On Morita Equivalences Between KLR Algebras and VV Algebras

A thesis submitted to the University of East Anglia in partial fulfilment of the requirements for the degree of Doctor of Philosophy

Ruari Walker

School of Mathematics, UEA, Norwich, NR4 7TJ England

March 2, 2016

Abstract

This thesis is investigative work into the properties of a recently defined family of graded algebras, which we call VV algebras. These algebras were defined by Varagnolo and Vasserot. We compare categories of modules over KLR algebras with categories of modules over VV algebras, establishing various Morita equivalences. Using these Morita equivalences we are able to prove several properties of certain classes of VV algebras such as affine cellularity and affine quasi-heredity. We also include a discussion about the problems encountered when trying to define finite-dimensional quotients of these algebras. Finally, in certain settings, we show how to define these quotients and explain how they relate to cyclotomic KLR algebras.

Contents

Acknowledgements

First and foremost, I want to thank my supervisor, Dr Vanessa Miemietz, for her constant support, guidance, and help throughout my time as a PhD student. I have learnt a vast amount from Vanessa, and working with her has been nothing but a pleasure. I would like to thank my second supervisor, Prof. Shaun Stevens, for many helpful discussions and for the advice and support he has given me during my time at the University of East Anglia. More generally, I thank the School of Mathematics and the University of East Anglia for the studentship and the financial support that came with it, which enabled me to complete this work. I would also like to offer my thanks to Prof. Steffen Koenig for inviting me to present my work in the research seminar at the University of Stuttgart, and for allowing me to finish this thesis at the start of my time in Stuttgart. I must also thank my examiners Dr Karin Erdmann and Dr Sinéad Lyle for reading this thesis, taking an interest in my work and for the interesting questions, comments and suggestions.

I want to thank my Mum and Dad for always encouraging me to pursue the things that make me happy and for teaching me the value of hard work. Their unwavering love and encouragement has been invaluable. Thanks also goes to my extended family who spur me on and always take an interest in my work. I would also like to thank Frankie for her support over the past three years. She has always found ways of making me laugh, regardless of my stress levels.

Last but not least, I want to thank my friends; in London, in Norwich, in mathematics and out of mathematics, for playing sports with me, having lunch with me, keeping me sane, and just generally putting up with me. I also thank Peter Ashford who took the time to read through a draft of this thesis.

Introduction

With applications in so many different areas, some of the most interesting and important objects in mathematics are symmetric groups and objects relating to them. The representation theory of these objects forms a fascinating area of research which contains many deep and intriguing results. For example, let $\mathcal{G}$ be a connected split reductive group over a p-adic field $\mathbf{k}$ with connected centre and let $\mathcal{I}$ be an Iwahori subgroup of $\mathcal{G}$. For $\mathcal{G} = GL_n(\mathbf{k})$ it has been shown that the category of admissible representations of $\mathcal{G}$ which have non-zero vectors fixed by $\mathcal{I}$ is equivalent to the category of finite-dimensional representations of H_n^A, an affine Hecke algebra of type A. These algebras are deformations of the group algebra of type A extended affine Weyl groups and are shown to be Morita equivalent to KLR algebras, on the level of finite length module categories. This shows that an interesting area of the representation theory of p-adic GL_n can often be reduced to that of H_n^A and also means that we have a graded structure on blocks of these categories of representations of $GL_n(\mathbf{k})$. Similarly in type B, the category of admissible representations of p-adic SO_{2n+1} which have non-zero vectors fixed by $\mathcal{I}$ is equivalent to the category of finite-dimensional representations of H_n^B, an affine Hecke algebra of type B. Let $\mathbf{k}$ be a field and fix elements $p, q \in \mathbf{k}^\times$, known as deformation parameters. H_n^B is a deformation of the group algebra $\mathbf{k}W_n^B$ of the extended affine Weyl group of type B. It has generators $T_0, \ldots, T_{n-1}$ which deform the finite Weyl group, and $X_1^{\pm 1}, \cdots, X_n^{\pm 1}$ which generate a Laurent polynomial subalgebra, splitting the category of finite length representations into blocks according to the eigenvalues the action of the generators take. These algebras are now essential objects in modern representation theory but also make appearances in a number of different areas of mathematics and theoretical physics including algebraic combinatorics, harmonic analysis, integrable quantum systems, quantum statistical mechanics, knot theory and string theory.

KLR algebras, or quiver Hecke algebras, are a relatively new family of graded algebras which have been introduced by Khovanov, Lauda and independently by Rouquier, the representation theory of which is intimately related to that of the affine Hecke algebras of type A. There are categorification theorems linking the representation theory of these algebras with quantum groups and Lie theory. Namely, the finite-dimensional projective modules over KLR algebras categorify the negative half of the quantised universal enveloping algebra of the Kac-Moody algebra $\mathfrak{g}$ associated to some Cartan datum. There exist finite-dimensional quotients of KLR algebras known as cyclotomic KLR algebras. In 2008, Brundan and Kleshchev showed in [BK09a] that there is an isomorphism between cyclotomic KLR algebras and blocks of cyclotomic Hecke algebras, H_m^Λ, where Λ is a dominant integral weight. When Λ is of level one, H_m^Λ is either the group algebra of the symmetric group, or a deformation thereof. This established a non-trivial grading on these Hecke algebras which was not previously known. It has been shown that the cyclotomic Hecke algebras categorify an irreducible highest weight module $V(\Lambda)$ for $\mathfrak{g}$. I will denote the KLR algebra corresponding to $\tilde{\nu} \in \mathbb{N}I$ by $\mathbf{R}_{\tilde{\nu}}$.

Varagnolo and Vasserot have also recently defined a new family of graded algebras whose representation theory is closely related to the representation theory of the affine Hecke algebras of type B. Indeed, they prove in [VV11a] that categories of finite-dimensional modules over these algebras are equivalent to categories of finite-dimensional modules over affine Hecke algebras of type B. They also use these algebras to prove a conjecture of Enomoto and Kashiwara which states that the representations of the affine Hecke algebra of type B categorify a simple highest weight module for a certain quantum group (see [EK09]). This conjecture of Enomoto and Kashiwara gives gradings, similar to the type A setting, on blocks of these categories. These are the main motivating reasons behind studying these algebras. Throughout this work I refer to these algebras as VV algebras, and denote the VV algebra corresponding to $\nu \in {}^\theta\mathbb{N}I$ by $\mathfrak{W}_\nu$. One of the advantages of working with VV algebras is that they are graded, whilst affine Hecke algebras of type B are not. They depend on two parameters, p and q, which correspond to the deformation parameters of the affine Hecke algebra of type B. It turns out that there are four different cases to consider when studying VV algebras; $p, q \notin I$, precisely one of $p, q \in I$ and both $p, q \in I$, where I is an index set associated to a quiver Γ.

In this thesis we compare categories of modules over KLR algebras with categories of modules over VV algebras. The main result of this work is that certain subclasses of VV algebras are affine cellular and even affine quasi-hereditary.

Theorem (Main Result) (3.4.22, 3.4.23, 3.4.29, 3.4.30)**.** *For any $\nu \in {}^\theta\mathbb{N}I$ when $p, q \notin I$ and p not a root of unity, or for $\nu \in {}^\theta\mathbb{N}I$ having multiplicity one when $q \in I$, $p \notin I$ and p not a root of unity, the algebras $\mathfrak{W}_\nu$ are affine cellular and affine quasi-hereditary.*

In Chapter 1, we cover some background material including finite reflection groups and minimal length coset representatives, which we will refer to throughout the thesis. We also prove a useful proposition which will be needed in Chapter 3. In Chapter 2, we first provide, in more detail, the motivating reasons behind the study of VV algebras. We then define KLR algebras of type A and the VV algebras, which are our main objects of study, and prove a proposition which shows how these algebras are related. Namely,

Proposition (2.3.12)**.** *Under certain conditions, KLR algebras are idempotent subalgebras* *of VV algebras. That is, there is an algebra isomorphism* $\boldsymbol{e}\mathfrak{W}_\nu\boldsymbol{e} \cong \boldsymbol{R}_{\tilde{\nu}}$*, for an idempotent* $\boldsymbol{e} \in \mathfrak{W}_\nu$*.*

Chapter 3 starts with some background material on Morita theory before going on to prove why we may assume the dimension vector ν defining the VV algebra $\mathfrak{W}_\nu$ has connected support.

Theorem (3.2.9)**.** *For* $\nu = \nu_1 + \nu_2 \in {}^\theta\mathbb{N}(I \cup J)$ *with* $\nu_1 \in {}^\theta\mathbb{N}I$ *and* $\nu_2 \in {}^\theta\mathbb{N}J$*,* $\mathfrak{W}_\nu$ *and* $\mathfrak{W}_{\nu_1} \otimes_{\boldsymbol{k}} \mathfrak{W}_{\nu_2}$ *are Morita equivalent.*

We then consider various classes of VV algebras and compare their module categories with module categories over KLR algebras. In particular, in the case $p, q \notin I$ we find Morita equivalence between VV algebras and KLR algebras.

Theorem (3.3.1)**.** *When* $p, q, \notin I$*, the algebras* $\mathfrak{W}_\nu$ *and* $\boldsymbol{R}_{\tilde{\nu}}$ *are Morita equivalent.*

Chapter 3 also introduces the notions of affine cellularity, as defined by Koenig and Xi in [KX12], and affine quasi-heredity as defined by Kleshchev in [Kle15]. We then prove some statements about affine quasi-hereditary algebras and show, in the case $q \in I$, $p \notin I$, when ν has multiplicity one, that the VV algebras are affine cellular and affine quasi-hereditary. To do this we prove the following result, where A and $\tilde{A}$ are the path algebras of a given quiver.

Theorem (3.4.8)**.** *When* $q \in I$*,* $p \notin I$*,* p *not a root of unity, and* ν *has multiplicity one, the algebras* $\mathfrak{W}_\nu$ *and* $A^{\otimes(m-1)} \otimes \tilde{A}$ *are Morita equivalent.*

At the end of Chapter 3 we show that when we relax this multiplicity condition imposed on ν we obtain the following Morita equivalence.

Theorem (3.4.34)**.** *When* $q \in I$*,* $p \notin I$*, and* q *has multiplicity one in* ν*, the algebras* $\mathfrak{W}_\nu$ *and* $\boldsymbol{R}^+_{\tilde{\nu}} \otimes_{\boldsymbol{k}[z]} \tilde{A}$ *are Morita equivalent.*

At the end of Chapter 3 we prove a Morita equivalence statement in the case $p \in I$, $q \notin I$, p not a root of unity, where p has multiplicity exactly two in ν.

Theorem (3.5.3)**.** *When* $p \in I$*,* $q \notin I$*,* p *not a root of unity, and* p *has multiplicity two in* ν*, the algebras* $\mathfrak{W}_\nu$ *and* $\boldsymbol{R}^+_{\tilde{\nu}} \otimes_{\boldsymbol{k}[z]} \tilde{A}$ *are Morita equivalent.*

Chapter 4 leads with a discussion on cyclotomic KLR algebras and their relationship to affine Hecke algebras of type A. We explore, and comment on, the various problems encountered when trying to define analogous quotients of VV algebras. We then define a suitable cyclotomic finite-dimensional quotient of VV algebras in the case $q \in I$, $p \notin I$, multiplicity one, and show that they contain cyclotomic KLR algebras as idempotent subalgebras.

Some of the results and proofs presented in this thesis are rather combinatorial in nature, partly due to the various relations imposed on the generators of these algebras. We try to make these results as clear as possible to the reader by scattering this thesis with examples where necessary.

Chapter 1

Background

1.1 Graded Algebras and Modules

In this thesis we study KLR algebras and VV algebras and compare categories of modules over these families of algebras. Both of these families of algebras are graded.

Throughout this thesis $\mathbf{k}$ will denote a field with characteristic not equal to 2.

Definition 1.1.1. A $\mathbf{k}$-algebra A is **graded** if there exists a monoid I and a family of $\mathbf{k}$-subspaces $\{A_i\}_{i\in I}$ such that

1. $A = \bigoplus_{i\in I} A_i$ as $\mathbf{k}$-subspaces.

2. $A_j A_k \subseteq A_{j+k}$, for all $j, k \in I$.

A non-zero element $a \in A_k$ is called a **homogeneous** element of degree k and we write $\deg(a) = k$. Note that any idempotent in A must be of degree 0. In particular, $0, 1 \in A_0$. In this thesis, when referring to a grading on an algebra, we will always mean a $\mathbb{Z}$-grading, i.e. where $I = \mathbb{Z}$. A homomorphism between graded algebras A and B is an algebra homomorphism $f : A \longrightarrow B$ such that $f(A_n) \subseteq B_n$ for all $n \in \mathbb{Z}$.

Example 1.1.2. Note that any $\mathbf{k}$-algebra can be given the trivial grading. Put $A_m = 0$, for all $m \neq 0$, and $A_0 = A$.

Example 1.1.3. Consider a polynomial algebra $A = \mathbf{k}[x_1 \ldots, x_n]$, over a field $\mathbf{k}$, in n variables. For each $m \in \mathbb{N}_0$, set

$$A_m = \left\langle x_1^{n_{i_1}} \cdots x_n^{n_{i_n}} \mid n_{i_j} \in \mathbb{N}_0 \text{ for all } j \text{ and } \sum_{j=1}^{n} n_{i_j} = m \right\rangle_{\mathbf{k}}.$$

Each A_m is an abelian group, closed under multiplication by $\mathbf{k}$, so that $A = \bigoplus_{m\in\mathbb{N}_0} A_m$ is a graded $\mathbf{k}$-algebra.

Definition 1.1.4. Let $A = \bigoplus_{i\in I} A_i$ be a graded $\mathbf{k}$-algebra and let M be an A-module. M is a **graded module** if there exists a family of $\mathbf{k}$-subspaces $\{M_j\}_{j\in I}$ such that

1. $M = \bigoplus_{j \in I} M_j$ as $\mathbf{k}$-subspaces.

2. $A_k M_l \subseteq M_{k+l}$, for all $k, l \in I$.

As with graded algebras, when referring to a graded module in this thesis, we mean a $\mathbb{Z}$-graded module. A homomorphism of graded A-modules M, N is an A-module morphism $\phi : M \longrightarrow N$ such that $\phi(M_i) \subseteq N_i$ for all $i \in \mathbb{Z}$.

Example 1.1.5. Any algebra A is trivially graded and so any A-module M is trivially graded by setting $M_l = 0$, for all $l \neq 0$, and $M_0 = M$. Since every A-module M is trivially graded, when referring to a graded module we usually mean a non-trivial grading.

Definition 1.1.6. A **graded submodule** of a graded A-module M is an A-submodule N which is itself a graded A-module such that the inclusion $i : N \hookrightarrow M$ is a homomorphism of graded modules. Explicitly, this means $N = \bigoplus_{j \in \mathbb{Z}} N_j$ is a submodule of M with $N_j = N \cap M_j$.

Definition 1.1.7. A **graded ideal** $I \subset A$ is a graded A-submodule of A.

Proposition 1.1.8. *Let $I \subset A$ be a graded ideal. Then A/I has a natural grading under which the natural map $A \longrightarrow A/I$ is a homomorphism of graded algebras.*

Proof. The ideal I is a graded A-submodule, so $I = \bigoplus_{k \in \mathbb{Z}} I_k$ with each $I_k = I \cap A_k$. Then A/I is graded with the k^{th} graded component given by $(A/I)_k = A_k/I_k$. Then it is clear that $A \longrightarrow A/I$ is a homomorphism of graded algebras. $\square$

We now introduce the notion of graded dimension for a graded module over a $\mathbf{k}$-algebra A. We write q to be both a formal variable and a degree shift functor, shifting the degree by 1. That is, for a graded A-module $M = \bigoplus_{n \in \mathbb{Z}} M_n$, qM is again a graded A-module with $(qM)_k = M_{k-1}$. The graded A-module M is said to be **locally finite-dimensional** if each graded component M_k is finite-dimensional. In this case the **graded dimension** of M is defined to be

$$\dim_q M = \sum_{n \in \mathbb{Z}} (\dim(M_n)) q^n$$

where $\dim(M_n)$ is the dimension of M_n over $\mathbf{k}$.

Example 1.1.9. Let $\mathbf{k}$ be a field and consider the polynomial ring $\mathbf{k}[x]$. As a $\mathbf{k}$-vector space, $\mathbf{k}[x]$ has a basis $\{1, x, x^2, x^3, \ldots\}$. Each graded component is of dimension 1. Then,

$$\dim_q \mathbf{k}[x] = \sum_{n \in \mathbb{Z}_{\geq 0}} q^n = \frac{1}{1-q}.$$

Example 1.1.10. Let $\mathbf{k}$ be a field and consider the polynomial ring $\mathbf{k}[x, y]$. Since $\mathbf{k}[x, y] \cong \mathbf{k}[x] \otimes \mathbf{k}[y]$ we find, using Example 1.1.9,

$$\dim_q \mathbf{k}[x, y] = \frac{1}{(1-q)^2}.$$

Similarly, for a polynomial ring $\mathbf{k}[x_1, \ldots, x_m]$ in m variables,

$$\dim_q \mathbf{k}[x_1, \ldots, x_m] = \frac{1}{(1-q)^m}.$$

1.2 Coxeter Groups

Definition 1.2.1. A **Coxeter group** is a group W generated by a set S of elements s_k which are subject to the relations

$$\begin{aligned} s_k^2 &= 1, \text{ for every } s_k \in S \\ (s_i s_j)^{m_{ij}} &= 1, \text{ for every } s_i \neq s_j \in S \end{aligned}$$

where 1 denotes the identity element, $m_{ij} \in \{2, 3, 4, \ldots, \infty\}$ and where $m_{ij} = \infty$ indicates that no such relation exists between s_i and s_j. The pair (W, S) is called a Coxeter system and S is called the set of distinguished generators.

Example 1.2.2. The symmetric group $\mathfrak{S}_n$ on n letters is an example of such a group.

$$\mathfrak{S}_n = \left\langle s_1, \ldots, s_{n-1} \mid s_i^2 = 1 \ \forall i, \ (s_i s_j)^2 = 1 \text{ if } |i-j| > 1, \ (s_i s_{i+1})^3 = 1 \ \forall i \ \right\rangle.$$

In this example, $m_{ij} = 2$ if $|i - j| > 1$ and $m_{ij} = 3$ if $j = i + 1$.

To every Coxeter system we can associate a Coxeter graph, which is defined as follows. The vertices are in one-to-one correspondence with the $s_i \in S$. Label the vertex corresponding to s_i by i. There is an edge between i and j if $m_{ij} \geq 3$. This edge is labelled by m_{ij} when $m_{ij} \geq 4$. Edges are allowed to be labelled by ∞. The Coxeter graph determines (W, S) up to isomorphism. A Coxeter group is said to be irreducible if, for any two generators $s_r, s_t \in S$ there exist generators $s_r = s_{i_1}, s_{i_2}, \ldots, s_{i_k} = s_t$ such that $m_{i_a i_{a+1}} \geq 3$ for all $1 \leq a \leq k-1$. This is equivalent to a Coxeter group having a connected Coxeter graph. Every Coxeter group is a direct product of irreducible Coxeter groups.

The most important Coxeter groups are the Weyl groups and the affine Weyl groups. They arise in the theory of Lie groups and Lie algebras as reflection groups of root systems. They are classified.

Example 1.2.3. The Weyl group of type B is an important example of such a Weyl group.

$$\begin{aligned} W_n^B = \langle s_0, s_1, \ldots, s_{n-1} \mid s_i^2 = 1 \ \forall i, (s_i s_{i+1})^3 = 1 \ \forall i > 0, \\ (s_0 s_1)^4 = 1, (s_i s_j)^2 = 1 \ \forall |i-j| > 1 \rangle. \end{aligned}$$

Example 1.2.4. The symmetric group $\mathfrak{S}_n$ and the Weyl group W_n^B have the following

Coxeter graphs, respectively.

$$A_n \quad (n \geq 1): \quad \bullet \!-\!\!-\!\!-\! \bullet \!-\!\!-\!\!-\! \cdots \!-\!\!-\!\!-\! \bullet \!-\!\!-\!\!-\! \bullet$$

$$B_n \quad (n \geq 1): \quad \bullet \overset{4}{\!-\!\!-\!\!-\!} \bullet \!-\!\!-\!\!-\! \cdots \!-\!\!-\!\!-\! \bullet \!-\!\!-\!\!-\! \bullet$$

Definition 1.2.5. Let (W, S) be any Coxeter system. Let 1 be the identity element in W. We define the **length function** $\ell : W \longrightarrow \mathbb{N}_0$ as follows. Set $\ell(1) = 0$. For any other element $w \in W$ there exist generators $s_1, s_2, \ldots, s_k \in S$ such that $w = s_1 s_2 \cdots s_k$. Pick these generators in such a way that k is minimal. Then $\ell(w) = k$. Such an expression for w is said to be a **reduced expression** for w.

Affine Weyl groups are generated by affine reflections which are reflections about some hyperplanes which do not necessarily pass through the origin. As with the finite case, the affine Hecke algebra is a deformation of a group algebra $\mathbf{k}W$, where W is now an affine Weyl group. Below are the two examples of affine Hecke algebras most relevant to the work in this thesis.

Definition 1.2.6. Fix an element $q \in \mathbf{k}^\times$. The **affine Hecke algebra** of type A_m is the $\mathbf{k}$-algebra H_m^A generated by

$$T_1, \ldots, T_{m-1}, X_1^{\pm 1}, \ldots, X_m^{\pm 1}$$

which are subject to the following relations.

1. $T_i^2 = (q-1)T_i + q$.
2. $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ for $1 \leq i < n-1$
   $T_i T_j = T_j T_i$ for $|i-j| > 1$.
3. $X_i X_j = X_j X_i$ for all i, j.
4. $T_i X_i T_i = q X_{i+1}$ for $1 \leq i < m$
   $T_i X_j = X_j T_i$ if $j \neq i, i+1$.

If $m = 0$ then $H_0^A = \mathbf{k}$.

Definition 1.2.7. Fix p, $q \in \mathbf{k}^\times$. The **affine Hecke algebra** of type B_m is the $\mathbf{k}$-algebra H_m^B generated by

$$T_0, \ldots, T_{m-1}, X_1^{\pm 1}, \ldots, X_m^{\pm 1}$$

which are subject to the following relations.

1. $(T_0 - p)(T_0 + p^{-1}) = 0$ and $(T_i - q)(T_i + q^{-1}) = 0$ for $1 \leq i < m$.
2. $T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0$
   $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ for $1 \leq i < m-1$
   $T_i T_j = T_j T_i$ for $|i-j| > 1$.

3. $X_i X_j = X_j X_i$ for all i, j.

4. $T_0 X_1^{-1} T_0 = X_1$
   $T_i X_i T_i = X_{i+1}$ for $1 \leq i < n$
   $T_i X_j = X_j T_i$ if $j \neq i, i+1$.

If $m = 0$ then $H_0^B = \mathbf{k}$.

1.3 Parabolic Subgroups

Let (W, S) be a Coxeter system and $J \subseteq S$ a subset of the generating set S. Consider $W_J := \langle J \rangle \subset W$, the subgroup of W generated by J.

Definition 1.3.1. W_J is called a **parabolic subgroup** of W.

Each generator $s_i \in S = \{s_1, \ldots, s_n\}$ is either included in J or is excluded. Hence there are 2^n parabolic subgroups of a Coxeter group W. The pair (W_J, J) is a Coxeter system in its own right and one can show that the length functions on W and W_J coincide. See, for example, [GP00].

Example 1.3.2. Let $W = W_n^B$ be the Weyl group of type B_n, with distinguished generating set $\{s_0, s_1, \ldots, s_{n-1}\}$. The symmetric group $\mathfrak{S}_n$ is a parabolic subgroup of W, taking $J = \{s_1, \ldots, s_{n-1}\}$.

Example 1.3.3. Let $W = \mathfrak{S}_n$, the symmetric group on n letters. It has generating set $S = \{s_1, \ldots, s_{n-1}\}$. For some a, with $1 < a \leq n$, let $J = \{s_1, \ldots, s_{a-1}\}$. J generates the symmetric group $\mathfrak{S}_a$ on a letters which is a parabolic subgroup of $\mathfrak{S}_n$.

The following two well-known results will be used throughout this thesis and demonstrate how we can manipulate and decrease the length of an expression which is not reduced. Again, $(W, S = \{s_1, \ldots, s_n\})$ is a Coxeter system.

Theorem 1.3.4 (Cancellation Law)**.** *Let $s_1, \ldots, s_k \in S$ (for some $k \geq 2$) and suppose the expression $s_1 \cdots s_k$ is not reduced, i.e. $\ell(s_1 \cdots s_k) < k$. Then there exist indices $1 \leq i < j \leq k$ such that $s_1 \cdots s_k = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k$, where the hat denotes omission.*

Proof. See, for example, [Hum92] Section 1.7. $\square$

The Exchange Condition follows from the theorem.

Corollary 1.3.5 (Exchange Condition)**.** *Let $w = s_1 \cdots s_k$ be an element of W. If $\ell(ws) < \ell(w)$ for some $s \in S$, then there exists an index i for which $ws = s_1 \cdots \hat{s}_i \cdots s_k$. In other words, $w = s_1 \cdots \hat{s}_i \cdots s_k s$, and we have exchanged s_i for s. In particular, w has a reduced expression ending in s if and only if $\ell(ws) < \ell(w)$.*

Proof. See, for example, [Hum92] Section 1.7. $\square$

1.4 Minimal Length Coset Representatives

Let (W, S) be a Coxeter system and $J \subseteq S$ a subset of the generating set. We can consider left cosets, wW_J, of W_J in W. Each left coset is of the same size, this size being equal to $|W_J|$, and the left cosets partition W. The following Proposition shows that, for any parabolic subgroup W_J in W, we can choose left coset representatives of W_J in W in a particularly nice way.

Proposition 1.4.1 ([GP00], Proposition 2.1.1). *Let*

$$\mathcal{D}(W/W_J) := \{w \in W \mid \ell(ws) > \ell(w) \text{ for all } s \in J\}.$$

1. *For each $w \in W$ there exists a unique $\omega \in W_J$ and $\eta \in \mathcal{D}(W/W_J)$ such that $w = \eta\omega$. Moreover, $\ell(w) = \ell(\eta) + \ell(\omega)$.*

2. *For any $w \in W$ the following are equivalent:*

   (i) $w \in \mathcal{D}(W/W_J)$;

   (ii) $\ell(w\omega) = \ell(w) + \ell(\omega)$ for all $\omega \in W_J$;

   (iii) w is the unique element of minimal length in wW_J.

   In particular, $\mathcal{D}(W/W_J)$ is a complete set of distinguished left coset representatives of W_J in W.

Elements in $\mathcal{D}(W/W_J)$ will be referred to as **minimal length left coset representatives** of W_J in W. An analogous statement can be made for right coset representatives of W_J in W.

1.4.1 Deodhar's Algorithm

Deodhar's algorithm is a way of calculating the minimal length left coset representatives of a parabolic subgroup W_J in a Coxeter group W. This algorithm is defined inductively on the length ℓ of the representative:

The only left coset representative of length 0 is the identity element, 1. We proceed as follows.

Let η be a left coset representative of length k. Then a left coset representative of length $k+1$, call it η', is obtained from η as follows.

Set $\eta' = s_i\eta$ subject to the conditions

1. $\ell(s_i\eta) > \ell(\eta)$ and

2. $\ell(s_i\eta s_j) > \ell(s_i\eta)$ for all $s_j \in J$.

Example 1.4.2. Let $\mathcal{D}_m$ denote the set of minimal length left coset representatives of the symmetric group $\mathfrak{S}_m$ in the Weyl group W_m^B. Consider the case when $m = 3$. We use Deodhar's algorithm to calculate the minimal length left coset representatives of $\mathfrak{S}_3$ in W_3^B. If a generator s_i satisfies conditions (1) and (2) then we place a tick in the corresponding column:

k	$\eta \in \mathcal{D}_3$	s_0	s_1	s_2
0	1	✓	×	×
1	s_0	×	✓	×
2	s_1s_0	✓	×	✓
3	$s_0s_1s_0$	×	×	✓
	$s_2s_1s_0$	✓	×	×
4	$s_2s_0s_1s_0$	×	✓	×
5	$s_1s_2s_0s_1s_0$	✓	×	×
6	$s_0s_1s_2s_0s_1s_0$	×	×	×

Remark 1.4.3. Take any $w \in \mathcal{D}_m$. Then $w \in W_m^B$ and $\ell(ws) > \ell(w)$ for every $s \in \{s_1, \ldots, s_{m-1}\}$. Note that $w \in W_{m+1}^B$ and, since s_m is not a factor of w (because $w \in W_m^B$), $\ell(ws_m) > \ell(w)$. Then $\ell(ws) > \ell(w)$ for every $s \in \{s_1, \ldots, s_m\}$, meaning $w \in \mathcal{D}_{m+1}$. This proves,

$$\mathcal{D}_1 \subsetneq \mathcal{D}_2 \subsetneq \cdots \subsetneq \mathcal{D}_m \subsetneq \mathcal{D}_{m+1} \subsetneq \cdots$$

Lemma 1.4.4 (Deodhar's Lemma)**.** *Suppose we have $J \subseteq S$, $\eta \in \mathcal{D}(W/W_J)$ and $s_i \in S$. Then either $s_i\eta \in \mathcal{D}(W/W_J)$ or $s_i\eta = \eta s_j$ for some $s_j \in J$.*

Proof. See Chapter 2 of [GP00]. □

Definition 1.4.5. Let $J \subset S$ so that W_J is a parabolic subgroup of W. Let $\eta \in \mathcal{D}(W/W_J)$ be a minimal length left coset representative. An element $s_{i_k} \cdots s_{i_n}$ is called a **prefix** of η if there exists $s_{i_1}, \ldots, s_{i_{k-1}} \in S$ such that $\eta = s_{i_1} \cdots s_{i_{k-1}} s_{i_k} \cdots s_{i_n}$ is a reduced expression of η. Prefixes for minimal length right coset representatives are defined similarly.

Let w_J denote the longest element in W_J.
Let w_0 denote the longest element in W.

Lemma 1.4.6 ([GP00], Lemma 2.2.1)**.** *Let $d_J := w_0 w_J$. Then,*

(a) d_J is the unique element of maximal length in $\mathcal{D}(W/W_J)$.

(b) $\mathcal{D}(W/W_J) = \{w \in W \mid w$ is a prefix of $d_J\}$.

As in Example 1.4.2, let $\mathcal{D}_m = \mathcal{D}(W_m^B/\mathfrak{S}_m)$. Let d_{J_m} denote the longest element in $\mathcal{D}_m$.

Example 1.4.7. We calculated $\mathcal{D}_3$ in Example 1.4.2. The longest element in $\mathcal{D}_3$ is $d_{J_3} = s_0s_1s_2s_0s_1s_0 = s_0s_1s_0s_2s_1s_0$. From the table we can see that indeed every $\eta \in \mathcal{D}_3$ is a prefix of d_{J_3}.

Lemma 1.4.8. *Every $\eta \in \mathcal{D}_m \setminus \mathcal{D}_{m-1}$ is of the form $\bar{\eta} s_{m-1} \cdots s_1 s_0$, for some $\bar{\eta} \in \mathcal{D}_{m-1}$.*

Proof. From Remark 1.4.3, $\mathcal{D}_{m-1} \subset \mathcal{D}_m$. Since $|\mathcal{D}_m| = 2|\mathcal{D}_{m-1}|$, it is enough to show that $\bar{\eta} s_{m-1} \cdots s_1 s_0 \in \mathcal{D}_m$, for all $\bar{\eta} \in \mathcal{D}_{m-1}$, i.e. then we would have

$$\mathcal{D}_m = \mathcal{D}_{m-1} \sqcup \mathcal{D}_{m-1} \cdot s_{m-1} \cdots s_1 s_0.$$

Using the notation in Lemma 1.4.6 for $W = W_m^B$, $W_J = \mathfrak{S}_m$ it is well-known that,

$$\begin{aligned} w_J &= s_1 s_2 s_3 \cdots s_{m-1} s_1 s_2 s_3 \cdots s_{m-2} \cdots s_1 s_2 s_3 s_1 s_2 s_1 \\ w_0 &= s_0 s_1 s_2 \cdots s_{m-1} s_0 s_1 s_2 \cdots s_{m-1} \cdots s_0 s_1 s_2 \cdots s_{m-1} \quad (m \text{ times}) \end{aligned}$$

are the longest elements in $\mathfrak{S}_m$ and W_m^B, respectively. Then one can verify that,

$$d_{J_m} = s_0 s_1 s_0 s_2 s_1 s_0 \cdots s_{m-1} \cdots s_1 s_0.$$

Using Lemma 1.4.6, every $\bar{\eta} \in \mathcal{D}_{m-1}$ is a prefix of $d_{J_{m-1}} = s_0 s_1 s_0 s_2 s_1 s_0 \cdots s_{m-2} \cdots s_1 s_0$. Then it is clear that $\bar{\eta} s_{m-1} \cdots s_1 s_0$ is a prefix of d_{J_m} and so $\bar{\eta} s_{m-1} \cdots s_1 s_0 \in \mathcal{D}_m$. □

1.5 Roots and Reflections

Let V be a Euclidean vector space with a positive definite symmetric bilinear form $(\cdot\,,\cdot) : V \times V \longrightarrow \mathbb{R}$.

Definition 1.5.1. A **reflection** in V is a linear operator $s : V \longrightarrow V$ defined by, for some $\lambda \in V$,

$$s_\lambda(w) = s(w) := w - \frac{2(\lambda, w)}{(\lambda, \lambda)}\lambda.$$

Then s sends λ to $-\lambda$ and fixes pointwise the hyperplane H_λ orthogonal to λ. Since $V = \mathbb{R}\lambda \oplus H_\lambda$, this is indeed a reflection in V. We sometimes write $s = s_\lambda$ to indicate that the reflection is about the hyperplane H_λ. It is clear that $s_{r\lambda} = s_\lambda$, for any non-zero $r \in \mathbb{R}$. One can check that, for every $\lambda \in V$, $(s_\lambda(v), s_\lambda(w)) = (v, w)$ for every $v, w \in V$ so that s_λ is an orthogonal transformation.

Definition 1.5.2. A **finite reflection group** is a group generated by finitely many reflections s_λ in V. It is a subgroup of $O(V)$.

The most interesting finite reflection groups are the Weyl groups. They are particularly important objects in Lie theory because they classify simple finite-dimensional complex Lie algebras. Every finite reflection group is a Coxeter group.

Definition 1.5.3. A **root system** is a finite set of non-zero vectors $\Phi \subseteq V$ such that

1. $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for all $\alpha \in \Phi$.

2. $s_\alpha \Phi = \Phi$ for all $\alpha \in \Phi$.

The elements of Φ are called **roots** and are typically denoted $\alpha, \beta, \gamma, \ldots$ etc. Let W denote the finite reflection group generated by the reflections s_α, $\alpha \in \Phi$.

Given a root system Φ we want to find a subset $\Pi \subseteq \Phi$ such that each root is a linear combination of elements in Π with coefficients all of the same sign (either all positive or all negative). First, we partition our roots into what are called positive and negative roots.

Definition 1.5.4. A **total order** on V, denoted $<$, is a relation on V satisfying the following conditions.

1. For every pair $v, w \in V$ either $v < w$, $w < v$ or $v = w$.

2. For all $v_1, v_2, w \in V$, whenever $v_1 < v_2$ then $v_1 + w < v_2 + w$.

3. If $v < w$ and r is a non-zero real number then $rv < rw$ if $r > 0$, and $rw < rv$ if $r < 0$.

4. If $v_1 < v_2$ and $v_2 < v_3$ then $v_1 < v_3$.

Definition 1.5.5. Given a root system $\Phi \subseteq V$ and a total order $<$ on V, a root α is a **positive root** if $0 < \alpha$ and is a **negative root** if $\alpha < 0$. The set of positive (resp. negative) roots is called a **positive system** (resp. **negative system**) and is denoted by Φ^+ (resp. Φ^-).

Then $\Phi = \Phi^- \sqcup \Phi^+$ since $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$.

Remark 1.5.6. We can always put a total order on V as follows. Let $v_1, \ldots, v_n$ be a basis of V. Then define $<$ to be the lexicographic order on V; for $v = \sum_i \mu_i v_i$, $w = \sum_j \eta_j v_j \in V$, $v < w$ if $\mu_k < \eta_k$ where k is the largest index such that $\mu_i \neq \eta_i$. This shows that positive systems always exist.

Definition 1.5.7. A subset $\Pi \subseteq \Phi^+$ is called a **simple system** if it is a basis for the $\mathbf{k}$-span of Φ in V such that each $\alpha \in \Phi$ is an $\mathbb{R}$-linear combination of elements in Π with coefficients all of the same sign.

The following theorem shows that a simple system always exists and that any simple system must be a subset of the positive roots.

Theorem 1.5.8 ([Hum92], Section 1.3)**.**

1. *Any simple system* Π *is contained in a unique positive system.*

2. *Every positive system* Φ^+ *in* Φ *contains a unique simple system.*

Since we have seen that positive systems always exist, this theorem shows that simple systems always exist. This theorem shows that positive systems and simple systems determine each other uniquely.

Theorem 1.5.9. *For a fixed simple system* Π*, the finite refection group* W *is generated by* s_α*,* $\alpha \in \Pi$.

We define the length function $\ell : W \longrightarrow \mathbb{N}_0$ on W as in Definition 1.2.5.

Definition 1.5.10. The s_α, with $\alpha \in \Pi$, are called **simple reflections**.

Example 1.5.11. Let $V = \mathbb{R}^3$ with standard basis $\{e_1, e_2, e_3\}$ and the usual inner product $(e_i, e_j) = \delta_{ij}$. Then one can check that $\Phi = \{e_i - e_j \mid 1 \leq i, j \leq 3, i \neq j\}$ is a root system. Using the total order described in Remark 1.5.6 we have

$$e_1 - e_3 < e_2 - e_3 < e_1 - e_2 < 0 < e_2 - e_1 < e_3 - e_2 < e_3 - e_1$$

and we obtain a partition of Φ into positive and negative roots

$$\Phi^- = \{e_1 - e_3, e_2 - e_3, e_1 - e_2\} \qquad \Phi^+ = \{e_2 - e_1, e_3 - e_2, e_3 - e_1\}.$$

It is easy to check that $\Pi = \{e_2 - e_1, e_3 - e_2\}$ is the unique simple system in Φ^+. The finite reflection group W associated to this root system is generated by $\{s_{e_2-e_1}, s_{e_3-e_2}\}$ and this group is isomorphic to $\mathfrak{S}_3 = \langle s_1, s_2\rangle$, the symmetric group on 3 letters. The isomorphism is given by sending $s_{e_2-e_1}$ to s_1 and $s_{e_3-e_2}$ to s_2.

More generally,

Example 1.5.12. Let $V = \mathbb{R}^n$ with standard basis $\{e_1, \ldots, e_n\}$ and the usual inner product, $(e_i, e_j) = \delta_{ij}$. Then $\Phi = \{e_i - e_j \mid 1 \leq i, j \leq n, i \neq j\}$ is a root system and $\Phi^+ = \{e_i - e_j \mid 1 \leq j < i \leq n\}$ is the positive system with respect to the ordering described in Remark 1.5.6. The unique simple system in Φ^+ is $\Pi = \{e_{i+1} - e_i \mid 1 \leq i \leq n-1\}$. The corresponding finite reflection group is isomorphic to $\mathfrak{S}_n$ via $s_{e_{i+1}-e_i} \mapsto s_i$, for $1 \leq i \leq n-1$, where the s_i are the generators of $\mathfrak{S}_n$.

Example 1.5.13. Let $V = \mathbb{R}^2$ again with the standard basis and usual inner product. Then $\Phi = \{e_1, e_2, e_1 + e_2, e_1 - e_2, -e_1, -e_2, -e_1 - e_2, e_2 - e_1\}$ is a root system. With the total ordering described in Remark 1.5.6 we have

$$-e_1 - e_2 < -e_2 < e_1 - e_2 < -e_1 < 0 < e_1 < e_2 - e_1 < e_2 < e_1 + e_2$$

so that the positive and negative systems are

$$\Phi^+ = \{e_1, e_2 - e_1, e_2, e_1 + e_2\} \qquad \Phi^- = \{-e_1 - e_2, -e_2, e_1 - e_2, -e_1\}.$$

Then one finds that $\Pi = \{e_2 - e_1, e_1\} \subseteq \Phi^+$ is the unique simple system corresponding to this positive system. The finite reflection group W is generated by $\{s_{e_2-e_1}, s_{e_1}\}$, and it turns out that W is isomorphic to $W_2^B = \langle s_0, s_1\rangle$, the Weyl group of type B_2, via $s_{e_2-e_1} \mapsto s_1$, $s_{e_1} \mapsto s_0$.

More generally,

Example 1.5.14. Let $V = \mathbb{R}^n$ with standard basis. Then $\Phi^+ = \{e_i \pm e_j, e_k \mid 1 \leq j < i \leq n, 1 \leq k \leq n\}$ is a positive system. The corresponding simple system is $\Pi = \{e_{i+1} - e_i, e_1 \mid 1 \leq i \leq n-1\}$ and the corresponding finite reflection group W is isomorphic to the Weyl group of type B_n.

Example 1.5.15. Let $V = \mathbb{R}^n$ with standard basis. Let Φ be the root system with positive roots

$$\Phi^+ = \{e_i \pm e_j, e_k \mid 1 \leq j < i \leq s \text{ or } s+1 \leq j < i \leq n \text{ and } 1 \leq k \leq n\}.$$

The corresponding simple roots are

$$\Pi = \{e_{i+1} - e_i, e_1, e_{s+1} \mid 1 \leq i < s \text{ or } s+1 \leq i < n\}.$$

The simple reflections generate a finite reflection group isomorphic to $W_s^B \times W_{n-s}^B$.

Let V be a real Euclidean vector space, $\Phi \subseteq V$ a root system with positive root system $\Phi^+ \subseteq \Phi$. Let $\Pi \subseteq \Phi^+$ be the simple roots and W the corresponding finite reflection group. Now let $J \subseteq \Pi$, a subset of the simple roots, and denote by V_J the subspace of V spanned by J. Let $\Phi_J := \Phi \cap V_J$ and write W_J for the subgroup of W generated by the simple reflections s_j, $j \in J$.

Proposition 1.5.16 ([Car89])**.** *Φ_J is a root system in V_J. J is a simple system in Φ_J and the finite reflection group of Φ_J is W_J.*

Let D_J be the set of elements $w \in W$ such that $w(j) \in \Phi^+$, for all $j \in J$. We have the following theorem, taken from [Car89].

Theorem 1.5.17 ([Car89])**.** *Each element of W has a unique expression of the form $w = \eta_J \omega_J$, for $\eta_J \in D_J$ and $\omega_J \in W_J$. Furthermore, we have $\ell(w) = \ell(\eta_J) + \ell(\omega_J)$.*

Corollary 1.5.18. *In each coset wW_J there is a unique element of D_J. The length of this element is smaller than the length of any other element in wW_J.*

So D_J is the set of minimal length coset representatives of W_J in W. Using the notation as in Section 1.4,

$$D_J = \{w \in W \mid w(j) \in \Phi^+ \text{ for all } j \in J\} = \mathcal{D}(W/W_J).$$

Example 1.5.19. From Example 1.5.12, let

$$J = \{e_2 - e_1, \ldots, \widehat{e_{s+1} - e_s}, \ldots, e_n - e_{n-1}\}$$

for some $1 \leq s < n$, where the hat denotes omission. Then $W_J \cong \mathfrak{S}_s \times \mathfrak{S}_{n-s}$. Using the characterisation of D_J above we have,

$$\mathcal{D}(\mathfrak{S}_n/(\mathfrak{S}_s \times \mathfrak{S}_{n-s})) = \left\{ w \in W \;\middle|\; \begin{array}{l} w(j) < w(i) \text{ whenever} \\ 1 \leq j < i \leq s \text{ or } s+1 \leq j < i \leq n \end{array} \right\}.$$

Example 1.5.20. From Example 1.5.15, consider the following subset of Π,

$$J = \{e_1, e_{i+1} - e_i \mid 1 \leq i < s \text{ and } s+1 \leq i < n\}.$$

Then $W_J \cong W_s^B \times \mathfrak{S}_{n-s}$ and

$$\mathcal{D}((W_s^B \times W_{n-s}^B)/(W_s^B \times \mathfrak{S}_{n-s})) = \left\{ w \in W \;\middle|\; \begin{array}{l} 1 \leq w(j) < w(i) \leq s \text{ or} \\ s+1 \leq w(j) < w(i) \leq n \text{ whenever} \\ 1 \leq j < i \leq s \text{ or } s+1 \leq j < i \leq n. \\ w(1) \geq 1 \end{array} \right\}.$$

Example 1.5.21. From Example 1.5.14, let

$$J = \{e_1, e_2 - e_1, \ldots, \widehat{e_{s+1} - e_s}, \ldots, e_n - e_{n-1}\}$$

for some $1 \leq s < n$, where the hat denotes omission again. Then $W_J \cong W_s^B \times \mathfrak{S}_{n-s}$. Again using the characterisation of D_J given above,

$$\mathcal{D}(W_n^B/(W_s^B \times \mathfrak{S}_{n-s})) = \left\{ w \in W \;\middle|\; \begin{array}{l} w(j) < w(i) \text{ whenever } 1 \leq j < i \leq s \\ \text{or } s+1 \leq j < i \leq n. \\ w(1) \geq 1 \end{array} \right\}.$$

Note that the root system corresponding to $\mathfrak{S}_n$ and the root system corresponding to $W_n^B \times W_{n-s}^B$ are both subsets of the roots associated with W_n^B. So we can consider all reflections defined by these roots as being reflections in $W \cong W_n^B$.

Take $w_1 \in \mathcal{D}(\mathfrak{S}_n/(\mathfrak{S}_s \times \mathfrak{S}_{n-s}))$ and $w_2 \in \mathcal{D}((W_s^B \times W_{n-s}^B)/(W_s^B \times \mathfrak{S}_{n-s}))$. Then $w_1 w_2 \in \mathcal{D}(W_n^B/(W_s \times \mathfrak{S}_{n-s}))$; take i, j such that $1 \leq j < i \leq s$. Then $w_2(j) < w_2(i)$, with either $1 \leq w_2(j) < w_2(i) \leq s$ or $s+1 \leq w_2(j) < w_2(i) \leq n$, and so $w_1 w_2(j) < w_1 w_2(i)$. Also, $w_2(1) \geq 1$, and so $w_1 w_2(1) \geq 1$. This shows,

Proposition 1.5.22. *For any $n \geq 2$ and $1 \leq s \leq n-1$*

$$\mathcal{D}(\mathfrak{S}_n/(\mathfrak{S}_s \times \mathfrak{S}_{n-s}))\mathcal{D}((W_s^B \times W_{n-s}^B)/(W_s^B \times \mathfrak{S}_{n-s})) \subseteq \mathcal{D}(W_n^B/(W_s^B \times \mathfrak{S}_{n-s})).$$

Proof. See the discussion above. $\square$

1.6 Lie Theoretic Notation

Let $\Gamma = \Gamma_I$ be a quiver with vertex set I. Associated to this data is a Cartan matrix $(a_{ij})_{i,j \in I}$ defined by

$$a_{i,j} := \begin{cases} 2 & \text{if } i = j \\ 0 & \text{if } i \not\leftrightarrow j \\ -1 & \text{if } i \to j \text{ or } i \leftarrow j \\ -2 & \text{if } i \leftrightarrows j. \end{cases}$$

Let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realisation of the Cartan matrix $(a_{ij})_{i,j\in I}$. We have simple roots $\{\alpha_i \mid i \in I\}$, and fundamental weights $\{\Lambda_i \mid i \in I\}$. Let

$$P := \bigoplus_{i\in I} \mathbb{Z}\Lambda_i \quad \text{and} \quad Q := \bigoplus_{i\in I} \mathbb{Z}\alpha_i$$

denote the weight lattice and root lattice, respectively. Let P_+ and Q_+ denote the subsets of P and Q, respectively, consisting of elements which have non-negative coefficients when written in terms of the above bases. Let $(\cdot,\cdot) : P \times Q \longrightarrow \mathbb{Z}$ be the bilinear pairing defined by $(\Lambda_i, \alpha_j) = \delta_{ij}$. For $\alpha \in Q_+$ we can define the height of α, denoted $\mathrm{ht}(\alpha)$ or sometimes $|\alpha|$, by $\mathrm{ht}(\alpha) := \sum_{i\in I}(\Lambda_i, \alpha)$. For $\Lambda \in P_+$ we can define the level of Λ, denoted $\ell(\Lambda)$, by $\ell(\Lambda) := \sum_{i\in I}(\Lambda, \alpha_i)$.

1.6.1 Root Partitions

Now suppose $I = \mathbb{Z}$. Let $\Gamma = \Gamma_{\mathbb{Z}}$ have arrows $i \longrightarrow i+1$, for every $i \in \mathbb{Z}$. As discussed above, given this data we can define a Cartan matrix $(a_{ij})_{i,j\in I}$. Consider the Lie algebra associated to this Cartan matrix. We have simple roots in type A given by $\{\alpha_i \mid i \in I\}$, and positive roots $\alpha_{i,i+k} := \alpha_i + \alpha_{i+1} + \cdots + \alpha_{i+k}$.

We order the roots as follows.

$$\alpha_i + \alpha_{i+2} + \cdots + \alpha_{i+2k} > \alpha_j + \alpha_{j+2} + \cdots + \alpha_{j+2l} \iff i > j \text{ or } i = j \text{ and } k > l.$$

Definition 1.6.1. A **root partition** of $\alpha \in Q_+$ is a tuple of positive roots, $(\beta_1, \beta_2, \beta_3, \cdots, \beta_r)$ such that $\beta_1 \geq \beta_2 \geq \beta_3 \geq \cdots \geq \beta_r$ and such that $\alpha = \beta_1 + \beta_2 + \beta_3 + \cdots + \beta_r$.

A root partition of α will often be denoted by π and the set of all root partitions of α will be denoted by $\Pi(\alpha)$.

Example 1.6.2. Take $\alpha = \alpha_1 + \alpha_2 + \alpha_3 \in Q_+$. Taking $\beta_1 = \alpha_2 + \alpha_3$ and $\beta_2 = \alpha_1$, we have (β_1, β_2) a root partition of α.

In general, for a given $\alpha \in Q_+$, root partitions are not unique. In the above example, setting $\beta_1 = \alpha_3$ and $\beta_2 = \alpha_1 + \alpha_2$, we have another root partition (β_1, β_2) of α.

To each positive root $\alpha_{i,i+k} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{i+k}$ we associate a sequence of integers,

$$\mathbf{i}_{\alpha_{i,i+k}} := (i, i+1, \ldots, i+k).$$

To a root partition $(\beta_1, \cdots, \beta_r)$ of α, denoted by π, we associate the sequence

$$\mathbf{i}_\pi := \mathbf{i}_{\beta_1}\mathbf{i}_{\beta_2}\cdots\mathbf{i}_{\beta_r}$$

which is the concatenation of the r sequences $\mathbf{i}_{\beta_1}, \mathbf{i}_{\beta_2}, \cdots, \mathbf{i}_{\beta_r}$.

We can put a total order on $\Pi(\alpha)$ in a natural way, using the lexicographic order; if $(\beta_1, \beta_2, \cdots, \beta_r)$ and $(\gamma_1, \gamma_2, \cdots, \gamma_s)$ are root partitions of α then $(\beta_1, \beta_2, \cdots, \beta_r) > (\gamma_1, \gamma_2, \cdots, \gamma_s)$ if $\beta_k > \gamma_k$ where k is the least index such that $\beta_i \neq \gamma_i$.

Example 1.6.3. Take $\alpha = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \in Q_+$. Then $(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)$, $(\alpha_4, \alpha_1 + \alpha_2 + \alpha_3)$, $(\alpha_4, \alpha_2 + \alpha_3, \alpha_1)$ are root partitions of α. Denote them as π_1, π_2, π_3, respectively. We have

$$\pi_1 < \pi_2 < \pi_3$$

and $\mathbf{i}_{\pi_1} = (1, 2, 3, 4)$, $\mathbf{i}_{\pi_2} = (4, 1, 2, 3)$, $\mathbf{i}_{\pi_3} = (4, 2, 3, 1)$.

Chapter 2

KLR and VV Algebras

2.1 Motivation For Studying VV Algebras

The main motivating reasons for studying VV algebras lie in the categorification of quantum groups, thereby connecting the representation theory of affine Hecke algebras of type B with Lie theory. We first give a brief description of this picture for affine Hecke algebras of type A and discuss the Lascoux-Leclerc-Thibon conjecture (LLT conjecture).

Let $\mathbf{k}$ be a field with $q \in \mathbf{k}^\times$. The Iwahori-Hecke algebra associated to $\mathfrak{S}_m$, denoted $\mathrm{H}_m = \mathrm{H}_m(\mathbf{k}, q)$, is a q-deformation of the group algebra of the symmetric group $\mathbf{k}\mathfrak{S}_m$. When $q = 1$ we have $\mathrm{H}_m(\mathbf{k}, 1) = \mathbf{k}\mathfrak{S}_m$. The LLT conjecture asserts a connection between canonical bases of modules over affine Kac-Moody algebras $\mathfrak{g} = \widehat{\mathfrak{sl}}_e$, $e \in \mathbb{N} \cup \{\infty\}$, and projective indecomposable H_m-modules. Ariki was able to prove a more general statement involving finite-dimensional quotients of affine Hecke algebras of type A known as cyclotomic Hecke algebras, see Chapter 4. We start with some Cartan datum. In particular, we have a quiver Γ with vertex set I. For a dominant integral weight Λ one can define a finite-dimensional quotient of H_m^A, an affine Hecke algebra of type A, known as a cyclotomic Hecke algebra which is denoted by H_m^Λ. For Λ of level one, this is either the group algebra of the symmetric group or an Iwahori-Hecke algebra, depending on the deformation parameter. It turns out that Λ is a dominant integral weight for $\mathfrak{g} = \widehat{sl}_e$ and the finite-dimensional H_m^Λ-modules, for all $m > 0$, categorify the irreducible highest weight module $V(\Lambda)$ over $\widehat{sl}_e$ in such a way that the Chevalley generators of $\widehat{sl}_e$ correspond to induction and restriction functors on the cyclotomic Hecke algebra side. In this categorification, canonical basis elements of $V(\Lambda)$ correspond to isomorphism classes of projective indecomposable H_m^Λ-modules. Specialising Λ to level one meant Ariki had proved the LLT conjecture and, among other applications, this enabled the computation of decomposition numbers of Specht modules in characteristic 0.

Let $\mathbf{f}$ denote Lusztig's algebra associated to some Cartan datum. In particular, we have a quiver Γ with vertex set I. The algebra $\mathbf{f}$ is a $\mathbb{Q}(q)$-algebra, q an indeterminate, with generators θ_i, for $i \in I$, satisfying the Serre relations. Let $\mathscr{A} = \mathbb{Z}[q, q^{-1}]$ and ${}_{\mathscr{A}}\mathbf{f}$ be the $\mathscr{A}$-subalgebra of $\mathbf{f}$. For a given dimension vector $\tilde{\nu} \in \mathbb{N}I$ we denote the associated KLR algebra by $\mathbf{R}_{\tilde{\nu}}$. Let $\mathbf{R}_{\tilde{\nu}}$-proj denote the subcategory of $\mathbf{R}_{\tilde{\nu}}$-mod$^{\mathrm{fd}}$ consisting of all finite-dimensional projective graded $\mathbf{R}_{\tilde{\nu}}$-modules. In [KL09], [KL11] Khovanov and Lauda proved the following categorification result.

$$\mathbf{K}_0(\bigoplus_{\tilde{\nu} \in \mathbb{N}I} \mathbf{R}_{\tilde{\nu}}\text{-proj}) \cong {}_{\mathscr{A}}\mathbf{f}.$$

The left hand side of this isomorphism is an $\mathscr{A}$-algebra; the action of q and q^{-1} being grading shifts up and down, respectively. Khovanov and Lauda then conjectured a connection between Lusztig's canonical basis and the isomorphism classes of projective indecomposable modules. This was proved for all type A quivers by Brundan and Kleshchev in [BK09b] and for arbitrary simply-laced type by Varagnolo and Vasserot in [VV11b].

One can vary the type of Lie algebra involved. There is an algebra isomorphism $\mathbf{f} \cong U_q(\mathfrak{g})^-$, where $U_q(\mathfrak{g})^-$ is the negative part of the quantised universal enveloping algebra of the Lie algebra $\mathfrak{g}$ associated to the given Cartan datum. It can be shown that there is a vector space isomorphism between $\mathbf{f}$ and the Grothendieck group of the category of finite-dimensional representations of affine Hecke algebras of type A, H^A_m, in which isomorphism classes of simple modules correspond to elements of the canonical basis of $\mathbf{f}$. Moreover, Ariki proved that the action of the Chevalley generators on $\mathbf{f}$ correspond to the action of the induction and restriction functors on H^A_m-Mod$^{\mathrm{fd}}$.

Alternatively, one can vary the type of Hecke algebra. Up to this point, we have given the type A results. We now discuss the analogues to these results in a type B setting on the affine Hecke algebra side. Let H^B_m denote the affine Hecke algebra of type B. Again suppose we are given Cartan datum including a quiver Γ, with vertex set $I \subset \mathbf{k}^\times$, which depends on whether the deformation parameters $p, q \in \mathbf{k}^\times$ of H^B_m, lie in I or not. In [EK06], Enomoto and Kashiwara define a $\mathbb{Q}(q)$-algebra $\mathcal{B}_\theta(\mathfrak{g})$, where θ is an involution on I. This algebra is generated by elements E_i, F_i and invertible elements T_i, $i \in I$, which are subject to a list of relations which include the Serre relations. They then define an irreducible highest weight module $V_\theta(\lambda)$ over $\mathcal{B}_\theta(\mathfrak{g})$ and conjecture the following $\mathcal{B}_\theta(\mathfrak{g})$-module isomorphism,

$$\mathbf{K}_0(\bigoplus_n \mathrm{H}^B_n\text{-Mod}^{\mathrm{fd}}) \cong V_\theta(\lambda).$$

This can be considered as a type B analogue of the LLT conjecture. In [EK09], Enomoto and Kashiwara prove this conjecture in the case $p \in I$. In [VV11a], Varagnolo and Vasserot introduce a family of Ext-algebras associated to a quiver with an involution θ; the VV algebras. They compute these algebras explicitly and give a presentation of VV algebras in terms of generators and relations, see Section 2.3. These algebras depend on a dimension vector ν satisfying certain properties. The set of such dimension vectors is denoted ${}^{\theta}\mathbb{N}I$. They prove that categories of modules over VV algebras are equivalent to certain categories of modules over affine Hecke algebras of type B. More precisely, let

$$\mathfrak{W} := \bigoplus_{\substack{\nu \in {}^{\theta}\mathbb{N}I \\ |\nu| = 2m}} \mathfrak{W}_{\nu}.$$

Let H_m^B-Mod$_I$ denote the category of modules over H_m^B in which all eigenvalues of X_i, $1 \leq i \leq m$, lie in I. That is, $M \in H_m^B$-Mod$_I$ if and only if whenever $X_i n = \lambda n$ for some $n \in M$, $\lambda \in \mathbf{k}$ and for some $1 \leq i \leq m$, then $\lambda \in I$. In this case we say that M is of type I. Let $\mathfrak{W}$-mod$_0$ denote the category of finitely generated modules over $\mathfrak{W}$ such that the elements $x_1, \ldots, x_m$ act locally nilpotently.

Theorem 2.1.1 ([VV11a], Theorem 8.5)**.** *There is an equivalence of categories*

$$\mathfrak{W}\text{-}mod_0 \sim H_m^B\text{-}mod_I.$$

Varagnolo and Vasserot then use this equivalence of categories to prove the conjecture of Enomoto and Kashiwara in complete generality, providing a type B analogue to the type A picture.

Theorem 2.1.1 implies that in order to study categories of modules over type B affine Hecke algebras it suffices to study categories of modules over VV algebras. This is another reason motivating the study of VV algebras. The VV algebras are graded algebras whereas affine Hecke algebras are not. Furthermore, the combinatorial description of the generators and relations makes the study of VV algebras more accessible and easier to work with in comparison to affine Hecke algebras.

2.2 KLR Algebras

In this section we define a family of algebras which were introduced by Khovanov, Lauda and independently by Rouquier. They are known as KLR algebras, or sometimes as quiver Hecke algebras.

Fix an element $p \in \mathbf{k}^{\times}$. Define an action of $\mathbb{Z}$ on $\mathbf{k}^{\times}$ as follows,

$$n \cdot \lambda = p^{2n}\lambda.$$

Let $\tilde{I}$ be a $\mathbb{Z}$-orbit. So $\tilde{I} = \tilde{I}_{\lambda}$ is the $\mathbb{Z}$-orbit of λ,

$$\tilde{I} = \tilde{I}_{\lambda} = \{p^{2n}\lambda \mid n \in \mathbb{Z}\}.$$

To $\tilde{I}$ we associate a quiver $\tilde{\Gamma} = \tilde{\Gamma}_{\tilde{I}}$. The vertices of $\tilde{\Gamma}$ are the elements $i \in \tilde{I}$ and we have arrows $p^2 i \longrightarrow i$ for every $i \in \tilde{I}$.

We always assume that $\pm 1 \notin \tilde{I}$ and that $p \neq \pm 1$.

If p is not a root of unity then $\tilde{\Gamma}_{\tilde{I}}$ has the following form.

$$\cdots \longrightarrow p^4\lambda \longrightarrow p^2\lambda \longrightarrow \lambda \longrightarrow p^{-2}\lambda \longrightarrow p^{-4}\lambda \longrightarrow \cdots$$

If $p^n = 1$, for some positive integer n, then $\tilde{\Gamma}_{\tilde{I}}$ is of the following form.

$$\begin{tikzcd}
 & \lambda \arrow[dr] & \\
p^2\lambda \arrow[ur] & & p^{n-2}\lambda \arrow[d] \\
\cdots \arrow[u] & & \cdots
\end{tikzcd}$$

Now define $\mathbb{N}\tilde{I} = \{\tilde{\nu} = \sum_{i \in \tilde{I}} \tilde{\nu}_i i \mid \tilde{\nu} \text{ has finite support}, \tilde{\nu}_i \in \mathbb{Z}_{\geq 0} \quad \forall i\}$. Elements $\tilde{\nu} \in \mathbb{N}\tilde{I}$ are called **dimension vectors**. For $\tilde{\nu} \in \mathbb{N}\tilde{I}$, the **height** of $\tilde{\nu}$ is defined to be

$$|\tilde{\nu}| = \sum_{i \in \tilde{I}} \tilde{\nu}_i.$$

For $\tilde{\nu} \in \mathbb{N}\tilde{I}$ with $|\tilde{\nu}| = m$, define

$$\tilde{I}^{\tilde{\nu}} := \{\mathbf{i} = (i_1, \ldots, i_m) \in \tilde{I}^m \mid \sum_{k=1}^{m} i_k = \tilde{\nu}\}.$$

Definition 2.2.1. $\tilde{\nu} = \sum_{i \in \tilde{I}} \tilde{\nu}_i i \in \mathbb{N}\tilde{I}$ is said to have **multiplicity one** if $\tilde{\nu}_i \leq 1$ for every $i \in \tilde{I}$. We say that $j \in \tilde{I}$ has multiplicity one in $\tilde{\nu}$, or j appears with multiplicity one in $\tilde{\nu}$, if the coefficient of j in $\tilde{\nu}$ is 1, i.e. if $\tilde{\nu}_j = 1$.

Example 2.2.2. $\tilde{\nu}_1 = \lambda + p^2\lambda \in \mathbb{N}\tilde{I}_\lambda$ has multiplicity one, while $\tilde{\nu}_2 = 2\lambda + p^2\lambda$ does not have multiplicity one. In the latter example, $p^2\lambda$ appears with multiplicity one in $\tilde{\nu}_2$.

Definition 2.2.3. For $\tilde{\nu} \in \mathbb{N}\tilde{I}$ with $|\tilde{\nu}| = m$ the **KLR algebra**, denoted by $\mathbf{R}_{\tilde{\nu}}$, is the graded $\mathbf{k}$-algebra generated by elements

$$\{x_1, \ldots, x_m\} \cup \{\sigma_1, \ldots, \sigma_{m-1}\} \cup \{\mathbf{e}(\mathbf{i}) \mid \mathbf{i} \in \tilde{I}^{\tilde{\nu}}\}$$

which are subject to the following relations.

1. $\mathbf{e}(\mathbf{i})\mathbf{e}(\mathbf{j}) = \delta_{\mathbf{ij}}\mathbf{e}(\mathbf{i}), \quad \sigma_k\mathbf{e}(\mathbf{i}) = \mathbf{e}(s_k\mathbf{i})\sigma_k, \quad x_l\mathbf{e}(\mathbf{i}) = \mathbf{e}(\mathbf{i})x_l, \quad \sum_{\mathbf{i} \in \tilde{I}^{\tilde{\nu}}} \mathbf{e}(\mathbf{i}) = 1.$

2. The x_l's commute.

3. $\sigma_k^2\mathbf{e}(\mathbf{i}) = \begin{cases} \mathbf{e}(\mathbf{i}) & i_k \not\leftrightarrow i_{k+1} \\ (x_{k+1} - x_k)\mathbf{e}(\mathbf{i}) & i_k \leftarrow i_{k+1} \\ (x_k - x_{k+1})\mathbf{e}(\mathbf{i}) & i_k \to i_{k+1} \\ (x_{k+1} - x_k)(x_k - x_{k+1})\mathbf{e}(\mathbf{i}) & i_k \leftrightarrow i_{k+1} \\ 0 & i_k = i_{k+1} \end{cases}$

$\sigma_j\sigma_k = \sigma_k\sigma_j \quad \text{for } j \neq k \pm 1$

$$(\sigma_{k+1}\sigma_k\sigma_{k+1} - \sigma_k\sigma_{k+1}\sigma_k)\mathbf{e}(\mathbf{i}) = \begin{cases} 0 & i_k \neq i_{k+2} \text{ or } i_k \not\leftrightarrow i_{k+1} \\ \mathbf{e}(\mathbf{i}) & i_k = i_{k+2} \text{ and } i_k \to i_{k+1} \\ -\mathbf{e}(\mathbf{i}) & i_k = i_{k+2} \text{ and } i_k \leftarrow i_{k+1} \\ (2x_{k+1} - x_{k+2} - x_k)\mathbf{e}(\mathbf{i}) & i_k = i_{k+2} \text{ and } i_k \leftrightarrow i_{k+1}. \end{cases}$$

4. $(\sigma_k x_l - x_{s_k(l)}\sigma_k)\mathbf{e}(\mathbf{i}) = \begin{cases} -\mathbf{e}(\mathbf{i}) & \text{if } l = k, i_k = i_{k+1} \\ \mathbf{e}(\mathbf{i}) & \text{if } l = k+1, i_k = i_{k+1} \\ 0 & \text{else.} \end{cases}$

The grading on $\mathbf{R}_{\tilde{\nu}}$ is given as follows.

$$\deg(\mathbf{e}(\mathbf{i})) = 0$$
$$\deg(x_l\mathbf{e}(\mathbf{i})) = 2$$
$$\deg(\sigma_k\mathbf{e}(\mathbf{i})) = \begin{cases} |i_k \to i_{k+1}| + |i_{k+1} \to i_k| & \text{if } i_k \neq i_{k+1} \\ -2 & \text{if } i_k = i_{k+1} \end{cases}$$

where $|i_k \to i_{k+1}|$ denotes the number of arrows from i_k to i_{k+1} in the quiver $\tilde{\Gamma}$.

If $\tilde{\nu} = 0$ we set $\mathbf{R}_{\tilde{\nu}} = \mathbf{k}$ as a graded $\mathbf{k}$-algebra.

Remark 2.2.4. In this thesis the underlying quiver $\tilde{\Gamma}$ for KLR algebras is always of type A. In the literature however, KLR algebras are defined more generally. In general, one starts with any loop-free quiver $\tilde{\Gamma}$ which has vertex set $\tilde{I}$. Then KLR algebras can be defined as above with the relations depending upon the choice of underlying quiver $\tilde{\Gamma}$. Here, we are restricting ourselves to the quiver $\tilde{\Gamma}$ that is described above.

Examples

Example 2.2.5. Take $\tilde{\nu} = \lambda \in \mathbb{N}\tilde{I}$. $\mathbf{R}_{\tilde{\nu}}$ is generated by $\mathbf{e}(\lambda)$, x_1. So the KLR algebra associated to $\tilde{\nu}$ in this example is just a polynomial ring $\mathbf{k}[x]$.

Example 2.2.6. Take $\tilde{\nu} = \lambda + p^2\lambda \in \mathbb{N}\tilde{I}$. $\mathbf{R}_{\tilde{\nu}}$ has generators $\mathbf{e}(\lambda, p^2\lambda)$, $\mathbf{e}(p^2\lambda, \lambda)$, x_1, x_2, σ_1. We can represent this algebra with the following quiver.

$$\begin{tikzcd}
\mathbf{e}(\lambda, p^2\lambda) \arrow[loop left, "x_1, x_2"] \arrow[r, bend left, "\sigma_1"] & \mathbf{e}(p^2\lambda, \lambda) \arrow[loop right, "x_1, x_2"] \arrow[l, bend left, "\sigma_1"]
\end{tikzcd}$$

Indeed, the path algebra of this quiver modulo the given relations is the KLR algebra $\mathbf{R}_{\tilde{\nu}}$.

Example 2.2.7. Take $\tilde{\nu} = \lambda + p^2\lambda + p^4\lambda \in \mathbb{N}\tilde{I}$. $\mathbf{R}_{\tilde{\nu}}$ is the path algebra of the following quiver modulo the relations. To reduce notation the paths of x_1, x_2 and x_3 at each $\mathbf{e}(\mathbf{i})$ have been omitted.

$$\begin{tikzcd}
\mathbf{e}(\lambda, p^2\lambda, p^4\lambda) \arrow[r, bend left, "\sigma_2"] \arrow[d, bend left, "\sigma_1"] & \mathbf{e}(\lambda, p^4\lambda, p^2\lambda) \arrow[l, bend left, "\sigma_2"] \arrow[d, bend left, "\sigma_1"] \\
\mathbf{e}(p^2\lambda, \lambda, p^4\lambda) \arrow[u, bend left, "\sigma_1"] \arrow[d, bend left, "\sigma_2"] & \mathbf{e}(p^4\lambda, \lambda, p^2\lambda) \arrow[u, bend left, "\sigma_1"] \arrow[d, bend left, "\sigma_2"] \\
\mathbf{e}(p^2\lambda, p^4\lambda, \lambda) \arrow[u, bend left, "\sigma_2"] \arrow[r, bend left, "\sigma_1"] & \mathbf{e}(p^4\lambda, p^2\lambda, \lambda) \arrow[u, bend left, "\sigma_2"] \arrow[l, bend left, "\sigma_1"]
\end{tikzcd}$$

Example 2.2.8. In each of the examples above, $\tilde{\nu}$ has multiplicity one. Now take $\tilde{\nu} = 2\lambda + p^2\lambda \in \mathbb{N}\tilde{I}$, in which λ appears with multiplicity two. The corresponding quiver is, again with the generators x_i omitted at each idempotent $\mathbf{e}(\mathbf{i})$,

$$\begin{tikzcd}
\mathbf{e}(\lambda, \lambda, p^2\lambda) \arrow[loop left, "\sigma_1"] \arrow[r, bend left, "\sigma_2"] & \mathbf{e}(\lambda, p^2\lambda, \lambda) \arrow[l, bend left, "\sigma_2"] \arrow[r, bend left, "\sigma_1"] & \mathbf{e}(p^2\lambda, \lambda, \lambda) \arrow[l, bend left, "\sigma_1"] \arrow[loop right, "\sigma_2"]
\end{tikzcd}$$

Example 2.2.9. Take $\tilde{\nu} = n\lambda \in \mathbb{N}\tilde{I}_\lambda$ for some $n \in \mathbb{N}$. The associated KLR algebra $\mathbf{R}_{\tilde{\nu}}$ has generators $\mathbf{e}(n\lambda), x_1, \ldots, x_n, \sigma_1, \ldots, \sigma_{n-1}$, with the following relations.

$$x_i x_j = x_j x_i, \qquad \sigma_i^2 = 0,$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i - j| > 1, \qquad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1},$$

$$(\sigma_k x_l - x_{s_k(l)} \sigma_k)\mathbf{e}(n\lambda) = \begin{cases} -\mathbf{e}(n\lambda) & \text{if } l = k \\ \mathbf{e}(n\lambda) & \text{if } l = k+1 \\ 0 & \text{else.} \end{cases}$$

In this case, $\mathbf{R}_{\tilde{\nu}}$ is isomorphic to ${}^0H_n$, the affine nil Hecke algebra of type A_n.

At this point we introduce some notation. Take any $\tilde{\nu} \in \mathbb{N}\tilde{I}$, of height $|\tilde{\nu}| = n$, and the associated KLR algebra $\mathbf{R}_{\tilde{\nu}}$. Whenever we work with the KLR algebra $\mathbf{R}_{\tilde{\nu}}$, for each $w \in \mathfrak{S}_n$, we must choose and fix a reduced expression of w, say $w = s_{i_1} \cdots s_{i_r}$ where $1 \leq i_k < n$, for all k. Then define $\sigma_{\dot{w}} \in \mathbf{R}_{\tilde{\nu}}$ as follows.

$$\sigma_{\dot{w}}\mathbf{e}(\mathbf{i}) = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_r}\mathbf{e}(\mathbf{i})$$
$$\sigma_{\dot{1}}\mathbf{e}(\mathbf{i}) = \mathbf{e}(\mathbf{i})$$

where 1 denotes the identity element in $\mathfrak{S}_n$. Note that reduced expressions of w are not always unique and so $\sigma_{\dot{w}}$ depends upon the choice of reduced expression of w. For example, $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ in $\mathfrak{S}_n$, whereas the defining relations for the KLR algebra state that we do not always have $\sigma_i\sigma_{i+1}\sigma_i\mathbf{e}(\mathbf{i}) = \sigma_{i+1}\sigma_i\sigma_{i+1}\mathbf{e}(\mathbf{i})$. In particular, this means that for elements $w_1, w_2 \in \mathfrak{S}_n$, $w_1 = w_2$ does not necessarily imply $\sigma_{\dot{w}_1}\mathbf{e}(\mathbf{i}) = \sigma_{\dot{w}_2}\mathbf{e}(\mathbf{i})$.

Note 2.2.10. If we have fixed a reduced expression for w we will usually omit the dot and write σ_w for $\sigma_{\dot{w}}$.

Lemma 2.2.11 (Basis Theorem for KLR Algebras)**.** *[[KL09], Theorem 2.5] Take $\tilde{\nu} \in \mathbb{N}\tilde{I}$ with $|\tilde{\nu}| = m$. The elements*

$$\{\sigma_{\dot{w}} x_1^{n_1} \cdots x_m^{n_m} \boldsymbol{e}(\boldsymbol{i}) \mid w \in \mathfrak{S}_m, \boldsymbol{i} \in I^{\tilde{\nu}}, n_i \in \mathbb{N}_0 \ \forall i\}$$

form a $\boldsymbol{k}$-basis for $\boldsymbol{R}_{\tilde{\nu}}$.

2.2.1 Root Partitions Associated to $\mathbf{R}_{\tilde{\nu}}$

To each $i \in \tilde{I}$ we associate an integer n; the power of p at that vertex. For example, to $p^2\lambda$ we associate 2. To $p^{-4}\lambda$ we associate -4. If $p \in \tilde{I}$; to the vertex p^{2k+1}, $k \in \mathbb{Z}$ we associate $2k+1$. So $i \in \tilde{I}$ is identified with an integer and refers to the power of p at that vertex. The order on $\tilde{I}$ is the natural order on $\mathbb{Z}$.

As discussed in Section 1.6, given this data we can define a Cartan matrix $(a_{ij})_{i,j\in\tilde{I}}$. Consider the Lie algebra associated to this Cartan matrix. We have simple roots in type A given by $\{\alpha_i \mid i \in \tilde{I}\}$, and positive roots $\alpha_{i,i+2k} := \alpha_i + \alpha_{i+2} + \cdots + \alpha_{i+2k}$, $k \in \mathbb{Z}_{\geq 0}$.

Given $\tilde{\nu} \in \mathbb{N}\tilde{I}$ it is now clear what we mean by a root partition of $\tilde{\nu}$. A **root partition** of $\tilde{\nu}$ is a tuple of positive roots $(\beta_1, \beta_2, \beta_3, \ldots, \beta_r)$ such that $\tilde{\nu} = \beta_1 + \beta_2 + \beta_3 + \cdots + \beta_r$ and such that $\beta_1 \geq \beta_2 \geq \beta_3 \geq \cdots \geq \beta_r$.

A root partition of $\tilde{\nu}$ will often be denoted by π and the set of all root partitions of $\tilde{\nu}$ will be denoted by $\Pi(\tilde{\nu})$.

Example 2.2.12. Take $\tilde{\nu} = \lambda + p^2\lambda + p^4\lambda \in \mathbb{N}\tilde{I}_\lambda$. Put $\beta_1 = p^2\lambda + p^4\lambda$ and $\beta_2 = \lambda$. Then (β_1, β_2) is a root partition of $\tilde{\nu}$. Setting $\beta_1 = p^4\lambda$ and $\beta_2 = \lambda + p^2\lambda$ gives another root partition of $\tilde{\nu}$.

Example 2.2.13. Take $\tilde{\nu} = \lambda+p^2\lambda+p^4\lambda+p^6\lambda \in \mathbb{N}\tilde{I}_\lambda$. $(\lambda+p^2\lambda+p^4\lambda+p^6\lambda)$, $(p^6\lambda, \lambda+p^2\lambda+p^4\lambda)$, $(p^6\lambda, p^2\lambda+p^4\lambda, \lambda)$, are root partitions of $\tilde{\nu}$. Denote them as π_1, π_2, π_3, respectively. We have

$$\pi_1 < \pi_2 < \pi_3$$

and $\mathbf{i}_{\pi_1} = (\lambda, p^2\lambda, p^4\lambda, p^6\lambda)$, $\mathbf{i}_{\pi_2} = (p^6\lambda, \lambda, p^2\lambda, p^4\lambda)$, $\mathbf{i}_{\pi_3} = (p^6\lambda, p^2\lambda, p^4\lambda, \lambda)$.

Given $\tilde{\nu} \in \mathbb{N}\tilde{I}$ and the associated KLR algebra $\mathbf{R}_{\tilde{\nu}}$, we have a set $\{\mathbf{e}(\mathbf{i}) \mid \mathbf{i} \in \tilde{I}^{\tilde{\nu}}\}$ of idempotents. Each $\mathbf{e}(\mathbf{i})$ is labelled by a sequence of integers $\mathbf{i} = (i_1, \ldots, i_m) \in \tilde{I}^m$ which may or may not correspond to a root partition in the way we describe above. So, associated to each KLR algebra $\mathbf{R}_{\tilde{\nu}}$ we obtain a set of root partitions of $\tilde{\nu}$. In fact, since all permutations of $(i_1, \ldots, i_m)$ lie in $\tilde{I}^m$, we obtain a complete set of root partitions of $\tilde{\nu}$. If $\mathbf{i} = (i_1, \ldots, i_m)$ corresponds to a root partition $\pi \in \Pi(\tilde{\nu})$ then we will sometimes write $\mathbf{e}(\mathbf{i}_\pi) = \mathbf{e}(\mathbf{i})$ to emphasise this.

Example 2.2.14. Consider the KLR algebra associated to the dimension vector $\tilde{\nu} = \lambda + p^2\lambda + p^4\lambda$. It has the following quiver presentation (with the generators x_i omitted):

$$\begin{tikzcd}
(\lambda, p^2\lambda, p^4\lambda) \arrow[r, bend left, "\sigma_2"] \arrow[d, bend left, "\sigma_1"] & (\lambda, p^4\lambda, p^2\lambda) \arrow[l, bend left, "\sigma_2"] \arrow[d, bend left, "\sigma_1"] \\
(p^2\lambda, \lambda, p^4\lambda) \arrow[u, bend left, "\sigma_1"] \arrow[d, bend left, "\sigma_2"] & (p^4\lambda, \lambda, p^2\lambda) \arrow[u, bend left, "\sigma_1"] \arrow[d, bend left, "\sigma_2"] \\
(p^2\lambda, p^4\lambda, \lambda) \arrow[u, bend left, "\sigma_2"] \arrow[r, bend left, "\sigma_1"] & (p^4\lambda, p^2\lambda, \lambda) \arrow[u, bend left, "\sigma_2"] \arrow[l, bend left, "\sigma_1"]
\end{tikzcd}$$

The set of root partitions associated to $\mathbf{R}_{\tilde{\nu}}$ is,

$$\Pi(\tilde{\nu}) = \{(0, 2, 4), (2, 4, 0), (4, 0, 2), (4, 2, 0)\}$$

with $(0, 2, 4) < (2, 4, 0) < (4, 0, 2) < (4, 2, 0)$.

Example 2.2.15. Consider the KLR algebra associated to the dimension vector $\tilde{\nu} = 2\lambda + p^2\lambda$. It has the following quiver presentation (with the generators x_i omitted):

$$\begin{tikzcd}
\mathbf{e}(\lambda, \lambda, p^2\lambda) \arrow[loop left, "\sigma_1"] \arrow[r, bend left, "\sigma_2"] & \mathbf{e}(\lambda, p^2\lambda, \lambda) \arrow[l, bend left, "\sigma_2"] \arrow[r, bend left, "\sigma_1"] & \mathbf{e}(p^2\lambda, \lambda, \lambda) \arrow[l, bend left, "\sigma_1"] \arrow[loop right, "\sigma_2"]
\end{tikzcd}$$

The set of root partitions associated to $\mathbf{R}_{\tilde{\nu}}$ is

$$\Pi(\tilde{\nu}) = \{(0,2,0),(2,0,0)\}$$

with $(0,2,0) < (2,0,0)$.

2.3 VV Algebras

In this section we define the family of algebras of which this thesis is concerned; the VV algebras. They were introduced by Varagnolo and Vasserot in [VV11a].

Fix elements $p, q \in \mathbf{k}^{\times}$. Assume that p is not a power of q and that q is not a power of p. Define an action of $\mathbb{Z} \rtimes \{\pm 1\} = \mathbb{Z} \rtimes \mathbb{Z}_2$ on $\mathbf{k}^{\times}$ as follows.

$$(n, \varepsilon) \cdot \lambda = p^{2n} \lambda^{\varepsilon}.$$

Let I be a $\mathbb{Z} \rtimes \{\pm 1\}$-orbit. So $I = I_\lambda$ is the $\mathbb{Z} \rtimes \{\pm 1\}$-orbit of λ.

$$I = I_\lambda = \{p^{2n}\lambda^{\pm 1} \mid n \in \mathbb{Z}\}.$$

To I we associate a quiver $\Gamma = \Gamma_I$ together with an involution θ. The vertices of Γ are the elements $i \in I$ and we have arrows $p^2 i \longrightarrow i$ for every $i \in I$. The involution θ is defined by

$$\begin{aligned} \theta(i) &= i^{-1} \\ \theta(p^2 i \longrightarrow i) &= p^{-2} i^{-1} \longleftarrow i^{-1}, \qquad \text{for all } i \in I. \end{aligned}$$

We always assume that $\pm 1 \notin I$ and that $p \neq \pm 1$. This implies that θ has no fixed points and that Γ has no loops (1-cycles).

Now define ${}^{\theta}\mathbb{N}I := \{\nu = \sum_{i \in I} \nu_i i \mid \nu \text{ has finite support}, \nu_i \in \mathbb{Z}_{\geq 0}, \nu_i = \nu_{\theta(i)} \ \ \forall i\}$. In particular, for each $\nu \in {}^{\theta}\mathbb{N}I$, the coefficients of i and i^{-1} in ν must be equal. Elements $\nu \in {}^{\theta}\mathbb{N}I$ are called **dimension vectors**. For $\nu \in {}^{\theta}\mathbb{N}I$ the **height** of ν is defined to be,

$$|\nu| = \sum_{i \in I} \nu_i.$$

The shape of Γ depends on whether $p \in I$ or $p \notin I$, as well as whether or not p is a root of unity.

- Suppose $p \notin I$. Let $I_\lambda^+ := \{p^{2n}\lambda \mid n \in \mathbb{Z}\}$, $I_\lambda^- := \{p^{2n}\lambda^{-1} \mid n \in \mathbb{Z}\}$. So $I_\lambda = I_\lambda^- \sqcup I_\lambda^+$. If p is not a root of unity then Γ_I has the form,

$$\cdots \longrightarrow p^2\lambda \longrightarrow \lambda \longrightarrow p^{-2}\lambda \longrightarrow \cdots$$

$$\cdots \longleftarrow p^{-2}\lambda^{-1} \longleftarrow \lambda^{-1} \longleftarrow p^{2}\lambda^{-1} \longleftarrow \cdots$$

If $p^n = 1$ for some positive integer m then Γ_I is of the form,

$$\begin{tikzcd} & \lambda \arrow[dr] & \\ p^2\lambda \arrow[ur] & & p^{n-2}\lambda \arrow[d] \\ \ddots \arrow[u] & & \cdots \end{tikzcd} \qquad \begin{tikzcd} & \lambda^{-1} \arrow[dl] & \\ p^{-2}\lambda^{-1} \arrow[d] & & p^{2-n}\lambda^{-1} \arrow[ul] \\ \ddots & & \cdots \arrow[u] \end{tikzcd}$$

- Now suppose $p \in I$. Let $I_p^+ := \{p^{2n+1} \mid n \in \mathbb{Z}_{\geq 0}\}$, $I_p^- := \{p^{2n-1} \mid n \in \mathbb{Z}_{\leq 0}\}$. So $I_p = I_p^- \sqcup I_p^+$, provided p is not a root of unity in which case we have $I_p^- = I_p^+$. If p is not a root of unity then Γ_I is of the form,

$$\begin{tikzcd} \cdots \arrow[r] & p^5 \arrow[r] & p^3 \arrow[r] & p \arrow[d] \\ \cdots & p^{-5} \arrow[l] & p^{-3} \arrow[l] & p^{-1} \arrow[l] \end{tikzcd}$$

If $p^m = 1$ for some positive integer m then Γ_I has the form,

$$\begin{tikzcd} & p \arrow[dr] & \\ p^3 \arrow[ur] & & p^{m-1} \arrow[d] \\ \ddots \arrow[u] & & \cdots \end{tikzcd}$$

Note that for $\nu \in {}^\theta\mathbb{N}I$, $|\nu| = 2m$ for some positive integer m. For $\nu \in {}^\theta\mathbb{N}I$ with $|\nu| = 2m$, define

$${}^\theta I^\nu := \{\mathbf{i} = (i_1, \ldots, i_m) \in I^m \mid \sum_{k=1}^{m} i_k + \sum_{k=1}^{m} i_k^{-1} = \nu\}.$$

Definition 2.3.1. A dimension vector $\nu = \sum_{i\in I} \nu_i i \in {}^\theta\mathbb{N}I$ is said to have **multiplicity one** if $\nu_i \leq 1$ for every $i \in I$. We say that $j \in I$ has multiplicity one in ν, or j appears with multiplicity one in ν, if the coefficient of j in ν is 1, i.e. if $\nu_j = 1$.

Example 2.3.2. The dimension vector $\nu_1 = p^{-2}q^{-1} + q^{-1} + q + p^2q \in {}^\theta\mathbb{N}I_q$ has multiplicity one, while $\nu_2 = p^{-2}q^{-1} + 2q^{-1} + 2q + p^2q$ does not have multiplicity one. In the latter example, p^2q appears with multiplicity one in ν_2.

Remark 2.3.3. We remark here that, given this data, we can again define KLR algebras. That is, given a $\mathbb{Z} \rtimes \mathbb{Z}_2$-orbit I_λ and associated quiver Γ_{I_λ} we can pick $\tilde{\nu} \in \mathbb{N}I_\lambda$ which yields a KLR algebra $\mathbf{R}_{\tilde{\nu}}$. This KLR algebra is defined by the generators and relations given in 2.2 with the understanding that there are no arrows between vertices belonging to different branches of Γ_{I_λ}.

Definition 2.3.4. For $\nu \in {}^\theta\mathbb{N}I$ with $|\nu| = 2m$ the **VV algebra**, denoted by $\mathfrak{W}_\nu$, is the graded **k**-algebra generated by elements

$$\{x_1, \dots, x_m\} \cup \{\sigma_1, \dots, \sigma_{m-1}\} \cup \{\mathbf{e}(\mathbf{i}) \mid \mathbf{i} \in {}^\theta I^\nu\} \cup \{\pi\}$$

which are subject to the following relations.

1. $\mathbf{e}(\mathbf{i})\mathbf{e}(\mathbf{j}) = \delta_{\mathbf{ij}}\mathbf{e}(\mathbf{i}), \quad \sigma_k\mathbf{e}(\mathbf{i}) = \mathbf{e}(s_k\mathbf{i})\sigma_k, \quad \sum_{\mathbf{i}\in{}^\theta I^\nu}\mathbf{e}(\mathbf{i}) = 1,$

   $x_l\mathbf{e}(\mathbf{i}) = \mathbf{e}(\mathbf{i})x_l, \quad \pi\mathbf{e}(i_1, \dots, i_m) = \mathbf{e}(\theta(i_1), i_2 \dots, i_m)\pi = \mathbf{e}(i_1^{-1}, i_2 \dots, i_m)\pi.$

2. $\pi^2\mathbf{e}(\mathbf{i}) = \begin{cases} x_1\mathbf{e}(\mathbf{i}) & i_1 = q \\ -x_1\mathbf{e}(\mathbf{i}) & i_1 = q^{-1} \\ \mathbf{e}(\mathbf{i}) & i_1 \neq q^{\pm 1}. \end{cases}$

3. The x_l's commute.

4. $\sigma_k^2\mathbf{e}(\mathbf{i}) = \begin{cases} \mathbf{e}(\mathbf{i}) & i_k \not\leftrightarrow i_{k+1} \\ (x_{k+1} - x_k)\mathbf{e}(\mathbf{i}) & i_k \leftarrow i_{k+1} \\ (x_k - x_{k+1})\mathbf{e}(\mathbf{i}) & i_k \rightarrow i_{k+1} \\ (x_{k+1} - x_k)(x_k - x_{k+1})\mathbf{e}(\mathbf{i}) & i_k \leftrightarrow i_{k+1} \\ 0 & i_k = i_{k+1} \end{cases}$

   $\sigma_j\sigma_k = \sigma_k\sigma_j \quad \text{for } j \neq k \pm 1$

   $(\sigma_{k+1}\sigma_k\sigma_{k+1} - \sigma_k\sigma_{k+1}\sigma_k)\mathbf{e}(\mathbf{i}) = \begin{cases} 0 & i_k \neq i_{k+2} \text{ or } i_k \not\leftrightarrow i_{k+1} \\ \mathbf{e}(\mathbf{i}) & i_k = i_{k+2} \text{ and } i_k \rightarrow i_{k+1} \\ -\mathbf{e}(\mathbf{i}) & i_k = i_{k+2} \text{ and } i_k \leftarrow i_{k+1} \\ (2x_{k+1} - x_{k+2} - x_k)\mathbf{e}(\mathbf{i}) & i_k = i_{k+2} \text{ and } i_k \leftrightarrow i_{k+1}. \end{cases}$

5. $\pi x_1 = -x_1\pi$

   $\pi x_l = x_l\pi \quad \text{for all} \quad l > 1.$

6. $(\sigma_1\pi)^2\mathbf{e}(\mathbf{i}) - (\pi\sigma_1)^2\mathbf{e}(\mathbf{i}) = \begin{cases} 0 & i_1^{-1} \neq i_2 \text{ or if } i_1 \neq q^{\pm 1} \\ \sigma_1\mathbf{e}(\mathbf{i}) & i_1^{-1} = i_2 = q^{-1} \\ -\sigma_1\mathbf{e}(\mathbf{i}) & i_1^{-1} = i_2 = q \end{cases}$

   $\pi\sigma_k = \sigma_k\pi$ for all $k \neq 1$.

7. $(\sigma_k x_l - x_{s_k(l)}\sigma_k)\mathbf{e}(\mathbf{i}) = \begin{cases} -\mathbf{e}(\mathbf{i}) & \text{if } l = k, i_k = i_{k+1} \\ \mathbf{e}(\mathbf{i}) & \text{if } l = k+1, i_k = i_{k+1} \\ 0 & \text{else.} \end{cases}$

The grading on $\mathfrak{W}_\nu$ is defined as follows.

$$\begin{aligned}
&\deg(\mathbf{e}(\mathbf{i})) = 0\\
&\deg(x_l\mathbf{e}(\mathbf{i})) = 2\\
&\deg(\pi\mathbf{e}(\mathbf{i})) = \begin{cases} 1 & \text{if } i_1 = q^{\pm 1}\\ 0 & \text{if } i_1 \neq q^{\pm 1}\end{cases}\\
&\deg(\sigma_k\mathbf{e}(\mathbf{i})) = \begin{cases} |i_k \to i_{k+1}| + |i_{k+1} \to i_k| & \text{if } i_k \neq i_{k+1}\\ -2 & \text{if } i_k = i_{k+1}.\end{cases}
\end{aligned}$$

where $|i_k \to i_{k+1}|$ denotes the number of arrows from i_k to i_{k+1} in the quiver Γ.

If $\nu = 0$ we set $\mathfrak{W}_\nu = \mathbf{k}$ as a graded $\mathbf{k}$-algebra.

These relations indicate that VV algebras are closely related to KLR algebras. Each VV algebra has an additional generator π which we can think of as being analogous to the Weyl group of type B_n having one more generator s_0 than the symmetric group $\mathfrak{S}_n$.

Examples

Example 2.3.5. Take $\nu = \lambda^{-1} + \lambda \in {}^\theta\mathbb{N}I$. $\mathfrak{W}_\nu$ is generated by $\mathbf{e}(\lambda)$, $\mathbf{e}(\lambda^{-1})$, x_1, π. $\mathfrak{W}_\nu$ is the path algebra of the following quiver, modulo the defining relations.

$$\begin{tikzcd}
\mathbf{e}(\lambda) \arrow[loop left, "x_1"] \arrow[r, bend left, "\pi"] & \mathbf{e}(\lambda^{-1}) \arrow[l, bend left, "\pi"] \arrow[loop right, "x_1"]
\end{tikzcd}$$

Example 2.3.6. Take $\nu = p^{-2}\lambda^{-1} + \lambda^{-1} + \lambda + p^2\lambda \in {}^\theta\mathbb{N}I$. $\mathfrak{W}_\nu$ is the path algebra of the following quiver, modulo the defining relations. Note that we have omitted the loops x_1 and x_2 at each idempotent $\mathbf{e}(\mathbf{i})$.

$$
\begin{tikzcd}
\mathbf{e}(\lambda, p^2\lambda) \arrow[r, bend left, "\sigma_1"] \arrow[d, bend left, "\pi"] & \mathbf{e}(p^2\lambda, \lambda) \arrow[l, bend left, "\sigma_1"] \arrow[d, bend left, "\pi"] \\
\mathbf{e}(\lambda^{-1}, p^2\lambda) \arrow[u, bend left, "\pi"] \arrow[d, bend left, "\sigma_1"] & \mathbf{e}(p^{-2}\lambda^{-1}, \lambda) \arrow[u, bend left, "\pi"] \arrow[d, bend left, "\sigma_1"] \\
\mathbf{e}(p^2\lambda, \lambda^{-1}) \arrow[u, bend left, "\sigma_1"] \arrow[d, bend left, "\pi"] & \mathbf{e}(\lambda, p^{-2}\lambda^{-1}) \arrow[u, bend left, "\sigma_1"] \arrow[d, bend left, "\pi"] \\
\mathbf{e}(p^{-2}\lambda^{-1}, \lambda^{-1}) \arrow[u, bend left, "\pi"] \arrow[r, bend left, "\sigma_1"] & \mathbf{e}(\lambda^{-1}, p^{-2}\lambda^{-1}) \arrow[u, bend left, "\pi"] \arrow[l, bend left, "\sigma_1"]
\end{tikzcd}
$$

Example 2.3.7. Take $\nu = 2p^{-1} + 2p \in {}^{\theta}\mathbb{N}I_p$, so p appears with multiplicity two. The corresponding quiver is, again with paths x_1 and x_2 omitted,

$$
\begin{tikzcd}
\mathbf{e}(p,p) \arrow[loop left, "\sigma_1"] \arrow[r, bend left, "\pi"] & \mathbf{e}(p^{-1},p) \arrow[l, bend left, "\pi"] \arrow[r, bend left, "\sigma_1"] & \mathbf{e}(p,p^{-1}) \arrow[l, bend left, "\sigma_1"] \arrow[r, bend left, "\pi"] & \mathbf{e}(p^{-1},p^{-1}) \arrow[l, bend left, "\pi"] \arrow[loop right, "\sigma_1"]
\end{tikzcd}
$$

Remark 2.3.8. Every $\nu \in {}^{\theta}\mathbb{N}I$ can be written as

$$\nu = \sum_{i \in I^+} \nu_i i + \sum_{i \in I^-} \nu_i i.$$

Setting $\tilde{\nu} = \sum_{i \in I^+} \nu_i i \in \mathbb{N}I^+$ defines a KLR algebra. Denote the KLR algebra associated to $\tilde{\nu} \in \mathbb{N}I^+$ by $\mathbf{R}^+_{\tilde{\nu}}$. For the remainder of this thesis, for $\nu = \sum_{i \in I^+} \nu_i i + \sum_{i \in I^-} \nu_i i \in {}^{\theta}\mathbb{N}I$, $\tilde{\nu}$ will be used to denote $\sum_{i \in I^+} \nu_i i \in \mathbb{N}I^+$. Sometimes we write $\tilde{\nu} = \tilde{\nu}^+$ to make this explicit.

Similarly, setting $\tilde{\nu}^- = \sum_{i \in I^-} \nu_i i \in \mathbb{N}I^-$ also defines a KLR algebra. Denote the KLR algebra associated to $\tilde{\nu}^- \in \mathbb{N}I^-$ by $\mathbf{R}^-_{\tilde{\nu}}$.

The relations above, and therefore the algebras $\mathfrak{W}_\nu$, depend on the following four cases.

1. **The case p, q $\notin$ I.** In this setting Γ_{I_λ} is of type $A^\infty_\infty \sqcup A^\infty_\infty$ if p is not a root of unity, as shown below. If p^2 is an r^{th} primitive root of unity then Γ_{I_λ} is of type $A^{(1)}_r \sqcup A^{(1)}_r$. Note that we do not allow $p^{2n+1} = 1$ since $\pm 1 \notin I$. For now, we name this particular setting the **ME setting**.

$$\begin{tikzcd} \cdots \arrow[r] & p^2\lambda \arrow[r] & \lambda \arrow[r] & p^{-2}\lambda \arrow[r] & \cdots \end{tikzcd}$$

$$\begin{tikzcd} \cdots & p^{-2}\lambda^{-1} \arrow[l] & \lambda^{-1} \arrow[l] & p^2\lambda^{-1} \arrow[l] & \cdots \arrow[l] \end{tikzcd}$$

2. **The case $\mathbf{p \in I}$ and $\mathbf{q \notin I}$.** If $p \in I$ and p is not a root of unity then Γ_{I_p} has type A_∞:

$$\begin{tikzcd} \cdots \arrow[r] & p^5 \arrow[r] & p^3 \arrow[r] & p \arrow[d] \\ \cdots & p^{-5} \arrow[l] & p^{-3} \arrow[l] & p^{-1} \arrow[l] \end{tikzcd}$$

Γ_{I_p} is of type $A_r^{(1)}$ if p^2 is an r$^{\text{th}}$ root of unity. For example, if $p^6 = 1$ then Γ_{I_p} is of the form,

$$\begin{tikzcd} p^3 \arrow[dr] & \\ & p \arrow[dl] \\ p^5 \arrow[uu] & \end{tikzcd}$$

3. **The case $\mathbf{q \in I}$ and $\mathbf{p \notin I}$.** In this setting Γ_{I_q} is of type $A_\infty^\infty \sqcup A_\infty^\infty$ if p is not a root of unity;

$$\begin{tikzcd} \cdots \arrow[r] & p^2 q \arrow[r] & q \arrow[r] & p^{-2} q \arrow[r] & \cdots \end{tikzcd}$$

$$\begin{tikzcd} \cdots & p^{-2}q^{-1} \arrow[l] & q^{-1} \arrow[l] & p^2 q^{-1} \arrow[l] & \cdots \arrow[l] \end{tikzcd}$$

If p is a root of unity then Γ_{I_q} is of type $A_r^{(1)} \sqcup A_r^{(1)}$;

$$\begin{tikzcd}[column sep=small] & q \arrow[dr] & & & & q^{-1} \arrow[dl] & \\ p^2 q \arrow[ur] & & p^{m-2} q \arrow[d] & & p^{-2}q^{-1} \arrow[d] & & p^{2-m}q^{-1} \arrow[ul] \\ {} \arrow[u] \arrow[dr, dotted, no head] & & {} \arrow[dl, dotted, no head] & & {} \arrow[dr, dotted, no head] & & {} \arrow[u] \arrow[dl, dotted, no head] \\ & {} & & & & {} & \end{tikzcd}$$

4. **The case $\mathbf{p}, \mathbf{q} \in \mathbf{I}$**. In this case $I_p = I_q$. That is,

$$\{p^{2n+1} \mid n \in \mathbb{Z}\} = \{p^{2n}q^{\pm 1} \mid n \in \mathbb{Z}\}.$$

If $q = p^{2n}$, for some $n \in \mathbb{Z}$, then $\pm 1 \in I$, which we have ruled out. So we must have $q = p^{2n+1}$ for some $n \in \mathbb{Z}$.

Here we introduce some notation similar to the notation used for KLR algebras. Take any $\nu \in {}^{\theta}\mathbb{N}I$, of height $|\nu| = 2m$, and the associated VV algebra $\mathfrak{W}_\nu$. Whenever we work with the VV algebra $\mathfrak{W}_\nu$, for each $w \in W_m^B$, we must choose and fix a reduced expression, say $w = s_{i_1} \cdots s_{i_r}$ where $0 \leq i_k < m$, for all k. Then define $\sigma_{\dot{w}} \in \mathfrak{W}_\nu$ as follows.

$$\begin{aligned}\sigma_{\dot{w}}\mathbf{e}(\mathbf{i}) &= \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_r}\mathbf{e}(\mathbf{i})\\ \sigma_{\dot{1}}\mathbf{e}(\mathbf{i}) &= \mathbf{e}(\mathbf{i})\end{aligned}$$

where 1 is the identity element in W_m^B. As with elements of the symmetric group, reduced expressions of $w \in W_m^B$ are not always unique and so $\sigma_{\dot{w}}$ depends upon the choice of reduced expression of w.

Note 2.3.9. If we have fixed a reduced expression for $w \in W_m^B$ we will usually omit the dot and write σ_w instead of $\sigma_{\dot{w}}$.

For each $\nu \in {}^{\theta}\mathbb{N}I$, with $|\nu| = 2m$, define ${}^{\theta}\mathbf{F}_\nu$ to be a polynomial ring in the x_k at each $\mathbf{e}(\mathbf{i})$. More precisely,

$${}^{\theta}\mathbf{F}_\nu := \bigoplus_{\mathbf{i} \in {}^{\theta}I^\nu} \mathbf{k}[x_1\mathbf{e}(\mathbf{i}), \ldots, x_m\mathbf{e}(\mathbf{i})].$$

Proposition 2.3.10 ([VV11a], Proposition 7.5)**.** *The $\boldsymbol{k}$-algebra $\mathfrak{W}_\nu$ is a free (left or right) ${}^{\theta}\boldsymbol{F}_\nu$-module on basis $\{\sigma_{\dot{w}} \mid w \in W_n^B\}$. It has rank $2^m m!$. The operator $\sigma_{\dot{w}}\boldsymbol{e}(\boldsymbol{i})$ is homogeneous and its degree is independent of the choice of reduced expression of $\dot{w}$.*

That is,

$$\mathfrak{W}_\nu = \bigoplus_{\{\sigma_{\dot{w}} \mid w \in W_m^B\}} {}^{\theta}\mathbf{F}_\nu.$$

Then we have a $\mathbf{k}$-basis for VV algebras, as follows.

Lemma 2.3.11 (Basis Theorem for VV Algebras)**.** *Take $\nu \in \mathbb{N}I$ with $|\nu| = 2m$. The elements*

$$\{\sigma_{\dot{w}}x_1^{n_1}\cdots x_m^{n_m}\boldsymbol{e}(\boldsymbol{i}) \mid w \in W_m^B, \boldsymbol{i} \in {}^{\theta}I^\nu, n_k \in \mathbb{N}_0\ \forall k\}$$

form a $\boldsymbol{k}$-basis for $\mathfrak{W}_\nu$.

2.3.1 KLR Algebras as Idempotent Subalgebras

The following lemma provides more of an understanding of the relationship between KLR algebras and VV algebras.

Proposition 2.3.12. *For any $\nu \in {}^{\theta}\mathbb{N}I$ we can write $\nu = \sum_{i\in I} \nu_i i + \sum_{i\in I} \nu_i i^{-1}$ and set $\tilde{\nu} = \sum_{i\in I} \nu_i i$. If $i + i^{-1}$ is not a summand of $\tilde{\nu}$, for any $i \in I$, then $\tilde{\nu}$ yields an idempotent subalgebra $\boldsymbol{R}_{\tilde{\nu}}$ of $\mathfrak{W}_\nu$.*

Proof. We will show that there is an algebra isomorphism

$$\mathbf{e}\mathfrak{W}_\nu\mathbf{e} \cong \mathbf{R}_{\tilde{\nu}} \text{ where } \mathbf{e} = \sum_{\mathbf{i}\in I^{\tilde{\nu}}} \mathbf{e}(\mathbf{i}).$$

Let $\mathcal{D}_m := \mathcal{D}(W_m^B/\mathfrak{S}_m)$ denote the minimal length left coset representatives of $\mathfrak{S}_m$ in W_m^B. Fix a reduced expression $\dot{s}$ for every $s \in \mathfrak{S}_m$. It is well-known (see [GP00] Proposition (2.1.1), for example) that every $w \in W_m^B$ can be written uniquely in the form ηs, for $\eta \in \mathcal{D}_m$, $s \in \mathfrak{S}_m$, with $\ell(\eta s) = \ell(\eta) + \ell(s)$. For every $w \in W_m^B$ fix a reduced expression $\dot{w} = \dot{\eta}\dot{s}$. By the basis theorem for VV algebras 2.3.11, any element $v \in \mathfrak{W}_\nu$ can be expressed in the following form.

$$v = \sum_{\substack{w\in W_m^B \\ \mathbf{i}\in {}^{\theta}I^{\nu}}} \sigma_w p_{\mathbf{i}}(\underline{x})\mathbf{e}(\mathbf{i}) = \sum_{\substack{s_{i_k}\in\mathfrak{S}_m \\ \eta\in\mathcal{D}_m \\ \mathbf{i}\in {}^{\theta}I^{\nu}}} \sigma_\eta \sigma_{s_{i_1}\cdots s_{i_r}} p_{\mathbf{i}}(\underline{x})\mathbf{e}(\mathbf{i})$$

where $\underline{x} = (x_1, \dots, x_m)$, and $p_{\mathbf{i}}(\underline{x}) \in \mathbf{k}[x_1\mathbf{e}(\mathbf{i}), \dots, x_m\mathbf{e}(\mathbf{i})]$. Then

$$\mathbf{e}v\mathbf{e} = \sum_{\substack{s_{i_k}\in\mathfrak{S}_m \\ \eta\in\mathcal{D}_m \\ \mathbf{i}\in I^{\tilde{\nu}}}} \mathbf{e}\sigma_\eta \mathbf{e}\sigma_{s_{i_1}\cdots s_{i_r}} p_{\mathbf{i}}(\underline{x}).$$

Claim 2.3.13. *For $\eta \in \mathcal{D}(W_m^B/\mathfrak{S}_m)$,*

$$\boldsymbol{e}\sigma_\eta\boldsymbol{e} = \begin{cases} \boldsymbol{e} & \text{if } \eta = 1 \\ 0 & \text{else.} \end{cases}$$

This is clear when $\eta = 1$. So suppose that $\eta \neq 1$. We prove by induction on m that $\mathbf{e}\sigma_\eta\mathbf{e} = 0$.

For $m = 1$ we have $\mathcal{D}_1 = \{1, s_0\}$ and $\tilde{\nu} = a \in \mathbb{N}I$. Clearly $\mathbf{e}\sigma_0\mathbf{e} = \mathbf{e}\pi\mathbf{e} = 0$. Now for any $\tilde{\nu} \in \mathbb{N}I$ of height $k < m$ assume $\mathbf{e}\sigma_\eta\mathbf{e} = 0$ for all $\eta \in \mathcal{D}_k$.

Take $\tilde{\nu} \in \mathbb{N}I$ of height m and $\eta \in \mathcal{D}_m \setminus \mathcal{D}_{m-1}$. So, by Lemma 1.4.8, $\eta = \bar{\eta}s_{m-1}\cdots s_1 s_0$, for $\bar{\eta} \in \mathcal{D}_{m-1}$. Consider one summand of $\mathbf{e}$, say $e(a_1, \dots, a_m)$. Then,

$$\begin{aligned} \mathbf{e}\sigma_\eta e(a_1, \dots, a_m) &= \mathbf{e}\sigma_{\bar{\eta}}\sigma_{m-1}\cdots\sigma_1\pi e(a_1, \dots, a_m) \\ &= \mathbf{e}\sigma_{\bar{\eta}} e(a_2, \dots, a_m, a_1^{-1})\sigma_{m-1}\cdots\sigma_1\pi. \end{aligned}$$

Since a_1 is a summand of $\tilde{\nu}$, by assumption, a_1^{-1} is not an entry of any $(i_1, \dots, i_m) \in I^{\tilde{\nu}}$ (otherwise $a_1 + a_1^{-1}$ would be a summand of $\tilde{\nu}$). Also, $\sigma_{\bar{\eta}}$ does not affect the entry a_1^{-1}, or its position, in $e(a_1, \ldots, a_m, a_1^{-1})$. Hence we have

$$\mathbf{e}\sigma_{\tilde{\eta}} e(a_2, \ldots, a_m, a_1^{-1}) = 0, \text{ and so } \mathbf{e}\sigma_\eta e(a_1, \ldots, a_m) = 0.$$

This is true for every summand of $\mathbf{e}$ and so $\mathbf{e}\sigma_\eta \mathbf{e} = 0$, for $\eta \neq 1$.

Then,

$$\mathbf{e}v\mathbf{e} = \begin{cases} \sum_{\substack{s \in \mathfrak{S}_m \\ \mathbf{i} \in I^{\tilde{\nu}}}} \sigma_s p_{\mathbf{i}}(\underline{x})\mathbf{e} & \text{if } \eta = 1 \\ 0 & \text{if } \eta \neq 1. \end{cases}$$

Now define a map

$$\begin{aligned} f : \mathbf{e}\mathfrak{W}_\nu \mathbf{e} &\longrightarrow \mathbf{R}_{\tilde{\nu}} \\ \mathbf{e}v\mathbf{e} &\mapsto \sum_{\substack{s \in \mathfrak{S}_m \\ \mathbf{i} \in I^{\tilde{\nu}}}} \sigma_s p_{\mathbf{i}}(\underline{x})\mathbf{e}. \end{aligned}$$

From 2.2.11 we know $\{\sigma_s x_1^{n_1} \cdots x_m^{n_m} \mathbf{e}(\mathbf{i}) \mid s \in \mathfrak{S}_m, \mathbf{i} \in I^{\tilde{\nu}}, n_k \in \mathbb{N}_0 \, \forall k\}$ is a basis for $\mathbf{R}_{\tilde{\nu}}$ and so f is an isomorphism of $\mathbf{k}$-vector spaces. On inspection of the defining relations of the KLR algebras and VV algebras it follows that f is in fact a morphism of algebras. This completes the proof. □

Remark 2.3.14. Using the notation from Remark 2.3.8, every VV algebra $\mathfrak{W}_\nu$ has idempotent subalgebras $\mathbf{R}_{\tilde{\nu}}^-$ and $\mathbf{R}_{\tilde{\nu}}^+$.

Example 2.3.15. Take $\nu = p^{-2}\lambda^{-1} + \lambda^{-1} + \lambda + p^2\lambda \in {}^\theta\mathbb{N}I$ from Example 2.3.6. $\mathbf{R}_{\tilde{\nu}}^+$, corresponding to $\tilde{\nu} = \lambda + p^2\lambda$, is always an idempotent subalgebra of $\mathfrak{W}_\nu$, as is $\mathbf{R}_{\tilde{\nu}}^-$ which is the KLR algebra corresponding to $\tilde{\nu} = \lambda^{-1} + p^{-2}\lambda^{-1}$. But $\mathbf{R}_{\nu_1}, \mathbf{R}_{\nu_2}$ are also idempotent subalgebras for $\nu_1 = \lambda^{-1} + p^2\lambda$ and for $\nu_2 = \lambda + p^{-2}\lambda^{-1}$. In this example $\mathfrak{W}_\nu$ has four KLR algebras appearing as idempotent subalgebras.

Example 2.3.16. Take $\nu = 2p^{-1} + 2p \in {}^\theta\mathbb{N}I_p$, as in Example 2.3.7. Then $\mathbf{R}_{\tilde{\nu}}$, for $\tilde{\nu} = p + p^{-1}$, is not an idempotent subalgebra of $\mathfrak{W}_\nu$. In this example, the only idempotent subalgebras isomorphic to KLR algebras are $\mathbf{R}_{\tilde{\nu}}^+$ and $\mathbf{R}_{\tilde{\nu}}^-$.

2.3.2 $\mathfrak{W}_\nu \mathbf{e}$ is a Free $\mathbf{R}_{\tilde{\nu}}$-Module

Take $\nu \in {}^\theta\mathbb{N}I$, $|\nu| = 2m$ and the corresponding VV algebra $\mathfrak{W}_\nu$. Write $\nu = \sum_{i \in I^+} \nu_i i + \sum_{i \in I^-} \nu_i i$ and set $\tilde{\nu} = \sum_{i \in I^+} \nu_i i \in \mathbb{N}I^+$, as in Remark 2.3.8, so that $\mathbf{R}_{\tilde{\nu}}^+$ is an idempotent subalgebra of $\mathfrak{W}_\nu$.

Let $\mathbf{e} = \sum_{\mathbf{i} \in I^{\tilde{\nu}}} \mathbf{e}(\mathbf{i})$. Then $\mathfrak{W}_\nu \mathbf{e}$ has the structure of a right $\mathbf{R}_{\tilde{\nu}}^+$-module; $\mathbf{e}\mathfrak{W}_\nu \mathbf{e} \cong \mathbf{R}_{\tilde{\nu}}^+$ and the action of $\mathbf{R}_{\tilde{\nu}}^+$ on $\mathfrak{W}_\nu \mathbf{e}$ is given by multiplication from the right.

As before let $\mathcal{D}_m := \mathcal{D}(W_m^B/\mathfrak{S}_m)$, the set of minimal length left coset representatives of $\mathfrak{S}_m$ in W_m^B.

Proposition 2.3.17. *As a right $\boldsymbol{R}_{\tilde{\nu}}^+$-module, $\mathfrak{W}_\nu\mathbf{e}$ is free on a basis given by minimal length left coset representatives of $\mathfrak{S}_m$ in W_m^B.*

Proof. First fix reduced expressions $\dot{s}$, $\dot{\eta}$ for every $s \in \mathfrak{S}_m$, $\eta \in \mathcal{D}_m$. Then for every $w \in W_m^B$ fix a reduced expression of the form $\dot{w} = \dot{\eta}\dot{s}$.

Recall from Lemma 2.2.11, that the elements

$$\{\sigma_{\dot{s}} x_1^{n_1} \cdots x_m^{n_m} \mathbf{e}(\mathbf{i}) \mid s \in \mathfrak{S}_m, \mathbf{i} \in I^{\tilde{\nu}}, n_i \in \mathbb{N}_0 \ \forall i\}$$

form a $\mathbf{k}$-basis for $\mathbf{R}_{\tilde{\nu}}$. Recall from 2.3.11 that the elements

$$\{\sigma_{\dot{w}} x_1^{n_1} \cdots x_m^{n_m} \mathbf{e}(\mathbf{i}) \mid w \in W_m^B, \mathbf{i} \in {}^\theta I^\nu, n_i \in \mathbb{N}_0 \ \forall k\}$$

form a $\mathbf{k}$-basis for $\mathfrak{W}_\nu$.

Using that $\dot{w} = \dot{\eta}\dot{s}$ with $\ell(\dot{w}) = \ell(\dot{\eta}) + \ell(\dot{s})$, $\mathfrak{W}_\nu\mathbf{e}$ has a $\mathbf{k}$-basis

$$\{\sigma_{\dot{\eta}}\sigma_{\dot{s}} x_1^{n_1} \cdots x_m^{n_m} \mathbf{e}(\mathbf{i}) \mid w \in W_m^B, \mathbf{i} \in {}^\theta I^\nu, n_k \in \mathbb{N}_0 \ \forall k, \mathbf{e}(\mathbf{i})\mathbf{e} \neq 0\}.$$

Consider the map

$$\phi : \mathfrak{W}_\nu\mathbf{e} \quad \longrightarrow \quad \bigoplus_{\dot{\eta} \in \mathcal{D}_m} \mathbf{R}_{\tilde{\nu}}$$

which is defined on basis elements as follows. The map ϕ maps $\sigma_{\dot{\eta}}\sigma_{\dot{s}} x_1^{n_1} \cdots x_m^{n_m} \mathbf{e}(\mathbf{i})$ to the element $\sigma_{\dot{s}} x_1^{n_1} \cdots x_m^{n_m} \mathbf{e}(\mathbf{i})$ lying in the copy of $\mathbf{R}_{\tilde{\nu}}$ indexed by $\dot{\eta}$. Extend this map linearly. This is an isomorphism of $\mathbf{k}$-vector spaces. It follows from the defining relations of $\mathfrak{W}_\nu$ and $\mathbf{R}_{\tilde{\nu}}$ that this is also a right $\mathbf{R}_{\tilde{\nu}}$-module morphism. $\square$

Chapter 3

Morita Equivalences Between KLR and VV Algebras

3.1 Morita Theory

Morita equivalence is an equivalence relation between rings and is one which preserves many ring-theoretic properties, such as semisimplicity. It is named after the mathematician Kiita Morita who introduced the idea.

Definition 3.1.1. Rings A and B are said to be **Morita equivalent** if their respective module categories are equivalent, i.e. there exists functors $F : A\text{-Mod} \longrightarrow B\text{-Mod}$ and $G : B\text{-Mod} \longrightarrow A\text{-Mod}$ such that $FG \cong 1_{B\text{-Mod}}$ and $GF \cong 1_{A\text{-Mod}}$.

Rings are often studied in terms of their modules and so the existence of a Morita equivalence between rings A and B may provide a wealth of information about ring A if ring B is one which is already well-understood. It can be shown that the left module categories A-Mod and B-Mod are equivalent if and only if the right module categories Mod-A and Mod-B are equivalent, so one does not need to specify left or right Morita equivalence. If F is an equivalence of categories then F is an exact functor since α is a monomorphism (resp. epimorphism, isomorphism) if and only if $F\alpha$ is a monomorphism (resp. epimorphism, isomorphism).

Definition 3.1.2. Let A and B be rings and $M \in A\text{-Mod-}B$. We say that M is **faithfully balanced** if the natural maps $A \longrightarrow \mathrm{End}(M_B)$ and $B \longrightarrow \mathrm{End}({}_AM)$ are both ring isomorphisms.

Lemma 3.1.3 ([Lam99], Lemma 18.41)**.** *Let ${}_AM_B$ be faithfully balanced. Then $Z(A) \cong Z(B)$ and both rings are isomorphic to $End({}_AM_B)$.*

Proof. Define $f : Z(A) \longrightarrow \mathrm{End}({}_AM_B)$ by $z \mapsto f(z)$, where $f(z)m = zm$. Since $z \in Z(A)$ it is clear that $f(z) \in \mathrm{End}({}_AM_B)$. One can also easily check that f is a ring homomorphism.

Since $\mathrm{End}({}_AM_B) \subset \mathrm{End}(M_B)$ we know that f is injective. To show surjectivity, take $g \in \mathrm{End}({}_AM_B)$. Since $g \in \mathrm{End}(M_B) \cong A$, we know that g corresponds to left multiplication by some $a \in A$, i.e. $g(m) = am$ for all $m \in M$. But we also know that $g \in \mathrm{End}({}_AM)$ so that $g(a_1m) = a_1g(m)$, for every $a_1 \in A, m \in M$. But then,

$$g(a_1m) = a_1g(m) \Longrightarrow aa_1m = a_1am$$

for all $a_1 \in A, m \in M$. Since $aa_1, a_1a \in A$ and $A \cong \mathrm{End}(M_B)$ we have that $aa_1 = a_1a$ for every $a_1 \in A$. In other words, $a \in Z(A)$ and f is surjective, meaning that we have a ring isomorphism $Z(A) \cong \mathrm{End}({}_AM_B)$. By symmetry, we also have $Z(B) \cong \mathrm{End}({}_AM_B)$ and hence $Z(A) \cong Z(B)$. $\square$

Definition 3.1.4. $P \in R$-Mod is a **generator** for R-Mod if $\mathrm{Hom}_R(P, -)$ is a faithful functor from R-Mod to **Ab**, the category of abelian groups. That is, when $f_* = g_*$ if and only if $f = g$.

Example 3.1.5. An example of a generator is the left regular module ${}_RR$. In this case, $\mathrm{Hom}_R(R, M) \cong M$ for any R-module M. So $\mathrm{Hom}_R(R, -)$ is the forgetful functor, which is faithful.

Theorem 3.1.6 ([Lam99], Theorem 18.8)**.** *The following are equivalent for $P \in R$-Mod.*

1. *P is a generator for R-Mod.*

2. *$tr_R(P) = R$, where $tr_RP = \sum_{g \in \mathrm{Hom}_R(P,R)} g(P)$ is called the* ***trace ideal****.*

3. *There exists $n \in \mathbb{N}$ such that $R \mid P^{\oplus n}$.*

4. *$R \mid \bigoplus_{i\in I} P$ for some set I.*

5. *For every $M \in R$-Mod there exists a surjection $\psi : \bigoplus_{i\in I} P \twoheadrightarrow M$.*

So, for example, $R \oplus M$ is a generator for any $M \in R$-Mod.

Definition 3.1.7. $P \in R$-Mod is a **progenerator** for R-Mod if P is a finitely generated projective generator. So then, $P \mid \bigoplus_{i\in I} R$ for some I and $R \mid \bigoplus_{i\in J} P$ for some J.

Example 3.1.8. It is clear that ${}_RR$ is a progenerator in R-Mod. This means that, since being a progenerator is a categorical property, $F({}_RR)$ is a progenerator in S-Mod, for $F : R\text{-Mod} \longrightarrow S\text{-Mod}$ an equivalence of categories.

Remark 3.1.9. Let $1 = e_1 + \cdots + e_n \in R$ be a decomposition of 1 into primitive orthogonal idempotents. Suppose that $Re_{i_1}, \ldots, Re_{i_k}$ is a complete list of non-isomorphic indecomposable projectives. Then, using Theorem 3.1.6, an arbitrary finitely generated projective module $P = (Re_{i_1})^{\oplus n_{i_1}} \oplus \cdots \oplus (Re_{i_k})^{\oplus n_{i_k}}$ is a progenerator if and only if $n_{i_r} > 0$ for all $1 \leq r \leq k$.

Remark 3.1.10. Let R be a ring and $e = e^2 \in R$ a non-trivial idempotent. Then Re is a finitely generated projective R-module since

$$_RR = Re \oplus R(1-e),$$

i.e. Re is a direct summand of a free R-module. Consider $\operatorname{tr}_R(Re) = \sum_{g \in \operatorname{Hom}_R(Re,R)} g(Re)$. Take any $g \in \operatorname{Hom}_R(Re, R)$ and any $r \in R$. Then, $g(re) = reg(e) = res$, for some $s \in R$, so $g(Re) \subseteq ReR$. Hence, $\operatorname{tr}_R(Re) \subseteq ReR$. Conversely, for $res \in ReR$ define $g_{res} : Re \longrightarrow R$ as multiplication on the right by res, i.e. $g_{res}(te) = teres$. Then $g_{res} \in \operatorname{Hom}_R(Re, R)$ and there is an injection $ReR \hookrightarrow \operatorname{tr}_R(Re)$. So $ReR = \operatorname{tr}_R(Re)$. By Theorem 3.1.6 (2), it follows that Re is a generator, and hence a progenerator, if and only if $ReR = R$, i.e. if and only if e is a full idempotent in R.

3.1.1 Morita Contexts

Let R be a ring. For $P \in R$-Mod, let $Q = P^* = \operatorname{Hom}_R(P, R)$ and $S = \operatorname{End}_R(P)$. Then P is an R-S-bimodule where the action of S is given by $p \cdot g = g(p)$, for $p \in P$ and $g \in S$, and Q is an S-R-bimodule via

$$\begin{aligned} S \times Q &\longrightarrow Q \\ (s, f) &\mapsto sf : p \mapsto f(s(p)) \end{aligned}$$

and

$$\begin{aligned} Q \times R &\longrightarrow Q \\ (f, r) &\mapsto fr : p \mapsto rf(p) \end{aligned}$$

Lemma 3.1.11 ([Lam99], Lemma 18.15)**.** *We have the following bimodule morphisms.*

1. $\alpha : P \otimes_S Q \longrightarrow R$, $p \otimes f \mapsto f(p)$ *defines an R-R-bimodule morphism.*

2. $\beta : Q \otimes_R P \longrightarrow S$, $f \otimes p \mapsto fp : p' \mapsto f(p)p'$ *defines an S-S-bimodule morphism.*

Let

$$M = \begin{pmatrix} R & Q \\ P & S \end{pmatrix}$$

M is a ring, called the **Morita ring** associated with P. The multiplication and addition is given by usual matrix multiplication and matrix addition. The 6-tuple $(R, P, Q, S, \alpha, \beta)$ is the **Morita context** associated with P. Fixing such a Morita context, we have the following results.

Proposition 3.1.12 ([Lam99], Proposition 18.17)**.**

1. *P is a generator $\Longleftrightarrow$ α is onto.*

2. *Assume P is a generator. Then,*

(a) $\alpha : P \otimes_S Q \longrightarrow R$ *is an R-R-isomorphism.*

(b) $Q = Hom_R(P, R) \cong Hom_S(P, S)$ *as S-R-bimodules.*

(c) $P \cong Hom_S(Q, S)$ *as R-S-bimodules.*

(d) $R \cong End_S(P) \cong End_S(Q)$ *as rings.*

Proposition 3.1.13 ([Lam99], Proposition 18.19)**.**

1. *P is a finitely generated projective module if and only if* β *is onto.*

2. *Assume P is a finitely generated projective module. Then,*

   (a) $\beta : Q \otimes_R P \longrightarrow S$ *is an S-S-isomorphism.*

   (b) $Q = Hom_R(P, R)$ *as S-R-bimodules.*

   (c) $P \cong Hom_R(Q, R)$ *as R-S-bimodules.*

   (d) $S \cong End_R(P) \cong End_R(Q)$ *as rings.*

Proposition 3.1.14 ([Lam99], Proposition 18.22)**.** *Suppose* ${}_RP$ *is a progenerator. Then* P_S*,* ${}_SQ$ *and* Q_R *are also progenerators and* α*,* β *are isomorphisms.*

Proof. This follows immediately from Proposition 3.1.13 and Proposition 3.1.12 above. □

We now state two of Morita's Theorems.

Theorem 3.1.15 (Morita's First Theorem)**.** *Let* ${}_RP \in R\text{-}Mod$ *be a progenerator and* $(R, P, Q, S, \alpha, \beta)$ *the Morita context associated to* ${}_RP$*. Then,*

1. $Q \otimes_R - : R\text{-}Mod \longrightarrow S\text{-}Mod$ *and* $P \otimes_S - : S\text{-}Mod \longrightarrow R\text{-}Mod$ *are mutually inverse equivalences of categories.*

2. $- \otimes_R P : Mod\text{-}R \longrightarrow Mod\text{-}S$ *and* $- \otimes_S Q : Mod\text{-}S \longrightarrow Mod\text{-}R$ *are mutually inverse equivalences of categories.*

Proof. See [Lam99], Theorem 18.24. □

Remark 3.1.16. For any left R-module, M one can check that

$$\beta_M : Q \otimes_R M \longrightarrow \mathrm{Hom}_R(P, M)$$
$$q \otimes m \mapsto qm \ \ (p \mapsto q(p)m)$$

is an isomorphism of left R-modules. So we have a natural isomorphism

$$Q \otimes_R - \cong \mathrm{Hom}_R(P, -).$$

Similarly, $P \otimes_S - \cong \mathrm{Hom}_S(Q, -)$. A similar note can be made for the functors of right module categories.

Theorem 3.1.17 (Morita's Second Theorem)**.** *Let $F : R\text{-Mod} \longrightarrow S\text{-Mod}$ and $G : S\text{-Mod} \longrightarrow R\text{-Mod}$ be mutually inverse equivalences of categories, with $F({}_RR) = Q$ and $G({}_SS) = P$. Then we have bimodule structures ${}_RP_S$ and ${}_SQ_R$. Using these we have natural isomorphisms $F \cong Q \otimes_R -$ and $G \cong P \otimes_S -$.*

Proof. See [Lam99], Theorem 18.26. □

Corollary 3.1.18. *Rings R and S are Morita equivalent if and only if $S \cong End_R(P)$, for some progenerator $P \in R\text{-Mod}$.*

Proof. Suppose first that R and S are Morita equivalent. Let $F : R\text{-Mod} \longrightarrow S\text{-Mod}$ and $G : S\text{-Mod} \longrightarrow R\text{-Mod}$ be mutually inverse category equivalences and let $F(R) = Q$ and $G(S) = P$. P and Q are progenerators in their respective categories. By Morita's Second Theorem together with Remark 3.1.16, we have $F \cong Q \otimes_R - \cong \mathrm{Hom}_R(P, -)$. Notice then that $Q \cong \mathrm{Hom}_R(P, R)$ so that using Proposition 3.1.13 for the third isomorphism,

$$F(P) \cong Q \otimes_R P \cong \mathrm{End}_R(P) \cong S.$$

Suppose now that $S \cong \mathrm{End}_R(P)$ for some progenerator $P \in R\text{-Mod}$. Since P is projective, it is a direct summand of a free R-module, i.e. $R^n = P \oplus P'$. We also identify $\mathrm{End}(R^n)$ with $M_n(R)$, the ring of $n \times n$ matrices with entries in R. Let e be the map,

$$\begin{aligned} e : R^n &\longrightarrow R^n \\ (r_1, r_2) &\mapsto (r_1, 0) \end{aligned}$$

for $r_1 \in P$ and $r_2 \in P'$, using that $R^n = P \oplus P'$. Then $e(R^n) = P$, e is an idempotent element of $M_n(R)$, and one can show that $M_n(tr_R(P)) = M_n(R)eM_n(R)$. But $tr_R(P) = R$ since P is a generator, and so we must have that e is a full idempotent in $M_n(R)$. Now define

$$\begin{aligned} \lambda : \mathrm{End}_R(P) &\longrightarrow e \cdot \mathrm{End}_R(R^n) \cdot e = eM_n(R)e \\ f &\mapsto \lambda(f) \end{aligned}$$

where $\lambda(f)|_P = f$ and $\lambda(f)|_{P'} = 0$. One can check that λ is an isomorphism so that $S \cong \mathrm{End}_R(P) \cong eM_n(R)e$. Since e is full in $M_n(R)$, S is Morita equivalent to $M_n(R)$ which is Morita equivalent to R. □

We now know that if ${}_RP_S$ is a progenerator then $R \cong \mathrm{End}(P_S)$ and $S \cong \mathrm{End}({}_RP)$. So we have the following corollary of Lemma 3.1.3.

Corollary 3.1.19. *If R and S are Morita equivalent then $Z(R) \cong Z(S)$. In particular, for commutative rings R and S we have, $R \sim_{ME} S \iff R \cong S$.*

A **generalized Morita context**, or pre-equivalence, is nothing but a ring with a distinguished idempotent. This can be seen in the Morita context above when arranging the data into a 2×2-matrix.

Let R be a ring with an idempotent $e \in R$. Let $f = 1 - e$ be its complementary idempotent. The Pierce decomposition $R = eRe \oplus eRf \oplus fRe \oplus fRf$ allows us to write R as an algebra of 2×2-matrices

$$R \cong \begin{pmatrix} eRe & eRf \\ fRe & fRf \end{pmatrix}$$

The diagonal entries $A = eRe$ and $B = fRf$ are rings with multiplicative identities $1_A = e$ and $1_B = f$, respectively. Note that they are not unital subrings of R. Then $Q = eRf$ is an A-B-bimodule and $P = fRe$ is a B-A-bimodule. To emphasise this let us write again

$$R \cong \begin{pmatrix} eRe & eRf \\ fRe & fRf \end{pmatrix} \cong \begin{pmatrix} A & Q \\ P & B \end{pmatrix}.$$

Multiplication in R induces a pair of bimodule homomorphisms,

$$f : Q \otimes_B P \longrightarrow A$$
$$g : P \otimes_A Q \longrightarrow B.$$

Conversely, as we saw above, we can arrange a Morita context into a 2×2-matrix to obtain a ring R with the distinguished pair of complementary idempotents $e = 1_A$, $f = 1_B$. We will also make use of the following results.

Proposition 3.1.20. *Let e, f be idempotents in a ring R. Then the following statements are equivalent:*

1. *$eR \cong fR$ as right R-modules.*
2. *$Re \cong Rf$ as left R-modules.*
3. *There exists $a \in eRf$ and $b \in fRe$ such that $e = ab$ and $f = ba$.*
4. *There exists $a, b \in R$ such that $e = ab$ and $f = ba$.*

Proof. See [Lam99], Proposition (21.20). □

If any two idempotents $e, f \in R$ satisfy the above conditions then we say that they are isomorphic idempotents and write $e \cong f$. Note that the isomorphism of left R-modules is given by

$$\begin{aligned} \theta :\ & R\mathbf{e}_1 \longrightarrow R\mathbf{e}_2 & \qquad \theta^{-1} :\ & R\mathbf{e}_2 \longrightarrow R\mathbf{e}_1 \\ & r \mapsto ra & & s \mapsto sb \end{aligned}$$

Lemma 3.1.21. *Let A be a finite-dimensional algebra over a field k. Suppose $A = Ae_1 \oplus \cdots \oplus Ae_n$ is a decomposition of A where each submodule Ae_i is indecomposable. Then every simple left A-module is isomorphic to one of $S_1 = top(Ae_1), \ldots, S_n = top(Ae_n)$ and $S_i \cong S_j$ if and only if $e_i \cong e_j$.*

Proof. See [ASS06]. □

Proposition 3.1.22 ([Lam01], Proposition 21.6). *Let $e, e' \in A$ be idempotents, and M a left A-module. There is a natural additive group isomorphism $\lambda : Hom_R(Re, M) \longrightarrow eM$. In particular, there is a natural group isomorphism $Hom_R(Re, Re') \cong eRe'$.*

Corollary 3.1.23 ([Lam01], Proposition 21.7). *For any idempotent $e \in A$, there is a natural ring isomorphism $End_A(Ae)^{\mathrm{op}} \cong eAe$.*

3.2 Morita Equivalence in the Separated Case

Let R and S be $\mathbf{k}$-algebras with $M \in R$-Mod, $N \in S$-Mod. The outer tensor product of M and N is denoted $M \boxtimes N$ and is defined to be $M \otimes N$ as a $\mathbf{k}$-vector space, considered as an $R \otimes S$-module. Recall that a H_m^B-module is said to be of type I if all X_i-eigenvalues, $1 \leq i \leq m$, lie in I.

Lemma 3.2.1 ([EK06], Lemma 3.5).

1. *Let N' be a simple $\mathrm{H}_{n'}^B$-module of type I and N'' a simple $\mathrm{H}_{n''}^B$-module of type J. Then $N' \boxtimes N''$ is a simple $\mathrm{H}_{n'}^B \otimes \mathrm{H}_{n''}^B$-module and $\mathrm{Ind}_{\mathrm{H}_{n'}^B \otimes \mathrm{H}_{n''}^B}^{\mathrm{H}_{n'+n''}^B} N' \boxtimes N''$ is a simple $\mathrm{H}_{n'+n''}^B$-module of type $I \cup J$.*

2. *Conversely if M is a simple H_m^B-module of type $I \cup J$ then there exists a simple H_n^B-module N' of type I and a simple H_{m-n}^B-module N'' of type J such that $M \cong \mathrm{Ind}_{\mathrm{H}_n^B \otimes \mathrm{H}_{m-n}^B}^{\mathrm{H}_m^B} N' \boxtimes N''$.*

In other words, for $N' \in \mathrm{H}_{n'}^B$-Mod_I simple and for $N'' \in \mathrm{H}_{n''}^B$-$\mathrm{Mod}_J$ simple, $\mathrm{Ind}_{\mathrm{H}_{n'}^B \otimes \mathrm{H}_{n''}^B}^{\mathrm{H}_{n'+n''}^B} N' \boxtimes N'' \in \mathrm{H}_{n'+n''}^B$-$\mathrm{Mod}_{I \cup J}$ is simple. Moreover, every simple $\mathrm{H}_{n'+n''}^B$-module of type $I \cup J$ arises in this way. In [EK06] they conclude that it suffices to study H_m^B-modules of type I. In this chapter we provide more of a categorical justification of this fact.

Recall, from Section 2.3, we fixed an element $p \in \mathbf{k}^\times$ in order to define an action of $\mathbb{Z} \rtimes \mathbb{Z}_2$ on $\mathbf{k}^\times$. We then fixed a $\mathbb{Z} \rtimes \mathbb{Z}_2$-invariant subset I_λ of $\mathbf{k}^\times$, for some $\lambda \in \mathbf{k}^\times$, and defined a family of VV algebras from this data. In this section we will define families of VV algebras in a slightly more general setting and then show that in fact it suffices to study the families of VV algebras as defined in Section 2.3. We again fix elements $p, q \in \mathbf{k}^\times$ and keep the action of $\mathbb{Z} \rtimes \mathbb{Z}_2$ on $\mathbf{k}^\times$ as before, namely,

$$(n, \varepsilon) \cdot \lambda = p^{2n} \lambda^{\varepsilon}.$$

Let $I, J \subset \mathbf{k}^\times$ be $\mathbb{Z} \rtimes \mathbb{Z}_2$-orbits with $I \cap J = \emptyset$. To $I \cup J$ we associate a quiver $\Gamma = \Gamma_{I \cup J}$ together with an involution θ. The vertices of Γ are the elements $i \in I \cup J$ and we have arrows $p^2 i \longrightarrow i$ for every $i \in I \cup J$. The involution θ is defined as before;

$$\theta(i) = i^{-1}$$
$$\theta(p^2 i \longrightarrow i) = p^{-2} i^{-1} \longleftarrow i^{-1} \quad \text{for all } i \in I \cup J.$$

Define ${}^{\theta}\mathbb{N}(I \cup J) := \{\nu = \sum_{i \in I \cup J} \nu_i i \mid \nu \text{ has finite support}, \nu_i \in \mathbb{Z}_{\geq 0}, \nu_i = \nu_{\theta(i)} \quad \forall i\}$. For $\nu \in {}^{\theta}\mathbb{N}(I \cup J)$, the height of ν is defined to be

$$|\nu| = \sum_{i \in I \cup J} \nu_i$$

and is equal to $2m$, for some positive integer m. For $\nu \in {}^{\theta}\mathbb{N}(I \cup J)$ of height $2m$, define

$${}^{\theta}(I \cup J)^{\nu} := \{\mathbf{i} = (i_1, \ldots, i_m) \in (I \cup J)^m \mid \sum_{k=1}^{m} i_k + \sum_{k=1}^{m} i_k^{-1} = \nu\}.$$

Definition 3.2.2. For $\nu \in {}^{\theta}\mathbb{N}(I \cup J)$ with $|\nu| = 2m$ the **separated VV algebra**, denoted $\mathfrak{W}_{\nu}$, is the graded $\mathbf{k}$-algebra generated by elements

$$\{x_1, \ldots, x_m\} \cup \{\sigma_1, \ldots, \sigma_{m-1}\} \cup \{\mathbf{e}(\mathbf{i}) \mid \mathbf{i} \in {}^{\theta}(I \cup J)^{\nu}\} \cup \{\pi\}$$

which are subject to the following relations.

1. $\mathbf{e}(\mathbf{i})\mathbf{e}(\mathbf{j}) = \delta_{\mathbf{ij}}\mathbf{e}(\mathbf{i}), \quad \sigma_k \mathbf{e}(\mathbf{i}) = \mathbf{e}(s_k \mathbf{i})\sigma_k, \quad \sum_{\mathbf{i} \in {}^{\theta}(I \cup J)^{\nu}} \mathbf{e}(\mathbf{i}) = 1,$

   $x_l \mathbf{e}(\mathbf{i}) = \mathbf{e}(\mathbf{i}) x_l, \quad \pi \mathbf{e}(i_1, \ldots, i_m) = \mathbf{e}(\theta(i_1), i_2 \ldots, i_m)\pi = \mathbf{e}(i_1^{-1}, i_2 \ldots, i_m)\pi$

2. $\pi^2 \mathbf{e}(\mathbf{i}) = \begin{cases} x_1 \mathbf{e}(\mathbf{i}) & i_1 = q \\ -x_1 \mathbf{e}(\mathbf{i}) & i_1 = q^{-1} \\ \mathbf{e}(\mathbf{i}) & i_1 \neq q^{\pm 1} \end{cases}$

3. The x_l's commute.

4. $\sigma_k^2 \mathbf{e}(\mathbf{i}) = \begin{cases} \mathbf{e}(\mathbf{i}) & i_k \not\leftrightarrow i_{k+1} \\ (x_{k+1} - x_k)\mathbf{e}(\mathbf{i}) & i_k \leftarrow i_{k+1} \\ (x_k - x_{k+1})\mathbf{e}(\mathbf{i}) & i_k \rightarrow i_{k+1} \\ (x_{k+1} - x_k)(x_k - x_{k+1})\mathbf{e}(\mathbf{i}) & i_k \leftrightarrow i_{k+1} \\ 0 & i_k = i_{k+1} \end{cases}$

   $\sigma_j \sigma_k = \sigma_k \sigma_j \quad \text{for } j \neq k \pm 1$

   $(\sigma_{k+1}\sigma_k\sigma_{k+1} - \sigma_k\sigma_{k+1}\sigma_k)\mathbf{e}(\mathbf{i}) = \begin{cases} 0 & i_k \neq i_{k+2} \text{ or } i_k \not\leftrightarrow i_{k+1} \\ \mathbf{e}(\mathbf{i}) & i_k = i_{k+2} \text{ and } i_k \rightarrow i_{k+1} \\ -\mathbf{e}(\mathbf{i}) & i_k = i_{k+2} \text{ and } i_k \leftarrow i_{k+1} \\ (2x_{k+1} - x_{k+2} - x_k)\mathbf{e}(\mathbf{i}) & i_k = i_{k+2} \text{ and } i_k \leftrightarrow i_{k+1} \end{cases}$

5. $\pi x_1 = -x_1 \pi$

   $\pi x_l = x_l \pi \quad \text{for all} \quad l > 1$

6. $(\sigma_1\pi)^2\mathbf{e}(\mathbf{i}) - (\pi\sigma_1)^2\mathbf{e}(\mathbf{i}) = \begin{cases} 0 & i_0 \neq i_2 \text{ or if } i_1 \neq q^{\pm 1} \\ \sigma_1\mathbf{e}(\mathbf{i}) & i_0 = i_2 \text{ and } i_1 = q \\ -\sigma_1\mathbf{e}(\mathbf{i}) & i_0 = i_2 \text{ and } i_1 = q^{-1} \end{cases}$

   $\pi\sigma_k = \sigma_k\pi$ for all $k \neq 1$

7. $(\sigma_k x_l - x_{s_k(l)}\sigma_k)\mathbf{e}(\mathbf{i}) = \begin{cases} -\mathbf{e}(\mathbf{i}) & \text{if } l = k, i_k = i_{k+1} \\ \mathbf{e}(\mathbf{i}) & \text{if } l = k+1, i_k = i_{k+1} \\ 0 & \text{else} \end{cases}$

The grading on $\mathfrak{W}_\nu$ is defined as follows.

$$\deg(\mathbf{e}(\mathbf{i})) = 0$$
$$\deg(x_l\mathbf{e}(\mathbf{i})) = 2$$
$$\deg(\pi\mathbf{e}(\mathbf{i})) = \begin{cases} 1 & \text{if } i_1 = q^{\pm 1} \\ 0 & \text{if } i_1 \neq q^{\pm 1} \end{cases}$$
$$\deg(\sigma_k\mathbf{e}(\mathbf{i})) = \begin{cases} |i_k \to i_{k+1}| + |i_{k+1} \to i_k| & \text{if } i_k \neq i_{k+1} \\ -2 & \text{if } i_k = i_{k+1} \end{cases}$$

where $|i_k \to i_{k+1}|$ denotes the number of arrows from i_k to i_{k+1} in the quiver $\Gamma_{I\cup J}$.

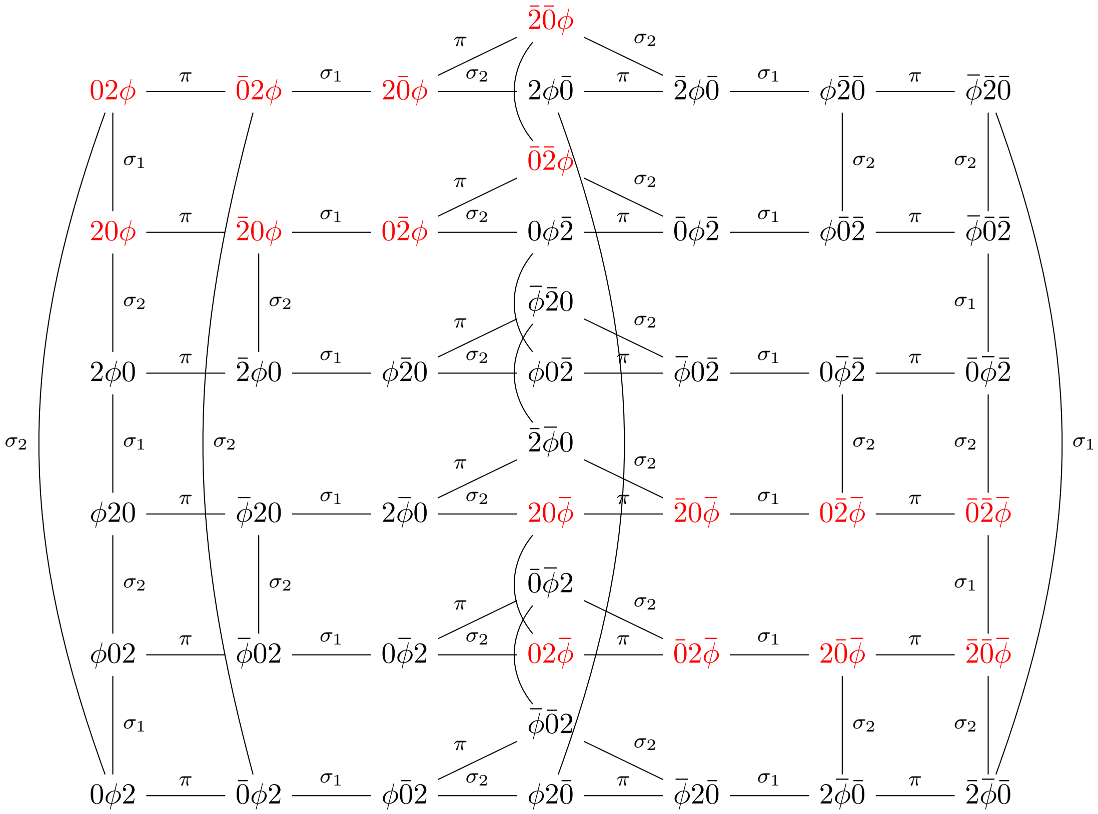
If $\nu = 0$ we set $\mathfrak{W}_\nu = \mathbf{k}$ as a graded $\mathbf{k}$-algebra.

Remark 3.2.3. Take $\nu \in {}^\theta\mathbb{N}(I \cup J)$ and the separated VV algebra $\mathfrak{W}_\nu$. We remark here that there are no arrows between any $i \in I$ and $j \in J$. So, for some $\mathbf{e}(\mathbf{i})$ with $i_k = i$ and $i_{k+1} = j$, it is always the case that $\sigma_k^2\mathbf{e}(\mathbf{i}) = \mathbf{e}(\mathbf{i})$.

Note 3.2.4. Fix $\nu \in {}^\theta\mathbb{N}(I \cup J)$. We can write $\nu = \nu_1 + \nu_2$, for $\nu_1 \in {}^\theta\mathbb{N}I$ and $\nu_2 \in {}^\theta\mathbb{N}J$. We note here that $(i_1, \dots, i_{m_1}, j_1, \dots, j_{m_2}) \in {}^\theta(I \cup J)^\nu$ for any $(i_1, \dots, i_{m_1}) \in {}^\theta I^{\nu_1}$ and any $(j_1, \dots, j_{m_2}) \in {}^\theta J^{\nu_2}$. We will write $(\mathbf{ij})$ for such a tuple, where $\mathbf{i} = (i_1, \dots, i_{m_1}) \in {}^\theta I^{\nu_1}$ and $\mathbf{j} = (j_1, \dots, j_{m_2}) \in {}^\theta J^{\nu_2}$.

Example 3.2.5. Fix $\mathbb{Z} \rtimes \{\pm 1\}$-orbits I_λ and J_μ. Take $\nu_1 = p^{-2}\lambda^{-1} + \lambda^{-1} + \lambda + p^2\lambda \in {}^\theta\mathbb{N}I_\lambda$, $\nu_2 = \mu^{-1} + \mu \in {}^\theta\mathbb{N}J_\mu$ and put $\nu = \nu_1 + \nu_2$.

In an effort to reduce notation let us write 0 for λ, 2 for $p^2\lambda$, ϕ for μ, $\bar{0}$ for λ^{-1}, $\bar{2}$ for $p^{-2}\lambda^{-1}$ and $\bar{\phi}$ for μ^{-1}. The vertices $\mathbf{e}(i_1, i_2, i_3)$ in the following quiver will be written $i_1i_2i_3$. The arrows will be represented by a labelled line with the understanding that each line represents two arrows; the source of each one being the target of the other. As usual the x_i arrows have been omitted. Then the quiver associated to $\mathfrak{W}_\nu$ is below.



Let $\mathbf{e} = \sum_{\substack{\mathbf{i}\in {}^\theta I^{\nu_1} \\ \mathbf{j}\in {}^\theta J^{\nu_2}}} \mathbf{e}(\mathbf{ij})$. So, in the example above, $\mathbf{e}$ is the sum of the idempotents highlighted in red in the quiver. In this section we will see that $\mathfrak{W}_\nu \mathbf{e}$ is a progenerator in $\mathfrak{W}_\nu$-Mod such that $\mathbf{e}\mathfrak{W}_\nu\mathbf{e} \cong \mathfrak{W}_{\nu_1} \otimes_{\mathbf{k}} \mathfrak{W}_{\nu_2}$, from which it follows that $\mathfrak{W}_\nu$ and $\mathfrak{W}_{\nu_1} \otimes_{\mathbf{k}} \mathfrak{W}_{\nu_2}$ are Morita equivalent.

From here until the end of this section we fix the following notation. Let $I, J \subset \mathbf{k}^\times$ be $\mathbb{Z} \rtimes \{\pm 1\}$-orbits with $I \cap J = \emptyset$. Take $\nu \in {}^\theta\mathbb{N}(I \cup J)$ and write $\nu = \nu_1 + \nu_2$, for $\nu_1 \in {}^\theta\mathbb{N}I$ with $|\nu_1| = m_1$ and $\nu_2 \in {}^\theta\mathbb{N}J$ with $|\nu_2| = m_2$. Let $m = m_1 + m_2$. Set $\mathbf{e} := \sum_{\substack{\mathbf{i}\in {}^\theta I^{\nu_1} \\ \mathbf{j}\in {}^\theta J^{\nu_2}}} \mathbf{e}(\mathbf{ij})$, using the notation as in Note 3.2.4.

Let $Q := \langle s_0, s_1, \ldots, s_{m_1-1}, s_{m_1} \cdots s_1 s_0 s_1 \cdots s_{m_1}, s_{m_1+1}, \ldots, s_{m_1+m_2-1} \rangle \subset W_m^B$, a quasi-parabolic subgroup of W_m^B.

Lemma 3.2.6. *There is a group isomorphism $W_{m_1}^B \times W_{m_2}^B \cong Q$.*

Proof. Let

$$\phi : W_{m_1}^B \times W_{m_2}^B \longrightarrow Q$$

be the group homomorphism defined on generators as follows.

$$\begin{aligned}
(e_1, e_2) &\mapsto e \\
(s_i, e_2) &\mapsto s_i \quad 0 \le i \le m_1 - 1 \\
(e_1, s_j) &\mapsto s_{j+m_1} \quad 1 \le j \le m_2 - 1 \\
(e_1, s_0) &\mapsto s_{m_1} \cdots s_1 s_0 s_1 \cdots s_{m_1}.
\end{aligned}$$

One can check that ϕ really is a well-defined group homomorphism by checking the relations. In particular,

$$\phi\big((e_1, s_0 s_1 s_0 s_1)\big) = \phi\big((e_1, s_1 s_0 s_1 s_0)\big).$$

The homomorphism ϕ is surjective since it is surjective on the generators of Q. Then, since $|W^B_{m_1} \times W^B_{m_2}| = |Q|$, it follows that ϕ is bijective and we have a group isomorphism $W^B_{m_1} \times W^B_{m_2} \cong Q$. $\square$

Proposition 3.2.7. *There is a* $\boldsymbol{k}$*-algebra isomorphism* $\mathfrak{W}_{\nu_1} \otimes_{\boldsymbol{k}} \mathfrak{W}_{\nu_2} \cong \boldsymbol{e}\mathfrak{W}_{\nu}\boldsymbol{e}$*.*

Proof. Define a map $\psi : \mathfrak{W}_{\nu_1} \otimes_{\mathbf{k}} \mathfrak{W}_{\nu_2} \longrightarrow \mathbf{e}\mathfrak{W}_{\nu_1+\nu_2}\mathbf{e}$ by,

$$\begin{aligned}
\mathbf{e}(\mathbf{i}) \otimes \mathbf{e}(\mathbf{j}) &\mapsto \mathbf{e}(\mathbf{ij}) \\
x_k \mathbf{e}(\mathbf{i}) \otimes \mathbf{e}(\mathbf{j}) &\mapsto x_k \mathbf{e}(\mathbf{ij}) \quad 1 \le k \le m_1 \\
\sigma_k \mathbf{e}(\mathbf{i}) \otimes \mathbf{e}(\mathbf{j}) &\mapsto \sigma_k \mathbf{e}(\mathbf{ij}) \quad 1 \le k \le m_1 - 1 \\
\pi \mathbf{e}(\mathbf{i}) \otimes \mathbf{e}(\mathbf{j}) &\mapsto \pi \mathbf{e}(\mathbf{ij}) \\
\mathbf{e}(\mathbf{i}) \otimes x_k \mathbf{e}(\mathbf{j}) &\mapsto x_{m_1+k} \mathbf{e}(\mathbf{ij}) \quad 1 \le k \le m_2 \\
\mathbf{e}(\mathbf{i}) \otimes \sigma_k \mathbf{e}(\mathbf{j}) &\mapsto \sigma_{m_1+k} \mathbf{e}(\mathbf{ij}) \quad 1 \le k \le m_2 - 1 \\
\mathbf{e}(\mathbf{i}) \otimes \pi \mathbf{e}(\mathbf{j}) &\mapsto \sigma_{m_1} \cdots \sigma_1 \pi \sigma_1 \cdots \sigma_{m_1} \mathbf{e}(\mathbf{ij}),
\end{aligned}$$

extending $\mathbf{k}$-linearly and multiplicatively. Then, by inspection of the defining relations, one can see that ψ is well-defined and is therefore a morphism of $\mathbf{k}$-algebras. Let $Q = \big\langle s_0, s_1, \ldots, s_{m_1-1}, s_{m_1} \cdots s_1 s_0 s_1 \cdots s_{m_1}, s_{m_1+1}, \ldots, s_{m-1} \big\rangle \subset W^B_m$ be the subgroup of W^B_m as in Lemma 3.2.6. For each $w \in Q$, fix a reduced expression and consider

$$\mathcal{B}' = \{\sigma_w x_1^{n_1} \cdots x_m^{n_m} \mathbf{e}(\mathbf{ij}) \mid w \in Q, n_k \in \mathbb{N}_0 \ \forall k, \mathbf{i} \in {}^{\theta}I^{\nu_1}, \mathbf{j} \in {}^{\theta}J^{\nu_2}\}.$$

This set is linearly independent because it is a subset of the basis given for VV algebras in Lemma 2.3.11. $\mathcal{B}'$ spans $\mathbf{e}\mathfrak{W}_{\nu_1+\nu_2}\mathbf{e}$ because the $w \in Q$ are precisely the elements of W^B_m that permute the $(i_1, \ldots, i_{m_1})$ and the $(j_1, \ldots, j_{m_2})$, and which do not intertwine elements $i_r \in I$ with elements $j_t \in J$. So $\mathcal{B}'$ is a $\mathbf{k}$-basis for $\mathbf{e}\mathfrak{W}_{\nu_1+\nu_2}\mathbf{e}$.

We can now calculate the graded dimension of these algebras and show that they are indeed equal. First we calculate the graded dimension of $\mathbf{e}\mathfrak{W}_\nu\mathbf{e}$ using $\mathcal{B}'$. Referring back to Example 1.1.10, we know that the polynomial part of this basis contributes a factor of

$\frac{1}{(1-q^2)^m}$ to this graded dimension. Then,

$$\dim_q(\mathbf{e}\mathfrak{W}_\nu\mathbf{e}) = \frac{1}{(1-q^2)^{m_1+m_2}} \sum_{\substack{w\in Q \\ \mathbf{i}\in^\theta I^{\nu_1} \\ \mathbf{j}\in^\theta J^{\nu_2}}} q^{\deg(\sigma_w\mathbf{e}(\mathbf{ij}))}.$$

Consider $\sigma_w\mathbf{e}(\mathbf{ij})$, for some $w \in Q$. Note that

$$\begin{array}{lcl} \deg(\sigma_{s_{m_1}\cdots s_1 s_0 s_1\cdots s_{m_1}}\mathbf{e}(\mathbf{ij})) & = & \deg(\sigma_{s_0}\mathbf{e}(\mathbf{j})) \\ \deg(\sigma_{s_{m_1+1}}\mathbf{e}(\mathbf{ij})) & = & \deg(\sigma_{s_1}\mathbf{e}(\mathbf{j})) \\ & \vdots & \\ \deg(\sigma_{s_{m_1+m_2-1}}\mathbf{e}(\mathbf{ij})) & = & \deg(\sigma_{s_{m_2-1}}\mathbf{e}(\mathbf{j})). \end{array}$$

Similarly, $\deg(\sigma_{s_i}\mathbf{e}(\mathbf{ij})) = \deg(\sigma_{s_i}\mathbf{e}(\mathbf{i}))$ for all $0 \le i \le m_1 - 1$. Hence,

$$\deg(\sigma_w\mathbf{e}(\mathbf{ij})) = \deg(\sigma_u\mathbf{e}(\mathbf{i})) + \deg(\sigma_v\mathbf{e}(\mathbf{j}))$$

for some $u \in W^B_{m_1}$, $v \in W^B_{m_2}$. Then,

$$\sum_{\substack{w\in Q \\ \mathbf{i}\in^\theta I^{\nu_1} \\ \mathbf{j}\in^\theta J^{\nu_2}}} q^{\deg(\sigma_w\mathbf{e}(\mathbf{ij}))} = \sum_{\substack{u\in W^B_{m_1}, v\in W^B_{m_2} \\ \mathbf{i}\in^\theta I^{\nu_1} \\ \mathbf{j}\in^\theta J^{\nu_2}}} q^{\deg(\sigma_u\mathbf{e}(\mathbf{i}))+\deg(\sigma_v\mathbf{e}(\mathbf{j}))}.$$

On the other hand, for $\mathfrak{W}_{\nu_1} \otimes \mathfrak{W}_{\nu_2}$,

$$\begin{aligned} \dim_q(\mathfrak{W}_{\nu_1} \otimes \mathfrak{W}_{\nu_2}) &= \frac{1}{(1-q^2)^{m_1}} \sum_{\substack{u\in W^B_{m_1} \\ \mathbf{i}\in^\theta I^{\nu_1}}} q^{\deg(\sigma_u\mathbf{e}(\mathbf{i}))} \frac{1}{(1-q^2)^{m_2}} \sum_{\substack{v\in W^B_{m_2} \\ \mathbf{j}\in^\theta J^{\nu_2}}} q^{\deg(\sigma_v\mathbf{e}(\mathbf{j}))} \\ &= \frac{1}{(1-q^2)^{m_1+m_2}} \sum_{\substack{u\in W^B_{m_1}, v\in W^B_{m_2} \\ \mathbf{i}\in^\theta I^{\nu_1} \\ \mathbf{j}\in^\theta J^{\nu_2}}} q^{\deg(\sigma_u\mathbf{e}(\mathbf{i}))+\deg(\sigma_v\mathbf{e}(\mathbf{j}))}. \end{aligned}$$

Hence, we have shown

$$\dim_q(\mathbf{e}\mathfrak{W}_\nu\mathbf{e}) = \dim_q(\mathfrak{W}_{\nu_1} \otimes \mathfrak{W}_{\nu_2}).$$

To prove the claimed result it now suffices to prove that ψ is surjective.

Rename the generators of Q as follows. Put,

$$c_{i+1} = \begin{cases} s_{m_1}\cdots s_1 s_0 s_1 \cdots s_{m_1} & i = m_1 \\ s_i & i \neq m_1 \end{cases}$$

so that Q is generated by $\{c_1, c_2, \dots, c_{m_1+m_2}\}$. Define $\ell_Q : Q \longrightarrow \mathbb{N}_0$ in the following way. For the identity element $1 \in Q$ put $\ell_Q(1) = 0$. Any $w \in Q$ can be written as a product $w = c_{i_1}\cdots c_{i_k}$. Pick these generators in such a way that k is minimal. Then $\ell_Q(w) = k$.

For example, $\ell_Q(c_i) = 1$ for every i. We say that $w = c_{i_1} \cdots c_{i_k} \in Q$ is an ℓ_Q-reduced expression for w if $\ell_Q(w) = k$.

Let $\ell : W_m^B \longrightarrow \mathbb{N}_0$ be the usual length function on W_m^B, as defined in Section 1.2. The isomorphism ϕ from Lemma 3.2.6 demonstrates that any ℓ_Q-reduced expression $c_{i_1} \cdots c_{i_k}$ is also a reduced expression with respect to ℓ. For every $w \in Q$ fix an ℓ_Q-reduced expression $w = c_{i_1} \cdots c_{i_k}$. Then $\sigma_w \mathbf{e}(\mathbf{ij}) = \sigma_{c_{i_1}} \cdots \sigma_{c_{i_k}} \mathbf{e}(\mathbf{ij})$ for each $\mathbf{i} \in {}^\theta I^{\nu_1}, \mathbf{j} \in {}^\theta J^{\nu_2}$.

It is clear that ψ is surjective on elements $\mathbf{e}(\mathbf{ij})$ and $x_i\mathbf{e}(\mathbf{ij})$, for all $\mathbf{i} \in {}^\theta I^{\nu_1}$, $\mathbf{j} \in {}^\theta J^{\nu_2}$. Notice that

$$\begin{aligned} \psi(\sigma_k \mathbf{e}(\mathbf{i}) \otimes \mathbf{e}(\mathbf{j})) &= \sigma_{c_{k+1}} \mathbf{e}(\mathbf{ij}) \quad \text{for } 0 \leq k \leq m_1 - 1 \\ \psi(\mathbf{e}(\mathbf{i}) \otimes \sigma_k \mathbf{e}(\mathbf{j})) &= \sigma_{c_{m_1+k+1}} \mathbf{e}(\mathbf{ij}) \quad \text{for } 0 \leq k \leq m_2 - 1. \end{aligned}$$

Therefore ψ is surjective on the basis $\mathcal{B}'$ and hence on $\mathbf{e}\mathfrak{W}_{\nu_1+\nu_2}\mathbf{e}$. $\square$

Lemma 3.2.8. *Let* $\mathbf{e} = \sum_{\substack{\boldsymbol{i} \in {}^\theta I^{\nu_1} \\ \boldsymbol{j} \in {}^\theta J^{\nu_2}}} \boldsymbol{e}(\boldsymbol{ij})$. *Then* $\mathbf{e}$ *is full in the separated VV algebra* $\mathfrak{W}_\nu$.

Proof. We must show $\mathfrak{W}_\nu = \mathfrak{W}_\nu \mathbf{e} \mathfrak{W}_\nu$. We take any idempotent $\mathbf{e}(\mathbf{k}) \in \mathfrak{W}_\nu$, not a summand of $\mathbf{e}$, and show that $\mathbf{e}(\mathbf{k}) \in \mathfrak{W}_\nu \mathbf{e} \mathfrak{W}_\nu$. Suppose $k_1 = i \in I$ (the same argument holds if $k_1 = j \in J$). Let $\varepsilon_1, \ldots, \varepsilon_r$ denote the positions of entries belonging to I, and assume $\varepsilon_1 < \cdots < \varepsilon_r$. Since $k_1 = i \in I$, we have $\varepsilon_1 = 1$. Let $w_1 = s_2 s_3 s_4 \cdots s_{\varepsilon_2 - 1} \in \mathfrak{S}_m$, where the s_j are the generators of $\mathfrak{S}_m$. Then,

$$\sigma_{w_1}^\rho \sigma_{w_1} \mathbf{e}(\mathbf{k}) = \mathbf{e}(\mathbf{k})$$

because $\sigma_r^2 \mathbf{e}(\mathbf{i}) = \mathbf{e}(\mathbf{i})$ when there are no arrows between i_r and i_{r+1}.

Suppose $\sigma_{w_1} \mathbf{e}(\mathbf{k}) = \mathbf{e}(\mathbf{k}_1) \sigma_{w_1}$. Then the first two entries of $\mathbf{k}_1$ are elements of I. Let $w_2 = s_3 s_4 s_5 \cdots s_{\varepsilon_3 - 1} \in \mathfrak{S}_m$. Then,

$$\sigma_{w_2}^\rho \sigma_{w_2} \mathbf{e}(\mathbf{k}_1) = \mathbf{e}(\mathbf{k}_1)$$

for the same reasoning as above. Suppose $\sigma_{w_2} \mathbf{e}(\mathbf{k}_1) = \mathbf{e}(\mathbf{k}_2) \sigma_{w_2}$. Then the first three entries of $\mathbf{k}_2$ are elements of I.

Continuing like this we obtain $\sigma_{w_2}, \ldots, \sigma_{w_{r-1}}$ with $\sigma_{w_t}^\rho \sigma_{w_t} \mathbf{e}(\mathbf{k}_{t-1}) = \mathbf{e}(\mathbf{k}_{t-1})$, for t with $2 \leq t \leq r - 1$. Then we have,

$$\sigma_{w_1}^\rho \sigma_{w_2}^\rho \cdots \sigma_{w_{r-1}}^\rho \mathbf{e}(\mathbf{ij}) \sigma_{w_{r-1}} \cdots \sigma_{w_2} \sigma_{w_1} \mathbf{e}(\mathbf{k}) = \mathbf{e}(\mathbf{k}),$$

and $\mathbf{e}(\mathbf{ij})$ is a summand of $\mathbf{e}$. Hence $\mathbf{e}(\mathbf{k}) \in \mathfrak{W}_\nu \mathbf{e} \mathfrak{W}_\nu$ so that $\mathfrak{W}_\nu = \mathfrak{W}_\nu \mathbf{e} \mathfrak{W}_\nu$, as required. $\square$

Corollary 3.2.9. *$\mathfrak{W}_\nu$ and $\mathfrak{W}_{\nu_1} \otimes_{\boldsymbol{k}} \mathfrak{W}_{\nu_2}$ are Morita equivalent.*

Proof. Using Lemma 3.2.8 and Remark 3.1.10, we have that $\mathfrak{W}_\nu\mathbf{e}$ is a progenerator in $\mathfrak{W}_\nu$-Mod. By Corollary 3.1.18, it remains to show $\mathrm{End}_{\mathfrak{W}_\nu}(\mathfrak{W}_\nu\mathbf{e}) \cong \mathfrak{W}_{\nu_1} \otimes \mathfrak{W}_{\nu_2}$. But, by Proposition 3.2.7 and the fact that $\mathrm{End}_{\mathfrak{W}_\nu}(\mathfrak{W}_\nu\mathbf{e}) \cong \mathbf{e}\mathfrak{W}_\nu\mathbf{e}$, we are done. □

Remark 3.2.10. In this section we have defined families of VV algebras arising from unions of $\mathbb{Z} \rtimes \{\pm 1\}$-orbits. But in fact Corollary 3.2.9 shows that in order to study these VV algebras it suffices to study the families of VV algebras arising from a single $\mathbb{Z} \rtimes \{\pm 1\}$-orbit, as we defined in Section 2.3. A slightly modified version of this theorem, using the same proof, explains why we may assume that ν has connected support.

An Alternative Proof

Originally, we wanted to provide more of a categorical proof of 3.2.9. However, we could only obtain a sketch of proof for one of the results leading up to this proof. We include this approach here because we like the methods involved.

Let A be any algebra over a field $\mathbf{k}$, and let $0 \neq e \in A$ be an idempotent. Let $F : A\text{-Mod} \longrightarrow eAe\text{-Mod}$ be the following functor. On modules we have $F : M \mapsto eM$, and if $g : M \longrightarrow N$ is a module morphism then $F(g) : eM \longrightarrow eN$ is the restriction of g. Note that eM is a well-defined eAe-module. We now state and prove the following well-known result which can be found in the literature.

Proposition 3.2.11. *If $S \in A$-Mod is simple then $eS \in eAe$-Mod is either a simple module or is zero.*

Proof. Let L be any non-zero eAe-submodule of eS. Then $L = eL$. Therefore $AL = AeL$, which is a non-zero A-submodule of S, and so $S = AL = AeL$. So

$$eS = e(AeL) = (eAe)L \subseteq L.$$

So, if eS is non-zero then it is a simple eAe-module. □

We can also define a functor in the opposite direction $G : eAe\text{-Mod} \longrightarrow A\text{-Mod}$ sending M to $Ae \otimes_{eAe} M$. This is well-defined since Ae is a left A-module and a right eAe-module, and $M \in eAe$-Mod. In general, for $S \in eAe$-Mod simple, $G(S)$ is not a simple module. The following well-known result can be found in most algebra textbooks. We state it here and will make use of it in the lemma to follow.

Lemma 3.2.12 (The 5-Lemma)**.** *Let A be a ring. In any commutative diagram of A-modules*

$$\begin{tikzcd}
A' \arrow[r] \arrow[d, "a"', "\sim"] & B' \arrow[r] \arrow[d, "b"', "\sim"] & C' \arrow[r] \arrow[d, "c"'] & D' \arrow[r] \arrow[d, "d"', "\sim"] & E' \arrow[d, "e"', "\sim"] \\
A \arrow[r] & B \arrow[r] & C \arrow[r] & D \arrow[r] & E
\end{tikzcd}$$

with exact rows, if a, b, d and e are isomorphisms then c is also an isomorphism.

Lemma 3.2.13. *Let A,B be $\boldsymbol{k}$-algebras. Suppose we have exact functors*

$$A\text{-}Mod \underset{F}{\overset{G}{\leftrightarrows}} B\text{-}Mod$$

each of which send simple modules to simple modules and such that (F, G) is an adjoint pair. Then A and B have equivalent categories of finite length modules.

Proof. Let $\mathcal{A} = A$-Mod and $\mathcal{B} = B$-Mod. Since (F, G) is an adjoint pair we have, for any $M \in \mathcal{A}$, $N \in \mathcal{B}$, the following isomorphism.

$$\mathrm{Hom}_{\mathcal{B}}(FM, N) \xrightarrow{\alpha_{M,N}} \mathrm{Hom}_{\mathcal{A}}(M, GN).$$

In particular,

$$\mathrm{Hom}_{\mathcal{B}}(FGN, N) \cong \mathrm{Hom}_{\mathcal{A}}(GN, GN) \text{ and } \mathrm{Hom}_{\mathcal{B}}(FM, FM) \cong \mathrm{Hom}_{\mathcal{A}}(M, GFM)$$

For any $M \in \mathcal{A}$-Mod and any $N \in \mathcal{B}$-Mod let $\alpha_{M,FM}(1_{FM}) = g_M$ and let $\alpha^{-1}_{GN,N}(1_{GN}) = h_N$. By induction on the length of M we prove that these morphisms are isomorphisms.

If M is simple then GFM is simple and $g_M \in \mathrm{Hom}_{\mathcal{A}}(M, GFM)$. By Schur's Lemma and the fact that $g_M \neq 0$, we have that g_M is indeed an isomorphism. Now assume that they are isomorphisms for all modules of length less than r. Take $M \in \mathcal{A}$ with $\ell(M) = r$.

For any maximal submodule L of M and simple module $S = M/L$, we have the following diagram of morphisms, where the top row is a short exact sequence.

$$\begin{tikzcd}
L \arrow[r, hook, "i"] \arrow[d, "g_L"', "\sim"] & M \arrow[r, two heads, "s"] \arrow[d, "g_M"'] & S \arrow[d, "g_S"', "\sim"] \\
GFL \arrow[r, hook, "GFi"'] & GFM \arrow[r, two heads, "GFs"'] & GFS
\end{tikzcd}$$

In fact, since GF is exact, both rows in this diagram are short exact sequences. By the inductive hypothesis, g_L and g_S are isomorphisms. We can prove that this diagram commutes (see below). Hence, by the 5-Lemma, we know that g_M is also an isomorphism.

Now, for any morphism $B_1 \xrightarrow{b} B_2$ in $\mathcal{B}$

$$\begin{tikzcd}
B_1 \arrow[r, "b"] \arrow[d, "k_{B_1}"'] & B_2 \arrow[d, "k_{B_1}"] \\
FGB_1 \arrow[r, "FGb"'] & FGB_2
\end{tikzcd}$$

is a commutative diagram and for any morphism $A_1 \xrightarrow{a} A_2$ in $\mathcal{A}$

$$\begin{tikzcd}
A_1 \arrow[r, "a"] \arrow[d, "g_{A_1}"'] & A_2 \arrow[d, "g_{A_1}"] \\
GFA_1 \arrow[r, "GFa"'] & GFA_2
\end{tikzcd}$$

is a commutative diagram. See below for proofs of this. Hence $FG \cong 1_{B\text{-mod}}$ and $GF \cong 1_{A\text{-mod}}$ so that A and B are Morita equivalent.

It remains to show that the above diagrams commute. Since (F, G) is an adjoint pair we have the following commutative diagrams.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{B}}(FM, FM) \arrow[r, "(Fi)^*"] \arrow[d, "\alpha_{M,FM}"'] & \mathrm{Hom}_{\mathcal{B}}(FL, FM) \arrow[d, "\alpha_{L,FM}"] \\
\mathrm{Hom}_{\mathcal{A}}(M, GFM) \arrow[r, "i^*"'] & \mathrm{Hom}_{\mathcal{A}}(L, GFM)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{B}}(FL, FL) \arrow[r, "(Fi)_*"] \arrow[d, "\alpha_{L,FL}"'] & \mathrm{Hom}_{\mathcal{B}}(FL, FM) \arrow[d, "\alpha_{L,FM}"] \\
\mathrm{Hom}_{\mathcal{A}}(L, GFL) \arrow[r, "(GFi)_*"'] & \mathrm{Hom}_{\mathcal{A}}(L, GFM)
\end{tikzcd}$$

The first square gives

$$\begin{aligned}
i^* \circ \alpha_{M,FM}(1_{FM}) &= \alpha_{L,FM} \circ (Fi)^*(1_{FM}) \\
\Longrightarrow g_M \circ i &= \alpha_{L,FM}(Fi).
\end{aligned}$$

From the second square we get

$$\begin{aligned}
(GFi)_* \circ \alpha_{L,FL}(1_{FL}) &= \alpha_{L,FM} \circ (Fi)_*(1_{FL}) \\
\Longrightarrow GFi \circ g_L &= \alpha_{L,FM}(Fi).
\end{aligned}$$

Together, these give $GFi \circ g_L = g_M \circ i$. Similarly, we can show that $GFs \circ g_M = g_S \circ s$. Hence all aforementioned squares commute and the proof is complete. $\square$

The following discussion follows [GP00], Chapter 1. For $m \geq 1$, let W_m be the subgroup of $\mathrm{GL}_m(\mathbb{R})$ consisting of matrices with exactly one non-zero entry in each row and column. Let this entry be either 1 or -1. Then W_m is a finite group of order $2^m m!$.
Let $S_m \subset W_m$ be the subgroup of all permutation matrices. That is, those matrices with precisely one non-zero entry in each row and column, that entry being 1. For $1 \leq i \leq m-1$, let $s_i \in S_m$ be the matrix obtained from the identity matrix by swapping the i^{th} and $(i+1)^{\text{th}}$ columns. Then $\{s_1, \ldots, s_{m-1}\}$ generate S_m and it is clear that S_m can be identified with $\mathfrak{S}_m$.

Let $N_m \subset W_m$ be the subgroup consisting of all diagonal matrices. Then N_m is an abelian group with $|N_m| = 2^m$. Moreover, since $N_m \cap S_m = \{1\}$, we have

$$|N_m \cdot S_m| = \frac{|N_m||S_m|}{|N_m \cap S_m|} = 2^m m! = |W_m|$$

so $N_m \cdot S_m = W_m$.

For $0 \leq i \leq m-1$, let τ_i be the diagonal matrix whose diagonal entries are all 1 except the $(i+1)^{\text{th}}$ diagonal entry which is -1. Then $\tau_0, \tau_1, \ldots, \tau_{m-1}$ generate N_m. Note also that the elements τ_i are conjugate in W_m:

$$\tau_i = s_i \tau_{i-1} s_i = s_i \cdots s_1 \tau_0 s_1 \cdots s_i \quad \text{for } 1 \leq i \leq m-1.$$

Thus, the elements $\{\tau_0, s_1, \ldots, s_{m-1}\}$ generate W_m and one can check that the following relations hold

$$\begin{aligned}
&\tau_0^2 = 1 = s_i^2 \quad \text{for } 1 \leq i \leq m-1 \\
&\tau_0 s_1 \tau_0 s_1 = s_1 \tau_0 s_1 \tau_0 \\
&\tau_0 s_i = s_i \tau_0 \quad \text{for } i > 1 \\
&s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad \text{for } 1 \leq i \leq m-1 \\
&s_i s_j = s_j s_i \quad \text{for } |i-j| > 1.
\end{aligned}$$

So W_m is a Weyl group of type B_m. This shows that every element $w \in W_m^B$ can be expressed uniquely as $w = s\tau_0^{\varepsilon_0}\tau_1^{\varepsilon_1} \cdots \tau_{m-1}^{\varepsilon_{m-1}}$, for some $s \in \mathfrak{S}_m$ and $\varepsilon_i \in \{0,1\}$, for all $0 \leq i \leq m-1$. Let (m_1, m_2) be a partition of m. Since every $s \in \mathfrak{S}_m$ can be written uniquely as $s = \eta\omega$, for $\eta \in \mathcal{D}(\mathfrak{S}_m/(\mathfrak{S}_{m_1} \times \mathfrak{S}_{m_2}))$ and $\omega \in \mathfrak{S}_{m_1} \times \mathfrak{S}_{m_2}$, every $w \in W_m^B$ can be expressed uniquely in the form

$$w = \eta\omega\tau_0^{\varepsilon_0}\tau_1^{\varepsilon_1} \cdots \tau_{m-1}^{\varepsilon_{m-1}}$$

for $\eta \in \mathcal{D}(\mathfrak{S}_m/(\mathfrak{S}_{m_1} \times \mathfrak{S}_{m_2}))$, $\omega \in \mathfrak{S}_{m_1} \times \mathfrak{S}_{m_2}$ and $\varepsilon_i \in \{0,1\}$ $\forall i$. But, since $(\mathfrak{S}_{m_1} \times \mathfrak{S}_{m_2}) \cdot N_m = W_{m_1} \times W_{m_2}$, we can write, for any $w \in W_m^B$, $w = \eta\theta$ for $\eta \in \mathcal{D}(\mathfrak{S}_m/(\mathfrak{S}_{m_1} \times \mathfrak{S}_{m_2}))$

and $\theta \in W_{m_1} \times W_{m_2}$.

Proposition 3.2.14. *$\mathfrak{W}_\nu \boldsymbol{e}$ is free as a right $\boldsymbol{e}\mathfrak{W}_\nu \boldsymbol{e}$-module on basis $\mathcal{D}(\mathfrak{S}_m/(\mathfrak{S}_{m_1} \times \mathfrak{S}_{m_2}))$.*

Proof. From the discussion above, we have seen that every $w \in W_m^B$ can be expressed in uniquely the form $w = \eta\theta$ for some $\eta \in \mathcal{D}(\mathfrak{S}_m/(\mathfrak{S}_{m_1} \times \mathfrak{S}_{m_2}))$ and some $\theta \in W_{m_1} \times W_{m_2}$. By Proposition 1.4.1, every $\theta \in W_{m_1}^B \times W_{m_2}^B$ can be expressed in the form $\theta = \chi\tau$, for some $\chi \in \mathcal{D}((W_{m_1}^B \times W_{m_2}^B)/(W_{m_1}^B \times \mathfrak{S}_{m_2}))$ and $\tau \in W_{m_1}^B \times \mathfrak{S}_{m_2}$. For every $\theta \in W_{m_1}^B \times W_{m_2}^B$, fix a reduced expression of this form. For every $w \in W_m^B$ fix a reduced expression of the form $w = \eta\theta$. Then every $w \in W_m^B$ has a reduced expression of the form,

$$w = \eta\theta = \eta\chi\tau$$

where $\ell(\chi\tau) = \ell(\chi) + \ell(\tau)$.

Claim 3.2.15. *$\sigma_{\eta\theta}\boldsymbol{e}(\boldsymbol{ij}) = \sigma_\eta\sigma_\theta\boldsymbol{e}(\boldsymbol{ij})$.*

Proof. Note that, in the expression $\sigma_\eta\sigma_\theta\mathbf{e}(\mathbf{ij})$, σ_η does not depend upon the choice of reduced expression of $\eta \in \mathcal{D}(\mathfrak{S}_m/(\mathfrak{S}_{m_1} \times \mathfrak{S}_{m_2}))$. To see this, first note that $\sigma_\theta\mathbf{e}(\mathbf{ij}) = \mathbf{e}(\mathbf{i}'\mathbf{j}')\sigma_\theta$, for some $\mathbf{i}' \in {}^\theta I^{\nu_1}, \mathbf{j}' \in {}^\theta I^{\nu_2}$, since $\theta \in W_{m_1} \times W_{m_2}$. So $\sigma_\eta\sigma_\theta\mathbf{e}(\mathbf{ij}) = \sigma_\eta\mathbf{e}(\mathbf{i}'\mathbf{j}')\sigma_\theta\mathbf{e}(\mathbf{ij})$, and we know that σ_η interchanges elements $i_r \in I$ with elements $j_t \in J$, since $\eta \in \mathcal{D}(\mathfrak{S}_m/(\mathfrak{S}_{m_1} \times \mathfrak{S}_{m_2}))$.

We write e instead of $\mathbf{e}(\mathbf{ij})$ to reduce notation.

From Proposition 1.5.22 we have

$$\mathcal{D}(\mathfrak{S}_m/(\mathfrak{S}_{m_1} \times \mathfrak{S}_{m_2}))\mathcal{D}((W_{m_1}^B \times W_{m_2}^B)/(W_{m_1}^B \times \mathfrak{S}_{m_2})) \subseteq \mathcal{D}(W_m^B/(W_{m_1}^B \times \mathfrak{S}_{m_2})).$$

Hence, $\eta\chi = \xi \in \mathcal{D}(W_m^B/(W_{m_1}^B \times \mathfrak{S}_{m_2}))$ and $\eta\theta = \eta\chi\tau = \xi\tau$. Note that $\ell(\xi\tau) = \ell(\xi) + \ell(\tau)$. So, it suffices to prove $\sigma_{\eta\chi}\mathbf{e}(\mathbf{ij}) = \sigma_\eta\sigma_\chi\mathbf{e}(\mathbf{ij})$.

Subclaim 3.2.16.

(i) $\sigma_\chi\boldsymbol{e}(\boldsymbol{ij})$ does not depend on the choice of reduced expression of $\chi \in \mathcal{D}((W_{m_1}^B \times W_{m_2}^B)/(W_{m_1}^B \times \mathfrak{S}_{m_2}))$.

(ii) $\sigma_\xi\boldsymbol{e}(\boldsymbol{ij})$ does not depend on the choice of reduced expression of $\xi \in \mathcal{D}(W_m^B/(W_{m_1}^B \times \mathfrak{S}_{m_2}))$.

Sketched Proof of Subclaim. For a parabolic subgroup W_J in a Coxeter group W let w_0 denote the longest element in W, and let w_J denote the longest element in W_J. By Lemma 1.4.6, $d_J := w_0w_J$ is the longest element in $\mathcal{D}(W/W_J)$ and every element in $\mathcal{D}(W/W_J)$ is a prefix of d_J.

(i) Applying Lemma 1.4.6 to $W = W^B_{m_1} \times W^B_{m_2}$ and $W_J = W^B_{m_1} \times \mathfrak{S}_{m_2}$ we find the longest element in $\mathcal{D}((W^B_{m_1} \times W^B_{m_2})/(W^B_{m_1} \times \mathfrak{S}_{m_2}))$ to be

$$d_J = s_{t_0} s_{m_1+1} s_{t_0} s_{m_1+2} s_{m_1+1} s_{t_0} \cdots s_{m-1} \cdots s_{m_1+1} s_{t_0}$$

where $t_0 = s_{m_1} \cdots s_1 s_0 s_1 \cdots s_{m_1}$. Every other element of $\mathcal{D}((W^B_{m_1} \times W^B_{m_2})/(W^B_{m_1} \times \mathfrak{S}_{m_2}))$ is a prefix of d_J. Thus it suffices to prove that $\sigma_{d_J}\mathbf{e}(\mathbf{ij})$ is independent of the choice of reduced expression of d_J. We make three observations at this point:

- $t_0 = s_{m_1} \cdots s_1 s_0 s_1 \cdots s_{m_1}$ is the unique reduced expression of t_0.
- Sequences of the form $s_{m_1+k} \cdots s_{m_1+1}$, for some $k > 1$, are also unique reduced expressions.
- $t_0 s_r = s_r t_0$ unless $r = m_1 + 1$.

Hence, the only non-trivial braiding arises from $t_0 s_{m_1+1} t_0$.

$$\begin{aligned}
t_0 s_{m_1+1} t_0 &= s_{m_1} \cdots s_1 s_0 s_1 \cdots s_{m_1-1} s_{m_1} \cdot s_{m_1+1} \cdot s_{m_1} s_{m_1-1} \cdots s_1 s_0 s_1 \cdots s_{m_1} \\
&= s_{m_1} \cdots s_1 s_0 s_1 \cdots s_{m_1-1} s_{m_1+1} s_{m_1} s_{m_1+1} s_{m_1-1} \cdots s_1 s_0 s_1 \cdots s_{m_1} \\
&= s_{m_1} \cdots s_1 s_0 s_1 \cdots s_{m_1+1} s_{m_1-1} s_{m_1} s_{m_1-1} s_{m_1+1} \cdots s_1 s_0 s_1 \cdots s_{m_1} \\
&= s_{m_1} \cdots s_1 s_0 s_1 \cdots s_{m_1+1} s_{m_1} s_{m_1-1} s_{m_1} s_{m_1+1} \cdots s_1 s_0 s_1 \cdots s_{m_1} \\
&\;\;\vdots \\
&= s_{m_1} \cdots s_1 s_0 s_{m_1+1} \cdots s_2 s_1 s_2 \cdots s_{m_1+1} s_0 s_1 \cdots s_{m_1}.
\end{aligned}$$

We now check that elements $\sigma_{t_0}\sigma_{m_1+1}\sigma_{t_0}\mathbf{e}(\mathbf{ij}) \in \mathfrak{W}_\nu$ also braid in this way.

$$\begin{aligned}
&\sigma_{t_0}\sigma_{m_1+1}\sigma_{t_0} \\
=&\sigma_{m_1} \cdots \sigma_1 \pi \sigma_1 \cdots \sigma_{m_1-1}\sigma_{m_1} \cdot \sigma_{m_1+1} \cdot \sigma_{m_1}\sigma_{m_1-1} \cdots \sigma_1 \pi \sigma_1 \cdots \sigma_{m_1}\mathbf{e}(\mathbf{ij}).
\end{aligned} \tag{3.2.1}$$

Consider $\sigma_{m_1}\sigma_{m_1+1}\sigma_{m_1}\mathbf{e}(i_1, \ldots, i_{m_1-1}, j_1^{-1}, i_{m_1}, j_2, \ldots, j_b)$, where j_1^{-1} lies in the $m_1{}^{\text{th}}$ entry of the tuple. Since i_{m_1-1} and j_1^{-1} belong to different orbits I and J respectively, there are no arrows between them. Then, according to the relations, we can rebraid:

$$\begin{aligned}
&\sigma_{m_1}\sigma_{m_1+1}\sigma_{m_1}\mathbf{e}(i_1, \ldots, i_{m_1-1}, j_1^{-1}, i_{m_1}, j_2, \ldots, j_b) \\
=&\sigma_{m_1+1}\sigma_{m_1}\sigma_{m_1+1}\mathbf{e}(i_1, \ldots, i_{m_1-1}, j_1^{-1}, i_{m_1}, j_2, \ldots, j_b).
\end{aligned}$$

Now equation 3.2.1 becomes

$$\begin{aligned}
\sigma_{t_0}\sigma_{m_1+1}\sigma_{t_0} &= \sigma_{m_1} \cdots \sigma_1 \pi \sigma_1 \cdots \sigma_{m_1-1}\sigma_{m_1+1}\sigma_{m_1}\sigma_{m_1+1}\sigma_{m_1-1} \cdots \sigma_1 \pi \sigma_1 \cdots \sigma_{m_1}\mathbf{e}(\mathbf{ij}) \\
&= \sigma_{m_1} \cdots \sigma_1 \pi \sigma_1 \cdots \sigma_{m_1+1}\sigma_{m_1-1}\sigma_{m_1}\sigma_{m_1-1}\sigma_{m_1+1} \cdots \sigma_1 \pi \sigma_1 \cdots \sigma_{m_1}\mathbf{e}(\mathbf{ij})
\end{aligned}$$

where, in the second equality, we have used $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i - j| > 1$.

Consider $\sigma_{m_1-1}\sigma_{m_1}\sigma_{m_1-1}\mathbf{e}(i_1, \ldots, j_1^{-1}, i_{m_1-1}, j_2, i_{m_1}, \ldots, j_b)$, where j_1^{-1} now lies in the $(m_1-1)^{\text{th}}$ entry of the tuple. Again, since i_{m_1-1} and j_1^{-1} belong to different orbits I and J respectively, there are no arrows between them. Then, according to the relations, we can rebraid:

$$\begin{aligned}&\sigma_{m_1-1}\sigma_{m_1}\sigma_{m_1-1}\mathbf{e}(i_1,\ldots,j_1^{-1},i_{m_1-1},j_2,i_{m_1},\ldots,j_b)\\ =&\sigma_{m_1}\sigma_{m_1-1}\sigma_{m_1}\mathbf{e}(i_1,\ldots,j_1^{-1},i_{m_1-1},j_2,i_{m_1},\ldots,j_b).\end{aligned}$$

Now equation 3.2.1 becomes

$$\begin{aligned}\sigma_{t_0}\sigma_{m_1+1}\sigma_{t_0}&=\sigma_{m_1}\cdots\sigma_1\pi\sigma_1\cdots\sigma_{m_1-1}\sigma_{m_1+1}\sigma_{m_1}\sigma_{m_1+1}\sigma_{m_1-1}\cdots\sigma_1\pi\sigma_1\cdots\sigma_{m_1}\mathbf{e}(\mathbf{ij})\\ &=\sigma_{m_1}\cdots\sigma_1\pi\sigma_1\cdots\sigma_{m_1+1}\sigma_{m_1-1}\sigma_{m_1}\sigma_{m_1-1}\sigma_{m_1+1}\cdots\sigma_1\pi\sigma_1\cdots\sigma_{m_1}\mathbf{e}(\mathbf{ij})\\ &=\sigma_{m_1}\cdots\sigma_1\pi\sigma_1\cdots\sigma_{m_1+1}\sigma_{m_1}\sigma_{m_1-1}\sigma_{m_1}\sigma_{m_1+1}\cdots\sigma_1\pi\sigma_1\cdots\sigma_{m_1}\mathbf{e}(\mathbf{ij})\\ &=\sigma_{m_1}\cdots\sigma_1\pi\sigma_1\cdots\sigma_{m_1+1}\sigma_{m_1}\sigma_{m_1-2}\sigma_{m_1-1}\sigma_{m_1-2}\sigma_{m_1}\sigma_{m_1+1}\cdots\\ &\qquad\qquad\cdots\sigma_1\pi\sigma_1\cdots\sigma_{m_1}\mathbf{e}(\mathbf{ij}).\end{aligned}$$

Continuing in this way we see that we can always rebraid and so $\sigma_\chi\mathbf{e}(\mathbf{ij})$ does not depend on the choice of reduced expression of χ.

(ii) Applying Lemma 1.4.6 to $W=W_m^B$ and $W_J=W_{m_1}^B\times\mathfrak{S}_{m_2}$ we find the longest element in $\mathcal{D}(W_m^B/(W_{m_1}^B\times\mathfrak{S}_{m_2}))$ to be

$$d_J=s_{t_0}s_{m_1+1}s_{m_1+2}\cdots s_{n-1}s_{t_0}s_{m_1+1}s_{m_1+2}\cdots s_{n-2}\cdots s_{t_0}s_{m_1+1}s_{m_1+2}s_{t_0}s_{m_1+1}s_{t_0}$$

where $t_0=s_{m_1}\cdots s_1s_0s_1\cdots s_{m_1}$. Every other element of $\mathcal{D}((W_{m_1}^B\times W_{m_2}^B)/(W_{m_1}^B\times\mathfrak{S}_{m_2}))$ is a prefix of d_J. Thus it suffices to prove that $\sigma_{d_J}\mathbf{e}(\mathbf{ij})$ is independent of the choice of reduced expression of d_J. The argument now follows in the same way as in (i).

□

We now proceed by induction on $\ell(\eta)$. When $\eta=1$, clearly $\sigma_\eta\sigma_\chi\mathbf{e}(\mathbf{ij})=\sigma_{\eta\chi}\mathbf{e}(\mathbf{ij})$. Now assume $\sigma_\eta\sigma_\chi\mathbf{e}(\mathbf{ij})=\sigma_{\eta\chi}\mathbf{e}(\mathbf{ij})$ for all η with $\ell(\eta)<k$.

Take η with $\ell(\eta)=k$. Let $\eta=s_{i_1}\cdots s_{i_k}$ be a reduced expression. Let $\chi=s_{h_1}\cdots s_{h_u}$ be a reduced expression of χ. Note that $\eta'=s_{i_1}\eta=s_{i_2}\cdots s_{i_k}\in\mathcal{D}(\mathfrak{S}_m/(\mathfrak{S}_{m_1}\times\mathfrak{S}_{m_2}))$. We have,

$$\begin{aligned}\sigma_\eta\sigma_\chi\mathbf{e}(\mathbf{ij})&=\sigma_{i_1}\sigma_{\eta'}\sigma_\chi\mathbf{e}(\mathbf{ij})\\ &=\sigma_{i_1}\sigma_{\eta'\chi}\mathbf{e}(\mathbf{ij})\end{aligned}$$

where the second equality uses the inductive hypothesis. If $\ell(s_{i_1}\eta'\chi)=\ell(s_{i_1})+\ell(\eta'\chi)$ then we are done, so assume otherwise.

By the Exchange Condition 1.3.5, $\eta'\chi$ has a reduced expression beginning with s_{i_1}, say

$\eta'\chi = s_{i_1}s_{j_1}\cdots s_{j_t}$. Then,

$$\begin{aligned}\eta\chi &= s_{i_1}\cdots s_{i_k}s_{h_1}\cdots s_{h_u}\\ &= s_{i_1}\eta'\chi\\ &= s_{j_1}\cdots s_{j_t}.\end{aligned}$$

By Subclaim 3.2.16, $\sigma_{\eta'\chi}\mathbf{e}(\mathbf{ij})$ is independent of the reduced expression of $\xi' = \eta'\chi \in \mathcal{D}(W_m^B/(W_{m_1}^B \times \mathfrak{S}_{m_2}))$ so that $\sigma_{\eta'\chi}\mathbf{e}(\mathbf{ij}) = \sigma_{i_1}\sigma_{j_1}\cdots\sigma_{j_t}\mathbf{e}(\mathbf{ij})$. Since σ_{i_1} interchanges vertices from different orbits,

$$\begin{aligned}\sigma_\eta\sigma_\chi\mathbf{e}(\mathbf{ij}) &= \sigma_{i_1}\sigma_{\eta'}\sigma_\chi\mathbf{e}(\mathbf{ij})\\ &= \sigma_{i_1}\sigma_{\eta'\chi}\mathbf{e}(\mathbf{ij})\\ &= \sigma_{j_1}\cdots\sigma_{j_t}\mathbf{e}(\mathbf{ij})\\ &= \sigma_{\eta\chi}\mathbf{e}(\mathbf{ij}).\end{aligned}$$

We are now in a position to prove the original claim.

$$\begin{aligned}\sigma_\eta\sigma_\theta e &= \sigma_\eta\sigma_\chi\sigma_\tau e\\ &= \sigma_{\eta\chi}\sigma_\tau e\\ &= \sigma_{\eta\chi\tau} e\\ &= \sigma_{\eta\theta} e\end{aligned}$$

where, in the third equality, we use the fact that $\ell(\eta\chi\tau) = \ell(\eta\chi) + \ell(\tau)$. □

Writing $\mathcal{D}$ for $\mathcal{D}(\mathfrak{S}_m/(\mathfrak{S}_{m_1} \times \mathfrak{S}_{m_2}))$, define the map,

$$\psi : \mathfrak{W}_\nu\mathbf{e} \longrightarrow \bigoplus_{\eta\in\mathcal{D}} \mathbf{e}\mathfrak{W}_\nu\mathbf{e}$$

as follows. For each basis element $\sigma_{\eta\theta}x_1^{n_1}\cdots x_m^{n_m}\mathbf{e}(\mathbf{ij})$ of $\mathfrak{W}_\nu\mathbf{e}$, let

$$\psi(\sigma_{\eta\theta}x_1^{n_1}\cdots x_m^{n_m}\mathbf{e}(\mathbf{ij})) = (\sigma_\theta x_1^{n_1}\cdots x_m^{n_m}\mathbf{e}(\mathbf{ij}))\Big|_\eta$$

where $|_\eta$ indicates that the element lies in the η^{th} copy of $\mathbf{e}\mathfrak{W}_\nu\mathbf{e}$. Extend this linearly. So ψ is an isomorphism of $\mathbf{k}$-vector spaces. By Claim 3.2.15, ψ is also a right $\mathbf{e}\mathfrak{W}_\nu\mathbf{e}$-module morphism and is clearly bijective. □

Note 3.2.17. Proposition 3.2.14 shows that $\mathfrak{W}_\nu\mathbf{e}$ is a right projective $\mathbf{e}\mathfrak{W}_\nu\mathbf{e}$-module so that $\mathfrak{W}_\nu\mathbf{e}$ is flat. That is, $\mathfrak{W}_\nu\mathbf{e} \otimes_{\mathbf{e}\mathfrak{W}_\nu\mathbf{e}} (-)$ is an exact functor. Since $\mathfrak{W}_\nu\mathbf{e}$ is also a left projective $\mathfrak{W}_\nu$-module, the functor $\mathrm{Hom}_{\mathfrak{W}_\nu}(\mathfrak{W}_\nu\mathbf{e}, -)$ is exact.

We now collect these results and again prove the main result of this section.

Theorem 3.2.18. *$\mathfrak{W}_\nu$ and $\mathfrak{W}_{\nu_1} \otimes_{\boldsymbol{k}} \mathfrak{W}_{\nu_2}$ are Morita equivalent.*

Proof. Define functors

$$\mathbf{e}\mathfrak{W}_\nu\mathbf{e}\text{-Mod} \underset{F}{\overset{G}{\leftrightarrows}} \mathfrak{W}_\nu\text{-Mod}$$

as follows. Set $F(-) = \mathfrak{W}_\nu\mathbf{e} \otimes_{\mathbf{e}\mathfrak{W}_\nu\mathbf{e}} (-)$ to be the induction functor and set $G(-) = \mathbf{e} \cdot (-)$ to be restriction. Note that, since $\mathfrak{W}_{\nu_1} \otimes \mathfrak{W}_{\nu_2} \cong \mathbf{e}\mathfrak{W}_\nu\mathbf{e}$ as shown in Proposition 3.2.7, it suffices to prove that these functors are equivalences of categories. These functors are both exact and, since induction is left adjoint to restriction, by Lemma 3.2.13, it suffices to prove that F and G take simples to simples.

This follows immediately for G using Proposition 3.2.11.

We now go on to show that $F(S)$ is simple, for $S \in \mathbf{e}\mathfrak{W}_\nu\mathbf{e}$-Mod a simple module. Suppose S has $\mathbf{k}$-basis $\mathcal{B} = \{b_1, \ldots, b_n\}$. Then, by Proposition 3.2.14, $F(S)$ has a $\mathbf{k}$-basis

$$\mathcal{B}' = \{\sigma_\eta\mathbf{e} \otimes b_i \mid \eta \in \mathcal{D}(\mathfrak{S}_m/(\mathfrak{S}_{m_1} \times \mathfrak{S}_{m_2})), 1 \leq i \leq n\}$$

Note that the $\mathbf{e} \otimes b_i$ generate $\mathfrak{W}_\nu\mathbf{e} \otimes S$ as a $\mathfrak{W}_\nu$-module. Let $S' = F(S) = \mathfrak{W}_\nu\mathbf{e} \otimes_{\mathbf{e}\mathfrak{W}_\nu\mathbf{e}} S$, and suppose $\overline{S} \subseteq S'$ is a non-zero $\mathfrak{W}_\nu$-submodule. Pick a non-zero $x \in \overline{S}$. Since $\mathcal{B}'$ is a $\mathbf{k}$-basis of S' we have $x = \sum_{\eta,i} a_{\eta,i}\sigma_\eta\mathbf{e} \otimes b_i$, for $a_{\eta,i} \in \mathbf{k}$ and such that each $\eta \in \mathcal{D}(\mathfrak{S}_m/(\mathfrak{S}_{m_1} \times \mathfrak{S}_{m_2}))$.

Take $\mathbf{e}(\mathbf{j})$ such that $\mathbf{e}(\mathbf{j})x \neq 0$. Then, for some η,

$$\mathbf{e}(\mathbf{j})x = \sum_i a_{\eta,i}\sigma_\eta\mathbf{e} \otimes b_i = \sigma_\eta\mathbf{e} \otimes \sum_i a_{\eta,i}b_i \in \overline{S}.$$

σ_η is invertible in $\mathfrak{W}_\nu$, since $\eta \in \mathcal{D}(\mathfrak{S}_m/(\mathfrak{S}_{m_1} \times \mathfrak{S}_{m_2}))$, and so

$$(\sigma_\eta)^{-1}\mathbf{e}(\mathbf{j})x = \mathbf{e} \otimes \sum_i a_{\eta,i}b_i \in \overline{S}$$

Since $S \in \mathbf{e}\mathfrak{W}_\nu\mathbf{e}$-Mod is simple, it follows that

$$\{\mathbf{e} \otimes b_i \mid 1 \leq i \leq n\} \subseteq \overline{S} \Longrightarrow S' \subseteq \overline{S} \Longrightarrow \overline{S} = S'$$

so that $F(S) = S'$ is simple, as claimed. Then by Lemma 3.2.13 the result is proved. □

3.3 Morita Equivalence in the Case $p, q \notin I$

In this section we assume $p, q \notin I$. Then $I = I_\lambda$ is the $\mathbb{Z} \rtimes \{\pm 1\}$-orbit of $\lambda \in \mathbf{k}^\times$, for some $\lambda \neq p, q$.

For the convenience of the reader let us briefly recall the form of Γ_{I_λ} from Section 2.3. We have $I_\lambda^+ := \{p^{2n}\lambda \mid n \in \mathbb{Z}\}$, $I_\lambda^- := \{p^{2n}\lambda^{-1} \mid n \in \mathbb{Z}\}$. So $I_\lambda = I_\lambda^- \sqcup I_\lambda^+$. If p is not a root of unity Γ_{I_λ} has the form,

$$\cdots \longrightarrow p^2\lambda \longrightarrow \lambda \longrightarrow p^{-2}\lambda \longrightarrow \cdots$$

$$\cdots \longleftarrow p^{-2}\lambda^{-1} \longleftarrow \lambda^{-1} \longleftarrow p^2\lambda^{-1} \longleftarrow \cdots$$

If $p^m = 1$ for some positive integer m then Γ_{I_λ} has the form,

$$\begin{tikzcd}
 & \lambda \arrow[dr] & \\
p^2\lambda \arrow[ur] & & p^{m-2}\lambda \arrow[d] \\
\cdots \arrow[u] & & \cdots
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \lambda^{-1} \arrow[dl] & \\
p^{-2}\lambda^{-1} \arrow[d] & & p^{2-m}\lambda^{-1} \arrow[ul] \\
\cdots & & \cdots \arrow[u]
\end{tikzcd}$$

The defining relations of $\mathfrak{W}_\nu$ are dependent on whether or not q is an element of I. In particular, in this case, for every idempotent $\mathbf{e}(\mathbf{i})$, we have

$$\pi^2\mathbf{e}(\mathbf{i}) = \mathbf{e}(\mathbf{i})$$
$$\pi\sigma_1\pi\sigma_1\mathbf{e}(\mathbf{i}) = \sigma_1\pi\sigma_1\pi\mathbf{e}(\mathbf{i})$$
$$\deg(\pi\mathbf{e}(\mathbf{i})) = 0$$

For any VV algebra $\mathfrak{W}_\nu$ we can consider various idempotent subalgebras $\mathbf{e}\mathfrak{W}_\nu\mathbf{e}$, for different choices of $\mathbf{e}$. Each of these idempotent subalgebras may or may not be isomorphic to a KLR algebra (see Proposition 2.3.12). Among these KLR algebras we can always distinguish $\mathbf{R}^+_{\tilde{\nu}}$ and $\mathbf{R}^-_{\tilde{\nu}}$, as mentioned in Remark 2.3.14. Here we show that the VV algebras arising from the setting $p, q \notin I$ are Morita equivalent to KLR algebras of type A. Namely, for any $\nu \in {}^\theta\mathbb{N}I$, $\mathfrak{W}_\nu$ and $\mathbf{R}^+_{\tilde{\nu}}$ are Morita equivalent, as are $\mathfrak{W}_\nu$ and $\mathbf{R}^-_{\tilde{\nu}}$. We show this here for $\mathbf{R}^+_{\tilde{\nu}}$. Let $|\nu| = 2m$. Again, we are using the notation as in Remark 2.3.8. To stress this point; for $\nu = \sum_{i \in I^+} \nu_i i + \sum_{i \in I^-} \nu_i i$ we set $\tilde{\nu} = \sum_{i \in I^+} \nu_i i \in \mathbb{N}I^+$.

Theorem 3.3.1. *$\mathfrak{W}_\nu$ and $\boldsymbol{R}^+_{\tilde{\nu}}$ are Morita equivalent.*

Proof. Using Corollary 3.1.18, it suffices to find a progenerator $P \in \mathfrak{W}_\nu$-Mod such that $\mathbf{R}^+_{\tilde{\nu}} \cong \mathrm{End}_{\mathfrak{W}_\nu}(P)$.

Let

$$\mathbf{e} := \sum_{\mathbf{i} \in I^{\tilde{\nu}}} \mathbf{e}(\mathbf{i})$$

From Remark 3.1.10, $\mathfrak{W}_\nu\mathbf{e}$ is a progenerator if and only $\mathbf{e}$ is full in $\mathfrak{W}_\nu$, i.e. if and only if $\mathfrak{W}_\nu\mathbf{e}\mathfrak{W}_\nu = \mathfrak{W}_\nu$.

Clearly $\mathfrak{W}_\nu\mathbf{e}\mathfrak{W}_\nu \subseteq \mathfrak{W}_\nu$. So it remains to show $\mathfrak{W}_\nu \subseteq \mathfrak{W}_\nu\mathbf{e}\mathfrak{W}_\nu$. We do this by showing that every idempotent $\mathbf{e}(\mathbf{i})$ lies in $\mathfrak{W}_\nu\mathbf{e}\mathfrak{W}_\nu$. Then, since $\sum_{\mathbf{i} \in {}^\theta I^\nu} \mathbf{e}(\mathbf{i}) = 1$, it follows that $1 \in \mathfrak{W}_\nu \mathbf{e} \mathfrak{W}_\nu$ and so $\mathfrak{W}_\nu \mathbf{e} \mathfrak{W}_\nu \subseteq \mathfrak{W}_\nu$.

If $\mathbf{e}(\mathbf{i})$ is a summand of $\mathbf{e}$ then it is clear that $\mathbf{e}(\mathbf{i}) \in \mathfrak{W}_\nu \mathbf{e} \mathfrak{W}_\nu$. Take $\mathbf{e}(\mathbf{i}) \in \mathfrak{W}_\nu$ not a summand of $\mathbf{e}$. Then there are finitely many entries of $\mathbf{i}$, say $i_{k_1}, \ldots, i_{k_r}$, with $i_{k_s} \in I^-$ for all $1 \leq s \leq r$, and with $k_1 < k_2 < \cdots < k_r$. Let η be the minimal length left coset representative of $\mathfrak{S}_m$ in W_m^B given by

$$\eta = s_{k_1-1} \cdots s_1 s_0 \cdots \cdots s_{k_{r-1}-1} \cdots s_1 s_0 s_{k_r-1} \cdots s_1 s_0$$

Fix this reduced expression of η. Fix the reduced expression

$$\eta^{-1} = s_0 s_1 \cdots s_{k_r-1} s_0 s_1 \cdots s_{k_{r-1}-1} \cdots \cdots s_0 s_1 \cdots s_{k_1-1}.$$

Then $\mathbf{e}(\mathbf{i})\sigma_\eta = \sigma_\eta \mathbf{e}(\mathbf{j})$ for some $\mathbf{e}(\mathbf{j})$, where $\mathbf{j} \in I^{\tilde{\nu}}$ so that $\mathbf{e}(\mathbf{j})$ is a summand of $\mathbf{e}$, and

$$\sigma_\eta \mathbf{e}(\mathbf{j}) \sigma_{\eta^{-1}} = \mathbf{e}(\mathbf{i}).$$

Hence $\mathbf{e}(\mathbf{i}) \in \mathfrak{W}_\nu \mathbf{e} \mathfrak{W}_\nu$ as required. It follows now that $\mathbf{e}$ is full in $\mathfrak{W}_\nu$ so that $\mathfrak{W}_\nu \mathbf{e}$ is a progenerator in $\mathfrak{W}_\nu$-Mod.

It remains to show $\mathbf{R}_{\tilde{\nu}}^+ \cong \mathrm{End}_{\mathfrak{W}_\nu}(\mathfrak{W}_\nu \mathbf{e})$. Using Corollary 3.1.23, $\mathrm{End}_{\mathfrak{W}_\nu}(\mathfrak{W}_\nu \mathbf{e}) \cong \mathbf{e}\mathfrak{W}_\nu \mathbf{e}$ and, by Proposition 2.3.12, $\mathbf{e}\mathfrak{W}_\nu \mathbf{e} \cong \mathbf{R}_{\tilde{\nu}}^+$. By Corollary 3.1.18, $\mathfrak{W}_\nu$ and $\mathbf{R}_{\tilde{\nu}}^+$ are Morita equivalent. □

Note that the dual proof shows Morita equivalence between $\mathfrak{W}_\nu$ and $\mathbf{R}_{\tilde{\nu}}^-$, for $\tilde{\nu} \in \mathbb{N}I^-$.

Remark 3.3.2. The functors between the two module categories are given by,

$$\mathfrak{W}_\nu \mathbf{e} \otimes - : \mathbf{R}_{\tilde{\nu}}\text{-Mod} \longrightarrow \mathfrak{W}_\nu\text{-Mod}$$
$$\mathbf{e} \cdot - : \mathfrak{W}_\nu\text{-Mod} \longrightarrow \mathbf{R}_{\tilde{\nu}}\text{-Mod}.$$

KLR algebras of type A have been studied extensively over the past decade or so. For example; they are Morita equivalent to affine Hecke algebras of type A on the level of finite length modules, there exists a parametrisation of simple modules (see [KR11]), and they are affine cellular (see [KLM13]). Much of the information and many of the properties that we have for KLR algebras can be taken across the Morita equivalence and applied to this family of VV algebras. For this reason it is perhaps more interesting for us to focus on other cases, where either p or q lie in I.

Note that the proof of $\mathbf{e}\mathfrak{W}_\nu \mathbf{e} \cong \mathbf{R}_{\tilde{\nu}}$ does not use the fact that $p, q \notin I$. One can therefore ask why this proof fails for either p or q in I.

A Note on the Cases: $p \in I$, $q \in I$

Case: $\mathbf{q} \in \mathbf{I}$

Consider first the case $q \in I$, $p \notin I$. Take $\nu \in {}^{\theta}\mathbb{N}I_q$ and consider the associated VV algebra $\mathfrak{W}_\nu$. Note that many of the relations now depend on whether or not $i_1 = q^{\pm 1}$. If we pick ν so that q is not a summand then of course we always have $i_1 \neq q^{\pm 1}$. Hence, in this case, the relations are exactly the same as those in the ME setting so that $\mathfrak{W}_\nu$ is Morita equivalent to $\mathbf{R}^+_{\tilde{\nu}}$. From now on, when we work in the setting $q \in I$, we assume q (and hence q^{-1}) is a summand of ν. That is, $\nu_q \geq 1$.

The proof of Theorem 3.3.1 fails to show Morita equivalence in the case $q \in I$ because, in this case, $\mathbf{e}$ is not full. In particular, we no longer have $\mathbf{e}(\mathbf{i}) = \sigma_\eta \mathbf{e}(\mathbf{j}) \sigma_{\eta^{-1}}$, for $\mathbf{e}(\mathbf{j}) \in \mathbf{R}^+_{\tilde{\nu}}$ and some minimal length left coset representative $\eta \in \mathcal{D}(W^B_m / \mathfrak{S}_m)$. Let us see this explicitly with an example.

Example 3.3.3. Take $\nu = p^{-2}q^{-1} + q^{-1} + q + p^2 q \in {}^{\theta}\mathbb{N}I_q$. The corresponding quiver is, again with the generators x_i omitted,

$$
\begin{tikzcd}
\mathbf{e}(q,p^2q) \arrow[r, bend left, "\sigma_1"] \arrow[d, bend left, "\pi"] & \mathbf{e}(p^2q,q) \arrow[l, bend left, "\sigma_1"] \arrow[d, bend left, "\pi"] \\
\mathbf{e}(q^{-1},p^2q) \arrow[u, bend left, "\pi"] \arrow[d, bend left, "\sigma_1"] & \mathbf{e}(p^{-2}q^{-1},q) \arrow[u, bend left, "\pi"] \arrow[d, bend left, "\sigma_1"] \\
\mathbf{e}(p^2q,q^{-1}) \arrow[u, bend left, "\sigma_1"] \arrow[d, bend left, "\pi"] & \mathbf{e}(q,p^{-2}q^{-1}) \arrow[u, bend left, "\sigma_1"] \arrow[d, bend left, "\pi"] \\
\mathbf{e}(p^{-2}q^{-1},q^{-1}) \arrow[u, bend left, "\pi"] \arrow[r, bend left, "\sigma_1"] & \mathbf{e}(q^{-1},p^{-2}q^{-1}) \arrow[u, bend left, "\pi"] \arrow[l, bend left, "\sigma_1"]
\end{tikzcd}
$$

From the defining relations specified in 2.3.4, we have

$$
\begin{aligned}
\pi^2 \mathbf{e}(q, p^2 q) &= x_1 \mathbf{e}(q, p^2 q) \\
\pi^2 \mathbf{e}(q^{-1}, p^2 q) &= -x_1 \mathbf{e}(q^{-1}, p^2 q) \\
\pi^2 \mathbf{e}(q, p^{-2} q^{-1}) &= x_1 \mathbf{e}(q, p^{-2} q^{-1}) \\
\pi^2 \mathbf{e}(q^{-1}, p^{-2} q^{-1}) &= -x_1 \mathbf{e}(q^{-1}, p^{-2} q^{-1})
\end{aligned}
$$

Set $\mathbf{e} := \mathbf{e}(q, p^2 q) + \mathbf{e}(p^2 q, q)$. We want to show $\mathbf{e}(\mathbf{i}) \in \mathfrak{W}_\nu \mathbf{e} \mathfrak{W}_\nu$ for every $\mathbf{e}(\mathbf{i}) \in \mathfrak{W}_\nu$. Take, for example, $\mathbf{e}(q, p^{-2}q^{-1})$ and $\eta = s_1 s_0 \in \mathcal{D}(W_2^B/\mathfrak{S}_2)$. Then,

$$\begin{aligned}
\sigma_1 \pi \mathbf{e}(p^2 q, q) \pi \sigma_1 &= \sigma_1 \pi^2 \mathbf{e}(p^{-2}q^{-1}, q)\sigma_1 \\
&= \sigma_1 \mathbf{e}(p^{-2}q^{-1}, q)\sigma_1 \\
&= \sigma_1^2 \mathbf{e}(q, p^{-2}q^{-1}) \\
&= \mathbf{e}(q, p^{-2}q^{-1})
\end{aligned}$$

Since $\mathbf{e}(p^2q, q)$ is a summand of $\mathbf{e}$, we have $\mathbf{e}(q, p^{-2}q^{-1}) \in \mathfrak{W}_\nu \mathbf{e} \mathfrak{W}_\nu$. Let us now take $\mathbf{e}(q^{-1}, p^{-2}q^{-1})$ and try to show $\mathbf{e}(q^{-1}, p^{-2}q^{-1}) \in \mathfrak{W}_\nu \mathbf{e} \mathfrak{W}_\nu$. The minimal length left coset representative we need to take is $\eta = s_0 s_1 s_0$, with $\mathbf{e}(\mathbf{j}) = \mathbf{e}(p^2q, q)$. But,

$$\begin{aligned}
\pi \sigma_1 \pi \mathbf{e}(p^2 q, q) \pi \sigma_1 \pi &= \pi^2 \mathbf{e}(q^{-1}, p^{-2}q^{-1}) \\
&= -x_1 \mathbf{e}(q^{-1}, p^{-2}q^{-1})
\end{aligned}$$

This example demonstrates that $\mathbf{e}$ is not full in $\mathfrak{W}_\nu$, which is why Morita equivalence fails in this case.

Case: $\mathbf{p} \in \mathbf{I}$

Now consider the case $p \in I$, $q \notin I$. Take $\nu \in {}^\theta\mathbb{N}I_p$ and the associated VV algebra $\mathfrak{W}_\nu$. The defining relations do not explicitly depend on whether or not $p \in I$, but the subtle difference arises when we examine the underlying quiver Γ_{I_p}. Locally, with regards to the relations, Γ_{I_p} is exactly the same as Γ_{I_λ}, for some $\lambda \neq p, q$, except in the following neighbourhood of Γ_{I_p}.

$$\cdots \longrightarrow p \longrightarrow p^{-1} \longrightarrow \cdots$$

In other words, $\Gamma_{I_p^+}$ and $\Gamma_{I_p^-}$ are not two disjoint connected components of Γ_{I_p}. If we choose $\nu \in {}^\theta\mathbb{N}I$ with $\nu_p \leq 1$ the defining relations of $\mathfrak{W}_\nu$ are exactly those in the ME setting. So again, in this case, $\mathfrak{W}_\nu$ and $\mathbf{R}_\nu^+$ are Morita equivalent. From now on, when we work in the setting $p \in I$, we assume $\nu_p \geq 2$ (and hence $\nu_{p^{-1}} \geq 2$).

The proof of Theorem 3.3.1 fails to show Morita equivalence in the case $p \in I$ because $\mathbf{e}$ is not full in $\mathfrak{W}_\nu$. We see this explicitly in the following example.

Example 3.3.4. Take $\nu = 2p^{-1} + 2p \in {}^\theta\mathbb{N}I_p$ as in Example 2.3.7. The corresponding quiver is, again with paths x_1 and x_2 omitted,

$$\begin{tikzcd}
\mathbf{e}(p,p) \arrow[loop left, "\sigma_1"] \arrow[r, bend left, "\pi"] & \mathbf{e}(p^{-1},p) \arrow[l, bend left, "\pi"] \arrow[r, bend left, "\sigma_1"] & \mathbf{e}(p,p^{-1}) \arrow[l, bend left, "\sigma_1"] \arrow[r, bend left, "\pi"] & \mathbf{e}(p^{-1},p^{-1}) \arrow[l, bend left, "\pi"] \arrow[loop right, "\sigma_1"]
\end{tikzcd}$$

Set $\mathbf{e} := \mathbf{e}(p,p)$. Take $\mathbf{e}(p,p^{-1})$ and $\eta = s_1 s_0 \in \mathcal{D}(W_2^B/\mathfrak{S}_2)$. Then,

$$\begin{aligned}
\sigma_1 \pi \mathbf{e}(p,p) \pi \sigma_1 &= \sigma_1 \pi^2 \mathbf{e}(p^{-1},p)\sigma_1 \\
&= \sigma_1 \mathbf{e}(p^{-1},p)\sigma_1 \\
&= \sigma_1^2 \mathbf{e}(p,p^{-1}) \\
&= (x_1 - x_2)\mathbf{e}(p,p^{-1})
\end{aligned}$$

This demonstrates why $\mathbf{e}$ is not full in $\mathfrak{W}_\nu$ and hence why the Morita equivalence does not hold in this setting.

3.4 Morita Equivalences in the Case $q \in I$

For the remainder of this chapter, unless stated otherwise, we assume that p is not a root of unity.

3.4.1 Affine Cellularity of Classes of VV Algebras

Graham and Lehrer defined the notion of cellularity for finite-dimensional algebras in 1996 [GL96]. They defined these algebras in terms of a basis satisfying various combinatorial properties. Showing that an algebra is cellular gives rise to a parametrisation of its irreducible modules. Typical examples of algebras which are cellular include Ariki-Koike algebras and Temperley-Lieb algebras. Cyclotomic KLR algebras, which will be discussed later in this thesis (see Chapter 4), are also cellular. In [KX98] Koenig and Xi give an equivalent definition of a cellular algebra in terms of so-called cell ideals and a cell chain, which they show is equivalent to the original definition of Graham and Lehrer.

Here we define what it means for an algebra to be affine cellular. The notion of affine cellularity was introduced by Koenig and Xi in [KX12], and extends the notion of cellularity to algebras which need not be finite-dimensional. Analogous to the finite case, showing affine cellularity enables one to classify irreducible modules. There are also interesting homological features of affine cellular algebras; for example, if the cell chain (to be defined) of an affine cellular algebra over a field consists of idempotent ideals then the algebra has finite global dimension. Examples of algebras which are currently known to be affine cellular include affine Temperley-Lieb algebras, affine Birman-Murakami-Wenzl algebras and KLR algebras of finite type.

Although throughout this thesis we have fixed $\mathbf{k}$ to be a field of characteristic not equal to 2, we note here that the following definition is valid for any Noetherian domain $\mathbf{k}$. The main results of this chapter will be that certain classes of VV algebras are affine cellular.

Definition 3.4.1. A **$\mathbf{k}$-involution** of a $\mathbf{k}$-algebra A is a $\mathbf{k}$-linear anti-automorphism ω with $\omega^2 = \mathrm{id}_A$.

By an **affine algebra** we mean a commutative $\mathbf{k}$-algebra B which is a quotient of a polynomial ring $\mathbf{k}[x_1, \dots, x_n]$ in finitely many variables.

Definition 3.4.2 ([KX12], Definition 2.1)**.** Let A be a unitary $\mathbf{k}$-algebra with a $\mathbf{k}$-involution ω on A. A two-sided ideal $J \subseteq A$ is called an **affine cell ideal** if and only if the following data are given and the following conditions are satisfied.

1. $\omega(J) = J$.
2. There exists a free $\mathbf{k}$-module V of finite rank and an affine commutative $\mathbf{k}$-algebra B with identity and with a $\mathbf{k}$-involution i such that $\Delta := V \otimes_{\mathbf{k}} B$ is an A-B-bimodule, where the right B-module structure is induced by that of the right regular B-module.
3. There is an A-A-bimodule isomorphism $\alpha : J \longrightarrow \Delta \otimes_B \Delta'$, where $\Delta' = B \otimes_{\mathbf{k}} V$ is a B-A-bimodule with the left B-structure induced by the left regular B-module. The right A-structure is induced via ω. That is,
$$(b \otimes v)a := s(\omega(a)(v \otimes b)) \text{ for } a \in A, b \in B, v \in V$$
where $s : V \otimes B \longrightarrow B \otimes V$, $v \otimes b \mapsto b \otimes v$ is the switch map, such that the following diagram commutes:
$$\begin{tikzcd} J \arrow[r, "\alpha"] \arrow[d, "\omega"'] & \Delta \otimes_B \Delta' \arrow[d, "v \otimes b \otimes b' \otimes w \mapsto w \otimes i(b') \otimes i(b) \otimes v"] \\ J \arrow[r, "\alpha"'] & \Delta \otimes_B \Delta' \end{tikzcd}$$

Definition 3.4.3 ([KX12], Definition 2.1)**.** A $\mathbf{k}$-algebra A, with a $\mathbf{k}$-involution ω, is called **affine cellular** if there is a $\mathbf{k}$-module decomposition $A = J_1' \oplus \cdots \oplus J_n'$, for some n, with $\omega(J_j') = J_j'$ for each j and such that setting $J_j = \bigoplus_{l=1}^{j} J_l'$ gives a chain of two-sided ideals of A,
$$(0) = J_0 \subset J_1 \subset \cdots \subset J_n = A,$$
and for each j the quotient $J_j' \cong J_j/J_{j-1}$ is an affine cell ideal of A/J_{j-1} (with respect to the involution induced by ω on the quotient). This chain is called a **cell chain** for the affine cellular algebra A. The module Δ is called a **cell lattice** for the affine cell ideal J.

Remark 3.4.4. In [Kle15], Kleshchev gives graded versions of Definitions 3.4.2 and 3.4.3 in which all algebras, ideals, etc. are graded and the maps ω, i are homogeneous.

In this section we fix the following setting. Assume $q \in I$, $p \notin I$, p not a root of unity and take $\nu \in {}^{\theta}\mathbb{N}I_q$ with multiplicity one and $|\nu| = 2m$, for some $m \in \mathbb{N}$. Suppose we have fixed a reduced expression $s_{i_1} \cdots s_{i_k}$ for some $w \in W_m^B$ so that $\sigma_w = \sigma_{i_1} \cdots \sigma_{i_k}$. Define
$$\sigma_w^{\rho} := \sigma_{i_k} \cdots \sigma_{i_1}.$$

Let Π^+ denote the root partitions of the idempotent subalgebra $\mathbf{R}^+_{\tilde{\nu}} \subseteq \mathfrak{W}_\nu$ and let Π^- denote the root partitions of the idempotent subalgebra $\mathbf{R}^-_{\tilde{\nu}} \subseteq \mathfrak{W}_\nu$. Put $\Pi^\pm = \Pi^+ \cup \Pi^-$. Throughout this section let $\mathbf{e} := \sum_{\lambda \in \Pi^\pm} \mathbf{e}(\mathbf{i}_\lambda)$ where $\mathbf{e}(\mathbf{i}_\lambda)$ denotes the idempotent associated to the root partition $\lambda \in \Pi^\pm$, as discussed in 2.2.1.

Define $\Pi(m)$ to be the following set. $\Pi(m) := \big\{(a_1, \dots, a_{m-1}) \mid a_i \in \{1,2\}\ \forall i\big\}$. There is a bijection,

$$\begin{aligned} \theta : \Pi^+ &\longrightarrow \Pi(m) \\ \lambda &\mapsto (a_1, \dots, a_{m-1}) \end{aligned} \tag{3.4.2}$$

where $a_i = \begin{cases} 1 & \text{if } p^{2i-2}q \text{ appears before } p^{2i}q \text{ in } \mathbf{i}_\lambda \\ 2 & \text{if } p^{2i-2}q \text{ appears after } p^{2i}q \text{ in } \mathbf{i}_\lambda. \end{cases}$

Similarly there is a one-to-one correspondence between Π^- and $\Pi(m)$.

The next lemma states that every idempotent $\mathbf{e}(\mathbf{i}) \in \mathfrak{W}_\nu$ is isomorphic to either an idempotent in $\mathbf{R}^+_{\tilde{\nu}}$ or to an idempotent in $\mathbf{R}^-_{\tilde{\nu}}$.

Lemma 3.4.5. *Take $\mathbf{e}(\boldsymbol{i}) \in \mathfrak{W}_\nu$ with $\boldsymbol{i} \notin I^{\tilde{\nu}^+} \cup I^{\tilde{\nu}^-}$. Note that precisely one of q, q^{-1} will appear as an entry in $\boldsymbol{i}$.*

(i) If q is in position k (i.e. $i_k = q$) then $\mathbf{e}(\boldsymbol{i}) \cong \mathbf{e}(\boldsymbol{j})$ for some $\boldsymbol{j} \in I^{\tilde{\nu}^+}$.

(ii) If q^{-1} is in position k (i.e. $i_k = q^{-1}$) then $\mathbf{e}(\boldsymbol{i}) \cong \mathbf{e}(\boldsymbol{j})$ for some $\boldsymbol{j} \in I^{\tilde{\nu}^-}$.

Proof. For (i), assume q is in position k, i.e. $i_k = q$. By Proposition 3.1.20, we require $a \in \mathbf{e}(\mathbf{i})\mathfrak{W}_\nu\mathbf{e}(\mathbf{j})$, $b \in \mathbf{e}(\mathbf{j})\mathfrak{W}_\nu\mathbf{e}(\mathbf{i})$ such that $ab = \mathbf{e}(\mathbf{i})$ and $ba = \mathbf{e}(\mathbf{j})$, for some $\mathbf{j} \in I^{\tilde{\nu}^+}$.

Since $\mathbf{i} \notin I^{\tilde{\nu}^+} \cup I^{\tilde{\nu}^-}$ there is a non-empty subset $\{\varepsilon_1, \dots, \varepsilon_d\} \subsetneq \{1, \dots, m\}$, with $\varepsilon_r \leq \varepsilon_{r+1}$ for all r, such that $i_{\varepsilon_r} = p^{2n_r}q^{-1}$, where $n_r \in \mathbb{Z} \setminus \{0\}$ for all r. Note also that $\varepsilon_r \neq k$ for all r since $i_k = q$. Put,

$$\begin{aligned} a &= \mathbf{e}(\mathbf{i})\sigma_{\varepsilon_1 - 1} \cdots \sigma_1 \pi \sigma_{\varepsilon_2 - 1} \cdots \sigma_1 \pi \cdots \sigma_{\varepsilon_d - 1} \cdots \sigma_1 \pi \mathbf{e}(\mathbf{j}) \in \mathbf{e}(\mathbf{i})\mathfrak{W}_\nu\mathbf{e}(\mathbf{j}) \\ b &= \mathbf{e}(\mathbf{j})\pi\sigma_1 \cdots \sigma_{\varepsilon_d - 1}\pi\sigma_1 \cdots \sigma_{\varepsilon_{d-1} - 1} \cdots \pi\sigma_1 \cdots \sigma_{\varepsilon_1 - 1}\mathbf{e}(\mathbf{i}) \in \mathbf{e}(\mathbf{j})\mathfrak{W}_\nu\mathbf{e}(\mathbf{i}). \end{aligned}$$

Then $\mathbf{e}(\mathbf{j}) \in \mathfrak{W}_\nu$ is such that $\mathbf{j} \in I^{\tilde{\nu}^+}$ and, from the relations, we know that $ab = \mathbf{e}(\mathbf{i})$ and $ba = \mathbf{e}(\mathbf{j})$ so that $\mathbf{e}(\mathbf{i}) \cong \mathbf{e}(\mathbf{j})$. A similar argument is used for (ii) when $i_k = q^{-1}$. □

Corollary 3.4.6. *Every idempotent $\mathbf{e}(\boldsymbol{i}) \in \mathfrak{W}_\nu$ is isomorphic to some $\mathbf{e}(\boldsymbol{j}_\lambda)$, $\lambda \in \Pi^\pm$.*

Proof. By Lemma 3.4.5, it suffices to prove that every $\mathbf{e}(\mathbf{i})$, with $\mathbf{i} \in I^{\tilde{\nu}^+}$ or with $\mathbf{i} \in I^{\tilde{\nu}^-}$, is isomorphic to some $\mathbf{e}(\mathbf{j}_\lambda)$, $\lambda \in \Pi^\pm$. Suppose $\mathbf{i} \in I^{\tilde{\nu}^+}$ (the proof for when $\mathbf{i} \in I^{\tilde{\nu}^-}$ is the same). Let $|\nu| = 2m$ so that $|\tilde{\nu}^+| = m$.

Take any $\mathbf{i} \in I^{\tilde{\nu}+}$. Associate to $\mathbf{i}$ the $(m-1)$-tuple $(a_1, \dots, a_{m-1}) \in \Pi(m)$, where

$$a_j = \begin{cases} 1 & \text{if } p^{2j-2}q \text{ appears before } p^{2j}q \text{ in } \mathbf{i} \\ 2 & \text{if } p^{2j-2}q \text{ appears after } p^{2j}q \text{ in } \mathbf{i}. \end{cases}$$

Then there exists a surjection,

$$\begin{aligned} g : I^{\tilde{\nu}+} &\twoheadrightarrow \Pi(m) \\ \mathbf{i} &\mapsto (a_1, \dots, a_m). \end{aligned}$$

We have seen already from 3.4.2 that there is a one-to-one correspondence between $\Pi(m)$ and root partitions associated to $\tilde{\nu}$. Now, $g(\mathbf{i}) \in \Pi(m)$ and therefore corresponds to some root partition $\lambda \in \Pi^+$, i.e. $g(\mathbf{i}) = \theta(\lambda)$. This means we have $\mathbf{e}(\mathbf{i}) \cong \mathbf{e}(\mathbf{i}_\lambda)$, because when we permute entries of $\mathbf{i}$ which do not affect $g(\mathbf{i})$, we are permuting entries of $\mathbf{i}$ in such a way that neighbouring vertices of Γ_I never switch position, by definition. Hence we can permute entries of $\mathbf{i}$ in this way to obtain a root partition, $\mathbf{e}(\mathbf{i}_\lambda)$. In other words, there exists an element $w \in \mathfrak{S}_m$ such that $\sigma_w^\rho \mathbf{e}(\mathbf{i}_\lambda)\sigma_w \mathbf{e}(\mathbf{i}) = \mathbf{e}(\mathbf{i})$. □

Corollary 3.4.7. *The idempotent $\boldsymbol{e} = \sum_{\lambda \in \Pi^\pm} \boldsymbol{e}(\boldsymbol{i}_\lambda)$ is full in $\mathfrak{W}_\nu$.*

Proof. It is clear that $\mathfrak{W}_\nu \mathbf{e} \mathfrak{W}_\nu \subseteq \mathfrak{W}_\nu$. To show $\mathfrak{W}_\nu \subseteq \mathfrak{W}_\nu \mathbf{e} \mathfrak{W}_\nu$ it suffices to show $\mathbf{e}(\mathbf{i}) \in \mathfrak{W}_\nu \mathbf{e} \mathfrak{W}_\nu$, for every idempotent $\mathbf{e}(\mathbf{i})$ not a summand of $\mathbf{e}$. Let $\mathbf{e}(\mathbf{i})$ be such an idempotent. By Corollary 3.4.6, $\mathbf{e}(\mathbf{i}) \cong \mathbf{e}(\mathbf{i}_\lambda)$ for some $\lambda \in \Pi^\pm$. That is, there exists $a \in \mathbf{e}(\mathbf{i})\mathfrak{W}_\nu \mathbf{e}(\mathbf{i}_\lambda)$, $b \in \mathbf{e}(\mathbf{i}_\lambda)\mathfrak{W}_\nu \mathbf{e}(\mathbf{i})$ such that

$$\mathbf{e}(\mathbf{i}) = ab = a\mathbf{e}(\mathbf{i}_\lambda)b \in \mathfrak{W}_\nu \mathbf{e} \mathfrak{W}_\nu.$$

Then $\mathfrak{W}_\nu = \mathfrak{W}_\nu \mathbf{e} \mathfrak{W}_\nu$ as required. □

Let A and $\tilde{A}$ be the isomorphic path algebras of the following quivers, respectively.

$$\begin{tikzcd} e_1 \arrow[r, bend left, "u_1"] & e_2 \arrow[l, bend left, "u_2"] \end{tikzcd} \qquad \begin{tikzcd} a_1 \arrow[r, bend left, "v_1"] & a_2 \arrow[l, bend left, "v_2"] \end{tikzcd}$$

A and $\tilde{A}$ are graded $\mathbf{k}$-algebras via

$$\begin{aligned} \deg(e_i) &= \deg(a_i) = 0 \\ \deg(u_j e_j) &= \deg(v_j a_j) = 1 \end{aligned}$$

for $i, j \in \{1, 2\}$.

Theorem 3.4.8. *$\mathfrak{W}_\nu$ and $A^{\otimes(m-1)} \otimes \tilde{A}$ are Morita equivalent.*

Proof. Define a map

$$\phi : A^{\otimes(m-1)} \otimes \tilde{A} \longrightarrow \mathbf{e}\mathfrak{W}_\nu\mathbf{e}$$

as follows.

$$e_{j_1} \otimes \cdots \otimes e_{j_{m-1}} \otimes a_i \mapsto \mathbf{e}(\mathbf{i}_\lambda) \quad \text{for } \lambda \in \begin{cases} \Pi^+ \text{ if } i = 1 \\ \Pi^- \text{ if } i = 2 \end{cases}$$
$$\text{and } \theta(\lambda) = (j_1, \ldots, j_{m-1})$$

$$e_{j_1} \otimes \cdots \otimes e_{j'_k} u_\ell e_{j_k} \otimes \ldots \otimes e_{j_{m-1}} \otimes a_i \mapsto \mathbf{e}(\mathbf{i}_{\lambda'})\sigma_w \mathbf{e}(\mathbf{i}_\lambda) \quad \begin{array}{l} \text{for the unique} \\ w \in \mathfrak{S}_m \text{ such that} \\ \theta(\lambda) = (j_1, \ldots, j_k, \ldots, j_{m-1}) \\ \theta(\lambda') = (j_1, \ldots, j'_k, \ldots, j_{m-1}) \end{array}$$

$$e_{j_1} \otimes \cdots \otimes e_{j_{m-1}} \otimes v_i a_i \mapsto \sigma_\eta \mathbf{e}(\mathbf{i}_\lambda) \quad \begin{array}{l} \text{where } \eta \in \mathcal{D}(W^B_m/\mathfrak{S}_m) \text{ is the longest element} \\ \text{and } \theta(\lambda) = (j_1, \ldots, j_{m-1}). \end{array}$$

Extend this map $\mathbf{k}$-linearly and multiplicatively. Since ϕ is defined on the generators of $A^{\otimes(m-1)} \otimes \tilde{A}$, and the commutativity of elements in each tensor factor is preserved, the map ϕ is well-defined and is an algebra morphism.

We claim that ϕ is an isomorphism of $\mathbf{k}$-algebras. We first check surjectivity. Since there is a bijection between $\Pi(m)$ and Π^+ we have that ϕ is surjective on idempotents $\mathbf{e}(\mathbf{i}_\lambda)$. The map ϕ is also surjective on elements $\mathbf{e}(\mathbf{i}_{\lambda'})\sigma_w\mathbf{e}(\mathbf{i}_\lambda)$, $w \in W^B_m$. To see this, take any $\mathbf{e}(\mathbf{i}_{\lambda'})\sigma_w\mathbf{e}(\mathbf{i}_\lambda) \in \mathbf{e}\mathfrak{W}_\nu\mathbf{e}$. We have

$$\theta(\lambda) = (j_1, \ldots, j_{m-1})$$
$$\theta(\lambda^{'}) = (j^{'}_1, \ldots, j^{'}_{m-1})$$

for some $(j_1, \ldots, j_{m-1})$, $(j^{'}_1, \ldots, j^{'}_{m-1}) \in \Pi(m)$. Then

$$\phi(u^{\varepsilon_1}_{j_1} e_{j_1} \otimes u^{\varepsilon_2}_{j_2} e_{j_2} \otimes \cdots \otimes u^{\varepsilon_{m-1}}_{j_{m-1}} e_{j_{m-1}} \otimes v^{\varepsilon_i}_i a_i) = \mathbf{e}(\mathbf{i}_{\lambda'})\sigma_w\mathbf{e}(\mathbf{i}_\lambda),$$

where, for $1 \leq j \leq m-1$,

$$\varepsilon_j = \begin{cases} 1 & \text{if } j_1 \neq j^{'}_1 \\ 0 & \text{if } j_1 = j^{'}_1 \end{cases}$$

$$\varepsilon_i = \begin{cases} 0 & \text{if } \lambda, \lambda^{'} \in \Pi^+ \text{ or } \lambda, \lambda^{'} \in \Pi^- \\ 1 & \text{else.} \end{cases}$$

It remains to check surjectivity on the $x_j\mathbf{e}(\mathbf{i}_\lambda)$, $1 \leq j \leq m$.

Take any $\lambda \in \Pi^{\pm}$ with $\theta(\lambda) = (j_1, \dots, j_{m-1})$. For an entry $a \in \mathbf{i}_\lambda$ let $\psi(a)$ denote its position in $\mathbf{i}_\lambda$. Let us assume that q is an entry of $\mathbf{i}_\lambda$ since the same argument holds when q^{-1} is an entry of $\mathbf{i}_\lambda$. Then

$$\phi(e_{j_1} \otimes \cdots \otimes e_{j_{m-1}} \otimes \pi^2 a_i) = x_{\psi(q)}\mathbf{e}(\mathbf{i}_\lambda).$$

Now consider $x_j\mathbf{e}(\mathbf{i}_\lambda)$, for any $j \neq \psi(q)$. Let i_j be the j^{th} entry of $\mathbf{i}_\lambda$. Note that

$$x_j\mathbf{e}(\mathbf{i}_\lambda) = (x_j - x_k + x_k)\mathbf{e}(\mathbf{i}_\lambda)$$

where $k = \psi(p^{-2}i_j)$ is the position of $p^{-2}i_j$ in $\mathbf{i}_\lambda$.

But note that

$$(x_j - x_k)\mathbf{e}(\mathbf{i}_\lambda) = \sigma_w^\rho \mathbf{e}(\mathbf{i}_{\lambda'})\sigma_w\mathbf{e}(\mathbf{i}_\lambda)$$

for some $w \in \mathfrak{S}_m$, where $\lambda' \in \Pi^{\pm}$ is the root partition in which simple roots appear in the same order (possibly with different position) as in λ, except for i_j and $p^{-2}i_j$. In particular, $\theta(\lambda) = (j_1, \dots, j_k, \dots, j_{m-1})$ and $\theta(\lambda') = (j_1, \dots, j'_k, \dots j_{m-1})$ for some $1 \leq k \leq m-1$. We can repeat this until we get

$$x_j\mathbf{e}(\mathbf{i}_\lambda) = (x_j - x_k + x_k - \cdots - x_\ell + x_\ell)\mathbf{e}(\mathbf{i}_\lambda)$$

where $\ell = \psi(q)$. So $x_j\mathbf{e}(\mathbf{i}_\lambda) = (\sigma_{w_1}^\rho \sigma_{w_1} + \cdots + \sigma_{w_k}^\rho \sigma_{w_k} + x_{\psi(q)})\mathbf{e}(\mathbf{i}_\lambda)$. Since ϕ is surjective on the $\mathbf{e}(\mathbf{i}_{\lambda'})\sigma_w\mathbf{e}(\mathbf{i}_\lambda)$ we have ϕ surjective on the $x_j\mathbf{e}(\mathbf{i}_\lambda)$.

We complete the proof by comparing the graded dimensions of $A^{\otimes(m-1)} \otimes \tilde{A}$ and $\mathbf{e}\mathfrak{W}_\nu\mathbf{e}$ and showing that they are equal.

Consider first the graded dimension of A. There are two elements in degree 0: e_1 and e_2. There are two elements in degree 1: u_1e_1 and u_2e_2. Indeed we have two elements in each degree and so $\dim_q A = 2 + 2q + 2q^2 + 2q^3 + \cdots = \frac{2}{1-q}$. Noting that $A \cong \tilde{A}$, we therefore have

$$\dim_q(A^{\otimes(m-1)} \otimes \tilde{A}) = \frac{2^m}{(1-q)^m}.$$

Claim 3.4.9. *$dim_q(\boldsymbol{e}\mathfrak{W}_\nu\boldsymbol{e}) = dim_q(A^{\otimes(m-1)} \otimes \tilde{A})$.*

For every $w \in W_m^B$ fix a reduced expression of the form $w = \eta s$, for $\eta \in \mathcal{D}(W_m^B/\mathfrak{S}_m)$ and $s \in \mathfrak{S}_m$.

First we show that ϕ is degree-preserving. This is clear on idempotents and $x_k\mathbf{e}(\mathbf{i}_\lambda)$. Now take $\mathbf{e}(\mathbf{i}_{\lambda'})\sigma_w\mathbf{e}(\mathbf{i}_\lambda)$, for some $w \in W_m^B$.

- Suppose first that λ, λ' both lie either in Π^+ or in Π^-. Then $w = s \in \mathfrak{S}_m$. Let $s = s_{i_1} \cdots s_{i_k}$ be the fixed reduced expression for s so that $\sigma_s\mathbf{e}(\mathbf{i}_\lambda) = \sigma_{i_1} \cdots \sigma_{i_k}\mathbf{e}(\mathbf{i}_\lambda)$.

Now, $\deg(\sigma_{i_j}\mathbf{e}(\mathbf{i}))$ is 1 if σ_{i_j} swaps i_j and $p^{\pm 2}i_j$, and is 0 otherwise. Then $\deg(\sigma_s\mathbf{e}(\mathbf{i}_\lambda))$ is the number of pairs $(i_j, p^{\pm 2}i_j)$ in $\mathbf{e}(\mathbf{i}_\lambda)$ which appear in the opposite order in $\mathbf{e}(\mathbf{i}_{\lambda'})$. Letting $(a_1, \ldots, a_{m-1})_\lambda$, $(a_1, \ldots, a_{m-1})_{\lambda'}$ be the elements of $\Pi(m)$ corresponding to λ, $\lambda' \in \Pi^\pm$ respectively, we find $\deg(\sigma_s\mathbf{e}(\mathbf{i}_\lambda))$ is the number of entries in $(a_1, \ldots, a_{m-1})_\lambda$ different to entries in $(a_1, \ldots, a_{m-1})_{\lambda'}$, i.e. the number of u_ℓ appearing as tensorands in $\phi^{-1}(\mathbf{e}(\mathbf{i}_{\lambda'})\sigma_w\mathbf{e}(\mathbf{i}_\lambda))$.

- Now suppose one of λ, λ' lies in Π^+ and the other in Π^-. Then $w = \eta s$, where $s \in \mathfrak{S}_m$ and $\eta \in \mathcal{D}(W_m^B/\mathfrak{S}_m)$ is the longest element, and $\sigma_w\mathbf{e}(\mathbf{i}_\lambda) = \sigma_\eta\sigma_s\mathbf{e}(\mathbf{i}_\lambda)$. Since we always have $\deg(\sigma_\eta\mathbf{e}(\mathbf{i}_\lambda)) = 1$ it follows that $\deg(\mathbf{e}(\mathbf{i}_{\lambda'})\sigma_w\mathbf{e}(\mathbf{i}_\lambda)) = \deg(\sigma_\eta\mathbf{e}(\mathbf{i}_{\lambda''}))+ \deg(\sigma_s\mathbf{e}(\mathbf{i}_\lambda)) = 1 + \deg(\sigma_s\mathbf{e}(\mathbf{i}_\lambda))$, for some root partition λ''.

Hence $\deg(\mathbf{e}(\mathbf{i}_{\lambda'})\sigma_w\mathbf{e}(\mathbf{i}_\lambda)) = \#\{u_j, v_k \text{ appearing in } \phi^{-1}(\sigma_w\mathbf{e}(\mathbf{i}_\lambda))\}$ and so ϕ is degree-preserving.

This means we have a bijection

$$\{\mathbf{e}(\mathbf{i}_{\lambda'})\sigma_w\mathbf{e}(\mathbf{i}_\lambda) \mid \lambda', \lambda \in \Pi^\pm\} \longleftrightarrow \{\gamma_1 \otimes \cdots \otimes \gamma_m \in A^{\otimes(m-1)} \otimes \tilde{A} \mid \deg \gamma_i \leq 1 \quad \forall i\}.$$

Put $\mathscr{A} := A^{\otimes(m-1)} \otimes \tilde{A}$ and let $\mathscr{A}_{\deg\leq 1}$ be the vector space $\langle \gamma_1 \otimes \cdots \otimes \gamma_m \mid \deg\gamma_i \leq 1\rangle$. Each tensorand has two elements in each degree. Then the graded dimension of $\mathscr{A}_{\deg\leq 1}$ has a factor of 2^m. In each degree k we choose k tensorands from a possible m. There are $2^m\binom{m}{0}$ elements in degree 0. There are $2^m\binom{m}{1}$ elements in degree 1. In degree k there are $2^m\binom{m}{k}$ elements.

Then,

$$\dim_q \mathscr{A}_{\deg\leq 1} = 2^m \sum_{k=0}^{m} \binom{m}{k} q^k = 2^m(1+q)^m = \sum_{\lambda,\lambda'} q^{\deg(\mathbf{e}(\mathbf{i}_{\lambda'})\sigma_w\mathbf{e}(\mathbf{i}_\lambda))}$$

where we have used the above bijection for the third equality. Then we have

$$\dim_q(\mathbf{e}\mathfrak{W}_\nu\mathbf{e}) = \sum_{\lambda,\lambda'} q^{\deg(\mathbf{e}(\mathbf{i}_{\lambda'})\sigma_w\mathbf{e}(\mathbf{i}_\lambda))} \cdot \frac{1}{(1-q^2)^m} = \frac{2^m}{(1-q)^m}$$

and hence $\dim_q(\mathbf{e}\mathfrak{W}_\nu\mathbf{e}) = \dim_q(A^{\otimes(m-1)} \otimes \tilde{A})$ so that $\mathbf{e}\mathfrak{W}_\nu\mathbf{e} \cong A^{\otimes(m-1)} \otimes \tilde{A}$. This, together with the fact that $\mathbf{e}$ is full in $\mathfrak{W}_\nu$ by Corollary 3.4.7, proves Morita equivalence between $\mathfrak{W}_\nu$ and $A^{\otimes(m-1)} \otimes \tilde{A}$ using Corollary 3.1.18. □

We compare this result to a result of Brundan and Kleshchev. Pick $\tilde{\nu} \in \mathbb{N}I$, with multiplicity one, and $|\tilde{\nu}| = m$. Let $\mathbf{R}_{\tilde{\nu}}$ be the corresponding KLR algebra.

Theorem 3.4.10 ([Bru13], Theorem 3.13). *$\boldsymbol{R}_{\tilde{\nu}}$ and $A^{\otimes(m-1)} \otimes_{\boldsymbol{k}} \boldsymbol{k}[x]$ are Morita equivalent.*

3.4.2 Affine Quasi-Heredity

The notion of quasi-heredity for finite-dimensional algebras was first defined by Cline, Parshall and Scott in 1987. The motivating reasons came from the study of highest weight categories arising in the representation theory of semisimple complex Lie algebras and algebraic groups. They proved that every quasi-hereditary algebra has finite global dimension and showed that a finite-dimensional algebra A is quasi-hereditary if and only if A-mod is a highest weight category. A quasi-hereditary algebra with an involution i has been shown to be cellular, a notion defined for finite-dimensional algebras by Graham and Lehrer in [GL96]. Showing that an algebra is cellular gives rise to a parametrisation of simple modules, as discussed in 3.4.1.

In [KLM13], Kleshchev, Loubert and Miemietz proved that KLR algebras $\mathbf{R}_{\tilde{\nu}}$ in type A are affine cellular. Following this, Kleshchev and Loubert generalised this result for all finite types in [KL]. Kato and Brundan, Kleshchev, McNamara showed that the category of finitely generated $\mathbf{R}_{\tilde{\nu}}$-modules has features similar to that of a highest weight category, indicating that one could define the notion of an affine highest weight category. Indeed, in 2014, Kleshchev introduced affine quasi-heredity and the definition of an affine highest weight category, see [Kle15]. In particular, he proves an affine analogue of the Cline-Parshall-Scott Theorem; an algebra A is affine quasi-hereditary if and only if the category A-mod of finitely generated graded A-modules is an affine highest weight category. Kleshchev also shows that if A is an affine quasi-hereditary algebra together with an anti-involution ω, then A is affine cellular. In [Kle15], Kleshchev defines these notions in a more general setting, but for our purposes we take $\mathscr{B}$ to be the class of all positively graded polynomial algebras. The definitions that follow are taken from [Kle15].

A graded vector space V is said to be locally finite-dimensional if each graded component V_n is finite-dimensional.

Definition 3.4.11. A graded vector space V is called **Laurentian** if it is locally finite-dimensional and bounded below. A graded algebra A is Laurentian if it is Laurentian as a graded vector space.

Lemma 3.4.12 ([Kle], Lemma 2.2)**.** *Let H be a Laurentian algebra. Then,*

(i) All irreducible H-modules are finite-dimensional.

(ii) H is semiperfect (every finitely generated (graded) H-module has a (graded) projective cover); in particular, there are finitely many irreducible H-modules up to isomorphism and degree shift.

Definition 3.4.13. Let A be a Noetherian Laurentian graded $\mathbf{k}$-algebra. A is said to be **connected** if $A_n = 0$ for all $n < 0$ and $A_0 = \mathbf{k} \cdot 1$.

For example, all algebras in $\mathscr{B}$ are connected.

For a left Noetherian Laurentian algebra A let

$$\{L(\pi) \mid \pi \in \Pi\}$$

be a complete irredundant set of simple A-modules up to isomorphism and degree shift. For each $\pi \in \Pi$, let $P(\pi)$ be the projective cover of $L(\pi)$. That is, $P(\pi)$ is a projective A-module and there is a surjection $\theta : P(\pi) \twoheadrightarrow L(\pi)$ with $\ker(\theta)$ negligible, i.e. whenever $N \subset P(\pi)$ is a submodule with $N + \ker(\theta) = P(\pi)$, then $N = P(\pi)$.

We let q be both a formal variable and also a degree shift functor. If $V = \bigoplus_{n\in\mathbb{Z}} V_n$ then $(qV)_n = V_{n-1}$. For the remainder of this chapter all algebras will be Noetherian, Laurentian and graded and we will only consider finitely generated modules over these algebras.

Definition 3.4.14. A two-sided ideal $J \subseteq A$ is an **affine heredity ideal** if

(SI1) $\mathrm{Hom}_A(J, A/J) = 0$.

(SI2) As a left module, $J \cong m(q)P(\pi)$ for some graded multiplicity $m(q) \in \mathbb{Z}[q, q^{-1}]$ and some $\pi \in \Pi$, such that $B_\pi := \mathrm{End}_A(P(\pi))^{\mathrm{op}} \in \mathscr{B}$.

(PSI) As a right B_π-module, $P(\pi)$ is finitely generated and flat.

Remark 3.4.15. For a connected algebra, a finitely generated module is flat if and only if it is free. For B_π connected, the (PSI) condition can be reformulated: as a right B_π-module, $P(\pi)$ is free finite rank.

Lemma 3.4.16 ([Kle15], Lemma 6.5)**.** *Let J be an ideal in the algebra A such that the left A-module ${}_AJ$ is projective. Then the condition (SI1) is equivalent to the condition $J^2 = J$, which in turn is equivalent to $J = AeA$ for an idempotent $e \in A$.*

Lemma 3.4.17 ([Kle15], Lemma 6.6)**.** *Let $J \subseteq A$ be an affine heredity ideal. Write $J = AeA$ for an idempotent e, according to Lemma 3.4.16. Then the natural map $Ae \otimes_{eAe} eA \longrightarrow J$ is an isomorphism. Moreover, we may choose an idempotent e to be primitive so that, using the notation of Definition 3.4.14, we have $Ae \cong P(\pi)$ and $B_\pi \cong eAe$.*

Definition 3.4.18. An algebra A is **affine quasi-hereditary** if there exists a finite chain of ideals

$$(0) = J_0 \subsetneq J_1 \subsetneq \cdots \subsetneq J_n = A$$

with J_{i+1}/J_i an affine heredity ideal in A/J_i, for all $0 \leq i < n$. Such a chain of ideals is called an **affine heredity chain**.

Proposition 3.4.19. *If A and B are two affine quasi-hereditary $\boldsymbol{k}$-algebras then the tensor product $A \otimes_{\boldsymbol{k}} B$ is affine quasi-hereditary.*

Proof. Let,

$$\begin{aligned} (0) = A_0 \subset A_1 \subset \cdots \subset A_n = A \\ (0) = B_0 \subset B_1 \subset \cdots \subset B_m = B \end{aligned} \tag{3.4.3}$$

be affine heredity chains for A and B, respectively. We claim that

$$(0) \subset A_1 \otimes B_1 \subset A_1 \otimes B_2 \subset \cdots \subset A_1 \otimes B \subset A_2 \otimes B_1 + A_1 \otimes B \subset A_2 \otimes B_2 + A_1 \otimes B \subset \cdots \subset A \otimes B$$

is an affine heredity chain. We fix $m \geq 1$ and proceed by induction on n.

When $n = 1$, A has affine heredity chain $(0) = A_0 \subset A_1 = A$ and we claim that $A \otimes B$ has affine heredity chain,

$$(0) \subset A \otimes B_1 \subset A \otimes B_2 \subset \cdots \subset A \otimes B_m = A \otimes B.$$

We must show that each $(A \otimes B_i)/(A \otimes B_{i-1}) \subset (A \otimes B)/(A \otimes B_{i-1})$, for $1 \leq i \leq m$, is an affine heredity ideal. Note that, $(A \otimes B_i)/(A \otimes B_{i-1}) \cong A \otimes B_i/B_{i-1}$ and $(A \otimes B)/(A \otimes B_{i-1}) \cong A \otimes B/B_{i-1}$. Since A and B_i/B_{i-1} are both idempotent ideals, we know that $A \otimes B_i/B_{i-1}$ is an idempotent ideal. Then, by Lemma 3.4.16, to show (SI1) it suffices to show that $A \otimes B_i/B_{i-1}$ is projective as a left $A \otimes B/B_{i-1}$-module.

Since $A \subset A$ is an affine heredity ideal we have $A \cong m(q)P(\pi) \in A$-proj, for some graded multiplicity $m(q) \in \mathbb{Z}[q, q^{-1}]$, such that $B_\pi := \mathrm{End}(P(\pi))^{\mathrm{op}} \in \mathscr{B}$. By Lemma 3.4.17, $P(\pi) \cong Ae$ for some idempotent $e \in A$. Then we have $B_\pi \cong eAe \in \mathscr{B}$.

Similarly, since $B_i/B_{i-1} \subset B/B_{i-1}$ is an affine heredity ideal we have $B_i/B_{i-1} \cong n(q)Q(\sigma) \in B/B_{i-1}$-proj, for some graded multiplicity $n(q) \in \mathbb{Z}[q, q^{-1}]$, such that $B_\sigma := \mathrm{End}(Q(\sigma))^{\mathrm{op}} \in \mathscr{B}$. Again by Lemma 3.4.17, $Q(\sigma) \cong B/B_{i-1}\overline{f}$ for some idempotent $\overline{f} \in B/B_{i-1}$. Then we have $B_\sigma \cong \overline{f}B/B_{i-1}\overline{f} \in \mathscr{B}$.

Hence, $A \otimes B_i/B_{i-1} \cong m(q)n(q) \cdot P(\pi) \otimes Q(\sigma) \in A \otimes B/B_{i-1}$-proj, so that (SI1) holds. Moreover,

$$P(\pi) \otimes Q(\sigma) \cong Ae \otimes B/B_{i-1}\overline{f} = A \otimes B/B_{i-1} \cdot e \otimes \overline{f}.$$

So $B_{\pi,\sigma} := \mathrm{End}_{A \otimes B/B_{i-1}}(A \otimes B/B_{i-1} \cdot e \otimes \overline{f})^{\mathrm{op}} \cong eAe \otimes \overline{f}B/B_{i-1}\overline{f} \in \mathscr{B}$, so (SI2) also holds. We have so far shown that (SI1) and (SI2) hold. It remains to show (PSI). We know that $P(\pi)$ is free finite rank as a right B_π-module and that $Q(\sigma)$ is free finite rank as a right B_σ-module. That is, $P(\pi) \cong (eAe)^k$ and $Q(\sigma) \cong (\overline{f}B/B_{i-1}\overline{f})^l$, for some $k, l \in \mathbb{N}$. Hence,

$$P(\pi) \otimes Q(\sigma) \cong (eAe \otimes \overline{f}B/B_{i-1}\overline{f})^{kl}.$$

Then, as a right $B_{\pi,\sigma}$-module, $P(\pi) \otimes Q(\sigma)$ is free finite rank and the statement of the theorem holds for $n = 1$.

Suppose now that the result holds whenever A has an affine heredity chain of length $k < n$. Then let A and B be affine quasi-hereditary with affine heredity chains 3.4.3. One

can easily verify that
$$(0) \subset A_2/A_1 \subset \cdots \subset A_n/A_1 = A/A_1$$
is an affine heredity chain. It is a chain of length $n-1 < n$ so, using induction, the statement of the theorem holds for $(A/A_1) \otimes B$. Hence it suffices to prove that $(A_1 \otimes B_i)/(A_1 \otimes B_{i-1}) \subset (A \otimes B)/(A_1 \otimes B_{i-1})$, for $1 \leq i \leq m$, is an affine heredity ideal. Note that $(A_1 \otimes B_i)/(A_1 \otimes B_{i-1}) \cong A_1 \otimes (B_i/B_{i-1})$.

A_1 and B_i/B_{i-1} are both idempotent ideals so that $A_1 \otimes B_i/B_{i-1}$ is an idempotent ideal. Then, by Lemma 3.4.16, to show (SI1) it suffices to show that $A_1 \otimes B_i/B_{i-1}$ is a projective $(A \otimes B)/(A_1 \otimes B_{i-1})$-module. Firstly, note that $A_1 \otimes (B/B_{i-1}) \in (A \otimes B)/(A_1 \otimes B_{i-1})$-proj because
$$A_1 \otimes (B/B_{i-1}) \cong ((A \otimes B)/(A_1 \otimes B_{i-1})) \otimes_{A \otimes B} A_1 \otimes B$$
and, since $A_1 \otimes B \in A \otimes B$-proj, it follows that $A_1 \otimes (B/B_{i-1})$ is a direct summand of a free $(A \otimes B)/(A_1 \otimes B_{i-1})$-module. That is,
$$(A_1 \otimes (B/B_{i-1})) \oplus N \cong \bigoplus_{i \in I} ((A \otimes B)/(A_1 \otimes B_{i-1})),$$
for some module N. We also know that $B_i/B_{i-1} \in B/B_{i-1}$-proj, so that $B_i/B_{i-1} \oplus M \cong \bigoplus_{k \in K} B/B_{i-1}$, for some module M. Then,
$$\bigoplus_{k \in K} (A_1 \otimes B/B_{i-1}) \cong (A_1 \otimes B_i/B_{i-1}) \oplus (A_1 \otimes M).$$
Then we have,
$$\bigoplus_{k \in K} \bigoplus_{i \in I} ((A \otimes B)/(A_1 \otimes B_{i-1})) \cong (A_1 \otimes B_i/B_{i-1}) \oplus (A_1 \otimes M) \oplus \bigoplus_{k \in K} N.$$
That is, $A_1 \otimes (B_i/B_{i-1}) \in (A \otimes B)/(A_1 \otimes B_{i-1})$-proj. It follows that (SI1) is satisfied. We now show (SI2) holds.

Note first that $A_1 \subset A$ is an affine heredity ideal. So
$$A_1 \cong m(q)P(\pi) \cong m(q)Ae,$$
for some $\pi \in \Pi$, graded multiplicity $m(q) \in \mathbb{Z}[q, q^{-1}]$, and $e \in A$ a primitive idempotent, where we have used Lemma 3.4.17, for the second isomorphism. Similarly, $B_i/B_{i-1} \subset B/B_{i-1}$ is an affine heredity ideal so
$$B_i/B_{i-1} \cong n(q)Q(\sigma) \cong n(q)B/B_{i-1} \cdot \bar{f},$$
for some $\sigma \in \Sigma$, graded multiplicity $n(q) \in \mathbb{Z}[q, q^{-1}]$, and $\bar{f} = f + B_{i-1}$ a primitive

idempotent, again using Lemma 3.4.17, for the second isomorphism. Then,

$$\begin{aligned} A_1 \otimes B_i/B_{i-1} &\cong m(q)n(q)P(\pi) \otimes Q(\sigma) \\ &\cong m(q)n(q)A \otimes B/B_{i-1} \cdot (e \otimes \overline{f}) \in (A \otimes B)/(A \otimes B_{i-1})\text{−proj}, \end{aligned}$$

and is the projective cover of a simple $A \otimes (B/B_i)$-module. However, we need these facts for $(A \otimes B)/(A_1 \otimes B_{i-1})$. There is a short exact sequence of $A \otimes B$-bimodules,

$$0 \longrightarrow ker(g) \longrightarrow (A \otimes B)/(A_1 \otimes B_{i-1}) \xrightarrow{g} (A \otimes B)/(A \otimes B_{i-1}) \longrightarrow 0,$$

where $ker(g) \cong (A \otimes B_{i-1})/(A_1 \otimes B_{i-1}) \cong (A/A_1) \otimes B_{i-1}$. So the short exact sequence is

$$0 \longrightarrow (A/A_1) \otimes B_{i-1} \longrightarrow (A \otimes B)/(A_1 \otimes B_{i-1}) \xrightarrow{g} (A \otimes B)/(A \otimes B_{i-1}) \longrightarrow 0.$$

We now apply the functor $- \otimes_{A \otimes B} (Ae \otimes Bf)$, which we know to be right exact. But $(A/A_1) \otimes B_{i-1}$ vanishes under this functor, since $Ae \subset A_1$. This means we obtain an isomorphism,

$$(A \otimes B)/(A_1 \otimes B_{i-1}) \cdot (e \otimes f) \cong (A \otimes B)/(A \otimes B_{i-1}) \cdot (e \otimes f).$$

So,

$$\begin{aligned} A_1 \otimes (B_i/B_{i-1}) &\cong m(q)n(q)(A \otimes B)/(A_1 \otimes B_{i-1}) \cdot (e \otimes f) \\ &= m(q)n(q)(A \otimes B)/(A_1 \otimes B_{i-1}) \cdot (\overline{e \otimes f}), \end{aligned}$$

which is the projective cover of a simple $(A \otimes B)/(A_1 \otimes B_{i-1})$-module. Let $P(\pi, \sigma) := (A \otimes B)/(A_1 \otimes B_{i-1}) \cdot (e \otimes f)$. For (SI2) it remains to check that $B_{\pi,\sigma} := \text{End}(P(\pi, \sigma))^{\text{op}} \in \mathscr{B}$.

$$\begin{aligned} B_{\pi,\sigma} := \text{End}(P(\pi, \sigma))^{\text{op}} &\cong (e \otimes f) \cdot (A \otimes B)/(A_1 \otimes B_{i-1}) \cdot (e \otimes f) \\ &\cong (e \otimes f) \cdot (A \otimes B)/(A \otimes B_{i-1}) \cdot (e \otimes f) \\ &\cong eAe \otimes f(B/B_{i-1})f \in \mathscr{B}, \end{aligned}$$

since eAe, $f(B/B_{i-1})f \in \mathscr{B}$. This proves (SI2). We now show (PSI).

Ae is free finite rank as a right eAe-module, i.e. $Ae \cong (eAe)^k$. Similarly, $B/B_{i-1}f$ is free finite rank as a right $f(B/B_{i-1})f$-module, i.e. $(B/B_{i-1})f \cong (f(B/B_{i-1})f)^l$. This implies,

$$A \otimes (B/B_{i-1})(e \otimes f) \cong (eAe \otimes f(B/B_{i-1})f)^{kl}.$$

This proves (PSI) so that $(A_1 \otimes B_i)/(A_1 \otimes B_{i-1}) \subset (A \otimes B)/(A_1 \otimes B_{i-1})$, for $1 \leq i \leq m$, is an affine heredity ideal. Thus, using induction, we have shown that $A \otimes B$ is affine quasi-hereditary. $\square$

Let A be the path algebra of the following quiver, as in 3.4.1.

$$\begin{tikzcd} e_1 \arrow[r, bend left, "u_1"] & e_2 \arrow[l, bend left, "u_2"] \end{tikzcd}$$

A is a graded algebra with $\deg(e_i) = 0$, $\deg(u_i e_i) = 1$, for $i \in \{1, 2\}$. In this case, $\Pi = \{1, 2\}$. There are two simple graded modules denoted $L(1)$ and $L(2)$, with projective covers $P(1) = Ae_1$ and $P(2) = Ae_2$, respectively. As a $\mathbf{k}$-vector space, $A = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} A_n$, where A_n is the subspace of A spanned by the two degree n elements. So A is a Laurentian algebra. Consider Ae_1 as a left A-module. Since it is a uniserial module, every submodule is finitely generated and so Ae_1 is left Noetherian. Similarly, Ae_2 is left Noetherian. Hence $A = Ae_1 \oplus Ae_2$ is left Noetherian, using the fact that the direct sum of two left Noetherian modules is again left Noetherian.

Proposition 3.4.20. *A is affine quasi-hereditary.*

Proof. Consider the following chain of ideals

$$(0) \subsetneq Ae_1A \subsetneq A.$$

We first take $Ae_1A \subsetneq A$ and show it is an affine heredity ideal.

(SI1) As a $\mathbf{k}$-vector space $A/Ae_1A = \langle e_2 \rangle$. We have $\mathrm{Hom}_A(Ae_1A, A/Ae_1A) = 0$; for any $f \in \mathrm{Hom}_A(Ae_1A, A/Ae_1A)$ and any $a_1, a_2 \in A$, $f(a_1e_1a_2) = a_1e_1f(e_1a_2) = 0$, so that $\mathrm{Hom}_A(Ae_1A, A/Ae_1A) = 0$.

(SI2) We have $Ae_1A \subseteq Ae_1 + Au_2e_2$. But clearly $Ae_1 + Au_2e_2 \subseteq Ae_1A$ and so $Ae_1A = Ae_1 + Au_2e_2$. Since $Ae_1 \cap Au_2e_2 = \{0\}$, we have $Ae_1A = Ae_1 \oplus Au_2e_2$. So $m_1(q) = 1 + q$ and $P(1) = Ae_1$.

$P(1) = Ae_1$ so $B_1 = \mathrm{End}_A(Ae_1)^{\mathrm{op}} \cong e_1Ae_1 \cong \mathbf{k}[x]$, a polynomial algebra. So indeed we have $B_1 \in \mathscr{B}$.

(PSI) As a right e_1Ae_1-module, $Ae_1 = e_1 \cdot e_1Ae_1 + u_1e_1 \cdot e_1Ae_1$ and so is finitely generated. Also, $e_1 \cdot e_1Ae_1 \cap u_1e_1 \cdot e_1Ae_1 = \{0\}$. Then, $Ae_1 = \bigoplus_{\{e_1, u_1e_1\}} e_1Ae_1$ as a right e_1Ae_1-module and so is projective, which implies Ae_1 is flat.

This shows that Ae_1A is an affine heredity ideal in A. It remains to show that A/Ae_1A is an affine heredity ideal in A/Ae_1A. But this is immediate: for any ring R, $\mathrm{Hom}_R(R, R/R) = 0$ so that (SI1) is satisfied. A/Ae_1A has one simple module with $\Pi = \{2\}$. In fact, since

$$A/Ae_1A \cong \mathbf{k}$$

we have that the regular representation A/Ae_1A is a simple A/Ae_1A-module with projective cover $P(2) = A/Ae_1A$. So $m_2(q) = 1$. Also, $\mathrm{End}_{A/Ae_1A}(A/Ae_1A)^{\mathrm{op}} \cong A/Ae_1A \cong \mathbf{k}$ which lies in $\mathscr{B}$ so (SI2) is satisfied. Since $A/Ae_1A = \mathbf{k}e_2 = e_2\mathbf{k}$, A/Ae_1A is finitely generated as a right $\mathbf{k}$-module and clearly A/Ae_1A is flat.

Then $(0) \subsetneq Ae_1A \subsetneq A$ is an affine heredity chain and hence A is affine quasi-hereditary as claimed. $\square$

Combining Proposition 3.4.19 with Proposition 3.4.20 we obtain the following Corollary.

Corollary 3.4.21. *The $\boldsymbol{k}$-algebras $A^{\otimes(m-1)} \otimes_{\boldsymbol{k}} \tilde{A}$, $m \in \mathbb{N}$ are affine quasi-hereditary.*

Corollary 3.4.22. *For any $\nu \in {}^{\theta}\mathbb{N}I_q$ with multiplicity one the VV algebras $\mathfrak{W}_\nu$ are affine quasi-hereditary.*

Proof. Kleshchev proves, in [Kle15], Theorem 6.7, that an algebra R is affine quasi-hereditary if and only if R-Mod satisfies certain properties, in which case he calls R-Mod an affine highest weight category. This characterisation of affine quasi-heredity is purely categorical, meaning that affine quasi-heredity is a Morita invariant property. Morita equivalence between $A^{\otimes(m-1)} \otimes \tilde{A}$ and $\mathfrak{W}_\nu$, together with Corollary 3.4.21, proves the claim. $\square$

Corollary 3.4.23. *Suppose now that $p, q \notin I$. For any $\nu \in {}^{\theta}\mathbb{N}I$ the VV algebras $\mathfrak{W}_\nu$ are affine quasi-hereditary.*

Proof. Using Theorem 3.3.1 together with the fact that the algebras $\mathbf{R}_{\tilde{\nu}}$ are affine quasi-hereditary, when p is not a root of unity, (see [Kle15], Section 10.1) immediately gives the result. $\square$

3.4.3 Balanced Involution

Let A be a left Noetherian Laurentian algebra and let $\tau : A \longrightarrow A$ be a homogeneous anti-involution on A. That is, τ is an anti-automorphism of A such that $\tau^2 = id_A$. Given a left A-module $M \in A$-Mod, we can define a right A-module M^τ via $ma := \tau(a)m$. Given a graded module $M \in A$-Mod with finite-dimensional graded components M_n we can define its graded dual $M^{\circledast} \in A$-Mod. As a graded vector space, $M^{\circledast}_n := M^*_{-n}$ for all $n \in \mathbb{Z}$, and the action is given by $af(m) := f(\tau(a)m)$, for $f \in M^{\circledast}$, $m \in M$ and $a \in A$. Note that $(q^nV)^{\circledast} \cong q^{-n}V^{\circledast}$ and $\dim_q V^{\circledast} = \dim_{q^{-1}} V$.

Definition 3.4.24. The homogeneous anti-involution τ of A is called a **balanced involution** if, for every $\pi \in \Pi$, we have that $L(\pi)^{\circledast} \cong q^n L(\pi)$ for some even integer n.

As before, let A be the path algebra $A = \mathbf{k}(e_1 \rightleftarrows e_2)$. Define the following algebra morphism on A.

$$\begin{aligned} \tau : A &\longrightarrow A \\ e_i &\mapsto e_i \quad \text{for } i = 1, 2 \\ u_1 &\mapsto u_2 \\ u_2 &\mapsto u_1. \end{aligned}$$

The map τ is an anti-involution on A. Now consider $L(\pi)^{\circledast}$ for $\pi = 1$.

$$L(1)^{\circledast}_n = L(1)^*_{-n} = \begin{cases} \mathrm{Hom}(L(1), \mathbf{k}) & \text{if } n = 0 \\ 0 & \text{if } n \neq 0. \end{cases}$$

As a $\mathbf{k}$-vector space $\mathrm{Hom}(L(1), \mathbf{k}) = \langle f \rangle$ where $f(e_1) = 1$. $\mathrm{Hom}(L(1), \mathbf{k})$ is an A-module with the action given by $(af)(m) = f(\tau(a)m)$, for all $a \in A$ and $m \in L(1)$. Then, $(e_1 f)(e_1) = f(e_1)$ so that $e_1 f = f$. $(e_2 f)(e_1) = f(e_2 e_1) = 0$ so $e_2 f = 0$. Similarly, $u_1 f = 0 = u_2 f$. Then $\mathrm{Hom}(L(1), \mathbf{k}) \cong L(1)$, under the A-module isomorphism $f \mapsto e_1$, and hence $L(\pi)^{\circledast} \cong L(\pi)$ so that τ is a balanced involution.

Remark 3.4.25. This also applies to $\tilde{A}$ so we can extend τ to a homogeneous anti-involution $\tau^{\otimes m}$ on $A^{\otimes(m-1)} \otimes_{\mathbf{k}} \tilde{A}$, for any $m \in \mathbb{N}$. Indeed, $\tau^{\otimes m}$ is also a balanced involution.

Now we have shown the algebras $A^{\otimes(m-1)} \otimes_{\mathbf{k}} \tilde{A}$, $m \in \mathbb{N}$, are affine quasi-hereditary we can use the following result of Kleshchev to deduce that $A^{\otimes(m-1)} \otimes_{\mathbf{k}} \tilde{A}$, $m \in \mathbb{N}$, are affine cellular.

Proposition 3.4.26 ([Kle15], Proposition 9.8)**.** *Let B be an affine quasi-hereditary algebra with a balanced involution τ. Then B is an affine cellular algebra.*

Corollary 3.4.27. *The algebras $A^{\otimes(m-1)} \otimes_{\mathbf{k}} \tilde{A}$, $m \in \mathbb{N}$ are affine cellular with respect to $\tau^{\otimes m}$.*

Lemma 3.4.28 ([Yan14], Lemma 3.4)**.** *Let A be an algebra with an idempotent $e \in A$ and a $\mathbf{k}$-involution i such that $i(e) = e$. Suppose that $AeA = A$ and eAe is an affine cellular algebra with respect to the restriction of i to eAe. Then A is affine cellular with respect to i.*

Let us now recall the following notation. Suppose we have fixed a reduced expression $s_{i_1} \cdots s_{i_k}$ for some $w \in W^B_m$ so that $\sigma_w = \sigma_{i_1} \cdots \sigma_{i_k}$. We have defined $\sigma^{\rho}_w := \sigma_{i_k} \cdots \sigma_{i_1}$.

Corollary 3.4.29. *For any $\nu \in {}^{\theta}\mathbb{N}I_q$ with multiplicity one the VV algebras $\mathfrak{W}_\nu$ are affine cellular.*

Proof. Fix $\nu \in {}^{\theta}\mathbb{N}I_q$ with multiplicity one and take the associated VV algebra $\mathfrak{W}_\nu$. Fix $\mathbf{e} := \sum_{\lambda \in \Pi^{\pm}} \mathbf{e}(\mathbf{i}_\lambda)$ and define a $\mathbf{k}$-involution on $\mathfrak{W}_\nu$ by,

$$\begin{aligned} i : \mathfrak{W}_\nu &\longrightarrow \mathfrak{W}_\nu \\ x_k &\mapsto x_k \\ \sigma_w &\mapsto \sigma^{\rho}_w \\ \mathbf{e}(\mathbf{i}) &\mapsto \mathbf{e}(\mathbf{i}) \end{aligned}$$

for all $1 \leq k \leq m$, $w \in W^B_m$, $\mathbf{i} \in {}^{\theta}I^{\nu}$. Then $i(\mathbf{e}) = \mathbf{e}$ and, from Corollary 3.4.7, $\mathfrak{W}_\nu \mathbf{e} \mathfrak{W}_\nu = \mathfrak{W}_\nu$. Using Corollary 3.4.27 and Theorem 3.4.8, $\mathbf{e}\mathfrak{W}_\nu\mathbf{e}$ is affine cellular with respect to the restriction of i. Then, by Lemma 3.4.28, $\mathfrak{W}_\nu$ is affine cellular with respect to i. $\square$

Corollary 3.4.30. *The VV algebras $\mathfrak{W}_\nu$ in the setting $p, q \notin I$ are affine cellular.*

Proof. Fix $\nu \in {}^\theta\mathbb{N}I_\lambda$ and take the associated VV algebra $\mathfrak{W}_\nu$. Fix $\mathbf{e} := \sum_{\mathbf{i} \in I^{\tilde{\nu}}} \mathbf{e}(\mathbf{i})$ and let i be the $\mathbf{k}$-involution on $\mathfrak{W}_\nu$ from Corollary 3.4.29. Then $i(\mathbf{e}) = \mathbf{e}$ and, from Theorem 3.3.1, $\mathfrak{W}_\nu \mathbf{e} \mathfrak{W}_\nu = \mathfrak{W}_\nu$. From [KLM13] we know that KLR algebras of type A are affine cellular. This means that $\mathbf{e}\mathfrak{W}_\nu\mathbf{e}$ is affine cellular with respect to the restriction of i. We now use Lemma 3.4.28 to conclude that $\mathfrak{W}_\nu$ is affine cellular with respect to i. □

3.4.4 Conjecture - An Affine Cellular Basis of $\mathfrak{W}_\nu$

In this small subsection we conjecture an affine cellular basis of $\mathfrak{W}_\nu$ in the setting $\nu \in {}^\theta\mathbb{N}I_q$ with multiplicity one.

Let $|\nu| = 2m$, $m \in \mathbb{N}$. The VV algebra $\mathfrak{W}_\nu$ has idempotent subalgebras $\mathbf{R}^+_{\tilde{\nu}}$ and $\mathbf{R}^-_{\tilde{\nu}}$. Denote the corresponding root partitions by Π^+ and Π^-. Since ν has multiplicity one, we have $|\Pi^+ \cup \Pi^-| = 2^m$. Order these root partitions lexicographically, i.e. for root partitions $\pi_1 = (\beta_1, \dots, \beta_r)$, $\pi_2 = (\gamma_1, \dots, \gamma_s)$ we have $\pi_1 < \pi_2$ if and only if $\beta_j < \gamma_j$, where j is smallest index with $\beta_i \neq \gamma_i$.

For $\mathbf{i} \in {}^\theta I^\nu$ and $a \in I$ an entry of $\mathbf{i}$, let $\varphi(a)$ denote the position of a in $\mathbf{i}$.

Every root partition $\pi = (\beta_1, \dots, \beta_r)$ defines a parabolic subgroup of W_m^B;

$$\mathfrak{S}_\pi \cong \mathfrak{S}_{|\beta_1|} \times \cdots \times \mathfrak{S}_{|\beta_r|}$$

and a subalgebra $\Lambda_\pi \subseteq \mathbf{k}[x_1, \dots, x_m]$, where

$$\Lambda_\pi := \mathbf{k}[x_{|\beta_1|}, x_{|\beta_1|+|\beta_2|}, \dots, x_{|\beta_1|+\cdots+|\beta_r|}].$$

For $\pi \in \Pi^+$ define

$$\mathcal{B}^+_\pi := \{\sigma_\eta \mathbf{e}(\mathbf{i}_\pi) \mid \eta \in \mathcal{D}(W_m^B/\mathfrak{S}_\pi)\}.$$

For $\pi \in \Pi^-$ define

$$\mathcal{B}^-_\pi := \mathcal{D}'_{\mathbf{e}(\mathbf{i}_\pi)}\mathbf{e}(\mathbf{i}_\pi) \cup \mathcal{D}'_{\mathbf{e}(\eta_1\mathbf{i}_\pi)}\mathbf{e}(\eta_1\mathbf{i}_\pi) \cup \cdots \cup \mathcal{D}'_{\mathbf{e}(\eta_k\mathbf{i}_\pi)}\mathbf{e}(\eta_k\mathbf{i}_\pi)$$

where $\eta_1, \dots, \eta_k \in \mathcal{D}(\mathfrak{S}_m/\mathfrak{S}_\pi)$ and where

$$\mathcal{D}'_{\mathbf{e}(\mathbf{i})} := \left\{ \sigma_\eta \mathbf{e}(\mathbf{i}) \;\middle|\; \begin{array}{l} \eta \in \mathcal{D}(W_m^B/\mathfrak{S}_m) \text{ such that } s_0 \text{ appears at most} \\ \varphi(q^{-1}) - 1 \text{ times in a reduced expression} \end{array} \right\}.$$

Take the following chain of ideals

$$(0) \subseteq I_{>\pi_{2^m-1}} \subseteq \cdots \subseteq I_{>\pi_2} \subseteq I_{>\pi_1} \subseteq \mathfrak{W}_\nu$$

where $I_{>\pi_i} = \mathfrak{W}_\nu(\mathbf{e}(\mathbf{i}_{\pi_{i+1}}) + \mathbf{e}(\mathbf{i}_{\pi_{i+2}}) + \cdots + \mathbf{e}(\mathbf{i}_{\pi_{2m}}))\mathfrak{W}_\nu$. Then set

$$I'_{\pi_i} := I_{>\pi_i}/I_{>\pi_{i+1}} \text{ for each } i.$$

Conjecture 3.4.31. *The given chain of ideals is a cell chain for $\mathfrak{W}_\nu$ and each I'_π has a basis given by*

$$\mathscr{B}_\pi = \left\{ \sigma_\eta p \boldsymbol{e}(\boldsymbol{i}_\pi)\sigma_\xi^\rho \middle| \sigma_\eta, \sigma_\xi \in \left\{ \begin{array}{l} \mathcal{B}_\pi^+ \textit{ if } \pi \in \Pi^+ \\ \mathcal{B}_\pi^- \textit{ if } \pi \in \Pi^- \end{array} \right., p \in \Lambda_\pi \right\}.$$

3.4.5 Morita Equivalence Between $\mathfrak{W}_\nu$ and $\mathbf{R}_{\tilde{\nu}}^+ \otimes_{\mathbf{k}[z]} \tilde{A}$

In this subsection we loosen the restriction on multiplicity by enforcing a multiplicity restriction only on q; we take $\nu \in {}^\theta\mathbb{N}I_q$, $|\nu| = 2m$, in which q appears with multiplicity one. We also allow p be to a root of unity in this subsection.

Let $\tilde{A}$ be the path algebra of the quiver described in 3.4.1. Then $\tilde{A}$ is a left $\mathbf{k}[z]$-module via

$$\begin{aligned} z \cdot a_1 &= v_2 v_1 a_1 \\ z \cdot a_2 &= -v_1 v_2 a_2. \end{aligned}$$

We can also put the structure of a right $\mathbf{k}[z]$-module on $\mathbf{R}_{\tilde{\nu}}^+$, the action of z on $\mathbf{R}_{\tilde{\nu}}^+$ being multiplication by $\sum_{\mathbf{i} \in I^{\tilde{\nu}}} x_{\varphi_\mathbf{i}(q)} \mathbf{e}(\mathbf{i})$, where $\varphi_\mathbf{i}(q)$ denotes the position of q in $\mathbf{i}$.

Lemma 3.4.32. *For every $\nu \in {}^\theta\mathbb{N}I$ with $|\nu| = 2m$ there is an isomorphism of KLR algebras, $\boldsymbol{R}_{\tilde{\nu}}^+ \cong \boldsymbol{R}_{\tilde{\nu}}^-$, given by*

$$\begin{aligned} \psi : \boldsymbol{R}_{\tilde{\nu}}^+ &\longrightarrow \boldsymbol{R}_{\tilde{\nu}}^- \\ \boldsymbol{e}(\boldsymbol{i}) &\mapsto \boldsymbol{e}(\eta \boldsymbol{i}) \\ x_j \boldsymbol{e}(\boldsymbol{i}) &\mapsto -x_{m-j+1} \boldsymbol{e}(\eta \boldsymbol{i}) \\ \sigma_k \boldsymbol{e}(\boldsymbol{i}) &\mapsto \sigma_{m-k} \boldsymbol{e}(\eta \boldsymbol{i}) \end{aligned}$$

and extending $\boldsymbol{k}$-linearly and multiplicatively, where $\eta \in \mathcal{D}(W_m^B/\mathfrak{S}_m)$ is the longest element.

Proof. One can use the relations to check that ψ is indeed an algebra morphism and it is clear that this map is a bijection. □

Let $\mathbf{e} := \sum_{\mathbf{i} \in I^{\tilde{\nu}+}} \mathbf{e}(\mathbf{i}) + \sum_{\mathbf{i} \in I^{\tilde{\nu}-}} \mathbf{e}(\mathbf{i})$. Considering $\mathbf{e}\mathfrak{W}_\nu \mathbf{e}$ yields the following result.

Lemma 3.4.33. *In the setting described above, $dim_q(\boldsymbol{e}\mathfrak{W}_\nu \boldsymbol{e}) = dim_q(\boldsymbol{R}_{\tilde{\nu}}^+ \otimes_{\boldsymbol{k}[z]} \tilde{A})$.*

Proof. From Lemma 2.2.11 we have a basis

$$\mathcal{B} = \{\sigma_{\dot{w}} x_1^{n_1} \cdots x_m^{n_m} \mathbf{e}(\mathbf{i}) \mid w \in \mathfrak{S}_m, \mathbf{i} \in I^{\tilde{\nu}}, n_i \in \mathbb{N}_0 \; \forall i\}$$

of $\mathbf{R}^+_{\tilde{\nu}}$, and $\tilde{A}$ has basis $\{a_1, a_2, v_1a_1, v_2a_2, v_2v_1a_1, v_1v_2a_2, \ldots\}$. Since $za_1 = v_2v_1a_1$ and $za_2 = v_1v_2a_2$, we have the following basis for $\mathbf{R}^+_{\tilde{\nu}} \otimes_{\mathbf{k}[z]} \tilde{A}$.

$$\mathcal{B}' = \{b \otimes a_1, b \otimes a_2, b \otimes v_1a_1, b \otimes v_2a_2 \mid b \in \mathcal{B}\}.$$

So the graded dimension of $\mathbf{R}^+_{\tilde{\nu}} \otimes_{\mathbf{k}[z]} \tilde{A}$ is

$$\begin{aligned} \dim_q(\mathbf{R}^+_{\tilde{\nu}} \otimes_{\mathbf{k}[z]} \tilde{A}) &= 2\dim_q(\mathbf{R}^+_{\tilde{\nu}}) + 2q\dim_q(\mathbf{R}^+_{\tilde{\nu}}) \\ &= 2(1+q)\dim_q(\mathbf{R}^+_{\tilde{\nu}}). \end{aligned}$$

For each $w \in W^B_m$ fix a reduced expression $w = \eta s$, for $\eta \in \mathcal{D}(W^B_m/\mathfrak{S}_m)$, $s \in \mathfrak{S}_m$. Take any basis element $\mathbf{e}\sigma_w x_1^{n_1} \cdots x_m^{n_m}\mathbf{e} \in \mathbf{e}\mathfrak{W}_\nu\mathbf{e}$. Note that

$$\mathbf{e}\sigma_w x_1^{n_1} \cdots x_m^{n_m}\mathbf{e} = \mathbf{e}\sigma_\eta\mathbf{e}\sigma_s x_1^{n_1} \cdots x_m^{n_m}\mathbf{e}.$$

For this expression to be non-zero either $\eta = 1$ or η is the longest element in $\mathcal{D}(W^B_m/\mathfrak{S}_m)$. If $\eta = 1$ then $\mathbf{e}\sigma_\eta\mathbf{e}$ has degree 0. If η is the longest element in $\mathcal{D}(W^B_m/\mathfrak{S}_m)$ then $\mathbf{e}\sigma_\eta\mathbf{e}$ has degree 1. This is because σ_η starts at an idempotent in $\mathbf{R}^+_{\tilde{\nu}}$ and ends at one in $\mathbf{R}^-_{\tilde{\nu}}$ so that there is precisely one factor of σ_η which contributes to the degree of σ_η, namely $\pi\mathbf{e}(q^{\pm 1}, \ldots)$. So,

$$\begin{aligned} \dim_q(\mathbf{e}\mathfrak{W}_\nu\mathbf{e}) &= 2\frac{\sum_{\mathbf{i}\in I^{\tilde{\nu}}}\left(\sum_{s\in\mathfrak{S}_m} q^{\deg\sigma_s\mathbf{e}(\mathbf{i})} + q\sum_{s\in\mathfrak{S}_m} q^{\deg\sigma_s\mathbf{e}(\mathbf{i})}\right)}{(1-q^2)^m} \\ &= 2\dim_q(\mathbf{R}^+_{\tilde{\nu}}) + 2q\dim_q(\mathbf{R}^+_{\tilde{\nu}}) \\ &= 2(1+q)\dim_q(\mathbf{R}^+_{\tilde{\nu}}) \\ &= \dim_q(\mathbf{R}^+_{\tilde{\nu}} \otimes_{\mathbf{k}[z]} \tilde{A}) \end{aligned}$$

where we have used

$$\dim_q(\mathbf{R}^+_{\tilde{\nu}}) = \frac{\sum_{\mathbf{i}\in I^{\tilde{\nu}}}\sum_{s\in\mathfrak{S}_m} q^{\deg\sigma_s\mathbf{e}(\mathbf{i})}}{(1-q^2)^m}$$

in the second equality. Hence we have shown $\dim_q(\mathbf{e}\mathfrak{W}_\nu\mathbf{e}) = \dim_q(\mathbf{R}^+_{\tilde{\nu}} \otimes_{\mathbf{k}[z]} \tilde{A})$. □

Theorem 3.4.34. *$\mathfrak{W}_\nu$ and $\mathbf{R}^+_{\tilde{\nu}} \otimes_{\mathbf{k}[z]} \tilde{A}$ are Morita equivalent.*

Proof. Let $\mathbf{e} := \sum_{\mathbf{i}\in I^{\tilde{\nu}+}} \mathbf{e}(\mathbf{i}) + \sum_{\mathbf{i}\in I^{\tilde{\nu}-}} \mathbf{e}(\mathbf{i})$ and define a map

$$\phi : \mathbf{R}^+_{\tilde{\nu}} \otimes_{\mathbf{k}[z]} \tilde{A} \longrightarrow \mathbf{e}\mathfrak{W}_\nu\mathbf{e}$$

as,

$$\mathbf{e}(\mathbf{i}) \otimes a_i \mapsto \begin{cases} \mathbf{e}(\mathbf{i}) \text{ if } i = 1 \\ \mathbf{e}(\eta\mathbf{i}) \text{ if } i = 2 \end{cases}$$
$$x_j\mathbf{e}(\mathbf{i}) \otimes a_i \mapsto \begin{cases} x_j\mathbf{e}(\mathbf{i}) \text{ if } i = 1 \\ -x_{m-j+1}\mathbf{e}(\eta\mathbf{i}) \text{ if } i = 2 \end{cases}$$
$$\sigma_k\mathbf{e}(\mathbf{i}) \otimes a_i \mapsto \begin{cases} \sigma_k\mathbf{e}(\mathbf{i}) \text{ if } i = 1 \\ \sigma_{m-k}\mathbf{e}(\eta\mathbf{i}) \text{ if } i = 2 \end{cases}$$
$$\mathbf{e}(\mathbf{i}) \otimes v_i a_i \mapsto \begin{cases} \sigma_\eta\mathbf{e}(\mathbf{i}) \text{ if } i = 1 \\ \sigma_\eta\mathbf{e}(\eta\mathbf{i}) \text{ if } i = 2, \end{cases}$$

for $1 \leq j \leq m$, $1 \leq k \leq m-1$ and where $\eta \in \mathcal{D}(W_m^B/\mathfrak{S}_m)$ is the longest element, extending $\mathbf{k}$-linearly and multiplicatively. We must show that ϕ is well-defined so that ϕ is a morphism of $\mathbf{k}$-algebras.

Clearly ϕ preserves relations when we restrict the second tensorand to a_1. That is, $\phi|_{\mathbf{R}_{\tilde{\nu}}^+ \otimes a_1}$ yields an injective morphism $\mathbf{R}_{\tilde{\nu}}^+ \hookrightarrow \mathbf{e}\mathfrak{W}_\nu\mathbf{e}$, so the relations are preserved.

Now consider restricting the second tensorand to a_2. The restriction of ϕ to $\mathbf{R}_{\tilde{\nu}}^+ \otimes_{\mathbf{k}[z]} a_2$ also yields an injection $\mathbf{R}_{\tilde{\nu}}^+ \hookrightarrow \mathbf{e}\mathfrak{W}_\nu\mathbf{e}$, with image $\mathbf{R}_{\tilde{\nu}}^-$, using Lemma 3.4.32. So the relations are also preserved in this case.

We must also show $\phi(\mathbf{e}(\mathbf{i}) \otimes za_1) = \phi(\mathbf{e}(\mathbf{i})z \otimes a_1)$ and $\phi(\mathbf{e}(\mathbf{i}) \otimes za_2) = \phi(\mathbf{e}(\mathbf{i})z \otimes a_2)$. Firstly,

$$\begin{aligned}
\phi(\mathbf{e}(\mathbf{i}) \otimes za_1) &= \phi(\mathbf{e}(\mathbf{i}) \otimes v_2v_1a_1) \\
&= \phi(\mathbf{e}(\mathbf{i}) \otimes v_2a_2)\phi(\mathbf{e}(\mathbf{i}) \otimes v_1a_1) \\
&= \sigma_\eta^\rho\sigma_\eta\mathbf{e}(\mathbf{i}) \\
&= x_{\varphi_{\mathbf{i}}(q)}\mathbf{e}(\mathbf{i}) \\
&= \phi(x_{\varphi_{\mathbf{i}}(q)}\mathbf{e}(\mathbf{i}) \otimes a_1) \\
&= \phi(\mathbf{e}(\mathbf{i})z \otimes a_1).
\end{aligned}$$

Secondly, noting that $\varphi_{\eta\mathbf{i}}(q^{-1}) = m - \varphi_{\mathbf{i}}(q) + 1$,

$$\begin{aligned}
\phi(\mathbf{e}(\mathbf{i}) \otimes za_2) &= \phi(\mathbf{e}(\mathbf{i}) \otimes -v_1v_2a_2) \\
&= -\phi(\mathbf{e}(\mathbf{i}) \otimes v_1a_1)\phi(\mathbf{e}(\mathbf{i}) \otimes v_2a_2) \\
&= -\sigma_\eta^\rho\sigma_\eta\mathbf{e}(\eta\mathbf{i}) \\
&= -x_{\varphi_{\eta\mathbf{i}}(q^{-1})}\mathbf{e}(\eta\mathbf{i}) \\
&= -x_{m-\varphi_{\mathbf{i}}(q)+1}\mathbf{e}(\eta\mathbf{i}) \\
&= \phi(x_{\varphi_{\mathbf{i}}(q)}\mathbf{e}(\mathbf{i}) \otimes a_2) \\
&= \phi(\mathbf{e}(\mathbf{i})z \otimes a_2).
\end{aligned}$$

The multiplicative identity element in $\mathbf{R}^+_{\tilde{\nu}} \otimes_{\mathbf{k}[z]} \tilde{A}$ is $\big(\sum_{\mathbf{i}\in I^{\tilde{\nu}+}} \mathbf{e}(\mathbf{i})\big) \otimes (a_1 + a_2)$ and $\mathbf{e}\mathfrak{W}_\nu\mathbf{e}$ has identity element $\mathbf{e}$. We have,

$$\begin{aligned}
\phi\Big(\big(\sum_{\mathbf{i}\in I^{\tilde{\nu}+}} \mathbf{e}(\mathbf{i})\big) \otimes (a_1 + a_2)\Big) &= \sum_{\mathbf{i}\in I^{\tilde{\nu}+}} \phi(\mathbf{e}(\mathbf{i}) \otimes a_1) + \sum_{\mathbf{i}\in I^{\tilde{\nu}+}} \phi(\mathbf{e}(\mathbf{i}) \otimes a_2) \\
&= \sum_{\mathbf{i}\in I^{\tilde{\nu}+}} \mathbf{e}(\mathbf{i}) + \sum_{\mathbf{i}\in I^{\tilde{\nu}+}} \mathbf{e}(\eta\mathbf{i}) \\
&= \sum_{\mathbf{i}\in I^{\tilde{\nu}+}} \mathbf{e}(\mathbf{i}) + \sum_{\mathbf{i}\in I^{\tilde{\nu}-}} \mathbf{e}(\mathbf{i}) \\
&= \mathbf{e}.
\end{aligned}$$

So ϕ is indeed a $\mathbf{k}$-algebra homomorphism. Now to show that ϕ is surjective. This is clear for idempotents and for elements $x_j\mathbf{e}(\mathbf{i}) \in \mathbf{e}\mathfrak{W}_\nu\mathbf{e}$. For every $w \in W_m^B$ fix a reduced expression of the form $w = s\eta$, for $s \in \mathfrak{S}_m$ and $\eta \in \mathcal{D}(\mathfrak{S}_m\backslash W_m^B)$ the longest element in the set of minimal length right coset representatives of $\mathfrak{S}_m$ in W_m^B. Take $\mathbf{e}(\mathbf{i})\sigma_w\mathbf{e}(\mathbf{j}) \in \mathbf{e}\mathfrak{W}_\nu\mathbf{e}$, for some $w = s\eta \in W_m^B$.

- If $\mathbf{e}(\mathbf{i}), \mathbf{e}(\mathbf{j}) \in \mathbf{R}^+_{\tilde{\nu}}$ then $\eta = 1$ so that $w = s \in \mathfrak{S}_m$ and $\phi(\sigma_w\mathbf{e}(\mathbf{j}) \otimes a_1) = \sigma_w\mathbf{e}(\mathbf{j}) = \mathbf{e}(\mathbf{i})\sigma_w\mathbf{e}(\mathbf{j})$.

- Similarly if $\mathbf{e}(\mathbf{i}), \mathbf{e}(\mathbf{j}) \in \mathbf{R}^-_{\tilde{\nu}}$ then $\eta = 1$ and $w = s \in \mathfrak{S}_m$. Let $s = s_{i_1}\cdots s_{i_k}$ be a reduced expression for s. Note that $\mathbf{e}(\eta\mathbf{i}) \in \mathbf{R}^+_{\tilde{\nu}}$ since $\mathbf{e}(\mathbf{j}) \in \mathbf{R}^-_{\tilde{\nu}}$. Then $\sigma_s\mathbf{e}(\mathbf{j}) = \sigma_{i_1}\cdots\sigma_{i_k}\mathbf{e}(\mathbf{j})$ and

$$\begin{aligned}
\phi(\sigma_{m-i_1}\cdots\sigma_{m-i_k}\mathbf{e}(\eta\mathbf{j}) \otimes a_2) &= \sigma_{i_1}\cdots\sigma_{i_k}\mathbf{e}(\mathbf{j}) \\
&= \sigma_s\mathbf{e}(\mathbf{j}).
\end{aligned}$$

- Suppose now that $\mathbf{e}(\mathbf{i}) \in \mathbf{R}^+_{\tilde{\nu}}$ and $\mathbf{e}(\mathbf{j}) \in \mathbf{R}^-_{\tilde{\nu}}$. Then $w = s\eta$ and $\sigma_w\mathbf{e}(\mathbf{j}) = \sigma_s\sigma_\eta\mathbf{e}(\mathbf{j})$. Suppose s has fixed reduced expression $s = s_{i_1}\cdots s_{i_k}$. Again note that since $\mathbf{e}(\mathbf{j}) \in$

$\mathbf{R}^-_{\tilde{\nu}}$, we have $\mathbf{e}(\eta\mathbf{j}) \in \mathbf{R}^+_{\tilde{\nu}}$. Then

$$\begin{aligned}
\phi(\sigma_{i_1}\cdots\sigma_{i_k}\mathbf{e}(\eta\mathbf{j}) \otimes v_2a_2) &= \phi(\sigma_{i_1}\cdots\sigma_{i_k}\mathbf{e}(\eta\mathbf{j}) \otimes a_1)\phi(\mathbf{e}(\eta\mathbf{j}) \otimes v_2a_2) \\
&= \sigma_{i_1}\cdots\sigma_{i_k}\mathbf{e}(\eta\mathbf{j})\sigma_\eta\mathbf{e}(\mathbf{j}) \\
&= \sigma_s\sigma_\eta\mathbf{e}(\mathbf{j}) \\
&= \mathbf{e}(\mathbf{i})\sigma_w\mathbf{e}(\mathbf{j}).
\end{aligned}$$

The case $\mathbf{e}(\mathbf{i}) \in \mathbf{R}^-_{\tilde{\nu}}$ and $\mathbf{e}(\mathbf{j}) \in \mathbf{R}^+_{\tilde{\nu}}$ is similar.

It remains to show that ϕ is injective. Since ϕ is surjective and, by Lemma 3.4.33, $\dim_q(\mathbf{e}\mathfrak{W}_\nu\mathbf{e}) = \dim_q(\mathbf{R}^+_{\tilde{\nu}} \otimes_{\mathbf{k}[z]} \tilde{A})$ injectivity follows immediately and we have a $\mathbf{k}$-algebra isomorphism $\mathbf{e}\mathfrak{W}_\nu\mathbf{e} \cong \mathbf{R}^+_{\tilde{\nu}} \otimes_{\mathbf{k}[z]} \tilde{A}$.

The proof of $\mathbf{e}$ full in $\mathfrak{W}_\nu$ follows precisely the same reasoning as in the proof of Lemma 3.4.5, so we omit this here and instead refer the reader to Lemma 3.4.5. Then $\mathfrak{W}_\nu\mathbf{e}$ is a progenerator for $\mathfrak{W}_\nu$-Mod and, by Corollary 3.1.18, we have Morita equivalence between $\mathfrak{W}_\nu$ and $\mathbf{R}^+_{\tilde{\nu}} \otimes_{\mathbf{k}[z]} \tilde{A}$. □

3.5 Morita Equivalence in the Case $p \in I$

In this section we fix the following setting; assume $p \in I$, $q \notin I$, p is not a root of unity. As noted in 3.3, we take $\nu \in {}^\theta\mathbb{N}I_p$ so that p always appears with multiplicity greater than 1, i.e. $\nu_p > 1$. In this section we impose a further restriction on our choice of $\nu \in {}^\theta\mathbb{N}I_p$; we pick ν so that p appears with multiplicity exactly 2, i.e. $\nu_p = 2$. As before, Π^+ denotes the root partitions of $\mathbf{R}^+_{\tilde{\nu}}$ and Π^- denotes the root partitions of $\mathbf{R}^-_{\tilde{\nu}}$. Put $\Pi^\pm := \Pi^+ \cup \Pi^-$.

As before, let $\tilde{A}$ be the path algebra of the following quiver.

$$\begin{tikzcd} a_1 \arrow[r, bend left, "v_1"] & a_2 \arrow[l, bend left, "v_2"] \end{tikzcd}$$

We define the structure of a left $\mathbf{k}[z]$-module on $\tilde{A}$ via

$$\begin{aligned}
z \cdot a_1 &= v_2v_1a_1 \\
z \cdot a_2 &= v_1v_2a_2.
\end{aligned}$$

We also define the structure of a right $\mathbf{k}[z]$-module on $\mathbf{R}^+_{\tilde{\nu}}$. The action of z on $\mathbf{R}^+_{\tilde{\nu}}$ is multiplication by $\sum_{\mathbf{i} \in I^{\tilde{\nu}+}}(x_{\varphi_{\mathbf{i},1}(p)} + x_{\varphi_{\mathbf{i},2}(p)})\mathbf{e}(\mathbf{i})$, where $\varphi_{\mathbf{i},1}(p)$ denotes the position of the first p appearing in $\mathbf{i}$ and $\varphi_{\mathbf{i},2}(p)$ denote the position of the second p appearing in $\mathbf{i}$.

Using the definition of ${}^\theta I^\nu$, we note here that for any given $\mathbf{i} \in {}^\theta I^\nu$ we have either p appearing twice in $\mathbf{i}$, p^{-1} appearing before p in $\mathbf{i}$, p appearing before p^{-1} in $\mathbf{i}$, or p^{-1} appearing twice in $\mathbf{i}$.

Lemma 3.5.1. *Take $\boldsymbol{e}(\boldsymbol{i}) \in \mathfrak{W}_\nu$ with $\boldsymbol{i} \notin I^{\tilde{\nu}+} \cup I^{\tilde{\nu}-}$.*

(i) If p appears twice in $\boldsymbol{i}$ or if p^{-1} appears before p in $\boldsymbol{i}$ then $\boldsymbol{e}(\boldsymbol{i}) \cong \boldsymbol{e}(\boldsymbol{j})$ for some $\boldsymbol{j} \in I^{\tilde{\nu}+}$.

(ii) If p^{-1} appears twice in $\boldsymbol{i}$ or if p appears before p^{-1} in $\boldsymbol{i}$ then $\boldsymbol{e}(\boldsymbol{i}) \cong \boldsymbol{e}(\boldsymbol{j})$ for some $\boldsymbol{j} \in I^{\tilde{\nu}-}$.

Proof. For (i) assume first that p appears twice in $\mathbf{i}$, say $i_{k_1} = i_{k_2} = p$. By Proposition 3.1.20, we need $a \in \mathbf{e}(\mathbf{i})\mathfrak{W}_\nu\mathbf{e}(\mathbf{j})$, $b \in \mathbf{e}(\mathbf{j})\mathfrak{W}_\nu\mathbf{e}(\mathbf{i})$ such that $ab = \mathbf{e}(\mathbf{i})$ and $ba = \mathbf{e}(\mathbf{j})$, for some $\mathbf{j} \in I^{\tilde{\nu}+}$.

Since $\mathbf{i} \notin I^{\tilde{\nu}+} \cup I^{\tilde{\nu}-}$ there is a non-empty subset $\{\varepsilon_1, \ldots, \varepsilon_d\} \subsetneq \{1, \ldots, m\}$, with $\varepsilon_r < \varepsilon_{r+1}$ for all r, such that $i_{\varepsilon_r} = p^{-(2n_r+1)}$, where $n_r \in \mathbb{Z}_{>0}$ for all r. Note also that $\varepsilon_r \neq k_1, k_2$ for all r, since we start with the assumption that p appears twice in $\mathbf{i}$. Put,

$$\begin{aligned} a &= \mathbf{e}(\mathbf{i})\sigma_{\varepsilon_1-1}\cdots\sigma_1\pi\sigma_{\varepsilon_2-1}\cdots\sigma_1\pi\cdots\sigma_{\varepsilon_d-1}\cdots\sigma_1\pi\mathbf{e}(\mathbf{j}) \in \mathbf{e}(\mathbf{i})\mathfrak{W}_\nu\mathbf{e}(\mathbf{j}) \\ b &= \mathbf{e}(\mathbf{j})\pi\sigma_1\cdots\sigma_{\varepsilon_d-1}\pi\sigma_1\cdots\sigma_{\varepsilon_{d-1}-1}\cdots\pi\sigma_1\cdots\sigma_{\varepsilon_1-1}\mathbf{e}(\mathbf{i}) \in \mathbf{e}(\mathbf{j})\mathfrak{W}_\nu\mathbf{e}(\mathbf{i}). \end{aligned}$$

Then $\mathbf{e}(\mathbf{j}) \in \mathfrak{W}_\nu$ is such that $\mathbf{j} \in I^{\tilde{\nu}+}$ and, from the relations, we know that $ab = \mathbf{e}(\mathbf{i})$ and $ba = \mathbf{e}(\mathbf{j})$ so that $\mathbf{e}(\mathbf{i}) \cong \mathbf{e}(\mathbf{j})$. The same argument can be used for the case when p^{-1} appears before p in $\mathbf{i}$. A similar argument is used for (ii). $\square$

Lemma 3.5.2. *For every $\boldsymbol{i} \in I^{\tilde{\nu}+}$ and $\eta \in \mathcal{D}(W_m^B/\mathfrak{S}_m)$, the longest element, we have*

(i) $\sigma_\eta^\rho\sigma_\eta\boldsymbol{e}(\boldsymbol{i}) = (x_{\varphi_{i,1}(p)} + x_{\varphi_{i,2}(p)})\boldsymbol{e}(\boldsymbol{i})$

(ii) $\sigma_\eta^\rho\sigma_\eta\boldsymbol{e}(\eta\boldsymbol{i}) = -(x_{\varphi_{\eta i,1}(p^{-1})} + x_{\varphi_{\eta i,2}(p^{-1})})\boldsymbol{e}(\eta\boldsymbol{i})$.

Proof. These equalities are immediate consequences of the relations and the fact that Γ_{I_p} has an arrow $p \longrightarrow p^{-1}$. $\square$

Theorem 3.5.3. *$\mathfrak{W}_\nu$ and $\boldsymbol{R}_{\tilde{\nu}}^+ \otimes_{\boldsymbol{k}[z]} \tilde{A}$ are Morita equivalent.*

Proof. Let $\mathbf{e} = \sum_{\mathbf{i}\in I^{\tilde{\nu}+}} \mathbf{e}(\mathbf{i}) + \sum_{\mathbf{i}\in I^{\tilde{\nu}-}} \mathbf{e}(\mathbf{i})$. Define a map

$$\phi : \mathbf{R}_{\tilde{\nu}}^+ \otimes_{\mathbf{k}[z]} \tilde{A} \longrightarrow \mathbf{e}\mathfrak{W}_\nu\mathbf{e}$$

as,

$$\mathbf{e}(\mathbf{i}) \otimes a_i \mapsto \begin{cases} \mathbf{e}(\mathbf{i}) \text{ if } i = 1 \\ \mathbf{e}(\eta\mathbf{i}) \text{ if } i = 2 \end{cases}$$

$$x_j\mathbf{e}(\mathbf{i}) \otimes a_i \mapsto \begin{cases} x_j\mathbf{e}(\mathbf{i}) \text{ if } i = 1 \\ -x_{m-j+1}\mathbf{e}(\eta\mathbf{i}) \text{ if } i = 2 \end{cases}$$

$$\sigma_k\mathbf{e}(\mathbf{i}) \otimes a_i \mapsto \begin{cases} \sigma_k\mathbf{e}(\mathbf{i}) \text{ if } i = 1 \\ \sigma_{m-k}\mathbf{e}(\eta\mathbf{i}) \text{ if } i = 2 \end{cases}$$

$$\mathbf{e}(\mathbf{i}) \otimes v_i a_i \mapsto \begin{cases} \sigma_\eta\mathbf{e}(\mathbf{i}) \text{ if } i = 1 \\ \sigma_\eta\mathbf{e}(\eta\mathbf{i}) \text{ if } i = 2 \end{cases}$$

for $1 \leq j \leq m$, $1 \leq k \leq m-1$ and where $\eta \in \mathcal{D}(W_m^B/\mathfrak{S}_m)$ is the longest element, extending $\mathbf{k}$-linearly and multiplicatively. We must show that ϕ is well-defined so that ϕ is a morphism of $\mathbf{k}$-algebras.

Firstly, when we restrict the second tensorand to a_1 or to a_2, ϕ preserves the relations using the same argument as in the proof of Theorem 3.4.34. It remains to check $\phi(\mathbf{e}(\mathbf{i}) \otimes z a_k) = \phi(\mathbf{e}(\mathbf{i})z \otimes a_k)$, for $k = 1, 2$. We have, using Lemma 3.5.2 in the fourth equality,

$$\begin{aligned}
\phi(\mathbf{e}(\mathbf{i}) \otimes z a_1) &= \phi(\mathbf{e}(\mathbf{i}) \otimes v_2 v_1 a_1) \\
&= \phi(\mathbf{e}(\mathbf{i}) \otimes v_2 a_2)\phi(\mathbf{e}(\mathbf{i}) \otimes v_1 a_1) \\
&= \sigma_\eta^\rho \sigma_\eta \mathbf{e}(\mathbf{i}) \\
&= (x_{\varphi_{\mathbf{i},1}(p)} + x_{\varphi_{\mathbf{i},2}(p)})\mathbf{e}(\mathbf{i}) \\
&= \phi((x_{\varphi_{\mathbf{i},1}(p)} + x_{\varphi_{\mathbf{i},2}(p)})\mathbf{e}(\mathbf{i}) \otimes a_1) \\
&= \phi(\mathbf{e}(\mathbf{i})z \otimes a_1).
\end{aligned}$$

Secondly, noting again from Lemma 3.5.2 that $\sigma_\eta^\rho \sigma_\eta \mathbf{e}(\eta\mathbf{i}) = -(x_{\varphi_{\eta\mathbf{i},1}(p^{-1})} + x_{\varphi_{\eta\mathbf{i},2}(p^{-1})})\mathbf{e}(\eta\mathbf{i})$ and $m - \varphi_{\eta\mathbf{i},k}(p^{-1}) + 1 = \varphi_{\mathbf{i},k}(p)$ for $k = 1, 2$ we have

$$\begin{aligned}
\phi(\mathbf{e}(\mathbf{i}) \otimes z a_2) &= \phi(\mathbf{e}(\mathbf{i}) \otimes v_1 v_2 a_2) \\
&= \phi(\mathbf{e}(\mathbf{i}) \otimes v_1 a_1)\phi(\mathbf{e}(\mathbf{i}) \otimes v_2 a_2) \\
&= \sigma_\eta^\rho \sigma_\eta \mathbf{e}(\eta\mathbf{i}) \\
&= -(x_{\varphi_{\eta\mathbf{i},1}(p^{-1})} + x_{\varphi_{\eta\mathbf{i},2}(p^{-1})})\mathbf{e}(\eta\mathbf{i}) \\
&= \phi((x_{\varphi_{\mathbf{i},1}(p)} + x_{\varphi_{\mathbf{i},2}(p)})\mathbf{e}(\mathbf{i}) \otimes a_2) \\
&= \phi(\mathbf{e}(\mathbf{i})z \otimes a_2).
\end{aligned}$$

The same calculation from the proof of Theorem 3.4.34 shows

$$\phi((\sum_{\mathbf{i} \in I^{\tilde{\nu}+}} \mathbf{e}(\mathbf{i})) \otimes (a_1 + a_2)) = \mathbf{e}.$$

Then ϕ is indeed a well-defined $\mathbf{k}$-algebra morphism.

Now to show that ϕ is surjective. This is clear for idempotents $\mathbf{e}(\mathbf{i})$ and for $x_j\mathbf{e}(\mathbf{i}) \in \mathbf{e}\mathfrak{W}_\nu\mathbf{e}$. For every $w \in W_m^B$ fix a reduced expression of the form $w = s\eta$, for $s \in \mathfrak{S}_m$ and $\eta \in \mathcal{D}(\mathfrak{S}_m \backslash W_m^B)$ the longest element in the set of minimal length right coset representatives of $\mathfrak{S}_m$ in W_m^B. Take $\mathbf{e}(\mathbf{i})\sigma_w\mathbf{e}(\mathbf{j}) \in \mathbf{e}\mathfrak{W}_\nu\mathbf{e}$, for some $w = s\eta \in W_m^B$.

- If $\mathbf{e}(\mathbf{i}), \mathbf{e}(\mathbf{j}) \in \mathbf{R}_{\tilde{\nu}}^+$ then $\eta = 1$ so that $w = s \in \mathfrak{S}_m$ and $\phi(\sigma_w\mathbf{e}(\mathbf{j}) \otimes a_1) = \sigma_w\mathbf{e}(\mathbf{j}) = \mathbf{e}(\mathbf{i})\sigma_w\mathbf{e}(\mathbf{j})$.
- Similarly if $\mathbf{e}(\mathbf{i}), \mathbf{e}(\mathbf{j}) \in \mathbf{R}_{\tilde{\nu}}^-$ then $\eta = 1$ and $w = s \in \mathfrak{S}_m$. Let $s = s_{i_1} \cdots s_{i_k}$ be a reduced expression for s. Then $\sigma_s\mathbf{e}(\mathbf{j}) = \sigma_{i_1} \cdots \sigma_{i_k}\mathbf{e}(\mathbf{j})$ and

$$\begin{aligned}\phi(\sigma_{m-i_1} \cdots \sigma_{m-i_k}\mathbf{e}(\eta\mathbf{j}) \otimes a_2) &= \sigma_{i_1} \cdots \sigma_{i_k}\mathbf{e}(\mathbf{j}) \\ &= \sigma_s\mathbf{e}(\mathbf{j}).\end{aligned}$$

- Suppose now that $\mathbf{e}(\mathbf{i}) \in \mathbf{R}_{\tilde{\nu}}^+$ and $\mathbf{e}(\mathbf{j}) \in \mathbf{R}_{\tilde{\nu}}^-$. Then $w = s\eta$ and $\sigma_w\mathbf{e}(\mathbf{j}) = \sigma_s\sigma_\eta\mathbf{e}(\mathbf{j})$. Suppose s has fixed reduced expression $s = s_{i_1} \cdots s_{i_k}$. Again note that since $\mathbf{e}(\mathbf{j}) \in \mathbf{R}_{\tilde{\nu}}^-$, we have $\mathbf{e}(\eta\mathbf{j}) \in \mathbf{R}_{\tilde{\nu}}^+$. Then

$$\begin{aligned}\phi(\sigma_{i_1} \cdots \sigma_{i_k}\mathbf{e}(\eta\mathbf{j}) \otimes v_2a_2) &= \phi(\sigma_{i_1} \cdots \sigma_{i_k}\mathbf{e}(\eta\mathbf{j}) \otimes a_1)\phi(\mathbf{e}(\eta\mathbf{j}) \otimes v_2a_2) \\ &= \sigma_{i_1} \cdots \sigma_{i_k}\mathbf{e}(\eta\mathbf{j})\sigma_\eta\mathbf{e}(\mathbf{j}) \\ &= \sigma_s\sigma_\eta\mathbf{e}(\mathbf{j}) \\ &= \mathbf{e}(\mathbf{i})\sigma_w\mathbf{e}(\mathbf{j}).\end{aligned}$$

  The case $\mathbf{e}(\mathbf{i}) \in \mathbf{R}_{\tilde{\nu}}^-$ and $\mathbf{e}(\mathbf{j}) \in \mathbf{R}_{\tilde{\nu}}^+$ is similar.

It remains to show that ϕ is injective. Since ϕ is surjective we can show injectivity by showing equality between the graded dimensions of $\mathbf{R}_{\tilde{\nu}}^+ \otimes_{\mathbf{k}[x]} \tilde{A}$ and $\mathbf{e}\mathfrak{W}_\nu\mathbf{e}$. This follows from the analogue of Lemma 3.4.33 applied to the current setting. Hence ϕ is an isomorphism.

By Lemma 3.5.1 we have $\mathbf{e}$ full in $\mathfrak{W}_\nu$. Morita equivalence between $\mathbf{e}\mathfrak{W}_\nu\mathbf{e}$ and $\mathbf{R}_{\tilde{\nu}}^+ \otimes_{\mathbf{k}[x]} \tilde{A}$ then follows from Corollary 3.1.18. $\square$

Chapter 4

Finite-Dimensional Quotients of $\mathfrak{W}_\nu$

In this chapter we aim to define certain finite-dimensional quotients of VV algebras. We first discuss finite-dimensional quotients of KLR algebras, known as cyclotomic KLR algebras, and explain why they are important. We then go on to describe the various difficulties encountered when trying to define analogous quotients of VV algebras. Finally, in 4.3, we are able to define a family of finite-dimensional quotients of VV algebras in a particular case and show how they relate to cyclotomic KLR algebras.

4.1 Cyclotomic KLR Algebras

Recall from Section 2.2, when defining KLR algebras, we started with a loop-free quiver $\tilde{\Gamma}_{\tilde{I}}$, of type A, with vertex set $\tilde{I}$ and arrows $p^2 i \longrightarrow i$, for all $i \in \tilde{I}$. Fix such a quiver $\tilde{\Gamma}_{\tilde{I}}$. To $\tilde{I}$ we can associate a lattice

$$\tilde{P} = \bigoplus_{i \in \tilde{I}} \mathbb{Z}\Lambda_i.$$

Let $\tilde{P}_+$ denote the subset of $\tilde{P}$ consisting of elements with non-negative coefficients with respect to basis elements. Let $(\cdot, \cdot) : \tilde{P} \times \mathbb{N}\tilde{I} \longrightarrow \mathbb{Z}$ be the bilinear map defined by $(\Lambda_i, j) = \delta_{ij}$, for $i, j \in \tilde{I}$. For $\tilde{\nu} \in \mathbb{N}\tilde{I}$ and $\Lambda \in \tilde{P}_+$, the **level** of Λ, denoted $\ell(\Lambda)$, is defined by

$$\ell(\Lambda) := \sum_{j \in \tilde{I}} (\Lambda, j).$$

The **height** of $\tilde{\nu} \in \mathbb{N}\tilde{I}$, defined in Section 2.2, is given by

$$|\tilde{\nu}| = \sum_{i \in \tilde{I}} (\Lambda_i, \tilde{\nu}).$$

Definition 4.1.1. The **cyclotomic KLR algebra** of type $\tilde{\Gamma}_{\tilde{I}}$ associated to $\Lambda \in \tilde{P}_+$ of level l and $\tilde{\nu} \in \mathbb{N}\tilde{I}$ of height m is the $\mathbf{k}$-algebra, denoted $\mathbf{R}_{\tilde{\nu}}^{\Lambda}$, generated by elements

$$\{x_1, \ldots, x_m\} \cup \{\sigma_1, \ldots, \sigma_{m-1}\} \cup \{\mathbf{e}(\mathbf{i}) \mid \mathbf{i} \in \tilde{I}^{\tilde{\nu}}\},$$

subject to the relations given in Definition 2.2.3 together with the following additional relation.

$$x_1^{(\Lambda, i_1)} \mathbf{e}(\mathbf{i}) = 0 \text{ for all } \mathbf{i} \in \tilde{I}^{\tilde{\nu}}. \tag{4.1.1}$$

That is, $\mathbf{R}_{\tilde{\nu}}^{\Lambda} = \mathbf{R}_{\tilde{\nu}}/J^{\Lambda}$ where J^{Λ} is the **cyclotomic ideal**,

$$J^{\Lambda} = \langle x_1^{(\Lambda, i_1)} \mathbf{e}(\mathbf{i}) \mid \mathbf{i} \in \tilde{I}^{\tilde{\nu}} \rangle.$$

One can think of this quotient in the following way. We start with a KLR algebra $\mathbf{R}_{\tilde{\nu}}$ to which we associate a quiver whose vertices are idempotents of the form $\mathbf{e}(\mathbf{i})$. A cyclotomic KLR algebra $\mathbf{R}_{\tilde{\nu}}^{\Lambda}$ is obtained from $\mathbf{R}_{\tilde{\nu}}$ by choosing nilpotency degrees of x_1 at each vertex $\mathbf{e}(\mathbf{i})$. This nilpotency is dependent upon i_1 at each $\mathbf{e}(\mathbf{i})$.

Example 4.1.2. Consider the KLR algebra from Example 2.2.7. So $\tilde{\nu} = \lambda + p^2\lambda + p^4\lambda \in \mathbb{N}\tilde{I}$ and $\mathbf{R}_{\tilde{\nu}}$ can be represented by the following quiver, where the x_i paths have been omitted to reduce notation.

$$\begin{tikzcd}
\mathbf{e}(\lambda, p^2\lambda, p^4\lambda) \arrow[rr, bend left, "\sigma_2"] \arrow[d, bend left, "\sigma_1"] & & \mathbf{e}(\lambda, p^4\lambda, p^2\lambda) \arrow[ll, bend left, "\sigma_2"] \arrow[d, bend left, "\sigma_1"] \\
\mathbf{e}(p^2\lambda, \lambda, p^4\lambda) \arrow[u, bend left, "\sigma_1"] \arrow[d, bend left, "\sigma_2"] & & \mathbf{e}(p^4\lambda, \lambda, p^2\lambda) \arrow[u, bend left, "\sigma_1"] \arrow[d, bend left, "\sigma_2"] \\
\mathbf{e}(p^2\lambda, p^4\lambda, \lambda) \arrow[u, bend left, "\sigma_2"] \arrow[rr, bend left, "\sigma_1"] & & \mathbf{e}(p^4\lambda, p^2\lambda, \lambda) \arrow[u, bend left, "\sigma_2"] \arrow[ll, bend left, "\sigma_1"]
\end{tikzcd}$$

Let $\Lambda = \Lambda_{p^4\lambda} + 3\Lambda_{p^2\lambda} + 9\Lambda_{p^{-6}\lambda}$, so $\ell(\Lambda) = 13$. Then,

$$\begin{aligned}
(\Lambda, \lambda) &= 0 \\
(\Lambda, p^2\lambda) &= 3 \\
(\Lambda, p^4\lambda) &= 1
\end{aligned}$$

and the cyclotomic ideal is

$$J^{\Lambda} = \left\langle \begin{array}{ll} \mathbf{e}(\lambda, p^2\lambda, p^4\lambda), & \mathbf{e}(\lambda, p^4\lambda, p^2\lambda), \\ x_1^3\mathbf{e}(p^2\lambda, \lambda, p^4\lambda), & x_1^3\mathbf{e}(p^2\lambda, p^4\lambda, \lambda), \\ x_1\mathbf{e}(p^4\lambda, \lambda, p^2\lambda), & x_1\mathbf{e}(p^4\lambda, p^2\lambda, \lambda) \end{array} \right\rangle.$$

Brundan and Kleshchev proved that in fact all x_i, $1 \leq i \leq m$, in $\mathbf{R}^{\Lambda}_{\tilde{\nu}}$ are nilpotent and used this to show that the algebras $\mathbf{R}^{\Lambda}_{\tilde{\nu}}$ are finite-dimensional.

Lemma 4.1.3 ([BK09a], Lemma 2.1)**.** *The elements x_i are nilpotent, for all $1 \leq i \leq m$.*

Corollary 4.1.4 ([BK09a], Corollary 2.2)**.** *$\boldsymbol{R}^{\Lambda}_{\tilde{\nu}}$ is a finite-dimensional algebra.*

Proof. For each $s \in \mathfrak{S}_m$ fix a reduced expression. Using Lemma 2.2.11 we know that

$$\{\sigma_s x_1^{n_1} \cdots x_m^{n_m} \mathbf{e}(\mathbf{i}) \mid s \in \mathfrak{S}_m, \mathbf{i} \in \tilde{I}^{\tilde{\nu}}, n_i \in \mathbb{N}_0 \ \forall i\}$$

is a basis for $\mathbf{R}_{\tilde{\nu}}$, and hence a spanning set for $\mathbf{R}^{\Lambda}_{\tilde{\nu}}$. Now using Lemma 4.1.3 we have that this spanning set is finite so that $\mathbf{R}^{\Lambda}_{\tilde{\nu}}$ is finite-dimensional. $\square$

Let H^{A}_{m} denote the affine Hecke algebra of type A, from Definition 1.2.6. Fix $q \in \mathbf{k}^{\times}$.

Definition 4.1.5. The **cyclotomic Hecke algebra** associated to $\Lambda \in \tilde{P}_{+}$ of level l, denoted H^{Λ}_{m}, is defined to be the following quotient of H^{A}_{m}.

$$\mathrm{H}^{\Lambda}_{m} := \mathrm{H}^{A}_{m} / \big\langle \prod_{i \in \tilde{I}} (X_1 - q^i)^{(\Lambda, i)} \big\rangle.$$

Remark 4.1.6. There also exists a degenerate affine Hecke algebra when $q = 1$, with analogous cyclotomic quotients.

It can be shown that H^{Λ}_{m} is a finite-dimensional algebra and that there exists a set $\{\mathbf{e}(\mathbf{i}) \mid \mathbf{i} \in \tilde{I}^{m}\}$ of mutually orthogonal idempotents in H^{Λ}_{m}. Fix $\tilde{\nu} \in \mathbb{N}\tilde{I}$ with $|\tilde{\nu}| = m$. Setting $e_{\tilde{\nu}} := \sum_{\mathbf{i} \in \tilde{I}^{\tilde{\nu}}} \mathbf{e}(\mathbf{i}) \in \mathrm{H}^{\Lambda}_{m}$ it follows that $e_{\tilde{\nu}}$ is either zero or is a primitive central idempotent in H^{Λ}_{m} and so

$$\mathrm{H}^{\Lambda}_{\tilde{\nu}} := e_{\tilde{\nu}} \mathrm{H}^{\Lambda}_{m}$$

is a block of H^{Λ}_{m} or is zero.

In [BK09a] Brundan and Kleshchev prove that these blocks of cyclotomic Hecke algebras are isomorphic to cyclotomic KLR algebras in type A.

Theorem 4.1.7 ([BK09a], Main Theorem)**.** *The algebra $H^{\Lambda}_{\tilde{\nu}}$ is generated by elements*

$$\{x_1, \ldots, x_m\} \cup \{\sigma_1, \ldots, \sigma_{m-1}\} \cup \{\boldsymbol{e}(\boldsymbol{i}) \mid \boldsymbol{i} \in \tilde{I}^{\tilde{\nu}}\}$$

subject to the relations given in Definition 2.2 together with 4.1.1, for $\tilde{\Gamma} = \tilde{\Gamma}_{\tilde{I}}$.

Using this isomorphism, together with the fact that KLR algebras are naturally $\mathbb{Z}$-graded, we now have a grading on the blocks of cyclotomic Hecke algebras which was not known prior to this work of Brundan and Kleshchev. When $\ell(\Lambda) = 1$, H^{Λ}_{m} is isomorphic to the group algebra of the symmetric group $\mathfrak{S}_m$ when $q = 1$ or the associated Iwahori-Hecke algebra for arbitrary q. Hence we have an interesting $\mathbb{Z}$-grading on blocks of symmetric groups and Iwahori-Hecke algebras. This also provides a link between KLR algebras, the LLT conjecture and the categorification theorems discussed in Section 2.1 of this thesis.

Another interesting feature of cyclotomic KLR algebras is the cellular structure which they possess. In 2010 Hu and Mathas constructed an explicit cellular basis for $\mathbf{R}^{\Lambda}_{\tilde{\nu}}$, proving that cyclotomic KLR algebras are cellular algebras. Among other properties, this gives a parametrisation of simple modules over $\mathbf{R}^{\Lambda}_{\tilde{\nu}}$. See [HM10] for details.

Because of the close relationship between KLR algebras and VV algebras, it is a natural question to ask whether one can define a quotient of a given VV algebra $\mathfrak{W}_\nu$ analogous to the cyclotomic quotient of $\mathbf{R}_{\tilde{\nu}}$. In the following sections we investigate the result of taking quotients of VV algebras by certain ideals with the overall aim of trying to find a finite-dimensional algebra which, in some sense, is induced from a cyclotomic KLR algebra. That is, we hope to find a quotient that agrees with the quotient of the KLR algebra.

4.2 Cyclotomic VV Algebras

Recall from Chapter 2.3 that in order to define VV algebras we first fix a quiver Γ_I with vertex set I. To I we can associate a lattice

$$P = \bigoplus_{i \in I} \mathbb{Z}\Lambda_i.$$

Let P_+ denote the subset of P consisting of elements with non-negative coefficients with respect to basis elements. As in Section 4.1, let $(\cdot,\cdot) : P \times \mathbb{N}I \longrightarrow \mathbb{Z}$ be the bilinear map defined by $(\Lambda_i, j) = \delta_{ij}$, for $i, j \in I$. In particular, this defines a map $(\cdot,\cdot) : P \times {}^{\theta}\mathbb{N}I \longrightarrow \mathbb{Z}$. Each $\nu \in {}^{\theta}\mathbb{N}I$ defines a VV algebra, $\mathfrak{W}_\nu$. Fix such a ν. Let $\mathfrak{W}^N_\nu$ be the $\mathbf{k}$-algebra generated by elements

$$\{x_1, \ldots, x_m\} \cup \{\sigma_1, \ldots, \sigma_{m-1}\} \cup \{\mathbf{e}(\mathbf{i}) \mid \mathbf{i} \in {}^{\theta}I^{\nu}\} \cup \{\pi\}$$

subject to the relations given in Definition 2.3.4 together with the additional relation

$$x_1^N \mathbf{e}(\mathbf{i}) = 0 \text{ for all } \mathbf{i} \in {}^{\theta}I^{\nu},$$

for some non-negative integer N. This is the $\mathbf{k}$-algebra constructed as the quotient of $\mathfrak{W}_\nu$ by the ideal $\langle x_1^N \mathbf{e}(\mathbf{i}) \mid \mathbf{i} \in {}^{\theta}I^{\nu}\rangle$.

The proof of the following lemma is a replica of the proof of [BK09a], Lemma 2.1. We use it to prove the analogous result for VV algebras; imposing a nilpotency degree on x_1 at every idempotent $\mathbf{e}(\mathbf{i}) \in \mathfrak{W}_\nu$ yields a finite-dimensional algebra.

Lemma 4.2.1. *The elements $x_i \in \mathfrak{W}^N_\nu$ are nilpotent, for all $1 \leq i \leq m$.*

Proof. It suffices to show that each $x_r\mathbf{e}(\mathbf{j})$ is nilpotent since $1 = \sum_{\mathbf{j}} \mathbf{e}(\mathbf{j})$. We do this by induction on r. We know this holds when $r = 1$ since

$$(x_1\mathbf{e}(\mathbf{i}))^N = x_1^N\mathbf{e}(\mathbf{i}) = 0$$

where we have used the relation $x_1\mathbf{e}(\mathbf{i}) = \mathbf{e}(\mathbf{i})x_1$, $\forall l$. Assume now that $x_r\mathbf{e}(\mathbf{i})$ is nilpotent for each idempotent $\mathbf{e}(\mathbf{i})$. This is equivalent to assuming x_r nilpotent. We will show that $x_{r+1}\mathbf{e}(\mathbf{i})$ is nilpotent. There are two cases.

If $i_r = i_{r+1}$, set $\tau_r := \sigma_r(x_r - x_{r+1}) + 1$. Then, $\tau_r^2\mathbf{e}(\mathbf{i}) = \mathbf{e}(\mathbf{i})$ and $\tau_r x_r \tau_r\mathbf{e}(\mathbf{i}) = x_{r+1}\mathbf{e}(\mathbf{i})$. Then we have that $x_{r+1}\mathbf{e}(\mathbf{i})$ nilpotent because $(x_{r+1}\mathbf{e}(\mathbf{i}))^N = x_{r+1}^N\mathbf{e}(\mathbf{i}) = \tau_r x_r^N \tau_r\mathbf{e}(\mathbf{i}) = 0$.

If $i_r \neq i_{r+1}$ then multiplying the equation $x_r\sigma_r\mathbf{e}(\mathbf{i}) = \sigma_r x_{r+1}\mathbf{e}(\mathbf{i})$ on the left by σ_r we get $\sigma_r x_r \sigma_r\mathbf{e}(\mathbf{i}) = \pm(x_r - x_{r+1})^k x_{r+1}\mathbf{e}(\mathbf{i})$ for some $k \in \{0, 1, 2\}$ and some choice of sign. Then,

$$x_{r+1}^{k+1}\mathbf{e}(\mathbf{i}) = x_r f\mathbf{e}(\mathbf{i}) \pm \sigma_r x_r \sigma_r\mathbf{e}(\mathbf{i})$$

for some polynomial $f \in F[x_r, x_{r+1}]$ and some sign. x_r is nilpotent and hence, using the relations, we see that $\sigma_r x_r \sigma_r\mathbf{e}(\mathbf{i})$ is also nilpotent. Note also that $x_r f$ is also nilpotent. Since we find that $x_r f\mathbf{e}(\mathbf{i})$ and $\sigma_r x_r \sigma_r\mathbf{e}(\mathbf{i})$ commute, we have that $x_{r+1}^{k+1}\mathbf{e}(\mathbf{i})$ is nilpotent too. Hence, $x_{r+1}\mathbf{e}(\mathbf{i})$ is nilpotent as required. □

Corollary 4.2.2. *$\mathfrak{W}_\nu^N$ is a finite-dimensional algebra.*

Proof. For every $w \in W_m^B$, fix a reduced expression, say $w = s_{i_1} \cdots s_{i_r}$, so that $\sigma_w = \sigma_{i_1} \cdots \sigma_{i_r}$. From Lemma 2.3.11 we have the following $\mathbf{k}$-basis of $\mathfrak{W}_\nu$,

$$\{\sigma_w x_1^{n_1} \cdots x_m^{n_m}\mathbf{e}(\mathbf{i}) \mid w \in W_m^B, \mathbf{i} \in {}^\theta I^\nu, n_k \in \mathbb{N}_0\ \forall k\}.$$

Then we know that this set is a $\mathbf{k}$-linear spanning set of $\mathfrak{W}_\nu^N$. By Lemma 4.2.1, all but finitely many elements of this set are zero so that $\mathfrak{W}_\nu^N$ is finite-dimensional. □

Remark 4.2.3. Although Lemma 4.2.1 proves that all x_i are nilpotent, it does not provide a way of calculating the minimal nilpotency degrees. So, although we know that this basis of $\mathfrak{W}_\nu^N$ must be finite, we do not know the dimension of $\mathfrak{W}_\nu^N$.

The two results above show that imposing a nilpotency degree on x_1, at every idempotent $\mathbf{e}(\mathbf{i}) \in \mathfrak{W}_\nu$, yields a finite-dimensional quotient of $\mathfrak{W}_\nu$. There is no particular reason why x_1 is chosen. One can show, using the same arguments as above, that for any $1 \leq k \leq m$, putting a nilpotency degree on x_k at every $\mathbf{e}(\mathbf{i})$ implies that all x_i are nilpotent and hence the algebra is finite-dimensional.

Note also that the ideal which we took in the above is generated by $x_1^N\mathbf{e}(\mathbf{i})$, $\mathbf{i} \in {}^\theta I^\nu$. Lemma 4.2.1 and Corollary 4.2.2 clearly also hold when N depends upon the idempotent $\mathbf{e}(\mathbf{i})$ because, in that case, $\max\{N_\mathbf{i} \mid \mathbf{i} \in {}^\theta I^\nu\}$ is a nilpotency degree for every $x_1\mathbf{e}(\mathbf{i})$,

$\mathbf{i} \in {}^\theta I^\nu$, so we can apply the same arguments with $N = \max\{N_{\mathbf{i}} \mid \mathbf{i} \in {}^\theta I^\nu\}$. This means we can attempt to mimic the ideal used to define the cyclotomic KLR algebra and insist that the nilpotency degree of $x_1\mathbf{e}(\mathbf{i})$ depends upon i_1.

Example 4.2.4. Consider the setting $q \in I$, $p \notin I$. Take $\nu = 2q + 2q^{-1}$ and the VV algebra $\mathfrak{W}_\nu$. The associated quiver, with paths x_i omitted, is

$$\begin{tikzcd} \mathbf{e}(q,q) \arrow[r, bend left, "\pi"] & \mathbf{e}(q^{-1},q) \arrow[l, bend left, "\pi"] \arrow[r, bend left, "\sigma_1"] & \mathbf{e}(q,q^{-1}) \arrow[l, bend left, "\sigma_1"] \arrow[r, bend left, "\pi"] & \mathbf{e}(q^{-1},q^{-1}) \arrow[l, bend left, "\pi"] \end{tikzcd}$$

Let us now impose nilpotency degrees on x_1 at each idempotent $\mathbf{e}(\mathbf{i})$ which depend upon i_1. Let the nilpotency degree of $x_1\mathbf{e}(\mathbf{i})$ be 2 when $i_1 = q$ and let the nilpotency degree of $x_1\mathbf{e}(\mathbf{i})$ be 4 when $i_1 = q^{-1}$. That is,

$$\begin{array}{ll} x_1^2\mathbf{e}(q,q) = 0 & x_1^4\mathbf{e}(q^{-1},q^{-1}) = 0 \\ x_1^2\mathbf{e}(q,q^{-1}) = 0 & x_1^4\mathbf{e}(q^{-1},q) = 0. \end{array}$$

According to Lemma 4.2.1 and Corollary 4.2.2, all x_i are nilpotent and the resulting algebra is finite-dimensional. However, note that

$$0 = x_1^2\mathbf{e}(q,q) = \pi x_1^2\mathbf{e}(q,q)\pi = \pi^2 x_1^2\mathbf{e}(q^{-1},q) = -x_1^3\mathbf{e}(q^{-1},q).$$

This demonstrates that the nilpotency degrees we put on the $x_1\mathbf{e}(\mathbf{i})$ are not necessarily minimal nilpotency degrees.

We want to extend the notion of a cyclotomic quotient to VV algebras. We start in the setting $p, q \notin I$, where we have established Morita equivalence between KLR algebras and VV algebras. So, until stated otherwise, we assume $p, q \notin I$. In this case we have $\pi^2\mathbf{e}(\mathbf{i}) = \mathbf{e}(\mathbf{i})$ for all $\mathbf{e}(\mathbf{i}) \in {}^\theta I^\nu$. Recall that the cyclotomic ideal in $\mathbf{R}_{\tilde{\nu}}$ is defined by $J^\Lambda = \left\langle x_1^{(\Lambda,i_1)}\mathbf{e}(\mathbf{i}) \mid \mathbf{i} \in \tilde{I}^{\tilde{\nu}} \right\rangle$. The nilpotency degrees at each idempotent $\mathbf{e}(\mathbf{i}) = \mathbf{e}(i_1, \ldots, i_m)$ are dependent upon i_1. We want a function $f : I \longrightarrow \mathbb{N}_0$ which picks a nilpotency degree based on i_1 at the chosen idempotent $\mathbf{e}(\mathbf{i})$. But note that, if $x_1^{f(i_1)}\mathbf{e}(i_1, i_2, \ldots, i_m) = 0$ then,

$$0 = \pi x_1^{f(i_1)}\mathbf{e}(i_1, i_2, \ldots, i_m)\pi = -x_1^{f(i_1)}\mathbf{e}(i_1^{-1}, i_2, \ldots, i_m).$$

This suggests that the natural way of extending this cyclotomic ideal to VV algebras is by defining a function,

$$f : I \longrightarrow \mathbb{N}_0 \text{ such that } f(i) = f(i^{-1}) \; \forall i \in I,$$

and then defining the cyclotomic ideal for $\mathfrak{W}_\nu$ as follows.

$$J^f := \langle x_1^{f(i_1)}\mathbf{e}(\mathbf{i}) \mid \mathbf{i} \in {}^\theta I^\nu \rangle.$$

Let ${}^{\theta}P_{+}$ denote the subset of P_{+} consisting of elements $\Lambda = \sum_{i \in I} n_i \Lambda_i \in P_{+}$ with the property $n_i = n_{\theta(i)}$, for all i.

Note 4.2.5. The function f is equivalent to $(\Lambda, -) : I \longrightarrow \mathbb{N}_0$, for a unique $\Lambda \in {}^{\theta}P_{+}$. So, choosing a function f with the property $f(i) = f(\theta(i))$, for all $i \in I$, is equivalent to choosing $\Lambda \in {}^{\theta}P_{+}$. Note also that any $\Lambda \in {}^{\theta}P_{+}$ can be expressed as,

$$\Lambda = \sum_{i \in I^{+}} n_i \Lambda_i + \sum_{i \in I^{-}} n_i \Lambda_i.$$

Let the corresponding ideal in $\mathfrak{W}_{\nu}$ be denoted by J^{Λ} and let $\mathfrak{W}_{\nu}^{\Lambda} := \mathfrak{W}_{\nu}/J^{\Lambda}$. We fix the notation, $\Lambda^{+} := \sum_{i \in I^{+}} n_i \Lambda_i$.

However, defining the quotient in this way poses the following problem. This definition allows us to choose a non-zero $\Lambda \in {}^{\theta}P_{+}$ such that the associated quotient of $\mathfrak{W}_{\nu}$ is zero, whereas when we consider restricting Λ to Λ^{+}, taking the quotient of $\mathbf{R}_{\tilde{\nu}}^{+}$ with $J^{\Lambda^{+}} = \left\langle x_1^{(\Lambda^{+}, i_1)} \mathbf{e}(\mathbf{i}) \mid \mathbf{i} \in I^{\tilde{\nu}} \right\rangle$ yields a non-zero quotient. The following example demonstrates this.

Example 4.2.6. Take the algebra used in Example 2.3.6. We are in the setting $p, q \notin I$ and we take $\nu = p^{-2}\lambda^{-1} + \lambda^{-1} + \lambda + p^{2}\lambda \in {}^{\theta}\mathbb{N}I_{\lambda}$. The associated VV algebra, $\mathfrak{W}_{\nu}$, is the path algebra of the following quiver modulo the defining relations. As usual, we have omitted the x_i paths at each $\mathbf{e}(\mathbf{i})$.

$$\begin{tikzcd}
\mathbf{e}(\lambda, p^{2}\lambda) \arrow[rr, bend left, "\sigma_1"] \arrow[d, bend left, "\pi"] & & \mathbf{e}(p^{2}\lambda, \lambda) \arrow[ll, bend left, "\sigma_1"] \arrow[d, bend left, "\pi"] \\
\mathbf{e}(\lambda^{-1}, p^{2}\lambda) \arrow[u, bend left, "\pi"] \arrow[d, bend left, "\sigma_1"] & & \mathbf{e}(p^{-2}\lambda^{-1}, \lambda) \arrow[u, bend left, "\pi"] \arrow[d, bend left, "\sigma_1"] \\
\mathbf{e}(p^{2}\lambda, \lambda^{-1}) \arrow[u, bend left, "\sigma_1"] \arrow[d, bend left, "\pi"] & & \mathbf{e}(\lambda, p^{-2}\lambda^{-1}) \arrow[u, bend left, "\sigma_1"] \arrow[d, bend left, "\pi"] \\
\mathbf{e}(p^{-2}\lambda^{-1}, \lambda^{-1}) \arrow[u, bend left, "\pi"] \arrow[rr, bend left, "\sigma_1"] & & \mathbf{e}(\lambda^{-1}, p^{-2}\lambda^{-1}) \arrow[u, bend left, "\pi"] \arrow[ll, bend left, "\sigma_1"]
\end{tikzcd}$$

Take $\Lambda = 10\Lambda_{\lambda} + 10\Lambda_{\lambda^{-1}} \in {}^{\theta}P_{+}$.

Let $J^{\Lambda} = \langle x_1^{(\Lambda, i_1)} \mathbf{e}(\mathbf{i}) \mid \mathbf{i} \in {}^{\theta}I^{\nu} \rangle$, an ideal of $\mathfrak{W}_{\nu}$. Now consider the quotient $\mathfrak{W}_{\nu}^{\Lambda} = \mathfrak{W}_{\nu}/J^{\Lambda}$. For $a \in \mathfrak{W}_{\nu}$, let $\bar{a}$ denote the image of a in $\mathfrak{W}_{\nu}^{\Lambda}$. In this quotient we have,

$$\bar{\mathbf{e}}(p^{2}\lambda, \lambda) = \bar{\mathbf{e}}(p^{-2}\lambda^{-1}, \lambda) = \bar{\mathbf{e}}(p^{2}\lambda, \lambda^{-1}) = \bar{\mathbf{e}}(p^{-2}\lambda^{-1}, \lambda^{-1}) = 0.$$

But, using these equalities, this means that we also have, for example,

$$\bar{\mathbf{e}}(\lambda^{-1}, p^2\lambda) = \bar{\sigma}^2\bar{\mathbf{e}}(\lambda^{-1}, p^2\lambda) = \bar{\sigma}\bar{\mathbf{e}}(p^2\lambda, \lambda^{-1})\bar{\sigma} = 0.$$

Similarly, we find that all other remaining idempotents vanish in this quotient and hence, $\mathfrak{W}_\nu^\Lambda = 0$.

In this example,

$$\Lambda^+ = 10\Lambda_\lambda.$$

Recalling Remark 2.3.8, we can define a KLR algebra $\mathbf{R}_{\tilde{\nu}}^+$, which is in fact an idempotent subalgebra of $\mathfrak{W}_\nu$. Consider the quotient of $\mathbf{R}_{\tilde{\nu}}^+$ by $J^{\Lambda^+} = \langle x_1^{(\Lambda^+, i_1)}\mathbf{e}(\mathbf{i}) \mid \mathbf{i} \in I^{\tilde{\nu}} \rangle$. We have $\bar{x}_1^{10}\bar{\mathbf{e}}(\lambda, p^2\lambda) = 0$ and $\bar{\mathbf{e}}(p^2\lambda, \lambda) = 0$ in the quotient $\mathbf{R}_{\tilde{\nu}}^{\Lambda^+} := \mathbf{R}_{\tilde{\nu}}^+/J^{\Lambda^+}$. Since $\bar{\mathbf{e}}(\lambda, p^2\lambda) \neq 0$ we have $\mathbf{R}_{\tilde{\nu}}^{\Lambda^+} \neq 0$.

We do not want this situation to arise. Our aim is then to define a finite-dimensional, cyclotomic quotient of $\mathfrak{W}_\nu$ which is compatible with the cyclotomic KLR algebra in the following way. Recall from 2.3.12, $\mathbf{R}_{\tilde{\nu}}^+$ is an idempotent subalgebra of $\mathfrak{W}_\nu$. We would like to define a cyclotomic quotient $\mathfrak{W}_\nu^\Lambda$ in which $\mathbf{R}_{\tilde{\nu}}^{\Lambda^+}$ lies as an idempotent subalgebra. More specifically, take $\Lambda \in {}^\theta P_+$ and let

$$\mathbf{e} := \sum_{\mathbf{i} \in I^{\tilde{\nu}}} (\mathbf{e}(\mathbf{i}) + J^\Lambda) \in \mathfrak{W}_\nu^\Lambda.$$

With $\Lambda^+ = \sum_{i \in I^+} n_i\Lambda_i$, we want to define $\mathfrak{W}_\nu^\Lambda$ such that there is an algebra isomorphism $\mathbf{e}\mathfrak{W}_\nu^\Lambda\mathbf{e} \cong \mathbf{R}_{\tilde{\nu}}^{\Lambda^+}$.

Conversely, one could start with a KLR algebra $\mathbf{R}_{\tilde{\nu}}$, take a cyclotomic ideal $I^\Lambda \subseteq \mathbf{R}_{\tilde{\nu}}$ and induce this up to an ideal in $\mathfrak{W}_\nu$ using the following theorem.

Theorem 4.2.7 ([Lam01], (21.11))**.** *Let e be an idempotent in the ring R and let I be an ideal in eRe. Then $e(RIR)e = I$. In particular, $I \longmapsto RIR$ defines an injective map from ideals of eRe to ideals of R. This map respects multiplication of ideals, and is surjective if e is full in R.*

Fix a dimension vector $\tilde{\nu} = \sum_{i \in I} \tilde{\nu}_i i \in \mathbb{N}I$, such that $ni + ni^{-1}$ is not a summand of $\tilde{\nu}$, together with the corresponding KLR algebra $\mathbf{R}_{\tilde{\nu}}$. By Proposition 2.3.12, $\mathbf{R}_{\tilde{\nu}}$ is an idempotent subalgebra of $\mathfrak{W}_\nu$, where $\nu = \sum_{i \in I} \tilde{\nu}_i i + \sum_{i \in I} \tilde{\nu}_i i^{-1} \in {}^\theta\mathbb{N}I$. Take a cyclotomic ideal J^Λ of $\mathbf{R}_{\tilde{\nu}}$ as described in Section 4.1. Theorem 4.2.7 applied here tells us that $\mathfrak{W}_\nu J^\Lambda \mathfrak{W}_\nu$ is an ideal in $\mathfrak{W}_\nu$, such that $\mathbf{e}(\mathfrak{W}_\nu J^\Lambda \mathfrak{W}_\nu)\mathbf{e} = J^\Lambda$. It turns out that taking the quotient of $\mathfrak{W}_\nu$ by the ideal $\mathfrak{W}_\nu J^\Lambda \mathfrak{W}_\nu$ in the setting $p, q \notin I$ yields a finite-dimensional algebra. Note that,

$$\begin{aligned} J^\Lambda &= \mathbf{R}_{\tilde{\nu}} x_1^{(\Lambda, i_{11})}\mathbf{e}(\mathbf{i}_1)\mathbf{R}_{\tilde{\nu}} + \cdots + \mathbf{R}_{\tilde{\nu}} x_1^{(\Lambda, i_{k1})}\mathbf{e}(\mathbf{i}_k)\mathbf{R}_{\tilde{\nu}} \\ &\cong \mathbf{e}\mathfrak{W}_\nu\mathbf{e} x_1^{(\Lambda, i_{11})}\mathbf{e}(\mathbf{i}_1)\mathbf{e}\mathfrak{W}_\nu\mathbf{e} + \cdots + \mathbf{e}\mathfrak{W}_\nu\mathbf{e} x_1^{(\Lambda, i_{k1})}\mathbf{e}(\mathbf{i}_k)\mathbf{e}\mathfrak{W}_\nu\mathbf{e}. \end{aligned}$$

From this we have, using the fact that $\mathbf{e}$ is full in $\mathfrak{W}_\nu$ from Section 3.3,

$$\begin{aligned}\mathfrak{W}_\nu J^\Lambda \mathfrak{W}_\nu &= \mathfrak{W}_\nu \mathbf{e} \mathfrak{W}_\nu \mathbf{e} x_1^{(\Lambda,i_{11})}\mathbf{e}(\mathbf{i}_1)\mathbf{e}\mathfrak{W}_\nu \mathbf{e}\mathfrak{W}_\nu + \cdots + \mathfrak{W}_\nu \mathbf{e} \mathfrak{W}_\nu \mathbf{e} x_1^{(\Lambda,i_{k1})}\mathbf{e}(\mathbf{i}_k)\mathbf{e}\mathfrak{W}_\nu \mathbf{e}\mathfrak{W}_\nu \\ &= \mathfrak{W}_\nu x_1^{(\Lambda,i_{11})}\mathbf{e}(\mathbf{i}_1)\mathfrak{W}_\nu + \cdots + \mathfrak{W}_\nu x_1^{(\Lambda,i_{k1})}\mathbf{e}(\mathbf{i}_k)\mathfrak{W}_\nu.\end{aligned}$$

where $\mathbf{i}_j = (i_{j1}, \ldots, i_{jm})$, for $1 \leq j \leq k$.

Proposition 4.2.8. *$\mathfrak{W}_\nu^\Lambda = \mathfrak{W}_\nu / \mathfrak{W}_\nu J^\Lambda \mathfrak{W}_\nu$ is finite-dimensional.*

Proof. To free up notation, let us omit the bar above elements which belong to the quotient $\mathfrak{W}_\nu^\Lambda$. That is, all elements are considered modulo the ideal $\mathfrak{W}_\nu J^\Lambda \mathfrak{W}_\nu$. Using Lemma 4.2.1 and Corollary 4.2.2, it suffices to show that $x_k\mathbf{e}(\mathbf{i})$ is nilpotent, for all $\mathbf{i} \in {}^\theta I^\nu$, $1 \leq k \leq m$. By definition of $\mathfrak{W}_\nu^\Lambda$, the elements $x_k\mathbf{e}(\mathbf{i})$, $\mathbf{i} \in I^{\tilde{\nu}^+}$, $1 \leq k \leq m$, are nilpotent. So we want to show $x_k\mathbf{e}(\mathbf{i})$ is nilpotent, for some $\mathbf{i} \notin I^{\tilde{\nu}^+}$, and any k, with $1 \leq k \leq m$. In this setting we know that $\mathbf{e}(\mathbf{i})$ is isomorphic to some $\mathbf{e}(\mathbf{j})$, $\mathbf{j} \in I^{\tilde{\nu}^+}$, by the following reasoning. Since $\mathbf{i} \notin I^{\tilde{\nu}^+}$, there exists $\{\varepsilon_1, \ldots, \varepsilon_d\} \subseteq \{1, \ldots, m\}$ with $\varepsilon_r < \varepsilon_{r+1}$ and $i_{\varepsilon_r} \in I^{\tilde{\nu}^-}$ for all r. Then,

$$\begin{aligned}&\sigma_{\varepsilon_1-1}\cdots\sigma_1\pi\cdots\cdots\sigma_{\varepsilon_d-1}\cdots\sigma_1\pi\mathbf{e}(\mathbf{j})\pi\sigma_1\cdots\sigma_{\varepsilon_d-1}\cdots\cdots\pi\sigma_1\cdots\sigma_{\varepsilon_1-1}\mathbf{e}(\mathbf{i}) = \mathbf{e}(\mathbf{i})\\ &\pi\sigma_1\cdots\sigma_{\varepsilon_d-1}\cdots\cdots\pi\sigma_1\cdots\sigma_{\varepsilon_1-1}\mathbf{e}(\mathbf{i})\sigma_{\varepsilon_1-1}\cdots\sigma_1\pi\cdots\cdots\sigma_{\varepsilon_d-1}\cdots\sigma_1\pi\mathbf{e}(\mathbf{j}) = \mathbf{e}(\mathbf{j}),\end{aligned}$$

where we are using that $\pi^2\mathbf{e}(\mathbf{k}) = \mathbf{e}(\mathbf{k})$, for every $\mathbf{k} \in {}^\theta I^\nu$, and that each σ_r interchanges vertices from $I^{\tilde{\nu}^+}$ and $I^{\tilde{\nu}^-}$ so therefore squares to the appropriate idempotent. This shows that $\mathbf{e}(\mathbf{i})$ is isomorphic to some $\mathbf{e}(\mathbf{j})$, $\mathbf{j} \in I^{\tilde{\nu}^+}$. Then, using the defining relations of $\mathfrak{W}_\nu$, we have

$$x_k\mathbf{e}(\mathbf{i}) = \sigma_{\varepsilon_1-1}\cdots\sigma_1\pi\cdots\cdots\sigma_{\varepsilon_d-1}\cdots\sigma_1\pi x_{k'}\mathbf{e}(\mathbf{j})\pi\sigma_1\cdots\sigma_{\varepsilon_d-1}\cdots\cdots\pi\sigma_1\cdots\sigma_{\varepsilon_1-1}\mathbf{e}(\mathbf{i}),$$

for some k', $1 \leq k' \leq m$. Since $x_{k'}\mathbf{e}(\mathbf{j})$ is nilpotent, it follows that $x_k\mathbf{e}(\mathbf{i})$ is nilpotent. This proves the claim. ☐

The following example should provide a clearer picture of what happens when we induce a cyclotomic ideal in a KLR algebra up to an ideal in the corresponding VV algebra.

Example 4.2.9. We remind the reader that we are in the setting $p, q \notin I$. Let $\tilde{\nu} = \lambda + p^2\lambda$ so that $\mathbf{R}_{\tilde{\nu}}$ is an idempotent subalgebra of $\mathfrak{W}_\nu$, for $\nu = p^{-2}\lambda^{-1} + \lambda^{-1} + \lambda + p^2\lambda$. The VV algebra $\mathfrak{W}_\nu$ has the following quiver associated to it. As usual in this thesis, the x_i paths have been omitted. The idempotents in $\mathbf{R}_{\tilde{\nu}}$ are coloured red here, to emphasise that $\mathbf{R}_{\tilde{\nu}}$ is an idempotent subalgebra.

$$
\begin{tikzcd}
\mathbf{e}(\lambda, p^2\lambda) \arrow[rr, bend left, "\sigma_1"] \arrow[d, bend left, "\pi"] & & \mathbf{e}(p^2\lambda, \lambda) \arrow[ll, bend left, "\sigma_1"] \arrow[d, bend left, "\pi"] \\
\mathbf{e}(\lambda^{-1}, p^2\lambda) \arrow[u, bend left, "\pi"] \arrow[d, bend left, "\sigma_1"] & & \mathbf{e}(p^{-2}\lambda^{-1}, \lambda) \arrow[u, bend left, "\pi"] \arrow[d, bend left, "\sigma_1"] \\
\mathbf{e}(p^2\lambda, \lambda^{-1}) \arrow[u, bend left, "\sigma_1"] \arrow[d, bend left, "\pi"] & & \mathbf{e}(\lambda, p^{-2}\lambda^{-1}) \arrow[u, bend left, "\sigma_1"] \arrow[d, bend left, "\pi"] \\
\mathbf{e}(p^{-2}\lambda^{-1}, \lambda^{-1}) \arrow[u, bend left, "\pi"] \arrow[rr, bend left, "\sigma_1"] & & \mathbf{e}(\lambda^{-1}, p^{-2}\lambda^{-1}) \arrow[u, bend left, "\pi"] \arrow[ll, bend left, "\sigma_1"]
\end{tikzcd}
$$

Suppose we take the cyclotomic ideal in $\mathbf{R}_{\tilde{\nu}}$ to be $J^\Lambda = \langle x_1\mathbf{e}(\lambda, p^2\lambda), x_1^2\mathbf{e}(p^2\lambda, \lambda)\rangle$. Using the relation $\sigma_1 x_2 \mathbf{e}(\mathbf{i}) = x_1\sigma_1\mathbf{e}(\mathbf{i})$, for $\mathbf{i} \in \{(p^2\lambda, \lambda), (\lambda, p^2\lambda)\}$, we find $x_2^3\mathbf{e}(\lambda, p^2\lambda) = 0$ and $x_2^3\mathbf{e}(p^2\lambda, \lambda) = 0$. Inducing this ideal up to one in $\mathfrak{W}_\nu$ and then taking the quotient we have, with elements of the quotient written in the same way as those in $\mathfrak{W}_\nu$,

$$
\begin{aligned}
0 &= \pi x_1\mathbf{e}(\lambda, p^2\lambda)\pi = -\pi^2 x_1\mathbf{e}(\lambda^{-1}, p^2\lambda) = -x_1\mathbf{e}(\lambda^{-1}, p^2\lambda) \\
0 &= \sigma_1\pi x_2^3\mathbf{e}(\lambda, p^2\lambda)\pi\sigma_1 = x_1^3\mathbf{e}(p^2\lambda, \lambda^{-1}) \\
0 &= \pi\sigma_1\pi x_2^3\mathbf{e}(\lambda, p^2\lambda)\pi\sigma_1\pi = \pi x_1^3\mathbf{e}(p^2\lambda, \lambda^{-1})\pi = -x_1^3\mathbf{e}(p^{-2}\lambda^{-1}, \lambda^{-1})
\end{aligned}
$$

Similarly, using the nilpotency degrees of x_1 and x_2 at $\mathbf{e}(p^2\lambda, \lambda)$ one can check that x_1 is nilpotent at the remaining idempotents. Then x_1 is nilpotent at every idempotent and, using Lemma 4.2.1 and Corollary 4.2.2, $\mathfrak{W}_\nu/\mathfrak{W}_\nu J^\Lambda \mathfrak{W}_\nu$ is finite-dimensional.

The following examples should give some idea of why, in other settings, we cannot use these methods to show that the induced quotient is finite-dimensional.

Example 4.2.10. Suppose now that $q \in I$. Let us take $\tilde{\nu} = q + p^2 q$ so that $\mathbf{R}_{\tilde{\nu}}$ is an idempotent subalgebra of $\mathfrak{W}_\nu$ where $\nu = p^{-2}q^{-1} + q^{-1} + q + p^2 q$. The quiver associated to $\mathfrak{W}_\nu$ is similar to the one above in Example 4.2.9. Let $J^\Lambda = \langle x_1\mathbf{e}(q, p^2 q), x_1^2\mathbf{e}(p^2 q, q)\rangle$ and consider $\mathfrak{W}_\nu^\Lambda := \mathfrak{W}_\nu/\mathfrak{W}_\nu J^\Lambda \mathfrak{W}_\nu$. Note that $x_2^3\mathbf{e}(q, p^2 q) = 0 = x_2^3\mathbf{e}(p^2 q, q)$. In the same way as in Example 4.2.9, we can show that $x_1\mathbf{e}(\mathbf{i})$ is nilpotent, but here we can do this only at some of the idempotents $\mathbf{e}(\mathbf{i})$. For example,

$$
\begin{aligned}
x_1^2\mathbf{e}(p^2 q, q) = 0 &\Longrightarrow \pi x_1^2\mathbf{e}(p^2 q, q)\pi = 0 \\
&\Longrightarrow -x_1^2\mathbf{e}(p^{-2}q^{-1}, q) = 0.
\end{aligned}
$$

However, this does not work for every idempotent since, in this example, we now have 4 isomorphism classes of idempotents in $\mathfrak{W}_\nu$ and the idempotents associated to $\mathbf{R}_{\tilde{\nu}}$ lie in only 2 of these isomorphism classes. For example,

$$\begin{aligned}
x_2^3\mathbf{e}(q,p^2q) = 0 &\implies \sigma_1\pi x_2^3\mathbf{e}(q,p^2q)\pi\sigma_1 = 0 \\
&\implies \sigma_1 x_1 x_2^3\mathbf{e}(q^{-1},p^2q)\sigma_1 = 0 \\
&\implies x_2x_1^3\mathbf{e}(p^2q,q^{-1}) = 0,
\end{aligned}$$

so that we do not have x_1 nilpotent at $\mathbf{e}(p^2q,q^{-1})$. It is a similar story for other idempotents in this quotient.

Example 4.2.11. Suppose now that $p \in I$. Take $\tilde{\nu} = 2p$ so that $\mathbf{R}_{\tilde{\nu}}$ is an idempotent subalgebra of $\mathfrak{W}_\nu$, $\nu = 2p^{-1} + 2p$, which has the following associated quiver. As usual in this thesis, the x_i paths have been omitted. The idempotent in $\mathbf{R}_{\tilde{\nu}}$ is coloured red here.

$$\begin{tikzcd}
\color{red}{\mathbf{e}(p,p)} \arrow[loop left, "\sigma_1"] \arrow[r, bend left, "\pi"] & \mathbf{e}(p^{-1},p) \arrow[l, bend left, "\pi"] \arrow[r, bend left, "\sigma_1"] & \mathbf{e}(p,p^{-1}) \arrow[l, bend left, "\sigma_1"] \arrow[r, bend left, "\pi"] & \mathbf{e}(p^{-1},p^{-1}) \arrow[l, bend left, "\pi"] \arrow[loop right, "\sigma_1"]
\end{tikzcd}$$

Let $J^\Lambda = \langle x_1\mathbf{e}(p,p)\rangle$ be the cyclotomic ideal in $\mathbf{R}_{\tilde{\nu}}$. One can show that $x_2\mathbf{e}(p,p) = 0$ in $\mathbf{R}_{\tilde{\nu}}$. We again induce this up to an ideal in $\mathfrak{W}_\nu$ and take the quotient. Now if we try to show that $x_1\mathbf{e}(p,p^{-1})$ is nilpotent we find the following.

$$\begin{aligned}
x_2\mathbf{e}(p,p) = 0 &\implies \sigma_1\pi x_2\mathbf{e}(p,p)\pi\sigma_1 = 0 \\
&\implies \sigma_1^2 x_1\mathbf{e}(p,p^{-1}) = 0 \\
&\implies x_1(x_1 - x_2)\mathbf{e}(p,p^{-1}) = 0
\end{aligned}$$

Similarly, x_1 is not nilpotent at $\mathbf{e}(p^{-1},p^{-1})$ and so $\mathfrak{W}_\nu^\Lambda$ is not a finite-dimensional algebra.

4.3 Defining $\mathfrak{W}_\nu^\Lambda$

Case: $\mathbf{q} \in \mathbf{I}$

In this section, until stated otherwise, we fix the following setting. Assume $q \in I$, $p \notin I$ and, whenever we take $\nu \in {}^\theta\mathbb{N}I_q$, we assume that q has multiplicity one in ν. We assume $|\nu| = 2m$, for some $m \in \mathbb{N}$.

Here we show how one can define finite-dimensional quotients of the algebras $\mathfrak{W}_\nu$ such that truncating at a certain idempotent $\mathbf{e}$ yields a cyclotomic KLR algebra, i.e. we define a finite-dimensional algebra $\mathfrak{W}_\nu^\Lambda$ such that $\mathbf{e}\mathfrak{W}_\nu^\Lambda\mathbf{e} \cong \mathbf{R}_\nu^{\Lambda^+}$. We do this using the Morita equivalence in Subsection 3.4.5. In this subsection we have seen that $\mathfrak{W}_\nu$ and $\mathbf{R}_\nu^+ \otimes_{\mathbf{k}[z]} \tilde{A}$ are Morita equivalent. Our strategy is to choose the ideal $J^\Lambda \subset \mathfrak{W}_\nu$ by working backwards; we define a finite-dimensional quotient of $\mathbf{R}_\nu^+ \otimes_{\mathbf{k}[z]} \tilde{A}$ and pass this information through the Morita equivalence. We show that this defines a finite-dimensional quotient $\mathfrak{W}_\nu^\Lambda$ of $\mathfrak{W}_\nu$ such that $\mathbf{e}\mathfrak{W}_\nu^\Lambda\mathbf{e} \cong \mathbf{R}^{\Lambda^+}$, where $\mathbf{e} = \sum_{\mathbf{i}\in I^{\tilde{\nu}^+}}(\mathbf{e}(\mathbf{i}) + J^\Lambda)$.

Fix $\nu \in {}^\theta\mathbb{N}I_q$, with $|\nu| = 2m$, so that the multiplicity of q in ν is 1. By Lemma 2.3.14, we have an algebra $\mathfrak{W}_\nu$ and a distinguished idempotent subalgebra $\mathbf{R}_{\tilde{\nu}}^+$. Take a cyclotomic ideal J^{Λ^+} of $\mathbf{R}_{\tilde{\nu}}^+$, for some $\Lambda^+ \in P_+$ as described in Section 4.1, and take the quotient of $\mathbf{R}_{\tilde{\nu}}^+$ by J^{Λ^+} to obtain a cyclotomic KLR algebra $\mathbf{R}_{\tilde{\nu}}^{\Lambda^+}$. There is a canonical surjection,

$$\mathbf{R}_{\tilde{\nu}}^+ \twoheadrightarrow \mathbf{R}_{\tilde{\nu}}^{\Lambda^+}.$$

The functor $-\otimes_{\mathbf{k}[z]} \tilde{A}$ is right exact so we get a surjection,

$$\mathbf{R}_{\tilde{\nu}}^+ \otimes_{\mathbf{k}[z]} \tilde{A} \twoheadrightarrow \mathbf{R}_{\tilde{\nu}}^{\Lambda^+} \otimes_{\mathbf{k}[z]} \tilde{A},$$

so that $\mathbf{R}_{\tilde{\nu}}^{\Lambda^+} \otimes_{\mathbf{k}[z]} \tilde{A}$ is a quotient of $\mathbf{R}_{\tilde{\nu}}^+ \otimes_{\mathbf{k}[z]} \tilde{A}$. It has a spanning set,

$$\mathcal{B}' = \{b\otimes a_1, b\otimes a_2, b\otimes v_1a_1, b\otimes v_2a_2 \mid b\in\mathcal{B}\}$$

where $\mathcal{B}$ is the spanning set of $\mathbf{R}_{\tilde{\nu}}^\Lambda$ given in Corollary 4.1.4. Finitely many of the elements in $\mathcal{B}$ are non-zero so that finitely many of the elements in $\mathcal{B}'$ are non-zero. Then $\mathbf{R}_{\tilde{\nu}}^\Lambda \otimes_{\mathbf{k}[z]} \tilde{A}$ is finite-dimensional.

From Theorem 3.4.34 we have $\mathbf{R}_{\tilde{\nu}}^+ \otimes_{\mathbf{k}[z]} \tilde{A} \cong \mathbf{e}\mathfrak{W}_\nu\mathbf{e}$, where $\mathbf{e} := \sum_{\mathbf{i}\in I^{\tilde{\nu}^+}} \mathbf{e}(\mathbf{i}) + \sum_{\mathbf{i}\in I^{\tilde{\nu}^-}} \mathbf{e}(\mathbf{i})$. We now show how to pick $J^\Lambda \subset \mathfrak{W}_\nu$, which defines $\mathfrak{W}_\nu^\Lambda$, using this isomorphism. In [BK09a] Brundan and Kleshchev showed that, in $\mathbf{R}_{\tilde{\nu}}^\Lambda$, all x_i, $1 \le i \le m$ are nilpotent. Write the minimal nilpotency degrees of $x_1\mathbf{e}(\mathbf{i}), x_2\mathbf{e}(\mathbf{i}), \ldots, x_m\mathbf{e}(\mathbf{i})$ as an m-tuple,

$$((n_1)_{\mathbf{e}(\mathbf{i})}, (n_2)_{\mathbf{e}(\mathbf{i})}, \ldots, (n_m)_{\mathbf{e}(\mathbf{i})}).$$

It should be noted here that, for any given example, we only know $(n_1)_{\mathbf{e}(\mathbf{i})} = (\Lambda, i_1)$ at each idempotent $\mathbf{e}(\mathbf{i})$. Minimal nilpotency degrees $(n_2)_{\mathbf{e}(\mathbf{i})}, \ldots, (n_m)_{\mathbf{e}(\mathbf{i})}$ exist at each $\mathbf{e}(\mathbf{i})$ but it is unknown how to calculate them. In the proof of Theorem 3.4.34 we defined a map ϕ such that,

$$\begin{aligned}\phi &: x_j\mathbf{e}(\mathbf{i}) \otimes a_1 \mapsto x_j\mathbf{e}(\mathbf{i})\\ \phi &: x_j\mathbf{e}(\mathbf{i}) \otimes a_2 \mapsto -x_{m-j+1}\mathbf{e}(\eta\mathbf{i}).\end{aligned}$$

This map tells us how to pick nilpotency degrees of the x_i in $\mathfrak{W}_\nu$. It follows that we should define the ideal $J^\Lambda \subset \mathfrak{W}_\nu$ as,

$$J^\Lambda = \left\langle x_1^{(\Lambda,i_1)}\mathbf{e}(\mathbf{i}), x_m^{(\Lambda,j_m^{-1})}\mathbf{e}(\mathbf{j}) \mid \mathbf{i}\in I^{\tilde{\nu}^+}, \mathbf{j}\in I^{\tilde{\nu}^-}\right\rangle.$$

Note that $(n_k)_{\mathbf{e}(\eta\mathbf{i})} = (n_{m-k+1})_{\mathbf{e}(\mathbf{i})}$.

Proposition 4.3.1. $\mathfrak{W}_\nu^\Lambda = \mathfrak{W}_\nu / J^\Lambda$ *is finite-dimensional.*

Proof. Using Lemma 4.2.1 and Corollary 4.2.2, it suffices to show that $x_k\mathbf{e}(\mathbf{i})$ is nilpotent, for all $\mathbf{i} \in {}^\theta I^\nu$, $1 \le k \le m$. First note that all $x_1\mathbf{e}(\mathbf{i})$, $\mathbf{i} \in I^{\tilde{v}^+}$, are nilpotent, by definition of $\mathfrak{W}_\nu^\Lambda$. Using the same argument as in Lemma 4.2.1, it follows that all $x_k\mathbf{e}(\mathbf{i})$ are nilpotent for $1 \le k \le m$, $\mathbf{i} \in I^{\tilde{\nu}^+}$. Similarly all $x_m\mathbf{e}(\mathbf{i})$, $\mathbf{i} \in I^{\tilde{v}^-}$ are nilpotent, by definition of $\mathfrak{W}_\nu^\Lambda$, and again using the argument from Lemma 4.2.1, it follows that all $x_k\mathbf{e}(\mathbf{i})$ are nilpotent for $1 \le k \le m$ and $\mathbf{i} \in I^{\tilde{\nu}^-}$. Now take any idempotent $\mathbf{e}(\mathbf{i}) \in \mathfrak{W}_\nu^\Lambda$ with $\mathbf{i} \notin I^{\tilde{\nu}^+} \cup I^{\tilde{\nu}^-}$. By Lemma 3.4.5 we know that if q lies in position k in $\mathbf{i} \in {}^\theta I^\nu$ then $\mathbf{e}(\mathbf{i})$ is isomorphic to an idempotent $\mathbf{e}(\mathbf{j})$, $\mathbf{j} \in I^{\tilde{v}^+}$. In the proof of Lemma 3.4.5 we saw that there exists $w \in W_m^B$ with $\sigma_w^\rho \mathbf{e}(\mathbf{j})\sigma_w = \mathbf{e}(\mathbf{i})$. In fact, $w \in \mathcal{D}(W_m^B/\mathfrak{S}_m)$. Then,

$$x_k^{n_k}\mathbf{e}(\mathbf{j}) = 0 \quad \Longrightarrow \quad \sigma_w^\rho x_k^{n_k}\mathbf{e}(\mathbf{j})\sigma_w = 0.$$

Using that $\sigma_w^\rho \mathbf{e}(\mathbf{j})\sigma_w = \mathbf{e}(\mathbf{i})$ together with the relations,

$$\sigma_i x_k \mathbf{e}(\mathbf{j}) = x_{s_i(k)}\sigma_i\mathbf{e}(\mathbf{j})$$
$$\pi x_k \mathbf{e}(\mathbf{j}) = \begin{cases} x_k\pi\mathbf{e}(\mathbf{j}) & \text{if } k > 1 \\ -x_k\pi\mathbf{e}(\mathbf{j}) & \text{if } k = 1, \end{cases}$$

it follows that, since all x_k are nilpotent at $\mathbf{e}(\mathbf{j})$, all x_k are nilpotent at our chosen $\mathbf{e}(\mathbf{i})$, which was chosen arbitrarily. A similar line of reasoning applies when q^{-1} lies in position k. This proves that $\mathfrak{W}_\nu^\Lambda$ is finite-dimensional. □

We remind the reader here that for any $\Lambda \in {}^\theta P_+$ we can write $\Lambda = \sum_{i\in I^+} n_i\Lambda_i + \sum_{i\in I^-} n_i\Lambda_i$, and we set the notation $\Lambda^+ := \sum_{i\in I^+} n_i\Lambda_i$. The next result states that this finite-dimensional quotient of $\mathfrak{W}_\nu$ contains the cyclotomic KLR algebra $\mathbf{R}_{\tilde{\nu}}^{\Lambda^+}$ as an idempotent subalgebra.

Proposition 4.3.2. *Let* $\boldsymbol{e} = \sum_{i\in I^{\tilde{v}^+}} (\boldsymbol{e}(\boldsymbol{i}) + J^\Lambda)$. *Then,* $\boldsymbol{e}\mathfrak{W}_\nu^\Lambda\boldsymbol{e} \cong \boldsymbol{R}_{\tilde{\nu}}^{\Lambda^+}$.

Proof. To reduce notation, we again omit the bar above elements which belong to the quotient $\mathfrak{W}_\nu^\Lambda$. All elements are considered modulo the ideal J^Λ. For every element $w \in W_m^B$ we fix a reduced expression $w = \eta s$, where $\eta \in \mathcal{D}(W_m^B/\mathfrak{S}_m)$, $s \in \mathfrak{S}_m$. We want to prove this result in the same way as in Proposition 2.3.12. That is, we know that every element $\mathbf{e}v\mathbf{e} \in \mathbf{e}\mathfrak{W}_\nu^\Lambda\mathbf{e}$ has the form,

$$\mathbf{e}v\mathbf{e} = \begin{cases} \sum_{\substack{s\in\mathfrak{S}_m \\ \mathbf{i}\in I^{\tilde{\nu}^+}}} \sigma_s p_{\mathbf{i}}(\underline{x})\mathbf{e} & \text{if } \eta = 1 \\ 0 & \text{if } \eta \neq 1. \end{cases}$$

Then we want to define a map,

$$\begin{aligned} f : \mathbf{e}\mathfrak{W}_\nu^\Lambda\mathbf{e} &\longrightarrow \mathbf{R}_{\tilde{\nu}}^{\Lambda^+} \\ \mathbf{e}v\mathbf{e} &\mapsto \sum_{\substack{s\in\mathfrak{S}_m \\ \mathbf{i}\in I^{\tilde{\nu}^+}}} \sigma_s p_{\mathbf{i}}(\underline{x})\mathbf{e}, \end{aligned}$$

and show that this is an isomorphism. Except now we must ensure that $a = 0$ in $\mathbf{e}\mathfrak{W}_\nu^\Lambda\mathbf{e}$ if and only if $f(a) = 0$ in $\mathbf{R}_{\tilde{\nu}}^{\Lambda^+}$. It is clear from the way Λ^+ is defined that, if $f(a) = 0$ then $a = 0$. This $a \in \mathbf{e}\mathfrak{W}_\nu^\Lambda\mathbf{e}$ will be of the form $\mathbf{e}(\mathbf{i}_1)\sigma_{w_1}x_k^n\mathbf{e}(\mathbf{h})\sigma_{w_2}\mathbf{e}(\mathbf{i}_2)$, some $n \in \mathbb{N}$ and $\mathbf{e}(\mathbf{i}_1), \mathbf{e}(\mathbf{i}_2)$ summands of $\mathbf{e}$. Let $\eta_1 s_1$ and $\eta_2 s_2$ be our fixed reduced expressions for $w_1, w_2 \in W_m^B$, respectively. There are two cases to consider; either q is an entry of $\mathbf{h}$ or q^{-1} is an entry of $\mathbf{h}$.

- Suppose first that q is an entry of $\mathbf{h}$. We know from Lemma 3.4.5 (i) that $\mathbf{e}(\mathbf{h}) \cong \mathbf{e}(\mathbf{j})$, for some $\mathbf{j} \in I^{\tilde{\nu}^+}$. That is, there exists $\eta \in \mathcal{D}(W_m^B/\mathfrak{S}_m)$ such that $\sigma_\eta^\rho\mathbf{e}(\mathbf{h})\sigma_\eta\mathbf{e}(\mathbf{j}) = \mathbf{e}(\mathbf{j})$ and $\sigma_\eta\mathbf{e}(\mathbf{j})\sigma_\eta^\rho\mathbf{e}(\mathbf{h}) = \mathbf{e}(\mathbf{h})$. In this case, the nilpotency degrees of the x_i at $\mathbf{e}(\mathbf{h})$ arise from nilpotency degrees of x_i at $\mathbf{e}(\mathbf{j})$, $\mathbf{j} \in I^{\tilde{\nu}^+}$. Hence, if $a = 0$ in $\mathbf{e}\mathfrak{W}_\nu^\Lambda\mathbf{e}$ then $a = 0 \in \mathbf{R}_{\tilde{\nu}}^{\Lambda^+}$. Explicitly, we have,

$$\begin{aligned} \mathbf{e}(\mathbf{i}_1)\sigma_{w_1}x_k^n\mathbf{e}(\mathbf{h})\sigma_{w_2}\mathbf{e}(\mathbf{i}_2) &= \mathbf{e}(\mathbf{i}_1)\sigma_{w_1}x_k^n\mathbf{e}(\mathbf{h})\sigma_{\eta_2}\sigma_{s_2}\mathbf{e}(\mathbf{i}_2) \\ &= \mathbf{e}(\mathbf{i}_1)\sigma_{w_1}x_k^n\mathbf{e}(\mathbf{h})\sigma_{\eta_2}\mathbf{e}(\mathbf{j})\sigma_{s_2} \\ &= \mathbf{e}(\mathbf{i}_1)\sigma_{w_1}\mathbf{e}(\mathbf{h})\sigma_{\eta_2}x_{\eta_2\cdot k}^n\mathbf{e}(\mathbf{j})\sigma_{s_2}. \end{aligned}$$

  But $x_{\eta_2\cdot k}^n\mathbf{e}(\mathbf{j}) = 0$ in $\mathbf{R}_{\tilde{\nu}}^{\Lambda^+}$, i.e. the nilpotency degree of $x_k\mathbf{e}(\mathbf{h})$ was induced from the nilpotency degree of $x_{\eta_2\cdot k}$ at $\mathbf{e}(\mathbf{j}) \in \mathbf{R}_{\tilde{\nu}}^{\Lambda^+}$. Hence we have $\mathbf{e}\sigma_{w_1}x_k^n\mathbf{e}(\mathbf{h})\sigma_{w_2}\mathbf{e} = 0$ in $\mathbf{R}_{\tilde{\nu}}^{\Lambda^+}$.

- Now suppose that q^{-1} is an entry of $\mathbf{h}$. Using Lemma 3.4.5 (ii), we know that $\mathbf{e}(\mathbf{h}) \cong \mathbf{e}(\mathbf{j})$, for some $\mathbf{j} \in I^{\tilde{\nu}^-}$. So, without loss of generality, we can assume that $\mathbf{h} = \mathbf{j}$. This means that $\eta_1 = \eta_2 = \eta$, where η is the longest element in $\mathcal{D}(W_m^B/\mathfrak{S}_m)$. Let $((n_1)_{\mathbf{e}(\eta\mathbf{j})}, \ldots, (n_m)_{\mathbf{e}(\eta\mathbf{j})})$ be the minimal nilpotency degrees at $\mathbf{e}(\eta\mathbf{j})$ so that $((n_m)_{\mathbf{e}(\mathbf{j})}, \ldots, (n_1)_{\mathbf{e}(\mathbf{j})})$ are the minimal nilpotency degrees at $\mathbf{e}(\mathbf{j})$. Then, for any $1 \le k \le m$ and any summands of $\mathbf{e}$, $\mathbf{e}(\mathbf{i}_1)$ and $\mathbf{e}(\mathbf{i}_2)$, we have

$$\begin{aligned} \mathbf{e}(\mathbf{i}_1)\sigma_{w_1}x_k^{n_{m-(k-1)}}\mathbf{e}(\mathbf{j})\sigma_{w_2}\mathbf{e}(\mathbf{i}_2) &= \mathbf{e}(\mathbf{i}_1)\sigma_{w_1}x_k^{n_{m-(k-1)}}\mathbf{e}(\mathbf{j})\sigma_\eta\sigma_{s_2}\mathbf{e}(\mathbf{i}_2) \\ &= \mathbf{e}(\mathbf{i}_1)\sigma_{w_1}\sigma_\eta x_{m-(k-1)}^{n_{m-(k-1)}}\mathbf{e}(\eta\mathbf{j})\sigma_{s_2}\mathbf{e}(\mathbf{i}_2), \end{aligned}$$

  where we have used that $x_i\sigma_\eta = \sigma_\eta x_{m-(i-1)}$, $\eta \in \mathcal{D}(W_m^B/\mathfrak{S}_m)$ is the longest element, in the last equality. But we know, by definition of J^Λ, that $x_{m-(k-1)}^{n_{m-(k-1)}}\mathbf{e}(\eta\mathbf{j}) = 0$ in $\mathbf{R}_{\tilde{\nu}}^{\Lambda^+}$. So $\mathbf{e}(\mathbf{i}_2)\sigma_{w_1}x_k^{n_{m-(k-1)}}\mathbf{e}(\mathbf{j})\sigma_{w_2}\mathbf{e}(\mathbf{i}_1) = 0$ in $\mathbf{R}_{\tilde{\nu}}^{\Lambda^+}$.

We can now prove that the map f is an isomorphism, as in Proposition 2.3.12. $\square$

Case: $\mathbf{p} \in \mathbf{I}$

We now fix the following setting for the remainder of this section; assume $p \in I$, $q \notin I$, p is not a root of unity. As in 3.5, we take $\nu \in {}^\theta\mathbb{N}I_p$ so that p always appears with multiplicity 2, i.e. $\nu_p = 2$. Then, from Theorem 3.5.3, we have that $\mathfrak{W}_\nu$ and $\mathbf{R}_{\tilde{\nu}}^+ \otimes_{\mathbf{k}[z]} \tilde{A}$ are Morita equivalent. In exactly the same way as in the subsection above, we use this Morita equivalence to define the following ideal $J^\Lambda \subset \mathfrak{W}_\nu$. We define

$$J^\Lambda = \left\langle x_1^{(\Lambda, i_1)}\mathbf{e}(\mathbf{i}), x_m^{(\Lambda, j_m^{-1})}\mathbf{e}(\mathbf{j}) \mid \mathbf{i} \in I^{\tilde{\nu}^+}, \mathbf{j} \in I^{\tilde{\nu}^-} \right\rangle.$$

Proposition 4.3.3. $\mathfrak{W}_\nu^\Lambda = \mathfrak{W}_\nu / J^\Lambda$ *is finite-dimensional.*

Proof. Using Lemma 4.2.1 and Corollary 4.2.2, it suffices to show that $x_k\mathbf{e}(\mathbf{i})$ is nilpotent, for all $\mathbf{i} \in {}^\theta I^\nu$, $1 \leq k \leq m$. Using the same argument from Proposition 4.3.1, the $x_k\mathbf{e}(\mathbf{i})$ are nilpotent, for all $\mathbf{i} \in {}^\theta I^\nu$ and $1 \leq k \leq m$. Now take any idempotent $\mathbf{e}(\mathbf{i}) \in \mathfrak{W}_\nu^\Lambda$ with $\mathbf{i} \notin I^{\tilde{\nu}^+} \cup I^{\tilde{\nu}^-}$. By Lemma 3.5.1 we know that if p appears twice in $\mathbf{i}$ or if p^{-1} appears before p in $\mathbf{i}$ then $\mathbf{e}(\mathbf{i}) \cong \mathbf{e}(\mathbf{j})$ for some $\mathbf{j} \in I^{\tilde{\nu}^+}$. In the proof of Lemma 3.5.1 we saw that there exists $w \in W_m^B$ with $\sigma_w^\rho \mathbf{e}(\mathbf{j})\sigma_w = \mathbf{e}(\mathbf{i})$. In fact, $w \in \mathcal{D}(W_m^B/\mathfrak{S}_m)$. Then,

$$x_k^{n_k}\mathbf{e}(\mathbf{j}) = 0 \quad \Longrightarrow \quad \sigma_w^\rho x_k^{n_k}\mathbf{e}(\mathbf{j})\sigma_w = 0.$$

Using that $\sigma_w^\rho \mathbf{e}(\mathbf{j})\sigma_w = \mathbf{e}(\mathbf{i})$ together with the relations,

$$\sigma_i x_k \mathbf{e}(\mathbf{j}) = x_{s_i(k)}\sigma_i\mathbf{e}(\mathbf{j})$$
$$\pi x_k \mathbf{e}(\mathbf{j}) = \begin{cases} x_k\pi\mathbf{e}(\mathbf{j}) & \text{if } k > 1 \\ -x_k\pi\mathbf{e}(\mathbf{j}) & \text{if } k = 1, \end{cases}$$

it follows that, since all x_k are nilpotent at $\mathbf{e}(\mathbf{j})$, all x_k are nilpotent at our chosen $\mathbf{e}(\mathbf{i})$, which was chosen arbitrarily. A similar line of reasoning applies when p^{-1} appears twice in $\mathbf{i}$ or when p appears before p^{-1} in $\mathbf{i}$. This proves that $\mathfrak{W}_\nu^\Lambda$ is finite-dimensional. □

The next proposition states that, as in the $q \in I$ setting, the algebra $\mathfrak{W}_\nu^\Lambda$ contains a cyclotomic KLR algebra as an idempotent subalgebra.

Proposition 4.3.4. *Let $\boldsymbol{e} = \sum_{i \in I^{\tilde{\nu}^+}} (\boldsymbol{e}(\boldsymbol{i}) + J^\Lambda)$. Then, $\boldsymbol{e}\mathfrak{W}_\nu^\Lambda\boldsymbol{e} \cong \boldsymbol{R}_{\tilde{\nu}}^{\Lambda^+}$.*

Proof. The proof uses the same method as in Proposition 4.3.2 together with Lemma 3.5.1. □

Final Remarks

In this thesis we have studied various classes of VV algebras and have shown how their module categories relate to module categories over KLR algebras. We have proved that certain classes of VV algebras are affine cellular and affine quasi-hereditary. Ultimately, we would like to show that all VV algebras exhibit these properties. We have considered various cases, depending on the parameters p and q. In many cases we imposed restrictions on these parameters. It is a natural question to ask how these algebras behave when we remove these restrictions. We did not treat the case $p, q \in I$ here and we hope that this will form part of our future work.

Question 1. In the case $p, q \in I$, what is the relationship between module categories over VV algebras and module categories over KLR algebras?

We have also defined finite-dimensional quotients of VV algebras in certain cases. We would like to be able to do this for VV algebras in all settings. Another very interesting question to ask is whether we have a type B analogue of the famous Brundan-Kleshchev isomorphism.

Question 2. How does this quotient relate to the affine Hecke algebra of type B? Can we find an explicit isomorphism between $\mathfrak{W}_\nu^\Lambda$ and a quotient of the affine Hecke algebra of type B, as Brundan and Kleshchev do in the type A setting?

In 2007 Kashiwara and Miemietz stated conjectures relating to canonical bases and branching rules of affine Hecke algebras of type D. In particular they introduce a $\mathcal{B}_\theta(\mathfrak{g})$-module V_θ, where $\mathcal{B}_\theta(\mathfrak{g})$ is an algebra defined previously by Enomoto and Kashiwara, which, for a specialization of V_θ, they conjecture the finite-dimensional H_n^D-modules categorify. In 2010 Shan, Varagnolo and Vasserot proved this conjecture by introducing a graded algebra ${}^\circ\mathbf{R}(\Gamma)_\nu$ associated to a quiver Γ, with involution θ, and a dimension vector ν (see [SVV11]). For a particular choice of Γ and θ they prove that ${}^\circ\mathbf{R}(\Gamma)_\nu$ is Morita equivalent to the affine Hecke algebra of type D. Similarly to VV algebras, the algebras ${}^\circ\mathbf{R}(\Gamma)_\nu$ are defined by generators and relations and depend upon Γ and the choice of dimension vector.

Question 3. What is the connection between ${}^\circ\mathbf{R}(\Gamma)_\nu$ and VV algebras? How are the module categories over these algebras related?

Bibliography

[ASS06] I. Assem, D. Simson, and A. Skowronski, *Elements of the representation theory of associative algebras: Volume 1: Techniques of representation theory*, Elements of the Representation Theory of Associative Algebras, Cambridge University Press, 2006.

[BK09a] J. Brundan and A. Kleshchev, *Blocks of cyclotomic Hecke algebras and Khovanov-Lauda algebras*, Invent. Math. 178, no. 3, 451–484 (2009).

[BK09b] ———, *Graded decomposition numbers for cyclotomic Hecke algebras*, Adv. Math. **222** (2009), no. 6, 1883–1942. MR 2562768 (2011i:20002)

[Bou02] N. Bourbaki, *Lie groups and Lie algebras. Chapters 4–6*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 2002, Translated from the 1968 French original by Andrew Pressley. MR 1890629 (2003a:17001)

[Bru13] J. Brundan, *Quiver Hecke algebras and categorification*, Advances in representation theory of algebras, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2013, pp. 103–133. MR 3220535

[Car89] R.W. Carter, *Simple groups of Lie type*, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1989, Reprint of the 1972 original, A Wiley-Interscience Publication. MR 1013112 (90g:20001)

[DR89] V. Dlab and C.M. Ringel, *Quasi-hereditary algebras*, Illinois J. Math. **33** (1989), no. 2, 280–291. MR 987824 (90e:16023)

[EK06] N. Enomoto and M. Kashiwara, *Symmetric crystals and affine Hecke algebras of type B*, Proc. Japan Acad. Ser. A Math. Sci. 82 (2006), no. 8, 131–136. (2006).

[EK09] ———, *Symmetric crystals for* $\mathfrak{gl}_\infty$, Publ. Res. Inst. Math. Sci. 44, no. 3, 837–891 (2009).

[GL96] J. J. Graham and G. I. Lehrer, *Cellular algebras*, Invent. Math. **123** (1996), no. 1, 1–34. MR 1376244 (97h:20016)

[GP00] M. Geck and G. Pfeiffer, *Characters of Finite Coxeter Groups and Iwahori-Hecke Algebras*, Oxford University Press (2000).

[HM10] J. Hu and A. Mathas, *Graded cellular bases for the cyclotomic Khovanov-Lauda-Rouquier algebras of type A*, Adv. Math. **225** (2010), no. 2, 598–642. MR 2671176 (2011g:20006)

[Hum92] J.E. Humphreys, *Reflection groups and coxeter groups*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1992.

[KL] A. Kleshchev and J. Loubert, *Affine cellularity of Khovanov-Lauda-Rouquier algebras of finite types*, preprint arXiv:1310.4467v1.

[KL09] M. Khovanov and A. Lauda, *A diagrammatic approach to categorification of quantum groups I*, Represent. Theory 13, 309–347. (2009).

[KL11] ———, *A diagrammatic approach to categorification of quantum groups II*, Trans. Amer. Math. Soc. **363** (2011), no. 5, 2685–2700. MR 2763732 (2012a:17021)

[Kle] A. Kleshchev, *Representation theory and cohomology of Khovanov-Lauda-Rouquier algebras*, preprint arXiv:1401.6151v1.

[Kle10] ———, *Representation theory of symmetric groups and related Hecke algebras*, Bull. Amer. Math. Soc. (N.S.) 47, no. 3, 419–481 (2010).

[Kle15] ———, *Affine highest weight categories and affine quasihereditary algebras*, Proc. Lond. Math. Soc. (3) **110** (2015), no. 4, 841–882. MR 3335289

[KLM13] A. Kleshchev, J. Loubert, and V. Miemietz, *Affine cellularity of Khovanov-Lauda-Rouquier algebras in type A*, J. Lond. Math. Soc. (2) **88** (2013), no. 2, 338–358. MR 3106725

[KR10] A. Kleshchev and A. Ram, *Homogeneous representations of Khovanov-Lauda algebras*, J. Eur. Math. Soc. (JEMS) 12, no. 5, 1293–1306 (2010).

[KR11] ———, *Representations of Khovanov-Lauda-Rouquier algebras and combinatorics of Lyndon words*, Math. Ann. 349, no. 4, 943–975 (2011).

[KX98] S. Koenig and C. Xi, *On the structure of cellular algebras*, Algebras and modules, II (Geiranger, 1996), CMS Conf. Proc., vol. 24, Amer. Math. Soc., Providence, RI, 1998, pp. 365–386. MR 1648638 (2000a:16011)

[KX12] ———, *Affine cellular algebras*, Adv. Math. **229** (2012), no. 1, 139–182. MR 2854173

[Lam99] T.Y. Lam, *Lectures on modules and rings*, Graduate Texts in Mathematics, Springer New York, 1999.

[Lam01] ———, *A first course in noncommutative rings*, A First Course in Noncommutative Rings, Springer, 2001.

[Mak01] C.K. Mak, *Quasi-parabolic subgroups of $G(m,1,r)$*, J. Algebra **246** (2001), no. 2, 471–490. MR 1872111 (2002j:20072)

[SVV11] P. Shan, M. Varagnolo, and E. Vasserot, *Canonical bases and affine Hecke algebras of type D*, Adv. Math. **227** (2011), no. 1, 267–291. MR 2782195 (2012a:20007)

[VV11a] M. Varagnolo and E. Vasserot, *Canonical bases and affine Hecke algebras of type B*, Invent. Math. 185, no. 3, 593–693 (2011).

[VV11b] ______, *Canonical bases and KLR-algebras*, J. Reine Angew. Math. **659** (2011), 67–100. MR 2837011

[Wei95] C.A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1995.

[Yan14] G. Yang, *Affine cellular algebras and Morita equivalences*, Arch. Math. (Basel) **102** (2014), no. 4, 319–327. MR 3196959