CLOSED ORBITS IN QUOTIENT SYSTEMS

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Abstract

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If we have topological conjugacy between two continuous maps, $T : X \to X$ and $T' : X' \to X'$, then counts of closed orbits and periodic points are preserved. However, if we only have topological semi-conjugacy between T and T', then anything is possible, and there is, in general, no relationship between closed orbits (or periodic points) of T and T'. However, if we let a finite group G act on X, where the action of G commutes with T and where we let $X' = G\backslash X$ be the quotient of the action, then it is indeed possible to say a bit more about the relationship between the count of closed orbits of (X, T) and its quotient system (X', T'). In this thesis, we will describe the behaviour of closed orbits in quotient systems, and we will show that there exists a wide but restricted range of what growth rates can be achieved for these orbits. Moreover, we will examine the analytic properties of the dynamical zeta function in quotient systems.

Contents

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Chapter 1

Introduction

Let $T : X \to X$ be a map defined on some set X. We define a closed orbit of $x \in X$ which has length k to be any set of the form

$$\mathfrak{O}_T(x) = \{x, T(x), T^2(x), T^3(x), ..., T^{k-1}(x)\}\,,$$

where $T^k(x) = x$ such that $\mathfrak{O}_T(x)$ has cardinality $|\mathfrak{O}_T(x)| = k$, for k a natural number.

The study of closed orbits is useful for understanding the behaviour and complexity of the dynamical system (X, T), and useful analogies have been drawn between the study of closed orbits of (X, T) and the study of prime numbers. For example, dynamical analogues of the Prime Number Theorem concern the study of the asymptotic behaviour of quantities like

$$\pi_T(y) = \#\{\mathfrak{O}_T(x) : |\mathfrak{O}_T(x)| \le y\}\,, \tag{1.1}$$

where y is a real number. Then, for example, Waddington states in [18] that, for T an ergodic automorphism of a torus, we have that

$$\pi_T(y) \sim \frac{e^{h(T)(y+1)}}{y}\rho(y) \quad \text{as } y \to \infty\,,$$

where $\rho : \mathbb{N} \to \mathbb{R}^+$ is an explicit almost periodic function which is bounded away from zero and infinity, and where $h(T)$ denotes the topological entropy of T. Moreover, dynamical analogues of Mertens' Theorem concern the study of the asymptotic behaviour of quantities like

$$M_T(y) = \sum_{|\mathfrak{O}_T(x)|=1}^{y} \frac{1}{e^{h(T)|\mathfrak{O}_T(x)|}}, \tag{1.2}$$

and, for example, [6] and [9] show that, for T an ergodic quasi-hyperbolic toral automorphism, we have

$$M_T(y) \sim c\log(y) \quad \text{as } y \to \infty\,,$$

for some constant c.

A motivation for this thesis was the study of the growth properties of both (1.1) and (1.2) in flows. General statements, which will not be discussed here, are proved by Parry in [10], in by Parry and Pollicott in [11], by Sharp in [15]. In [16], Sharp obtains results for the asymptotic behaviour of (1.1) in the context of finite group extensions of Axiom A flows. Assuming that $T : \mathbb{R} \times M \to M$ is an Axiom A flow, where M is a compact smooth Riemannian manifold, Sharp considers a finite group G acting freely on M by the action commuting with T. Here, T induces an Axiom A flow T' on the quotient space $M' = G\backslash M$, and we obtain a semi-conjugacy between T and T'. Each closed orbit $\mathfrak{O}_{T'}(x)$ gives rise to a unique conjugacy class in G, and we denote this conjugacy class by $\mathcal{C}_{\mathfrak{O}_{T'}(x)}$. A special case of the expression which Sharp studies is the quantity

$$\pi_{\mathcal{C}}(y) = \#\{\mathfrak{O}_{T'}(x) : |\mathfrak{O}_{T'}(x)| \le y, \mathcal{C}_{\mathfrak{O}_{T'}(x)} = \mathcal{C}\}\,, \tag{1.3}$$

where T' (and therefore T) is topologically weak-mixing. Topologically weak-mixing flows are defined as the flows whose product flow is topologically transitive, that is every non-empty invariant open set is dense. Then for (1.3), Sharp obtains the result that

$$\pi_{\mathcal{C}}(y) \sim \frac{|\mathcal{C}|}{|G|}\,\pi_{T'}(y) \quad \text{as } y \to \infty\,.$$

These results for flows were motivational in studying a discrete analogue. Here, we consider a finite group G acting on a set X where the action of G commutes with the map $T : X \to X$. Note that we do not require the action to be free here. We denote by $X' = G\backslash X$ the quotient space and by $T' : X' \to X'$ the induced map on the quotient space. The system (X', T') is called the quotient system of the dynamical system (X, T), and we have defined a topological semi-conjugacy between T and T' given by the quotient map itself.

Here, it is essential to note that there is, in general, no relationship between the count of closed orbits of two topologically semi-conjugate maps T and T'. For example, it is possible for T to have no closed orbits while T' has many closed orbits as illustrated in Figure 1.1. On the other hand, it is possible for T to have many

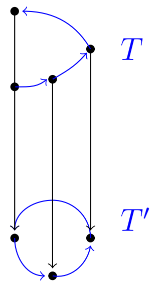


Figure 1.1: Creating closed orbits from non-closed orbits

closed orbits while T' only has few as illustrated in Figure 1.2.

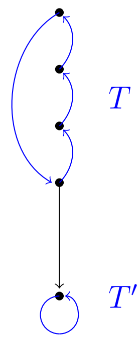


Figure 1.2: Squashing closed orbits to a single orbit of length 1

However, if we consider the quotient system (X', T') of the dynamical system (X, T) under the action of a finite group G, it is then possible to establish a relationship between the count closed orbits of T and T'. A motivation for this study were the following two examples:

Example 1.1. Let $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be the circle doubling map defined by

$$T(x) = 2x \pmod{1},$$

for all $x \in \mathbb{R}/\mathbb{Z}$, and let C_2 be the cyclic group of two elements, where we define the action of the generator $g \in C_2$ by

$$g(x) = 1 - x \pmod 1,$$

for all $x \in \mathbb{R}/\mathbb{Z}$. Then g commutes with T. The induced map on the quotient space, which can be identified with $\widetilde{X} = [0, \frac{1}{2}]$, is given by the tent map $\widetilde{T}$. Denoting by $F_k(T)$ and $F_k(\widetilde{T})$ the number of period k points of T and $\widetilde{T}$, respectively, we have that

$$F_k(T) = 2^k - 1 \quad \text{and} \quad F_k(\widetilde{T}) = 2^k,$$

for all $k \geq 1$. Then

$$F_k(T) \sim F_k(\widetilde{T}) \quad \text{as } k \to \infty,$$

showing the same asymptotic growth rates for the periodic points (and therefore closed orbits) of the circle doubling map T and the tent map $\widetilde{T}$.

Example 1.2. Let $T : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ be the doubling map defined by

$$T(x, y) = (2x, 2y) \pmod 1,$$

for all $(x, y) \in \mathbb{R}^2/\mathbb{Z}^2$, and let D_8 be the dihedral group of eight elements, where D_8 is acting as the symmetries of the square $[0, 1]^2$, so that the action commutes with T. We call the induced map on the quotient space, which can be identified with $\widehat{X} = \{(x, y) : 0 \leq y \leq x \leq \frac{1}{2}\}$, the triangle map $\widehat{T}$. Then

$$F_k(T) = (2^k - 1)^2 \quad \text{and} \quad F_k(\widehat{T}) = 4^k,$$

for all $k \geq 1$, where a proof for the formula of $F_k(\widehat{T})$ is given in Section 2.4.2 using Markov partitions. Then

$$F_k(T) \sim F_k(\widehat{T}) \quad \text{as } k \to \infty,$$

showing the same asymptotic growth rates for the periodic points (and therefore closed orbits) of the doubling map T and the triangle map $\widehat{T}$.

Example 1.1 and Example 1.2 both show the same asymptotic growth rates for closed orbits of (X, T) and its quotient system (X', T'). Are these examples representative for the general case though? We find that in a general setting, orbit behaviour can be much more complicated, but nevertheless we are able to analyse the possible behaviours by partitioning the space X according to the action of the group G. In particular, there are two numbers $\delta = \delta(G)$ and $\kappa = \kappa(G)$ associated to the group G which determine the extremal behaviour as follows:

Theorem 1.3. *Let G be a finite group acting on a set X, and let $T : X \to X$ be any map commuting with the action of G. Denote by $T' : X' \to X'$ the induced map defined on the quotient space $X' = G\backslash X$. Suppose that $F_k(T) \sim \lambda^k$ as $k \to \infty$, for some $1 < \lambda \in \mathbb{R}$. Then*

$$\liminf_{k\to\infty} \left(\frac{|G|\, F_k(T')}{\lambda^k} \right) \geq 1 \quad \textit{and} \quad \limsup_{k\to\infty} \left(\frac{\delta\kappa\, F_k(T')}{\lambda^{\delta k}} \right) \leq 1 \, .$$

Moreover, the following theorem shows that any growth rate in between the bounds given in Theorem 1.3 can be achieved:

Theorem 1.4. *Let G be a finite group. Suppose $1 < \lambda \in \mathbb{R}$, $\gamma \in \mathbb{R}$, and $c \in \mathbb{R}^+$ are such that either*

(i) $\gamma = \lambda$ *and* $c \geq \frac{1}{|G|}$, *or*

(ii) $\lambda < \gamma < \lambda^\delta$, *or*

(iii) $\gamma = \lambda^\delta$ *and* $c \leq \frac{1}{\delta\kappa}$.

Then there exist a system (X, T) and an action of the group G on the set X which commutes with the map $T : X \to X$ such that

$$F_k(T) \sim \lambda^k \quad \textit{as } k \to \infty \, ,$$

and

$$F_k(T') \sim c\gamma^k \quad \textit{as } k \to \infty \, ,$$

where (X', T') is the quotient of (X, T) under the action of G. Moreover, we can find such (X, T) with X a compact metric space and T a homeomorphism.

Remark 1.5. Theorem 1.4 follows from the more precise result given in Lemma 4.21.

Remark 1.6. Note that if the growth rate for periodic points is given by

$$F_k(T) \sim \lambda^k \quad \text{as } k \to \infty\,,$$

then the corresponding growth rate for orbits of length k, denoted by $O_k(T)$, is given by

$$O_k(T) \sim \frac{\lambda^k}{k} \quad \text{as } k \to \infty\,.$$

Hence, one can easily deduce results concerning growth rates for closed orbits from Theorem 1.3 and Theorem 1.4.

In conclusion, the quotient systems given in Example 1.1 and Example 1.2 are in fact not representative for the general case. They only show one of the many (but restricted) possibilities shown in Theorem 1.4 of what growth rates can be achieved for closed orbits in quotient systems.

Next, in analogy with number theory, we have another useful invariant given by the dynamical zeta function. For any map $T : X \to X$, such that $F_k(T) < \infty$ for all $k \geq 1$, the dynamical zeta function is defined by

$$\zeta_T(z) = \exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k}\, F_k(T)\right)\,.$$

The analytic properties of the dynamical zeta function give information regarding the complexity of the dynamical system (X, T): If a dynamical zeta function is rational, then nice and well behaved growth rates of the sequence $F = (F_k(T))_{k=1}^{\infty}$ can be expected, while with an irrational dynamical zeta function a more irregular behaviour of F should be anticipated. Bowen and Lanford show in [1] that there are only countably many rational dynamical zeta functions. Then by Theorem 1.4, one can easily deduce that there exist dynamical systems (X, T) where T has a rational dynamical zeta function while in the quotient system (X', T') we find T' to have an irrational dynamical zeta function. In this thesis, we will give explicit examples of quotient systems showing one map to have a rational dynamical zeta function while the other map has an irrational dynamical zeta function.

Example 1.7. Let (X', T') be a dynamical system with

$$F_k(T') = \begin{cases} 0, & \text{if } k = 1; \\ 2^k, & \text{if } k \text{ is even;} \\ \sum_{\substack{d|k \\ d\neq 1}} d\, 2^{(d-1)/2}, & \text{if } k > 1 \text{ is odd,} \end{cases}$$

for all $k \geq 1$. Let $X = X' \times \mathbb{Z}/2\mathbb{Z}$ and define $T : X \to X$ by

$$T(x, n) = (T'(x), n + 1 \,(\mathrm{mod}\ 2)) \,,$$

for all $(x, n) \in X$. Choose $G = \mathbb{Z}/2\mathbb{Z}$, where we define an action of G on X by

$$g(x, n) = (x, n + 1 \,(\mathrm{mod}\ 2)) \,,$$

for all $(x, n) \in X$, so that the quotient space $G\backslash X$ is naturally identified with X' and where we have the quotient map T'. Then

$$F_k(T) = \begin{cases} 0, & \text{if } k \text{ is odd;} \\ 2F_k(T'), & \text{if } k \text{ is even.} \end{cases}$$

Now, the dynamical zeta function of T given by

$$\zeta_T(z) = \frac{1}{1 - 4z^2}$$

is rational while the logarithmic derivative of the dynamical zeta function of T' is given by

$$z \left[\frac{\zeta'_{T'}(z)}{\zeta_{T'}(z)} \right] = \frac{4z^2}{1 - 4z^2} + \frac{6z^3 - 4z^5}{(1 - 2z^2)^2} + \varphi(z) \,,$$

where $\varphi(z)$ is irrational, showing the dynamical zeta function of T' to be irrational.

Remark 1.8. The reverse of Example 1.7 is possible: T could have an irrational dynamical zeta function while T' has a rational dynamical zeta function. Please refer to Chapter 5 for an example.

Moreover, it is possible for a dynamical zeta function to have a natural boundary. A natural boundary occurs whenever there is no analytic continuation beyond the radius of convergence of the dynamical zeta function, and it implies very complex behaviour of the sequence F, showing more irregular behaviour of T than with an irrational dynamical zeta function.

Example 1.9. Define

$$F_k = 2^k \quad \text{and} \quad F'_k = 2^k + a_k\, 2^k\,,$$

for all $k \geq 1$, where $a = (a_k)_{k=1}^{\infty}$ is a sequence of integers defined in Section 5.3. Setting $G = C_2$, we have that $F_k = F_k(T)$, for some system (X,T), and $F'_k = F_k(T')$, where (X',T') is the quotient system of (X,T) under some action of G. Then the dynamical zeta function of T given by

$$\zeta_T(z) = \frac{1}{1-2z}$$

is rational while the logarithmic derivative of the dynamical zeta function of T' is given by

$$z\left[\frac{\zeta'_{T'}(z)}{\zeta_{T'}(z)}\right] = \frac{2z}{1-2z} + \sum_{k=1}^{\infty} a_k\,(2z)^k\,,$$

where a has been chosen such that $\sum_{k=1}^{\infty} a_k\,(2z)^k$ has a natural boundary at $z = \frac{1}{2}$, showing the dynamical zeta function of T' to have a natural boundary at $z = \frac{1}{2}$.

Remark 1.10. Please refer to Chapter 5 for the details of Examples 1.7 and 1.9.

Last but not least, open problems for future work include the question whether or not naturally arising examples of quotient systems (compared to the more abstract constructions used for the proof of Theorem 1.4) such as Example 1.1 and Example 1.2 always exhibit the same asymptotic growth rates for closed orbits of (X,T) and its quotient system (X',T'). More ideas for future work are discussed in Chapter 6.

Here, please note that throughout this thesis, when we refer to orbits of the dynamical system (X,T), we refer to closed orbits.

1.1 Notation

Definition 1.11. We define $\mathbb{N}$ to be the set of positive natural numbers and $\mathbb{N}_0$ to be the set of non-negative natural numbers.

Definition 1.12. A real valued function $f(x)$ is asymptotic to a real valued function $g(x)$, denoted by

$$f(x) \sim g(x) \quad \text{as } k \to \infty\,,$$

if $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 1$.

Definition 1.13. Let $f(x)$ and $g(x)$ be two real-valued functions. Then

$$f(x) = \mathrm{O}\left(g(x)\right) \quad \text{as } x \to \infty$$

if and only if there exists a positive real value c such that $|f(x)| \leq c|g(x)|$, for all sufficiently large x.

Chapter 2

Preliminaries

2.1 Periodic Points and Orbits

We will now introduce periodic points and orbits. Let X be a set and $T : X \to X$ be a function that maps X to itself. If we repeatedly apply T to a point $x \in X$, we obtain the orbit of x given by the set

$$\{x, T(x), T^2(x), T^3(x), T^4(x), \ldots\}.$$

An orbit is called **periodic** or **closed** if there exists $k \in \mathbb{N}$ such that $T^k(x) = x$, that is the point x returns to its starting position. We denote such an orbit by $\mathfrak{O}_T(x)$. Further, we call the point x a **period $\boldsymbol{k}$ point**. We say x has **least period $\boldsymbol{k}$** if k is the smallest natural number such that $T^k(x) = x$, that is $T^j(x) \neq x$ for all $j \in \mathbb{N}$ such that $j < k$. Then $\mathfrak{O}_T(x)$ is a closed orbit of length k if x is a point of least period k, that is

$$\mathfrak{O}_T(x) = \{x, T(x), T^2(x), T^3(x), \ldots, T^{k-1}(x)\},$$

and we have $|\mathfrak{O}_T(x)| = k$. Further, we define the set of all orbits which have length k under T, denoted by $\mathfrak{O}_k(T)$, by

$$\mathfrak{O}_k(T) = \{\mathfrak{O}_T(x) : |\mathfrak{O}_T(x)| = k\}.$$

Now, denote by $\mathfrak{L}_k(T)$ the set of all points of *least period* k under T and by $L_k(T)$ the number of points of *least period* k under T, so that $\#\mathfrak{L}_k(T) = L_k(T)$. Then the number of orbits of length k under T is given by

$$O_k(T) = \#\,\mathfrak{O}_k(T) = \frac{\#\,\mathfrak{L}_k(T)}{k} = \frac{L_k(T)}{k}\,. \tag{2.1}$$

Hence, in order to obtain the number of closed orbits which have length k under T, we will now introduce the Möbius function– a function which counts only points of *least* period k and excludes all points of period less than k. The Möbius function is based on the following Inclusion-Exclusion Principle:

Theorem 2.1 (Inclusion-Exclusion Principle)**.** *For a finite collection of subsets $A_1, A_2, A_3, \ldots, A_k$ of A, we have*

$$\begin{aligned}\left|\bigcup_{i=1}^{k} A_i\right| = \sum_{i=1}^{k}|A_i| - \sum_{i,j:1\le i<j\le k}|A_i\cap A_j| + \sum_{i,j,s:1\le i<j<s\le k}|A_i\cap A_j\cap A_s| - \cdots\\ - (-1)^k|A_1\cap A_2\cap A_3\cap\cdots\cap A_k|\,.\end{aligned}$$

Now, if we denote by $\mathfrak{F}_k(T)$ the set of period k points under T, then

$$\mathfrak{F}_k(T) = \{x \in X : T^k(x) = x\}\,.$$

Thus, the number of period k points under T, denoted by $F_k(T)$, is given by

$$F_k(T) = \#\,\mathfrak{F}_k(T)\,.$$

Here, note that if a point has period less than k, it has period $\frac{k}{d}$ such that $d \mid k$. Then choosing $A_i = F_{k/d}$ in Theorem 2.1, where d runs through all the divisors of k which are greater than 1, and $d_1, d_2, d_3, ..., d_m$ are all distinct, we obtain

$$\begin{aligned}\left|\bigcup_{i=1}^{m}\mathfrak{F}_{k/d_i}(T)\right| &= \sum_{i=1}^{m}|\mathfrak{F}_{k/d_i}(T)| - \sum_{i,j:1\le i<j\le m}|\mathfrak{F}_{k/d_i}(T)\cap\mathfrak{F}_{k/d_j}(T)|\\ &\quad + \sum_{i,j,s:1\le i<j<s\le m}|\mathfrak{F}_{k/d_i}(T)\cap\mathfrak{F}_{k/d_j}(T)\cap\mathfrak{F}_{k/d_s}(T)|\\ &\quad - \cdots - (-1)^m|\mathfrak{F}_{k/d_1}(T)\cap\mathfrak{F}_{k/d_2}(T)\cap\mathfrak{F}_{k/d_3}(T)\cap\cdots\cap\mathfrak{F}_{k/d_m}(T)|\,.\end{aligned} \tag{2.2}$$

Now, if we have any divisors that are not square-free, then we will find that the corresponding points will cancel in (2.2). This is because each $F_{k/p^{n_p}}(T) \subseteq F_{k/p}(T)$, for p prime and $n_p \in \mathbb{N}$. For example, $F_2(T) \subseteq F_4(T) \subseteq F_8(T)$, $F_2(T)\cap F_4(T) =$

$F_2(T)$, and $F_4(T) \cap F_8(T) = F_4(T)$, etcetera. Therefore, if $k = p_1^{n_1} p_2^{n_2} p_3^{n_3} \cdots p_l^{n_l}$, for some $l \in \mathbb{N}$, then (2.2) is equal to

$$\begin{aligned}\left|\bigcup_{i=1}^{l} \mathfrak{F}_{k/p_i}(T)\right| &= \sum_{i=1}^{l} |\mathfrak{F}_{k/p_i}(T)| - \sum_{i,j:1\le i<j\le l} |\mathfrak{F}_{k/p_i}(T) \cap \mathfrak{F}_{k/p_j}(T)| \\ &\quad + \sum_{i,j,s:1\le i<j<s\le l} |\mathfrak{F}_{k/p_i}(T) \cap \mathfrak{F}_{k/p_j}(T) \cap \mathfrak{F}_{k/p_s}(T)| \\ &\quad - \cdots - (-1)^l |\mathfrak{F}_{k/p_1}(T) \cap \mathfrak{F}_{k/p_2}(T) \cap \mathfrak{F}_{k/p_3}(T) \cap \cdots \cap \mathfrak{F}_{k/p_l}(T)| \,,\end{aligned} \tag{2.3}$$

and if we look at each intersection of (2.3), we find that

$$\begin{aligned}&|\mathfrak{F}_{k/p_i}(T) \cap \mathfrak{F}_{k/p_j}(T)| = F_{k/p_ip_j}(T)\,,\\ &|\mathfrak{F}_{k/p_i}(T) \cap \mathfrak{F}_{k/p_j}(T) \cap \mathfrak{F}_{k/p_s}(T)| = F_{k/p_ip_jp_s}(T)\,,\\ &\vdots\\ &|\mathfrak{F}_{k/p_1}(T) \cap \mathfrak{F}_{k/p_2}(T) \cap \mathfrak{F}_{k/p_3}(T) \cap \cdots \cap \mathfrak{F}_{k/p_l}(T)| = F_{k/p_1p_2\cdots p_l}(T)\,.\end{aligned}$$

This is because $\gcd\left(\frac{k}{p_i}, \frac{k}{p_j}\right) = \frac{k}{p_ip_j}$, $\gcd\left(\frac{k}{p_i}, \frac{k}{p_j}, \frac{k}{p_s}\right) = \frac{k}{p_ip_jp_s}$, and so on. Then (2.3) is equal to

$$\begin{aligned}\left|\bigcup_{i=1}^{l} \mathfrak{F}_{k/p_i}(T)\right| &= \sum_{i=1}^{l} F_{k/p_i}(T) - \sum_{i,j:1\le i<j\le l} F_{k/p_ip_j}(T) + \sum_{i,j,s:1\le i<j<s\le l} F_{k/p_ip_jp_s}(T) \\ &\quad - \cdots - (-1)^l F_{k/p_1p_2\cdots p_l}(T)\,.\end{aligned}$$

It follows that

$$\begin{aligned}L_k(T) &= F_k(T) - \left|\bigcup_{p|k} \mathfrak{F}_{k/p_i}(T)\right| \\ &= F_k(T) - \sum_{i=1}^{l} |F_{k/p}(T)| + \sum_{i,j:1\le i<j\le l} F_{k/p_ip_j}(T) \\ &\quad - \sum_{i,j,s:1\le i<j<s\le l} F_{k/p_ip_jp_s}(T) + \cdots + (-1)^l F_{k/p_1p_2\cdots p_l}(T)\,.\end{aligned} \tag{2.4}$$

We define the **Möbius function** μ as follows:

Definition 2.2.

$$\mu(k) = \begin{cases} 1\,, & \text{if } k \text{ is square-free and has an even number of prime factors}\,;\\ -1\,, & \text{if } k \text{ is square-free and has an odd number of prime factors}\,;\\ 0\,, & \text{if } k \text{ is not square-free}\,.\end{cases}$$

Here, note that

$$\sum_{d|k} \mu(d) = \begin{cases} 1\,, & \text{if } k = 1\,; \\ 0\,, & \text{if } k > 1\,. \end{cases} \tag{2.5}$$

Then by (2.4), we have that

$$L_k(T) = \sum_{d|k} \mu(d) F_{k/d}(T)\,.$$

This holds since $\mu(d) = 0$ if d is not square-free. By substituting $d = \frac{k}{l}$, we obtain

$$L_k(T) = \sum_{d|k} \mu(d) F_{k/d}(T) = \sum_{l|k} \mu\left(\tfrac{k}{l}\right) F_l(T)\,. \tag{2.6}$$

Last but not least, combining (2.6) with (2.1) and letting $l = d$, we have that the formula for the number of orbits of length k under T is given by

$$O_k(T) = \tfrac{1}{k} \sum_{d|k} \mu\left(\tfrac{k}{d}\right) F_d(T)\,.$$

Now that we have defined periodic points and orbits for a given map, we want to show how, for a given sequence $F = (F_k)_{k=1}^{\infty}$, we can verify that there exists a map T such that $F_k = F_k(T)$, for all $k \geq 1$. We need the following definition and theorem:

Definition 2.3. Let $F = (F_k)_{k=1}^{\infty}$ be a sequence of non-negative integers. Then F is **realizable** if there exists a set X and a map $T : X \to X$ such that $F_k = F_k(T)$, for all $k \geq 1$.

Theorem 2.4 ([14, The Basic Lemma])**.** *Let $F = (F_k)_{k=1}^{\infty}$ be a sequence of non-negative integers. Then F is realizable if and only if*

$$O_k = \tfrac{1}{k} \sum_{d|k} \mu\left(\tfrac{k}{d}\right) F_d \in \mathbb{N}_0\,,$$

for all $k \geq 1$.

Hence, in order to prove that there exists a map T such that $F_k = F_k(T)$, we can use the following two methods:

(i) We notice a map T such that $F_k = F_k(T)$, for all $k \geq 1$, or

(ii) we use Theorem 2.4.

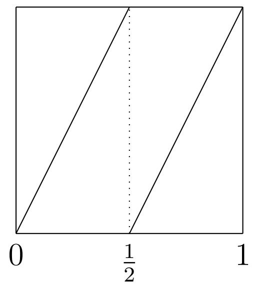


Figure 2.1: The circle doubling map

2.2 Some Dynamical Systems

We will now introduce the following three dynamical systems which were motivational for this study: The circle doubling map, the tent map, and the triangle map.

2.2.1 The Circle Doubling Map

We will define the circle doubling map T on $X = \mathbb{R}/\mathbb{Z}$, where $\mathbb{R}/\mathbb{Z}$ is the space whose points are equivalence classes of real numbers up to integers, that is

$$\mathbb{R}/\mathbb{Z} = \{x + \mathbb{Z} : x \in \mathbb{R}\}\,.$$

Two real numbers x_1 and x_2 are in the same equivalence class if and only if there exists an integer l such that $x_1 = x_2 + l$. Now, by $[0,1]/\!\sim$, we denote the unit interval whose endpoints are identified, that is $0 \sim 1$. Then $[0,1]/\!\sim$ is a circle. Further, since $[0,1]/\!\sim$ contains exactly one representative from each equivalence class, with the only exception of 0 and 1 (since these belong to the same equivalence class), we have $\mathbb{R}/\mathbb{Z} = [0,1]/\!\sim\; = [0,1)$.

We define the circle doubling map $T : X \to X$ by $T(x) = 2x \pmod 1$, that is

$$T(x) = \begin{cases} 2x\,, & \text{if } 0 \le x < \frac{1}{2}\,; \\ 2x - 1\,, & \text{if } \frac{1}{2} \le x \le 1\,. \end{cases}$$

The graph of this map is shown in Figure 2.1. To check that T is well defined, we need to check that the endpoints, that is 0 and 1, have equivalent images. We have $T(0) = 0$ and $T(1) = 1$, but $0 \sim 1$, so $T(0) \sim T(1)$, and the map is well defined.

Now, we want to look at the periodic points of T. Suppose there exists some k such that $T^k(x) = x$. Then

$$T^k(x) = 2^k x = x \qquad (\bmod 1)\,.$$

Hence, there exists an integer l such that $2^k x = x + l$. Solving for x, we obtain

$$x = \frac{l}{2^k - 1}\,.$$

Note that we have distinct values of x in $[0,1]/\sim$ for $l = 1, 2, 3, \ldots, 2^k - 1$. Therefore, the number of period k points of the circle doubling map T is equal to

$$F_k(T) = 2^k - 1\,,$$

for all $k \geq 1$.

Next, we want to look at orbits of T. We have that the number of orbits which have length k under T is given by

$$\begin{aligned} O_k(T) &= \tfrac{1}{k} \sum_{d|k} \mu\left(\tfrac{k}{d}\right) F_d(T) \\ &= \tfrac{1}{k} \sum_{d|k} \mu\left(\tfrac{k}{d}\right) \left(2^d - 1\right)\,, \end{aligned}$$

for all $k \geq 1$. Here, note that for $k \geq 2$, we have that

$$\begin{aligned} O_k(T) &= \tfrac{1}{k} \sum_{d|k} \mu\left(\tfrac{k}{d}\right) \left(2^d - 1\right) \\ &= \frac{1}{k} \sum_{d|k} \left[\mu\left(\tfrac{k}{d}\right) 2^d - \mu\left(\tfrac{k}{d}\right)\right] \\ &= \tfrac{1}{k} \sum_{d|k} \mu\left(\tfrac{k}{d}\right) 2^d - \tfrac{1}{k} \sum_{d|k} \mu\left(\tfrac{k}{d}\right) \\ &= \tfrac{1}{k} \sum_{d|k} \mu\left(\tfrac{k}{d}\right) 2^d\,, \end{aligned}$$

by (2.5).

2.2.2 The Tent Map

Let $\widetilde{X}$ be the interval $\widetilde{X} = [0,1]$. Then we define the tent map $\widetilde{T} : \widetilde{X} \to \widetilde{X}$ by

$$\widetilde{T}(x) = \begin{cases} 2x\,, & \text{if} \quad 0 \leq x \leq \frac{1}{2}\,; \\ 2 - 2x\,, & \text{if} \quad \frac{1}{2} \leq x \leq 1\,. \end{cases}$$

Note that the graph of this map, shown in Figure 2.2, resembles a 'tent', therefore giving the map its name.

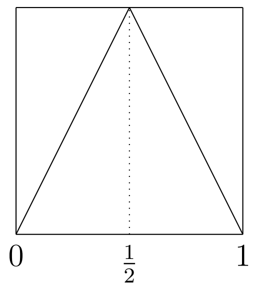
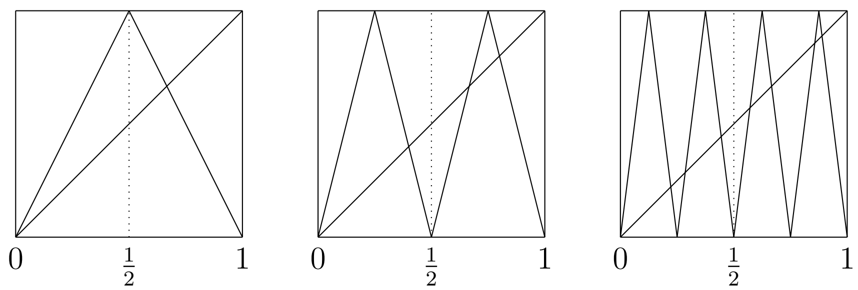


Figure 2.2: The tent map

We want to look at the periodic points of $\widetilde{T}$. We will start by studying the first three iterates of $\widetilde{T}$. We have the following graphs, where the intersection of the graphs with the line $\widetilde{T}(x) = x$ gives the periodic points:



From this, we might guess that there are 2^k points of period k, for $k \geq 1$, and indeed, in Section 2.4.2 on Markov partitions, we will give a proof to show that

$$F_k(\widetilde{T}) = 2^k\,,$$

for all $k \geq 1$. It follows that the number of orbits which have length k under the tent map $\widetilde{T}$ is given by

$$\begin{aligned} O_k(\widetilde{T}) &= \tfrac{1}{k} \sum_{d|k} \mu\left(\tfrac{k}{d}\right) F_d(\widetilde{T}) \\ &= \tfrac{1}{k} \sum_{d|k} \mu\left(\tfrac{k}{d}\right) 2^d\,, \end{aligned}$$

for all $k \geq 2$. Here, note that when $k \geq 2$, $O_k(\widetilde{T}) = O_k(T)$, where T denotes the circle doubling map.

2.2.3 The Triangle Map

Let $\widehat{X}$ be the space $\widehat{X} = \{(x, y) : 0 \leq y \leq x \leq \frac{1}{2}\}$, where the graph of $\widehat{X}$ is given by a triangle, therefore giving the map its name. We will divide the space into the following four regions:

$$R_1 = \{(x, y) : 0 \leq y \leq x \leq \tfrac{1}{2}\};$$

$$R_2 = \{(x, y) : \tfrac{1}{4} \leq x \leq \tfrac{1}{2}, 0 \leq y \leq \tfrac{1}{4}, y \leq \tfrac{1}{2} - x\};$$

$$R_3 = \{(x, y) : \tfrac{1}{4} \leq x \leq \tfrac{1}{2}, 0 \leq y \leq \tfrac{1}{4}, y \geq \tfrac{1}{2} - x\};$$

$$R_4 = \{(x, y) : \tfrac{1}{4} \leq y \leq x \leq \tfrac{1}{2}\}.$$

Then the graph of $\widehat{X}$ is shown in Figure 2.3. Next, we define the triangle map

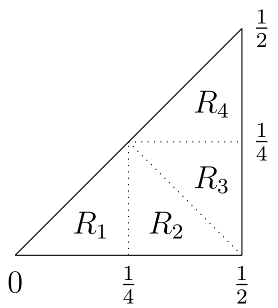


Figure 2.3: The triangle space $\widehat{X}$

$\widehat{T} : \widehat{X} \to \widehat{X}$ by

$$\widehat{T}(x, y) = \begin{cases} (2x, 2y)\,, & \text{if} \quad (x, y) \in R_1\,; \\ (1 - 2x, 2y)\,, & \text{if} \quad (x, y) \in R_2\,; \\ (2y, 1 - 2x)\,, & \text{if} \quad (x, y) \in R_3\,; \\ (1 - 2y, 1 - 2x)\,, & \text{if} \quad (x, y) \in R_4\,. \end{cases}$$

To check that this map is well defined, we need to check that the boundary points have equivalent images under T. First, consider

$$R_1 \cap R_2 = \left\{ \left(\tfrac{1}{4}, y\right) : 0 \leq y \leq \tfrac{1}{4} \right\}\,.$$

Letting $a = \frac{1}{4}$ and $b = y$, we have

$$\widehat{T}(a, b) = (2a, 2b) = (1 - 2a, 2b)\,.$$

Second, consider

$$R_2 \cap R_3 = \left\{ \left(x, \tfrac{1}{2} - x\right) : \tfrac{1}{4} \leq x \leq \tfrac{1}{2} \right\} .$$

Letting $a = x$ and $b = \frac{1}{2} - x$, we have

$$\widehat{T}(a,b) = (1-2a, 2b) = (2b, 1-2a) .$$

Third, consider

$$R_3 \cap R_4 = \left\{ \left(x, \tfrac{1}{4}\right) : \tfrac{1}{4} \leq x \leq \tfrac{1}{2} \right\} .$$

Letting $a = x$ and $b = \frac{1}{4}$, we have

$$\widehat{T}(a,b) = (2b, 1-2a) = (1-2b, 1-2a) .$$

Last, but not least consider

$$R_1 \cap R_2 \cap R_3 \cap R_4 = \left\{ \left(\tfrac{1}{4}, \tfrac{1}{4}\right) : \tfrac{1}{4} \leq x \leq \tfrac{1}{2} \right\} .$$

Then letting $a = b = \frac{1}{4}$, we have

$$\widehat{T}(a,b) = (2a, 2b) = (1-2a, 2b) = (2b, 1-2a) = (1-2b, 1-2a) .$$

Hence, the map is well defined.

Further, in Section 2.4.2 on Markov partitions, we will show that the number of period k points of the triangle map $\widehat{T}$ is given by

$$F_k(\widehat{T}) = 4^k ,$$

for all $k \geq 1$. It follows that

$$\begin{aligned} O_k(\widehat{T}) &= \frac{1}{k} \sum_{d|k} \mu\left(\frac{k}{d}\right) F_d(\widehat{T}) \\ &= \frac{1}{k} \sum_{d|k} \mu\left(\frac{k}{d}\right) 4^d , \end{aligned}$$

for all $k \geq 1$.

2.3 Topological Conjugacy and Semi-conjugacy

We will now give some basic background information on topological conjugacy and semi-conjugacy. We will start by introducing metric spaces.

2.3.1 Metric Spaces

We have the following definition:

Definition 2.5. A **metric** d on a set X is a map $d : X \times X \to \mathbb{R}$ such that, for any $x, y, z \in X$, the following hold:

(i) $d(x, y) \geq 0$,

(ii) $d(x, y) = 0$ if and only if $x = y$,

(iii) $d(x, y) = d(y, x)$,

(iv) $d(x, z) \leq d(x, y) + d(y, z)$.

The pair (X, d) is called a **metric space**.

Example 2.6. Let $X = \mathbb{R}/\mathbb{Z}$ be the space of the circle doubling map T. Then if we define a metric d by

$$d(x, y) = \min\{|x - y|, 1 - |x - y|\},$$

for all $x, y \in X$, we have that (X, d) is a metric space.

Example 2.7. Let $\widetilde{X} = [0, 1]$ be the space of the tent map $\widetilde{T}$. Then if we define a metric $\widetilde{d}$ by

$$\widetilde{d}(x, y) = |x - y|,$$

for all $x, y \in \widetilde{X}$, we have that $(\widetilde{X}, \widetilde{d})$ is a metric space.

Example 2.8. Let $\widehat{X} = \{(x, y) : 0 \leq y \leq x \leq \frac{1}{2}\}$ be the space of the triangle map $\widehat{T}$. Then if we define a metric $\widehat{d}$ by

$$\widehat{d}((x, y), (x', y')) = \sqrt{(x' - x)^2 + (y' - y)^2},$$

for all $(x, y), (x', y') \in \widehat{X}$, we have that $(\widehat{X}, \widehat{d})$ is a metric space.

Next, in order to study continuous functions between two metric spaces, we need the following definition:

Definition 2.9. Let (X, d) and (X', d') be two metric spaces. A map $T : X \to X$ is **continuous** if, for all $x, y \in X$ and all $\epsilon > 0$, there exists $\delta > 0$ such that when $d(x, y) < \delta$ then $d'(T(x), T(y)) < \epsilon$.

Example 2.10. The circle doubling map T is continuous on (X, d), where d is defined as in Example 2.6. Let $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{2}$, and assume that $x, y \in X$ are such that $d(x, y) < \delta$. We will consider four cases:
(i) If $0 \leq x, y \leq \frac{1}{2}$, then $d(x, y) = |x - y|$, and we have

$$\begin{aligned} d(T(x), T(y)) &= d(2x, 2y) \\ &= \min\{2\,|x - y|, 1 - 2\,|x - y|\} \\ &\leq 2|x - y| \\ &= 2\,d(x, y) < 2\delta = \epsilon\,. \end{aligned}$$

(ii) If $0 \leq x \leq \frac{1}{2} \leq y \leq 1$, then $d(x, y) = \min\{|x - y|, 1 - |x - y|\}$. Here, note that since $y - \frac{1}{4} \geq 0$ and $x + \frac{1}{4} \geq 0$, we have that

$$\begin{aligned} |2y - 1 - 2x| &= 2\,\left|y - \tfrac{1}{4} - \left(x + \tfrac{1}{4}\right)\right| \\ &\leq 2\,\left|y - \tfrac{1}{4}\right| + 2\,\left|x + \tfrac{1}{4}\right| \\ &= 2y - \tfrac{1}{2} + 2x + \tfrac{1}{2} \\ &= 2y - 2x \\ &= 2\,|x - y|\,. \end{aligned}$$

Further, we have

$$1 - |2y - 1 - 2x| \leq 1 - 2\,|x - y| \leq 2 - 2\,|x - y|\,.$$

Then it follows that

$$\begin{aligned} d(T(x), T(y)) &= d(2x, 2y - 1) = d(2y - 1, 2x) \\ &= \min\{|2y - 1 - 2x|, 1 - |2y - 1 - 2x|\} \\ &\leq \min\{2|x - y|, 2 - 2|x - y|\} \\ &= 2\,d(x, y) < 2\delta = \epsilon\,. \end{aligned}$$

(iii) If $0 \leq y \leq \frac{1}{2} \leq x \leq 1$, then by symmetry of (ii) we have that

$$d(T(x), T(y)) = d(2x-1, 2y) < \epsilon\,.$$

(iv) If $\frac{1}{2} \leq x, y \leq 1$, then $d(x, y) = |x - y|$, and

$$\begin{aligned} d(T(x), T(y)) &= d(2x-1, 2y-1) \\ &= \min\{2\,|x-y|, 1-2\,|x-y|\} \\ &\leq 2|x-y| \\ &= 2\,d(x,y) < 2\delta = \epsilon\,. \end{aligned}$$

It follows that T is continuous on (X, d).

Example 2.11. By a similar argument as in Example 2.10, taking $\delta = \frac{\epsilon}{2}$, the tent map $\widetilde{T}$ is continuous on $(\widetilde{X}, \widetilde{d})$, where $\widetilde{d}$ is as defined in Example 2.7.

Example 2.12. By a similar argument as in Example 2.10, taking $\delta = \frac{\epsilon}{2}$, the triangle map $\widehat{T}$ is continuous on $(\widehat{T}, \widehat{d})$, where $\widehat{d}$ is as defined in Example 2.8.

2.3.2 1-Point Compactification

Since we are interested in studying topologically conjugate and topologically semi-conjugate maps defined on *compact* metric spaces, we will now introduce 1-point (Alexandroff) compactification as a way of compactifying topological spaces, specifically metric spaces.

Remark 2.13. Note that not all topological spaces can be compactified through 1-point compactification. However, for the purpose of this thesis, 1-point compactification can be used where needed.

We will start by defining what a topological space is and by introducing the notion of open sets in topological spaces and, more specifically, in metric spaces:

Definition 2.14. A **topological space** (X, Θ) is a set X together with a collection Θ of subsets of X, called **open sets**, which satisfy the following conditions:

(i) Both $\emptyset$ and X are open. That is $\emptyset, X \subseteq \Theta$;

(ii) The union of any open sets is open. That is, if $\{U_j\}_{j\in J} \subseteq \Theta$, then $\bigcup_{j\in J} U_j \subseteq \Theta$, for any set J;

(iii) The finite intersection of any open sets is open. That is, if $\{U_j\}_{j=0}^{r} \subseteq \Theta$, then $\bigcap_{j=0}^{r} U_j \subseteq \Theta$.

Definition 2.15. A subset U of a metric space (X, d) is **open** if for all $x \in U$, there exists $\epsilon > 0$ such that the open ball of radius ϵ centred at x, denoted by $B_\epsilon(x)$, is contained in U. That is, we have

$$B_\epsilon(x) = \{y \in X : d(x, y) < \epsilon\} \subseteq U\,.$$

Hence, if (X, d) is a metric space and Θ the set of all open subsets of X (as defined in Definition 2.15), then (X, Θ) is a topological space as defined in Definition 2.14.

Now, before we continue with the definition of compactness and compactification, we will first introduce boundary points and dense sets:

Definition 2.16. Let D be an open set such that $D \subseteq X$. Then $z \in X$ is a **boundary point** if, for all $\epsilon > 0$, there exist $x, y \in B_\epsilon(z)$ such that $x \in D$ and $y \in X\backslash D$. The set of all boundary points is called the **boundary** of D, denoted by $\mathfrak{B}(D)$. Then the **closure** of D, denoted by $\overline{D}$, is given by

$$\overline{D} = \mathfrak{B}(D) \cup D\,.$$

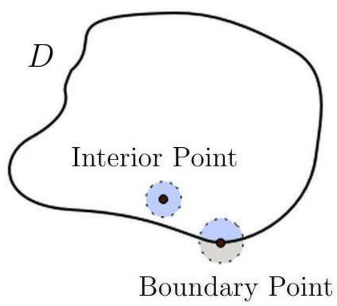


Figure 2.4: Interior and boundary points

Example 2.17. If $X = \mathbb{R}$, and $D = (0,1)$, then $\overline{D} = \overline{(0,1)} = [0,1]$.

Definition 2.18. A set D is dense in X if $\overline{D} = X$.

We have the following definitions of compactness and compactification:

Definition 2.19. An **open cover** of a topological space (X, Θ) is a collection $\{U_j\}_{j\in J}$ of open sets such that $X \subseteq \bigcup_{j\in J} U_j$. Then (X, Θ) is a **compact space** if every open cover of (X, Θ) has a finite subcover.

Definition 2.20. A **compactification** of a topological space (X, Θ) is a compact topological space (X', Θ') together with a continuous injection $f : X \to X'$ such that $f(X)$ is dense in X'.

Hence, in order to compactify a given topological space (X, Θ), we must find a topological space (X', Θ'), such as in Definition 2.20, which satisfies Definition 2.19. Then in our situation, the general idea is the following: We take a topological space (X', Θ'), where we define $X' = X \cup \{\infty\}$ to be the space X with the added point at infinity and where all the open sets $U_j \in \Theta'$ are either of the form

(i) $\infty \notin U_j$ and $U_j \subseteq \Theta$, or

(ii) $\infty \in U_j$ and $X \backslash U_j$ is finite.

Now, we take $\{U_j\}_{j\in J}$ to be an arbitrary open cover of X', so that

$$X' = \bigcup_{j\in J} U_j \,.$$

Then there exists an open set $U_0 \subseteq \Theta'$ such that $\infty \in U_0$. But then $X \backslash U_0$ must be finite. Say

$$X \backslash U_0 = \{x_1, x_2, x_3, \ldots, x_r\}$$

Now, if we choose $j \in \{1, 2, 3, \ldots, r\}$ such that $x_j \in U_j$, then $\{U_j\}_{j=0}^{r}$ is a finite subcover of X'. Hence, (X', Θ') is compact.

2.3.3 Topological Conjugacy and Semi-Conjugacy

Let (X, d) and (X', d') be two compact metric spaces with continuous maps $T : X \to X$ and $T' : X' \to X'$. We have the following definitions:

Definition 2.21. We say that T is **topologically semi-conjugate** to T' if there exists a continuous surjection $\pi : X \to X'$ such that

$$\pi(T(x)) = T'(\pi(x)),$$

for all $x \in X$. Equivalently, we can say that T' is a **topological factor** of T.

Then for T and T' topologically semi-conjugate, the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ X' \arrow[r, "T'"'] & X' \end{tikzcd}$$

Definition 2.22. A function $\pi : X \to X'$ between two metric spaces is called a **homeomorphism** if it has the following properties:

(i) π is a bijection,

(ii) π is continuous, and

(ii) π^{-1} is continuous.

Definition 2.23. T is **topologically conjugate** to T' if $\pi : X \to X'$ is a homeomorphism such that $\pi(T(x)) = T'(\pi(x))$, for all $x \in X$. Equivalently, we can say that π is a **topological conjugacy**.

Here, note that if T is topologically conjugate to T', then periodic points and orbits are preserved, that is, we have $F_k(T) = F_k(T')$ and $O_k(T) = O_k(T')$, for all $k \geq 1$.

Example 2.24. Let $X = [0, \frac{1}{2}]$ and $d(x, y) = |x - y|$, for all $x, y \in X$. Then (X, d) is a compact metric space. Now, if we define $T : X \to X$ by

$$T(x) = \begin{cases} 2x\,, & \text{if} \quad 0 \leq x \leq \frac{1}{4}\,; \\ 1 - 2x\,, & \text{if} \quad \frac{1}{4} \leq x \leq \frac{1}{2}\,, \end{cases}$$

then we have that T is topologically conjugate to the tent map $\widetilde{T} : \widetilde{X} \to \widetilde{X}$. We will start by letting $\pi : X \to \widetilde{X}$ be the map defined by $\pi(x) = 2x$, for all $x \in X$, and notice that π satisfies $\pi(T(x)) = \widetilde{T}(\pi(x))$, for all $x \in X$. In order to show that π is a homeomorphism, we will first show that π and π^{-1} are continuous, where we define $\pi^{-1}(x) = \frac{x}{2}$, for all $x \in \widetilde{X}$.

We will start by showing that π is continuous. Let $\epsilon > 0$ and $\delta = \frac{\epsilon}{2}$. Suppose $x, y \in X$ are such that $d(x, y) < \delta$. Then

$$\widetilde{d}(\pi(x), \pi(y)) = \widetilde{d}(2x, 2y) = 2\,|x - y| = 2\,d(x, y) < 2\delta = \epsilon\,.$$

Next, we will show that π^{-1} is continuous. Let $\epsilon > 0$, $\delta = 2\epsilon$, and assume that $x, y \in X$ are such that $d(x, y) < \delta$. Then

$$\widetilde{d}(\pi^{-1}(x), \pi^{-1}(y)) = \widetilde{d}\left(\tfrac{x}{2}, \tfrac{y}{2}\right) = \tfrac{1}{2}\,|x - y| = \tfrac{1}{2}\,d(x, y) < \tfrac{\delta}{2} = \epsilon\,.$$

Second, we need to show that π is a bijection. Here, note that $\pi^{-1}(\pi(x)) = x$, for all $x \in X$, and $\pi(\pi^{-1}(x')) = x'$, for all $x' \in \widetilde{X}$. Hence, π is bijective. Then π is a homeomorphism. Therefore, the map T is topologically conjugate to the tent map $\widetilde{T}$.

Remark 2.25. From now on, we will refer to the map T on $[0, \frac{1}{2}]$ also as a tent map since it is topologically conjugate to the tent map on $[0, 1]$. Due to the topological conjugacy, we have that the periodic points and orbits are preserved.

Next, note that in contrast to topological conjugacy, if we only have topological semi-conjugacy between two maps T and T', then there is, in general, no relationship between the count of periodic points and orbits of T and T'. We have the following examples:

Example 2.26. It is possible for T to have few periodic points while its topological factor T' has many. Let T' be the circle doubling map. Let

$$\begin{aligned} \mathbb{S}^1 &= \{z \in \mathbb{C} : |z| = 1\}, \\ \widehat{\mathbb{Z}} &= \{\text{group homomorphisms } \chi : \mathbb{Z} \to \mathbb{S}^1\}, \end{aligned}$$

and $T'' : \widehat{\mathbb{Z}} \to \widehat{\mathbb{Z}}$ be defined by

$$(T''(\chi))(n) = \chi(2n)\,,$$

for all $n \in \mathbb{Z}$. Then T'' is topologically conjugate to the circle doubling map T'. Next, we let

$$\widehat{\mathbb{Q}} = \{\text{group homomorphisms } \chi : \mathbb{Q} \to \mathbb{S}^1\}\,,$$

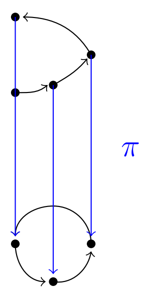
and $T : \widehat{\mathbb{Q}} \to \widehat{\mathbb{Q}}$ be defined by

$$(T(\chi))(q) = \chi(2q)\,,$$

for all $q \in \mathbb{Q}$. Further, we define $\pi : \widehat{\mathbb{Q}} \to \widehat{\mathbb{Z}}$ by

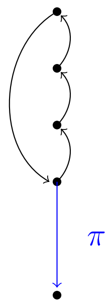
$$\pi(\chi) = \chi|_{\mathbb{Z}}\,.$$

Then π is a restriction of characters from $\widehat{\mathbb{Q}}$ to $\widehat{\mathbb{Z}}$, and the map T is topologically semi-conjugate to T''. Since T'' is topologically conjugate to T', and T is topologically semi-conjugate to T'', then T is also topologically semi-conjugate to T' (see [4]). We have $F_k(T) = 1$ while $F_k(T') = 2^k - 1$, for all $k \geq 1$. Hence, for $k \geq 2$, we have that $O_k(T) = 0$, while T' has many orbits of length k. This is possible since π is surjective but not injective. Hence, we have created closed orbits of the topological factor T' from non-closed orbits of T as illustratd in the following figure:



Example 2.27. The reverse of Example 2.26 could happen. That is, T could have many periodic points while its topological factor T' could have few. Take T to be the circle doubling map. Let X' be a space consisting of one single point, and define $T' : X' \to X'$ to be the identity map. Take $\pi : X \to X'$ to be the map that sends all points in X to the single point in X'. Then here, we have $F_k(T) = 2^k - 1$, for all

$k \geq 1$, and we have many orbits of length k, but X' consists of one single point, so we have $F_k(T') = 1$, for all $k \geq 1$, and $O_k(T') = 0$, for all $k \geq 2$. Hence, π takes the many orbits of T and reduces them to a single point on T', that is one orbit of length 1, as illustrated in the following figure:



In both examples, we have the equivalence relation $x \sim y$ if and only if $\pi(x) = \pi(y)$, for all $x, y \in X$. The equivalence classes are the **fibres** of π:

$$\pi^{-1}(x') := \{x \in X : \pi(x) = x'\},$$

for $x' \in X'$. In Examples 2.26 and 2.27, our equivalence classes are **not** finite. Now, if we consider a finite group G acting on a set X, where the action of G commutes with the map $T : X \to X$, then this defines a topological semi-conjugacy $\pi : X \to X'$ onto the factor system (X', T'), where $X' = G \backslash X$. The focus of this thesis is the following question: In the case of factor systems, is it possible to establish a relationship between the count of periodic points and orbits of T and T'?

2.4 Subshifts of Finite Type and Markov Partitions

In order to count periodic points of certain maps, we will now introduce one-sided shift and subshifts of finite type and Markov partitions. Good references for this material are [2] and [8]. In addition to giving basic background information, we will give two examples: a Markov partition of the triangle map and a Markov partition of the tent map.

2.4.1 Shifts and Subshifts of Finite Type

Let $S = \{0, 1, 2, \ldots, n-1\}$ be a set of n symbols, for n a natural number, and set $s = (s_i)_{i=0}^{\infty}$. Then we define

$$\Sigma_n^+ = \{(s_i)_{i=0}^{\infty} : s_i \in S\}$$

to be the space of all infinite one-sided sequences on n symbols. Here, note that (Σ_n^+, d^*) is a compact metric space, where the notion of distance is defined by

$$d^*(s, s') = \frac{1}{2^i}\,,$$

for $s, s' \in \Sigma_n^+$, where i is the smallest natural number such that $s_i \neq s_i'$. Further, we let $\sigma : \Sigma_n^+ \to \Sigma_n^+$ be the one-sided shift map defined by

$$(\sigma\,(s))_i = s_{i+1}\,,$$

for all $s \in \Sigma_n^+$. Then (Σ_n^+, σ) is the **full one-sided n-shift**.

The full one-sided n-shift has a natural class of closed σ-invariant subsets which we call one-sided subshifts of finite type. One-sided subshifts of finite type can be described in terms of adjacency matrices and their associated directed graphs. Here, by directed graphs, we are refering to directed graphs with no parallel edges.

An adjacency matrix $A = (a_{ij})$ is an $n \times n$ matrix whose entries consist of 0s and 1s. Here, we have $0 \leq i, j \leq n-1$. To A, we can associate a directed graph $\Gamma = \Gamma_A$ with n vertices $v_0, v_1, \ldots, v_{n-1}$, where the (i, j)-entry of A corresponds to the number of edges from v_i to v_j. Conversely, to a directed graph Γ with vertices $v_0, v_1, \ldots, v_{n-1}$, we can associate an $n \times n$ adjacency matrix A, where the number of edges from v_i to v_j gives the (i, j)-entry of A, and $\Gamma = \Gamma_A$.

Now, given such an adjacency matrix A and its associated directed graph Γ_A, we say that an infinite one-sided sequence $s \in \Sigma_n^+$ is **allowed** if $a_{s_i s_{i+1}} > 0$, for every $i \geq 0$. Equivalently, s is allowed if there is a directed edge from v_{s_i} to $v_{s_{i+1}}$ on the directed graph Γ_A, for any i. Then we define

$$\Sigma_A^+ = \{(s_i)_{i=0}^{\infty} : s_i \in S, a_{s_i s_{i+1}} > 0\}$$

to be the space of all infinite one-sided allowed sequences consisting of n symbols. Hence, we can view a sequence $s \in \Sigma_A^+$ as a one-sided infinite walk along directed edges on the graph Γ_A, where s_i gives the index of the vertex v_{s_i} visited at time i. Then we call the pair (Σ_A^+, σ) a **one-sided subshift of finite type**. We will show that the number of periodic points under the shift map σ is given by the trace of our adjacency matrix A, that is we will show that $F_k(\sigma) = \text{trace}(A^k)$, for all $k \geq 1$.

Now, note that a sequence $s \in \Sigma_A^+$ has period k under σ if and only if $(\sigma(s))_i = s_{i+k}$, for all $i \geq 0$. Hence, s is an infinite one-sided walk along directed edges on Γ_A that visits the vertex v_{s_i} at time i, and comes back to it at time $i+k$.

By $\gamma_i^k(s)$, we denote the **path** of length k given by

$$s_i \to s_{i+1} \to s_{i+2} \to \cdots \to s_{i+k-1} \to s_{i+k}\,.$$

Then $\gamma_i^k(s)$ is a **cycle** of length k if and only if $s_{i+k} = s_i$. Then if s has period k under σ, it is a repetition of cycles of length k, that is, $\gamma_0^k(s) = \gamma_k^k(s) = \gamma_{2k}^k(s) = \ldots = \gamma_{ik}^k(s)$. Hence, we have the map $s \mapsto \gamma_0^k(s)$ from the set of period k points under Σ_A^+ to the set of cycles of length k on Γ_A.

Conversely, given a cycle η of length k along directed edges on Γ_A, we define $s = s(\eta) \in \Sigma_A^+$ as a repetition of cycles η, that is, $s_{i+jk} = \eta_i$, for $0 \leq i \leq k-1$ and $j \geq 0$. Then s is periodic with period k under σ and we have a map $\eta \mapsto s(\eta)$ from the set of cycles of length k to the set of period k points under Σ_A^+ which, by construction, is the inverse of $s \mapsto \gamma_0^k(s)$. It follows that we have a bijection between the set $\mathfrak{F}_k(\sigma)$ and the set of all cycles γ of length k. We will continue to show that $F_k(\sigma) = \text{trace}(A^k)$, for all $k \geq 1$.

For $k = 1$, $F_1(\sigma)$ gives the number of all cycles of length 1. Then

$$F_1(\sigma) = \#\{s_i \in S : a_{s_i s_i} > 0\}\,.$$

Hence, $F_1(\sigma)$ gives the number of all (i,i)-entries of A whose entry is 1. It follows that

$$F_1(\sigma) = \text{trace}(A)\,.$$

Now, $F_k(\sigma)$ gives the number of all cycles of length k on Γ_A. That is, $F_k(\sigma)$ gives the number of all cycles γ of length k that contain the path

$$s_i \to s_{i+1} \to s_{i+2} \to \cdots \to s_{i+k-1} \,,$$

where $s_{i+k} = s_i$. But this path exists if and only if $a_{s_i s_{i+1}} \, a_{s_{i+1} s_{i+2}} \cdots a_{s_{i+k-1} s_i} = 1$. Summing over all possibilities, we obtain

$$\sum_{s_{i+1}=1}^{n} \sum_{s_{i+2}=1}^{n} \cdots \sum_{s_{i+k-1}=1}^{n} a_{s_i s_{i+1}} \, a_{s_{i+1} s_{i+2}} \cdots a_{s_{i+k-1} s_i} \,,$$

the number of all cycles γ_i of length k that start and end at the vertex v_{s_i} on the directed graph Γ_A. Note that this sum gives the (i,i)-entry of A^k. Again, summing over all possible $s_i \in S$, that is over all (i,i)-entries of A^k, we obtain the cardinality of the set of all cycles of length k on Γ_A. It follows that

$$F_k(\sigma) = \#\{\text{cycles of length } k \text{ on } \Gamma_A\} = \mathrm{trace}(A^k) \,,$$

for all $k \geq 1$.

2.4.2 Markov Partitions

We will now introduce Markov partitions which we can use to show that for certain continuous maps $T : X \to X$ defined on a compact metric space (X, d), we can find an adjacency matrix A such that there is a bijection between the periodic points of T and the one sided subshift of finite type (Σ_A^+, σ) (as defined in the previous section), so that

$$F_k(T) = F_k(\sigma) = \mathrm{trace}(A^k) \,,$$

for all $k \geq 1$. We have the following definition:

Definition 2.28. A **topological partition** of a metric space (X, d) is a finite collection $P = \{P_0, P_1, P_2, \ldots, P_{n-1}\}$ of open disjoint sets $P_0, P_1, P_2, \ldots, P_{n-1}$, such that

$$X = \bigcup_{i=0}^{n-1} \overline{P_i} \,.$$

Example 2.29. Let $\widehat{X} = \{(x, y) : 0 \le y \le x \le \frac{1}{2}\}$ be the space of the triangle map $\widehat{T}$, and let $P = \{P_0, P_1, P_2, P_3\}$ be the following open disjoints sets in $\widehat{X}$ shown in Figure 2.5:

$$
\begin{aligned}
P_0 &= \{(x, y) : 0 \le y \le x < \tfrac{1}{2}\}\,; \\
P_1 &= \{(x, y) : \tfrac{1}{4} < x < \tfrac{1}{2}, 0 \le y < \tfrac{1}{4}, y < \tfrac{1}{2} - x\}\,; \\
P_2 &= \{(x, y) : \tfrac{1}{4} < x \le \tfrac{1}{2}, 0 < y < \tfrac{1}{4}, y > \tfrac{1}{2} - x\}\,; \\
P_3 &= \{(x, y) : \tfrac{1}{4} < y \le x \le \tfrac{1}{2}\}\,.
\end{aligned}
$$

Here, we note that $\widehat{X} = \bigcup_{i=0}^{3} \overline{P_i}$. It follows that P is a topological partition of $(\widehat{X}, \widehat{d})$.

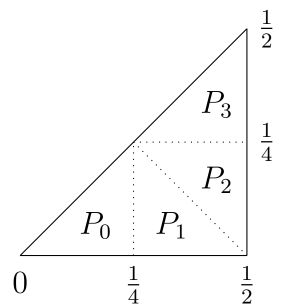


Figure 2.5: The partition of the triangle space $\widehat{X}$

Now, let $T : X \to X$ be a continuous map defined on a compact metric space (X, d), and let $P = \{P_0, P_1, P_2, \ldots, P_{n-1}\}$ be a topological partition of X. Set S to be the set of n symbols $\{0, 1, \ldots, n-1\}$, and let $s = (s_i)_{i=0}^{\infty}$. Suppose Σ_n^+ is the space of all infinite one-sided sequences consisting of n symbols and $\sigma : \Sigma_n^+ \to \Sigma_n^+$ is the one-sided shift map. Given m a natural number, a sequence $s \in \Sigma_n^+$ that starts with $(s_0 s_1 s_2 \cdots s_{m-1})$ determines the following intersection:

$$
\begin{aligned}
I_m(s) &= P_{s_0} \cap T^{-1}(P_{s_1}) \cap T^{-2}(P_{s_2}) \cap \cdots \cap T^{-(m-1)}(P_{s_{m-1}}) \\
&= \bigcap_{j=0}^{m-1} T^{-j}(P_{s_j})\,.
\end{aligned}
$$

We say a sequence $s \in \Sigma_n^+$ is **allowed for** P if

$$I_m(s) \ne \emptyset\,, \text{ for all } m \ge 1\,,$$

and we define

$$\Sigma_P^+ = \{(s_i)_{i=0}^{\infty} : s_i \in S, I_m(s) \neq \emptyset \; \forall \, m \geq 1\},$$

to be the space of all one-sided infinite allowed sequences for P. Here, observe that Σ_P^+ is closed under the one-sided shift map σ since

$$\begin{aligned} T(I_m(s)) &= T\left[P_{s_0} \cap T^{-1}(P_{s_1}) \cap T^{-2}(P_{s_2}) \cap \cdots \cap T^{-(m-1)}(P_{s_{m-1}})\right] \\ &\subseteq T(P_{s_0}) \cap \left[(P_{s_1}) \cap T^{-1}(P_{s_2}) \cap \cdots \cap T^{-(m-2)}(P_{s_{m-1}})\right] \\ &= T(P_{s_0}) \cap I_{m-1}(\sigma(s)). \end{aligned} \tag{2.7}$$

It follows that if $I_m(s) \neq \emptyset$, then $I_{m-1}(\sigma(s)) \neq \emptyset$, for all $m \geq 1$. We will need the following definitions:

Definition 2.30. Let $T : X \to X$ be a continuous map defined on the compact metric space (X, d). A topological partition $P = \{P_0, P_1, P_2, \ldots, P_{n-1}\}$ of X gives a one-sided **symbolic representation** of X if, for every $s \in \Sigma_P^+$, the intersection

$$\bigcap_{m=1}^{\infty} \overline{I_m(s)}$$

consists of exactly one point.

Definition 2.31. A one-sided **Markov partition** $\mathfrak{P}$ is a topological partition which gives a one-sided symbolic representation of X such that $(\Sigma_{\mathfrak{P}}^+, \sigma)$ is a one-sided subshift of finite type, and it satisfies that

$$T^{-j}\left(\bigcup_{i=0}^{n-1} \mathfrak{P}_i\right) \text{ is dense in } X, \text{ for all } j \geq 0. \tag{2.8}$$

Remark 2.32. The typical definition in literature omits (2.8) since, for T a homeomorphism, (2.8) always holds. However, since, in this work, we are also interested in non-invertible maps (such as the triangle map and tent map) which do not, in general, satisfy (2.8), it is necessary to add this condition.

Remark 2.33. By the above definition we have that if $\mathfrak{P}$ is a Markov partition, then $(\Sigma_{\mathfrak{P}}^+, \sigma) = (\Sigma_A^+, \sigma)$, where (Σ_A^+, σ) is the one-sided subshift of finite type as described in the previous section. Hence, we have an $n \times n$ adjacency matrix and its associated graph Γ_A with n vertices $v_0, v_1, \ldots, v_{n-1}$, where each vertex v_i corresponds to a partition piece $\mathfrak{P}_i$ and where we have a directed edge from v_{s_i} to v_{s_j} if and only if there exists $x \in \mathfrak{P}_{s_i}$ such that $T(x) \in \mathfrak{P}_{s_j}$.

Now, suppose that $\mathfrak{P}$ is a one-sided Markov partition for X. Then the intersection $\bigcap_{m=1}^{\infty} \overline{I_m(s)}$ consists of only one point. We call this unique point $\pi(s)$. Hence, we have defined a map $\pi : \Sigma_{\mathfrak{P}}^+ \to X$. Here, note that, for $s \in \Sigma_{\mathfrak{P}}^+$, we have

$$\begin{aligned}
T(\{\pi(s)\}) &= T\left[\bigcap_{m=1}^{\infty} (\overline{I_m(s)})\right] \\
&\subseteq \bigcap_{m=1}^{\infty} T(\overline{I_m(s)}) \\
&\subseteq \bigcap_{m=2}^{\infty} \left[T(\overline{\mathfrak{P}_{s_0}}) \cap \overline{I_{m-1}(\sigma(s))}\right] \qquad \text{(by (2.7))} \\
&= T(\overline{\mathfrak{P}_{s_0}}) \cap \left[\bigcap_{m=1}^{\infty} \overline{I_m(\sigma(s))}\right] \\
&= T(\overline{\mathfrak{P}_{s_0}}) \cap \{\pi(\sigma(s))\} \\
&\subseteq \{\pi(\sigma(s))\},
\end{aligned}$$

and since $T(\{\pi(s)\}) \neq \emptyset$, we must have equality, so that $T(\pi(s)) = \pi(\sigma(s))$, for all $s \in \Sigma_{\mathfrak{P}}^+$. Then the following diagram commutes:

$$\begin{tikzcd}
\Sigma_{\mathfrak{P}}^+ \arrow[r, "\sigma"] \arrow[d, "\pi"'] & \Sigma_{\mathfrak{P}}^+ \arrow[d, "\pi"] \\
X \arrow[r, "T"'] & X
\end{tikzcd}$$

We will show that π is continuous and surjective. We will start by showing continuity. First, note that since $\mathfrak{P}$ is a one-sided symbolic representation of X, we have that $\bigcap_{m=1}^{\infty} \overline{I_m(s)}$ consists of exactly one point. Now, let

$$\mathrm{Diam}(\overline{I_m(s)}) = \sup\{d(x, y) : x, y \in \overline{I_m(s)}\}.$$

Then combining the fact that $\overline{I_1(s)} \supseteq \overline{I_2(s)} \supseteq \overline{I_3(s)} \supseteq \cdots$ and that the intersection of theses sets consists of exactly one point, we have that $\mathrm{Diam}\left(\overline{I_m(s)}\right) \to 0$ as $m \to \infty$.

Now, let $\epsilon > 0$, and let m be such that $\mathrm{Diam}(\overline{I_m(s)}) < \epsilon$. Set $\delta = \frac{1}{2^{m+1}}$, and let $s, s' \in \Sigma_{\mathfrak{P}}^{+}$ be such that $d^*(s, s') < \delta$. Then $s_i = s'_i$, for $0 \leq i \leq m$, so that both $\pi(s), \pi(s') \in \overline{I_m(s)}$. Hence, we have that

$$d(\pi(s), \pi(s')) \leq \mathrm{Diam}(\overline{I_m(s)}) < \epsilon\,.$$

It follows that π is continuous.

We will continue to show that π is surjective. We need the following theorem:

Theorem 2.34. *[8, Baire Category Theorem] Let X be a compact metric space, and suppose that $I_1, I_2, I_3, \ldots$ are all open sets which are dense in X. Then $\bigcap_{i=1}^{\infty} I_i$ is dense in X.*

Now, let

$$U = \bigcup_{i=0}^{n-1} \mathfrak{P}_i\,.$$

Then since $\mathfrak{P}$ is a topological partition, we have that U is open and dense in X. Now, note that since T is continuous, we have that the inverse image of an open set under T is open. Hence, $T^{-i}(U)$ is open, for all $i \geq 0$. Further, since $\mathfrak{P}$ is a Markov partition, we have that $T^{-i}(U)$ is dense in X, for all $i \geq 0$. Then taking

$$U_\infty = \bigcap_{i=0}^{\infty} T^{-i}(U)\,,$$

we have that U_∞ is dense in X by the Baire Category Theorem. Now, take $x \in U_\infty$. Then $x \in T^{-i}(U)$, for all $i \geq 0$. Hence, for each i, there exists a unique $s_i \in \{0, 1, \ldots, n-1\}$ such that $x \in T^{-i}(\mathfrak{P}_{s_i})$. Then taking $s \in \Sigma_n^{+}$, we have that $x \in \bigcap_{m=1}^{\infty} I_m(s)$, so that $s \in \Sigma_{\mathfrak{P}}^{+}$. Further, since $\mathfrak{P}$ is a one-sided symbolic representation of X, we have that, for all $s \in \Sigma_{\mathfrak{P}}^{+}$, $\bigcap_{m=1}^{\infty} \overline{I_m(s)}$ consists of exactly one point. Then since

$$x \in \bigcap_{m=1}^{\infty} I_m(s) \subseteq \bigcap_{m=1}^{\infty} \overline{I_m(s)}\,,$$

we have that

$$\{x\} = \bigcap_{m=1}^{\infty} \overline{I_m(s)} = \bigcap_{m=1}^{\infty} I_m(s)\,.$$

Hence, $\pi(s) = x$, and, since $x \in U_\infty$ was arbitrary, we have that $U_\infty \subseteq \mathrm{Im}(\pi)$. Since $\Sigma_{\mathfrak{P}}^{+}$ is compact (since Σ_n^{+} is compact) and π is continuous, we have that $\mathrm{Im}(\pi)$ is compact. But $\mathrm{Im}(\pi) \subseteq X$ and X is a compact metric space, so $\mathrm{Im}(\pi)$ is a closed set. Now, recall that U_∞ is dense in X. Then since $\mathrm{Im}(\pi)$ is a closed set and $U_\infty \subseteq \mathrm{Im}(\pi)$, we have that $\mathrm{Im}(\pi)$ is dense in X, that is $\mathrm{Im}(\pi) = X$. It follows that π is surjective.

However, π is clearly not necessarily injective: if x is a boundary point, then x may lie in the closure of more than one partition. Then in order to obtain a bijection between the periodic points of T and σ, we need $\mathfrak{P}$ to be a **bijective Markov partition**. A bijective Markov partition is a one-sided Markov partition $\mathfrak{P}$ such that the pairwise intersections $\overline{\mathfrak{P}_i} \cap \overline{\mathfrak{P}_j}$, do not contain any periodic points, for all $i \neq j$.

Now, suppose $\mathfrak{P}$ is a bijective Markov partition. Then if we restrict $\Sigma_{\mathfrak{P}}^+$ to sequences which are periodic, the map $\pi' = \pi \mid_{\mathfrak{F}(\sigma)}$, where $\mathfrak{F}(\sigma) = \bigcup_{k=1}^{\infty} \mathfrak{F}_k(\sigma)$, is injective. Hence, letting $\mathfrak{F}(T) = \bigcup_{k=1}^{\infty} \mathfrak{F}_k(T)$, we have that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{F}(\sigma) & \xrightarrow{\sigma} & \mathfrak{F}(\sigma) \\ \pi' \Big\updownarrow & & \Big\updownarrow \pi' \\ \mathfrak{F}(T) & \xrightarrow{T} & \mathfrak{F}(T) \end{array}$$

It follows that we have a bijection between the periodic points of T and the periodic points of σ, that is $F_k(T) = F_k(\sigma)$, for all $k \geq 1$. Now, recall from the previous section, that

$$F_k(\sigma) = \mathrm{trace}(A^k)\,,$$

for all $k \geq 1$, where A is the $n \times n$ adjacency matrix of the subshift of finite type $\left(\Sigma_A^+, \sigma\right)$. Then if is a bijective Markov partition of X, we have

$$F_k(T) = \mathrm{trace}(A^k)\,,$$

for all $k \geq 1$.

A Markov Partition of the Triangle Map

We will now give a detailed example of a bijective Markov partition for the triangle map $\widehat{T}$. We will start by partitioning the triangle space into the four subsets $P = \{P_0$, P_1, P_2, $P_3\}$ as given in Example 2.29 and shown in Figure 2.6.

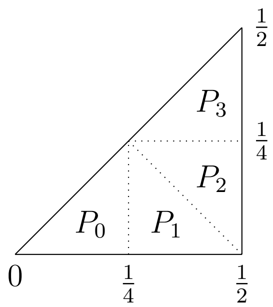
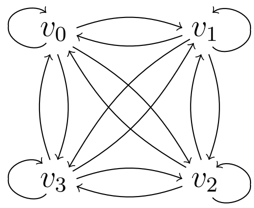


Figure 2.6: The partition of the triangle space $\widehat{X}$

To $\widehat{T}$, we can associate a directed graph Γ with four vertices v_0, v_1, v_2, and v_3, where the vertex v_i corresponds to the partition piece P_i. Here, note that the closure of each subset maps to the entire space $\widehat{X}$, that is $\widehat{T}(\overline{P_i}) = \widehat{X}$, for each $i \in \{0, 1, 2, 3\}$. Hence, we have a directed edge from any vertex v_i to any vertex v_j (including itself) and we obtain the following directed graph Γ:



Moreover, we can associate to Γ the 4×4 adjacency matrix $A = (a_{ij})$, where $0 \leq i, j \leq 3$, and where the number of edges from the vertex v_i to the vertex v_j gives the (i, j)-entry of A. Then $\Gamma = \Gamma_A$, and we have that

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}. \tag{2.9}$$

Now, let $S = \{0, 1, 2, 3\}$ be the set of 4 symbols, and let Σ_A^+ be the space of all one-sided infinite allowed sequences consisting of 4 symbols in S, which gives the full shift on S. Hence, a sequence $s = (s_i)_{i=0}^{\infty}$ in Σ_A^+ is an infinite walk along directed edges on the graph Γ_A, where v_{s_i} is the vertex visited at time i. Further, we let $\sigma : \Sigma_A^+ \to \Sigma_A^+$ be the one-sided shift map. Hence, $\left(\Sigma_A^+, \sigma\right)$ is a one-sided subshift of finite type, and we have that $\left(\Sigma_A^+, \sigma\right) = \left(\Sigma_P^+, \sigma\right)$.

Next, we will note the itinerary for each point $x \in \widehat{X}$, that is we note in which partition $\widehat{T}^i(x)$ lies, for $i \geq 0$. We denote by P_{s_i} the partition piece visited at time i. Hence, we obtain a code $s \in \Sigma_P^+$, where

$$s_i = \begin{cases} 0\,, & \text{if } \widehat{T}^i(x) \in P_0\,; \\ 1\,, & \text{if } \widehat{T}^i(x) \in P_1\,; \\ 2\,, & \text{if } \widehat{T}^i(x) \in P_2\,; \\ 3\,, & \text{if } \widehat{T}^i(x) \in P_3\,. \end{cases}$$

Note that, for $i \neq j$, the pairwise intersections $\overline{P_i} \cap \overline{P_j}$ (given by the dotted lines in Figure 2.6) do not contain any periodic points. Then in order to show that P is a bijective Markov partition, we must show the following:

(i) The intersection $\bigcap_{m=0}^{\infty} \overline{I_m(s)} = \bigcap_{m=1}^{\infty} \overline{\left[\bigcap_{j=0}^{m-1} \widehat{T}^{-j}(P_{s_j})\right]}$ consists of exactly one point, and

(ii) $\widehat{T}^{-j}\left(\bigcup_{i=0}^{3} P_i\right)$ is dense, for all $j \geq 0$.

First, let us consider (i). Here, recall that for $s \in \Sigma_P^+$, we have that $I_m(s) \neq \emptyset$. Then since $I_m(s) \subseteq \overline{I_m(s)}$, we have that $\overline{I_m(s)} \neq \emptyset$, for all $m \geq 1$. Further, since $\overline{I_1(s)} \supseteq \overline{I_2(s)} \supseteq \overline{I_3(s)} \supseteq \cdots$ is a decreasing nested sequence of non-empty compact subsets of X, we have that $\bigcap_{m=0}^{\infty} \overline{I_m(s)} \neq \emptyset$ by Cantor's Intersection Theorem. Hence, if we can show that $\mathrm{Diam}(\overline{I_m(s)}) \to 0$ as $m \to \infty$, then it follows that $\bigcap_{m=0}^{\infty} \overline{I_m(s)}$ consists of exactly one point. This is because if $\mathrm{Diam}(\overline{I_m(s)}) \to 0$ as $m \to \infty$, it follows that $\mathrm{Diam}\left(\bigcap_{m=0}^{\infty} \overline{I_m(s)}\right) = 0$ so that either $\bigcap_{m=0}^{\infty} \overline{I_m(s)}$ is empty (which we know not to be true) or consists of exactly one point.

We will start by using induction to show that $\mathrm{Diam}(\overline{I_m(s)}) = \frac{1}{2^m}\,\mathrm{Diam}(\widehat{X})$, for all $m \geq 2$. Let $m = 2$. Then

$$\overline{I_2(s)} = \overline{P_{s_0} \cap T^{-1}(P_{s_1})} \subseteq \overline{P_{s_0}} \cap T^{-1}(\overline{P_{s_1}})\,.$$

Since, $\overline{I_2(s)}$ is non-empty, we can take some points $x, y \in \overline{I_2(s)}$. Then $x, y \in \overline{P_{s_0}}$ and $x, y \in \widehat{T}^{-1}(\overline{P_{s_1}})$. It follows that $\widehat{T}(x), \widehat{T}(y) \in \overline{P_{s_1}}$. Hence, for $x, y \in \overline{I_2(s)}$, we have

$$\widehat{d}(\widehat{T}(x), \widehat{T}(y)) \leq \sup_{x,y \in \overline{P_{s_0}}} \widehat{d}(\widehat{T}(x), \widehat{T}(y)) = \mathrm{Diam}(\overline{P_{s_1}}) = \mathrm{Diam}(\overline{P_{s_0}})\,.$$

Further, since $\widehat{d}(\widehat{T}(x), \widehat{T}(y)) = 2\,\widehat{d}(x,y)$ for $x, y \in \overline{P_{s_i}}$, for all $i \geq 0$, we obtain that

$$\sup_{x,y \in \overline{P_{s_0}}} \widehat{d}(x,y) = \tfrac{1}{2} \sup_{x,y \in \overline{P_{s_0}}} \widehat{d}(\widehat{T}(x), \widehat{T}(y)) = \tfrac{1}{2}\mathrm{Diam}(\overline{P_{s_0}})\,.$$

Now, note that $\mathrm{Diam}(\overline{P_{s_i}}) = \frac{\sqrt{2}}{4} = (\frac{1}{2})(\frac{\sqrt{2}}{2}) = \frac{1}{2}\ \mathrm{Diam}(\widehat{X})$, for all $i \geq 0$. Hence, it follows that

$$\sup_{x,y \in \overline{P_{s_0}}} \widehat{d}(x,y) = \tfrac{1}{2}\mathrm{Diam}(\overline{P_{s_0}}) = \tfrac{1}{2}\left(\tfrac{1}{2}\mathrm{Diam}(\widehat{X})\right) = \tfrac{1}{2^2}\mathrm{Diam}(\widehat{X})\,,$$

for $x, y \in \overline{I_2(s)}$. Then $\mathrm{Diam}(\overline{I_2(s)}) = \frac{1}{2^2}\ \mathrm{Diam}(\widehat{X})$. Now, assume that

$$\mathrm{Diam}(\overline{I_m(s)}) = \tfrac{1}{2^m}\mathrm{Diam}(\widehat{X})\,,$$

for some $m \in \mathbb{N}$. We have that

$$\begin{aligned}
\overline{I_{m+1}(s)} &= \overline{P_{s_0} \cap \widehat{T}^{-1}(P_{s_1}) \cap \widehat{T}^{-2}(P_{s_2}) \cap \cdots \cap \widehat{T}^{-m}(P_{s_m})} \\
&\subseteq \overline{P_{s_0}} \cap \widehat{T}^{-1}(\overline{P_{s_1}}) \cap \widehat{T}^{-2}(\overline{P_{s_2}}) \cap \cdots \cap \widehat{T}^{-n}(\overline{P_{s_m}}) \\
&= \overline{P_{s_0}} \cap \widehat{T}^{-1}\left[(\overline{P_{s_1}}) \cap \widehat{T}^{-1}(\overline{P_{s_2}}) \cap \cdots \cap \widehat{T}^{-(m-1)}(\overline{P_{s_m}})\right]\,.
\end{aligned}$$

Here, note that

$$\mathrm{Diam}\left[(\overline{P_{s_1}}) \cap \widehat{T}^{-1}(\overline{P_{s_2}}) \cap \cdots \cap \widehat{T}^{-(m-1)}(\overline{P_{s_m}})\right] = \mathrm{Diam}(\overline{I_m(s)}) = \tfrac{1}{2^m}\ \mathrm{Diam}(\widehat{X})\,.$$

Taking $x, y \in \overline{I_{m+1}(s)}$, we have that $x, y \in \widehat{T}^{-m}(\overline{P_{s_m}})$. Then $\widehat{T}(x), \widehat{T}(y) \in \widehat{T}^{-(m-1)}(\overline{P_{s_m}})$. Hence, we obtain

$$\begin{aligned}
\sup_{x,y \in \overline{I_{m+1}(s)}} \widehat{d}(x,y) &= \tfrac{1}{2} \sup_{x,y \in \overline{I_m(s)}} \widehat{d}(\widehat{T}(x), \widehat{T}(y)) \\
&= \tfrac{1}{2}\mathrm{Diam}(\overline{I_m(s)}) = \tfrac{1}{2}\left(\tfrac{1}{2^m}\mathrm{Diam}(\widehat{X})\right) = \tfrac{1}{2^{m+1}}\mathrm{Diam}(\widehat{X})\,.
\end{aligned}$$

Therefore, $\mathrm{Diam}(\overline{I_{m+1}(s)}) = \frac{1}{2^{m+1}}\ \mathrm{Diam}(\widehat{X})$. Then $\mathrm{Diam}(\overline{I_m(s)}) = \frac{1}{2^m}\ \mathrm{Diam}(\widehat{X})$, for all $m \geq 2$. Morover, since

$$\mathrm{Diam}(\overline{I_m(s)}) = \tfrac{1}{2^m}\mathrm{Diam}(\widehat{X}) = \left(\tfrac{1}{2^m}\right)\left(\tfrac{\sqrt{2}}{2}\right) = \tfrac{\sqrt{2}}{2^{m+1}}\,,$$

it follows that $\mathrm{Diam}(\overline{I_m(s)}) \to 0$ as $m \to \infty$. It follows that the intersection of all sets $\overline{I_m(s)}$ consists of exactly one point.

Second, let us consider (ii). Again, we will use a proof by induction. Let $j = 0$. Then

$$\overline{\widehat{T}^0\left(\bigcup_{i=0}^{3} P_i\right)} = \overline{\bigcup_{i=0}^{3} P_i} = \bigcup_{i=0}^{3} \overline{P_i} = \widehat{X}\,.$$

Next, note that for $j = 1$, we have that $T^{-1}\left(\bigcup_{i=0}^{3} P_i\right)$ splits each triangle P_i into four more triangles P_{i0}, P_{i1}, P_{i2} and P_{i3}, where we define $P_{ij} = \{x \in P_i : \widehat{T}(x) \in P_j\}$. Each triangle P_{ij} is $\frac{1}{4}$ of the area of the triangle P_i, and we have that $\widehat{T}(\overline{P_{ij}}) = \overline{P_j}$.

Now, assume that $\widehat{T}^{-j}\left(\bigcup_{i=0}^{3} P_i\right)$ is dense in $\widehat{X}$ for some $j \in \mathbb{N}$. We have that $\widehat{T}^{-j}\left(\bigcup_{i=0}^{3} P_i\right)$ splits each triangle P_i into 4^j triangles $P_{i_0 i_1 i_2 \cdots i_j}$, where we define

$$P_{i_0 i_1 i_2 \cdots i_j} = \{x \in P_{i_0} : \widehat{T}(x) \in P_{i_1}, \widehat{T}^2(x) \in P_{i_2}, \ldots, \widehat{T}^j(x) \in P_{i_j}\}\,,$$

for $i_j \in \{0, 1, 2, 3\}$. Each triangle $P_{i_0 i_1 i_2 \cdots i_j}$ is $\frac{1}{4^j}$ the area of the triangle P_i, and we have that $\widehat{T}(\overline{P_{i_0 i_1 i_2 \cdots i_j}}) = \overline{P_{i_1 i_2 \cdots i_j}}$.

Now, consider $\widehat{T}^{-(j+1)}\left(\bigcup_{i=0}^{3} P_i\right) = \widehat{T}^{-1}\left(\widehat{T}^{-j}\left(\bigcup_{i=0}^{3} P_i\right)\right)$. Here, we have that each triangle $P_{i_0 i_1 i_2 \cdots i_j}$ in P_i is split into four further triangles $P_{i_0 i_1 i_2 \cdots i_j i_{j+1}}$, where we have that

$$\bigcup_{i_0=0}^{3} \overline{P_{i_0 i_1 i_2 \ldots i_j i_{j+1}}} = \overline{P_{i_0 i_1 i_2 \ldots i_j}}\,. \tag{2.10}$$

Then

$$\overline{\widehat{T}^{-(j+1)}\left(P_{i_0}\right)} = \overline{\bigcup_{i_0 i_1 i_2 \ldots i_j i_{j+1}} P_{i_0 i_1 i_2 \ldots i_j i_{j+1}}} = \overline{\bigcup_{i_0 i_1 i_2 \ldots i_j} P_{i_0 i_1 i_2 \ldots i_j}} = \overline{\widehat{T}^{-j}(P_{i_0})}\,,$$

by Equation (2.10). It follows that

$$\overline{\widehat{T}^{-(j+1)}\left(\bigcup_{i=0}^{3} P_i\right)} = \overline{\bigcup_{i_0=0}^{3} \widehat{T}^{-(j+1)}\left(P_{i_0}\right)} = \overline{\bigcup_{i_0=0}^{3} \widehat{T}^{-j}\left(P_{i_0}\right)} = \overline{\widehat{T}^{-j}\left(\bigcup_{i=0}^{3} P_i\right)} = \widehat{X}\,.$$

Therefore, $\widehat{T}^{-j}\left(\bigcup_{i=0}^{3} \overline{P_i}\right)$ is dense in $\widehat{X}$, for every $j \geq 0$.

Combining (i) and (ii) and the fact that, for $i \neq j$, the pairwise intersections $\overline{P_i} \cap \overline{P_j}$ do not contain any periodic points, we have that P satisfies all criteria of a bijective Markov partition. It follows that we have a bijection between the periodic points under σ and $\widehat{T}$, that is $F_k(\sigma) = F_k(\widehat{T})$, for all $k \geq 1$.

Further, from Section (2.4), we know that $F_k(\sigma) = \mathrm{trace}(A^k)$, for all $k \geq 1$, and from (2.9), we obtain that

$$A^k = \begin{pmatrix} 4^{k-1} & 4^{k-1} & 4^{k-1} & 4^{k-1} \\ 4^{k-1} & 4^{k-1} & 4^{k-1} & 4^{k-1} \\ 4^{k-1} & 4^{k-1} & 4^{k-1} & 4^{k-1} \\ 4^{k-1} & 4^{k-1} & 4^{k-1} & 4^{k-1} \end{pmatrix},$$

for all $k \geq 1$. Then $\mathrm{trace}(A^k) = 4 \cdot 4^{k-1} = 4^k$, for all $k \geq 1$. It follows that the number of period k points under the triangle map $\widehat{T}$ is given by

$$F_k(\widehat{T}) = 4^k\,,$$

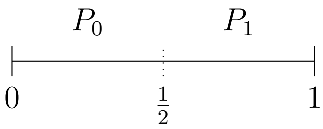
for all $k \geq 1$. ★

A Markov Partition of the Tent Map

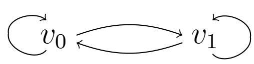
We will now give a detailed example of a bijective Markov partition of the tent map $\widetilde{T}$ defined on the interval $\widetilde{X} = [0,1]$. We will start by partitioning $\widetilde{X}$ into the following two subsets:

$$P_0 = [0, \tfrac{1}{2})\,;$$
$$P_1 = (\tfrac{1}{2}, 1]\,.$$

Hence, we obtain the following partition:



To $\widetilde{T}$, we can associate a directed graph Γ with two vertices v_0 and v_1, where the vertex v_i corresponds to the partition piece P_i. Now, note that for each i, we have $\widetilde{T}(\overline{P_i}) = \widetilde{X}$. Hence, we have a directed edge from the vertex v_i to the vertex v_i and v_j, for $i, j \in \{0,1\}$ and $i \neq j$. We obtain the following directed graph Γ:



Moreover, we can associate to Γ the 2×2 adjacency matrix $A = (a_{ij})$, where $0 \leq i, j \leq 1$, and where the number of edges from the vertex v_i to the vertex v_j gives the (i, j)-entry of A. Then $\Gamma = \Gamma_A$, and we obtain

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} . \tag{2.11}$$

Now, let $S = \{0, 1\}$ be the set of 2 symbols, and let Σ_A^+ be the space of all one-sided infinite allowed sequences consisting of 2 symbols in S, which is the full shift on S. Hence, a sequence $s = (s_i)_{i=0}^{\infty}$ in Σ_A^+ is an infinite walk along directed edges on the graph Γ_A, where v_{s_i} is the vertex visited at time i. Further, we let $\sigma : \Sigma_A^+ \to \Sigma_A^+$ be the one-sided shift map. Hence, $\left(\Sigma_A^+, \sigma\right)$ is a one-sided subshift of finite type, and we have $\left(\Sigma_A^+, \sigma\right) = \left(\Sigma_P^+, \sigma\right)$.

Next, we will note the itinerary for each point $x \in \widetilde{X}$, that is we note in which partition $\widetilde{T}^i(x)$ lies, for $i \geq 0$. We denote by P_{s_i} the partition piece visited at time i. Hence, we obtain a code $s \in \Sigma_P^+$, where

$$s_i = \begin{cases} 0\,, & \text{if } \widetilde{T}^i(x) \in P_0\,; \\ 1\,, & \text{if } \widetilde{T}^i(x) \in P_1\,. \end{cases}$$

Note that the pairwise intersection $\overline{P_0} \cap \overline{P_1} = \{\frac{1}{2}\}$, and therefore it does not contain any periodic points. Hence, in order to show that P is a bijective Markov partition, it is sufficient to show that P is a Markov Partition. That is, we must satisfy the following:

(i) The intersection $\bigcap_{m=0}^{\infty} \overline{I_m(s)} = \bigcap_{m=1}^{\infty} \overline{\left[\bigcap_{i=0}^{m-1} \widetilde{T}^{-i}(P_{s_i})\right]}$ consists of exactly one point, and

(ii) $T^{-j}\left(P_0 \cup P_1\right)$ is dense, for all $j \geq 0$.

First, consider (i). Note that by the same argument as with the previous example of the triangle map, we have that the intersection $\bigcap_{m=0}^{\infty} \overline{I_m(s)}$ is non-empty. Then in order to show this intersection consists of exactly one point it is sufficient to show that $\text{Diam}(\overline{I_m(s)}) \to 0$ as $m \to \infty$. We will give a proof by induction to show that $\text{Diam}(\overline{I_m(s)}) = \frac{1}{2^m}$.

Let $m = 1$. Then

$$\mathrm{Diam}(\overline{I_m(s)}) = \mathrm{Diam}(\overline{P_{s_0}}) = \tfrac{1}{2}\,.$$

Now, assume that $\mathrm{Diam}(\overline{I_m(s)}) = \frac{1}{2^m}$ for some $m \in \mathbb{N}$. Then

$$\mathrm{Diam}(\overline{I_{m+1}(s)}) = \mathrm{Diam}\left(\overline{P_{s_0} \cap \widetilde{T}^{-1}[P_{s_1} \cap \cdots \cap T^{m-1}(P_{s_m})]}\right)\,.$$

Here, note that

$$\mathrm{Diam}\left(\overline{P_{s_1} \cap \cdots \cap T^{m-1}(P_{s_m})}\right) = \mathrm{Diam}(\overline{I_m(s)})$$

and that, for all $x, y \in P_i$, we have $d(\widehat{T}(x), \widehat{T}(x)) = 2\,d(x,y)$. It follows that

$$\mathrm{Diam}(\overline{I_{m+1}(s)}) = \mathrm{Diam}\left(\overline{\widetilde{T}^{-1}(I_m(s))}\right) = \tfrac{1}{2}\mathrm{Diam}(\overline{I_m(s)}) = \frac{1}{2^{m+1}}\,.$$

Hence, $\mathrm{Diam}(\overline{I_m(s)}) = \frac{1}{2^m}$, for all $m \in \mathbb{N}$. Therefore, $\mathrm{Diam}(\overline{I_m(s)}) \to 0$ as $m \to \infty$.

For (ii), by a similar argument as for the triangle map, we have that $\widetilde{T}^{-j}(P_0 \cap P_1)$ is dense in $\widetilde{X}$, for all $j \geq 0$.

Combining (i) and (ii), we have that P is a bijective Markov partition. It follows that we have a bijection between the periodic points under σ and $\widetilde{T}$. Then $F_k(\sigma) = F_k(\widetilde{T}) = \mathrm{trace}(A^k)$, for all $k \geq 1$. Further, from (2.11), we obtain

$$A^k = \begin{pmatrix} 2^{k-1} & 2^{k-1} \\ 2^{k-1} & 2^{k-1} \end{pmatrix}\,,$$

for all $k \geq 1$. Then $\mathrm{trace}(A^k) = 2 \cdot 2^{k-1} = 2^k$, for all $k \geq 1$. It follows that the number of period k points under the tent map $\widetilde{T}$ is given by

$$F_k(\widetilde{T}) = 2^k\,,$$

for all $k \geq 1$. ★

Chapter 3

Quotient Systems

We will now introduce quotient systems. Let (X, d) be a compact metric space and $T : X \to X$ be a homeomorphism such that $F_k(T) < \infty$, for all $k \geq 1$. Then the pair (X, T) is called a **topological dynamical system**. Now, if we let G be a finite group acting on X where the action of G commutes with T, then we call $X' = G\backslash X$ the quotient space and $T' : X' \to X'$ the induced map on the quotient space. The pair (X', T') is called the **quotient system** of (X, T). Hence, we have a topological semi-conjugacy between T and T'. Now, recall from Section 2.3 that there is, in general, no relationship between the count of orbits of two topological semi-conjugate maps. However, in this chapter, we will give two examples of quotient systems, where there indeed exists a relationship. Are these examples representative though?

3.1 Group Actions

Before we continue to discuss quotient systems (topological factor maps), we need a basic understanding of group actions.

In general, a group action is a description of symmetries of objects. The elements of the object are described by a set X, and the symmetries of the object are described by the symmetry group of X or a subgroup of the symmetry group. We have the following definitions:

Definition 3.1. Let X be a set. A bijection of X onto itself is called a **permutation**

of X.

Definition 3.2. The set of all permutations of X forms a group under composition of maps called the **symmetry group** on X, denoted by $\mathrm{Sym}(X)$.

Definition 3.3. If G is a group and X is a set, then a (left) **group action** of G on X is a map $G \times X \to X$, $(g, x) \mapsto g(x)$, that satisfies the following:

(i) $(gh)(x) = g(h(x))$, for all $g, h \in G$ and $x \in X$;

(ii) $e(x) = x$, for all $x \in X$,

where e denotes the identity element of G. The set X is called a (left) ***G*-Set**. We also say that ***G* acts on *X*** (on the left).

Clearly, from the definition of a group action, we can see that for all $g \in G$ the function which maps $x \in X$ to $g(x)$ is bijective (here the inverse is the function which maps x to $g^{-1}(x)$). Therefore, we can alternatively define a group action of G on X to be a group homomorphism from G into the symmetry group $\mathrm{Sym}(X)$.

Example 3.4. Let X be a set. Let H be a subgroup of $\mathrm{Sym}(X)$. Then H acts on X, where the action of $h \in H$ on X is its action as an element of $\mathrm{Sym}(X)$, so that $hx = h(x)$, for all $x \in X$. Then both conditions for a group action are satisfied. (i) follows from the definition of permutation multiplication as function composition. (ii) follows immediately from the definition of the identity permutation as the identity function.

Example 3.5. The trivial action for any group G is defined by $g(x) = x$, for all $g \in G$ and for all $x \in X$. That is, every group element induces the identity permutation on X.

Since we want to study orbits under the action of G, we will start by defining the orbit of a point $x \in X$ under the action of G, denoted by $\mathfrak{O}_G(x)$, by

$$\mathfrak{O}_G(x) = \{g(x) : g \in G\}.$$

Here, the general properties of the group guarantee that the set of orbits of X under the action of G form a partition of X into equivalence classes. We define the associated equivalence relation by letting $x \sim y$ if and only if there exists $g \in G$ such that $g(x) = y$. Then two elements x and y are in the same equivalence class if and only if their orbits are the same, that is $\mathfrak{O}_G(x) = \mathfrak{O}_G(y)$. We have the following definition:

Definition 3.6. The **quotient** of the action of G, denoted by $G\backslash X$, is the set of all orbits of X under the action of G and is defined by

$$G\backslash X = \{\mathfrak{O}_G(x) : x \in X\}.$$

3.2 Topological Factor Maps

Let (X, d) be a compact metric space, $T : X \to X$ be a homeomorphism, and G be a finite group acting on X where the action of G commutes with T. Define $X' = G\backslash X$, to be the quotient space and let $\pi : X \to X'$ be the canonical map $\pi(x) = \mathfrak{O}_G(x)$, for all $x \in X$. Note that since G is finite, then all orbits are finite. It follows that π is surjective. Since we also want π to be continuous, we need to define a metric d' on X': We let

$$d'(\mathfrak{O}_G(x), \mathfrak{O}_G(y)) = \min\{d(a, b) : a \in \mathfrak{O}_G(x), b \in \mathfrak{O}_G(y)\}.$$

We will check that d' satisfies all criteria of a metric:

(i) $d'(\mathfrak{O}_G(x), \mathfrak{O}_G(y)) \geq 0$:

This holds since $d(a, b) \geq 0$ and d' gives the minimum of d.

(ii) $d'(\mathfrak{O}_G(x), \mathfrak{O}_G(y)) = 0$ if and only if $\mathfrak{O}_G(x) = \mathfrak{O}_G(y)$:

$d'(\mathfrak{O}_G(x), \mathfrak{O}_G(y)) = 0$ implies that the minimum distance $d(a, b) = 0$ and this is true if and only if there exists $a \in \mathfrak{O}_G(x)$ and $b \in \mathfrak{O}_G(y)$ such that $a = b$. But then $\mathfrak{O}_G(x) = \mathfrak{O}_G(y)$. If $a = b$, then $d(a, b) = 0$, and we have that $\mathfrak{O}_G(x) = \mathfrak{O}_G(y)$. Since $d(a, b) \geq 0$, then the minimum distance is given by $d(a, b) = 0$. Hence, $d'(\mathfrak{O}_G(x), \mathfrak{O}_G(y)) = 0$.

(iii) $d'(\mathfrak{O}_G(x), \mathfrak{O}_G(y)) = d'(\mathfrak{O}_G(y), \mathfrak{O}_G(y)x)$:

This follows since $d(a, b) = d(b, a)$.

(iv) $d'(\mathfrak{O}_G(x), \mathfrak{O}_G(z)) \leq d'(\mathfrak{O}_G(x), \mathfrak{O}_G(y)) + d'(\mathfrak{O}_G(y), \mathfrak{O}_G(z))$:

We know that $d(a, c) \leq d(a, b) + d(b, c)$ since d is a metric. Then

$$\begin{aligned}
&\min\{d(x, z) : a \in \mathfrak{O}_G(x), c \in \mathfrak{O}_G(z)\} \\
&\leq \min\{d(a, b) + d(b, c) : a \in \mathfrak{O}_G(x), b \in \mathfrak{O}_G(y), c \in \mathfrak{O}_G(z)\} \\
&\leq \min\{d(a, b) : a \in \mathfrak{O}_G(x), b \in \mathfrak{O}_G(y)\} + \min\{d(b, c) : b \in \mathfrak{O}_G(y), c \in \mathfrak{O}_G(z)\}\,.
\end{aligned}$$

Now that we have shown that d' is a metric, we need to show that π is continuous. Let $\epsilon > 0$. Choose $\delta = \epsilon$ and assume that $a, b \in X$ are such that $d(x, y) < \delta$. Then

$$\begin{aligned}
d'(\pi(x), \pi(y)) &= d'(\mathfrak{O}_G(x), \mathfrak{O}_G(y)) \\
&= \min\{d(a, b) : a \in \mathfrak{O}_G(x), b \in \mathfrak{O}_G(y)\} \\
&< \delta = \epsilon\,.
\end{aligned}$$

Hence, π is continuous.

Next, we define the induced map $T' : X' \to X'$ on the quotient space by $T'(\mathfrak{O}_G(x)) = \mathfrak{O}_G(T(x))$, for all $x \in X$. Then $\pi(T(x)) = \mathfrak{O}_G(T(x)) = T'(\mathfrak{O}_G(x)) = T'(\pi(x))$, for all $x \in X$, so that T and T' are topologically semi-conjugate. Then π is called a **topological factor map**. To show that T' is well defined we need to show that if $\mathfrak{O}_G(x) = \mathfrak{O}_G(y)$ then $T'(\mathfrak{O}_G(x)) = T'(\mathfrak{O}_G(y))$. We first note that if $\mathfrak{O}_G(x) = \mathfrak{O}_G(y$, then there exists some $g \in G$ such that $g(x) = y$. Hence, we have that

$$g(T(x)) = T(g(x)) = T(y)$$

if and only if

$$\mathfrak{O}_G(T(x)) = \mathfrak{O}_G(T(g(x))) = \mathfrak{O}_G(T(y))$$

if and only if

$$T'(\mathfrak{O}_G(x)) = T'(\mathfrak{O}_G(g(x))) = T'(\mathfrak{O}_G(y))\,.$$

This holds since g commutes with T. Thus, T' is well defined. It remains to show that T' is continuous.

Let $\epsilon > 0$, and suppose that $a \in \mathfrak{O}_G(x)$ and $b \in \mathfrak{O}_G(y)$ are such that $d'(\mathfrak{O}_G(x), \mathfrak{O}_G(y)) < \delta$. Since T is continuous with respect to d, it follows that $d(T(a), T(b)) < \epsilon$. Here, we note that $T(a) \in \mathfrak{O}_G(T(x)) = T'(\mathfrak{O}_G(x))$ and that $T(b) \in \mathfrak{O}_G(T(y)) = T'(\mathfrak{O}_G(y))$. Then

$$\begin{aligned} d'(T'(\mathfrak{O}_G(x)), T'(\mathfrak{O}_G(y))) &= \min\{d(T(a), T(b)) : T(a) \in T'(\mathfrak{O}_G(x)), T(b) \in T'(\mathfrak{O}_G(y))\} \\ &\leq d(T(a), T(b)) < \epsilon. \end{aligned}$$

Hence, T' is continuous.

The Tent Map: A Quotient of the Circle Doubling Map

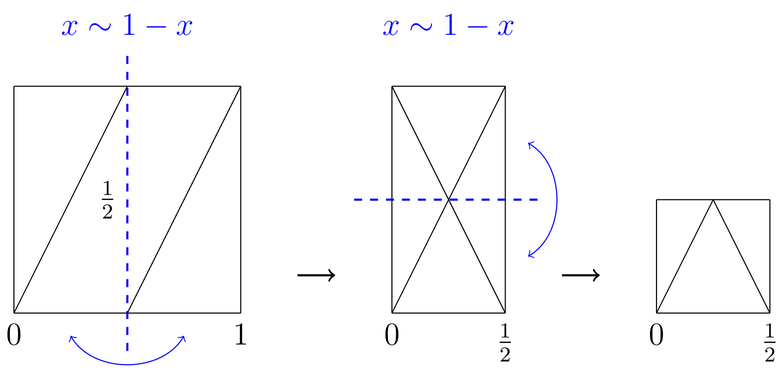
Take T to be the circle doubling map defined on the circle $X = \mathbb{R}/\mathbb{Z}$. Let G be the cyclic group of two elements $C_2 = \{1, \sigma\}$ acting on X, where the action of C_2 is defined by

$$\sigma(x) = 1 - x\,,$$

for all $x \in X$. Then σ acts on X through rotation by $\frac{\pi}{2}$. Hence, we have

$$\mathfrak{O}_G(x) = \{x, 1 - x\}\,,$$

for all $x \in X$, so that $x \sim 1 - x$. Then $\widetilde{X} = [0, \frac{1}{2}]$, and the induced map is given by the tent map as illustrated in the following figure:



We have that three phenomena occur in the quotient system:

(i) Orbits survive. Take, for example, $x = 0$. Then on the circle doubling map we have the orbit

$$0 \to 1 = 0$$

which is the same orbit on the tent map. As a matter of fact, this is the only example of an orbit which survives on the tent map.

(ii) Orbits glue together in pairs. For example, take $x = \frac{1}{7}$ and $y = \frac{6}{7}$. then we have the following two orbits on the circle doubling map:

$$\frac{1}{7} \to \frac{2}{7} \to \frac{4}{7} \to \frac{1}{7} \qquad\qquad \frac{6}{7} \to \frac{5}{7} \to \frac{3}{7} \to \frac{6}{7}$$

On the tent map, however, we have the one orbit

$$\frac{1}{7} \to \frac{2}{7} \to \frac{3}{7} \to \frac{1}{7}$$

since $\left[\frac{1}{7}\right] = \left\{\frac{1}{7}, \frac{6}{7}\right\}$, $\left[\frac{2}{7}\right] = \left\{\frac{2}{7}, \frac{5}{7}\right\}$, and $\left[\frac{3}{7}\right] = \left\{\frac{3}{7}, \frac{4}{7}\right\}$.

(iii) Orbits of even length shorten in length by a factor of $\frac{1}{2}$. For example, take $x = \frac{1}{5}$. Then on the circle doubling map, we have the orbit

$$\frac{1}{5} \to \frac{2}{5} \to \frac{4}{5} \to \frac{3}{5} \to \frac{1}{5}$$

while on the tent map we have the orbit

$$\frac{1}{5} \to \frac{2}{5} \to \frac{1}{5}$$

since $\left[\frac{1}{5}\right] = \left\{\frac{1}{5}, \frac{4}{5}\right\}$, and $\left[\frac{2}{5}\right] = \left\{\frac{2}{5}, \frac{3}{5}\right\}$.

Now, recall from Section 2.2 that we have the following number of period k points for the circle doubling map T and the tent map $\widetilde{T}$:

$$F_k(T) = 2^k - 1 \quad \text{and} \quad F_k(\widetilde{T}) = 2^k\,,$$

for all $k \geq 1$. Then

$$F_k(T) \sim F_k(\widetilde{T}) \quad \text{as } k \to \infty\,,$$

giving the same growth rates for periodic points under (X, T) and its quotient system $(\widetilde{X}, \widetilde{T})$. We can conclude that almost no orbits shorten in length. But is this true in greater generality? We will examine one more example. ★

The Triangle Map: A Quotient of the Doubling Map

Take T to be the doubling map defined on the two-dimensional torus $X = \mathbb{R}^2/\mathbb{Z}^2$. Let G be the dihedral group of eight elements

$$D_8 = \{1, \sigma, \sigma^2, \sigma^3, \tau, \sigma\tau, \sigma^2\tau, \sigma^3\tau\}$$

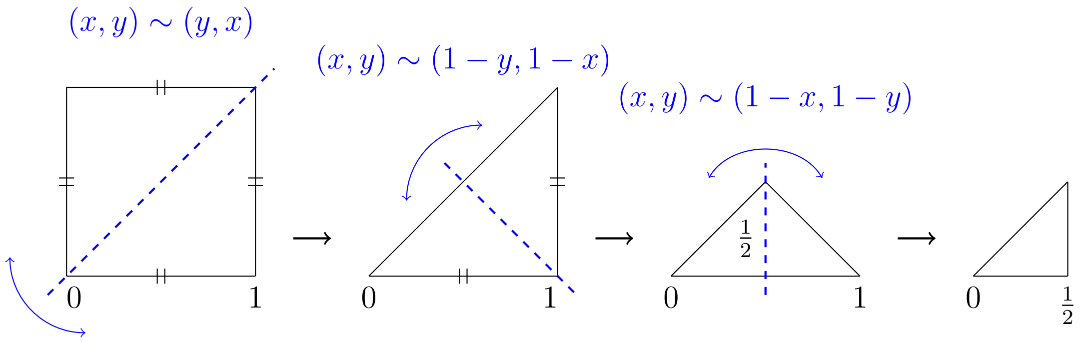
acting on X, where we define the action of D_8 by

$$\sigma(x, y) = (1 - y, x)\,,$$
$$\tau(x, y) = (y, x)\,,$$

for all $(x, y) \in X$. Then σ acts on X through rotation by $\frac{\pi}{2}$, and τ acts on X by reflection. Hence, we have that

$$\mathfrak{O}_{D_8}(x, y) = \{(x, y), (y, x), (1 - y, 1 - x), (1 - x, y), (x, 1 - y),$$
$$(1 - x, 1 - y), (1 - y, x), (y, 1 - x)\}\,,$$

so when D_8 acts on X, we have that all points in $\mathfrak{O}_G(x)$ identify with each other on the torus. Then the induced map is the triangle map defined on the quotient space $\widehat{X} = \{(x, y) : 0 \leq y \leq x \leq \frac{1}{2}\}$, as illustrated in the figure below:



Again, we have that three phenomena occur in the quotient system: surviving orbits, several orbits glueing together, and orbits that shorten in length. Here, orbit behaviour is much more complex then with the previous example of the circle doubling map, and it is possible for an orbit to both glue and shorten at the same time. For example, take $(x, y) = \left(\frac{1}{5}, \frac{2}{5}\right)$ and $(x', y') = \left(\frac{2}{5}, \frac{1}{5}\right)$. Then on the doubling map, we have the two orbits

$$\left(\frac{1}{5},\frac{2}{5}\right) \to \left(\frac{2}{5},\frac{4}{5}\right) \to \left(\frac{4}{5},\frac{3}{5}\right) \to \left(\frac{3}{5},\frac{1}{5}\right) \to \left(\frac{1}{5},\frac{2}{5}\right);$$
$$\left(\frac{2}{5},\frac{1}{5}\right) \to \left(\frac{4}{5},\frac{2}{5}\right) \to \left(\frac{3}{5},\frac{4}{5}\right) \to \left(\frac{1}{5},\frac{3}{5}\right) \to \left(\frac{2}{5},\frac{1}{5}\right).$$

On the triangle map, however, we have the one orbit

$$\left(\frac{2}{5},\frac{1}{5}\right) \to \left(\frac{2}{5},\frac{1}{5}\right)$$

since $\left[\left(\frac{1}{5},\frac{2}{5}\right)\right] = \left\{\left(\frac{1}{5},\frac{2}{5}\right), \left(\frac{2}{5},\frac{4}{5}\right), \left(\frac{4}{5},\frac{3}{5}\right), \left(\frac{3}{5},\frac{1}{5}\right), \left(\frac{2}{5},\frac{1}{5}\right), \left(\frac{4}{5},\frac{2}{5}\right), \left(\frac{3}{5},\frac{4}{5}\right), \left(\frac{1}{5},\frac{3}{5}\right)\right\}$.

Now, note that T maps $(x,y) \in X$ to $(T^*(x), T^*(y))$, where T^* is the circle doubling map defined on $\mathbb{R}/\mathbb{Z}$, and recall that $F_k(T^*) = 2^k - 1$, for all $k \geq 1$. It follows that

$$F_k(T) = (2^k - 1)^2\,,$$

for all $k \geq 1$. Next, recall from Section 2.2 that

$$F_k(\widehat{T}) = 4^k\,,$$

for all $k \geq 1$. It follows that

$$F_k(T) \sim F_k(\widehat{T}) \quad \text{as } k \to \infty\,,$$

giving the same growth rates for periodic points under (X,T) and its quotient system $(\widehat{X},\widehat{T})$. Again, we can conclude that almost no orbits shorten in length. It seems that a pattern is emerging. Are these examples representative for the general case though? This question will be examined in detail in the following chapter on orbit behaviour and orbit growth rates in quotient systems. ★

Chapter 4

Orbit Behaviour and Orbit Growth Rates in Quotient Systems

We will now give a detailed study of orbit behaviour and orbit growth rates occurring in quotient systems (X', T'), where (X', T') is the quotient system of a dynamical system (X, T) under the action of a finite group G.

4.1 Orbit Behaviour in Quotient Systems

Let G be a finite group acting on a set X where the action of G commutes with $T : X \to X$. In order to study orbit behaviour in the quotient system (X', T') of the system (X, T), we will partition X according to the G-conjugacy classes of subgroups of G, so that each partition piece exhibits uniform (but not necessarily unique) orbit behaviour. We have the following definition:

Definition 4.1. Let X be a set, let G be a group acting on X, and let $x \in X$. Then the set of all $g \in G$ which fix x is called the **stabilizer** G_x of x and is defined by

$$G_x = \{g \in G : g(x) = x\}.$$

By $P(G)$, we denote the set of all subgroups of G, and by $\overline{P}(G)$, we denote the set of all G-conjugacy classes of subgroups of G. Then for $H \in P(G)$, we write

$$[H] = \{gHg^{-1} : g \in G\}$$

for its conjugacy class in $\overline{P}(G)$. Further, we let X_H be the set

$$X_H = \{x \in X : G_x = H\}$$

and $X_{[H]}$ be the set

$$X_{[H]} = \{x \in X : G_x \in [H]\} .$$

Here, note that if $g(x) = x$ then $g(T(x)) = T(x)$, for all $x \in X$ and $g \in G$, so that $G_x \subseteq G_{T(x)}$. Moreover, if x is periodic, then if $g(T(x)) = T(x)$, we have $g(x) = x$. Hence, if we let $x \in \mathbb{X}$, where $\mathbb{X}$ is the **periodic set**, that is $\mathbb{X}$ is the subset of all periodic points of X under T, then $G_x = G_{T(x)}$. Then T preserves each set $\mathbb{X}_{[H]} = X_{[H]} \cap \mathbb{X}$, and we have

$$\mathbb{X} = \bigsqcup_{[H] \in \overline{P}(G)} \mathbb{X}_{[H]} .$$

For the induced map $T' : \mathbb{X}' \to \mathbb{X}'$, the quotient set $\mathbb{X}'$ is partitioned as follows:

$$\mathbb{X}' = \bigsqcup_{[H] \in \overline{P}(G)} G \backslash \mathbb{X}_{[H]} = \bigsqcup_{[H] \in \overline{P}(G)} \mathbb{X}'_{[H]} .$$

Again T' preserves each set $X'_{[H]}$.

The Tent Map: A Quotient of the Circle Doubling Map

Take T to be the circle doubling map defined on the circle $X = \mathbb{R}/\mathbb{Z}$. Choose G to be the finite cyclic group of two elements

$$C_2 = \{1, \sigma\}$$

acting on X, where we define the action of C_2 by

$$\sigma(x) = 1 - x ,$$

for all $x \in X$. Then σ acts on X through rotation by $\frac{\pi}{2}$. C_2 has two subgroups: $\{1, \sigma\}$ and $\{1\}$. Hence, we have the following two conjugacy classes:

$$[H_1] = [\{1, \sigma\}] = \{\{1, \sigma\}\} ;$$

$$[H_2] = [\{1\}] = \{\{1\}\} .$$

Then X is partitioned as follows:

$$X_{[H_1]} = \left\{0, \tfrac{1}{2}\right\} \; ;$$
$$X_{[H_2]} = \left(0, \tfrac{1}{2}\right) \cup \left(\tfrac{1}{2}, 1\right) \; .$$

Now, recall that the circle X is equivalent to the interval $[0,1]/\sim$. Then the partition of X is shown in Figure 4.1, where the coloured points give the indicated set.

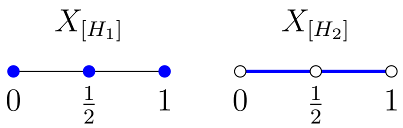


Figure 4.1: The partition of X

Next, recall from Chapter 3 that the quotient system of the circle doubling map under the action of C_2 is given by the tent map $\widetilde{T}$ defined on the interval $\widetilde{X} = \left[0, \frac{1}{2}\right]$. Then $\widetilde{X}_{[H]} = C_2 \backslash X_{[H]}$, and we partition $\widetilde{X}$ as follows, where the partition of $\widetilde{X}$ is shown in Figure 4.2:

$$\widetilde{X}_{[H_1]} = \left\{0, \tfrac{1}{2}\right\} ;$$
$$\widetilde{X}_{[H_2]} = \left(0, \tfrac{1}{2}\right) \; ,$$

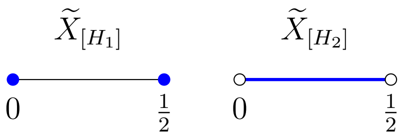


Figure 4.2: The partition of $\widetilde{X}$

Last but not least, we must intersect each partition piece $X_{[H]}$ with the periodic set $\mathbb{X}$. Then, for example, $\widetilde{\mathbb{X}}_{[H_1]} = \{0\}$ since the point $x = \frac{1}{2}$ is not periodic. ★

The Triangle Map: A Quotient of the Doubling Map

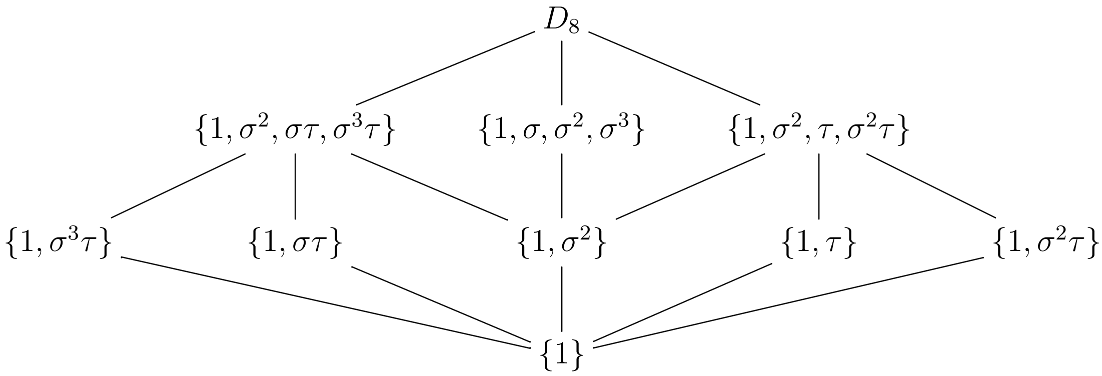
Take T to be the doubling map defined on the two-dimensional torus $X = \mathbb{R}^2/\mathbb{Z}^2$. Choose G to be the dihedral group of 8 elements

$$D_8 = \{1, \sigma, \sigma^2, \sigma^3, \tau, \sigma\tau, \sigma^2\tau, \sigma^3\tau\}$$

acting on X. We define the action of D_8 as follows:

$$\sigma(x, y) = (1 - y, x)\,,$$
$$\tau(x, y) = (y, x)\,,$$

for all $(x, y) \in X$. Then σ acts on X through rotation by $\frac{\pi}{2}$, and τ acts on X by reflection. We have the following diagram showing all subgroups of D_8:



Hence, we have the following conjugacy classes:

$$[H_1] = [\{1, \sigma, \sigma^2, \sigma^3, \tau, \sigma\tau, \sigma^2\tau, \sigma^3\tau\}] = \{\{1, \sigma, \sigma^2, \sigma^3, \tau, \sigma\tau, \sigma^2\tau, \sigma^3\tau\}\}\,;$$
$$[H_2] = [\{1, \sigma^2, \sigma\tau, \sigma^3\tau\}] = \{\{1, \sigma^2, \sigma\tau, \sigma^3\tau\}\}\,;$$
$$[H_3] = [\{1, \sigma, \sigma^2, \sigma^3\}] = \{\{1, \sigma, \sigma^2, \sigma^3\}\}\,;$$
$$[H_4] = [\{1, \sigma^2, \tau, \sigma^2\tau\}] = \{\{1, \sigma^2, \tau, \sigma^2\tau\}\}\,;$$
$$[H_5] = [\{1, \sigma^3\tau\}] = [\{1, \sigma\tau\}] = \{\{1, \sigma^3\tau\}, \{1, \sigma\tau\}\}\,;$$
$$[H_6] = [\{1, \sigma^2\}] = \{\{1, \sigma^2\}\}\,;$$
$$[H_7] = [\{1, \tau\}] = [\{1, \sigma^2\tau\}] = \{\{1, \tau\}, \{1, \sigma^2\tau\}\}\,;$$
$$[H_8] = [\{1\}] = \{\{1\}\}\,.$$

Then

$$
\begin{aligned}
X_{[H_3]} =& X_{[H_4]} = X_{[H_6]} = \emptyset\,; \\
X_{[H_1]} =& \left\{(0,0)\,,\left(\tfrac{1}{2},\tfrac{1}{2}\right)\right\}\,; \\
X_{[H_2]} =& \left\{\left(0,\tfrac{1}{2}\right),\left(\tfrac{1}{2},0\right)\right\}\,; \\
X_{[H_5]} =& \left\{(x,0) : x\in\left(0,\tfrac{1}{2}\right)\cup\left(\tfrac{1}{2},1\right)\right\}\bigcup\left\{\left(x,\tfrac{1}{2}\right) : x\in\left(0,\tfrac{1}{2}\right)\cup\left(\tfrac{1}{2},1\right)\right\}\bigcup \\
& \left\{(0,y) : y\in\left(0,\tfrac{1}{2}\right)\cup\left(\tfrac{1}{2},1\right)\right\}\bigcup\left\{\left(\tfrac{1}{2},y\right) : y\in\left(0,\tfrac{1}{2}\right)\cup\left(\tfrac{1}{2},1\right)\right\}\,; \\
X_{[H_7]} =& \left\{(x,x) : x\in\left(0,\tfrac{1}{2}\right)\cup\left(\tfrac{1}{2},1\right)\right\}\bigcup\left\{(x,-x) : x\in\left(0,\tfrac{1}{2}\right)\cup\left(\tfrac{1}{2},1\right)\right\}\,; \\
X_{[H_8]} =& \left\{(x,y) : x,y\not\equiv 0\,(\bmod 1), x,y\not\equiv \tfrac{1}{2}\,(\bmod 1), x\pm y\equiv 0\,(\bmod 1)\right\}\,,
\end{aligned}
$$

and we partition the space X as shown in Figure 4.3, where the coloured points give the indicated set.

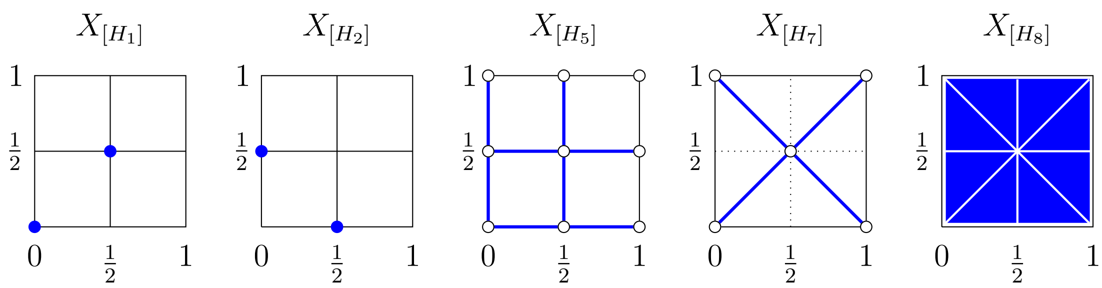


Figure 4.3: The partition of X

Next, recall from Chapter 3 that the quotient map of the doubling map (defined on the two-torus) under the action of D_8 is given by the triangle map $\widehat{T}$. Then $\widehat{X}_{[H]} = D_8\backslash X_{[H]}$, and we partition $\widehat{X}$ as follows, where the partition of $\widehat{X}$ is shown in Figure 4.4:

$$
\begin{aligned}
\widehat{X}_{[H_1]} &= \left\{(0,0)\,,\left(\tfrac{1}{2},\tfrac{1}{2}\right)\right\}\,; \\
\widehat{X}_{[H_2]} &= \left\{\left(\tfrac{1}{2},0\right)\right\}\,; \\
\widehat{X}_{[H_5]} &= \left\{\left(x,\tfrac{1}{2}\right) : 0<x<\tfrac{1}{2}\right\}\bigcup\left\{\left(\tfrac{1}{2},y\right) : 0<y<\tfrac{1}{2}\right\}\,; \\
\widehat{X}_{[H_6]} &= \left\{(x,x) : 0<x<\tfrac{1}{2}\right\}\,; \\
\widehat{X}_{[H_8]} &= \left\{0<x,y<\tfrac{1}{2}, x,y\not\equiv 0\,(\bmod 1), x,y\not\equiv \tfrac{1}{2}\,(\bmod 1), x\pm y\equiv 0\,(\bmod 1)\right\}\,,
\end{aligned}
$$

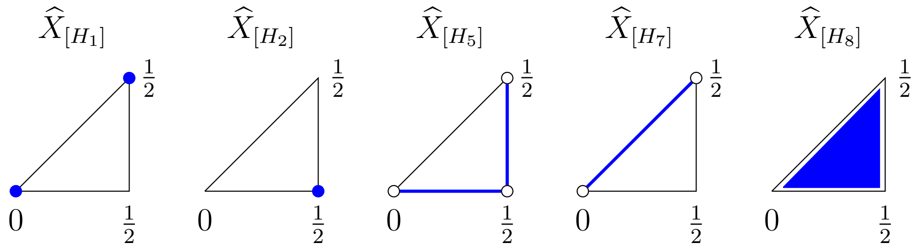


Figure 4.4: The partition of $\widehat{X}$

Last but not least, we must intersect each partition piece $X_{[H]}$ with the periodic set $\mathbb{X}$. Then, for example, $\widehat{\mathbb{X}}_{[H_1]} = \{(0,0)\}$ and $\widehat{\mathbb{X}}_{[H_2]} = \emptyset$. ★

Now, in order to understand orbit behaviour in each partition piece $X_{[H]}$ of X, we need to first compute which elements of G act on X_H and which permute the $X_H \subseteq X_{[H]}$. We need the following definition and lemma:

Definition 4.2. Let H be a subset of a group G. Then the **normalizer** of H in G is defined by

$$N_G(H) = \{g \in G : gHg^{-1} = H\}\,.$$

Lemma 4.3 (Orbit-Stabilizer-Theorem, [7, Proposition 5.1])**.** *Let G be a group acting on a set X. Then for $x \in X$, we have*

$$|\mathfrak{O}_G(x)| = [G : G_x]\,.$$

Here, note that

$$X_{[H]} = \bigsqcup_{j=1}^{|[H]|} X_{H_j}\,,$$

for $H_j \in [H]$. Then for $x \in X_{[H]}$, we have that $x \in X_{H_j}$, for some $H_j \in [H]$. Hence, by the Orbit Stabilizer Theorem, we have that, for $x \in X_{[H]}$,

$$|\mathfrak{O}_G(x)| = [G : G_x] = [G : H_j] = [G : H]\,. \tag{4.1}$$

It follows that

$$|\mathfrak{O}_G(x)| = [G : N_G(H)][N_G(H) : H]\,. \tag{4.2}$$

Now, we have two orbit behaviour phenomena that occur: glueing together of orbits and shortening of orbits in length. **Glueing** orbits are orbits that will identify with other orbits of the same length to form into one orbit. Note that it is possible for both phenomena, glueing and shortening, to occur at the same time.

We will start by discussing when exactly orbits shorten. We have the following proposition:

Proposition 4.4. *Let $T : X \to X$ be a map defined on a set X. Let G be a finite group acting on X where the action of G commutes with T, and let $n \in \mathbb{N}$. Then an orbit $\mathfrak{O}_T(x)$ of $x \in X$ shortens in length by a factor of $\frac{1}{n}$ if and only if $|\mathfrak{O}_T(x) \cap \mathfrak{O}_G(x)| = n$. Moreover, if $\mathfrak{O}_T(x)$ shortens in length by a factor of $\frac{1}{n}$, then* *there exists an element $g \in G$ with order n such that $\mathfrak{O}_T(x) \cap \mathfrak{O}_G(x) = \{g^i(x) : i \in \mathbb{N}_0\} = \mathfrak{O}_{\langle g \rangle}(x)$.*

Proof. First, we will show that if an orbit $\mathfrak{O}_T(x)$ shortens in length by a factor of $\frac{1}{n}$, then $|\mathfrak{O}_T(x) \cap \mathfrak{O}_G(x)| = n$ and there exists an element $g \in G$ with minimal order n. Assume that $\mathfrak{O}_T(x)$ shortens in length by a factor of $\frac{1}{n}$. An orbit can only shorten in length by a factor of $\frac{1}{n}$ if the length is divisible by n since $|\mathfrak{O}_{T'}(x)| \in \mathbb{N}$, where T' denotes the induced map on the quotient space. Thus, we can write that $|\mathfrak{O}_T(x)| = nm$, for some $m \in \mathbb{N}$. Then

$$\mathfrak{O}_T(x) = \{x, T(x), T^2(x), \ldots, T^m(x), \ldots, T^{nm-1}(x)\} .$$

Since the orbit shortens by a factor of $\frac{1}{n}$, we have that $|\mathfrak{O}_{T'}(x)| = m$. Hence, we have that

$$T^m(x) = g(x) ,$$

for some $g \in G$, and m is the least integer for which there is such a g. Further,

$$\begin{aligned}
g^2(x) &= g\,(\,g(x)) = g\,(\,T^m(x)) = T^m(\,g(x)) = T^m(\,T^m(x)) = T^{2m}(x) \\
g^3(x) &= T^{3m}(x) \\
&\ \vdots \\
g^n(x) &= T^{nm}(x) = x ,
\end{aligned}$$

and it follows that

$$\{x, T^m(x), T^{2m}(x), \ldots, T^{(n-1)m}(x)\} \subseteq [\mathfrak{O}_T(x) \cap \mathfrak{O}_G(x)] .$$

Now, assume that there exists $j \geq 1$ such that

$$T^j(x) \in [\mathfrak{O}_T(x) \cap \mathfrak{O}_G(x)] .$$

Then

$$T^j(x) = h(x) ,$$

for some $h \in G$. If $m \nmid j$, then $j = lm + s$, where $l \in \mathbb{N}_0$ and $0 < s < m$. Then

$$\begin{aligned}
T^s(x) = T^{j-lm}(x) \quad &= T^{-lm}(\,T^j(x)) = T^{-lm}(h(x)) \\
= h(\,T^{-lm}(x)) = h(\,g^{-l}(x)) \quad &= h'(x) ,
\end{aligned} \tag{4.3}$$

where $h' = gh^{-l} \in G$.

Remark 4.5. Here, note that T^{-lm} in Equation (4.3) does not imply invertibility of T. We are simply using the fact that T is invertible on $\mathfrak{O}_T(x)$ since $\mathfrak{O}_T(x)$ is finite.

Since $s < m$, Equation (4.3) contradicts the minimality of m. Hence, $m \mid j$, and $h(x) = g^i(x)$, for some $i \in \mathbb{N}_0$. It follows that

$$[\mathfrak{O}_T(x) \cap \mathfrak{O}_G(x)] = \left\{x, T^m(x), T^{2m}(x), \ldots, T^{(n-1)m}(x)\right\}.$$

Then $|\mathfrak{O}_T(x) \cap \mathfrak{O}_G(x)| = n$.

Second, we will show that if $|\mathfrak{O}_T(x) \cap \mathfrak{O}_G(x)| = n$, then $\mathfrak{O}_T(x)$ shortens in length by a factor of $\frac{1}{n}$. Let $|\mathfrak{O}_T(x) \cap \mathfrak{O}_G(x)| = n$. Since this intersection is non-empty, there exists some $m \in \mathbb{N}$ such that

$$T^m(x) = g(x),$$

for some $g \in G$. Assume that m is minimal with this property. It follows that

$$\left\{x, T^m(x), T^{2m}(x), \ldots, T^{(n-1)m}(x), \ldots\right\} \subseteq [\mathfrak{O}_T(x) \cap \mathfrak{O}_G(x)].$$

Now, assume that there exists some $m < j \in \mathbb{N}$ such that $T^j(x) \in [\mathfrak{O}_T(x) \cap \mathfrak{O}_G(x)]$. Then $T^j(x) = h(x)$, for some $h \in G$. If $m \nmid j$, then $j = lm + s$, where $l \in \mathbb{N}$ and $0 < s < m$. By Remark (4.5), we have that

$$\begin{aligned} T^s(x) = T^{j-lm}(x) \quad &= T^{-lm}(T^j(x)) = T^{-lm}(h(x)) \\ = h(T^{-lm}(x)) = h(g^{-l}(x)) \quad &= h'(x), \end{aligned}$$

where $h' = hg^{-l} \in G$. But since $s < m$, this contradicts the minimality of m. Hence, $m \mid j$, and $h(x) = g^i(x)$, for some $i \in \mathbb{N}_0$. Thus,

$$\begin{aligned} [\mathfrak{O}_T(x) \cap \mathfrak{O}_G(x)] &= \left\{x, T^m(x), T^{2m}(x), \ldots, T^{(n-1)m}(x), \ldots\right\} \\ &= \left\{x, g(x), g^2(x), \ldots, g^{(n-1)}(x), \ldots\right\}. \end{aligned}$$

Since $|\mathfrak{O}_T(x) \cap \mathfrak{O}_G(x)| = n$, it follows that $g^n(x) = x$, where n is minimal. Then $g^n(x) = T^{nm}(x) = x$, so $|\mathfrak{O}_T(x)| \leq nm$. Further, since $g^i(x) \neq x$, for $i < n$, $|\mathfrak{O}_T(x)| = nm$, and, by the minimality of m, $|\mathfrak{O}_{T'}(x)| = m$. Hence, $\mathfrak{O}_T(x)$ shortens in length by a factor of $\frac{1}{n}$. $\square$

Now, suppose $x \in X_H$, for some $H \subseteq G$, such that the orbit of x has length $|\mathfrak{O}_T(x)| = mn$ and $|\mathfrak{O}_T(x) \cap \mathfrak{O}_G(x)| = n$, for some $m, n \in \mathbb{N}$. Then by Proposition 4.4, we have that

$$|\mathfrak{O}_T(x) \cap \mathfrak{O}_G(x)| = \{x, g(x), g^2(x), \ldots, g^{n-1}(x)\},$$

where $g(x) = T^m(x)$ and $g^n(x) = x$ and where n is minimal such that $g^n(x) = x$. Now, note that since x is a periodic point in X_H, we have that $G_x = H = G_{T(x)}$. Then $G_{T(x)} = G_{T^m(x)} = G_{g(x)}$. But $G_{g(x)} = gHg^{-1}$, so that $gHg^{-1} = H$, and we have that $g \in N_G(H)$.

Further, since $g^n(x) = x$, we have that $g^n \in G_x = H$ by definition, and since n is minimal such that $g^n(x) = x$, then n is the order of the coset gH in $N_G(H)/H$. Hence, we have that, for $x \in X_H$,

$$|\mathfrak{O}_T(x) \cap \mathfrak{O}_G(x)| = n \in \{i : |\langle h \rangle| = i, \text{ for some } h \in N_G(H)/H\}. \tag{4.4}$$

Then the maximal n such that an orbit of $x \in X_H$ shortens in length by a factor of $\frac{1}{n}$ is given by

$$\delta(N_G(H)/H),$$

where $\delta(N_G(H)/H)$ denotes the largest order of an element of $N_G(H)/H$.

Example 4.6. Let $C_2 = \{1, \sigma\}$. Then σ is the element of C_2 with the largest order, that is order 2. Therefore, $\delta(C_2) = 2 = |C_2|$.

Example 4.7. Let $D_8 = \{1, \sigma, \sigma^2, \sigma^3, \tau, \sigma\tau, \sigma^2\tau, \sigma^3\tau\}$. Here, no single element has order 8, and the element with the largest order is σ which has order 4. Therefore, $\delta(D_8) = 4$.

Example 4.8. Let V_4 be the Klein 4-group consisting of the following permutations: $V_4 = \{(), (12)(34), (13)(24), (14)(23)\}$. Clearly, each element has order 2. Therefore, $\delta(V_4) = 2$.

Example 4.9. Let S_3 be the symmetric group consisting of the following permutations: $S_3 = \{(), (12), (13), (23), (123), (132)\}$. Then both (123) and (132) are elements of order 3, which is the largest order of any element in S_3. Therefore, $\delta(S_3) = 3$.

Next, we will discuss when exactly orbits glue together. We have the following proposition:

Proposition 4.10. *Let $T : X \to X$ be a map defined on a set X. Let G be a finite group acting on X where the action of G commutes with T, and let $x \in X$. Then the number of orbits that glue to $\mathfrak{O}_T(x)$ (including itself) is given by*

$$\frac{|\mathfrak{O}_G(x)|}{|\mathfrak{O}_T(x) \cap \mathfrak{O}_G(x)|}\,.$$

Proof. Set $m = |\mathfrak{O}_G(x)|$. Then $1 \leq |\mathfrak{O}_T(x) \cap \mathfrak{O}_G(x)| \leq m$, say $|\mathfrak{O}_T(x) \cap \mathfrak{O}_G(x)| = n$, for some $n \in \{i : |<h>| = i,$ for some $h \in N_G(H)/H\}$ such that $1 \leq n \leq m$. Then by Theorem 4.4, this implies that there exists an element $h \in N_G(H)/H$ with minimal order n, and since n divides $[N_G(H) : H]$ by definition and $[N_G(H) : H]$ divides $|\mathfrak{O}_G(x)|$ by Equation (4.2), then it follows that n divides $|\mathfrak{O}_G(x)|$. Hence, we have that $m = nl$, for some $l \in \mathbb{N}$. Then when G acts on X, we identify m elements of $\mathfrak{O}_G(x)$. Since $|\mathfrak{O}_T(x) \cap \mathfrak{O}_G(x)| = n$, we have l different orbits glueing together to give a total of m elements of $\mathfrak{O}_G(x)$ glueing together. Hence, the number of glueing orbits is equal to

$$l = \frac{|\mathfrak{O}_G(x)|}{|\mathfrak{O}_T(x) \cap \mathfrak{O}_G(x)|}\,.$$

□

Remark 4.11. Here, note that when

$$\frac{|\mathfrak{O}_G(x)|}{|\mathfrak{O}_T(x) \cap \mathfrak{O}_G(x)|} = 1\,,$$

we have one orbit glueing to itself, that is we have trivial glueing. We say that an orbit with no shortening and trivial glueing is a **surviving** orbit.

Now, let $x \in X_{[H]}$. Then combining Proposition 4.10 and Equations (4.1) and (4.4), we have the following number of glueing orbits in $X_{[H]}$:

$$\frac{|\mathfrak{O}_G(x)|}{|\mathfrak{O}_T(x) \cap \mathfrak{O}_G(x)|} = \frac{[G : H]}{n}\,,$$

where $n \in \{i : |<h>| = i,$ for some $h \in N_G(H)/H\}$. We have the following proposition:

Proposition 4.12. *Given the orbit* $\mathfrak{O}_T(x)$ *of* $x \in X_{[H]}$*, there exists* $n \in \{i : |<h>| = i,\ \text{for some } h \in N_G(H)/H\}$ *such that*

($\mathscr{B}_1$) $\mathfrak{O}_T(x)$ *shortens in length by a factor of* $\frac{1}{n}$*, and*

($\mathscr{B}_2$) $\mathfrak{O}_T(x)$ *glues to* $\frac{[G:H]}{n}$ *orbits (including itself).*

Proof. (i) follows from Proposition 4.4 and Equation (4.4). (ii) follows from (i), Proposition 4.10, and Equation (4.1). □

Remark 4.13. It is possible for orbits to glue without shortening, to shorten without glueing, to do both, or to do neither. For example, if $n = 1$ and $\frac{[G:H]}{n} > 1$, then all orbits glue together without shortening. However, if, for example, $n > 1$ and $\frac{[G:H]}{n} = 1$, then we only have shortening orbits that do not glue together with any other orbits. Moreover, if both $n = 1$ and $\frac{[G:H]}{n} = 1$, then we have no glueing or shortening, in which case we only have surviving orbits.

The Tent Map: A Quotient of the Circle Doubling Map

Let T be the circle doubling map defined on the circle $X = \mathbb{R}/\mathbb{Z}$. Let $(\widetilde{X}, \widetilde{T})$ be the quotient system of (X, T) under the action of C_2, where $\widetilde{T}$ is the tent map defined on the interval $\widetilde{X} = [0, \frac{1}{2}]$. We will study orbit behaviour in each partition piece $X_{[H]}$ of X.

$X_{[H_1]}$ We have $N_{C_2}(H_1) = H_1$. Then by Proposition 4.12, we have that

$$(\mathscr{B}_1) \quad |\mathfrak{O}_{C_2}(x) \cap \mathfrak{O}_T(x)| = 1 \qquad \text{and} \qquad (\mathscr{B}_2) \quad [C_2 : H_1] = 1\,.$$

Hence, we only have surviving orbits on $X_{[H_1]}$. That is, we have

$$O_k^{[H_1]}(\widetilde{T}) = O_k^{[H_1]ns}(T)\,,$$

for all $k \geq 1$, where $O_k^{[H_1]ns}(T)$ denotes the number of non-shortening orbits of length k.

$X_{[H_2]}$ We have $N_{C_2}(H_2) = H_1$. Then by Proposition 4.12, we have that

$$(\mathscr{B}_1) \quad |\mathfrak{O}_{C_2}(x) \cap \mathfrak{O}_T(x)| = n \in \{1, 2\} \qquad \text{and} \qquad (\mathscr{B}_2) \frac{[C_2:H_2]}{n} = \frac{2}{n}\,.$$

Hence, we have the following two orbit behaviours on $X_{[H_2]}$:

(i) $|\mathfrak{O}_{C_2}(x) \cap \mathfrak{O}_T(x)| = 2$. Then $\frac{[C_2:H_2]}{n} = 1$. Thus, orbits will shorten in length by a factor of $\frac{1}{2}$ but will not glue together with any other orbits.

(ii) $|\mathfrak{O}_{C_2}(x) \cap \mathfrak{O}_T(x)| = 1$. Then $\frac{[C_2:H_2]}{n} = 2$. Thus, orbits will not shorten in length but glue together in pairs.

It follows that

$$O_k^{[H_2]}(\widetilde{T}) = \tfrac{1}{2}\, O_k^{[H_2]ns}(T) + O_{2k}^{[H_2]2s}(T)\,,$$

for all $k \geq 1$, where $O_k^{[H_2]ns}(T)$ denotes the number of non-shortening orbits of length k, and where $O_{2k}^{[H_2]2s}(T)$ denotes the number of orbits of length $2k$ which will shorten in length by a factor of $\frac{1}{2}$.

Combining orbit behaviours in $X_{[H_1]}$ and $X_{[H_2]}$, we have the following general relationship between the orbits of the circle doubling map T and the tent map $\widetilde{T}$:

$$O_k(\widetilde{T}) = O_k^{[H_1]}(\widetilde{T}) + O_k^{[H_2]}(\widetilde{T})\,,$$

for all $k \geq 1$.

Please refer to Appendix A for more details including a list of all the points on the circle doubling map that will give rise to shortening orbits on the tent map and a formula for the number of shortening orbits. ★

The Triangle Map: A Quotient of the Doubling Map

Let T be the doubling map defined on the two-dimensional torus $X = \mathbb{R}^2/\mathbb{Z}^2$. Let $(\widehat{X}, \widehat{T})$ be the quotient system of (X, T) under the action of D_8, where $\widehat{T}$ is the triangle map. We will study orbit behaviour in each partition piece $X_{[H]}$ of X. Here, note that we will not consider $X_{[H_2]}$ since $\widehat{\mathbb{X}}_{[H_2]} = \emptyset$.

$X_{[H_1]}$ Here, we have $N_{D_8}(H_1) = D_8$. Then by Proposition 4.12, we have that

$$(\mathscr{B}_1) \quad |\mathfrak{O}_{D_8}(x) \cap \mathfrak{O}_T(x)| = 1 \qquad \text{and} \qquad (\mathscr{B}_2) \quad [D_8 : H_1] = 1\,.$$

Hence, we only have surviving orbits on $X_{[H_1]}$. That is, we have

$$O_k^{[H_1]}(\widetilde{T}) = O_k^{[H_1]_{ns}}(T)\,,$$

for all $k \geq 1$, where $O_k^{[H_1]_{ns}}(T)$ denotes the number of non-shortening orbits of length k.

$X_{[H_5]}$ We have $N_{D_8}(H_5) = H_2$. Then by Proposition 4.12, we have that

$$(\mathscr{B}_1) \quad |\mathfrak{O}_{D_8}(x) \cap \mathfrak{O}_T(x)| = n \in \{1,2\} \qquad \text{and} \qquad (\mathscr{B}_2) \quad \tfrac{[D_8:H_5]}{n} = \tfrac{4}{n}\,.$$

Hence, we have the following two orbit behaviours on $X_{[H_5]}$:

(i) $|\mathfrak{O}_{D_8}(x) \cap \mathfrak{O}_T(x)| = 2$. Then $\frac{[D_8:H_5]}{n} = 2$. Thus, orbits will shorten in length by a factor of $\frac{1}{2}$ and glue together in pairs.

(ii) $|\mathfrak{O}_{D_8}(x) \cap \mathfrak{O}_T(x)| = 1$. Then $\frac{[D_8:H_5]}{n} = 4$. Thus, orbits will not shorten in length but glue together in quadruplets.

It follows that

$$O_k^{[H_5]}(\widehat{T}) = \tfrac{1}{4}\, O_k^{[H_5]_{ns}}(T) + \tfrac{1}{2}\, O_{2k}^{[H_5]_{2s}}(T)\,,$$

for all $k \geq 1$, where $O_k^{[H_5]_{ns}}(T)$ denotes the number of non-shortening orbits of length k and where $O_{2k}^{[H_5]_{2s}}(T)$ denotes the number of orbits of length $2k$ which will shorten in length by a factor of $\frac{1}{2}$.

$X_{[H_7]}$ We have $N_{D_8}(H_7) = H_4$. Then by Proposition 4.12, we have that

$$(\mathscr{B}_1) \quad |\mathfrak{O}_{D_8}(x) \cap \mathfrak{O}_T(x)| = n \in \{1,2\} \qquad \text{and} \qquad (\mathscr{B}_2) \quad \tfrac{[D_8:H_7]}{n} = \tfrac{4}{n}\,.$$

Here, note that $(\mathscr{B}_1)$ and $(\mathscr{B}_2)$ give the same two orbit behaviours as for partition $X_{[H_5]}$. It follows that

$$O_k^{[H_7]}(\widehat{T}) = \tfrac{1}{4}\, O_k^{[H_7]_{ns}}(T) + \tfrac{1}{2}\, O_{2k}^{[H_7]_{2s}}(T)\,,$$

for all $k \geq 1$, where $O_k^{[H_7]_{ns}}(T)$ denotes the number of non-shortening orbits of length k and where $O_{2k}^{[H_7]_{2s}}(T)$ denotes the number of orbits of length $2k$ which will shorten in length by a factor of $\frac{1}{2}$.

$X_{[H_8]}$ We have $N_{D_8}(H_8) = H_1$. Then by Proposition 4.12, we have that

($\mathscr{B}_1$) $|\mathfrak{O}_{D_8}(x) \cap \mathfrak{O}_T(x)| = n \in \{1, 2, 4\}$ (since $\delta(N_{D_8}(H_8)/H_8) = 4$) and

($\mathscr{B}_2$) $\frac{[D_8:H_8]}{n} = \frac{8}{n}$.

Hence, we have the following three orbit behaviours on $X_{[H_8]}$:

(i) $|\mathfrak{O}_{D_8}(x) \cap \mathfrak{O}_T(x)| = 4$. Then $\frac{[D_8:H_8]}{n} = 2$. Thus, orbits shorten in length by a factor of $\frac{1}{4}$ and glue together in pairs.

(ii) $|\mathfrak{O}_{D_8}(x) \cap \mathfrak{O}_T(x)| = 2$. Then $\frac{[D_8:H_8]}{n} = 4$. Thus, orbits shorten in length by a factor of $\frac{1}{2}$ and glue together in quadruplets.

(iii) $|\mathfrak{O}_{D_8}(x) \cap \mathfrak{O}_T(x)| = 1$. Then $\frac{[D_8:H_8]}{n} = 8$. Thus, orbits will not shorten in length but glue together in octuplets.

It follows that

$$O_k^{[H_8]}(\widehat{T}) = \tfrac{1}{8}\, O_k^{[H_8]_{ns}}(T) + \tfrac{1}{4}\, O_{2k}^{[H_8]_{2s}}(T) + \tfrac{1}{2}\, O_{4k}^{[H_8]_{4s}}(T)\,,$$

for all $k \geq 1$, where $O_k^{[H_8]_{ns}}(T)$ denotes the number of non-shortening orbits of length k, where $O_{2k}^{[H_8]_{2s}}(T)$ denotes the number of orbits of length $2k$ which will shorten in length by a factor of $\frac{1}{2}$, and where $O_{4k}^{[H_8]_{4s}}(T)$ denotes the number of orbits of length $4k$ which will shorten in length by a factor of $\frac{1}{4}$.

Combining orbit behaviours in all partition pieces of X, we have the following general relationship between the orbits of the doubling map T and the triangle map $\widehat{T}$:

$$O_k(\widehat{T}) = O_k^{[H_1]}(\widehat{T}) + O_k^{[H_5]}(\widehat{T}) + O_k^{[H_7]}(\widehat{T}) + O_k^{[H_8]}(\widehat{T})\,,$$

for all $k \geq 1$.

Please refer to Appendix B for a list of all the points on the doubling map that will give rise to shortening orbits on the triangle map and a formula for the number of shortening orbits. ★

4.2 Orbit Growth Rates in Quotient Systems

Now, given a finite group G, we will discuss what growth rates can be achieved for orbits in quotient systems.

We will start by defining κ by

$$\kappa(G) = \frac{[G : H_\delta]}{\delta(G)},$$

where H_δ is a maximal subgroup of G such that $N_G(H_\delta)/H_\delta$ has an element of order $\delta(G)$.

Example 4.14. Let $C_2 = \{1, \sigma\}$. Then $\delta(C_2) = 2$ and $H_\delta = \{1\}$ is the largest subgroup of C_2 such that $N_{C_2}(H_\delta)/H_\delta$ has an element of order 2. Therefore, $\kappa(C_2) = 1$.

Example 4.15. Let $D_8 = \{1, \sigma, \sigma^2, \sigma^3, \tau, \sigma\tau, \sigma^2\tau, \sigma^3\tau\}$. Then $\delta(D_8) = 4$ and $H_\delta = \{1\}$ is the largest subgroup of D_8 such that $N_{D_8}(H_\delta)/H_\delta$ has an element of order 4. Therefore, $\kappa(D_8) = 2$.

Example 4.16. Let V_4 be the Klein 4-group consisting of the following permutations: $V_4 = \{(), (12)(34), (13)(24), (14)(23)\}$. Then $\delta(V_4) = 2$ and $H_\delta = \{(), (12)(34)\}$, $H_\delta = \{(), (13)(24)\}$ or $H_\delta = \{(), (14)(23)\}$, where any such H_δ is a maximal subgroup of V_4 such that $N_{V_4}(H_\delta)/H_\delta$ has an element of order 2. Therefore, $\kappa(V_4) = 1$.

Example 4.17. Let S_3 be the symmetric group consisting of the following permutations: $S_3 = \{(), (12), (13), (23), (123), (132)\}$. Then $\delta(S_3) = 3$ and $H_\delta = \{1\}$ is the largest subgroup of S_3 such that $N_{S_3}(H_\delta)/H_\delta$ has an element of order 3. Therefore, $\kappa(S_3) = 2$.

We have the following theorems concerning orbit growth rates:

Theorem 4.18. *Let G be a finite group acting on a set X, and let $T : X \to X$ be any map commuting with the action of G. Denote by $T' : X' \to X'$ the induced map defined on the quotient space $X' = G\backslash X$. Suppose $F_k(T) \sim \lambda^k$ as $k \to \infty$, for some $1 < \lambda \in \mathbb{R}$. Then setting $\delta = \delta(G)$ and $\kappa = \kappa(G)$, we have*

$$\liminf_{k\to\infty} \left(\frac{|G|\, F_k(T')}{\lambda^k} \right) \geq 1 \quad \textit{and} \quad \limsup_{k\to\infty} \left(\frac{\delta\kappa\, F_k(T')}{\lambda^{\delta k}} \right) \leq 1\,.$$

Theorem 4.19. *Let G be a finite group, $\delta = \delta(G)$ and $\kappa = \kappa(G)$. Suppose $1 < \lambda \in \mathbb{R}$, $\gamma \in \mathbb{R}$, and $c \in \mathbb{R}^+$ are such that either*

(i) $\gamma = \lambda$ *and* $c \geq \frac{1}{|G|}$, *or*

(ii) $\lambda < \gamma < \lambda^\delta$, *or*

(iii) $\gamma = \lambda^\delta$ *and* $c \leq \frac{1}{\delta\kappa}$.

Then there exist a system (X, T) and an action of the group G on the set X which commutes with the map $T : X \to X$ such that

$$F_k(T) \sim \lambda^k \quad \textit{as } k \to \infty\,,$$

and

$$F_k(T') \sim c\gamma^k \quad \textit{as } k \to \infty\,,$$

where (X', T') is the quotient system of (X, T) under the action of G. Moreover, we can find such (X, T) with (X, T) a topological dynamical system.

Remark 4.20. Note that Theorem 4.19 shows that any growth rate in between the bounds shown in Theorem 4.18 can be achieved. Further, note that if the growth rate for periodic points is given by

$$F_k(T) \sim \lambda^k \quad \text{as } k \to \infty\,,$$

for some $1 < \lambda \in \mathbb{R}$, then

$$O_k(T) \sim \frac{\lambda^k}{k} \quad \text{as } k \to \infty$$

gives the corresponding growth rate for orbits. Hence, one can easily deduce results concerning orbit growth rates from Theorem 4.19.

Proof of Theorem 4.18. Let G be a finite group. Let (X', T') be the quotient system of (X, T) under the action of G. The lower bound for $F_k(T')$ comes from the fact that the fibres of the topological factor map $\pi : X \to X'$, which are given by

$$\pi^{-1}(\mathfrak{O}_G(x)) = \{x \in X : \pi(x) = \mathfrak{O}_G(x)\}\,,$$

for all $\mathfrak{O}_G(x) \in X'$, have cardinality at most $|G|$. Hence, we can deduce that

$$\frac{F_k(T)}{|G|} \leq F_k(T')\,. \tag{4.5}$$

This lower bound will be achieved with maximal glueing, that is whenever we have that

$$(\mathscr{B}_1) \quad |\mathfrak{O}_G(x) \cap \mathfrak{O}_T(x)| = 1 \qquad \text{and} \qquad (\mathscr{B}_2) \quad \tfrac{|\mathfrak{O}_G(x)|}{n} = |\mathfrak{O}_G(x)| = |G|$$

in Proposition 4.12. Further, by assumption, we have

$$F_k(T) \sim \lambda^k \qquad \text{as } k \to \infty\,.$$

Hence, for $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that when $k > N$, we have that

$$(1-\epsilon)\,\lambda^k < F_k(T) < (1+\epsilon)\,\lambda^k\,. \tag{4.6}$$

Combining (4.5) with (4.6), we obtain

$$\frac{|G|\,F_k(T')}{\lambda^k} > 1-\epsilon\,.$$

It follows that

$$\liminf_{k\to\infty}\left(\frac{|G|\,F_k(T')}{\lambda^k}\right) \geq 1\,.$$

Next, note that for $x \in X$, we have that x lies in exactly one of the partitions X_H, for some $H \subseteq G$. For the upper bound of $F_k(T')$ we note that if $x \in \mathfrak{F}_{nk}(T)$, for some $n \in \{i : |\langle h\rangle| = i, \text{ for some } h \in N_G(H)/H\}$, then $\pi(x) \in \mathfrak{F}_k(T')$ if and only if x lies in an orbit which shortens in length by a factor of $\frac{1}{n}$. Hence, we deduce that

$$F_k(T') \leq \sum_{n=1}^{\delta(G)} \frac{F_{nk}(T)}{j(n)n}\,, \tag{4.7}$$

where $j(n)$ is the minimal possible glueing which can occur when an orbit shortens in length by a factor of $\frac{1}{n}$. This upper bound will be achieved with maximal shortening. That is, for $|\mathfrak{O}_T(x)| = nk$, $n \in \{i : |\langle h\rangle| = i, \text{ for some } h \in N_G(H)/H\}$, we have

$$(\mathscr{B}_1) \quad |\mathfrak{O}_G(x) \cap \mathfrak{O}_T(x)| = n \qquad \text{and} \qquad (\mathscr{B}) \quad \tfrac{|\mathfrak{O}_G(x)|}{n} = j(n)$$

in Proposition 4.12. Now, let $\delta = \delta(G)$ and $\kappa = \kappa(G)$, and note that

$$\begin{aligned}\frac{\delta\kappa F_k(T')}{\lambda^{\delta k}} &\leq \sum_{n=1}^{\delta} \frac{\delta\kappa F_{nk}(T)}{j(n)n\lambda^{\delta k}} \\ &= \frac{F_{\delta k}(T)}{\lambda^{\delta k}} + \sum_{n=1}^{\delta-1} \frac{\delta\kappa F_{nk}(T)}{j(n)n\lambda^{\delta k}}\,,\end{aligned}$$

by (4.7). Further, since $nk < \delta k$ and $j(\delta) = \kappa$, we have that

$$\sum_{n=1}^{\delta-1} \frac{\delta\kappa F_{nk}(T)}{j(n)n\lambda^{\delta k}} \to 0 \quad \text{as } k \to \infty\,.$$

Hence, by (4.6), we have that

$$\limsup_{k\to\infty} \left(\frac{\delta\kappa F_k(T')}{\lambda^{\delta k}}\right) \leq \limsup_{k\to\infty} \left(\frac{F_{\delta k}(T)}{\lambda^{\delta k}}\right) \leq 1\,.$$

$\square$

Before we can continue with the proof of Theorem 4.19, we must first prove the following two lemmas:

Lemma 4.21. *Let G be a finite group. Let $(b_k^v)_{k=1}^\infty$, $(b_k^g)_{k=1}^\infty$, and $(b_k^{\delta s})_{k=1}^\infty$ be sequences of non-negative integers, and set $\delta = \delta(G)$ and $\kappa = \kappa(G)$. Define $(a_k^v)_{k=1}^\infty$, $(a_k^g)_{k=1}^\infty$, and $(a_k^{\delta s})_{k=1}^\infty$ by*

(i) $a_k^v = b_k^v$,

(ii) $a_k^g = |G|b_k^g$, *and*

(iii) $a_k^{\delta s} = \begin{cases} \kappa\, b_{k/\delta}^{\delta s} & \text{if } \delta \mid k\,; \\ 0 & \text{otherwise}\,. \end{cases}$

Further, define $a_k = a_k^v + a_k^g + a_k^{\delta s}$ and $b_k = b_k^v + b_k^g + b_k^{\delta s}$, for all $k \geq 1$. Then there exist a system (X, T) and an action of G on the set X which commutes with $T : X \to X$, such that

$$O_k(T) = a_k\,,$$

and

$$O_k(T') = b_k\,,$$

for all $k \geq 1$, where (X', T') is the quotient system of (X, T) under the action of G.

Lemma 4.22. *Assume that b_1^v, as defined in Lemma 4.21, is such that $b_1^v \geq 1$. Then we can find (X, T), such as in Lemma 4.21, with (X, T) a topological dynamical system.*

Remark 4.23. Note that Lemma 4.21 only considers extreme cases of orbit behaviour: surviving, maximal glueing, and maximal shortening with minimal glueing. Hence, Lemma 4.21 only covers a restricted range of possibilities. However, for the proof of Theorem 4.19 and the purpose of this thesis, the cases considered are sufficient.

Proof of Lemma 4.21. Let G be a finite group, and let H_δ be a maximal subgroup of G such that $N_G(H_\delta)/H_\delta$ contains an element of order $\delta(G)$. Suppose $h_\delta \in N_G(H_\delta)$ is such that $h_\delta H_\delta$ has order $\delta(G)$ in $N_G(H_\delta)/H_\delta$. Set

$$J = \langle h_\delta \rangle \, H_\delta \, .$$

Then

$$[J : H_\delta] = | \, \langle h_\delta H_\delta \rangle \, | = \delta(G) \, ,$$

and

$$[G : J] = \frac{[G : H_\delta]}{[J : H_\delta]} = \frac{[G : H_\delta]}{\delta(G)} = \kappa(G) \, .$$

Now, choose S to be a set of representatives for the coset space G/J. Then $|S| = \kappa(G)$.

Next, set $\delta = \delta(G)$ and $\kappa = \kappa(G)$, and let $(b_k^v)_{k=1}^\infty$, $(b_k^g)_{k=1}^\infty$, and $(b_k^{\delta s})_{k=1}^\infty$ be sequences of non-negative integers. Define $(a_k^v)_{k=1}^\infty$, $(a_k^g)_{k=1}^\infty$, and $(a_k^{\delta s})_{k=1}^\infty$ by

(i) $a_k^v = b_k^v$,

(ii) $a_k^g = |G| b_k^g$, and

(iii) $a_k^{\delta s} = \begin{cases} \kappa \, b_{k/\delta}^{\delta s} & \text{if } \delta \mid k \, ; \\ 0 & \text{otherwise} \, . \end{cases}$

Further, we let $a_k = a_k^v + a_k^g + a_k^{\delta s}$ and $b_k = b_k^v + b_k^g + b_k^{\delta s}$, for all $k \geq 1$. Next, we define

$$X = \bigsqcup_{k \geq 1} X_k \, ,$$

where X_k is the union of closed orbits of length k. We set

$$X_k = X_k^v \sqcup X_k^g \sqcup X_k^{\delta s},$$

for all $k \geq 1$, where

$$\begin{cases} X_k^v = & \mathbb{Z}/k\mathbb{Z} \times \{1, 2, 3, \ldots, a_k^v\}; \\ X_k^g = & \mathbb{Z}/k\mathbb{Z} \times G \times \{1, 2, 3, \ldots, b_k^g\}; \\ X_k^{\delta s} = & \mathbb{Z}/k\mathbb{Z} \times S \times \{1, 2, 3, \ldots, b_{k/\delta}^{\delta s}\}. \end{cases}$$

We define T as follows:

(i) Let $n \in \mathbb{Z}/k\mathbb{Z}$ and $1 \leq i \leq a_k^v$. Then, for $x = (n, i) \in X_k^v$, we define $T(x)$ by

$$T(n, i) = T(n + 1 \,(\mathrm{mod}\ k), i).$$

(ii) Let $n \in \mathbb{Z}/k\mathbb{Z}$, $g' \in G$, and $1 \leq i \leq b_k^g$. Then, for $x = (n, g', i) \in X_k^g$, we define $T(x)$ by

$$T(n, g', i) = T(n + 1 \,(\mathrm{mod}\ k), g', i).$$

(iii) Let $n \in \mathbb{Z}/k\mathbb{Z}$, $s \in S$, and $1 \leq i \leq b_{k/2}^{\delta s}$. Then, for $x = (n, s, i) \in X_k^{\delta s}$, we define $T(x)$ by

$$T(n, s, i) = T(n + 1 \,(\mathrm{mod}\ k), s, i).$$

Now, note that for $s \in S$, we can write $gs = s'h_\delta^{r'}h'$, where $s' \in S$, $r' \in \mathbb{N}_0$, and $h' \in H_\delta$. Here, we have that s' is uniquely determined and r' is taken modulo δ (since $h_\delta H$ has order δ). Hence, we can define an action for $g \in G$ by

$$g(n, i) = (n, i),$$

$$g(n, g', i) = (n, gg', i),$$

$$g(n, s, i) = (n + \tfrac{r'k}{\delta}, s', i),$$

where $g' \in G$. This action is well defined since $\frac{r'k}{\delta}$ is unique modulo k. Now, note that, for $g'', g''' \in G$, we have that

$$(gg'')(n, g', i) = g'''(n, g', i) = (n, g'''g', i),$$

and

$$g(g''(n,g',i) = g(n,g''g',i) = (n,gg''g',i) = (n,g'''g',i)\,.$$

Hence, we have that $(gg'')(n,g',i) = g(g''(n,g',i)$. Further, note that if $g's' = s''h_\delta^{r''}h''$, where $s'' \in S$, $r'' \in \mathbb{N}_0$, and $h'' \in H_\delta$, then we have $(g'g)s = s''h_\delta^{r''}h''h_\delta^{r'}h' = s''h_\delta^{r''+r'}h'''$, where $h''' \in H_\delta$, since h_δ normalizes H_δ, and it follows that

$$(g'g)(n,s,i) = (n + \tfrac{(r'+r'')k}{\delta}, s'', i)\,.$$

Then

$$g'(g(n,s,i)) = g'(n + \tfrac{r'k}{\delta}, s', i) = (n + \tfrac{r'k}{\delta} + \tfrac{r''k}{\delta}, s'', i) = (n + \tfrac{(r'+r'')k}{\delta}, s'', i)\,.$$

Hence, we have that $(g'g)(n,s,i) = g'(g(n,s,i))$, and, therefore, g satisfies all criteria to be an action of G. Moreover, since

$$\begin{aligned} T(g(n,g',i)) &= T(n,gg',i) \\ &= (n+1 \pmod{k}, gg', i) \\ &= g(n+1 \pmod{k}, g', i) \\ &= g(T(n,g',i))\,, \end{aligned}$$

and

$$\begin{aligned} T(g(n,s,i)) &= T(n + \tfrac{r'k}{\delta}, s', i) \\ &= (n + 1 + \tfrac{r'k}{\delta} \pmod{k}, s', i) \\ &= g(n+1 \pmod{k}, s', i) \\ &= g(T(n,s,i))\,, \end{aligned}$$

we have that g commutes with T.

Next, we notice that, for $x \in X_k^v$, we have

$$(\mathscr{B}_1) \quad |\mathfrak{O}_G(x) \cap \mathfrak{O}_T(x)| = 1 \qquad \text{and} \qquad (\mathscr{B}_2) \quad [G:H] = 1\,,$$

so that, by Proposition 4.12, X_k^v only contains surviving orbits. For $x \in X_k^g$, we have that

$$(\mathscr{B}_1) \quad |\mathfrak{O}_G(x) \cap \mathfrak{O}_T(x)| = 1 \qquad \text{and} \qquad (\mathscr{B}_2) \quad [G:H] = |G|\,,$$

so that, by Proposition 4.12, X_k^g only contains maximally glueing orbits, that is all orbits glue together in $|G|$-tuplets. Last but not least, for $x \in X_k^{\delta s}$, we have that

$$(\mathscr{B}_1) \quad |\mathfrak{O}_G(x) \cap \mathfrak{O}_T(x)| = \delta \qquad \text{and} \qquad (\mathscr{B}_2) \quad \tfrac{[G:H]}{\delta} = \kappa\,,$$

so that, by Proposition 4.12, $X_k^{\delta s}$ only contains maximally shortening orbits, that is all orbits shorten in length by a factor of $\frac{1}{\delta}$ and minimally glue together in κ-tuplets. It follows that

$$\begin{aligned} O_k(T) &= a_k^v + |G| b_k^g + \kappa\, b_{k/\delta}^{\delta s} \\ &= a_k^v + a_k^g + a_k^{\delta s} \\ &= a_k\,, \end{aligned}$$

and

$$\begin{aligned} O_k(T') &= a_k^v + b_k^g + b_{k/\delta}^{\delta s} \\ &= b_k^v + b_k^g + b_k^{\delta s} \\ &= b_k\,, \end{aligned}$$

for all $k \geq 1$. □

Proof of Lemma 4.22. Assume that b_1^v and (X, T), as defined in Lemma 4.21, are such that $b_1^v \geq 1$. Then in order to prove that (X, T) is a topological dynamical system, we must show that X can be given a compact metric structure with respect to which T is a homeomorphism.

Now, let

$$X' = X \backslash (\mathbb{Z}/\mathbb{Z})\,.$$

Then X' is equal to X with one copy of $\mathbb{Z}/\mathbb{Z}$ taken away, that is we take away one surviving orbit of length 1. We call this orbit (point) the point at infinity and denote it by ∞. Then

$$X = X' \cup \{\infty\}\,.$$

Next, we define the length m of an orbit $\mathfrak{O}_T(x)$ of $x \in X$ to be equal to

$$m = \begin{cases} |\mathfrak{O}_T(x)| & \text{if } x \neq \infty; \\ \infty & \text{if } x = \infty. \end{cases}$$

Then if we let $\mathfrak{O}_T(y)$ be an orbit of $y \in X$ such that $|\mathfrak{O}_T(y)| = n$, we can define a metric on X by

$$d(x,y) = \begin{cases} \frac{1}{\min\{m,n\}} & \text{if } \mathfrak{O}_T(x) \neq \mathfrak{O}_T(y); \\ 0 & \text{otherwise.} \end{cases}$$

The metric d does indeed satisfy all conditions of a metric: If we let $x, y, z \in X$ be such that $|\mathfrak{O}_T(x)| = m$, $|\mathfrak{O}_T(y)| = n$, and $|\mathfrak{O}_T(z)| = l$, then we have that

$$\frac{\min\{m,l\}}{\min\{m,n\}} + \frac{\min\{m,l\}}{\min\{n,l\}} \geq 1\,. \tag{4.8}$$

This follows since if $\min\{m,l\} = m$ and $\min\{m,n\} = m$, then $\frac{\min\{m,l\}}{\min\{m,n\}} \geq 1$, and if $\min\{m,n\} = n$, then $m > n$, so that $\frac{\min\{m,l\}}{\min\{m,n\}} \geq 1$. By a similar argument, (4.8) holds, for $\min\{m,l\} = l$. Hence, we have that

$$d(x,y) + d(y,z) \geq d(x,z)\,.$$

Now, for $x \in X$ and $\epsilon > 0$, we define the open ball of radius ϵ centered at x to be

$$\begin{aligned} B_\epsilon(x) &= \{y \in X : d(x,y) < \epsilon\} \\ &= \left\{y \in X : \tfrac{1}{\min\{m,n\}} < \epsilon\right\}\,. \end{aligned}$$

Then $B_\epsilon(x)$ contains all orbits whose length is greater than $\frac{1}{\epsilon}$. Taking the union over all $\epsilon > 0$, we have

$$X = \bigcup_{\epsilon>0} B_\epsilon(x)\,.$$

Then X has an open cover $\{U_j\}_{j\in J}$, for $J = \{j_0, j_1, j_2, \ldots, j_r, \ldots\}$, so that

$$X = \bigcup_{j\in J} U_j\,.$$

Now, let j_0 be such that $\infty \in U_{j_0}$. Then since U_{j_0} is open, there exists $\epsilon' > 0$ such that

$$B_{\epsilon'}(\infty) = \{y \in X : d(\infty, y) < \epsilon'\} = \left\{y \in X : \tfrac{1}{|\mathfrak{O}_T(y)|} < \epsilon'\right\} \subseteq U_{j_0}\,.$$

Hence, $B_{\epsilon'}(\infty)$ contains all orbits whose length is greater than $\frac{1}{\epsilon'}$. It follows that $X \backslash B_\epsilon(\infty)$ consists of all orbits whose length is less than or equal to $\frac{1}{\epsilon'}$. Thus, the set $X \backslash B_\epsilon(\infty)$ is finite, and since $X \backslash U_{j_0} \subseteq X \backslash B_\epsilon(\infty)$, then $X \backslash U_{j_0}$ is finite, say

$$X \backslash U_{j_0} = \{x_1, x_2, \ldots, x_r\}\,.$$

Now, for each $i \in \{1, 2, \dots, r\}$, we choose j_i such that $x_i \in U_{j_i}$. Then

$$X = \bigcup_{i=0}^{r} U_{j_i} .$$

Hence, each cover of X has a finite subcover, and X is compact.

It remains to show that T is a homeomorphism. We will start by noting that, for $x \in X_k^v$, we have

$$T^{-1}(x) = T^{-1}(n, i) = (n - 1 \,(\text{mod } k), i) ,$$

for $x \in X_k^g$, we have

$$T^{-1}(x) = T^{-1}(n, g', i) = (n - 1 \,(\text{mod } k), g', i) ,$$

and for $x \in X_k^{\delta s}$, we have

$$T^{-1}(x) = T^{-1}(n, s, i) = (n - 1 \,(\text{mod } k), s, i) .$$

It follows that, for $x \in X_k$,

$$T(T^{-1}(x)) = T^{-1}(T(x)) = x ,$$

so that T is bijective.

Next, if $x \in \mathfrak{O}_T(x)$, then $T(x), T^{-1}(x) \in \mathfrak{O}_T(x)$, so that $|\mathfrak{O}_T(x)| = |\mathfrak{O}_T(T(x))| = |\mathfrak{O}_T(T^{-1}(x))|$. Similarly, if $y \in \mathfrak{O}_T(y)$, then $T(y), T^{-1}(y) \in \mathfrak{O}_T(y)$, so that $|\mathfrak{O}_T(y)| = |\mathfrak{O}_T(T(y))| = |\mathfrak{O}_T(T^{-1}(y))|$. Now, if we set $\delta = \epsilon$, then, for all $\epsilon > 0$, there exists δ such that when $d(x, y) < \delta$, we have that

$$d(T(x), T(y)) = d(T^{-1}(x), T^{-1}(y)) = d(x, y) < \delta = \epsilon .$$

Hence, T and T^{-1} are continuous on X, and it follows that T is a homeomorphism. □

Now that we have proved Lemma 4.21 and Lemma 4.22, we can continue to prove Theorem 4.19:

Proof of Theorem 4.19. Let G be a finite group, and set $\delta = \delta(G)$ and $\kappa = \kappa(G)$. Let $(b_k^v)_{k=1}^{\infty}$, $(b_k^g)_{k=1}^{\infty}$, and $(b_k^{\delta s})_{k=1}^{\infty}$ be sequences of non-negative integers which we will specify later. Define $(a_k^v)_{k=1}^{\infty}$, $(a_k^g)_{k=1}^{\infty}$, and $(a_k^{\delta s})_{k=1}^{\infty}$ by

(i) $a_k^v = b_k^v$,

(ii) $a_k^g = |G| b_k^g$, and

(iii) $a_k^{\delta s} = \begin{cases} \kappa\, b_{k/\delta}^{\delta s} & \text{if } \delta \mid k\,; \\ 0 & \text{otherwise}\,. \end{cases}$

Further, define $a_k = a_k^v + a_k^g + a_k^{\delta s}$ and $b_k = b_k^v + b_k^g + b_k^{\delta s}$, for all $k \geq 1$. Then by Lemma 4.21, there exist a system (X, T) and an action of G on the set X which commutes with $T : X \to X$, such that

$$O_k(T) = a_k\,, \qquad \text{and} \qquad O_k(T') = b_k\,,$$

for all $k \geq 1$, where (X', T') is the quotient system of (X, T) under the action of G.

First, let $\gamma = \lambda$ and $c_1 \in \mathbb{R}$ such that $\frac{1}{|G|} \leq c_1 \leq 1$. Choose $b_k^{\delta s} = 0$,

$$b_k^g = \left\lceil \frac{(c_1 - 1)\lambda^k}{k(1 - |G|)} \right\rceil\,,$$

and $b_k^v = \left\lceil \frac{\lambda^k}{k} \right\rceil - |G| b_k^g$, for all $k \geq 1$. Then

$$\begin{aligned} a_k &= a_k^v + a_k^g \\ &= b_k^v + a_k^g \\ &= \left\lceil \frac{\lambda^k}{k} \right\rceil\,, \end{aligned}$$

and

$$\begin{aligned} b_k &= b_k^v + b_k^g \\ &= \left\lceil \frac{\lambda^k}{k} \right\rceil - |G| b_k^g + b_k^g \\ &= \left\lceil \frac{\lambda^k}{k} \right\rceil + (1 - |G|) b_k^g \\ &= \left\lceil \frac{\lambda^k}{k} \right\rceil + (1 - |G|) \left\lceil \frac{(c_1 - 1)\lambda^k}{k(1 - |G|)} \right\rceil \\ &= \frac{\lambda^k}{k} + \frac{(c_1 - 1)\lambda^k}{k} + r_1 + (1 - |G|) r_2\,, \end{aligned}$$

for all $k \geq 1$ and some $r_1, r_2 \in [0,1)$. Then

$$\begin{aligned}\frac{kb_k}{c_1\lambda^k} &= \frac{1}{c_1} + \frac{c_1 - 1}{c_1} + \frac{k(r_1 + (1 - |G|)r_2)}{c_1\lambda^k} \\ &\to \frac{1}{c_1} + \frac{c_1 - 1}{c_1} \\ &= 1 \qquad \text{as } k \to \infty\,,\end{aligned}$$

so that $O_k(T') \sim \frac{c_1\gamma^k}{k}$ as $k \to \infty$. It follows that

$$F_k(T') \sim c_1\gamma^k \qquad \text{as } k \to \infty\,.$$

Second, let $\gamma \in [\lambda, \lambda^\delta]$. Choose $b_k^g = 0$,

$$b_k^{\delta s} = \left\lceil \frac{c_2'\gamma^k}{k} \right\rceil\,,$$

where

$$c_2' = \begin{cases} c_2 & \text{if } \gamma \in (\lambda, \lambda^\delta]\,; \\ c_2 - 1 & \text{if } \gamma = \lambda\,, \end{cases}$$

and $b_k^v = \left\lceil \frac{\lambda^k}{k} \right\rceil - a_k^{\delta s}$, for all $k \geq 1$. Then

$$\begin{aligned}a_k &= a_k^v + a_k^{\delta s} \\ &= b_k^v + a_k^{\delta s} \\ &= \left\lceil \frac{\lambda^k}{k} \right\rceil\,.\end{aligned}$$

Moreover,

$$\begin{aligned}b_k &= b_k^v + b_k^{\delta s} \\ &= \left\lceil \frac{\lambda^k}{k} \right\rceil - a_k^{\delta s} + b_k^{\delta s} \\ &= \left\lceil \frac{\lambda^k}{k} \right\rceil - a_k^{\delta s} + \left\lceil \frac{c_2'\gamma^k}{k} \right\rceil \\ &= \frac{\lambda^k}{k} - a_k^{\delta s} + \frac{c_2'\gamma^k}{k} + r_3 + r_4\,,\end{aligned}$$

for all $k \geq 1$ and some $r_3, r_4 \in [0,1)$. Then

$$\frac{kb_k}{c_2\gamma^k} = \frac{\lambda^k}{c_2\gamma^k} - \frac{ka_k^{\delta s}}{c_2\gamma^k} + \frac{c_2'}{c_2} + \frac{k(r_3 + r_4)}{c_2\gamma^k}\,.$$

Here, note that

$$\frac{ka_k^{\delta s}}{c_2\gamma^k} = \begin{cases} \left\lceil \frac{\delta c_2' \gamma^{k/\delta}}{k} \right\rceil \left(\frac{k\kappa}{c_2\gamma^k}\right) & \text{if } \delta \mid k\,; \\ 0 & \text{otherwise}\,. \end{cases}$$

Hence, $\frac{ka_k^{\delta s}}{c_2\gamma^k} \to 0$ as $k \to \infty$, for any $\gamma \in [\lambda, \lambda^\delta]$. Now, we will consider two cases:

(i) Let $\gamma = \lambda$. Then $c_2' = c_2 - 1$. Hence, we have that

$$\begin{aligned} \frac{kb_k}{c_2\lambda^k} &= \frac{1}{c_2} - \frac{ka_k^{\delta s}}{c_2\lambda^k} + \frac{c_2 - 1}{c_2} + \frac{k(r_3 + r_4)}{c_2\lambda^k} \\ &\to \frac{1}{c_2} + \frac{c_2 - 1}{c_2} \\ &= 1 \qquad \text{as } k \to \infty\,. \end{aligned}$$

(ii) Let $\gamma \in (\lambda, \lambda^\delta]$. Then $c_2' = c_2$. Hence, we have that

$$\begin{aligned} \frac{kb_k}{c_2\gamma^k} &= \frac{\lambda^k}{c_2\gamma^k} - \frac{ka_k^{\delta s}}{c_2\lambda^k} + 1 + \frac{k(r_3 + r_4)}{c_2\gamma^k} \\ &\to 1 \qquad \text{as } k \to \infty\,, \end{aligned}$$

since $\gamma > \lambda$.

Then $O_k(T') \sim \frac{c_2\gamma^k}{k}$ as $k \to \infty$, and it follows that

$$F_k(T') \sim c_2\gamma^k \qquad \text{as } k \to \infty\,.$$

Summarising, we have found a system (X, T) such that

$$F_k(T) \sim \lambda^k \qquad \text{as } k \to \infty\,,$$

and

$$F_k(T') \sim c\gamma^k \qquad \text{as } k \to \infty\,,$$

where (X', T') is the quotient system of (X, T) under the action of G. Now, since we chose $b_k^v \neq 0$, then we may assume that $b_1^v \geq 1$. Hence, by Lemma 4.22, it follows that (X, T) is a topological dynamical system. $\square$

The Tent Map: A Quotient of the Circle Doubling Map

Let T be the circle doubling map defined on the circle $X = \mathbb{R}/\mathbb{Z}$. Let $(\widetilde{X}, \widetilde{T})$ be the quotient system of (X, T) under the action of C_2, where $\widetilde{T}$ is the tent map defined on the interval $\widetilde{X} = [0, \frac{1}{2}]$.

Recall from Chapter 3 that we have the same asymptotic growth rates for orbits under the circle doubling map T and the tent map $\widetilde{T}$, that is

$$O_k(T) \sim O_k(\widetilde{T}) \qquad \text{as } k \to \infty\,,$$

and we concluded that almost no orbits of the circle doubling map shorten in length on the tent map. We will now give a proof to verify this statement.

We will start by noting that

$$O_{2k}(T) = \tfrac{1}{2k} \sum_{d|2k} \mu\left(\tfrac{2k}{d}\right) 2^d\,, \tag{4.9}$$

for all $k \geq 1$. We have the following propositions:

Proposition 4.24. *Denote by $O^{2s}_{2k}(T)$ the number of orbits of length $2k$ on the circle doubling map T which will shorten in length by a factor of $\frac{1}{2}$ on the tent map $\widetilde{T}$. Then*

$$O^{2s}_{2k}(T) = \tfrac{1}{2k} \sum_{\substack{d|k \\ k/d \text{ odd}}} \mu\left(\tfrac{k}{d}\right) 2^d\,,$$

for all $k \geq 1$.

Proof. Please refer to Appendix A. ∎

Proposition 4.25. *Denote by $O^{2s}_{2k}(T)$ the number of orbits of length $2k$ on the circle doubling map T which will shorten in length by a factor of $\frac{1}{2}$ on the tent map $\widetilde{T}$. Then*

$$\frac{O^{2s}_{2k}(T)}{O_{2k}(T)} \sim \frac{1}{2^k} \qquad \text{as } k \to \infty\,.$$

Proof of Proposition 4.25. From (4.9), we obtain

$$O_{2k}(T) = \frac{1}{2k}\left[2^{2k} + \mathrm{O}\Bigg(\sum_{\substack{d|2k \\ d\leq k}} 2^d\Bigg)\right] = \frac{2^{2k} + \mathrm{O}\left(2^k\right)}{2k}\,.$$

Further, from Proposition 4.24, we have

$$O_{2k}^{2s}(T) = \frac{1}{2k}\left[2^k + \mathrm{O}\Bigg(\sum_{\substack{d|k \\ k/d \text{ odd} \\ d \le k/3}} 2^d\Bigg)\right] = \frac{2^k + \mathrm{O}\left(2^{k/3}\right)}{2k}.$$

Then

$$\frac{O_{2k}^{2s}(T)}{O_{2k}(T)} = \frac{2^k + \mathrm{O}\big(2^{k/3}\big)}{2^{2k} + \mathrm{O}\big(2^k\big)},$$

and

$$2^k\left[\frac{O_{2k}^{2s}(T)}{O_{2k}(T)}\right] = \frac{2^{2k} + \mathrm{O}\big(2^{4k/3}\big)}{2^{2k} + \mathrm{O}\big(2^k\big)} = \frac{1 + \mathrm{O}\big(2^{-2k/3}\big)}{1 + \mathrm{O}\big(2^{-k}\big)}$$
$$\to 1 \qquad \text{as } k \to \infty\,.$$

□

Hence, we have shown that the number of shortening orbits on the circle doubling map is indeed asymptotically zero. In conclusion, this quotient system only shows one of the many (but restricted) possibilities shown in Theorem 4.19 of what orbit growth rates can be achieved in quotient systems. ★

The Triangle Map: A Quotient of the Doubling Map

Let T be the doubling map defined on the two-dimensional torus $X = \mathbb{R}^2/\mathbb{Z}^2$. Let $(\widehat{X}, \widehat{T})$ be the quotient system of (X, T) under the action of D_8, where $\widehat{T}$ is the triangle map.

Now, recall from Chapter 3 that we have the same asymptotic growth rates for orbits under the doubling map T and the triangle map $\widehat{T}$, that is

$$O_k(T) \sim O_k(\widehat{T}) \qquad \text{as } k \to \infty\,,$$

and we concluded that almost no orbits of the doubling map shorten in length on the triangle map. We will now give a proof to verify this statement.

We will start by noting that

$$O_{2k}(T) = \tfrac{1}{2k}\sum_{d|2k} \mu\left(\tfrac{2k}{d}\right)\left(2^d - 1\right)^2, \tag{4.10}$$

and that

$$O_{4k}(T) = \tfrac{1}{4k} \sum_{d|4k} \mu\left(\tfrac{4k}{d}\right)\left(2^d - 1\right)^2 , \tag{4.11}$$

for all $k \geq 1$. We have the following propositions:

Proposition 4.26. *Denote by $O_{2k}^{2s}(T)$ the number of orbits of length $2k$ on the doubling map T which will shorten in length by a factor of $\frac{1}{2}$ on the triangle map $\widehat{T}$. Then, for $k > 1$ odd, we have that*

$$O_{2k}^{2s}(T) = \tfrac{1}{2k} \sum_{\substack{d|k \\ k/d \text{ odd}}} \mu\left(\tfrac{k}{d}\right)\left(5 \cdot 2^{2d} - 4 \cdot 2^d - 4\right) ,$$

and, for k even, we have that

$$O_{2k}^{2s}(T) = \tfrac{1}{2k} \sum_{\substack{d|k \\ k/d \text{ odd}}} \mu\left(\tfrac{k}{d}\right)\left(5 \cdot 2^{2d} - 6 \cdot 2^d - 4\right) .$$

Proof. Please refer to Appendix B. □

Proposition 4.27. *Denote by $O_{4k}^{4s}(T)$ the number of orbits of length $4k$ on the doubling map T which will shorten in length by a factor of $\frac{1}{4}$ on the triangle map $\widehat{T}$. Then*

$$O_{4k}^{4s}(T) = \tfrac{1}{4k} \sum_{\substack{d|k \\ k/d \text{ odd}}} \mu\left(\tfrac{k}{d}\right) 2^{2d+1} ,$$

for all $k \geq 1$.

Proof. Please refer to Appendix B. □

Proposition 4.28. *Denote by $O_{2k}^{2s}(T)$ the number of orbits of length $2k$ on the doubling map T which will shorten in length by a factor of $\frac{1}{2}$ on the triangle map $\widehat{T}$. Then*

$$\frac{O_{2k}^{2s}(T)}{O_{2k}(T)} \sim \frac{5}{2^{2k}} \qquad \text{as } k \to \infty .$$

Proposition 4.29. *Denote by $O_{4k}^{4s}(T)$ the number of orbits of length $4k$ on the doubling map T which will shorten in length by a factor of $\frac{1}{4}$ on the triangle map $\widehat{T}$. Then*

$$\frac{O_{4k}^{4s}(T)}{O_{4k}(T)} \sim \frac{2}{2^{6k}} \qquad \text{as } k \to \infty .$$

Proof of Proposition 4.28. From (4.10), we obtain

$$O_{2k}(T) = \frac{1}{2k}\left[\left(2^{2k}-1\right)^2 + \mathrm{O}\left(\sum_{\substack{d|2k\\ d\le k}} \left(2^d-1\right)^2\right)\right] = \frac{2^{4k}+\mathrm{O}\left(2^{2k}\right)}{2k}.$$

Further, from Proposition 4.26, we have that, for $k>1$ odd,

$$\begin{aligned} O_{2k}^{2s}(T) &= \frac{1}{2k}\left[5\cdot 2^{2k} - 4\cdot 2^k - 4 + \mathrm{O}\left(\sum_{\substack{d|2k\\ k/d \text{ odd}\\ d\le k/3}} 5\cdot 2^{2d} - 4\cdot 2^d - 4\right)\right] \\ &= \frac{5\cdot 2^{2k}+\mathrm{O}\left(2^{2k/3}\right)}{2k}, \end{aligned}$$

and, for even k, we have that

$$\begin{aligned} O_{2k}^{2s}(T) &= \frac{1}{2k}\left[5\cdot 2^{2k} - 6\cdot 2^k - 4 + \mathrm{O}\left(\sum_{\substack{d|2k\\ k/d \text{ odd}\\ d\le k/3}} 5\cdot 2^{2d} - 6\cdot 2^d - 4\right)\right] \\ &= \frac{5\cdot 2^{2k}+\mathrm{O}\left(2^{2k/3}\right)}{2k}. \end{aligned}$$

Then, for all $k>1$, we have

$$\begin{aligned} \frac{2^{2k}}{5}\left[\frac{O_{2k}^{2s}(T)}{O_{2k}(T)}\right] &= \frac{5\cdot 2^{4k}+\mathrm{O}\left(2^{14k/3}\right)}{5\cdot 2^{4k}+\mathrm{O}\left(2^{4k}\right)} = \frac{1+\mathrm{O}\left(2^{-4k/3}\right)}{1+\mathrm{O}\left(2^{-2k}\right)} \\ &\to 1 \quad \text{as } k\to\infty. \end{aligned}$$

□

Proof of Proposition 4.29. From (4.11), we obtain

$$O_{4k}(T) = \tfrac{1}{4k}\left[\left(2^{4k}-1\right)^2 + \mathrm{O}\left(\sum_{\substack{d|4k\\ d\le 2k}} (2^d-1)^2\right)\right] = \frac{2^{8k}+\mathrm{O}\left(2^{4k}\right)}{4k}.$$

Further, from Proposition 4.27, we have

$$O_{4k}^{4s}(T) = \frac{1}{4k}\left[2^{2k+1} + \mathrm{O}\left(\sum_{\substack{d|k\\ k/d \text{ odd}\\ d\le k/3}} 2^{2d+1}\right)\right] = \frac{2^{2k+1}+\mathrm{O}\left(2^{2k/3+1}\right)}{4k}.$$

Then

$$\begin{aligned} \frac{2^{6k}}{2}\left[\frac{O_{4k}^{4s}(T)}{O_{4k}(T)}\right] &= \frac{2^{8k+1}+\mathrm{O}\left(2^{20k/3+1}\right)}{2^{8k+1}+\mathrm{O}\left(2^{4k+1}\right)} = \frac{1+\mathrm{O}\left(2^{-4k/3}\right)}{1+\mathrm{O}\left(2^{-4k}\right)} \\ &\to 1 \quad \text{as } k\to\infty. \end{aligned}$$

□

Hence, we have shown that the number of shortening orbits on the doubling map is indeed asymptotically zero. Again, this quotient system only gives one of the possibilities shown in Theorem 4.19 and, therefore, is not representative for the general case. However, it is worth noting that both quotient systems from Chapter 3 are naturally arising examples rather than the more abstract constructions used in Lemma 4.21 for the proof of Theorem 4.19. Here, we come across an interesting question for future work: Is it true, perhaps, that naturally arising examples always have the property that almost no orbits shorten? Please refer to Chapter 6 on Future Work for a further discussion. ★

Chapter 5

The Dynamical Zeta Function for Quotient Systems

We will now introduce the dynamical zeta function and discuss the significance of its analytic properties.

For any map $T : X \to X$, such that $F_k(T) < \infty$ for all $k \geq 1$, the dynamical zeta function is defined by

$$\zeta_T(z) = \exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k} F_k(T)\right) .$$

Here, note that the dynamical zeta function has a formal expansion as an Euler product given by

$$\zeta_T(z) = \prod_{\mathfrak{O}_T(x)} \left(\frac{1}{1 - z^{|\mathfrak{O}_T(x)|}}\right) .$$

Now, recall that topological conjugacy between two maps preserves the count of periodic points and orbits. Hence, it also preserves the dynamical zeta function. It follows that the dynamical zeta function is an invariant for topological conjugacy. However, if we only have topological semi-conjugacy between two maps, then anything is possible. In particular, it would be possible for one map to have a rational dynamical zeta function while the other map has an irrational dynamical zeta function. When the dynamical zeta function is rational, then we only have a finite number of invariants (namely the poles and the zeros of the dynamical zeta function) that determine the numbers $F_k(T)$, for all $k \geq 1$. Further, Bowen and Lanford showed in [1] that there are only countably many rational dynamical zeta functions. Rationality of the dynamical zeta function implies nice and well behaved growth rates for the sequence $F = (F_k(T))_{k=1}^{\infty}$ while irrationality suggests more complex and irregular behaviour of F.

In this chapter, we will look at examples of quotient systems where one map has a rational dynamical zeta function and the other map has an irrational dynamical zeta function. In order to prove irrationality, we will study the dynamical zeta function in terms of linear recurrence sequences. We have the following definition:

Definition 5.1. A linear recurrence sequence of order at most n is a sequence $a = (a_k)_{k=1}^{\infty}$ of complex numbers where each term is a linear combination of the n preceding terms with fixed coefficients $b_1, b_2, \ldots, b_n$, that is we have

$$a_{m+n} = b_n\, a_{m+n-1} + b_{n-1}\, a_{m+n-2} + \cdots + b_1\, a_m\,, \tag{5.1}$$

for all $m \in \mathbb{N}_0$.

Example 5.2. The Fibonacci numbers are one of the best known linear recurrence sequences. Each number is defined by the sum of the previous two numbers, so that we have the recurrence relation

$$F_{m+2} = F_{m+1} + F_m\,,$$

for all $m \geq 0$, where $F_0 = 0$ and $F_1 = 1$. Hence, we obtain the following sequence of numbers:

$$F = (0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots)$$

An interesting fact shown by Puri-Ward in [13] is that among all the solutions to the Fibonacci recurrence of the form

$$F_a = (1, a, 1 + a, 1 + 2a, 2 + 3a, 3 + 5a, 5 + 8a, \ldots)$$

the only solution that counts periodic points for any map is $a = 3$.

To the recurrence relation (5.1), we can associate the polynomial

$$f(z) = z^n - b_n\, z^{n-1} - \cdots - b_3\, z^2 - b_2\, z - b_1\,,$$

and we call this polynomial a characteristic polynomial of order n. Any linear recurrence relation satisfies a recurrence relation of minimal length. The characteristic polynomial of the minimal length relation is the minimal polynomial of the sequence a. The degree of this minimal polynomial gives the order of the linear recurrence sequence. The minimal polynomial divides the characteristic polynomial. Linear recurrence sequences arise naturally as sequences of Taylor coefficients of rational functions, which can be seen in [17]. We have the following results:

Theorem 5.3 ([5, Theorem 1.5])**.** *A sequence $a = (a_k)_{k=1}^{\infty}$ is a linear recurrence sequence if and only if it is the sequence of Taylor coefficients of a power series representing a rational function.*

Theorem 5.4 ([5, Theorem 2.1, Skolem–Mahler–Lech Theorem])**.** *If a sequence $a = (a_k)_{k=1}^{\infty}$ is a linear recurrence sequence, then the set of zeros, that is the set of all k such that $a_k = 0$, is the union of a finite set together with a finite number of arithmetic progressions.*

Hence, one way to show that a dynamical zeta function is irrational is to first show that its sequence of coefficients does not satisfy the criteria of being a linear recurrence sequence by Skolem-Mahler-Lech, and then use Theorem 5.3 to conclude that the dynamical zeta function is irrational.

Further, it is possible for a dynamical zeta function to have a natural boundary. The dynamical zeta function has a natural boundary if it does not have an analytic continuation beyond its radius of convergence, where we define the radius of convergence, denoted by R, by

$$R = \frac{1}{\limsup\limits_{k\to\infty} \left(\frac{F_k(T)}{k}\right)^{1/k}}.$$

Then at $|z| = R$, we have a dense set of singularities or zeros. Therefore, a dynamical zeta function with a natural boundary implies even stronger results concerning the irregularity and complex behaviour of F compared to a dynamical zeta function which is irrational. An example of a natural boundary is shown in Figure 5.1.

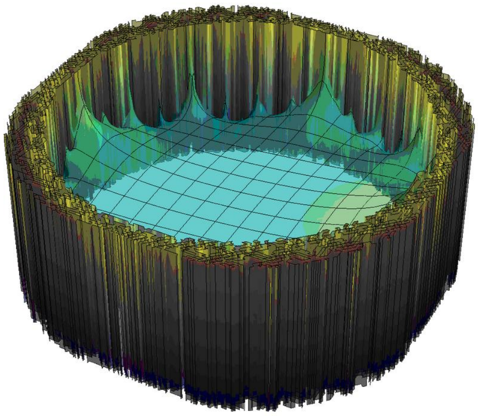


Figure 5.1: (Approximate) plot of the modulus of $\sum_{k=0}^{\infty} z^{2^k}$ showing a natural boundary at the unit circumference, [19] .

In this chapter, we will give an example of a quotient system where one map has a rational dynamical zeta function while the other map has a natural boundary. We have the following result:

Theorem 5.5 (Pólya-Carlson Theorem, [3],[12]). *A power series with integer coefficients and radius of convergence* 1 *is either rational or has the unit circle as a natural boundary.*

Hence, by Theorem 5.5, one way to show that a dynamical zeta function has a natural boundary is to first show that its logarithmic derivative $f(z) = \sum_{k=1}^{\infty} z^k F_k(T)$ has radius of convergence 1 and then prove $f(z)$ to be irrational.

5.1 Irrationality as a Factor of Rationality

We will now give an example of a quotient system where (X, T) has a rational dynamical zeta function while its quotient (X', T') has an irrational dynamical zeta function.

Let $G = \mathbb{Z}/2\mathbb{Z}$, and let (X', T') be a dynamical system. We define $X = X' \times \mathbb{Z}/2\mathbb{Z}$ and $T : X \to X$ by $T(x, n) = (T'(x), n + 1 \pmod 2)$, for $n \in \{0, 1\}$ and $(x, n) \in X$. Further, we define an action of G by

$$g(x, n) = (x, n + 1 \pmod 2)\,,$$

for all $(x, n) \in X$. Then (X', T') is topologically conjugate to the quotient of (X, T). Now, note that T can only have points of period k if $2 \mid k$. Then

$$F_k(T) = \begin{cases} 0\,, & \text{if} \quad k \ \text{is odd}\,; \\ 2F_k(T')\,, & \text{if} \quad k \ \text{is even}\,. \end{cases} \tag{5.2}$$

Next, we choose

$$F_k = \begin{cases} 0\,, & \text{if} \quad k \ \text{is odd}\,; \\ 2^{k+1}\,, & \text{if} \quad k \ \text{is even}\,, \end{cases} \tag{5.3}$$

giving the following sequence of periodic points:

$$F = \left(0, 2^3, 0, 2^5, 0, 2^7, 0, 2^9, 0, 2^{11}, 0, 2^{13}, 0, 2^{15}, 0, \ldots\right)$$

Hence, in order to satisfy Equation (5.2), we choose

$$F'_k = \begin{cases} 0\,, & \text{if} \quad k = 1\,; \\ 2^k\,, & \text{if} \quad k \ \text{is even}\,; \\ \sum\limits_{\substack{d \mid k \\ d \neq 1}} d\, 2^{(d-1)/2}\,, & \text{if} \quad k = 2n+1, n \geq 1\,, \end{cases} \tag{5.4}$$

giving the following sequence of periodic points:

$$F' = (0, 4, 6, 16, 20, 65, 56, 256, 150, 1024, 352, 4096, 832, 16384, 1920, 65536, \ldots)$$

We will continue to show that F' is realizable, that is there exists a map T' such that $F'_k = F_k(T')$, for all $k \geq 1$. Here, note that if F' is realizable, then it follows that F is realizable, by construction.

Now, recall from Theorem 2.4, that F' is realizable if and only if

$$O'_k = \tfrac{1}{k} \sum_{d \mid k} \mu\left(\tfrac{k}{d}\right) F'_d \in \mathbb{N}_0\,,$$

for all $k \geq 1$. Here, we note that since

$$F'_k = \sum_{l \mid k} l\, O'_l = \sum_{\substack{d \mid k \\ d \neq 1}} d\, 2^{(d-1)/2}\,,$$

we have that $O'_k = 2^{(k-1)/2} = 2^n$ whenever $k = 2n+1$, for $n \geq 1$. Further, for $k = 1$, $O'_k = 0$. Hence, $O'_k \in \mathbb{N}_0$, for k odd. It remains to show that $O'_k \in \mathbb{N}_0$, for k even.

We will have to consider two cases. Let $k = 2^l m$, where $l \in \mathbb{N}$ and $m \in \mathbb{N}$ is odd. First, let $l = 1$. We will start by defining a map $T^* : [0,1]^2 \to [0,1]^2$ by $(x,y) \mapsto (\widetilde{T}(x), \widetilde{T}(y))$, where $\widetilde{T}$ is the tent map defined on the interval $[0,1]$. Then $F_k(T^*) = 2^{2k}$, so that $O_k(T^*) \in \mathbb{N}_0$. It follows that

$$\begin{aligned}
O'_k &= \tfrac{1}{k} \sum_{d|k} \mu\left(\tfrac{k}{d}\right) F'_d \\
&= \tfrac{1}{2m} \sum_{\substack{d|2m \\ d \text{ even}}} \mu\left(\tfrac{2m}{d}\right) F'_d + \tfrac{1}{2m} \sum_{\substack{d|2m \\ d \text{ odd}}} \mu\left(\tfrac{2m}{d}\right) F'_d \\
&= \tfrac{1}{2m} \sum_{d|m} \mu\left(\tfrac{m}{d}\right) 2^{2d} - \tfrac{1}{2m} \sum_{d|m} \mu\left(\tfrac{m}{d}\right) F'_d \\
&= \tfrac{1}{2}\, O_m(T^*) - \tfrac{1}{2}\, O'_m\,, \qquad (5.5)
\end{aligned}$$

for all $k \geq 1$. Note that since $O_m(T^*) \in \mathbb{N}_0$, we have that $\frac{1}{m}\sum_{d|m} \mu\left(\frac{m}{d}\right) 2^{2d} \in \mathbb{N}_0$. Then since $\sum_{d|m} \mu(\frac{m}{d}) 2^{2d}$ is even (since every term of the sum is even), we have that $\frac{1}{m}\sum_{d|m} \mu\left(\frac{m}{d}\right) 2^{2d}$ must be even since m is odd. Hence, $\frac{1}{2}\, O'_m(T^*) \in \mathbb{N}_0$. Further, for $m = 1$, we have $O'_1 = 0$, and for $m \geq 3$, we have $m = 2n+1$, for some n, so that $O'_m = 2^n$. Hence, $\frac{1}{2}\, O'_m \in \mathbb{N}_0$. It follows that equation (5.5) gives an integer.

It remains to show non-negativity, that is we need to show that

$$\tfrac{1}{2} O_m(T^*) \geq \tfrac{1}{2} O'_m\,.$$

First, note that

$$\begin{aligned}
\sum_{d|m} \mu\left(\tfrac{m}{d}\right) 2^{2d} &= 2^{2m} + \sum_{\substack{d|m \\ d<m}} \mu\left(\tfrac{m}{d}\right) 2^{2d} \\
&\geq 2^{2m} - m\, 2^{2m/3}
\end{aligned}$$

since all divisors $d < m$ of m are at most $\frac{m}{3}$ since m is odd. Further, since $2^m \geq 2m$, we have that

$$2^{2m} \geq 2m\, 2^m = m\, \left(2^m + 2^m\right) \geq m\, 2^{(m-1)/2} + m\, 2^{(2m)/3}$$

since $m \geq \frac{m-1}{2}$ and $m \geq \frac{2m}{3}$. Hence, we have that

$$\begin{aligned}
\tfrac{1}{2} O_m(T^*) &= \tfrac{1}{2m} \sum_{d|m} \mu\left(\tfrac{m}{d}\right) 2^{2d} \\
&\geq \frac{2^{2m} - m\, 2^{(2m)/3}}{2m} \\
&\geq \frac{m\, 2^{(m-1)/2}}{2m} \\
&= \frac{2^{(m-1)/2}}{2} \\
&= \tfrac{1}{2} O'_m \,.
\end{aligned}$$

Second, consider the case when $l > 1$. We have

$$\begin{aligned}
\tfrac{1}{k} \sum_{d|k} \mu\left(\tfrac{k}{d}\right) F'_d &= \tfrac{1}{k} \sum_{\substack{d|k \\ d \text{ even}}} \mu\left(\tfrac{k}{d}\right) F'_d + \tfrac{1}{k} \sum_{\substack{d|k \\ d \text{ odd}}} \mu\left(\tfrac{k}{d}\right) F'_d \\
&= \tfrac{1}{k} \sum_{\substack{d|k \\ d \text{ even}}} \mu\left(\tfrac{k}{d}\right) 2^d + \tfrac{1}{k} \sum_{\substack{d|k \\ d \text{ odd}}} \mu\left(\tfrac{k}{d}\right) F'_d \,,
\end{aligned} \tag{5.6}$$

for all $k \geq 1$. Here, note that $\mu\left(\frac{k}{d}\right) = 0$, whenever d is odd. Hence, the second term on the right hand side of Equation (5.6) is equal to 0. Further, we have that

$$\tfrac{1}{k} \sum_{\substack{d|k \\ d \text{ even}}} \mu\left(\tfrac{k}{d}\right) 2^d = \tfrac{1}{k} \sum_{d|k} \mu\left(\tfrac{k}{d}\right) 2^d - \tfrac{1}{k} \sum_{\substack{d|k \\ d \text{ odd}}} \mu\left(\tfrac{k}{d}\right) 2^d \,, \tag{5.7}$$

and since the second term of the right hand side of Equation (5.7) is equal to 0 by the same argument as for Equation (5.6), we have that (5.7) is equal to $O_k(\widetilde{T})$, for all $k \geq 1$. It follows that

$$\tfrac{1}{k} \sum_{d|k} \mu\left(\tfrac{k}{d}\right) F'_d = O_k(\widetilde{T}) \in \mathbb{N}_0 \,,$$

for all $k \geq 1$.

Remark 5.6. Here, note that in order to show that F' is the sequence of periodic points of a quotient system by a G-action of a system which has F as its sequence of periodic points, we could have also used Lemma 4.21.

Choose $b_k^v = 0$, and set

$$b_k^g = \begin{cases} O_k(T')\,, & \text{for } k \text{ even}\,; \\ 0\,, & \text{otherwise}\,, \end{cases}$$

and

$$b_k^{2s} = \begin{cases} O_k(T')\,, & \text{for } k \text{ odd}\,; \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Then setting

(i) $a_k^v = 0$,

(ii) $a_k^g = 2O_k(T')$, and

(iii) $a_k^{2s} = \begin{cases} b_{k/2}^{2s}\,, & \text{for } k \text{ even}\,; \\ 0\,, & \text{otherwise}\,, \end{cases}$

and $a_k = a_k^v + a_k^g + a_k^{2s}$ and $b_k = b_k^v + b_k^g + b_k^{2s}$, for all $k \geq 1$, we have that $a_k = O_k$ and $b_k = O'_k$. Then by Lemma 4.21, there exists (X, T) such that $F_k = F_k(T)$ and $F'_k = F_k(T')$, for all $k \geq 1$, where (X', T') is the quotient system of (X, T) under a C_2 action.

Last but not least, we will show that the dynamical zeta functions for T is rational while the dynamical zeta function for T' is irrational. We will start by looking at the dynamical zeta function for T. Here, we have that, for $|z| < \frac{1}{2}$, the dynamical zeta function is given by

$$\begin{aligned} \zeta_T(z) &= \exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k} F_k(T)\right) \\ &= \exp\left(\sum_{k=1}^{\infty} \frac{z^{2k}}{2k} 2^{2k+1}\right) \\ &= \exp\left(\sum_{k=1}^{\infty} \frac{(2z)^{2k}}{k}\right) \\ &= \exp\left(-\log\left(1-(2z)^2\right)\right) \\ &= \frac{1}{1-4z^2}\,. \end{aligned}$$

It follows that $\zeta_T(z)$ is rational.

Next, we will study the dynamical zeta function for T'. First note that $\zeta_{T'}(z)$ is rational only if the logarithmic derivative $z\left[\frac{\zeta'_{T'}(z)}{\zeta_{T'}(z)}\right]$ is rational, where

$$
\begin{aligned}
z\left[\frac{\zeta'_{T'}(z)}{\zeta_{T'}(z)}\right] &= \sum_{k=1}^{\infty} z^k\, F_k(T') \\
&= \sum_{k=1}^{\infty} z^{2k}\, F_{2k}(T') + \sum_{k=1}^{\infty} z^{2k-1}\, F_{2k-1}(T')\,. \qquad (5.8)
\end{aligned}
$$

Hence, to show that the zeta function of T' is irrational, it is sufficient to show that exactly one of the terms of Equation (5.8) is irrational. We will start by showing that the first term of the right hand side of Equation (5.8) is rational. For $|z| < \frac{1}{2}$, we have that

$$
\begin{aligned}
\sum_{k=1}^{\infty} z^{2k}\, F_{2k}(T') &= \sum_{k=1}^{\infty} z^{2k}\, 2^{2k} \\
&= (2z)^2\left[1 + (2z)^2 + \left[(2z)^2\right]^2 + \left[(2z)^2\right]^3 + \cdots\right] \\
&= \frac{4z^2}{1-4z^2}\,,
\end{aligned}
$$

which is rational. It remains to show that the second term on the right hand side of Equation (5.8) is irrational. For $|z| < \frac{1}{2}$, we have that

$$
\begin{aligned}
\sum_{k=1}^{\infty} z^{2k-1}\, F_{2k-1}(T') &= z\, F_1(T') + \sum_{k=2}^{\infty} z^{2k-1} F_{2k-1}(T') \\
&= \sum_{k=1}^{\infty} z^{2k+1} F_{2k+1}(T') \\
&= \sum_{k=1}^{\infty} z^{2k+1} \sum_{\substack{d|2k+1 \\ d\neq 1}} d\, 2^{(d-1)/2} \\
&= \sum_{k=1}^{\infty} (2k+1)\, 2^k z^{2k+1} + \varphi(z)\,, \qquad (5.9)
\end{aligned}
$$

where

$$
\varphi(z) = \sum_{k=1}^{\infty} z^{2k+1} \sum_{\substack{d|2k+1 \\ d\neq 1 \\ d\neq 2k+1}} d\, 2^{(d-1)/2}\,.
$$

Now, if we take

$$
f(z) = \sum_{k=1}^{\infty} 2^k z^{2k+1} = \frac{2z^3}{1-2z^2}\,,
$$

then the first term of the right hand side of Equation (5.9) is equal to $z\,f'(z)$ which is given by

$$\sum_{k=1}^{\infty}(2k+1)\,2^k z^{2k+1} = \frac{6z^3-4z^5}{(1-2z^2)^2}\,,$$

which is rational. We are left with the term $\varphi(z)$. Note that since all other terms of $\zeta_{T'}(z)$ are rational, it follows that $\zeta_{T'}(z)$ is irrational if and only if $\varphi(z)$ is irrational. We will study $\varphi(z)$ in terms of linear recurrence sequences.

Let $a = (a_k)_{k=1}^{\infty}$ be the sequence of coefficients of $\varphi(z)$ given by $a_{2k} = 0$ and

$$a_{2k+1} = \sum_{\substack{d|2k+1\\ d\neq 1\\ d\neq 2k+1}} d\,2^{(d-1)/2}\,.$$

Then by Theorem 5.3,

$$\varphi(z) = \sum_{k=1}^{\infty} a_k z^k$$

is rational if and only if a is a linear recurrence sequence.

Now, set $Z = \{k : a_k = 0\}$ to be the set of zeros, and assume a to be a linear recurrence sequence. Note that, since $a_k = 0$ whenever k is an odd prime, we have that

$$\{p : p \text{ is prime}, p > 2\} \subseteq Z\,.$$

Since the set of primes is infinite, then Z must contain an arithmetic progression containing an odd prime by Theorem 5.4. But any arithmetic progression containing an odd prime p also contains an odd composite number $2k+1$ since, if the common difference in the progression is d, then $p + (2p)d = (2d+1)p$ is odd and composite. But $a_{2k+1} \neq 0$ since $2k+1 = (2d+1)p$ is odd but not prime, which is a contradiction. It follows that a does not satisfy the criteria to be a linear recurrence sequence by Theorem 5.4. Then by Theorem 5.3, $\varphi(z)$ is irrational. It follows that the dynamical zeta function of T' is irrational.

5.2 Rationality as a Factor of Irrationality

We will now give an example of a quotient system where the topological dynamical system (X, T) has an irrational dynamical zeta function while the quotient system

(X', T') has a rational dynamical zeta function.

First, we choose

$$F_k = 2^k + \sigma(k)\,,$$

for all $k \geq 1$, where $\sigma(k)$ is the sum of all the divisors of k defined by

$$\sigma(k) = \sum_{d|k} d\,.$$

We will show that $F = (F_k(T))_{k=1}^{\infty}$ is realizable. First, we note that $F_k = F_k(\widetilde{T}) + \sigma(k)$, where $\widetilde{T}$ is the tent map. Then by the definition of $\sigma(k)$, we have that $F_k \in \mathbb{N}_0$, for all $k \geq 1$. Further, we have that

$$\begin{aligned} O_k &= \tfrac{1}{k} \sum_{d|k} \mu\left(\tfrac{k}{d}\right)\left(2^d + \sigma(d)\right) \\ &= \tfrac{1}{k} \sum_{d|k} \mu\left(\tfrac{k}{d}\right) 2^d + \frac{1}{k} \sum_{d|k} \mu\left(\tfrac{k}{d}\right) \sigma(d) \\ &= O_k(\widetilde{T}) + 1\,, \end{aligned}$$

for all $k \geq 1$, since $\sigma(k) = \sum_{d|k} d = \sum_{d|k} dO_d$ for $O_d = 1$, for all d. Then $O_k \in \mathbb{N}_0$, for all $k \geq 1$. Hence, F is realizable.

Next, we choose

$$F'_k = 2^k\,,$$

for all $k \geq 1$. Then since $F'_k = F_k(\widetilde{T})$, for all k, we have that F' is realizable.

Now, we will use Lemma 4.21 and Lemma 4.22 to show that F' is the sequence of periodic points of a quotient system by a C_2 action of a system which has F as its sequence of periodic points.

Choose $b_k^v = O'_k - 1 = \left[\tfrac{1}{k} \sum_{d|k} \mu\left(\tfrac{k}{d}\right) F'_d\right] - 1$, $b_k^g = 1$, and $b_k^{2s} = 0$, for all $k \geq 1$. Then setting

(i) $a_k^v = b_k^v$,

(ii) $a_k^g = 2$, and

(iii) $a_k^{2s} = 0$,

and

$$a_k = a_k^v + a_k^g + a_k^{2s} = O'_k + 1 = O_k \,,$$

and

$$b_k = b_k^v + b_k^g + b_k^{2s} = O'_k \,,$$

for all $k \geq 1$, we have that by Lemma 4.21 there exists (X, T) such that $F_k = F_k(T)$ and $F'_k = F_k(T')$, for all $k \geq 1$, where (X', T') is the quotient system of (X, T) under a C_2 action. Further, since $b_1^v \geq 1$, we have that (X, T) is a topological dynamical system by Lemma 4.22.

Next, we would like to show that T has an irrational dynamical zeta function while T' has a rational dynamical zeta function. We will start by showing that T' has a rational dynamical zeta function. Here, for $|z| < \frac{1}{2}$, we have

$$\begin{aligned}
\zeta_{T'}(z) &= \exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k} F_k(T')\right) \\
&= \exp\left(\sum_{k=1}^{\infty} \frac{(2z)^k}{k}\right) \\
&= \frac{1}{1-2z} \,.
\end{aligned}$$

It follows that $\zeta_{T'}(z)$ is rational.

Next, let us consider the dynamical zeta funtion of T. First, note that $\zeta_T(z)$ is rational only if the logarithmic derivative $z\left[\frac{\zeta'_T(z)}{\zeta_T(z)}\right]$ is rational, where, for $|z| < \frac{1}{2}$,

$$\begin{aligned}
z\left[\frac{\zeta'_T(z)}{\zeta_T(z)}\right] &= \sum_{k=1}^{\infty} z^k F_k(T) \\
&= \sum_{k=1}^{\infty} (2z)^k + \sum_{k=1}^{\infty} \sigma(k) z^k \\
&= \sum_{k=1}^{\infty} (2z)^k + \sum_{k=1}^{\infty} (1+k) z^k + \sum_{k=1}^{\infty} \sum_{\substack{d \mid k \\ d \neq 1,k}} d z^k \\
&= \frac{2z}{1-2z} + \frac{2z}{1-z} + \sum_{k=1}^{\infty} \sum_{\substack{d \mid k \\ d \neq 1,k}} d z^k \\
&= \frac{2z}{1-2z} + \frac{2z}{1-z} + \varphi(z) \,.
\end{aligned}$$

Hence, to show that $\zeta_T(z)$ is irrational, it is sufficient to show that $\varphi(z)$ is irrational. Now, let $a = (a_k)_{k=1}^{\infty}$ be the sequence of coefficients of $\varphi(z)$ given by

$$a_k = \sum_{\substack{d|k \\ d\neq 1,k}} d\,.$$

Then by Theorem 5.3,

$$\varphi(z) = \sum_{k=1}^{\infty} a_k z^k$$

is rational if and only if a is a linear recurrence sequence. We will prove by contradiction that $\varphi(z)$ is irrational.

Assume that a is a linear recurrence sequence. Set $Z = \{k : a_k = 0\}$ to be the set of zeros. Here, note that $a_k = 0$ whenever k is prime (or $k = 1$). Hence, we have that

$$\{p : p \text{ is prime}\} \subseteq Z\,.$$

Since the set of primes is infinite, then it must contain an arithmetic progression containing a prime p by Theorem 5.4. But any arithmetic progression containing a prime p also contains a composite number since if the common difference in the arithmetic progression is d, then $p + pd = p(1 + d)$ is composite. But $a_k \neq 0$ for $k = p(1 + d)$ composite, which is a contradiction. It follows that a does not satisfy the criteria to be a linear recurrence sequence by Theorem 5.4. Therefore, by Theorem 5.3, $\varphi(z)$ is irrational. It follows that the dynamical zeta function of T is irrational.

5.3 A Natural Boundary as a Factor of Rationality

We will now give an example of a quotient system where the topological dynamical system (X, T) has a rational dynamical zeta function while the quotient system (X', T') has a natural boundary.

Choose

$$F_k = 2^k \qquad \text{and} \qquad F'_k = 2^k + a_k 2^k\,,$$

for all $k \in\geq 1$, where we define a_k as follows:

- $a_1 = a_2 = a_p = 0$;

- $a_{q^m} = q^2$ (for $m \geq 2$);
- $a_{2p^m} = p$ (for $m \geq 1$);
- $a_{2^n p^m} = 2^2$ (for $n \geq 2$ and $m \geq 1$);
- $a_k = a_{k/p} \pmod{p^{\text{order}_p(k)}}$, for $p \mid k$, where we choose a_k such that $\frac{k}{2^n} \leq a_k \leq \frac{2k}{2^n}$ for $n = \text{order}_2(k)$, $n \in \mathbb{N}_0$,

where q is any prime and p is an odd prime. Note here that in order to solve for $a_k = a_{k/p} \pmod{p^{\text{order}_p(k)}}$, we must use the Chinese Remainder Theorem.

Example 5.7. For a_{15}, we have

$$\begin{aligned} a_{15} &\equiv 3 \pmod{5} \\ &\equiv 5 \pmod{3}. \end{aligned}$$

Then by the Chinese Remainder Theorem, we have

$$a_{15} \equiv 8 \pmod{15}.$$

Hence, to satisfy $15 \leq a_{15} \leq 30$, we choose $a_{15} = 23$.

Example 5.8. For a_{60}, we have

$$\begin{aligned} a_{60} &\equiv a_{12} \pmod{5} \\ &\equiv a_{20} \pmod{3}, \end{aligned}$$

where $a_{12} = a_{20} = 4$. Then by the Chinese Remainder Theorem, we have

$$a_{60} \equiv 4 \pmod{15}.$$

Now, note that $\text{order}_2(60) = 2$. Hence, in order to satisfy $15 \leq a_{60} \leq 30$, we choose $a_{60} = 19$.

We obtain the following sequences of periodic points:

$$F = (2, 2^2, 2^3, 2^4, 2^5, 2^6, 2^7, 2^8, 2^9, 2^{10}, 2^{11}, 2^{12}, \ldots),$$

and

$$F' = (2, 2^2, 2^3, 5 \cdot 2^4, 2^5, 4 \cdot 2^6, 2^7, 5 \cdot 2^8, 10 \cdot 2^9, 6 \cdot 2^{10}, 2^{11}, 5 \cdot 2^{12}, \dots)\,.$$

Letting $G = C_2$, we will show that there exists a quotient system (X', T') of a topological dynamical system (X, T) under a C_2-action such that $F_k = F_k(T)$ and $F'_k = F_k(T')$, for all $k \geq 1$. Moreover, we will show that T has a rational zeta function while T' has a natural boundary.

We will start by showing that F and F' are realizable. Here, note that $F = F(\widetilde{T})$, where $\widetilde{T}$ is the tent map, so that F is realizable. It remains to show that F' is realizable. We will start by showing divisibility.

Let $k = p_1^{n_1} p_2^{n_2} \cdots p_l^{n_l}$, for $l \geq 1$, where $p_1, p_2, \dots$ and p_l are all distinct primes and $n_i \geq 1$, for $i\{1, 2, \dots, l\}$. Now, if we can show that

$$\sum_{d|k} \mu\left(\tfrac{k}{d}\right) a_d \, 2^d \equiv 0 \pmod{p_i^{n_i}}\,,$$

for all $i \in \{1, 2, \dots, l\}$, it follows that

$$\sum_{d|k} \mu\left(\tfrac{k}{d}\right) a_d \, 2^d \equiv 0 \pmod{k}$$

by the Chinese Remainder Theorem.

First, let us look at the decomposition of $\sum_{d|k} \mu\left(\frac{k}{d}\right) a_d \, 2^d$. Since $\mu\left(\frac{k}{d}\right) = 0$ whenever $\frac{k}{d}$ contains a square factor, we have that

$$\begin{aligned}\sum_{d|k} \mu\left(\tfrac{k}{d}\right) a_d \, 2^d =& \mu(1) a_k \, 2^k + \mu(p_i) a_{k/pi} \, 2^{k/p_p} + \cdots + \mu(p_i p_j \cdots p_r) a_{k/p_i p_j \cdots p_r} \, 2^{k/p_i p_j \cdots p_r} \\ &+ \cdots + \mu(p_1 p_2 \cdots p_l) a_{k/p_1 p_2 \cdots p_l} \, 2^{k/p_1 p_2 \cdots p_l}\,,\end{aligned}$$

for $l > \cdots > r > \cdots > j > i$. In order to show divisibility, we will consider two cases. First, let p_1 be an odd prime. By Fermat's Little Theorem, we have that, for an odd prime p,

$$2^{p^n} \equiv 2^{p^{n-1}} \pmod{p^n}\,.$$

Then it follows that

$$2^k \equiv 2^{k/p_1} \pmod{p_1^{n_1}}\,.$$

Further, by the definition of a_k, we have that

$$a_k \equiv a_{k/p_1} \pmod{p_1^{n_1}}.$$

Note that this holds true for any p_1 such that $p_1 \mid k$. Hence, we have

$$\mu(1)a_k 2^k \equiv -\mu(p_1)a_{k/p_1}\, 2^{k/p_1} \pmod{p_1^{n_1}}.$$

Further, for $i > 1$ and $2 \le r \le l$, we have that

$$2^{k/p_ip_j\cdots p_r} \equiv 2^{k/p_1p_ip_j\cdots p_r} \pmod{p_1^{n_1}},$$

and

$$a_{k/p_ip_j\cdots p_r} \equiv a_{k/p_1p_ip_j\cdots p_r} \pmod{p_1^{n_1}}.$$

Hence,

$$\mu(p_ip_j\cdots p_r)a_{k/p_ip_j\cdots p_j}\, 2^{k/p_ip_j\cdots p_r} \equiv -\mu(p_1p_ip_j\cdots p_r)a_{k/p_1p_ip_j\cdots p_j}\, 2^{k/p_1p_ip_j\cdots p_r} \pmod{p_1^{n_1}}.$$

It follows that

$$\sum_{d\mid k} \mu\left(\tfrac{k}{d}\right) a_d\, 2^d \equiv 0 \pmod{p_1^{n_1}}. \tag{5.10}$$

Since p_1 is an arbitrary odd prime, this holds true for any odd prime.

Second, suppose $p_1 = 2$. Then for $0 \le r \le l$, we have that

$$2^k \equiv 0 \equiv 2^{k/2p_ip_j\cdots p_r} \pmod{2^{n_1}}$$

since $k/2p_ip_j\cdots p_r = 2^{n_1-1}\prod_{s\in\{i,j,\dots,r\}} p_s^{n_s-1} \ge n_1$. Hence,

$$\sum_{d\mid k} \mu\left(\tfrac{k}{d}\right) a_d\, 2^d \equiv 0 \pmod{2^{n_1}}. \tag{5.11}$$

Combining (5.10) and (5.11), we have that

$$\sum_{d\mid k} \mu\left(\tfrac{k}{d}\right) a_d\, 2^d \equiv 0 \pmod{p_i^{n_i}},$$

for all $i \in \{1, 2, \dots, l\}$ and any prime p.

Next, we must show positivity. We will have to consider several cases.

(i) Let $k = p$. Then

$$\sum_{d|k} \mu\left(\tfrac{k}{d}\right) a_d \, 2^d = \mu(1) a_p \, 2^p + \mu(p) a_1 \, 2 = 0\,.$$

(ii) Let $k = q^m$ (for $m \geq 2$). Then

$$\begin{aligned}\sum_{d|k} \mu\left(\tfrac{k}{d}\right) a_d \, 2^d &= \mu(1) a_{q^m} \, 2^{q^m} + \mu(q) a_{q^{m-1}} \, 2^{q^{m-1}} \\ &\geq q^2 \cdot 2^{q^m} - q^2 \cdot 2^{q^{m-1}} \\ &\geq 0\,.\end{aligned}$$

(iii) Let $k = 2p^m$ (for $m \geq 1$). Then

$$\begin{aligned}&\sum_{d|k} \mu\left(\tfrac{k}{d}\right) a_d \, 2^d \\ &= \mu(1) a_{2p^m} \, 2^{2p^m} + \mu(2) a_{p^m} \, 2^{p^m} + \mu(p) a_{2p^{m-1}} \, 2^{2p^{m-1}} + \mu(2p) a_{p^{m-1}} \, 2^{p^{m-1}} \\ &\geq p \cdot 4^{p^m} - p^2 \cdot 2^{p^m} - p \cdot 2^{p^{m-1}} \\ &\geq p \cdot 4^{p^m} - (p^2 + p)\, 2^{p^m} \\ &\geq p \cdot 4^{p^m} - 2p^2 \cdot 2^{p^m} \\ &\geq p \cdot 4^{p^m} - p\,(2^{p^m})^2 \\ &\geq 0\,.\end{aligned}$$

(iv) Let $k = 2^n p^m$ (for $m = 2, n = 1$). Then

$$\begin{aligned}\sum_{d|k} \mu\left(\tfrac{k}{d}\right) a_d \, 2^d &= \mu(1) a_{4p} \, 2^{4p} + \mu(2) a_{2p} \, 2^{2p} + \mu(p) a_4 \, 2^4 + \mu(2p) a_2 \, 2^2 \\ &= 4 \cdot 16^p - p \cdot 4^p - 4 \cdot 2^4 \\ &\geq 4 \cdot 16^p - 2 \cdot 4^{2p} \\ &= 4 \cdot 16^p - 2 \cdot 16^p \\ &= 2 \cdot 16^{2p} \\ &\geq 0\,.\end{aligned}$$

(v) Let $k = 2^n p^m$ (for $m = 2, n \geq 2$). Then

$$\begin{aligned}\sum_{d|k} \mu\left(\tfrac{k}{d}\right) a_d\, 2^d &= \mu(1)a_{4p^m}\, 2^{4p^m} + \mu(2)a_{2p^m}\, 2^{2p^m} + \mu(p)a_{4p^{m-1}}\, 2^{4p^{m-1}} \\ &\quad + \mu(2p)a_{2p^{m-1}}\, 2^{2p^{m-1}} \\ &= 4\cdot 16^{p^m} - p\cdot 4^{p^m} - 4\cdot 16^{p^{m-1}} - p\cdot 4^{p^{m-1}} \\ &= 4\cdot 16^{p^{m-1}}\,(16^{p^m - p^{m-1}} - 1) - p\cdot 4^{p^{m-1}}\,(4^{p^m-p^{m-1}} - 1) \\ &\geq 0\end{aligned}$$

since for $m \geq 2$, we have $4 \geq \frac{p}{4^{p^{m-1}}}$ and therefore

$$4\cdot 16^{p^{m-1}} \geq p\cdot 4^{p^{m-1}}\,.$$

Similarly, for any m, we have $1 \geq \frac{1}{4^{p^m-p^{m-1}}}$ and therefore

$$4\cdot 16^{p^m-p^{m-1}} - 1 \geq p\cdot 4^{p^m-p^{m-1}} - 1\,.$$

(vi) Let $k = 2^n p^m$ (for $m \geq 3, n \geq 1$). Then

$$\begin{aligned}\sum_{d|k} \mu\left(\tfrac{k}{d}\right) a_d\, 2^d &= \mu(1)a_{2^n p^m}\, 2^{2^n p^m} + \mu(2)a_{2^{n-1}p^m}\, 2^{2^{n-1}p^m} + \mu(p)a_{2^n p^{m-1}}\, 2^{2^n p^{m-1}} \\ &\quad + \mu(2p)a_{2^{n-1}p^{m-1}}\, 2^{2^{n-1}p^{m-1}} \\ &= 4\cdot 2^{2^n p^m} - 4\cdot 2^{2^{n-1}p^m} - 4\cdot 2^{2^n p^{m-1}} - 4\cdot 2^{2^{n-1}p^{m-1}} \\ &= 4\cdot 2^{2^{n-1}p^m}\,(2^{2^n p^m - 2^{n-1}p^m} - 1) \\ &\quad - 4\cdot 2^{2^{n-1}p^{m-1}}\,(2^{2^n p^m - 2^{n-1}p^{m-1}} - 1) \\ &\geq 0\,.\end{aligned}$$

(vii) Let $k = p_1^{n_1} p_2^{n_2} \cdots p_l^{n_l}$, where p_i are all distinct primes in ascending order with $n_i \geq 1$, for $i \in \{1, 2, \ldots, l\}$. We have that

$$\begin{aligned}\sum_{d|k} \mu\left(\tfrac{k}{d}\right) a_d\, 2^d &\geq a_k\, 2^k - \sum_{\substack{d|k \\ d<k}} a_d\, 2^d \\ &\geq a_k\, 2^k - a_{\max} \sum_{\substack{d|k \\ d<k}} 2^d \\ &\geq a_k\, 2^k - \frac{a_{\max}\, k\cdot 2^{k/2}}{2}\end{aligned}$$

where we define $a_{\max} = \max\{a_d : d \mid k, d < k\}$. We will consider two cases. First, suppose $p_1 = 2$. Then

$$p_2^{n_2} \cdots p_l^{n_l} \leq a_k, a_{\max} \leq 2p_2^{n_2} \cdots p_l^{n_l} ,$$

and we have that

$$\begin{aligned}\frac{a_{\max}\, k \cdot 2^{k/2}}{2} &\leq \frac{(2p_2^{n_2} \cdots p_l^{n_l})\,(k \cdot 2^{k/2})}{2} \\ &= (p_2^{n_2} \cdots p_l^{n_l})\,(k \cdot 2^{k/2}) \\ &\leq a_k\, 2^k .\end{aligned}$$

Second, suppose $p_1 \neq 2$. Then

$$p_1^{n_1-1} p_2^{n_2} \cdots p_l^{n_l} \leq a_{\max} \leq 2p_1^{n_1-1} p_2^{n_2} \cdots p_l^{n_l} ,$$

and $k \leq a_k \leq 2k$. It follows that

$$\begin{aligned}\frac{a_{\max}\, k \cdot 2^{k/2}}{2} &\leq \frac{(2p_1^{n_1-1} p_2^{n_2} \cdots p_l^{n_l})(k \cdot 2^{k/2})}{2} \\ &= (p_1^{n_1-1} p_2^{n_2} \cdots p_l^{n_l})(k \cdot 2^{k/2}) \\ &\leq k^2 \cdot 2^{k/2} \\ &\leq a_k\, 2^k .\end{aligned}$$

Hence, we have shown that $O'_k = \frac{1}{k} \sum_{d|k} \mu\left(\frac{k}{d}\right) F'_d \in \mathbb{N}_0$, for all $k \geq 1$. We conclude that F' is realizable.

Next, we will use Lemma 4.21 and Lemma 4.22 to show that F' is the sequence of periodic points of a quotient system by a C_2 action of a system which has F as its sequence of periodic points.

Choose $b_k^g = 0$,

$$b_k^{2s} = \sum_{n=0}^{\text{order}_2(k)} \left(O'_{k/2^n} - O_{k/2^n}\right) ,$$

and

$$b_k^v = \begin{cases} O_k - b_{k/2}^{2s} , & \text{for } k \text{ even} ; \\ O_k , & \text{for } k \text{ odd} , \end{cases}$$

for all $k \geq 1$. Clearly, by the definition of F_k and F'_k, we have that $b_k^{2s} \geq 0$. Also, $b_k^v \geq 0$ since $b_{k/2}^{2s} = 2^{k/2} + \mathrm{O}(2^{k/4}) \leq O_k$. We set

(i) $a_k^v = b_k^v$,

(ii) $a_k^g = 0$, and

(iii) $a_k^{2s} = \begin{cases} b_{k/2}^{2s}, & \text{for } k \text{ even}; \\ 0, & \text{otherwise}, \end{cases}$

and $a_k = a_k^v + a_k^g + a_k^{2s}$ and $b_k = b_k^v + b_k^g + b_k^{2s}$, for all $k \geq 1$. Then

$$\begin{aligned} a_k &= a_k^v + a_k^{2s} \\ &= b_k^v + a_k^{2s} \\ &= O_k , \end{aligned}$$

and

$$b_k = b_k^v + b_k^{2s} ,$$

for all $k \geq 1$. We will consider two cases. Suppose k is odd. Then

$$b_k = O_k + (O'_k - O_k) = O'_k$$

since $\text{order}_2(k) = 0$. Next, suppose k is even, say $\text{order}_2(k) = j$, for some $j \in \mathbb{N}$. Then

$$\begin{aligned} b_k &= O_k - b_{k/2}^{2s} + b_k^{2s} \\ &= O_k - \Big[\big(O'_{k/2} - O_{k/2}\big) + \big(O'_{k/4} - O_{k/4}\big) + \cdots + \Big(O'_{k/2^j} - O_{k/2^j}\Big)\Big] \\ &\quad + \Big[(O'_k - O_k) + \big(O'_{k/2} - O_{k/2}\big) + \cdots + \Big(O'_{k/2^j} - O_{k/2^j}\Big)\Big] \\ &= O_k + (O'_k - O_k) \\ &= O'_k . \end{aligned}$$

Hence, by Lemma 4.21, there exists (X, T) such that $F_k = F_k(T)$ and $F'_k = F_k(T')$, for all $k \geq 1$, where (X', T') is the quotient system under a C_2 action. Further, since $b_1^v \geq 1$, we have that (X, T) is a topological dynamical system by Lemma 4.22.

Last but not least, we will show that the dynamical zeta function of T is rational while the zeta function of T' has a natural boundary. We will start by studying the dynamical zeta function for T. For $|z| < \frac{1}{2}$, the dynamical zeta function of T is given by

$$\begin{aligned}\zeta_T(z) &= \exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k} F_k(T)\right) \\ &= \exp\left(\sum_{k=1}^{\infty} \frac{(2z)^k}{k}\right) \\ &= \frac{1}{1-2z}\,,\end{aligned}$$

which is rational.

Next, we will look at the dynamical zeta function of T'. First, note that, for all $k \geq 1$, we have that $a_k \geq 1$ infinitely many often, while $a_k \leq 2k$ always. Then

$$\limsup_{k\to\infty} a_k = 1\,,$$

and it follows that

$$\limsup_{k\to\infty} \left(\frac{a_k 2^k}{k}\right)^{1/k} = 2.$$

Hence, for $|z| < \frac{1}{2}$, the dynamical zeta function of T' is given by

$$\begin{aligned}\zeta_{T'}(z) &= \exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k} F_k(T')\right) \\ &= \exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k} \left(2^k + a_k 2^k\right)\right) \\ &= \exp\left(\sum_{k=1}^{\infty} \frac{(2z)^k}{k}\right) \exp\left(\sum_{k=1}^{\infty} \frac{a_k (2z)^k}{k}\right) \\ &= \left(\frac{1}{1-2z}\right) \exp\left(\sum_{k=1}^{\infty} \frac{a_k (2z)^k}{k}\right)\,.\end{aligned}$$

Then in order to show that T' has a natural boundary at $z = \frac{1}{2}$, we must show that $\exp\left(\sum_{k=1}^{\infty} \frac{a_k(2z)^k}{k}\right)$ has a natural boundary at $z = \frac{1}{2}$. If we consider

$$f(z) = \sum_{k=1}^{\infty} a_k z^k\,,$$

which has radius of convergence 1, then

$$f(2z) = \sum_{k=1}^{\infty} a_k (2z)^k = z\left[\frac{\zeta'_{T'}(z)}{\zeta_{T'}(z)}\right]\,,$$

which has radius of convergence $\frac{1}{2}$. It follows that if we can show that $f(z)$ has a natural boundary at 1, then $f(2z)$, and therefore the dynamical zeta function of T', has a natural boundary at $\frac{1}{2}$.

First we will show that $f(z)$ is irrational. By Theorem 5.3, $f(z)$ is irrational if and only if $a = (a_k)_{k=1}^{\infty}$ does not satisfy the criteria to be a linear recurrence sequence.

Set $Z = \{k : a_k = 0\}$ to be the set of zeros, and assume, to the contrary, that a is a linear recurrence sequence. Note that since $a_k = 0$ whenever k is prime (or $k = 1$), we have that

$$\{p : p \text{ is prime}\} \subseteq Z\,.$$

Since the set of primes is infinite, Z must then contain an arithmetic progression containing a prime number by Theorem 5.4. But any arithmetic progression containing a prime number also contains a composite number since, if d is the common difference in the progression, then $p + pd = p(1 + d)$, which is composite. But $a_k \neq 0$ for $k = p(1+d)$ composite. Hence, a does not satisfy the criteria to be a linear recurrence sequence by Theorem 5.4, and therefore, $f(z)$ is irrational. Then by Theorem 5.5, we have that $f(z)$ has a natural boundary at 1. It follows that $f(2z)$ and, therefore the dynamical zeta function of T', has a natural boundary at $\frac{1}{2}$.

Chapter 6

Future Work

We will now discuss some ideas for future work.

Recall that if we let T be the circle doubling map defined on the circle $X = \mathbb{R}/\mathbb{Z}$ and $(\widetilde{X}, \widetilde{T})$ be the quotient system of (X, T) under the action of C_2, where $\widetilde{T}$ is the tent map defined on the interval $\widetilde{X} = [0, \frac{1}{2}]$, then we have the same asymptotic growth rates for orbits of the circle doubling map T and the tent map $\widetilde{T}$, that is

$$O_k(T) \sim O_k(\widetilde{T}) \quad \text{as } k \to \infty \, .$$

Indeed, here we have the sharp bound

$$|O_k(T) - O_k(\widetilde{T})| = \mathrm{O}(1) \, .$$

Further, recall that if we let T be the doubling map defined on the two-dimensional torus $X = \mathbb{R}^2/\mathbb{Z}^2$ and $(\widehat{X}, \widehat{T})$ be the quotient system of (X, T) under the action of D_8, where $\widehat{T}$ is the triangle map, then we have the same asymptotic growth rates for orbits of the doubling map T and the triangle map $\widehat{T}$.

Both of the above examples occur in a natural setting: They have easy to define spaces and maps, and the groups chosen are natural (non-trivial) choices for the given space and map. Now, from Theorem 4.19, we know that in a general setting, where (X', T') is the quotient system of a dynamical system (X, T) under the action of a finite group G, then we have a wide (but restricted) range of possibilities of what orbit growth rates can be achieved in the quotient system (X', T'). However, the proof of Theorem 4.19 required a more abstract combinatorial construction of (X, T) (detailed in the proof of Lemma 4.21) very unlike the more natural constructions of the previous two examples. Then one question to ask is, do examples occurring in a natural setting, for examples, classes of systems like group automorphisms, subshifts of finite type, or expanding map on an interval, always exhibit the same growth rates for orbits in the quotient system?

Another set of questions to ask is, given a space X and a map T, for example, a non-trivial expanding map, which group (or groups), if any, commute with T? And given a space X and an action of a group G, which maps commute with the action of G? Here, for a given family of systems (X, T), the aim is to find as many natural examples as possible. Will we find the choices of groups to be limited, implying limited behaviour?

Next, we have the following result:

Theorem 6.1 ([20])**.** *If $F = (F_k)_{k=1}^{\infty}$ is a realizable sequence of non-negative integers, then there exists a C^{∞} map $\mathbb{T}^2 \to \mathbb{T}^2$ such that $F = F_k(T)$.*

Thus, we can find a smooth model (X, T) for the sequence F of periodic points described in Theorem 4.19. Now, note that if (X, T) is a topological dynamical system, then it follows that (X', T') is a topological dynamical system since G is finite. However, do the same implications follow if we have a smooth model (X, T) for F? That is, if we have a smooth model (X, T) for the sequence F of periodic points, does it follow that there exists a smooth model (X', T') for the sequence F' of periodic points in the quotient system?

Appendix A

The Tent Map: A Quotient of the Circle Doubling Map

Let T be the circle doubling map defined on the circle $X = \mathbb{R}/\mathbb{Z}$. Let $(\widetilde{X}, \widetilde{T})$ be the quotient system of (X, T) under the action of C_2, where $\widetilde{T}$ is the tent map defined on the interval $\widetilde{X} = [0, \frac{1}{2}]$.

We will give a list of all the points on the circle doubling map which will give rise to shortening orbits on the tent map, a formula for the number of shortening orbits, and a result for the general relationship between the count of orbits on the circle doubling map and the tent map. We have the following propositions:

Proposition A.1. *Let T be the circle doubling map and $\widetilde{T}$ be the tent map. Then orbits of T which have odd length $2k+1$, will not shorten in length on $\widetilde{T}$.*

Proposition A.2. *Let T be the circle doubling map and $\widetilde{T}$ be the tent map. Then points on T which have the form*

$$\left\{ \frac{c}{2^k+1} : 1 \leq c \leq 2^k, c \in \mathbb{N} \right\}$$

will give rise to orbits on T which will shorten on $\widetilde{T}$ by a factor of $\frac{1}{2}$ and which have length $2d \leq 2k$, such that $2d \mid 2k$ but $2d \nmid k$.

Proposition A.3. *Let T be the circle doubling map and $\widetilde{T}$ be the tent map. Then* *orbits of T which have even length and do <u>not</u> contain points of the form*

$$\left\{\frac{c}{2^k+1} : 1 \leq c \leq 2^k, c \in \mathbb{N}\right\}$$

will not shorten in length on $\widetilde{T}$.

Proposition A.4. *Let T be the circle doubling map and $\widetilde{T}$ be the tent map. Then the number of orbits of T that have even length $2k$, and that will shorten into orbits of length k on $\widetilde{T}$ is given by*

$$O_{2k}^{2s}(T) = \frac{1}{2k} \sum_{\substack{d|k \\ k/d \text{ odd}}} \mu\left(\tfrac{k}{d}\right) 2^d \,,$$

for all $k \geq 1$.

Proposition A.5. *Let T be the circle doubling map, $\widetilde{T}$ be the tent map, and $k \geq 1$. Then*

(i) $O_k(T) = O_k^{2s}(T) + O_k^{ns}(T)$;

(ii) $O_k^{2s}(T) = 0$ *(for k odd)*;

(iii) $O_k(\widetilde{T}) = \frac{1}{2} O_k^{ns}(T) + O_{2k}^{2s}(T)$ *(for $k \geq 2$)*;

(iv) $O_1(\widetilde{T}) = O_1^{ns}(T) + O_2^{2s}(T)$.

Proof of Proposition A.1. Given an orbit of T which has odd length $2k+1$, there exists a natural number a, $1 \leq a \leq 2^{2k+1} - 2$, such that the orbit consists of the distinct points

$$\frac{a}{2^{2k+1}-1}, \quad \frac{2a}{2^{2k+1}-1}, \quad \frac{2^2 a}{2^{2k+1}-1}, \ldots, \quad \frac{2^k a}{2^{2k+1}-1}, \ldots, \quad \frac{2^{2k} a}{2^{2k+1}-1} \pmod{1}.$$

Now, assume that the orbit will shorten in length on $\widetilde{T}$. Then there exists some $2k+1 > m \in \mathbb{N}$ such that

$$\frac{2^m a}{2^{2k+1}-1} \equiv 1 - \frac{a}{2^{2k+1}-1} \pmod{1}, \tag{A.1}$$

that is, for $x = \frac{a}{2^{2k+1}-1}$,

$$T^m(x) \equiv 1 - x \pmod{1}.$$

Then

$$\begin{aligned}
T^m(x) &\equiv -x \pmod 1 \\
T^{2m}(x) &\equiv T^m(T^m(x)) \pmod 1 \\
&\equiv T^m(-x) \pmod 1 \\
&\equiv -T^m(x) \pmod 1 \\
&\equiv -(-x) \pmod 1 \\
&\equiv x \pmod 1 .
\end{aligned}$$

This means that the T-period of x divides $2m$. However, since by definition this is an orbit of period $2k+1$, we also have

$$T^{2k+1}(x) \equiv x \pmod 1 ,$$

so the T-period also divides $2k+1$. Then the T-period of x must divide $\gcd(2m, 2k+1)$ and this is equal to $\gcd(m, 2k+1)$, so the T-period of x must divide m, that is we must have

$$T^m(x) \equiv x \pmod 1 .$$

However, we also have

$$T^m(x) \equiv -x \pmod 1 ,$$

so we must have

$$T^m(x) \equiv x \equiv -x \pmod 1 .$$

Since x is neither equal to 0 (by definition) nor $\frac{1}{2}$ (since $2 \nmid 2^{2k+1}-1$), this is impossible. Therefore, orbits of T of odd length $2k+1$ will not shorten on $\widetilde{T}$. □

Proof of Proposition A.2. Suppose we have an orbit of T which contains points of the form

$$\left\{ \frac{c}{2^k+1} : 1 \leq c \leq 2^k, c \in \mathbb{N} \right\} .$$

We will consider two cases. First, assume that $\frac{c}{2^k+1}$ has least period $2k$. Then our orbits consists of the distinct points

$$\frac{c}{2^k+1}, \quad \frac{2c}{2^k+1}, \quad \frac{2^2c}{2^k+1}, \ldots, \quad \frac{2^kc}{2^k+1}, \ldots, \quad \frac{2^{2k-1}c}{2^k+1} \quad (\bmod 1) ,$$

where the $(k+1)$-th term of the orbit is given by

$$\frac{2^k c}{2^k+1},$$

and this term is equivalent to

$$\begin{aligned}\frac{2^k c}{2^k+1} &\equiv \frac{2^k c}{2^k+1} - (c-1)\left(\frac{2^k+1}{2^k+1}\right) && (\bmod 1)\\ &\equiv \frac{2^k+1-c}{2^k+1} && (\bmod 1)\\ &\equiv 1-\frac{c}{2^k+1} && (\bmod 1)\,.\end{aligned}$$

Hence, we have that, for $x = \frac{c}{2^k+1}$,

$$T^k(x) \equiv 2^k x \equiv 1-x \pmod 1\,.$$

Then when C_2 acts on X, we identify x with $1-x$ on the tent map $\widetilde{T}$. It follows that the length of the orbit shortens by a factor of $\frac{1}{2}$.

Second, assume that $\frac{c}{2^k+1}$ has period $n < 2k$ such that $n \mid 2k$, for some $n \in \mathbb{N}$. Since orbits can only shorten in length by a factor of $\frac{1}{2}$, then n must have even period, say $n = 2d$. Hence, we have that

$$T^{2d}(x) \equiv x \pmod 1\,. \tag{A.2}$$

Now, suppose $k/d = 2m$, for some $m \in \mathbb{N}$. Then by (A.2), we have that

$$T^k(x) \equiv T^{2md}(x) \equiv x \pmod 1\,,$$

which is a contradiction. Hence, k/d must be odd, so that $2 \nmid (k/d)$, that is $2d \nmid k$. $\square$

Proof of Proposition A.3. Assume that some orbit of T which has even length $2k$ shortens in length on $\widetilde{T}$. Hence, there exists some $2k > m \in \mathbb{N}$ such that

$$T^m(x) \equiv 1-x \pmod 1\,.$$

It follows that

$$T^{2m}(x) \equiv x \pmod 1\,.$$

Then $2k \mid 2m$ (since $2k$ gives the least period), so $k \mid m$ but $m < 2k$, so $m = k$.

Next, suppose we have an orbit of even length $2d$ such that $2d \mid 2k$ and which shortens in length on $\widetilde{T}$. Then

$$T^{2d}(x) \equiv x \pmod 1 . \tag{A.3}$$

Now, if $k/d = 2m$, for some $m \in \mathbb{N}$, then, by (A.3), we have that

$$T^k(x) \equiv T^{2md}(x) \equiv x \pmod 1 ,$$

which is a contradiction. Hence, k/d must be odd.

Moreover, since

$$T^k(x) \equiv 1 - x \equiv -x \pmod 1 ,$$

we have that

$$2^k x \equiv -x \pmod 1 ,$$

and it follows that

$$x(2^k + 1) \equiv 0 \pmod 1 .$$

Then we have that $1 \mid x(2^k+1)$, so that $x(2^k+1) = c$, for some $c \in \mathbb{N}$. But then $x = \frac{c}{2^k+1}$. In conclusion, orbits which have even length $2d$, such that $2d \mid 2k$ but $2d \nmid k$, on T and which shorten in length on $\widetilde{T}$ consist of points of the form

$$\left\{ \frac{c}{2^k+1} : 1 \leq c \leq 2^k, c \in \mathbb{N} \right\} .$$

□

Proof of Proposition A.4. By Proposition A.1, A.2, and A.3, we have that

$$\begin{aligned} \mathfrak{F}^{2s}_{2k}(T) = \{&\text{points of period dividing } 2k \text{ but not dividing } k \text{ which will give} \\ &\text{rise to orbits on } T \text{ which will shorten by a factor of } \tfrac{1}{2} \text{ on } \widetilde{T}\} \\ = &\left\{ \frac{c}{2^k+1} : 1 \leq c \leq 2^k, c \in \mathbb{N} \right\} . \end{aligned}$$

Now, in order to only count points of *least* period $2k$, we must exclude points of period $2d < 2k$ such that $d \mid k$ with k/d odd, by Proposition A.2. Then using the

Inclusion-Exclusion Principle and the Möbius function, we have that

$$\begin{aligned}
L_{2k}^{2s}(T) &= \#\{\text{points of } \textit{least} \text{ period } 2k \text{ which will give rise to orbits} \\
&\qquad \text{on } T \text{ which will shorten by a factor of } \tfrac{1}{2} \text{ on } \widetilde{T}\} \\
&= F_{2k}^{2s}(T) - \left| \bigcup_{\substack{p|k \\ k/p \text{ odd}}} F_{2k/p}^{2s}(T) \right| \\
&= \sum_{\substack{d|k \\ k/d \text{ odd}}} \mu\left(\tfrac{k}{d}\right) F_{2d}^{2s}(T) \,,
\end{aligned}$$

for all $k \geq 1$. It follows that

$$\begin{aligned}
O_{2k}^{2s}(T) &= \#\{\text{orbits of } T \text{ which have length } 2k \text{ on } T \text{ and} \\
&\qquad \text{which will shorten by a factor of } \tfrac{1}{2} \text{ on } \widetilde{T}\} \\
&= \tfrac{1}{2k} \sum_{\substack{d|k \\ k/d \text{ odd}}} \mu\left(\tfrac{k}{d}\right) F_{2d}^{2s}(T) \\
&= \tfrac{1}{2k} \sum_{\substack{d|k \\ k/d \text{ odd}}} \mu\left(\tfrac{k}{d}\right) 2^d \,,
\end{aligned}$$

for all $k \geq 1$. □

Proof of Proposition A.5. First, note that (ii) follows from Proposition A.1, and (i) follows since all orbits of length k either shorten in length or the length remains the same. It is consistent with (ii) since, in the case where k is odd and we do not have any shortening orbits, we only have non-shortening orbits. So we are left to prove (iii) and (iv).

For (iii), we need to show that $O_k(\widetilde{T}) = \frac{1}{2}\, O_k^{ns}(T) + O_{2k}^{2s}(T)$. Assume that $k \geq 2$. Now, when G acts on X, we identify the periodic points x and $1-x$ of T. Then when x and $1-x$ lie in different orbits of length k on T, we have that two orbits of length k will form into one orbit of length k on $\widetilde{T}$. The length of the orbits remains the same since x and $1-x$ give rise to orbits on T that are of equal length. Hence, each pair of non-shortening orbits which have length k on T will give rise to one orbit of length k on $\widetilde{T}$. Now, when x and $1-x$ lie in the same orbit of length k on T, then the length of the orbit shortens by a factor of $\frac{1}{2}$ on $\widetilde{T}$. This is due to x and $1-x$

coming in pairs for orbits of length k on T, for $k \geq 2$. It follows that in order to count the number of orbits of length k on $\widetilde{T}$, we need to consider the number of shortening orbits of length $2k$ on T, where each shortening orbit of length $2k$ shortens in length by a factor of $\frac{1}{2}$ and gives rise to one orbit of length k on $\widetilde{T}$.

Further, for (iv), we need to show that $O_1(\widetilde{T}) = O_1^{ns}(T) + O_2^{2s}(T)$. Here, note that when $x = 0$ or $x = \frac{1}{2}$, then $x = 1 - x$. But $x = \frac{1}{2}$ does not lie in a closed orbit, and $x = 0$ gives rise to a non-shortening orbit of length 1 on T which gets preserved on $\widetilde{T}$. Moreover, we need to consider orbits of length 2 on T that will shorten into orbits of length 1 on $\widetilde{T}$, as seen in the argument for (iii). $\square$

Appendix B

The Triangle Map: A Quotient of the Doubling Map

Let T be the doubling map defined on the two-dimensional torus $X = \mathbb{R}^2/\mathbb{Z}^2$. Let $(\widehat{X}, \widehat{T})$ be the quotient system of (X, T) under the action of D_8, where $\widehat{T}$ is the triangle map.

We will give a list of all the points on the doubling map which will give rise to shortening orbits on the triangle map, and we will give formulae for the number of shortening orbits. Here, note that throughout this appendix, whenever we refer to the doubling map T we refer to the doubling map defined on the two-dimensional torus.

We have the following propositions:

Proposition B.1. *Let T be the doubling map and $\widehat{T}$ be the triangle map. Then points on T which have the form*

$$
\begin{aligned}
&(i) \left\{ \left(\frac{a}{2^{2k}+1}, \frac{2^k a}{2^{2k}+1} \right) : 1 \leq a \leq 2^{2k}, a \in \mathbb{N} \right\} ; \\
&(ii) \left\{ \left(\frac{a}{2^{2k}+1}, \frac{-2^k a}{2^{2k}+1} \right) : 1 \leq a \leq 2^{2k}, a \in \mathbb{N} \right\} ,
\end{aligned}
$$

will give rise to orbits on $\widehat{T}$ which will shorten in length by a factor of $\frac{1}{4}$ and which have length $4d \leq 4k$ such that $4d \mid 4k$ but $4d \nmid 2k$. Moreover, these points are the only points which will give rise to such shortening orbits.

Proposition B.2. *Let T be the doubling map and $\widehat{T}$ be the triangle map. Then points on T which have the form*

(i) $\left\{\left(\frac{a}{2^{2k}-1}, \frac{2^k a}{2^{2k}-1}\right) : 1 \leq a \leq 2^{2k}-2, a \in \mathbb{N}\right\}$;

(ii) $\left\{\left(\frac{a}{2^{k}+1}, \frac{b}{2^{k}-1}\right) : 1 \leq a \leq 2^{k}, 0 \leq b \leq 2^k-2, a, b \in \mathbb{N}\right\}$;

(iii) $\left\{\left(\frac{a}{2^{2k}-1}, \frac{-2^k a}{2^{2k}-1}\right) : 1 \leq a \leq 2^{2k}-2, a \in \mathbb{N}\right\}$;

(iv) $\left\{\left(\frac{a}{2^{k}-1}, \frac{b}{2^{k}+1}\right) : 0 \leq a \leq 2^{k}-2, 1 \leq b \leq 2^k, a, b \in \mathbb{N}\right\}$;

(v) $\left\{\left(\frac{a}{2^{k}+1}, \frac{b}{2^{k}+1}\right) : 1 \leq a, b \leq 2^{k}, b \not\equiv \pm a, \pm 2^{k/2} a \,(mod\; 2^k+1), a, b \in \mathbb{N}\right\}$,

will give rise to orbits on $\widehat{T}$ which will shorten in length by a factor of $\frac{1}{2}$ and which have length $2d \leq 2k$ such that $2d \mid 2k$ but $2d \nmid k$. Moreover, these points are the only points which will give rise to such shortening orbits.

Proposition B.3. *Denote by $O^{4s}_{4k}(T)$ the number of orbits of length $4k$ on the doubling map T which will shorten in length by a factor of $\frac{1}{4}$ on the triangle map $\widehat{T}$. Then*

$$O^{4s}_{4k}(T) = \tfrac{1}{4k} \sum_{\substack{d \mid k \\ k/d \text{ odd}}} \mu\left(\tfrac{k}{d}\right) 2^{2d+1} ,$$

for all $k \geq 1$.

Proposition B.4. *Denote by $O^{2s}_{2k}(T)$ the number of orbits of length $2k$ on the doubling map T which will shorten in length by a factor of $\frac{1}{2}$ on the triangle map $\widehat{T}$. Then, for $k > 1$ odd, we have*

$$O^{2s}_{2k}(T) = \tfrac{1}{2k} \sum_{\substack{d \mid k \\ k/d \text{ odd}}} \mu\left(\tfrac{k}{d}\right) \left(5 \cdot 2^{2d} - 4 \cdot 2^d - 4\right) ,$$

and, for k even, we have

$$O^{2s}_{2k}(T) = \tfrac{1}{2k} \sum_{\substack{d \mid k \\ k/d \text{ odd}}} \mu\left(\tfrac{k}{d}\right) \left(5 \cdot 2^{2d} - 6 \cdot 2^d - 4\right) .$$

Proof of Proposition B.1. First note that orbits on T can only shorten in length by a factor of $\frac{1}{4}$ if there exists $g(x, y) \in \mathfrak{O}_{D_8}(x, y)$ such that $g^4(x, y) = (x, y)$, for some

$g \in D_8$ and $(x, y) \in X$, where (x, y), $g(x, y)$, and $g^2(x, y)$ are all distinct. Then

$$g \in \{\sigma, \sigma^3\}.$$

Now, suppose we have an orbit of length $4k$ which will shorten by a factor of $\frac{1}{4}$. Then we must have

$$T^{2k}(x, y) \equiv \sigma^2(x, y) \equiv (-x, -y) \pmod 1$$

and either

$$T^k(x, y) \equiv \sigma(x, y) \equiv (-y, x) \pmod 1,$$
$$T^{3k}(x, y) \equiv \sigma^3(x, y) \equiv (y, -x) \pmod 1,$$

or

$$T^k(x, y) \equiv \sigma^3(x, y) \equiv (y, -x) \pmod 1,$$
$$T^{3k}(x, y) \equiv \sigma(x, y) \equiv (-y, x) \pmod 1.$$

Now, let $(x, y) = (\frac{a}{2^{4k}-1}, \frac{b}{2^{4k}-1})$ be a point of period $4k$, where $0 \le a, b \le 2^{4k} - 2$, $a, b \in \mathbb{N}$, such that $a \neq b$ for $a, b = 0$.

First, assume that

$$T^{2k}(x, y) \equiv \sigma^2(x, y) \equiv (-x, -y) \pmod 1.$$

Then

$$(2^{2k}a, 2^{2k}b) \equiv (-a, -b) \pmod{2^{4k} - 1}.$$

It follows that

$$(2^{2k}a + a, 2^{2k}b + b) \equiv 0 \pmod{2^{4k} - 1},$$

so that $2^{4k} - 1 \mid (2^{2k}a + a)$ and $2^{4k} - 1 \mid (2^{2k}b + b)$. Hence, we have $u_1(2^{4k} - 1) = 2^{2k}a + a \pmod{2^{4k} - 1}$, for some $u_1 \in \mathbb{N}$. Solving for a, we obtain $a = u_1(2^{2k} - 1) \pmod{2^{4k} - 1}$. Then

$$\frac{a}{2^{4k} - 1} \equiv \frac{u_1}{2^{2k} + 1} \pmod 1.$$

Similarly, for b, we obtain

$$\frac{b}{2^{4k} - 1} \equiv \frac{u_2}{2^{2k} + 1} \pmod 1,$$

for some $u_2 \in \mathbb{N}$. Hence, replacing u_1 and u_2 by a and b, we have the following set of points:

$$\left\{\left(\frac{a}{2^{2k}+1}, \frac{b}{2^{2k}+1}\right) : 0 \leq a, b \leq 2^{2k}+1, a \neq b \text{ for } a, b = 0\right\}$$

We will now consider two cases:

(i) Assume that

$$T^k(x,y) \equiv \sigma(x,y) \equiv (-y,x) \pmod{1}.$$

Then

$$(2^k a, 2^k b) \equiv (-b, a) \pmod{2^{2k}+1}.$$

It follows that

$$(2^k a + b, 2^k b - a) \equiv 0 \pmod{2^{2k}+1},$$

so that $2^{2k}+1 \mid (2^k a + b)$ and $2^{2k}+1 \mid (2^k b - a)$. Hence, we have $u_3(2^{2k}+1) = 2^k a + b \,(\text{mod } 2^{2k}+1)$, for some $u_3 \in \mathbb{N}$. Solving for b, we obtain $b = -2^k a + u_3(2^{2k}+1) \,(\text{mod } 2^{2k}+1)$. Then

$$\frac{b}{2^{2k}+1} \equiv \frac{-2^k a}{2^{2k}+1} \pmod{1}.$$

Hence, we have the following set of points:

$$\left\{\left(\frac{a}{2^{2k}+1}, \frac{-2^k a}{2^{2k}+1}\right) : 1 \leq a \leq 2^{2k}\right\} \tag{B.1}$$

Similarly, from $2^{2k}+1 \mid (2^k b - a)$, we obtain that $u_4(2^{2k}+1) = 2^k b - a \,(\text{mod } 2^{2k}+1)$, for some $u_4 \in \mathbb{N}$. Solving for a, we obtain $a = 2^k b + u_4(2^{2k}+1) \,(\text{mod } 2^{2k}+1)$. Then

$$\frac{a}{2^{2k}+1} \equiv \frac{2^k b}{2^{2k}+1} \pmod{1}.$$

Hence, we have the following set of points:

$$\left\{\left(\frac{2^k b}{2^{2k}+1}, \frac{b}{2^{2k}+1}\right) : 1 \leq b \leq 2^{2k}\right\} \tag{B.2}$$

Now, note that if $a = 2^k b$, then $-2^k a = -2^{2k} b$, where $-2^{2k} b \equiv b \,(\text{mod } 2^{2k}+1)$. It follows that the sets (B.1) and (B.2) are equivalent.

(ii) Assume that

$$T^k(x,y) \equiv \sigma^3(x,y) \equiv (y,-x) \pmod 1 .$$

Then

$$(2^k a, 2^k b) \equiv (b, -a) \pmod{2^{2k}+1} .$$

Then by a similar argument as for (i), we obtain the following set of points:

$$\left\{ \left(\frac{a}{2^{2k}+1}, \frac{2^k a}{2^{2k}+1} \right) : 1 \le a \le 2^{2k} \right\} \tag{B.3}$$

Next, we will show that the sets (B.1) and (B.3) satisfy the necessary conditions. First, consider the set (B.1). Here, we have

$$\begin{aligned}
T^k(x,y) &\equiv \left(\frac{2^k a}{2^{2k}+1}, \frac{-2^{2k} a}{2^{2k}+1} \right) && \pmod 1 \\
&\equiv \left(\frac{2^k a}{2^{2k}+1}, \frac{-2^{2k} a + a(2^{2k}+1)}{2^{2k}+1} \right) && \pmod 1 \\
&\equiv \left(\frac{2^k a}{2^{2k}+1}, \frac{a}{2^{2k}+1} \right) && \pmod 1 \\
&\equiv (-y, x) && \pmod 1 \\
&\equiv \sigma(x,y) && \pmod 1 ,
\end{aligned}$$

$$\begin{aligned}
T^{2k}(x,y) &\equiv \left(\frac{2^{2k} a}{2^{2k}+1}, \frac{-2^{3k} a}{2^{2k}+1} \right) && \pmod 1 \\
&\equiv \left(\frac{2^{2k} a - a(2^{2k}+1)}{2^{2k}+1}, \frac{-2^{3k} a + 2^k a(2^{2k}+1)}{2^{2k}+1} \right) && \pmod 1 \\
&\equiv \left(\frac{-a}{2^{2k}+1}, \frac{2^k a}{2^k+1} \right) && \pmod 1 \\
&\equiv (-x, -y) && \pmod 1 \\
&\equiv \sigma^2(x,y) && \pmod 1 ,
\end{aligned}$$

and

$$
\begin{aligned}
T^{3k}(x,y) &\equiv \left(\frac{2^{3k}a}{2^{2k}+1}, \frac{-2^{4k}a}{2^{2k}+1}\right) && (\bmod 1)\\
&\equiv \left(\frac{2^{3k}a - 2^{k}a(2^{2k}+1)}{2^{2k}+1}, \frac{-a}{2^{2k}+1}\right) && (\bmod 1)\\
&\equiv \left(\frac{-2^{k}a}{2^{2k}+1}, \frac{-a}{2^{k}+1}\right) && (\bmod 1)\\
&\equiv (y,-x) && (\bmod 1)\\
&\equiv \sigma^3(x,y) && (\bmod 1)\,.
\end{aligned}
$$

Next, consider the set (B.3). Here, we have

$$
\begin{aligned}
T^{k}(x,y) &\equiv \left(\frac{2^{k}a}{2^{2k}+1}, \frac{2^{2k}a}{2^{2k}+1}\right) && (\bmod 1)\\
&\equiv \left(\frac{2^{k}a}{2^{2k}+1}, \frac{2^{2k}a - a(2^{2k}+1)}{2^{2k}+1}\right) && (\bmod 1)\\
&\equiv \left(\frac{2^{k}a}{2^{2k}+1}, \frac{-a}{2^{2k}+1}\right) && (\bmod 1)\\
&\equiv (y,-x) && (\bmod 1)\\
&\equiv \sigma^3(x,y) && (\bmod 1)\,,
\end{aligned}
$$

$$
\begin{aligned}
T^{2k}(x,y) &\equiv \left(\frac{2^{2k}a}{2^{2k}+1}, \frac{2^{3k}a}{2^{2k}+1}\right) && (\bmod 1)\\
&\equiv \left(\frac{2^{2k}a - a(2^{2k}+1)}{2^{2k}+1}, \frac{2^{3k}a - 2^{k}a(2^{2k}+1)}{2^{2k}+1}\right) && (\bmod 1)\\
&\equiv \left(\frac{-a}{2^{2k}+1}, \frac{-2^{k}a}{2^{k}+1}\right) && (\bmod 1)\\
&\equiv (-x,-y) && (\bmod 1)\\
&\equiv \sigma^2(x,y) && (\bmod 1)\,,
\end{aligned}
$$

and

$$\begin{aligned}
T^{3k}(x,y) &\equiv \left(\frac{2^{3k}a}{2^{2k}+1}, \frac{2^{4k}a}{2^{2k}+1}\right) && (\bmod 1)\\
&\equiv \left(\frac{2^{3k}a - 2^k a(2^{2k}+1)}{2^{2k}+1}, \frac{a}{2^{2k}+1}\right) && (\bmod 1)\\
&\equiv \left(\frac{-2^k a}{2^{2k}+1}, \frac{a}{2^k+1}\right) && (\bmod 1)\\
&\equiv (-y, x) && (\bmod 1)\\
&\equiv \sigma(x,y) && (\bmod 1)\,.
\end{aligned}$$

Last but not least, suppose (x,y) is a point in either (B.1) or (B.3) which lies in an orbit of length $4d$ such that $4d \mid 4k$. Then

$$T^{4d}(x,y) \equiv (x,y) \quad (\bmod 1)\,. \tag{B.4}$$

Now, when $k/d = 2m$, for some $m \in \mathbb{N}$, then by (B.4), we have that

$$T^{2k}(x,y) \equiv T^{4md}(x,y) \equiv (x,y) \equiv (-x,-y) \quad (\bmod 1)\,,$$

which is a contradiction. Hence, we must have k/d odd, so that $2 \nmid (k/d)$, that is $4d \nmid 2k$. □

Proof of Proposition B.2. First note that orbits on T can only shorten in length by a factor of $\frac{1}{2}$ if there exists $g(x,y) \in \mathfrak{O}_{D_8}(x,y)$ such that $g^2(x,y) = (x,y)$, for some $g \in D_8$ and $(x,y) \in X$, where (x,y) and $g(x,y)$ are distinct. Then

$$g \in \{\tau, \sigma\tau, \sigma^2\tau, \sigma^3\tau, \sigma^2\}\,.$$

Now, suppose we have an orbit of length $2k$ which will shorten by a factor of $\frac{1}{2}$. We let $(x,y) = (\frac{a}{2^{2k}-1}, \frac{b}{2^{2k}-1})$, where $0 \le a, b \le 2^{2k}-2$, $a, b \in \mathbb{N}$, such that $a \ne b$ for $a, b = 0$. We will consider several cases.

(i) Assume that

$$T^k(x,y) \equiv \tau(x,y) \equiv (y,x) \quad (\bmod 1)\,.$$

Then

$$(2^k a, 2^k b) \equiv (b,a) \quad (\bmod\ 2^{2k}-1)\,.$$

It follows that

$$(2^k a - b, 2^k b - a) \equiv 0 \pmod{2^{2k} - 1},$$

so that $2^{2k} - 1 \mid (2^k a - b)$ and $2^{2k} - 1 \mid (2^k b - a)$. Hence, we have that $u_1(2^{2k} - 1) = 2^k a - b \,(\mathrm{mod}\ 2^{2k} - 1)$, for some $u_1 \in \mathbb{N}$. Solving for b, we obtain $b = 2^k a - u_1(2^{2k} - 1) \,(\mathrm{mod}\ 2^{2k} - 1)$. Then

$$\begin{aligned} \frac{b}{2^{2k} - 1} &\equiv \frac{2^k a - u_1(2^{2k} - 1)}{2^{2k} - 1} \quad (\mathrm{mod}\, 1) \\ &\equiv \frac{2^k a}{2^{2k} - 1} \quad (\mathrm{mod}\, 1). \end{aligned}$$

Hence, we obtain the following set of points:

$$\left\{ \left(\frac{a}{2^{2k} - 1}, \frac{2^k a}{2^{2k} - 1} \right) : 1 \le a \le 2^{2k} - 1 \right\} \tag{B.5}$$

Similarly, from $2^{2k} - 1 \mid (2^k b - a)$, we obtain the following set of points:

$$\left\{ \left(\frac{2^k b}{2^{2k} - 1}, \frac{b}{2^{2k} - 1} \right) : 1 \le b \le 2^{2k} - 1 \right\} \tag{B.6}$$

Now, note that if $a = 2^k b$, then $2^k a = 2^{2k} b$, where $\frac{2^{2k} b}{2^{2k} - 1} \equiv \frac{b}{2^{2k} - 1} \,(\mathrm{mod}\, 1)$. It follows that the sets (B.5) and (B.6) are equivalent.

(ii) Assume that

$$T^k(x, y) \equiv \sigma\tau(x, y) \equiv (-x, y) \quad (\mathrm{mod}\, 1).$$

Then

$$(2^k a, 2^k b) \equiv (-a, b) \pmod{2^{2k} - 1}.$$

It follows that

$$(2^k a + a, 2^k b - b) \equiv 0 \pmod{2^{2k} - 1},$$

so that $2^{2k} - 1 \mid (2^k a + a)$ and $2^{2k} - 1 \mid (2^k b - b)$. Hence, we have $u_2(2^{2k} - 1) = 2^k a + a \,(\mathrm{mod}\ 2^{2k} - 1)$, for some $u_2 \in \mathbb{N}$. Solving for a, we obtain

$$a = \frac{u_2(2^{2k} - 1)}{2^k + 1} = u_2(2^k - 1) \pmod{2^{2k} - 1}.$$

Then

$$\begin{aligned} \frac{a}{2^{2k} - 1} &\equiv \frac{u_2(2^k - 1)}{2^{2k} - 1} \quad (\mathrm{mod}\, 1) \\ &\equiv \frac{u_2}{2^k + 1} \quad (\mathrm{mod}\, 1). \end{aligned}$$

Next, from $2^{2k}-1 \mid (2^k b - b)$, we obtain that $u_3(2^{2k}-1) = 2^k b - b \pmod{2^{2k}-1}$, for some $u_3 \in \mathbb{N}$. Solving for b, we obtain

$$b = \frac{u_3(2^{2k}-1)}{2^k-1} = u_3(2^k+1) \pmod{2^{2k}-1}\,.$$

Then

$$\begin{aligned}\frac{b}{2^{2k}-1} &\equiv \frac{u_3(2^k+1)}{2^{2k}-1} \pmod 1\\ &\equiv \frac{u_3}{2^k-1} \pmod 1\,.\end{aligned}$$

Hence, replacing u_2 and u_3 by a and b, we obtain the following set of points:

$$\left\{\left(\frac{a}{2^k+1}, \frac{b}{2^k-1}\right) : 1 \le a \le 2^k, 0 \le b \le 2^k-2\right\}$$

Here, note that if $a=0$, then $\sigma\tau(x,y) = (x,y) \pmod 1$, which is a contradiction to our initial assumption.

(iii) Assume that

$$T^k(x,y) \equiv \sigma^2\tau(x,y) \equiv (-y,-x) \pmod 1\,.$$

Then

$$(2^k a, 2^k b) \equiv (-b,-a) \pmod{2^{2k}-1}\,.$$

It follows that

$$(2^k a + b, 2^k b + a) \equiv 0 \pmod{2^{2k}-1}\,,$$

so that $2^{2k}-1 \mid (2^k a + b)$ and $2^{2k}-1 \mid (2^k b + a)$. Hence, we have that $u_4(2^{2k}-1) = 2^k a + b \pmod{2^{2k}-1}$, for some $u_4 \in \mathbb{N}$. Solving for b, we obtain $b = -2^k a - u_4(2^{2k}-1) \pmod{2^{2k}-1}$. Then

$$\begin{aligned}\frac{b}{2^{2k}-1} &\equiv \frac{-2^k a - u_4(2^k-1)}{2^{2k}-1} \pmod 1\\ &\equiv \frac{-2^k a}{2^{2k}-1} \pmod 1\,.\end{aligned}$$

Hence, we obtain the following set of points:

$$\left\{\left(\frac{a}{2^{2k}-1}, \frac{-2^k a}{2^{2k}-1}\right) : 1 \le a \le 2^{2k}-1\right\} \tag{B.7}$$

Similarly, from $2^{2k}-1 \mid (2^k b + a)$, we obtain the following set of points:

$$\left\{ \left(\frac{-2^k b}{2^{2k}-1}, \frac{b}{2^{2k}-1} \right) : 1 \leq b \leq 2^{2k}-1 \right\} \tag{B.8}$$

Now, note that if $a = -2^k b$, then $-2^k a = 2^{2k} b$, where $\frac{2^{2k} b}{2^{2k}-1} \equiv \frac{b}{2^{2k}-1} \pmod 1$. It follows that the sets (B.7) and (B.8) are equivalent.

(iv) Assume that

$$T^k(x,y) \equiv \sigma^3 \tau(x,y) \equiv (x,-y) \pmod 1 .$$

Then

$$(2^k a, 2^k b) \equiv (-a, b) \pmod{2^{2k}-1} .$$

Then by a similar argument as for (ii), we have the following set of points:

$$\left\{ \left(\frac{a}{2^k-1}, \frac{b}{2^k+1} \right) : 0 \leq a \leq 2^k, 1 \leq b \leq 2^k - 2 \right\}$$

(v) Assume that

$$T^k(x,y) \equiv \sigma^2(x,y) \equiv (-x,-y) \pmod 1 .$$

Then

$$(2^k a, 2^k b) \equiv (-a, -b) \pmod{2^{2k}-1} ,$$

and by a similar argument as for (iv), we have the following set of points:

$$\left\{ \left(\frac{a}{2^k+1}, \frac{b}{2^k+1} \right) : 0 \leq a, b \leq 2^k, a \neq b \text{ for } a, b = 0 \right\}$$

Next, we need to ensure that (i) through (v) and the points given in Proposition B.1 are pairwise disjoint. Here, note that 2^k+1 and 2^k-1 are co-prime since if $d \mid 2^k+1$ and $d \mid 2^k-1$, for some d, then

$$d \mid (2^k+1) - (2^k-1) ,$$

so that $d \mid 2$, but since d must be odd, then $d = 1$. Hence, if we add the condition that $a, b \neq 0$ to (v), it is sufficient to show that (i),(iii),(iv), and the points given in Proposition B.1 are pairwise disjoint.

First, consider (i) and (iii). If (x, y) is in both (i) and (iii), then we have that

$$\tau(x,y) \equiv T^k(x,y) \equiv \sigma^2\tau(x,y) \pmod 1 .$$

Applying τ, we obtain

$$(x,y) \equiv \sigma^2(x,y) \pmod 1 ,$$

but then either $x, y = 0$ or $x, y = \frac{1}{2}$, which is a contradiction. Hence, (i) and (iii) must be disjoint.

Second, consider (i) and (v). If (x, y) is in both (i) and (v), then

$$\tau(x,y) \equiv T^k(x,y) \equiv \sigma^2(x,y) \pmod 1 .$$

Applying σ^2, we obtain

$$\sigma^2\tau(x,y) \equiv (x,y) \pmod 1 ,$$

so that we must have $(x, y) = (-y, -x) \pmod 1$. Hence, for (i) and (v) to be disjoint, we must add the condition that $b \neq -a \,(\mathrm{mod}\ 2^k + 1)$ to (v). Similarly, to satisfy that (iii) and (v) are pairwise disjoint, we must add the condition that $b \neq a \,(\mathrm{mod}\ 2^k + 1)$.

Next, we will check that (i), (iii), (v) and the sets of points of Proposition B.1 are pairwise disjoint. Here, note first that $2^{2k} - 1$ and $2^{2k} + 1$ are co-prime. Hence, it is sufficient to show that (v) and the sets of points in Proposition B.1 are pairwise disjoint. Here note that if $k = \frac{d}{2}$, then

$$\left(\frac{a}{2^{2k}+1}, \frac{\pm 2^k a}{2^{2k}+1}\right) = \left(\frac{a}{2^d+1}, \frac{\pm 2^{d/2}}{2^d+1}\right) \pmod 1 .$$

Hence, for k even, we must add the condition that $b \neq \pm 2^{k/2} a$ in (v) to satisfy that the sets of points in Proposition B.1 and (v) are disjoint. Hence, for (v), we have the following set of points:

$$\left\{\left(\frac{a}{2^k+1}, \frac{b}{2^k+1}\right) : 1 \leq a, b \leq 2^k, b \neq \pm a, \pm 2^{k/2} a \,(\mathrm{mod}\ 2^k + 1)\right\}$$

Last but not least, suppose (x, y) is a point in $(i), (ii), \ldots,$ or (v) which lies in an orbit of length $2d < 2k$ such that $2d \mid 2k$. Then

$$T^{2d}(x,y) \equiv (x,y) \pmod 1 . \tag{B.9}$$

Now, when $k/d = 2m$, for some $m \in \mathbb{N}$, then by (B.9), we have that

$$T^k(x,y) \equiv T^{2dm}(x,y) \equiv (x,y) \pmod 1,$$

which is a contradiction since $T^k(x,y) = g(x,y)$, for some $g \in \{\tau, \sigma\tau, \sigma^2\tau, \sigma^3\tau, \sigma^2\}$, where $g(x,y) \neq (x,y)$. Hence, k/d must be odd, so that $2 \nmid (k/d)$, that is $2d \nmid k$. $\square$

Proof of Proposition B.3. By Proposition B.1 we have that

$$\begin{aligned}
\mathfrak{F}^{4s}_{4k}(T) = \{&\text{points of period dividing } 4k \text{ but not dividing } 2k \text{ which will give} \\
&\text{rise to orbits on } T \text{ which will shorten by a factor of } \tfrac{1}{4} \text{ on } \widehat{T}\} \\
= &\left\{\left(\frac{a}{2^{2k}+1}, \frac{2^k a}{2^{2k}+1}\right) : 1 \leq a \leq 2^{2k}, a \in \mathbb{N}\right\} \bigsqcup \\
&\left\{\left(\frac{a}{2^{2k}+1}, \frac{-2^k a}{2^{2k}+1}\right) : 1 \leq a \leq 2^{2k}, a \in \mathbb{N}\right\}.
\end{aligned}$$

Then

$$F^{4s}_{4k} = 2 \cdot 2^{2k} = 2^{2k+1},$$

for all $k \geq 1$. Now, in order to only count points of *least* period $4k$, we must exclude points of period $4d < 4k$ such that $4d \mid 4k$ with k/d odd, by Proposition B.1. Then using the Inclusion-Exclusion Principle and the Möbius function, we have that

$$\begin{aligned}
L^{4s}_{4k}(T) = \#\{&\text{points of } \textit{least} \text{ period } 4k \text{ which will give rise to orbits} \\
&\text{on } T \text{ which will shorten by a factor of } \tfrac{1}{4} \text{ on } \widehat{T}\} \\
= &F^{4s}_{4k}(T) - \left| \bigcup_{\substack{p \mid k \\ k/p \text{ odd}}} F^{4s}_{4k/p}(T) \right| \\
= &\sum_{\substack{d \mid k \\ k/d \text{ odd}}} \mu\left(\tfrac{k}{d}\right) F^{4s}_{4d}(T),
\end{aligned}$$

for all $k \geq 1$. It follows that

$$\begin{aligned}O_{4k}^{4s}(T) &= \#\{\text{orbits of } T \text{ which have length } 4k \text{ and} \\ &\qquad \text{which will shorten by a factor of } \tfrac{1}{4}\} \\ &= \tfrac{1}{4k} \sum_{\substack{d|k \\ k/d \text{ odd}}} \mu\left(\tfrac{k}{d}\right) F_{4d}^{4s}(T) \\ &= \tfrac{1}{4k} \sum_{\substack{d|k \\ k/d \text{ odd}}} \mu\left(\tfrac{k}{d}\right) 2^{2d+1}\,,\end{aligned}$$

for all $k \geq 1$. □

Proof of Proposition B.4. By Proposition B.2, we have that

$$\begin{aligned}\mathfrak{F}_{2k}^{2s}(T) &= \{\text{points of period dividing } 2k \text{ but not dividing } k \text{ which will give} \\ &\qquad \text{rise to orbits on } T \text{ which will shorten by a factor of } \tfrac{1}{2} \text{ on } \widehat{T}\} \\ &= \left\{\left(\frac{a}{2^{2k}-1}, \frac{2^k a}{2^{2k}-1}\right) : 1 \leq a \leq 2^{2k} - 2\, a \in \mathbb{N}\right\} \bigsqcup \\ &\quad \left\{\left(\frac{a}{2^k+1}, \frac{b}{2^k-1}\right) : 1 \leq a \leq 2^k, 0 \leq b \leq 2^k - 2,\ a, b \in \mathbb{N}\right\} \bigsqcup \\ &\quad \left\{\left(\frac{a}{2^{2k}-1}, \frac{-2^k a}{2^{2k}-1}\right) : 1 \leq a \leq 2^{2k} - 2,\ a \in \mathbb{N}\right\} \bigsqcup \\ &\quad \left\{\left(\frac{a}{2^k-1}, \frac{b}{2^k+1}\right) : 0 \leq a \leq 2^k - 2, 1 \leq b \leq 2^k,\ a, b \in \mathbb{N}\right\} \bigsqcup \\ &\quad \left\{\left(\frac{a}{2^k+1}, \frac{b}{2^k+1}\right) : 1 \leq a, b \leq 2^k, b \neq \pm a, \pm 2^{k/2} a \pmod{2^k+1},\ a, b \in \mathbb{N}\right\}.\end{aligned}$$

Then, for k odd, we have that

$$F_{2k}^{2s} = 5 \cdot 2^{2k} - 4 \cdot 2^k - 4\,.$$

Now, in order to only count points of *least* period $2k$, we must exclude points of period $2d < 2k$ such that $d \mid k$ with k/d odd, by Proposition B.2. Then using the

Inclusion-Exclusion Principle and the Möbius function, we have that

$$\begin{aligned}
L_{2k}^{2s}(T) &= \#\{\text{points of } \textit{least} \text{ period } 2k \text{ which will give rise to orbits} \\
&\qquad \text{on } T \text{ which will shorten by a factor of } \tfrac{1}{2} \text{ on } \widehat{T}\} \\
&= F_{2k}^{2s}(T) - \left| \bigcup_{\substack{p|k \\ k/p \text{ odd}}} F_{2k/p}^{2s}(T) \right| \\
&= \sum_{\substack{d|k \\ k/d \text{ odd}}} \mu\left(\tfrac{k}{d}\right) F_{2d}^{2s}(T)\,,
\end{aligned}$$

for all odd k. It follows that

$$\begin{aligned}
O_{2k}^{2s}(T) &= \#\{\text{orbits of } T \text{ which have length } 2k \text{ and} \\
&\qquad \text{which will shorten by a factor of } \tfrac{1}{2}\} \\
&= \tfrac{1}{2k} \sum_{\substack{d|k \\ k/d \text{ odd}}} \mu\left(\tfrac{k}{d}\right) F_{2d}^{2s}(T) \\
&= \tfrac{1}{2k} \sum_{\substack{d|k \\ k/d \text{ odd}}} \mu\left(\tfrac{k}{d}\right) \left(5 \cdot 2^{2d} - 4 \cdot 2^{d} - 4\right)\,,
\end{aligned}$$

for all odd k.

Next, for k even, we have

$$F_{2k}^{2s} = 5 \cdot 2^{2k} - 6 \cdot 2^{k} - 4\,.$$

Then

$$\begin{aligned}
O_{2k}^{2s}(T) &= \#\{\text{orbits of } T \text{ which have length } 2k \text{ and} \\
&\qquad \text{which will shorten by a factor of } \tfrac{1}{2}\} \\
&= \tfrac{1}{2k} \sum_{\substack{d|k \\ k/d \text{ odd}}} \mu\left(\tfrac{k}{d}\right) F_{2d}^{2s}(T) \\
&= \tfrac{1}{2k} \sum_{\substack{d|k \\ k/d \text{ odd}}} \mu\left(\tfrac{k}{d}\right) \left(5 \cdot 2^{2d} - 6 \cdot 2^{d} - 4\right)\,,
\end{aligned}$$

for all even k. □

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