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Communities in University Mathematics

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This paper regards communities of learners and teachers that are formed, develop and interact in university mathematics environments through the *Communities of Practice* theoretical lenses. In this perspective learning is described as a process of participation and reification in a community in which individuals belong and form their identity through engagement, imagination and alignment. In addition, when inquiry is considered as a fundamental way of participation, through critical alignment, the community becomes a *Community of Inquiry.* We discuss the above theoretical underpinnings with examples of their application in research in university mathematics education and, in more detail, in two Research Cases on mathematics students' and teachers' perspectives about proof and on engineering students' conceptual understanding of mathematics. The paper concludes with a critical reflection on the theorising of the role of communities at university level teaching and learning as well as ways forward for future research.

Keywords: community of practice; community of inquiry; identity, critical alignment, university mathematics education

Introduction

Experience in university mathematics teaching indicates that there is no clear consensus between university teachers[1] and students on the meaning and the value of mathematics (e.g. Solomon, 2006). This observation has attracted the interest of mathematics education researchers to investigate the *takes* on the meaning and values of mathematics in different communities – such as researcher mathematicians, teachers of mathematics, undergraduate and postgraduate students – that are involved in practices within university especially in relation to the teaching and learning (e.g., Burton, 2004; Herzig,

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[1] We use (university) teacher to describe all those involved in the teaching of mathematics at university level. We describe other identities with specific characterisations, such as research mathematicians or mathematics educator, when it is necessary.

2002; Solomon, 2007). In this endeavour, research has drawn on the theory of *Communities of Practice* (henceforth, *CoP*) based on the work of Lave and Wenger (1991) and Wenger (1998) and *Communities of Inquiry* (henceforth, *CoI*) based on the work of Jaworski, Goodchild, and others (e.g., Goodchild, Fuglestad & Jaworski, 2013).

Our aim in this paper is to give more theoretical insight into the role of these communities in the learning and teaching of mathematics at university level and to take this theorisation forward in future research. Our point is that mathematical practices at university level are distinguished from those at secondary or primary level for reasons related to the mathematical content, the teachers and the students involved. In many countries, it is at university level that the mathematical theory involves communication with very specific and rigorous rules and processes (such as theorems, definitions and proofs). Teachers who are very often researchers of mathematics become learners themselves and experience the double identity of the teacher and the researcher in the same institutional environment. Students are adults who are accountable for their choices, including their choice of studying mathematics; belong to multiple communities; often have to learn individually and may consider their studies as a step towards their professional future.

In the following sections we present the main theoretical underpinnings of *CoP* and *CoI* and we exemplify how their theoretical constructs have been used in university mathematics education research. We conclude with a discussion on the potentialities, limitations and ways forward of *CoP* and *CoI* in university mathematics education research, also in (dis)connection with other sociocultural theories.

Theoretical perspective

In taking a perspective on knowledge, learning and teaching within university mathematics we start from the position of Vygotsky that cognition arises through participation in sociocultural contexts (Vygotsky, 1978). We see learning as taking place through interactions in social settings, specifically within the communities in which university students, their teachers, research students and researchers interact. A *community* is a group of individuals identifiable by who they are in terms of how they relate to each other, their common activities and ways of thinking, beliefs and values. Such communities of course extend beyond the university boundaries and into wider cultures and systems of which individuals are a part. In taking a community perspective, we are focusing on specific *practices* within a university, especially those that include the teaching and learning of mathematics. We draw specifically on the work of Lave and Wenger (1991) and Wenger (1998) who introduced the idea of Communities of Practice (*CoP*) and we apply their theory to the study of learning and teaching of mathematics at university.

These two principal sources take different positions on a *CoP* and its constitution. Lave and Wenger focus on the concept of *legitimate peripheral participation* by which newcomers to a practice are drawn into the practice and, potentially, become old-timers, around whom the practice is based. The transition from newcomer to old-timer involves differing trajectories of identity. Kanes and Lerman (2008) characterise such transition as the active process of an individual who wants to move from the periphery to the centre. Wenger (1998), on the other hand, focuses on the community as a whole and on the practice that takes place in it:

> The concept of practice connotes doing, but not just doing in and of itself. It is doing in a historical and social context that gives structure and meaning to what we do. In this sense practice is always social practice. (p. 47)

We recognise the long history of the practices of mathematics, learning of mathematics and research into mathematics that has led to where we are today and which is ever present in the 'doing' in which we engage at university level. According to Wenger (1998) identities form trajectories, both within and across *CoPs*, including the *inbound trajectories* from the periphery to the centre. A trajectory can be seen as a continuous motion that connects the past, the present and the future. Kanes and Lerman (2008) describe Wenger's (1998) perspective as passive and inductive, and we acknowledge that Wenger (1998) does not put so much attention on how trajectories are influenced and operationalized in the context of the community.

Within a *CoP*, Wenger (1998, p. 55) introduces two key processes through which an individual make meanings (through which they learn): *participation* and *reification*. Participation involves *being* within a *CoP*, taking part in its activities, interacting, negotiating, agreeing, disagreeing, formulating and making sense. The last two of these, formulating and making sense link participation to *reification*. Reification means "making into a thing … the process of giving form to our experience by producing objects that congeal this experience into thingness" (p. 58). Wenger states, "We project our meanings onto the world and then we perceive them as existing in the world, as having a reality of their own" (p. 58). This has particular resonance with mathematics in which abstract entities and relationships are formed through negotiation in mathematical communities and over time take on a nature of objects in mathematics. In conceptualizing *CoP*, Wenger talks of three dimensions of practice: *mutual engagement* – establishing norms, expectations, ways of working and social relationships; *joint enterprise* – developing common understandings of what the enterprise is about and where it is going, its aims and ideals; and *shared repertoire* – the objects that we use and how we use them, resources such as technology, symbols, abstract forms. We can see these dimensions encompassing lectures and lecturing, definitions and theorems, symbolisation and proof, graphing and the technology of graphing, mathematical software and so on. These dimensions help us to characterise and analyse practice in university mathematics. We need also to interpret the various roles of practitioners within this practice – how do they define themselves, and are there differences between groups such as researchers, teachers, students, graduate students? Wenger talks of *learning* as "a process of becoming" (p. 215). This, he claims, is "an experience of identity" (p. 215), where identity "serves as a pivot between the social and the individual, so that each can be talked about in terms of the other" (p. 145). He offers again three dimensions which he calls this time "modes of belonging" in which *identity* is conceptualised in terms of "belonging" to a *CoP* involving "engagement", "imagination" and "alignment" (p. 173). An individual *engages* with practice, alongside co-practitioners, uses *imagination* to weave a personal trajectory within the practice and *aligns* with the norms and expectations of the practice. Thus individual identity is defined in relation to the individual's *(non-)participation* in the *CoP* and of course other *CoPs* to which the individual belongs.

In a following section, we offer case studies of university practice in which theory of *CoP* has been used to make sense of characteristics and issues. However, before doing so, we will address what we see as a limitation of Wenger's *CoP* theory.

The mode of belonging designated as *alignment* describes ways in which the person-in-practice 'lines up with' the norms and expectations that hold sway within the *CoP*. This can be seen to perpetuate/sustain forms of practice whether or not they are the best for achieving the goals of the practice (Jaworski, 2008). There is a scepticism and sometimes critique on certain traditional practices (e.g., chalk and talk, some lecturing styles) as effective teaching methods for students' learning (Biggs, 2003). Students experience mathematics as something 'done to them', rather than 'done by them', and do not share in the ownership of meaning – let alone meaning making; they are excluded from vital aspects of participation (Solomon, 2007, p. 90). Thus, alignment with traditional practices can leave something to be desired in relation to students' mathematical understanding.

In most practices, alignment of some kind is unavoidable; however, it does not have to be uncritical. A *critical alignment* would imply a questioning of the status quo. For example, the teacher who recognises that students are suffering serious problems with the traditional mode of lecturing might seek to modify her practice to support the students in some way. Asking questions about one's practice is a form of *inquiry*, inquiring into the teaching-learning process to achieve 'better' outcomes from it, taking an 'inquiry stance' in practice; inquiry develops as 'a way of being' for the teachers (and students) involved (Cochran-Smith & Lytle, 1999; Jaworski, 2004; Wells, 1999). Thus we might say that teacher and students working together in inquiry ways form a 'community of inquiry'.

Wells (1999) writes:

> inquiry does not refer to a method … still less to a generic set of procedures for carrying out activities. Rather it indicates a stance towards experiences and ideas – a willingness to wonder, to ask questions and to seek to understand by collaborating with others in the attempt to make answers to them. (p. 122)

Inquiry is also fundamental in all research processes (Stenhouse, 1984), so research which seeks to promote the development of mathematics teaching, as well as to document its characteristics and issues, is a process of systematic inquiry. Such inquiry has resonance too with the use of inquiry-based tasks to engage students with mathematics and foster learning (Abdulwahed, Jaworski & Crawford, 2012).

The idea of inquiry community can be seen to transform the idea of *Community of Practice*. A *Community of Inquiry* (*CoI*) is a *CoP* in which inquiry is a fundamental way of being in practice. So the *CoI* encompasses Wenger's three dimensions: *mutual engagement* is an inquiry-based process; *joint enterprise* involves the goals of inquiry which are to reach better understanding of what is being questioned; and, *shared repertoire* includes such resources as inquiry-based tasks and inquiry approaches in exploring mathematical concepts. Identities of participants within a *CoI* develop through trajectories of *engagement* and *imagination* as for a *CoP*; however, the crucial difference is with *alignment*. In a *CoI*, *alignment* is *always critical alignment*. As a 'normal' part of their participation, participants question the practices in which they engage. Such questioning leads to new forms of practice and new ways of awareness of the problems and issues in developing effective ways of working and good outcomes for students learning.

We now exemplify how the core ideas of *CoP* have been deployed in university mathematics education research studies.

The *CoP* in university mathematics education research

Research in university mathematics education has used *CoP* and *CoI* theoretical constructs in order to gain insight into how learners, teachers and researchers act and interact within specific institutional and sociocultural contexts. In this section we present indicative cases from university mathematics education research with focus on undergraduate and postgraduate learning as well as on teaching.

In studies conducted by Solomon and colleagues in the English undergraduate mathematics context, students participate in several communities: the general undergraduate student community, the mathematics undergraduate community and the first-year student community. Additionally students belong to the classroom community of learners and tutors. These communities are different from the community of research mathematicians of which students may not be aware or of which may not aspire to be a part (Solomon, 2007). Students' *participation* (or *non-participation*) in multiple *CoP*s, and sometimes communities with opposing rules of engagement, may result in differential experiences of *identity* and *belonging* and generate identities of *not belonging* among students. For example, students may experience *non-participation* to the mathematical discipline *CoP* – a teaching-learning community of students and teachers – which emphasises deep learning of mathematical rules and the justification behind these rules. This *non-participation* can lead students to *marginality* from this *CoP*. This *marginalisation,* according to Solomon's (2007) study, might mean *alignment* to the rules of the community of undergraduates, which emphasises summative assessment and surface learning. Solomon (2006), also, discusses in what extent undergraduate students share the same epistemic values of mathematics with the community of research mathematicians. She mentions that the way undergraduate mathematics is taught and portrayed in the lectures of most English universities is disjoint with the practitioner's/lecturer's tacit knowledge and practice in mathematics research. Students are introduced to a predefined structure of definition-theorem-proof that hides research approaches such as intuition, trial and error, building and testing conjectures. As a result students develop *identities* and beliefs about mathematics and learning of mathematics, which are not aligned with the practices and epistemic values of the mathematics community.

Similarly to the study above, a substantial part of the research on *CoP*s in university mathematics focuses on an important element in mathematical practice, *proof* (e.g. Hemmi, 2006, 2008, 2010; Solomon, 2006). In these studies, the introduction to proof and proving processes aims to be part of the process of students' enculturation to the mathematical way of thinking and working. *Research Case 1* in the following section elaborates this further. However, not all students aim to become mathematicians and, especially in the first year of their studies, do not have access to the practices of experienced mathematicians (cf. Solomon, 2006). Additionally, not all teachers of mathematics are researchers of mathematics (cf. Biza, 2014; Jaworski & Matthews, 2011a).

At postgraduate studies in mathematics, as opposed to undergraduate level, we can assume that students intend to be involved in research. So, we can see them as *legitimate peripheral participants* in the community of research mathematicians. For example, in a study conducted in the US (Herzig, 2002) on doctoral mathematics students and faculty experiences in the mathematical community of their department, doctoral students encounter two communities: firstly, the course-taking community with its relevant assessment (coursework and examination); and then, the research community. Students who become integrated in the first community have little access to mathematics research practices and, as a result, they are prevented from peripheral participation into what is necessary for their integration later into the research community. For the faculty, these obstacles to participation are often intentional. They are meant as challenges: force the students to work hard and make them prove that they are able to complete their doctoral studies before the institution invests important resources in them. To students, the lack of opportunities for participation into mathematical research practices is frustrating and interferes with their learning of mathematics.

The main focus of our research examples so far was on the students' role in communities formed in university mathematical *practice*. If we shift now the focus to the teaching, there is not always a consensus on the *joint enterprise* in mathematical teaching. Jaworski and Matthews (2011a) studied cases of university teachers' lecturing in an English mathematics department. The analysis of their discourses, where teaching was concerned, indicated that the claim for the *joint enterprise* of teaching was hard to justify. Teachers demonstrated different understandings regarding the meaning and the aim of teaching mathematics. For example, some teachers seem to not care about students' attendance in lectures and transfer the responsibility of participation to the students. Others provide inspiration and structure to students and want students to attend and gain from this experience.

From the perspective of the new lecturer, Biza (2014) discussed the existence of multiple communities (research mathematicians/statisticians, mathematics educators, users of statistics etc.) practising in the teaching of mathematics and statistics in an English mathematics department and the influence of these communities in the experiences of a new university teacher. From a similar standpoint, Blanton and Stylianou (2009) see the new teacher of mathematics as a newcomer who learns from the experienced teachers, the old-timers, through her *legitimate peripheral participation* in the *CoP* that has already been established in the institution she is entering.

All aforementioned examples are using the theoretical construct of the *CoP* with a focus either on the trajectory from the periphery to the centre (*legitimate peripheral participation*, as in Lave and Wenger, 1991) or with emphasis on the community (the practices and the identities, as in Wenger, 1998). In the next section we present in detail two *Research Cases. Research Case 1* discusses teachers' and students' perspectives about proof through the theoretical lenses of *CoP*. *Research Case 2* concerns the example of one study that employs the concepts of *CoI* on engineering students' conceptual understanding of mathematics (Jaworski & Matthews, 2011b).

Research Case 1 (CoP): Proof in the process of entering the mathematical community

The study we draw on here is a research application of *CoP* that combines both Lave and Wenger's and Wenger's positions in an investigation of university teachers' pedagogical perspectives on, and students' experiences of, mathematical proof in a mathematics department in Sweden (Hemmi, 2006, 2008, 2010). Both qualitative and quantitative data were collected consisting of interviews with teachers; questionnaires and focus group interviews with students in different levels of their studies; observations of lectures; and, examination papers and textbooks. Here we discuss how the *CoP* theory shaped the focus of the study and its data analysis. From Wenger's perspective on *CoP* the study deployed constructs such as *mutual engagement*, *joint enterprise*, *shared repertoire*, *participation/non-participation, identity building*, *negotiation/ownership of meaning* to give insight into: the mathematical community of the department; the mathematical practices and the role of proof in these; and, participants' positions and engagement in this practice. From Lave and Wenger's perspective on *CoP*, the study deployed constructs such as *legitimate peripheral participation* and *transparency of mediating artefacts* to illustrate students' *peripheral participation* and tensions and conflicts in their trajectories.

"Proof is the soul of mathematics", a university teacher in the study claimed; it is a multi-faceted notion that permeates all mathematics and has been defined in many and varied ways. The term *proof* can refer both to a process of proving and the product of proving; this duality reflects the complex process of working with and creating proofs. The balance between intuitive and formal aspects, and between inductive and deductive modes of reasoning, can be connected to proof as a process of *reification* and there is an on-going *negotiation of meaning* along with these interacting aspects of proof, in which both teachers and students participate (Hemmi, 2006, 2008, 2010).

The newcomers (students) have to involve into their practice mathematical theory which comes from outside and *reifications* coming from outside, have to be *reappropriated* into a local process of students in order to become meaningful. Also, proof can be seen as an artefact with several important functions in the mathematical practice (Hanna & Barbeau, 2008). Lave and Wenger introduce the concept of *transparency of the artefacts* in connection to technology but in this study it is used for describing proof as a symbolic and intellectual artefact in the teaching and learning of mathematics. The term *transparency* refers to the way in which using artefacts and understanding their significance interacts with the learning process (cf. Hemmi, 2008).

In this study, all individuals who are involved in university level mathematics (*the practice*) at the department are members of the same *CoP*. The *mutual engagement* consists of studying, teaching/explaining, learning and communicating mathematics. The learning defines this community and the enhancement of this learning can be seen as the *joint enterprise* for both teachers and students. Learning is conceived as increasing *participation* in the community which leads to changing *identities*. All the members of this *CoP* are engaged in the learning of mathematics in various ways and all of them use partly the same tools, even if the learning of mathematics occurs on very different levels. In this sense researching new mathematics can, also, be seen as learning; since it leads to increasing *participation* with changing *identities* and extends the collective knowledge of mathematics (Hemmi, 2006, pp. 34-36). The *shared repertoire* includes routines like organising courses, seminars and examinations, but, also, words and symbols specific for the mathematical language and criteria for justifying knowledge in mathematics (including proof).

According to Wenger one's *identity* is always changing and building an *identity* consists of negotiating meanings of the experiences in social communities. Not only the students but also the teachers constitute a heterogeneous group concerning their *identity* building as some of them devote more time for research and work with graduate students while others focus more on teaching and the development of undergraduate courses.

Only a small part of the students will become mathematicians but many of them leave the practice after a while and some of them may become *brokers* between the mathematical practice and some other practices (Wenger, 1998, p. 105). Yet, the students need to use the established tools and *reifications*, such as mathematical theories and language with specific symbols and, particularly, become accustomed to a rigorous and systematic way of presenting mathematics with definitions and proofs that are acceptable in the mathematical practice at the university. The process of students' *identity* forming can be seen through Wenger's terms of *participation/non-participation* and their interaction.

The analysis shows that the newcomers (students) eventually started to talk about the role of proof in mathematics in a similar manner as the old-timers (teachers) did. The following example shows that some students, already from the first term, associated proof with "real mathematics" and "understanding" in contrast to school mathematics, which was connected to rule learning and applications of formulas without understanding:

> I think it's another thing here. In upper secondary school we had a lot of rules, you learn a lot of rules and then you just go ahead. There is nothing to understand. But here it's more like…he [the teacher] stresses it all the time, to count is not mathematics but mathematics is the understanding of it and that is exactly the point. (Student – Basic course, 2004)[2]

In the above excerpt, the student has constructed a meaning that shares old-timers' values and what is a part of their identity: "he [the teacher] stresses it all the

[2] All the excerpts presented in *Research Case 1* are translated from Swedish.

time”. In this way, students have the possibility to make the old-timers’ practice their own practice. In particular, after the first assessment on proof in the second term students in the focus groups started to talk about school mathematics as “doing sums and applying formulas”, and university mathematics as proof connected to “questioning the evident”, “derivation of formulas” and “the understanding of mathematics by seeing how everything is related”. These are aspects that also the teachers connected to proof. The students expressing themselves in this manner were considered as developing an *identity of participation*. In contrast with those students who seemed to be developing an *identity of non-participation*, these students talk about the advantages of studying proof:

> I think that if you go through the proofs and understand them you get a lot for free, since you can always go back, I mean a proof is often a rather concentrated piece and if you have understood it you hardly have to cram at all. No, I mean that then you don’t have to sit with everything else that takes so much time if you want to spare some time. It is clear it can be hard to work through them and really acquaint yourself with them but it can actually be worthwhile. (Student – Intermediate course, 2004)

The students who developed an *identity of non-participation* stopped listening to the teachers when they proved theorems, skipped the proofs in the textbooks and could not see any meaning in activities involving proofs and proving. They experienced the teachers’ proofs during the lectures as an obligatory ritual, without any real purpose:

> I often feel that they have to give the proof whether or not someone understands it, that’s how it feels. (Student – Intermediate course, 2004)

Also they did not see any meaning of studying the proofs as they felt they had no use of them in problem solving or applications.

> Most often you don’t have to be able to know anything of the proofs in order to solve problems. (Student – Intermediate course, 2004)

Wenger states that it is the way information can be integrated within an *identity of participation* that transforms information into knowledge and makes this empowering. The way in which the students in the previous excerpts talked shows that the information about proof they got in the lectures did not build up to an *identity of participation* but remained alien, fragmented and unnegotiable to them (Hemmi, 2006).

Peripheral participation involves a mix of *participation* and *non-participation* where the participation aspect is dominating. The following excerpt, in which a student talks about her first lectures, indicates that students who manage to accept *non-participation* as an 'adventure' may experience the encounter with proof as a challenge that can lead to *participation*:

> But I know that there were protests at the lectures sometimes and there were very many who said: 'How can we understand delta and epsilon; help, this is tough!' Most of the students thought it was enormously difficult and tough to understand where all this would lead. I didn't perhaps understand very much myself all the time but I thought it was so very fascinating, very fun, for me it was more like a spur; I want to learn more about this. (Student – Intermediate course, 2003)

The study shows that, besides the possibility of *participating* in various kinds of activities involving proof, students' learning enhancement is also related to the access students had to various aspects of proof such as: the meaning of proof in mathematics; the formal demands of proofs; and, the logical structure of the proofs that are included in the courses. For example, students struggle with questions about what proof is. The lack of discussions about the issues led them to feel that they do not know while all the others know what is going on: "How do you define a proof? Because we have never been informed about that, so you think: OK, the rest of the class knows what a proof is". An assumption that someone else understands what is going on resonates with

Wenger’s account of an *identity of non-participation* in relation to *ownership of meaning* (pp. 200-202).

Lave and Wenger’s metaphor of *transparency of artefacts* illustrates a dilemma of balancing between using an artefact (proof) and focusing on the artefact with some extended information (importance of proof) (Hemmi, 2008). The condition of *transparency*, in this study, is considered both from the teaching and the learning perspective. The analysis revealed several discrepancies between teachers’ intentions and expectations, on the one hand, and students’ experiences on the other hand (Hemmi, 2010). For example, almost all students wanted to learn more about proof from the very beginning of their studies while teachers in general expected them not to be interested in proofs. Several teachers, also, avoided proof in order to not frighten the students. Yet, the study shows that leaving something very central aside, only because it is expected to be experienced as difficult, may not always be the best way to facilitate learning. As Wenger points out, demanding alignments by a *CoP* does not need to imply lack of negotiability; in fact the demanding of alignment itself can be a means of sharing *ownership of meaning*.

Research Case 2 (CoI): Seeking conceptual understanding of mathematics

“The mathematics problem” whereby students entering university for mathematics or mathematics-related courses are ill-prepared for the nature of mathematics they will encounter at university level is well documented (e.g., Hawkes and Savage, 2000). In the English educational context in which the study presented in *Research Case 2* was conducted, school mathematics in the final years is highly procedural in both teaching and learning and few students are given the opportunity to reach conceptual understandings of the mathematics they learn (Minards, 2012).

The ESUM project (Engineering Students Understanding Mathematics)[3] involved the design and operationalization of an innovation in teaching in a first year mathematics module for engineering students. The innovation had the aim of enabling students' more conceptual understandings of mathematics. A team of four – three experienced mathematics-teacher-researchers and one research officer – designed the project and taught, monitored, collected and analysed data, and published results from it. One member was 'the teacher' conducting lectures and tutorials with students. One member was 'the researcher' collecting data from teaching activity. All were involved variously in design of materials and approaches, in monitoring of activity and in analysis of data (Jaworski & Matthews, 2011b).

Methodologically, the project involved *developmental research* in which research both studied the practices and processes involved and acted as a tool for development of teaching and learning (Jaworski, 2003; Goodchild, 2008). Through an iterative, cyclic process, the team designed materials and approaches to teaching; the teacher used the designed materials with students, reflected on their use, often with the rest of the team, and modified teaching practice accordingly.

The innovation aimed to engage students in mathematics in ways which encouraged them to *think mathematically* (Mason, 1988) and which developed an *inquiry stance* or *inquiry ways of being* in practice (Cochran-Smith and Lytle, 1999; Jaworski, 2004). Tasks and teaching approaches were designed to draw students into inquiry in mathematics through which they would engage with mathematical concepts more deeply than at their familiar procedural levels. For example, the following

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[3] With financial support from the UK HE STEM programme via *The Royal Academy of Engineering*.

questions were part of a series of tasks designed to engage students in the concept of *function*:

> Consider the function $f(x) = x^2 + 2x$ (x is real)
> a) Give an equation of a line that intersects the graph of this function
> (i) Twice (ii) Once (iii) Never (Adapted from Pilzer *et al*., 2003, p. 7)
> b) If we have the function $f(x) = ax^2 + bx + c$.
> What can you say about lines which intersect this function twice?

Students were expected to be familiar with quadratic functions, albeit, perhaps, in procedural ways. They were expected to visualise $f(x)$, sketch its graph and be able to think about what lines would cross it twice, once or never. By writing down equations of possible lines, and asking *why* these are possible but not others, they would engage (conceptually) with mathematics: be drawn into graphical representations of linear and quadratic functions, relate the functions to each other through inspecting intersecting graphs, and start to consider more general cases of such intersections. Their engagement would require them to consider characteristics of such functions and to relate algebraic and graphical forms.

The inquiry nature of the task can be seen in its invitation to explore relationships at a more general level in part (b), drawing on use of established knowledge in part (a). The language of "expected to" and "would" above indicates the design stage of *developmental research*. Tasks such as this were designed to contribute to the aims of the innovation. They were used in lectures or tutorials (Part (a) was used in a lecture and Part (b) in a tutorial following the lecture). In the lecture the teacher posed the question, gave students five minutes to work on it (circulating, viewing, and listening in to their dialogue) and invited responses from a range of students. Such tasks in a lecture aimed to enculturate students into mathematical engagement and oral response – students were expected to participate overtly and, with encouragement from the teacher, many did offer responses. In the tutorial, students were grouped in fours in a computer laboratory, using graphing software (GeoGebra, http://www.geogebra.org/cms/en/) and expected to use the GeoGebra environment to work investigatively on given tasks (such as (b)) and agree on their findings. The teacher circulated, encouraging and discussing with groups their exploration, thinking and findings. The researcher observed and audio-recorded the activity of lecture and tutorial.

The outcomes of this activity were studied in two ways. The teacher reflected on the activity of the students as they engaged with the task and on her perceptions of outcomes of the task for the students. Teacher and researcher discussed the teacher's reflections, the researcher feeding in from her observations, and periodic meetings of the whole team reviewed the ongoing teaching process. Modifications were made to practice based on these reflections and team discussions.

Teacher and students can be seen as part of *a community of mathematical practice* in which the practice was the teaching-learning of mathematical concepts. This is somewhat problematic since teacher (teaching) and students (learning) cannot be considered as engaging in the "same" practice. However, conceptualising the practice as *teaching-learning* allows us to circumvent this objection: we think of the whole *practice* of creating joint *participation* through which students (and teacher) *reify* mathematical concepts. Dialogue in *engagement* contributes to *reification* of concepts as part of *participation*. Teacher and students play different, but highly interactive, roles and develop *identities* through their *engagement*, use of *imagination*, and *alignment* with the norms and expectations in the setting.

The *community of mathematical practice* transforms into a *Community of Inquiry* when inquiry becomes a part of the practice. This happens in several layers relating to the differing roles of participants in the community. Inquiry-based tasks engage students and teacher in *inquiry in mathematics*; the teacher asks, and encourages students to ask mathematical questions which take them more deeply into the concepts. The teacher engages in *inquiry into teaching*, asking questions about the joint practice as she reflects on interactions with students and hears the researcher's observations. Researcher and teacher and the others in the team engage in *research inquiry* in the developmental process. All participants engage in *critical alignment*: rather than expecting to be told by the teacher, students are encouraged to ask mathematical questions and seek their own way of expressing mathematical ideas; the teacher looks critically at her own practice, with evidence from the research, and seeks to modify it to be more aligned with the aims of the innovation; the research team explores the situation as a whole, collecting and analysing data, seeking outcomes of students' engagement, and recognising issues. As an example, we quote from the teacher's reflection written after a lecture and following discussion with the researcher:

> In the first example [in the lecture] on Tuesday, I asked students to draw a triangle of given dimensions before going on to consider use of sine or cosine rules. In fact two triangles were possible for the given dimensions. This turned out to be a very good question, since different students wanted to approach it in different ways and we achieved a discussion across the lecture with students in different parts of the room arguing their approach. (Teacher's reflection, Week 1, Jaworski & Matthews, 2011b, p. 182)

A seemingly simple task emerged as valuable in engaging students in asking questions and noticing differences, and in alerting the teacher to the nature of tasks that promote student inquiry. Precious lecture time was given to this discussion, so that other plans had to be modified and the consequences assessed. We see *critical alignment* in student recognition of alternative ways of seeing a mathematical object and in the teacher's necessary adjustments to facilitate the student dialogue.

A *CoI* transforms a *CoP* to promote development. Through *critical alignment* students develop their understandings of mathematics, teachers develop their understandings of teaching and the researchers their understandings of research-based developmental practice. Such development is rarely straightforward, however. The development that is sought, through the innovation, is specified through the *joint enterprise* of engaging with inquiry-based tasks, GeoGebra, small group investigation, dialogue and questioning. The outcomes are hugely dependent on the actions and interactions of the participants in inquiry-based practice. These outcomes do not relate only to the inquiry-based nature of the enterprise: they are influenced by a range of factors in the sociocultural settings of the practice. For example, the students' expectations deriving from their school learning lead to some resistance to learning through exploration; lectures and tutorials are influenced by the physical environment where they take place: inflexible lecture theatre space, pressures of curriculum, assessment, timetables and time itself constrain what is possible for the teacher. Inquiry-based practice has to take into account of all of these factors and work with them to achieve the aims of the enterprise. Such 'working-with' can be seen as part of an overt process of *critical alignment* which is the key element of a *CoI*.

Analysis of data from students provided insights into students' perceptions of their engagement in the module. Two quotations reveal some of these perceptions:

> As a group we looked at many different functions using GeoGebra and found that having a visual representation of graphs in front of us gave a better understanding of the functions and how they worked. In this project the ability to be able to see the graphs that were talked about helped us to spot patterns and trends that would have been impossible to spot without the use of GeoGebra." (Group Project Report)

> Understanding maths – that was the point of Geogebra wasn't it? Just because I understand maths better doesn't mean I'll do better in the exam. I have done less past paper practice. (Focus group interview)

The first quotation was written by a group of students in their project report which was assessed. In writing in this way, we suggest, they entered into a *repertoire* of assessment in which they wrote what they perceived would be likely to gain good marks – a positive appreciation of GeoGebra. Nevertheless, what they write gives some indication of their appreciation of value in using GeoGebra to "spot patterns and trends" in understanding functions. The second quotation came from a focus group interview after the end of the module and its assessment. This was typical of comments about the nature of understanding and its relation to assessment. The fact that the module had an exam at the end was hugely influential on students' overall activity and perceptions. Such comments revealed tensions in the inquiry-based *enterprise* in relation to the norms of university practice which required an end of module examination. *Alignment* with these norms contradicted the development of inquiry-based norms.

Thus, although there was evidence of student understanding, and some appreciation of how aspects of the innovation contributed to understanding, the various influences on the practice, and especially the assessment by examination (despite the more formative project assessment) proved overwhelming. We conducted an activity theory analysis to gain further insights into these evident contradictions (Jaworski, Robinson, Matthews & Croft, 2012).

Theorising communities in university mathematics and ways forward into research

In this paper we consider university mathematics learning as a social activity and specifically as participation in communities each one of them shares common practices. With this as a theoretical perspective we aimed to gain more insight into the nature of teaching and learning by using the theoretical lenses of the Community of Practice (*CoP*) (Lave & Wenger, 1991; Wenger, 1998) and the Community of Inquiry (*CoI*) (e.g., Goodchild *et al.*, 2013). To this endeavour we revisited the main theoretical underpinnings of *CoP* and *CoI* and we exemplified how these have been used in research in university mathematics education – especially through their implementation in two *Research Cases*. In this concluding section we reflect on the ways communities have been theorised in university mathematics education research; we discuss the use and the analytical power of both *CoP* and *CoI*; and, we suggest ways forward in future research.

According to Wenger the community in a *CoP* is defined by the *practice* that gives coherence to this community and *identity* is formed through the participation in this community. In the research examples we presented, we identified a spectrum of different ways in which research sees the community formation and the *practice* that take place in these communities. Hemmi (2006), for example, sees students standing at the periphery of a community consisting of all the individuals engaged with university mathematics at any level and she states that the *mutual engagement* in this community includes studying, teaching/explaining, learning and communicating mathematics. From this viewpoint students' *identities* are seen in terms of their *participation/non-participation* and the interaction between these two. In contrast, Solomon (2007), rather than conceptualising students as *legitimate peripheral participants*, sees students belonging to multiple *CoPs* – including that of the mathematical discipline – and she identifies *identities* of *non-participation* and/or *marginalisation*. Such differences beg further reflection on a positioning of students with respect to mathematical practices.

Jaworski and Matthews (2011b) identified conflicts between students' previous experience on procedural learning with summative assessment and the more conceptual learning through inquiry-based activity and formative assessment that was desired by the teacher. Additionally, in their study, it became clear that the interactions between students and teacher were influenced importantly by their *identities*. These studies draw attention to the complexities (and tensions) inherent in teaching-learning practices in university education, particularly the *multimembership* of students or/and teacher. Although Wenger's model of identity attempts to capture complexity in its definition, we agree with Solomon (2007) that "it neglects to explore in detail the nature of identity in multiple, and possibly conflicting, communities of practice" (p. 88). To this already complicated picture we add the *negotiation* of *identities* especially students' and teacher's *alignment* to the community structure and rules. Wenger argues that *negotiability* is the process in which members gain control over the meaning and, through this, form their *identity*. However, the theory of *CoP* offers little insight into how this negotiation takes place, especially in terms of members' *alignment* to the community rules, and how rules are defined, sustained and developed in the context of *CoP* – see, for example, about *discursive rules* at (Nardi, Ryve, Stadler & Viirman, 2014) in this Special Issue. With the introduction of *critical alignment* and *inquiry* as a tool for *negotiation* (of meanings in mathematics and in mathematics teaching) contradiction and tensions can be revealed and addressed (Jaworski & Matthews, 2011b). Reflecting on the studies we reviewed we can see the teaching and learning university mathematics practice as a practice in which teacher and students are initially engaged at the boundary of their own communities (e.g. undergraduate students, research mathematicians, mathematics educators, etc.) with their *joint enterprise* being the development of mathematical learning. If this joint enterprise involves the maintenance of this community and the establishment of shared rules through *critical alignment* and realignment, gradually this community will gain its own status and its own *economy of meanings*, i.e. the social configuration in which negotiation of meaning takes place (Wenger, 1998, pp. 198-200). We contend that this developmental process that addresses conflicts, reconciling perspectives and seeking of resolutions can be theorised through the *CoI* lenses.

The developmental process we described above cannot been described by newcomer/old-timer relationships that are interested only in the trajectory towards the centre, as one of the criticisms to the *CoP* theory claims (for a critical view on these issues, see Barton & Tusting, 2005; Hughes, Jewson & Unwin, 2007). Engeström (2007) argues that the newcomers/old-timer relationship is "a foundationally conservative choice" (p. 42) that marginalises creativity and novelty. Furthermore, although Wenger suggested other types of *trajectories*, including the *inbound trajectory*, he did not explain how these trajectories affect or are influenced by the community (Kanes & Lerman, 2008). As we mentioned earlier, in university mathematics practices, the interaction of and the tension between different communities are very important, thus their analysis seeks a theoretical tool that can offer a refined insight and go further than their description. Wenger suggested the complimentary concept of *constellation of practices* (pp. 126-131) to describe multiple communities which are somehow connected to a specific community. However, the vague definition of *constellation* challenges the stability of its explanatory power (Engeström, 2007). Other approaches – such as Activity Theory – are perhaps more solid in this respect: for example, Jaworski *et al*. (2012) applied activity theory to gain further insights in the contradictions occurred in an inquiry-based lesson for mathematics to engineers.

There are one more criticisms on the *CoP* theory that we would like to discuss in our reflection: *individuals* and their role is undermined in a *CoP*. For example *agency* is not addressed in the *CoP* theory, namely how self-directed individuals respond and affect the learning environment within which they practice (see more at Hughes, Jewson & Unwin, 2007). Also, the role of *power* and the *interest* are implicit and undervalued in the *CoP* theory (Kanes and Lerman, 2008). According to Kanes and Lerman (2008) a view that can assist in the identification of the individual in the social context can be offered by a deeper analysis of the elements that constitute this practice. These elements can be the used tools, the technologies and the discursive practices. In Wenger's theory, tools, artefacts, discourses are part of the *shared repertoire* – in terms of practice – and part of the *alignment* – in terms of the modes of belonging, but their value is implicitly assumed in the overall structure of the community. However, in university mathematics education both resources (tools, artefacts, technology) and discourses are very important and their understanding is crucial in the understanding of how communities are formed. *Trajectories*, for example in a community, can be seen as discursive formations whereas shifting *identity* of an individual can be identified through the shifting of discourse. Analysis of *discursive patterns* and their development have the potential to give us more insight in our understanding of the establishment, the maintenance and the development of a *CoP* in university mathematics (see also Nardi et al., 2014). On the other hand, development and introduction of new resources or alternative use of existing ones (*documentation genesis*, see also (Gueudet, Buteau, Mesa & Misfeldt, 2014) in the same special issue) can be seen as the *critical alignment* of the teacher under the influence of the feedback she receives from students.

Reflecting on the research affordances *CoP* and *CoI* can offer, we can say that both suggest useful theoretical lenses through which the teaching and learning of mathematics at university level can be examined. In the steps forward we can see research that will regard communities in university mathematics in their complexity (e.g. *multimembership*, interaction of communities, *boundary practices*, *brokers*); embedded in the overall social context (e.g. technological and economic rapid changes); with distinguished role of individuals (e.g. *authority*, *power*, personal *interest*); and, be shaped for mathematical practices that have their own *discursive rules* and *resources*. Finally, we believe that more effort should be put in design *for* a mathematical learning community of practice that has the potential to develop its own *economy* of mathematical meanings.

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