Automorphism groups of countable algebraically closed graphs and endomorphisms of the random graph

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Abstract

We establish links between countable algebraically closed graphs and the endomorphisms of the countable universal graph R. As a consequence we show that, for any countable graph Γ, there are uncountably many maximal subgroups of the endomorphism monoid of R isomorphic to the automorphism group of Γ. Further structural information about $\operatorname{End} R$ is established including that $\operatorname{Aut} \Gamma$ arises in uncountably many ways as a Schützenberger group. Similar results are proved for the countable universal directed graph and the countable universal bipartite graph.

1 Introduction

Existentially closed relational structures have been widely considered with the example of the *countable universal homogeneous graph* (also known as the *random graph* or the *Rado graph*) probably the most studied (see, for one example of a survey, [2]). It was established by Truss [20] that the automorphism group of the countable universal homogeneous graph is simple and this was placed in a general setting by Macpherson and Tent [15]. The work in the present paper arose when attempting to establish what can be said about other naturally arising groups acting (in some sense) upon the countable universal graph R. To be more precise, we present information about the maximal subgroups of the endomorphism monoid of R. We note that this is not the first work to focus on endomorphisms in the context of homogeneous structures. For example, Cameron and Nešetřil [3] consider homomorphism-homogeneous structures and there are various links between their results and our work, particularly [3, Section 2]. More recently, Lockett and Truss [14] examine generic endomorphisms of homogeneous structures and in their concluding remarks propose that there should be a counterpart to the literature on automorphism groups of such structures applying to the monoids of endomorphisms. This paper may be thought of as part of the study suggested by Lockett and Truss.

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A maximal subgroup of the endomorphism monoid of R is determined by the idempotent endomorphism that plays the role of its identity element. Indeed, it is the $\mathscr{H}$-class of that endomorphism, as we summarise in Section 2 below. Bonato and Delić [1, Proposition 4.2] show that the images of idempotent endomorphisms of the countable universal graph are characterised as being *algebraically closed* (a property weaker than existentially closed) and this is discussed in detail by Dolinka [5, especially Theorem 3.2]. We make the same observation and also the corresponding result for images of idempotent endomorphisms of the countable universal directed graph and countable universal bipartite graph in the course of our work. Indeed, we observe that there are, except for one case, $2^{\aleph_0}$ idempotent endomorphisms with image isomorphic to a given algebraically closed graph, directed graph, or bipartite graph (Theorems 3.6, 4.4 and 5.6, respectively). These observations are, however, merely the first steps in establishing the results herein.

We shall establish the same types of theorem for the classes of graphs (that is, undirected graphs), directed graphs, and bipartite graphs. These classes of relational structure are treated in turn in separate sections below. The proofs for (undirected) graphs are the archetypes and so the sections relating to directed graphs and bipartite graphs are concerned mostly with explaining what modifications are required to establish the analogous results. One needs particular care with bipartite graphs, in the first instance to ensure that the correct definition is chosen so that the class of bipartite graphs does indeed have a Fraïssé limit, as noted in [8]. However, a second wrinkle occurs since there are examples of algebraically closed bipartite graphs that are finite (for example, the complete bipartite graph $K_{m,n}$ on two parts of cardinality m and n respectively), unlike the situation for graphs and directed graphs where algebraically closed structures are necessarily infinite, and this has some surprising consequences for our results (compare Theorems 5.6 and 5.9 with their graph analogues). We therefore need to introduce a stronger condition, that we term *strongly algebraically closed*, in order to establish some of the analogues for the countable universal bipartite graph. These issues are discussed in detail in Section 5. Homogeneous bipartite graphs were, for example, also considered by Goldstern, Grossberg and Kojman [11], but they only permit what we term part-fixing automorphisms whereas our automorphisms will be allowed to interchange the parts.

In the summary of our results that follows, we use the term "any group" to mean a group isomorphic to the automorphism group of a countable graph. The extension of Frucht's Theorem [10] to infinite groups established by de Groot [7] and by Sabidussi [19] tells us this includes every countable group. We note in the course of our work that this class of groups is the same as those arising as the automorphism group of countable directed graphs (Proposition 4.2) and of countable bipartite graphs (Theorem 5.3).

Let $\mathcal{C}$ denote either the class of countable graphs, countable directed graphs, or countable bipartite graphs and let Ω denote the universal homogeneous structure in $\mathcal{C}$. Then

- any group arises in $2^{\aleph_0}$ ways as the automorphism group of an algebraically closed structure in $\mathcal{C}$ (Theorems 3.4, 4.3 and 5.4);

- any group arises in $2^{\aleph_0}$ ways as a maximal subgroup of the endomorphism monoid of Ω (Theorems 3.7, 4.5 and 5.7);

- any group arises in $2^{\aleph_0}$ ways as the Schützenberger group of a non-regular $\mathscr{H}$-class in the endomorphism monoid of Ω (Theorems 3.14, 4.10 and 5.11).

The maximal subgroups of the endomorphism monoid of Ω are the group $\mathscr{H}$-classes of regular $\mathscr{D}$-classes of $\mathrm{End}\,\Omega$ (that is, $\mathscr{H}$-classes that inherit the structure of a group from $\mathrm{End}\,\Omega$). For

general $\mathscr{H}$-classes (including all those in $\mathscr{D}$-classes that are not regular), there is an alternative group that one can use instead. This is the Schützenberger group referred to above (and which we expand upon in Section 2) and generalises the concept of a group $\mathscr{H}$-class (not least because the Schützenberger group is isomorphic to the $\mathscr{H}$-class when the latter happens to be a group).

Theorems 3.7, 4.5 and 5.7 say more about the structure of the endomorphism monoid of Ω, namely every group arises as a group $\mathscr{H}$-class in $2^{\aleph_0}$ many $\mathscr{D}$-classes and, except for one case for the countable universal bipartite graph, every regular $\mathscr{D}$-class contains $2^{\aleph_0}$ group $\mathscr{H}$-classes. From the first of these facts, it follows there are $2^{\aleph_0}$ regular $\mathscr{D}$-classes in $\mathrm{End}\,\Omega$. We also describe how many $\mathscr{L}$- and $\mathscr{R}$-classes there are (usually $2^{\aleph_0}$) in each of these regular $\mathscr{D}$-classes (see Theorems 3.9, 4.6 and 5.9, the latter containing the exceptions and illustrating the surprising behaviour of the countable universal bipartite graph). For non-regular $\mathscr{D}$-classes, we observe in Theorems 3.11, 4.7 and 5.8 that there exist non-regular injective endomorphisms with specified image and whose $\mathscr{D}$-class contains both $2^{\aleph_0}$ many $\mathscr{L}$- and $\mathscr{R}$-classes. By varying the image, we shall deduce there are $2^{\aleph_0}$ non-regular $\mathscr{D}$-classes in the endomorphism monoid of Ω.

A number of questions remain about the endomorphism monoid of each of our universal structures. For example, is it true that every $\mathscr{D}$-class of the endomorphism monoid of the countable universal graph contains $2^{\aleph_0}$ many $\mathscr{L}$- and $\mathscr{R}$-classes? This question has a positive answer for regular $\mathscr{D}$-classes (Theorem 3.9) and some of the non-regular $\mathscr{D}$-classes (by Theorem 3.11). It is unclear whether the latter can be extended to all non-regular $\mathscr{D}$-classes. One reason for the difficulty in making further progress is that we have a necessary condition for endomorphisms to be $\mathscr{D}$-related in terms of the isomorphism class of the images (in Lemma 2.3(iii) below) but only for regular endomorphisms can we reverse the condition to be also sufficient.

One could also consider endomorphisms of the countable universal linearly ordered set (that is, the rationals $\mathbb{Q}$ under $\leqslant$) or the countable universal partially ordered set. Indeed, the third author's PhD thesis [17] contains information, including an analogue of Theorem 3.7, about $\mathrm{End}(\mathbb{Q}, \leqslant)$. The methods are inevitably a little different and this will appear in a subsequent publication.

2 Preliminaries

In this section, we establish the terminology used throughout the paper. We summarise the basic facts about relational structures, including what it means for them to be algebraically closed, and the semigroup theory needed when discussing their endomorphism monoids.

A *relational structure* is a pair $\Gamma = (V, \mathcal{E})$ consisting of a non-empty set V and a sequence $\mathcal{E} = (E_i)_{i \in I}$ of relations on V. In general, one permits the E_i to have arbitrary arity, but as we are principally concerned with (various types of) graphs it will be sufficient to deal only with binary relations. For convenience then we shall make this assumption throughout. When Γ is a graph, we shall then also call V the set of *vertices* of Γ. The definitions of graph, directed graph and bipartite graph with this viewpoint are given at the beginning of Sections 3–5, respectively. A *relational substructure* of Γ is a relational structure $\Delta = (U, \mathcal{D})$, where U is a non-empty subset of V and where $\mathcal{D} = (D_i)_{i \in I}$ satisfies $D_i \subseteq E_i$ for all i. If U is a subset of V, we write $\langle U \rangle$ for the substructure $(U, \mathcal{D})$ where $\mathcal{D} = (D_i)_{i \in I}$ is defined by $D_i = E_i \cap (U \times U)$ for each i. We shall call $\langle U \rangle$ the *relational substructure induced by* U.

If $\Gamma = (V, (E_i)_{i \in I})$ and $\Delta = (W, (F_i)_{i \in I})$ are relational structures (with relations indexed by the same set I), a *homomorphism* $f \colon \Gamma \to \Delta$ is a map $f \colon V \to W$ such that $(uf, vf) \in F_i$

whenever $(u, v) \in E_i$. The map $f \colon V \to W$ then induces $f \colon E_i \to F_i$, for each $i \in I$, and we call the substructure $\operatorname{im} f = (Vf, \mathcal{E}f)$, where $\mathcal{E}f = (E_i f)_{i \in I}$, of Δ the *image* of f. We define the *kernel* of f to be the relation $\{(u, v) \mid uf = vf\}$ on the vertex set V. An *embedding* is an injective homomorphism $f \colon \Gamma \to \Delta$ such that, for each i, $(u, v) \in E_i$ if and only if $(uf, vf) \in F_i$.

In order to describe what it means for a relational structure to be algebraically closed we shall need a little model theory. We refer to Hodges [12] for the basic terminology.

Let L be a signature and $\mathbf{K}$ be a class of L-structures. A structure A in $\mathbf{K}$ is called *algebraically closed* (in $\mathbf{K}$) if given a formula $\Phi(\boldsymbol{x})$ of the form

$$(\exists \boldsymbol{y}) \bigwedge_{i=1}^{k} \Psi_i(\boldsymbol{x}, \boldsymbol{y}), \tag{1}$$

where $k \in \mathbb{N}$ and each Ψ_i is an atomic formula, and a finite sequence $\boldsymbol{a}$ of elements of A such that there exists an extension A' of A with $A' \models \Phi(\boldsymbol{a})$, then already $A \models \Phi(\boldsymbol{a})$. (As an aside, we mention that the formula $\Phi(\boldsymbol{x})$ given in (1) is called a *positive primitive formula*, see, for example, [12, page 50].) In certain cases, this definition of algebraic closure can often be simplified. For example, it is easy to see that a graph Γ is algebraically closed if and only if, given any finite set A of vertices in Γ, there exists some vertex w that is adjacent to every one of the vertices in A. We shall similarly interpret below what algebraically closed means for directed and bipartite graphs in Sections 4 and 5. The concept of an *existentially closed* relational structure is defined similarly but for this we permit each Ψ_i to be an atomic formula or its negation.

Algebraically closed structures for our classes of relational structures can be characterised as follows. Part (i) of this result is [3, Proposition 2.1(a)], which is established by a back-and-forth argument. The proof is easily adjusted to cover directed graphs and bipartite graphs, though one necessarily needs to use the strongly algebraically closed condition for the latter. This condition is defined in Section 5 just before Theorem 5.4 where it is first used.

Proposition 2.1   (i) *Let $\Gamma = (V, E)$ be a countable graph or directed graph. Then Γ is algebraically closed (in the class of graphs or directed graphs, respectively) if and only if there exists $F \subseteq E$ such that (V, F) is existentially closed.*

(ii) *Let $\Gamma = (V, E, P)$ be a countable bipartite graph. Then Γ is strongly algebraically closed if and only if there exists $F \subseteq E$ such that (V, F, P) is existentially closed.*

The classes of finite graphs, of finite directed graphs and of finite bipartite graphs each possess what is known as the hereditary property, the joint embedding property and the amalgamation property. (Indeed, the reason for our particular way of defining the term bipartite graph below is to ensure that the class of such graphs has these properties.) Consequently, each class has a unique Fraïssé limit [9], referred to as the *countable universal homogeneous structure* of the class (see, for example, [12, Theorem 6.1.2]). We shall follow Truss [20] and others and abbreviate the terminology to refer to the *countable universal graph*, the *countable universal directed graph*, and the *countable universal bipartite graph*. Furthermore, these Fraïssé limits are the unique countable existentially closed structures in the classes of graphs, of directed graphs, and of bipartite graphs (see [12, page 185]). The following is now an immediate corollary of Proposition 2.1 and is used in the proofs of Theorems 3.11, 4.7 and 5.8.

Corollary 2.2 *Let Γ be countable and either an algebraically closed graph, algebraically closed directed graph, or strongly algebraically closed bipartite graph. Let Ω be, correspondingly, the countable universal graph, countable universal directed graph, or countable universal bipartite graph. Then there is a homomorphism from Ω into Γ given by a bijection between the vertices.*
$\square$

Since we shall be concerned with maximal subgroups (that is, the group $\mathscr{H}$-classes) of endomorphism monoids, we need to recall Green's relations and their properties. We refer to Howie's monograph [13] for a general background on semigroups.

Let $M = \operatorname{End}\Gamma$ be the endomorphism monoid of a relational structure $\Gamma = (V, \mathcal{E})$. Two elements f and g of M are $\mathscr{L}$-*related* if f and g generate the same left ideal (that is, $Mf = Mg$), while they are $\mathscr{R}$-*related* if $fM = gM$. Green's $\mathscr{H}$-*relation* is the intersection of the binary relations $\mathscr{L}$ and $\mathscr{R}$, while the $\mathscr{D}$-*relation* is their composite $\mathscr{L} \circ \mathscr{R}$ (which can be shown also to be an equivalence relation). Finally, but less central to our work, f and g are $\mathscr{J}$-*related* if $MfM = MgM$. We shall use the notation $f\mathscr{L}g$ to denote that f and g are $\mathscr{L}$-related and similarly for the other relations. If $f \in M$, we write H_f for the $\mathscr{H}$-class of f. If e is an idempotent in M (that is, $e^2 = e$), the $\mathscr{H}$-class H_e is a subgroup of M [13, Corollary 2.2.6] and the maximal subgroups of our monoid M are precisely the $\mathscr{H}$-classes of idempotents of M.

The $\mathscr{L}$-, $\mathscr{R}$- and $\mathscr{D}$-classes in the full transformation monoid $\mathcal{T}_V$, of all maps $V \to V$, are fully described in terms of the images and kernels of the maps involved (see [13, Exercise 2.6.16]). We may view the endomorphism monoid M of $\Gamma = (V, \mathcal{E})$ as a submonoid of $\mathcal{T}_V$ and if f and g are, for example, $\mathscr{L}$-related in $\operatorname{End}\Gamma$, they are certainly $\mathscr{L}$-related in $\mathcal{T}_V$. Consequently, parts (i) and (ii) of the following lemma follow immediately.

Lemma 2.3 *Let f and g be endomorphisms of the relational structure $\Gamma = (V, \mathcal{E})$.*

(i) *If f and g are $\mathscr{L}$-related, then $Vf = Vg$.*

(ii) *If f and g are $\mathscr{R}$-related, then $\ker f = \ker g$.*

(iii) *If f and g are $\mathscr{D}$-related, then the induced substructures $\langle Vf \rangle$ and $\langle Vg \rangle$ are isomorphic.*

PROOF: (iii) Write $\mathcal{E} = (E_i)_{i \in I}$. By assumption, there exists $h \in \operatorname{End}\Gamma$ such that $f\mathscr{R}h$ and $h\mathscr{L}g$. By (i), it follows $Vh = Vg$. As $f\mathscr{R}h$, there exist endomorphisms s and t of Γ with $h = fs$ and $f = ht$. As $f = fst$ and $h = hts$, the map s induces a bijection from Vf to Vh. Moreover, as s and t are endomorphisms, s induces, for each $i \in I$, a bijection from $E_i \cap (Vf \times Vf)$ to $E_i \cap (Vh \times Vh)$ with inverse t. Hence s induces an isomorphism from $\langle Vf \rangle$ to $\langle Vh \rangle = \langle Vg \rangle$. $\square$

An element f of M is called *regular* if there exists $g \in M$ such that $fgf = f$. An idempotent endomorphism e is regular since $e^3 = e$ and if f is regular, then every element in the $\mathscr{D}$-class of f is also regular [13, Proposition 2.3.1]. We refer to such $\mathscr{D}$-classes as *regular $\mathscr{D}$-classes*. We are particularly concerned with idempotent endomorphisms and their $\mathscr{H}$- and $\mathscr{D}$-classes and so the observation in Lemma 2.5 below that the implications in Lemma 2.3 reverse for regular elements is useful.

If f is any endomorphism of $\Gamma = (V, \mathcal{E})$, where $\mathcal{E} = (E_i)_{i \in I}$, then immediately $E_i f \subseteq E_i \cap (Vf \times Vf)$ for all i. On the other hand, if f is regular, say $fgf = f$ for $g \in M$, then gf is idempotent and it is easy to check that $Vgf = Vf$. Hence if $(x, y) \in E_i \cap (Vf \times Vf)$, then $x, y \in Vgf$ and $(x, y) = (x, y)gf \in E_i f$. Consequently $E_i \cap (Vf \times Vf) = E_i f$, which establishes that for regular endomorphisms our two possible definitions of image coincide.

Proposition 2.4 *Let f be a regular endomorphism of the relational structure $\Gamma = (V, \mathcal{E})$. Then the image $\operatorname{im} f = (Vf, \mathcal{E}f)$ and the induced substructue $\langle Vf \rangle$ of Γ are equal.* $\square$

Lemma 2.5 *Let f and g be regular elements in the endomorphism monoid of the relational structure $\Gamma = (V, \mathcal{E})$. Then*

(i) *f and g are $\mathscr{L}$-related if and only if $Vf = Vg$;*

(ii) *f and g are $\mathscr{R}$-related if and only if $\ker f = \ker g$;*

(iii) *f and g are $\mathscr{D}$-related if and only if the images of f and g are isomorphic.*

PROOF: It suffices to establish the "if" versions of each part. For (i) and (ii) this follows immediately since, for example, if $Vf = Vg$, then f and g are $\mathscr{L}$-related in $\mathcal{T}_V$ (by [13, Exercise 2.6.16]) and hence are $\mathscr{L}$-related in $\operatorname{End} \Gamma$ (see [18, Proposition A.1.16]).

(iii) A more general version of this result is Theorem 2.6 in [16], but what we require can be completed easily. If α is an isomorphism from $\operatorname{im} f$ to $\operatorname{im} g$, then $f\alpha$ is an endomorphism of Γ with image $\operatorname{im} g$. As g is regular, there is an idempotent endomorphism e that is $\mathscr{L}$-related to g, by [13, Proposition 2.3.2]. By (i), $Ve = Vg$ and therefore $\operatorname{im} g = \operatorname{im} e$, using Proposition 2.4. Hence the restriction $e|_{\operatorname{im} g}$ is the identity. Then $e\alpha^{-1}$ is an endomorphism of Γ such that $f\alpha \cdot e\alpha^{-1} = f\alpha\alpha^{-1} = f$ and we conclude that $f \mathscr{R} f\alpha$. It follows, by [13, Proposition 2.3.1], that $f\alpha$ is also a regular element and so $f\alpha \mathscr{L} g$ by (i). Hence $f \mathscr{D} g$, as required. $\square$

One might ask what happens in the case that the $\mathscr{H}$-class $H = H_f$, of some endomorphism f in $M = \operatorname{End} \Gamma$, is not a group. In such a case, one can associate to H the *Schützenberger group*, which we shall denote $\mathcal{S}_H$. This consists of the permutations of the $\mathscr{H}$-class H induced by certain elements of M. Specifically, we define $T_H = \{ t \in M \mid Ht \subseteq H \}$ and $\mathcal{S}_H = \{ \gamma_t \mid t \in T_H \}$, where the map $\gamma_t \colon H \to H$ is given by $h \mapsto ht$. It is known (see, for example, [4, Theorem 2.22]) that $\mathcal{S}_H$ is a group and if H is itself a group (for example, when f is an idempotent) then $\mathcal{S}_H \cong H$. Moreover, two $\mathscr{H}$-classes in the same $\mathscr{D}$-class have isomorphic Schützenberger groups (see [4, Theorem 2.25]), which is why we produce distinct $\mathscr{D}$-classes in Theorems 3.14, 4.10 and 5.11. Amongst other things, we shall observe that in our context this group can be expressed as a subgroup of the automorphism group of the image of f.

Proposition 2.6 *Let f be an endomorphism of the relational structure $\Gamma = (V, \mathcal{E})$ and H be the $\mathscr{H}$-class of f in $\operatorname{End} \Gamma$. Then*

(i) *if $t \in T_H$, the restriction of t to the set Vf induces an automorphism of both $\langle Vf \rangle$ and $\operatorname{im} f$;*

(ii) *the mapping $\phi \colon \gamma_t \mapsto t|_{Vf}$ (for $t \in T_H$) defines an injective homomorphism from the Schützenberger group $\mathcal{S}_H$ into $\operatorname{Aut}\langle Vf \rangle \cap \operatorname{Aut}(\operatorname{im} f)$;*

(iii) *if f is an idempotent endomorphism, then $H \cong \operatorname{Aut}(\operatorname{im} f)$ (as groups);*

(iv) *if $f \colon V \to V$ is injective and defines an endomorphism of Γ and g is an automorphism of $\operatorname{im} f$, then fg is $\mathscr{L}$-related to f.*

Part (iii) of this lemma can also be shown directly without reference to the Schützenberger group; see [17, Theorem 2.7]. When Γ is one of the existentially closed structures that we are interested in and f arises in a specific way, we shall extend part (iv) to show that fg is actually $\mathscr{H}$-related to f and hence that the image of ϕ in part (ii) is $\mathrm{Aut}\langle Vf\rangle \cap \mathrm{Aut}(\mathrm{im}\, f)$ (see Propositions 3.13, 4.9 and 5.10 below).

PROOF: (i) If $t \in T_H$, then ft is $\mathscr{H}$-related to f and so $Vft = Vf$ and $\ker ft = \ker f$, by Lemma 2.3. It follows that t induces a bijection on the set Vf and hence an endomorphism of the substructure $\langle Vf\rangle$. Also there exists some endomorphism s such that $fts = f$. Hence if $(uft, vft) \in E_i$ for some relation $E_i \in \mathcal{E}$, then so is $(uf, vf) = (ufts, vfts)$ and we conclude that t induces an automorphism of $\langle Vf\rangle$.

Now ft is, in particular, $\mathscr{L}$-related to f and so there exist endomorphisms g and h of Γ such that $ft = gf$ and $hft = f$. Let $(v_1, v_2) \in E_i f$ for some $E_i \in \mathcal{E}$, so $v_j = u_j f$ for some points $u_j \in V$, for $j = 1, 2$, with $(u_1, u_2) \in E_i$. Then $v_j t = u_j ft = u_j gf$ and we conclude $(v_1 t, v_2 t)$ is the image of (u_1, u_2) under the endomorphism gf and so $(v_1 t, v_2 t) \in E_i f$. Thus t induces an endomorphism of $\mathrm{im}\, f$. In addition, the endomorphism s in the previous paragraph is the inverse of t on the set Vf and, since $fs = hfts = hf$, we similarly conclude s induces an endomorphism of $\mathrm{im}\, f$. Hence t induces an automorphism of the image.

(ii) Note that if $\gamma_s = \gamma_t$ for some $s, t \in T_H$, then in particular $fs = ft$ and so the restrictions of s and t to Vf coincide. Conversely, if these restrictions coincide then $fs = ft$ and we conclude that $hs = ht$ for all h that are $\mathscr{L}$-related to f. Hence $\gamma_s = \gamma_t$. Therefore, using (i), we observe that ϕ is a injective map and it is straightforward to see that it is also a homomorphism.

(iii) Since f is an idempotent endomorphism, $\langle Vf\rangle = \mathrm{im}\, f$, by Proposition 2.4. Then, given an automorphism g of $\mathrm{im}\, f$, note that fg is $\mathscr{H}$-related to f, because $(fg)(fg^{-1}) = f$ since f acts as the identity on its image. We then see that $\gamma_{fg}\phi$ has the same effect on points in Vf as g does. Hence $\phi\colon \mathcal{S} \to \mathrm{Aut}(\mathrm{im}\, f)$ is surjective, as required to establish the isomorphism.

(iv) If g is an automorphism of $\mathrm{im}\, f$, define $h\colon V \to V$ by setting vh to be the unique point satisfying $vhf = vfg$. Since f is injective and g is a bijection on the set Vf, we conclude that h is a permutation of V. As both g and g^{-1} are automorphisms of $\mathrm{im}\, f$, we deduce that h is an automorphism of Γ. Then, from $f = h^{-1}fg$, we conclude that fg and f are $\mathscr{L}$-related. $\qquad\square$

3 Graphs

In this paper, a *graph* will have its usual definition; that is, a relational structure $\Gamma = (V, E)$ where V is the set of vertices and E is an irreflexive symmetric binary relation on V. Thus the term graph refers to an undirected graph without loops or multiple edges. If (u, v) is an edge in E, we then say that the vertices u and v are *adjacent* in Γ. Recall that a graph Γ is *algebraically closed* if for every finite subset A of its vertices, there exists some vertex v such that v is adjacent to every member of A.

In order to establish our results, we introduce a number of constructions. If $\Gamma = (V, E)$ is any graph, we define the *complement* of Γ to be the graph $\Gamma^\dagger$ with vertex set V and edge set $(V \times V) \setminus (E \cup \{(v, v) \mid v \in V\})$. Thus $\Gamma^\dagger$ is the graph containing precisely all the edges that are not present in Γ. We observe immediately:

Lemma 3.1 *Let Γ and Δ be any graphs. Then* (i) $\mathrm{Aut}\,\Gamma^\dagger = \mathrm{Aut}\,\Gamma$; (ii) $\Gamma \cong \Delta$ *if and only if* $\Gamma^\dagger \cong \Delta^\dagger$. $\qquad\square$
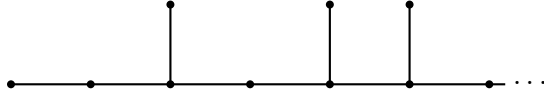
Figure 1: The graph $L_{\{2,4,5,\dots\}}$

Recall that a graph is *locally finite* if every vertex is adjacent to a finite number of vertices. If $\Gamma = (V, E)$ and $\Delta = (W, F)$ are two graphs (where the vertex sets V and W are assumed disjoint), then the *disjoint union* $\Gamma \,\dot\cup\, \Delta$ is the graph with vertex set $V \cup W$ and edge set $E \cup F$. These two concepts may be used to construct an algebraically closed graph as follows:

Lemma 3.2 *Let Γ be any graph and Λ be an infinite locally finite graph. Then $(\Gamma \,\dot\cup\, \Lambda)^\dagger$ is algebraically closed.*

PROOF: Let V and W denote the vertex sets of Γ and Λ respectively and let $\Delta = (\Gamma \,\dot\cup\, \Lambda)^\dagger$. If A is a finite subset of $V \cup W$ then, since Λ is locally finite, there exists some vertex $v \in W$ that is not adjacent in Λ to any vertex in $A \cap W$. Consequently, v is not adjacent in $\Gamma \,\dot\cup\, \Lambda$ to any vertex in A and so by construction v is adjacent in Δ to every vertex in A. $\qquad\square$

We shall use here, and also in later sections, the locally finite graphs L_S defined as follows. Let S be any subset of $\mathbb{N} \setminus \{0, 1\}$. The set of vertices of L_S is $\{\, \ell_n \mid n \in \mathbb{N} \,\} \cup \{\, v_n \mid n \in S \,\}$. For every $n \in \mathbb{N}$, vertex ℓ_n is adjacent to ℓ_{n+1}, while for every $n \in S$, vertex ℓ_n is also adjacent to v_n. See Figure 1 for a diagram of an example of L_S. The following presents the basic information we need about the graphs L_S. If $S \subseteq \mathbb{N} \setminus \{0, 1\}$, then we write $S + k$ for the set $\{\, n + k \mid n \in S \,\}$.

Lemma 3.3 *Let $S, T \subseteq \mathbb{N} \setminus \{0, 1\}$. Then*

(i) $\operatorname{Aut} L_S = \mathbf{1}$;

(ii) *there exists a graph homomorphism $f \colon L_S \to L_T$ defined by an injective map on the sets of vertices if and only if there exists some $k \in \mathbb{N}$ such that $S + k \subseteq T$;*

(iii) $L_S \cong L_T$ *if and only if $S = T$.*

PROOF: (i) By construction, ℓ_0 is the only vertex of degree 1 in L_S that is adjacent to a vertex of degree 2. All other vertices ℓ_n (for $n \geqslant 1$) have degree at least 2. All vertices v_n (for $n \in S$) have degree 1 and are adjacent to vertices ℓ_n of degree 3. It follows that $\operatorname{Aut} L_S = \mathbf{1}$.

(ii) Suppose $f \colon L_S \to L_T$ is a graph homomorphism given by an injective map on the sets of vertices. Then f must map the infinite path $\{(\ell_0, \ell_1), (\ell_1, \ell_2), \dots\}$ in L_S to an infinite path of distinct vertices in L_T. Hence there exists $k \in \mathbb{N}$ such that $\ell_n f = \ell_{n+k}$ for all $n \in \mathbb{N}$. In order that edges of the form (ℓ_n, v_n) in L_S are mapped to edges in L_T, it follows that $S + k \subseteq T$.

Conversely, if $S + k \subseteq T$, then the map $f \colon L_S \to L_T$ given by $\ell_n f = \ell_{n+k}$ for $n \in \mathbb{N}$ and $v_n f = v_{n+k}$ for $n \in S$ is a graph homomorphism.

(iii) follows immediately from (ii). $\qquad\square$

We now establish the first of our main theorems for graphs.

Theorem 3.4 *Let Γ be a countable graph. Then there exist $2^{\aleph_0}$ pairwise non-isomorphic countable algebraically closed graphs whose automorphism group is isomorphic to that of Γ.*

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PROOF: Fix the countable graph Γ. This has at most countably many connected components and so, by Lemma 3.3(iii), there are $2^{\aleph_0}$ choices of subsets S of $\mathbb{N} \setminus \{0,1\}$ such that L_S is isomorphic to no component of Γ. For such a choice of S,

$$\mathrm{Aut}(\Gamma \mathbin{\dot{\cup}} L_S)^\dagger = \mathrm{Aut}(\Gamma \mathbin{\dot{\cup}} L_S) \cong \mathrm{Aut}\,\Gamma \times \mathrm{Aut}\,L_S \cong \mathrm{Aut}\,\Gamma.$$

Hence there are $2^{\aleph_0}$ choices of S such that $\Delta_S = (\Gamma \mathbin{\dot{\cup}} L_S)^\dagger$ has automorphism group isomorphic to that of Γ. Lemma 3.2 tells us that each Δ_S is algebraically closed.

Finally, if S and T are distinct subsets of $\mathbb{N} \setminus \{0,1\}$ such that neither L_S nor L_T are isomorphic to a connected component of Γ, then $L_S \not\cong L_T$ by Lemma 3.3(iii). It then follows that $\Gamma \mathbin{\dot{\cup}} L_S \not\cong \Gamma \mathbin{\dot{\cup}} L_T$ and hence $\Delta_S \not\cong \Delta_T$ by Lemma 3.1(ii). This completes the proof. $\qquad\square$

To establish the required information about images of endomorphisms of the countable universal graph R, we shall remind the reader of a standard way to construct R by building it around any countable graph. For a countable graph $\Gamma = (V, E)$, construct a new graph $\mathcal{G}(\Gamma)$ as follows. Enumerate the finite subsets of V as $(A_i)_{i \in I}$ where $I \subseteq \mathbb{N}$. For each $i \in I$, let v_i be a new vertex. Define $\mathcal{G}(\Gamma)$ to be the graph with vertex set $V \cup \{\, v_i \mid i \in I \,\}$ and edge set

$$E \cup \{\, (v_i, a), (a, v_i) \mid a \in A_i,\ i \in I \,\};$$

thus, we have, for each i, added a new vertex v_i that is adjacent to every vertex in A_i but to no other vertex in $\mathcal{G}(\Gamma)$. Now construct a sequence of graphs Γ_n by defining $\Gamma_0 = \Gamma$ and $\Gamma_{n+1} = \mathcal{G}(\Gamma_n)$ for each $n \geqslant 0$. Since each Γ_n is naturally a subgraph of Γ_{n+1}, we can define Γ_∞ to be the limit of this sequence of graphs. The resulting graph is countable and is, by construction, existentially closed and so isomorphic to the countable universal graph R.

Suppose now that we also have a graph homomorphism $f \colon \Gamma \to \Gamma_\infty$. As before, enumerate the finite subsets of V as $(A_i)_{i \in \mathbb{N}}$. We shall define an extension $\tilde{f} \colon \mathcal{G}(\Gamma) \to \Gamma_\infty$. Indeed, suppose that a graph homomorphism f_n has been defined with domain equal to the subgraph of $\mathcal{G}(\Gamma)$ induced by $V \cup \{v_1, v_2, \ldots, v_n\}$ and such that the restriction of f_n to Γ equals f. As Γ_∞ is in particular algebraically closed, there exists a vertex w within it that is adjacent to every vertex in $(A_{n+1} \cup \{v_1, v_2, \ldots, v_n\})f_n$. We extend to a function f_{n+1} with domain equal to the subgraph of $\mathcal{G}(\Gamma)$ induced by $V \cup \{v_1, v_2, \ldots, v_{n+1}\}$ by defining $v_{n+1}f_{n+1} = w$. The choice of w ensures that f_{n+1} is a graph homomorphism. Note that, if $\mathrm{im}\,f$ was originally an algebraically closed subgraph of Γ_∞, then we could at every stage choose $w_{n+1} \in \mathrm{im}\,f$. Consequently, in this case we can arrange for $\mathrm{im}\,f_n = \mathrm{im}\,f$ for all n.

Since each f_{n+1} extends f_n, we may define $\tilde{f} = \lim_{n \to \infty} f_n = \bigcup_{n=0}^\infty f_n$. Then $\tilde{f}$ is a graph homomorphism $\mathcal{G}(\Gamma) \to \Gamma_\infty$ whose restriction to Γ equals f. Moreover, if $\mathrm{im}\,f$ is algebraically closed, we can arrange that $\mathrm{im}\,\tilde{f} = \mathrm{im}\,f$. We use this contruction to establish the first two parts of the following result (a variant of which appears in a more general setting as [6, Theorem 4.1]):

Lemma 3.5 *Let Γ be a countable graph, let Γ_∞ be the copy of the countable universal graph constructed around Γ as described above, and let $f \colon \Gamma \to \Gamma_\infty$ be a graph homomorphism.*

(i) *There exist $2^{\aleph_0}$ endomorphisms $\hat{f} \colon \Gamma_\infty \to \Gamma_\infty$ such that the restriction of $\hat{f}$ to Γ equals f.*

(ii) *If $\mathrm{im}\,f$ is algebraically closed, then there are $2^{\aleph_0}$ such extensions $\hat{f}$ of f with $\mathrm{im}\,\hat{f} = \mathrm{im}\,f$.*

(iii) *If f is an automorphism of Γ, then there is an automorphism $\hat{f}$ of Γ_∞ such that the restriction of $\hat{f}$ to Γ equals f.*

PROOF: (i), (ii): Our recipe above describes how to extend a graph homomorphism $f\colon \Gamma \to \Gamma_\infty$ to $\tilde{f}\colon \mathcal{G}(\Gamma) \to \Gamma_\infty$. As Γ_∞ is defined as the limit of the sequence given by $\Gamma_0 = \Gamma$, $\Gamma_{n+1} = \mathcal{G}(\Gamma_n)$, repeated use of this recipe constructs one example of the required endomorphism $\hat{f}$. Now observe that algebraic closure ensures there are infinitely many suitable vertices w adjacent to every vertex in $(A_{n+1} \cup \{v_1, v_2, \ldots, v_n\})f_n$. This freedom at each stage ensures that there are uncountably many possible extensions $\hat{f}$.

(iii) This is achieved by a variant construction. The automorphism f induces a permutation of the finite subsets A_i. If $A_i f = A_j$, then define the extension $\tilde{f}\colon \mathcal{G}(\Gamma) \to \mathcal{G}(\Gamma)$ by setting $v_i \tilde{f} = v_j$. This is an automorphism of $\mathcal{G}(\Gamma)$ and repeated use of this construction yields the required extension to Γ_∞. $\qquad\square$

We can now observe that images of idempotent endomorphisms of the countable universal graph R are characterized by being algebraically closed. This was established by Bonato and Delić [1, Proposition 4.2] and is part (i) of the following theorem. However, our proof shows there are in fact uncountably many idempotents with specified (algebraically closed) image.

Theorem 3.6 *Let Γ be a countable graph. Then*

(i) *there exists an idempotent endomorphism f of the countable universal graph R such that $\operatorname{im} f \cong \Gamma$ if and only if Γ is algebraically closed;*

(ii) *if Γ is algebraically closed, there are $2^{\aleph_0}$ idempotent endomorphisms f of R such that $\operatorname{im} f \cong \Gamma$.*

PROOF: An existentially closed graph is certainly also algebraically closed and it is easy to see that this latter property is inherited by images of an endomorphism f.

Conversely, if Γ is algebraically closed, let f be one of the extensions to Γ_∞, given by Lemma 3.5(ii), of the identity map $\Gamma \to \Gamma$. We identify Γ_∞ with R. Then the restriction of f to $\operatorname{im} f = \Gamma$ is the identity and so f is an idempotent endomorphism of R with image isomorphic to Γ. Moreover, as observed in Lemma 3.5, there are actually $2^{\aleph_0}$ many such idempotent endomorphisms. Consequently, part (ii) also follows. $\qquad\square$

We may now establish that any suitable group arises in $2^{\aleph_0}$ ways as a maximal subgroup of $\operatorname{End} R$.

Theorem 3.7 *Let R denote the countable universal graph.*

(i) *Let Γ be a countable graph. Then there exist $2^{\aleph_0}$ distinct regular $\mathscr{D}$-classes of $\operatorname{End} R$ whose group $\mathscr{H}$-classes are isomorphic to $\operatorname{Aut} \Gamma$.*

(ii) *Every regular $\mathscr{D}$-class of $\operatorname{End} R$ contains $2^{\aleph_0}$ distinct group $\mathscr{H}$-classes.*

PROOF: (i) By Theorem 3.4, there are $2^{\aleph_0}$ pairwise non-isomorphic countable algebraically closed graphs with automorphism group isomorphic to that of Γ. For each such graph Δ, there is an idempotent endomorphism f_Δ of R with $\operatorname{im} f_\Delta \cong \Delta$ by Theorem 3.6(i). The idempotents f_Δ belong to distinct $\mathscr{D}$-classes, by Lemma 2.5(iii). By Proposition 2.6(iii), the corresponding group $\mathscr{H}$-class satisfies $H_{f_\Delta} \cong \operatorname{Aut} \Delta \cong \operatorname{Aut} \Gamma$. This establishes part (i).

(ii) Let D_f be a regular $\mathscr{D}$-class in $\operatorname{End} R$ with f an idempotent endomorphism belonging to this class. Let $\Gamma = \operatorname{im} f$. Then by Theorem 3.6, there exist $2^{\aleph_0}$ idempotent endomorphisms of R with image isomorphic to Γ. Each such endomorphism is $\mathscr{D}$-related to f by Lemma 2.5(iii) but lies in a distinct $\mathscr{H}$-class by parts (i) and (ii) of that lemma. $\qquad\square$

We now turn to the $\mathscr{L}$- and $\mathscr{R}$-classes in the endomorphism monoid of R. We use the graph $\Gamma^\sharp$ constructed from a countably infinite graph Γ by, loosely speaking, replacing every edge in Γ by a copy of the complete bipartite graph $K_{2,2}$. More precisely, if $\Gamma = (V, E)$ and $V = \{\, v_i \mid i \in \mathbb{N} \,\}$, then define $\Gamma^\sharp = (V^\sharp, E^\sharp)$ where

$$V^\sharp = \{\, v_{i,r} \mid i \in \mathbb{N}, \ r \in \{0,1\} \,\}$$

and

$$E^\sharp = \{\, (v_{i,r}, v_{j,s}) \mid (v_i, v_j) \in E, \ r,s \in \{0,1\} \,\}.$$

The following observations are straightforward.

Lemma 3.8 *Let Γ be any countably infinite graph.*

(i) *If Γ is algebraically closed, then so is $\Gamma^\sharp$.*

(ii) *For any sequence $(b_i)_{i \in \mathbb{N}}$ with $b_i \in \{0,1\}$ for all i, the subgraph of $\Gamma^\sharp$ induced by the vertices $\{\, v_{i,b_i} \mid i \in \mathbb{N} \,\}$ is isomorphic to Γ.* $\qquad\square$

Theorem 3.9 *Every regular $\mathscr{D}$-class of the endomorphism monoid of the countable universal graph R contains $2^{\aleph_0}$ many $\mathscr{L}$- and $\mathscr{R}$-classes.*

PROOF: Fix an idempotent endomorphism f of R and let Γ be the image of f, which is algebraically closed by Theorem 3.6. First assume that R is constructed as Γ_∞ as described above by taking $\Gamma_0 = \Gamma$. Lemma 3.5(ii) tells us that there $2^{\aleph_0}$ extensions to R of the identity map on Γ with the same image and all such extensions of $\mathscr{D}$-related to f by Lemma 2.5(iii). However, as idempotents with the same image, they have distinct kernels and so are not $\mathscr{R}$-related.

On the other hand, we may start with the same graph Γ, form $\Gamma^\sharp$ as described above and then construct $\Gamma_\infty \cong R$ now taking $\Gamma_0 = \Gamma^\sharp$. As $\Gamma^\sharp$ is also algebraically closed, we can extend the identity map on $\Gamma^\sharp$ to an idempotent endomorphism g of R with image equal to $\Gamma^\sharp$. Now let $\mathbf{b} = (b_i)_{i \in I}$ be an arbitrary sequence with $b_i \in \{0,1\}$ for each i and define $\phi_{\mathbf{b}} \colon \Gamma^\sharp \to \Gamma^\sharp$ by

$$v_{i,r}\phi_{\mathbf{b}} = v_{i,b_i}.$$

Then $\phi_{\mathbf{b}}$ is an endomorphism of $\Gamma^\sharp$ with image equal to the subgraph $\Lambda_{\mathbf{b}} \cong \Gamma$ induced by the set of vertices $\{\, v_{i,b_i} \mid i \in \mathbb{N} \,\}$. Note that $g\phi_{\mathbf{b}}$ is an idempotent endomorphism of R with image equal to $\Lambda_{\mathbf{b}}$ and hence is $\mathscr{D}$-related to f by Lemma 2.5(iii). As we permit $\mathbf{b}$ to vary, we produce endomorphisms that are not $\mathscr{L}$-related, by Lemma 2.5(i), since if $\mathbf{b} \neq \mathbf{c}$ then $\Lambda_{\mathbf{b}} \neq \Lambda_{\mathbf{c}}$.

This establishes that the $\mathscr{D}$-class of f has both $2^{\aleph_0}$ many $\mathscr{L}$- and $\mathscr{R}$-classes within it. $\qquad\square$

Comment 3.10 It is possible to construct a collection $\mathcal{P}$ of $2^{\aleph_0}$ subsets of $\mathbb{N}$ such that for all distinct pairs $S, T \in \mathcal{P}$ and all positive integers k, the translate $S + k$ is not contained in T. Consequently, the graph L_S cannot be embedded in any L_T for $S, T \in \mathcal{P}$. Let us write f_S for an idempotent with image isomorphic to the graph Δ_S as defined in the proof of Theorem 3.4. It then follows that $\mathrm{im}\, f_S$ cannot be embedded in $\mathrm{im}\, f_T$ for distinct $S, T \in \mathcal{P}$ and this is sufficient to establish that $\mathrm{End}\, R$ has $2^{\aleph_0}$ many $\mathscr{J}$-classes. See [17, Theorem 3.32] for more details.

Theorem 3.11 *Let Γ be any countable algebraically closed graph that is not isomorphic to the countable universal graph R. Then there exists a non-regular injective endomorphism f of R such that the subgraph induced by the images of the vertices under f is isomorphic to Γ and such that the $\mathscr{D}$-class of f contains $2^{\aleph_0}$ many $\mathscr{R}$- and $\mathscr{L}$-classes.*

PROOF: From the graph Γ, first build $\Gamma^\sharp$ as above and take $\Gamma_0 = \Gamma^\sharp$ when building $\Gamma_\infty \cong R$ as described earlier. We may thus assume that the countable universal graph R contains amongst its vertices the $v_{i,r}$. Write V for the set of vertices of R. Let Λ_0 denote the subgraph of $\Gamma^\sharp$ induced by the set of vertices $V_0 = \{\, v_{i,0} \mid i \in \mathbb{N} \,\}$. Then, by Lemma 3.8(ii), $\Lambda_0 \cong \Gamma \not\cong R$. There is therefore, by Corollary 2.2, a bijection $V \to V_0$ defining a graph homomorphism $f \colon R \to \Lambda_0$. By construction, $\langle Vf \rangle = \langle V_0 \rangle \cong \Gamma$. We shall view f as an endomorphism of R via the constructed embedding of Λ_0 in R. Since $\Lambda_0 \not\cong R$, there must exist a pair of vertices u and v in R that are not adjacent but such that (uf, vf) is an edge in Λ_0. Consequently, f is not regular by Proposition 2.4.

A variant of the argument used in Theorem 3.9 shows that the $\mathscr{R}$-class of f contains $2^{\aleph_0}$ many $\mathscr{H}$-classes, i.e., that it intersects $2^{\aleph_0}$ $\mathscr{L}$-classes in the $\mathscr{D}$-class of f. Let $\mathbf{b} = (b_i)_{i \in \mathbb{N}}$ be an arbitrary sequence with $b_i \in \{0,1\}$ for each i and define $\psi_{\mathbf{b}} \colon \Gamma^\sharp \to \Gamma^\sharp$ by

$$v_{i,j}\psi_{\mathbf{b}} = v_{i,j+b_i}$$

(where, in the subscript, we perform addition in $\{0,1\}$ modulo 2). It follows from the definition of $\Gamma^\sharp$ that $\psi_{\mathbf{b}}$ is an automorphism of this graph. By Lemma 3.5(iii), $\psi_{\mathbf{b}}$ can be extended to an automorphism $\hat{\psi}_{\mathbf{b}}$ of R. Certainly $f\hat{\psi}_{\mathbf{b}}$ is $\mathscr{R}$-related to f. Now $Vf\hat{\psi}_{\mathbf{b}} = V_0\psi_{\mathbf{b}}$ and so, using Lemma 2.3(i), $f\hat{\psi}_{\mathbf{b}}$ and $f\hat{\psi}_{\mathbf{c}}$ are not $\mathscr{L}$-related if $\mathbf{b}$ and $\mathbf{c}$ are different sequences, since $V_0\psi_{\mathbf{b}} \neq V_0\psi_{\mathbf{c}}$. It follows that the $\mathscr{R}$-class of f contains $2^{\aleph_0}$ non-$\mathscr{L}$-related elements, as required.

Now let Δ denote the graph that is the disjoint union of a copy R' of the countable universal graph and the empty graph E (i.e., with no edges) on a countably infinite set of vertices. By taking $\Gamma_0 = \Delta$ in the initial step of our standard construction, we may assume that $\Delta = R' \mathbin{\dot\cup} E$ occurs as a subgraph of R. Let $g \colon R' \to R$ be a fixed isomorphism and let $h \colon E \to R$ be any map. Then using Lemma 3.5(i) we find an endomorphism $\xi_h \colon R \to R$ that simultaneously extends both g and h. We continue to use the endomorphism f constructed above. Note that $\xi_h f$ and f are $\mathscr{L}$-related, for any choice of h, since $g^{-1}\xi_h f = f$ (where by g^{-1} we mean the endomorphism of R corresponding to the inverse $R \to R'$ of g).

Observe $\ker \xi_h f = \ker \xi_h$ since f is injective. Therefore, if $h, k \colon E \to R$ are chosen with $\ker h \neq \ker k$, then $\xi_h f$ and $\xi_k f$ are not $\mathscr{R}$-related, by Lemma 2.3(ii). As there are $2^{\aleph_0}$ possible kernels for the map h, we conclude the $\mathscr{D}$-class of f indeed contains $2^{\aleph_0}$ many $\mathscr{R}$-classes. $\qquad\square$

Corollary 3.12 There are $2^{\aleph_0}$ non-regular $\mathscr{D}$-classes in $\operatorname{End} R$.

PROOF: By Theorem 3.4 there are $2^{\aleph_0}$ isomorphism types of countable algebraically closed graphs. By the previous theorem, for each such graph Γ, with $\Gamma \not\cong R$, there is a non-regular injective endomorphism f with $\langle Vf \rangle \cong \Gamma$ and each Γ determines a distinct $\mathscr{D}$-class of f by Lemma 2.3(iii). $\qquad\square$

Finally, we turn to the Schützenberger groups of $\mathscr{H}$-classes of non-regular endomorphisms. As mentioned in Section 2, for specific injective endomorphisms of the countable universal graph we are able to make Proposition 2.6 more precise.

Let $\Gamma_0 = (V_0, E_0)$ be a countable algebraically closed graph. Then, by Proposition 2.1(i), there exists some $F_0 \subseteq E_0$ such that (V_0, F_0) is isomorphic to the countable universal graph R. Use Γ_0 in the initial step of the construction of R. Hence we can assume that $R = (V, E)$ contains $\Gamma_0 = (V_0, E_0)$ as a subgraph. Let $f \colon R \to R$ be the endomorphism that realises the isomorphism $(V_0, F_0) \cong R$; that is, f is given by a bijection from V to V_0 and from E to F_0.

Consider a bijection $g\colon V_0 \to V_0$ such that g is an automorphism both of $\operatorname{im} f = (V_0, F_0)$ and of $\langle V f \rangle = \langle V_0 \rangle = (V_0, E_0)$. By Proposition 2.6(iv), fg is $\mathscr{L}$-related to f. However, since g is an automorphism of Γ_0, we can, by Lemma 3.5(iii), extend it to an automorphism $\hat{g}$ of R. Then we observe that fg and f are also $\mathscr{R}$-related since $fg = f\hat{g}$ and $(fg)\hat{g}^{-1} = f$. We can now similarly establish that if h is an element of the $\mathscr{H}$-class H of f, then $h\hat{g}$ is also $\mathscr{H}$-related to h. Hence $\hat{g} \in T_H$ (in the notation introduced in Section 2). Now returning to Proposition 2.6 we see that

$$\gamma_{\hat{g}} \phi = \hat{g}|_{\langle V_0 \rangle} = g.$$

Hence we conclude that the image of ϕ is $\operatorname{Aut}(V_0, F_0) \cap \operatorname{Aut}(V_0, E_0)$; that is:

Proposition 3.13 *Let f be an injective endomorphism of the countable universal graph R of the form specified above and let $H = H_f$. Then $\mathcal{S}_H \cong \operatorname{Aut}\langle V f \rangle \cap \operatorname{Aut}(\operatorname{im} f)$.* $\qquad\square$

To construct $\mathscr{H}$-classes with Schützenberger group isomorphic to a particular group, we need to specify the particular graph to select as Γ_0 in the above argument. We shall again make use of the graphs L_S, for $S \subseteq \mathbb{N} \setminus \{0, 1\}$, defined earlier. For such a subset S, define M_S to be the graph whose vertices are those of L_S together with new vertices x_n (for $n \in \mathbb{N}$) and whose edges are those of L_S together with additional edges

$$\{\, (y, x_n), (x_n, y), (x_m, x_n), (x_n, x_m) \mid y \in V(L_S),\ m, n \in \mathbb{N},\ m \neq n \,\}.$$

Note that the x_n are joined to every other vertex in M_S, while no other vertex has this property. It follows that any automorphism of M_S must induce an automorphism of L_S and permute the vertices x_n. As $\operatorname{Aut} L_S = \mathbf{1}$, we conclude $\operatorname{Aut} M_S$ is isomorphic to the symmetric group on a countably infinite set. Similarly, using Lemma 3.3(iii), if S and T are subsets of $\mathbb{N} \setminus \{0, 1\}$ then $M_S \cong M_T$ if and only if $S = T$.

Now let Γ be an arbitrary countable graph and let S_n, for $n \in \mathbb{N}$, be a sequence of distinct subsets of $\mathbb{N} \setminus \{0, 1\}$ such that the graph M_{S_n} is not isomorphic to any connected component of Γ. We perform the following construction: Define $\Gamma_0^* = \Gamma^\dagger$ (the complement of Γ, as previously). Then, assuming that Γ_n^* has been defined, enumerate the finite subsets of vertices of Γ_n^* as $(A_i)_{i \in \mathbb{N}}$. Let the vertices of Γ_{n+1}^* be the union of the vertices of Γ_n^*, the vertices of L_{S_n} and new vertices $\{\, x_i^{(n)} \mid i \in \mathbb{N} \,\}$. Define the edges of Γ_{n+1}^* to be the edges of Γ_n^* together with edges between a and $x_i^{(n)}$ for all $a \in A_i$ and all i. Having constructed the graphs Γ_n^*, we let $\Gamma^* = (V^*, E^*)$ be the limit of this sequence of graphs. By construction, Γ^* is existentially closed and therefore isomorphic to the countable universal graph R.

Now let $\Gamma_0 = (V^*, E_0)$ be the graph whose edges are all possible edges between pairs of vertices except the following are *not* included:

 (i) the edges in Γ;

 (ii) for each $n \in \mathbb{N}$, all edges between distinct vertices of $\{\, x_i^{(n)} \mid i \in \mathbb{N} \,\}$;

 (iii) for each $n \in \mathbb{N}$, the edges in L_{S_n};

 (iv) for each $n \in \mathbb{N}$, all edges between a vertex in L_{S_n} and a vertex $x_i^{(n)}$.

Note then that $E^* \subseteq E_0$. Therefore Γ_0 is algebraically closed and it is this graph that we use in the argument employed above to establish Proposition 3.13. Let $f\colon R \to R$ be the endomorphism given by an injective map on the set V of vertices of R and whose image is (V^*, E^*). Note

that f is necessarily not regular by Proposition 2.4, since $\operatorname{im} f = (V^*, E^*) \neq \langle Vf \rangle = (V^*, E_0)$. Then the Schützenberger group of the $\mathscr{H}$-class of f is as specified by Proposition 3.13, namely $\mathcal{S}_{H_f} \cong \operatorname{Aut}(V^*, E_0) \cap \operatorname{Aut}(V^*, E^*)$.

To apply this result, we first determine the automorphism group of Γ_0. Note that $\Gamma_0^\dagger$ is the disjoint union of the graphs Γ and M_{S_n} for $n \in \mathbb{N}$. Hence

$$\operatorname{Aut} \Gamma_0 \cong \operatorname{Aut} \Gamma_0^\dagger \cong \operatorname{Aut} \Gamma \times \prod_{n \in \mathbb{N}} \operatorname{Aut} M_{S_n} \cong \operatorname{Aut} \Gamma \times (\operatorname{Sym} \mathbb{N})^{\aleph_0}$$

by our earlier observations. It follows that if g is a bijection $V \to V$ that is simultaneously an automorphism of both (V^*, E_0) and (V^*, E^*), then g induces an automorphism of Γ, fixes all vertices of L_{S_n} (for all $n \in \mathbb{N}$) and, for each $n \in \mathbb{N}$, permutes the vertices in $\{\, x_i^{(n)} \mid i \in \mathbb{N} \,\}$. However, given an automorphism of Γ, there is precisely one choice for these permutations that defines an automorphism of (V^*, E^*) since, at each stage n, the vertex $x_i^{(n)}$ must be mapped to the vertex adjoined to the finite set $A_i g$. We conclude that mapping g to its restriction to the vertices of Γ yields an isomorphism from $\operatorname{Aut}(V^*, E_0) \cap \operatorname{Aut}(V^*, E^*)$ to $\operatorname{Aut} \Gamma$.

Finally, note that this method also constructs for us $2^{\aleph_0}$ many $\mathscr{D}$-classes where the Schützenberger group is isomorphic to the automorphism group of Γ. First fix the subsets S_n for $n \geqslant 2$ as above. There remain $2^{\aleph_0}$ possible choices now for S_1 in order to follow the above construction. Each such S_1 determines an (injective) non-regular endomorphism $f = f_{S_1}$ of R with $\mathcal{S}_{H_f} \cong \operatorname{Aut} \Gamma$. Moreover, since $\langle Vf \rangle^\dagger = \Gamma_0^\dagger$ is the disjoint union of Γ and the M_{S_n}, when $S_1 \neq S_1'$ there can exist no isomorphism from $\langle Vf_{S_1} \rangle$ to $\langle Vf_{S_1'} \rangle$ since $M_{S_1} \not\cong M_{S_1'}$. Hence f_{S_1} and $f_{S_1'}$ belong to distinct $\mathscr{D}$-classes if $S_1 \neq S_1'$ by Lemma 2.3(iii).

We have now established our final result about the endomorphism monoid of R:

Theorem 3.14 *Let Γ be any countable graph. There are $2^{\aleph_0}$ non-regular $\mathscr{D}$-classes of the countable universal graph R such that the Schützenberger groups of $\mathscr{H}$-classes within them are isomorphic to* $\operatorname{Aut} \Gamma$. $\qquad\square$

4 Directed graphs

A *directed graph* is a relational structure $\Gamma = (V, E)$ where E is an irreflexive binary relation on V. This ensures that every graph is, in particular, a directed graph. We can therefore use all the graphs constructed in Section 3 but now viewed as directed graphs. Consequently, our methods in this section are almost identical to those for undirected graphs and we therefore omit most of the details. Furthermore, the class of groups arising as the automorphism group of a graph are amongst those that arise as the automorphism group of a directed graph. Our first step is to show that these two classes are actually the same.

Let $\Gamma = (V, E)$ be a countable directed graph. Enumerate the vertices of Γ as $V = \{\, v_i \mid i \in I \,\}$ (where $I \subseteq \mathbb{N}$). We construct an (undirected) graph $\Gamma^\dashv = (V', E')$ as follows. Set

$$V' = V \cup \{\, x_{jk}, y_{jk}, z_{jk} \mid (v_j, v_k) \in E \,\}$$

and define the (undirected) edges in $\Gamma^\dashv$ to be all (v_j, x_{jk}), (x_{jk}, y_{jk}), (y_{jk}, z_{jk}) and (y_{jk}, v_k) for all $(v_j, v_k) \in E$. This has the effect of replacing the "arrow" from v_j to v_k in Γ by the shape shown in Figure 2.

For our directed graph $\Gamma = (V, E)$, let us also define, for $v \in V$,

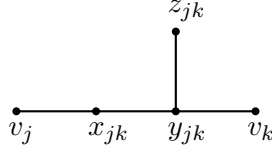$$\Gamma_+(v) = \{\, x \in V \mid (v, x) \in E \,\} \qquad \text{and} \qquad \Gamma_-(v) = \{\, x \in V \mid (x, v) \in E \,\}.$$

Figure 2: Replacement edges in $\Gamma^{\dashv}$

Figure 3: $L_{\{2,4,5,\dots\}}$ as a directed graph

Lemma 4.1 *Let $\Gamma = (V, E)$ be a directed graph. Suppose that $|\Gamma_+(v)| + |\Gamma_-(v)| > 3$ for all $v \in V$. Then $\operatorname{Aut}\Gamma^{\dashv} \cong \operatorname{Aut}\Gamma$.*

PROOF: We observe that, by construction, in $\Gamma^{\dashv}$, every vertex in $X = \{\, x_{jk} \mid (v_j, v_k) \in E \,\}$ has degree 2, every vertex in $Y = \{\, y_{jk} \mid (v_j, v_k) \in E \,\}$ has degree 3, every vertex in $Z = \{\, z_{jk} \mid (v_j, v_k) \in E \,\}$ has degree 1 and (by assumption) every v_i has degree greater than 3. Consequently, if $f \in \operatorname{Aut}\Gamma^{\dashv}$ then $Xf = X$, $Yf = Y$, $Zf = Z$ and $Vf = V$.

Let us define a map $\phi \colon \operatorname{Aut}\Gamma^{\dashv} \to \operatorname{Aut}\Gamma$ by $f\phi = f|_V$ for all $f \in \operatorname{Aut}\Gamma^{\dashv}$. For such an automorphism f, from the above observation, $f\phi$ defines a bijection $V \to V$. If (u, v) is a (directed) edge in Γ, then there exists $x \in X$ and $y \in Y$ such that (u, x), (x, y) and (y, v) are (undirected) edges in $\Gamma^{\dashv}$. Then (uf, xf), (xf, yf) and (yf, vf) are edges in $\Gamma^{\dashv}$ and necessarily $xf \in X$ and $yf \in Y$. It follows that (uf, vf) must be an edge in Γ. Similarly, if $(u, v) \notin E$, then $(uf, vf) \notin E$. Hence $f\phi$ is indeed a graph automorphism of Γ.

It is straightforward to see that ϕ is a homomorphism and, since the images of x_{jk}, y_{jk} and z_{jk} (for $(v_j, v_k) \in E$) under f are completely determined by $v_j f$ and $v_k f$, it follows that ϕ is injective. Finally, if $h \in \operatorname{Aut}\Gamma$, define $v_i f = v_i h$ for all i and if $(v_j, v_k) \in E$ with $v_j h = v_\ell$ and $v_k h = v_m$ define $x_{jk}f = x_{\ell m}$, $y_{jk}f = y_{\ell m}$ and $z_{jk}f = z_{\ell m}$. This defines $f \in \operatorname{Aut}\Gamma^{\dashv}$ with the property that $f\phi = f|_V = h$. Consequently, ϕ is an isomorphism as required. $\qquad\square$

In addition to the above lemma, the other tools we require are the constructions used in Section 3. If S is a subset of $\mathbb{N} \setminus \{0, 1\}$, we define the graph L_S as earlier, but we now view it as a directed graph. So, for any pair of vertices u and v joined in L_S, there is both an edge from u to v and from v to u (see Figure 3). The disjoint union of two directed graphs and the complement $\Gamma^{\dagger}$ of a directed graph Γ are defined exactly as earlier. In particular, there is an edge (u, v) from u to v in $\Gamma^{\dagger}$ if and only if there is no edge from u to v in Γ. Using these we now observe that the classes of groups arising as the automorphism group of a directed graph and as the automorphism group of an undirected graph coincide.

Proposition 4.2 *Let Γ be a countable directed graph. Then there exists an (undirected) countable graph Λ such that $\operatorname{Aut}\Lambda \cong \operatorname{Aut}\Gamma$.*

PROOF: As Γ is countable, we can choose S such that the directed graph L_S is not isomorphic to any component of Γ. Take Δ_S to be the directed graph $(\Gamma \,\dot\cup\, L_S)^{\dagger}$ and Λ to be the undirected graph $\Delta_S^{\dashv}$ as constructed above. By construction, each vertex in Δ_S has infinite degree and so, by Lemmas 4.1, 3.1(i) and 3.3(i), $\operatorname{Aut}\Lambda \cong \operatorname{Aut}\Delta_S \cong \operatorname{Aut}(\Gamma \,\dot\cup\, L_S) \cong \operatorname{Aut}\Gamma$, as required. $\qquad\square$

For the class of directed graphs, the condition to be algebraically closed is easily seen to be equivalent to the following: a directed graph $\Gamma = (V, E)$ is *algebraically closed* if for any finite subset A of its vertices, there exists some vertex v such that $(u, v), (v, u) \in E$ for all $u \in A$. If we start with a directed graph Γ and perform the same constructions as in Section 3, then we observe that $\Delta_S = (\Gamma \dot{\cup} L_S)^\dagger$ is algebraically closed (as a directed graph) and, provided L_S is not isomorphic to any (weakly) connected component of Γ, that $\operatorname{Aut} \Delta_S \cong \operatorname{Aut} \Gamma$. This establishes the analogue of Theorem 3.4 for the class of directed graphs:

Theorem 4.3 *Let Γ be a countable (undirected) graph. Then there exist $2^{\aleph_0}$ pairwise non-isomorphic countable algebraically closed directed graphs whose automorphism group is isomorphic to that of Γ.* □

As in the previous section, we shall make use of a standard method to construct a copy of the countable universal directed graph. If $\Gamma = (V, E)$ is any countable directed graph, first construct a new directed graph $\mathcal{H}(\Gamma)$ as follows. Enumerate the set of all triples of finite and pairwise disjoint subsets of Γ as $(A_i, B_i, C_i)_{i \in I}$ for $I \subseteq \mathbb{N}$. Define $\mathcal{H}(\Gamma)$ to be the directed graph with vertex set $V \cup \{ v_i \mid i \in I \}$ (where each v_i is a new vertex) and edge set

$$E \cup \{ (v_i, a), (b, v_i), (v_i, c), (c, v_i) \mid a \in A_i, b \in B_i, c \in C_i, i \in I \}.$$

Thus each new vertex v_i has the property that there is an edge from v_i to every vertex in A_i, from every vertex in B_i to v_i, and from v_i to every vertex in C_i and *vice versa*.

Now construct a sequence of directed graphs Γ_n by defining $\Gamma_0 = \Gamma$ and $\Gamma_{n+1} = \mathcal{H}(\Gamma_n)$ for each $n \geqslant 0$. We define Γ_∞ to be the limit of this sequence of graphs, which is countable and, by construction, existentially closed. Therefore Γ_∞ is isomorphic to the countable universal directed graph D.

The same arguments apply to constructing extensions of homomorphisms and an analogue of Lemma 3.5 transfers straight across to the setting of directed graphs. We then establish, by identical methods as used in Section 3, first the characterization of graphs that arise as images of idempotent endomorphisms of the countable universal directed graph D and second observations about $\mathscr{H}$- and $\mathscr{D}$-classes of regular elements of the endomorphism monoid D:

Theorem 4.4 *Let Γ be a countable directed graph. Then*

(i) *there exists an idempotent endomorphism f of the countable universal directed graph D such that $\operatorname{im} f \cong \Gamma$ if and only if Γ is algebraically closed;*

(ii) *if Γ is algebraically closed, there are $2^{\aleph_0}$ idempotent endomorphisms f of D such that $\operatorname{im} f \cong \Gamma$.* □

Theorem 4.5 *Let D denote the countable universal directed graph.*

(i) *Let Γ be a countable (undirected) graph. Then there exist $2^{\aleph_0}$ distinct regular $\mathscr{D}$-classes of $\operatorname{End} D$ whose group $\mathscr{H}$-classes are isomorphic to $\operatorname{Aut} \Gamma$.*

(ii) *Every regular $\mathscr{D}$-class of $\operatorname{End} D$ contains $2^{\aleph_0}$ distinct group $\mathscr{H}$-classes.* □

It is a consequence of this theorem that the maximal subgroups of the endomorphism monoid of the countable universal directed graph are, up to isomorphism, the same as the maximal subgroups of the endomorphism monoid of the countable universal (undirected) graph R.

The formulae used in Section 3 to define $\Gamma^\sharp$ make perfect sense when Γ is a directed graph. They provide us with a directed graph satisfying analogous properties to those stated in Lemma 3.8. This is the primary tool in establishing the following results, which are the directed graph analogues of 3.9, 3.11 and 3.12. The proofs are identical to those for undirected graphs, except for judicious insertion of the word "directed", and so we omit them.

Theorem 4.6 *Every regular $\mathscr{D}$-class of the endomorphism monoid of the countable universal directed graph D contains $2^{\aleph_0}$ many $\mathscr{L}$- and $\mathscr{R}$-classes.*

Theorem 4.7 *Let Γ be any algebraically closed directed graph that is not isomorphic to the countable universal directed graph D. Then there exists a non-regular injective endomorphism f of D such that the subgraph induced by the images of the vertices under f is isomorphic to Γ and such that the $\mathscr{D}$-class of f contains $2^{\aleph_0}$ $\mathscr{R}$- and $\mathscr{L}$-classes.* $\qquad\square$

Corollary 4.8 *There are $2^{\aleph_0}$ non-regular $\mathscr{D}$-classes in $\operatorname{End} D$.* $\qquad\square$

For the Schützenberger groups of $\mathscr{H}$-classes of non-regular endomorphisms of the countable universal directed graph $D = (V, E)$, we proceed once more as in Section 3. If $\Gamma_0 = (V_0, E_0)$ is an algebraically closed graph, let $F_0 \subseteq E_0$ be such that $(V_0, F_0) \cong D$ (as provided by Proposition 2.1(i)). Assume that D has been constructed using Γ_0 in the initial step of our standard method and let $f \colon D \to D$ be the endomorphism that realises this isomorphism; that is, f is given by a bijection $V \to V_0$ that induces a bijection from E to F_0. Then the same argument as used to establish Proposition 3.13 gives:

Proposition 4.9 *Let f be an injective endomorphism of the countable universal directed graph D of the form specified above and $H = H_f$. Then $\mathcal{S}_H \cong \operatorname{Aut}\langle Vf \rangle \cap \operatorname{Aut}(\operatorname{im} f)$.* $\qquad\square$

To apply this proposition, we make minor changes in the argument used to establish Theorem 3.14. Let Γ be an arbitrary countable (undirected) graph and let S_n, for $n \in \mathbb{N}$, be a sequence of distinct subsets of $\mathbb{N} \setminus \{0, 1\}$ such that the graph M_{S_n} (as defined towards the end of Section 3) is not isomorphic to any connected component of Γ. Define $\Gamma_0^* = \Gamma^\dagger$ (the complement of Γ) and view this as a directed graph. Then, assuming that the directed graph Γ_n^* has been defined, enumerate the triples of finite pairwise disjoint subsets of Γ_n^* as $(A_i, B_i, C_i)_{i \in \mathbb{N}}$. Let the vertices of Γ_{n+1}^* be the union of the vertices of Γ_n^*, the vertices of the graph L_{S_n} and new vertices $\{x_i^{(n)}, y_i^{(n)}, z_i^{(n)} \mid i \in \mathbb{N}\}$. Define the edges of Γ_{n+1}^* to be the edges of Γ_n^* together with, for all $i \in \mathbb{N}$, $(x_i^{(n)}, a)$ for $a \in A_i$, $(b, y_i^{(n)})$ for $b \in B_i$, and both $(z_i^{(n)}, c)$ and $(c, z_i^{(n)})$ for $c \in C_i$. We define $\Gamma^* = (V^*, E^*)$ to be the limit of this sequence of directed graphs, which is existentially closed by construction and so isomorphic to the countable universal directed graph D.

Then let $\Gamma_0 = (V^*, E_0)$ be the directed graph whose edges are all possible edges between pairs of vertices except the following are *not* included:

(i) the edges of Γ;

(ii) for each $n \in \mathbb{N}$, all edges between distinct vertices of $\{x_i^{(n)}, y_i^{(n)}, z_i^{(n)} \mid i \in \mathbb{N}\}$;

(iii) for each $n \in \mathbb{N}$, the edges in L_{S_n};

(iv) for each $n \in \mathbb{N}$, all edges between a vertex in L_{S_n} and a vertex $x_i^{(n)}$, $y_i^{(n)}$ or $z_i^{(n)}$ and *vice versa*.

We have, of course, constructed an undirected graph but in this context we shall view Γ_0 as a directed graph. By construction, $E^* \subseteq E_0$. Therefore Γ_0 is an algebraically closed directed graph and we use this graph when applying Proposition 4.9. The endomorphism $f: D \to D$ whose image is (V^*, E^*) is not regular since $E^* \neq E_0$.

The complement $\Gamma_0^\dagger$ is the disjoint union of Γ and copies of M_{S_n} (namely the graph comprising the vertices of L_{S_n} and the vertices $x_i^{(n)}$, $y_i^{(n)}$ and $z_i^{(n)}$ for $i \in \mathbb{N}$). Hence

$$\operatorname{Aut}\Gamma_0 \cong \operatorname{Aut}\Gamma_0^\dagger \cong \operatorname{Aut}\Gamma \times \prod_{n\in\mathbb{N}} \operatorname{Aut} M_{S_n} \cong \operatorname{Aut}\Gamma \times (\operatorname{Sym}\mathbb{N})^{\aleph_0}.$$

The same argument as employed in Section 3 shows that $\operatorname{Aut}(V^*, E_0) \cap \operatorname{Aut}(V^*, E^*) \cong \operatorname{Aut}\Gamma$. Equally, by varying the subset S_1 we produce $2^{\aleph_0}$ many $\mathscr{D}$-classes of the endomorphism f. Therefore, the analogue of Theorem 3.14 holds for the countable universal directed graph:

Theorem 4.10 *Let Γ be any countable (undirected) graph. There are $2^{\aleph_0}$ non-regular $\mathscr{D}$-classes of the countable universal directed graph D such that the Schützenberger groups of $\mathscr{H}$-classes within them are isomorphic to $\operatorname{Aut}\Gamma$.* $\qquad\square$

5   Bipartite graphs

One usually defines an (undirected) graph $\Gamma = (V, E)$ to be bipartite if there exists a function $c: V \to \{0, 1\}$ such that $c(u) \neq c(v)$ whenever $(u, v) \in E$. However, as noted by Evans in [8, Section 2.2.2], it is easy to observe that the class of finite graphs satisfying this condition does not have the amalgamation property and so we cannot speak of the Fraïssé limit of such graphs. The solution is to encode the partition of the vertex set V via an additional equivalence relation. Accordingly, we define a *bipartite graph* to be a relational structure $\Gamma = (V, E, P)$ such that E is an irreflexive symmetric binary relation on V, $P = (V_0 \times V_0) \cup (V_1 \times V_1)$ for some partition $V = V_0 \,\dot\cup\, V_1$ of the vertex set, and if $(u, v) \in E$ then $(u, v) \notin P$ (which means one of u and v belongs to V_0 and the other belongs to V_1).

Let $\Gamma = (V, E, P)$ and $\Delta = (W, F, Q)$ be bipartite graphs in this sense, where $V = V_0 \,\dot\cup\, V_1$ and $W = W_0 \,\dot\cup\, W_1$ are the partitions of the vertex sets given by P and Q, respectively. It follows from the definition that if $f: \Gamma \to \Delta$ is a homomorphism, then $V_0 f$ is contained in one of W_0 or W_1 and similarly for $V_1 f$. Moreover, if f is an embedding then either $V_0 f \subseteq W_0$ and $V_1 f \subseteq W_1$, or $V_0 f \subseteq W_1$ and $V_1 f \subseteq W_0$. This enables one to show that the class of bipartite graphs (according to our definition) satisfies the amalgamation property and hence there is a unique Fraïssé limit of the finite bipartite graphs, the *countable universal bipartite graph B*.

A particular observation from the previous paragraph is that if f is an automorphism of the bipartite graph $\Gamma = (V, E, P)$, where $P = (V_0 \times V_0) \cup (V_1 \times V_1)$, then either $V_0 f = V_0$ and $V_1 f = V_1$, or $V_0 f = V_1$ and $V_1 f = V_0$. We call f *part-fixing* if $V_0 f = V_0$ and $V_1 f = V_1$. The following observation is straightforward.

Lemma 5.1 *Let $\Gamma = (V, E, P)$ be a bipartite graph such that the graph (V, E) is connected. Then $\operatorname{Aut}\Gamma = \operatorname{Aut}(V, E)$.*

To address which groups could arise as the group $\mathscr{H}$-classes and Schützenberger groups of the countable universal bipartite graph, we shall first observe that any group arising as the automorphism group of a graph also arises as that of a bipartite graph, and *vice versa*. To

achieve the first of these, let $\Gamma = (V, E)$ be any countable graph. We now associate a countable bipartite graph $\Gamma' = (V', E', P')$ to Γ. Enumerate the vertices of Γ as $V = \{\, v_i \mid i \in I \,\}$ (where $I \subseteq \mathbb{N}$). Whenever there is an edge joining vertices v_i and v_j in Γ (with $i < j$), choose a new vertex x_{ij} so that the set $X = \{\, x_{ij} \mid (v_i, v_j) \in E,\ i < j \,\}$ is a set disjoint from V. Set $V' = V \cup X$, $P' = (V \times V) \cup (X \times X)$ and

$$E' = \{\, (v_i, x_{ij}), (v_j, x_{ij}), (x_{ij}, v_i), (x_{ij}, v_j) \mid (v_i, v_j) \in E,\ i < j \,\}.$$

(Intuitively, we have added a new vertex x_{ij} in the middle of each original edge (v_i, v_j) and the partition of our new graph Γ' is into "old vertices" and "new middle-edge vertices".)

Define $\Gamma(v)$ to be the set of vertices in Γ that are joined by an edge to v.

Lemma 5.2 *Let $\Gamma = (V, E)$ be a countable graph such that $|\Gamma(v)| \geqslant 3$ for all $v \in V$. Then $\operatorname{Aut}\Gamma' \cong \operatorname{Aut}\Gamma$ and every automorphism of Γ' is part-fixing.*

PROOF: If $f \in \operatorname{Aut}\Gamma'$, then f defines an automorphism of the graph (V', E'). Since each $x \in X$ has degree 2 and each vertex $v \in V$ has the same degree in Γ' as in Γ, we conclude from the hypothesis that $Xf = X$ and $Vf = V$. In other words, each automorphism of Γ' is part-fixing.

Now define a map $\phi\colon \operatorname{Aut}\Gamma' \to \operatorname{Aut}\Gamma$ by defining $f\phi$ to be the restriction of f to the elements of V. We have observed that f restricts to a bijection $V \to V$. If $(u, v) \in E$, then there is a unique $x \in X$ such that $(u, x), (x, v) \in E'$. Then $(uf, xf), (xf, vf) \in E'$, as f is an automorphism of Γ', and the definition of our bipartite graph Γ' then tells us that $(uf, vf) \in E$. Similarly, if $(uf, vf) \in E$, then we deduce $(u, v) \in E$ and we conclude that $f\phi$ does indeed define an automorphism of the graph Γ.

Then ϕ is a homomorphism from $\operatorname{Aut}\Gamma'$ to $\operatorname{Aut}\Gamma$. It is injective, since the effect of f on the vertices in V completely determines the effect on the vertices in X. If $h \in \operatorname{Aut}\Gamma$, we may extend h to an automorphism $\tilde{h}$ of Γ' by mapping a vertex $x \in X$ satisfying $(u, x), (x, v) \in E$ to the unique vertex $y \in X$ satisfying $(uh, y), (y, vh) \in E$. Then $\tilde{h} \in \operatorname{Aut}\Gamma'$ and $\tilde{h}\phi = h$, completing the proof. $\qquad\square$

We shall also need analogues of the constructions in Section 3 for bipartite graphs. First, if $\Gamma = (V, E, P)$ is a bipartite graph, with $P = (V_0 \times V_0) \cup (V_1 \times V_1)$, define the *bipartite complement* $\Gamma^{\ddagger}$ to be $\Gamma^{\ddagger} = (V, E^{\ddagger}, P)$ where

$$E^{\ddagger} = (V \times V) \setminus (E \cup P).$$

Then by construction, $\Gamma^{\ddagger}$ is a bipartite graph with the same vertex partition as Γ such that, for $u \in V_0$ and $v \in V_1$, (u, v) is an edge in $\Gamma^{\ddagger}$ if and only if (u, v) is not an edge in Γ.

If $\Gamma = (V, E, P)$, where $P = (V_0 \times V_0) \cup (V_1 \times V_1)$, and $\Delta = (V', E', P')$, where $P' = (V_0' \times V_0') \cup (V_1' \times V_1')$, are bipartite graphs, then we define the *bipartite disjoint union* of Γ and Γ' to be

$$\Gamma \mathbin{\dot\sqcup} \Delta = (V \cup V', E \cup E', Q)$$

where $Q = (V_0 \cup V_0') \times (V_0 \cup V_0') \cup (V_1 \cup V_1') \times (V_1 \cup V_1')$. This definition depends upon our choice for V_0, V_1, V_0' and V_1' (i.e., which way round we pair the parts of the vertex sets) so strictly speaking we must refer to *a* bipartite disjoint union of Γ and Δ. This choice, however, does not affect the results in this section.

Finally, we shall use the following adjustment to produce a bipartite analogue of the graph L_S defined in Section 3. For $S \subseteq \mathbb{N} \setminus \{0, 1\}$, take $\Lambda_S = (V_S, E_S, P_S)$, where V_S and E_S are the vertices and edges of L_S as defined in Section 3 and $P_S = (V_0 \times V_0) \cup (V_1 \times V_1)$ where

$$V_0 = \{ \ell_n \mid n \text{ is even} \} \cup \{ v_n \mid n \in S \text{ is odd} \}$$
$$V_1 = \{ \ell_n \mid n \text{ is odd} \} \cup \{ v_n \mid n \in S \text{ is even} \}.$$

With these constructions, we easily establish analogues of Lemmas 3.1 and 3.3: If Γ and Δ are bipartite graphs then $\operatorname{Aut} \Gamma^\ddagger = \operatorname{Aut} \Gamma$ and $\Gamma \cong \Delta$ if and only if $\Gamma^\ddagger \cong \Delta^\ddagger$. Also if S and T are subsets of $\mathbb{N} \setminus \{0, 1\}$, then $\operatorname{Aut} \Lambda_S = \mathbf{1}$ and $\Lambda_S \cong \Lambda_T$ if and only if $S = T$.

Theorem 5.3 *The class of groups arising as automorphism groups of countable graphs is precisely the same as the class of groups arising as automorphism groups of countable bipartite graphs.*

PROOF: Let Γ be a countable graph. Choose $S \subseteq \mathbb{N} \setminus \{0, 1\}$ such that the graph L_S, as in Section 3 is not isomorphic to any connected component of Γ. Let $\Delta_S = (\Gamma \mathbin{\dot{\cup}} L_S)^\dagger$, so that $\operatorname{Aut} \Delta_S \cong \operatorname{Aut} \Gamma$ (see Theorem 3.4). Then each vertex v of Δ_S has infinite degree and so, taking $\Lambda = (\Delta_S)'$ as defined above, we conclude from Lemma 5.2 that $\operatorname{Aut} \Lambda \cong \operatorname{Aut} \Delta_S \cong \operatorname{Aut} \Gamma$.

Conversely, if $\Gamma = (V, E, P)$ is a bipartite graph, choose $S \subseteq \mathbb{N} \setminus \{0, 1\}$ such that L_S is not isomorphic to any connected component of the graph (V, E). Take $\Delta_S = (W, F, Q)$ to be the bipartite graph $(\Gamma \mathbin{\sqcup} \Lambda_S)^\ddagger$. Then (W, F) is a connected graph and so, by Lemma 5.1, $\operatorname{Aut}(W, F) = \operatorname{Aut} \Delta_S \cong \operatorname{Aut} \Gamma$. $\qquad\square$

The definition of algebraic closure for bipartite graphs is soon established to be equivalent to the following. A bipartite graph $\Gamma = (V, E, P)$ is *algebraically closed* if, given a finite collection of vertices $\{v_1, v_2, \ldots, v_k\}$ such that $(v_i, v_j) \in P$ for all i and j, there exists some vertex w that is connected by an edge to each of the v_i. Equivalently, if $V = V_0 \mathbin{\dot{\cup}} V_1$ is the partition of the vertices determined by the relation P, then if given finite subsets $A_0 \subseteq V_0$ and $A_1 \subseteq V_1$, there exist vertices $w_0 \in V_0$ and $w_1 \in V_1$ such that w_0 is joined by an edge for each vertex in A_1 and w_1 is joined by an edge to each vertex in A_0.

There are some potentially unexpected consequences of this observation. The first is that, for $m, n \in \mathbb{N}$, the complete bipartite graph $K_{m,n}$ on two parts of cardinalities m and n, respectively, is algebraically closed. There are also examples of algebraically closed infinite bipartite graphs but with only one witness w of the algebraic closure condition. This stands in contrast to the observation that in both algebraically closed graphs and algebraically closed directed graphs, there are always infinitely many witnesses. Accordingly, we define a bipartite graph $\Gamma = (V, E, P)$ to be *strongly algebraically closed* if for every finite collection A of vertices satisfying $(v, w) \in P$ for all distinct $v, w \in V$, there exist infinitely many vertices x connected to every vertex of A by an edge. Our analogue of Theorem 3.4 for bipartite graphs makes use of this stronger condition.

Theorem 5.4 *Let Γ be a countable graph. Then there exist $2^{\aleph_0}$ pairwise non-isomorphic strongly algebraically closed bipartite graphs whose automorphism group is isomorphic to that of Γ.*

PROOF: By Theorem 5.3, there is a countable bipartite graph Δ such that $\operatorname{Aut} \Delta \cong \operatorname{Aut} \Gamma$. Now there are $2^{\aleph_0}$ choices for subsets S of $\mathbb{N} \setminus \{0, 1\}$ such that Λ_S is not isomorphic to any connected component of Δ. Take $\Pi_S = (\Delta \mathbin{\sqcup} \Lambda_S)^\ddagger$. Then, by construction, $\operatorname{Aut} \Pi_S \cong \operatorname{Aut} \Delta \times \operatorname{Aut} \Lambda_S \cong$

Aut Γ, while $\Pi_S \cong \Pi_T$ if and only if $S = T$. Moreover, each Π_S is strongly algebraically closed: given any finite subset A of vertices of Π_S that are related under the partition relation, there are in fact infinitely many vertices in Λ_S that are joined to every vertex in A. $\qquad\square$

The standard method to construct a copy of the countable universal bipartite graph is as follows. If $\Gamma = (V, E, P)$ is any countable bipartite graph with corresponding vertex partition $V = V_0 \,\dot\cup\, V_1$, enumerate the set of all finite subsets of V_0 as $(A_i)_{i \in I}$ and all finite subsets of V_1 as $(B_j)_{j \in J}$ for some $I, J \subseteq \mathbb{N}$. Set $W_0 = V_0 \cup \{\, v_j \mid j \in J \,\}$ and $W_1 = V_1 \cup \{\, w_i \mid i \in I \,\}$, where the v_j and w_i are new vertices. Define $\mathcal{I}(\Gamma)$ to be the bipartite graph with vertex set $W_0 \cup W_1$, edge set to consist of E and new edges joining each v_j to every element of B_j (for $j \in J$) and each w_i to every element of A_i (for $i \in I$), and partition relation

$$Q = (W_0 \times W_0) \cup (W_1 \times W_1).$$

As in the previous sections, we construct a sequence of bipartite graphs by setting $\Gamma_0 = \Gamma$ and $\Gamma_{n+1} = \mathcal{I}(\Gamma_n)$. Then the limit Γ_∞ of this sequence is existentially closed and therefore isomorphic to the countable universal bipartite graph B.

Many of our arguments transfer from Section 3 but throughout one needs to be careful when, for example, finite algebraically closed bipartite graphs arise. The first example of this occurs in our analogue of Lemma 3.5 for bipartite graphs.

Lemma 5.5 *Let Γ be a countable bipartite graph, let Γ_∞ be the copy of the countable universal bipartite graph constructed around Γ as described above, and let $f \colon \Gamma \to \Gamma_\infty$ be a homomorphism of bipartite graphs.*

(i) *There exist $2^{\aleph_0}$ endomorphisms $\hat{f} \colon \Gamma_\infty \to \Gamma_\infty$ such that the restriction of $\hat{f}$ to Γ equals f.*

(ii) *If $\operatorname{im} f$ is an algebraically closed bipartite graph, then there is an extension $\hat{f}$ of f with $\operatorname{im} \hat{f} = \operatorname{im} f$. Moreover, if $\operatorname{im} f$ is not isomorphic to $K_{1,1}$, the countable universal bipartite graph Γ_∞ may be constructed so that there are $2^{\aleph_0}$ such extensions $\hat{f}$.*

(iii) *If f is an automorphism of Γ, then there is an automorphism $\hat{f}$ of Γ_∞ such that the restriction of $\hat{f}$ to Γ equals f.*

PROOF: For (i) and (ii), the argument is essentially the same as in Lemma 3.5: having defined an extension f_n of f to Γ_n, we extend to $f_{n+1} \colon \Gamma_{n+1} \to \Gamma_\infty$, where $\Gamma_{n+1} = \mathcal{I}(\Gamma_n)$ by mapping each new vertex w_n to a vertex joined to every vertex of $B_n f_n$ and each v_n to a vertex joined to every vertex of $A_n f_n$. In Γ_∞, there are always infinitely many choices and hence in the end we obtain $2^{\aleph_0}$ extensions $\hat{f}$ in (i).

The only place where extra care is required in the argument is in part (ii) since, although we can at every stage arrange that $\operatorname{im} f_n = \operatorname{im} f$, we can no longer guarantee there are infinitely many such extensions. If $\operatorname{im} f$ is not isomorphic to $K_{1,1}$, there is some vertex x in $\operatorname{im} f$ that is joined to at least two vertices in this image. Suppose $x = yf$. At each stage, when constructing $\mathcal{I}(\Gamma_n)$ we enumerate the finite subsets as above by always choosing $A_1 = \{y\}$. This ensures that there are at least two choices for the image of the new vertex w_1, namely any of the vertices joined to x. We then continue as before. We conclude that there are at least two extensions of the endomorphism f_n of Γ_n to endomorphisms f_{n+1} of Γ_{n+1} preserving the image being equal to $\operatorname{im} f$. Hence there are $2^{\aleph_0}$ extensions $\hat{f}$ of f to an endomorphism of Γ_∞ with $\operatorname{im} \hat{f} = \operatorname{im} f$.

Part (iii) is proved exactly as in the case for graphs. $\qquad\square$

Relying on the above lemma, but otherwise proceeding in exactly the same way as in Section 3, we establish parts (i) and (ii) of the result that classifies images of idempotent endomorphisms of the countable universal bipartite graph.

Theorem 5.6 *Let Γ be a countable bipartite graph. Then*

(i) *there exists an idempotent endomorphism f of the countable universal bipartite graph B such that $\operatorname{im} f \cong \Gamma$ if and only if Γ is algebraically closed;*

(ii) *if Γ is algebraically closed and is not isomorphic to the finite complete bipartite graph $K_{1,1}$, then there are $2^{\aleph_0}$ idempotent endomorphisms f of B such that $\operatorname{im} f \cong \Gamma$;*

(iii) *if $\Gamma \cong K_{1,1}$, there are $\aleph_0$ idempotent endomorphisms f of B such that $\operatorname{im} f \cong \Gamma$.*

PROOF: (iii) If $V = V_0 \,\dot\cup\, V_1$ is the partition of the vertex set associated to the partition relation on Γ, then idempotent endomorphisms of f with $\operatorname{im} f \cong K_{1,1}$ are determined by choosing any edge (u,v) present in E where $u \in V_0$ and $v \in V_1$, and then mapping vertices in V_0 to u and vertices in V_1 to v. As B has $\aleph_0$ many edges, there are $\aleph_0$ such idempotent endomorphisms. $\square$

By following the same steps as used to establish Theorem 3.7, we can then establish parts (i) and (ii) of the following. We rely on Theorem 5.4 and note that strongly algebraically closed bipartite graphs are certainly not isomorphic to $K_{1,1}$. For the final part, two idempotent endomorphisms of B with image isomorphic to $K_{1,1}$ are $\mathscr{D}$-related by Lemma 2.5 and each of the $\aleph_0$ such idempotent endomorphisms determines a group $\mathscr{H}$-class.

Theorem 5.7 *Let B denote the countable universal bipartite graph.*

(i) *Let Γ be a countable graph. Then there exist $2^{\aleph_0}$ distinct regular $\mathscr{D}$-classes of $\operatorname{End} B$ whose group $\mathscr{H}$-classes are isomorphic to $\operatorname{Aut}\Gamma$.*

(ii) *Let f be an idempotent endomorphism of B whose image is not isomorphic to the finite complete bipartite graph $K_{1,1}$. Then the $\mathscr{D}$-class of f in $\operatorname{End} B$ contains $2^{\aleph_0}$ distinct group $\mathscr{H}$-classes.*

(iii) *There is a single $\mathscr{D}$-class of B containing the idempotent endomorphisms with image isomorphic to $K_{1,1}$. This $\mathscr{D}$-class contains precisely $\aleph_0$ group $\mathscr{H}$-classes, each of which is isomorphic to the cyclic group of order 2.* $\square$

We now turn to establishing information about $\mathscr{L}$- and $\mathscr{R}$-classes in the endomorphism monoid of B. We deal first with the bipartite graph version of Theorem 3.11, since it involves fewer changes. We use essentially the same construction $\Gamma^\sharp$ as in Section 3. Let $\Gamma = (V, E, P)$ be a bipartite graph with $P = (V_0 \times V_0) \cup (V_1 \times V_1)$. Assume that $V_k = \{\, v_i^{(k)} \mid i \in \mathbb{N} \,\}$ for $k = 0, 1$. Then define $\Gamma^\sharp = (V^\sharp, E^\sharp, P^\sharp)$, where $V^\sharp = V_0^\sharp \cup V_1^\sharp$,

$$
\begin{aligned}
V_k^\sharp &= \{\, v_{i,r}^{(k)} \mid i \in \mathbb{N},\ r \in \{0,1\} \,\}, \qquad \text{for } k = 0, 1, \\
E^\sharp &= \{\, (v_{i,r}^{(k)}, v_{j,s}^{(1-k)}) \mid (v_i^{(k)}, v_j^{(1-k)}) \in E,\ r, s \in \{0,1\} \,\}, \\
P^\sharp &= (V_0^\sharp \times V_0^\sharp) \cup (V_1^\sharp \times V_1^\sharp).
\end{aligned}
$$

The analogue of Lemma 3.8 holds for this bipartite version of $\Gamma^\sharp$ and, indeed, if Γ is strongly algebraically closed, then so is $\Gamma^\sharp$.

We now mimic the proof of Theorem 3.11, using our bipartite construction $\Gamma^\sharp = (V^\sharp, E^\sharp, P^\sharp)$. We need Γ to be strongly algebraically closed in order to be able to apply Corollary 2.2 to construct the injective homomorphism $f \colon B \to \Lambda_0$ that appears. In addition, in the last half of the proof we must take $E = (W_0 \cup W_1, \varnothing, (W_0 \times W_0) \cup (W_1 \times W_1))$ as the empty bipartite graph where W_0 and W_1 are countably infinite disjoint sets. This then allows us to establish:

Theorem 5.8 *Let Γ be a countable strongly algebraically closed bipartite graph that is not isomorphic to the countable universal bipartite graph B. Then there exists a non-regular injective endomorphism f of R such that the subgraph induced by the images of the vertices under f is isomorphic to Γ and such that the $\mathscr{D}$-class of f contains $2^{\aleph_0}$ $\mathscr{R}$- and $\mathscr{L}$-classes.* $\qquad\square$

The bipartite analogue of Theorem 3.9 involves some surprising differences and reflects the fact that there can be finite algebraically closed bipartite graphs.

Theorem 5.9 *Let f be a regular endomorphism of the countable universal bipartite graph B.*

(i) *If the image of f is infinite, then the $\mathscr{D}$-class of f contains $2^{\aleph_0}$ many $\mathscr{L}$- and $\mathscr{R}$-classes.*

(ii) *If the image of f is finite but not isomorphic to $K_{1,1}$, then the $\mathscr{D}$-class of f contains $\aleph_0$ many $\mathscr{L}$-classes and $2^{\aleph_0}$ many $\mathscr{R}$-classes.*

(iii) *If $\operatorname{im} f \cong K_{1,1}$, then the $\mathscr{D}$-class of f contains $\aleph_0$ many $\mathscr{L}$-classes and one $\mathscr{R}$-class.*

PROOF: Suppose that $B = (V, E, P)$, where $V = V_0 \;\dot\cup\; V_1$ is the partition of the vertices determined by the relation P. Let $\Gamma = \operatorname{im} f$. For the $\mathscr{R}$-classes in the $\mathscr{D}$-class D_f of f, we can, for (i) and (ii), argue exactly as in Theorem 3.9: build B as Γ_∞ around Γ as described above and use Lemma 5.5(ii) to extend the identity map on Γ to $2^{\aleph_0}$ idempotent endomorphisms of B with image Γ. Each such extension is $\mathscr{D}$-related to f but they have distinct kernels and so are not $\mathscr{R}$-related to each other by Lemma 2.5.

For the $\mathscr{R}$-classes in Case (iii), note that when $\Gamma \cong K_{1,1}$, the endomorphisms in D_f map all the vertices in V_0 to some fixed vertex v and all the vertices in V_1 to some fixed vertex w joined to v (and necessarily v and w lie in different parts of the partition). Thus the kernel of such an endomorphism equals the partition relation P and we conclude that all endomorphisms in D_f are $\mathscr{R}$-related by Lemma 2.5(ii).

When Γ is infinite (i.e., Case (i)), we build a copy of B around the bipartite graph $\Gamma^\sharp$ and use the same argument as in Theorem 3.9 to show that D_f contains $2^{\aleph_0}$ many $\mathscr{L}$-classes. When Γ is finite (i.e., Cases (ii) and (iii)), write $n = |Vf|$. If g is $\mathscr{D}$-related to f, then $\operatorname{im} g \cong \operatorname{im} f$ and so $|Vg| = n$. There are $\aleph_0$ many subsets of V of cardinality n and so at most $\aleph_0$ many $\mathscr{L}$-classes in D_f by Lemma 2.5(i). However, we can construct infinitely many non-$\mathscr{L}$-related endomorphisms by adjusting the construction $\Gamma^\sharp$ as follows.

Assume $\Gamma = (W, F, Q)$ with associated partition $W = W_0 \;\dot\cup\; W_1$ of its vertices. Write $W_k = \{\, w_i^{(k)} \mid i \in I_k \,\}$ for finite subsets I_0 and I_1 of $\mathbb{N}$. Define $\Gamma^\natural = (W^\natural, F^\natural, Q^\natural)$, where $W^\natural = W_0^\natural \cup W_1^\natural$,

$$
\begin{aligned}
W_k^\natural &= \{\, w_{i,r}^{(k)} \mid i \in I_k,\ r \in \mathbb{N} \,\}, \qquad \text{for } k = 0, 1,\\
F^\natural &= \{\, (w_{i,r}^{(k)}, w_{j,s}^{(1-k)}) \mid (v_i^{(k)}, v_j^{(1-k)}) \in F,\ r, s \in \mathbb{N} \,\},\\
Q^\natural &= (W_0^\natural \times W_0^\natural) \cup (W_1^\natural \times W_1^\natural).
\end{aligned}
$$

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Thus we are now in effect replacing each edge in Γ by a copy of the infinite complete bipartite graph $K_{\aleph_0,\aleph_0}$. The remainder of the argument is similar to Theorem 3.9. We build a copy of B around $\Gamma^\natural$ and extend the identity map on $\Gamma^\natural$ to an idempotent endomorphism g of B with image $\Gamma^\natural$. For any $\mathbf{b} = (b_i^{(k)})$ with $b_i^{(k)} \in \mathbb{N}$ for each $i \in I_k$, we define an endomorphism $\phi_{\mathbf{b}} \colon \Gamma^\natural \to \Gamma^\natural$ by $w_{i,r}^{(k)} \phi_{\mathbf{b}} = w_{i,b_i^{(k)}}^{(k)}$. Then $g\phi_{\mathbf{b}}$ has image isomorphic to Γ and so is $\mathscr{D}$-related to f, but as $\mathbf{b}$ varies we obtain infinitely many distinct images and so these endomorphisms are not $\mathscr{L}$-related. This completes the proof. $\qquad\square$

We can perform the same arguments for the Schützenberger groups of $\mathscr{H}$-classes of non-regular endomorphisms of the countable universal bipartite graph $B = (V, E, P)$ as in Section 3. If $\Gamma_0 = (V_0, E_0, P_0)$ is a strongly algebraically closed bipartite graph, let $F_0 \subseteq E_0$ be such that $(V_0, F_0, P_0) \cong B$ (as provided by Proposition 2.1(ii)). Assume that B has been constructed using Γ_0 in the initial step of our method and let $f \colon B \to B$ be the endomorphism that realises this isomorphism. Then we establish:

Proposition 5.10 *Let f be an injective endomorphism of the countable universal bipartite graph B of the form specified above and $H = H_f$. Then $\mathcal{S}_H \cong \mathrm{Aut}\langle Vf \rangle \cap \mathrm{Aut}(\mathrm{im}\, f)$.* $\qquad\square$

To complete the work on the Schützenberger group, we shall need a bipartite analogue of the graphs M_S appearing in Section 3. For $S \subseteq \mathbb{N} \setminus \{0, 1\}$, recall L_S contains vertices ℓ_n (for $n \in \mathbb{N}$) and v_n (for $n \in S$). Let x_n and y_n (for $n \in \mathbb{N}$) be new vertices and set

$$V_0 = \{\, \ell_n \mid n \text{ is even} \,\} \cup \{\, v_n \mid n \in S \text{ is odd} \,\} \cup \{\, x_n \mid n \in \mathbb{N} \,\},$$
$$V_1 = \{\, \ell_n \mid n \text{ is odd} \,\} \cup \{\, v_n \mid n \in S \text{ is even} \,\} \cup \{\, y_n \mid n \in \mathbb{N} \,\}.$$

Let N_S be the bipartite graph with vertex set $V_0 \cup V_1$, partition relation $(V_0 \times V_0) \cup (V_1 \times V_1)$ and edges consisting of all edges present in L_S, together with an edge between each x_n and every vertex in V_1 and an edge between each y_n and every vertex in V_0.

Let f be any automorphism of N_S. Since the x_n and y_n are the only vertices adjacent to all vertices in the other part of the partition, either f fixes the parts and then must permute the x_n and permute the y_n, or f interchanges the parts and then it maps the x_n to the y_n and *vice versa*. Therefore f induces an automorphism of the bipartite graph L_S. Since $\mathrm{Aut}\, L_S = \mathbf{1}$, we conclude that f actually does fix the parts of the partition and simply permutes the x_n and permutes the y_n. Hence $\mathrm{Aut}\, N_S \cong (\mathrm{Sym}\, \mathbb{N})^2$. Similarly $N_S \cong N_T$ if and only if $S = T$.

Now let Γ be an arbitrary countable (undirected and not necessarily bipartite) graph. Apply Theorem 5.3 to construct a countable bipartite graph Λ satisfying $\mathrm{Aut}\, \Lambda \cong \mathrm{Aut}\, \Gamma$. Let S_n, for $n \in \mathbb{N}$, be a sequence of distinct subsets of $\mathbb{N} \setminus \{0, 1\}$ such that the bipartite graph N_{S_n} is not isomorphic to any connected component of Λ. (Indeed, note that the Λ occurring in Theorem 5.3 is connected, so we simply require $N_{S_n} \not\cong \Lambda$.) Define $\Gamma_0^* = \Lambda^\ddagger$ (the bipartite complement of Λ, as described above). Then, assuming that the bipartite graph Γ_n^* has been defined with partition relation $(W_0 \times W_0) \cup (W_1 \times W_1)$, enumerate the finite subsets of W_0 as $(A_i)_{i \in \mathbb{N}}$ and the finite subsets of W_1 as $(B_i)_{i \in \mathbb{N}}$. Let the vertices of Γ_{n+1}^* be the union of the vertices of Γ_n^*, the vertices of L_{S_n} and new vertices $\{\, x_i^{(n)}, y_i^{(n)} \mid i \in \mathbb{N} \,\}$. Define the edges of Γ_{n+1}^* to be the edges of Γ_n^* together with edges between a and $y_i^{(n)}$ for all $a \in A_i$ and between b and $x_i^{(n)}$ for all $b \in B_i$. The partition relation on Γ_{n+1}^* is the one that groups together all the vertices in W_0 with all the $x_i^{(n)}$ and all the vertices in W_1 with the $y_i^{(n)}$. Having constructed the bipartite graphs Γ_n^*, we let

$\Gamma^* = (V^*, E^*, P^*)$ be the limit of this sequence of graphs. By construction, Γ^* is existentially closed and therefore isomorphic to the countable universal bipartite graph B.

Now let $\Gamma_0 = (V^*, E_0, P^*)$ be the bipartite graph whose edges are all possible edges permitted by the bipartite relation P^*, except the following are *not* included:

(i) the edges in Λ;

(ii) for each $n \in \mathbb{N}$, all edges between an $x_i^{(n)}$ and a $y_j^{(n)}$;

(iii) for each $n \in \mathbb{N}$, the edges in L_{S_n};

(iv) for each $n \in \mathbb{N}$, all (permitted) edges between a vertex in L_{S_n} and a vertex $x_i^{(n)}$ or $y_i^{(n)}$.

As in previous sections, we have arranged that $E^* \subseteq E_0$. Therefore Γ_0 is algebraically closed and we use this when applying Proposition 5.10. The endomorphism $f\colon B \to B$ is not regular since $E_0 \neq E^*$.

The bipartite complement $\Gamma_0^\ddagger$ is the disjoint union of the bipartite graphs Λ and N_{S_n} (for all $n \in \mathbb{N}$). Hence

$$\operatorname{Aut}\Gamma_0 \cong \operatorname{Aut}\Lambda \times \prod_{n \in \mathbb{N}} \operatorname{Aut} N_{S_n} \cong \operatorname{Aut}\Gamma \times (\operatorname{Sym}\mathbb{N})^{\aleph_0}.$$

As in the previous sections, we observe that $\operatorname{Aut}(V^*, E_0, P^*) \cap \operatorname{Aut}(V^*, E^*, P^*)$ is isomorphic to $\operatorname{Aut}\Gamma$, and by varying S_1 we construct $2^{\aleph_0}$ many $\mathscr{D}$-classes of such endomorphisms f. This completes our final step in establishing the analogue of Theorem 3.14 for bipartite graphs.

Theorem 5.11 *Let Γ be any countable graph. There are $2^{\aleph_0}$ non-regular $\mathscr{D}$-classes of the countable universal bipartite graph B such that the Schützenberger group of $\mathscr{H}$-classes within them are isomorphic to $\operatorname{Aut}\Gamma$.* $\qquad\square$

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