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Adding many Baumgartner clubs

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Abstract I define a homogeneous $\aleph_2$–c.c. proper product forcing for adding many clubs of ω_1 with finite conditions. I use this forcing to build models of $\mathfrak{b}(\omega_1) = \aleph_2$, together with $\mathfrak{d}(\omega_1)$ and $2^{\aleph_0}$ large and with very strong failures of club guessing at ω_1.

Keywords Baumgartner clubs · Strong failures of club guessing · Cardinal characteristics for ω_1 · Generalizations of Martin's Axiom

Mathematics Subject Classification 03E05 · 03E35 · 03E50 · 03E17

1 Introduction

Cohen's forcing $2^{<\omega}$ for adding a real, usually called Cohen forcing, is perhaps the simplest non-trivial forcing notion one can think of (and it was also the first to be discovered). There is a very simple and nicely behaved forcing for adding an arbitrary number θ of Cohen reals. This forcing is of course $\mathrm{Add}(\omega, \theta)$, where $\mathrm{Add}(\omega, X)$, for a set of ordinals X, is the partial order of finite functions $p \subseteq X \times \omega \times 2$, ordered by reverse inclusion. For every $\mathrm{Add}(\omega, X)$-generic G and every $\alpha \in X$, $r^G_\alpha := \bigcup_{p \in G}\{(n, \epsilon) \mid (\alpha, n, \epsilon) \in p\}$ is a Cohen real over $\mathbf{V}$ and $r^G_\alpha \neq r^G_{\alpha'}$ for $\alpha \neq \alpha'$ in X. Also, $\mathrm{Add}(\omega, X)$ is nicely behaved: It has the countable chain condition (c.c.c.), it is homogeneous—in the sense that given any $p, p' \in \mathrm{Add}(\omega, X)$ there are $q \leq p$ and $q' \leq p'$ such that $\mathrm{Add}(\omega, X) \restriction q \cong \mathrm{Add}(\omega, X) \restriction q'$ –, and it can be naturally represented as the product of $\mathrm{Add}(\omega, X_0)$ and $\mathrm{Add}(\omega, X_1)$ for any partition (X_0, X_1) of X. In particular, for every G as above and all $\alpha \neq \alpha'$ in X, $r^G_{\alpha'}$ is Cohen generic

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over $\mathbf{V}[r_\alpha^G]$. Cohen forcing and $\mathrm{Add}(\omega, \theta)$ have of course been extensively studied for more than 50 years now. For example, $\mathrm{Add}(\omega, \theta)$ is the forcing that Cohen used to prove the consistency of $\neg$ CH (by forcing over L).

There is a particularly simple forcing notion for adding a club subset of ω_1. This forcing was first defined and studied by Baumgartner [4] and will be denoted in this note by $\mathbb{B}$. $\mathbb{B}$ is the set, ordered by reverse inclusion, of all finite functions $f \subseteq \omega_1 \times \omega_1$ that can be extended to a strictly increasing and continuous function $F : \omega_1 \longrightarrow \omega_1$. $\mathbb{B}$ has size $\aleph_1$, and the union of any generic filter for $\mathbb{B}$ is the enumerating function of a new club of ω_1, which I will call a *Baumgartner club* (over $\mathbf{V}$). $\mathbb{B}$ is often described as the forcing for adding a club of ω_1 with finite conditions. It can be presented in other appealing ways too (see for example [1] or [16]). $\mathbb{B}$ is proper and so, since it has size $\aleph_1$, preserves all cardinals; in fact, given any countable $N \preccurlyeq H(\omega_2)$ and any $p \in \mathbb{B} \cap N$, $p^* := p \cup \{\langle \delta_N, \delta_N \rangle\}$ is an $(N, \mathbb{B})$-generic condition extending p (see [4]) where, here and throughout the note, δ_X denotes $X \cap \omega_1$ whenever X is a set and $X \cap \omega_1$ is an ordinal. The proof of properness for $\mathbb{B}$ coming from this choice of p^* can in fact be seen as perhaps the simplest possible proof of properness of a partial order using submodels as side conditions: N is 'added' to p by declaring δ_N to be a fixed point of p^*.

There are certainly some similarities between Cohen forcing, in the context of adding a real, and $\mathbb{B}$, in the context of adding a club of ω_1 not including any club from the ground model. To point out some obvious examples, both forcing notions add the relevant new object by finite approximations, both are homogeneous (in the above sense) and of the least possible size, both have a simple definition,[1] and in fact both are absolute in a strong sense (in the sense that any two transitive models of ZF have the same Cohen forcing, and have the same $\mathbb{B}$ in case they agree on ω_1). It is therefore natural to ask if there is version of $\mathrm{Add}(\omega, X)$ for $\mathbb{B}$. The main purpose of this note is to present such a forcing. More precisely, I will show that, given a set X of ordinals, there is a forcing, which I will denote by $\mathrm{Add}_{\mathbb{B}}(X)$, which is quite simple to define and which has the following properties.

(1) For every $\mathrm{Add}_{\mathbb{B}}(X)$-generic G and every $\alpha \in X$ one can naturally extract a Baumgartner club C_α^G from G. Furthermore, $C_\alpha^G \neq C_{\alpha'}^G$ for all distinct α, α' in X.
(2) $\mathrm{Add}_{\mathbb{B}}(X)$ is proper and has the $\aleph_2$–c.c.
(3) $\mathrm{Add}_{\mathbb{B}}(X)$ is homogeneous.
(4) For every partition (X_0, X_1) of X, $\mathrm{Add}_{\mathbb{B}}(X)$ can be naturally represented as the product $\mathrm{Add}_{\mathbb{B}}(X_0) \times \mathrm{Add}_{\mathbb{B}}(X_1)$. In particular, if G is as in (1) and $\alpha \neq \alpha'$ are in X, then $C_{\alpha'}^G$ is $\mathbb{B}$-generic over $\mathbf{V}[C_\alpha^G]$.

In the next section I will discuss the forcing axiom for $\mathbb{B}$ for collections of λ-many dense sets (for arbitrary λ), its consistency and, in Sect. 2.1, its consequences at the level of club guessing at ω_1. Subsection 2.2 relates this forcing axiom to the covering number of the meagre ideal $\mathcal{M}_{\mathcal{C}_{\omega_1}}$ for the set of all clubs on ω_1 endowed with a natural topology. Section 3 introduces the forcing $\mathrm{Add}_{\mathbb{B}}(X)$ and presents its

[1] For a way to make this precise see for example Zapletal's result on forcings of size ω_1 in $L[x]$, x a real, cited right after Definition 2.1.

basic theory. Finally, Sect. 3.1 presents the main new consistency result in the note, which is the joint consistency of $\mathfrak{b}(\omega_1) = \mathbf{non}(\mathcal{M}_{\mathcal{C}_{\omega_1}}) = \aleph_2$ with $\mathfrak{d}(\omega_1)$ large and with very strong failures of club guessing at ω_1; these conclusions hold after forcing with $\mathrm{Add}_{\mathbb{B}}(\theta)$ for large θ.

2 $\mathrm{FA}(\mathbb{B})_\lambda$, for arbitrary λ, and strong failures of club guessing

Given a partial order $\mathcal{P}$ and a cardinal λ, the forcing axiom for $\mathcal{P}$ and for λ-many dense sets, denoted by $\mathrm{FA}(\mathcal{P})_\lambda$, is the statement that for every $\{D_i \mid i < \lambda\}$, if each D_i is a dense subset of $\mathcal{P}$, then there is a filter $G \subseteq \mathcal{P}$ such that $G \cap D_i \neq \emptyset$ for all i. Also, given a class Γ of partial orders, $\mathrm{FA}(\Gamma)_\lambda$ means $\mathrm{FA}(\mathcal{P})_\lambda$ for all $\mathcal{P} \in \Gamma$. Since Cohen forcing is c.c.c., the forcing axiom for it and for λ-many dense sets follows from the (consistent) axiom MA_λ, and the same is of course true for $\mathrm{Add}(\omega, X)$. For the same reason, the forcing axiom PFA, i.e., $\mathrm{FA}(\{\mathcal{P} \mid \mathcal{P}\text{ proper}\})_{\aleph_1}$—and in fact $\mathrm{FA}(\{\mathcal{P} \mid \mathcal{P}\text{ a proper poset of size }\aleph_1\})_{\aleph_1}$, which, unlike PFA, can be forced over any ZFC model –, implies $\mathrm{FA}(\mathbb{B})_{\aleph_1}$.

A natural question arises at this point: $\mathrm{FA}(\{\mathcal{P} \mid \mathcal{P}\text{ proper}\})_{\aleph_2}$ is false. But is $\mathrm{FA}(\mathbb{B})_{\aleph_2}$ consistent? Note that we cannot use countable-support proper iterated forcing to answer this question as every such forcing will produce a model of $2^{\aleph_0} \leq \aleph_2$ and $\mathrm{FA}(\mathbb{B})_{\aleph_2}$ clearly implies $2^{\aleph_0} \geq \aleph_3$.

The method of finite-support iterated forcing with symmetric systems of structures as side conditions (see [2] and [3]) can be used to answer this question. This method enables one to force without collapsing cardinals in such a way that the resulting model satisfies $\mathrm{FA}(\mathcal{P})_\lambda$, for an arbitrarily fixed λ, for various well-behaved proper partial orders $\mathcal{P}$ with the $\aleph_2$–c.c. Background information on this method can be found in [2] and [3], so I will not say anything about it here. It turns out that $\mathbb{B}$ is such a well-behaved forcing. More specifically, there is a class of proper posets, which in [3] we refer to as the posets having the $\aleph_{1.5}$–c.c., to which $\mathbb{B}$ belongs, and such that the forcing axiom $\mathrm{MA}(\{\mathcal{P} \mid \mathcal{P}\text{ has the }\aleph_{1.5}\text{–c.c.}\})_\lambda$—which in [3] we call $\mathrm{MA}^{1.5}_\lambda$— is consistent for arbitrarily chosen λ (cf. Theorem 3.11). The main reason for this terminology is that every c.c.c. poset is in our class and every poset in our class is $\aleph_2$–c.c. It follows that, for any λ, $\mathrm{MA}^{1.5}_\lambda$ implies both MA_λ and $\mathrm{FA}(\mathbb{B})_\lambda$. The definition of the $\aleph_{1.5}$–c.c. is the following.

Definition 2.1 ([3]) A poset $\mathcal{P}$ has the $\aleph_{1.5}$*–c.c.* if and only if for every regular cardinal θ such that $\mathcal{P} \in H(\theta)$ there is a club $D \subseteq [H(\theta)]^{\aleph_0}$ such that for every finite $\mathcal{N} \subseteq D$ and every $p \in \mathcal{P}$, if $p \in N$ for some $N \in \mathcal{N}$ such that $\delta_N = \min\{\delta_M \mid M \in \mathcal{N}\}$, then there is some condition extending p and $(N, \mathcal{P})$-generic for all $N \in \mathcal{N}$.

Even if $\mathrm{FA}(\mathbb{B})_\lambda$ is consistent, for arbitrary λ, it is not known whether $\mathrm{FA}(\{\mathcal{P} \mid \mathcal{P}\text{ a proper poset of size }\aleph_1\})_{\aleph_2}$ is consistent.

$\mathbb{B}$ is a very prominent poset with the $\aleph_{1.5}$–c.c., at least in the sense that it is the simplest such poset which does not have the c.c.c. One way to make mathematical sense of this assertion is to quote the following fact due to Zapletal: If $x \in \mathbb{R}$, $x^\sharp$ exists, and $\mathcal{P} \in L[x]$ is a non-atomic partial order on $\omega_1^{\mathbf{V}}$, then $\mathcal{P}$ is forcing-equivalent to the disjoint sum of some number of copies of forcings in

$\{2^{<\omega}, \mathrm{Add}(\omega, \omega_1), \mathbb{B}, \mathrm{Coll}(\omega, \omega_1)\}$, where $\mathrm{Coll}(\omega, \omega_1)$ is the collapse of ω_1 to ω with finite conditions ([17]). It is also worth pointing out that, on the other hand, $\mathbb{B}$ is nowhere c.c.c. (i.e., it is not c.c.c. below any condition) and, as Zapletal proved in [18], under PFA it is a minimal nowhere c.c.c. poset, in the sense that every nowhere c.c.c. poset adds a generic for $\mathbb{B}$. Also, if $P = \{p_\alpha \mid \alpha < \omega_1\}$ is a proper nowhere c.c.c. forcing notion adding a club $C \subseteq \omega_1$ such that for all $\alpha \in C$, $\dot{G} \cap \{p_\beta \mid \beta < \alpha\}$ is generic for $\{p_\beta \mid \beta < \alpha\}$ (where $\dot{G}$ denotes the generic filter), then $\mathrm{RO}(P) = \mathrm{RO}(\mathbb{B})$ ([16]).

2.1 Some weakenings of club guessing at ω_1

Club-guessing principles are well studied weakenings of Jensen's diamond principle $\diamondsuit_\kappa$, for a cardinal κ, in which the guessing device is a club-sequence $(C_\delta \mid \delta \in S)$ for some $S \subseteq \kappa \cap \mathrm{Lim}$ (i.e., every C_δ is a club of δ) and in which the relevant guessing applies to clubs of κ rather than arbitrary subsets of κ. Unlike $\diamondsuit_\kappa$, they are consistent with 2^μ large for any given $\mu < \kappa$, simply because they are preserved by κ–c.c. forcing (since every club of κ in any extension by a κ–c.c. forcing includes a club from the ground model). This is of course the reason why MA_λ, for any cardinal λ, is consistent with such guessing principles. Also, whereas club-guessing principles for ω_2 or higher regular cardinals are often outright true in ZFC (cf. [14], III, §2), the truth value of their versions at ω_1 can be easily changed by forcing (see e.g. [7,8,15]). Let us see a couple of examples:

A C-sequence—also known as a ladder system—is a sequence $(C_\delta \mid \delta \in \mathrm{Lim}(\omega_1))$ such that C_δ is a cofinal subset of δ of order type ω for every δ. *Club Guessing at* ω_1 (CG) says that there is a C-sequence $(C_\delta \mid \delta \in \mathrm{Lim}(\omega_1))$ which guesses clubs in the sense that for every club $D \subseteq \omega_1$ there is some $\delta \in \mathrm{Lim}(\omega_1)$ with $C_\delta \backslash D$ finite. *Kunen's Axiom* (KA), also known as *Interval Hitting Principle* (see e.g. [7]; see also [11]), is the following statement, first considered by Kunen: There is a C-sequence $(C_\delta \mid \delta \in \mathrm{Lim}(\omega_1))$ with the property that for every club $D \subseteq \omega_1$ there is some δ such that $[C_\delta(n), C_\delta(n+1)) \cap D \neq \emptyset$ for a tail of $n < \omega$ (where $X(\xi)$ denotes the ξ-th member of X if X is a set of ordinals). CG clearly implies KA, follows from $\diamondsuit$, and is preserved by c.c.c. forcing. On the other hand, it is easy to see and a well-known fact that $\mathrm{FA}(\{\mathcal{P} \mid \mathcal{P}$ a proper poset of size $\aleph_1\})_{\aleph_1}$ implies $\neg$ KA.

Consider the following weak versions of CG and KA, respectively: Given a cardinal λ, let CG_λ be the assertion that there is a set $\mathcal{C}$ of subsets of ω_1 of order type ω such that $|\mathcal{C}| \leq \lambda$ and such that for every club $D \subseteq \omega_1$ there is some $X \in \mathcal{C}$ such that $X \backslash D$ is finite. Let also KA_λ be the assertion that there is a set $\mathcal{C}$ of subsets of ω_1 of order type ω such that $|\mathcal{C}| \leq \lambda$ and such that for every club $D \subseteq \omega_1$ there is some $X \in \mathcal{C}$ such that $[X(n), X(n+1)) \cap D \neq \emptyset$ for a tail of $n < \omega$. By a straightforward density argument, if C is $\mathbb{B}$-generic and $X \in \mathbf{V}$ is a subset of ω_1 of order type ω, then $[X(n), X(n+1)) \cap C$ is empty for infinitely many $n < \omega$ (see [4]), which immediately implies the following proposition.

Proposition 2.2 *For every cardinal* $\lambda \geq \omega_1$, $\mathrm{FA}(\mathbb{B})_\lambda$ *implies* $\neg\, \mathrm{KA}_\lambda$.

It is worth pointing out that the stronger forcing axiom $\mathrm{MA}^{1.5}_\lambda$ implies even stronger failures of club guessing ([3]).

2.2 FA($\mathbb{B}$)$_\lambda$ and the covering number of the meagre ideal of $\mathcal{C}_{\omega_1}$

Recall that for an ideal $\mathcal{I}$ on a set of X, the covering number of $\mathcal{I}$, denoted by $\mathbf{cov}(\mathcal{I})$, is the minimal size of a collection of members of $\mathcal{I}$ whose union is X. Given a topological space X and a cardinal κ, let $\mathcal{M}^\kappa_X$ denote the (possibly improper) ideal on X κ-generated by the nowhere dense sets, i.e., the collection of all unions of less than κ-many nowhere dense sets. Let us refer to the members of $\mathcal{M}^\kappa_X$ as *κ–meagre subsets of* X. This way, the classical meagre ideal on X is $\mathcal{M}^{\omega_1}_X$. I will also denote it by $\mathcal{M}_X$. The following is a straightforward but useful observation in this general context.

Fact 2.3 $\mathcal{M}^\kappa_X \subseteq \mathcal{M}^{\kappa'}_X$ *and therefore* $\mathbf{cov}(\mathcal{M}^{\kappa'}_X) \leq \mathbf{cov}(\mathcal{M}^\kappa_X)$ *whenever* $\kappa \leq \kappa'$, *and* $\mathbf{cov}(\mathcal{M}^\omega_X) = \mathbf{cov}(\mathcal{M}^\kappa_X)$ *for every* $\kappa < \mathbf{cov}(\mathcal{M}^\omega_X)$.

In the following definition and throughout the note, if C is a club of ω_1, $\tilde{C}$ will denote its enumerating function.

Definition 2.4 Consider the topology $\tau_\mathbb{B}$ on the set of all clubs of ω_1 whose basis is given by the conditions in $\mathbb{B}$; in other words, a basis for this topology is given by $\{B_p \mid p \in \mathbb{B}\}$, where B_p is the set of all clubs $C \subseteq \omega_1$ such that $p \subseteq \tilde{C}$. Then $\mathcal{C}_{\omega_1}$ denotes the set of all clubs of ω_1 endowed with the topology $\tau_\mathbb{B}$.

One first observation is that the usual Baire category theorem holds for $\mathcal{C}_{\omega_1}$:

Lemma 2.5 $\mathbf{cov}(\mathcal{M}_{\mathcal{C}_{\omega_1}}) \geq \aleph_1$.

Proof Let $(D_n)_{n<\omega}$ be a sequence of dense subsets of $\mathbb{B}$. It suffices to show that there is a club $C \subseteq \omega_1$ such that for all n, $p \subseteq \tilde{C}$ for some $p \in D_n$.

Let $(C_\delta : \delta \in \mathrm{Lim}(\omega_1))$ be a C-sequence. For each n, let D^*_n denote the set of $\mathbb{B}$-conditions p extending some condition in D_n and such that for every limit ordinal $\delta \in \mathrm{dom}(p)$ there is some $\delta' \in \mathrm{dom}(p) \cap \delta$ such that $p(\delta') > C_{p(\delta)}(n)$, where $(C_{p(\delta)}(i))_{i<\omega}$ is the increasing enumeration of $C_{p(\delta)}$. Each D^*_n is clearly dense in $\mathbb{B}$. Let N be a countable elementary substructure of $H(\omega_2)$ containing each D^*_n and let $(\delta_n)_{n<\omega}$ be an enumeration of δ_N. Now we build a decreasing sequence $(p_n)_{n<\omega}$ of $\mathbb{B}$-conditions such that for each n, $p_n \in N \cap D^*_n$ and $\delta_n \in \mathrm{dom}(p_n)$. This is possible since each D^*_n is dense. In the end, $p^* = \bigcup_n p_n : \delta_N \longrightarrow \delta_N$ is a strictly increasing and continuous function by the choice of $(D^*_n)_{n<\omega}$ and of course range(p^*) is cofinal in δ_N. Hence range$(p^*) \cup (\omega_1 \backslash \delta_N)$ is a club as desired. □

Given a partial order $\mathcal{P}$, let $\mathfrak{m}(\mathcal{P})$ be the minimal size of a family $\mathcal{D}$ of dense subsets of $\mathcal{P}$ such that there is no filter $G \subseteq \mathcal{P}$ intersecting all members of $\mathcal{D}$ (i.e., $\mathfrak{m}(\mathcal{P})$ is the least cardinal λ such that FA$(\mathcal{P})_\lambda$ fails). If $\mathcal{P}$ is infinite, $\mathfrak{m}(\mathcal{P})$ is of course at least $\aleph_1$.

Proposition 2.6 $\mathbf{cov}(\mathcal{M}_{\mathcal{C}_{\omega_1}}) = \mathfrak{m}(\mathbb{B})$

Proof The proof is a standard translation exercise between topological notions and order-theoretical notions, and is essentially identical to the proof that $\mathfrak{m}$(Cohen) is the covering number of the meagre ideal for the Baire space (see for example [10], Theorem 16.1). It is easy to see that $\mathbf{cov}(\mathcal{M}_{\mathcal{C}_{\omega_1}}) \leq \mathfrak{m}(\mathbb{B})$. In fact, if $\mathcal{D}$ is a collection of dense subsets of $\mathbb{B}$ and there is no filter of $\mathbb{B}$ intersecting all members of $\mathcal{D}$, then $\{X_D \mid D \in \mathcal{D}\}$ is a collection of nowhere dense sets covering $\mathcal{C}_{\omega_1}$, where X_D is, for every $D \in \mathcal{D}$, the collection of $C \in \mathcal{C}_{\omega_1}$ such that $p \not\subseteq \tilde{C}$ for any $p \in D$. Hence, in order to prove the equality we may assume, by the previous lemma, that $\mathfrak{m}(\mathbb{B}) > \aleph_1$. Now, given $\lambda < \mathfrak{m}(\mathbb{B})$ and a collection $\{X_i \mid i < \lambda\}$ of closed nowhere dense subsets of $\mathcal{C}_{\omega_1}$, $D_i = \{p \in \mathbb{B} \mid B_p \cap X_i = \emptyset\}$ is a dense open subset of $\mathbb{B}$ for all i. Let $\{E_\nu \mid \nu < \omega_1\}$ be a set of dense subsets of $\mathbb{B}$ such that $\bigcup \mathrm{range}(G)$ is a club of ω_1 for every filter $G \subseteq \mathbb{B}$ meeting all E_ν. Since $\mathfrak{m}(\mathbb{B}) > \aleph_1$, we can find a filter $G \subseteq \mathbb{B}$ meeting all E_ν and all D_i. It follows then that $C = \bigcup \mathrm{range}(G)$ is a club of ω_1 such that $C \notin X_i$ of all i. □

Of course there is nothing special about Cohen forcing or $\mathbb{B}$ in any of these translations; in fact, a similar characterisation can be always obtained for $\mathfrak{m}(\mathcal{P})$ for any poset $\mathcal{P}$. What is nice about Cohen forcing and $\mathbb{B}$ is the appealing appearance of the topological side of the translation.

3 Adding many Baumgartner clubs

The following proposition shows that some nontrivial move is necessary in order to add many Baumgartner clubs by a product forcing not collapsing ω_1.

Proposition 3.1 *Both the finite support product of $\aleph_0$ copies of $\mathbb{B}$ and the countable support product of $\aleph_0$ copies of $\mathbb{B}$ collapse ω_1.*

Proof Let $\mathcal{P}_0$ be the finite support product of copies of $\mathbb{B}$ indexed by ω and let $\mathcal{P}_1$ be the countable support product of copies of $\mathbb{B}$ indexed by ω. The first part is trivial: Given any $f \in \mathcal{P}_0$ and any $\alpha < \omega_1$ we can find an extension f' of f and some $n \in \mathrm{dom}(f')$ such that $f'(n)(0) > \alpha$. This gives a cofinal function $\dot{f} : \omega \longrightarrow \omega_1^{\mathbf{V}}$ in the extension by $\mathcal{P}_0$.

The second part can be argued for by a variation of a standard argument for showing the well-known fact that the full support product of countably many copies of Cohen forcing collapses $\mathbb{R}^{\mathbf{V}}$ to ω (see for example [6]): We start out by fixing a ladder system $(C_\delta \mid \delta \in \mathrm{Lim}(\omega_1))$. Given a $\mathcal{P}_1$-generic filter G and $n < \omega$ let $F^n : \omega_1^{\mathbf{V}} \longrightarrow \omega_1^{\mathbf{V}}$ be the strictly increasing and continuous function added by G on the n-th coordinate and let $\delta_n = F^n(\omega) \in \mathrm{Lim}(\omega_1)$. Now, given any $k < \omega$, look at the biggest $n_0 < \omega$ such that the interval $[F^0(k+i),\ F^0(k+i+1))$ contains exactly one element of C_{δ_0} for every $i < n_0$, which exists by density. Now write $a_0 = 0$ or $a_0 = 1$ depending on whether or not $|[F^1(n_0),\ F^1(n_0+1)) \cap C_{\delta_1}| = 1$. Let $n_1 < \omega$ be biggest such that $|[F^1(n_0+1+i),\ F^1(n_0+2+i)) \cap C_{\delta_1}| = 1$ for all $i < n_1$, write $a_1 = 0$ or $a_1 = 1$ depending on whether or not $|[F^2(n_1),\ F^2(n_1+1)) \cap C_{\delta_2}| = 1$, and keep going. This way we associate to k a unique $(a_n)_{n<\omega} \in {}^{\omega}2$, and it is easy to check that, by density, every real in $\mathbf{V}$ is the image of some $k < \omega$ under this mapping. □

The following notation will be used in the remainder of this section. Given two functions $\mathcal{F}$ and $\mathcal{G}$, I will denote by $\mathcal{F} \oplus \mathcal{G}$ the function $\mathcal{H}$ with domain $\mathrm{dom}(\mathcal{F}) \cup \mathrm{dom}(\mathcal{G})$ such that

- $\mathcal{H}(x) = \mathcal{F}(x)$ for every $x \in \mathrm{dom}(\mathcal{F}) \setminus \mathrm{dom}(\mathcal{G})$,
- $\mathcal{H}(x) = \mathcal{G}(x)$ for every $x \in \mathrm{dom}(\mathcal{G}) \setminus \mathrm{dom}(\mathcal{F})$, and
- $\mathcal{H}(x) = \mathcal{F}(x) \cup \mathcal{G}(x)$ for every $x \in \mathrm{dom}(\mathcal{F}) \cap \mathrm{dom}(\mathcal{G})$.

Note that $(\mathcal{F} \oplus \mathcal{G}) \oplus \mathcal{H} = \mathcal{F} \oplus (\mathcal{G} \oplus \mathcal{H})$ for all functions $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$. I will denote $(\mathcal{F} \oplus \mathcal{G}) \oplus \mathcal{H}$ simply by $\mathcal{F} \oplus \mathcal{G} \oplus \mathcal{H}$.

The following simple fact will be used repeatedly.

Fact 3.2 $\mathrm{ot}(A \cup B) < \delta$ *whenever* δ *is an indecomposable ordinal and* A, B *are sets of ordinals with* $\mathrm{ot}(A)$, $\mathrm{ot}(B) < \delta$.

Proof Since $\mathrm{ot}(A \cup B) \leq \mathrm{ot}(A) + \mathrm{ot}(B)$. □

I will now define $\mathrm{Add}_{\mathbb{B}}(X)$. $\mathrm{Add}_{\mathbb{B}}(X)$ can be seen as a forcing adding Baumgartner clubs, indexed by ordinals in X, by using finite supports and with countable subsets of X as side conditions. The definition of $\mathrm{Add}_{\mathbb{B}}(X)$ is a streamlined version of the constructions from [2] or [3]. In fact, it is simple enough that it is an actual product (Lemma 3.5), whereas the constructions in [2] and [3] certainly are not. The fact that $\mathrm{Add}_{\mathbb{B}}(X)$ is a product is crucial for the proof (in Theorem 3.15) that there is a Luzin set of clubs of size $\aleph_2$ in the extension.

Definition 3.3 Let X be a set of ordinals. $\mathrm{Add}_{\mathbb{B}}(X)$ is the following forcing notion: Conditions in $\mathrm{Add}_{\mathbb{B}}(X)$ are pairs of the form $p = (f, \mathcal{F})$ with the following properties.

(1) f is a finite function with $\mathrm{dom}(f) \subseteq X$ and such that $f(\alpha) \in \mathbb{B}$ for every $\alpha \in \mathrm{dom}(f)$.
(2) $\mathcal{F}$ is a finite function with $\mathrm{dom}(\mathcal{F}) \subseteq \omega_1$ such that for every $\delta \in \mathrm{dom}(\mathcal{F})$,
  (a) δ is a countable indecomposable ordinal,
  (b) $\mathcal{F}(\delta)$ is a countable subset of X,
  (c) $\delta \in \mathrm{dom}(f(\alpha))$ and $f(\alpha)(\delta) = \delta$ for all $\alpha \in \mathrm{dom}(f) \cap \mathcal{F}(\delta)$, and
  (d) $\mathrm{ot}(\mathcal{F}(\delta')) < \delta$ for every $\delta' \in \mathrm{dom}(\mathcal{F} \restriction \delta)$.

Given $\mathrm{Add}_{\mathbb{B}}(X)$ conditions $(f_0, \mathcal{F}_0)$, $(f_1, \mathcal{F}_1)$, $(f_1, \mathcal{F}_1)$ extends $(f_0, \mathcal{F}_0)$ iff

- $\mathrm{dom}(f_0) \subseteq \mathrm{dom}(f_1)$ and $f_0(\alpha) \subseteq f_1(\alpha)$ for every $\alpha \in \mathrm{dom}(f_0)$, and
- $\mathrm{dom}(\mathcal{F}_0) \subseteq \mathrm{dom}(\mathcal{F}_1)$ and $\mathcal{F}_0(\delta) \subseteq \mathcal{F}_1(\delta)$ for every $\delta \in \mathrm{dom}(\mathcal{F}_0)$.

Let us fix a set X of ordinals and let us prove the relevant facts about $\mathrm{Add}_{\mathbb{B}}(X)$.

Given $\alpha \in X$ and a generic filter G for $\mathrm{Add}_{\mathbb{B}}(X)$, let $F^\alpha_G = \{f(\alpha) \mid (f, \mathcal{F}) \in G$ for some $\mathcal{F}\}$. The following fact is clear.

Fact 3.4 *Given* $\alpha \in X$ *and a generic filter* G *for* $\mathrm{Add}_{\mathbb{B}}(X)$, F^α_G *is a generic filter for* $\mathbb{B}$.

Given a condition $p = (f, \mathcal{F})$ in $\mathrm{Add}_{\mathbb{B}}(X)$ and $Y \subseteq X$, let $p \restriction Y = (f \restriction Y, d, N \cap Y\rangle \mid \langle \delta, N\rangle \in \mathcal{F},\ \delta \in \mathrm{dom}(\mathcal{F})\})$.

Lemma 3.5 follows from Fact 3.2.

Lemma 3.5 *Let* X_0, X_1 *be disjoint sets of ordinals. Then, the function sending a pair* $((f_0, \mathcal{F}_0), (f_1, \mathcal{F}_1)) \in \mathrm{Add}_{\mathbb{B}}(X_0) \times \mathrm{Add}_{\mathbb{B}}(X_1)$ *to* $(f_0 \cup f_1, \mathcal{F}_0 \oplus \mathcal{F}_1)$ *is an isomorphism between the posets* $\mathrm{Add}_{\mathbb{B}}(X_0) \times \mathrm{Add}_{\mathbb{B}}(X_1)$ *and* $\mathrm{Add}_{\mathbb{B}}(X_0 \cup X_1)$. *The inverse of this function is the function sending* $p \in \mathrm{Add}_{\mathbb{B}}(X)$ *to* $(p \restriction X_0, p \restriction X_1)$.

Lemma 3.6 $\mathrm{Add}_{\mathbb{B}}(X)$ *has the* $\aleph_2$*–c.c.*

Proof Given $\mathrm{Add}_{\mathbb{B}}(X)$ conditions $(f_\xi, \mathcal{F}_\xi)$ for $\xi < \omega_2$, we may assume that $\{\mathrm{dom}(f_\xi) \mid \xi < \omega_2\}$ forms a Δ-system with root R and that $f_\xi(\alpha) = f_{\xi'}(\alpha)$ for all $\xi, \xi' < \omega_2$ and all $\alpha \in R$. We may also assume that $\mathrm{dom}(\mathcal{F}_\xi) = \mathrm{dom}(\mathcal{F}_{\xi'})$ for all $\xi, \xi' < \omega_2$. Since $\{\mathrm{dom}(f_\xi) \mid \xi < \omega_2\}$ forms a Δ-system, there is a club $D \subseteq \omega_2$ such that for all $\xi < \xi'$ in D, $\bigcup \mathrm{range}(\mathcal{F}_\xi) \cap \mathrm{dom}(f_{\xi'}) \subseteq R$. In fact we may assume that for all $\xi < \xi'$ in D, $\bigcup \mathrm{range}(\mathcal{F}_\xi) \cap \mathrm{dom}(f_{\xi''}) \subseteq R$ for every $\xi'' < \omega_2$ such that $\xi' \leq \xi''$: Let $\bar{\xi} < \omega_2$ and assume $D \cap \bar{\xi}$ has been defined. We may clearly assume that $D \cap \bar{\xi}$ has a maximum, ξ_0. There must then be a least $\zeta < \omega_2$, $\zeta \geq \bar{\xi}$, such that $\bigcup \mathrm{range}(F_{\xi_0}) \cap \mathrm{dom}(f_{\zeta'}) \subseteq R$ for all $\zeta' \geq \zeta$. Otherwise there are $\aleph_2$-many $\zeta < \omega_2$ such that $\bigcup \mathrm{range}(F_{\xi_0}) \cap \mathrm{dom}(f_\zeta) \nsubseteq R$. But then, since $|\bigcup \mathrm{range}(F_{\xi_0})| \leq \aleph_0$, there must be $\zeta \neq \zeta'$ and some $\alpha \in \mathrm{dom}(f_\zeta) \cap \mathrm{dom}(f_{\zeta'})$ such that $\alpha \in \bigcup \mathrm{range}(F_{\xi_0}) \setminus R$. This is of course impossible since $\mathrm{dom}(f_\zeta) \cap \mathrm{dom}(f_{\zeta'}) = R$. Now we can of course set such a ζ to be the least member of D above ξ_0.

Let ξ^* be the ω_1-th member of D. Since $\bigcup \mathrm{range}(\mathcal{F}_{\xi^*})$ is countable, again by the fact that $\{\mathrm{dom}(f_\xi) \mid \xi < \omega_1\}$ forms a Δ-system and therefore the sets $\mathrm{dom}(f_\xi) \setminus R$ (for $\xi < \omega_1$) are pairwise disjoint, there must be some $\xi^{**} < \xi^*$ such that $\mathrm{dom}(f_{\xi^{**}}) \cap \bigcup \mathrm{range}(\mathcal{F}_{\xi^*}) \subseteq R$. Using Fact 3.2 it follows then immediately that $(f_{\xi^{**}} \cup f_{\xi^*}, \mathcal{F}_{\xi^{**}} \oplus \mathcal{F}_{\xi^*})$ is a condition in $\mathrm{Add}_{\mathbb{B}}(X)$ and that it extends both $(f_{\xi^{**}}, \mathcal{F}_{\xi^{**}})$ and $(f_{\xi^*}, \mathcal{F}_{\xi^*})$. □

Note that Lemma 3.6 is true in ZFC. This is in contrast, for example, with corresponding lemmas in [2] and [3], for which CH is needed.

Next comes the properness lemma. As we will see, the proof of the lemma proceeds quite naturally by induction on the initial segments of X. Most features of the definition of $\mathrm{Add}_{\mathbb{B}}(X)$ are there precisely to make the proof of Lemma 3.7 go through.

Lemma 3.7 $\mathrm{Add}_{\mathbb{B}}(X)$ *is proper.*

Proof Let M be a countable elementary substructure of any large enough $H(\lambda)$ containing X, let $p = (f, \mathcal{F}) \in \mathrm{Add}_{\mathbb{B}}(X) \cap M$, and let $N = M \cap X$. Let f^* be the function with the same domain as f such that $f^*(\alpha) = f(\alpha) \cup \{\langle \delta_M, \delta_M \rangle\}$ for every $\alpha \in \mathrm{dom}(f)$, and let $p^* = (f^*, \mathcal{F} \cup \{\langle \delta_M, N \rangle\})$. For every $N' \in \mathcal{F}$, since $N' \in M$ and N' is countable in M, $\mathrm{ot}(N') < \delta_M$. It follows that p^* is a condition in $\mathrm{Add}_{\mathbb{B}}(X)$. Hence, it suffices to show by induction on the ordinals γ that if $\gamma \in M$, then $p^* \restriction (X \cap \gamma)$ is $(M, \mathrm{Add}_{\mathbb{B}}(X \cap \gamma))$-generic.

For this, we may clearly assume $\gamma > 0$. Let A be any maximal antichain of $\mathrm{Add}_{\mathbb{B}}(X \cap \gamma)$, $A \in M$, and let $p' = (f', \mathcal{F}')$ be any condition extending $p^* \restriction (X \cap \gamma)$ and extending some condition $\tilde{p} = (\tilde{f}, \tilde{\mathcal{F}})$ in A. It suffices to see that $\tilde{p}$ is compatible with a condition in $M \cap A$ as then it will of course follow that $\tilde{p} \in M$.

Suppose $\gamma = \gamma_0 + 1$. We may assume without loss of generality that $\gamma_0 \in \mathrm{dom}(f')$. Let G be $\mathrm{Add}_{\mathbb{B}}(X \cap \gamma_0)$-generic such that $p' \restriction (X \cap \gamma_0) \in G$. In $M[G]$ there is a condition $q = (g, \mathcal{G})$ with the following properties.

(i) q extends a condition in A.
(ii) $q \restriction (X \cap \gamma_0) \in G$.

(iii) $\gamma_0 \in \mathrm{dom}(g)$ and $g(\gamma_0)$ end-extends $f'(\gamma_0) \restriction \delta_M$ (i.e., $f'(\gamma_0) \restriction \delta_M \subseteq g(\gamma_0)$ and, for all $\nu \in \mathrm{dom}(f'(\gamma_0) \restriction \delta_M)$, $g(\gamma_0) \restriction \nu = f'(\gamma_0) \restriction \nu$).
(iv) $\mathrm{ot}(\mathcal{G}(\delta')) < \delta$ for every $\delta \in \mathrm{dom}(\mathcal{F}' \restriction \delta_M)$ and every $\delta' \in \mathrm{dom}(\mathcal{G} \restriction \delta)$.
(v) $\mathrm{ot}(\mathcal{F}'(\delta')) < \delta$ for every $\delta \in \mathrm{dom}(\mathcal{G})$ and every $\delta' \in \mathrm{dom}(\mathcal{F}' \restriction \delta)$.

The existence of such a q is witnessed by p' and can be expressed by a sentence with G, $f'(\gamma_0) \restriction \delta_M$ and $\{\langle \delta, \mathrm{ot}(\mathcal{F}'(\delta))\rangle \mid \delta \in \mathrm{dom}(\mathcal{F}' \restriction \delta_M)\}$ as parameters. Hence there is such a q in $M[G]$ as these parameters are in $M[G] \preccurlyeq H(\lambda)[G]$. Note that $q \in M$ since $M[G] \cap \mathbf{V} = M$ by the induction hypothesis. Since $p' \restriction (X \cap \gamma_0) \in G$, it follows that $p' \restriction (X \cap \gamma_0)$ and $q \restriction (X \cap \gamma_0)$ are compatible as $\mathrm{Add}_{\mathbb{B}}(X \cap \gamma_0)$ conditions. Let $r = (h, \mathcal{H})$ be a lower bound for them. Then

$$\left(h \cup \left\{\langle \gamma_0, f'(\gamma_0) \cup g(\gamma_0)\rangle\right\}, \mathcal{H} \oplus \mathcal{F}' \oplus \mathcal{G}\right)$$

is a common extension of p' and q using Fact 3.2.

Next suppose γ is a nonzero limit ordinal with $\mathrm{cf}(\gamma) \neq \omega_1$. By either the fact that $Z \subseteq M$ for some cofinal subset Z of γ—when $\mathrm{cf}(\gamma) = \omega$—or the fact that there is some $\sigma_0 \in M$ bounding $\mathrm{dom}(h \restriction \gamma)$ for every $(h, \mathcal{H}) \in A$—when $\mathrm{cf}(\gamma) \geq \omega_2$, as $|A| \leq \aleph_1$ by Lemma 3.6, there is some $\sigma \in M \cap \gamma$ bounding $\mathrm{dom}(\tilde{f} \restriction \gamma)$. Let G be $\mathrm{Add}_{\mathbb{B}}(X \cap \sigma)$-generic such that $p' \restriction (X \cap \sigma) \in G$. In $M[G]$ we may then find some $q = (g, \mathcal{G})$ in A with the following properties.

(i) $q \restriction (X \cap \sigma) \in G$.
(ii) $\mathrm{dom}(g) \subseteq \sigma$.
(iii) $\mathrm{ot}(\mathcal{G}(\delta')) < \delta$ for every $\delta \in \mathrm{dom}(\mathcal{F}' \restriction \delta_M)$ and every $\delta' \in \mathrm{dom}(\mathcal{G} \restriction \delta)$.
(iv) $\mathrm{ot}(\mathcal{F}'(\delta')) < \delta$ for every $\delta \in \mathrm{dom}(\mathcal{G})$ and every $\delta' \in \mathrm{dom}(\mathcal{F}' \restriction \delta)$.

As in the previous case, the existence of such a q is witnessed by $\tilde{p}$ and can be expressed by a sentence with parameters in $M[G]$. Also as in the previous case, using that $M[G] \cap \mathbf{V} = M$, which is true by the induction hypothesis, we may assume that q is in fact in M. Again, we may find a common extension $(h, \mathcal{H})$ of $\tilde{p} \restriction (X \cap \sigma)$ of $q \restriction (X \cap \sigma)$. But then, $(h, \mathcal{H} \oplus \mathcal{G} \oplus \mathcal{F}')$ is a common extension of $\tilde{p}$ and q. Note that in the case $\mathrm{cf}(\gamma) \geq \omega_2$ the proof does not produce a common extension of p' and of some condition in $A \cap M$, but only of $\tilde{p}$ and some condition in $A \cap M$. This is of course enough for our purposes.

Finally suppose γ is a nonzero limit ordinal with $\mathrm{cf}(\gamma) = \omega_1$. Let $\sigma \in \gamma \cap M$ be such that $\mathrm{dom}(f' \restriction \mathsf{sup}(M \cap \gamma)) < \sigma$ and let $(\gamma_\nu)_{\nu < \omega_1} \in M$ be a strictly increasing and continuous sequence of ordinals converging to γ. Let D be the set of all conditions in $\mathrm{Add}_{\mathbb{B}}(X \cap \gamma)$ extending some condition in A. Let G be $\mathrm{Add}_{\mathbb{B}}(X \cap \sigma)$-generic with $p' \restriction (X \cap \sigma) \in G$, and let $C \in M[G]$ be the club of $\nu \in \omega_1 = \omega_1^{V[G]}$—where the equality holds by the induction hypothesis—such that for every $\nu' < \nu$ there is some $q = (g, \mathcal{G}) \in D$ such that

(i) $q \restriction (X \cap \sigma) \in G$,
(ii) $\mathrm{dom}(g) \backslash \sigma \subseteq [\gamma_{\nu'}, \gamma_\nu)$,
(iii) $\mathrm{ot}(\mathcal{G}(\delta')) < \delta$ for every $\delta \in \mathrm{dom}(\mathcal{F}' \restriction \delta_M)$ and every $\delta' \in \mathrm{dom}(\mathcal{G} \restriction \delta)$, and
(iv) $\mathrm{ot}(\mathcal{F}'(\delta')) < \delta$ for every $\delta \in \mathrm{dom}(\mathcal{G})$ and every $\delta' \in \mathrm{dom}(\mathcal{F}' \restriction \delta)$.

C, being defined with $\{\langle\delta, \text{ot}(\mathcal{F}'(\delta))\rangle \mid \delta \in \text{dom}(\mathcal{F}' \restriction \delta_M)\}$, $G \in M[G]$ as parameters, is in $M[G]$, and it is clearly closed by definition.

To see that C is unbounded, note that for every $\nu < \omega_1$ there is some $q = (g, \mathcal{G}) \in D$ such that $\text{dom}(g)\backslash\sigma \subseteq [\gamma_\nu, \gamma)$, $q \restriction (X \cap \sigma) \in G$, $\text{ot}(\mathcal{G}(\delta')) < \delta$ for every $\delta \in \text{dom}(\mathcal{F}' \restriction \delta_M)$ and every $\delta' \in \text{dom}(\mathcal{G} \restriction \delta)$, and such that $\text{ot}(\mathcal{F}'(\delta')) < \delta$ for every $\delta \in \text{dom}(\mathcal{G})$ and every $\delta' \in \text{dom}(\mathcal{F}' \restriction \delta)$, as this is witnessed by p'. Now, since $\text{ot}(C \cap \delta_M) = \delta_M$ and $\text{ot}(\mathcal{F}'(\delta)) < \delta_M$ for every $\delta \in \text{dom}(\mathcal{F}' \restriction \delta_M)$, using Fact 3.2 we may find some $\nu \in C \cap \delta_M$ and some $\nu' < \nu$ such that $[\nu', \nu)$ has empty intersection with $\bigcup\{\mathcal{F}'(\delta) \mid \delta \in \text{dom}(\mathcal{F}' \restriction \delta_M)\}$.

Let $q = (g, \mathcal{G}) \in D \cap M[G]$ be such that (i)–(iv) above hold for q with this particular choice of ν and ν', and note that $q \in M$ again since $M[G] \cap \mathbf{V} = M$ by the induction hypothesis.

Let $(\delta_i)_{i<n}$ be an enumeration of $\text{dom}(\mathcal{F}')\backslash\delta_M$ and let g' be the function with domain $\text{dom}(g)\backslash\sigma$ such that $g'(\alpha) = g(\alpha) \cup \{\langle\delta_i, \delta_i\rangle \mid i < n\}$ for all $\alpha \in \text{dom}(g')$. As in the previous two cases we may find a condition $(h, \mathcal{H})$ extending $p' \restriction (X \cap \sigma)$ and $q \restriction (X \cap \sigma)$. Then $(h \cup g', \mathcal{H} \oplus \mathcal{F}' \oplus \mathcal{G})$ is a common extension of p' and q. This completes the proof of the Lemma. □

It would be nice to have that $\text{Add}_{\mathbb{B}}(X)$ has the $\aleph_{1.5}$–c.c. (in the same way that both Cohen forcing and $\text{Add}(\omega, X)$ have the c.c.c.). Unfortunately this does not seem to be the case with the official definition of $\aleph_{1.5}$–c.c. (Definition 2.1). Nevertheless, $\text{Add}_{\mathbb{B}}(X)$ belongs to a slightly bigger class Γ such that a model of the corresponding forcing axiom $\text{FA}(\Gamma)_\lambda$ for every $\lambda < 2^{\aleph_0}$ (and $2^{\aleph_0}$ arbitrarily large) can be built by the same construction as in [3] (replacing of course everywhere in that construction $\aleph_{1.5}$–c.c. by the following slightly more general class). The definition of this relaxed $\aleph_{1.5}$–c.c. is obtained by restricting a bit the collections of finite families $\mathcal{N} \subseteq D$ under consideration. The families we will be considering are what I call here[2] $\text{Add}_{\mathbb{B}}$-friendly.

Definition 3.8 A set $\mathcal{N}$ of countable sets such that δ_N exists for every $N \in \mathcal{N}$ is $\text{Add}_{\mathbb{B}}$*–friendly* if and only if for all $N, N' \in \mathcal{N}$, if $\delta_{N'} < \delta_N$, then $\text{ot}(N' \cap \text{Ord}) < \delta_N$.

The relevant weakening of the $\aleph_{1.5}$–c.c. is the following.

Definition 3.9 A partial order $\mathcal{P}$ has the $\text{Add}_{\mathbb{B}}$–$\aleph_{1.5}$*–c.c.* if for every regular θ such that $\mathcal{P} \in H(\theta)$ there is a club $D \subseteq [H(\theta)]^{\aleph_0}$ such that for every finite $\text{Add}_{\mathbb{B}}$-friendly $\mathcal{N} \subseteq D$ and every $p \in \mathcal{P}$, if $p \in N$ for some $N \in \mathcal{N}$ such that $\delta_N = \min\{\delta_M \mid M \in \mathcal{N}\}$, then there is an extension of p which is $(N, \mathcal{P})$-generic for every $N \in \mathcal{N}$.

Every poset with the $\aleph_{1.5}$–c.c. obviously has the $\text{Add}_{\mathbb{B}}$–$\aleph_{1.5}$–c.c., and every poset with the $\text{Add}_{\mathbb{B}}$–$\aleph_{1.5}$–c.c. is proper and has the $\aleph_2$–c.c.

It is easy to see that the proof of Lemma 3.7 can be adapted to show the following.

Proposition 3.10 *For every set X of ordinals,* $\text{Add}_{\mathbb{B}}(X)$ *has the* $\text{Add}_{\mathbb{B}}$–$\aleph_{1.5}$*–c.c.*

The proof of the following theorem is essentially contained in [3].

[2] For the purpose of this note, only.

Theorem 3.11 (essentially [3]) *(*CH*) Let $\kappa \geq \omega_2$ be a regular cardinal such that $\mu^{\aleph_0} < \kappa$ for all $\mu < \kappa$ and $\diamondsuit(\{\alpha < \kappa \mid \mathrm{cf}(\alpha) \geq \omega_1\})$ holds. Then there is a proper forcing notion $\mathcal{P}$ of size κ with the $\aleph_2$–c.c. such that the following statements hold in the generic extension by $\mathcal{P}$.*

(1) $2^{\aleph_0} = \kappa$
(2) $\mathrm{FA}(\{\mathcal{P} \mid \mathcal{P}$ *has the* $\mathrm{Add}_{\mathbb{B}}$–$\aleph_{1.5}$–*c.c.*$\})_\lambda$ *for every* $\lambda < \kappa$.

3.1 A model of $\mathfrak{b}(\omega_1) = \aleph_2$, $\mathfrak{d}(\omega_1)$ large, and $\neg\,\mathrm{KA}_\lambda$ for large λ

The classical cardinal invariants for the continuum can be naturally extended to $\mathcal{P}(\kappa)$ or ${}^\kappa\kappa$, for higher κ. For example, given a regular cardinal κ one can define $\mathfrak{b}(\kappa)$ as the minimal size of a family $\mathcal{F}$ of functions $f : \kappa \longrightarrow \kappa$ such that no function $g : \kappa \longrightarrow \kappa$ dominates all $f \in \mathcal{F}$ modulo the ideal of bounded sets, which means that no g as above is such that $\{\nu < \kappa \mid g(\nu) \leq f(\nu)\}$ is bounded for all $f \in \mathcal{F}$. With this definition, $\mathfrak{b}(\omega)$ is just the familiar bounding number $\mathfrak{b}$ on the Baire space. Similarly, we can define $\mathfrak{d}(\kappa)$ as the minimal size of a family $\mathcal{F}$ of functions $f : \kappa \longrightarrow \kappa$ such that for every $g : \kappa \longrightarrow \kappa$ there is some $f \in \mathcal{F}$ such that $\{\nu < \kappa \mid f(\nu) \leq g(\nu)\}$ is bounded in κ. $\mathfrak{d}$ is then $\mathfrak{d}(\omega)$. Obviously, $\mathfrak{b}(\kappa) \leq \mathfrak{d}(\kappa)$ holds always for every κ.

Recall that, given an ideal $\mathcal{I}$ on a set X, $\mathbf{non}(\mathcal{I})$ is the least size of a subset of X not in $\mathcal{I}$. An old observation of Rothberger ([12], see also [5], Theorem 2.8) is that $\mathfrak{b} = \mathbf{non}(\mathcal{K}_\sigma)$ and $\mathfrak{d} = \mathbf{cov}(\mathcal{K}_\sigma)$, where $\mathcal{K}_\sigma$ denotes the ideal on the Baire space σ-generated by the compact sets. Similar characterisations can be derived in general for $\mathfrak{b}(\kappa)$ and $\mathfrak{d}(\kappa)$ by the same proof:

Proposition 3.12 *Given an infinite cardinal κ, $\mathfrak{b} = \mathbf{non}(\mathcal{K}_{\kappa,\kappa^+})$ and $\mathfrak{d}(\kappa) = \mathbf{cov}(\mathcal{K}_{\kappa,\kappa^+})$ where, for all cardinals κ, λ, $\mathcal{K}_{\kappa,\lambda}$ denotes the ideal on ${}^\kappa\kappa$ λ-generated by the sets of the form $\{f \in {}^\kappa\kappa \mid f \leq g\}$ (for $g \in {}^\kappa\kappa$).*

Given $f \in {}^{\omega_1}\omega_1$, let C_f be the club of ω_1 whose enumerating function $\tilde{C}_f$ is such that $\tilde{C}_f(0) = f(0)$, $\tilde{C}_f(n+1) = \tilde{C}_f(n) + f(n+1)$ for all $n < \omega$, and such that $\tilde{C}_f(\nu+1) = \tilde{C}_f(\nu) + f(\nu)$ if $\nu \geq \omega$. Clearly, the map Φ sending f to C_f is a bijection between ${}^{\omega_1}\omega_1$ and $\mathcal{C}_{\omega_1}$ and $\{C_f \ : \ f \leq g\}$ is nowhere dense in $\tau_{\mathbb{B}}$ for all $g \in {}^{\omega_1}\omega_1$. In fact, for every $p \in \mathbb{B}$ let $p' \in \mathbb{B}$ extend p and such that $\tilde{C}_g(\nu) = p'(\nu)$ and $\tilde{C}_g(\nu+1) < p'(\nu)$ for some ν. Then no club of ω_1 whose enumerating function extends p' belongs to $\{C_f \ : \ f \leq g\}$. But then, for every κ, $\Phi^{-1}(X) \notin \mathcal{K}_{\omega_1,\kappa}$ if $X \subseteq \mathcal{C}_{\omega_1}$ and $X \notin \mathcal{M}^\kappa_{\mathcal{C}_{\omega_1}}$. From this we immediately obtain the following inequalities (cf. for example [5], Proposition 5.5 for the corresponding $\mathfrak{b} = \mathbf{non}(\mathcal{K}_\sigma) \leq \mathbf{non}(\mathcal{M})$ and $\mathbf{cov}(\mathcal{M}) \leq \mathbf{cov}(\mathcal{K}_\sigma) = \mathfrak{d}$).

Proposition 3.13 $\mathfrak{b}(\omega_1) = \mathbf{non}(\mathcal{K}_{\omega_1,\omega_2}) \leq \mathbf{non}(\mathcal{M}^{\omega_2}_{\mathcal{C}_{\omega_1}})$ *and* $\mathfrak{d} = \mathbf{cov}(\mathcal{K}_{\omega_1,\omega_2}) \geq \mathbf{cov}(\mathcal{M}^{\omega_2}_{\mathcal{C}_{\omega_1}})$.

A set X of reals is called a Luzin set if it is uncountable and has countable intersection with all meagre sets of reals (see [9]). The same definition can be generalised to subsets of $\mathcal{C}_{\omega_1}$: Let us say that $X \subseteq \mathcal{C}_{\omega_1}$ is *Luzin* if and only if it has cardinality at least $\aleph_2$ and has intersection of cardinality at most $\aleph_1$ with every ω_2-meagre subset of $\mathcal{C}_{\omega_1}$. It is a well-known fact that if G is $\mathrm{Add}(\omega_1)$-generic, then $\{r_\alpha^G \mid \alpha < \omega_1\}$ is a Luzin set in the extension (see for example [9]). This follows from the fact that $\mathrm{Add}(\omega_1)$ satisfies the relevant forms of clauses (2) and (4) from the introduction. By essentially the same argument we obtain the following.

Proposition 3.14 *If G is an $\mathrm{Add}_{\mathbb{B}}(\omega_2)$-generic, then $\{\bigcup \mathrm{range}(F_\alpha^G) : \alpha < \omega_2\}$ is a Luzin set of clubs.*

Proof Let $\{D_\xi \mid \xi < \omega_1\}$ be a collection of dense open subsets of $\mathcal{C}_{\omega_1}$. For every ξ let $(p_i^\xi)_{i<\omega_1}$ be a sequence of conditions in $\mathbb{B}$ such that $D_\xi = \bigcup_{i<\omega_1}\{C \in \mathcal{C}_{\omega_1} \mid p_i^\xi \subseteq \tilde{C}\}$. It suffices to show that there is some $\beta < \omega_2$ with the property that for $\alpha > \beta$ and all ξ, $p_i^\xi \subseteq F_\alpha^G$ for some i. But this is true since, by the $\aleph_2$–c.c. of $\mathrm{Add}_{\mathbb{B}}(\omega_2)$, we may fix $\beta < \omega_2$ such that $(p_i^\xi)_{\xi,i<\omega_1} \in \mathbf{V}[G \cap \mathrm{Add}_{\mathbb{B}}(\beta)]$ and since each F_α^G, for $\alpha > \beta$, is $\mathbb{B}$-generic over $\mathbf{V}[G \cap \mathrm{Add}_{\mathbb{B}}(\beta)]$ by Lemma 3.5 and Fact 3.4. □

The main new consistency result in this note is the joint consistency of clauses (1)–(3) in the statement of Theorem 3.15.

Theorem 3.15 *Let $\theta \geq \omega_2$ be a cardinal. Then the following statements hold after forcing with $\mathrm{Add}_{\mathbb{B}}(\theta)$.*

(1) *There is a Luzin subset of $\mathcal{C}_{\omega_1}$ of cardinality $\aleph_2$.*
(2) $\mathfrak{m}(\mathbb{B}) \geq \theta$
(3) $\theta \leq 2^{\aleph_0} \leq 2^{\aleph_1} \leq \theta^{\aleph_1}$

Proof (1) follows immediately from Proposition 3.14 together with Lemma 3.5. (2) follows immediately from Lemma 3.6 together with Proposition 3.14, Lemma 3.5 and Fact 3.4: Given an $\mathrm{Add}_B(\theta)$-generic filter G, $\lambda < \theta$, and a collection $\mathcal{D} = \{D_i \mid i < \lambda\} \in \mathbf{V}[G]$ of dense subsets of $\mathbb{B}$ we may find $X \subseteq \theta$, $|X| = \max\{\lambda, \aleph_1\}$, such that $\mathcal{D} \in \mathbf{V}[G \cap \mathrm{Add}_B(X)]$. But then, if $\alpha \in \theta \backslash X$, F_G^α extends a condition in D_i for every i.

$\mathrm{Add}_{\mathbb{B}}(\theta)$ adds at least θ-many Cohen reals. In fact, let $(C_\delta \mid \delta \in \mathrm{Lim}(\omega_1))$ be a ladder system and, for given $\mathrm{Add}_{\mathbb{B}}(\theta)$-generic filter G and $\alpha < \theta$, let $f_G^\alpha : \omega \longrightarrow \{0, 1\}$ be such that $f_G^\alpha(n) = 0$ if and only if $|[F_G^\alpha(n),\ F_G^\alpha(n+1)) \cap C_{F_G^\alpha(\omega)}|$ is even. Then, a simple density argument shows that f_G^α is a Cohen real over $V[(f_G^{\alpha'} \mid \alpha' < \theta,\ \alpha' \neq \alpha)]$. That $2^{\aleph_1} \leq \theta^{\aleph_1}$ holds in the extension with $\mathrm{Add}_{\mathbb{B}}(\theta)$ follows easily from a simple counting argument of nice names for subsets of ω_1 together with $|\mathrm{Add}_{\mathbb{B}}(\theta)| = \theta^{\aleph_0}$ and the $\aleph_2$–c.c. of $\mathrm{Add}_{\mathbb{B}}(\theta)$.

Corollary 3.16 follows immediately from Theorem 3.15 together with the second part of Proposition 3.13 (for conclusion (1)), Proposition 2.6 and the first part of Proposition 3.13 (for conclusion (2)), and Proposition 2.2 (for conclusion (3)).

Corollary 3.16 *Let $\theta \geq \omega_2$ be a cardinal. Then the following holds after forcing with $\mathrm{Add}_{\mathbb{B}}(\theta)$.*

(1) $\mathfrak{d}(\omega_1) \geq \theta$
(2) $\mathbf{non}(\mathcal{M}^{\omega_2}_{\mathcal{C}_{\omega_1}}) = \aleph_2$. *In particular,* $\mathfrak{b}(\omega_1) = \aleph_2$.

(3) $\neg\,\mathrm{KA}_\lambda$ *for every* $\lambda < \theta$.

As we have seen in the above corollary, $\mathfrak{b}(\omega_1) = \aleph_2$ holds after forcing with $\mathrm{Add}_{\mathbb{B}}(X)$, for any set of ordinals X of order type at least ω_2. The equality $\mathfrak{d}(\omega_1) = \aleph_2$—and therefore also $\mathfrak{b}(\omega_1) = \aleph_2$—follows of course from $2^{\aleph_1} = \aleph_2$ and can be easily forced in other ways too. For example it holds after forcing with any c.c.c. forcing over any model of $\mathfrak{d}(\omega_1) = \aleph_2$, simply because every function $f : \omega_1 \longrightarrow \omega_1$ in the extension is dominated everywhere by a function $g : \omega_1 \longrightarrow \omega_1$ in the ground model. Also, $\mathfrak{b}(\omega_1) = \aleph_2$ holds always after forcing with $\mathrm{Add}(\omega_1, \theta)$, for any $\theta \geq \omega_2$, where $\mathrm{Add}(\omega_1, \theta)$ is the forcing for adding θ-many Cohen subsets of ω_1: We may assume CH in $\mathbf{V}$ since $\mathrm{Add}(\omega_1, \theta) \cong \mathrm{Add}(\omega_1, 1) \times \mathrm{Add}(\omega_1, \theta)$ and since $\mathrm{Add}(\omega_1, 1)$ forces CH and does not change $\mathrm{Add}(\omega_1, \theta)$. But then $\mathrm{Add}(\omega_1, \theta)$ has the $\aleph_2$–c.c., i.e., the relevant form of clause (2) from the introduction holds. This, together with the relevant form of clause (4) for $\mathrm{Add}(\omega_1, \theta)$, shows $\mathfrak{b}(\omega_1) = \aleph_2$ in the extension. Of course, $\diamondsuit$ holds also in this extension. This, by Corollary 3.16, is in stark contrast to what holds after forcing with $\mathrm{Add}_{\mathbb{B}}(\theta)$ for $\theta \geq \omega_2$.

By essentially the same argument as in [13]—showing that $\neg\,\mathrm{CG}$ is preserved after adding Cohen reals—one can prove that $\mathrm{Add}(\omega, \theta)$ preserves $\neg\,\mathrm{CG}_\lambda$. Also, by refining the argument from [13], one can establish the following preservation result:

Lemma 3.17 *For every cardinal* θ*,* $\mathrm{Add}(\omega, \theta)$ *preserves* $\neg\,\mathrm{KA}_\lambda$.

Proof Let X be a set of ordinals and suppose that $\langle \dot{A}_i \mid i < \lambda \rangle$ is a sequence of $\mathrm{Add}(\omega, X)$-names for subsets of ω_1 of order type ω such that some $p \in \mathrm{Add}(\omega, X)$ forces that $\{\dot{A}_i \mid i < \lambda\}$ witnesses KA_λ. By homogeneity we may assume p is the empty condition. The first observation is that for every i there is a countable $Y_i \subseteq X$ such that $\dot{A}_i$ is in fact an $\mathrm{Add}(\omega, Y_i)$-name and such that for every $\alpha < \omega_1$ there is some $p \in \mathrm{Add}(\omega, X)$ such that $p \Vdash_{\mathrm{Add}(\omega, X)} \check{\alpha} \in \dot{A}_i$ if and only if there is some $p \in \mathrm{Add}(\omega, Y_i)$ such that $p \Vdash_{\mathrm{Add}(\omega, Y_i)} \check{\alpha} \in \dot{A}_i$.

For every i let $(p^i_n)_{n<\omega}$ be an enumeration of $\mathrm{Add}(\omega, Y_i)$. Also, for every $n < \omega$, if there is some $\sigma \in \omega_1$ such that $p^i_n \Vdash_{\mathrm{Add}(\omega, Y_i)} \mathsf{sup}(\dot{A}_i) = \sigma$, then let $\mathcal{X}^i_n$ be a set of pairwise compatible $\mathrm{Add}(\omega, Y_i)$-conditions extending p^i_n and such that $\{\xi < \sigma \mid p \Vdash_{\mathrm{Add}(\omega, Y_i)} \xi \in \dot{A}_i$ for some $p \in \mathcal{X}^i_n\}$ is cofinal in σ. The introduction of the $\mathcal{X}^i_n$'s is the new ingredient with respect to the proof in [13].

Given any club $C \subseteq \omega_1$ there are $i < \lambda$, $\sigma < \omega_1$, $\gamma < \sigma$ and $n^C < \omega$ such that $p^i_{n^C}$ forces in $\mathrm{Add}(\omega, Y_i)$ that $\mathsf{sup}(\dot{A}_i) = \sigma$ and that $[\alpha, \alpha') \cap C \neq \emptyset$ for every two consecutive points α, α' of $\dot{A}_i$ above γ. Let $n^* < \omega$ be such that the set $\mathcal{C}$ of clubs C such that $n^C = n^*$ is $\subseteq$-dense in the set of all clubs of ω_1. For every $i < \lambda$, if there is a σ such that $p^i_{n^*} \Vdash_{\mathrm{Add}(\omega, Y_i)} \mathsf{sup}(\dot{A}_i) = \sigma$, then let $B_i = \{\xi < \sigma \mid p \Vdash_{\mathrm{Add}(\omega, Y_i)} \xi \in \dot{A}_i$ for some $p \in \mathcal{X}^i_{n^*}\}$.

By $\subseteq$-density of $\mathcal{C}$ it suffices to show that if $C \in \mathcal{C}$, then there is some $i < \lambda$ such that $p^i_{n^*} \Vdash \mathsf{sup}(\dot{A}_i) = \sigma$ for some σ, and such that $[\beta, \beta') \cap C \neq \emptyset$ for every two consecutive points $\beta < \beta'$ in B_i above some $\gamma < \sigma$. But this is true by the previous paragraph and the definition of B_i since all conditions in $\mathcal{X}^i_{n^*}$ extend $p^i_{n^*}$ and are pairwise compatible. $\square$

It follows from the above lemma that if we start with a model of $\neg\,\mathrm{KA}_{\aleph_1} + 2^{\aleph_1} = \aleph_2$—which can be easily obtained by a countable support iteration of proper forcing or by [3]—and add any amount of Cohen reals to it, we will preserve both $\neg\,\mathrm{KA}_{\aleph_1}$ and $\mathfrak{d}(\omega_1) = \aleph_2$. In terms of obtaining $\mathfrak{b}(\omega_1)$ small together with strong failures of club guessing at ω_1 (and together with $2^{\aleph_1}$ large), this is the best I can do without resorting to Theorem 3.15.

To finish this note, I will mention that I do not have at the moment any use for the homogeneity of $\mathrm{Add}_{\mathbb{B}}(X)$.

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