1 \otimes f_1)$$

we have

$$\begin{aligned} A + B = \ & m_2(m_2(f_1 \otimes f_1) \otimes f_2 - f_2 \otimes m_2(f_1 \otimes f_1) \\ & -m_2(m_2(f_1 \otimes f_2 - f_2 \otimes f_1) \otimes \mathbf{1} + \mathbf{1} \otimes m_2(f_1 \otimes f_2 - f_2 \otimes f_1)) \\ = \ & m_2(m_2 \otimes \mathbf{1} - \mathbf{1} \otimes m_2)(f_1 \otimes f_1 \otimes f_2 + f_1 \otimes f_2 \otimes f_1 + f_2 \otimes f_1 \otimes f_1) \end{aligned}$$

and by **SI(3)**, we have

$$m_2(m_2 \otimes \mathbf{1} - \mathbf{1} \otimes m_2) = 0,$$

which means $A + B = 0$ and **SI(4)** holds for m_3.

□

5.4 Tensor product of A_∞-algebras

5.4.1 A naive approach

Since the extension algebra we are interested in appears as a subalgebra in the tensor product of $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})$ with itself, it is natural to ask if the A_∞-structure of this tensor product can be described in terms of the A_∞-structure of the tensorands.

Considering that dg-algebras are a subclass of A_∞-algebras and that it is well-known how to tensor two such algebras, it looks like there is a natural way to define the tensor product of two A_∞-algebras.

Let (A, m_n^A) and (B, m_n^B) be two A_∞-algebras. Naively attempting to define an A_∞-structure on $A \otimes B$ by setting the multiplication maps to be

$$m_n^{A\otimes B} := m_n^A \otimes m_n^B,$$

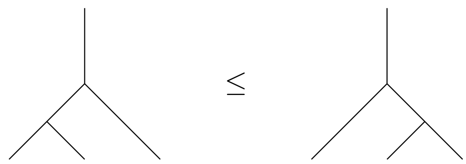
does not work since the degree of $m_n^{A\otimes B}$ is equal to $2 - n + 2 - n = 2(2 - n)$, and unless $n = 0$, we see that it differs from the requested degree of higher multiplications in the definition of an A_∞-algebra. Therefore, it is not possible to tensor A_∞-algebras in the same naive way as to tensor dg-algebras.

5.4.2 The solution

The right way to define such an A_∞-algebra structure on the tensor product of two A_∞-algebras is a lot more involved as we can see in Chapter 2 of [Amo12]. Let us introduce some of the notation used in the theorem of interest to us.

Let G_n be the set of planar rooted trees with n leaves satisfying the following conditions: if we denote by $val(v)$ the number of incoming edges of a vertex v, then $val(v) \geq 2$ for each internal vertex v, by which we mean any vertex that is not one of the leaves or the root.

Denote by $G_n^{\text{bin}} \subset G_n$ the subset of binary trees, i.e. those trees such that all internal vertices have valency 2. There is a natural partial order on the set G_n^{bin} which is generated by the relation on the following picture:



Given $T \in G_n$, we define a binary tree $T_{\max} \in G_n^{\text{bin}}$ as the maximal (with respect to the partial order) binary tree that resolves T, by which we mean that T can be obtained by collapsing several edges in $T_{\max}$. For instance

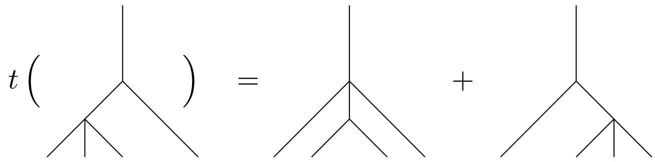
T $\qquad\qquad$ $T_{\max}$

We denote by $c_n \in G_n$ the tree with only one internal vertex and call it the *n-corolla*. Finally, let L_n be the subset of G_n of trees obtained by grafting corollas together but avoiding the last leaf.

Let $V \in L_n$ be a tree with k internal edges. There is a natural correspondence between internal edges of V and right leaning edges of $V_{\max}$. Denote by $R(B)$ the set of right leaning edges of a binary tree B and by $|R(B)|$ the order of this set. We can define some map $t : L_n \to k[G_n]$ which assigns a formal sum of trees of G_n to a tree of L_n, more precisely

$$t(V) := \sum_{\substack{S \in G^{\mathrm{bin}_n} \\ S \geq V_{\max} \\ |R(S)| = k}} S/R(S),$$

where $S/R(S)$ is the tree obtained by collapsing all the edges of S in $R(S)$. For example,



Let (A, m_n) be an A$_\infty$-algebra. Let $T \in G_n$. Then we can produce a map

$$T^A : A^{\otimes n} \to A$$

by assigning $m_{val(v)}$ to each vertex and using T as a flow chart. For example, the map corresponding to the tree in Figure 5.2 is $m_2(m_3 \otimes \mathbf{1})$ where $\mathbf{1}$ denotes the identity map.

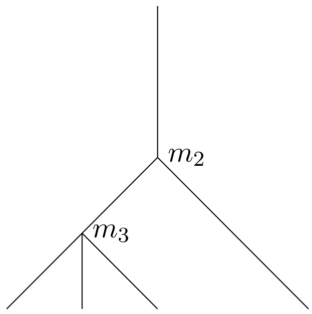
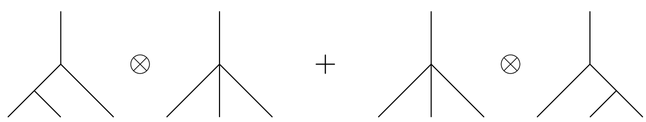


Figure 5.2: Example of a tree of G_n with vertices labeled by $m_{val(v)}$

Theorem 5.4.1 (Theorem 2.21, [Amo12])**.** *Let A and B be (unital) A_∞-algebras. Then the tensor product $A \otimes_\infty B$ is quasi-isomorphic to the A_∞-algebra $\{A \otimes B, m_k^\otimes\}$ given by*

$$\begin{aligned} m_1^\otimes &= m_1^A \otimes id + id \otimes m_1^B, \\ m_2^\otimes &= m_2^A \otimes m_2^B, \\ m_n^\otimes &= \sum_{U \in L_n} U^A \otimes t(U)^B, \text{ for } n \geq 3. \end{aligned}$$

For example, we can write the map $m_3^\otimes$ explicitly:

$m_3^\otimes =$ 

which we interpret as the map

$$m_3^\otimes = \left(m_2^A(m_2^A \otimes \mathbf{1}^A)\right) \otimes m_3^B + m_3^A \otimes \left(m_2^B(\mathbf{1}^B \otimes m_2^B)\right).$$

Lemma 5.4.2. *Let (A, m_n^A) and (B, m_n^B) be two formal A_∞-algebras. Their tensor product $(A \otimes B, m_n^\otimes)$ is formal. In particular, any finite tensor product of formal A_∞-algebras is formal.*

Proof. Let $n \geq 3$. Suppose that given $U \in L_n$, U has, or all the trees of $t(U)$ have, a vertex of valency greater or equal to 3. Then the map U^A or the maps $t(U)^B$ are zero by formality of A or B. If this is true for all $U \in L_n$, we obtain that $m_n^{\otimes}$ is zero.

Let $U \in L_n$. We then want to show that U has a vertex of valency greater or equal to 3 or all the trees of $t(U)$ have a vertex of valency greater or equal to 3. Suppose U does not have a vertex of valency greater or equal to 3. That means that all the internal vertices v and the root of U satisfy $\mathrm{val}(v) = 2$, i.e. U is a binary tree. In particular, U has $n-2$ internal edges. Indeed, let us prove that by induction on the number of leaves n. If $n = 2$, then the edges linking the root to the leaves are not internal, namely there are no internal edge. Suppose that a binary tree with n leaves has $n-2$ internal edges. To obtain a binary tree with $n+1$ leaves, we would need to connect an additional leaf to the binary tree with n leaves, and to respect the fact that it is a binary tree, we cannot link the additional leaf to any internal vertex nor to the root of the tree. The only thing we can do to create an internal vertex of valency 2 is to connect the additional leaf to a non-internal edge. In doing so, we create an internal edge. Hence the number of internal edges become $n-2+1 = (n+1)-2$.

Now, recall the expression of $t(U)$:

$$t(U) := \sum_{\substack{S \in G^{\mathrm{bin}_n} \\ S \geq U_{\max} \\ |R(S)| = n-2}} S/R(S).$$

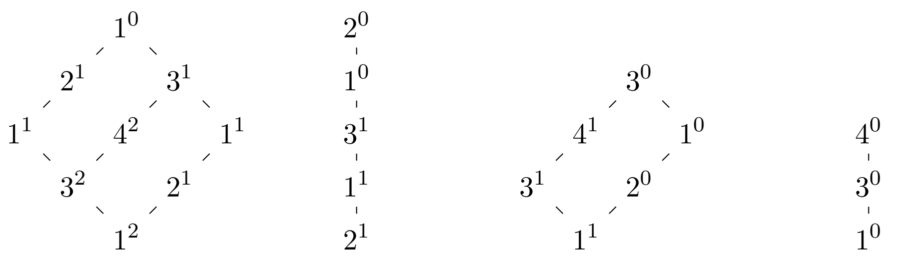
Since U is binary, we have that $U = U_{\max}$. Besides, there is only one binary tree S with n leaves and $n-2$ right leaning edges. As a result, the corresponding tree obtained by collapsing right leaning edges $S/R(S)$ is an n-corolla. In particular, the unique vertex of $t(U)$ has valency n, which is greater or equal to 3 by assumption. □

5.5 Examples

5.5.1 The $\mathbf{A}_\infty$-structure of $\mathrm{Ext}(\Delta, \Delta)$ for $\mathbf{c}_2(\mathbf{c}_2, \mathbf{t}_2)$

5.5.1.1 Computation of $\mathrm{Ext}(\Delta, \Delta)$ for $\mathbf{c}_2(\mathbf{c}_2, \mathbf{t}_2)$

Recall that we obtained the following decomposition of $\mathbf{c}_2(\mathbf{c}_2, \mathbf{t}_2)$ into indecomposable projective modules $P(i)$, $i = 1, 2, 3, 4$ (cf. Figure 5.3).



(a) $P(1)$ (b) $P(2)$ (c) $P(3)$ (d) $P(4)$

Figure 5.3: Projective modules

It is easily seen that the standard module $\Delta(i)$ corresponds to the 0-d-degree part of the projective module $P(i)$ (cf. Figure 5.4).

$$\begin{array}{cccc} & & 3 & 4 \\ & & | & | \\ & 2 & 1 & 3 \\ & | & | & | \\ 1 & 1 & 2 & 1 \end{array}$$

(a) $\Delta(1)$ (b) $\Delta(2)$ (c) $\Delta(3)$ (d) $\Delta(4)$

Figure 5.4: Standard modules

To compute $\mathrm{Ext}_{\mathbf{c}_2(\mathbf{c}_2,\mathbf{t}_2)}(\Delta,\Delta)$ where $\Delta = \oplus_{i=1}^{4}\Delta(i)$, we need to find projective resolutions of the standard modules and then use that

$$\mathrm{Ext}^n_{\mathbf{c}_2(\mathbf{c}_2,\mathbf{t}_2)}(\Delta,\Delta) \cong \mathrm{Hom}_{\mathcal{D}^b(\mathbf{c}_2(\mathbf{c}_2,\mathbf{t}_2))}(P, P[n]),$$

where P is a projective resolution of Δ and $P[n]$ is the complex P shifted n times to the left ($n \in \mathbb{N}$). We have:

$$P_1: \; 0 \to \begin{smallmatrix}4\\3\\1\end{smallmatrix} \to \begin{smallmatrix} & & 3 & \\ & 4 & & 1 \\ 3 & & 2 & \\ & 1 & & \end{smallmatrix} \to \begin{smallmatrix}2\\1\\3\\1\\2\end{smallmatrix} \oplus \begin{smallmatrix} & & 3 & \\ & 4 & & 1 \\ 3 & & 2 & \\ & 1 & & \end{smallmatrix} \to \begin{smallmatrix} & & 1 & & \\ & 2 & & 3 & \\ 1 & & 4 & & 1 \\ & 3 & & 2 & \\ & & 1 & & \end{smallmatrix} \to 1$$

$$P_2: \; 0 \to \begin{smallmatrix}4\\3\\1\end{smallmatrix} \to \begin{smallmatrix} & & 3 & \\ & 4 & & 1 \\ 3 & & 2 & \\ & 1 & & \end{smallmatrix} \to \begin{smallmatrix}2\\1\\3\\1\\2\end{smallmatrix} \to \begin{smallmatrix}2\\1\end{smallmatrix}$$

$$P_3: \; 0 \to \begin{smallmatrix}4\\3\\1\end{smallmatrix} \to \begin{smallmatrix} & & 3 & \\ & 4 & & 1 \\ 3 & & 2 & \\ & 1 & & \end{smallmatrix} \to \begin{smallmatrix}3\\1\\2\end{smallmatrix}$$

$$P_4: \; 0 \to \begin{smallmatrix}4\\3\\1\end{smallmatrix} \to \begin{smallmatrix}4\\3\\1\end{smallmatrix}$$

and $P = \oplus_{i=1}^{4} P_i$.

5.5.1.1.1 $\mathrm{Ext}^i(\Delta(1),\Delta)$ We will only need to go through the cases $i = 0,\dots,3$ since the projective resolutions have length up to 4.

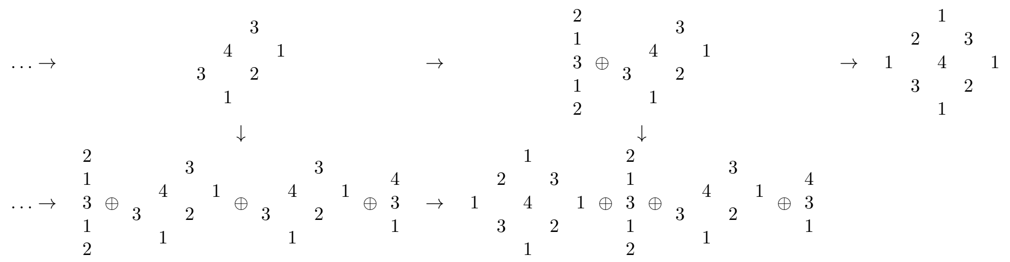
- $i = 0$: In this case, $\mathrm{Ext}^0(\Delta(1),\Delta) \cong \mathrm{Hom}(\Delta(1),\Delta)$. So we have:

$$1 \to \begin{smallmatrix}1\\ \\ \end{smallmatrix} \oplus \begin{smallmatrix}2\\1\\ \end{smallmatrix} \oplus \begin{smallmatrix}3\\1\\2\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\\1\end{smallmatrix}.$$

We get three different generators:

$$\begin{aligned} &e_1 = \mathrm{id}_{\Delta(1)} \in \mathrm{Hom}(\Delta(1),\Delta(1)) \\ &a \in \mathrm{Hom}(\Delta(1),\Delta(2)) \\ &d \in \mathrm{Hom}(\Delta(1),\Delta(4)). \end{aligned}$$

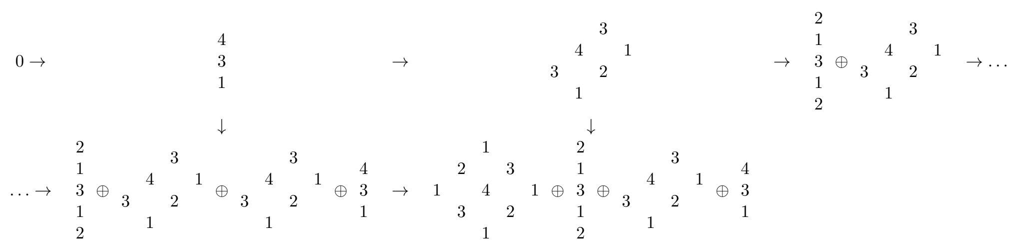
- $i = 1$: We need to understand the non null-homotopic maps between the two complexes:



We get three different generators:

$$\begin{aligned} \alpha &\in \mathrm{Ext}^1(\Delta(1), \Delta(2)) \\ \alpha_1 &\in \mathrm{Ext}^1(\Delta(1), \Delta(3)) \\ \alpha_2 &\in \mathrm{Ext}^1(\Delta(1), \Delta(4)). \end{aligned}$$

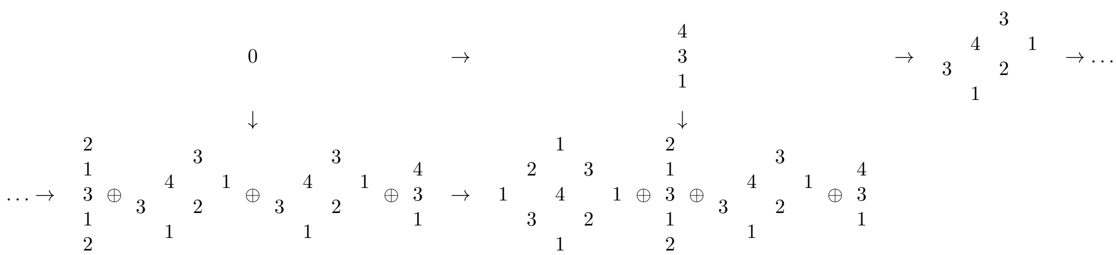
- $i = 2$: We need to understand the non null-homotopic maps between the two complexes:



We get two different generators:

$$\begin{aligned} \alpha_3 &\in \mathrm{Ext}^2(\Delta(1), \Delta(3)) \\ \alpha_4 &\in \mathrm{Ext}^2(\Delta(1), \Delta(4)). \end{aligned}$$

- $i = 3$: We need to understand the non null-homotopic maps between the two complexes:



We get one generator:

$$\alpha_5 \in \mathrm{Ext}^3(\Delta(1), \Delta(4)).$$

5.5.1.1.2 Computing $\mathrm{Ext}^i(\Delta(2), \Delta)$ We will only need to go through the cases $i = 0, \dots, 2$ since the projective resolution of $\Delta(2)$ has length up to 3.

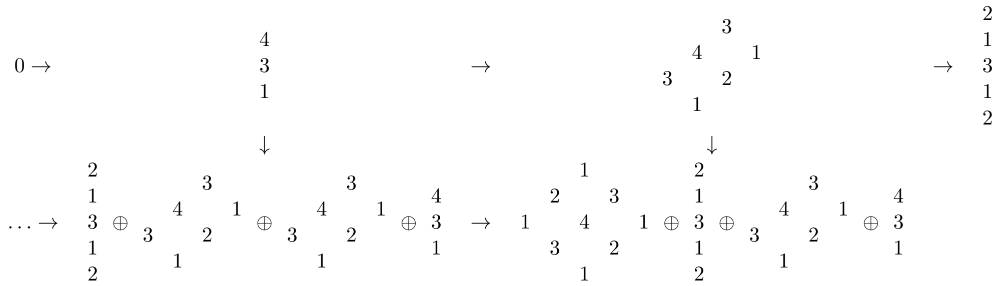
- $i = 0$: In this case, $\mathrm{Ext}^0(\Delta(2), \Delta) \cong \mathrm{Hom}(\Delta(2), \Delta)$. So we have:

$$\begin{array}{c}2\\1\end{array} \to 1 \oplus \begin{array}{c}2\\1\end{array} \oplus \begin{array}{c}3\\1\\2\end{array} \oplus \begin{array}{c}4\\3\\1\end{array}.$$

We get two different generators:

$$e_2 = \mathrm{id}_{\Delta(2)} \in \mathrm{Hom}(\Delta(2), \Delta(2))$$
$$b \in \mathrm{Hom}(\Delta(2), \Delta(3)).$$

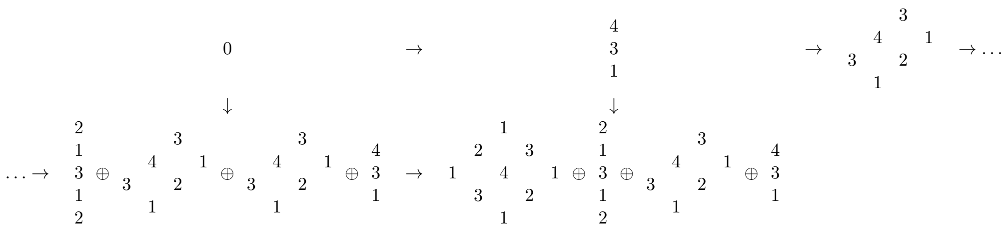
- $i = 1$: We need to understand the non null-homotopic maps between the two complexes:



We get two different generators:

$$\beta \in \mathrm{Ext}^1(\Delta(2), \Delta(3))$$
$$\beta_1 \in \mathrm{Ext}^1(\Delta(2), \Delta(4)).$$

- $i = 2$: We need to understand the non null-homotopic maps between the two complexes:



We get one generator:

$$\beta_2 \in \mathrm{Ext}^2(\Delta(2), \Delta(4)).$$

5.5.1.1.3 Computing $\mathrm{Ext}^i(\Delta(3), \Delta)$ We will only need to go through the cases $i = 0, 1$ since the projective resolution of $\Delta(3)$ has length up to 2.

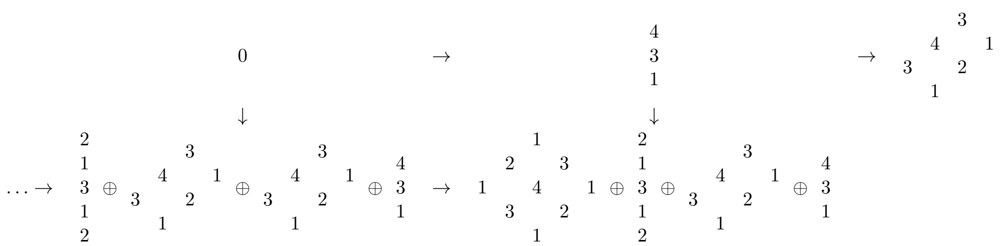
- $i = 0$: In this case, $\mathrm{Ext}^0(\Delta(3), \Delta) \cong \mathrm{Hom}(\Delta(3), \Delta)$. So we have:

$$\begin{matrix} 3 \\ 1 \\ 2 \end{matrix} \to \begin{matrix} 1 \\ \\ \\ \end{matrix} \oplus \begin{matrix} 2 \\ 1 \\ \\ \end{matrix} \oplus \begin{matrix} 3 \\ 1 \\ 2 \end{matrix} \oplus \begin{matrix} 4 \\ 3 \\ 1 \end{matrix}.$$

We get two different generators:

$$e_3 = \mathrm{id}_{\Delta(3)} \in \mathrm{Hom}(\Delta(3), \Delta(3))$$
$$c \in \mathrm{Hom}(\Delta(3), \Delta(4)).$$

- $i = 1$: We need to understand the non null-homotopic maps between the two complexes:



We get one generator:

$$\gamma \in \mathrm{Ext}^1(\Delta(3), \Delta(4)).$$

5.5.1.1.4 Computing $\mathrm{Ext}^i(\Delta(4), \Delta)$ We will only need to go through the cases $i = 0$ since the projective resolution of $\Delta(3)$ has length up to 1.

- $i = 0$: In this case, $\mathrm{Ext}^0(\Delta(4), \Delta) \cong \mathrm{Hom}(\Delta(4), \Delta)$. So we have:

$$\begin{matrix} 4 \\ 3 \\ 1 \end{matrix} \to \begin{matrix} 1 \\ \\ \end{matrix} \oplus \begin{matrix} 2 \\ 1 \\ \end{matrix} \oplus \begin{matrix} 3 \\ 1 \\ 2 \end{matrix} \oplus \begin{matrix} 4 \\ 3 \\ 1 \end{matrix}.$$

We get one generator:

$$e_4 = \mathrm{id}_{\Delta(4)} \in \mathrm{Hom}(\Delta(3), \Delta(3)).$$

5.5.1.1.5 A basis for $\mathrm{Ext}(\Delta, \Delta)$ Composing the different maps found above, we see that a minimal generating set is given by:

$$\begin{aligned} &e_1 \in \mathrm{Hom}(\Delta(1), \Delta(1));\\ &e_2 \in \mathrm{Hom}(\Delta(2), \Delta(2));\\ &e_3 \in \mathrm{Hom}(\Delta(3), \Delta(3));\\ &e_4 \in \mathrm{Hom}(\Delta(4), \Delta(4));\\ &a \in \mathrm{Hom}(\Delta(1), \Delta(2));\\ &b \in \mathrm{Hom}(\Delta(2), \Delta(3));\\ &c \in \mathrm{Hom}(\Delta(3), \Delta(4));\\ &d \in \mathrm{Hom}(\Delta(1), \Delta(4));\\ &\alpha \in \mathrm{Ext}^1(\Delta(1), \Delta(2));\\ &\beta \in \mathrm{Ext}^1(\Delta(2), \Delta(3));\\ &\gamma \in \mathrm{Ext}^1(\Delta(3), \Delta(4)); \end{aligned}$$

and that they satisfy the following relations:

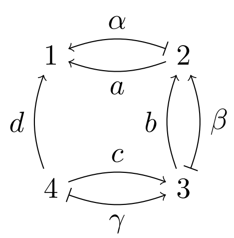
$$b \circ a = c \circ b = c \circ \beta \circ \alpha - \gamma \circ \beta \circ a = b \circ \alpha = \gamma \circ b = 0.$$

Since we want the $\mathbf{c}_2(\mathbf{c}_2, \mathbf{t}_2)$-action on Δ to match (hence the redundant notation for the e_i's), we will actually consider $\mathrm{Ext}(\Delta, \Delta)^{op}$, so that for instance:

$$b \in \mathrm{Hom}(\Delta(2), \Delta(3)) = \mathrm{Hom}(\Delta e_2, \Delta e_3) = e_2 \,\mathrm{Hom}(\Delta, \Delta) e_3,$$

where $b = e_2 b e_3$, which is the same as $e_3 \circ b \circ e_2$.

We can finally represent $\mathrm{Ext}(\Delta, \Delta)^{op}$ as the path algebra of the following quiver:



modulo relations $(ab, bc, \alpha\beta c - a\beta\gamma, \alpha b, b\gamma)$. It is an 18-dimensional k-algebra, with basis:

Degree 0:

$$e_1 \in \mathrm{Hom}(\Delta(1), \Delta(1));$$
$$a \in \mathrm{Hom}(\Delta(1), \Delta(2));$$
$$e_2 \in \mathrm{Hom}(\Delta(2), \Delta(2));$$
$$b \in \mathrm{Hom}(\Delta(2), \Delta(3));$$
$$e_3 \in \mathrm{Hom}(\Delta(3), \Delta(3));$$
$$c \in \mathrm{Hom}(\Delta(3), \Delta(4));$$
$$e_4 \in \mathrm{Hom}(\Delta(4), \Delta(4));$$
$$d \in \mathrm{Hom}(\Delta(1), \Delta(4));$$

Degree 1:

$$\alpha \in \mathrm{Ext}^1(\Delta(1), \Delta(2));$$
$$\beta a \in \mathrm{Ext}^1(\Delta(1), \Delta(3));$$
$$c\beta a \in \mathrm{Ext}^1(\Delta(1), \Delta(4));$$
$$\beta \in \mathrm{Ext}^1(\Delta(2), \Delta(3));$$
$$c\beta \in \mathrm{Ext}^1(\Delta(2), \Delta(4));$$
$$\gamma \in \mathrm{Ext}^1(\Delta(3), \Delta(4));$$

Degree 2:

$$\beta\alpha \in \mathrm{Ext}^2(\Delta(1), \Delta(3));$$
$$c\beta\alpha \in \mathrm{Ext}^2(\Delta(1), \Delta(4));$$
$$\gamma\beta \in \mathrm{Ext}^2(\Delta(2), \Delta(4));$$

Degree 3:

$$\gamma\beta\alpha \in \mathrm{Ext}^3(\Delta(1), \Delta(4)).$$

5.5.1.2 Computation of the A_∞-structure

We can now compute the A_∞-structure of this algebra (which exists by Kadeishvili's theorem, $\mathrm{Ext}(\Delta, \Delta)^{op}$ being the homology of the dg-algebra $A = \mathrm{Hom}(P, P)^{op}$). We follow a recipe found in [Mad02] or in [Kel99]. We want to construct the compositions $m_i : H^*(A)^{\otimes i} \to H^*(A)$ and a quasi-isomorphism of A_∞-algebras $f : H^*(A) \to A$. Since A is a dg-algebra, we have:

- $m_1^A : A \to A$ is the differential of A;
- $m_2^A : A \otimes A \to A$ is the multiplication;
- $m_i^A : A^{\otimes i} \to A$ vanishes for all $i \geq 3$.

Since $H^*(A)$ is the homology of A, we can choose the differential $m_1 : H^*(A) \to H^*(A)$ to be zero. We choose the map of complexes $f_1 : (H^*(A), 0) \to (A, m_1^A)$ such that Hf_1 is the identity. Considering how we computed $H^*(A)$, it is quite obvious how to define it.

Next, we need to consider $\Phi_2 = f_1 m_2 - m_2^A(f_1 \otimes f_1)$. This expression is going to be zero when we evaluate it on pairs of basis elements of $H^*(A)$ if and only if their multiplication was already zero or it becomes zero in homology. Therefore, we only need to check on the relations of length two to find pairs of basis elements which will yield non-zero Φ_2 when evaluating.

It turns out that there are only two pairs with non-zero Φ_2:

$$\Phi_2(a \otimes b) = -g_1$$

where g_1 is the following map of complexes:

$$\begin{array}{ccccccc}
P(4) & \to & P(3)\oplus P(4) & \to & P(2)\oplus P(3)\oplus P(3)\oplus P(4) & \to & P(1)\oplus P(2)\oplus P(3)\oplus P(4) \\
\downarrow 0 & & \downarrow 0 & & \downarrow 0 & & \downarrow g_1 \\
P(4) & \to & P(3)\oplus P(4) & \to & P(2)\oplus P(3)\oplus P(3)\oplus P(4) & \to & P(1)\oplus P(2)\oplus P(3)\oplus P(4)
\end{array}$$

which sends the top of $P(1)$ to the socle of $P(3)$, and

$$\Phi_2(\alpha \otimes b) = -g_2$$

where g_2 is the following map of complexes:

$$\begin{array}{ccccccc}
P(4) & \to & P(3)\oplus P(4) & \to & P(2)\oplus P(3)\oplus P(3)\oplus P(4) & \to & P(1)\oplus P(2)\oplus P(3)\oplus P(4) \\
 & \searrow^{0} & & \searrow^{0} & & \searrow^{g_2} & \\
P(4) & \to & P(3)\oplus P(4) & \to & P(2)\oplus P(3)\oplus P(3)\oplus P(4) & \to & P(1)\oplus P(2)\oplus P(3)\oplus P(4)
\end{array}$$

which sends the top of $P(2)$ to the composition factor 2 of $P(3)$.

It is now possible to choose $f_2 : H^*(A)^{\otimes 2} \to A$ as a morphism of complexes of degree -1 so that $m_1 f_2 = \Phi_2$. We may choose $f_2(x \otimes y) = 0$ for all $x \otimes y \in H^*(A) \otimes H^*(A)$ such that $\Phi_2(x \otimes y) = 0$, and for $a \otimes b$ and $\alpha \otimes b$, we choose:

$$f_2(a \otimes b) = -g_3$$

where g_3 is the following map of complexes:

$$\begin{array}{ccccccc}
P(4) & \to & P(3)\oplus P(4) & \to & P(2)\oplus P(3)\oplus P(3)\oplus P(4) & \to & P(1)\oplus P(2)\oplus P(3)\oplus P(4) \\
 & \swarrow_{0} & & \swarrow_{0} & & \swarrow_{g_3} & \\
P(4) & \to & P(3)\oplus P(4) & \to & P(2)\oplus P(3)\oplus P(3)\oplus P(4) & \to & P(1)\oplus P(2)\oplus P(3)\oplus P(4)
\end{array}$$

which sends the top of $P(1)$ to the socle of $P(4)$, and

$$f_2(\alpha \otimes b) = -g_4$$

where g_4 is the following map of complexes:

$$\begin{array}{ccccccc}
P(4) & \to & P(3)\oplus P(4) & \to & P(2)\oplus P(3)\oplus P(3)\oplus P(4) & \to & P(1)\oplus P(2)\oplus P(3)\oplus P(4) \\
\downarrow 0 & & \downarrow 0 & & \downarrow 0 & & \downarrow g_4 \\
P(4) & \to & P(3)\oplus P(4) & \to & P(2)\oplus P(3)\oplus P(3)\oplus P(4) & \to & P(1)\oplus P(2)\oplus P(3)\oplus P(4)
\end{array}$$

which sends the top of $P(1)$ to the composition factor 1 of d-degree 0 of $P(3)$.

Since we have found non-zero f_2, we continue the construction. Consider $\Phi_3 := m_2^A(f_1 \otimes f_2 - f_2 \otimes f_1) + f_2(\mathbf{1} \otimes m_2 - m_2 \otimes \mathbf{1})$ and let us evaluate it for triple $x \otimes y \otimes z$ of basis elements of $H^*(A)$. It turns out it is non-zero for $a \otimes b \otimes \gamma$ and $\alpha \otimes b \otimes c$, and

$$\Phi_3(a \otimes b \otimes \gamma) = \Phi_3(\alpha \otimes b \otimes c) = d.$$

We must choose maps m_3 and f_3 such that:

$$f_1 m_3 = m_1^A f_3 + \Phi_3.$$

Taking f_3 to be identically zero, $m_3(a \otimes b \otimes \gamma) = m_3(\alpha \otimes b \otimes c) = d$ and $m_3(x \otimes y \otimes z) = 0$ otherwise, the required relation is satisfied on $H^*(A)^{\otimes 3}$. Since f_3 is identically zero, the construction finishes and all other m_i's are zero for $i \geq 4$.

5.5.2 A partial A_∞-structure on $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})$

We want to compute a partial A_∞-structure on $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})$; more specifically, we are interested in knowing $m_3 : \mathbf{d} \otimes \mathbb{H}(\mathbf{u}) \otimes \mathbf{d} \to \mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})$, and we need to build a quasi-isomorphism:

$$f : \mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}}) \to \mathbb{T}_{\mathbf{d}}(\underline{\mathbf{u}}) \to \mathbb{T}_{\mathcal{K}(\mathbf{c})}\left(\mathcal{K}(\tilde{\mathbf{t}})\right).$$

To compute m_3, we have the following formula derived from the relations defining an A_∞-structure:

$$f_1 m_3 = m_1 f_3 + m_2(f_1 \otimes f_2 - f_2 \otimes f_1) + f_2(\mathbf{1} \otimes m_2 - m_2 \otimes \mathbf{1}),$$

so we need to know the following maps:

- $f_1 : \mathbf{d} \to \mathbb{T}_{\mathcal{K}(\mathbf{c})}\left(\mathcal{K}(\tilde{\mathbf{t}})\right)$;
- $f_1 : \mathbb{H}(\mathbf{u}) \to \mathbb{T}_{\mathcal{K}(\mathbf{c})}\left(\mathcal{K}(\tilde{\mathbf{t}})\right)$;
- $f_2 : \mathbf{d} \otimes \mathbb{H}(\mathbf{u}) \to \mathbb{T}_{\mathcal{K}(\mathbf{c})}\left(\mathcal{K}(\tilde{\mathbf{t}})\right)$;
- $f_2 : \mathbb{H}(\mathbf{u}) \otimes \mathbf{d} \to \mathbb{T}_{\mathcal{K}(\mathbf{c})}\left(\mathcal{K}(\tilde{\mathbf{t}})\right)$;
- $m_2 : \mathbf{d} \otimes \mathbb{H}(\mathbf{u}) \to \mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})$;
- $m_2 : \mathbb{H}(\mathbf{u}) \otimes \mathbf{d} \to \mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})$;
- $m_2 : \mathbb{T}_{\mathcal{K}(\mathbf{c})}\left(\mathcal{K}(\tilde{\mathbf{t}})\right) \otimes \mathbb{T}_{\mathcal{K}(\mathbf{c})}\left(\mathcal{K}(\tilde{\mathbf{t}})\right) \to \mathbb{T}_{\mathcal{K}(\mathbf{c})}\left(\mathcal{K}(\tilde{\mathbf{t}})\right)$.

5.5.2.1 Making f_1's explicit

From [MT13], we have:

- $f_1 : \mathbf{d} \overset{q.i.}{\to} \mathcal{K}(\mathbf{c})$;
- $f_1 : \mathbb{H}(\mathbf{u}) \overset{q.i.}{\to} \mathcal{K}(\mathbf{t})$.

We want to make those two maps explicit.

Recall $\mathcal{K}(M) = \mathrm{Hom}_A(P, A) \otimes_A M \otimes_A P \cong \mathrm{Hom}_A(P, M) \otimes_A P$. In particular, $\mathcal{K}(\mathbf{c}) \cong \mathrm{End}_{\mathbf{c}}(P)$, where P is a projective resolution of the standard modules of $\mathbf{c}$:

$$\begin{array}{cccccccc} P_1 : & 0 & \to & \begin{smallmatrix}2\\1\end{smallmatrix} & \to & \begin{smallmatrix}1\\2\\1\end{smallmatrix} & \to & 1 \\ P_2 : & & & 0 & \to & \begin{smallmatrix}2\\1\end{smallmatrix} & \to & \begin{smallmatrix}2\\1\end{smallmatrix} \end{array}$$

We have the following:

$$\mathrm{End}_{\mathbf{c}}(P) = \mathrm{Hom}_{\mathbf{c}}\left(\mathbf{c}e_2 \to \mathbf{c}, \mathbf{c}e_2 \to \mathbf{c}\right).$$

It is a chain complex with differential:

$$\partial(f)(-) = d^P(f(-)) + (-1)^{|f|} f(d^P(-)),$$

which decomposes as

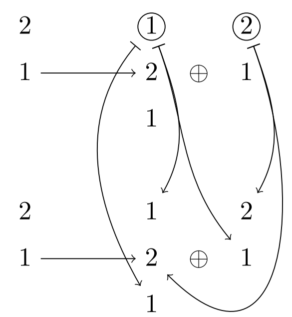
$$\begin{aligned}
\mathrm{End}_{\mathbf{c}}(P) \;=\;& \mathrm{Hom}^0_{\mathbf{c}}\left(\begin{array}{c}1\\2\\1\end{array}, \begin{array}{c}1\\2\\1\end{array}\right) \oplus \mathrm{Hom}^0_{\mathbf{c}}\left(\begin{array}{c}1\\2\\1\end{array}, \begin{array}{c}2\\1\end{array}\right) \oplus \mathrm{Hom}^0_{\mathbf{c}}\left(\begin{array}{c}2\\1\end{array}, \begin{array}{c}1\\2\\1\end{array}\right)\\
\oplus\;& \mathrm{Hom}^0_{\mathbf{c}}\left(\begin{array}{c}2\\1\end{array}, \begin{array}{c}2\\1\end{array}\right) \oplus \mathrm{Hom}^0_{\mathbf{c}}\left(\begin{array}{c}2\\1\end{array}, \begin{array}{c}2\\1\end{array}\right) \oplus \mathrm{Hom}^{-1}_{\mathbf{c}}\left(\begin{array}{c}1\\2\\1\end{array}, \begin{array}{c}2\\1\end{array}\right)\\
\oplus\;& \mathrm{Hom}^{-1}_{\mathbf{c}}\left(\begin{array}{c}2\\1\end{array}, \begin{array}{c}2\\1\end{array}\right) \oplus \mathrm{Hom}^1_{\mathbf{c}}\left(\begin{array}{c}2\\1\end{array}, \begin{array}{c}1\\2\\1\end{array}\right) \oplus \mathrm{Hom}^1_{\mathbf{c}}\left(\begin{array}{c}2\\1\end{array}, \begin{array}{c}2\\1\end{array}\right)
\end{aligned}$$

where the superscripts on the Hom's indicate the degree of the maps. We obtain 10 basis elements:

$$\begin{aligned}
\mathrm{End}_{\mathbf{c}}(P) \;=\;& \langle e_1 \mapsto e_1, e_1 \mapsto \beta\alpha\rangle^0 \oplus \langle e_1 \mapsto \beta\rangle^0 \oplus \langle e_2^{II} \mapsto \alpha\rangle^0\\
\oplus\;& \langle e_2^I \mapsto e_2^I\rangle^0 \oplus \langle e_2^{II} \mapsto e_2^{II}\rangle^0 \oplus \langle e_1 \mapsto \beta\rangle^{-1}\\
\oplus\;& \langle e_2^{II} \mapsto e_2^I\rangle^{-1} \oplus \langle e_2^I \mapsto \alpha\rangle^1 \oplus \langle e_2^I \mapsto e_2^{II}\rangle^1.
\end{aligned}$$

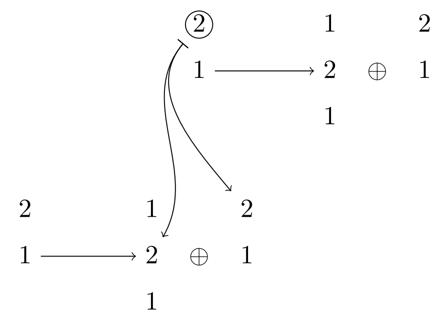
For these to appear in homology, they must vanish under the differential; this exactly means that they must be maps of chain complexes, i.e. commute with the differential of the complex P. We see that they are the following maps:

$\mathrm{Hom}^0_{Ch(\mathbf{c})}(P,P)$:



$\langle e_1 \mapsto e_1\rangle^0, \langle e_2 \mapsto e_2\rangle^0$
$\langle e_1 \mapsto \beta\rangle^0, \langle e_2 \mapsto \alpha\rangle^0$
$\langle e_1 \mapsto \beta\alpha\rangle^0$

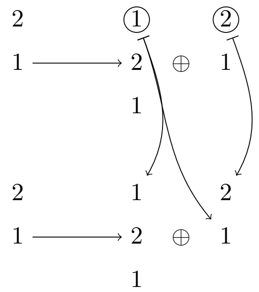
$\mathrm{Hom}^1_{Ch(\mathbf{c})}(P,P)$:



$\langle e_2 \mapsto e_2\rangle^1$
$\langle e_2 \mapsto \alpha\rangle^1$

In particular, the homology is positively graded. Therefore, it comes that:

$\mathrm{Ext}^0_{\mathbf{c}}(\Delta,\Delta)$:

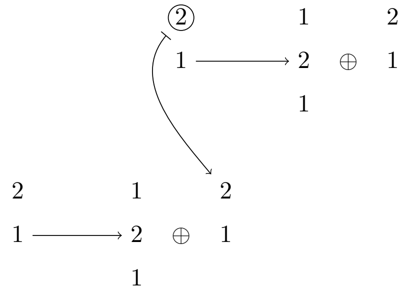


$\langle e_1 \mapsto e_1\rangle^0 = \mathrm{id}_{P_1}$

$\langle e_1 \mapsto \beta\rangle^0 : P_1 \to P_2, e_1 \mapsto \beta$

$\langle e_2 \mapsto e_2\rangle^0 = \mathrm{id}_{P_2}$

$\mathrm{Ext}^1_{\mathbf{c}}(\Delta,\Delta)$:



$\langle e_2 \mapsto e_2\rangle^1 : P_1 \to P_2, e_2 \mapsto e_2[1]$

We conclude that $f_1 : \mathbf{d} \to \mathrm{End}_{\mathbf{c}}(P)^{\mathrm{op}}$ is given by

$$\begin{array}{rcl} e_1 & \mapsto & \langle e_1 \mapsto e_1\rangle^0 + \langle e_2^I \mapsto e_2^I\rangle^0 \\ e_2 & \mapsto & \langle e_2^{II} \mapsto e_2^{II}\rangle^0 \\ \xi & \mapsto & \langle e_1 \mapsto \beta\rangle^0 : P_1 \to P_2, e_1 \mapsto \beta \\ x & \mapsto & \langle e_2 \mapsto e_2\rangle^1 : P_1 \to P_2, e_2^I \mapsto e_2^{II}[1] \end{array}$$

where e_2^I, resp. e_2^{II}, refers to the basis element at the top of the second projective module in the first, resp. second, projective resolution.

This map induces the identity in homology and it satisfies

$$m_1 f_1 = f_1 m_1 : \mathbf{d} \to \mathrm{End}_{\mathbf{c}}(P),$$

which rewrites as $m_1 f_1 = 0$, because the differential $m_1 : \mathbf{d} \to \mathbf{d}$ is the zero map. This equation means that all lifts of elements of $\mathbf{d}$ are mapped to elements in the kernel of the differential on $\mathrm{End}_{\mathbf{c}}(P)$, which is true because those elements appear in homology (and are non-zero).

We now want the map $f_1 : \mathbb{H}(\mathbf{u}) \to \mathcal{K}(\mathbf{t})$ to be explicit. We work with a $(\mathbf{c},\mathbf{c})$-bimodule resolution $\tilde{\mathbf{t}}$ of $\mathbf{t}$ which is projective both on the left and on the right:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{c} & \overset{f}{\longrightarrow} & \mathbf{c}e_1 \otimes e_1\mathbf{c} & \longrightarrow & \mathbf{t} & \longrightarrow & 0 \\ & & 1 & \longmapsto & e_1 \otimes \beta\alpha + \alpha \otimes \beta - \beta\alpha \otimes e_1 & & & & \end{array}$$

We would like to express $\mathcal{K}(\tilde{\mathbf{t}})$ in the form of a complex and compute its differentials, so that we may then compute its homology. It writes:

$$\mathcal{K}(\tilde{\mathbf{t}}) = \mathrm{Hom}_{\mathbf{c}}\left(\mathbf{c}e_2 \overset{\cdot\alpha}{\to} \mathbf{c}, \mathbf{c} \overset{f}{\to} \mathbf{c}e_1 \otimes e_1\mathbf{c}\right) \otimes_{\mathbf{c}} \left(\mathbf{c}e_2 \overset{\cdot\alpha}{\to} \mathbf{c}\right).$$

Focusing on the first tensorand, we have the following:

$$
\begin{array}{rcl}
& & \mathrm{Hom}_{\mathbf{c}}\left(\mathbf{c}e_2 \overset{\cdot\alpha}{\to} \mathbf{c}, \mathbf{c} \overset{f}{\to} \mathbf{c}e_1 \otimes e_1\mathbf{c}\right) \\[2ex]
= & & \mathrm{Hom}_{\mathbf{c}}\left(\mathbf{c}, \mathbf{c} \overset{f}{\to} \mathbf{c}e_1 \otimes e_1\mathbf{c}\right) \xrightarrow{(\cdot\alpha)^*} \mathrm{Hom}_{\mathbf{c}}\left(\mathbf{c}e_2, \mathbf{c} \overset{f}{\to} \mathbf{c}e_1 \otimes e_1\mathbf{c}\right) \\[2ex]
= & & \begin{array}{ccc}
\mathrm{Hom}_{\mathbf{c}}(\mathbf{c}, \mathbf{c}) & \xrightarrow{\quad(\cdot\alpha)^*\quad} & \mathrm{Hom}_{\mathbf{c}}(\mathbf{c}e_2, \mathbf{c}) \\
\big\downarrow f_* & & \big\downarrow f_* \\
\mathrm{Hom}_{\mathbf{c}}(\mathbf{c}, \mathbf{c}e_1 \otimes e_1\mathbf{c}) & \xrightarrow{\quad(\cdot\alpha)^*\quad} & \mathrm{Hom}_{\mathbf{c}}(\mathbf{c}e_2, \mathbf{c}e_1 \otimes e_1\mathbf{c})
\end{array}
\end{array}
$$

This can be considered trivially as a double complex, and using the sign trick [Wei95, p. 8], we take the total complex:

$$
\mathrm{Hom}_{\mathbf{c}}\left(\mathbf{c}e_2 \overset{\cdot\alpha}{\to} \mathbf{c}, \mathbf{c} \overset{f}{\to} \mathbf{c}e_1 \otimes e_1\mathbf{c}\right) =
$$

$$
\mathrm{Hom}_{\mathbf{c}}(\mathbf{c}, \mathbf{c}) \overset{\begin{pmatrix} (\cdot\alpha)^* \\ -f_* \end{pmatrix}}{\longrightarrow} \mathrm{Hom}_{\mathbf{c}}(\mathbf{c}e_2, \mathbf{c}) \oplus \mathrm{Hom}_{\mathbf{c}}(\mathbf{c}, \mathbf{c}e_1 \otimes e_1\mathbf{c}) \overset{(f_*,(\cdot\alpha)^*)}{\longrightarrow} \mathrm{Hom}_{\mathbf{c}}(\mathbf{c}e_2, \mathbf{c}e_1 \otimes e_1\mathbf{c})
$$

It is now possible to include the second tensorand, and $\mathcal{K}(\tilde{\mathbf{t}})$ becomes:

$$
\begin{array}{ccccc}
\mathrm{Hom}_{\mathbf{c}}(\mathbf{c}, \mathbf{c}) \otimes_{\mathbf{c}} \mathbf{c}e_2 & \xrightarrow{\begin{pmatrix} (\cdot\alpha)^* \otimes \mathbf{1} \\ -f_* \otimes \mathbf{1} \end{pmatrix}} & \mathrm{Hom}_{\mathbf{c}}(\mathbf{c}e_2, \mathbf{c}) \otimes_{\mathbf{c}} \mathbf{c}e_2 \oplus \mathrm{Hom}_{\mathbf{c}}(\mathbf{c}, \mathbf{c}e_1 \otimes e_1\mathbf{c}) \otimes_{\mathbf{c}} \mathbf{c}e_2 & \xrightarrow{(f_* \otimes \mathbf{1}, (\cdot\alpha)^* \otimes \mathbf{1})} & \mathrm{Hom}_{\mathbf{c}}(\mathbf{c}e_2, \mathbf{c}e_1 \otimes e_1\mathbf{c}) \otimes_{\mathbf{c}} \mathbf{c}e_2 \\
\mathbf{1} \otimes (\cdot\alpha) \big\downarrow & & \big\downarrow \begin{pmatrix} \mathbf{1} \otimes (\cdot\alpha) & 0 \\ 0 & \mathbf{1} \otimes (\cdot\alpha) \end{pmatrix} & & \big\downarrow \mathbf{1} \otimes (\cdot\alpha) \\
\mathrm{Hom}_{\mathbf{c}}(\mathbf{c}, \mathbf{c}) \otimes_{\mathbf{c}} \mathbf{c} & \xrightarrow{\begin{pmatrix} (\cdot\alpha)^* \otimes \mathbf{1} \\ -f_* \otimes \mathbf{1} \end{pmatrix}} & \mathrm{Hom}_{\mathbf{c}}(\mathbf{c}e_2, \mathbf{c}) \otimes_{\mathbf{c}} \mathbf{c} \oplus \mathrm{Hom}_{\mathbf{c}}(\mathbf{c}, \mathbf{c}e_1 \otimes e_1\mathbf{c}) \otimes_{\mathbf{c}} \mathbf{c} & \xrightarrow{(f_* \otimes \mathbf{1}, (\cdot\alpha)^* \otimes \mathbf{1})} & \mathrm{Hom}_{\mathbf{c}}(\mathbf{c}e_2, \mathbf{c}e_1 \otimes e_1\mathbf{c}) \otimes_{\mathbf{c}} \mathbf{c}
\end{array}
$$

This is a double complex, and taking the total complex yields the following complex:

$$
\mathrm{Hom}_{\mathbf{c}}(\mathbf{c}, \mathbf{c}) \otimes_{\mathbf{c}} \mathbf{c}e_2 \xrightarrow{\partial_3} \mathrm{Hom}_{\mathbf{c}}(\mathbf{c}e_2, \mathbf{c}) \otimes_{\mathbf{c}} \mathbf{c}e_2 \oplus \mathrm{Hom}_{\mathbf{c}}(\mathbf{c}, \mathbf{c}e_1 \otimes e_1\mathbf{c}) \otimes_{\mathbf{c}} \mathbf{c}e_2 \oplus \mathrm{Hom}_{\mathbf{c}}(\mathbf{c}, \mathbf{c}) \otimes_{\mathbf{c}} \mathbf{c}
$$

$$
\xrightarrow{\partial_2} \mathrm{Hom}_{\mathbf{c}}(\mathbf{c}e_2, \mathbf{c}) \otimes_{\mathbf{c}} \mathbf{c} \oplus \mathrm{Hom}_{\mathbf{c}}(\mathbf{c}, \mathbf{c}e_1 \otimes e_1\mathbf{c}) \otimes_{\mathbf{c}} \mathbf{c} \oplus \mathrm{Hom}_{\mathbf{c}}(\mathbf{c}e_2, \mathbf{c}e_1 \otimes e_1\mathbf{c}) \otimes_{\mathbf{c}} \mathbf{c}e_2
$$

$$
\xrightarrow{\partial_1} \mathrm{Hom}_{\mathbf{c}}(\mathbf{c}e_2, \mathbf{c}e_1 \otimes e_1\mathbf{c}) \otimes_{\mathbf{c}} \mathbf{c}
$$

where

$$
\begin{aligned}
\partial_3 &= \begin{pmatrix} (\cdot\alpha)^* \otimes \mathbf{1} \\ -f_* \otimes \mathbf{1} \\ -\mathbf{1} \otimes (\cdot\alpha) \end{pmatrix} \\
\partial_2 &= \begin{pmatrix} \mathbf{1} \otimes (\cdot\alpha) & 0 & (\cdot\alpha)^* \otimes \mathbf{1} \\ 0 & \mathbf{1} \otimes (\cdot\alpha) & -f_* \otimes \mathbf{1} \\ f_* \otimes \mathbf{1} & (\cdot\alpha)^* \otimes \mathbf{1} & 0 \end{pmatrix} \\
\partial_1 &= (f_* \otimes \mathbf{1}, (\cdot\alpha)^* \otimes \mathbf{1}, -\mathbf{1} \otimes (\cdot\alpha))
\end{aligned}
$$

We can simplify the complex by means of classical isomorphisms, and we obtain:

$$
\mathbf{c}e_2 \overset{\tilde{\partial}_3}{\longrightarrow} e_2\mathbf{c}e_2 \oplus \mathbf{c}e_1 \otimes e_1\mathbf{c}e_2 \oplus \mathbf{c} \overset{\tilde{\partial}_2}{\longrightarrow} e_2\mathbf{c} \oplus \mathbf{c}e_1 \otimes e_1\mathbf{c} \oplus e_2\mathbf{c}e_1 \otimes e_1\mathbf{c}e_2 \overset{\tilde{\partial}_1}{\longrightarrow} e_2\mathbf{c}e_1 \otimes e_1\mathbf{c}
$$

where

$$\begin{aligned}
\tilde{\partial}_3 &= \begin{pmatrix} \alpha\cdot \\ -(\cdot\alpha\otimes\beta) \\ -(\cdot\alpha) \end{pmatrix} \\
\tilde{\partial}_2 &= \begin{pmatrix} \cdot\alpha & 0 & \alpha\cdot \\ 0 & \cdot\alpha & \cdot(\beta\alpha\otimes e_1 - \alpha\otimes\beta - e_1\otimes\beta\alpha) \\ \cdot\alpha\otimes\beta & \alpha\cdot & 0 \end{pmatrix} \\
\tilde{\partial}_1 &= (\cdot(e_1\otimes\beta\alpha + \alpha\otimes\beta - \beta\alpha\otimes e_1), \quad \alpha\cdot, \quad -(\cdot\alpha))
\end{aligned}$$

Let us compute the homology of this complex. We know that $\mathbf{c}e_2 = \langle e_2, \beta\rangle$, and we have:

$$\begin{aligned}
\tilde{\partial}_3(e_2) &= \begin{pmatrix} 0 \\ -\alpha\otimes\beta \\ -\alpha \end{pmatrix}, \\
\tilde{\partial}_3(\beta) &= \begin{pmatrix} 0 \\ -\beta\alpha\otimes\beta \\ -\beta\alpha \end{pmatrix},
\end{aligned}$$

and hence we see that $\operatorname{Ker}\tilde{\partial}_3 = 0$ and $\operatorname{Im}\tilde{\partial}_3 = \left\langle \begin{pmatrix} 0 \\ -\alpha\otimes\beta \\ -\alpha \end{pmatrix}, \begin{pmatrix} 0 \\ -\beta\alpha\otimes\beta \\ -\beta\alpha \end{pmatrix} \right\rangle$.

We know that $e_2\mathbf{c}e_2 = \langle e_2\rangle$, so:

$$\tilde{\partial}_2\left(\begin{pmatrix} e_2 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} \alpha \\ 0 \\ \alpha\otimes\beta \end{pmatrix},$$

and $\mathbf{c}e_1\otimes e_1\mathbf{c}e_2 = \langle e_1\otimes\beta, \alpha\otimes\beta, \beta\alpha\otimes\beta\rangle$, so:

$$\begin{aligned}
\tilde{\partial}_2\left(\begin{pmatrix} 0 \\ e_1\otimes\beta \\ 0 \end{pmatrix}\right) &= \begin{pmatrix} 0 \\ e_1\otimes\beta\alpha \\ \alpha\otimes\beta \end{pmatrix}, \\
\tilde{\partial}_2\left(\begin{pmatrix} 0 \\ \alpha\otimes\beta \\ 0 \end{pmatrix}\right) &= \begin{pmatrix} 0 \\ \alpha\otimes\beta\alpha \\ 0 \end{pmatrix}, \\
\tilde{\partial}_2\left(\begin{pmatrix} 0 \\ \beta\alpha\otimes\beta \\ 0 \end{pmatrix}\right) &= \begin{pmatrix} 0 \\ \beta\alpha\otimes\beta\alpha \\ 0 \end{pmatrix},
\end{aligned}$$

and $\mathbf{c} = \langle e_1, \alpha, \beta\alpha, e_2, \beta\rangle$, so:

$$\begin{aligned}
\tilde{\partial}_2\left(\begin{pmatrix} 0 \\ 0 \\ e_1 \end{pmatrix}\right) &= \begin{pmatrix} \alpha \\ \beta\alpha\otimes e_1 - e_1\otimes\beta\alpha \\ 0 \end{pmatrix}, & \tilde{\partial}_2\left(\begin{pmatrix} 0 \\ 0 \\ e_2 \end{pmatrix}\right) &= \begin{pmatrix} 0 \\ -\alpha\otimes\beta \\ 0 \end{pmatrix}, \\
\tilde{\partial}_2\left(\begin{pmatrix} 0 \\ 0 \\ \alpha \end{pmatrix}\right) &= \begin{pmatrix} 0 \\ -\alpha\otimes\beta\alpha \\ 0 \end{pmatrix}, & \tilde{\partial}_2\left(\begin{pmatrix} 0 \\ 0 \\ \beta \end{pmatrix}\right) &= \begin{pmatrix} 0 \\ -\beta\alpha\otimes\beta \\ 0 \end{pmatrix}, \\
\tilde{\partial}_2\left(\begin{pmatrix} 0 \\ 0 \\ \beta\alpha \end{pmatrix}\right) &= \begin{pmatrix} 0 \\ -\beta\alpha\otimes\beta\alpha \\ 0 \end{pmatrix},
\end{aligned}$$

and we see that $\operatorname{Ker}\tilde{\partial}_2 = \left\langle \begin{pmatrix} 0 \\ \alpha\otimes\beta \\ \alpha \end{pmatrix}, \begin{pmatrix} 0 \\ \beta\alpha\otimes\beta \\ \beta\alpha \end{pmatrix} \right\rangle$ and

$$\begin{aligned} \operatorname{Im}\tilde{\partial}_2 \quad = \quad & \left\langle \begin{pmatrix} \alpha \\ 0 \\ \alpha\otimes\beta \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta\alpha\otimes e_1 - e_1\otimes\beta\alpha \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e_1\otimes\beta\alpha \\ \alpha\otimes\beta \end{pmatrix}, \right. \\ & \left. \begin{pmatrix} 0 \\ \alpha\otimes\beta\alpha \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \beta\alpha\otimes\beta\alpha \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -\alpha\otimes\beta \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -\beta\alpha\otimes\beta \\ 0 \end{pmatrix} \right\rangle. \end{aligned}$$

We know that $e_2\mathbf{c} = \langle e_2, \alpha\rangle$, so:

$$\begin{aligned} \tilde{\partial}_1\left(\begin{pmatrix} e_2 \\ 0 \\ 0 \end{pmatrix}\right) &= \alpha\otimes\beta, \\ \tilde{\partial}_1\left(\begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix}\right) &= \alpha\otimes\beta\alpha. \end{aligned}$$

We also know that $\mathbf{c}e_1\otimes e_1\mathbf{c} = \langle e_1\otimes e_1, \alpha\otimes e_1, \beta\alpha\otimes e_1, e_1\otimes\beta, \alpha\otimes\beta, \beta\alpha\otimes\beta, e_1\otimes\beta\alpha,$ $\alpha\otimes\beta\alpha, \beta\alpha\otimes\beta\alpha\rangle$, so:

$$\begin{aligned} \tilde{\partial}_1\left(\begin{pmatrix} 0 \\ e_1\otimes e_1 \\ 0 \end{pmatrix}\right) &= \alpha\otimes e_1, & \tilde{\partial}_1\left(\begin{pmatrix} 0 \\ e_1\otimes\beta \\ 0 \end{pmatrix}\right) &= \alpha\otimes\beta, \\ \tilde{\partial}_1\left(\begin{pmatrix} 0 \\ \alpha\otimes e_1 \\ 0 \end{pmatrix}\right) &= 0, & \tilde{\partial}_1\left(\begin{pmatrix} 0 \\ \alpha\otimes\beta \\ 0 \end{pmatrix}\right) &= 0, \\ \tilde{\partial}_1\left(\begin{pmatrix} 0 \\ \beta\alpha\otimes e_1 \\ 0 \end{pmatrix}\right) &= 0, & \tilde{\partial}_1\left(\begin{pmatrix} 0 \\ \beta\alpha\otimes\beta \\ 0 \end{pmatrix}\right) &= 0, \\ \tilde{\partial}_1\left(\begin{pmatrix} 0 \\ e_1\otimes\beta\alpha \\ 0 \end{pmatrix}\right) &= \alpha\otimes\beta\alpha, \\ \tilde{\partial}_1\left(\begin{pmatrix} 0 \\ \alpha\otimes\beta\alpha \\ 0 \end{pmatrix}\right) &= 0, \\ \tilde{\partial}_1\left(\begin{pmatrix} 0 \\ \beta\alpha\otimes\beta\alpha \\ 0 \end{pmatrix}\right) &= 0, \end{aligned}$$

and $e_2\mathbf{c}e_1\otimes e_1\mathbf{c}e_2 = \langle\alpha\otimes\beta\rangle$, so:

$$\tilde{\partial}_1\left(\begin{pmatrix} 0 \\ 0 \\ \alpha\otimes\beta \end{pmatrix}\right) = -\alpha\otimes\beta\alpha.$$

Hence

$$\begin{aligned}\operatorname{Ker}\tilde{\partial}_1 &= \left\langle \begin{pmatrix}\alpha\\0\\\alpha\otimes\beta\end{pmatrix}, \begin{pmatrix}-e_2\\e_1\otimes\beta\\0\end{pmatrix}, \begin{pmatrix}0\\e_1\otimes\beta\alpha\\\alpha\otimes\beta\end{pmatrix}, \begin{pmatrix}0\\\alpha\otimes e_1\\0\end{pmatrix}, \begin{pmatrix}0\\\alpha\otimes\beta\\0\end{pmatrix},\right.\\ &\left.\begin{pmatrix}0\\\beta\alpha\otimes e_1\\0\end{pmatrix}, \begin{pmatrix}0\\\beta\alpha\otimes\beta\\0\end{pmatrix}, \begin{pmatrix}0\\\alpha\otimes\beta\alpha\\0\end{pmatrix}, \begin{pmatrix}0\\\beta\alpha\otimes\beta\alpha\\0\end{pmatrix}\right\rangle\end{aligned}$$

and $\operatorname{Im}\tilde{\partial}_1 = \langle \alpha\otimes e_1, \alpha\otimes\beta, \alpha\otimes\beta\alpha\rangle$.

We can compute homology; to sum up we have:

$$\begin{aligned}
\operatorname{Ker}\tilde{\partial}_3 &= 0 && \Rightarrow\ H^3(\mathcal{K}(\tilde{\mathbf{t}}))=0\\
\operatorname{Im}\tilde{\partial}_3 &= \left\langle \begin{pmatrix}0\\-\alpha\otimes\beta\\-\alpha\end{pmatrix}, \begin{pmatrix}0\\-\beta\alpha\otimes\beta\\-\beta\alpha\end{pmatrix}\right\rangle && \\
\operatorname{Ker}\tilde{\partial}_2 &= \left\langle \begin{pmatrix}0\\\alpha\otimes\beta\\\alpha\end{pmatrix}, \begin{pmatrix}0\\\beta\alpha\otimes\beta\\\beta\alpha\end{pmatrix}\right\rangle && \Rightarrow\ H^2(\mathcal{K}(\tilde{\mathbf{t}}))=0\\
\operatorname{Im}\tilde{\partial}_2 &= \left\langle \begin{pmatrix}\alpha\\0\\\alpha\otimes\beta\end{pmatrix}, \begin{pmatrix}\alpha\\\beta\alpha\otimes e_1 - e_1\otimes\beta\alpha\\0\end{pmatrix}, \begin{pmatrix}0\\e_1\otimes\beta\alpha\\\alpha\otimes\beta\end{pmatrix},\right.\\
&\left.\begin{pmatrix}0\\\alpha\otimes\beta\alpha\\0\end{pmatrix}, \begin{pmatrix}0\\\beta\alpha\otimes\beta\alpha\\0\end{pmatrix}, \begin{pmatrix}0\\-\alpha\otimes\beta\\0\end{pmatrix}, \begin{pmatrix}0\\-\beta\alpha\otimes\beta\\0\end{pmatrix}\right\rangle && \Rightarrow\ H^1(\mathcal{K}(\tilde{\mathbf{t}}))=\langle v_1,v_2\rangle\\
\operatorname{Ker}\tilde{\partial}_1 &= \left\langle \begin{pmatrix}\alpha\\0\\\alpha\otimes\beta\end{pmatrix}, \begin{pmatrix}-e_2\\e_1\otimes\beta\\0\end{pmatrix}, \begin{pmatrix}0\\e_1\otimes\beta\alpha\\\alpha\otimes\beta\end{pmatrix}, \begin{pmatrix}0\\\alpha\otimes e_1\\0\end{pmatrix}, \begin{pmatrix}0\\\alpha\otimes\beta\\0\end{pmatrix},\right.\\
&\left.\begin{pmatrix}0\\\beta\alpha\otimes e_1\\0\end{pmatrix}, \begin{pmatrix}0\\\beta\alpha\otimes\beta\\0\end{pmatrix}, \begin{pmatrix}0\\\alpha\otimes\beta\alpha\\0\end{pmatrix}, \begin{pmatrix}0\\\beta\alpha\otimes\beta\alpha\\0\end{pmatrix}\right\rangle
\end{aligned}$$

where $v_1 = \begin{pmatrix}e_2\\-e_1\otimes\beta\\0\end{pmatrix}$ and $v_2 = \begin{pmatrix}0\\\alpha\otimes e_1\\0\end{pmatrix}$.

There is one non-zero homology group, and there is a natural **d**-**d**-bimodule structure on it, e.g. there is a left action of e_1 on $\begin{pmatrix}e_2\\-e_1\otimes\beta\\0\end{pmatrix}$, since using the map $f_1 : \mathbf{d}\to\mathcal{K}(\mathbf{c})$, we know that:

$$\begin{aligned}
e_1\cdot\begin{pmatrix}e_2\\-e_1\otimes\beta\\0\end{pmatrix} &= (\langle e_1\mapsto e_1\rangle^0 + \langle e_2^I\mapsto e_2^I\rangle^0)\begin{pmatrix}e_2\\-e_1\otimes\beta\\0\end{pmatrix}\\
&= \begin{pmatrix}\langle e_2^I\mapsto e_2^I\rangle^0(e_2)\\\langle e_1\mapsto e_1\rangle^0(-e_1\otimes\beta)\\0\end{pmatrix}\\
&= \begin{pmatrix}e_2\\-e_1\otimes\beta\\0\end{pmatrix}
\end{aligned}$$

The only part of **d** acting non trivially on both sides is the semi-simple top, thus this homology group really is a $\mathbf{d}^0$-$\mathbf{d}^0$-bimodule.

Recall that the homology of $\mathbf{u} := \mathbf{d} \otimes^{\sigma}_{\mathbf{d}^0} \mathbf{d}^*$ is given by $\mathbb{H}(\mathbf{u}) = \langle e_1 \otimes e_2^*, e_2 \otimes e_1^* \rangle$, which is isomorphic to ${}^{\sigma}(\mathbf{d}^0)$. We can thus define the following map:

$$\begin{aligned} f_1 : \mathbb{H}(\mathbf{u}) &\to \mathcal{K}(\mathbf{t}) \\ e_1 \otimes e_2^* &\mapsto \begin{pmatrix} e_2 \\ -e_1 \otimes \beta \\ 0 \end{pmatrix} \\ e_2 \otimes e_1^* &\mapsto \begin{pmatrix} 0 \\ \alpha \otimes e_1 \\ 0 \end{pmatrix} \end{aligned}$$

In addition, if we want homological degrees to match, we can shift the complex $\mathcal{K}(\tilde{\mathbf{t}})$.

5.5.2.2 Making m_2's explicit

We are now concerned with the following maps:

- $m_2 : \mathbf{d} \otimes \mathbb{H}(\mathbf{u}) \to \mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})$;
- $m_2 : \mathbb{H}(\mathbf{u}) \otimes \mathbf{d} \to \mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})$;
- $m_2 : \mathbb{T}_{\mathcal{K}(\mathbf{c})}\left(\mathcal{K}(\tilde{\mathbf{t}})\right) \otimes \mathbb{T}_{\mathcal{K}(\mathbf{c})}\left(\mathcal{K}(\tilde{\mathbf{t}})\right) \to \mathbb{T}_{\mathcal{K}(\mathbf{c})}\left(\mathcal{K}(\tilde{\mathbf{t}})\right)$.

The first two consist in concatenating the tensors and using the multiplication in $\mathbf{d}$.

The third multiplication map comes from the natural multiplication in the tensor algebra, that is the multiplication that arises from the $\mathcal{K}(\mathbf{c})$-bimodule structure on $\mathcal{K}(\tilde{\mathbf{t}})$. It is given in figures 5.5 and 5.6.

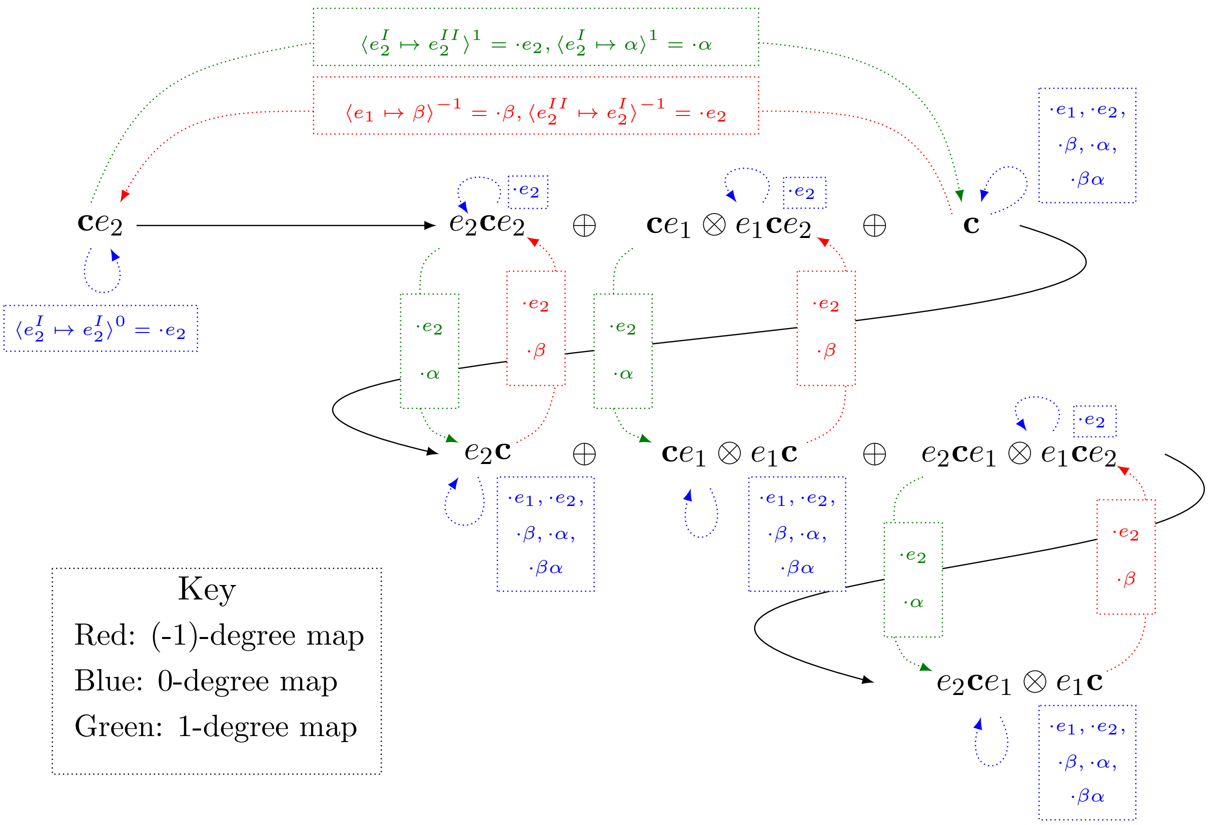


Figure 5.5: Right action of $\mathcal{K}(\mathbf{c})$ on $\mathcal{K}(\tilde{\mathbf{t}})$

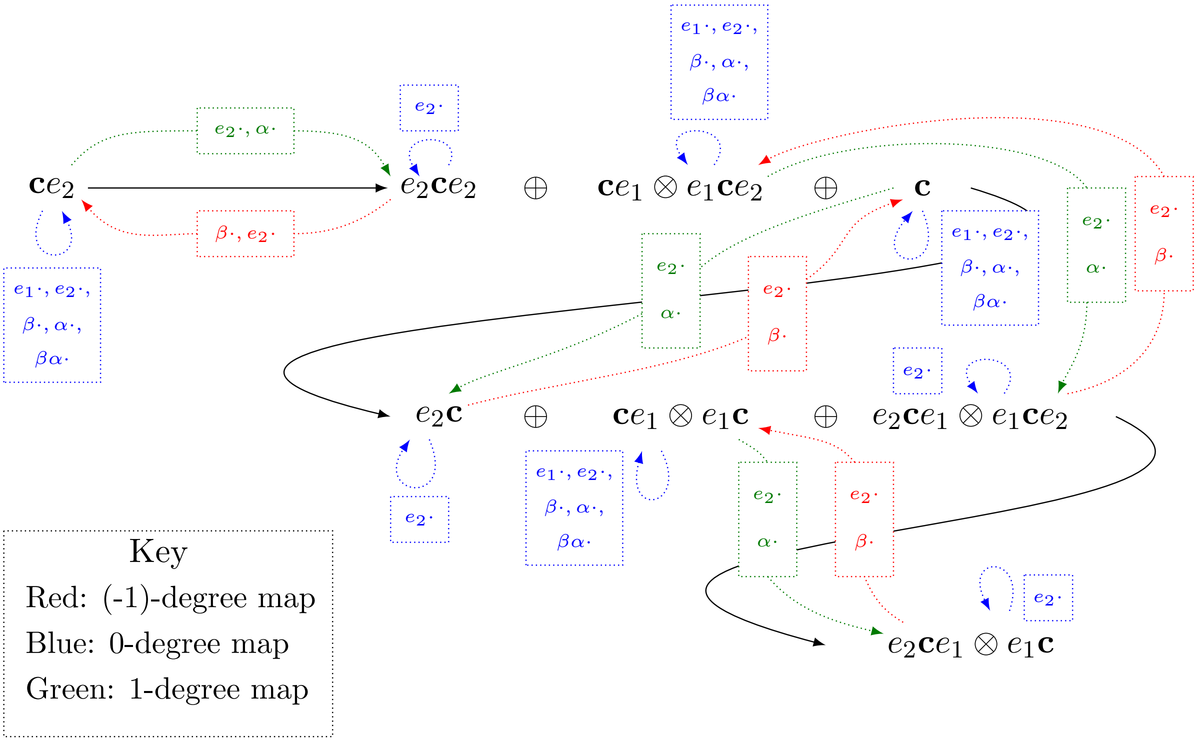


Figure 5.6: Left action of $\mathcal{K}(\mathbf{c})$ on $\mathcal{K}(\tilde{\mathbf{t}})$

5.5.2.3 Making f_2's explicit

We can now tackle the description of the maps:

- $f_2 : \mathbf{d} \otimes \mathbb{H}(\mathbf{u}) \to \mathbb{T}_{\mathcal{K}(\mathbf{c})}\left(\mathcal{K}(\tilde{\mathbf{t}})\right)$;
- $f_2 : \mathbb{H}(\mathbf{u}) \otimes \mathbf{d} \to \mathbb{T}_{\mathcal{K}(\mathbf{c})}\left(\mathcal{K}(\tilde{\mathbf{t}})\right)$.

To achieve this, we need to evaluate the following expression Φ_2, derived from the A_∞-structure relations, on $\mathbf{d} \otimes \mathbb{H}(\mathbf{u})$ and $\mathbb{H}(\mathbf{u}) \otimes \mathbf{d}$, and then find preimages of non-zero elements under the differential:

$$\Phi_2 = f_1 m_2 - m_2(f_1 \otimes f_1).$$

Note that we do not need to pay attention to hypothetical signs coming from the Koszul sign rule when evaluating Φ_2: the maps f_1 and m_2 are both of degree 0.

$\mathbf{d} \otimes \mathbb{H}(\mathbf{u})$	m_2	$f_1 m_2$	$f_1 \otimes f_1$	$m_2(f_1 \otimes f_1)$	Φ_2
$e_1 \otimes (e_1 \otimes e_2^*)$	$e_1 \otimes e_2^*$	$\begin{pmatrix} e_2 \\ -e_1 \otimes \beta \\ 0 \end{pmatrix}$	$\begin{matrix}(\langle e_1 \mapsto e_1\rangle^0 \\ +\langle e_2^I \mapsto e_2^I\rangle^0)\end{matrix} \otimes \begin{pmatrix} e_2 \\ -e_1 \otimes \beta \\ 0 \end{pmatrix}$	$\begin{pmatrix} e_2 \\ -e_1 \otimes \beta \\ 0 \end{pmatrix}$	0
$e_1 \otimes (e_2 \otimes e_1^*)$	0	0	$\begin{matrix}(\langle e_1 \mapsto e_1\rangle^0 \\ +\langle e_2^I \mapsto e_2^I\rangle^0)\end{matrix} \otimes \begin{pmatrix} 0 \\ \alpha \otimes e_1 \\ 0 \end{pmatrix}$	0	0
$e_2 \otimes (e_1 \otimes e_2^*)$	0	0	$\langle e_2^{II} \mapsto e_2^{II}\rangle^0 \otimes \begin{pmatrix} e_2 \\ -e_1 \otimes \beta \\ 0 \end{pmatrix}$	0	0
$e_2 \otimes (e_2 \otimes e_1^*)$	$e_2 \otimes e_1^*$	$\begin{pmatrix} 0 \\ \alpha \otimes e_1 \\ 0 \end{pmatrix}$	$\langle e_2^{II} \mapsto e_2^{II}\rangle^0 \otimes \begin{pmatrix} 0 \\ \alpha \otimes e_1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ \alpha \otimes e_1 \\ 0 \end{pmatrix}$	0
$\xi \otimes (e_1 \otimes e_2^*)$	0	0	$\langle e_1 \mapsto \beta\rangle^0 \otimes \begin{pmatrix} e_2 \\ -e_1 \otimes \beta \\ 0 \end{pmatrix}$	0	0
$\xi \otimes (e_2 \otimes e_1^*)$	0	0	$\langle e_1 \mapsto \beta\rangle^0 \otimes \begin{pmatrix} 0 \\ \alpha \otimes e_1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ \beta\alpha \otimes e_1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ -\beta\alpha \otimes e_1 \\ 0 \end{pmatrix}$
$x \otimes (e_1 \otimes e_2^*)$	0	0	$\langle e_2^I \mapsto e_2^{II}\rangle^1 \otimes \begin{pmatrix} e_2 \\ -e_1 \otimes \beta \\ 0 \end{pmatrix}$	0	0
$x \otimes (e_2 \otimes e_1^*)$	0	0	$\langle e_2^I \mapsto e_2^{II}\rangle^1 \otimes \begin{pmatrix} 0 \\ \alpha \otimes e_1 \\ 0 \end{pmatrix}$	$\alpha \otimes e_1$	$-\alpha \otimes e_1$

We have:

$$
\begin{aligned}
m_1 \begin{pmatrix} e_2 \\ -e_1 \otimes \beta \\ -e_1 \end{pmatrix} &= \begin{pmatrix} 0 \\ -\beta\alpha \otimes e_1 \\ 0 \end{pmatrix} \\
m_1 \begin{pmatrix} 0 \\ -e_1 \otimes e_1 \\ 0 \end{pmatrix} &= -\alpha \otimes e_1
\end{aligned}
$$

Therefore, we choose:

$$
\begin{aligned}
f_2(\xi \otimes (e_2 \otimes e_1^*)) &= \begin{pmatrix} e_2 \\ -e_1 \otimes \beta \\ -e_1 \end{pmatrix} \\
f_2(x \otimes (e_2 \otimes e_1^*)) &= \begin{pmatrix} 0 \\ -e_1 \otimes e_1 \\ 0 \end{pmatrix}
\end{aligned}
$$

$\mathbb{H}(\mathbf{u})\otimes\mathbf{d}$	m_2	f_1m_2	$f_1\otimes f_1$	$m_2(f_1\otimes f_1)$	Φ_2
$(e_1\otimes e_2^*)\otimes e_1$	0	0	$\begin{pmatrix} e_2 \\ -e_1\otimes\beta \\ 0 \end{pmatrix}\otimes \begin{matrix}(\langle e_1\mapsto e_1\rangle^0 \\ +\langle e_2^I\mapsto e_2^I\rangle^0)\end{matrix}$	0	0
$(e_2\otimes e_1^*)\otimes e_1$	$e_2\otimes e_1^*$	$\begin{pmatrix} 0 \\ \alpha\otimes e_1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ \alpha\otimes e_1 \\ 0 \end{pmatrix}\otimes \begin{matrix}(\langle e_1\mapsto e_1\rangle^0 \\ +\langle e_2^I\mapsto e_2^I\rangle^0)\end{matrix}$	$\begin{pmatrix} 0 \\ \alpha\otimes e_1 \\ 0 \end{pmatrix}$	0
$(e_1\otimes e_2^*)\otimes e_2$	$e_1\otimes e_2^*$	$\begin{pmatrix} e_2 \\ -e_1\otimes\beta \\ 0 \end{pmatrix}$	$\begin{pmatrix} e_2 \\ -e_1\otimes\beta \\ 0 \end{pmatrix}\otimes\langle e_2^{II}\mapsto e_2^{II}\rangle^0$	$\begin{pmatrix} e_2 \\ -e_1\otimes\beta \\ 0 \end{pmatrix}$	0
$(e_2\otimes e_1^*)\otimes e_2$	0	0	$\begin{pmatrix} 0 \\ \alpha\otimes e_1 \\ 0 \end{pmatrix}\otimes\langle e_2^{II}\mapsto e_2^{II}\rangle^0$	0	0
$(e_1\otimes e_2^*)\otimes \xi$	0	0	$\begin{pmatrix} e_2 \\ -e_1\otimes\beta \\ 0 \end{pmatrix}\otimes\langle e_1\mapsto \beta\rangle^0$	0	0
$(e_2\otimes e_1^*)\otimes \xi$	0	0	$\begin{pmatrix} 0 \\ \alpha\otimes e_1 \\ 0 \end{pmatrix}\otimes\langle e_1\mapsto \beta\rangle^0$	$\begin{pmatrix} 0 \\ \alpha\otimes\beta \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ -\alpha\otimes\beta \\ 0 \end{pmatrix}$
$(e_1\otimes e_2^*)\otimes x$	0	0	$\begin{pmatrix} e_2 \\ -e_1\otimes\beta \\ 0 \end{pmatrix}\otimes\langle e_2^{I}\mapsto e_2^{II}\rangle^1$	0	0
$(e_2\otimes e_1^*)\otimes x$	0	0	$\begin{pmatrix} 0 \\ \alpha\otimes e_1 \\ 0 \end{pmatrix}\otimes\langle e_2^{I}\mapsto e_2^{II}\rangle^1$	0	0

We have:

$$m_1\begin{pmatrix} 0 \\ 0 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -\alpha\otimes\beta \\ 0 \end{pmatrix}$$

Therefore, we choose:

$$f_2((e_2\otimes e_1^*)\otimes\xi) = \begin{pmatrix} 0 \\ 0 \\ e_2 \end{pmatrix}$$

5.5.2.4 Computing m_3

As stated in the introduction, to compute m_3 on $\mathbf{d}\otimes\mathbb{H}(\mathbf{u})\otimes\mathbf{d}$, we have the following formula derived from the relations defining an A_∞-structure:

$$f_1m_3 = m_1f_3 + m_2(f_1\otimes f_2 - f_2\otimes f_1) + f_2(\mathbf{1}\otimes m_2 - m_2\otimes\mathbf{1}),$$

which means we must consider:

$$\Phi_3 := m_2(f_1\otimes f_2 - f_2\otimes f_1) + f_2(\mathbf{1}\otimes m_2 - m_2\otimes\mathbf{1}).$$

Let us evaluate it on $\mathbf{d}\otimes\mathbb{H}(\mathbf{u})\otimes\mathbf{d}$. Note that we need to be careful with signs:

$$(f_1\otimes f_2)(x\otimes y) = (-1)^{|f_2||x|}f_1(x)\otimes f_2(y).$$

Instead of writing all basis elements, we shall only write those which yield non-trivial results:

$\mathbf{d}\otimes\mathbb{H}(\mathbf{u})\otimes d$	$f_1\otimes f_2$	$-f_2\otimes f_1$	$m_2(f_1\otimes f_2 - f_2\otimes f_1)$	$f_2(\mathbf{1}\otimes m_2 - m_2\otimes\mathbf{1})$	Φ_3
$\xi\otimes(e_2\otimes e_1^*)\otimes e_1$	0	$-\begin{pmatrix} e_2\\ -e_1\otimes\beta\\ -e_1\end{pmatrix}\otimes \begin{matrix}(\langle e_1\mapsto e_1\rangle^0\\ +\langle e_2^I\mapsto e_2^I\rangle^0)\end{matrix}$	$\begin{pmatrix}-e_2\\ e_1\otimes\beta\\ e_1\end{pmatrix}$	$f_2(\xi\otimes(e_2\otimes e_1^*)) = \begin{pmatrix}e_2\\ -e_1\otimes\beta\\ -e_1\end{pmatrix}$	0
$x\otimes(e_2\otimes e_1^*)\otimes e_1$	0	$-\begin{pmatrix} 0\\ -e_1\otimes e_1\\ 0\end{pmatrix}\otimes \begin{matrix}(\langle e_1\mapsto e_1\rangle^0\\ +\langle e_2^I\mapsto e_2^I\rangle^0)\end{matrix}$	$\begin{pmatrix}0\\ e_1\otimes e_1\\ 0\end{pmatrix}$	$f_2(x\otimes(e_2\otimes e_1^*)) = \begin{pmatrix}0\\ -e_1\otimes e_1\\ 0\end{pmatrix}$	0
$\xi\otimes(e_2\otimes e_1^*)\otimes e_2$	0	$-\begin{pmatrix} e_2\\ -e_1\otimes\beta\\ -e_1\end{pmatrix}\otimes\langle e_2^{II}\mapsto e_2^{II}\rangle^0$	0	0	0
$x\otimes(e_2\otimes e_1^*)\otimes e_2$	0	$-\begin{pmatrix} 0\\ -e_1\otimes e_1\\ 0\end{pmatrix}\otimes\langle e_2^{II}\mapsto e_2^{II}\rangle^0$	0	0	0
$e_2\otimes(e_2\otimes e_1^*)\otimes\xi$	$\langle e_2^{II}\mapsto e_2^{II}\rangle^0\otimes\begin{pmatrix}0\\ 0\\ e_2\end{pmatrix}$	0	$\begin{pmatrix}0\\ 0\\ e_2\end{pmatrix}$	$f_2(-(e_2\otimes e_1^*)\otimes\xi) = -\begin{pmatrix}0\\ 0\\ e_2\end{pmatrix}$	0
$\xi\otimes(e_2\otimes e_1^*)\otimes\xi$	$\langle e_1\mapsto\beta\rangle^0\otimes\begin{pmatrix}0\\ 0\\ e_2\end{pmatrix}$	$-\begin{pmatrix} e_2\\ -e_1\otimes\beta\\ -e_1\end{pmatrix}\otimes\langle e_1\mapsto\beta\rangle^0$	$\begin{pmatrix}0\\ 0\\ 2\beta\end{pmatrix} = \mathbf{0}$	0	0
$x\otimes(e_2\otimes e_1^*)\otimes\xi$	$-\langle e_2^I\mapsto e_2^{II}\rangle^1\otimes\begin{pmatrix}0\\ 0\\ e_2\end{pmatrix}$	$-\begin{pmatrix} 0\\ -e_1\otimes e_1\\ 0\end{pmatrix}\otimes\langle e_1\mapsto\beta\rangle^0$	$\begin{pmatrix}-e_2\\ 0\\ 0\end{pmatrix}+\begin{pmatrix}0\\ e_1\otimes\beta\\ 0\end{pmatrix}$	0	$\begin{pmatrix}-e_2\\ e_1\otimes\beta\\ 0\end{pmatrix}$
$\xi\otimes(e_2\otimes e_1^*)\otimes x$	0	$-\begin{pmatrix} e_2\\ -e_1\otimes\beta\\ -e_1\end{pmatrix}\otimes\langle e_2^I\mapsto e_2^{II}\rangle^1$	$\begin{pmatrix}-e_2\\ e_1\otimes\beta\\ 0\end{pmatrix}$	0	$\begin{pmatrix}-e_2\\ e_1\otimes\beta\\ 0\end{pmatrix}$
$x\otimes(e_2\otimes e_1^*)\otimes x$	0	$-\begin{pmatrix} 0\\ -e_1\otimes e_1\\ 0\end{pmatrix}\otimes\langle e_2^I\mapsto e_2^{II}\rangle^1$	0	0	0

In a nutshell, we find:

$$
\begin{aligned}
\Phi_3(\xi\otimes(e_2\otimes e_1^*)\otimes\xi) &= \begin{pmatrix}0\\ 0\\ 2\beta\end{pmatrix} && = 0 \text{ since } \operatorname{char} k = 2,\\
\Phi_3(x\otimes(e_2\otimes e_1^*)\otimes\xi) &= \begin{pmatrix}-e_2\\ e_1\otimes\beta\\ 0\end{pmatrix} && = f_1(-e_1\otimes e_2^*)\\
\Phi_3(\xi\otimes(e_2\otimes e_1^*)\otimes x) &= \begin{pmatrix}-e_2\\ e_1\otimes\beta\\ 0\end{pmatrix} && = f_1(-e_1\otimes e_2^*)
\end{aligned}
$$

We then choose:

$$m_3(x\otimes(e_2\otimes e_1^*)\otimes\xi) = m_3(\xi\otimes(e_2\otimes e_1^*)\otimes x) = -e_1\otimes e_2^*\in\mathbb{H}(\mathbf{u}),$$

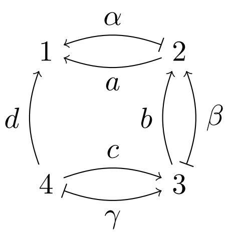
and take f_3 to be the zero map. The construction therefore finishes.

Chapter 6

Formality Result for $p = 2$

6.1 Non-trivial A_∞-structure on $\mathrm{Ext}(\Delta, \Delta)$ and $\mathbf{w}_2$

We recall the A_∞-structure obtained on $\mathrm{Ext}_{\mathbf{c}_2(\mathbf{c}_2,\mathbf{t}_2)}(\Delta, \Delta)$ in Example 5.5.1. We can represent $\mathrm{Ext}(\Delta, \Delta)^{op} \cong \mathbf{w}_2$ as the path algebra of the following quiver:



modulo relations $(ab, bc, \alpha\beta c - a\beta\gamma, \alpha b, b\gamma)$.

We showed that the only non-zero higher multiplication is m_3, and the only non-zero terms are

$$m_3(a \otimes b \otimes \gamma) = m_3(\alpha \otimes b \otimes c) = d. \tag{6.1}$$

Now, we can replace a, b, γ, α, c, and d by their expression in terms of x's and ξ's thanks to Chapter 3. We obtained

$$\begin{aligned} a &= e_1 \otimes \xi \\ \alpha &= e_1 \otimes x \\ b &= \xi \otimes (e_2 \otimes e_1^*) \\ c &= e_2 \otimes \xi \\ \gamma &= e_2 \otimes x \\ d &= \xi \otimes (e_1 \otimes e_2^*). \end{aligned}$$

Rewriting Equation (6.1), we see

$$\begin{aligned} m_3((e_1 \otimes \xi) \otimes (\xi \otimes (e_2 \otimes e_1^*)) \otimes (e_2 \otimes x)) &= (\xi \otimes (e_1 \otimes e_2^*)) \\ m_3((e_1 \otimes x) \otimes (\xi \otimes (e_2 \otimes e_1^*)) \otimes (e_2 \otimes \xi)) &= (\xi \otimes (e_1 \otimes e_2^*)) . \end{aligned}$$

A possible interpretation is that it encodes the fact that acting on $\mathbf{u}$ on the left or on the right is not trivial.

6.2 Computation of an A_∞-structure on $\mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})$

We need to restrict the scope of our computation to $\mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})$ in order to apply Kadeishvili's theorem and use the recipe to construct an A_∞-structure (as can be seen in [Mad02, Appendix B.] or Keller in [Kel99]). Indeed, we saw in Chapter 2 that $\mathbb{T}_{\mathbf{d}}(\underline{\mathbf{u}})$ is not an algebra, and it is enough to consider the subspace $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})^{\leq 1}$ in homology to obtain the alternative description of $\mathbf{w}_q$. Thus, we will focus on the subspace $\mathbb{T}_{\mathbf{d}}(\mathbf{u}^{-1})$ of $\mathbb{T}_{\mathbf{d}}(\underline{\mathbf{u}})$ which is an algebra and such that its homology is a subspace of the necessary part to construct $\mathbf{w}_q$.

We can decompose this vector space according to the k-grading:

$$\mathbb{T}_{\mathbf{d}}(\mathbf{u}^{-1}) = \bigoplus_{k\in\mathbb{Z}} \mathbb{T}_{\mathbf{d}}(\mathbf{u}^{-1})^k.$$

As mentioned in Chapter 2, on each $\mathbf{u}^{-i}$, there is a differential δ_i of k-degree 1 obtained by internal multiplication by some elements (cf. [MT13]), and therefore, $\mathbb{T}_{\mathbf{d}}(\mathbf{u}^{-1})$ can be turned into a differential complex. We denote by $m_1 : \mathbb{T}_{\mathbf{d}}(\mathbf{u}^{-1}) \to \mathbb{T}_{\mathbf{d}}(\mathbf{u}^{-1})$ the differential obtained in this way. As a consequence, $\mathbb{T}_{\mathbf{d}}(\mathbf{u}^{-1})$ is a dg-algebra, and by Kadeishvili's theorem, its homology $\mathbb{H}(\mathbb{T}_{\mathbf{d}}(\mathbf{u}^{-1}))$ carries an A_∞-structure. We now wish to make it explicit and will use the recipe mentioned previously.

We assume that the characteristic p is equal to 2 in the rest of this section. In [MT13, pp. 188-189], the homology $\mathbb{H}(\mathbf{u}^{-i})$ of $\mathbf{u}^{-i}$ has been computed and using the same notation, we have $\mathbb{H}(\mathbf{u}^{-i}) \cong V_i$, for $i > 0$, and $\mathbb{H}(\mathbf{d}) = \mathbf{d}$ since $\delta_0 = 0$ (recall that $\mathbf{d} = \mathrm{Ext}_{\mathbf{c}_2}(\Delta, \Delta)$). Finally, we know that $\mathbb{H}(\mathbf{u}) = (\mathbf{d}^0)^\sigma$.

It is then possible to represent basis elements of $\mathbb{H}(\mathbf{u}^{-i})$ in the form:

$$e_2^{\otimes\epsilon} \otimes \xi^{\otimes m} \otimes x^{\otimes l} \otimes e_1^{\otimes\eta}, \tag{6.2}$$

with $\epsilon, \eta \in \{0, 1\}$, $m, l \in \mathbb{N}_0$, and such that $\epsilon + m + l + \eta = i + 1$, and its k-degree is $1 - \epsilon - m - \eta$. This description is also valid if $i = 0$. As expressions involving tensors will get bigger, we will more often than not notationally simplify those by omitting the tensor product symbol; e.g. expression 6.2 is equivalent to $e_2^\epsilon \xi^m x^l e_1^\eta$.

1. Let $m_1 : \mathbb{H}(\mathbb{T}_{\mathbf{d}}(\mathbf{u}^{-1})) \to \mathbb{H}(\mathbb{T}_{\mathbf{d}}(\mathbf{u}^{-1}))$ be zero.

   We need to choose f_1 as a morphism of complexes $\big(\mathbb{H}(\mathbb{T}_{\mathbf{d}}(\mathbf{u}^{-1})), 0\big) \to \big(\mathbb{T}_{\mathbf{d}}(\mathbf{u}^{-1}), \delta_1\big)$ such that $\mathbb{H}f_1$ identifies with the identity of $\mathbb{H}(\mathbb{T}_{\mathbf{d}}(\mathbf{u}^{-1}))$. There is an obvious choice by taking lifts of basis elements of $\mathbb{H}(\mathbb{T}_{\mathbf{d}}(\mathbf{u}^{-1}))$ in $\mathbb{T}_{\mathbf{d}}(\mathbf{u}^{-1})$.

2. Let $m_2 : \mathbb{H}(\mathbb{T}_{\mathbf{d}}(\mathbf{u}^{-1})) \otimes \mathbb{H}(\mathbb{T}_{\mathbf{d}}(\mathbf{u}^{-1})) \to \mathbb{H}(\mathbb{T}_{\mathbf{d}}(\mathbf{u}^{-1}))$ be the multiplication map induced from the multiplication of $\mathbb{T}_{\mathbf{d}}(\mathbf{u}^{-1})$.

   Then, by definition, the morphisms $m_2(f_1 \otimes f_1)$ and $f_1 m_2$ are homotopic as morphisms of complexes from $\mathbb{H}(\mathbb{T}_{\mathbf{d}}(\mathbf{u}^{-1})) \otimes \mathbb{H}(\mathbb{T}_{\mathbf{d}}(\mathbf{u}^{-1}))$ to $\mathbb{T}_{\mathbf{d}}(\mathbf{u}^{-1})$.

   We need to choose $f_2 : \mathbb{H}(\mathbb{T}_{\mathbf{d}}(\mathbf{u}^{-1})) \otimes \mathbb{H}(\mathbb{T}_{\mathbf{d}}(\mathbf{u}^{-1})) \to \mathbb{T}_{\mathbf{d}}(\mathbf{u}^{-1})$ as a graded map of degree -1 such that

   $$f_1 m_2 = m_1 f_2 + m_2(f_1 \otimes f_1).$$

   Let $\Phi_2 = f_1 m_2 - m_2(f_1 \otimes f_1)$. We need to compute the image of each basis elements of $\mathbb{H}(\mathbb{T}_{\mathbf{d}}(\mathbf{u}^{-1})) \otimes \mathbb{H}(\mathbb{T}_{\mathbf{d}}(\mathbf{u}^{-1}))$ under Φ_2.

We have:

$$\begin{aligned}
&\Phi_2\left(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1}, e_2^{\epsilon_2}\xi^{m_2}x^{l_2}e_1^{\eta_2}\right)\\
= \; & f_1(m_2\left(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1}, e_2^{\epsilon_2}\xi^{m_2}x^{l_2}e_1^{\eta_2}\right))\\
&-m_2((f_1\otimes f_1)\left(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1}, e_2^{\epsilon_2}\xi^{m_2}x^{l_2}e_1^{\eta_2}\right))\\
= \; & f_1((1-\delta_{\eta_1,\epsilon_2})(-1)^{m_2l_1}e_2^{\epsilon_1}\xi^{m_1+m_2}x^{l_1+l_2}e_1^{\eta_2})\\
&-m_2(\left(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1}, e_2^{\epsilon_2}\xi^{m_2}x^{l_2}e_1^{\eta_2}\right))\\
= \; & (1-\delta_{\eta_1,\epsilon_2})\left((-1)^{m_2l_1}e_2^{\epsilon_1}\xi^{m_1+m_2}x^{l_1+l_2}e_1^{\eta_2} - e_2^{\epsilon_1}\xi^{m_1}x^{l_1}\xi^{m_2}x^{l_2}e_1^{\eta_2}\right).
\end{aligned}$$

To choose f_2, we need a good understanding of the image of $\mathbf{u}^{-i}$ under the differential δ_i. For $i = 0, 1$, they are defined as follows ([MT13]):

$$\begin{aligned}
\delta_{-1}: \mathbf{u}\to\mathbf{u}, &\qquad a\otimes b\mapsto (-1)^{|a|_k}(ax\otimes\xi b + a\xi\otimes xb)\\
\delta_0: \mathbf{d}\to\mathbf{d}, &\qquad a\otimes b\mapsto 0\\
\delta_1: \mathbf{u}^{-1}\to\mathbf{u}^{-1}, &\qquad a\otimes b\mapsto (-1)^{|a|_k}(ax\otimes\xi b + a\xi\otimes xb)
\end{aligned}$$

and more generally, we have:

Lemma 6.2.1. *For $i\geq 1$, $\delta_i : \mathbf{u}^{-i}\to\mathbf{u}^{-i}$ sends $a_1\otimes\ldots\otimes a_{i+1}$ to*

$$\sum_{l=1}^{i}(-1)^{\sum_{j=1}^{l}|a_i|_k}\left(\ldots\otimes a_lx\otimes\xi a_{l+1}\otimes\ldots+\ldots\otimes a_l\xi\otimes xa_{l+1}\otimes\ldots\right).$$

Idea of proof. Recall the description of $\mathbf{u}^{-i} = \mathbf{u}^{-1}\otimes_{\mathbf{d}}\ldots\otimes_{\mathbf{d}}\mathbf{u}^{-1}$ and use induction on $i\geq 1$ together with the usual way to define a differential on the tensor product of two complexes. □

For $i\geq 1$ and $1\leq l\leq i$, we define $\delta_{i,l}(a_1\otimes\ldots\otimes a_{i+1})$ as:

$$(-1)^{\sum_{j=1}^{l}|a_i|_k}\left(\ldots\otimes a_lx\otimes\xi a_{l+1}\otimes\ldots+\ldots\otimes a_l\xi\otimes xa_{l+1}\otimes\ldots\right).$$

It turns out that $\delta_{i,l}(a_1\otimes\ldots\otimes a_{i+1})\neq 0$ if and only if $a_l = e_1$ and $a_{l+1} = e_2$; this comes from the multiplication rule in $\mathbf{d}$.

The following comes rather immediately:

Lemma 6.2.2. *The map $f_2 : \mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})^{\otimes 2}\to\mathbb{T}_{\mathbf{d}}(\mathbf{u}^{-1})$ defined by:*

$$f_2\left(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1}\otimes e_2^{\epsilon_2}\xi^{m_2}x^{l_2}e_1^{\eta_2}\right) := \sum_{i_1=1}^{m_2}\sum_{i_2=1}^{l_1}(-1)^{i_1l_1}e_2^{\epsilon_1}\xi^{m_1+i_1-1}x^{l_1-i_2}e_1e_2x^{i_2-1}\xi^{m_2-i_1}x^{l_2}e_1^{\eta_2},$$

if $\eta_1\neq\epsilon_2$, and zero otherwise, is a graded map of k-degree -1 such that $m_1f_2 = \Phi_2$.

Proof. Let $e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1}, e_2^{\epsilon_2}\xi^{m_2}x^{l_2}e_1^{\eta_2}\in\mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})$. Their tensor product is of k-degree l_1+l_2 and all elements in the sum are of the form

$$e_2^{\epsilon_1}\xi^{m_1+i_1-1}x^{l_1-i_2}e_1e_2x^{i_2-1}\xi^{m_2-i_1}x^{l_2}e_1^{\eta_2}$$

and these have k-degree $l_1 - i_2 + i_2 - 1 + l_2 = l_1 + l_2 - 1$, so f_2 is a graded map of k-degree -1. Let us now compute $m_1 f_2\left(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1} \otimes e_2^{\epsilon_2}\xi^{m_2}x^{l_2}e_1^{\eta_2}\right)$:

$$
\begin{aligned}
& m_1\left(\sum_{i_1=1}^{m_2}\sum_{i_2=1}^{l_1}(-1)^{i_1 l_1} e_2^{\epsilon_1}\xi^{m_1+i_1-1}x^{l_1-i_2}e_1e_2x^{i_2-1}\xi^{m_2-i_1}x^{l_2}e_1^{\eta_2}\right)\\
= & \sum_{i_1=1}^{m_2}\sum_{i_2=1}^{l_1}(-1)^{i_1 l_1} m_1\left(e_2^{\epsilon_1}\xi^{m_1+i_1-1}x^{l_1-i_2}e_1e_2x^{i_2-1}\xi^{m_2-i_1}x^{l_2}e_1^{\eta_2}\right)\\
= & \sum_{i_1=1}^{m_2}\sum_{i_2=1}^{l_1}(-1)^{i_1 l_1+l_1-i_2} e_2^{\epsilon_1}\xi^{m_1+i_1-1}x^{l_1-i_2}\left(x\xi+\xi x\right)x^{i_2-1}\xi^{m_2-i_1}x^{l_2}e_1^{\eta_2}\\
= & \sum_{i_1=1}^{m_2}\sum_{i_2=1}^{l_1}(-1)^{i_1 l_1+l_1-i_2} e_2^{\epsilon_1}\xi^{m_1+i_1-1}x^{l_1-i_2+1}\xi x^{i_2-1}\xi^{m_2-i_1}x^{l_2}e_1^{\eta_2}\\
& +\sum_{i_1=1}^{m_2}\sum_{i_2=1}^{l_1}(-1)^{i_1 l_1+l_1-i_2} e_2^{\epsilon_1}\xi^{m_1+i_1-1}x^{l_1-i_2}\xi x^{i_2}\xi^{m_2-i_1}x^{l_2}e_1^{\eta_2}\\
= & \sum_{i_1=1}^{m_2}\sum_{i_2=1}^{l_1}(-1)^{i_1 l_1+l_1-i_2} e_2^{\epsilon_1}\xi^{m_1+i_1-1}x^{l_1-i_2+1}\xi x^{i_2-1}\xi^{m_2-i_1}x^{l_2}e_1^{\eta_2}\\
& +\sum_{i_1=1}^{m_2}\sum_{i_2=2}^{l_1+1}(-1)^{i_1 l_1+l_1-i_2+1} e_2^{\epsilon_1}\xi^{m_1+i_1-1}x^{l_1-i_2+1}\xi x^{i_2-1}\xi^{m_2-i_1}x^{l_2}e_1^{\eta_2}\\
= & \sum_{i_1=1}^{m_2}(-1)^{i_1 l_1+l_1-1} e_2^{\epsilon_1}\xi^{m_1+i_1-1}x^{l_1}\xi^{m_2-i_1+1}x^{l_2}e_1^{\eta_2}\\
& +\sum_{i_1=1}^{m_2}(-1)^{i_1 l_1} e_2^{\epsilon_1}\xi^{m_1+i_1}x^{l_1}\xi^{m_2-i_1}x^{l_2}e_1^{\eta_2}\\
= & \sum_{i_1=1}^{m_2}(-1)^{i_1 l_1+l_1-1} e_2^{\epsilon_1}\xi^{m_1+i_1-1}x^{l_1}\xi^{m_2-i_1+1}x^{l_2}e_1^{\eta_2}\\
& +\sum_{i_1=2}^{m_2+1}(-1)^{i_1 l_1-l_1} e_2^{\epsilon_1}\xi^{m_1+i_1-1}x^{l_1}\xi^{m_2-i_1+1}x^{l_2}e_1^{\eta_2}\\
= & -e_2^{\epsilon_1}\xi^{m_1}x^{l_1}\xi^{m_2}x^{l_2}e_1^{\eta_2} + (-1)^{m_2 l_1} e_2^{\epsilon_1}\xi^{m_1+m_2}x^{l_1+l_2}e_1^{\eta_2},
\end{aligned}
$$

which is exactly expression Φ_2 if we assume $\eta_1 \neq \epsilon_2$. □

Remark 6.2.3. When working with specific examples, one really sees that this formula encodes how to 'entangle' the configuration $e_2^{\epsilon_1}\xi^{m_1}x^{l_1}\xi^{m_2}x^{l_2}e_1^{\eta_2}$ to our chosen representation of elements $e_2^{\epsilon_1}\xi^{m_1+m_2}x^{l_1+l_2}e_1^{\eta_2}$.

3. Let us move on to the next step of the construction: we try to determine the higher multiplication $m_3 : \mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})^{\otimes 3} \to \mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})$. We consider the following expression, which comes from the relations defining A_∞-morphisms:

$$m_1 f_3 - m_2(f_1 \otimes f_2 - f_2 \otimes f_1) = f_1 m_3 - f_2(\mathbf{1} \otimes m_2 - m_2 \otimes \mathbf{1}),$$

bearing in mind that this equality is an equality as graded maps from $\mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})^{\otimes 3} \to \mathbb{T}_{\mathbf{d}}(\mathbf{u}^{-1})$; we use the same notation for higher multiplications of $\mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})$ and those of $\mathbb{T}_{\mathbf{d}}(\mathbf{u}^{-1})$. We define

$$\Phi_3 := m_2(f_1 \otimes f_2 - f_2 \otimes f_1) + f_2(\mathbf{1} \otimes m_2 - m_2 \otimes \mathbf{1}),$$

so that we have the equality: $\Phi_3 = -m_1 f_3 + f_1 m_3$. We evaluate Φ_3 on basis elements of $\mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})^{\otimes 3}$. Note that since f_2 is of k-degree -1, the Koszul sign rule will apply.

We have:

$$\Phi_3\left(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1}\otimes e_2^{\epsilon_2}\xi^{m_2}x^{l_2}e_1^{\eta_2}\otimes e_2^{\epsilon_3}\xi^{m_3}x^{l_3}e_1^{\eta_3}\right)=A+B+C+D,$$

where

- $A=m_2(f_1\otimes f_2)\left(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1}\otimes e_2^{\epsilon_2}\xi^{m_2}x^{l_2}e_1^{\eta_2}\otimes e_2^{\epsilon_3}\xi^{m_3}x^{l_3}e_1^{\eta_3}\right)$,
- $B=-m_2(f_2\otimes f_1)\left(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1}\otimes e_2^{\epsilon_2}\xi^{m_2}x^{l_2}e_1^{\eta_2}\otimes e_2^{\epsilon_3}\xi^{m_3}x^{l_3}e_1^{\eta_3}\right)$,
- $C=f_2(\mathbf{1}\otimes f_2)\left(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1}\otimes e_2^{\epsilon_2}\xi^{m_2}x^{l_2}e_1^{\eta_2}\otimes e_2^{\epsilon_3}\xi^{m_3}x^{l_3}e_1^{\eta_3}\right)$,
- $D=-f_2(m_2\otimes \mathbf{1})\left(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1}\otimes e_2^{\epsilon_2}\xi^{m_2}x^{l_2}e_1^{\eta_2}\otimes e_2^{\epsilon_3}\xi^{m_3}x^{l_3}e_1^{\eta_3}\right)$.

Let us write down those expressions explicitly. We assume that $\eta_1\neq\epsilon_2$ and $\eta_2\neq\epsilon_3$.

$$\begin{aligned}
A &= m_2((-1)^{l_1}f_1(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1})\otimes f_2(e_2^{\epsilon_2}\xi^{m_2}x^{l_2}e_1^{\eta_2}\otimes e_2^{\epsilon_3}\xi^{m_3}x^{l_3}e_1^{\eta_3}))[\text{Koszul sign rule}]\\
&= \sum_{i_1=1}^{m_3}\sum_{i_2=1}^{l_2}(-1)^{i_1l_2-l_1}e_2^{\epsilon_1}\xi^{m_1}x^{l_1}\xi^{m_2+i_1-1}x^{l_2-i_2}e_1e_2x^{i_2-1}\xi^{m_3-i_1}x^{l_3}e_1^{\eta_3}
\end{aligned}$$

$$\begin{aligned}
B &= -m_2(f_2(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1}\otimes e_2^{\epsilon_2}\xi^{m_2}x^{l_2}e_1^{\eta_2})\otimes f_1(e_2^{\epsilon_3}\xi^{m_3}x^{l_3}e_1^{\eta_3}))\\
&= \sum_{i_1=1}^{m_2}\sum_{i_2=1}^{l_1}(-1)^{i_1l_1+1}e_2^{\epsilon_1}\xi^{m_1+i_1-1}x^{l_1-i_2}e_1e_2x^{i_2-1}\xi^{m_2-i_1}x^{l_2}\xi^{m_3}x^{l_3}e_1^{\eta_3}
\end{aligned}$$

$$\begin{aligned}
C &= f_2(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1}\otimes(-1)^{m_3l_2}e_2^{\epsilon_2}\xi^{m_2+m_3}x^{l_2+l_3}e_1^{\eta_3}))\\
&= \sum_{i_1=1}^{m_2+m_3}\sum_{i_2=1}^{l_1}(-1)^{i_1l_1+m_3l_2}e_2^{\epsilon_1}\xi^{m_1+i_1-1}x^{l_1-i_2}e_1e_2x^{i_2-1}\xi^{m_2+m_3-i_1}x^{l_2+l_3}e_1^{\eta_3}
\end{aligned}$$

$$\begin{aligned}
D &= -f_2((-1)^{m_2l_1}e_2^{\epsilon_1}\xi^{m_1+m_2}x^{l_1+l_2}e_1^{\eta_2}\otimes e_2^{\epsilon_3}\xi^{m_3}x^{l_3}e_1^{\eta_3}))\\
&= \sum_{i_1=1}^{m_3}\sum_{i_2=1}^{l_1+l_2}(-1)^{i_1(l_1+l_2)+m_2l_1+1}e_2^{\epsilon_1}\xi^{m_1+m_2+i_1-1}x^{l_1+l_2-i_2}e_1e_2x^{i_2-1}\xi^{m_3-i_1}x^{l_3}e_1^{\eta_3}
\end{aligned}$$

We see that there are some cancellations appearing between C and D:

$$C\quad i_1=m_2+m_3\quad \sum_{i_2=1}^{l_1}(-1)^{(m_2+m_3)l_1+m_3l_2}e_2^{\epsilon_1}\xi^{m_1+m_2+m_3-1}x^{l_1-i_2}e_1e_2x^{i_2-1+l_2+l_3}e_1^{\eta_3}$$

$$\begin{aligned}
D\quad i_1=m_3\quad &\sum_{i_2=1+\mathbf{l_2}}^{l_1+l_2}(-1)^{m_3(l_1+l_2)+m_2l_1+1}e_2^{\epsilon_1}\xi^{m_1+m_2+m_3-1}x^{l_1+l_2-i_2}e_1e_2x^{i_2-1+l_3}e_1^{\eta_3}\\
&=\sum_{i_2=1}^{l_1}(-1)^{m_3(l_1+l_2)+m_2l_1+1}e_2^{\epsilon_1}\xi^{m_1+m_2+m_3-1}x^{l_1-i_2}e_1e_2x^{i_2-1+l_2+l_3}e_1^{\eta_3}
\end{aligned}$$

They have opposite signs and cancel each other. Therefore, there is still one summand of D corresponding to $i_1=m_3$, however, i_2 runs between 1 and l_2.

The task is to try and find a preimage under the differential of Φ_3. It can be achieved in the following way.

Lemma 6.2.4. *There is a map* $f_3 : \mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})^{\otimes 3} \to \mathbb{T}_{\mathbf{d}}(\mathbf{u}^{-1})$ *defined by:*

$$f_3\left(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1}\otimes e_2^{\epsilon_2}\xi^{m_2}x^{l_2}e_1^{\eta_2}\otimes e_2^{\epsilon_3}\xi^{m_3}x^{l_3}e_1^{\eta_3}\right) :=$$
$$\sum_{i_1=1}^{m_3}\sum_{i_2=1}^{l_2}\sum_{i_3=1}^{m_2+i_1-1}\sum_{i_4=1}^{l_1}(-1)^{i_1l_2+i_3l_1-l_1}e_2^{\epsilon_1}\xi^{m_1+i_3-1}x^{l_1-i_4}e_1e_2x^{i_4-1}\xi^{m_2+i_1-1-i_3}\dots$$
$$\dots x^{l_2-i_2}e_1e_2x^{i_2-1}\xi^{m_3-i_1}x^{l_3}e_1^{\eta_3},$$

if $\eta_1\neq\epsilon_2$ *and* $\eta_2\neq\epsilon_3$*, and zero otherwise. Moreover, it is a graded map of k-degree -2 such that* $-m_1f_3=\Phi_3$*. In particular, the higher multiplication* $m_3 : \mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})^{\otimes 3}\to\mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})$ *can be chosen as identically zero.*

Proof. Similarly to the proof of Lemma 6.2.2, we count the number of x's in the expression to compute the k-degree. The argument has k-degree $l_1+l_2+l_3$, and elements in the expression have k-degree $l_1-i_4+i_4-1+l_2-i_2+i_2-1+l_3=l_1+l_2+l_3-2$. Therefore, the map f_3 has k-degree -2. We apply the differential to f_3 and we obtain:

$$\begin{aligned}
&-m_1f_3\left(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1}\otimes e_2^{\epsilon_2}\xi^{m_2}x^{l_2}e_1^{\eta_2}\otimes e_2^{\epsilon_3}\xi^{m_3}x^{l_3}e_1^{\eta_3}\right)\\
=\;&-m_1\Bigg(\sum_{i_1=1}^{m_3}\sum_{i_2=1}^{l_2}\sum_{i_3=1}^{m_2+i_1-1}\sum_{i_4=1}^{l_1}(-1)^{i_1l_2+i_3l_1-l_1}e_2^{\epsilon_1}\xi^{m_1+i_3-1}x^{l_1-i_4}e_1e_2x^{i_4-1}\xi^{m_2+i_1-1-i_3}\dots\\
&\qquad\dots x^{l_2-i_2}e_1e_2x^{i_2-1}\xi^{m_3-i_1}x^{l_3}e_1^{\eta_3}\Bigg)\\
=\;&\sum_{i_1=1}^{m_3}\sum_{i_2=1}^{l_2}\sum_{i_3=1}^{m_2+i_1-1}\sum_{i_4=1}^{l_1}(-1)^{i_1l_2+i_3l_1-l_1+1+l_1-i_4}e_2^{\epsilon_1}\xi^{m_1+i_3-1}x^{l_1-i_4}(x\xi+\xi x)x^{i_4-1}\xi^{m_2+i_1-1-i_3}\dots\\
&\qquad\dots x^{l_2-i_2}e_1e_2x^{i_2-1}\xi^{m_3-i_1}x^{l_3}e_1^{\eta_3}\\
&+\sum_{i_1=1}^{m_3}\sum_{i_2=1}^{l_2}\sum_{i_3=1}^{m_2+i_1-1}\sum_{i_4=1}^{l_1}(-1)^{i_1l_2+i_3l_1-l_1+1+l_1-i_4+i_4-1+l_2-i_2}e_2^{\epsilon_1}\xi^{m_1+i_3-1}x^{l_1-i_4}e_1e_2x^{i_4-1}\xi^{m_2+i_1-1-i_3}\dots\\
&\qquad\dots x^{l_2-i_2}(x\xi+\xi x)x^{i_2-1}\xi^{m_3-i_1}x^{l_3}e_1^{\eta_3}\\
=\;&\sum_{i_1=1}^{m_3}\sum_{i_2=1}^{l_2}\sum_{i_3=1}^{m_2+i_1-1}\sum_{i_4=1}^{l_1}(-1)^{i_1l_2+i_3l_1+1-i_4}e_2^{\epsilon_1}\xi^{m_1+i_3-1}x^{l_1-i_4+1}\xi x^{i_4-1}\xi^{m_2+i_1-1-i_3}\dots\\
&\qquad\dots x^{l_2-i_2}e_1e_2x^{i_2-1}\xi^{m_3-i_1}x^{l_3}e_1^{\eta_3}\\
&+\sum_{i_1=1}^{m_3}\sum_{i_2=1}^{l_2}\sum_{i_3=1}^{m_2+i_1-1}\sum_{i_4=1}^{l_1}(-1)^{i_1l_2+i_3l_1+1-i_4}e_2^{\epsilon_1}\xi^{m_1+i_3-1}x^{l_1-i_4}\xi x^{i_4}\xi^{m_2+i_1-1-i_3}\dots\\
&\qquad\dots x^{l_2-i_2}e_1e_2x^{i_2-1}\xi^{m_3-i_1}x^{l_3}e_1^{\eta_3}\\
&+\sum_{i_1=1}^{m_3}\sum_{i_2=1}^{l_2}\sum_{i_3=1}^{m_2+i_1-1}\sum_{i_4=1}^{l_1}(-1)^{i_1l_2+i_3l_1+l_2-i_2}e_2^{\epsilon_1}\xi^{m_1+i_3-1}x^{l_1-i_4}e_1e_2x^{i_4-1}\xi^{m_2+i_1-1-i_3}\dots\\
&\qquad\dots x^{l_2-i_2+1}\xi x^{i_2-1}\xi^{m_3-i_1}x^{l_3}e_1^{\eta_3}\\
&+\sum_{i_1=1}^{m_3}\sum_{i_2=1}^{l_2}\sum_{i_3=1}^{m_2+i_1-1}\sum_{i_4=1}^{l_1}(-1)^{i_1l_2+i_3l_1+l_2-i_2}e_2^{\epsilon_1}\xi^{m_1+i_3-1}x^{l_1-i_4}e_1e_2x^{i_4-1}\xi^{m_2+i_1-1-i_3}\dots\\
&\qquad\dots x^{l_2-i_2}\xi x^{i_2}\xi^{m_3-i_1}x^{l_3}e_1^{\eta_3}
\end{aligned}$$

Cancellations occur in this sum of sums. In the second sum, make the change of variable $i_4 = i_4' - 1$, and in the last sum, make the change of variable $i_2 = i_2' - 1$:

$$\begin{aligned}
= &\sum_{i_1=1}^{m_3}\sum_{i_2=1}^{l_2}\sum_{i_3=1}^{m_2+i_1-1}\sum_{i_4=1}^{l_1}(-1)^{i_1l_2+i_3l_1+1-i_4}e_2^{\epsilon_1}\xi^{m_1+i_3-1}x^{l_1-i_4+1}\xi x^{i_4-1}\xi^{m_2+i_1-1-i_3}\dots \\
&\qquad\qquad\qquad\dots x^{l_2-i_2}e_1e_2x^{i_2-1}\xi^{m_3-i_1}x^{l_3}e_1^{\eta_3} \\
&+\sum_{i_1=1}^{m_3}\sum_{i_2=1}^{l_2}\sum_{i_3=1}^{m_2+i_1-1}\sum_{i_4'=2}^{l_1+1}(-1)^{i_1l_2+i_3l_1+1-i_4'+1}e_2^{\epsilon_1}\xi^{m_1+i_3-1}x^{l_1-i_4'+1}\xi x^{i_4'-1}\xi^{m_2+i_1-1-i_3}\dots \\
&\qquad\qquad\qquad\dots x^{l_2-i_2}e_1e_2x^{i_2-1}\xi^{m_3-i_1}x^{l_3}e_1^{\eta_3} \\
&+\sum_{i_1=1}^{m_3}\sum_{i_2=1}^{l_2}\sum_{i_3=1}^{m_2+i_1-1}\sum_{i_4=1}^{l_1}(-1)^{i_1l_2+i_3l_1+l_2-i_2}e_2^{\epsilon_1}\xi^{m_1+i_3-1}x^{l_1-i_4}e_1e_2x^{i_4-1}\xi^{m_2+i_1-1-i_3}\dots \\
&\qquad\qquad\qquad\dots x^{l_2-i_2+1}\xi x^{i_2-1}\xi^{m_3-i_1}x^{l_3}e_1^{\eta_3} \\
&+\sum_{i_1=1}^{m_3}\sum_{i_2'=2}^{l_2+1}\sum_{i_3=1}^{m_2+i_1-1}\sum_{i_4=1}^{l_1}(-1)^{i_1l_2+i_3l_1+l_2-i_2'+1}e_2^{\epsilon_1}\xi^{m_1+i_3-1}x^{l_1-i_4}e_1e_2x^{i_4-1}\xi^{m_2+i_1-1-i_3}\dots \\
&\qquad\qquad\qquad\dots x^{l_2-i_2'+1}\xi x^{i_2'-1}\xi^{m_3-i_1}x^{l_3}e_1^{\eta_3}
\end{aligned}$$

We see that the first two sums have opposite signs and yield one term when $i_4 = 1$ and another one when $i_4' = l_1 + 1$; similarly, the last two sums have opposite signs and yield one term when $i_2 = 1$ and another one when $i_2' = l_2 + 1$:

$$\begin{aligned}
= &\sum_{i_1=1}^{m_3}\sum_{i_2=1}^{l_2}\sum_{i_3=1}^{m_2+i_1-1}(-1)^{i_1l_2+i_3l_1}e_2^{\epsilon_1}\xi^{m_1+i_3-1}x^{l_1}\xi^{m_2+i_1-i_3}\dots \\
&\qquad\qquad\qquad\dots x^{l_2-i_2}e_1e_2x^{i_2-1}\xi^{m_3-i_1}x^{l_3}e_1^{\eta_3} \\
&+\sum_{i_1=1}^{m_3}\sum_{i_2=1}^{l_2}\sum_{i_3=1}^{m_2+i_1-1}(-1)^{i_1l_2+i_3l_1+1+l_1}e_2^{\epsilon_1}\xi^{m_1+i_3}x^{l_1}\xi^{m_2+i_1-1-i_3}\dots \\
&\qquad\qquad\qquad\dots x^{l_2-i_2}e_1e_2x^{i_2-1}\xi^{m_3-i_1}x^{l_3}e_1^{\eta_3} \\
&+\sum_{i_1=1}^{m_3}\sum_{i_3=1}^{m_2+i_1-1}\sum_{i_4=1}^{l_1}(-1)^{i_1l_2+i_3l_1+l_2-1}e_2^{\epsilon_1}\xi^{m_1+i_3-1}x^{l_1-i_4}e_1e_2x^{i_4-1}\xi^{m_2+i_1-1-i_3}\dots \\
&\qquad\qquad\qquad\dots x^{l_2}\xi^{m_3-i_1+1}x^{l_3}e_1^{\eta_3} \\
&+\sum_{i_1=1}^{m_3}\sum_{i_3=1}^{m_2+i_1-1}\sum_{i_4=1}^{l_1}(-1)^{i_1l_2+i_3l_1}e_2^{\epsilon_1}\xi^{m_1+i_3-1}x^{l_1-i_4}e_1e_2x^{i_4-1}\xi^{m_2+i_1-i_3}\dots \\
&\qquad\qquad\qquad\dots x^{l_2}\xi^{m_3-i_1}x^{l_3}e_1^{\eta_3}
\end{aligned}$$

Further cancellations occur. In the second sum, we make the change of variable

$i_3 = i_3' - 1$; in the last sum, set $i_1 = i_1' - 1$:

$$\begin{aligned}
= & \sum_{i_1=1}^{m_3}\sum_{i_2=1}^{l_2}\sum_{i_3=1}^{m_2+i_1-1}(-1)^{i_1l_2+i_3l_1}e_2^{\epsilon_1}\xi^{m_1+i_3-1}x^{l_1}\xi^{m_2+i_1-i_3}\dots \\
& \qquad \dots x^{l_2-i_2}e_1e_2x^{i_2-1}\xi^{m_3-i_1}x^{l_3}e_1^{\eta_3} \\
& +\sum_{i_1=1}^{m_3}\sum_{i_2=1}^{l_2}\sum_{i_3'=2}^{m_2+i_1}(-1)^{i_1l_2+i_3'l_1+1}e_2^{\epsilon_1}\xi^{m_1+i_3'-1}x^{l_1}\xi^{m_2+i_1-i_3'}\dots \\
& \qquad \dots x^{l_2-i_2}e_1e_2x^{i_2-1}\xi^{m_3-i_1}x^{l_3}e_1^{\eta_3} \\
& +\sum_{i_1=1}^{m_3}\sum_{i_3=1}^{m_2+i_1-1}\sum_{i_4=1}^{l_1}(-1)^{i_1l_2+i_3l_1+l_2-1}e_2^{\epsilon_1}\xi^{m_1+i_3-1}x^{l_1-i_4}e_1e_2x^{i_4-1}\xi^{m_2+i_1-1-i_3}\dots \\
& \qquad \dots x^{l_2}\xi^{m_3-i_1+1}x^{l_3}e_1^{\eta_3} \\
& +\sum_{i_1'=2}^{m_3+1}\sum_{i_3=1}^{m_2+i_1'-2}\sum_{i_4=1}^{l_1}(-1)^{i_1'l_2+i_3l_1-l_2}e_2^{\epsilon_1}\xi^{m_1+i_3-1}x^{l_1-i_4}e_1e_2x^{i_4-1}\xi^{m_2+i_1'-1-i_3}\dots \\
& \qquad \dots x^{l_2}\xi^{m_3-i_1'+1}x^{l_3}e_1^{\eta_3}
\end{aligned}$$

We see that the first two sums have opposite signs and yield one term when $i_3 = 1$ and another one when $i_3' = m_2 + i_1$. Although the last two sums have opposite signs as well, we must be more careful: in the third one, we have $1 \leq i_3 \leq m_2 + i_1 - 1$, while in the fourth we have $1 \leq i_3 \leq m_2 + i_1 - 2$. Therefore, three sums will be produced: one corresponding to $i_1 = 1$, another corresponding to $i_1' = m_3 + 1$, and one last corresponding to $i_3 = m_2 + i_1 - 1$ (where $2 \leq i_1 \leq m_3$):

$$\begin{aligned}
= & \sum_{i_1=1}^{m_3}\sum_{i_2=1}^{l_2}(-1)^{i_1l_2+l_1}e_2^{\epsilon_1}\xi^{m_1}x^{l_1}\xi^{m_2+i_1-1}x^{l_2-i_2}e_1e_2x^{i_2-1}\xi^{m_3-i_1}x^{l_3}e_1^{\eta_3} \\
& +\sum_{i_1=1}^{m_3}\sum_{i_2=1}^{l_2}(-1)^{i_1l_2+(m_2+i_1)l_1+1}e_2^{\epsilon_1}\xi^{m_1+m_2+i_1-1}x^{l_1+l_2-i_2}e_1e_2x^{i_2-1}\xi^{m_3-i_1}x^{l_3}e_1^{\eta_3} \\
& +\sum_{i_3=1}^{m_2}\sum_{i_4=1}^{l_1}(-1)^{l_2+i_3l_1+l_2-1}e_2^{\epsilon_1}\xi^{m_1+i_3-1}x^{l_1-i_4}e_1e_2x^{i_4-1}\xi^{m_2-i_3}x^{l_2}\xi^{m_3}x^{l_3}e_1^{\eta_3} \\
& +\sum_{i_3=1}^{m_2+m_3-1}\sum_{i_4=1}^{l_1}(-1)^{(m_3+1)l_2+i_3l_1-l_2}e_2^{\epsilon_1}\xi^{m_1+i_3-1}x^{l_1-i_4}e_1e_2x^{i_4-1}\xi^{m_2+m_3-i_3}x^{l_2}x^{l_3}e_1^{\eta_3} \\
& +\sum_{i_1=2}^{m_3}\sum_{i_4=1}^{l_1}(-1)^{i_1l_2+(m_2+i_1-1)l_1+l_2-1}e_2^{\epsilon_1}\xi^{m_1+m_2+i_1-2}x^{l_1-i_4}e_1e_2x^{i_4-1+l_2}\xi^{m_3-i_1+1}x^{l_3}e_1^{\eta_3}
\end{aligned}$$

We perform some changes of variable and some relabellings:

- in the third sum, relabel i_3 by i_1, relabel i_4 by i_2;
- in the fourth sum, relabel i_3 by i_1 and i_4 by i_2;
- in the last sum, make the change of variable $i_1 = i_1' + 1$ and $i_4 = i_2 - l_2$.

After simplifying the exponents governing the sign, we obtain:

$$\begin{aligned}
= & \sum_{i_1=1}^{m_3}\sum_{i_2=1}^{l_2}(-1)^{i_1l_2+l_1}e_2^{\epsilon_1}\xi^{m_1}x^{l_1}\xi^{m_2+i_1-1}x^{l_2-i_2}e_1e_2x^{i_2-1}\xi^{m_3-i_1}x^{l_3}e_1^{\eta_3}\\
& +\sum_{i_1=1}^{m_3}\sum_{i_2=1}^{l_2}(-1)^{i_1(l_1+l_2)+m_2l_1+1}e_2^{\epsilon_1}\xi^{m_1+m_2+i_1-1}x^{l_1+l_2-i_2}e_1e_2x^{i_2-1}\xi^{m_3-i_1}x^{l_3}e_1^{\eta_3}\\
& +\sum_{i_1=1}^{m_2}\sum_{i_2=1}^{l_1}(-1)^{i_1l_1+1}e_2^{\epsilon_1}\xi^{m_1+i_1-1}x^{l_1-i_2}e_1e_2x^{i_2-1}\xi^{m_2-i_1}x^{l_2}\xi^{m_3}x^{l_3}e_1^{\eta_3}\\
& +\sum_{i_1=1}^{m_2+m_3-1}\sum_{i_2=1}^{l_1}(-1)^{m_3l_2+i_1l_1}e_2^{\epsilon_1}\xi^{m_1+i_1-1}x^{l_1-i_2}e_1e_2x^{i_2-1}\xi^{m_2+m_3-i_1}x^{l_2}x^{l_3}e_1^{\eta_3}\\
& +\sum_{i_1'=1}^{m_3-1}\sum_{i_2=1+l_2}^{l_1+l_2}(-1)^{i_1'(l_1+l_2)+m_2l_1-1}e_2^{\epsilon_1}\xi^{m_1+m_2+i_1'-1}x^{l_1+l_2-i_2}e_1e_2x^{i_2-1}\xi^{m_3-i_1'}x^{l_3}e_1^{\eta_3}
\end{aligned}$$

We can see that:

- the first sum corresponds to term A in Φ_3;
- the third sum corresponds to term B;
- the fourth sum corresponds to term C;
- the second and fifth sums combined together give exactly term D.

This shows that $-m_1f_3 = \Phi_3$ and $m_3 : \mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})^{\otimes 3} \to \mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})$ can thus be chosen to be identically zero. □

Remark 6.2.5. We have constructed f_3 in the following manner. Consider expression A in Φ_3:

$$\begin{aligned}
A &= m_2((-1)^{l_1}f_1(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1})\otimes f_2(e_2^{\epsilon_2}\xi^{m_2}x^{l_2}e_1^{\eta_2}\otimes e_2^{\epsilon_3}\xi^{m_3}x^{l_3}e_1^{\eta_3}))\\
&= \sum_{i_1=1}^{m_3}\sum_{i_2=1}^{l_2}(-1)^{i_1l_2-l_1}e_2^{\epsilon_1}\xi^{m_1}x^{l_1}\xi^{m_2+i_1-1}x^{l_2-i_2}e_1e_2x^{i_2-1}\xi^{m_3-i_1}x^{l_3}e_1^{\eta_3}
\end{aligned}$$

and 'entangle' the first occurrence of $x^a\xi^b$ using f_2:

$$\sum_{i_1=1}^{m_3}\sum_{i_2=1}^{l_2}(-1)^{i_1l_2-l_1}e_2^{\epsilon_1}\xi^{m_1}f_2(x^{l_1}e_1^{\eta_1}\otimes e_2^{\epsilon_2}\xi^{m_2+i_1-1})x^{l_2-i_2}e_1e_2x^{i_2-1}\xi^{m_3-i_1}x^{l_3}e_1^{\eta_3}$$

We obtain:

$$\begin{aligned}
&\sum_{i_1=1}^{m_3}\sum_{i_2=1}^{l_2}\sum_{i_3=1}^{m_2+i_1-1}\sum_{i_4=1}^{l_1}(-1)^{i_1l_2-l_1+i_3l_1}e_2^{\epsilon_1}\xi^{m_1+i_3-1}x^{l_1-i_4}e_1e_2x^{i_4-1}\xi^{m_2+i_1-1-i_3}\dots\\
&\qquad\qquad\dots x^{l_2-i_2}e_1e_2x^{i_2-1}\xi^{m_3-i_1}x^{l_3}e_1^{\eta_3}
\end{aligned}$$

which is exactly f_3.

We want to generalise this construction for f_n, $n \geq 3$. Let $n > 3$. We make the following assumption: for all $2 < r < n$, the higher multiplication $m_r : \mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})^{\otimes r} \to \mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})$ can be chosen to be identically zero. Note that we also have that $m_1 : \mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1}) \to \mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})$ is the zero map. Under this hypothesis, we can write down the n-analogue Φ_n of Φ_3:

$$\Phi_n := \sum_{s=1}^{n-2}(-1)^{s+1}f_{n-1}(\mathbf{1}^{\otimes s}\otimes m_2\otimes \mathbf{1}^{\otimes n-2-s}) + \sum_{t=1}^{n-1}(-1)^{t-1}m_2(f_t\otimes f_{n-t}),$$

so that Φ_n satisfies $\Phi_n = -m_1 f_n + f_1 m_n$. We want to show that m_n can be chosen zero.

Proposition 6.2.6. *Under the assumption that for all $2 < r < n$, the higher multiplication $m_r : \mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})^{\otimes r} \to \mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})$ can be chosen to be identically zero, there is a map $f_n : \mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})^{\otimes n} \to \mathbb{T}_{\mathbf{d}}(\mathbf{u}^{-1})$ defined by:*

$$\begin{aligned}
&f_n\left(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1}\otimes\ldots\otimes e_2^{\epsilon_n}\xi^{m_n}x^{l_n}e_1^{\eta_n}\right) := \\
&\sum_{i_1=1}^{m_n}\sum_{i_2=1}^{l_{n-1}}\cdots\sum_{i_{2(n-j)-1}=1}^{m_{j+1}+i_{2(n-(j+1))-1}-1}\ \sum_{i_{2(n-j)}=1}^{l_j}\ \sum_{i_{2(n-(j-1))-1}=1}^{m_j+i_{2(n-j)-1}-1}\ \sum_{i_{2(n-(j-1))}=1}^{l_{j-1}}\cdots\sum_{i_{2n-3}=1}^{m_2+i_{2n-5}-1}\ \sum_{i_{2n-2}=1}^{l_1} \\
&(-1)^{\sum_{k=1}^{n-1}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}}e_2^{\epsilon_1}\xi^{m_1+i_{2n-3}-1}x^{l_1-i_{2n-2}}e_1e_2x^{i_{2n-2}-1}\xi^{m_2+i_{2n-5}-1-i_{2n-3}}\ldots \\
&\qquad\ldots x^{l_2-i_{2n-4}}e_1e_2x^{i_{2n-4}-1}\xi^{m_3+i_{2n-7}-1-i_{2n-5}}\ldots \\
&\qquad\ldots \\
&\qquad\ldots x^{l_j-i_{2(n-j)}}e_1e_2x^{i_{2(n-j)}-1}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}\ldots \\
&\qquad\ldots \\
&\qquad\ldots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n},
\end{aligned}$$

if $\eta_i \neq \epsilon_{i+1}$ for all $1 \leq i \leq n-1$, and zero otherwise. Moreover, it is a graded map of k-degree $1-n$ such that $-m_1 f_n = \Phi_n$. In particular, the higher multiplication $m_n : \mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})^{\otimes n} \to \mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})$ can be chosen as identically zero.

Proof. Counting the number of x's in the expression gives us the k-degree; it is given by $\sum_{j=1}^{n-1}(l_j - 1) + l_n = \sum_{j=1}^{n} l_j + 1 - n$. Hence, f_n is of k-degree $1-n$. Although very similar to the computation for f_3, the one leading to the result is given in Appendix A because of its length. □

Corollary 6.2.7. *The algebra $\mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})$ is formal.*

6.3 A formal subalgebra of $\mathbf{w}_q$

6.3.1 Existence of $\boldsymbol{\omega}_q$

Since $\mathbb{T}_{\mathbf{d}}(\underline{\mathbf{u}})$ is not an algebra, we cannot use Kadeishvili's recipe. Therefore, we restrict ourselves to the subspace $\mathbb{T}_{\mathbf{d}}(\mathbf{u}^{-1})$, which is a dg-algebra.

By Corollary 6.2.7, we know that $\mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})$ is formal. In addition, by Lemma 5.4.2, we obtain that $\mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})^{\otimes q-1}$ is formal since it is a finite tensor product of formal A_∞-algebras. Finally, since $\mathbf{d}$ is a dg-algebra with 0-differential, it is formal and the tensor product $\mathbf{d} \otimes \mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})^{\otimes q-1}$ is as well.

We know that $\mathbf{w}_q$ is a subspace of $\mathbf{d} \otimes (\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})^{\leq 1})^{\otimes q-1}$, so we can consider its intersection with $\mathbf{d} \otimes \mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})^{\otimes q-1}$. Since $\mathbf{d} \otimes \mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})^{\otimes q-1}$ is formal, the subalgebra of $\mathbf{w}_q$ generated by basis elements in $\mathbf{d} \otimes \mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})^{\otimes q-1}$ is formal. This shows:

Proposition 6.3.1. *Let $p = 2$. There is a large subalgebra of $\mathbf{w}_q$ which is formal. We denote it by $\boldsymbol{\omega}_q$.*

In particular, those basis elements cannot "involve $\mathbf{u}$", in a sense that needs to be made precise. Recall that $\mathbb{H}(\mathbf{u})$ is the i-degree 1 part of $\mathbb{HT}_{\mathbf{d}}(\underline{\mathbf{u}})^{\leq 1}$.

Definition 6.3.2. Let $v = v_1 \otimes \ldots \otimes v_q$ be a monomial basis element of $\mathbf{w}_q$. We say that v *involves* $\mathbf{u}$ if there exists an index $1 \leq l \leq q$ such that v_l has i-degree 1.

By [BLM13, Lemma 11], all the elements $v = v_1 \otimes \ldots \otimes v_q$ of $\mathbf{w}_q$ of type $\mathbf{1}^h\mathbf{2}\mathbf{3}^{q-h-1}$ for $1 \leq h \leq q-2$, or of type $\mathbf{1}^h\mathbf{3}^{q-h}$ for $1 \leq h \leq q-1$ satisfy that their last component v_q is of type $\mathbf{3}$, namely v_q has i-degree 1 and it is of the form

$$e_{s_q} \otimes e^*_{3-s_q}.$$

Thus v is in the formal subalgebra $\boldsymbol{\omega}_q$ if it is of type $\mathbf{1}^q$ or $\mathbf{1}^{q-1}\mathbf{2}$.

6.3.2 Quiver of $\boldsymbol{\omega}_q$

In Chapter 3, we determined the quiver of $\mathbf{w}_q$ for any value of the parameter q. To obtain the arrows of $\boldsymbol{\omega}_q$, we now need to remove all the arrows involving $\mathbf{u}$.

Lemma 6.3.3. *Let $v \in V_q$. Then v involves $\mathbf{u}$ if it is of the form*

$$e_{s_1} \otimes \ldots \otimes e_{s_l} \otimes \xi \otimes (e_{s_{l+1}} \otimes e^*_{3-s_{l+1}}) \otimes \ldots \otimes (e_{s_q} \otimes e^*_{3-s_q}),$$

with $l < q$.

In particular, amongst the arrows involving ξ, we only keep those of the form

$$e_{s_1} \otimes \ldots \otimes e_{s_{q-1}} \otimes \xi.$$

However, that means that some new elements might be irreducible. Indeed, we must trace back in the proofs determining the irreducible monomials of $\mathbf{w}_q$ which elements were decomposed using an element of the form described in Lemma 6.3.3. In fact, we only need to consider the Corollary describing the irreducible arrows of $\mathbf{w}_q$ starting with x:

Corollary (Corollary 3.3.9)**.** *Let $a_1 \otimes \ldots \otimes a_q$ be an irreducible monomial of $\mathbf{w}_q$ such that $a_1 = x$. Then $a_1 \otimes \ldots \otimes a_q$ has one of the following forms:*

- $x \otimes (e_2 \otimes e_1) \otimes \ldots \otimes (e_2 \otimes e_1)$;
- $x \otimes (e_2 \otimes e_1) \otimes \ldots \otimes (e_2 \otimes e_1) \otimes (\xi \otimes e_1) \otimes e_{l_1} \otimes e_{l_2} \otimes \ldots e_{l_s}$ *if there exists $1 \leq i \leq s$ such that $l_i = 2$ $(s > 1)$;*
- $x \otimes (e_2 \otimes e_1) \otimes \ldots \otimes (e_2 \otimes e_1) \otimes (e_2 \otimes \xi) \otimes e_{l_1} \otimes e_{l_2} \otimes \ldots e_{l_r}$ *if there exists $1 \leq i \leq r$ such that $l_i = 1$ $(r > 1)$.*

To obtain the condition on

$$x \otimes (e_2 \otimes e_1) \otimes \ldots \otimes (e_2 \otimes e_1) \otimes (\xi \otimes e_1) \otimes e_{l_1} \otimes e_{l_2} \otimes \ldots e_{l_s}$$

and on

$$x \otimes (e_2 \otimes e_1) \otimes \ldots \otimes (e_2 \otimes e_1) \otimes (e_2 \otimes \xi) \otimes e_{l_1} \otimes e_{l_2} \otimes \ldots e_{l_r},$$

we tried to factor an element of the form

$$e_{l_1} \otimes e_{l_2} \otimes \ldots \otimes e_{l_r} \otimes \xi \otimes (e_1 \otimes e^*_2) \otimes \ldots \otimes (e_1 \otimes e^*_2).$$

However, that element does not appear in our subalgebra, so these decompositions do not happen there. Hence we must add the missing arrows

$$e_{s_1} \otimes \ldots \otimes e_{s_l} \otimes x \otimes (e_2 \otimes e_1) \otimes \ldots \otimes (e_2 \otimes e_1) \otimes (\xi \otimes e_1) \otimes e_1 \otimes \ldots \otimes e_1$$

and

$$e_{s_1} \otimes \ldots \otimes e_{s_l} \otimes x \otimes (e_2 \otimes e_1) \otimes \ldots \otimes (e_2 \otimes e_1) \otimes (e_2 \otimes \xi) \otimes e_2 \otimes \ldots \otimes e_2.$$

Note that since the element

$$e_{s_1} \otimes \ldots \otimes e_{s_{q-1}} \otimes \xi,$$

still features in $\boldsymbol{\omega}_q$, it is easily seen that the elements

$$e_{s_1} \otimes \ldots \otimes e_{s_l} \otimes x \otimes (e_2 \otimes e_1) \otimes \ldots \otimes (e_2 \otimes e_1) \otimes (\xi \otimes e_1)$$

and

$$e_{s_1} \otimes \ldots \otimes e_{s_l} \otimes x \otimes (e_2 \otimes e_1) \otimes \ldots \otimes (e_2 \otimes e_1) \otimes (e_2 \otimes \xi)$$

are still reducible.

We have proved the following

Proposition 6.3.4. *Let $v = v_1 \otimes \ldots \otimes v_q$ be an irreducible monomial of $\boldsymbol{\omega}_q$. Then it is of the form*

- $e_{s_1} \otimes \ldots \otimes e_{s_q}$;
- $e_{s_1} \otimes \ldots \otimes e_{s_{q-1}} \otimes \xi$;
- $e_{s_1} \otimes \ldots \otimes e_{s_n} \otimes x \otimes (e_2 \otimes e_1) \otimes \ldots \otimes (e_2 \otimes e_1)$;
- $e_{s_1} \otimes \ldots \otimes e_{s_n} \otimes x \otimes (e_2 \otimes e_1) \otimes \ldots \otimes (e_2 \otimes e_1) \otimes (\xi \otimes e_1) \otimes e_{l_1} \otimes e_{l_2} \otimes \ldots e_{l_s}$, $s > 1$;
- $e_{s_1} \otimes \ldots \otimes e_{s_n} \otimes x \otimes (e_2 \otimes e_1) \otimes \ldots \otimes (e_2 \otimes e_1) \otimes (e_2 \otimes \xi) \otimes e_{l_1} \otimes e_{l_2} \otimes \ldots e_{l_r}$, $r > 1$.

We can compare the quiver obtained in Chapter 3 for $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$ and the one for $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$ and $\boldsymbol{\omega}_3$ below.

Example 6.3.5. 1. The quiver of $\mathbf{w}_1 = \mathbf{d}$ and $\boldsymbol{\omega}_1$ are given in Figure 6.1.

Figure 6.1: Comparison between the quiver of $\mathbf{w}_1$ and that of $\boldsymbol{\omega}_1$

2. The quiver of $\mathbf{w}_2$ and $\boldsymbol{\omega}_2$ are given in Figure 6.2.

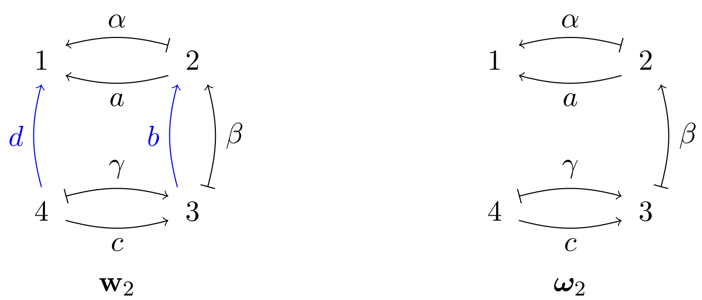


Figure 6.2: Comparison between the quiver of $\mathbf{w}_2$ and that of $\boldsymbol{\omega}_2$

Recall that the label of the arrows correspond to the following elements of V_2:

$$\begin{aligned} a &= e_1 \otimes \xi \\ \alpha &= e_1 \otimes x \\ b &= \xi \otimes (e_2 \otimes e_1^*) \\ \beta &= x \otimes (e_2 \otimes e_1) \\ c &= e_2 \otimes \xi \\ \gamma &= e_2 \otimes x \\ d &= \xi \otimes (e_1 \otimes e_2^*) \end{aligned}$$

so we see that we need to remove arrows b and d, and there aren't any arrows to add.

3. The quiver of $\mathbf{w}_3$ and $\boldsymbol{\omega}_3$ are given in Figure 6.3.

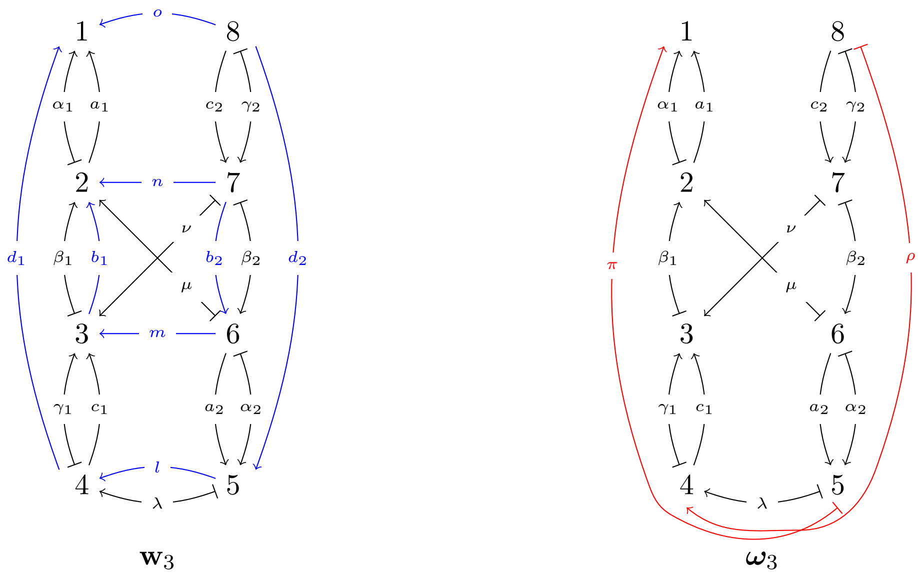


Figure 6.3: Comparison between the quiver of $\mathbf{w}_3$ and that of $\boldsymbol{\omega}_3$

Recall that the label of the arrows correspond to the following elements of V_3:

$$\begin{aligned} a_i &= e_i \otimes e_1 \otimes \xi \\ \alpha_i &= e_i \otimes e_1 \otimes x \\ b_i &= e_i \otimes \xi \otimes (e_2 \otimes e_1^*) \\ \beta_i &= e_i \otimes x \otimes (e_2 \otimes e_1) \\ c_i &= e_i \otimes e_2 \otimes \xi \\ \gamma_i &= e_i \otimes e_2 \otimes x \\ d_i &= e_i \otimes \xi \otimes (e_1 \otimes e_2^*) \\ l &= \xi \otimes (e_2 \otimes e_1^*) \otimes (e_2 \otimes e_1^*) \\ \lambda &= x \otimes (e_2 \otimes e_1) \otimes (e_2 \otimes e_1) \\ m &= \xi \otimes (e_2 \otimes e_1^*) \otimes (e_1 \otimes e_2^*) \\ \mu &= x \otimes (\xi \otimes e_1) \otimes e_2 \\ \nu &= x \otimes (e_2 \otimes \xi) \otimes e_1 \\ n &= \xi \otimes (e_1 \otimes e_2^*) \otimes (e_2 \otimes e_1^*) \\ o &= \xi \otimes (e_1 \otimes e_2^*) \otimes (e_1 \otimes e_2^*) \end{aligned}$$

where $i \in \{1, 2\}$. Hence we need to remove $b_1, b_2, d_1, d_2, l, m, n, o$ and we must add $\pi = x \otimes (\xi \otimes e_1) \otimes e_1$, $\rho = x \otimes (e_2 \otimes \xi) \otimes e_2$.

6.3.3 Properties of $\boldsymbol{\omega}_q$

We first recall a few definitions and explain why some of these properties might be expected for $\boldsymbol{\omega}_q$.

Definition (Definition 1.2.1 [Bei96])**.** A Koszul algebra is a positively graded algebra $A = \oplus_{j \geq 0} A_j$ such that

1. A_0 is semisimple;
2. Considering A_0 as a graded left A-module, it admits a graded projective resolution

$$\ldots \to P^2 \to P^1 \to P^0 \twoheadrightarrow A_0$$

such that P^j is generated by its component of degree j, i.e. $P^j = AP^j_j$.

In particular, Koszul algebras are quadratic algebras (cf. Proposition 1.2.3 in [Bei96]), which means A is generated by A_1 over A_0 with relations of degree 2. They can be represented as the quotient of the tensor algebra

$$A = \mathbb{T}_{A_0}(A_1)/I$$

by some homogeneous ideal $I \subset A_1 \otimes_{A_0} A_1$.

In the setting of graded A-modules, we consider graded extensions and denote by $\mathrm{ext}_A(M, N)$ the graded extension algebra of M by N. We have the following result:

Proposition (Proposition 2.1.3, [Bei96])**.** *Let $A = \oplus_{j \geq 0} A_j$ be a positively graded algebra and suppose A_0 is semisimple. The following conditions are equivalent*

1. *A is Koszul;*

3. *$ext^i_A(A_0, A_0\langle n \rangle) = 0$ unless $i = n$.*

This result means in particular that A is Koszul if and only if $\mathrm{ext}_A(A_0, A_0) = \mathrm{ext}^0_A(A_0, A_0)$ is concentrated in degree 0, i.e. if and only if $\mathrm{Ext}_A(A_0, A_0)$ is concentrated in degree 0 as a graded A-module. Thus, we have the following

Proposition 6.3.6. *Let $A = \oplus_{j \geq 0} A_j$ be a positively graded algebra and suppose A_0 is semisimple. The following conditions are equivalent*

1. *A is Koszul;*
2. $\mathrm{Ext}_A(A_0, A_0)$ *is formal.*

Due to Koszul duality, there exists a Koszul algebra B such that

$$\mathrm{Ext}_B(B_0, B_0)^{\mathrm{op}} \cong A$$

and in fact,

$$B = \mathrm{Ext}_A(A_0, A_0)^{\mathrm{op}} =: E(A),$$

so that $A \cong E(E(A))$. The algebra B is called the Koszul dual of A.

Since we found a formal subalgebra of $\mathbf{w}_q$, the extension algebra of Weyl modules of the principal block of rational representations of $GL_2(\overline{\mathbb{F}_2})$, one could hope that $\boldsymbol{\omega}_q$ would have some nice properties such as Koszulity.

As for $\mathbf{w}_q$, the algebra $\boldsymbol{\omega}_q$ contains two copies of the previous iteration $\boldsymbol{\omega}_{q-1}$, which are both subalgebras.

Let us write down the Loewy structure of $\boldsymbol{\omega}_3$ (cf. Figure 6.4). We do not provide a list of relations between the generators as it is straightforward knowing the expression of the generators under x and ξ form.

The algebra $\boldsymbol{\omega}_q$ is not quadratic since $\boldsymbol{\omega}_3$ is a non-quadratic subalgebra; we have the following relation $\alpha_1\beta_1c_1 = a_1\beta_1\gamma_1$, which is of length 3. In particular, $\boldsymbol{\omega}_q$ is not Koszul.

Since it is not Koszul, we could look at some generalisation of Koszulity such as N-Koszulity:

Definition ([GMMVZ04]). Let $A = \oplus_{j\geq 0}A_j$ be a positively graded algebra such that A_0 is a semisimple algebra and A_1 is finite-dimensional, and denote by $\mathcal{P}^\bullet$ is a minimal graded A-projective resolution of A_0. Then A is called a d-Koszul algebra if for each $n \geq 0$, the n-th term P^n of $\mathcal{P}^\bullet$ is generated in exactly one degree $\delta(n)$, where

$$\delta(n) = \begin{cases} \frac{n}{2}d & \text{if } n \text{ is even,} \\ \left(\frac{n-1}{2}d\right) + 1 & \text{if } n \text{ is odd.} \end{cases}$$

Note that if $d = 2$, then we recover the definition of a Koszul algebra.

Looking at the projective resolutions of the simples, we see that $\boldsymbol{\omega}_q$ is not d-Koszul for any $d \geq 3$. Indeed, simple 1 up to simple 6 have the first projective in the resolution generated in degree 1, and the second projective in the resolution (if any) is generated in degree 3. However, the 7th simple has its second projective in the resolution generated in degree 2. Hence we cannot find a suitable $d \geq 3$ for $\boldsymbol{\omega}_q$ to be d-Koszul.

We have shown:

Proposition 6.3.7. *The algebra $\boldsymbol{\omega}_q$ is not d-Koszul for any $d \geq 2$.*

6.3.4 The case $p > 2$

For $p > 2$, the complexity of the formulae increases dramatically when computing an A_∞-structure on $\mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})$ and it looks very difficult to apply the same method as before to obtain a similar result concerning the formality of $\mathbb{HT}_{\mathbf{d}}(\mathbf{u}^{-1})$ for $p = 2$.

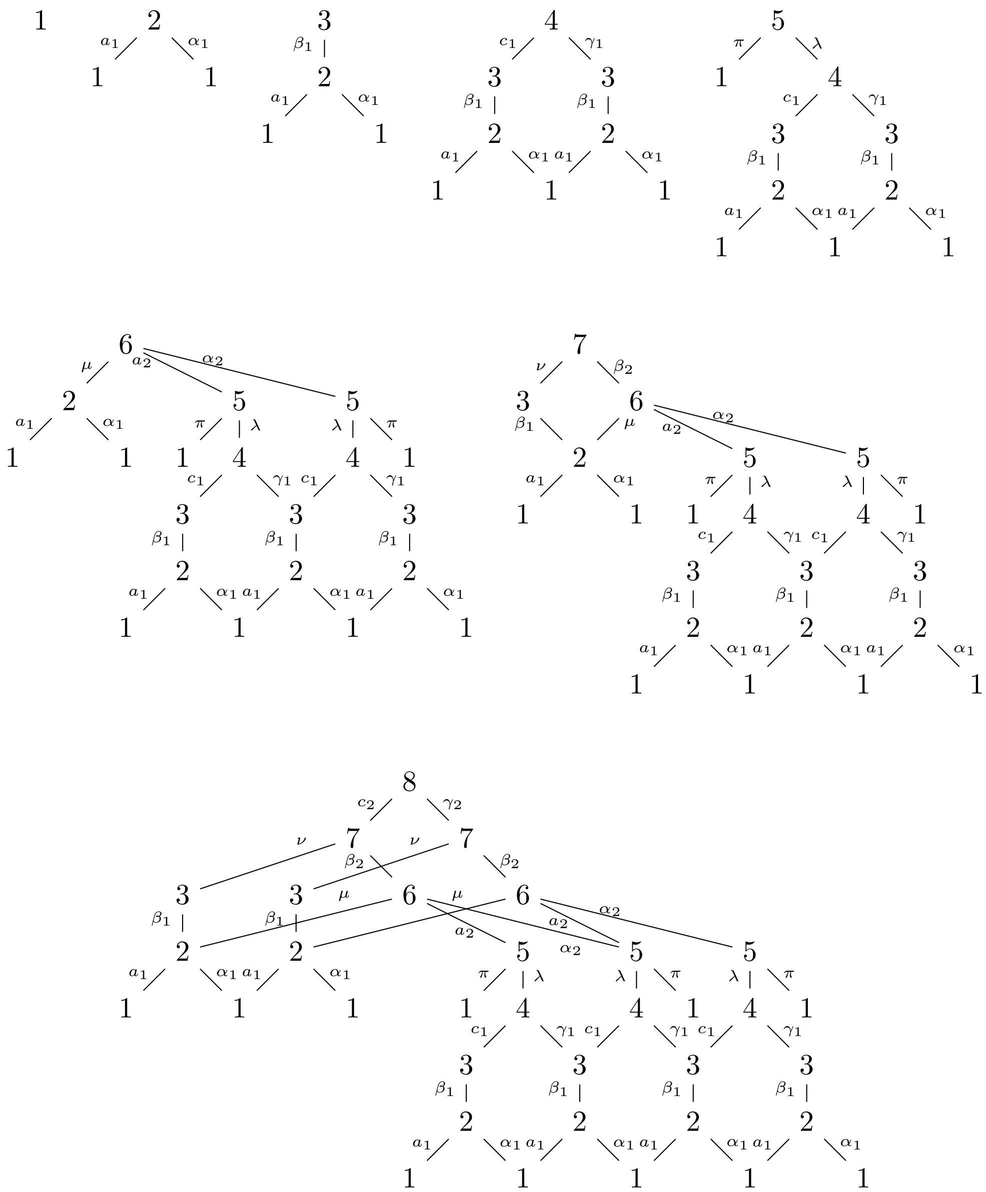


Figure 6.4: Loewy structure of the left projectives of $\boldsymbol{\omega}_3$

Appendix A

Computation for the proof of Proposition 6.2.6

The following completes the proof of Proposition 6.2.6. Let us compute $-m_1 f_n$:

$$-m_1 f_n \left(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1}\otimes\ldots\otimes e_2^{\epsilon_n}\xi^{m_n}x^{l_n}e_1^{\eta_n}\right)$$

$$= -m_1 \left(\sum_{i_1=1}^{m_n}\sum_{i_2=1}^{l_{n-1}}\cdots\sum_{i_{2(n-j)-1}=1}^{m_{j+1}+i_{2(n-(j+1))-1}-1}\ \sum_{i_{2(n-j)}=1}^{l_j}\ \sum_{i_{2(n-(j-1))-1}=1}^{m_j+i_{2(n-j)-1}-1}\ \sum_{i_{2(n-(j-1))}=1}^{l_{j-1}}\cdots\sum_{i_{2n-3}=1}^{m_2+i_{2n-5}-1}\ \sum_{i_{2n-2}=1}^{l_1}\right.$$

$$(-1)^{\sum_{k=1}^{n-1}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}}e_2^{\epsilon_1}\xi^{m_1+i_{2n-3}-1}x^{l_1-i_{2n-2}}e_1e_2x^{i_{2n-2}-1}\xi^{m_2+i_{2n-5}-1-i_{2n-3}}\ldots$$

$$\ldots$$

$$\ldots\mathbf{x^{l_j-i_{2(n-j)}}e_1e_2x^{i_{2(n-j)}-1}}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}\ldots$$

$$\ldots$$

$$\left.\ldots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}\right)$$

$$=\sum_{j=1}^{n-1}\sum_{\underline{i}}(-1)^{\sum_{k=1}^{n-1}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+1+\sum_{a=1}^{j-1}(l_a-1)+l_j-i_{2(n-j)}}$$

$$e_2^{\epsilon_1}\xi^{m_1+i_{2n-3}-1}x^{l_1-i_{2n-2}}e_1e_2x^{i_{2n-2}-1}\xi^{m_2+i_{2n-5}-1-i_{2n-3}}\ldots$$

$$\ldots$$

$$\ldots\mathbf{x^{l_j-i_{2(n-j)}}}\left(\mathbf{x}\xi+\xi\mathbf{x}\right)\mathbf{x^{i_{2(n-j)}-1}}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}\ldots$$

$$\ldots$$

$$\ldots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}$$

$$=\sum_{j=1}^{n-1}\sum_{\underline{i}}(-1)^{\sum_{k=1}^{n-1}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+1+\sum_{a=1}^{j-1}(l_a-1)+l_j-i_{2(n-j)}}$$

$$e_2^{\epsilon_1}\xi^{m_1+i_{2n-3}-1}x^{l_1-i_{2n-2}}e_1e_2x^{i_{2n-2}-1}\xi^{m_2+i_{2n-5}-1-i_{2n-3}}\ldots$$

$$\ldots$$

$$\ldots\mathbf{x^{l_j-i_{2(n-j)}+1}}\xi\mathbf{x^{i_{2(n-j)}-1}}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}\ldots$$

$$\ldots$$

$$\ldots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}$$

$$+\sum_{j=1}^{n-1}\sum_{\underline{i}}(-1)^{\sum_{k=1}^{n-1}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+1+\sum_{a=1}^{j-1}(l_a-1)+l_j-i_{2(n-j)}}$$

$$e_2^{\epsilon_1}\xi^{m_1+i_{2n-3}-1}x^{l_1-i_{2n-2}}e_1e_2x^{i_{2n-2}-1}\xi^{m_2+i_{2n-5}-1-i_{2n-3}}\ldots$$

$$\ldots$$

$$\ldots\mathbf{x^{l_j-i_{2(n-j)}}}\xi\mathbf{x^{i_{2(n-j)}}}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}\ldots$$

$$\ldots$$

$$\ldots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}$$

For $1\leq j\leq n-1$, we perform the change of variable $i_{2(n-j)}=i'_{2(n-j)}-1$ in the second sum:

$$= \sum_{j=1}^{n-1} \sum_{\underline{i}\setminus\{i_{2(n-j)}\}} \sum_{i_{2(n-j)}=1}^{l_j} (-1)^{\sum_{k=1}^{n-1}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+1+\sum_{a=1}^{j-1}(l_a-1)+l_j-i_{2(n-j)}}$$
$$e_2^{\epsilon_1}\xi^{m_1+i_{2n-3}-1}x^{l_1-i_{2n-2}}e_1e_2x^{i_{2n-2}-1}\xi^{m_2+i_{2n-5}-1-i_{2n-3}}\dots$$
$$\dots$$
$$\dots\boxed{x^{l_j-i_{2(n-j)}+1}\xi x^{i_{2(n-j)}-1}}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}\dots$$
$$\dots$$
$$\dots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}$$
$$+ \sum_{j=1}^{n-1} \sum_{\underline{i}\setminus\{i_{2(n-j)}\}} \sum_{i'_{2(n-j)}=2}^{l_j+1} (-1)^{\sum_{k=1}^{n-1}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+1+\sum_{a=1}^{j-1}(l_a-1)+l_j-i'_{2(n-j)}+1}$$
$$e_2^{\epsilon_1}\xi^{m_1+i_{2n-3}-1}x^{l_1-i_{2n-2}}e_1e_2x^{i_{2n-2}-1}\xi^{m_2+i_{2n-5}-1-i_{2n-3}}\dots$$
$$\dots$$
$$\dots\boxed{x^{l_j-i'_{2(n-j)}+1}\xi x^{i'_{2(n-j)}-1}}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}\dots$$
$$\dots$$
$$\dots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}$$

Since they those two expressions have opposite signs, they cancel out, except for $i_{2(n-j)} = 1$ and $i'_{2(n-j)} = l_j + 1$. We have:

$$= \sum_{j=1}^{n-1} \sum_{\underline{i}\setminus\{i_{2(n-j)}\}} (-1)^{\sum_{k=1}^{n-1}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+1+\sum_{a=1}^{j-1}(l_a-1)+l_j-1}$$
$$e_2^{\epsilon_1}\xi^{m_1+i_{2n-3}-1}x^{l_1-i_{2n-2}}e_1e_2x^{i_{2n-2}-1}\xi^{m_2+i_{2n-5}-1-i_{2n-3}}\dots$$
$$\dots$$
$$\dots\boxed{x^{l_j}\xi}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}\dots$$
$$\dots$$
$$\dots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}$$
$$+ \sum_{j=1}^{n-1} \sum_{\underline{i}\setminus\{i_{2(n-j)}\}} (-1)^{\sum_{k=1}^{n-1}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+1+\sum_{a=1}^{j-1}(l_a-1)}$$
$$e_2^{\epsilon_1}\xi^{m_1+i_{2n-3}-1}x^{l_1-i_{2n-2}}e_1e_2x^{i_{2n-2}-1}\xi^{m_2+i_{2n-5}-1-i_{2n-3}}\dots$$
$$\dots$$
$$\dots\boxed{\xi x^{l_j}}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}\dots$$
$$\dots$$
$$\dots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}$$

Let us write down the neighbourhoods of the boxes; to do so, we need to consider the cases $j = 1, n-1$ separately:

$$= \sum_{\underline{i}\setminus\{i_{2n-2}\}} (-1)^{\sum_{k=1}^{n-1}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+1+l_1-1}$$
$$e_2^{\epsilon_1}\boxed{\xi^{m_1+i_{2n-3}-1}x^{l_1}\xi^{1+m_2+i_{2n-5}-1-i_{2n-3}}}\dots$$
$$\dots$$
$$\dots x^{l_j-i_{2(n-j)}}e_1e_2x^{i_{2(n-j)}-1}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}\dots$$
$$\dots$$
$$\dots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}$$
$$+ \sum_{\underline{i}\setminus\{i_{2n-2}\}} (-1)^{\sum_{k=1}^{n-1}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+1}$$
$$e_2^{\epsilon_1}\boxed{\xi^{m_1+i_{2n-3}-1+1}x^{l_1}\xi^{m_2+i_{2n-5}-1-i_{2n-3}}}\dots$$
$$\dots$$
$$\dots x^{l_j-i_{2(n-j)}}e_1e_2x^{i_{2(n-j)}-1}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}\dots$$
$$\dots$$
$$\dots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}$$

$$+\sum_{j=2}^{n-2}\sum_{\underline{i}\backslash\{i_{2(n-j)}\}}(-1)^{\sum_{k=1}^{n-1}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+1+\sum_{a=1}^{j-1}(l_a-1)+l_j-1}$$
$$\dots e_2^{\epsilon_1}\xi^{m_1+i_{2n-3}-1}x^{l_1-i_{2n-2}}e_1e_2x^{i_{2n-2}-1}\xi^{m_2+i_{2n-5}-1-i_{2n-3}}\dots$$
$$\dots$$
$$\dots\boxed{\xi^{m_j+i_{2(n-j)-1}-1-i_{2(n-(j-1))-1}}x^{l_j}\xi^{1+m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}}\dots$$
$$\dots$$
$$\dots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}$$
$$+\sum_{j=2}^{n-2}\sum_{\underline{i}\backslash\{i_{2(n-j)}\}}(-1)^{\sum_{k=1}^{n-1}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+1+\sum_{a=1}^{j-1}(l_a-1)}$$
$$\dots e_2^{\epsilon_1}\xi^{m_1+i_{2n-3}-1}x^{l_1-i_{2n-2}}e_1e_2x^{i_{2n-2}-1}\xi^{m_2+i_{2n-5}-1-i_{2n-3}}\dots$$
$$\dots$$
$$\dots\boxed{\xi^{m_j+i_{2(n-j)-1}-1-i_{2(n-(j-1))-1}+1}x^{l_j}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}}\dots$$
$$\dots$$
$$\dots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}$$
$$+\sum_{\underline{i}\backslash\{i_2\}}(-1)^{\sum_{k=1}^{n-1}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+1+\sum_{a=1}^{n-2}(l_a-1)+l_{n-1}-1}$$
$$e_2^{\epsilon_1}\xi^{m_1+i_{2n-3}-1}x^{l_1-i_{2n-2}}e_1e_2x^{i_{2n-2}-1}\xi^{m_2+i_{2n-5}-1-i_{2n-3}}\dots$$
$$\dots$$
$$\dots x^{l_j-i_{2(n-j)}}e_1e_2x^{i_{2(n-j)}-1}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}\dots$$
$$\dots$$
$$\dots\boxed{\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}}\xi^{1+m_n-i_1}}x^{l_n}e_1^{\eta_n}$$
$$+\sum_{\underline{i}\backslash\{i_2\}}(-1)^{\sum_{k=1}^{n-1}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+1+\sum_{a=1}^{n-2}(l_a-1)}$$
$$e_2^{\epsilon_1}\xi^{m_1+i_{2n-3}-1}x^{l_1-i_{2n-2}}e_1e_2x^{i_{2n-2}-1}\xi^{m_2+i_{2n-5}-1-i_{2n-3}}\dots$$
$$\dots$$
$$\dots x^{l_j-i_{2(n-j)}}e_1e_2x^{i_{2(n-j)}-1}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}\dots$$
$$\dots$$
$$\dots\boxed{\xi^{m_{n-1}+i_1-1-i_3+1}x^{l_{n-1}}\xi^{m_n-i_1}}x^{l_n}e_1^{\eta_n}$$

We make the changes of variable $i_{2n-3}=i'_{2n-3}-1$ in the second sum, $i_{2(n-j)-1}=i'_{2(n-j)-1}-1$ in the fourth sum, and $i_1=i'_1-1$ in the sixth sum so that:

$$=\sum_{\underline{i}\backslash\{i_{2n-2},i_{2n-3}\}}\sum_{i_{2n-3}=1}^{m_2+i_{2n-5}-1}(-1)^{\sum_{k=1}^{n-1}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+1+l_1-1}$$
$$e_2^{\epsilon_1}\boxed{\xi^{m_1+i_{2n-3}-1}x^{l_1}\xi^{1+m_2+i_{2n-5}-1-i_{2n-3}}}\dots$$
$$\dots$$
$$\dots x^{l_j-i_{2(n-j)}}e_1e_2x^{i_{2(n-j)}-1}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}\dots$$
$$\dots$$
$$\dots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}$$
$$+\sum_{\underline{i}\backslash\{i_{2n-2},i'_{2n-3}\}}\sum_{i'_{2n-3}=2}^{m_2+i_{2n-5}}(-1)^{\sum_{k=1}^{n-2}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+(i'_{2n-3}-1+\frac{1+(-1)^{n-1}}{2})l_1+1}$$
$$e_2^{\epsilon_1}\boxed{\xi^{m_1+i'_{2n-3}-1}x^{l_1}\xi^{m_2+i_{2n-5}-1-i'_{2n-3}+1}}\dots$$
$$\dots$$
$$\dots x^{l_j-i_{2(n-j)}}e_1e_2x^{i_{2(n-j)}-1}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}\dots$$
$$\dots$$
$$\dots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}$$

$$+\sum_{j=2}^{n-2}\sum_{\underline{i}\setminus\{i_{2(n-j)},i_{2(n-j)-1},i_{2(n-(j-1))-1}\}}\sum_{i_{2(n-j)-1}=1}^{m_{j+1}+i_{2(n-(j+1))-1}-1}\sum_{i_{2(n-(j-1))-1}=1}^{m_j+i_{2(n-j)-1}-1}$$

$$(-1)^{\sum_{k=1}^{n-1}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+1+\sum_{a=1}^{j-1}(l_a-1)+l_j-1}$$

$$\dots e_2^{\epsilon_1}\xi^{m_1+i_{2n-3}-1}x^{l_1-i_{2n-2}}e_1e_2x^{i_{2n-2}-1}\xi^{m_2+i_{2n-5}-1-i_{2n-3}}\dots$$

$$\dots$$

$$\dots\boxed{\xi^{m_j+i_{2(n-j)-1}-1-i_{2(n-(j-1))-1}}x^{l_j}\xi^{1+m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}}\dots$$

$$\dots$$

$$\dots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}$$

$$+\sum_{j=2}^{n-2}\sum_{\underline{i}\setminus\{i_{2(n-j)},i'_{2(n-j)-1},i_{2(n-(j-1))-1}\}}\sum_{i'_{2(n-j)-1}=2}^{m_{j+1}+i_{2(n-(j+1))-1}}\sum_{i_{2(n-(j-1))-1}=1}^{m_j+i'_{2(n-j)-1}-2}$$

$$(-1)^{\sum_{k=1,k\neq n-j}^{n-1}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+(i'_{2(n-j)-1}-1+\frac{1+(-1)^{n-j}}{2})l_j+1+\sum_{a=1}^{j-1}(l_a-1)}$$

$$\dots e_2^{\epsilon_1}\xi^{m_1+i_{2n-3}-1}x^{l_1-i_{2n-2}}e_1e_2x^{i_{2n-2}-1}\xi^{m_2+i_{2n-5}-1-i_{2n-3}}\dots$$

$$\dots$$

$$\dots\boxed{\xi^{m_j+i'_{2(n-j)-1}-1-i_{2(n-(j-1))-1}}x^{l_j}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i'_{2(n-j)-1}+1}}\dots$$

$$\dots$$

$$\dots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}$$

$$+\sum_{\underline{i}\setminus\{i_2,i_1,i_3\}}\sum_{i_1=1}^{m_n}\sum_{i_3=1}^{m_{n-1}+i_1-1}(-1)^{\sum_{k=1}^{n-1}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+1+\sum_{a=1}^{n-2}(l_a-1)+l_{n-1}-1}$$

$$e_2^{\epsilon_1}\xi^{m_1+i_{2n-3}-1}x^{l_1-i_{2n-2}}e_1e_2x^{i_{2n-2}-1}\xi^{m_2+i_{2n-5}-1-i_{2n-3}}\dots$$

$$\dots$$

$$\dots x^{l_j-i_{2(n-j)}}e_1e_2x^{i_{2(n-j)}-1}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}\dots$$

$$\dots$$

$$\dots\boxed{\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}}\xi^{1+m_n-i_1}}x^{l_n}e_1^{\eta_n}$$

$$+\sum_{\underline{i}\setminus\{i_2,i'_1,i_3\}}\sum_{i'_1=2}^{m_n+1}\sum_{i_3=1}^{m_{n-1}+i'_1-2}(-1)^{\sum_{k=2}^{n-1}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+(i'_1-1+0)l_{n-1}+1+\sum_{a=1}^{n-2}(l_a-1)}$$

$$e_2^{\epsilon_1}\xi^{m_1+i_{2n-3}-1}x^{l_1-i_{2n-2}}e_1e_2x^{i_{2n-2}-1}\xi^{m_2+i_{2n-5}-1-i_{2n-3}}\dots$$

$$\dots$$

$$\dots x^{l_j-i_{2(n-j)}}e_1e_2x^{i_{2(n-j)}-1}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}\dots$$

$$\dots$$

$$\dots\boxed{\xi^{m_{n-1}+i'_1-1-i_3}x^{l_{n-1}}\xi^{m_n-i'_1+1}}x^{l_n}e_1^{\eta_n}$$

Rearranging the exponents of the signs and of the ξ's in the boxes, we obtain:

$$=\sum_{\underline{i}\setminus\{i_{2n-2},i_{2n-3}\}}\sum_{i_{2n-3}=1}^{m_2+i_{2n-5}-1}(-1)^{\sum_{k=1}^{n-1}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+l_1}$$

$$e_2^{\epsilon_1}\boxed{\xi^{m_1+i_{2n-3}-1}x^{l_1}\xi^{m_2+i_{2n-5}-i_{2n-3}}}\dots$$

$$\dots$$

$$\dots x^{l_j-i_{2(n-j)}}e_1e_2x^{i_{2(n-j)}-1}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}\dots$$

$$\dots$$

$$\dots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}$$

$$+\sum_{\underline{i}\setminus\{i_{2n-2},i_{2n-3}\}}\sum_{i_{2n-3}=2}^{m_2+i_{2n-5}}(-1)^{\sum_{k=1}^{n-1}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}-l_1+1}$$

$$e_2^{\epsilon_1}\boxed{\xi^{m_1+i_{2n-3}-1}x^{l_1}\xi^{m_2+i_{2n-5}-i_{2n-3}}}\dots$$

$$\dots$$

$$\dots x^{l_j-i_{2(n-j)}}e_1e_2x^{i_{2(n-j)}-1}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}\dots$$

$$\dots$$

$$\dots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}$$

$$+\sum_{j=2}^{n-2}\sum_{\underline{i}\backslash\{i_{2(n-j)},i_{2(n-j)-1},i_{2(n-(j-1))-1}\}}\sum_{i_{2(n-j)-1}=1}^{m_{j+1}+i_{2(n-(j+1))-1}-1}\sum_{i_{2(n-(j-1))-1}=1}^{m_j+i_{2(n-j)-1}-1}$$
$$(-1)^{\sum_{k=1}^{n-1}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+\sum_{a=1}^{j-1}(l_a-1)+l_j}$$
$$\dots e_2^{\epsilon_1}\xi^{m_1+i_{2n-3}-1}x^{l_1-i_{2n-2}}e_1e_2x^{i_{2n-2}-1}\xi^{m_2+i_{2n-5}-1-i_{2n-3}}\dots$$
$$\dots$$
$$\dots\boxed{\xi^{m_j+i_{2(n-j)-1}-1-i_{2(n-(j-1))-1}}x^{l_j}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-i_{2(n-j)-1}}}\dots$$
$$\dots$$
$$\dots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}$$

$$+\sum_{j=2}^{n-2}\sum_{\underline{i}\backslash\{i_{2(n-j)},i_{2(n-j)-1},i_{2(n-(j-1))-1}\}}\sum_{i_{2(n-j)-1}=2}^{m_{j+1}+i_{2(n-(j+1))-1}}\sum_{i_{2(n-(j-1))-1}=1}^{m_j+i_{2(n-j)-1}-2}$$
$$(-1)^{\sum_{k=1}^{n-1}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+\sum_{a=1}^{j-1}(l_a-1)-l_j+1}$$
$$\dots e_2^{\epsilon_1}\xi^{m_1+i_{2n-3}-1}x^{l_1-i_{2n-2}}e_1e_2x^{i_{2n-2}-1}\xi^{m_2+i_{2n-5}-1-i_{2n-3}}\dots$$
$$\dots$$
$$\dots\boxed{\xi^{m_j+i_{2(n-j)-1}-1-i_{2(n-(j-1))-1}}x^{l_j}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-i_{2(n-j)-1}}}\dots$$
$$\dots$$
$$\dots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}$$

$$+\sum_{\underline{i}\backslash\{i_2,i_1,i_3\}}\sum_{i_1=1}^{m_n}\sum_{i_3=1}^{m_{n-1}+i_1-1}(-1)^{\sum_{k=1}^{n-1}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+\sum_{a=1}^{n-2}(l_a-1)+l_{n-1}}$$
$$e_2^{\epsilon_1}\xi^{m_1+i_{2n-3}-1}x^{l_1-i_{2n-2}}e_1e_2x^{i_{2n-2}-1}\xi^{m_2+i_{2n-5}-1-i_{2n-3}}\dots$$
$$\dots$$
$$\dots x^{l_j-i_{2(n-j)}}e_1e_2x^{i_{2(n-j)}-1}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}\dots$$
$$\dots$$
$$\dots\boxed{\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}}\xi^{1+m_n-i_1}}x^{l_n}e_1^{\eta_n}$$

$$+\sum_{\underline{i}\backslash\{i_2,i_1,i_3\}}\sum_{i_1=2}^{m_n+1}\sum_{i_3=1}^{m_{n-1}+i_1-2}(-1)^{\sum_{k=1}^{n-1}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+\sum_{a=1}^{n-2}(l_a-1)-l_{n-1}+1}$$
$$e_2^{\epsilon_1}\xi^{m_1+i_{2n-3}-1}x^{l_1-i_{2n-2}}e_1e_2x^{i_{2n-2}-1}\xi^{m_2+i_{2n-5}-1-i_{2n-3}}\dots$$
$$\dots$$
$$\dots x^{l_j-i_{2(n-j)}}e_1e_2x^{i_{2(n-j)}-1}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}\dots$$
$$\dots$$
$$\dots\boxed{\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}}\xi^{1+m_n-i_1}}x^{l_n}e_1^{\eta_n}$$

Similarly to the proof of Lemma 6.2.4, these three pairs of sums have opposite signs and a lot of care must be taken when simplifying them. The first two sums give two sums, one if we set $i_{2n-3}=1$ in the first, and one if $i_{2n-3}=m_2+i_{2n-5}$ in the second. The other two pairs will give three sums each: set $i_{2(n-j)-1}=1$ in the third sum, set $i_{2(n-j)-1}=m_{j+1}+i_{2(n-(j+1))-1}$ in the fourth and set $i_{2(n-(j-1))-1}=m_j+i_{2(n-j)-1}-1$ in the third sum (where $2\le i_{2(n-j)-1}\le m_{j+1}+i_{2(n-(j+1))-1}-1$). Finally, in the fifth sum set $i_1=1$ and in the sixth sum set $i_1=m_n+1$, and set $i_3=m_{n-1}+i_1-1$ in the fifth sum (where $2\le i_1\le m_n$).

$$
\begin{aligned}
= \sum_{\underline{i}\setminus\{i_{2n-2},i_{2n-3}\}} & (-1)^{\sum_{k=1}^{n-2}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+l_1+(1+\frac{1+(-1)^{n-1}}{2})l_1} \\
& e_2^{\epsilon_1}\boxed{\xi^{m_1}x^{l_1}\xi^{m_2+i_{2n-5}-1}}\dots \\
& \dots \\
& \dots x^{l_j-i_{2(n-j)}}e_1e_2x^{i_{2(n-j)}-1}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}\dots \\
& \dots \\
& \dots \xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n} \\
+ \sum_{\underline{i}\setminus\{i_{2n-2},i_{2n-3}\}} & (-1)^{\sum_{k=1}^{n-2}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}-l_1+1+(m_2+i_{2n-5}+\frac{1+(-1)^{n-1}}{2})l_1} \\
& e_2^{\epsilon_1}\boxed{\xi^{m_1+m_2+i_{2n-5}-1}x^{l_1}}\dots \\
& \dots \\
& \dots x^{l_j-i_{2(n-j)}}e_1e_2x^{i_{2(n-j)}-1}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}\dots \\
& \dots \\
& \dots \xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}
\end{aligned}
$$

$$
\begin{aligned}
+\sum_{j=2}^{n-2}\sum_{\underline{i}\setminus\{i_{2(n-j)},i_{2(n-j)-1},i_{2(n-(j-1))-1}\}}\sum_{i_{2(n-(j-1))-1}=1}^{m_j} & \\
(-1)^{\sum_{k=1,k\neq n-j}^{n-1}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+\sum_{a=1}^{j-1}(l_a-1)+l_j+(1+\frac{1+(-1)^{n-j}}{2})l_j} & \\
\dots e_2^{\epsilon_1}\xi^{m_1+i_{2n-3}-1}x^{l_1-i_{2n-2}}e_1e_2x^{i_{2n-2}-1}\xi^{m_2+i_{2n-5}-1-i_{2n-3}}\dots & \\
\dots & \\
\dots\boxed{\xi^{m_j-i_{2(n-(j-1))-1}}x^{l_j}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1}}\dots & \\
\dots & \\
\dots \xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n} &
\end{aligned}
$$

$$
\begin{aligned}
+\sum_{j=2}^{n-2}\sum_{\underline{i}\setminus\{i_{2(n-j)},i_{2(n-j)-1},i_{2(n-(j-1))-1}\}}\sum_{i_{2(n-(j-1))-1}=1}^{m_j+m_{j+1}+i_{2(n-(j+1))-1}-2} & \\
(-1)^{\sum_{k=1,k\neq n-j}^{n-1}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+\sum_{a=1}^{j-1}(l_a-1)-l_j+1+(m_{j+1}+i_{2(n-(j+1))-1}+\frac{1+(-1)^{n-j}}{2})l_j} & \\
\dots e_2^{\epsilon_1}\xi^{m_1+i_{2n-3}-1}x^{l_1-i_{2n-2}}e_1e_2x^{i_{2n-2}-1}\xi^{m_2+i_{2n-5}-1-i_{2n-3}}\dots & \\
\dots & \\
\dots\boxed{\xi^{m_j+m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-(j-1))-1}}x^{l_j}}\dots & \\
\dots & \\
\dots \xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n} &
\end{aligned}
$$

$$
\begin{aligned}
+\sum_{j=2}^{n-2}\sum_{\underline{i}\setminus\{i_{2(n-j)},i_{2(n-j)-1},i_{2(n-(j-1))-1},i_{2(n-(j-2))-1}\}}\sum_{i_{2(n-j)-1}=2}^{m_{j+1}+i_{2(n-(j+1))-1}-1}\sum_{i_{2(n-(j-2))-1}=1}^{m_{j-1}+m_j+i_{2(n-j)-1}-2} & \\
(-1)^{\sum_{k=1,k\neq n-(j-1)}^{n-1}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+\sum_{a=1}^{j-1}(l_a-1)+l_j+(m_j+i_{2(n-j)-1}-1+\frac{1+(-1)^{n-(j-1)}}{2})l_{j-1}} & \\
\dots e_2^{\epsilon_1}\xi^{m_1+i_{2n-3}-1}x^{l_1-i_{2n-2}}e_1e_2x^{i_{2n-2}-1}\xi^{m_2+i_{2n-5}-1-i_{2n-3}}\dots & \\
\dots & \\
\dots\boxed{\xi^{m_{j-1}+m_j+i_{2(n-j)-1}-1-1-i_{2(n-(j-2))-1}}x^{l_{j-1}-i_{2(n-(j-1))}}} & \\
\dots\boxed{e_1e_2x^{i_{2(n-(j-1))}-1+l_j}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-i_{2(n-j)-1}}}\dots & \\
\dots & \\
\dots \xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n} &
\end{aligned}
$$

$$+\sum_{\underline{i}\setminus\{i_2,i_1,i_3\}}\sum_{i_3=1}^{m_{n-1}}(-1)^{\sum_{k=2}^{n-1}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+\sum_{a=1}^{n-2}(l_a-1)+l_{n-1}+l_{n-1}}$$

$$e_2^{\epsilon_1}\xi^{m_1+i_{2n-3}-1}x^{l_1-i_{2n-2}}e_1e_2x^{i_{2n-2}-1}\xi^{m_2+i_{2n-5}-1-i_{2n-3}}\dots$$

$$\dots$$

$$\dots x^{l_j-i_{2(n-j)}}e_1e_2x^{i_{2(n-j)}-1}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}\dots$$

$$\dots$$

$$\dots\boxed{\xi^{m_{n-1}-i_3}x^{l_{n-1}}\xi^{m_n}}x^{l_n}e_1^{\eta_n}$$

$$+\sum_{\underline{i}\setminus\{i_2,i_1,i_3\}}\sum_{i_3=1}^{m_{n-1}+m_n-1}(-1)^{\sum_{k=2}^{n-1}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+\sum_{a=1}^{n-2}(l_a-1)-l_{n-1}+1+(m_n+1)l_{n-1}}$$

$$e_2^{\epsilon_1}\xi^{m_1+i_{2n-3}-1}x^{l_1-i_{2n-2}}e_1e_2x^{i_{2n-2}-1}\xi^{m_2+i_{2n-5}-1-i_{2n-3}}\dots$$

$$\dots$$

$$\dots x^{l_j-i_{2(n-j)}}e_1e_2x^{i_{2(n-j)}-1}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}\dots$$

$$\dots$$

$$\dots\boxed{\xi^{m_{n-1}+m_n-i_3}x^{l_{n-1}+l_n}}e_1^{\eta_n}$$

$$+\sum_{\underline{i}\setminus\{i_2,i_1,i_3,i_5\}}\sum_{i_1=2}^{m_n}\sum_{i_5=1}^{m_{n-2}+m_{n-1}+i_1-2}(-1)^{\sum_{k=1,k\neq 2}^{n-1}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+\sum_{a=1}^{n-2}(l_a-1)+l_{n-1}+(m_{n-1}+i_1-1+1)l_{n-2}}$$

$$e_2^{\epsilon_1}\xi^{m_1+i_{2n-3}-1}x^{l_1-i_{2n-2}}e_1e_2x^{i_{2n-2}-1}\xi^{m_2+i_{2n-5}-1-i_{2n-3}}\dots$$

$$\dots$$

$$\dots x^{l_j-i_{2(n-j)}}e_1e_2x^{i_{2(n-j)}-1}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}\dots$$

$$\dots$$

$$\dots\boxed{\xi^{m_{n-2}+m_{n-1}+i_1-2-i_5}x^{l_{n-2}-i_4}e_1e_2}$$

$$\dots\boxed{x^{i_4-1+l_{n-1}}\xi^{1+m_n-i_1}}x^{l_n}e_1^{\eta_n}$$

Considering the neighbouring terms of the boxes and simplifying the exponents of the signs, we obtain:

$$=\sum_{\underline{i}\setminus\{i_{2n-2},i_{2n-3}\}}(-1)^{\sum_{k=1}^{n-2}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+(\frac{1+(-1)^{n-1}}{2})l_1}$$

$$e_2^{\epsilon_1}\boxed{\xi^{m_1}x^{l_1}\xi^{m_2+i_{2n-5}-1}}\dots$$

$$\dots$$

$$\dots x^{l_j-i_{2(n-j)}}e_1e_2x^{i_{2(n-j)}-1}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}\dots$$

$$\dots$$

$$\dots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}$$

$$+\sum_{\underline{i}\setminus\{i_{2n-2},i_{2n-3}\}}(-1)^{\sum_{k=1}^{n-2}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+1+(m_2+i_{2n-5}+\frac{1+(-1)^{n-1}}{2}-1)l_1}$$

$$e_2^{\epsilon_1}\boxed{\xi^{m_1+m_2+i_{2n-5}-1}x^{l_1+l_2-i_{2n-4}}e_1e_2x^{i_{2n-4}-1}}\dots$$

$$\dots$$

$$\dots x^{l_j-i_{2(n-j)}}e_1e_2x^{i_{2(n-j)}-1}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}\dots$$

$$\dots$$

$$\dots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}$$

$$+\sum_{j=2}^{n-2}\sum_{\underline{i}\setminus\{i_{2(n-j)},i_{2(n-j)-1},i_{2(n-(j-1))-1}\}}\sum_{i_{2(n-(j-1))-1}=1}^{m_j}$$

$$(-1)^{\sum_{k=1,k\neq n-j}^{n-1}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+\sum_{a=1}^{j-1}(l_a-1)+(\frac{1+(-1)^{n-j}}{2})l_j}$$

$$\dots e_2^{\epsilon_1}\xi^{m_1+i_{2n-3}-1}x^{l_1-i_{2n-2}}e_1e_2x^{i_{2n-2}-1}\xi^{m_2+i_{2n-5}-1-i_{2n-3}}\dots$$

$$\dots$$

$$\dots\boxed{\xi^{m_j-i_{2(n-(j-1))-1}}x^{l_j}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1}}\dots$$

$$\dots$$

$$\dots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}$$

$$+\sum_{j=2}^{n-2}\sum_{\underline{i}\setminus\{i_{2(n-j)},i_{2(n-j)-1},i_{2(n-(j-1))-1}\}}\sum_{i_{2(n-(j-1))-1}=1}^{m_j+m_{j+1}+i_{2(n-(j+1))-1}-2}$$

$$(-1)^{\sum_{k=1,k\neq n-j}^{n-1}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+\sum_{a=1}^{j-1}(l_a-1)+1+(m_{j+1}+i_{2(n-(j+1))-1}+\frac{1+(-1)^{n-j}}{2}-1)l_j}$$

$$\dots e_2^{\epsilon_1}\xi^{m_1+i_{2n-3}-1}x^{l_1-i_{2n-2}}e_1e_2x^{i_{2n-2}-1}\xi^{m_2+i_{2n-5}-1-i_{2n-3}}\dots$$

$$\dots$$

$$\dots\boxed{\xi^{m_j+m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-(j-1))-1}}x^{l_j+l_{j+1}-i_{2(n-(j+1))}}e_1e_2x^{i_{2(n-(j+1))}-1}}\dots$$

$$\dots$$

$$\dots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}$$

$$+\sum_{j=2}^{n-2}\sum_{\underline{i}\setminus\{i_{2(n-j)},i_{2(n-j)-1},i_{2(n-(j-1))-1},i_{2(n-(j-2))-1}\}}\sum_{i_{2(n-j)-1}=2}^{m_{j+1}+i_{2(n-(j+1))-1}-1}\sum_{i_{2(n-(j-2))-1}=1}^{m_{j-1}+m_j+i_{2(n-j)-1}-2}$$

$$(-1)^{\sum_{k=1,k\neq n-(j-1)}^{n-1}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+\sum_{a=1}^{j-1}(l_a-1)+l_j+(m_j+i_{2(n-j)-1}-1+\frac{1+(-1)^{n-(j-1)}}{2})l_{j-1}}$$

$$\dots e_2^{\epsilon_1}\xi^{m_1+i_{2n-3}-1}x^{l_1-i_{2n-2}}e_1e_2x^{i_{2n-2}-1}\xi^{m_2+i_{2n-5}-1-i_{2n-3}}\dots$$

$$\dots$$

$$\dots\boxed{\xi^{m_{j-1}+m_j+i_{2(n-j)-1}-2-i_{2(n-(j-2))-1}}x^{l_{j-1}-i_{2(n-(j-1))}}}$$

$$\dots\boxed{e_1e_2x^{i_{2(n-(j-1))}-1+l_j}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-i_{2(n-j)-1}}}\dots$$

$$\dots$$

$$\dots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}$$

$$+\sum_{\underline{i}\backslash\{i_2,i_1,i_3\}}\sum_{i_3=1}^{m_{n-1}}(-1)^{\sum_{k=2}^{n-1}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+\sum_{a=1}^{n-2}(l_a-1)}$$

$$e_2^{\epsilon_1}\xi^{m_1+i_{2n-3}-1}x^{l_1-i_{2n-2}}e_1e_2x^{i_{2n-2}-1}\xi^{m_2+i_{2n-5}-1-i_{2n-3}}\dots$$
$$\dots$$
$$\dots x^{l_j-i_{2(n-j)}}e_1e_2x^{i_{2(n-j)}-1}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}\dots$$
$$\dots$$
$$\dots\boxed{\xi^{m_{n-1}-i_3}x^{l_{n-1}}\xi^{m_n}}x^{l_n}e_1^{\eta_n}$$

$$+\sum_{\underline{i}\backslash\{i_2,i_1,i_3\}}\sum_{i_3=1}^{m_{n-1}+m_n-1}(-1)^{\sum_{k=2}^{n-1}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+\sum_{a=1}^{n-2}(l_a-1)+1+m_nl_{n-1}}$$

$$e_2^{\epsilon_1}\xi^{m_1+i_{2n-3}-1}x^{l_1-i_{2n-2}}e_1e_2x^{i_{2n-2}-1}\xi^{m_2+i_{2n-5}-1-i_{2n-3}}\dots$$
$$\dots$$
$$\dots x^{l_j-i_{2(n-j)}}e_1e_2x^{i_{2(n-j)}-1}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}\dots$$
$$\dots$$
$$\dots\boxed{\xi^{m_{n-1}+m_n-i_3}x^{l_{n-1}+l_n}}e_1^{\eta_n}$$

$$+\sum_{\underline{i}\backslash\{i_2,i_1,i_3,i_5\}}\sum_{i_1=2}^{m_n}\sum_{i_5=1}^{m_{n-2}+m_{n-1}+i_1-2}(-1)^{\sum_{k=1,k\neq 2}^{n-1}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+\sum_{a=1}^{n-2}(l_a-1)+l_{n-1}+(m_{n-1}+i_1)l_{n-2}}$$

$$e_2^{\epsilon_1}\xi^{m_1+i_{2n-3}-1}x^{l_1-i_{2n-2}}e_1e_2x^{i_{2n-2}-1}\xi^{m_2+i_{2n-5}-1-i_{2n-3}}\dots$$
$$\dots$$
$$\dots x^{l_j-i_{2(n-j)}}e_1e_2x^{i_{2(n-j)}-1}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}\dots$$
$$\dots$$
$$\dots\boxed{\xi^{m_{n-2}+m_{n-1}+i_1-2-i_5}x^{l_{n-2}-i_4}e_1e_2}$$
$$\dots\boxed{x^{i_4-1+l_{n-1}}\xi^{1+m_n-i_1}}x^{l_n}e_1^{\eta_n}$$

Similarly to the proof of Lemma 6.2.4, we perform some relabellings and some changes of variable:

- in the third sum, relabel $i_{2(n-k)-1}$ by $i'_{2(j-k)-1}$ and relabel $i_{2(n-k)}$ by $i'_{2(j-k)}$ for all $1 \leq k \leq j-1$.
- in the fourth sum, relabel $i_{2(n-k)-1}$ by $i_{2(n-(k+1))-1}$ and relabel $i_{2(n-k)}$ by $i_{2(n-(k+1))}$ for all $k \leq j-1$;
- in the fifth sum, make the change of variable $i_{2(n-j)-1} = i_{2(n-j)-1}+1$, and relabel $i_{2(n-(j-1))}$ by $i_{2(n-j)}$, $i_{2(n-k)-1}$ by $i_{2(n-(k+1))-1}$ and $i_{2(n-k)}$ by $i_{2(n-(k+1))}$ for all $1 \leq k \leq j-2$.
- in the sixth sum, relabel $i_{2(n-k)-1}$ by $i_{2(n-(k+1))-1}$ and relabel $i_{2(n-k)}$ by $i_{2(n-(k+1))}$ for all $1 \leq k \leq n-2$.
- in the seventh sum, relabel $i_{2(n-k)-1}$ by $i_{2(n-(k+1))-1}$ and relabel $i_{2(n-k)}$ by $i_{2(n-(k+1))}$ for all $k \leq n-2$;
- in the eighth sum, make the change of variable $i_1 = i_1 + 1$, relabel i_4 by i_2, $i_{2(n-k)-1}$ by $i_{2(n-(k+1))-1}$ and $i_{2(n-k)}$ by $i_{2(n-(k+1))}$ for all $1 \leq k \leq n-3$.

$$=\sum_{\underline{i}\backslash\{i_{2n-2},i_{2n-3}\}}(-1)^{\sigma_1}e_2^{\epsilon_1}\boxed{\xi^{m_1}x^{l_1}\xi^{m_2+i_{2n-5}-1}}\dots$$
$$\dots$$
$$\dots x^{l_j-i_{2(n-j)}}e_1e_2x^{i_{2(n-j)}-1}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}\dots$$
$$\dots$$
$$\dots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}$$

$$+\sum_{\underline{i}\backslash\{i_{2n-2},i_{2n-3}\}}(-1)^{\sigma_2}e_2^{\epsilon_1}\boxed{\xi^{m_1+m_2+i_{2n-5}-1}x^{l_1+l_2-i_{2n-4}}e_1e_2x^{i_{2n-4}-1}}\dots$$
$$\dots$$
$$\dots x^{l_j-i_{2(n-j)}}e_1e_2x^{i_{2(n-j)}-1}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}\dots$$
$$\dots$$
$$\dots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}$$

$$+\sum_{j=2}^{n-2}\sum_{i'_1=1}^{m_j}\sum_{i'_2=1}^{l_{j-1}}\cdots\sum_{i'_{2j-3}=1}^{m_2+i'_{2j-5}-1}\sum_{i'_{2j-2}=1}^{l_1}\sum_{i_1=1}^{m_n}\sum_{i_2=1}^{l_{n-1}}\cdots\sum_{i_{2(n-(j+1))-1}=1}^{m_{j+2}-i_{2(n-(j+2))-1}-1}\sum_{i_{2(n-(j+1))}=1}^{l_{j+1}}$$

$$(-1)^{\sigma_3}e_2^{\epsilon_1}\xi^{m_1+i'_{2j-3}-1}x^{l_1-i'_{2j-2}}e_1e_2x^{i'_{2j-2}-1}\xi^{m_2+i'_{2j-5}-1-i'_{2j-3}}\cdots$$

$$\cdots$$

$$\cdots\xi^{m_j-i'_1}x^{l_j}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1}\cdots$$

$$\cdots$$

$$\cdots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}$$

$$+\sum_{j=2}^{n-2}\sum_{i_1=1}^{m_n}\sum_{i_2=1}^{l_{n-1}}\cdots\sum_{i_{2(n-(j+1))}=1}^{l_{j+1}}\sum_{i_{2(n-j)-1}=1}^{m_j+m_{j+1}+i_{2(n-(j+1))-1}-2}\sum_{i_{2(n-j)}=1}^{l_{j-1}}\cdots\sum_{i_{2n-5}=1}^{m_2+i_7-1}\sum_{i_{2n-4}=1}^{l_1}$$

$$(-1)^{\sigma_4}e_2^{\epsilon_1}\xi^{m_1+i_{2n-5}-1}x^{l_1-i_{2n-4}}e_1e_2x^{i_{2n-4}-1}\xi^{m_2+i_{2n-7}-1-i_{2n-5}}\cdots$$

$$\cdots$$

$$\cdots\boxed{\xi^{m_j+m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}x^{l_j+l_{j+1}-i_{2(n-(j+1))}}e_1e_2x^{i_{2(n-(j+1))}-1}}\cdots$$

$$\cdots$$

$$\cdots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}$$

$$+\sum_{j=2}^{n-2}\sum_{i_1=1}^{m_n}\sum_{i_2=1}^{l_{n-1}}\cdots\sum_{i_{2(n-j)-1}=1}^{m_{j+1}+i_{2(n-(j+1))-1}-2}\sum_{i_{2(n-j)}=1}^{l_{j-1}}\sum_{i_{2(n-(j-1))-1}=1}^{m_{j-1}+m_j+i_{2(n-j)-1}-1}\sum_{i_{2(n-(j-1))}=1}^{l_{j-2}}\cdots$$

$$\cdots\sum_{i_{2n-5}=1}^{m_2+i_{2n-7}-1}\sum_{i_{2n-4}=1}^{l_1}(-1)^{\sigma_5}e_2^{\epsilon_1}\xi^{m_1+i_{2n-5}-1}x^{l_1-i_{2n-4}}e_1e_2x^{i_{2n-4}-1}\xi^{m_2+i_{2n-7}-1-i_{2n-5}}\cdots$$

$$\cdots$$

$$\cdots\boxed{\xi^{m_{j-1}+m_j+i_{2(n-j)-1}-1-i_{2(n-(j-1))-1}}x^{l_{j-1}-i_{2(n-j)}}}$$

$$\cdots\boxed{e_1e_2x^{i_{2(n-j)}-1+l_j}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}}\cdots$$

$$\cdots$$

$$\cdots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}$$

$$6+\sum_{i_1=1}^{m_{n-1}}\sum_{i_2=1}^{l_{n-2}}\cdots\sum_{i_{2(n-(j+1))-1}=1}^{m_j+i_{2(n-(j+2)-1}-1}\sum_{i_{2(n-(j+1))}=1}^{l_j}\cdots\sum_{i_{2n-5}=1}^{m_2-i_{2n-7}-1}\sum_{i_{2n-4}=1}^{l_1}$$

$$(-1)^{\sigma_6}e_2^{\epsilon_1}\xi^{m_1+i_{2n-5}-1}x^{l_1-i_{2n-4}}e_1e_2x^{i_{2n-4}-1}\xi^{m_2+i_{2n-7}-1-i_{2n-5}}\cdots$$

$$\cdots$$

$$\cdots x^{l_j-i_{2(n-(j+1))}}e_1e_2x^{i_{2(n-(j+1))}-1}\xi^{m_{j+1}+i_{2(n-(j+2))-1}-1-i_{2(n-(j+1))-1}}\cdots$$

$$\cdots$$

$$\cdots\boxed{\xi^{m_{n-1}-i_1}x^{l_{n-1}}\xi^{m_n}}x^{l_n}e_1^{\eta_n}$$

$$7+\sum_{i_1=1}^{m_{n-1}+m_n-1}\sum_{i_2=1}^{l_{n-2}}\cdots\sum_{i_{2(n-(j+1))-1}=1}^{m_j+i_{2(n-(j+2)-1}-1}\sum_{i_{2(n-(j+1))}=1}^{l_j}\cdots\sum_{i_{2n-5}=1}^{m_2-i_{2n-7}-1}\sum_{i_{2n-4}=1}^{l_1}$$

$$(-1)^{\sigma_7}e_2^{\epsilon_1}\xi^{m_1+i_{2n-5}-1}x^{l_1-i_{2n-4}}e_1e_2x^{i_{2n-4}-1}\xi^{m_2+i_{2n-7}-1-i_{2n-5}}\cdots$$

$$\cdots$$

$$\cdots x^{l_j-i_{2(n-(j+1))}}e_1e_2x^{i_{2(n-(j+1))}-1}\xi^{m_{j+1}+i_{2(n-(j+2))-1}-1-i_{2(n-(j+1))-1}}\cdots$$

$$\cdots$$

$$\cdots\boxed{\xi^{m_{n-1}+m_n-i_1}x^{l_{n-1}+l_n}}e_1^{\eta_n}$$

$$8+\sum_{i_1=1}^{m_n-1}\sum_{i_2=1}^{l_{n-2}}\sum_{i_3=1}^{m_{n-2}+m_{n-1}+i_1-1}\sum_{i_4=1}^{l_{n-3}}\cdots\sum_{i_{2(n-(j+1))-1}=1}^{m_{j+1}+i_{2(n-(j+2))-1}-1}\sum_{i_{2(n-(j+1))}=1}^{l_j}\cdots\sum_{i_{2n-5}=1}^{m_2+i_{2n-7}-1}\sum_{i_{2n-4}=1}^{l_1}$$

$$(-1)^{\sigma_8}e_2^{\epsilon_1}\xi^{m_1+i_{2n-5}-1}x^{l_1-i_{2n-4}}e_1e_2x^{i_{2n-4}-1}\xi^{m_2+i_{2n-7}-1-i_{2n-5}}\cdots$$

$$\cdots$$

$$\cdots x^{l_j-i_{2(n-(j+1))}}e_1e_2x^{i_{2(n-(j+1))}-1}\xi^{m_{j+1}+i_{2(n-(j+2))-1}-1-i_{2(n-(j+1))-1}}\cdots$$

$$\cdots$$

$$\cdots\boxed{\xi^{m_{n-2}+m_{n-1}+i_1-1-i_3}x^{l_{n-2}-i_2}e_1e_2}$$

$$\cdots\boxed{x^{i_2-1+l_{n-1}}\xi^{m_n-i_1}}x^{l_n}e_1^{\eta_n}$$

where the exponents σ_i are given below (after applying the same changes):

$$\begin{aligned}
\sigma_1 &= \sum_{k=1}^{n-2}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+(\frac{1+(-1)^{n-1}}{2})l_1 \\
\sigma_2 &= \sum_{k=1}^{n-2}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+1+(m_2+i_{2n-5}+\frac{1+(-1)^{n-1}}{2}-1)l_1 \\
\sigma_3 &= \sum_{k=1}^{n-(j+1)}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+\sum_{k=1}^{j-1}(i'_{2k-1}+\frac{1+(-1)^{n-j+k}}{2})l_{j-k} \\
&\quad +\sum_{a=1}^{j-1}(l_a-1)+(\frac{1+(-1)^{n-j}}{2})l_j \\
\sigma_4 &= \sum_{k=1}^{n-(j+1)}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+\sum_{k=n-j}^{n-2}(i_{2k-1}+\frac{1+(-1)^{k+1}}{2})l_{n-k-1} \\
&\quad +\sum_{a=1}^{j-1}(l_a-1)+1+(m_{j+1}+i_{2(n-(j+1))-1}+\frac{1+(-1)^{n-j}}{2}-1)l_j \\
\sigma_5 &= \sum_{k=1}^{n-j}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+l_j+\sum_{k=n-(j-1)}^{n-2}(i_{2k-1}+\frac{1+(-1)^{k+1}}{2})l_{n-k-1} \\
&\quad +\sum_{a=1}^{j-1}(l_a-1)+l_j+(m_j+i_{2(n-j)-1}+\frac{1+(-1)^{n-(j-1)}}{2})l_{j-1} \\
\sigma_6 &= \sum_{k=1}^{n-2}(i_{2k-1}+\frac{1+(-1)^{k+1}}{2})l_{n-k-1}+\sum_{a=1}^{n-2}(l_a-1) \\
\sigma_7 &= \sum_{k=1}^{n-2}(i_{2k-1}+\frac{1+(-1)^{k+1}}{2})l_{n-k-1}+\sum_{a=1}^{n-2}(l_a-1)+1+m_nl_{n-1} \\
\sigma_8 &= \sum_{k=2}^{n-2}(i_{2k-1}+\frac{1+(-1)^{k+1}}{2})l_{n-k-1}+(i_1+1)l_{n-1}+\sum_{a=1}^{n-2}(l_a-1) \\
&\quad +l_{n-1}+(m_{n-1}+i_1+1)l_{n-2}
\end{aligned}$$

We need to pay close attention to those exponents governing the signs and transform them appropriately. We will thus work in $\mathbb{Z}/2\mathbb{Z}$. We note that the following equality holds:

$$\frac{1+(-1)^k}{2}=k+1[\bmod 2],$$

for all $k \in \mathbb{Z}$. We have:

$$\begin{aligned}
\sigma_1 = &\sum_{k=1}^{n-2}(i_{2k-1} + \frac{1+(-1)^k}{2})l_{n-k} + (\frac{1+(-1)^{n-1}}{2})l_1 \\
= &\sum_{k=1}^{n-2}(i_{2k-1} + \frac{1+(-1)^k}{2})l_{n-k} + (n-2)l_1
\end{aligned}$$

$$\begin{aligned}
\sigma_2 = &\sum_{k=1}^{n-2}(i_{2k-1} + \frac{1+(-1)^k}{2})l_{n-k} + 1 + (m_2 + i_{2n-5} + \frac{1+(-1)^{n-1}}{2} - 1)l_1 \\
= &\sum_{k=1}^{n-3}(i_{2k-1} + \frac{1+(-1)^k}{2})l_{n-k} + (i_{2n-5} + \frac{1+(-1)^{n-2}}{2})l_2 + 1 \\
&+(m_2 + i_{2n-5} + \frac{1+(-1)^{n-1}}{2} - 1)l_1 \\
= &\sum_{k=1}^{n-3}(i_{2k-1} + \frac{1+(-1)^k}{2})l_{n-k} + (i_{2n-5} + n - 1)l_2 + 1 + (m_2 + i_{2n-5} + n - 2 + 1)l_1 \\
= &\sum_{k=1}^{n-3}(i_{2k-1} + \frac{1+(-1)^k}{2})l_{n-k} + (i_{2n-5} + n - 1)(l_2 + l_1) + m_2 l_1 + 1
\end{aligned}$$

$$\begin{aligned}
\sigma_3 = &\sum_{k=1}^{n-(j+1)}(i_{2k-1} + \frac{1+(-1)^k}{2})l_{n-k} + \sum_{k=1}^{j-1}(i'_{2k-1} + \frac{1+(-1)^{n-j+k}}{2})l_{j-k} \\
&+\sum_{a=1}^{j-1}(l_a - 1) + (\frac{1+(-1)^{n-j}}{2})l_j \\
= &\sum_{k=1}^{n-(j+1)}(i_{2k-1} + \frac{1+(-1)^k}{2})l_{n-k} + \sum_{k=1}^{j-1}(i'_{2k-1} + n - j + k + 1)l_{j-k} \\
&+\sum_{a=1}^{j-1} l_a - j + 1 + (n-j+1)l_j \\
= &\sum_{k=1}^{n-(j+1)}(i_{2k-1} + \frac{1+(-1)^k}{2})l_{n-k} + \sum_{k=1}^{j-1}(i'_{2k-1} + k + 1)l_{j-k} \\
&+\sum_{k=1}^{j-1}(n-j)l_{j-k} + \sum_{a=1}^{j-1} l_a - j + 1 + (n-j+1)l_j \\
= &\sum_{k=1}^{n-(j+1)}(i_{2k-1} + \frac{1+(-1)^k}{2})l_{n-k} + \sum_{k=1}^{j-1}(i'_{2k-1} + \frac{1+(-1)^k}{2})l_{j-k} \\
&+(n-j)\sum_{k=1}^{j-1} l_k + \sum_{a=1}^{j-1} l_a - j + 1 + (n-j+1)l_j \\
= &\sum_{k=1}^{n-(j+1)}(i_{2k-1} + \frac{1+(-1)^k}{2})l_{n-k} + \sum_{k=1}^{j-1}(i'_{2k-1} + \frac{1+(-1)^k}{2})l_{j-k} \\
&+(n-j+1)\sum_{k=1}^{j} l_k - j + 1
\end{aligned}$$

$$
\begin{aligned}
\sigma_4 = \quad & \sum_{k=1}^{n-(j+1)} (i_{2k-1} + \frac{1+(-1)^k}{2}) l_{n-k} + \sum_{k=n-j}^{n-2} (i_{2k-1} + \frac{1+(-1)^{k+1}}{2}) l_{n-k-1} + \sum_{a=1}^{j-1} (l_a - 1) \\
& +1 + (m_{j+1} + i_{2(n-(j+1))-1} + \frac{1+(-1)^{n-j}}{2} - 1) l_j \\
= \quad & \sum_{k=1}^{n-(j+1)} (i_{2k-1} + \frac{1+(-1)^k}{2}) l_{n-k} + \sum_{k=n-j}^{n-2} (i_{2k-1} + k + 2) l_{n-k-1} + \sum_{a=1}^{j-1} l_a - j + 1 \\
& +1 + (m_{j+1} + i_{2(n-(j+1))-1} + n - j + 1 - 1) l_j \\
= \quad & \sum_{k=1}^{n-(j+1)} (i_{2k-1} + \frac{1+(-1)^k}{2}) l_{n-k} + \sum_{k=n-j}^{n-2} (i_{2k-1} + k + 1) l_{n-k-1} + \sum_{k=n-j}^{n-2} l_{n-k-1} \\
& + \sum_{a=1}^{j-1} l_a - j + 1 + 1 + (m_{j+1} + i_{2(n-(j+1))-1} + n - j - 1 + 1) l_j \\
= \quad & \sum_{k=1}^{n-(j+1)} (i_{2k-1} + \frac{1+(-1)^k}{2}) l_{n-k} + \sum_{k=n-j}^{n-2} (i_{2k-1} + \frac{1+(-1)^k}{2}) l_{n-k-1} + \sum_{k=1}^{j-1} l_k \\
& + \sum_{a=1}^{j-1} l_a - j + 1 + 1 + (m_{j+1} + i_{2(n-(j+1))-1} + \frac{1+(-1)^{n-(j+1)}}{2}) l_j \\
= \quad & \sum_{k=1}^{n-(j+2)} (i_{2k-1} + \frac{1+(-1)^k}{2}) l_{n-k} + (i_{2(n-(j+1))-1} + \frac{1+(-1)^{n-(j+1)}}{2})(l_{j+1} + l_j) \\
& + \sum_{k=n-j}^{n-2} (i_{2k-1} + \frac{1+(-1)^k}{2}) l_{n-k-1} + m_{j+1} l_j - j + 2 \left(1 + \sum_{k=1}^{j-1} l_k \right) \\
= \quad & \sum_{k=1}^{n-(j+2)} (i_{2k-1} + \frac{1+(-1)^k}{2}) l_{n-k} + (i_{2(n-(j+1))-1} + \frac{1+(-1)^{n-(j+1)}}{2})(l_{j+1} + l_j) \\
& + \sum_{k=n-j}^{n-2} (i_{2k-1} + \frac{1+(-1)^k}{2}) l_{n-k-1} + m_{j+1} l_j - j
\end{aligned}
$$

$$\begin{aligned}
\sigma_5 = \ & \sum_{k=1}^{n-j}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+l_j+\sum_{k=n-(j-1)}^{n-2}(i_{2k-1}+\frac{1+(-1)^{k+1}}{2})l_{n-k-1}+\sum_{a=1}^{j-1}(l_a-1) \\
& +l_j+(m_j+i_{2(n-j)-1}+\frac{1+(-1)^{n-(j-1)}}{2})l_{j-1} \\
= \ & \sum_{k=1}^{n-j}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+l_j+\sum_{k=n-(j-1)}^{n-2}(i_{2k-1}+k+1)l_{n-k-1}+\sum_{k=n-(j-1)}^{n-2}l_{n-k-1} \\
& +\sum_{a=1}^{j-1}l_a-j+1+l_j+m_jl_{j-1}(i_{2(n-j)-1}+(n-j+1)+1)l_{j-1} \\
= \ & \sum_{k=1}^{n-j}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+l_j+\sum_{k=n-(j-1)}^{n-2}(i_{2k-1}+k+1)l_{n-k-1}+\sum_{k=1}^{j-2}l_k+\sum_{a=1}^{j-1}l_a \\
& -j+1+l_j+l_{j-1}+m_jl_{j-1}+(i_{2(n-j)-1}+\frac{1+(-1)^{n-j}}{2})l_{j-1} \\
= \ & \sum_{k=1}^{n-(j+1)}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+(i_{2(n-j)-1}+\frac{1+(-1)^{n-j}}{2})(l_j+l_{j-1}) \\
& +\sum_{k=n-(j-1)}^{n-2}(i_{2k-1}+k+1)l_{n-k-1}+\sum_{k=1}^{j-2}l_k+l_{j-1}+l_j\sum_{a=1}^{j-1}l_a+l_j-j+1+m_jl_{j-1} \\
= \ & \sum_{k=1}^{n-(j+1)}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+(i_{2(n-j)-1}+\frac{1+(-1)^{n-j}}{2})(l_j+l_{j-1}) \\
& +\sum_{k=n-(j-1)}^{n-2}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k-1}+2\sum_{k=1}^{j}l_k-j+1+m_jl_{j-1} \\
= \ & \sum_{k=1}^{n-(j+1)}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+(i_{2(n-j)-1}+\frac{1+(-1)^{n-j}}{2})(l_j+l_{j-1}) \\
& +\sum_{k=n-(j-1)}^{n-2}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k-1}+m_jl_{j-1}-(j-1)
\end{aligned}$$

$$\begin{aligned}
\sigma_6 = \ & \sum_{k=1}^{n-2}(i_{2k-1}+\frac{1+(-1)^{k+1}}{2})l_{n-k-1}+\sum_{a=1}^{n-2}(l_a-1) \\
= \ & \sum_{k=1}^{n-2}(i_{2k-1}+(k+1)+1)l_{n-k-1}+\sum_{a=1}^{n-2}l_a-(n-2) \\
= \ & \sum_{k=1}^{n-2}(i_{2k-1}+(k+1))l_{n-k-1}+\sum_{k=1}^{n-2}l_{n-k-1}+\sum_{a=1}^{n-2}l_a-(n-2) \\
= \ & \sum_{k=1}^{n-2}(i_{2k-1}+(k+1))l_{n-k-1}+2\sum_{k=1}^{n-2}l_k-(n-2) \\
= \ & \sum_{k=1}^{n-2}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k-1}-(n-2)
\end{aligned}$$

$$\begin{aligned}
\sigma_7 = \ & \sum_{k=1}^{n-2}(i_{2k-1}+\frac{1+(-1)^{k+1}}{2})l_{n-k-1}+\sum_{a=1}^{n-2}(l_a-1)+1+m_nl_{n-1} \\
= \ & \sum_{k=1}^{n-2}(i_{2k-1}+(k+1)+1)l_{n-k-1}+\sum_{a=1}^{n-2}l_a-(n-2)+1+m_nl_{n-1} \\
= \ & \sum_{k=1}^{n-2}(i_{2k-1}+(k+1))l_{n-k-1}+\sum_{k=1}^{n-2}l_{n-k-1}+\sum_{a=1}^{n-2}l_a-(n-3)+m_nl_{n-1} \\
= \ & \sum_{k=1}^{n-2}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k-1}+2\sum_{k=1}^{n-2}l_k-(n-3)+m_nl_{n-1} \\
= \ & \sum_{k=1}^{n-2}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k-1}-(n-3)+m_nl_{n-1}
\end{aligned}$$

$$
\begin{aligned}
\sigma_8 = \;& \sum_{k=2}^{n-2}(i_{2k-1} + \frac{1+(-1)^{k+1}}{2})l_{n-k-1} + (i_1+1)l_{n-1} + \sum_{a=1}^{n-2}(l_a - 1) + l_{n-1} + (m_{n-1}+i_1+1)l_{n-2} \\
= \;& \sum_{k=2}^{n-2}(i_{2k-1} + (k+1)+1)l_{n-k-1} + i_1 l_{n-1} + l_{n-1} + \sum_{a=1}^{n-2} l_a - (n-2) + l_{n-1} + m_{n-1}l_{n-2} \\
& +i_1 l_{n-2} + l_{n-2} \\
= \;& i_1(l_{n-1}+l_{n-2}) + \sum_{k=2}^{n-2}(i_{2k-1} + (k+1))l_{n-k-1} + \sum_{k=2}^{n-2} l_{n-k-1} + l_{n-1} + \sum_{a=1}^{n-2} l_a - (n-2) \\
& +l_{n-2} + l_{n-1} + m_{n-1}l_{n-2} \\
= \;& i_1(l_{n-1}+l_{n-2}) + \sum_{k=2}^{n-2}(i_{2k-1} + (k+1))l_{n-k-1} + \sum_{k=1}^{n-3} l_k + l_{n-2} + l_{n-1} + \sum_{a=1}^{n-2} l_a + l_{n-1} \\
& -(n-2) + m_{n-1}l_{n-2} \\
= \;& i_1(l_{n-1}+l_{n-2}) + \sum_{k=2}^{n-2}(i_{2k-1} + \frac{1+(-1)^{k}}{2})l_{n-k-1} + 2\sum_{k=1}^{n-1} l_k \\
& +m_{n-1}l_{n-2} - (n-2) \\
= \;& i_1(l_{n-1}+l_{n-2}) + \sum_{k=2}^{n-2}(i_{2k-1} + \frac{1+(-1)^{k}}{2})l_{n-k-1} + m_{n-1}l_{n-2} - (n-2)
\end{aligned}
$$

Let us now analyse the following situations:

- Compare σ_4 with $i_{2(n-j)-1} = m_j + m_{j+1} + i_{2(n-(j+1))-1} - 1$ and σ_5 with $i_{2(n-j)-1} = m_{j+1} + i_{2(n-(j+1))-1} - 1$;
- Compare σ_7 with $i_1 = m_{n-1} + m_n$ and σ_8 with $i_1 = m_n$.

$$
\begin{aligned}
\sigma_4 = \;& \sum_{k=1}^{n-(j+2)} (i_{2k-1} + \frac{1+(-1)^{k}}{2})l_{n-k} + (i_{2(n-(j+1))-1} + \frac{1+(-1)^{n-(j+1)}}{2})(l_{j+1}+l_j) \\
& +(m_j + m_{j+1} + i_{2(n-(j+1))-1} - 1 + \frac{1+(-1)^{n-j}}{2})l_{j-1} \\
& + \sum_{k=n-(j-1)}^{n-2} (i_{2k-1} + \frac{1+(-1)^{k}}{2})l_{n-k-1} + m_{j+1}l_j - j \\
= \;& \sum_{k=1}^{n-(j+1)} (i_{2k-1} + \frac{1+(-1)^{k}}{2})l_{n-k} + (i_{2(n-(j+1))-1} + \frac{1+(-1)^{n-(j+1)}}{2})l_j \\
& +(m_j + m_{j+1} + i_{2(n-(j+1))-1} - 1 + n - j + 1)l_{j-1} \\
& + \sum_{k=n-(j-1)}^{n-2} (i_{2k-1} + \frac{1+(-1)^{k}}{2})l_{n-k-1} + m_{j+1}l_j - j \\
= \;& \sum_{k=1}^{n-(j+1)} (i_{2k-1} + \frac{1+(-1)^{k}}{2})l_{n-k} + (i_{2(n-(j+1))-1} + \frac{1+(-1)^{n-(j+1)}}{2})l_j \\
& +(m_j + m_{j+1} + i_{2(n-(j+1))-1} + \frac{1+(-1)^{n-(j+1)}}{2})l_{j-1} \\
& + \sum_{k=n-(j-1)}^{n-2} (i_{2k-1} + \frac{1+(-1)^{k}}{2})l_{n-k-1} + m_{j+1}l_j - j \\
= \;& \sum_{k=1}^{n-(j+1)} (i_{2k-1} + \frac{1+(-1)^{k}}{2})l_{n-k} + (i_{2(n-(j+1))-1} + \frac{1+(-1)^{n-(j+1)}}{2})(l_j + l_{j-1}) \\
& + \sum_{k=n-(j-1)}^{n-2} (i_{2k-1} + \frac{1+(-1)^{k}}{2})l_{n-k-1} + (m_j + m_{j+1})l_{j-1} + m_{j+1}l_j - j
\end{aligned}
$$

$$\begin{aligned}
\sigma_5 = &\sum_{k=1}^{n-(j+1)} (i_{2k-1} + \frac{1+(-1)^k}{2})l_{n-k} + (m_{j+1} + i_{2(n-(j+1))-1} - 1 + \frac{1+(-1)^{n-j}}{2})(l_j + l_{j-1}) \\
&+ \sum_{k=n-(j-1)}^{n-2} (i_{2k-1} + \frac{1+(-1)^k}{2})l_{n-k-1} + m_j l_{j-1} - (j-1) \\
= &\sum_{k=1}^{n-(j+1)} (i_{2k-1} + \frac{1+(-1)^k}{2})l_{n-k} + (i_{2(n-(j+1))-1} - 1 + n - j + 1)(l_j + l_{j-1}) \\
&+ \sum_{k=n-(j-1)}^{n-2} (i_{2k-1} + \frac{1+(-1)^k}{2})l_{n-k-1} + m_{j+1}(l_j + l_{j-1}) + m_j l_{j-1} - (j-1) \\
= &\sum_{k=1}^{n-(j+1)} (i_{2k-1} + \frac{1+(-1)^k}{2})l_{n-k} + (i_{2(n-(j+1))-1} + \frac{1+(-1)^{n-(j+1)}}{2})(l_j + l_{j-1}) \\
&+ \sum_{k=n-(j-1)}^{n-2} (i_{2k-1} + \frac{1+(-1)^k}{2})l_{n-k-1} + m_{j+1}(l_j + l_{j-1}) + m_j l_{j-1} - (j-1)
\end{aligned}$$

One sees that σ_4 and σ_5 in that case are opposite of each other.

$$\begin{aligned}
\sigma_7 = &\sum_{k=1}^{n-2}(i_{2k-1} + \frac{1+(-1)^k}{2})l_{n-k-1} - (n-3) + m_n l_{n-1} \\
= &(m_{n-1} + m_n)l_{n-2} + \sum_{k=2}^{n-2}(i_{2k-1} + \frac{1+(-1)^k}{2})l_{n-k-1} - (n-3) + m_n l_{n-1} \\
= &\sum_{k=2}^{n-2}(i_{2k-1} + \frac{1+(-1)^k}{2})l_{n-k-1}(m_{n-1} + m_n)l_{n-2} + m_n l_{n-1} - (n-3)
\end{aligned}$$

$$\begin{aligned}
\sigma_8 = &\ i_1(l_{n-1} + l_{n-2}) + \sum_{k=2}^{n-2}(i_{2k-1} + \frac{1+(-1)^k}{2})l_{n-k-1} + m_{n-1}l_{n-2} - (n-2) \\
= &\ m_n(l_{n-1} + l_{n-2}) + \sum_{k=2}^{n-2}(i_{2k-1} + \frac{1+(-1)^k}{2})l_{n-k-1} + m_{n-1}l_{n-2} - (n-2) \\
= &\sum_{k=2}^{n-2}(i_{2k-1} + \frac{1+(-1)^k}{2})l_{n-k-1} + m_n(l_{n-1} + l_{n-2}) + m_{n-1}l_{n-2} - (n-2)
\end{aligned}$$

One sees that σ_7 and σ_8 in that case are opposite of each other.

This means we can add two sums of opposite signs between the fourth and the fifth sums, and between the seventh and eighth sums:

$$\begin{aligned}
(4). \quad &\sum_{j=2}^{n-2}\sum_{i_1=1}^{m_n}\sum_{i_2=1}^{l_{n-1}}\cdots\sum_{i_{2(n-(j+1))}=1}^{l_{j+1}}\ \sum_{i_{2(n-j)-1}=1}^{m_j+m_{j+1}+i_{2(n-(j+1))-1}-2}\ \sum_{i_{2(n-j)}=1}^{l_{j-1}}\cdots\sum_{i_{2n-5}=1}^{m_2+i_7-1}\sum_{i_{2n-4}=1}^{l_1} \\
&(-1)^{\sigma_4} e_2^{\epsilon_1} \xi^{m_1+i_{2n-5}-1} x^{l_1-i_{2n-4}} e_1 e_2 x^{i_{2n-4}-1} \xi^{m_2+i_{2n-7}-1-i_{2n-5}} \ldots \\
&\ldots \\
&\ldots \boxed{\xi^{m_j+m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}} x^{l_j+l_{j+1}-i_{2(n-(j+1))}} e_1 e_2 x^{i_{2(n-(j+1))}-1}} \ldots \\
&\ldots \\
&\ldots \xi^{m_{n-1}+i_1-1-i_3} x^{l_{n-1}-i_2} e_1 e_2 x^{i_2-1} \xi^{m_n-i_1} x^{l_n} e_1^{\eta_n}
\end{aligned}$$

$$(i_{2(n-j)-1} = m_j + m_{j+1} + i_{2(n-(j+1))-1} - 1)$$

$$+\sum_{j=2}^{n-2}\sum_{i_1=1}^{m_n}\sum_{i_2=1}^{l_{n-1}}\cdots\sum_{i_{2(n-(j+1))}=1}^{l_{j+1}}\sum_{i_{2(n-j)}=1}^{l_{j-1}}\sum_{i_{2(n-(j-1))-1}=1}^{m_{j-1}+m_j+m_{j+1}+i_{2(n-(j+1))-1}-2}\cdots\sum_{i_{2n-5}=1}^{m_2+i_7-1}\sum_{i_{2n-4}=1}^{l_1}$$

$$(-1)^{\sigma_4}e_2^{\epsilon_1}\xi^{m_1+i_{2n-5}-1}x^{l_1-i_{2n-4}}e_1e_2x^{i_{2n-4}-1}\xi^{m_2+i_{2n-7}-1-i_{2n-5}}\cdots$$

$$\cdots$$

$$\cdots\boxed{\xi^{m_{j-1}+m_j+m_{j+1}+i_{2(n-(j+1))-1}-1-1-i_{2(n-(j-1))-1}}x^{l_{j-1}-i_{2(n-j)}}}\cdots$$

$$\cdots\boxed{e_1e_2x^{i_{2(n-j)}-1+l_j+l_{j+1}-i_{2(n-(j+1))}}e_1e_2x^{i_{2(n-(j+1))}-1}}\cdots$$

$$\cdots$$

$$\cdots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}$$

$$(i_{2(n-j)-1} = m_{j+1} + i_{2(n-(j+1))-1} - 1)$$

$$+\sum_{j=2}^{n-2}\sum_{i_1=1}^{m_n}\sum_{i_2=1}^{l_{n-1}}\cdots\sum_{i_{2(n-j)}=1}^{l_{j-1}}\sum_{i_{2(n-(j-1))-1}=1}^{m_{j-1}+m_j+m_{j+1}+i_{2(n-(j+1))-1}-2}\sum_{i_{2(n-(j-1))}=1}^{l_{j-2}}\cdots$$

$$\cdots\sum_{i_{2n-5}=1}^{m_2+i_{2n-7}-1}\sum_{i_{2n-4}=1}^{l_1}(-1)^{\sigma_5}e_2^{\epsilon_1}\xi^{m_1+i_{2n-5}-1}x^{l_1-i_{2n-4}}e_1e_2x^{i_{2n-4}-1}\xi^{m_2+i_{2n-7}-1-i_{2n-5}}\cdots$$

$$\cdots$$

$$\cdots\boxed{\xi^{m_{j-1}+m_j+m_{j+1}+i_{2(n-(j+1))-1}-1-1-i_{2(n-(j-1))-1}}x^{l_{j-1}-i_{2(n-j)}}}$$

$$\cdots\boxed{e_1e_2x^{i_{2(n-j)}-1+l_j+l_{j+1}-i_{2(n-(j+1))}}}\cdots$$

$$\cdots$$

$$\cdots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}$$

(5). $$+\sum_{j=2}^{n-2}\sum_{i_1=1}^{m_n}\sum_{i_2=1}^{l_{n-1}}\cdots\sum_{i_{2(n-j)-1}=1}^{m_{j+1}+i_{2(n-(j+1))-1}-2}\sum_{i_{2(n-j)}=1}^{l_{j-1}}\sum_{i_{2(n-(j-1))-1}=1}^{m_{j-1}+m_j+i_{2(n-j)-1}-1}\sum_{i_{2(n-(j-1))}=1}^{l_{j-2}}\cdots$$

$$\cdots\sum_{i_{2n-5}=1}^{m_2+i_{2n-7}-1}\sum_{i_{2n-4}=1}^{l_1}(-1)^{\sigma_5}e_2^{\epsilon_1}\xi^{m_1+i_{2n-5}-1}x^{l_1-i_{2n-4}}e_1e_2x^{i_{2n-4}-1}\xi^{m_2+i_{2n-7}-1-i_{2n-5}}\cdots$$

$$\cdots$$

$$\cdots\boxed{\xi^{m_{j-1}+m_j+i_{2(n-j)-1}-1-i_{2(n-(j-1))-1}}x^{l_{j-1}-i_{2(n-j)}}}$$

$$\cdots\boxed{e_1e_2x^{i_{2(n-j)}-1+l_j}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}}\cdots$$

$$\cdots$$

$$\cdots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}$$

(4′). $$=\sum_{j=2}^{n-2}\sum_{i_1=1}^{m_n}\sum_{i_2=1}^{l_{n-1}}\cdots\sum_{i_{2(n-(j+1))}=1}^{l_{j+1}}\sum_{i_{2(n-j)-1}=1}^{m_j+m_{j+1}+i_{2(n-(j+1))-1}-1}\sum_{i_{2(n-j)}=1}^{l_{j-1}}\cdots\sum_{i_{2n-5}=1}^{m_2+i_7-1}\sum_{i_{2n-4}=1}^{l_1}$$

$$(-1)^{\sigma_4}e_2^{\epsilon_1}\xi^{m_1+i_{2n-5}-1}x^{l_1-i_{2n-4}}e_1e_2x^{i_{2n-4}-1}\xi^{m_2+i_{2n-7}-1-i_{2n-5}}\cdots$$

$$\cdots$$

$$\cdots\boxed{\xi^{m_j+m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}x^{l_j+l_{j+1}-i_{2(n-(j+1))}}e_1e_2x^{i_{2(n-(j+1))}-1}}\cdots$$

$$\cdots$$

$$\cdots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}$$

(5′). $$+\sum_{j=2}^{n-2}\sum_{i_1=1}^{m_n}\sum_{i_2=1}^{l_{n-1}}\cdots\sum_{i_{2(n-j)-1}=1}^{m_{j+1}+i_{2(n-(j+1))-1}-1}\sum_{i_{2(n-j)}=1}^{l_{j-1}}\sum_{i_{2(n-(j-1))-1}=1}^{m_{j-1}+m_j+i_{2(n-j)-1}-1}\sum_{i_{2(n-(j-1))}=1}^{l_{j-2}}\cdots$$

$$\cdots\sum_{i_{2n-5}=1}^{m_2+i_{2n-7}-1}\sum_{i_{2n-4}=1}^{l_1}(-1)^{\sigma_5}e_2^{\epsilon_1}\xi^{m_1+i_{2n-5}-1}x^{l_1-i_{2n-4}}e_1e_2x^{i_{2n-4}-1}\xi^{m_2+i_{2n-7}-1-i_{2n-5}}\cdots$$

$$\cdots$$

$$\cdots\boxed{\xi^{m_{j-1}+m_j+i_{2(n-j)-1}-1-i_{2(n-(j-1))-1}}x^{l_{j-1}-i_{2(n-j)}}}$$

$$\cdots\boxed{e_1e_2x^{i_{2(n-j)}-1+l_j}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}}\cdots$$

$$\cdots$$

$$\cdots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}$$

and

$$(7).\quad \sum_{i_1=1}^{m_{n-1}+m_n-1}\sum_{i_2=1}^{l_{n-2}}\cdots\sum_{i_{2(n-(j+1))-1}=1}^{m_j+i_{2(n-(j+2)-1}-1}\sum_{i_{2(n-(j+1))}=1}^{l_j}\cdots\sum_{i_{2n-5}=1}^{m_2-i_{2n-7}-1}\sum_{i_{2n-4}=1}^{l_1}$$
$$(-1)^{\sigma_7}e_2^{\epsilon_1}\xi^{m_1+i_{2n-5}-1}x^{l_1-i_{2n-4}}e_1e_2x^{i_{2n-4}-1}\xi^{m_2+i_{2n-7}-1-i_{2n-5}}\ldots$$
$$\ldots$$
$$\ldots x^{l_j-i_{2(n-(j+1))}}e_1e_2x^{i_{2(n-(j+1))}-1}\xi^{m_{j+1}+i_{2(n-(j+2))-1}-1-i_{2(n-(j+1))-1}}\ldots$$
$$\ldots$$
$$\ldots\boxed{\xi^{m_{n-1}+m_n-i_1}x^{l_{n-1}+l_n}}e_1^{\eta_n}$$

$(i_1 = m_{n-1}+m_n)$

$$+\sum_{i_2=1}^{l_{n-2}}\sum_{i_3=1}^{m_{n-2}+m_{n-1}+m_n-1}\cdots\sum_{i_{2(n-(j+1))-1}=1}^{m_j+i_{2(n-(j+2)-1}-1}\sum_{i_{2(n-(j+1))}=1}^{l_j}\cdots\sum_{i_{2n-5}=1}^{m_2-i_{2n-7}-1}\sum_{i_{2n-4}=1}^{l_1}$$
$$(-1)^{\sigma_7}e_2^{\epsilon_1}\xi^{m_1+i_{2n-5}-1}x^{l_1-i_{2n-4}}e_1e_2x^{i_{2n-4}-1}\xi^{m_2+i_{2n-7}-1-i_{2n-5}}\ldots$$
$$\ldots$$
$$\ldots x^{l_j-i_{2(n-(j+1))}}e_1e_2x^{i_{2(n-(j+1))}-1}\xi^{m_{j+1}+i_{2(n-(j+2))-1}-1-i_{2(n-(j+1))-1}}\ldots$$
$$\ldots$$
$$\ldots\boxed{\xi^{m_{n-2}+m_{n-1}+m_n-1-i_3}x^{l_{n-2}-i_2}e_1e_2x^{i_2-1+l_{n-1}+l_n}}e_1^{\eta_n}$$

$(i_1 = m_n)$

$$+\sum_{i_2=1}^{l_{n-2}}\sum_{i_3=1}^{m_{n-2}+m_{n-1}+m_n-1}\sum_{i_4=1}^{l_{n-3}}\cdots\sum_{i_{2(n-(j+1))-1}=1}^{m_{j+1}+i_{2(n-(j+2))-1}-1}\sum_{i_{2(n-(j+1))}=1}^{l_j}\cdots\sum_{i_{2n-5}=1}^{m_2+i_{2n-7}-1}\sum_{i_{2n-4}=1}^{l_1}$$
$$(-1)^{\sigma_8}e_2^{\epsilon_1}\xi^{m_1+i_{2n-5}-1}x^{l_1-i_{2n-4}}e_1e_2x^{i_{2n-4}-1}\xi^{m_2+i_{2n-7}-1-i_{2n-5}}\ldots$$
$$\ldots$$
$$\ldots x^{l_j-i_{2(n-(j+1))}}e_1e_2x^{i_{2(n-(j+1))}-1}\xi^{m_{j+1}+i_{2(n-(j+2))-1}-1-i_{2(n-(j+1))-1}}\ldots$$
$$\ldots$$
$$\ldots\boxed{\xi^{m_{n-2}+m_{n-1}+m_n-1-i_3}x^{l_{n-2}-i_2}e_1e_2x^{i_2-1+l_{n-1}+l_n}}e_1^{\eta_n}$$

$$(8).\quad +\sum_{i_1=1}^{m_n-1}\sum_{i_2=1}^{l_{n-2}}\sum_{i_3=1}^{m_{n-2}+m_{n-1}+i_1-1}\sum_{i_4=1}^{l_{n-3}}\cdots\sum_{i_{2(n-(j+1))-1}=1}^{m_{j+1}+i_{2(n-(j+2))-1}-1}\sum_{i_{2(n-(j+1))}=1}^{l_j}\cdots\sum_{i_{2n-5}=1}^{m_2+i_{2n-7}-1}\sum_{i_{2n-4}=1}^{l_1}$$
$$(-1)^{\sigma_8}e_2^{\epsilon_1}\xi^{m_1+i_{2n-5}-1}x^{l_1-i_{2n-4}}e_1e_2x^{i_{2n-4}-1}\xi^{m_2+i_{2n-7}-1-i_{2n-5}}\ldots$$
$$\ldots$$
$$\ldots x^{l_j-i_{2(n-(j+1))}}e_1e_2x^{i_{2(n-(j+1))}-1}\xi^{m_{j+1}+i_{2(n-(j+2))-1}-1-i_{2(n-(j+1))-1}}\ldots$$
$$\ldots$$
$$\ldots\boxed{\xi^{m_{n-2}+m_{n-1}+i_1-1-i_3}x^{l_{n-2}-i_2}e_1e_2}$$
$$\ldots\boxed{x^{i_2-1+l_{n-1}}\xi^{m_n-i_1}}x^{l_n}e_1^{\eta_n}$$

$$(7').\quad = \sum_{i_1=1}^{m_{n-1}+m_n}\sum_{i_2=1}^{l_{n-2}}\cdots\sum_{i_{2(n-(j+1))-1}=1}^{m_j+i_{2(n-(j+2)-1}-1}\sum_{i_{2(n-(j+1))}=1}^{l_j}\cdots\sum_{i_{2n-5}=1}^{m_2-i_{2n-7}-1}\sum_{i_{2n-4}=1}^{l_1}$$
$$(-1)^{\sigma_7}e_2^{\epsilon_1}\xi^{m_1+i_{2n-5}-1}x^{l_1-i_{2n-4}}e_1e_2x^{i_{2n-4}-1}\xi^{m_2+i_{2n-7}-1-i_{2n-5}}\ldots$$
$$\ldots$$
$$\ldots x^{l_j-i_{2(n-(j+1))}}e_1e_2x^{i_{2(n-(j+1))}-1}\xi^{m_{j+1}+i_{2(n-(j+2))-1}-1-i_{2(n-(j+1))-1}}\ldots$$
$$\ldots$$
$$\ldots\boxed{\xi^{m_{n-1}+m_n-i_1}x^{l_{n-1}+l_n}}e_1^{\eta_n}$$

$$(8'). \quad + \sum_{i_1=1}^{m_n} \sum_{i_2=1}^{l_{n-2}} \sum_{i_3=1}^{m_{n-2}+m_{n-1}+i_1-1} \sum_{i_4=1}^{l_{n-3}} \cdots \sum_{i_{2(n-(j+1))-1}=1}^{m_{j+1}+i_{2(n-(j+2))-1}-1} \sum_{i_{2(n-(j+1))}=1}^{l_j} \cdots \sum_{i_{2n-5}=1}^{m_2+i_{2n-7}-1} \sum_{i_{2n-4}=1}^{l_1}$$
$$(-1)^{\sigma_8} e_2^{\epsilon_1} \xi^{m_1+i_{2n-5}-1} x^{l_1-i_{2n-4}} e_1 e_2 x^{i_{2n-4}-1} \xi^{m_2+i_{2n-7}-1-i_{2n-5}} \cdots$$
$$\cdots$$
$$\cdots x^{l_j-i_{2(n-(j+1))}} e_1 e_2 x^{i_{2(n-(j+1))}-1} \xi^{m_{j+1}+i_{2(n-(j+2))-1}-1-i_{2(n-(j+1))-1}} \cdots$$
$$\cdots$$
$$\cdots \boxed{\xi^{m_{n-2}+m_{n-1}+i_1-1-i_3} x^{l_{n-2}-i_2} e_1 e_2}$$
$$\cdots \boxed{x^{i_2-1+l_{n-1}} \xi^{m_n-i_1}} x^{l_n} e_1^{\eta_n}$$

For us to recognise Φ_n, we need to consider expressions $(4')$. for index $j-1$ and $(5')$. for index j. We make the change of variable $i_{2(n-j)} = i_{2(n-j)} - l_j$ in expression $(5')$. Note that this change of variable does not affect σ_5. The expressions write:

$(j-1)$

$$(4'). \quad \sum_{i_1=1}^{m_n} \sum_{i_2=1}^{l_{n-1}} \cdots \sum_{i_{2(n-((j-1)+1))}=1}^{l_{(j-1)+1}} \sum_{i_{2(n-(j-1))-1}=1}^{m_{(j-1)}+m_{(j-1)+1}+i_{2(n-((j-1)+1))-1}-1} \sum_{i_{2(n-(j-1))}=1}^{l_{(j-1)-1}} \cdots \sum_{i_{2n-5}=1}^{m_2+i_7-1} \sum_{i_{2n-4}=1}^{l_1}$$
$$(-1)^{\sigma_4} e_2^{\epsilon_1} \xi^{m_1+i_{2n-5}-1} x^{l_1-i_{2n-4}} e_1 e_2 x^{i_{2n-4}-1} \xi^{m_2+i_{2n-7}-1-i_{2n-5}} \cdots$$
$$\cdots$$
$$\cdots \boxed{\xi^{m_{(j-1)}+m_{(j-1)+1}+i_{2(n-((j-1)+1))-1}-1-i_{2(n-(j-1))-1}} x^{l_{(j-1)}+l_{(j-1)+1}-i_{2(n-((j-1)+1))}}}$$
$$\cdots \boxed{e_1 e_2 x^{i_{2(n-((j-1)+1))}-1}} \cdots$$
$$\cdots$$
$$\cdots \xi^{m_{n-1}+i_1-1-i_3} x^{l_{n-1}-i_2} e_1 e_2 x^{i_2-1} \xi^{m_n-i_1} x^{l_n} e_1^{\eta_n}$$
$$= \sum_{i_1=1}^{m_n} \sum_{i_2=1}^{l_{n-1}} \cdots \sum_{i_{2(n-j)}=1}^{l_j} \sum_{i_{2(n-(j-1))-1}=1}^{m_{j-1}+m_j+i_{2(n-j)-1}-1} \sum_{i_{2(n-(j-1))}=1}^{l_{j-2}} \cdots \sum_{i_{2n-5}=1}^{m_2+i_7-1} \sum_{i_{2n-4}=1}^{l_1}$$
$$(-1)^{\sigma_4} e_2^{\epsilon_1} \xi^{m_1+i_{2n-5}-1} x^{l_1-i_{2n-4}} e_1 e_2 x^{i_{2n-4}-1} \xi^{m_2+i_{2n-7}-1-i_{2n-5}} \cdots$$
$$\cdots$$
$$\cdots \boxed{\xi^{m_{j-1}+m_j+i_{2(n-j)-1}-1-i_{2(n-(j-1))-1}} x^{l_{j-1}+l_j-i_{2(n-j)}} e_1 e_2 x^{i_{2(n-j)}-1}} \cdots$$
$$\cdots$$
$$\cdots \xi^{m_{n-1}+i_1-1-i_3} x^{l_{n-1}-i_2} e_1 e_2 x^{i_2-1} \xi^{m_n-i_1} x^{l_n} e_1^{\eta_n}$$

(j) and $(i_{2(n-j)} = i_{2(n-j)} - l_j)$

$$(5''). \quad \sum_{i_1=1}^{m_n} \sum_{i_2=1}^{l_{n-1}} \cdots \sum_{i_{2(n-j)-1}=1}^{m_{j+1}+i_{2(n-(j+1))-1}-1} \sum_{i_{2(n-j)}=1+l_j}^{l_{j-1}+l_j} \sum_{i_{2(n-(j-1))-1}=1}^{m_{j-1}+m_j+i_{2(n-j)-1}-1} \sum_{i_{2(n-(j-1))}=1}^{l_{j-2}} \cdots$$
$$\cdots \sum_{i_{2n-5}=1}^{m_2+i_{2n-7}-1} \sum_{i_{2n-4}=1}^{l_1} (-1)^{\sigma_5} e_2^{\epsilon_1} \xi^{m_1+i_{2n-5}-1} x^{l_1-i_{2n-4}} e_1 e_2 x^{i_{2n-4}-1} \xi^{m_2+i_{2n-7}-1-i_{2n-5}} \cdots$$
$$\cdots$$
$$\cdots \boxed{\xi^{m_{j-1}+m_j+i_{2(n-j)-1}-1-i_{2(n-(j-1))-1}} x^{l_{j-1}+l_j-i_{2(n-j)}}}$$
$$\cdots \boxed{e_1 e_2 x^{i_{2(n-j)}-1} \xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}} \cdots$$
$$\cdots$$
$$\cdots \xi^{m_{n-1}+i_1-1-i_3} x^{l_{n-1}-i_2} e_1 e_2 x^{i_2-1} \xi^{m_n-i_1} x^{l_n} e_1^{\eta_n}$$

If $\sigma_4(j-1)$ is the same as $\sigma_5(j)$, we see that these two sums complement each other:

- in $(4')$., we have $1 \leq i_{2(n-j)} \leq l_j$;
- in $(5')$., we have $1 + l_j \leq i_{2(n-j)} \leq l_{j-1} + l_j$.

Let us compare $\sigma_4(j-1)$ with $\sigma_5(j)$.

$$\sigma_4(j-1) = \sum_{k=1}^{n-((j-1)+2)} (i_{2k-1} + \frac{1+(-1)^k}{2})l_{n-k} + (i_{2(n-((j-1)+1))-1} + \frac{1+(-1)^{n-((j-1)+1)}}{2})(l_{(j-1)+1} + l_{(j-1)})$$
$$+ \sum_{k=n-(j-1)}^{n-2} (i_{2k-1} + \frac{1+(-1)^k}{2})l_{n-k-1} + m_{(j-1)+1}l_{(j-1)} - (j-1)$$
$$= \sum_{k=1}^{n-(j+1)} (i_{2k-1} + \frac{1+(-1)^k}{2})l_{n-k} + (i_{2(n-j)-1} + \frac{1+(-1)^{n-j}}{2})(l_j + l_{j-1})$$
$$+ \sum_{k=n-(j-1)}^{n-2} (i_{2k-1} + \frac{1+(-1)^k}{2})l_{n-k-1} + m_j l_{j-1} - (j-1)$$
$$\sigma_5(j) = \sum_{k=1}^{n-(j+1)} (i_{2k-1} + \frac{1+(-1)^k}{2})l_{n-k} + (i_{2(n-j)-1} + \frac{1+(-1)^{n-j}}{2})(l_j + l_{j-1})$$
$$+ \sum_{k=n-(j-1)}^{n-2} (i_{2k-1} + \frac{1+(-1)^k}{2})l_{n-k-1} + m_j l_{j-1} - (j-1)$$

In addition, expression ($4'$) at $j = n-2$ complements expression ($8'$) after having made the change of variable $i_2 = i_2 - l_{n-1}$ in ($8'$), and expression ($5'$) at $j = 2$ complements expression (2).

Finally, we see that:

$$-m_1 f_n \left(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1} \otimes \ldots \otimes e_2^{\epsilon_n}\xi^{m_n}x^{l_n}e_1^{\eta_n}\right)$$
$$(1).\quad = \sum_{i_1=1}^{m_n}\sum_{i_2=1}^{l_{n-1}} \cdots \sum_{i_{2(n-j)-1}=1}^{m_{j+1}+i_{2(n-(j+1))-1}-1} \sum_{i_{2(n-j)}=1}^{l_j} \cdots \sum_{i_{2n-5}=1}^{m_3+i_7-1} \sum_{i_{2n-4}=1}^{l_2}$$
$$(-1)^{\sigma_1} e_2^{\epsilon_1} \boxed{\xi^{m_1}x^{l_1}\xi^{m_2+i_{2n-5}-1}} \ldots$$
$$\ldots$$
$$\ldots x^{l_j - i_{2(n-j)}} e_1 e_2 x^{i_{2(n-j)}-1} \xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}} \ldots$$
$$\ldots$$
$$\ldots \xi^{m_{n-1}+i_1-1-i_3} x^{l_{n-1}-i_2} e_1 e_2 x^{i_2-1} \xi^{m_n - i_1} x^{l_n} e_1^{\eta_n}$$
$$(2').\quad + \sum_{i_1=1}^{m_n}\sum_{i_2=1}^{l_{n-1}} \cdots \sum_{i_{2(n-j)-1}=1}^{m_{j+1}+i_{2(n-(j+1))-1}-1} \sum_{i_{2(n-j)}=1}^{l_j} \cdots \sum_{i_{2n-5}=1}^{m_3+i_7-1} \sum_{i_{2n-4}=1}^{l_1+l_2}$$
$$(-1)^{\sigma_2} e_2^{\epsilon_1} \boxed{\xi^{m_1+m_2+i_{2n-5}-1} x^{l_1+l_2-i_{2n-4}} e_1 e_2 x^{i_{2n-4}-1}} \ldots$$
$$\ldots$$
$$\ldots x^{l_j - i_{2(n-j)}} e_1 e_2 x^{i_{2(n-j)}-1} \xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}} \ldots$$
$$\ldots$$
$$\ldots \xi^{m_{n-1}+i_1-1-i_3} x^{l_{n-1}-i_2} e_1 e_2 x^{i_2-1} \xi^{m_n - i_1} x^{l_n} e_1^{\eta_n}$$
$$(3).\quad + \sum_{j=2}^{n-2} \sum_{i'_1=1}^{m_j} \sum_{i'_2=1}^{l_{j-1}} \cdots \sum_{i'_{2j-3}=1}^{m_2+i'_{2j-5}-1} \sum_{i'_{2j-2}=1}^{l_1} \sum_{i_1=1}^{m_n} \sum_{i_2=1}^{l_{n-1}} \cdots \sum_{i_{2(n-(j+1))-1}=1}^{m_{j+2}-i_{2(n-(j+2))-1}-1} \sum_{i_{2(n-(j+1))}=1}^{l_{j+1}}$$
$$(-1)^{\sigma_3} e_2^{\epsilon_1} \xi^{m_1+i'_{2j-3}-1} x^{l_1 - i'_{2j-2}} e_1 e_2 x^{i'_{2j-2}-1} \xi^{m_2+i'_{2j-5}-1-i'_{2j-3}} \ldots$$
$$\ldots$$
$$\ldots \xi^{m_j - i'_1} x^{l_j} \xi^{m_{j+1}+i_{2(n-(j+1))-1}-1} \ldots$$
$$\ldots$$
$$\ldots \xi^{m_{n-1}+i_1-1-i_3} x^{l_{n-1}-i_2} e_1 e_2 x^{i_2-1} \xi^{m_n - i_1} x^{l_n} e_1^{\eta_n}$$

(5″). $$+\sum_{j=3}^{n-2}\sum_{i_1=1}^{m_n}\sum_{i_2=1}^{l_{n-1}}\cdots\sum_{i_{2(n-j)-1}=1}^{m_{j+1}+i_{2(n-(j+1))-1}-1}\ \sum_{i_{2(n-j)}=1}^{l_{j-1}+l_j}\ \sum_{i_{2(n-(j-1))-1}=1}^{m_{j-1}+m_j+i_{2(n-j)-1}-1}\ \sum_{i_{2(n-(j-1))}=1}^{l_{j-2}}\cdots$$
$$\cdots\sum_{i_{2n-5}=1}^{m_2+i_{2n-7}-1}\ \sum_{i_{2n-4}=1}^{l_1}(-1)^{\sigma_5}e_2^{\epsilon_1}\xi^{m_1+i_{2n-5}-1}x^{l_1-i_{2n-4}}e_1e_2x^{i_{2n-4}-1}\xi^{m_2+i_{2n-7}-1-i_{2n-5}}\cdots$$
$$\cdots$$
$$\cdots\boxed{\xi^{m_{j-1}+m_j+i_{2(n-j)-1}-1-i_{2(n-(j-1))-1}}x^{l_{j-1}+l_j-i_{2(n-j)}}}$$
$$\cdots\boxed{e_1e_2x^{i_{2(n-j)}-1}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}}\cdots$$
$$\cdots$$
$$\cdots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}$$

(6). $$+\sum_{i_1=1}^{m_{n-1}}\sum_{i_2=1}^{l_{n-2}}\cdots\sum_{i_{2(n-(j+1))-1}=1}^{m_j+i_{2(n-(j+2)-1}-1}\ \sum_{i_{2(n-(j+1))}=1}^{l_j}\cdots\sum_{i_{2n-5}=1}^{m_2-i_{2n-7}-1}\ \sum_{i_{2n-4}=1}^{l_1}$$
$$(-1)^{\sigma_6}e_2^{\epsilon_1}\xi^{m_1+i_{2n-5}-1}x^{l_1-i_{2n-4}}e_1e_2x^{i_{2n-4}-1}\xi^{m_2+i_{2n-7}-1-i_{2n-5}}\cdots$$
$$\cdots$$
$$\cdots x^{l_j-i_{2(n-(j+1))}}e_1e_2x^{i_{2(n-(j+1))}-1}\xi^{m_{j+1}+i_{2(n-(j+2))-1}-1-i_{2(n-(j+1))-1}}\cdots$$
$$\cdots$$
$$\cdots\boxed{\xi^{m_{n-1}-i_1}x^{l_{n-1}}\xi^{m_n}}x^{l_n}e_1^{\eta_n}$$

(7′). $$+\sum_{i_1=1}^{m_{n-1}+m_n}\sum_{i_2=1}^{l_{n-2}}\cdots\sum_{i_{2(n-(j+1))-1}=1}^{m_j+i_{2(n-(j+2)-1}-1}\ \sum_{i_{2(n-(j+1))}=1}^{l_j}\cdots\sum_{i_{2n-5}=1}^{m_2-i_{2n-7}-1}\ \sum_{i_{2n-4}=1}^{l_1}$$
$$(-1)^{\sigma_7}e_2^{\epsilon_1}\xi^{m_1+i_{2n-5}-1}x^{l_1-i_{2n-4}}e_1e_2x^{i_{2n-4}-1}\xi^{m_2+i_{2n-7}-1-i_{2n-5}}\cdots$$
$$\cdots$$
$$\cdots x^{l_j-i_{2(n-(j+1))}}e_1e_2x^{i_{2(n-(j+1))}-1}\xi^{m_{j+1}+i_{2(n-(j+2))-1}-1-i_{2(n-(j+1))-1}}\cdots$$
$$\cdots$$
$$\cdots\boxed{\xi^{m_{n-1}+m_n-i_1}x^{l_{n-1}+l_n}}e_1^{\eta_n}$$

(8″). $$+\sum_{i_1=1}^{m_n}\ \sum_{i_2=1}^{l_{n-2}+l_{n-1}}\ \sum_{i_3=1}^{m_{n-2}+m_{n-1}+i_1-1}\ \sum_{i_4=1}^{l_{n-3}}\cdots\sum_{i_{2(n-(j+1))-1}=1}^{m_{j+1}+i_{2(n-(j+2))-1}-1}\ \sum_{i_{2(n-(j+1))}=1}^{l_j}\cdots\sum_{i_{2n-5}=1}^{m_2+i_{2n-7}-1}\ \sum_{i_{2n-4}=1}^{l_1}$$
$$(-1)^{\sigma_8}e_2^{\epsilon_1}\xi^{m_1+i_{2n-5}-1}x^{l_1-i_{2n-4}}e_1e_2x^{i_{2n-4}-1}\xi^{m_2+i_{2n-7}-1-i_{2n-5}}\cdots$$
$$\cdots$$
$$\cdots x^{l_j-i_{2(n-(j+1))}}e_1e_2x^{i_{2(n-(j+1))}-1}\xi^{m_{j+1}+i_{2(n-(j+2))-1}-1-i_{2(n-(j+1))-1}}\cdots$$
$$\cdots$$
$$\cdots\boxed{\xi^{m_{n-2}+m_{n-1}+i_1-1-i_3}x^{l_{n-2}+l_{n-1}-i_2}e_1e_2}$$
$$\cdots\boxed{x^{i_2-1}\xi^{m_n-i_1}}x^{l_n}e_1^{\eta_n}$$

Let us review each sum separately, and insert back the σ_i's.

$$
\begin{aligned}
(1).\quad & \sum_{i_1=1}^{m_n}\sum_{i_2=1}^{l_{n-1}}\cdots\sum_{i_{2(n-j)-1}=1}^{m_{j+1}+i_{2(n-(j+1))-1}-1}\ \sum_{i_{2(n-j)}=1}^{l_j}\cdots\sum_{i_{2n-5}=1}^{m_3+i_7-1}\ \sum_{i_{2n-4}=1}^{l_2}\\
& (-1)^{\sum_{k=1}^{n-2}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+(n-2)l_1}\\
& e_2^{\epsilon_1}\boxed{\xi^{m_1}x^{l_1}\xi^{m_2+i_{2n-5}-1}}\dots\\
& \dots\\
& \dots x^{l_j-i_{2(n-j)}}e_1e_2x^{i_{2(n-j)}-1}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}\dots\\
& \dots\\
& \dots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}\\
= \quad & (-1)^{(n-2)l_1}\sum_{i_1=1}^{m_n}\sum_{i_2=1}^{l_{n-1}}\cdots\sum_{i_{2(n-j)-1}=1}^{m_{j+1}+i_{2(n-(j+1))-1}-1}\ \sum_{i_{2(n-j)}=1}^{l_j}\cdots\sum_{i_{2n-5}=1}^{m_3+i_7-1}\ \sum_{i_{2n-4}=1}^{l_2}\\
& (-1)^{\sum_{k=1}^{n-2}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}}\\
& e_2^{\epsilon_1}\boxed{\xi^{m_1}x^{l_1}\xi^{m_2+i_{2n-5}-1}}\dots\\
& \dots\\
& \dots x^{l_j-i_{2(n-j)}}e_1e_2x^{i_{2(n-j)}-1}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}\dots\\
& \dots\\
& \dots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}\\
= \quad & (-1)^{(n-2)l_1}\left(f_1(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1})\otimes f_{n-1}(e_2^{\epsilon_2}\xi^{m_2}x^{l_2}e_1^{\eta_2}\otimes\dots\otimes e_2^{\epsilon_n}\xi^{m_n}x^{l_n}e_1^{\eta_n})\right)\\
= \quad & m_2(f_1\otimes f_{n-1})\left(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1}\otimes\dots\otimes e_2^{\epsilon_n}\xi^{m_n}x^{l_n}e_1^{\eta_n})\right) \qquad [\text{ Koszul sign rule}]
\end{aligned}
$$

$$
\begin{aligned}
(2').\quad & \sum_{i_1=1}^{m_n}\sum_{i_2=1}^{l_{n-1}}\cdots\sum_{i_{2(n-j)-1}=1}^{m_{j+1}+i_{2(n-(j+1))-1}-1}\ \sum_{i_{2(n-j)}=1}^{l_j}\cdots\sum_{i_{2n-5}=1}^{m_3+i_7-1}\ \sum_{i_{2n-4}=1}^{l_1+l_2}\\
& (-1)^{\sum_{k=1}^{n-3}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+(i_{2n-5}+n-1)(l_2+l_1)+m_2l_1+1}\\
& e_2^{\epsilon_1}\boxed{\xi^{m_1+m_2+i_{2n-5}-1}x^{l_1+l_2-i_{2n-4}}e_1e_2x^{i_{2n-4}-1}}\dots\\
& \dots\\
& \dots x^{l_j-i_{2(n-j)}}e_1e_2x^{i_{2(n-j)}-1}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}\dots\\
& \dots\\
& \dots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}\\
= \quad & (-1)^{m_2l_1+1}\sum_{i_1=1}^{m_n}\sum_{i_2=1}^{l_{n-1}}\cdots\sum_{i_{2(n-j)-1}=1}^{m_{j+1}+i_{2(n-(j+1))-1}-1}\ \sum_{i_{2(n-j)}=1}^{l_j}\cdots\sum_{i_{2n-5}=1}^{m_3+i_7-1}\ \sum_{i_{2n-4}=1}^{l_1+l_2}\\
& (-1)^{\sum_{k=1}^{n-3}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+(i_{2n-5}+n-1)(l_2+l_1)}\\
& e_2^{\epsilon_1}\boxed{\xi^{m_1+m_2+i_{2n-5}-1}x^{l_1+l_2-i_{2n-4}}e_1e_2x^{i_{2n-4}-1}}\dots\\
& \dots\\
& \dots x^{l_j-i_{2(n-j)}}e_1e_2x^{i_{2(n-j)}-1}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}\dots\\
& \dots\\
& \dots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}\\
= \quad & (-1)^{m_2l_1+1}f_{n-1}\left(e_2^{\epsilon_1}\xi^{m_1+m_2}x^{l_1+l_2}e_1^{\eta_2}\otimes e_2^{\epsilon_3}\xi^{m_3}x^{l_3}e_1^{\eta_3}\otimes\dots\otimes e_2^{\epsilon_n}\xi^{m_n}x^{l_n}e_1^{\eta_n})\right)\\
= \quad & f_{n-1}\left((-1)^{m_2l_1}e_2^{\epsilon_1}\xi^{m_1+m_2}x^{l_1+l_2}e_1^{\eta_2}\otimes e_2^{\epsilon_3}\xi^{m_3}x^{l_3}e_1^{\eta_3}\otimes\dots\otimes e_2^{\epsilon_n}\xi^{m_n}x^{l_n}e_1^{\eta_n})\right)\\
= \quad & -f_{n-1}\left(m_2(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1}\otimes e_2^{\epsilon_2}\xi^{m_2}x^{l_2}e_1^{\eta_2})\otimes e_2^{\epsilon_3}\xi^{m_3}x^{l_3}e_1^{\eta_3}\otimes\dots\otimes e_2^{\epsilon_n}\xi^{m_n}x^{l_n}e_1^{\eta_n})\right)\\
= \quad & -f_{n-1}(m_2\otimes\mathbf{1}^{n-2})\left(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1}\otimes\dots\otimes e_2^{\epsilon_n}\xi^{m_n}x^{l_n}e_1^{\eta_n})\right)
\end{aligned}
$$

$$
\begin{aligned}
(3).\quad & \sum_{j=2}^{n-2}\sum_{i'_1=1}^{m_j}\sum_{i'_2=1}^{l_{j-1}}\dots\sum_{i'_{2j-3}=1}^{m_2+i'_{2j-5}-1}\sum_{i'_{2j-2}=1}^{l_1}\sum_{i_1=1}^{m_n}\sum_{i_2=1}^{l_{n-1}}\dots\sum_{i_{2(n-(j+1))-1}=1}^{m_{j+2}-i_{2(n-(j+2))-1}-1}\sum_{i_{2(n-(j+1))}=1}^{l_{j+1}} \\
& (-1)^{\sum_{k=1}^{n-(j+1)}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+\sum_{k=1}^{j-1}(i'_{2k-1}+\frac{1+(-1)^k}{2})l_{j-k}+(n-j+1)\sum_{k=1}^{j}l_k-j+1} \\
& e_2^{\epsilon_1}\xi^{m_1+i'_{2j-3}-1}x^{l_1-i'_{2j-2}}e_1e_2x^{i'_{2j-2}-1}\xi^{m_2+i'_{2j-5}-1-i'_{2j-3}}\dots \\
& \dots \\
& \dots\xi^{m_j-i'_1}x^{l_j}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1}\dots \\
& \dots \\
& \dots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n} \\
= \quad & \sum_{j=2}^{n-2}(-1)^{(n-j+1)\sum_{k=1}^{j}l_k-j+1}\sum_{i'_1=1}^{m_j}\sum_{i'_2=1}^{l_{j-1}}\dots\sum_{i'_{2j-3}=1}^{m_2+i'_{2j-5}-1}\sum_{i'_{2j-2}=1}^{l_1}\sum_{i_1=1}^{m_n}\sum_{i_2=1}^{l_{n-1}}\dots\sum_{i_{2(n-(j+1))-1}=1}^{m_{j+2}-i_{2(n-(j+2))-1}-1}\sum_{i_{2(n-(j+1))}=1}^{l_{j+1}} \\
& (-1)^{\sum_{k=1}^{j-1}(i'_{2k-1}+\frac{1+(-1)^k}{2})l_{j-k}} \\
& e_2^{\epsilon_1}\xi^{m_1+i'_{2j-3}-1}x^{l_1-i'_{2j-2}}e_1e_2x^{i'_{2j-2}-1}\xi^{m_2+i'_{2j-5}-1-i'_{2j-3}}\dots \\
& \dots \\
& \dots\xi^{m_j-i'_1}x^{l_j} \\
& (-1)^{\sum_{k=1}^{n-(j+1)}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1}\dots \\
& \dots \\
& \dots\xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n} \\
= \quad & \sum_{j=2}^{n-2}(-1)^{(n-j+1)\sum_{k=1}^{j}l_k-j+1}f_j\left(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1}\otimes\dots\otimes e_2^{\epsilon_j}\xi^{m_j}x^{l_j}e_1^{\eta_j}\right)\otimes \\
& \quad f_j\left(e_2^{\epsilon_{j+1}}\xi^{m_{j+1}}x^{l_{j+1}}e_1^{\eta_{j+1}}\otimes\dots\otimes e_2^{\epsilon_n}\xi^{m_n}x^{l_n}e_1^{\eta_n}\right) \\
= \quad & \sum_{j=2}^{n-2}(-1)^{j-1}m_2(f_j\otimes f_{n-j})\left(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1}\otimes\dots\otimes e_2^{\epsilon_n}\xi^{m_n}x^{l_n}e_1^{\eta_n}\right)
\end{aligned}
$$

$$(5''). \quad \sum_{j=3}^{n-2} \sum_{i_1=1}^{m_n} \sum_{i_2=1}^{l_{n-1}} \cdots \sum_{i_{2(n-j)-1}=1}^{m_{j+1}+i_{2(n-(j+1))-1}-1} \sum_{i_{2(n-j)}=1}^{l_{j-1}+l_j} \sum_{i_{2(n-(j-1))-1}=1}^{m_{j-1}+m_j+i_{2(n-j)-1}-1} \sum_{i_{2(n-(j-1))}=1}^{l_{j-2}} \cdots$$

$$\cdots \sum_{i_{2n-5}=1}^{m_2+i_{2n-7}-1} \sum_{i_{2n-4}=1}^{l_1}$$

$$(-1)^{\sum_{k=1}^{n-(j+1)}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+(i_{2(n-j)-1}+\frac{1+(-1)^{n-j}}{2})(l_j+l_{j-1})+\sum_{k=n-(j-1)}^{n-2}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k-1}+m_jl_{j-1}-(j-1)}$$

$$e_2^{\epsilon_1}\xi^{m_1+i_{2n-5}-1}x^{l_1-i_{2n-4}}e_1e_2x^{i_{2n-4}-1}\xi^{m_2+i_{2n-7}-1-i_{2n-5}}\cdots$$

$$\cdots$$

$$\cdots \boxed{\xi^{m_{j-1}+m_j+i_{2(n-j)-1}-1-i_{2(n-(j-1))-1}}x^{l_{j-1}+l_j-i_{2(n-j)}}}$$

$$\cdots \boxed{e_1e_2x^{i_{2(n-j)}-1}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}}\cdots$$

$$\cdots$$

$$\cdots \xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}$$

$$= \sum_{j=3}^{n-2}(-1)^{m_jl_{j-1}-(j-1)} \sum_{i_1=1}^{m_n} \sum_{i_2=1}^{l_{n-1}} \cdots \sum_{i_{2(n-j)-1}=1}^{m_{j+1}+i_{2(n-(j+1))-1}-1} \sum_{i_{2(n-j)}=1}^{l_{j-1}+l_j} \sum_{i_{2(n-(j-1))-1}=1}^{m_{j-1}+m_j+i_{2(n-j)-1}-1} \sum_{i_{2(n-(j-1))}=1}^{l_{j-2}} \cdots$$

$$\cdots \sum_{i_{2n-5}=1}^{m_2+i_{2n-7}-1} \sum_{i_{2n-4}=1}^{l_1}$$

$$(-1)^{\sum_{k=1}^{n-(j+1)}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k}+(i_{2(n-j)-1}+\frac{1+(-1)^{n-j}}{2})(l_j+l_{j-1})+\sum_{k=n-(j-1)}^{n-2}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k-1}}$$

$$e_2^{\epsilon_1}\xi^{m_1+i_{2n-5}-1}x^{l_1-i_{2n-4}}e_1e_2x^{i_{2n-4}-1}\xi^{m_2+i_{2n-7}-1-i_{2n-5}}\cdots$$

$$\cdots$$

$$\cdots \boxed{\xi^{m_{j-1}+m_j+i_{2(n-j)-1}-1-i_{2(n-(j-1))-1}}x^{l_{j-1}+l_j-i_{2(n-j)}}}$$

$$\cdots \boxed{e_1e_2x^{i_{2(n-j)}-1}\xi^{m_{j+1}+i_{2(n-(j+1))-1}-1-i_{2(n-j)-1}}}\cdots$$

$$\cdots$$

$$\cdots \xi^{m_{n-1}+i_1-1-i_3}x^{l_{n-1}-i_2}e_1e_2x^{i_2-1}\xi^{m_n-i_1}x^{l_n}e_1^{\eta_n}$$

$$= \sum_{j=3}^{n-2}(-1)^{m_jl_{j-1}-(j-1)}f_{n-1}\left(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1}\otimes\cdots\otimes e_2^{\epsilon_{j-2}}\xi^{m_{j-2}}x^{l_{j-2}}e_1^{\eta_{j-2}}\otimes\right.$$

$$e_2^{\epsilon_{j-1}+\epsilon_j}\xi^{m_{j-1}+m_j}x^{l_{j-1}+l_j}e_1^{\eta_{j-1}\eta_j}\otimes$$

$$\left.e_2^{\epsilon_{j+1}}\xi^{m_{j+1}}x^{l_{j+1}}e_1^{\eta_{j+1}}\otimes\cdots\otimes e_2^{\epsilon_n}\xi^{m_n}x^{l_n}e_1^{\eta_n}\right)$$

$$= \sum_{j=3}^{n-2}(-1)^{j-1}f_{n-1}(\mathbf{1}^{j-2}\otimes m_2\otimes\mathbf{1}^{\otimes n-j})\left(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1}\otimes\cdots\otimes e_2^{\epsilon_n}\xi^{m_n}x^{l_n}e_1^{\eta_n}\right)$$

$$
\begin{aligned}
(6).\quad & \sum_{i_1=1}^{m_{n-1}} \sum_{i_2=1}^{l_{n-2}} \dots \sum_{i_{2(n-(j+1))-1}=1}^{m_j+i_{2(n-(j+2)-1}-1} \sum_{i_{2(n-(j+1))}=1}^{l_j} \dots \sum_{i_{2n-5}=1}^{m_2-i_{2n-7}-1} \sum_{i_{2n-4}=1}^{l_1} \\
& (-1)^{\sum_{k=1}^{n-2}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k-1}-(n-2)} \\
& e_2^{\epsilon_1}\xi^{m_1+i_{2n-5}-1}x^{l_1-i_{2n-4}}e_1e_2x^{i_{2n-4}-1}\xi^{m_2+i_{2n-7}-1-i_{2n-5}}\dots \\
& \dots \\
& \dots x^{l_j-i_{2(n-(j+1))}}e_1e_2x^{i_{2(n-(j+1))}-1}\xi^{m_{j+1}+i_{2(n-(j+2))-1}-1-i_{2(n-(j+1))-1}}\dots \\
& \dots \\
& \dots \boxed{\xi^{m_{n-1}-i_1}x^{l_{n-1}}\xi^{m_n}}x^{l_n}e_1^{\eta_n} \\
= \quad & (-1)^{n-2} \sum_{i_1=1}^{m_{n-1}} \sum_{i_2=1}^{l_{n-2}} \dots \sum_{i_{2(n-(j+1))-1}=1}^{m_j+i_{2(n-(j+2)-1}-1} \sum_{i_{2(n-(j+1))}=1}^{l_j} \dots \sum_{i_{2n-5}=1}^{m_2-i_{2n-7}-1} \sum_{i_{2n-4}=1}^{l_1} \\
& (-1)^{\sum_{k=1}^{n-2}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k-1}} \\
& e_2^{\epsilon_1}\xi^{m_1+i_{2n-5}-1}x^{l_1-i_{2n-4}}e_1e_2x^{i_{2n-4}-1}\xi^{m_2+i_{2n-7}-1-i_{2n-5}}\dots \\
& \dots \\
& \dots x^{l_j-i_{2(n-(j+1))}}e_1e_2x^{i_{2(n-(j+1))}-1}\xi^{m_{j+1}+i_{2(n-(j+2))-1}-1-i_{2(n-(j+1))-1}}\dots \\
& \dots \\
& \dots \boxed{\xi^{m_{n-1}-i_1}x^{l_{n-1}}\xi^{m_n}}x^{l_n}e_1^{\eta_n} \\
= \quad & (-1)^{n-2}\left(f_{n-1}(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1}\otimes\dots\otimes e_2^{\epsilon_{n-1}}\xi^{m_{n-1}}x^{l_{n-1}}e_1^{\eta_{n-1}})\otimes f_1(e_2^{\epsilon_n}\xi^{m_n}x^{l_n}e_1^{\eta_n})\right) \\
= \quad & (-1)^{n-2}m_2(f_{n-1}\otimes f_1)\left(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1}\otimes\dots\otimes e_2^{\epsilon_n}\xi^{m_n}x^{l_n}e_1^{\eta_n})\right)
\end{aligned}
$$

$$
\begin{aligned}
(7').\quad & \sum_{i_1=1}^{m_{n-1}+m_n} \sum_{i_2=1}^{l_{n-2}} \dots \sum_{i_{2(n-(j+1))-1}=1}^{m_j+i_{2(n-(j+2)-1}-1} \sum_{i_{2(n-(j+1))}=1}^{l_j} \dots \sum_{i_{2n-5}=1}^{m_2-i_{2n-7}-1} \sum_{i_{2n-4}=1}^{l_1} \\
& (-1)^{\sum_{k=1}^{n-2}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k-1}-(n-3)+m_nl_{n-1}} \\
& e_2^{\epsilon_1}\xi^{m_1+i_{2n-5}-1}x^{l_1-i_{2n-4}}e_1e_2x^{i_{2n-4}-1}\xi^{m_2+i_{2n-7}-1-i_{2n-5}}\dots \\
& \dots \\
& \dots x^{l_j-i_{2(n-(j+1))}}e_1e_2x^{i_{2(n-(j+1))}-1}\xi^{m_{j+1}+i_{2(n-(j+2))-1}-1-i_{2(n-(j+1))-1}}\dots \\
& \dots \\
& \dots \boxed{\xi^{m_{n-1}+m_n-i_1}x^{l_{n-1}+l_n}}e_1^{\eta_n} \\
= \quad & (-1)^{m_nl_{n-1}-(n-3)} \sum_{i_1=1}^{m_{n-1}+m_n} \sum_{i_2=1}^{l_{n-2}} \dots \sum_{i_{2(n-(j+1))-1}=1}^{m_j+i_{2(n-(j+2)-1}-1} \sum_{i_{2(n-(j+1))}=1}^{l_j} \dots \sum_{i_{2n-5}=1}^{m_2-i_{2n-7}-1} \sum_{i_{2n-4}=1}^{l_1} \\
& (-1)^{\sum_{k=1}^{n-2}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k-1}} \\
& e_2^{\epsilon_1}\xi^{m_1+i_{2n-5}-1}x^{l_1-i_{2n-4}}e_1e_2x^{i_{2n-4}-1}\xi^{m_2+i_{2n-7}-1-i_{2n-5}}\dots \\
& \dots \\
& \dots x^{l_j-i_{2(n-(j+1))}}e_1e_2x^{i_{2(n-(j+1))}-1}\xi^{m_{j+1}+i_{2(n-(j+2))-1}-1-i_{2(n-(j+1))-1}}\dots \\
& \dots \\
& \dots \boxed{\xi^{m_{n-1}+m_n-i_1}x^{l_{n-1}+l_n}}e_1^{\eta_n} \\
= \quad & (-1)^{m_nl_{n-1}-(n-3)} \\
& \quad f_{n-1}\left(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1}\otimes\dots\otimes e_2^{\epsilon_{n-2}}\xi^{m_{n-2}}x^{l_{n-2}}e_1^{\eta_{n-2}}\otimes e_2^{\epsilon_{n-1}+\epsilon_n}\xi^{m_{n-1}+m_n}x^{l_{n-1}+l_n}e_1^{\eta_{n-1}+\eta_n}\right) \\
= \quad & (-1)^{n-3}f_{n-1}(\mathbf{1}^{n-2}\otimes m_2)\left(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1}\otimes\dots\otimes e_2^{\epsilon_n}\xi^{m_n}x^{l_n}e_1^{\eta_n})\right)
\end{aligned}
$$

$$(8''). \quad \sum_{i_1=1}^{m_n}\sum_{i_2=1}^{l_{n-2}+l_{n-1}}\sum_{i_3=1}^{m_{n-2}+m_{n-1}+i_1-1}\sum_{i_4=1}^{l_{n-3}}\cdots\sum_{i_{2(n-(j+1))-1}=1}^{m_{j+1}+i_{2(n-(j+2))-1}-1}\sum_{i_{2(n-(j+1))}=1}^{l_j}\cdots\sum_{i_{2n-5}=1}^{m_2+i_{2n-7}-1}\sum_{i_{2n-4}=1}^{l_1}$$
$$(-1)^{i_1(l_{n-1}+l_{n-2})+\sum_{k=2}^{n-2}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k-1}+m_{n-1}l_{n-2}-(n-2)}$$
$$e_2^{\epsilon_1}\xi^{m_1+i_{2n-5}-1}x^{l_1-i_{2n-4}}e_1e_2x^{i_{2n-4}-1}\xi^{m_2+i_{2n-7}-1-i_{2n-5}}\cdots$$
$$\cdots$$
$$\cdots x^{l_j-i_{2(n-(j+1))}}e_1e_2x^{i_{2(n-(j+1))}-1}\xi^{m_{j+1}+i_{2(n-(j+2))-1}-1-i_{2(n-(j+1))-1}}\cdots$$
$$\cdots$$
$$\cdots \boxed{\xi^{m_{n-2}+m_{n-1}+i_1-1-i_3}x^{l_{n-2}+l_{n-1}-i_2}e_1e_2}$$
$$\cdots \boxed{x^{i_2-1}\xi^{m_n-i_1}}x^{l_n}e_1^{\eta_n}$$
$$= (-1)^{m_{n-1}l_{n-2}-(n-2)}$$
$$\sum_{i_1=1}^{m_n}\sum_{i_2=1}^{l_{n-2}+l_{n-1}}\sum_{i_3=1}^{m_{n-2}+m_{n-1}+i_1-1}\sum_{i_4=1}^{l_{n-3}}\cdots\sum_{i_{2(n-(j+1))-1}=1}^{m_{j+1}+i_{2(n-(j+2))-1}-1}\sum_{i_{2(n-(j+1))}=1}^{l_j}\cdots\sum_{i_{2n-5}=1}^{m_2+i_{2n-7}-1}\sum_{i_{2n-4}=1}^{l_1}$$
$$(-1)^{i_1(l_{n-1}+l_{n-2})+\sum_{k=2}^{n-2}(i_{2k-1}+\frac{1+(-1)^k}{2})l_{n-k-1}}$$
$$e_2^{\epsilon_1}\xi^{m_1+i_{2n-5}-1}x^{l_1-i_{2n-4}}e_1e_2x^{i_{2n-4}-1}\xi^{m_2+i_{2n-7}-1-i_{2n-5}}\cdots$$
$$\cdots$$
$$\cdots x^{l_j-i_{2(n-(j+1))}}e_1e_2x^{i_{2(n-(j+1))}-1}\xi^{m_{j+1}+i_{2(n-(j+2))-1}-1-i_{2(n-(j+1))-1}}\cdots$$
$$\cdots$$
$$\cdots \boxed{\xi^{m_{n-2}+m_{n-1}+i_1-1-i_3}x^{l_{n-2}+l_{n-1}-i_2}e_1e_2}$$
$$\cdots \boxed{x^{i_2-1}\xi^{m_n-i_1}}x^{l_n}e_1^{\eta_n}$$
$$= (-1)^{m_{n-1}l_{n-2}-(n-2)}f_{n-1}\left(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1}\otimes\ldots\otimes e_2^{\epsilon_{n-3}}\xi^{m_{n-3}}x^{l_{n-3}}e_1^{\eta_{n-3}}\otimes\right.$$
$$\left. e_2^{\epsilon_{n-2}+\epsilon_{n-1}}\xi^{m_{n-2}+m_{n-1}}x^{l_{n-2}+l_{n-1}}e_1^{\eta_{n-2}+\eta_{n-1}}\otimes e_2^{\epsilon_n}\xi^{m_n}x^{l_n}e_1^{\eta_n}\right)$$
$$= (-1)^{n-2}f_{n-1}(\mathbf{1}^{n-3}\otimes m_2\otimes\mathbf{1})\left(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1}\otimes\ldots\otimes e_2^{\epsilon_n}\xi^{m_n}x^{l_n}e_1^{\eta_n})\right)$$

Consequently, $-m_1f_n$ evaluated on $\left(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1}\otimes\ldots\otimes e_2^{\epsilon_n}\xi^{m_n}x^{l_n}e_1^{\eta_n}\right)$ yields:

$$-m_1f_n\left(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1}\otimes\ldots\otimes e_2^{\epsilon_n}\xi^{m_n}x^{l_n}e_1^{\eta_n}\right)$$
$$(1). \quad = m_2(f_1\otimes f_{n-1})\left(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1}\otimes\ldots\otimes e_2^{\epsilon_n}\xi^{m_n}x^{l_n}e_1^{\eta_n})\right)$$
$$(2'). \quad -f_{n-1}(m_2\otimes\mathbf{1}^{n-2})\left(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1}\otimes\ldots\otimes e_2^{\epsilon_n}\xi^{m_n}x^{l_n}e_1^{\eta_n})\right)$$
$$(3). \quad +\sum_{j=2}^{n-2}(-1)^{j-1}m_2(f_j\otimes f_{n-j})\left(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1}\otimes\ldots\otimes e_2^{\epsilon_n}\xi^{m_n}x^{l_n}e_1^{\eta_n}\right)$$
$$(5''). \quad +\sum_{j=3}^{n-2}(-1)^{j-1}f_{n-1}(\mathbf{1}^{j-2}\otimes m_2\otimes\mathbf{1}^{\otimes n-j})\left(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1}\otimes\ldots\otimes e_2^{\epsilon_n}\xi^{m_n}x^{l_n}e_1^{\eta_n}\right)$$
$$(6). \quad +(-1)^{n-2}m_2(f_{n-1}\otimes f_1)\left(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1}\otimes\ldots\otimes e_2^{\epsilon_n}\xi^{m_n}x^{l_n}e_1^{\eta_n}\right)$$
$$(7'). \quad +(-1)^{n-3}f_{n-1}(\mathbf{1}^{n-2}\otimes m_2)\left(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1}\otimes\ldots\otimes e_2^{\epsilon_n}\xi^{m_n}x^{l_n}e_1^{\eta_n})\right)$$
$$(8''). \quad +(-1)^{n-2}f_{n-1}(\mathbf{1}^{n-3}\otimes m_2\otimes\mathbf{1})\left(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1}\otimes\ldots\otimes e_2^{\epsilon_n}\xi^{m_n}x^{l_n}e_1^{\eta_n})\right)$$

which is exactly Φ_n evaluated on $\left(e_2^{\epsilon_1}\xi^{m_1}x^{l_1}e_1^{\eta_1}\otimes\ldots\otimes e_2^{\epsilon_n}\xi^{m_n}x^{l_n}e_1^{\eta_n}\right)$ after rearranging the terms.

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