Independence in Exponential Fields

A thesis submitted to the School of Mathematics
of the University of East Anglia in partial fulfilment
of the requirements for the degree of Doctor of Philosophy

By

Robert Stewart Henderson

October 2014

Abstract

Zilber constructed a class of exponential fields $\mathbf{ECF_{SK,CCP}}$ whose models have exponential-algebraic properties similar to the classical complex field with exponentiation $\mathbb{C}_{\exp}$. In this thesis we study this class and the more general classes $\mathbf{ECF_{SK}}$, also defined by Zilber, and **ECF**, studied by Zilber and Kirby. We investigate stable-like behaviour modulo arithmetic in these classes by developing a unique independence relation for each class, and in **ECF** we use this relation to examine types.

We provide an exposition of exponential fields that is more model theoretic and type-oriented than preceding work. We then investigate the types in **ECF** that are orthogonal to the kernel. New ideas presented include a characterisation of these types, and the definition of a grounding set; these results allow us find sufficient conditions to prove that a type over a set uniquely extends to a type over the smallest strong ELA-subfield containing that set.

For each class we define a ternary relation on subsets, and prove that these relations are independence relations, with properties akin to non-forking independence in first order theories. Applying work of Kangas, Hyttinen and Kesälä, we prove that in $\mathbf{ECF_{SK}}$ our independence notion is the unique independence relation for this class, and that our independence notion in $\mathbf{ECF_{SK,CCP}}$ is exactly the canonical independence relation for this class derived from the pregeometry. Assuming the conjecture known as CIT, we use our independence relation in **ECF** to prove that types orthogonal to the kernel are exactly the generically stable types.

Contents

Acknowledgements

Many thanks to my supervisor Jonathan Kirby, without whose guidance and inspiration this project would not have been possible. It has been a pleasure working with him. I have also had edifying conversations with many talented mathematicians, including Tapani Hyttinen, Kaisa Kangas, Meeri Kesälä, Charlotte Kestner, David Marker, Daniel Palacin, Anand Pillay, Amador Martin-Pizzarro, Frank Wagner, and Tim Zander. Thanks to my parents and siblings for their years of support, and to Scott Brown, Harry, Chrissy and others with whom I've worked on non-mathematical projects. Special thanks to Amy.

Chapter 1

Introduction

Consider the complex field with exponentiation $\mathbb{C}_{\exp} = (\mathbb{C}, +, \cdot, \exp, 0, 1)$. It is known that the integers are definable in this structure by

$$\mathbb{Z} = \{x \in \mathbb{C} \ : \ \forall y \ (\exp(y) = 1 \ \rightarrow \ \exp(xy) = 1 \ ) \ \}.$$

Therefore $\mathbb{C}_{\exp}$ is undecidable, and unstable. Wilkie proved that $\mathbb{R}_{\exp} = (\mathbb{R}; +, \cdot, \exp, 0, 1)$ is model complete [24, Second Main Theorem], but this is not the case for $\mathbb{C}_{\exp}$ [20, Proposition 1.1]. There are still many intriguing open questions about $\mathbb{C}_{\exp}$:

- Are the real numbers definable in this structure?
- (Zilber) Is $\mathbb{C}_{\exp}$ *quasi-minimal*, that is, are all of its definable subsets either countable or cocountable?
- (Mycielski) What are the non-trivial automorphisms other than complex conjugation?

We also would like to know if there are any strange exponential-algebraic relations between elements of $\mathbb{C}_{\exp}$, a question that can be more clearly described by a conjecture from transcendental number theory:

Schanuel's Conjecture: *Let $a_1, ..., a_n \in \mathbb{C}$ be $\mathbb{Q}$-linearly independent. Then*

$$\mathrm{td}(a_1, ..., a_n, \exp(a_1), ..., \exp(a_n)) \geq n$$

If this conjecture were true, it would minimize the exponential-algebraic relations between elements of $\mathbb{C}_{\exp}$. For instance it would imply that the following transcendental numbers are all algebraically independent over $\mathbb{Q}$ [22, p.326].

$$\pi, e, e^{\pi}, \pi^{e}, \pi^{\pi}, 2^{\pi}, 2^{i}, 2^{\sqrt{2}}, e^{i}, \log 2, \log 3, \log\log 2, (\log 2)^{\log 3}, e^{e}, e^{e^{e}}, e^{e^{e^{e}}} ...$$

Macintyre gave a description of an abstract algebraic exponential field in [18], defining an *E-field* to be a field F with a defined homomorphism $E : (F, +) \to (F^*, \cdot)$. For all E-fields that we consider, F will be an algebraically closed field of characteristic zero and E will be surjective; we call such an E-field an ELA-field. The class of all ELA-fields is still too general for us to characterise, so we shall work with more specific subclasses of E-fields. Macintyre also described a notion of E-algebraicity in terms of a non-singular set of solutions of polynomials in $\bar{x}, E(\bar{x})$ [18, Definition 5, Section 2.5]. From this notion we obtain a closure operator given by $\mathrm{ecl}_F(B) = \{\bar{a} \in F : \bar{a} \text{ is E-algebraic over } B\}$ for any subset $B \subseteq F$, and an associated dimension function, *exponential transcendence degree*, given by

$$\mathrm{etd}(\bar{a}/B) = \min\{|\bar{b}| : \bar{b} \subseteq \bar{a} \text{ and } (\mathrm{ecl}_F(\bar{b}B) = \mathrm{ecl}_F(\bar{a}B)\}.$$

We can in fact come to this dimension function another way. For an E-field F, consider the following Hrushovski predimension function for each finite $\bar{a} \subseteq F$

$$\delta(\bar{a}) = \mathrm{td}(\bar{a}, e^{\bar{a}}) - \mathrm{ldim}_{\mathbb{Q}}(\bar{a})$$

This predimension gives rise to a dimension function given by $d(\bar{a}) = \min\{\delta(\bar{a}\bar{b}) : \bar{b} \in F\}$ and a closure operator $\mathrm{cl}_F(\bar{a}) = \{\bar{b} \in F : d(\bar{b}\bar{a}) = d(\bar{a})\}$. Kirby proved that ecl_F agrees with cl_F and is always a pregeometry for every ELA-field F satisfying $\delta(\bar{a}) \geq 0$ for all $\bar{a} \in F$ [12, Theorem 1.1], and furthermore by [12, Theorem 1.3] it follows that

$$\mathrm{etd}_F(\bar{a}) = \min\{\delta(\bar{a}\bar{b}) \, : \, \bar{b} \in F\}.$$

Note that in $\mathbb{C}_{\exp}$ the statement $\delta(\bar{a}) \geq 0$ is equivalent to Schanuel's conjecture.

As a new method of studying exponential fields, in [26] Zilber constructed an $\mathcal{L}_{\omega_1,\omega}(Q)$-sentence Φ, where Q is the quantifier saying 'there exists uncountably many', that axiomatises all of the properties that we know of $\mathbb{C}_{\exp}$, as well as all those properties we desire it to have. Models of Φ are structures of the form $(K; +, \cdot, E, 0, 1)$, where $(K; +, \cdot, 0, 1)$ is an algebraically closed field of characteristic zero, $E : (K, +) \to (K^*, \cdot)$ is a surjective homomorphism, $\ker(E) = \tau\mathbb{Z}$ for some transcendental element $\tau \in K$, and Schanuel's conjecture in K holds. Φ also demands that these models are sufficiently existentially closed and have a countable closure property; that is, any system of exponential-algebraic equations that could have a solution in an extension of K already has a solution in K, and in fact K contains countably many such solutions. Assuming the other axioms, the countable closure property is equivalent to stating that $\mathrm{ecl}(\bar{a})$ is countable for all finite tuples $\bar{a} \in K$. Zilber proved that the class $\mathcal{K}$ of models of Φ, the class of *pseudo-exponential fields*, contains a unique model of cardinality κ for each uncountable cardinal κ. These models are quasi-minimal, and the model of cardinality κ has 2^κ automorphisms.

It is then natural to question whether or not the unique model of $\mathcal{K}$ of cardinality $2^{\aleph_0}$ is isomorphic to $\mathbb{C}_{\mathrm{exp}}$. However for this we would need to prove Schanuel's conjecture and more, so it is considered out of reach. Instead it is prudent to investigate properties of models in $\mathcal{K}$ and generalisations of $\mathcal{K}$, which we call exponential fields.

A significant issue with $\mathbb{C}_{\mathrm{exp}}$ is its combination of geometric and arithmetic structure. One reduct of $\mathbb{C}_{\mathrm{exp}}$ is the algebraically closed field $(\mathbb{C}; +, \cdot)$, which has a stable theory and is a well-behaved, strongly minimal, geometric structure. However, we also have definable arithmetic structure in the form of $(\mathbb{Z}; +, \cdot)$, giving rise to Gödel's phenomena, in particular wild definable sets. We therefore have the following question in mind: do exponential fields exhibit any stable-like behaviour modulo arithmetic? One definition of stability is that non-forking extensions of types should give rise to an independence relation with reasonable properties. So in particular we ask, do exponential fields allow for a useful notion of independence modulo arithmetic? If so, what does this relation tell us about their structure, and the structure of their types?

The main body of work in this thesis concerns defining and investigating independence relations that act over arithmetic in several different classes of exponential fields, with the intention of observing stable-like behaviour. We are especially interested in the class **ECF**, which was proved in [16] to be 'superstable over the kernel', and in fact an elementary class modulo a certain number theoretic conjecture known as CIT (the conjecture of intersections of tori with varieties). We would therefore like to know what other stable-like behaviour is exhibited in **ECF**. We use independence to study the types in **ECF** and investigate the meaning of the model theoretic property of generic stability in this setting.

The structure of the thesis is as follows. In Chapter 2 we define four classes of exponential fields, $\mathbf{ExpF}, \mathbf{ECF}, \mathbf{ECF_{SK}}$ and $\mathbf{ECF_{SK,CCP}}$. Zilber's class $\mathcal{K}$ is exactly $\mathbf{ECF_{SK,CCP}}$, and $\mathbf{ECF_{SK}}$ is the more general class containing models that may have uncountably many solutions of the form $(\bar{a}, e^{\bar{a}})$ for all algebraic varieties with sensible properties, as studied in [14]. **ECF** is even more general, and requires only that $(Z; +, \cdot)$ is a model of the theory of $(\mathbb{Z}; +, \cdot)$, rather than isomorphic to it. For each of the classes **ECF**, $\mathbf{ECF_{SK}}$, and $\mathbf{ECF_{SK,CCP}}$, we specify an appropriate embedding such that they are abstract elementary classes, admitting monster models. We define and describe the notions of hull and ELA-subfield in exponential fields from [15, Section 3,7], and with these we prove several useful statements about types in **ECF**. Types in exponential fields have been investigated before in [14] and implicitly throughout [15]; in this chapter we give an explicit characterisation of Galois types over strong ELA-closed subfields and show that these correspond exactly with the syntactic types over strong ELA-closed subfields. These types can be thought of as 'orthogonal to the kernel', in that they do not require new kernel elements in order to be realised. These types are our main objects of study, as if a model realising the type preserves the kernel of the base, the model must also preserve arithmetic of the base, and thus realising this type need not give rise to any additional arithmetic issues. We define the grounding set of a type orthogonal to the kernel, which is a finite set that fully characterises the type, comparable to the notion of a base in a first order theory. Using this definition we prove that, assuming CIT, any orthogonal type over a grounding set B uniquely extends to the Galois type over the strong ELA-subfield generated by B.

We begin Chapter 3 by providing an overview of the literature on independence and pre-independence relations in first order theories and abstract elementary classes. We then define and describe a new ternary relation on **ExpF** and prove that it is a pre-independence relation; we develop this relation to construct ternary relations specific to each of the classes $\mathbf{ECF}, \mathbf{ECF_{SK}}$ and $\mathbf{ECF_{SK,CCP}}$, and prove that each relation is indeed an independence relation for its class. Using work by Hyttinen, Kesälä and Kangas we prove that our independence relation for $\mathbf{ECF_{SK}}$ is the unique independence relation for that class (satisfying bounded free extensions of weak types). We also prove that for $\mathbf{ECF_{SK,CCP}}$ our independence relation is exactly the canonical pregeometric independence notion, and by work of Hyttinen and Kangas prove that additionally it is equivalent to non-splitting of weak types in this class.

In Chapter 4 we use our independence relation for **ECF** to prove that, assuming CIT, the global types orthogonal to the kernel are exactly the generically stable types, yielding the informal corollary that this independence relation is a useful notion of independence for **ECF**. We conclude with a brief discussion of potential future directions of research, suggesting other model-theoretic properties that could be investigated in **ECF** using exponential-algebraic techniques.

Throughout this thesis we write $A, B, C, ...$ to denote sets, $\bar{a}, \bar{b}, ...$ to denote tuples, and we write AB for $A \cup B$ and $\bar{a}\bar{b}$ for $\bar{a}^\frown\bar{b}$. We abuse notation and write $\bar{a} \in A$ to mean $\bar{a} \in A^{|\bar{a}|}$, and we also write $\bar{a}$ to mean the tuple $(a_1, a_2, ..., a_n)$ as well as the set $\{a_1, a_2, ..., a_n\}$; therefore if $\bar{b} = (b_1, ..., b_m)$ is any tuple and A is any set, $A\bar{b}$ denotes $A \cup \{b_1, ..., b_m\}$. We write e^a to mean $\exp(a)$ even when exp is not the standard analytic exponential function. We also allow the exponential of tuples by setting $e^{\bar{a}} = (e^{a_1}, ..., e^{a_n})$, and we allow the exponential of a subset by defining $e^A = \{e^a : a \in A\}$.

We recall that for a $\mathbb{Q}$-vector space V with subsets $A, B, C \subseteq V$ we say that A is $\mathbb{Q}$-linearly independent from B over C, written $A \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_C^{\mathbb{Q}\text{-lin}} B$, if for all $\bar{a} \in A$ we have $\mathrm{ldim}_{\mathbb{Q}}(\bar{a}/C) = \mathrm{ldim}_{\mathbb{Q}}(\bar{a}/BC)$. We recall also that for an algebraically closed field K with subsets $A, B, C \subseteq K$ we write $A \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_C^{\mathbf{ACF_0}} B$ and say A is field-theoretically algebraically independent from B over C if $\mathrm{td}(\bar{a}/C) = \mathrm{td}(\bar{a}/BC)$ for every $\bar{a} \in A$. Here $\mathrm{td}(X/Y)$ denotes the transcendence degree of $\mathbb{Q}(XY)$ over $\mathbb{Q}(Y)$. For $\bar{a} \in K$ and $B \subseteq K$ we define $\mathrm{Loc}(\bar{a}/B)$ the *locus* of $\bar{a}$ over B to be the intersection of all algebraic varieties defined over B containing $\bar{a}$.

Chapter 2

Exponential fields and types in ECF

In this chapter we define axiomatically four classes of exponential fields, namely **ExpF**, **ECF**, $\mathbf{ECF_{SK}}$, and $\mathbf{ECF_{SK,CCP}}$. The purpose of this chapter is to obtain tools that may be used in Chapter 3 to prove facts about independence relations in exponential fields, and to study types in **ECF**. We show that with certain associated embeddings, **ECF**, $\mathbf{ECF_{SK}}$ and $\mathbf{ECF_{SK,CCP}}$ are abstract elementary classes admitting monster models. We focus on investigating types in **ECF**, in particular types orthogonal to the kernel, which may be realised without extending the kernel. We show that in **ECF**, Galois types and syntactic types that are orthogonal to the kernel are equivalent over semi-strong ELA-subfields. Following this we introduce the notion of a grounding set, which can fully characterise a type orthogonal to the kernel defined over a semi-strong ELA-subfield, in particular, over a model. We show that assuming the Diophantine conjecture known as CIT, a type over a grounding set uniquely extends to a type over the smallest strong ELA-subfield containing the grounding set. Finally we comment on how a grounding set corresponds with the notion of a base in a first order theory.

2.1 Classes of exponential fields

In [26] Zilber constructed a class $\mathbf{ECF_{SK,CCP}}$ of pseudo-exponential fields which have all the properties we desire of $\mathbb{C}_{\exp}$. He showed that this class is axiomatisable in $\mathcal{L}_{\omega_1,\omega}(Q)$, where Q is a quantifier meaning 'there exist uncountably many', and further that it is κ-categorical for all uncountable κ. These models are quasiminimal, meaning that all definable sets are countable or cocountable, and the model of cardinality κ has an automorphism group of cardinality 2^κ. Zilber conjectured that the unique model of cardinality continuum is isomorphic to $\mathbb{C}_{\exp}$.

We now work towards defining $\mathbf{ECF_{SK,CCP}}$ as described above, and also the more general classes $\mathbf{ECF}$ and $\mathbf{ECF_{SK}}$ as investigated in [16] and [14] respectively. Appendix A provides a summary of the properties of these classes for the reader's reference.

We will consider structures of the form $(\mathcal{M}; +, \cdot, \exp)$ where one or more of the following axioms hold.

(I) $(\mathcal{M}; +, \cdot)$ is an algebraically closed field of characteristic 0, and $\exp : (\mathcal{M}, +) \to (\mathcal{M}^*, \cdot)$ is a surjective homomorphism.

We shall call a structure $\mathcal{M}$ satisfying axiom (I) an *ELA-field*, where the

'E' and the 'L' mean that every element in $\mathcal{M}$ has an exponential and a logarithm in $\mathcal{M}$, and the 'A' means that $\mathcal{M}$ is algebraically closed. In general, the exponential fields we shall study have stronger properties. For instance, we wish to have an axiom expressing that the kernel is an infinite cyclic group generated by a transcendental element; this axiom will include the statement that $Z = \mathbb{Z}$, where

$$Z = \{x : \forall y(\exp(y) = 1 \to \exp(xy) = 1)\}$$

is the multiplicative stabiliser of the kernel. This property is observable in $\mathbb{C}_{\exp}$, and is characterised by the following axiom.

(II) There is an element $\tau \in \mathcal{M}$ such that $\ker(\mathcal{M}) = \tau\mathbb{Z}$ and τ is transcendental.

If $\mathcal{M}$ is an ELA-field such that axiom (II) also holds, we say that $\mathcal{M}$ has *standard kernel*, that is $Z(\mathcal{M}) = \mathbb{Z}$. As seen in [14, Section 2.1], axiom (II) can be split into two parts, saying respectively that (a) the kernel is a cyclic Z-module and every element in the kernel is transcendental over Z, and (b) $Z = \mathbb{Z}$. Note that part (a) is first order expressible, while part (b) is given by an $\mathcal{L}_{\omega_1,\omega}$-sentence omitting the partial type of a non-standard integer. If we weaken the statement of part (b), we can allow for a wider range of exponential fields with non-standard kernels. We consider then the following first order axioms as a weakening of axiom (II).

(IIa) There is an element $\tau \in \mathcal{M}$ such that $\ker(\mathcal{M}) = \tau Z$ and τ is transcendental over Z.

(IIb) $(Z; +, \cdot) \models \mathrm{Th}(\mathbb{Z}; +, \cdot)$.

We also wish our structures to satisfy Schanuel's Conjecture over the kernel, which is expressed by the below axiom.

(III) **Schanuel Condition (SC)** If $\bar{a} \in \mathcal{M}^n$ is $\mathbb{Q}$-linearly independent over $\ker(\mathcal{M})$, then $\mathrm{td}(\bar{a}, e^{\bar{a}} / \ker(\mathcal{M})) \geq n$.

Here we say $\bar{a} \in \mathcal{M}^n$ is $\mathbb{Q}$-*linearly independent over a subset* $A \subseteq \mathcal{M}$ if for any non-zero tuple $\bar{\lambda} \in \mathbb{Q}^n$ we have $\sum_{i=1}^{n} \lambda_i a_i \notin \mathrm{span}_{\mathbb{Q}}(A)$.

The following class of exponential fields is the most general that we shall study.

Definition 2.1.1. Define **ExpF** to be the class of all models of axioms (I), (IIa), (IIb), and (III).

Further axioms require more terminology. We want axioms demanding that our models have a certain amount of saturation. Intuitively, axiom (IV) will say that any system of exponential algebraic equations that could have a solution in an exponential field extension of $\mathcal{M}$ already has a solution in $\mathcal{M}$, and axiom (V) will say that $\mathcal{M}$ contains only countably many such solutions. In order to describe these axioms precisely we need to define a particular matrix action on an element in G^n, where $G = \mathbb{G}_a(\mathcal{M}) \times \mathbb{G}_m(\mathcal{M})$, the product of a copy of the additive and multiplicative groups of the field $\mathcal{M}$. Suppose $(\bar{x}, \bar{y}) \in G^n$ and let M be a $k \times n$ integer matrix. We define the action $M \cdot (\bar{x}, \bar{y}) = (\bar{u}, \bar{v})$ where $u_i = \sum_{j=1}^{n} m_{ij} x_j$ and $v_i = \prod_{j=1}^{n} y_j^{m_{ij}}$ for $i = 1, ..., k$. For V an algebraic variety in G^n we define

$$M \cdot V = \{M \cdot (\bar{x}, \bar{y}) \mid (\bar{x}, \bar{y}) \in V\}$$

Definition 2.1.2. [15, Definition 5.1] Let $V \subseteq G^n$ be an algebraic variety. We say V is *rotund* if for any $k \times n$ integer matrix M with rk $M = k$, we have $\dim(M \cdot V) \geq k$. We say V is *free* if there do not exist $m_1, ..., m_n \in \mathbb{Z}$ not all zero and $b \in \mathcal{M}$ such that $V \subseteq \{(\bar{x}, \bar{y}) : \sum m_i x_i = b\}$ or $V \subseteq \{(\bar{x}, \bar{y}) : \prod y_i^{m_i} = b\}$.

(IV) **Strong Exponential Closure (SEC)** If $A \subset \mathcal{M}$ is any finite set, and $V \subset G^n$ is an irreducible, free, and rotund algebraic variety defined over A, then there exist $\bar{a} \in \mathcal{M}^n$ such that $(\bar{a}, \exp(\bar{a})) \in V$ is generic over A, that is $\operatorname{td}(\bar{a}, \exp(\bar{a})/A) = \dim V$.

(V) **Countable Closure Property (CCP)** If $A \subset \mathcal{M}$ is a finite subset and $V \subset G^n$ is an irreducible, free, rotund variety defined over A with $\dim V = n$, then

$$\{\bar{a} \in \mathcal{M}^n : (\bar{a}, \exp(\bar{a})) \in V \text{ is generic in } V \text{ over } A\}$$

is countable.

Definition 2.1.3.

- Define **ECF** to be the class containing all models of axioms (I), (IIa), (IIb), (III) and (IV). We call **ECF** the class of *exponentially closed fields.*
- Define $\mathbf{ECF_{SK}}$ to be the class containing all models of axioms (I), (II), (III) and (IV). We call $\mathbf{ECF_{SK}}$ the class of *exponentially closed fields with standard kernel.*
- Define $\mathbf{ECF_{SK,CCP}}$ to be the subclass of $\mathbf{ECF_{SK}}$ containing all models of axioms (I), (II), (III), (IV), and (V). We call $\mathbf{ECF_{SK,CCP}}$ the class

of *exponentially closed fields with standard kernel and the countable closure property.*

Lemma 2.1.4. $\boldsymbol{ECF_{SK,CCP}} \subseteq \boldsymbol{ECF_{SK}} \subseteq \boldsymbol{ECF} \subseteq \boldsymbol{ExpF}$.

Proof. The inclusions $\mathbf{ECF_{SK,CCP}} \subseteq \mathbf{ECF_{SK}}$ and $\mathbf{ECF} \subseteq \mathbf{ExpF}$ are clear. For $\mathbf{ECF_{SK}} \subseteq \mathbf{ECF}$, we observe that axioms (I), (III) and (IV) are common to both classes, and satisfaction of axioms (IIa) and (IIb) follows from axiom (II) as $Z = \mathbb{Z}$. □

By [26, Lemma 5.12] $\mathbb{C}_{\exp}$ satisfies axiom (V). However it is not known that $\mathbb{C}_{\exp}$ is in $\mathbf{ECF_{SK,CCP}}$ or even $\mathbf{ExpF}$, as axioms (III) and (IV) are unproven for $\mathbb{C}_{\exp}$. In the next section we show that these classes with certain associated embeddings are abstract elementary classes.

2.2 Abstract elementary classes

A class of structures $\mathcal{C}$ is called an elementary class if there is a first order theory T such that the models of T are exactly those structures contained in $\mathcal{C}$. Such a class has good model theoretic properties. Our classes $\mathbf{ECF_{SK}}$ and $\mathbf{ECF_{SK,CCP}}$ are not elementary classes, and whether or not $\mathbf{ECF}$ is elementary is dependent on the aforementioned conjecture known as CIT. However, we have the following generalisation of elementary classes due to Shelah.

Definition 2.2.1. [8, Definition 2.1] Let $\mathcal{L}$ be a countable language, $\mathcal{C}$ a class of $\mathcal{L}$-structures and let $\leq_{\mathcal{C}}$ be a partial order on $\mathcal{C}$. Then $(\mathcal{C}, \leq_{\mathcal{C}})$ is an *abstract elementary class* (or *AEC* for short) if the following properties hold.

(1) Both $\mathcal{C}$ and $\leq_{\mathcal{C}}$ are closed under isomorphisms.

(2) For all $\mathcal{M}, \mathcal{N} \in \mathcal{C}$, if $\mathcal{M} \leq_{\mathcal{C}} \mathcal{N}$ then $\mathcal{M}$ is a substructure of $\mathcal{N}$.

(3) Let $\mathcal{M}_1, \mathcal{M}_2$, and $\mathcal{M}_3$ be $\mathcal{L}$-structures in $\mathcal{C}$ with $\mathcal{M}_1 \subseteq \mathcal{M}_2$. If $\mathcal{M}_1 \leq_{\mathcal{C}} \mathcal{M}_3$ and $\mathcal{M}_2 \leq_{\mathcal{C}} \mathcal{M}_3$, then $\mathcal{M}_1 \leq_{\mathcal{C}} \mathcal{M}_2$.

(4) Suppose that $(\mathcal{M}_i : i < \omega)$ is a $\leq_{\mathcal{C}}$-chain of $\mathcal{L}$-structures in $\mathcal{C}$, and let $\mathcal{M}^* = \bigcup_{i<\omega} \mathcal{M}_i$. Then $\mathcal{M}^* \in \mathcal{C}$ and for each $i < \omega$ we have $\mathcal{M}_i \leq_{\mathcal{C}} \mathcal{M}^*$. Furthermore if $\mathcal{N} \in \mathcal{C}$ such that for each $i < \omega$ we have $\mathcal{M}_i \leq_{\mathcal{C}} \mathcal{N}$, then $\mathcal{M}^* \leq_{\mathcal{C}} \mathcal{N}$.

(5) *Downward Löwenheim-Skolem (DLS)* There is a cardinal $\mathrm{LS}(\mathcal{C}) \geq \aleph_0$ such that for every $\mathcal{M} \in \mathcal{C}$ and subset $A \subseteq \mathcal{M}$, there exists a model $A^* \in \mathcal{C}$ such that $A \subseteq A^* \leq_{\mathcal{C}} \mathcal{M}$ and $|A^*| = |A| + \mathrm{LS}(\mathcal{C})$.

Note that an elementary class $\mathcal{C}$ of models of a first order theory T is an abstract elementary class, with $\leq_{\mathcal{C}}$ being elementary embedding.

Definition 2.2.2. [8, Definition 2.3] Let $(\mathcal{C}, \leq_{\mathcal{C}})$ be an AEC and let $\mathcal{M}, \mathcal{N} \in \mathcal{C}$. We say a map $f : \mathcal{M} \to \mathcal{N}$ is a $\mathcal{C}$-*embedding* if $f(\mathcal{M}) \leq_{\mathcal{C}} \mathcal{N}$.

Definition 2.2.3. [6, Definitions 2.2-2.5] We say that an AEC $(\mathcal{C}, \leq_{\mathcal{C}})$ is *finitary* if the following hold.

1. $\mathrm{LS}(\mathcal{C}) = \aleph_0$.

2. *(ALM)* $\mathcal{C}$ has arbitrarily large models.

3. *Amalgamation Property (AP)* If $\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2 \in \mathcal{C}$ with $\mathcal{M}_0 \leq_{\mathcal{C}} \mathcal{M}_1$, $\mathcal{M}_0 \leq \mathcal{M}_2$, and $\mathcal{M}_0 = \mathcal{M}_1 \cap \mathcal{M}_2$, then there exists $\mathcal{N} \in \mathcal{C}$ such that $\mathcal{M}_1 \leq \mathcal{N}$ and a $\mathcal{C}$-embedding $f : \mathcal{M}_2 \to \mathcal{N}$ such that $f(\mathcal{M}_0) = \mathcal{M}_0$.

4. *Joint Embedding Property (JEP)* For every $\mathcal{M}_1, \mathcal{M}_2 \in \mathcal{C}$ there exists $\mathcal{N} \in \mathcal{C}$ such that $\mathcal{M}_1 \leq_{\mathcal{C}} \mathcal{N}$ and there exists a $\mathcal{C}$-embedding $f : \mathcal{M}_2 \to \mathcal{N}$.

5. *Finite Character* Let $\mathcal{M}, \mathcal{N} \in \mathcal{C}$ with $\mathcal{M} \subseteq \mathcal{N}$, and suppose that for each finite $\bar{a} \in \mathcal{M}$ we have $\mathrm{tp}^g_{\mathcal{M}}(\bar{a}) = \mathrm{tp}^g_{\mathcal{N}}(\bar{a})$, where $\mathrm{tp}^g_{\mathcal{M}}(\bar{a})$ refers to the Galois type. Then $\mathcal{M} \leq_{\mathcal{C}} \mathcal{N}$.

If an AEC $\mathcal{C}$ has AP, JEP and ALM, then $\mathcal{C}$ has a monster model $\mathbb{M}$ into which all models in $\mathcal{C}$ embed. When working in a monster model $\mathbb{M}$, every set we consider is a subset of $\mathbb{M}$, and every tuple we consider is a tuple in $\mathbb{M}$. We shall show that the classes $\mathbf{ECF}, \mathbf{ECF_{SK}}$ and $\mathbf{ECF_{SK,CCP}}$ with distinguished embeddings all have these properties, and so we may fix a monster model $\mathbb{M}$ for each class, which is saturated and of large cardinality. For our purposes we could just work in a model that is 'saturated over the kernel', that is, saturated with respect to extensions that do not extend the kernel (a precise description of this is Definition 2.5.4). This in particular in **ECF** would allow us to avoid the cardinality of the kernel equalling the cardinality of the model. However this distinction is generally not necessary to our study.

We now work towards proving these results about our classes of exponential fields. We define a predimension function, and use this function to define strong and semi-strong embeddings. We use these definitions to show that with certain distinguished embeddings, **ECF** and $\mathbf{ECF_{SK}}$ are finitary

AECs, and $\mathbf{ECF_{SK,CCP}}$ is a non-finitary AEC. As previously mentioned, see Appendix A for a summary of the properties of these classes.

Definition 2.2.4. Let $\mathcal{M}$ be a model in **ExpF**, and let $A \subseteq \mathcal{M}$ be a subset. We define $\langle A\rangle_{\mathcal{M}}$ to be the $\mathbb{Q}$-linear vector subspace of $\mathcal{M}$ generated by A, that is $\langle A\rangle_{\mathcal{M}} = \mathrm{span}_{\mathbb{Q}}(A)$. We also define the kernel of A as $\ker(A) = \langle A\rangle \cap \ker(\mathcal{M})$.

The subscript in $\langle A\rangle_{\mathcal{M}}$ means that the vector space $\mathrm{span}_{\mathbb{Q}}(A)$ is being considered as a subspace of $\mathcal{M}$, but for any vector space $\mathcal{N}$ containing A we have $\langle A\rangle_{\mathcal{M}} = \langle A\rangle_{\mathcal{N}}$, so we may omit the subscript when the context is clear.

Definition 2.2.5. [16, Definition 3.8] [15, Section 2.1] Let $\mathcal{M}$ be a model in **ExpF** and suppose A and B are subsets of $\mathcal{M}$. We define the relative predimension function

$$\Delta_{\mathcal{M}}(A/B) = \mathrm{td}(A, \exp(A)/B, \exp(B), \ker(\mathcal{M})) - \mathrm{ldim}_{\mathbb{Q}}(A/B, \ker(\mathcal{M}))$$

We also define $\Delta_{\mathcal{M}}(A) = \Delta_{\mathcal{M}}(A/\emptyset)$.

Note that axiom (III) states that $\Delta_{\mathcal{M}}(\bar{x}) \geq 0$ for all $\bar{x} \in \mathcal{M}$.

Lemma 2.2.6. *[16, Lemma 3.9] Let $\mathcal{M}$ be a model in **ExpF**.*

*(a) **Submodularity:** Let $\mathcal{M}$ be a model in **ExpF** and let A, B and C be $\mathbb{Q}$-vector subspaces of $\mathcal{M}$. Then*

$$\Delta_{\mathcal{M}}(A \cup B/C) + \Delta_{\mathcal{M}}(A \cap B/C) \leq \Delta_{\mathcal{M}}(A/C) + \Delta_{\mathcal{M}}(B/C)$$

(b) Let $\bar{a}$ and $\bar{b}$ be tuples from $\mathcal{M}$, and let C be a subset of $\mathcal{M}$. Then

(i) ***Additivity:*** $\Delta_{\mathcal{M}}(\bar{a}\bar{b}/C) = \Delta_{\mathcal{M}}(\bar{a}/\bar{b}C) + \Delta_{\mathcal{M}}(\bar{b}/C)$.

(ii) There exists a finite tuple $\bar{c}$ from C such that $\Delta_{\mathcal{M}}(\bar{a}/C) = \Delta_{\mathcal{M}}(\bar{a}/\bar{c})$.

(c) Suppose $A \subseteq \mathcal{M}$ is a subset and $\mathcal{N}$ is in ***ExpF*** *such that $\mathcal{M} \subseteq \mathcal{N}$ and $\mathcal{M} \mathop{\perp}^{\boldsymbol{ACF_0}}_{\ker(\mathcal{M})} \ker(\mathcal{N})$. Then for any $\bar{b} \in \mathcal{M}$ we have $\Delta_{\mathcal{N}}(\bar{b}/A) = \Delta_{\mathcal{M}}(\bar{b}/A)$.*

Definition 2.2.7. Let $\mathcal{M}$ be a model in **ExpF** and let $A \subseteq B$ be subsets of $\mathcal{M}$.

1. We say A is *semi-strong* in B and write $A \prec B$ iff $A, \exp(A) \mathop{\perp}^{\mathbf{ACF_0}}_{\ker(A)} \ker(B)$ and for every $\bar{b} \in B$ we have $\Delta_{\mathcal{M}}(\bar{b}/A) \geq 0$.
2. We say A is *strong* in B and write $A \triangleleft B$ if $A \prec B$ and $\ker(A) = \ker(B)$.

Remark 2.2.8. Let $\mathcal{M}$ be a model in **ExpF**. Suppose $A \subseteq \mathcal{M}$ such that A contains $\ker(\mathcal{M})$ and for all $\bar{x} \in \mathcal{M}$ we have $\Delta_{\mathcal{M}}(\bar{x}/A) \geq 0$. Then $\ker(\mathcal{M}) = \ker(A)$ and so $A \triangleleft \mathcal{M}$.

The above definition of semi-strong is from [16, Definition 3.10]. The definition of strong embedding appears to differ from the definition given by [16, Definition 3.7], which states that A is strong in B if for every $\bar{b} \in B$ we have $\delta(\bar{b}/A) = \operatorname{td}(\bar{b}, e^{\bar{b}}/Ae^{A}) - \operatorname{ldim}_{\mathbb{Q}}(\bar{b}/A) \geq 0$. Noting that $\Delta_{\mathcal{M}}(\bar{b}) = \delta(\bar{b}/\ker(\mathcal{M}))$, we see that any semi-strong kernel-preserving extension satisfies this definition of strong.

Conversely, suppose that we have models $\mathcal{M}$ and $\mathcal{N}$ of **ExpF** such that $\mathcal{M} \prec \mathcal{N}$ with $\ker(\mathcal{M}) \neq \ker(\mathcal{N})$, so there exists $\lambda \in Z(\mathcal{N}) \setminus Z(\mathcal{M})$.

Then $\lambda^r \in Z(\mathcal{N})$ for all $r \in \mathbb{N}$, so for τ a generator of $\ker(\mathcal{N})$ we have $\tau\lambda^r \in \ker(\mathcal{N})$ for all $r \in \mathbb{N}$. If $\mathrm{ldim}_{\mathbb{Q}}(1, \lambda, ..., \lambda^r) < r + 1$, then λ is algebraic, so $\mathrm{td}(\tau, \tau\lambda, ..., \tau\lambda^r) = 1$. By axiom (III) we have $\delta(\tau\lambda^0, ..., \tau\lambda^r) \geq 0$, and so $\mathrm{ldim}_{\mathbb{Q}}(\tau, \tau\lambda, ..., \tau\lambda^r) = 1$, so $\lambda \in \mathbb{Q}$, which is impossible for $\lambda \in Z(B) \setminus Z(A)$. Therefore $\mathrm{ldim}_{\mathbb{Q}}(1, \lambda, ..., \lambda^r) = r + 1$ for all $r \in \mathbb{N}$, but then $\mathrm{td}(\tau, \tau\lambda, ..., \tau\lambda^r) = r + 1$ which fails unless $r = 1$. Therefore for models of **ExpF** these definitions of strong are equivalent.

Lemma 2.2.9. *[16, Lemma 3.11][15, Lemma 2.3] Let $\mathcal{M}$ be a model in **ExpF** and suppose A, B, C are subsets of $\mathcal{M}$.*

(a) $\ker(\mathcal{M}) \lhd \mathcal{M}$.

(b) $A \prec A$.

(c) If $A \prec B$ and $B \prec C$ then $A \prec C$.

(d) $A \prec B$ if and only if for every finite tuple $\bar{b} \in B$ we have $A \prec A\bar{b}$.

(e) If $A \subseteq B$, $B \prec C$ and $A \prec C$ then $A \prec B$.

(f) Let γ be a limit ordinal, and suppose $A_1 \subseteq A_2 \subseteq \cdots$ is an γ-chain of subsets of $\mathcal{M}$ such that $A_\alpha \prec \mathcal{M}$ for each $\alpha < \gamma$. Then $\bigcup_{\alpha<\gamma} A_\alpha \prec \mathcal{M}$ and $A_\beta \prec \bigcup_{\alpha<\gamma} A_\alpha$ for all $\beta < \gamma$.

(g) The above properties (b)-(f) are also true of $\lhd$.

We shall use the following characterisation of exponential transcendence degree that was described in the introduction.

Fact 2.2.10. [16, Fact 3.16] Let $\mathcal{M} \in \mathbf{ExpF}$, A a semi-strong subset of $\mathcal{M}$ and let $\bar{b}$ be a finite tuple from $\mathcal{M}$. Then

$$\mathrm{etd}(\bar{b}/A) = \min\{\Delta_{\mathcal{M}}(\bar{b}\bar{c}/A) : \bar{c} \in \mathcal{M}\}$$

Corollary 2.2.11. *Let $\mathcal{M} \in$ **ECF**, let $A \prec \mathcal{M}$ be a semi-strong subset, and let $\bar{b} \in \mathcal{M}$ be a finite tuple. Then etd$(\bar{b}/A) = \Delta(\bar{b}/A)$ if and only if $\Delta(\bar{x}/A\bar{b}) \geq 0$ for all $\bar{x} \in \mathcal{M}$.*

In particular if $A \lhd \mathcal{M}$ is strong, then etd$(\bar{b}/A) = \Delta(\bar{b}/A)$ if and only if $A\bar{b} \lhd \mathcal{M}$.

Proof. If $\bar{x} \in \mathcal{M}$ then $\Delta(\bar{x}/A\bar{b}) = \Delta(\bar{x}\bar{b}/A) - \Delta(\bar{b}/A)$ by additivity. By Fact 2.2.10 we have $\Delta(\bar{x}\bar{b}/A) \geq \mathrm{etd}(\bar{b}/A)$. If $\mathrm{etd}(\bar{b}/A) = \Delta(\bar{b}/A)$ then

$$\Delta(\bar{x}/A\bar{b}) \geq \mathrm{etd}(\bar{b}/A) - \mathrm{etd}(\bar{b}/A) = 0.$$

Conversely, if for all $\bar{x} \in \mathcal{M}$ we have $\Delta(\bar{x}/A\bar{b}) \geq 0$, by additivity again $\Delta(\bar{x}\bar{b}/A) \geq \Delta(\bar{b}/A)$ for all $\bar{x} \in \mathcal{M}$. Then by Fact 2.2.10 we have $\mathrm{etd}(\bar{b}/A) = \Delta(\bar{b}/A)$ as desired. □

Definition 2.2.12. Let $\mathcal{M},\mathcal{N} \in \mathbf{ExpF}$ such that $\mathcal{M} \subseteq \mathcal{N}$. We define $\mathcal{M} \leq \mathcal{N}$ to mean that $\mathcal{M} \prec \mathcal{N}$ and $Z(\mathcal{M}) \preccurlyeq Z(\mathcal{N})$.

The below proposition uses similar ideas to [16, Proposition 3.13] to obtain a downward Lowenheim-Skolem result, with an additional kernel-preservation property.

Proposition 2.2.13. *Let $\mathcal{M}$ be a model in* ***ECF*** *and let $A \prec \mathcal{M}$ be any semi-strong subset. Then there exists a model $A^* \leq \mathcal{M}$ in* ***ECF*** *such that*

$A \subseteq A^*$, $|A^*| = |A| + \aleph_0$.

Furthermore, we can take $Z(A^*)$ *to be the smallest elementary submodel of* $Z(\mathcal{M})$ *containing* $\tau^{-1}\ker(A)$.

Proof. Let $\gamma = |A| + \aleph_0$. Since $\mathrm{Th}(\mathbb{Z}; +, \cdot)$ has definable Skolem functions, there exists a smallest elementary submodel Z of $Z(\mathcal{M})$ containing $\tau^{-1}\ker(A)$. Let $A_0 \subseteq \mathcal{M}$ be the $\mathbb{Q}$-vector space generated by AZ. Then $A_0 \prec \mathcal{M}$, $Z(A_0) = Z$, and since $|Z| \leq \gamma$ we have $|A_0| = \gamma$. Now enumerate $(A_0 \cup \exp(A_0))^{alg}$ as $(a_\alpha)_{\alpha<\gamma}$ where $(-)^{alg}$ denotes field-theoretic algebraic closure. Also enumerate all free irreducible rotund varieties $(V_\alpha)_{\alpha<\gamma}$ defined over $A_0 \cup \exp(A_0)$.

We define a chain of vector spaces $(A_\alpha)_{\alpha<\gamma}$ as follows. We have already defined A_0, so suppose we have A_α up to some $\alpha < \gamma$. Choose b from A_α, or from $\mathcal{M}$ if no such $b \in A_\alpha$ exists, such that $e^b = a_\alpha$ and $b \mathop{\smile\!\!\!\!\!\!\perp}^{\mathbf{ACF_0}}_{A_\alpha \exp(A_\alpha)} \ker(\mathcal{M})$. Choose also $(\bar{c}, e^{\bar{c}}) \in V_\alpha(\mathcal{M})$ generic over $A_\alpha a_\alpha b \cup \exp(A_\alpha a_\alpha b)\ker(\mathcal{M})$, so in particular $(\bar{c}, e^{\bar{c}})$ is generic in V_α over $A_0 \cup \exp(A_0)$. Define $A_{\alpha+1} = \langle A_\alpha a_\alpha b \bar{c} \rangle$. For each limit ordinal $\beta < \gamma$ we define $A_\beta = \bigcup_{\alpha<\beta} A_\alpha$. We now show that $A_\alpha \prec \mathcal{M}$ for each $\alpha < \gamma$.

We proceed by induction; we have $A_0 \prec \mathcal{M}$, so suppose that $A_\alpha \prec \mathcal{M}$ for some $\alpha < \gamma$. Consider $\Delta(a_\alpha, b/A_\alpha) = \mathrm{td}(a_\alpha, b, e^{a_\alpha}, e^b/A_\alpha \exp(A_\alpha)\ker(\mathcal{M})) - \mathrm{ldim}_{\mathbb{Q}}(a_\alpha, b/A_\alpha \ker(\mathcal{M}))$. Since a_α is algebraic over $A_0 \cup \exp(A_0)$ we have $\mathrm{td}(a_\alpha, b, e^{a_\alpha}, e^b/A_\alpha \exp(A_\alpha)\ker(\mathcal{M})) = \mathrm{td}(b, e^{a_\alpha}/A_\alpha \exp(A_\alpha)\ker(\mathcal{M})) \leq 2$, so $\Delta(a_\alpha, b/A_\alpha) \leq 2 - \mathrm{ldim}_{\mathbb{Q}}(a_\alpha, b/A_\alpha \ker(\mathcal{M}))$.

- If $a_\alpha, b \notin A_\alpha$, then $\Delta(a_\alpha, b/A_\alpha) \leq 2 - 2 = 0$.
- If $a_\alpha, b \in A_\alpha$ then $\mathrm{td}(b, e^{a_\alpha}/A_\alpha \exp(A_\alpha)\ker(\mathcal{M})) = 0$ and

$\mathrm{ldim}_{\mathbb{Q}}(a_\alpha, b/A_\alpha \ker(\mathcal{M})) = 0$ so we have $\Delta(a_\alpha, b/A_\alpha) = 0$.

- If exactly one of a_α and b is in A_α, then $\mathrm{td}(b, e^{a_\alpha}/A_\alpha \exp(A_\alpha) \ker(\mathcal{M})) \leq 1$ and $\mathrm{ldim}_{\mathbb{Q}}(b, a_\alpha/A_\alpha) = 1$, so $\Delta(a_\alpha, b/A_\alpha) \leq 0$.

Therefore we can be sure that $\Delta(a_\alpha, b/A_\alpha) \leq 0$. Since $A_\alpha \prec \mathcal{M}$, it follows that $\Delta(a_\alpha, b/A_\alpha) = 0$.

Now consider $\Delta(\bar{c}/A_\alpha, a_\alpha, b) = \mathrm{td}(\bar{c}, e^{\bar{c}}/A_\alpha, a_\alpha, b, \exp(A_\alpha, a_\alpha, b)) - \mathrm{ldim}_{\mathbb{Q}}(\bar{c}/A_\alpha, a_\alpha, b)$. Since $(\bar{c}, e^{\bar{c}})$ is generic for V_α over $A_\alpha a_\alpha b, \exp(A_\alpha, a_\alpha, b)$ and V_α is free, we have $\mathrm{ldim}_{\mathbb{Q}}(\bar{c}/A_\alpha, a_\alpha, b) = \mathrm{ldim}_{\mathbb{Q}}(\bar{c}/A_0) = |\bar{c}|$. Then

$$\begin{aligned}\Delta(\bar{c}/A_\alpha, a_\alpha, b) &= \mathrm{td}(\bar{c}, e^{\bar{c}}/A_\alpha, a_\alpha, b, \exp(A_\alpha, a_\alpha, b)) - \mathrm{ldim}_{\mathbb{Q}}(\bar{c}/A_\alpha, a_\alpha, b)\\ &= \dim V_\alpha - |\bar{c}| = 0\end{aligned}$$

By additivity of the predimension we have

$$\begin{aligned}\Delta(a_\alpha, b, \bar{c}/A_\alpha) &= \Delta(\bar{c}/A_\alpha, a_\alpha, b) + \Delta(a_\alpha, b/A_\alpha)\\ &= 0 + 0 = 0\end{aligned}$$

Now for any $\bar{d} \in \mathcal{M}$, by additivity again we have

$$\begin{aligned}\Delta(\bar{d}/A_{\alpha+1}) &= \Delta(\bar{d}/A_\alpha a_\alpha, b, \bar{c})\\ &= \Delta(\bar{d}a_\alpha, b, \bar{c}/A_\alpha) - \Delta(a_\alpha, b, \bar{c}/A_\alpha)\\ &= \Delta(\bar{d}a_\alpha, b, \bar{c}/A_\alpha) - 0\\ &= \Delta(\bar{d}a_\alpha, b, \bar{c}/A_\alpha) \geq 0\end{aligned}$$

since $A_\alpha \prec \mathcal{M}$.

We now need to prove that $A_{\alpha+1} \mathop{\smile\!\!\!\!\!\!|}^{\mathbf{ACF_0}}_{\ker(A_{\alpha+1})} \ker(\mathcal{M})$. Since $b \mathop{\smile\!\!\!\!\!\!|}^{\mathbf{ACF_0}}_{A_\alpha} \ker(\mathcal{M})$ and $(\bar{c}, e^{\bar{c}})$ is generic in V_α over $A_\alpha a_\alpha b \exp(A_\alpha a_\alpha b)$, we have $\ker(A_{\alpha+1}) = \ker(A_\alpha a_\alpha b \bar{c}) = \ker(A_\alpha)$. Note that in particular, $\ker(A_\alpha) = \ker(A_0)$ for all $\alpha < \gamma$. Now by the definitions of a_α and b, and by the definition and additivity of transcendence degree, we have

$$\begin{aligned}\operatorname{td}(A_{\alpha+1}\exp(A_{\alpha+1})/\ker(A_\alpha)) &= \operatorname{td}(A_\alpha b\bar{c}\exp(A_\alpha\bar{c})/\ker(A_\alpha))\\ &= \operatorname{td}(A_\alpha \exp(A_\alpha)/\ker(A_\alpha)) + \operatorname{td}(b, \bar{c}, e^{\bar{c}}/A_\alpha \exp(A_\alpha))\end{aligned}$$

By additivity again we have

$$\operatorname{td}(b, \bar{c}, e^{\bar{c}}/A_\alpha \exp(A_\alpha)) = \operatorname{td}(\bar{c}, e^{\bar{c}}/A_\alpha b \exp(A_\alpha b)) + \operatorname{td}(b/A_\alpha \exp(A_\alpha)).$$

Since $b \mathop{\smile\!\!\!\!\!\!|}^{\mathbf{ACF_0}}_{A_\alpha \exp(A_\alpha)} \ker(\mathcal{M})$ we have $\operatorname{td}(b/A_\alpha \exp(A_\alpha)) = \operatorname{td}(b/A_\alpha \exp(A_\alpha)\ker(\mathcal{M}))$. By definition of $\bar{c}$ we have $\operatorname{td}(\bar{c}, e^{\bar{c}}/A_\alpha \exp(A_\alpha) b) = \operatorname{td}(\bar{c}, e^{\bar{c}}/A_\alpha \exp(A_\alpha) b \ker(\mathcal{M}))$, and so applying additivity in the other direction we obtain

$$\operatorname{td}(b, \bar{c}, e^{\bar{c}}/A_\alpha \exp(A_\alpha)) = \operatorname{td}(b, \bar{c}, e^{\bar{c}}/A_\alpha \exp(A_\alpha)\ker(\mathcal{M})).$$

Since $A_\alpha \prec \mathcal{M}$ we have $\operatorname{td}(A_\alpha \exp(A_\alpha)/\ker(\mathcal{M})) = \operatorname{td}(A_\alpha \exp(A_\alpha)/\ker(A_\alpha))$, and so by substituting into the above expression for $\operatorname{td}(A_{\alpha+1}\exp(A_{\alpha+1}/\ker(A_\alpha))$ and additivity of transcendence degree we have

$$\begin{aligned}\mathrm{td}(A_{\alpha+1}\exp(A_{\alpha+1})/\ker(A_\alpha)) &= \mathrm{td}(A_\alpha\exp(A_\alpha)/\ker(\mathcal{M})) \\ &\quad + \mathrm{td}(b,\bar{c},e^{\bar{c}}/A_\alpha\exp(A_\alpha)\ker(\mathcal{M})) \\ &= \mathrm{td}(A_\alpha b\bar{c}\exp(A_\alpha\bar{c})/\ker(\mathcal{M})) \\ &= \mathrm{td}(A_{\alpha+1}\exp(A_{\alpha+1})/\ker(\mathcal{M}))\end{aligned}$$

and so $A_{\alpha+1}\exp(A_{\alpha+1}) \mathop{\downarrow}^{\mathbf{ACF_0}}_{\ker(A_{\alpha+1})} \ker(\mathcal{M})$. Therefore $A_{\alpha+1} \prec \mathcal{M}$.

By Lemma 2.2.9(f) we have $A_\beta = \bigcup_{\alpha<\beta} A_\alpha \prec \mathcal{M}$ for all limit ordinals $\beta < \gamma$, and also $A_\gamma = \bigcup_{\alpha<\gamma} A_\alpha \prec \mathcal{M}$. Write A_γ as $A_0^{(1)}$, and repeat the above argument replacing A_0 with $A_0^{(1)}$ to obtain $A_\gamma^{(1)} =: A_0^{(2)}$. Repeating this process ω times, we obtain a chain of vector spaces $(A_\gamma^{(n)})_{n<\omega}$ all semi-strong in $\mathcal{M}$, so in particular by Lemma 2.2.9(f) we have $\bigcup_{n<\omega} A_\gamma^{(n)} =: A^* \prec \mathcal{M}$. Since $\ker(A_\alpha) = \ker(A_0)$ for all $\alpha < \gamma$ we have $\ker(A^*) = \ker(A_0)$, and since there is a definable bijection between the multiplicative stabilizer and the kernel we have $Z(A^*) = Z(A_0) = Z$. By construction, A^* is an ELA-subfield of $\mathcal{M}$ satisfying axioms (I), (IIa),(IIb),(III),(IV) and so $A^* \in \mathbf{ECF}$. Since $A^* \prec \mathcal{M}$ and $Z(A^*) = Z \preccurlyeq Z(\mathcal{M})$ we have $A^* \leq \mathcal{M}$. Certainly $A \subseteq A^*$, and $|A^*| = \omega \cdot \gamma = \gamma$ as required. □

We also have the amalgamation property for each of our classes. To prove this we shall use the following definition.

Definition 2.2.14. [16, Definition 3.2] Let $\widehat{\mathbb{Z}} = \lim_{n\leftarrow} \mathbb{Z}/n\mathbb{Z}$ denote the profinite completion of the integers. Let $\mathcal{M}$ be a model in **ExpF**, and suppose that $(A;+,0)$ is a $\mathbb{Q}$-vector subspace of $\mathcal{M}$. Say that A has *very full kernel* iff $(\widehat{\mathbb{Z}};+,0)$ is contained in A as a pure subgroup; that is for each

$a \in \widehat{\mathbb{Z}}$, if there exists $b \in F$ such that $nb = a$ for some $n \in \mathbb{N}$, then there exists $b' \in \widehat{\mathbb{Z}}$ such that $nb' = a$.

Proposition 2.2.15. $(\boldsymbol{ExpF}, \leq)$ *has the amalgamation property.*

Proof. Let $A, B, C \in \mathbf{ExpF}$ with $A \leq B$ and $A \leq C$. Then $Z(A) \preccurlyeq Z(B)$ and $Z(A) \preccurlyeq Z(C)$, and since the elementary class of models of $\mathrm{Th}(\mathbb{Z}; +, \cdot)$ has the amalgamation property, we can find $Z \models \mathrm{Th}(\mathbb{Z}; +, \cdot)$ such that we have elementary embeddings $Z(B) \hookrightarrow Z$ and $Z(C) \hookrightarrow Z$ satisfying the following commutative diagram of elementary embeddings:

$$\begin{tikzcd} Z(A) \arrow[r, hook, "\preccurlyeq"] \arrow[d, hook, "\preccurlyeq"] & Z(B) \arrow[d, hook, "\preccurlyeq"] \\ Z(C) \arrow[r, hook, "\preccurlyeq"] & Z \end{tikzcd}$$

Furthermore we may extend Z if necessary to an $\aleph_0$-saturated model of $\mathrm{Th}(\mathbb{Z}; +, \cdot)$ so that $\langle BZ \rangle, \langle AZ \rangle$ and $\langle CZ \rangle$ have very full kernel. By [16, Corollary 3.6] we can take Z such that $B, \exp(B) \mathop{\smile\!\!\!\!\!\!\mid}_{\ker(B)} \ker(BZ)$, and since the extension of vector spaces $B \subseteq \langle BZ \rangle$ increases only the kernel, by the definition of Δ we have $B \prec \langle BZ \rangle$. Since $\langle BZ \rangle$ has very full kernel, by [16, Proposition 3.13 (1)] there exists a well-defined free strong extension of $\langle BZ \rangle$ to a model $B' \in \mathbf{ExpF}$ such that $\ker(B') = \tau Z$, and in particular $B \prec B'$. Similarly we can define semi-strong model extensions $A \prec A'$ and $C \prec C'$ such that $\ker(A') = \ker(B') = \ker(C') = \tau Z$. So we may assume that A, B and C are ELA-fields with very full kernel such that $A \triangleleft B$ and $A \triangleleft C$.

Consider $A \triangleleft B$. We construct a chain of submodels $(B_\alpha)_{\alpha<\gamma}$ of B such that

$$A = B_0 \triangleleft B_1 \triangleleft B_2 \triangleleft \cdots \triangleleft B_\alpha \triangleleft \cdots \triangleleft B_\gamma = B$$

and for each $\alpha < \gamma$ we have $B_{\alpha+1}$ finitely generated over B_α, that is, $B_{\alpha+1}$ is the ELA-subfield generated by B_α and a finite tuple.

Let $B_0 = A$. Suppose we have $A \triangleleft \cdots \triangleleft B_\alpha \triangleleft B$ for an ordinal $\alpha < \gamma$. Let $\bar{b}_\alpha$ be a tuple from B that is $\mathbb{Q}$-linearly independent over B_α such that $B_\alpha \bar{b}_\alpha \triangleleft B$. Note that if $\bar{b}_\alpha$ does not exist, then $B_\alpha = B$. Define $B_{\alpha+1}$ to be the ELA-subfield of B generated by $B_\alpha \bar{b}_\alpha$. Applying [16, Proposition 3.13 (2)] we have $B_\alpha \triangleleft B_{\alpha+1} \triangleleft B$. Defining $V_\alpha = \operatorname{Loc}(\bar{b}_\alpha e^{\bar{b}_\alpha}/B_\alpha)$, by [16, Proposition 3.13 (2)] we can write $B_{\alpha+1}$ as $B_\alpha | V_\alpha$, the free extension of B_α by a generic of V_α. For limit ordinals $\delta < \gamma$, by Lemma 2.2.9(f)(g) we can define $B_\delta = \bigcup_{\alpha<\gamma} B_\alpha$. We therefore have a chain of strong ELA-subfields of B given by $(B_\alpha)_{\alpha<\gamma}$. Set $D_0 = C$, and for each $\alpha < \gamma$ define $D_{\alpha+1} = D_\alpha | V_\alpha$. Then by [16, Lemma 3.18] we have a chain of embeddings $f_\alpha : B_\alpha \hookrightarrow D_\alpha$ over A such that $f_\alpha(B_\alpha) \triangleleft D_\alpha$ for each $\alpha < \gamma$, and noting that f_α extends f_β for all $\beta < \alpha$ we can define $f_\gamma = \bigcup_{\alpha<\gamma} f_\alpha$ for all limit ordinals $\delta < \gamma$. Therefore we obtain an embedding $f_\gamma : B \hookrightarrow D_\gamma$, where $D_\gamma \in \mathbf{ExpF}$. Setting $f = f_\gamma$ and $D = D_\gamma$ we have the following commutative diagram

$$\begin{tikzcd}
A \arrow[r, hook, "\triangleleft"] \arrow[d, hook, "\triangleleft"'] & f(B) \arrow[d, hook, "\triangleleft"] \\
C \arrow[r, hook, "\triangleleft"] & D
\end{tikzcd}$$

as required. □

Lemma 2.2.16. $(\boldsymbol{ECF}, \leq)$ *has the amalgamation property.*

Proof. By Proposition 2.2.15, for $A, B, C \in \mathbf{ECF}$ with $A \leq B$ and $A \leq C$

we have $D \in \mathbf{ExpF}$ such that $B \leq D$ and $C \leq D$ commute over A. Letting $\gamma = |D| + \aleph_0$, we may enumerate all irreducible, free, rotund varieties over D by $(V_\alpha)_{\alpha<\gamma}$. We construct a chain $(D_\alpha)_{\alpha<\gamma}$ of ELA-fields in the following way. Setting $D_0 = D$, by [16, Proposition 3.13(2)] for each $\alpha < |D|$ we can define an ELA-field extension $D_{\alpha+1} = D_\alpha | V_\alpha$ of D_α, and for limit ordinals $\delta < \gamma$ we set $D_\delta = \bigcup_{\beta<\delta} D_\beta$. By [16, Proposition 3.13(2)] again, for each $\alpha < \gamma$ we have $D_\alpha \lhd D_{\alpha+1}$ and by Lemma 2.2.9(f)(g) we have $D_\alpha \lhd D_\delta$ for each limit ordinal $\delta < \gamma$. Then defining $D^{(1)} = D_\gamma$ we have $D \lhd D^{(1)}$. Repeating the above procedure with $D^{(1)}$ instead of D we obtain $D^{(1)}_\gamma =: D^{(2)}$, and repeating this ω many times we obtain a chain of ELA-fields $(D^{(n)})_{n<\omega}$ such that $D \lhd D^{(n)} \lhd D^{(n+1)}$ for each $n < \omega$. Then $D' = \bigcup_{n<\omega} D^{(n)}$ is strongly exponentially closed and hence is a model in **ECF**, and by Lemma 2.2.9(f)(g) we have $D \lhd D'$. Therefore the following diagram of embeddings commutes,

$$\begin{tikzcd} A \arrow[r, hook, "\leq"] \arrow[d, hook, "\leq"'] & B \arrow[d, hook, "\leq"] \\ C \arrow[r, hook, "\leq"] & D' \end{tikzcd}$$

and so **ECF** has the amalgamation property. $\square$

Proposition 2.2.17. *$(\boldsymbol{ECF_{SK}}, \lhd)$ has the amalgamation property.*

Proof. Let $A, B, C \in \mathbf{ECF_{SK}}$ such that $A \lhd B$ and $A \lhd C$. Let K_0 denote the free field amalgam in $\mathbf{ACF_0}$ of B and C, and define a function $E : \mathrm{span}_{\mathbb{Q}}(BC) \to K_0^\times$ by $E(b + c) = \exp_B(b)\exp_C(c)$, where $\exp_B$ and $\exp_C$ denote the exponential functions in B and C respectively. Then $(K_0; +, \cdot, 0, 1, E)$ is a partial E-field in the sense of [15, Definition 2.1], so by [15, Construction 2.13] and [15, Lemma 2.14] there is an ELA-field K

freely generated by K_0 such that $B \lhd K$ and $C \lhd K$ by construction; in fact by [15, Theorem 2.18] if B, C are countable, there is a unique such K.

Let $\gamma = \aleph_0 + |K|$. Enumerate all irreducible free rotund varieties defined over K by $(V_\alpha)_{\alpha<\gamma}$. We define a chain of ELA-fields $(K_\alpha)_{\alpha<\gamma}$ in the following way. Set $K = K_0$ and for each $\alpha < \gamma$ let $K_{\alpha+1}$ be an ELA-field freely generated over K_α by a tuple $\bar{a}, e^{\bar{a}}$ generic in V_α over K_α, constructed as in [15, Construction 2.13]. For limit ordinals $\delta < \gamma$ define $K_\delta = \bigcup_{\alpha<\delta} K_\alpha$. By [15, Lemma 2.14] we have $K \lhd K_\alpha \lhd K_{\alpha+1}$ for every $\alpha < \gamma$ and by Lemma 2.2.9(f)(g) for each limit ordinal $\delta < \gamma$ we have $K_\alpha \lhd K_\delta$ for every $\alpha < \delta$. Defining $K^{(1)} = \bigcup_{\alpha<\gamma} K_\alpha$, by Lemma 2.2.9(f)(g) we have $K \lhd K^{(1)}$.

Repeating the above procedure with $K^{(1)}$ rather than K we obtain $K^{(2)} = \bigcup_{\alpha<\gamma} K_\alpha^{(1)}$ such that $K \lhd K^{(1)} \lhd K^{(2)}$, and repeating this ω many times we obtain a chain of ELA-fields $(K^{(n)})_{n<\omega}$ such that $K \lhd K^{(n)} \lhd K^{(n+1)}$ for each $n < \omega$. Then $D = \bigcup_{n<\omega} K^{(n)}$ is strongly exponentially algebraically closed and hence is a model of **ECF**. By Lemma 2.2.9(f)(g) we have $K \lhd D$ so $B \lhd D$ and $C \lhd D$. Also $\ker(D) = \ker(B) = \tau\mathbb{Z}$, so $D \in \mathbf{ECF_{SK}}$ as required. □

Definition 2.2.18. Let $\mathcal{M}, \mathcal{N} \in \mathbf{ECF_{SK,CCP}}$ such that $\mathcal{M} \subseteq \mathcal{N}$. We say $\mathcal{M} \subseteq \mathcal{N}$ is a *closed embedding*, written $\mathcal{M} \subseteq^{cl} \mathcal{N}$, if $\mathrm{ecl}_{\mathcal{N}}(\mathcal{M}) = \mathcal{M}$.

Since $\mathbf{ECF_{SK,CCP}}$ is a quasiminimal excellent class, the amalgamation property for $(\mathbf{ECF_{SK,CCP}}, \subseteq^{cl})$ follows from [13, Theorem 3.3].

Proposition 2.2.19. *$(\boldsymbol{ECF}, \leq)$, $(\boldsymbol{ECF_{SK}}, \lhd)$ and $(\boldsymbol{ECF_{SK,CCP}}, \subseteq^{cl})$ are abstract elementary classes. Each class has ALM, AP and JEP, and therefore each class admits its own monster model.*

Furthermore $(\boldsymbol{ECF_{SK}}, \lhd)$ *and* $(\boldsymbol{ECF}, \leq)$ *are finitary,* $(\boldsymbol{ECF_{SK,CCP}}, \subseteq^{cl})$ *is non-finitary.*

Proof. We first demonstrate that $(\mathbf{ECF}, \leq)$ is an AEC by proving each item of Definition 2.2.1.

(1) Suppose that $\mathcal{M}_1, \mathcal{M}_2, \mathcal{N}_1, \mathcal{N}_2 \in \mathbf{ECF}$ and $\mathcal{M}_1 \leq \mathcal{M}_2$ such that we have isomorphisms $f_i : \mathcal{M}_i \to \mathcal{N}_i$ for $i = 1, 2$ with $f_1 \subseteq f_2$. Transcendence degree and linear dimension are invariant under isomorphisms, so for each $\bar{b} \in \mathcal{N}_2$ we have $\Delta_{\mathcal{N}_2}(\bar{b}/\mathcal{N}_1) = \Delta_{\mathcal{M}_2}(f^{-1}(\bar{b})/\mathcal{M}_1) \geq 0$ since $\mathcal{M}_1 \preccurlyeq \mathcal{M}_2$. We also have $\mathcal{M}_1 \mathop{\smile\!\!\!\!\!\!\vert}^{\mathbf{ACF_0}}_{\ker(\mathcal{M}_1)} \ker(\mathcal{M}_2)$, so applying the isomorphism f_2 we obtain $\mathcal{N}_1 \mathop{\smile\!\!\!\!\!\!\vert}^{\mathbf{ACF_0}}_{\ker(\mathcal{N}_1)} \ker(\mathcal{N}_2)$, and so $\mathcal{N}_1 \preccurlyeq \mathcal{N}_2$. Since Z is a $\emptyset$-definable set, it too is preserved by f_2, and so $Z(\mathcal{N}_1) \preccurlyeq Z(\mathcal{N}_2)$.

(2) Immediate from the definition of semi-strong.

(3) By part (e) of Lemma 2.2.9.

(4) By parts (f) and (g) of Lemma 2.2.9.

(5) *(DLS)* By Proposition 2.2.13.

Hence $(\mathbf{ECF}, \leq)$ is an abstract elementary class. By Lemma 2.2.16 we have AP for $\mathbf{ECF}$, by [16, Theorem 1.1] we have arbitrarily large models in $\mathbf{ECF}$, and JEP follows from AP due the existence of prime models [14, Theorem 4]. We therefore have ALM, JEP and AP, so we may fix a monster model $\mathbb{M}$ in $\mathbf{ECF}$.

Suppose we have $\mathcal{M} \subseteq \mathcal{N}$ in $\mathbf{ECF}$ such that $\mathrm{tp}^g_{\mathcal{M}}(\bar{a}) = \mathrm{tp}^g_{\mathcal{N}}(\bar{a})$ for all $\bar{a} \in \mathcal{M}$. Equivalence of Galois types implies equivalence of syntactic types, therefore for any $\mathcal{L}$-formula $\phi(\bar{x})$ and $\bar{a} \in \mathcal{M}$ we have $\mathcal{M} \models \phi(\bar{a})$ iff $\mathcal{N} \models \phi(\bar{a})$, namely $\mathcal{M} \preccurlyeq \mathcal{N}$. If $\mathcal{M} \not\leq \mathcal{N}$ then $Z(\mathcal{M}) \not\preccurlyeq Z(\mathcal{N})$ or $\mathcal{M} \not\lhd \mathcal{N}$, which in either case will be witnessed by an $\mathcal{L}$-sentence with parameters from $\mathcal{M}$, implying that $\mathcal{M} \not\preccurlyeq \mathcal{N}$. Therefore we have finite character, and so $(\mathbf{ECF}, \leq)$ is a finitary AEC.

The proof that $(\mathbf{ECF_{SK}}, \lhd)$ is an AEC is the same as the above, where we replace $\prec$ with $\lhd$ and invoke part (g) of Lemma 2.2.9, and for the proof of DLS we may take $Z = Z(\mathcal{M}) = \mathbb{Z}$ as $|\mathbb{Z}| = \aleph_0$. We have AP for $(\mathbf{ECF_{SK}}, \lhd)$ by Proposition 2.2.17, and ALM and JEP for $(\mathbf{ECF_{SK}}, \lhd)$ follow by the same argument as for $(\mathbf{ECF}, \leq)$. $\mathbf{ECF_{SK}}$ is an $\mathcal{L}_{\omega_1,\omega}$-class, so by [14, Theorem 3] strong embeddings are exactly the elementary embeddings in $\mathbf{ECF_{SK}}$, so by the same reasoning as for $\mathbf{ECF}$ above, $(\mathbf{ECF_{SK}}, \lhd)$ is a finitary AEC.

$\mathbf{ECF_{SK,CCP}}$ is an uncountably categorical quasiminimal excellent class, so by [13, Theorem 4.2] it is an abstract elementary class, and by categoricity it has arbitrarily large models. For the Lowenheim-Skolem number of $\mathbf{ECF_{SK,CCP}}$ we observe that $\mathrm{ecl}_{\mathcal{M}}(A) \preccurlyeq \mathcal{M}$ and by the countable closure property axiom (V) we have $|\mathrm{ecl}_{\mathcal{M}}(A)| = |A| + \aleph_0$, so $\mathrm{LS}(\mathbf{ECF_{SK,CCP}}) = \aleph_0$. In Section 2.8 of [14] it is shown that $\mathbf{ECF_{SK,CCP}}$ is not $\mathcal{L}_{\omega_1,\omega}$-definable, and so by [17, Theorem 5.2] it is a non-finitary AEC. □

2.3 The hull in exponential fields

In this section we prove that for each model $\mathcal{M} \in \mathbf{ExpF}$ and subset $A \subseteq \mathcal{M}$ there exists a unique smallest $\mathbb{Q}$-linear vector space containing A that is strong in $\mathcal{M}$, called the hull of A, and furthermore that in extensions of

$\mathcal{M}$ this vector space will only extend by the span of the new kernel. Later in this chapter we will use these ideas to determine a correlation between Galois types and syntactic types. The hull will also be needed in the next chapter in order to define our independence relations.

Lemma 2.3.1. *Let $\mathcal{M}$ be a model of* ***ExpF****, and let $C \subseteq \mathcal{M}$ be any subset. Then there exists a unique smallest $\mathbb{Q}$-vector subspace $\lceil C \rceil_{\mathcal{M}}$, the* hull *of C in $\mathcal{M}$, such that $C \cup \ker(\mathcal{M}) \subseteq \lceil C \rceil_{\mathcal{M}}$ and $\lceil C \rceil_{\mathcal{M}} \triangleleft \mathcal{M}$.*

Proof. Suppose that C is finite. Then there exists a $\mathbb{Q}$-vector subspace A of $\mathcal{M}$ with minimal linear dimension over $\ker(\mathcal{M})$ such that $C \cup \ker(\mathcal{M}) \subseteq A$ and $\Delta_{\mathcal{M}}(A/C) = d$ is minimal. Suppose that $B \subseteq \mathcal{M}$ is another $\mathbb{Q}$-vector subspace with $\Delta_{\mathcal{M}}(B/C) = d$ and $C \cup \ker(\mathcal{M}) \subseteq B$. By submodularity we have

$$\Delta_{\mathcal{M}}(AB/C) + \Delta_{\mathcal{M}}(A \cap B/C) \leq \Delta_{\mathcal{M}}(A/C) + \Delta_{\mathcal{M}}(B/C) = 2d$$

and since $\Delta_{\mathcal{M}}(A/C)$ is minimal, we have $\Delta_{\mathcal{M}}(A \cap B/C) = d$. However, A has minimal linear dimension over $\ker(\mathcal{M})$, so $A \cap B = A$; that is, A is unique.

Suppose on the other hand C is infinite. Then there exists $\lceil C_0 \rceil_{\mathcal{M}} \triangleleft \mathcal{M}$ for each finite subset $C_0 \subseteq C$. Let $\widetilde{C} = \bigcup_{C_0 \subseteq^{fin} C} \lceil C_0 \rceil_{\mathcal{M}}$, and observe that $C \cup \ker(\mathcal{M}) \subseteq \widetilde{C}$. By Lemma 2.2.9 (g) we have $\widetilde{C} \triangleleft \mathcal{M}$ and so $\lceil C \rceil = \widetilde{C}$. $\square$

Definition 2.3.2. Let $\mathcal{M}$ be a model in **ExpF**. If A and B are subsets of $\mathcal{M}$, we say that a set $B' \in \mathcal{M}$ is a *basis for the hull of B over A in $\mathcal{M}$* if B' is $\mathbb{Q}$-linearly independent over $A \cup \ker(\mathcal{M})$ and $\langle B' A \ker(\mathcal{M}) \rangle_{\mathcal{M}} = \lceil B \rceil_{\mathcal{M}}$. If A is empty, we say B' is a basis for the hull of B (over the kernel).

Lemma 2.3.3. *Let $\mathcal{M}$ and $\mathcal{N}$ be in **ExpF** with $\mathcal{M} \prec \mathcal{N}$ and let $A \subseteq \mathcal{M}$. Then $\lceil A \rceil_{\mathcal{N}}$ is the $\mathbb{Q}$-linear vector space generated by $\lceil A \rceil_{\mathcal{M}}$ and* $\ker(\mathcal{N})$.

Proof. We observe that $\lceil A \rceil_{\mathcal{N}}$ contains $\ker(\mathcal{N}) \cup \lceil A \rceil_{\mathcal{M}}$, so we need only prove that the $\mathbb{Q}$-vector space generated by $\lceil A \rceil_{\mathcal{M}} \cup \ker(\mathcal{N})$ is strong in $\mathcal{N}$, namely for all $\bar{y} \in \mathcal{N}$ we have $\Delta_{\mathcal{N}}(\bar{y}/\lceil A \rceil_{\mathcal{M}}) \geq 0$. We can find $\bar{z} \in \mathcal{M}$ and $\bar{w} \in \mathcal{N} \setminus \mathcal{M}$ such that $\operatorname{ldim}_{\mathbb{Q}}(\bar{w}/\mathcal{M}) = |\bar{w}|$ and $\langle \bar{y} \rangle = \langle \bar{z}\bar{w} \rangle$. By additivity,

$$\Delta_{\mathcal{N}}(\bar{y}/\lceil A \rceil_{\mathcal{M}}) = \Delta_{\mathcal{N}}(\bar{z}/\lceil A \rceil_{\mathcal{M}}) + \Delta_{\mathcal{N}}(\bar{w}/\bar{z}\lceil A \rceil_{\mathcal{M}}).$$

We have $\mathcal{M} \prec \mathcal{N}$ and so $\mathcal{M} \mathop{\downarrow}^{\mathbf{ACF_0}}_{\ker(\mathcal{M})} \ker(\mathcal{N})$, therefore by Lemma 2.2.6(c) we have $\Delta_{\mathcal{N}}(\bar{z}/\lceil A \rceil_{\mathcal{M}}) = \Delta_{\mathcal{M}}(\bar{z}/\lceil A \rceil_{\mathcal{M}})$, which is greater than or equal to 0 since $\lceil A \rceil_{\mathcal{M}} \prec \mathcal{M}$. By the definition of $\bar{w}$ we have $\operatorname{ldim}_{\mathbb{Q}}(\bar{w}/\bar{z}\lceil A \rceil_{\mathcal{M}}) = \operatorname{ldim}_{\mathbb{Q}}(\bar{w}/\mathcal{M})$. Since $\mathcal{M} \prec \mathcal{N}$ we have

$$\begin{aligned}
\Delta_{\mathcal{N}}(\bar{w}/\bar{z}\lceil A \rceil_{\mathcal{M}}) &= \operatorname{td}(\bar{w}e^{\bar{w}}/\bar{z}\lceil A \rceil_{\mathcal{M}} \ker(\mathcal{N}) \exp(\bar{z}\lceil A \rceil_{\mathcal{M}})) - \operatorname{ldim}_{\mathbb{Q}}(\bar{w}/\bar{z}\lceil A \rceil_{\mathcal{M}} \ker(\mathcal{N})) \\
&= \operatorname{td}(\bar{w}e^{\bar{w}}/\bar{z}\lceil A \rceil_{\mathcal{M}} \ker(\mathcal{N}) \exp(\bar{z}\lceil A \rceil_{\mathcal{M}})) - \operatorname{ldim}_{\mathbb{Q}}(\bar{w}/\mathcal{M} \ker(\mathcal{N})) \\
&\geq \operatorname{td}(\bar{w}e^{\bar{w}}/\mathcal{M} \ker(\mathcal{N})) - \operatorname{ldim}_{\mathbb{Q}}(\bar{w}/\mathcal{M} \ker(\mathcal{N})) = \Delta_{\mathcal{N}}(\bar{w}/\mathcal{M}) \\
&\geq 0
\end{aligned}$$

and so $\Delta_{\mathcal{N}}(\bar{y}/\lceil A \rceil_{\mathcal{M}}) \geq 0$ as required. □

Immediately from the above lemma we have $\lceil A \rceil_{\mathcal{M}} = \lceil A \rceil_{\mathcal{N}}$ for any $\mathcal{M}, \mathcal{N}$ in **ExpF** such that $\mathcal{M} \triangleleft \mathcal{N}$. In particular for all extensions $\mathcal{M} \subseteq \mathcal{N}$ in $\mathbf{ECF_{SK}}$ we have $\lceil A \rceil_{\mathcal{M}} = \lceil A \rceil_{N}$. We may omit the subscript in $\lceil - \rceil_{\mathcal{M}}$ when the context is clear.

2.4 ELA-subfields and ELA-closure

In this section we define the ELA-closure of a subset of a model of **ExpF**, and show that the ELA-closure of any semi-strong set $B \subseteq \mathcal{M}$ is a union of definable sets in $\mathcal{M}$ with parameters in B. This will lead to a proof at the end of this chapter that a type over a set uniquely extends to a type over an ELA-field, which will ultimately be used to obtain a stationarity result.

Definition 2.4.1. Let $A \subseteq \mathcal{M}$ be a subset. We define $\langle A \rangle_{\mathcal{M}}^{\mathrm{ELA}}$, the *ELA-closure* of A in $\mathcal{M}$, as

$$\langle A \rangle_{\mathcal{M}}^{\mathrm{ELA}} = \bigcap \{F \subseteq \mathcal{M} : A \cup \ker(\mathcal{M}) \subseteq F \text{ and } F \text{ is an ELA-subfield}\}.$$

We also write $\lceil A \rceil_{\mathcal{M}}^{\mathrm{ELA}}$ for $\langle \lceil A \rceil_{\mathcal{M}} \rangle_{\mathcal{M}}^{\mathrm{ELA}}$, the *ELA-closure of the hull* of A in $\mathcal{M}$.

$\langle A \rangle^{\mathrm{ELA}}$ is an ELA-subfield due to its containment of the kernel; the intersection of two ELA-subfields is not necessarily an ELA-subfield, as it is possible for two ELA-subfields to have differing logarithms. To see this, consider the following example. Let $\mathcal{M} \in \mathbf{ECF}$ be a model with $\ker(\mathcal{M}) \neq \tau\mathbb{Z}$. We will determine two ELA-subfields of $\mathcal{M}$ whose intersection does not contain any logarithm of 2. Let $a, b \in \mathcal{M}$ be distinct elements such that $e^a = e^b = 2$ (so $a - b \in \ker(\mathcal{M})$) and choose b so that $a - b = \tau z$ for some $z \in Z \setminus \mathbb{Z}$. The standard prime model B_0 of $\mathbf{ECF_{SK}}$ embeds into $\mathcal{M}$, and by a method similar to the proof of Proposition 2.2.13, one may construct an $\mathbf{ECF}$-embedding $\theta_1 : B_0 \hookrightarrow \mathcal{M}$ such that $a \in \theta_1(B_0)$. Then the logarithms of 2 in $\theta_1(B_0)$ are $a + \tau\mathbb{Z}$. One may also construct another embedding $\theta_2 : B_0 \hookrightarrow \mathcal{M}$ such that $b \in \theta_2(B_0)$. However $a - b \notin \tau\mathbb{Z}$, so $a, b \notin$

$\theta_1(B_0) \cap \theta_2(B_0)$. More generally for any subset $A \subseteq \mathcal{M}$ we have no logarithm of 2 contained in $\bigcap\{F \subseteq \mathcal{M} : F \text{ is an ELA-subfield containing } A\}$. On the other hand $\langle A \rangle_{\mathcal{M}}^{\mathrm{ELA}}$ might not be the smallest ELA-subfield of $\mathcal{M}$ containing A; one can construct, as in the proof of $\mathrm{LS}(\mathbf{ECF}) = \aleph_0$, an ELA-subfield (in fact a model of **ECF**) of cardinality $|A| + \aleph_0$. However, demanding containment of the kernel ensures that $\langle A \rangle^{\mathrm{ELA}}$ will be an ELA-subfield.

Proposition 2.4.2. *[16, Proposition 3.13] Suppose that $\mathcal{M}$ and $\mathcal{N}$ are in **ECF**, and that A is a vector subspace of both $\mathcal{M}$ and $\mathcal{N}$ with very full kernel such that $A \triangleleft \mathcal{M}$ and $A \triangleleft \mathcal{N}$. Then $\langle A \rangle_{\mathcal{M}}^{ELA} \cong_A \langle A \rangle_{\mathcal{N}}^{ELA}$.*

Let $\mathcal{M}$, A and $\mathcal{N}$ be as in the above proposition and suppose that $\mathcal{M} \triangleleft \mathcal{N}$. We observe that $\lceil A \rceil_{\mathcal{M}}^{\mathrm{ELA}} \triangleleft \mathcal{N}$ by transitivity of strong embeddings and so $\lceil A \rceil_{\mathcal{M}}^{\mathrm{ELA}} = \lceil A \rceil_{\mathcal{N}}^{\mathrm{ELA}}$. Therefore we may drop the subscript when considering $\lceil - \rceil^{\mathrm{ELA}}$ in models of **ECF** that are strongly embedded.

Lemma 2.4.3. *Let $\mathcal{M}$ be a model in **ExpF** and suppose that $\bar{a}$ and $\bar{b}$ are finite tuples in $\mathcal{M}$ such that $\bar{a}\bar{b} \prec \mathcal{M}$. Then $\bar{a}\lceil \bar{b} \rceil_{\mathcal{M}}^{ELA} \triangleleft \mathcal{M}$.*

Proof. Since $\ker(\mathcal{M}) \subseteq \lceil \bar{b} \rceil_{\mathcal{M}}^{\mathrm{ELA}}$, it suffices to show that $\langle \bar{a} \lceil \bar{b} \rceil^{\mathrm{ELA}} \rangle \prec \mathcal{M}$. Note that $\lceil \bar{a} \lceil \bar{b} \rceil \rceil = \lceil \bar{a}\bar{b} \rceil = \langle \bar{a}\bar{b} \ker(\mathcal{M}) \rangle$ since $\bar{a}\bar{b} \prec \mathcal{M}$, so $\bar{a}\lceil \bar{b} \rceil \prec \mathcal{M}$. Enumerate $\bar{a}\lceil \bar{b} \rceil^{\mathrm{ELA}}$ as $\bar{a}\lceil \bar{b} \rceil \frown (b_n)_{n<\omega}$ such that for $n < \omega$ we have at least one of b_n, e^{b_n} field-theoretically algebraic over $\{\bar{a}, \lceil \bar{b} \rceil, b_0, ..., b_{n-1}, e^{\bar{a}}, e^{\lceil \bar{b} \rceil}, e^{b_0}, ..., e^{b_{n-1}}\}$. Set $B_0 = \langle \bar{a}\lceil \bar{b} \rceil \rangle$, and for each $n < \omega$ set $B_{n+1} = \langle B_n, b_n \rangle$. Note that $B_0 \prec \mathcal{M}$ by hypothesis, and suppose that $B_n \prec \mathcal{M}$ for some $n < \omega$. If $b_n \in B_n$ then $B_{n+1} = B_n$ so $B_{n+1} \prec \mathcal{M}$. Otherwise we have $\mathrm{ldim}_{\mathbb{Q}}(b_n/B_n) = 1$ and $\mathrm{td}(b_n, e^{b_n}/B_n, e^{B_n}) \leq 1$, so $\Delta(b_n/B_n) \leq 0$. However $B_n \prec \mathcal{M}$ so $\Delta(b_n/B_n) = 0$. Therefore for any

$\bar{x} \in \mathcal{M}$ we have $\Delta(\bar{x}/B_{n+1}) = \Delta(\bar{x}b_n/B_n) - \Delta(b_n/B_n) = \Delta(\bar{x}/B_n) \geq 0$ since $B_n \prec \mathcal{M}$. Applying Lemma 2.2.9(f), we have $\bar{a}\lceil\bar{b}\rceil^{\mathrm{ELA}}$ semi-strong in $\mathcal{M}$. $\square$

Before we investigate types over strong ELA-subfields, we shall show that for any subset $B \subseteq \mathcal{M}$, $\lceil B\rceil^{\mathrm{ELA}}_{\mathcal{M}}$ is a union of B-definable sets.

Lemma 2.4.4. *Let $\mathcal{M} \in$ **ECF**, $B \subseteq \mathcal{M}$ and suppose $a \in \langle B\rangle^{ELA}_{\mathcal{M}}$. Then there is a formula $\phi_a(w)$ with parameters from B such that $\mathcal{M} \models \phi_a(a)$ and $\phi_a(\mathcal{M}) \subseteq \langle B\rangle^{ELA}_{\mathcal{M}}$.*

Proof. By induction on the construction of $\langle B\rangle^{ELA}_{\mathcal{M}}$ from B. For the base case, if $a \in B$ then we define $\phi_a(w)$ to be $w = a$. Suppose $b \in \langle B\rangle^{ELA}_{\mathcal{M}}$ and we have such a formula $\phi_b(w)$. If $a = e^b$ then we can take $\phi_a(w)$ to be the formula $\exists y[w = e^y \wedge \phi_b(y)]$. If $e^a = b$ then we can take $\phi_a(w)$ to be $\exists y[y = e^w \wedge \phi_b(y)]$. Suppose $\bar{b} \in \langle B\rangle^{ELA}_{\mathcal{M}}$ is a finite tuple such that for each $b_i \in \bar{b}$ we have such a formula $\phi_{b_i}(y_i)$. Then if a is field-theoretically algebraic over $\bar{b}$ for some minimal polynomial $f(x, \bar{y}) \in \mathbb{Z}[x, \bar{y}]$ then we can take $\phi_a(w)$ to be $\exists \bar{y}\,[f(x, \bar{y}) = 0 \wedge \bigwedge_i \phi_{b_i}(y_i)]$. $\square$

Lemma 2.4.5. *Let $\bar{b}$ be a finite tuple from $\mathcal{M} \in$ **ECF** and let $\bar{b}' \in \mathcal{M}$ be a basis for the hull of $\bar{b}$ over the kernel. Then there is a formula $\psi(\bar{w})$ defined over $\bar{b}$ such that $\mathcal{M} \models \psi(\bar{b}')$ and any realisation of ψ in $\mathcal{M}$ is a basis for the hull of $\bar{b}$ over the kernel.*

Proof. Let $\bar{b}' \in \mathcal{M}^r$ be a basis for the hull of $\bar{b}$ over the kernel in $\mathcal{M}$. By definition of $\bar{b}'$ there is a unique $\bar{d}$ contained in the $\mathbb{Q}$-linear span of the kernel and a unique matrix $N \in \mathrm{Mat}_{r\times n}(\mathbb{Q})$ such that $\bar{b} = N\bar{b}' + \bar{d}$. Let $U = \mathrm{Loc}(\bar{b}', e^{\bar{b}'}/\bar{b}, e^{\bar{b}}, \ker(\mathcal{M})) = \mathrm{Loc}(\bar{b}', e^{\bar{b}'}/\bar{b}, e^{\bar{b}}, \bar{k})$ for some $\bar{k} \in \ker(\mathcal{M})$.

Let $\phi(\bar{x}_1, \bar{y}_1, \bar{x}_2, \bar{y}_2, \bar{z})$ be the $\mathcal{L}$-formula such that $\phi(\bar{x}, \bar{y}, \bar{b}, e^{\bar{b}}, \bar{k})$ defines U, and let $U(\bar{z}) = \{(\bar{x}, \bar{y}) : \mathcal{M} \models \phi(\bar{x}, \bar{y}, \bar{b}, e^{\bar{b}}, \bar{z})\}$. Define $\psi(\bar{w})$ to be the formula defined over $\bar{b}$ given by

$$(\exists \bar{z} \in \ker) \left\{ [(\bar{w}, e^{\bar{w}}) \in U(\bar{z})] \wedge \bar{b} = N\bar{w} + \bar{d} \wedge (\forall \bar{q} \in Q^{r+1})(\forall u \in \ker) \right.$$
$$\left. \left[\sum_{i=1}^{r} q_i w_i = q_{r+1} u \to \forall \bar{v} \left((\bar{v}, e^{\bar{v}}) \in U(\bar{z}) \to \sum_{i=1}^{r} q_i v_i = q_{r+1} u \right) \right] \right\}$$

where $Q = \{x : (\exists y, z \in \ker)[xz = y]\}$ denotes the (non-standard) rationals in $\mathcal{M}$ as ratios of kernel elements. For given $\bar{k} \in \ker$, the second line states that $\bar{w}$ satisfies only those Q-linear dependencies over the kernel that hold on $\mathrm{pr}(U(\bar{k}))$, the projection to the first r coordinates of $U(k)$. We now prove that if $\mathcal{M} \models \psi(\bar{c}')$ then $\bar{c}'$ is a basis for $\lceil \bar{b} \rceil_{\mathcal{M}}$.

First note that for a given $\bar{k} \in \ker$ we have $\mathrm{td}(\bar{c}', e^{\bar{c}'} / \ker(\mathcal{M})) \leq \dim U(\bar{k}) = r + \mathrm{etd}(\bar{b})$. By the second line of ψ it follows that $\bar{c}'$ satisfies only those $\mathbb{Q}$-linear dependences over the kernel satisfied by all $\bar{w}$ such that $(\bar{w}, e^{\bar{w}}) \in U(\bar{k})$. But $\bar{b}'$ was chosen $\mathbb{Q}$-linearly independent over $\ker(\mathcal{M})$ and so there are no such linear dependences, and thus $\mathrm{ldim}_{\mathbb{Q}}(\bar{c}' / \ker) = r$. Then $\Delta_{\mathcal{M}}(\bar{c}') = \mathrm{td}(\bar{c}' e^{\bar{c}'} / \ker) - \mathrm{ldim}_{\mathbb{Q}}(\bar{c}' / \ker) \leq \mathrm{etd}(\bar{b})$. Since $\bar{b} = N\bar{c}' + \bar{d}$ where $\bar{d} \in \ker(\mathcal{M})$ we have $\mathrm{etd}(\bar{b}) = \mathrm{etd}(\bar{c}')$, then by Fact 2.2.10 $\Delta_{\mathcal{M}}(\bar{c}') = \mathrm{etd}(\bar{c}')$, so $\bar{c}' \prec \mathcal{M}$.

Therefore the $\mathbb{Q}$-linear span of $\bar{c}'$ over the kernel is strong, and contains $\bar{b}$ since $\bar{b} = N\bar{c}' + \bar{d}$. Then $\langle \bar{c}' \ker(\mathcal{M}) \rangle \cap \langle \bar{b}' \ker(\mathcal{M}) \rangle$ is strong, but both $\bar{c}'$ and $\bar{b}'$ are of minimal length, so $\Delta_{\mathcal{M}}(\bar{c}') = \Delta_{\mathcal{M}}(\bar{b}')$. By Lemma 2.3.1 the hull is unique, so $\langle \bar{c}' \ker(\mathcal{M}) \rangle = \lceil \bar{b} \rceil_{\mathcal{M}}$. Since $\bar{c}'$ is of minimal length it is a basis for the hull of $\bar{b}$. $\square$

Proposition 2.4.6. *Let $\mathcal{M} \in$ **ECF** and $B \subseteq \mathcal{M}$ a subset. Then for any $a \in \lceil B \rceil_{\mathcal{M}}^{ELA}$ there exists a B-definable subset $X_a \subseteq \lceil B \rceil_{\mathcal{M}}^{ELA}$ containing a. Therefore, $\lceil B \rceil_{\mathcal{M}}^{ELA}$ is a union of definable sets with parameters from B.*

Proof. Let $a \in \lceil B \rceil^{\mathrm{ELA}}$. By finite character of $\lceil - \rceil^{ELA}$, $a \in \lceil B_0 \rceil^{ELA}$ for some finite subset $B_0 \in B$, so without loss of generality we assume that B is finite. Let $\bar{b}'$ be a basis for the hull of B in $\mathcal{M}$. By Lemma 2.4.4 there is a formula $\phi(w, \bar{u})$ and parameters $\bar{d} \in \lceil B \rceil$ such that $\mathcal{M} \models \phi(a, \bar{d})$ and for any $x \in \mathcal{M}$ such that $\mathcal{M} \models \phi(x, \bar{d})$ we have $x \in \lceil B \rceil^{\mathrm{ELA}}$. Then there exists $N \in \mathrm{Mat}_{r \times n}(\mathbb{Q})$ and $\bar{c} \in \ker(\mathcal{M})$ such that $\bar{d} = N\bar{b}' + \bar{c}$. Then by Lemma 2.4.5 there is a formula $\psi(\bar{v})$ defined over B, realised by $\bar{b}'$, and such that any realisation of ψ is a basis for the hull of B over the kernel. We define

$$\Psi(w) = \exists \bar{u} \exists \bar{v} \exists \bar{z} \left[\phi(w, \bar{u}) \wedge \psi(\bar{v}) \wedge \bar{u} = N\bar{v} + \bar{z} \wedge \bar{z} \in \ker \right]$$

By construction we have $\mathcal{M} \models \Psi(a)$. Suppose $x \in \mathcal{M}$ such that $\mathcal{M} \models \Psi(x)$ where $\bar{u}, \bar{v}$ and $\bar{z}$ are witnessed by $\bar{d}_0, \bar{b}_0$ and $\bar{c}_0$ respectively. Then by Lemma 2.4.5 $\bar{b}_0$ is a basis for the hull of B over the kernel, then $\bar{d}_0 = N\bar{b}_0 + \bar{c}_0$ where $\bar{c}_0 \in \ker$ so by the definition of the hull we have $\bar{d}_0 \in \lceil B \rceil$, and so by Lemma 2.4.4 we have $x \in \lceil B \rceil^{\mathrm{ELA}}$ as required. □

2.5 Types orthogonal to the kernel in ECF

In [15, Section 3] Kirby investigates types over ELA-fields in $\mathbf{ECF_{SK}}$ by defining an ELA-field extension $F \subseteq F|V$, where $F|V$ is the freely generated

ELA-subfield extending an ELA-subfield F by a generic tuple $(\bar{a}, e^{\bar{a}})$ over F of $V \subseteq G^n$, an irreducible, free, rotund and Kummer-generic variety. In fact by [15, Theorem 3.11] all finitely generated kernel-preserving extensions of ELA-fields are of this form for some free and Kummer-generic variety V, and the extension $F \subseteq F|V$ is strong if and only if V is also rotund [15, Proposition 5.2]. In the remainder of this chapter, we shall describe the types in **ECF** in a more explicit manner. In particular, since we intend to study stable-like behaviour modulo complications arising from the kernel, we focus on types over semi-strong ELA-subfields that are 'orthogonal to the kernel', that is, realised in a model $\mathcal{M}$ whose kernel does not extend from the kernel of the semi-strong ELA-field in the base. We construct tools that will ultimately allow us to define an independence notion that can be used to describe these types. In particular we introduce the notion of a type being grounded, and we show that this notion captures all the information needed to describe a type orthogonal to the kernel.

We fix a monster model $\mathbb{M}$ of **ECF**, and define the two kinds of type that we shall investigate.

Definition 2.5.1. Let $\bar{a}, \bar{b} \in \mathbb{M}$ and let C be a subset of $\mathbb{M}$.

- [19, Definition 4.1.1] The *(syntactic) type* of $\bar{a}$ over C, written $\mathrm{tp}(\bar{a}/C)$, is the set of all formulas $\phi(\bar{x}, \bar{c})$ such that $\mathbb{M} \models \phi(\bar{a}, \bar{c})$ and $\bar{c} \in C$. That is, $\bar{a}$ and $\bar{b}$ have the same syntactic type over C, written $\mathrm{tp}(\bar{a}/C) = \mathrm{tp}(\bar{b}/C)$, if for each $\mathcal{L}$-formula $\phi(\bar{x}, \bar{y})$ and every finite tuple $\bar{c} \in C$ we have $\mathbb{M} \models \phi(\bar{a}, \bar{c})$ if and only if $\mathbb{M} \models \phi(\bar{b}, \bar{c})$.

- [6, Definition 2.1] The *Galois type* of $\bar{a}$ over C, written $\mathrm{tp}^g(\bar{a}/C)$, is the automorphism orbit of $\bar{a}$ over C in $\mathbb{M}$. That is, $\bar{a}$ and $\bar{b}$ have the same Galois type over C, written $\mathrm{tp}^g(\bar{a}/C) = \mathrm{tp}^g(\bar{b}/C)$, if there exists an automorphism $\sigma \in \mathrm{Aut}(\mathbb{M})$ fixing C pointwise such that $\sigma(\bar{a}) = \bar{b}$.

When working in a saturated model we associate the set of $\mathcal{L}$-formulas $\mathrm{tp}(\bar{a}/C)$ with its set of realisations. If two tuples have the same Galois type, it is immediate that they have the same syntactic type, as automorphisms preserve formulas. In Proposition 2.5.10 we provide sufficient conditions for the converse to also hold.

Definition 2.5.2. Let F be a semi-strong ELA-subfield of $\mathbb{M}$ and let p be a complete type over F realised by $\bar{a} \in \mathbb{M}^n$. We say that p is *orthogonal to the kernel* if there exists a model $\mathcal{M} \in \mathbf{ECF}$ with $\bar{a} \in \mathcal{M}^r$ such that $\ker(\mathcal{M}) = \ker(F)$. We say that $\mathcal{M}$ *witnesses* the orthogonality of p.

Note that if p is a type over a semi-strong ELA-subfield $F \subseteq \mathbb{M}$ such that p is orthogonal to the kernel witnessed by $\mathcal{M} \in \mathbf{ECF}$, one could then view p as a type over a strong ELA-subfield $F \subseteq \mathcal{M}$.

Example 2.5.3. We give some examples and non-examples of types that are orthogonal to the kernel in **ECF**. We explain the orthogonality or non-orthogonality to the kernel for each example, saving explicit proofs until Example 2.6.3, by which point we will have developed more machinery. In each case we consider a type over a semi-strong ELA-subfield $F \subseteq \mathbb{M}$.

1. The type generated by $x \notin F$ and $x = e^x$ is orthogonal to the kernel. A realisation of this type is exponentially algebraic over F, but does not require a new kernel element to be realised.

2. For given $\lambda \in F$, the type generated by $x \notin F$ and $\lambda = e^x$ is not

orthogonal to the kernel. Realising this type requires the existence of a new logarithm, which necessitates a new kernel element.

3. Let $\lambda \in F^\times$ and $a, b, c \in \mathbb{M} \setminus F$ such that $Fabc \prec \mathbb{M}$, $a^2 = e^a + \lambda$, $e^b = a$ and $c + e^c = b$. Then the types $\mathrm{tp}(a/F)$, $\mathrm{tp}(c/F)$, $\mathrm{tp}(a, c/F)$ and $\mathrm{tp}(a, b, c/F)$ are all orthogonal to the kernel; for instance in a model realising the type $\mathrm{tp}(a, b, c/F)$, the existence of b requires a new logarithm, but only of a new element a, so the existence of b does not extend the kernel. However if F' is a semi-strong ELA-subfield containing Fa such that $\ker(F') = \ker(F)$, then $\mathrm{tp}(c/F')$ is not orthogonal to the kernel, since the existence of c implies the existence of $c + e^c = b$, which is a new logarithm of an element in F', namely a. This example demonstrates an interesting subtlety, that a type non-orthogonal to the kernel needn't always be realised by a new kernel element; instead, a model realising such a type could implicitly demand the existence of a new kernel element in that model.

Before we can characterise types that are orthogonal to the kernel, we first describe a stronger version of strong exponential closedness, saturation over the kernel.

Definition 2.5.4. [16, Definition 4.2] Let F be an ELA-field. We say F is *saturated over the kernel* iff $\mathrm{etd}(F) = |F|$ and whenever $V \subseteq \mathbb{G}_a(F)^n \times \mathbb{G}_m(F)^n$ is a rotund, free sub-variety of dimension n defined over F and A is a subset of F of cardinality strictly less than $|F|$, there exists $\bar{x} \in F$ such that $(\bar{x}, e^{\bar{x}}) \in V$ is generic in V over A.

Proposition 2.5.5. *[16, Proposition 4.3] Let $\mathcal{M} \in$ **ECF** and suppose $\mathcal{M}$ has very full kernel. Then there exists $\mathcal{N} \in$ **ECF** such that $|\mathcal{M}| = |\mathcal{N}|$, $\mathcal{M}$*

is strong in $\mathcal{N}$, and $\mathcal{N}$ is saturated over the kernel.

Theorem 2.5.6. *[16, Theorem 4.4] Suppose $\mathcal{M}_1, \mathcal{M}_2 \in$ **ECF** are models with the same cardinality, greater than $2^{\aleph_0}$, such that each $\mathcal{M}_i$ is saturated over the kernel. Suppose we have an isomorphism $\theta_Z : Z(\mathcal{M}_1) \to Z(\mathcal{M}_2)$ and a bijection $\theta_B : B_1 \to B_2$ between exponential transcendence bases of $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively. Suppose further that $A_1 \prec \mathcal{M}_1$ and $A_2 \prec \mathcal{M}_2$ are semi-strong $\mathbb{Q}$-vector subspaces of $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively such that $|A_1| = |A_2| < |\mathcal{M}_1|$ and $\theta_0 : F_1 \to F_2$ is a field isomorphism compatible with θ_Z and θ_B, where F_i is the field of fractions of $A_i \cup \exp_{\mathcal{M}_i}(A_i)$. Then there is an isomorphism $\theta : \mathcal{M}_1 \to \mathcal{M}_2$ extending $\theta_Z \cup \theta_B \cup \theta_0$.*

For each $r \in \mathbb{Z}$ we write $r \cdot (\bar{x}, \bar{y})$ for $(rI) \cdot (\bar{x}, \bar{y})$ where I is the identity matrix.

Definition 2.5.7. [2, Section 1] Let K be an algebraically closed field, and let $V \subseteq K^n \times (K^\times)^n$ be an algebraic variety. We say V is *Kummer-generic* if for every $r \in \mathbb{N}^+$ the variety $V_r = \{(\bar{x}, \bar{y}) \in K \,:\, r \cdot (\bar{x}, \bar{y}) \in V\}$ is irreducible.

Fact 2.5.8. The Thumbtack Lemma [15, Fact 2.16] Let $\mathcal{M}$ be a model of **ECF**, $\bar{a} \in \mathcal{M}$, and F an ELA-subfield of $\mathcal{M}$. Then there exists $m \in \mathbb{N}$ such that $\mathrm{Loc}(\frac{\bar{a}}{m}, e^{\frac{\bar{a}}{m}}/F)$ is Kummer-generic.

Lemma 2.5.9. *Let $\mathcal{M}$ be a model in **ECF**. Let F be a strong ELA-subfield of $\mathcal{M}$ and let $\bar{a} \in \mathcal{M}^r$ be a finite tuple. Let $\bar{a}' \in \mathcal{M}^n$ be a basis of $\lceil \bar{a}F \rceil_{\mathcal{M}}$ over F, and define the algebraic variety $V = Loc(\bar{a}', e^{\bar{a}'}/F)$. Then V is irreducible, free, and rotund. Furthermore, we may choose $\bar{a}'$ such that V is Kummer-generic.*

Proof. Let $\bar{a}'$ be any basis for the hull of $\bar{a}$ over F. Then in particular $\bar{a}'$ is $\mathbb{Q}$-linearly independent over F and so it is additively free over F. Therefore $\exp(\bar{a}')$ is multiplicatively free over F; otherwise we would have $\prod_{i=1}^{n} e^{\lambda_i a'_i} = c$ for some $c \in F$ and $\lambda_i \in \mathbb{Q}$, implying that $\sum_{i=1}^{n} \lambda_i a'_i = b$ for some b a logarithm of c in F. Then $V = \mathrm{Loc}(\bar{a}', e^{\bar{a}'}/F)$ is additively and multiplicatively free and irreducible, so since $\bar{a}'$ spans the hull of $\bar{a}$ over F we have V rotund. By Fact 2.5.8 we may replace $\bar{a}'$ with $\frac{\bar{a}'}{m}$ for some $m \in \mathbb{N}$ so that V is Kummer-generic. $\square$

Proposition 2.5.10. *Suppose that $F \subseteq \mathbb{M}$ is a semi-strong ELA-subfield and $\bar{a} \in \mathbb{M}^r$ is a tuple such that $tp(\bar{a}/F)$ is orthogonal to the kernel, witnessed by $\mathcal{M} \in$ **ECF**. Let $\bar{a}' \in \mathcal{M}^n$ be a basis for the hull of $\bar{a}$ over F in $\mathcal{M}$. Define $V = Loc(\bar{a}', e^{\bar{a}'}/F)$ and let $M \in Mat_{r\times n}(\mathbb{Q})$ and $\bar{c} \in F^r$ such that $M\bar{a}' + \bar{c} = \bar{a}$. Suppose that $\bar{b} \in \mathbb{M}^r$ such that there exists $\bar{b}' \in \mathbb{M}^n$ with*

- *$Loc(\bar{b}', e^{\bar{b}'}/F\ker(\mathbb{M})) = V$,*
- *$etd(\bar{b}/F) = \dim(V) - n$, and*
- *$M\bar{b}' + \bar{c} = \bar{b}$.*

Then $tp^g(\bar{b}/F) = tp^g(\bar{a}/F)$.

Proof. We have $\bar{a}'$ a basis for $\lceil \bar{a}F \rceil_{\mathcal{M}}$ over F so $F\bar{a}' \lhd \mathcal{M}$, and $\mathcal{M} \prec \mathbb{M}$, so therefore $F\bar{a}' \prec \mathbb{M}$. Since $\mathrm{Loc}(\bar{a}', e^{\bar{a}'}/F) = \mathrm{Loc}(\bar{b}', e^{\bar{b}'}/F\ker(\mathcal{M}))$ we have $\mathrm{ldim}_{\mathbb{Q}}(\bar{b}'/F\ker(\mathcal{M})) = n$. Noting also that $\dim V = \mathrm{td}(\bar{b}', e^{\bar{b}'}/F\ker(\mathcal{M}))$, we have $\mathrm{etd}(\bar{b}'/F) = \Delta_{\mathbb{M}}(\bar{b}'/F)$. Since F is semi-strong in $\mathbb{M}$, by Corollary 2.2.11 we have $\Delta_{\mathbb{M}}(\bar{x}/F\bar{b}') \geq 0$ for all $\bar{x} \in \mathbb{M}$. In order to prove $F\bar{b} \prec \mathbb{M}$, we need to show that $F\bar{b}'e^{\bar{b}'} \mathop{\smile\!\!\!\!\!\!\mid\;}^{\mathbf{ACF_0}}_{\ker(F)} \ker(\mathbb{M})$.

Since $F \prec \mathbb{M}$ we have $\operatorname{td}(F/\ker(\mathbb{M})) = \operatorname{td}(F/\ker(F))$, and since $\operatorname{Loc}(\bar{b}', e^{\bar{b}'}/F\ker(\mathcal{M})) = \operatorname{Loc}(\bar{a}', e^{\bar{a}'}/F) = \operatorname{Loc}(\bar{b}', e^{\bar{b}'}/F)$ it follows that $\operatorname{td}(\bar{b}', e^{\bar{b}'}/F) = \operatorname{td}(\bar{b}', e^{\bar{b}'}/F\ker(\mathcal{M}))$. By additivity of transcendence degree we have

$$\begin{aligned}\operatorname{td}(F\bar{b}', e^{\bar{b}'}/\ker(F)) &= \operatorname{td}(\bar{b}'e^{\bar{b}'}/F) - \operatorname{td}(F/\ker(F))\\ &= \operatorname{td}(\bar{b}'e^{\bar{b}'}/F\ker(\mathbb{M})) - \operatorname{td}(F/\ker(\mathbb{M}))\\ &= \operatorname{td}(F\bar{b}', e^{\bar{b}'}/\ker(\mathbb{M}))\end{aligned}$$

and so $F\bar{b}' \prec \mathbb{M}$ as required.

By Lemma 2.3.3 we have $\lceil F\bar{a}\rceil_{\mathbb{M}} = \langle F\bar{a}'\ker(\mathbb{M})\rangle_{\mathbb{M}}$, and similarly $\lceil F\bar{b}\rceil_{\mathbb{M}} = \langle F\bar{b}'\ker(\mathbb{M})\rangle_{\mathbb{M}}$. We may then define an isomorphism of strong vector subspaces of $\mathbb{M}$ by $\theta_0 : \langle F\bar{a}'\ker(\mathbb{M})\rangle_{\mathbb{M}} \to \langle F\bar{b}'\ker(\mathbb{M})\rangle_{\mathbb{M}}$ where $\theta_0(\bar{a}') = \bar{b}'$ and θ_0 fixes $F \cup \ker(\mathbb{M})$ pointwise. Note that we have $|F\bar{a}'| = |F\bar{b}'| < |\mathbb{M}|$, and since $\mathbb{M}$ is the monster model by Proposition 2.5.5 $\mathbb{M}$ is saturated over the kernel. We will apply Theorem 2.5.6, with $\mathcal{M}_1 = \mathcal{M}_2 = \mathbb{M}$, $A_1 = \langle F\bar{a}'\ker(\mathbb{M})\rangle_{\mathbb{M}}$ and $A_2 = \langle F\bar{b}'\ker(\mathbb{M})\rangle_{\mathbb{M}}$. We know θ_0 fixes the kernel pointwise, which fixes $Z(\mathbb{M})$ pointwise. Therefore define $\theta_Z = \mathrm{id}_{Z(\mathbb{M})}$. Since θ_0 is an isomorphism it maps an exponential transcendence basis for $F\bar{a}'$ to one for $F\bar{b}'$, which we can extend to a bijection θ_B of exponential transcendence bases of $\mathbb{M}$. Applying Theorem 2.5.6, θ_0 extends to an automorphism of $\mathbb{M}$ fixing F pointwise and sending $\bar{a}'$ to $\bar{b}'$. Therefore $\operatorname{tp}^g(\bar{a}'/F) = \operatorname{tp}^g(\bar{b}'/F)$, and so $\operatorname{tp}^g(\bar{a}/F) = \operatorname{tp}^g(\bar{b}/F)$. $\square$

Corollary 2.5.11. *Let F be a semi-strong ELA-subfield of $\mathbb{M}$. Suppose that we have $\bar{a}, \bar{b} \in \mathbb{M}^r$ such that $tp(\bar{a}/F) = tp(\bar{b}/F)$ is orthogonal to the kernel. Then $tp^g(\bar{a}/F) = tp^g(\bar{b}/F)$.*

Proof. Let $\mathcal{M}$ and $\mathcal{N} \in \mathbf{ECF}$ witness orthogonality of $\mathrm{tp}(\bar{a}/F)$ such that $\bar{a} \in \mathcal{M}^r$ and $\bar{b} \in \mathcal{N}^r$ respectively. Let $\bar{a}' \in \mathcal{M}^n$ be a basis for $\lceil \bar{a}F \rceil_{\mathcal{M}}$ over F, and let $\bar{c} \in F^n$ and $M \in \mathrm{Mat}_{r\times n}(\mathbb{Z})$ such that $M\bar{a}' + \bar{c} = \bar{a}$. Since $\mathrm{tp}(\bar{a}/F) = \mathrm{tp}(\bar{b}/F)$ we can find a basis $\bar{b}' \in \mathcal{N}^n$ for $\lceil \bar{b}F \rceil_{\mathcal{N}}$ such that $M\bar{b}' + \bar{c} = \bar{b}$, and we note that $\mathrm{Loc}(\bar{a}', e^{\bar{a}'}/F) = \mathrm{Loc}(\bar{b}', e^{\bar{b}'}/F)$. We also have $\mathrm{td}(\bar{a}', e^{\bar{a}'}/F) = \mathrm{td}(\bar{b}', e^{\bar{b}'}/F)$ where $\bar{a}'$ and $\bar{b}'$ are $\mathbb{Q}$-linearly independent over F, and so $\Delta(\bar{a}'/F) = \Delta(\bar{b}'/F)$. As these are hull bases, by Fact 2.2.10 we have $\mathrm{etd}(\bar{a}/F) = \mathrm{etd}(\bar{b}/F)$. Hence by Proposition 2.5.10 we have $\mathrm{tp}^g(\bar{a}/F) = \mathrm{tp}^g(\bar{b}/F)$. $\square$

We have shown that the Galois type of a tuple $\bar{a} \in \mathbb{M}$ over a semi-strong ELA-subfield $F \subseteq \mathbb{M}$ is equal to the set of realisations of the syntactic type $\mathrm{tp}(\bar{a}/F)$ when this type is orthogonal to the kernel. In the next section we shall use the characterisation of types given in Proposition 2.5.10 to show that types orthogonal the kernel may be fully described by a certain small subset of F.

2.6 Grounded types in ECF

Definition 2.6.1. Let $F \subseteq \mathbb{M}$ be a semi-strong ELA-subfield, p a type over F realised by $\bar{a}$ in some model $\mathcal{M} \in \mathbf{ECF}$ with $F \prec \mathcal{M}$. For a given subset $A \subseteq F$, we say p is *grounded at* A if $A \prec F$ and there is a basis $\bar{a}' \in \mathcal{M}^n$ of the hull of $\bar{a}$ over F such that

- $\mathrm{Loc}(\bar{a}', e^{\bar{a}'}/F) = \mathrm{Loc}(\bar{a}', e^{\bar{a}'}/A, e^A)$,
- $\mathrm{etd}(\bar{a}/A) = \mathrm{etd}(\bar{a}/F)$, and

- for the unique matrix $M \in \mathrm{Mat}_{r\times n}(\mathbb{Q})$ and $\bar{c} \in F$ such that $M\bar{a}' - \bar{a} = \bar{c}$, we have $\bar{c} \in A^r$.

We say such $\bar{a}'$ is an A-*basis* (or simply *grounding basis*) for the hull of $\bar{a}$ over F, and call A a *grounding set* for p.

A type being grounded in **ECF** is similar to the notion of 'based' for non-forking independence in first order theories. At the end of this chapter we shall specify just how closely they are related.

Lemma 2.6.2. *Let p be a type over a semi-strong ELA-subfield F. If p is grounded at A for some subset $A \subseteq F$, then p is orthogonal to the kernel.*

Proof. Suppose that p is not orthogonal to the kernel, and let $\mathcal{M} \in$ **ECF** be a model such that some $\bar{a} \in \mathcal{M}$ realises p, so in particular $\ker(F) \neq \ker(\mathcal{M})$. Let $\bar{a}'$ be a basis for $\lceil \bar{a}F \rceil_{\mathcal{M}}$ over F. Then there is a matrix $M \in \mathrm{Mat}_{r\times n}(\mathbb{Z})$, $\bar{c} \in F^r$ and $\bar{d} \in (\ker(\mathcal{M}) \setminus \ker(F))^r$ such that $\bar{a} = M\bar{a}' + \bar{c} + \bar{d}$. Then any set A containing $\bar{c} + \bar{d}$ cannot be a subset of F, as in particular $\bar{d} \notin F$. Therefore p cannot be grounded. $\square$

Example 2.6.3. We revisit the types described in Example 2.5.3, and use the machinery we have developed to prove their orthogonality (or non-orthogonality) to the kernel in **ECF**. As before we set F to be a semi-strong ELA-subfield of $\mathbb{M}$.

1. The type generated by $x \notin F$ and $x = e^x$ is grounded by $A = \emptyset$, and hence by Lemma 2.6.2 this type is orthogonal to the kernel.

2. Let $\lambda \in F^\times$ and consider the type generated by $x \notin F$ and $\lambda = e^x$. Suppose that this type is orthogonal to the kernel, witnessed by $\mathcal{M} \in$

ECF and realised by $b \in \mathcal{M}$. Then $\Delta_{\mathcal{M}}(b/F) = 1 - 1 = 0$, so by Corollary 2.2.11 we have $Fb \lhd \mathcal{M}$, and so b is a basis for $\lceil Fb \rceil_{\mathcal{M}}$ over F. By Lemma 2.5.9 the algebraic variety $\mathrm{Loc}(b, e^b/F)$ is irreducible, free, and rotund; however, this variety is defined by the formula $\lambda = y$ so it is not even free, which is a contradiction. Therefore no such $\mathcal{M}$ exists, and this type is not orthogonal to the kernel.

3. As before we set $\lambda \in F^{\times}$ and $a, b, c \in \mathbb{M} \setminus F$ such that $Fabc \prec \mathbb{M}$, $a^2 = e^a + \lambda$, $e^b = a$ and $c + e^c = b$. By Lemma 2.6.2, the following types are orthogonal to the kernel by the existence of their grounding sets:

   - $\mathrm{tp}(a/F)$ has grounding set $\{\lambda\}$,
   - $\mathrm{tp}(c/F)$ has grounding set $\emptyset$,
   - $\mathrm{tp}(a, c/F)$ has grounding set $\{\lambda\}$,
   - $\mathrm{tp}(a, b, c/F)$ has grounding set $\{\lambda\}$.

   Now let F' be a semi-strong ELA-subfield containing Fa such that $\ker(F') = \ker(F)$, and consider $\mathrm{tp}(c/F')$. Then $F'c \not\prec \mathbb{M}$ since $\Delta_{\mathbb{M}}(b/F'c) = -1$, but $F'cb \prec \mathbb{M}$ so we have cb a basis for $\lceil F'c \rceil_{\mathbb{M}}$ over F'. Suppose that $\mathrm{tp}(c/F')$ is orthogonal to the kernel witnessed by $\mathcal{M}$. Then $c + e^c = b \in \mathcal{M}$, so cb is a basis for $\lceil F'c \rceil_{\mathcal{M}}$ over F'. We have $\mathrm{Loc}(c, b, e^c, e^b/F') \subseteq \{y_2 = a\}$, so $\mathrm{Loc}(c, b, e^c, e^b/F')$ is not free. Therefore by Lemma 2.5.9 $\mathcal{M}$ does not exist, and so $\mathrm{tp}(c/F)$ is non-orthogonal to the kernel.

Henceforth in this section, unless otherwise stated, F is a semi-strong ELA-subfield of $\mathbb{M}$, p is a complete type over F such that p is orthogonal to the kernel witnessed by some model $\mathcal{M} \in \mathbf{ECF}$, and $\bar{a} \in \mathcal{M}^r$ realises p.

Lemma 2.6.4. *Let p be grounded at A with grounding basis $\bar{a}' \in \mathcal{M}^n$. Then $A\bar{a}' \prec \mathcal{M}$.*

Proof. We have $\mathrm{Loc}(\bar{a}', e^{\bar{a}'}/F) = \mathrm{Loc}(\bar{a}', e^{\bar{a}'}/A, e^A)$, and so $A\bar{a}'e^{A\bar{a}'} \mathop{\smile\!\!\!\!\!\!\mid}^{\mathbf{ACF_0}}_{A,e^A} F$, which by monotonicity of (field-theoretic) algebraic independence in $\mathbf{ACF_0}$ means that $A\bar{a}'e^{A\bar{a}'} \mathop{\smile\!\!\!\!\!\!\mid}^{\mathbf{ACF_0}}_{A,e^A} \ker(F)$. As $A \prec F$ we have $A, e^A \mathop{\smile\!\!\!\!\!\!\mid}^{\mathbf{ACF_0}}_{\ker(A)} \ker(F)$, so by (left) transitivity of algebraic independence in $\mathbf{ACF_0}$ we have $A\bar{a}', e^{A\bar{a}'} \mathop{\smile\!\!\!\!\!\!\mid}^{\mathbf{ACF_0}}_{\ker(A)} \ker(F)$. Since $\ker(A\bar{a}') \subseteq \ker(\mathcal{M}) = \ker(F)$ and $A\bar{a}', e^{A\bar{a}'} \mathop{\smile\!\!\!\!\!\!\mid}^{\mathbf{ACF_0}}_{\ker(A)} \ker(F)$ we have $\ker(A\bar{a}') = \ker(A)$, and so $A\bar{a}', e^{A\bar{a}'} \mathop{\smile\!\!\!\!\!\!\mid}^{\mathbf{ACF_0}}_{\ker(A\bar{a}')} \ker(\mathcal{M})$.

Secondly we need to show that for all $\bar{b} \in \mathcal{M}$ we have $\Delta_{\mathcal{M}}(\bar{b}/A\bar{a}') \geq 0$. We have $\mathrm{etd}(\bar{a}'/A) = \mathrm{etd}(\bar{a}'/F)$, and by Fact 2.2.10 and $F\bar{a}' \lhd \mathcal{M}$ we have $\Delta(\bar{a}'/F) = \mathrm{etd}(\bar{a}'/F)$. By finite character of the predimension function, for some finite tuple $\bar{d} \in F$ we have $\Delta(\bar{a}'/F) = \Delta(\bar{a}'/\bar{d})$. However $\mathrm{Loc}(\bar{a}', e^{\bar{a}'}/Ae^A) = \mathrm{Loc}(\bar{a}'e^{\bar{a}'}/F)$ so we can choose $\bar{d} \in A$. Therefore $\Delta(\bar{a}'/A) = \mathrm{etd}(\bar{a}'/A)$, so for any $\bar{b} \in \mathcal{M}$ by additivity of the predimension we have $\Delta(\bar{b}/A\bar{a}') = \Delta(\bar{b}\bar{a}'/A) - \mathrm{etd}(\bar{a}'/A) \geq 0$. Therefore $A\bar{a}' \prec \mathcal{M}$. □

Proposition 2.6.5. *Let $\bar{a}' \in \mathcal{M}^n$ be a basis for the hull of $\bar{a}$ over F in $\mathcal{M}$. Then there exists a finite subset $A \subseteq F$ such that p is grounded at A, and $\bar{a}'$ is a A-basis for $\bar{a}$ over F.*

Proof. As $\bar{a}'$ is a basis for $\lceil \bar{a}F \rceil_{\mathcal{M}}$ over F and $\ker(F) = \ker(\mathcal{M})$, we have $F\bar{a}' \lhd \mathcal{M}$, and there exists a matrix $M \in \mathrm{Mat}_{r\times n}(\mathbb{Q})$ and $\bar{c} \in F^r$ such that $\bar{a} = M\bar{a}' + \bar{c}$. We can find a finite subset $A \subseteq F$ containing $\bar{c}$ such that

$\text{Loc}(\bar{a}', e^{\bar{a}'}/F)$ is defined over $A \cup \exp(A)$ and $\text{etd}(\bar{a}/A) = \text{etd}(\bar{a}/F)$, and we may extend A in F so that $A \prec F$. Then p is grounded at A. $\square$

Lemma 2.6.6. *Suppose that p is grounded at $A \subseteq F$, and let $B \subseteq F$ be any semi-strong subset of F containing A. Then p is grounded at B.*

Proof. Let $\bar{a}' \in \mathcal{M}^n$ be an A-basis for $\bar{a}$ over F in $\mathcal{M}$. For a unique matrix $M \in \text{Mat}_{r\times n}(\mathbb{Q})$ and tuple $\bar{c} \in F^r$ we have $M\bar{a}' - \bar{a} = \bar{c}$, and by definition of A we have $\bar{c} \in A^r \subseteq B^r$. Since $A \subseteq B \subseteq F$ and $\text{etd}(\bar{a}/A) = \text{etd}(\bar{a}/F)$, it follows that $\text{etd}(\bar{a}/B) = \text{etd}(\bar{a}/F)$, and similarly $\text{Loc}(\bar{a}', e^{\bar{a}'}/A, e^A) = \text{Loc}(\bar{a}', e^{\bar{a}'}/F)$ implies that $\text{Loc}(\bar{a}', e^{\bar{a}'}/B, e^B) = \text{Loc}(\bar{a}', e^{\bar{a}'}/F)$. Therefore p is grounded at B. $\square$

The final technical result of this chapter implicitly uses a conjecture from Diophantine geometry formulated by Zilber in [25] known as the conjecture of intersections of tori with subvarieties, or CIT. Our result states that if a type over a set B is grounded, it uniquely extends to a type over $\lceil B \rceil^{\text{ELA}}$. We will not explicitly use the conjecture so we do not state it, however we shall now briefly describe its equivalence to **ECF** being an elementary class.

First, consider the axioms for **ECF**. In [16, Proposition 2.2] it was shown that assuming CIT, axiom (III) is first order expressible. Axioms (I),(IIa) and (IIb) are first order expressible, and assuming axioms (I),(IIa),(IIb) and (III), axiom (IV) is also first order expressible [16, Lemma 5.1]. Thus assuming CIT, **ECF** is an elementary class; in fact, the converse also holds [16, Theorem 1.4].

In [16, Theorem 6.1] it is shown that, assuming CIT, **ECF** has quantifier elimination in the language $(+, \cdot, \exp)$ expanded by predicates for every definable subset of $\mathbb{Z}$ and all existential formulas. Considering quantifier-free types in this expanded language $\mathcal{L}'$, if we have $\bar{a}, \bar{b} \in \mathbb{M}$ tuples and $C \subseteq \mathbb{M}$ a subset such that $\mathrm{qftp}_{\mathcal{L}'}(\bar{a}/C) = \mathrm{qftp}_{\mathcal{L}'}(\bar{b}/C)$, we see that $\mathrm{tp}^g(\bar{a}/C) = \mathrm{tp}^g(\bar{b}/C)$.

If CIT holds, then **ECF** is the class of models of a complete first order theory [16, Theorem 1.3]; this implies that Galois types are equivalent to syntactic types over sets, and allows us to use some first order techniques in the following theorem, which will lead to a stationarity result. The following theorem will later be used to show that if p is a global type, orthogonal to the kernel and grounded at A, then p is a definable type, definable over A.

Theorem 2.6.7. *Assume CIT, and suppose p is grounded at $A \subseteq F$, and $B \subset F$ is a semi-strong subset of F containing A. Then for any $\mathcal{N} \in$ **ECF** such that $\mathcal{M} \leq \mathcal{N}$, we have a set of formulas $\Theta(\bar{x})$ with parameters from B such that if $\bar{b} \in \mathcal{N}$ such that $\mathcal{N} \models \Theta(\bar{b})$, then $\bar{b}$ realises $p|\lceil B \rceil^{ELA}_{\mathcal{N}}$.*

Proof. Let $\mathcal{N} \in$ **ECF** such that $\mathcal{M} \leq \mathcal{N}$, so by CIT and [16, Theorem 6.1 (3)] $\mathcal{N}$ is an elementary extension of $\mathcal{M}$. Let $p = \mathrm{tp}(\bar{a}/F)$ for some $\bar{a} \in \mathcal{M}^r$, and set $n = \mathrm{ldim}_{\mathbb{Q}}(\lceil \bar{a} \rceil_{\mathcal{M}}/\ker)$. Let $\bar{a}' \in \mathcal{M}^n$ be an A-basis for the hull of $\bar{a}$ over F, and define $V = \mathrm{Loc}(\bar{a}', e^{\bar{a}'}/A, \exp(A))$. For each $s \in \mathbb{N}$, each formula $\Phi(\bar{w})$ with implicit parameters in B defining a subset of $(\lceil B \rceil^{\mathrm{ELA}})^s$ as found in Proposition 2.4.6, and each affine sub-variety $W \subseteq \mathbb{A}^{2n+s}$ defined over $\mathbb{Q}$, define $\theta_{W,\Phi}(\bar{x}) = \exists \bar{y}\, \phi_{W,\Phi}(\bar{x}, \bar{y})$ where

$$\begin{aligned}\phi_{W,\Phi}(\bar{x}, \bar{y}) = &\forall \bar{w}\, (\Phi(\bar{w}) \rightarrow [(\bar{y}, e^{\bar{y}}, \bar{w}) \in W \rightarrow \forall \bar{u}\, ((\bar{u}, e^{\bar{u}}) \in V \\ &\rightarrow (\bar{u}, e^{\bar{u}}, \bar{w}) \in W)]) \wedge (\bar{y}, e^{\bar{y}}) \in V \wedge \bar{x} = M\bar{y} + \bar{c}\end{aligned}$$

Let $\Theta(\bar{x})$ be the set of all such $\theta_{W,\Phi}(\bar{x})$, and note that $\Theta(\bar{x})$ is defined over B. Suppose that we have $\bar{b} \in \mathcal{N}$ such that $\mathcal{N} \models \theta_{W,\Phi}(\bar{b})$ for all formulae Φ defining subsets of $(\lceil B \rceil_{\mathcal{N}}^{\mathrm{ELA}})^s$ and affine sub-varieties $W \subseteq \mathbb{A}^{2n+s}$ for each $s > 0$. Let $q(\bar{y})$ be the partial type containing all formulae $\phi_{W,\Phi}(\bar{b}, \bar{y})$ for all formulae Φ defining subsets of $(\lceil B \rceil_{\mathcal{N}}^{\mathrm{ELA}})^s$ and affine sub-varieties $W \subseteq \mathbb{A}^{2n+s}$ for each $s > 0$, and let q_0 be a finite subset of q. Then $q_0(\bar{y}) = \{\phi_{W_1,\Phi_1}(\bar{b}, \bar{y}), ..., \phi_{W_k,\Phi_k}(\bar{b}, \bar{y})\}$ where for each $i = 1, ..., k$ we have $\Phi_i(\mathcal{N}^{s_i}) \subseteq (\lceil B \rceil_{\mathcal{N}}^{\mathrm{ELA}})^{s_i}$ for some $s_i > 0$ and W_i an affine variety of $\mathbb{A}^{2n+s_i}$. Let $\zeta_i(\bar{y}, \bar{z}, \bar{w}_i)$ be the formula defining W_i, with $|\bar{z}| = |\bar{w}| = n$ and $|\bar{w}_i| = s_i$. Then letting $s = \sum_{i=1}^{k} s_i$ and $\bar{w} = \bar{w}_1 ... \bar{w}_k$, define W be the affine subvariety of $\mathbb{A}^{2n+s}$ with defining formula $\varphi(\bar{y}, \bar{z}, \bar{u}, \bar{w}) = \bigwedge_{i=1}^{k} \zeta_i(\bar{y}, \bar{z}, \bar{u}_i, \bar{w}_i)$ with parameters in $\mathbb{Q}$, where $\bar{u} = \bar{u}_1 ... \bar{u}_k$. Define $\Phi(\bar{w}) = \bigwedge_{i=1}^{k} \Phi_i(\bar{w}_i)$ and note that $\Phi(\mathcal{N}^s)$ is a definable subset of $(\lceil B \rceil_{\mathcal{N}}^{\mathrm{ELA}})^s$. Then $\mathcal{N} \models \theta_{W,\Phi}(\bar{b})$ and there exists $\bar{b}_0 \in \mathcal{N}$ witnessing $\bar{y}$ in $\phi_{W,\Phi}(\bar{b}, \bar{y})$. Then for a given tuple $\bar{d} = (\bar{d}_1, ..., \bar{d}_k) \in \Phi(\mathcal{N}^s) \subseteq (\lceil B \rceil_{\mathcal{N}}^{\mathrm{ELA}})^s$, with $\bar{d}_i \in \Phi_i(\mathcal{N}^{s_i})$ for each $i = 1, ..., k$, we have $\mathcal{N} \models \varphi(\bar{b}_0, e^{\bar{b}_0}, \bar{d})$ if and only if for every $i = 1, ..., k$ we have $\mathcal{N} \models \zeta_i(\bar{b}_0, e^{\bar{b}_0}, \bar{d}_i)$. Therefore $\bar{b}_0$ also realises q_0, and so q is finitely satisfiable. By compactness there exists $\mathcal{N}' \succeq \mathcal{N}$ such that q is realised by some $\bar{b}' \in \mathcal{N}'$.

We have $\bar{b}' \in \mathcal{N}'$ witnessing $\bar{y}$ in all $\theta_{W,\Phi}(\bar{b})$ for all affine varieties W defined over $\mathbb{Q}$ and formulae Φ defining subsets of powers of $\lceil B \rceil_{\mathcal{N}}^{\mathrm{ELA}}$, so $\bar{b} = M\bar{b}' + \bar{c}$ and for all proper sub-varieties V' of V defined over $\lceil B \rceil_{\mathcal{N}}^{\mathrm{ELA}}$ we have $(\bar{b}', e^{\bar{b}'}) \in V \setminus V'$. Therefore $\mathrm{Loc}(\bar{b}', e^{\bar{b}'}/\lceil B \rceil_{\mathcal{N}}^{\mathrm{ELA}}) = V$.

We now wish to prove that $\bar{b}' \lceil B \rceil_{\mathcal{N}}^{\mathrm{ELA}} \prec \mathcal{N}'$. Since we also have $V =$

$\mathrm{Loc}(\bar{a}', e^{\bar{a}'}/\lceil B\rceil_{\mathcal{N}}^{\mathrm{ELA}})$, it follows that

$$\bar{b}'e^{\bar{b}'} \underset{\lceil B\rceil_{\mathcal{N}}^{\mathrm{ELA}}}{\overset{\mathbf{ACF_0}}{\perp}} \ker(\mathcal{N}'), \text{ and so } \bar{b}'e^{\bar{b}'}\lceil B\rceil_{\mathcal{N}}^{\mathrm{ELA}} \underset{\lceil B\rceil_{\mathcal{N}}^{\mathrm{ELA}}}{\overset{\mathbf{ACF_0}}{\perp}} \ker(\mathcal{N}').$$

We have $\lceil B\rceil_{\mathcal{N}}^{\mathrm{ELA}} \perp^{\mathbf{ACF_0}}_{\ker(\lceil B\rceil_{\mathcal{N}}^{\mathrm{ELA}})} \ker(\mathcal{N}')$ trivially since $\lceil B\rceil_{\mathcal{N}}^{\mathrm{ELA}} \prec \mathcal{N}'$. Therefore

$$\bar{b}'e^{\bar{b}'}\lceil B\rceil_{\mathcal{N}}^{\mathrm{ELA}} \underset{\ker(\lceil B\rceil_{\mathcal{N}}^{\mathrm{ELA}})}{\overset{\mathbf{ACF_0}}{\perp}} \ker(\mathcal{N}')$$

by transitivity of field-theoretic algebraic independence.

Since $\mathcal{N}' \models \phi_{W,\Phi}(\bar{b}, \bar{b}')$ for all affine varieties W defined over $\mathbb{Q}$ and all formulas Φ defining subsets of $\lceil B\rceil^{\mathrm{ELA}}$, we also know that $\bar{b}'$ satisfies only those $\mathbb{Q}$-linear dependencies over $\lceil B\rceil_{\mathcal{N}}^{\mathrm{ELA}}$ that hold on all $\mathrm{pr}(V)$, so $\mathrm{ldim}_{\mathbb{Q}}(\bar{b}'/\lceil B\rceil_{\mathcal{N}}^{\mathrm{ELA}}) = r$. Thence

$$\mathrm{td}(\bar{b}', e^{\bar{b}'}/\lceil B\rceil_{\mathcal{N}}^{\mathrm{ELA}}) \leq \dim(V) + \mathrm{etd}(\bar{b}'/\lceil B\rceil_{\mathcal{N}}^{\mathrm{ELA}}) = r + \mathrm{etd}(\bar{b}'/\lceil B\rceil_{\mathcal{N}}^{\mathrm{ELA}})$$

and so $\Delta_{\mathcal{N}'}(\bar{b}'/\lceil B\rceil_{\mathcal{N}}^{\mathrm{ELA}}) \leq \mathrm{etd}(\bar{b}'/\lceil B\rceil_{\mathcal{N}}^{\mathrm{ELA}})$. By Fact 2.2.10 $\Delta_{\mathcal{N}'}(\bar{b}'/\lceil B\rceil_{\mathcal{N}}^{\mathrm{ELA}}) = \mathrm{etd}(\bar{b}'/\lceil B\rceil_{\mathcal{N}}^{\mathrm{ELA}})$, and for any $\bar{x} \in \mathcal{N}'$ we have $\Delta_{\mathbb{M}}(\bar{x}\bar{b}'/\lceil B\rceil_{\mathcal{N}}^{\mathrm{ELA}}) \geq \mathrm{etd}(\bar{b}'/\lceil B\rceil_{\mathcal{N}}^{\mathrm{ELA}})$. By additivity,

$$\begin{aligned}\Delta_{\mathbb{M}}(\bar{x}/\bar{b}'\lceil B\rceil_{\mathcal{N}}^{\mathrm{ELA}}) &= \Delta_{\mathbb{M}}(\bar{x}\bar{b}'/\lceil B\rceil_{\mathcal{N}}^{\mathrm{ELA}}) - \Delta_{\mathbb{M}}(\bar{b}'/\lceil B\rceil_{\mathcal{N}}^{\mathrm{ELA}}) \\ &\geq \mathrm{etd}(\bar{b}'/\lceil B\rceil_{\mathcal{N}}^{\mathrm{ELA}}) - \mathrm{etd}(\bar{b}'/\lceil B\rceil_{\mathcal{N}}^{\mathrm{ELA}}) = 0\end{aligned}$$

and so $\bar{b}'\lceil B\rceil_{\mathcal{N}}^{\mathrm{ELA}} \prec \mathcal{N}'$. Applying Proposition 2.5.10, we have $\mathrm{tp}^g(\bar{a}/\lceil B\rceil_{\mathcal{N}}^{\mathrm{ELA}}) = \mathrm{tp}^g(\bar{b}/\lceil B\rceil_{\mathcal{N}}^{\mathrm{ELA}})$. □

Corollary 2.6.8. *Assume CIT. Let p be grounded at A for some subset $A \subseteq F$, and suppose that $B \subseteq F$ is a subset containing A. Then for any*

$\mathcal{N} \in$ ***ECF*** *such that* $\mathcal{M} \leq \mathcal{N}$, $p|B$ *uniquely extends to* $p|\lceil B \rceil_{\mathcal{N}}^{ELA}$.

Proof. Immediate from Theorem 2.6.7. □

Theorem 2.6.7 provides us with a useful set of formulas determining a complete type over a model. In the next chapter, the tools we have developed shall be used to prove that our proposed independence relations satisfy all necessary independence properties. Before we proceed to defining these independence relations, we use Theorem 2.6.7 to demonstrate the connection between a grounding set and a base in **ECF**.

2.7 A remark on bases and grounding sets

In a first order theory, the canonical base of a global type p is a definably closed tuple α such that for any automorphism σ we have $\sigma p = p$ iff $\sigma\alpha = \alpha$. If it exists, the canonical base is unique up to permutation. A canonical base may not exist in general for types in **ECF**, however we can obtain a result connecting a canonical base to the notion of a grounding set in **ECF**. As before we fix a monster model $\mathbb{M}$ for **ECF**.

Definition 2.7.1. Let p be a complete type in **ECF**. We say that a tuple α from $\mathbb{M}$ is a *base* for p if $\sigma p = p$ for any automorphism $\sigma \in \mathrm{Aut}(\mathbb{M}/\alpha)$.

The above definition is adapted from [3, p.223, Definition 5.1.9], which is a definition for stable theories stating that a syntactic type p is based at α if there exists a complete type q over α such that p and q have the same non-forking extension. Considering instead a Galois type q over A that uniquely extends to p, if an automorphism σ fixes A pointwise, then σ

must fix q. Therefore σ must fix the unique extension p of q setwise, so the definitions correspond.

Lemma 2.7.2. *Assuming CIT, let p be a complete type over a semi-strong ELA-subfield F, orthogonal to the kernel, grounded by $\alpha \prec F$. Then α is a base for p.*

Proof. Let the orthogonality to the kernel of p be witnessed by $\mathcal{M} \in \mathbf{ECF}$, and let $p = \mathrm{tp}(\bar{a}/F)$ for some tuple $\bar{a} \in \mathcal{M}^r$. Suppose that α is a grounding set for p. Then for some α-basis $\bar{a}'$ of p, the algebraic variety $V = \mathrm{Loc}(\bar{a}', e^{\bar{a}'}/F)$ is defined over α, and for a unique matrix $M \in \mathrm{Mat}_{r\times n}(\mathbb{Q})$ we have $M\bar{a}' - \bar{a} = \bar{c} \in F$ and $\bar{c} \subseteq \alpha$. For any $\bar{b} \in \mathbb{M}^r$ we have $\bar{b}$ realising p iff $\mathbb{M} \models \Theta(\bar{b})$ where $\Theta(\bar{x})$ is the set of formulas as obtained in Theorem 2.6.7, except that our parameters come from the ELA-subfield F rather than a subset B. By the definition of $\Theta(\bar{x})$, any automorphism fixing α pointwise will fix $\Theta(\bar{x})$ setwise, and thus fix p setwise. Therefore α is a base for p. $\square$

Lemma 2.7.3. *Assuming CIT, let p be a complete type over a semi-strong ELA-subfield F, orthogonal to the kernel. Let $\alpha \prec F$ be a base for p. Then $dcl(\alpha)$ is a grounding set for p.*

Proof. Let $\sigma \in \mathrm{Aut}(\mathbb{M})$ be an automorphism fixing α pointwise. Then σ fixes p setwise, so by Theorem 2.6.7, σ fixes $\Theta(\bar{x})$ setwise. Let $\bar{a}'$ be a basis for $\lceil \bar{a}F \rceil$ over F such that $\mathrm{Loc}(\bar{a}'e^{\bar{a}'}/F) = V$, where V is the algebraic variety defined in $\Theta(\bar{x})$ in the proof of Theorem 2.6.7, and we also have a unique matrix $M \in \mathrm{Mat}_{r\times n}(\mathbb{Q})$ and $\bar{c} \in F$ such that $M\bar{a}' - \bar{a} = \bar{c}$. Since $\Theta(\bar{x})$ is fixed setwise by σ, by the definition of $\Theta(\bar{x})$ we have $\bar{c} \subseteq \mathrm{dcl}(\alpha)$ and V is defined over $\mathrm{dcl}(\alpha)$. Therefore $\mathrm{Loc}(\bar{a}', e^{\bar{a}'}/F) = \mathrm{Loc}(\bar{a}', e^{\bar{a}'}/\mathrm{dcl}(\alpha))$.

By finite character of Δ we have $\Delta(\bar{a}'/F) = \Delta(\bar{a}'/\bar{b})$ for some $\bar{b} \in F$. Since $\mathrm{Loc}(\bar{a}'e^{\bar{a}'}/F) = \mathrm{Loc}(\bar{a}', e^{\bar{a}'}/\mathrm{dcl}(\alpha))$ we may assume $\bar{b} \in \mathrm{dcl}(\alpha)$, and so $\mathrm{etd}(\bar{a}'/F) = \mathrm{etd}(\bar{a}'/\mathrm{dcl}(\alpha))$. Therefore $\mathrm{dcl}(\alpha)$ is a grounding set for p. $\square$

Chapter 3

Independence in exponential fields

In this chapter we develop notions of independence for the classes $\mathbf{ECF}$, $\mathbf{ECF_{SK}}$ and $\mathbf{ECF_{SK,CCP}}$ that can be defined in terms of exponential algebra. We go on show that these relations are equivalent to natural model theoretic independence notions in $\mathbf{ECF_{SK}}$ and $\mathbf{ECF_{SK,CCP}}$. We demonstrate that our relation is a sensible and useful notion of independence in $\mathbf{ECF}$, where no appropriate natural notions exist. As $\mathbf{ECF_{SK}}$ is a finitary abstract elementary class, using work of Hyttinen and Kesala we prove that $\mathop{\smile\!\!\!\!\!\!|}^{\mathbf{ECF}}$ is exactly the independence notion coming from non-splitting of Lascar types. We also prove that $\mathop{\smile\!\!\!\!\!\!|}^{\mathbf{ECF_{SK,CCP}}}$ is the canonical independence notion coming from the pregeometry, so all the basic independence properties follow; furthermore by work of Hyttinen, Kesälä and Kangas we show that $\mathop{\smile\!\!\!\!\!\!|}^{\mathbf{ECF_{SK,CCP}}}$ is equivalent to non-splitting of weak types.

In [16, Theorem 6.1] it is shown that, assuming CIT, $\mathbf{ECF}$ is ‘superstable over the integers'. There can be no canonical independence notion in **ECF** due to its theory containing $\mathrm{Th}(\mathbb{Z})$, but we assert that our independence notion $\mathop{\smile\!\!\!\!\!\!|}\nolimits^{\mathbf{ECF}}$, defined as it is 'over the kernel', is a useful and appropriate independence relation for dealing with this class. In Chapter 4 we will use our independence notion for **ECF** to prove generic stability of types that are orthogonal to the kernel, corroborating this assertion.

3.1 Independence relations

Independence relations are fundamental to stability theory. They generalise key model theoretic concepts such as algebraic independence in fields and linear independence in vector spaces to make sense in more varied theories. They allow us to understand and relate types within a theory, and the existence and properties of independence relations within a theory provide us with structural understanding of the theory as a whole. We begin this chapter with a definition of an independence relation, and relate it to various theories and classes of structures from model theory.

Definition 3.1.1. [1, Definition 1.1] For a structure $\mathcal{M}$, we say that a ternary relation $\mathop{\smile\!\!\!\!\!\!|}$ is a *pre-independence relation* if the following hold for any small subsets $A, B, C \subseteq \mathcal{M}$:

1. **Monotonicity** If $A \mathop{\smile\!\!\!\!\!\!|}_C B$ and $X \subseteq B$ then $A \mathop{\smile\!\!\!\!\!\!|}_C X$.

2. **Transitivity** Suppose $C \subseteq X \subseteq B$. Then $A \mathop{\smile\!\!\!\!\!\!|}_C B$ if and only if $A \mathop{\smile\!\!\!\!\!\!|}_C X$ and $A \mathop{\smile\!\!\!\!\!\!|}_X B$.

3. **Invariance** If $\sigma \in \mathrm{Aut}(\mathcal{M})$ and $A \mathop{\smile\!\!\!\!\!\!|}_C B$, then $\sigma(A) \mathop{\smile\!\!\!\!\!\!|}_{\sigma(C)} \sigma(B)$.

4. **Finite Character** $A \mathop{\smile\!\!\!\!\!\!\perp}_C B$ if and only if for every $\bar{a} \in A$ we have $\bar{a} \mathop{\smile\!\!\!\!\!\!\perp}_C B$.

We say that $\mathop{\smile\!\!\!\!\!\!\perp}$ is an *independence relation* if the following also hold:

5. **Extension** If $\bar{a} \mathop{\smile\!\!\!\!\!\!\perp}_C B$ and $B' \supseteq B$, then there exists $\bar{b}$ such that $\mathrm{tp}(\bar{b}/BC) = \mathrm{tp}(\bar{a}/BC)$ and $\bar{b} \mathop{\smile\!\!\!\!\!\!\perp}_C B'$.

6. **Local character** There exists an ordinal κ such that for every finite tuple $\bar{a}$ and subset C there exists $C_0 \subseteq C$ with $|C_0| < \kappa$ such that $\bar{a} \mathop{\smile\!\!\!\!\!\!\perp}_{C_0} C$.

In [1] the above notions were defined for a model $\mathcal{M}$ of a first order theory T, but we want a definition that makes sense of an independence relation for models in a class of structures $\mathcal{C}$. Note also that the extension property necessitates a saturated model, but a pre-independence relation makes sense for any model.

We also have the following additional properties which an independence relation may or may not satisfy, dependent on the theory.

7. **Symmetry** If $A \mathop{\smile\!\!\!\!\!\!\perp}_C B$ then $B \mathop{\smile\!\!\!\!\!\!\perp}_C A$.

8. **Independence over models** Let $\mathcal{M}_0 \preccurlyeq \mathcal{M}$ be a submodel, $\mathrm{tp}(\bar{a}/\mathcal{M}_0) = \mathrm{tp}(\bar{b}/\mathcal{M}_0)$ and let $\bar{a}', \bar{b}'$ be such that $\bar{a}' \mathop{\smile\!\!\!\!\!\!\perp}_{\mathcal{M}_0} \bar{b}'$, $\bar{a} \mathop{\smile\!\!\!\!\!\!\perp}_{\mathcal{M}_0} \bar{a}'$, and $\bar{b} \mathop{\smile\!\!\!\!\!\!\perp}_{\mathcal{M}_0} \bar{b}'$. Then there exists $\bar{c}$ such that $\mathrm{tp}(\bar{c}/\mathcal{M}_0\bar{a}') = \mathrm{tp}(\bar{a}/\mathcal{M}_0\bar{a}')$ and $\mathrm{tp}(\bar{c}/\mathcal{M}_0\bar{b}') = \mathrm{tp}(\bar{b}/\mathcal{M}_0\bar{b}')$ and $\bar{c} \mathop{\smile\!\!\!\!\!\!\perp}_{\mathcal{M}_0} \bar{a}'\bar{b}'$.

9. **Stationarity** If $\mathrm{tp}(\bar{a}/C) = \mathrm{tp}(\bar{b}/C)$, $\bar{a} \mathop{\smile\!\!\!\!\!\!\perp}_C B$, and $\bar{b} \mathop{\smile\!\!\!\!\!\!\perp}_C B$, then $\mathrm{tp}(\bar{a}/BC) = \mathrm{tp}(\bar{b}/BC)$.

10. **Stationarity over models** The same statement as Stationarity, with the additional requirement that $C \preccurlyeq \mathcal{M}$ be a submodel.

11. **Pairs lemma** Let $A \subseteq B$, $\bar{a} \mathop{\smile\!\!\!\!\!|}_A B$, and $\bar{b} \mathop{\smile\!\!\!\!\!|}_{A\bar{a}} B$. Then $\bar{a}\bar{b} \mathop{\smile\!\!\!\!\!|}_A B$.

12. **Finite base** If $\bar{a} \mathop{\smile\!\!\!\!\!|}_C B$ then there exists a finite tuple $\bar{c} \in C$ such that $\bar{a} \mathop{\smile\!\!\!\!\!|}_{\bar{c}} B$.

We now give an overview of independence relations for various theories. The classic examples of independence are algebraic independence in $\mathbf{ACF_0}$ and linear independence in vector spaces.

Recall from the introduction that for A, B, C subsets of an algebraically closed field K of characteristic zero, we say A is field-theoretically algebraically independent from B over C if for any tuple $\bar{a} \in A$ we have $\mathrm{td}(\bar{a}/BC) = \mathrm{td}(\bar{a}/C)$, written as $A \mathop{\smile\!\!\!\!\!|}_C^{\mathbf{ACF_0}} B$. Similarly recall that for a $\mathbb{Q}$-vector space V with subsets X, Y, Z, we say that X is $\mathbb{Q}$-linearly independent from Y over Z if for any tuple $\bar{x} \in X$ we have $\mathrm{ldim}_{\mathbb{Q}}(\bar{x}/YZ) = \mathrm{ldim}_{\mathbb{Q}}(\bar{x}/Z)$, written $X \mathop{\smile\!\!\!\!\!|}_Z^{\mathbb{Q}\text{-lin}} Y$. These are in fact both examples of pregeometric independence, which we shall now define.

A *pregeometry* (X, cl) is a set X and a map $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ such that for any subset $A \subseteq X$ we have

- $A \subseteq \mathrm{cl}(A)$,
- $\mathrm{cl}(A) = \mathrm{cl}(\mathrm{cl}(A))$,
- $\mathrm{cl}(A) = \bigcup\{\mathrm{cl}(A_0) : A_0 \subseteq^{fin} A\}$, and
- *Steinitz Exchange Property* If $a \in \mathrm{cl}(Ab) \setminus \mathrm{cl}(A)$ then $b \in \mathrm{cl}(Aa)$.

There is a natural independence relation on a pregeometry (see for example [4, Section 2]) which we shall now briefly describe. Let $A \subseteq X$. We say that $B \subseteq A$ is a *basis* for A if $\mathrm{cl}(A) = \mathrm{cl}(B)$ and for every $b \in B$ we have $b \notin \mathrm{cl}(B \setminus \{b\})$. We define the *dimension* of A, written $\dim(A)$, to be the cardinality of a basis for A. This is well defined, and we also write $\dim(A/C) = \dim(AC) - \dim(C)$. We then have an independence relation given by

$$A \underset{C}{\overset{\mathrm{cl}}{\perp}} B \text{ if and only if for each finite tuple } \bar{a} \in A \text{ we have } \dim(\bar{a}/B) = \dim(\bar{a}/BC)$$

which we call the *pregeometric* independence relation derived from the closure operator $\mathrm{cl}(-)$. This is the canonical independence notion for pregeometry structures; in algebraically closed fields, pregeometric independence derived from $\mathrm{acl}(-)$ is field-theoretic algebraic independence, and in $\mathbb{Q}$-vector spaces the closure operator $\mathrm{span}_{\mathbb{Q}}(-)$ gives rise to $\mathbb{Q}$-linear independence.

The traditional example of independence in first order theories is nonforking independence. Let $\mathbb{M}$ be a monster model of a complete first order $\mathcal{L}$-theory T. For a subset $B \subset \mathbb{M}$ and an $\mathcal{L}$-formula $\phi(x, \bar{a})$ with parameters $\bar{a}$ in $\mathbb{M}$, we say $\phi(x, \bar{a})$ *divides over* B if there exist a sequence of indiscernibles $(\bar{c}_i)_{i<\omega}$ such that $\mathrm{tp}(\bar{a}/B) = \mathrm{tp}(\bar{c}_i/B)$ for all i, and the partial type given by $\{\phi(x, \bar{c}_i)\}_{i<\omega}$ is inconsistent. We say $\phi(x, \bar{a})$ *forks over* B if there exist some finite number of formulas $\phi_1, ..., \phi_n$ dividing over B such that $\phi \to \bigvee_{i=1}^{n} \phi_n$. We say that the type $\mathrm{tp}(\bar{a}/BC)$ *forks over* C if it contains a formula that forks over C. We say that $A \perp^f_C B$ if $\mathrm{tp}(\bar{a}/BC)$ does not fork over C for each $\bar{a} \in A$. Similarly we say that $A \perp^d_C B$ if $\mathrm{tp}(\bar{a}/BC)$ does not divide over C for each $\bar{a} \in A$.

In simple theories $\mathop{\smile\!\!\!\!\!\!|}^f = \mathop{\smile\!\!\!\!\!\!|}^d$, as proved in [9]. The theories of vector spaces and algebraically closed fields are simple, and in fact $\mathop{\smile\!\!\!\!\!\!|}^f = \mathop{\smile\!\!\!\!\!\!|}^{\mathbf{ACF_0}}$ in algebraically closed fields and $\mathop{\smile\!\!\!\!\!\!|}^f = \mathop{\smile\!\!\!\!\!\!|}^{\mathbb{Q}\text{-lin}}$ in vector spaces. A complete first order theory is simple if and only if $\mathop{\smile\!\!\!\!\!\!|}^f$ satisfies local character, which occurs if and only if $\mathop{\smile\!\!\!\!\!\!|}^f$ satisfies symmetry [10, Theorem 2.4].

Theorem 3.1.2. *[11, Theorem 4.2] Let T be a complete theory with monster model $\mathbb{M}$, and let $\mathop{\smile\!\!\!\!\!\!|}^*$ be a ternary relation on small subsets of $\mathbb{M}$ such that $\mathop{\smile\!\!\!\!\!\!|}^*$ is an independence relation satisfying independence over models and symmetry. Then T is simple and $\mathop{\smile\!\!\!\!\!\!|}^* = \mathop{\smile\!\!\!\!\!\!|}^f$.*

In particular, any simple theory has a unique independence relation. If T is simple and $\mathop{\smile\!\!\!\!\!\!|}^f$ satisfies stationarity over models, then T is stable. If T is simple and $\mathop{\smile\!\!\!\!\!\!|}^f$ also satisfies finite base, then T is supersimple. Similarly if T is stable and $\mathop{\smile\!\!\!\!\!\!|}^f$ satisfies finite base, then T is superstable.

It is possible to give sufficient conditions on a finitary AEC to have a unique independence relation.

Definition 3.1.3. [6, Definition 2.12] Let $(\mathcal{C}, \leq_{\mathcal{C}})$ be an AEC with a monster model $\mathbb{M}$. Suppose that $\bar{a}, \bar{b}$ are finite tuples of equal length in $\mathbb{M}$ and $C \subseteq \mathbb{M}$ is a subset. We say that $\bar{a}$ and $\bar{b}$ have the same *weak type* over C, written $\mathrm{tp}^w(\bar{a}/C) = \mathrm{tp}^w(\bar{b}/C)$, if for every finite subset $C_0 \subseteq C$ we have $\mathrm{tp}^g(\bar{a}/C_0) = \mathrm{tp}^g(\bar{b}/C_0)$.

Definition 3.1.4. [6, Definition 2.14, 4.2, 4.3] Let $(\mathcal{C}, \leq_{\mathcal{C}})$ be an abstract elementary class with AP, JEP and ALM, and fix $\mathbb{M}$ a monster model for $\mathcal{C}$.

- Let A be a subset of $\mathbb{M}$. We say that a sequence $(\bar{a}_i)_{i<\omega}$ in $\mathbb{M}$ is *strongly A-indiscernible* if for all cardinals $\kappa > \omega$ there exist $(\bar{a}_i)_{\omega<i<\kappa}$ such that $(\bar{a}_i)_{i<\kappa}$ is an indiscernible sequence over A.

- Let $\bar{a}$, $\bar{c}$ be tuples in $\mathbb{M}$ and let $B \subseteq \mathbb{M}$ be a subset such that $\bar{c} \in B$. We say that $\mathrm{tp}^w(\bar{a}/B)$ *Lascar-splits* over $\bar{c}$ if there is a strongly $\bar{c}$-indiscernible sequence $(\bar{b}_i)_{i<\omega}$ such that $\bar{b}_0, \bar{b}_1$ are in B and $\mathrm{tp}^g(\bar{b}_0/\bar{c}\bar{a}) \neq \mathrm{tp}^g(\bar{b}_1/\bar{c}\bar{a})$.

- Let A, B, C be subsets of $\mathbb{M}$. We say that A is *Lascar-independent* of B over C, written $A \mathop{\smile\!\!\!\!\!\!|}^{L}_{C} B$ if there exists a finite tuple $\bar{c} \in C$ such that for all $D \supseteq B \cup C$ there exists a finite tuple $\bar{b} \in \mathbb{M}$ such that $\mathrm{tp}^w(\bar{b}/BC) = \mathrm{tp}^w(\bar{a}/BC)$ and $\mathrm{tp}^w(\bar{b}/D)$ does not Lascar-split over $\bar{c}$.

Theorem 3.1.5. *[6, Theorem 4.9] Let $(\mathcal{C}, \leq_{\mathcal{C}})$ be a finitary abstract elementary class, and fix $\mathbb{M}$ a monster model for $\mathcal{C}$. Suppose that $\mathop{\smile\!\!\!\!\!\!|}^{*}$ is a ternary relation on small subsets of $\mathbb{M}$ satisfying the following:*

- *$\mathop{\smile\!\!\!\!\!\!|}^{*}$ is an independence relation with the extension property for weak types.*

- ***Bounded number of free extensions:** There is a cardinal κ such that if C is finite and $(\bar{a}_i)_{i<\kappa^+}$ is a sequence of realisations of $tp^w(\bar{a}/C)$ such that $\bar{a}_i \mathop{\smile\!\!\!\!\!\!|}^{*}_{C} B$ for each $i < \kappa^+$, then there are $i < j < \kappa^+$ such that $tp^w(\bar{a}_i/B) = tp^w(\bar{a}_j/B)$.*

Then $\mathop{\smile\!\!\!\!\!\!|}^{*} = \mathop{\smile\!\!\!\!\!\!|}^{L}$. Furthermore, $\mathop{\smile\!\!\!\!\!\!|}^{*}$ is symmetric, satisfying stationarity over $\aleph_0$-saturated models for weak types, and the Pairs lemma.

In this chapter we shall use this theorem and other facts about pregeometric structures to prove uniqueness results for independence relations in our classes of exponential fields.

3.2 Freeness in exponential fields

In this section we define a 'free from' relation for subsets of models in **ExpF**, and prove that this is a symmetric pre-independence relation. Later we will develop this relation into separate independence relations $\downarrow^{\mathbf{ECF}}$, $\downarrow^{\mathbf{ECF_{SK}}}$ and $\downarrow^{\mathbf{ECF_{SK,CCP}}}$ for the classes **ECF**, $\mathbf{ECF_{SK}}$ and $\mathbf{ECF_{SK,CCP}}$ respectively.

Definition 3.2.1. Let $\mathcal{M}$ be any model in **ExpF**, and let $A, B, C \subseteq \mathcal{M}$. We say A is *free from* B over C in $\mathcal{M}$, written $A \downarrow_C^{\mathcal{M}} B$, if

(i) $\lceil AC \rceil_{\mathcal{M}} \exp(\lceil AC \rceil_{\mathcal{M}}) \downarrow^{\mathbf{ACF_0}}_{\lceil C \rceil_{\mathcal{M}} \exp(\lceil C \rceil_{\mathcal{M}})} \lceil BC \rceil_{\mathcal{M}} \exp(\lceil BC \rceil_{\mathcal{M}})$,

(ii) $\lceil AC \rceil_{\mathcal{M}} \downarrow^{\mathbb{Q}\text{-lin}}_{\lceil C \rceil_{\mathcal{M}}} \lceil BC \rceil_{\mathcal{M}}$, and

(iii) $\langle \lceil AC \rceil_{\mathcal{M}}, \lceil BC \rceil_{\mathcal{M}} \rangle \triangleleft \mathcal{M}$.

Lemma 3.2.2. *Let $\mathcal{M} \subseteq \mathcal{N}$ be models in* ***ExpF*** *such that $\mathcal{M} \prec \mathcal{N}$. Let $A, B, C \subseteq \mathcal{M}$ be subsets. Then $A \downarrow_C^{\mathcal{M}} B$ if and only if $A \downarrow_C^{\mathcal{N}} B$.*

Proof. We consider each property (i),(ii) and (iii) from Definition 3.2.1 separately. We have

$$\lceil AC \rceil_{\mathcal{M}} e^{\lceil AC \rceil_{\mathcal{M}}} \underset{\lceil C \rceil_{\mathcal{M}} e^{\lceil C \rceil_{\mathcal{M}}}}{\overset{\mathbf{ACF_0}}{\downarrow}} \lceil BC \rceil_{\mathcal{M}} e^{\lceil BC \rceil_{\mathcal{M}}}$$

and so

$$\lceil AC \rceil_{\mathcal{M}} e^{\lceil AC \rceil_{\mathcal{M}}} \ker(\mathcal{N}) \underset{\lceil C \rceil_{\mathcal{M}} e^{\lceil C \rceil_{\mathcal{M}}} \ker(\mathcal{N})}{\overset{\mathbf{ACF_0}}{\downarrow}} \lceil BC \rceil_{\mathcal{M}} e^{\lceil BC \rceil_{\mathcal{M}}} \ker(\mathcal{N}).$$

By Lemma 2.3.3 for any $X \subseteq \mathcal{M}$ we have $\langle \lceil X \rceil_{\mathcal{M}} \cup \ker(\mathcal{N}) \rangle_{\mathcal{N}} = \lceil X \rceil_{\mathcal{N}}$, therefore

$$\lceil AC \rceil_{\mathcal{N}} e^{\lceil AC \rceil_{\mathcal{N}}} \mathop{\bigcup\!\!\!\!\!\!\downarrow}^{\mathbf{ACF_0}}_{\lceil C \rceil_{\mathcal{N}} e^{\lceil C \rceil_{\mathcal{N}}}} \lceil BC \rceil_{\mathcal{N}} e^{\lceil BC \rceil_{\mathcal{N}}}.$$

Similarly we have

$$\lceil AC \rceil_{\mathcal{M}} \mathop{\downarrow}^{\mathbb{Q}\text{-lin}}_{\lceil C \rceil_{\mathcal{M}}} \lceil BC \rceil_{\mathcal{M}} \Rightarrow \lceil AC \rceil_{\mathcal{M}} \ker(\mathcal{N}) \mathop{\downarrow}^{\mathbb{Q}\text{-lin}}_{\lceil C \rceil_{\mathcal{M}} \ker(\mathcal{N})} \lceil BC \rceil_{\mathcal{M}} \ker(\mathcal{N})$$

and so $\lceil AC \rceil_{\mathcal{N}} \downarrow^{\mathbb{Q}\text{-lin}}_{\lceil C \rceil_{\mathcal{N}}} \lceil BC \rceil_{\mathcal{N}}$. Finally we observe that

$$\begin{aligned} \langle \lceil AC \rceil_{\mathcal{N}}, \lceil BC \rceil_{\mathcal{N}} \rangle_{\mathcal{N}} &= \langle \lceil AC \rceil_{\mathcal{M}} \lceil BC \rceil_{\mathcal{M}} \ker(\mathcal{N}) \rangle_{\mathcal{N}} && \text{by Lemma 2.3.3} \\ &= \langle \langle \lceil AC \rceil_{\mathcal{M}} \lceil BC \rceil_{\mathcal{M}} \rangle_{\mathcal{M}} \ker(\mathcal{N}) \rangle_{\mathcal{N}} && \text{by construction} \\ &= \langle \lceil ABC \rceil_{\mathcal{M}} \ker(\mathcal{N}) \rangle_{\mathcal{N}} && \text{as } A \downarrow^{\mathcal{M}}_{C} B \\ &= \lceil ABC \rceil_{\mathcal{N}} && \text{by Lemma 2.3.3} \end{aligned}$$

and therefore $A \downarrow^{\mathcal{N}}_{C} B$.

Conversely, assume $A \downarrow^{\mathcal{N}}_{C} B$. Then $\lceil AC \rceil_{\mathcal{N}} e^{\lceil AC \rceil_{\mathcal{N}}} \downarrow^{\mathbf{ACF_0}}_{\lceil C \rceil_{\mathcal{N}} e^{\lceil C \rceil_{\mathcal{N}}}} \lceil BC \rceil e^{\lceil BC \rceil_{\mathcal{N}}}$, so by monotonicity of $\downarrow^{\mathbf{ACF_0}}$ we have

$$\lceil AC \rceil_{\mathcal{M}} e^{\lceil AC \rceil_{\mathcal{M}}} \mathop{\downarrow}^{\mathbf{ACF_0}}_{\lceil C \rceil_{\mathcal{N}} e^{\lceil C \rceil_{\mathcal{N}}}} \lceil BC \rceil_{\mathcal{M}} e^{\lceil BC \rceil_{\mathcal{M}}}.$$

Since $\mathcal{M} \prec \mathcal{N}$ implies that $\mathcal{M} \downarrow^{\mathbf{ACF_0}}_{\ker(\mathcal{M})} \ker(\mathcal{N})$, by monotonicity of $\downarrow^{\mathbf{ACF_0}}$ we have $\lceil AC \rceil_{\mathcal{M}} e^{\lceil AC \rceil_{\mathcal{M}}} \downarrow^{\mathbf{ACF_0}}_{\ker(\mathcal{M})} \ker(\mathcal{N})$. Then, again by Lemma 2.3.3, we have

$$\lceil AC \rceil_{\mathcal{M}} e^{\lceil AC \rceil_{\mathcal{M}}} \mathop{\downarrow}^{\mathbf{ACF_0}}_{\lceil C \rceil_{\mathcal{M}} e^{\lceil C \rceil_{\mathcal{M}}}} \lceil C \rceil_{\mathcal{N}} e^{\lceil C \rceil_{\mathcal{N}}}.$$

Therefore by transitivity in $\mathbf{ACF_0}$ we have

$$\lceil AC\rceil_{\mathcal{M}} e^{\lceil AC\rceil_{\mathcal{M}}} \mathop{\downarrow}^{\mathbf{ACF_0}}_{\lceil C\rceil_{\mathcal{M}} e^{\lceil C\rceil_{\mathcal{M}}}} \lceil BC\rceil e^{\lceil BC\rceil_{\mathcal{M}}}.$$

We also have $\lceil AC\rceil_{\mathcal{M}} \downarrow^{\mathbb{Q}\text{-lin}}_{\lceil C\rceil_{\mathcal{N}}} \lceil BC\rceil_{\mathcal{M}}$ by monotonicity of $\mathbb{Q}$-linear independence. Noting that $\mathcal{M} \downarrow^{\mathbf{ACF_0}}_{\ker(\mathcal{M})} \ker(\mathcal{N})$ implies $\mathcal{M} \downarrow^{\mathbb{Q}\text{-lin}}_{\ker(\mathcal{M})} \ker(\mathcal{N})$, by monotonicity of $\mathbb{Q}$-linear independence we obtain $\lceil AC\rceil_{\mathcal{M}} \downarrow^{\mathbb{Q}\text{-lin}}_{\ker(\mathcal{M})} \ker(\mathcal{N})$. Then $\lceil AC\rceil_{\mathcal{M}} \downarrow^{\mathbb{Q}\text{-lin}}_{\lceil C\rceil_{\mathcal{M}}} \lceil C\rceil_{\mathcal{N}}$, and so by transitivity of $\mathbb{Q}$-linear independence we obtain $\lceil AC\rceil_{\mathcal{M}} \downarrow^{\mathbb{Q}\text{-lin}}_{\lceil C\rceil_{\mathcal{M}}} \lceil BC\rceil_{\mathcal{N}}$.

Finally $\langle \lceil AC\rceil_{\mathcal{N}}, \lceil BC\rceil_{\mathcal{N}}\rangle = \lceil \lceil AC\rceil_{\mathcal{N}}, \lceil BC\rceil_{\mathcal{N}}\rceil_{\mathcal{N}}$, so by Lemma 2.3.3 we have $\langle \lceil AC\rceil_{\mathcal{M}} \lceil BC\rceil_{\mathcal{M}} \ker(\mathcal{N})\rangle = \langle \lceil ABC\rceil_{\mathcal{M}} \ker(\mathcal{N})\rangle$. Since $\mathcal{M} \downarrow^{\mathbb{Q}\text{-lin}}_{\ker(\mathcal{M})} \ker(\mathcal{N})$ it follows that $\langle \lceil AC\rceil_{\mathcal{M}}, \lceil BC\rceil_{\mathcal{M}}\rangle = \lceil ABC\rceil_{\mathcal{M}}$. Therefore $A \downarrow^{\mathcal{M}}_{C} B$ as required. $\square$

The above lemma demonstrates that our free-from relation is preserved under semi-strong embeddings and extensions of models, so we may omit the model superscript on $\downarrow$ when the context is clear. Next we give a lemma providing a nice consequence and a useful special case of the definition of $\downarrow$ that will make it easier to use the free-from notion.

Lemma 3.2.3. *Let $\mathcal{M}$ be a model in **ExpF**, and let A, B, C be subsets of $\mathcal{M}$.*

(a) Suppose that $A \downarrow_C B$. Then for all tuples $\bar{a} \in A$ we have $\Delta(\bar{a}/BC) = \Delta(\bar{a}/C)$.

(b) Suppose that C is a strong ELA-subfield of $\mathcal{M}$. Then

$$\lceil AC\rceil e^{\lceil AC\rceil} \mathop{\downarrow}_{C}^{\boldsymbol{ACF_0}} \lceil BC\rceil e^{\lceil BC\rceil} \Rightarrow \lceil AC\rceil \mathop{\downarrow}_{C}^{\mathbb{Q}\text{-}lin} \lceil BC\rceil$$

Proof. Note that $A \downarrow_C B$ if and only if $\lceil AC\rceil \downarrow_{\lceil C\rceil} \lceil BC\rceil$, so without loss of generality we may assume that $A = \lceil AC\rceil, B = \lceil BC\rceil$, and $C = \lceil C\rceil$.

(a) Let $\bar{a} \in A$. Then as $A \downarrow_C B$ we have $\operatorname{td}(\bar{a}, e^{\bar{a}}/Be^B \ker(\mathcal{M})) = \operatorname{td}(\bar{a}, e^{\bar{a}}/Ce^C \ker(\mathcal{M}))$ and $\operatorname{ldim}_{\mathbb{Q}}(\bar{a}/B \ker(\mathcal{M})) = \operatorname{ldim}_{\mathbb{Q}}(\bar{a}/C \ker(\mathcal{M}))$. By the definition of the predimension, the result follows.

(b) For $\mathbb{Q}$-vector spaces A, B, C we have $A \downarrow_C^{\mathbb{Q}\text{-lin}} B$ if and only if $A \cap B = C$, and we shall use the latter statement. Certainly $A \cap B$ contains C. If $a \in (A \cap B) \setminus C$ then $\operatorname{td}(a/C) = 1$ since C is an algebraically closed field. But also $a \in B$ which implies that $\operatorname{td}(a/Be^B) = 0$, and therefore by monotonicity $Ae^A \not\downarrow_C^{\mathbf{ACF_0}} Be^B$.

□

Proposition 3.2.4. *Let $\mathcal{M}$ be a model in **ExpF**. Then $\downarrow^{\mathcal{M}}$ is a symmetric pre-independence relation.*

Proof. We may assume that $A, B, C, X \subseteq \mathcal{M}$ are subsets such that $\lceil AC\rceil = A$, $\lceil BCX\rceil = B$, $\lceil C\rceil = C$ and $\lceil XC\rceil = X$ in $\mathcal{M}$.

7. *Symmetry* We have $A \downarrow_C B$ and so $Ae^A \downarrow_{Ce^C}^{\mathbf{ACF_0}} Be^B$. By symmetry of non-forking independence in $\mathbf{ACF_0}$ we have $Be^B \downarrow_{Ce^C}^{\mathbf{ACF_0}} Ae^A$. We also know that $A \downarrow_C B$ implies that $A \downarrow_C^{\mathbb{Q}\text{-lin}} B$, and by symmetry of non-forking independence for vector spaces we have $B \downarrow_C^{\mathbb{Q}\text{-lin}} A$. It is also clearly the case that $\langle B, A\rangle = \langle A, B\rangle \triangleleft \mathcal{M}$ and so $B \downarrow_C A$.

1. *Monotonicity* If $A \downarrow_C B$ then $Ae^A \mathop{\smile\!\!\!\!\!\!|}\,^{\mathbf{ACF_0}}_{Ce^C} Be^B$ by the definition of $A \downarrow_C B$. Given that $X \subseteq B$, by monotonicity of $\mathop{\smile\!\!\!\!\!\!|}\,^{\mathbf{ACF_0}}$ we have $Ae^A \mathop{\smile\!\!\!\!\!\!|}\,^{\mathbf{ACF_0}}_{Ce^C} Xe^X$. The definition also tells us that $A \mathop{\smile\!\!\!\!\!\!|}\,^{\mathbb{Q}\text{-lin}}_C B$, so by monotonicity of $\mathop{\smile\!\!\!\!\!\!|}\,^{\mathbb{Q}\text{-lin}}$ we have $A \mathop{\smile\!\!\!\!\!\!|}\,^{\mathbb{Q}\text{-lin}}_C X$.

   We now demonstrate that $\langle AX \rangle \lhd \langle AB \rangle$. Let $\bar{x} \in \langle AB \rangle$. Then there exist $\bar{a} \in A$ and $\bar{b} \in B$ $\mathbb{Q}$-linearly independent over $\bar{a}$ such that $\langle \bar{x} \rangle = \langle \bar{a}\bar{b} \rangle$. Then $\Delta(\bar{x}/AX) = \Delta(\bar{a}\bar{b}/AX) = \Delta(\bar{b}/AX)$. Since Δ has finite character, we have $\Delta(\bar{b}/AX) = \Delta(\bar{b}/\bar{d}A)$ for some finite tuple $\bar{d} \in X$. Extending $\bar{d}$ if necessary we may assume that $\bar{d}C \lhd X$. By additivity of the predimension again, we have

$$\Delta(\bar{x}/AX) = \Delta(\bar{b}\bar{d}/A) - \Delta(\bar{d}/A)$$

   By symmetry of $\downarrow$ and Lemma 3.2.3(a) we have $\Delta(\bar{y}/A) = \Delta(\bar{y}/C)$ for any $\bar{y} \in B$. But then $\Delta(\bar{b}\bar{d}/A) = \Delta(\bar{b}\bar{d}/C)$ and $\Delta(\bar{d}/A) = \Delta(\bar{d}/C)$. By the above, and the fact that $\bar{d}C \lhd X$, we have

$$\Delta(\bar{x}/AX) = \Delta(\bar{b}\bar{d}/C) - \Delta(\bar{d}/C) \geq 0$$

   Therefore $\langle AX \rangle \lhd \langle AB \rangle$ and since $\langle AB \rangle \lhd \mathcal{M}$, by transitivity of strong embeddings we have $\langle AX \rangle \lhd \mathcal{M}$. Thus, $A \downarrow_C X$.

2. *Transitivity* Suppose that $A \downarrow_C X$ and $A \downarrow_X B$. Then $Ae^A \mathop{\smile\!\!\!\!\!\!|}\,^{\mathbf{ACF_0}}_{Xe^X} Be^B$ and $Ae^A \mathop{\smile\!\!\!\!\!\!|}\,^{\mathbf{ACF_0}}_{Ce^C} Xe^X$, which by transitivity of non-forking independence in $\mathbf{ACF_0}$ implies that $Ae^A \mathop{\smile\!\!\!\!\!\!|}\,^{\mathbf{ACF_0}}_{Ce^C} Be^B$. Meanwhile we have $A \mathop{\smile\!\!\!\!\!\!|}\,^{\mathbb{Q}\text{-lin}}_X B$ and $A \mathop{\smile\!\!\!\!\!\!|}\,^{\mathbb{Q}\text{-lin}}_C X$, from which it follows that $A \mathop{\smile\!\!\!\!\!\!|}\,^{\mathbb{Q}\text{-lin}}_C B$ by transitivity of non-forking independence for vector

spaces. Also since $A \downarrow_X B$ we have $\langle AB \rangle \lhd \mathcal{M}$, and therefore $A \downarrow_C B$.

Conversely, suppose that $A \downarrow_C B$ with $C \subseteq X \subseteq B$. Then $Ae^A \mathop{\smile\!\!\!\!\!\!\mid}\,^{\mathbf{ACF_0}}_{Ce^C} Be^B$ and by transitivity of non-forking independence in $\mathbf{ACF_0}$ it follows that $Ae^A \mathop{\smile\!\!\!\!\!\!\mid}_{Ce^C} Xe^X$ and $Ae^A \mathop{\smile\!\!\!\!\!\!\mid}_{Xe^X} Be^B$. We also have $A \mathop{\smile\!\!\!\!\!\!\mid}\,^{\mathbb{Q}\text{-lin}}_C B$ which implies that $A \mathop{\smile\!\!\!\!\!\!\mid}\,^{\mathbb{Q}\text{-lin}}_C B$ and $A \mathop{\smile\!\!\!\!\!\!\mid}\,^{\mathbb{Q}\text{-lin}}_X B$ by transitivity of $\mathbb{Q}$-linear independence. Since $A \downarrow_C B$ we also have $A \downarrow_C X$ by monotonicity, which means that $\langle AX \rangle \lhd \mathcal{M}$. Hence we have $Ae^A \mathop{\smile\!\!\!\!\!\!\mid}\,^{\mathbf{ACF_0}}_{Ce^C} Xe^X$, $A \mathop{\smile\!\!\!\!\!\!\mid}\,^{\mathbb{Q}\text{-lin}}_C X$ and $\langle AX \rangle \lhd \mathcal{M}$, so $A \downarrow_C X$. We also have $Ae^A \mathop{\smile\!\!\!\!\!\!\mid}\,^{\mathbf{ACF_0}}_{Xe^X} Be^B$, $A \mathop{\smile\!\!\!\!\!\!\mid}\,^{\mathbb{Q}\text{-lin}}_X B$ and $\langle AB \rangle \lhd \mathcal{M}$, and therefore $A \downarrow_X B$.

3. *Invariance* By definition $A \downarrow_C B$ implies that $Ae^A \mathop{\smile\!\!\!\!\!\!\mid}\,^{\mathbf{ACF_0}}_{Ce^C} Be^B$. For any automorphism $\sigma \in \mathrm{Aut}(\mathcal{M})$, we have $\sigma(A)e^{\sigma(A)} \mathop{\smile\!\!\!\!\!\!\mid}\,^{\mathbf{ACF_0}}_{\sigma(C)e^{\sigma(C)}} \sigma(B)e^{\sigma(B)}$ by automorphism invariance of $\mathop{\smile\!\!\!\!\!\!\mid}\,^{\mathbf{ACF_0}}$. By definition, $A \downarrow_C B$ also implies that $A \mathop{\smile\!\!\!\!\!\!\mid}\,^{\mathbb{Q}\text{-lin}}_C B$ and so by automorphism invariance of $\mathop{\smile\!\!\!\!\!\!\mid}\,^{\mathbb{Q}\text{-lin}}$ we obtain $\sigma(A) \mathop{\smile\!\!\!\!\!\!\mid}\,^{\mathbb{Q}\text{-lin}}_{\sigma(C)} \sigma(B)$.

   Property (iii) of $A \downarrow_C B$ tells us that we have $\langle AB \rangle \lhd \mathcal{M}$. Suppose that σ is an automorphism of $\mathcal{M}$ such that $\langle \sigma(AB) \rangle$ is not strong in $\mathcal{M}$. Then we can find $\bar{c} \in \lceil \sigma(AB) \rceil \setminus \langle \sigma(AB) \rangle$, that is with $\Delta(\bar{c}/\sigma(AB)) < 0$. Then $\bar{c} = \sigma(\bar{d})$ for some $\bar{d} \in \mathcal{M}$ so $\Delta(\sigma(\bar{d})/\sigma(AB)) < 0$. Since transcendence degree and linear dimension are automorphism invariant, so is the predimension, so $\Delta(\bar{d}/AB) < 0$, contradicting that $\langle AB \rangle \lhd \mathcal{M}$. Therefore $\langle \sigma(AB) \rangle \lhd \mathcal{M}$, and so we have $\sigma(A) \downarrow_{\sigma(C)} \sigma(B)$.

4. *Finite character* Suppose $A \downarrow_C B$ and $\bar{a} \in A$. By symmetry and monotonicity we have left-monotonicity, so we have $\bar{a} \downarrow_C B$.

   Conversely, if $A \not\downarrow_C B$ then at least one of the three properties from the definition will fail. If (i) fails, then for some $\bar{a} \in A$ and $\bar{x} \in \lceil \bar{a}C \rceil$ we have $\bar{x}e^{\bar{x}} \not\perp^{\mathbf{ACF_0}}_{Ce^C} Be^B$ by finite character of $\perp^{\mathbf{ACF_0}}$. By left-monotonicity, $\lceil \bar{a}C \rceil e^{\lceil \bar{a}C \rceil} \not\perp^{\mathbf{ACF_0}}_{Ce^C} Be^B$ and so $\bar{a} \not\downarrow_C B$. Alternatively if (ii) fails, then there exists $\bar{a} \in A$ and $\bar{x} \in \lceil \bar{a}C \rceil$ such that $\bar{a} \not\perp^{\mathbb{Q}\text{-lin}}_C B$ by finite character of $\perp^{\mathbb{Q}\text{-lin}}$. By monotonicity of $\perp^{\mathbb{Q}\text{-lin}}$ we have $\lceil \bar{a}C \rceil \not\perp^{\mathbb{Q}\text{-lin}}_C B$ and hence $\bar{a} \not\downarrow_C B$. Finally if (iii) fails, then there exists $\bar{c} \in \mathcal{M}$ such that $\Delta(\bar{c}/AB) < 0$. Then by finite character of the predimension, $\Delta(\bar{c}/\bar{a}B) < 0$ for some $\bar{a} \in A$. Then $\langle \bar{a}B \rangle$ is not strong in $\mathcal{M}$, and so $\bar{a} \not\downarrow_C B$.

$\square$

We conclude this section with more useful results about our free-from relation.

Lemma 3.2.5. *Let $\mathcal{M}$ be a model in **ExpF** and let A and C be subsets of $\mathcal{M}$. Suppose that $B_0 \subseteq B_1 \subseteq \cdots \subseteq B_\gamma = \bigcup_{\alpha<\gamma} B_\alpha$ is a chain of subsets of $\mathcal{M}$ with $A \downarrow_C B_\alpha$ for each $\alpha < \gamma$, where γ is a limit ordinal. Then $A \downarrow_C B_\gamma$.*

Proof. If $A \not\downarrow_C B_\gamma$, then by finite character and symmetry of $\downarrow$ we have $A \not\downarrow_C \bar{d}$ for some finite $\bar{d} \in B_\gamma$. For some $\alpha < \gamma$ we have $\bar{d} \in B_\alpha$, therefore by monotonicity $A \not\downarrow_C B_\alpha$. $\square$

Proposition 3.2.6. *Let $\mathcal{M}$ be a model in **ExpF** and let A, B, C be subsets of $\mathcal{M}$. Then $A \downarrow_C B$ iff $A \downarrow_C \lceil B \rceil^{ELA}$.*

Proof. We may assume that $B = \lceil BC \rceil$, $A = \lceil AC \rceil$ and $C = \lceil C \rceil$. Right to left is immediate by monotonicity. For the other direction, let $\gamma = |B| + \aleph_0$ and enumerate $\lceil B \rceil^{\mathrm{ELA}}$ as $B \cup \{b_\alpha : \alpha < \gamma\}$ such that for all $\alpha < \gamma$ either

b_α or e^{b_α} is field-theoretically algebraic over $B \cup \exp(B) \cup \{b_\beta, e^{b_\beta} : \beta < \alpha\}$, which is possible by the definition of the ELA-closure. Define $B_0 = B$. For all ordinals $\alpha < \gamma$ define $B_{\alpha+1} = \langle B_\alpha, b_\alpha \rangle$. For $\delta < \gamma$ a limit ordinal define $B_\delta = \bigcup_{\beta<\delta} B_\beta$. By Lemma 3.2.5 it suffices to show that $A \downarrow_C B_\alpha$ for every ordinal $\alpha < \gamma$.

We shall prove that for every $\alpha < \gamma$ we have $B_\alpha \lhd \mathcal{M}$, $D_\alpha = \langle AB_\alpha \rangle \lhd \mathcal{M}$, and we will use these to show that $A \downarrow_C B_\alpha$. We proceed by induction. Note that $B_0 = \lceil B \rceil \lhd \mathcal{M}$ and by hypothesis $A \downarrow_C B_0$, which immediately implies that $D_0 \lhd \mathcal{M}$. Suppose that $A \downarrow_C B_\alpha$, and that $B_\alpha \lhd \mathcal{M}$ for some $\alpha < \gamma$. If $b_\alpha \in B_\alpha$ then set $B_{\alpha+1} = B_\alpha$ and so $A \downarrow_C B_{\alpha+1}$ is immediate. If $b_\alpha \notin B_\alpha$, then setting $B_{\alpha+1} = \langle B_\alpha b_\alpha \rangle$ we have $\mathrm{ldim}_{\mathbb{Q}}(B_{\alpha+1}/B_\alpha) = 1$. At least one of b_α, e^{b_α} is field-theoretically algebraic over $B_\alpha \cup \exp(B_\alpha)$, so $\mathrm{td}(B_{\alpha+1}e^{B_{\alpha+1}}/B_\alpha e^{B_\alpha}) = \mathrm{td}(b_{\alpha+1}, e^{b_{\alpha+1}}/B_\alpha e^{B_\alpha}) \leq 1$, and thus $\Delta(B_{\alpha+1}/B_\alpha) \leq 1 - 1 = 0$. But $B_\alpha \lhd \mathcal{M}$ so $\Delta(B_{\alpha+1}/B_\alpha) = 0$, and thus $\mathrm{td}(B_{\alpha+1}e^{B_{\alpha+1}}/B_\alpha e^{B_\alpha}) = 1$. If $\bar{d} \in \mathcal{M}$, then

$$\Delta(\bar{d}/B_{\alpha+1}) = \Delta(\bar{d}/b_\alpha, B_\alpha) = \Delta(\bar{d}b_\alpha/B_\alpha) - \Delta(b_\alpha/B_\alpha) = \Delta(\bar{d}b_\alpha/B_\alpha) \geq 0$$

since $B_\alpha \lhd \mathcal{M}$. Therefore $B_{\alpha+1}$ is strong in $\mathcal{M}$.

We prove by induction that if $D_\alpha \lhd \mathcal{M}$ then $D_{\alpha+1} \lhd \mathcal{M}$. Furthermore $D_{\alpha+1} = \langle D_\alpha, b_\alpha \rangle$, $\mathrm{td}(D_{\alpha+1}e^{D_{\alpha+1}}/D_\alpha e^{D_\alpha}) = \mathrm{ldim}_{\mathbb{Q}}(D_{\alpha+1}/D_\alpha) = 1$, and $\Delta(D_{\alpha+1}/D_\alpha) = 0$.

Suppose that $D_\alpha \lhd \mathcal{M}$. We have $B_{\alpha+1} \neq B_\alpha$, so $D_{\alpha+1} = \langle D_\alpha, b_\alpha \rangle$ so $\mathrm{ldim}_{\mathbb{Q}}(D_{\alpha+1}/D_\alpha) = 1$ and $\mathrm{td}(D_{\alpha+1}e^{D_{\alpha+1}}/D_\alpha e^{D_\alpha}) = \mathrm{td}(b_\alpha, e^{b_\alpha}/D_\alpha e^{D_\alpha}) \leq 1$ since one of b_α, e^b_α is algebraic over B_α and thus over D_α. Therefore

$\Delta(D_{\alpha+1}/D_\alpha) \leq 0$, but $D_\alpha \triangleleft \mathcal{M}$ so $\Delta(D_{\alpha+1}/D_\alpha) = 0$. If $\bar{d} \in \mathcal{M}$ then

$$\Delta(\bar{d}/D_{\alpha+1}) = \Delta(\bar{d}/D_\alpha, b_\alpha) = \Delta(\bar{d}, b_\alpha/D_\alpha) - \Delta(b_\alpha/D_\alpha) \geq 0$$

so we have $D_{\alpha+1} \triangleleft \mathcal{M}$.

We have $\mathrm{td}(D_{\alpha+1}e^{D_{\alpha+1}}/D_\alpha e^{D_\alpha}) = \mathrm{td}(B_{\alpha+1}e^{B_{\alpha+1}}/B_\alpha e^{B_\alpha})$ which implies that $Ae^A \downarrow^{\mathbf{ACF_0}}_{B_\alpha e^{B_\alpha}} B_{\alpha+1}e^{B_{\alpha+1}}$ by monotonicity and symmetry. Since $A \downarrow_C B_\alpha$ we have $Ae^A \downarrow^{\mathbf{ACF_0}}_{Ce^C} B_\alpha e^{B_\alpha}$, and so by transitivity of non-forking independence in $\mathbf{ACF_0}$ it follows that $Ae^A \downarrow^{\mathbf{ACF_0}}_{Ce^C} B_{\alpha+1}e^{B_{\alpha+1}}$. We also have $\mathrm{ldim}_{\mathbb{Q}}(D_{\alpha+1}/D_\alpha) = \mathrm{ldim}_{\mathbb{Q}}(B_{\alpha+1}/B_\alpha)$, so applying monotonicity again we obtain $A \downarrow^{\mathbb{Q}\text{-lin}}_{B_\alpha} B_{\alpha+1}$. By transitivity of non-forking independence for $\mathbb{Q}$-vector spaces we have $A \downarrow^{\mathbb{Q}\text{-lin}}_{C} B_{\alpha+1}$. Therefore $A \downarrow_C B_{\alpha+1}$ and we are done. □

3.3 Independence in ECF, $\mathbf{ECF_{SK}}$ and $\mathbf{ECF_{SK,CCP}}$

We now develop this pre-independence relation for **ExpF** into independence relations specific to the classes **ExpF**, **ECF**, $\mathbf{ECF_{SK}}$ and $\mathbf{ECF_{SK,CCP}}$. We prove that these relations are symmetric independence relations, some satisfying additional properties.

Definition 3.3.1. Let $\mathcal{C}$ be one of $\mathbf{ECF}, \mathbf{ECF_{SK}}$ or $\mathbf{ECF_{SK,CCP}}$. Let $\mathcal{M}$ be a model in $\mathcal{C}$ and let A, B and C be subsets of $\mathcal{M}$. Respective to the class $\mathcal{C}$, we define $A \downarrow^{\mathcal{C},\mathcal{M}}_C B$ in the following way.

- We write $A \downarrow^{\mathbf{ECF},\mathcal{M}}_C B$ if $A \downarrow^{\mathcal{M}}_{\lceil C \rceil^{\mathrm{ELA}}} B$.

- We write $A \downfree_C^{\mathbf{ECF_{SK}},\mathcal{M}} B$ if $A \downarrow^{\mathcal{M}}_{\lceil C\rceil^{\mathrm{ELA}}} B$.
- We write $A \downfree_C^{\mathbf{ECF_{SK,CCP}},\mathcal{M}} B$ if $A \downarrow^{\mathcal{M}}_{\mathrm{ecl}^{\mathcal{M}}(C)} B$.

We say A is $\mathcal{C}$-*independent* from B over C in $\mathcal{M}$.

We may drop the '$\mathcal{C}$' superscript and prefix if the context is clear, for example within proofs.

Proposition 3.3.2. *Let $\mathcal{M} \leq \mathcal{N}$ be models of* ***ECF****, and suppose that $A, B, C \subseteq \mathcal{M}$ are subsets. Then $A \downfree_C^{\boldsymbol{ECF},\mathcal{M}} B$ iff $A \downfree_C^{\boldsymbol{ECF},\mathcal{N}} B$.*

Proof. We observe that $A \downfree_C^{\mathbf{ECF},\mathcal{M}} B$ if and only if

$$\lceil A\lceil C\rceil^{\mathrm{ELA}}_{\mathcal{M}}\rceil_{\mathcal{M}} \underset{\lceil C\rceil^{\mathrm{ELA}}_{\mathcal{M}}}{\overset{\mathbf{ECF},\mathcal{M}}{\downfree}} \lceil B\lceil C\rceil^{\mathrm{ELA}}_{\mathcal{M}}\rceil_{\mathcal{M}},$$

so we may assume that $A = \lceil A\lceil C\rceil^{\mathrm{ELA}}_{\mathcal{M}}\rceil_{\mathcal{M}}$ and $B = \lceil B\lceil C\rceil^{\mathrm{ELA}}_{\mathcal{M}}\rceil$. By definition $A \downfree_C^{\mathbf{ECF},\mathcal{M}} B$ implies that $A \downarrow^{\mathcal{M}}_{\lceil C\rceil^{\mathrm{ELA}}_{\mathcal{M}}} B$, which by Lemma 3.2.2 means that $A \downarrow^{\mathcal{N}}_{\lceil C\rceil^{\mathrm{ELA}}_{\mathcal{M}}} B$. Equivalently

$$\lceil A\rceil_{\mathcal{N}} \downarrow^{\mathcal{N}}_{\lceil C\rceil^{\mathrm{ELA}}_{\mathcal{M}}} \lceil B\rceil_{\mathcal{N}}$$

from which it trivially follows that

$$\lceil A\rceil_{\mathcal{N}} \cup \lceil C\rceil^{\mathrm{ELA}}_{\mathcal{N}} \downarrow^{\mathcal{N}}_{\lceil C\rceil^{\mathrm{ELA}}_{\mathcal{N}}} \lceil B\rceil_{\mathcal{N}} \cup \lceil C\rceil^{\mathrm{ELA}}_{\mathcal{N}}$$

By symmetry and Proposition 3.2.6 this implies that

$$\lceil\lceil A\rceil_{\mathcal{N}} \cup \lceil C\rceil^{\mathrm{ELA}}_{\mathcal{N}}\rceil^{\mathrm{ELA}}_{\mathcal{N}} \downarrow^{\mathcal{N}}_{\lceil C\rceil^{\mathrm{ELA}}_{\mathcal{N}}} \lceil\lceil B\rceil_{\mathcal{N}} \cup \lceil C\rceil^{\mathrm{ELA}}_{\mathcal{N}}\rceil^{\mathrm{ELA}}_{\mathcal{N}}$$

However, $\lceil \lceil A \rceil_{\mathcal{N}} \cup \lceil C \rceil_{\mathcal{N}}^{\mathrm{ELA}} \rceil_{\mathcal{N}}^{\mathrm{ELA}} = \lceil A \rceil_{\mathcal{N}}^{\mathrm{ELA}}$ and $\lceil \lceil B \rceil_{\mathcal{N}} \cup \lceil C \rceil_{\mathcal{N}}^{\mathrm{ELA}} \rceil_{\mathcal{N}}^{\mathrm{ELA}} = \lceil B \rceil_{\mathcal{N}}^{\mathrm{ELA}}$. Therefore,

$$\lceil A \lceil C \rceil_{\mathcal{N}}^{\mathrm{ELA}} \rceil_{\mathcal{N}}^{\mathrm{ELA}} \downarrow^{\mathcal{N}}_{\lceil C \rceil_{\mathcal{N}}^{\mathrm{ELA}}} \lceil B \lceil C \rceil_{\mathcal{N}}^{\mathrm{ELA}} \rceil_{\mathcal{N}}^{\mathrm{ELA}}$$

hence by monotonicity of $\downarrow$ we have $\lceil A \lceil C \rceil_{\mathcal{N}}^{\mathrm{ELA}} \rceil_{\mathcal{N}} \downarrow^{\mathcal{N}}_{\lceil C \rceil_{\mathcal{N}}^{\mathrm{ELA}}} \lceil B \lceil C \rceil_{\mathcal{N}}^{\mathrm{ELA}} \rceil_{\mathcal{N}}$, and so $A \mathop{\smile\!\!\!\!\!\mid}_C^{\mathbf{ECF},\mathcal{N}} B$.

Conversely, if $A \mathop{\smile\!\!\!\!\!\mid}_C^{\mathbf{ECF},\mathcal{N}} B$ then by definition of $\downarrow$ and monotonicity we have

$$\lceil A \lceil C \rceil_{\mathcal{M}}^{\mathrm{ELA}} \rceil_{\mathcal{M}} \downarrow^{\mathcal{N}}_{\lceil C \rceil_{\mathcal{N}}^{\mathrm{ELA}}} \lceil B \lceil C \rceil_{\mathcal{M}}^{\mathrm{ELA}} \rceil_{\mathcal{M}}$$

Trivially $\lceil A \lceil C \rceil_{\mathcal{M}}^{\mathrm{ELA}} \rceil_{\mathcal{M}} \downarrow^{\mathcal{N}}_{\lceil C \rceil_{\mathcal{M}}^{\mathrm{ELA}}} \lceil C \rceil_{\mathcal{M}}^{\mathrm{ELA}}$ and we have $\lceil \lceil C \rceil_{\mathcal{M}}^{\mathrm{ELA}} \rceil_{\mathcal{N}}^{\mathrm{ELA}} = \lceil C \rceil_{\mathcal{N}}^{\mathrm{ELA}}$ so by Proposition 3.2.6 we observe that

$$\lceil A \lceil C \rceil_{\mathcal{M}}^{\mathrm{ELA}} \rceil_{\mathcal{M}} \downarrow^{\mathcal{N}}_{\lceil C \rceil_{\mathcal{M}}^{\mathrm{ELA}}} \lceil C \rceil_{\mathcal{N}}^{\mathrm{ELA}}$$

Applying transitivity of $\downarrow$ we obtain

$$\lceil A \lceil C \rceil_{\mathcal{M}}^{\mathrm{ELA}} \rceil_{\mathcal{M}} \downarrow^{\mathcal{N}}_{\lceil C \rceil_{\mathcal{M}}^{\mathrm{ELA}}} \lceil B \lceil C \rceil_{\mathcal{M}}^{\mathrm{ELA}} \rceil_{\mathcal{M}}$$

By Lemma 3.2.2 it follows that $\lceil A \lceil C \rceil_{\mathcal{M}}^{\mathrm{ELA}} \rceil_{\mathcal{M}} \downarrow^{\mathcal{M}}_{\lceil C \rceil_{\mathcal{M}}^{\mathrm{ELA}}} \lceil B \lceil C \rceil_{\mathcal{M}}^{\mathrm{ELA}} \rceil_{\mathcal{M}}$ and so $A \mathop{\smile\!\!\!\!\!\mid}_C^{\mathbf{ECF},\mathcal{M}} B$ as required. □

The above proposition tells us that **ECF**-independence is preserved under $\leq$-extensions of models in **ECF**. We may therefore drop the model superscript when the context is clear, writing $\mathop{\smile\!\!\!\!\!\mid}^{\mathbf{ECF}}$ for **ECF**-independence.

Corollary 3.3.3. *Let $\mathcal{M} \lhd \mathcal{N}$ be models in $\boldsymbol{ECF_{SK}}$, and let A, B, C be subsets of $\mathcal{M}$. Then $A \mathop{\smile\!\!\!\!\!\mid}_C^{\boldsymbol{ECF_{SK}},\mathcal{M}} B$ if and only if $A \mathop{\smile\!\!\!\!\!\mid}_C^{\boldsymbol{ECF_{SK}},\mathcal{N}} B$.*

Corollary 3.3.4. *Let $\mathcal{M} \subseteq^{cl} \mathcal{N}$ be models in $\boldsymbol{ECF_{SK,CCP}}$ and let A, B, C be subsets of $\mathcal{M}$. Then $A \mathop{\smile\!\!\!\!\!\!\,|}_C^{\boldsymbol{ECF_{SK,CCP}},\mathcal{M}} B$ if and only if $A \mathop{\smile\!\!\!\!\!\!\,|}_C^{\boldsymbol{ECF_{SK,CCP}},\mathcal{N}} B$.*

The proof of Corollary 3.3.3 is the same as that of Proposition 3.3.2 with the added simplification that $\lceil X \rceil_{\mathcal{M}} = \lceil X \rceil_{\mathcal{N}}$ for all subsets $X \subseteq \mathcal{M}$. It follows that $\mathbf{ECF_{SK}}$-independence is preserved under strong extensions in **ECF**, so we may drop the model subscript and write $\mathop{\smile\!\!\!\!\!\!\,|}^{\mathbf{ECF_{SK}}}$ when the context is clear. Corollary 3.3.4 follows immediately from Proposition 3.3.2 and the observation that for any subset $C \subseteq \mathcal{M}$ we have $\mathrm{ecl}^{\mathcal{M}}(C) = \mathrm{ecl}^{\mathcal{N}}(C)$ since $\mathrm{ecl}^{\mathcal{N}}(\mathcal{M}) = \mathcal{M}$. Hence $\mathbf{ECF_{SK,CCP}}$-independence is preserved under closed embeddings in $\mathbf{ECF_{SK,CCP}}$ so we may write $\mathop{\smile\!\!\!\!\!\!\,|}^{\mathbf{ECF_{SK,CCP}}}$ for $\mathop{\smile\!\!\!\!\!\!\,|}^{\mathbf{ECF_{SK,CCP}},\mathcal{M}}$.

Before we prove more facts about these relations we make the following observations which shall shorten future proofs.

Lemma 3.3.5. *Let $\mathcal{M}$ be a model in **ExpF** and suppose that we have A, B, C subsets of $\mathcal{M}$ such that $C = \lceil C \rceil^{ELA}$ and $\lceil AC \rceil e^{\lceil AC \rceil} \mathop{\smile\!\!\!\!\!\!\,|}_C^{\boldsymbol{ACF_0}} \lceil BC \rceil e^{\lceil BC \rceil}$. Then $\lceil AC \rceil \mathop{\smile\!\!\!\!\!\!\,|}_C^{\mathbb{Q}\text{-}lin} \lceil BC \rceil$.*

Proof. Since $\bar{a}, e^{\bar{a}} \mathop{\smile\!\!\!\!\!\!\,|}_C^{\mathbf{ACF_0}} \lceil BC \rceil e^{\lceil BC \rceil}$ for any $\bar{a} \in \lceil AC \rceil$, by monotonicity and symmetry we have $\bar{a} \mathop{\smile\!\!\!\!\!\!\,|}_C^{\mathbf{ACF_0}} \lceil BC \rceil$. By Lemma 3.2.3(ii) it follows that $\bar{a} \mathop{\smile\!\!\!\!\!\!\,|}_C^{\mathbb{Q}\text{-lin}} \lceil BC \rceil$. Therefore $\lceil AC \rceil \mathop{\smile\!\!\!\!\!\!\,|}_C^{\mathbb{Q}\text{-lin}} \lceil BC \rceil$ by finite character of $\mathop{\smile\!\!\!\!\!\!\,|}^{\mathbb{Q}\text{-lin}}$. □

Corollary 3.3.6. *Let $\mathcal{C}$ be one of **ECF** or $\boldsymbol{ECF_{SK}}$ and let $\mathcal{M}$ be a model in $\mathcal{C}$. Suppose A, B, C are subsets of $\mathcal{M}$ with $C = \lceil C \rceil^{ELA}$ and $\langle ABC \rangle \triangleleft \mathcal{M}$. Then $A \mathop{\smile\!\!\!\!\!\!\,|}_C^{\mathcal{C},\mathcal{M}} B$ if and only if $\lceil AC \rceil e^{\lceil AC \rceil} \mathop{\smile\!\!\!\!\!\!\,|}_C^{\boldsymbol{ACF_0}} \lceil BC \rceil e^{\lceil BC \rceil}$.*

Proof. Immediate from Lemma 3.3.5. □

3.4 Pregeometric independence in exponential fields

Let $\mathcal{C}$ be one of the classes $\mathbf{ECF}$, $\mathbf{ECF_{SK}}$, $\mathbf{ECF_{SK,CCP}}$ and let $\mathcal{M}$ be a model in $\mathcal{C}$. Then $(\mathcal{M}; \mathrm{ecl})$ is a pregeometry, and inherits an independence relation given by

$$A \mathop{\downarrow}_{C}^{\mathrm{ecl}} B \text{ if and only if for each finite tuple } \bar{a} \in A \text{ we have } \mathrm{etd}(\bar{a}/B) = \mathrm{etd}(\bar{a}/B \cup C).$$

Proposition 3.4.1. *Let $\mathcal{C}$ be one of **ECF**, **ECF_{SK}**, **$ECF_{SK,CCP}$** and let $\mathcal{M}$ be a model of $\mathcal{C}$. Then for subsets $A, B, C \subseteq \mathcal{M}$ we have*

$$A \mathop{\downarrow}_{C}^{ecl} B \textit{ if and only if } A \mathop{\downarrow}_{ecl(C)}^{\mathcal{C},\mathcal{M}} B$$

Proof. We may assume that $C = \mathrm{ecl}(C)$, $A = \lceil AC \rceil$ and $B = \lceil BC \rceil$, so in particular $C \supseteq \ker(\mathcal{M})$. The proof is then the same for $\mathcal{C}$ equal to any of the classes $\mathbf{ECF}, \mathbf{ECF_{SK}}$ or $\mathbf{ECF_{SK,CCP}}$.

First suppose that $A \not\mathop{\downarrow}_{C}^{\mathrm{ecl}} B$. Then there exists $\bar{a} \in A$ such that $\bar{a}B \triangleleft \mathcal{M}$ and $\mathrm{etd}(\bar{a}/B) < \mathrm{etd}(\bar{a}/C)$. By Fact 2.2.10 we have

$$\mathrm{etd}(\bar{a}/C) \leq \Delta(\bar{a}/C) = \mathrm{td}(\bar{a}, e^{\bar{a}}/C) - \mathrm{ldim}_{\mathbb{Q}}(\bar{a}/C).$$

Now $\bar{a}B$ is strong in $\mathcal{M}$, so by Corollary 2.2.11 we have $\mathrm{etd}(\bar{a}/B) = \Delta(\bar{a}/B)$. Therefore $\mathrm{td}(\bar{a}, e^{\bar{a}}/Be^{B}) - \mathrm{ldim}_{\mathbb{Q}}(\bar{a}/B) < \mathrm{td}(\bar{a}, e^{\bar{a}}/C) - \mathrm{ldim}_{\mathbb{Q}}(\bar{a}/C)$. Since $\mathrm{ldim}_{\mathbb{Q}}(\bar{a}/B) \leq \mathrm{ldim}_{\mathbb{Q}}(\bar{a}/C)$ it follows that $\mathrm{td}(\bar{a}, e^{\bar{a}}/Be^{B}) < \mathrm{td}(\bar{a}, e^{\bar{a}}/C)$. Therefore $A \not\mathop{\downarrow}_{C}^{\mathbf{ACF_0}} B$ and so $A \not\mathop{\downarrow}_{C} B$.

Conversely suppose that $A \not\downfree_C B$. Then by Corollary 3.3.6 either $Ae^A \not\downfree^{\mathbf{ACF_0}}_C Be^B$ or AB is not strong in $\mathcal{M}$. If the former, then there exists $\bar{a} \in A$ such that $\bar{a}C \lhd A$ and $\operatorname{td}(\bar{a}, e^{\bar{a}}/Be^B) < \operatorname{td}(\bar{a}, e^{\bar{a}}/C)$. If $\operatorname{ldim}_{\mathbb{Q}}(\bar{a}/C) > \operatorname{ldim}_{\mathbb{Q}}(\bar{a}/B)$ then $(\langle \bar{a}C \rangle \cap B) \setminus C$ is non-empty, containing say $d \in A$. But then $\operatorname{etd}(d/B) = 0$ and $\operatorname{etd}(d/C) = 1$ as $d \notin C$ and $C = \operatorname{ecl}(C)$, witnessing $A \not\downfree^{\mathrm{ecl}}_C B$. However if $\operatorname{ldim}_{\mathbb{Q}}(\bar{a}/C) = \operatorname{ldim}_{\mathbb{Q}}(\bar{a}/B)$ then $\Delta(\bar{a}/C) > \Delta(\bar{a}/B)$ by definition of Δ. By Fact 2.2.10 we know that $\operatorname{etd}(\bar{a}/B) \leq \Delta(\bar{a}/B)$, and since $\bar{a}C \lhd \mathcal{M}$ by Corollary 2.2.11 we have $\operatorname{etd}(\bar{a}/C) = \Delta(\bar{a}/C)$. Therefore $\operatorname{etd}(\bar{a}/B) < \operatorname{etd}(\bar{a}/C)$ as required.

Suppose then that $Ae^A \downfree^{\mathbf{ACF_0}}_C Be^B$ but AB is not strong in $\mathcal{M}$. Then there exists $\bar{a} \in A$ such that $\bar{a}C$ is strong in A but $\bar{a}B$ is not strong in $\mathcal{M}$, so by Corollary 2.2.11 we have $\operatorname{etd}(\bar{a}/B) < \Delta(\bar{a}/B)$. We know $\operatorname{td}(\bar{a}, e^{\bar{a}}/Be^B) = \operatorname{td}(\bar{a}, e^{\bar{a}}/C)$, so by Corollary 3.3.6 we have $\operatorname{ldim}_{\mathbb{Q}}(\bar{a}/B) = \operatorname{ldim}_{\mathbb{Q}}(\bar{a}/C)$ and hence $\Delta(\bar{a}/C) = \Delta(\bar{a}/B)$. Since $\bar{a}C$ is strong in $\mathcal{M}$, applying Corollary 2.2.11 we have $\operatorname{etd}(\bar{a}/C) = \Delta(\bar{a}/C)$, and so $\operatorname{etd}(\bar{a}/B) < \operatorname{etd}(\bar{a}/C)$ as required. □

Corollary 3.4.2. *Let $\mathcal{M}$ be a model for $\boldsymbol{ECF_{SK,CCP}}$, and let A, B, C be subsets of $\mathcal{M}$ such that $C = ecl(C)$. Then $A \downfree^{\boldsymbol{ECF_{SK,CCP}}}_C B$ if and only if $A \downfree^{ecl}_C B$. In particular $\downfree^{\boldsymbol{ECF_{SK,CCP}}}$ is a symmetric independence relation.*

Proof. Immediate from Proposition 3.4.1. □

Since $\downfree^{\mathbf{ECF_{SK,CCP}}}$ coincides with pregeometric independence in $\mathbf{ECF_{SK,CCP}}$, our independence notion is exactly the canonical model theoretic independence notion in this class. Before we move on to study independence in **ECF** and $\mathbf{ECF_{SK,CCP}}$, we observe some more pure model theoretic equivalences to $\mathbf{ECF_{SK,CCP}}$-independence.

Definition 3.4.3. Let $(\mathcal{C}, \leq_{\mathcal{C}})$ be an AEC with a monster model $\mathbb{M}$.

(a) [8, Definition 2.9] Let $\bar{a}$ be a tuple and $D \subseteq B$ subsets with D finite. We say that $\mathrm{tp}^w(\bar{a}/B)$ *splits over* D if there are $\bar{b}, \bar{c} \in B$ such that $\mathrm{tp}^w(\bar{b}/D) = \mathrm{tp}^w(\bar{c}/D)$ but $\mathrm{tp}^w(\bar{a}\bar{b}/D) \neq \mathrm{tp}^w(\bar{a}\bar{c}/D)$. We say a tuple $\bar{a}$ is *non-splitting free* from B over C and write $a \downarrow_C^{ns} B$ if there exists a finite subset $D \subseteq C$ such that $\mathrm{tp}^w(\bar{a}/BC)$ does not split over D. We write $A \downarrow_C^{ns} B$ if $\bar{a} \downarrow_C^{ns} B$ for all $\bar{a} \in A$.

(b) [8, Definition 2.28] For a tuple $\bar{a}$ and a model $\mathcal{M}$ define the *U-rank* of $\bar{a}$ over $\mathcal{M}$ inductively by

- $U(\bar{a}/\mathcal{M}) \geq 0$
- $U(\bar{a}/\mathcal{M}) \geq n+1$ if there is some model $\mathcal{N} \supseteq \mathcal{M}$ such that $\bar{a} \not\downarrow_{\mathcal{M}}^{ns} \mathcal{N}$ and $U(\bar{a}/\mathcal{N}) \geq n$
- $U(\bar{a}/\mathcal{M}) = n$ if n is maximal such that $U(\bar{a}/\mathcal{M}) \geq n$.

For a subset $B \subseteq \mathbb{M}$ we define $U(\bar{a}/B) = \max\{U(\bar{a}/\mathcal{M}) : \mathcal{M} \text{ is a model with } B \subseteq \mathcal{M}\}$.

Lemma 3.4.4. *[8, Lemma 2.29] Let $(\mathcal{C}, \leq_{\mathcal{C}})$ be an AEC with monster model $\mathbb{M}$. For models $\mathcal{M}, \mathcal{N}$ and a tuple $\bar{a}$ we have $\bar{a} \downarrow_{\mathcal{M}}^{ns} \mathcal{N}$ if and only if $U(\bar{a}/\mathcal{M}) = U(\bar{a}/\mathcal{N})$.*

In any quasiminimal pregeometry structure, the pregeometric dimension is equal to U-rank, as shown in [8, Lemma 2.92]. Fixing $\mathbb{M}$ a monster model in $\mathbf{ECF_{SK,CCP}}$, we have $(\mathbb{M}, \mathrm{ecl})$ a quasiminimal pregeometry structure, so by Lemma 3.4.4 above it follows that $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{ecl}} = \downarrow^{ns}$. Therefore

$\mathbf{ECF_{SK,CCP}}$-independence in $\mathbb{M}$ is equivalent to non-splitting independence of weak types.

We consider one final rephrasing of our independence notion.

Definition 3.4.5. [5, Definition 5.19] Let $\mathcal{C}$ be an AEC with AP, JEP and ALM, and let $\mathbb{M}$ denote the monster model of $\mathcal{C}$. For any subset $A \subseteq \mathbb{M}$ and tuple $\bar{a} \in \mathbb{M}$, define $r_{\bar{a}}(A)$ to be the set

$$r_{\bar{a}}(A) = \{\bar{c} \in \mathbb{M} \, : \, \mathrm{tp}^w(\bar{c}/A) = \mathrm{tp}^w(\bar{a}/A)\}$$

that is, $r_{\bar{a}}(A)$ denotes the set of realisations of $\mathrm{tp}^w(\bar{c}/A)$. Then for $A \subseteq \mathbb{M}$ we define the *bounded closure* of A in $\mathcal{C}$ by

$$\mathrm{bdd}_{\mathcal{C}}(A) = \{a \in \mathbb{M} : |r_a(A)| < |\mathbb{M}|\}$$

Lemma 3.4.6. *Let $\mathbb{M}$ be the monster model in $\boldsymbol{ECF_{SK,CCP}}$ and $A \subseteq \mathbb{M}$ a countable subset. Then $bdd_{\boldsymbol{ECF_{SK,CCP}}}(A) = ecl(A)$.*

Proof. If $x \in \mathrm{ecl}(A)$, then x is exponentially algebraic over A in the sense of Macintyre [18, Definition 5, Section 2.5], that is we have $\bar{f}$ polynomials defined over some $\bar{a}, e^{\bar{a}}$ where $\bar{a} \in A$ such that the $\bar{f}$ form a Khovanskii system, given by some formula $\chi_{\bar{f}}(x, \bar{a})$. By the countable closure property $\chi_{\bar{f}}(x, \bar{a})$ has only countably many realisations, so $x \in \mathrm{bdd}(A)$. If $x \notin \mathrm{ecl}(A)$ then $x \models q|A$ the unique complete exponentially transcendental type over A in $\mathbf{ECF_{SK,CCP}}$, which has unboundedly many realisations in $\mathbb{M}$. $\square$

By Lemma 3.4.6, $A \mathop{\smile\!\!\!\!\!\!\mid}_{C}^{\mathbf{ECF_{SK,CCP}},\mathcal{M}} B$ can also be defined as $A \downarrow^{\mathcal{M}}_{\mathrm{bdd}(C)} B$

since $\mathrm{bdd}(C) = \mathrm{ecl}(C)$. So in $\mathbf{ECF_{SK,CCP}}$ we also have

$$A \underset{\mathrm{bdd}(C)}{\overset{\mathrm{ecl}}{\downarrow}} B \text{ if and only if } A \underset{C}{\overset{\mathbf{ECF_{SK,CCP}}}{\downarrow}} B.$$

3.5 Independence properties in ECF and $\mathbf{ECF_{SK}}$

Proposition 3.5.1. $\downarrow^{\boldsymbol{ECF}}$ *is a symmetric pre-independence relation.*

Proof. Let $\mathcal{M}$ be a model in **ECF** and let A, B, C be subsets of $\mathcal{M}$. As before we may assume that $A = \lceil AC \rceil$, $B = \lceil BC \rceil$ and $C = \lceil C \rceil$. Symmetry, monotonicity and finite character for $\downarrow$ follow immediately from these properties for $\downarrow$ in Proposition 3.2.4. Observing that $\sigma(\lceil C \rceil^{\mathrm{ELA}}) = \lceil \sigma(C) \rceil^{\mathrm{ELA}}$ for any $C \subseteq \mathcal{M}$ and $\sigma \in \mathrm{Aut}(\mathcal{M})$, invariance for $\downarrow$ follows from invariance for $\downarrow$. For transitivity, by Proposition 3.2.6 and the definition of $\downarrow$ we have $A \downarrow_C B \Leftrightarrow A \downarrow_{\lceil C \rceil^{\mathrm{ELA}}} \lceil B \rceil^{\mathrm{ELA}}$. Then for $X \subseteq B$ we have $\lceil X \rceil^{\mathrm{ELA}} \subseteq \lceil B \rceil^{\mathrm{ELA}}$, and so by transitivity of $\downarrow$ it follows that $A \downarrow_{\lceil C \rceil^{\mathrm{ELA}}} \lceil B \rceil^{\mathrm{ELA}}$ iff $A \downarrow_{\lceil C \rceil^{\mathrm{ELA}}} \lceil X \rceil^{\mathrm{ELA}}$ and $A \downarrow_{\lceil X \rceil^{\mathrm{ELA}}} \lceil B \rceil^{\mathrm{ELA}}$. Applying Proposition 3.2.6 again, we obtain transitivity. □

Corollary 3.5.2. $\downarrow^{\boldsymbol{ECF_{SK}}}$ *is a symmetric pre-independence relation.*

The proof of the corollary is the same as that of Proposition 3.5.1. If we replace **ECF** with $\mathbf{ECF_{SK,CCP}}$ and $\lceil C \rceil^{\mathrm{ELA}}$ with $\mathrm{ecl}(C)$ in the proof of Proposition 3.5.1, we obtain an alternative proof of Corollary 3.4.2.

Lemma 3.5.3. *Let $\mathcal{C}$ be one of **ECF**, $\boldsymbol{ECF_{SK}}$ and $\boldsymbol{ECF_{SK,CCP}}$. Let $\mathcal{M}$ be a model in $\mathcal{C}$ and let A, B, C be subsets of $\mathcal{M}$. Then $A \downarrow_C^{\mathcal{C}} B$ iff $A \downarrow_C^{\mathcal{C}} \lceil B \rceil^{ELA}$.*

Proof. Immediate from Proposition 3.2.6 and the definition of independence for each class. $\square$

Lemma 3.5.4. *Let $\mathcal{M}$ be a model in **ECF** and let A, B, C be subsets of $\mathcal{M}$. Then $A \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_C^{\boldsymbol{ECF}} B$ if and only if $\lceil A\rceil^{ELA} \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_{\lceil C\rceil^{ELA}}^{\boldsymbol{ECF}} \lceil B\rceil^{ELA}$.*

Proof.

$$\begin{aligned}
A \mathop{\smile\!\!\!\!\!\!\mid}_{C} B &\text{ if and only if } A \mathop{\smile\!\!\!\!\!\!\mid}_{C} \lceil B\rceil^{\mathrm{ELA}} \text{ by Lemma 3.5.3}\\
&\text{ if and only if } \lceil B\rceil^{\mathrm{ELA}} \mathop{\smile\!\!\!\!\!\!\mid}_{C} \lceil A\rceil^{\mathrm{ELA}} \text{ by symmetry and Lemma 3.5.3}\\
&\text{ if and only if } \lceil B\rceil^{\mathrm{ELA}} \mathop{\smile\!\!\!\!\!\!\mid}_{\lceil C\rceil^{\mathrm{ELA}}} \lceil A\rceil^{\mathrm{ELA}} \text{ by the definition of } \mathop{\smile\!\!\!\!\!\!\mid}\\
&\text{ if and only if } \lceil A\rceil^{\mathrm{ELA}} \mathop{\smile\!\!\!\!\!\!\mid}_{\lceil C\rceil^{\mathrm{ELA}}} \lceil B\rceil^{\mathrm{ELA}} \text{ by symmetry.}
\end{aligned}$$

$\square$

Next we have a lemma that provides sufficient conditions for independence in terms of $\mathbb{Q}$-linear independence and exponential transcendence degree alone.

Lemma 3.5.5. *Let $\mathcal{M}$ be a model in **ECF**, let $A, B \subseteq \mathcal{M}$ be subsets and $C \subseteq \mathcal{M}$ be a strong ELA-subfield of $\mathcal{M}$. Suppose that etd$(A/C) =$ etd(A/BC) and $\lceil AC\rceil_{\mathcal{M}} \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_C^{\mathbb{Q}\text{-}lin} \lceil BC\rceil_{\mathcal{M}}$. Then $A \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_C B$.*

Proof. We may assume that $A \supseteq C \subseteq B$ and $\lceil AC\rceil = A, \lceil BC\rceil = B$. We first prove that $Ae^A \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_C^{\mathbf{ACF_0}} Be^B$. Let $\bar{x} \in A$. Then there exists $\bar{a} \in A$ such that $\bar{a}C \prec \mathcal{M}$ and $\bar{x} \in \mathbb{Q}(\bar{a}, e^{\bar{a}})$. Since $A \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_C^{\mathbb{Q}\text{-lin}} B$ we have $\mathrm{ldim}_{\mathbb{Q}}(\bar{a}/B) =$

$\mathrm{ldim}_{\mathbb{Q}}(\bar{a}/C)$. Then

$$\begin{aligned}\Delta(\bar{a}/B) &= \mathrm{td}(\bar{a}, e^{\bar{a}}/Be^{B}) - \mathrm{ldim}_{\mathbb{Q}}(\bar{a}/B) \\ &= \mathrm{td}(\bar{a}, e^{\bar{a}}/Be^{B}) - \mathrm{ldim}_{\mathbb{Q}}(\bar{a}/C) \\ &\leq \mathrm{td}(\bar{a}, e^{\bar{a}}/Ce^{C}) - \mathrm{ldim}_{\mathbb{Q}}(\bar{a}/C) = \Delta(\bar{a}/C)\end{aligned}$$

and by Fact 2.2.10 we have $\Delta(\bar{a}/C) = \mathrm{etd}(\bar{a}/C)$, so $\Delta(\bar{a}/B) \leq \mathrm{etd}(\bar{a}/C)$. By hypothesis $\mathrm{etd}(\bar{a}/B) = \mathrm{etd}(\bar{a}/C)$, so $\Delta(\bar{a}/B) \leq \mathrm{etd}(\bar{a}/B)$. However by Fact 2.2.10 again $\mathrm{etd}(\bar{a}/B) \leq \Delta(\bar{a}/B)$, and so $\mathrm{etd}(\bar{a}/B) = \Delta(\bar{a}/B)$. Therefore $\Delta(\bar{a}/B) = \Delta(\bar{a}/C)$, which combined with $\mathrm{ldim}_{\mathbb{Q}}(\bar{a}/B) = \mathrm{ldim}_{\mathbb{Q}}(\bar{a}/C)$ gives us that $\mathrm{td}(\bar{a}, e^{\bar{a}}/Be^{B}) = \mathrm{td}(\bar{a}, e^{\bar{a}}/C)$. Therefore $\bar{a}, e^{\bar{a}} \mathop{\smile\!\!\!\!\!\!|}\,_{C}^{\mathbf{ACF_0}} Be^{B}$, and by monotonicity $\bar{x} \mathop{\smile\!\!\!\!\!\!|}\,_{C}^{\mathbf{ACF_0}} Be^{B}$. By finite character, $Ae^{A} \mathop{\smile\!\!\!\!\!\!|}\,_{C}^{\mathbf{ACF_0}} Be^{B}$.

We also need to show that $\langle AB \rangle \triangleleft \mathcal{M}$. For any $\bar{x} \in \mathcal{M}$ there exist $\bar{a} \in A$ and $\bar{b} \in B$ strong over C such that $\Delta(\bar{x}/AB) = \Delta(\bar{x}/\bar{a}\bar{b}C)$. Since $\bar{a}, e^{\bar{a}} \mathop{\smile\!\!\!\!\!\!|}\,_{C}^{\mathbf{ACF_0}} Be^{B}$ we have $\mathrm{td}(\bar{a}, e^{\bar{a}}/\bar{b}C) = \mathrm{td}(\bar{a}, e^{\bar{a}}/C)$, and since $\bar{a} \mathop{\smile\!\!\!\!\!\!|}\,_{C}^{\mathbb{Q}\text{-lin}} B$ we have $\mathrm{ldim}_{\mathbb{Q}}(\bar{a}/C) = \mathrm{ldim}_{\mathbb{Q}}(\bar{a}/\bar{b}C)$. Therefore $\Delta(\bar{a}/\bar{b}C) = \Delta(\bar{a}/C)$, and by Fact 2.2.10 $\Delta(\bar{a}/\bar{b}C) = \mathrm{etd}(\bar{a}/C)$. By Fact 2.2.10 again, $\Delta(\bar{x}\bar{a}/\bar{b}C) \geq \mathrm{etd}(\bar{a}/\bar{b}C) \geq \mathrm{etd}(\bar{a}/C)$. By the addition property $\Delta(\bar{x}/\bar{a}\bar{b}C) = \Delta(\bar{x}\bar{a}/\bar{b}C) - \Delta(\bar{a}/\bar{b}C)$, and so

$$\Delta(\bar{x}/\bar{a}\bar{b}C) \geq \mathrm{etd}(\bar{a}/C) - \mathrm{etd}(\bar{a}/C) = 0,$$

hence $\langle AB \rangle \triangleleft \mathcal{M}$. $\square$

We now use pre-independence properties to show how $\mathop{\smile\!\!\!\!\!\!|}\,^{\mathbf{ECF}}$ is related to types orthogonal over the kernel, specifically to grounding sets.

Lemma 3.5.6. *Let p be a complete type over a semi-strong ELA-subfield B such that p is orthogonal to the kernel, witnessed by $\mathcal{M} \in$ **ECF**. Suppose that p is grounded at $A \subseteq B$, and $\bar{a}$ is a realisation for p in $\mathcal{M}$. Then $\bar{a} \mathop{\smile\!\!\!\!\!\mid}^{\boldsymbol{ECF}}_{A} B$.*

Proof. Let $\bar{a}'$ be a basis for $\lceil \bar{a}B \rceil_{\mathcal{M}}$ over B. By Lemma 2.6.4 $A\bar{a} \prec\!\!\!\!\lhd\, \mathcal{M}$. Let $\widehat{A} = \lceil A \rceil^{\mathrm{ELA}}_{\mathcal{M}}$. Then $\mathrm{Loc}(\bar{a}', e^{\bar{a}'}/B) = \mathrm{Loc}(\bar{a}', e^{\bar{a}'}/\widehat{A})$ and $\mathrm{etd}(\bar{a}'/\widehat{A}) = \mathrm{etd}(\bar{a}'/B)$. Therefore $\bar{a}', e^{\bar{a}'} \mathop{\smile\!\!\!\!\!\mid}^{\mathbf{ACF_0}}_{\widehat{A}} B$ and by Lemma 3.3.5 $\bar{a}' \mathop{\smile\!\!\!\!\!\mid}^{\mathbb{Q}\text{-lin}}_{\widehat{A}} B$. Since $A\bar{a}' \prec\!\!\!\!\lhd\, \mathcal{M}$, by Lemma 2.4.3 we have $\widehat{A}\bar{a}' \lhd \mathcal{M}$, and so $\bar{a}' \mathop{\smile\!\!\!\!\!\mid}_{\widehat{A}} B$. By the definition of independence, $\bar{a}' \mathop{\smile\!\!\!\!\!\mid}_{A} B$, and by symmetry and monotonicity of **ECF**-independence we have $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_{A} B$. □

Proposition 3.5.7. *Let $\mathcal{M}$ be a saturated model in **ECF**. Then $\mathop{\smile\!\!\!\!\!\mid}^{\boldsymbol{ECF},\mathcal{M}}$ is an independence relation, satisfying the following additional property.*

9'. ***Stationarity over strong ELA-subfields*** *Let $\bar{a}_1$ and $\bar{a}_2$ be finite tuples in $\mathcal{M}$, and suppose that $A \lhd \mathcal{M}$ is a strong ELA-subfield of $\mathcal{M}$. If $\bar{a}_1 \mathop{\smile\!\!\!\!\!\mid}_{A} B$ and $\bar{a}_2 \mathop{\smile\!\!\!\!\!\mid}_{A} B$ with $tp(\bar{a}_1/A) = tp(\bar{a}_2/A)$ then $tp(\bar{a}_1/B) = tp(\bar{a}_2/B)$.*

Property 9'. is close to stationarity, but has the additional requirement that A is a strong ELA-subfield. Note that we do not have stationarity in **ECF**, as the requirement that A is a strong ELA-subfield is unavoidable. For example, let $\mathbb{M}$ be the monster model in **ECF** and suppose we have a subset $A \subseteq \mathbb{M}$ such that $A \neq \lceil A \rceil^{\mathrm{ELA}}$. Suppose that a_1 and a_2 are two distinct elements in $Z(\mathbb{M}) \setminus \mathrm{bdd}_{\mathbf{ECF}}(A)$ such that $\mathrm{tp}(a_1/A) = \mathrm{tp}(a_2/A)$. Set $B = A \cup \{a_1, a_2\}$. Then $a_1, a_2 \in \lceil A \rceil^{\mathrm{ELA}} \setminus A$, so trivially $a_1 \mathop{\smile\!\!\!\!\!\mid}^{\mathbf{ECF}}_{A} B$ and $a_2 \mathop{\smile\!\!\!\!\!\mid}^{\mathbf{ECF}}_{A} B$. However, $\mathrm{tp}(a_1/B) \neq \mathrm{tp}(a_2/B)$, so stationarity fails.

We now prove Proposition 3.5.7.

Proof. By Proposition 3.5.1 $\mathop{\smile\!\!\!\!\!\!\;|}^{\mathbf{ECF}}$ satisfies the pre-independence properties of an independence relation in Definition 3.1.1.

5. *Extension:* Let $\bar{a} \mathop{\smile\!\!\!\!\!\!\;|}_C B$ and $B' \supseteq B$. By Lemma 3.5.3 may assume that $B = \lceil BC \rceil^{\mathrm{ELA}}$ and $B' = \lceil B' \rceil^{\mathrm{ELA}}$. We may also assume that $\bar{a}$ is a $\mathbb{Q}$-linear basis for $\lceil \bar{a}B \rceil$ over B, else we would proceed with such a basis $\bar{a}'$ and then apply monotonicity at the end of the proof. Let $V = \mathrm{Loc}(\bar{a}, e^{\bar{a}}/B)$. Since B' is an ELA-field, by strong exponential closure and saturation of $\mathcal{M}$ we can realise $B'|V$ as a strong ELA-subfield of $\mathcal{M}$, generated over B' by a $\mathbb{Q}$-linearly independent tuple $\bar{b} \in \mathcal{M}$. We have $B'\bar{b} \lhd \mathcal{M}$ and $\lceil \bar{b}C \rceil \mathop{\smile\!\!\!\!\!\!\;|}_C^{\mathbb{Q}\text{-lin}} B'$ by definition of $\bar{b}$. We also have $\mathrm{Loc}(\bar{b}, e^{\bar{b}}/B') = V$ and since $\bar{a} \mathop{\smile\!\!\!\!\!\!\;|}_C B$ it follows that V is defined over C, so $\mathrm{Loc}(\bar{b}, e^{\bar{b}}/B') = \mathrm{Loc}(\bar{b}, e^{\bar{b}}/C)$. Therefore $\bar{b}e^{\bar{b}} \mathop{\smile\!\!\!\!\!\!\;|}_C^{\mathbf{ACF_0}} B'$ and so $\bar{b} \mathop{\smile\!\!\!\!\!\!\;|}_C B'$. Finally $\Delta(\bar{b}/B') = \mathrm{td}(\bar{b}, e^{\bar{b}}/B') - \mathrm{ldim}_{\mathbb{Q}}(\bar{b}/B')$ is minimal and $\mathrm{td}(\bar{b}, e^{\bar{b}}/B') = \dim(V)$ since $\bar{b}$ is semi-strong over B', so we have $\mathrm{etd}(\bar{b}) = \dim(V) - |\bar{a}|$. By Proposition 2.5.10 we have $\mathrm{tp}^g(\bar{b}/B) = \mathrm{tp}^g(\bar{a}/B)$, and so $\mathrm{tp}(\bar{b}/B) = \mathrm{tp}(\bar{a}/B)$ as required.

6. *Local character:* We show that $\kappa = \omega$. Let $C \subseteq \mathcal{M}$ be a subset and let $\bar{a} \in \mathcal{M}$ be a tuple. By Lemma 3.5.3 we may take $C = \lceil C \rceil^{\mathrm{ELA}}_{\mathcal{M}}$. Therefore $\mathrm{tp}(\bar{a}/C)$ is orthogonal to the kernel witnessed by $\mathcal{M}$, so by Lemma 2.6.5 there exists $\bar{b} \in C$ a grounding set for $\mathrm{tp}(\bar{a}/C)$. Applying Lemma 3.5.6 the result follows.

9'. We have $A = \lceil A \rceil^{\mathrm{ELA}}$, and by Lemma 3.5.3 and Corollary 2.6.8(i) we may also assume that $B = \lceil B \rceil^{\mathrm{ELA}}$. By Corollary 2.5.11 it is sufficient to prove the result for Galois types. Furthermore we assume that $A\bar{a}_i \lhd \mathcal{M}$ and $\bar{a}_i$ are $\mathbb{Q}$-linearly independent over A, else we proceed with $\bar{a}_i'$ a basis of $\lceil \bar{a}_i A \rceil$ over A and then use monotonicity of $\downarrow$.

For each $i = 1, 2$ we have $\bar{a}_i \downarrow_A B$ and so $\lceil A\bar{a}_i \rceil e^{\lceil A\bar{a}_i \rceil} \downarrow_A^{\mathbf{ACF_0}} B$. By monotonicity, and since $A = \lceil A \rceil^{\mathrm{ELA}}$ is algebraically closed, we have $\mathrm{Loc}(\bar{a}_i, e^{\bar{a}_i}/A) = \mathrm{Loc}(\bar{a}_i, e^{\bar{a}_i}/B)$. Since $\mathrm{tp}^g(\bar{a}_1/A) = \mathrm{tp}^g(\bar{a}_2/A)$ it follows that $\mathrm{Loc}(\bar{a}_1, e^{\bar{a}_1}/A) = \mathrm{Loc}(\bar{a}_2, e^{\bar{a}_2}/A)$. Therefore $\mathrm{Loc}(\bar{a}_1, e^{\bar{a}_1}/B) = \mathrm{Loc}(\bar{a}_2, e^{\bar{a}_2}/B)$. Since also $\bar{a}_i \downarrow_A B$, we have $\mathrm{td}(\bar{a}_1, e^{\bar{a}_1}/B) = \mathrm{td}(\bar{a}_2, e^{\bar{a}_2}/B)$.

From property (ii) in the definition of $\downarrow^{\mathbf{ECF}}$, for $i = 1, 2$ we have $\lceil A\bar{a}_i \rceil \downarrow_A^{\mathbb{Q}\text{-lin}} B$ and so by monotonicity $\mathrm{ldim}_{\mathbb{Q}}(\bar{a}_i/A) = \mathrm{ldim}_{\mathbb{Q}}(\bar{a}_i/B)$. As $\mathrm{tp}^g(\bar{a}_1/A) = \mathrm{tp}^g(\bar{a}_2/A)$ it follows that $\mathrm{ldim}_{\mathbb{Q}}(\bar{a}_1/A) = \mathrm{ldim}_{\mathbb{Q}}(\bar{a}_2/A)$. Therefore $\mathrm{ldim}_{\mathbb{Q}}(\bar{a}_1/B) = \mathrm{ldim}_{\mathbb{Q}}(\bar{a}_2/B)$, and so $\Delta(\bar{a}_1/B) = \Delta(\bar{a}_2/B)$. Since $\bar{a}_i \downarrow_A B$ and $A\bar{a}_i \lhd \mathcal{M}$ we have $\langle \lceil \bar{a}_i A \rceil, B \rangle = \langle \bar{a}_i B \rangle \lhd \mathcal{M}$, that is we have $B\bar{a}_i$ strong in $\mathcal{M}$. Hence $\Delta(\bar{a}_i/B)$ is minimal, that is for any $\bar{x} \in \mathcal{M}$ we have $\Delta(\bar{x}\bar{a}_i/B) \geq \Delta(\bar{a}_i/B)$. By Fact 2.2.10 this means that $\Delta(\bar{a}_i/B) = \mathrm{etd}(\bar{a}_i/B)$, and so $\mathrm{etd}(\bar{a}_1/B) = \mathrm{etd}(\bar{a}_2/B)$.

Taking $\mathrm{ldim}_{\mathbb{Q}}(\bar{a}_1/B) = n$ and $V = \mathrm{Loc}(\bar{a}_1, e^{\bar{a}_1}/B)$, we observe that $\mathrm{etd}(\bar{a}_2/B) = \dim V - n$. Taking $M \in \mathrm{Mat}(\mathbb{Q})$ to be the identity matrix, we have satisfied all hypotheses of Proposition 2.5.10, and therefore $\mathrm{tp}^g(\bar{a}_1/B) = \mathrm{tp}^g(\bar{a}_2/B)$.

$\square$

We now demonstrate that under certain strong assumptions, non-forking independence in **ECF** implies **ECF**-independence. We shall need the following lemma.

Lemma 3.5.8. *Let $\mathcal{M}$ be a model in* ***ECF****, and let $B \subseteq \mathcal{M}$ be a strong ELA-subfield. Then B is model theoretically algebraically closed.*

Proof. By Proposition 2.5.5, we can find an elementary extension $\mathcal{N} \in$ **ECF** of $\mathcal{M}$ such that $\mathcal{N}$ is saturated over the kernel and $|\mathcal{N}| > |B|$. Suppose that $a \in \mathcal{N} \setminus B$. Let $\bar{a}$ be a $\mathbb{Q}$-linear basis for $\lceil a \rceil_{\mathcal{N}}$ over the kernel with first coordinate a, and define $W = \mathrm{Loc}(\bar{a}, e^{\bar{a}}/B)$. Since $\mathcal{N}$ is saturated over its kernel, setting $\bar{b}_0 = \bar{a}$ we can find a sequence $(\bar{b}_i)_{i<\omega}$ in $\mathcal{N}$ indiscernible over B, in particular for each $i < \omega$ we have $W_i = \mathrm{Loc}(\bar{b}_i, e^{\bar{b}_i}/B) = W$. We can construct a chain of strong ELA-field extensions, setting $F_0 = B$, such that each $\bar{b}_i$ generates an ELA-field extension $F_{i+1} \triangleleft \mathcal{N}$ with $F_{i+1} = F_i|W_i$ as in [16, Proposition 3.17]. Then for all $i < \omega$ we have $W_i = W$ and $\mathrm{ldim}_{\mathbb{Q}}(\bar{b}_i/B) = |\bar{a}|$, so $\mathrm{etd}(\bar{b}_i) = \dim W - |\bar{a}|$. Then by Proposition 2.5.10 we have $\mathrm{tp}^g(\bar{b}_i/B) = \mathrm{tp}^g(\bar{a}/B)$. Hence $\mathrm{tp}^g(b_i/B) = \mathrm{tp}^g(a/B)$ for all $i < \omega$, taking b_i to be the first coordinate of $\bar{b}_i$. Therefore if ϕ is a formula defined over B such that $\mathcal{N} \models \phi(a)$, then $\mathcal{N} \models \phi(b_i)$ for all $i < \omega$, so $\phi(\mathcal{N})$ is infinite. $\square$

Lemma 3.5.9. *Let $\mathcal{M}$ be a model of* ***ECF****. Suppose we have subsets $C \subseteq B \subseteq \mathcal{M}$ with $C = \lceil C \rceil^{ELA}_{\mathcal{M}}$, and a tuple $\bar{a} \in \mathcal{M}$ such that $etd(\bar{a}/C) = etd(\bar{a}/B)$. Then $\bar{a} \mathop{\smile\!\!\!\!\!\vert}^{f}_{C} B$ implies that $\bar{a} \mathop{\smile\!\!\!\!\!\vert}^{\boldsymbol{ECF}}_{C} B$.*

Proof. Suppose that $\bar{a} \not\mathop{\smile\!\!\!\!\!\vert}^{\mathbf{ECF}}_{C} B$. Then $\mathrm{etd}(\bar{a}/C) = \mathrm{etd}(\bar{a}/B)$, so by Lemma 3.5.5 we have $\mathrm{ldim}_{\mathbb{Q}}(\bar{a}/C) > \mathrm{ldim}_{\mathbb{Q}}(\bar{a}/\lceil B \rceil^{ELA}_{\mathcal{M}})$. That is, there exists $\lambda_1, ..., \lambda_k \in \mathbb{Q}$ such that $\sum_{i=1}^{k} \lambda_i a_i = b$ for some $b \in \lceil B \rceil_{\mathcal{M}}^{ELA} \setminus C$. Define $\phi(\bar{x}, y)$ to be the formula $\sum_{i=1}^{k} \lambda_i x_i = y$. Extending $\mathcal{N}$ if necessary, let $(b_i)_{i<\omega} \subseteq \mathcal{N}$ be an indiscernible sequence such that $\mathrm{tp}^g(b_i/C) = \mathrm{tp}^g(b/C)$. Since C is a strong ELA-subfield of $\mathcal{M}$ by Lemma 3.5.8 C is model theoretically algebraically closed. Since $b \notin C$ there exist infinitely many realisations of $tp(b/C)$, and in particular the b_i are distinct. Then $\phi(\bar{x}, b_1) \wedge \phi(\bar{x}, b_2)$ is inconsistent as $b_1 \neq b_2$, and so $\bar{a} \not\perp^{f}_{C} B$. □

We now show that we may freely extend grounded types in **ECF**, indicating that $\perp^{\mathbf{ECF}}$ is an ideal tool to use in order to study types in **ECF** that are orthogonal to the kernel.

Lemma 3.5.10. *Assume CIT, and work in the monster model $\mathbb{M}$ of* ***ECF****. Let p be a complete type over a subset C, and let B be a subset containing C. Suppose that we have two types p_1 and p_2 over B extending p, that is, $p_1|C = p_2|C$, such that p_1 and p_2 are grounded at C. Then $p_1 = p_2$.*

We say p_1 is the (unique) free extension *over B of p.*

Proof. For $i = 1, 2$, let the orthogonality to the kernel of p_i be witnessed by $\mathcal{M}_i \in \mathbf{ECF}$ and let $\bar{a}_i \in \mathcal{M}_i$ be a realisation of p_i. Since $\ker(\mathcal{M}_1) = \ker(\mathcal{M}_2)$, by following through the method of Lemma 2.2.16 we may amalgamate to a model $\mathcal{M} \in \mathbf{ECF}$ such that for $i = 1, 2$ we have $\ker(\mathcal{M}) = \ker(\mathcal{M}_i)$ and $\mathcal{M}_i \subseteq \mathcal{M}$, so in particular $\bar{a}_i \in \mathcal{M}$. Assuming CIT we may apply Corollary 2.6.8 so $p|C$ uniquely extends to $p|\lceil C \rceil_{\mathcal{M}}^{\mathrm{ELA}}$. By Lemma 3.5.6 since p is grounded at C we have $\bar{a}_i \perp^{\mathbf{ECF},\mathcal{M}}_{\lceil C \rceil_{\mathcal{M}}^{\mathrm{ELA}}} \lceil B \rceil_{\mathcal{M}}^{\mathrm{ELA}}$ for $i = 1, 2$. Since $p_1|\lceil C \rceil_{\mathcal{M}}^{\mathrm{ELA}} = p_2|\lceil C \rceil_{\mathcal{M}}^{\mathrm{ELA}}$, by stationarity over strong ELA-subfields from Proposition 3.5.7 we have $p_1 = p_2$. □

3.6 Lascar-independence in $\mathbf{ECF_{SK}}$

In this section we show that $\mathop{\smile\!\!\!\!\!\!|}\,^{\mathbf{ECF_{SK}}}$ is in fact the only independence relation on $\mathbf{ECF_{SK}}$, using work by Hyttinen and Kesälä [6].

Proposition 3.6.1. *Let $\mathbb{M}$ denote the monster model in $\boldsymbol{ECF_{SK}}$ and $A \subseteq \mathbb{M}$ a subset. Then $bdd_{\boldsymbol{ECF_{SK}}}(A) = \lceil A \rceil^{ELA}$.*

Proof. We first show that $\lceil A \rceil^{\mathrm{ELA}} \subseteq \mathrm{bdd}(A)$. Suppose $c \in \lceil A \rceil^{\mathrm{ELA}}$; then $c \in \lceil \bar{a} \rceil^{\mathrm{ELA}}$ for some finite tuple $\bar{a} \in A$. Let $\bar{b}$ be a $\mathbb{Q}$-linear basis for $\lceil \bar{a} \rceil$ over $\bar{a}$, let $W = \mathrm{Loc}(\bar{b}, \exp(\bar{b})/\bar{a}, \exp(\bar{a}))$, and let M denote the unique integer matrix such that $M\bar{b} = \bar{a}$. Let $\psi(\bar{y})$ be the formula given by

$$(\bar{y}, \exp(\bar{y})) \in W \wedge \bar{y} \text{ is } Q\text{-linearly independent} \wedge M\bar{y} = \bar{a}$$

where $Q = \{x \in \mathbb{M} : (\exists y, z \in \ker)[xz = y]\}$, and note that $Q(\mathbb{M}) = \mathbb{Q}$ and $\mathbb{M} \models \psi(\bar{b})$. We shall show that ψ is bounded. Suppose we have $\bar{b}' \in \mathbb{M}$ such that $\mathbb{M} \models \psi(\bar{b}')$. Then in particular $\bar{b}' \in \langle \bar{b} \rangle$, and since $|\bar{b}| = |\bar{b}'|$ and $\bar{b}'$ are $\mathbb{Q}$-linearly independent, it follows that $\langle \bar{b} \rangle = \langle \bar{b}' \rangle$. There are only countably many bases of a finite dimensional vector space, so ψ is bounded.

Suppose $a \in \mathrm{bdd}(A)$, so there exists some bounded formula $\phi(x)$ defined over A such that $\mathbb{M} \models \phi(a)$. Then e^a and $\log(a)$ satisfy the formulas $\exists y(x = e^y \wedge \phi(y))$ and $\exists z(z = e^x \wedge \phi(z))$ respectively, which both witness finite conjunctions of bounded formulas and therefore $e^a, \log(a) \in \mathrm{bdd}(A)$. Finally we need to show that if $b \in \mathbb{M}$ is field-theoretically algebraic over $\mathrm{bdd}(A)$ then $b \in \mathrm{bdd}(A)$. However,

$$\mathrm{acl}_{\mathbf{ACF_0}}(\mathrm{bdd}(A)) \subseteq \mathrm{acl}(\mathrm{bdd}(A)) \subseteq \mathrm{bdd}(\mathrm{bdd}(A)) = \mathrm{bdd}(A)$$

so this is true, and therefore $\lceil A \rceil^{\text{ELA}} \subseteq \text{bdd}(A)$.

If $x \notin \lceil A \rceil^{\text{ELA}}$, then either $x \in \text{ecl}(A) \setminus \lceil A \rceil^{\text{ELA}}$ or $x \notin \text{ecl}(A)$. We note that $\lceil A \rceil^{\text{ELA}} \subseteq \text{bdd}(A)$ and so $\text{bdd}(\lceil A \rceil^{\text{ELA}}) \subseteq \text{bdd}(\text{bdd}(A)) = \text{bdd}(A)$, but also $A \subseteq \lceil A \rceil^{\text{ELA}}$ so therefore $\text{bdd}(\lceil A \rceil^{\text{ELA}}) = \text{bdd}(A)$. Consequentially, for the remainder of this proof we may assume that $A = \lceil A \rceil^{\text{ELA}}$.

If $x \notin \text{ecl}(A)$ then $x \models q|A$ the unique exponentially transcendental type in $\mathbf{ECF_{SK}}$, which has unboundedly many realisations in $\mathbb{M}$, so $x \notin \text{bdd}(A)$. Suppose then that $x \in \text{ecl}(A) \setminus A$. By Lemma 2.5.9 we can choose a finite $\mathbb{Q}$-linear basis $\bar{x} \in \mathbb{M}$ for $\lceil Ax \rceil_{\mathbb{M}}$ over A such that $W = \text{Loc}(\bar{x}, e^{\bar{x}}/A)$ is additively and multiplicatively free, rotund, and Kummer-generic. Let $A_1 = \langle A\bar{x} \rangle_{\mathbb{M}}^{\text{ELA}}$ be the ELA-field extension of A by $(\bar{x}, e^{\bar{x}})$, and for any ordinal α let $A_{\alpha+1} = \langle A_\alpha \bar{x}_\alpha \rangle^{\text{ELA}}$, the ELA-field extension of A_α by $(\bar{x}_\alpha, e^{\bar{x}_\alpha}) \in V$ generic over A_α. Fix an arbitrarily large κ and let $A_\kappa = \bigcup_{\alpha<\kappa} A_\alpha$. For any given $\alpha < \kappa$, let $B_\alpha = \lceil A, \bar{x}, \bar{x}_\alpha \rceil_{\mathbb{M}}^{\text{ELA}}$. Applying [15, Lemma 5.9] we have an automorphism σ of B_α fixing A with $\sigma(\bar{x}) = \bar{x}_\alpha$ and $\sigma(\bar{x}_\alpha) = \bar{x}$. By [7, Theorem 8.2.1], any model in an inductive class $\mathcal{K}$ of $\mathcal{L}$-structures is contained within an existentially-closed model in $\mathcal{K}$, which means that the automorphism orbit of $\bar{x}$ over A in $\mathbb{M}$ contains $\{\bar{x}_\alpha : \alpha < \kappa\}$. Therefore $\text{tp}^g(\bar{x}/A) = \text{tp}^g(\bar{x}_\alpha/A)$ for all $\alpha < \kappa$, so in particular $x \notin \text{bdd}(A)$. $\square$

The above proposition immediately implies that $A \mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$\vert$}}_C^{\mathbf{ECF_{SK}},\mathbb{M}} B$ can also be defined as $A \downarrow_{\text{bdd}(C)}^{\mathbb{M}} B$.

Proposition 3.6.2. *Let $\mathbb{M}$ be the monster model of $\boldsymbol{ECF_{SK}}$ and suppose that A, B, C are subsets of $\mathbb{M}$. Then $A \mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$\vert$}}_C^{\boldsymbol{ECF_{SK}}} B$ if and only if $A \mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$\vert$}}_C^{L} B$. In particular, $\mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$\vert$}}^{\boldsymbol{ECF_{SK}}}$ is the unique independence notion for $\boldsymbol{ECF_{SK}}$ with bounded free extensions for weak types.*

Proof. $\mathbf{ECF_{SK}}$ is a finitary abstract elementary class, so it suffices to prove that $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}^{\mathbf{ECF_{SK}}}$ satisfies the hypotheses of Theorem 3.1.5. Proposition 3.5.7 tells us that $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}^{\mathbf{ECF}}$ is an independence relation, satisfying the first hypothesis.

We now need to prove that the number of free extensions of weak types over finite sets is bounded; we show this is true for Galois types and the result for weak types follows, since the number of weak types is bounded above by the number of Galois types. If κ bounds the number of free extension of Galois 1-types over finite sets, then the number of n-types will be bounded by $\kappa^n = \kappa$, so we need only prove the statement for 1-types.

Let B, C be subsets with C finite, and let $a \in \mathbb{M}$. We claim that if $(a_i)_{i<\omega^+}$ is a sequence of realisations of $\mathrm{tp}^g(a/C)$ such that $\bar{a} \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}_C B$ for each $i < \omega^+$, then there are $i < j < \omega^+$ such that $\mathrm{tp}^g(a_i/B) = \mathrm{tp}^g(a_j/B)$. That is, there are at most countably many extensions of $\mathrm{tp}^g(a/C)$ to a Galois type over B.

If $a \in \lceil C \rceil^{\mathrm{ELA}}$, then there are at most countably many extensions of $\mathrm{tp}^g(a/C)$ since $\lceil C \rceil^{\mathrm{ELA}}$ is countable. If $a \in \mathbb{M}$ is exponentially transcendental over C, then a is exponentially transcendental over B. Therefore there is only one extension of $\mathrm{tp}^g(a/C)$ to B, namely the exponentially transcendental type over B. Suppose then that $a \in \mathrm{ecl}(C) \setminus \lceil C \rceil^{\mathrm{ELA}}$. There are only countably many types over $\lceil C \rceil^{\mathrm{ELA}}$, as a 1-type $\mathrm{tp}^g(b/\lceil C \rceil^{\mathrm{ELA}})$ corresponds uniquely with the countable strong ELA-subfield $\lceil b \lceil C \rceil^{\mathrm{ELA}} \rceil^{\mathrm{ELA}}$. In particular there are at most countably many Galois types p over $\lceil C \rceil^{\mathrm{ELA}}$ such that $p|C = \mathrm{tp}^g(a/C)$. Since $a \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}_C^{\mathbf{ECF_{SK}}} B$, by stationarity over strong ELA-subfields $\mathrm{tp}^g(a/\lceil C \rceil^{\mathrm{ELA}})$ uniquely extends to $\mathrm{tp}^g(a/B)$. Therefore the number of free extensions of a Galois 1-type over a finite set is at most countable.

Applying Theorem 3.1.5, the result follows. $\square$

It may be possible to excise the final qualifier, that $\mathop{\smile\!\!\!\!\!\!|}^{\mathbf{ECF_{SK}}}$ is the unique independence relation with bounded free extensions for weak types, so that $\mathop{\smile\!\!\!\!\!\!|}^{\mathbf{ECF_{SK}}}$ is indeed the unique independence relation on $\mathbf{ECF_{SK}}$. In this chapter we have shown that $\mathop{\smile\!\!\!\!\!\!|}^{\mathbf{ECF_{SK,CCP}}}$ is the canonical independence relation on $\mathbf{ECF_{SK,CCP}}$, and that $\mathop{\smile\!\!\!\!\!\!|}^{\mathbf{ECF_{SK}}}$ is the unique independence relation on $\mathbf{ECF_{SK}}$ with bounded free extensions for weak types. An independence relation on **ECF** must be defined over the kernel, as for a given model $\mathcal{M}$ we have $Z(\mathcal{M})$ interpretable, so an independence relation on all of $\mathcal{M}$ would restrict to an independence relation on $(Z(\mathcal{M}); +, \cdot) \models \mathrm{Th}(\mathbb{Z}; +, \cdot)$ which is not simple. The main difference between structures in the classes **ECF** and $\mathbf{ECF_{SK}}$ is the variability of the kernel, which means that types in **ECF** are less predictable. Our **ECF**-independence notion works around issues from the kernel by keeping the kernel in the base. In particular, this independence notion allows us to investigate types over models that are realised in kernel preserving extensions, that is, types orthogonal to the kernel. In the next chapter we use **ECF**-independence to show that these types are exactly the generically stable types, and consequentially **ECF**-independence is a useful definition of independence in **ECF**.

Chapter 4

Generic stability in ECF

In this chapter we use our independence relation $\mathop{\smile\!\!\!\!\!\!\mid}\,^{\mathbf{ECF}}$ to show that, assuming CIT, the global types that are orthogonal to the kernel are exactly the generically stable types in **ECF**.

We observed in Corollary 2.5.11 that Galois types and syntactic types over semi-strong ELA-subfields coincide in **ECF**, so unconditionally they coincide over models. Assuming CIT, all Galois types and syntactic types over sets in **ECF** coincide.

4.1 Types orthogonal to the kernel revisited

Fix $\mathbb{M}$ a monster model for **ECF**. Let $\mathcal{M}$ be a model in **ECF**, and suppose that p is a complete type over $\mathcal{M}$ realised by $\bar{a} \in \mathbb{M}^r$. In particular $\mathcal{M}$ is a semi-strong ELA-subfield of $\mathbb{M}$, so recall from Definition 2.5.2 that p is *orthogonal to the kernel* if there exists some $\mathcal{N} \in \mathbf{ECF}$ such that $\bar{a} \in \mathcal{N}^r$ and $\mathcal{M} \leq \mathcal{N}$ with $\ker(\mathcal{M}) = \ker(\mathcal{N})$ (that is, $\mathcal{M} \lhd \mathcal{N}$). Recall also that by

Proposition 2.6.5, if p is orthogonal to the kernel then it is grounded over a finite set.

Proposition 4.1.1. *Assume CIT. Let $\mathcal{M}$ be a model in $\boldsymbol{ECF}$ and let p be a complete syntactic type over $\mathcal{M}$ such that p is realised in some $\mathcal{N} \in \boldsymbol{ECF}$ such that $\mathcal{M} \leq \mathcal{N}$. Suppose that for every countable submodel $\mathcal{M}' \subseteq \mathcal{M}$ in $\boldsymbol{ECF}$ we have $p|\mathcal{M}'$ orthogonal to the kernel. Then p is orthogonal to the kernel.*

Proof. Suppose that for every countable submodel $\mathcal{M}' \subseteq \mathcal{M}$ in **ECF** we have $p|\mathcal{M}'$ orthogonal to the kernel. Let $(\mathcal{M}_i)_{i \in I}$ denote the directed system of countable submodels of $\mathcal{M}$ in **ECF** for some indexing set I. For each $i \in I$ we have $p|\mathcal{M}_i$ orthogonal to the kernel, realised in some strong extension $\mathcal{N}_i$ of $\mathcal{M}_i$ by $\bar{b}_i \in \mathcal{N}_i$. By Proposition 2.6.5 there exists a finite subset $A_i \subseteq \mathcal{M}_i$ grounding $p|\mathcal{M}_i$ and a grounding basis $\bar{b}'_i$ for $\lceil \bar{b}_i \mathcal{M}_i \rceil$ over $\mathcal{M}_i$. Then $V_i = \operatorname{Loc}(\bar{b}'_i, e^{\bar{b}'_i}/\mathcal{M}_i)$ and A_i characterise $p|\mathcal{M}_i$ as in Proposition 2.5.10, where $\bar{b}_i \in \operatorname{dcl}(A_i \bar{b}'_i)$.

Let $i_0 \in I$ be such that $\dim V_{i_0} = \min\{\dim V_i : i \in I\}$. Set $J = \{i \in I : \mathcal{M}_i \supseteq \mathcal{M}_{i_0}\}$, and let $j \in J$. Then $\dim V_j = \dim V_{i_0}$, and since V_j is absolutely irreducible we must have $V_j = V_{i_0}$, which is defined over A_{i_0}. Therefore A_{i_0} is a grounding set for $p|\mathcal{M}_j$ with $\bar{b}'_{i_0}$ a grounding basis. By CIT and Lemma 3.5.10, $p|\mathcal{M}_j$ is the unique free extension of $p|\mathcal{M}_{i_0}$. Viewed syntactically, p is the union of $p|\mathcal{M}_j$ over all $j \in J$, consequentially p is the unique free extension of $p|\mathcal{M}_{i_0}$ and p is grounded at A_{i_0}. By Lemma 2.6.2 p must be orthogonal to the kernel. $\square$

Definition 4.1.2. A complete syntactic type p over a saturated model $\mathcal{M}$ is *A-invariant* if for any automorphism $\sigma \in \operatorname{Aut}(\mathcal{M}/A)$ we have $\sigma p = p$.

Proposition 4.1.3. *Assume CIT. Let p be a global type over a saturated model $\mathcal{M}$ such that p is orthogonal to the kernel, and let $A \subseteq \mathcal{M}$ be a finite subset such that p is grounded at A. Then p is A-invariant.*

Proof. As in the proof of Theorem 2.6.7 we can construct a set of formulas $\Theta(\bar{x})$ defined over $\mathcal{M}$ such that $\Theta(\bar{x}) \vdash p(\bar{x})$. Let $\theta_{W,\Psi}(\bar{x}) \in \Theta(\bar{x})$ for some affine variety W defined over $\mathbb{Q}$ and definable subset $\Psi(\mathcal{M})$ of $\mathcal{M}$. Then for any automorphism $\sigma \in \mathrm{Aut}(\mathcal{M}/A)$, we have $\sigma(\theta_{W,\Psi}(\bar{x})) = \theta_{W,\sigma(\Psi)}(\bar{x})$ as the parameters defining $\Theta_{W,\Psi}$ are comprised of the tuple $\bar{c}$ from the kernel and the parameters over which the algebraic variety V from Proposition 2.5.10 is defined, and the parameters from $\mathcal{M}$ over which the formula Ψ and the algebraic variety W are defined. The set of parameters for V is A and $\bar{c} \in A$, so the parameters of $\Theta_{W,\Psi}$ are fixed by σ apart from those in Ψ. However $\sigma(\Psi(\mathcal{M}))$ is simply another definable subset of $\mathcal{M}$, and so the scheme of formulae Θ is fixed set-wise by $\mathrm{Aut}(\mathcal{M}/A)$. Suppose $\mathcal{N}$ is an elementary extension of $\mathcal{M}$ with $\bar{b} \in \mathcal{N}$ such that $\mathcal{N} \models \Theta(\bar{b})$. By applying Theorem 2.6.7 again we see that $\bar{b}$ is a realisation of p. However it is also the case that for any automorphism $\sigma \in \mathrm{Aut}(\mathcal{M}/A)$ we have $\mathcal{N} \models \sigma\Theta(\bar{b})$, and so $\bar{b}$ satisfies σp. Therefore p is A-invariant. $\square$

4.2 Generic stability

We shall adapt the notion of generic stability of a type from its definition in an arbitrary complete first order theory by Pillay and Tanovic [21, Definition 1].

Definition 4.2.1. Let p be a complete type over a model $\mathcal{M}$, and let

$A \subseteq \mathcal{M}$ be a small subset.

1. A *Morley sequence* of p over A is a sequence $(\bar{a}_i)_{i<\omega}$ from $\mathcal{M}$ such that $p|A = \mathrm{tp}(\bar{a}_i/A)$ and $\bar{a}_i \mathop{\smile\!\!\!\!\!\!\!\;\mid}^{f}_{A} A\bar{a}_1...\bar{a}_{i-1}$ for all $i < \omega$.

2. We say p is *generically stable over A* if

   (†) p is A-invariant, and

   (‡) for any Morley sequence $(\bar{a}_i)_{i<\omega}$ from $\mathcal{M}$ for p over A, and for any formula $\phi(\bar{x})$ with parameters in $\mathcal{M}$, we have $\{i : \mathcal{M} \models \phi(\bar{a}_i)\}$ finite or cofinite.

   We say p is *generically stable* if it is generically stable over A for some subset A.

In particular we are interested in the case where $\mathcal{M}$ is a saturated model for **ECF** with very full kernel, which allows us to quantify over all Morley sequences.

Lemma 4.2.2. *Let p be a complete type over a model $\mathcal{M} \in$ **ECF**. Suppose that p satisfies (‡) for some finite subset $A \subseteq \mathcal{M}$. Then for every countable submodel $\mathcal{M}'$ of $\mathcal{M}$ in **ECF** containing A we have $p|\mathcal{M}'$ satisfying (‡).*

Proof. Let $\mathcal{M}' \subseteq \mathcal{M}$ be a countable submodel of $\mathcal{M}$ in **ECF** and let $\phi(\bar{x})$ be a formula with parameters in $\mathcal{M}'$. Any Morley sequence $(\bar{a}_i)_{i<\omega}$ from $\mathcal{M}'$ for $p|\mathcal{M}'$ over A is a Morley sequence from $\mathcal{M}$ for p over A. Since p satisfies (‡) we have $\{i : \mathcal{M}' \models \phi(\bar{a}_i)\}$ finite or cofinite. Therefore $p|\mathcal{M}'$ satisfies (‡). $\square$

Proposition 4.2.3. *Let $\mathcal{M}$ be a countable model in **ECF**, p a complete type over $\mathcal{M}$. If p satisfies (‡), then p is orthogonal to the kernel.*

Proof. Suppose p is not orthogonal to the kernel, that is if $\mathcal{N}$ is an elementary extension of $\mathcal{M}$ realising p, then $\ker(\mathcal{N}) \neq \ker(\mathcal{M})$. The kernel is in definable bijection with Z, so there exists a new integer $z \in Z(\mathcal{N}) \setminus Z(\mathcal{M})$, and since Z is a definable ring, by replacing z with $-z$ if necessary we may take $z > 0$. Define $\pi(x) = \{Z(x) \wedge x > 0 \wedge x \neq b : b \in Z(\mathcal{M})\}$ to be a partial type over $\mathcal{M}$ for a new non-standard positive integer.

Let $(c_i)_{i<\omega 2}$ be a Morley sequence for p over $\mathcal{M}$. For each $i < \omega 2$ define a new language $\mathcal{L}_i = \mathcal{L} \cup \{c_i, (m)_{m \in \mathcal{M}}\}$ where c_i is a new constant symbol and $(m)_{m\in\mathcal{M}}$ are new constant symbols for every element of $\mathcal{M}$. We also construct a new theory $T_i = \mathrm{Diag}(\mathcal{M}) \cup p(c_i)$ for each $i < \omega 2$, where $\mathrm{Diag}(\mathcal{M})$ is the diagram of $\mathcal{M}$. Any model of T_i must realise π, so by the Omitting Types Theorem [23, Section 4.10] $\pi(x)$ is isolated in T_i by $\psi(x, c_i)$ for some formula $\psi(x, y) \in \mathcal{L}(\mathcal{M})$; here we may choose $\psi(x, y)$ to be independent of i, as $(c_i)_{i<\omega}$ is an sequence of indiscernibles. Since $\mathrm{Th}(\mathbb{N}; +, \cdot)$ is definably well ordered, we may take $\psi'(x, y)$ to be the formula picking out the minimal $x > 0$ such that $\psi(x, c_i)$ holds. Then $T_i \models \exists! x\, \psi'(x, c_i)$ for each $i < \omega 2$. Take $\mathcal{N} \succeq \mathcal{M}$ to be an elementary extension such that $c_i \in \mathcal{N}$ for all $i < \omega 2$. Then for each i we have $\mathcal{N} \models \exists! x\, \psi'(x, c_i)$ witnessed by $b_i \in Z(\mathcal{N})$.

Since $(c_i)_{i<\omega 2}$ is a Morley sequence we have $\mathrm{tp}^g(c_1, c_2) = \mathrm{tp}^g(c_i, c_j)$ for all $i < j < \omega 2$. Let $\theta(y_1, y_2) = \exists x_1 x_2 \psi'(x_1, y_1) \wedge \psi'(x_2, y_2) \wedge x_1 < x_2$, so for $i < j < \omega 2$ we have $\mathcal{N} \models \theta(c_i, c_j)$ if and only if $b_i < b_j$. Either $\theta(y_1, y_2)$ or $\theta(y_2, y_1)$ is in $\mathrm{tp}^g(c_1, c_2)$, so without loss of generality say $\theta(y_1, y_2) \in \mathrm{tp}^g(c_1, c_2)$. Let $b = b_\omega$ and consider the formula $\varphi(y, b) = \exists x \psi(x, y) \wedge x < b$. Then $\{i : \mathcal{N} \models \varphi(c_i, b)\}$ is infinite and co-infinite, so $\varphi(x, \bar{b})$ witnesses the failure of (‡) for p. $\square$

Proposition 4.2.4. *Let p be a complete type over a model $\mathcal{M} \in$ **ECF**. If p satisfies (‡), then p is orthogonal to the kernel.*

Proof. If p satisfies (‡) then by Lemma 4.2.2 we have $p|\mathcal{M}'$ satisfying (‡) for all countable submodels $\mathcal{M}' \subseteq \mathcal{M}$. But then by Proposition 4.2.3 we have $p|\mathcal{M}'$ orthogonal to the kernel for every $\mathcal{M}'$. Therefore by Proposition 4.1.1 p is orthogonal to the kernel. $\square$

Theorem 4.2.5. *Assume CIT. Let $\mathcal{M}$ be a saturated model, and suppose that p is a complete type over $\mathcal{M}$. Then p is orthogonal to the kernel if and only if p is generically stable.*

Proof. If p is generically stable then p satisfies (‡), so by Proposition 4.2.4 p is orthogonal to the kernel. Suppose conversely that p is orthogonal to the kernel. By Proposition 2.6.5 we can find a finite subset $A_0 \subseteq \mathcal{M}$ such that p is grounded at A_0. We have $A_0 \prec \mathcal{M}$ so defining $A = \lceil A_0 \rceil^{\mathrm{ELA}}_{\mathcal{M}}$, by Corollary 2.6.8 we have $p|A$ the unique type extending $p|A_0$. Then for $\mathcal{N} \in \mathbf{ECF}$ a strong elementary extension of $\mathcal{M}$ with $\bar{a} \in \mathcal{N}$ realising p, and $\bar{a}' \in \mathcal{N}$ a $\mathbb{Q}$-linear basis for the hull of $\bar{a}$ over the kernel, we have $\mathrm{etd}(\bar{a}/A) = \mathrm{etd}(\bar{a}/\mathcal{M})$, $A \triangleleft \mathcal{M}$, and $A\bar{a}' \triangleleft \mathcal{N}$. By Proposition 4.1.3, p is A-invariant.

Let $\phi(\bar{x}, \bar{b})$ be a formula with $\bar{b} \in \mathcal{M}$. By changing parameters if necessary we may assume that $\bar{b}$ is $\mathbb{Q}$-linearly independent and $A\bar{b} \triangleleft \mathcal{M}$. Let $(\bar{a}_i)_{i<\omega}$ be a Morley sequence for p over A. Note that $\mathrm{etd}(\bar{a}_i/A)$ is fixed for all $i < \omega$, and set $d = \mathrm{etd}(\bar{a}_i/A)$. For any given $i_1, ..., i_n < \omega$ we have $\mathrm{etd}(\bar{a}_{i_1}, ..., \bar{a}_{i_n}/A) = nd$. Define $I = \{i : \mathrm{etd}(\bar{a}_i/A\bar{b}) < \mathrm{etd}(\bar{a}_i/A)\}$. We demonstrate that I must be finite. Let $i_1, ..., i_n \in I$ so $\mathrm{etd}(\bar{a}_{i_k}/Ab) \leq d-1$. Then by additivity

$$\begin{aligned}
\mathrm{etd}(\bar{a}_{i_1}, ..., \bar{a}_{i_n}\bar{b}/A) &= \mathrm{etd}(\bar{b}/A\bar{a}_{i_1}, ..., \bar{a}_{i_n}) + \mathrm{etd}(\bar{a}_{i_1}, ..., \bar{a}_{i_n}/A) \\
&= \mathrm{etd}(\bar{b}/A\bar{a}_{i_1}, ..., \bar{a}_{i_n}) + nd
\end{aligned}$$

Using additivity the other way we obtain

$$\begin{aligned}
\mathrm{etd}(\bar{a}_{i_1}, ..., \bar{a}_{i_n}\bar{b}/A) &= \mathrm{etd}(\bar{b}/A) + \mathrm{etd}(\bar{a}_{i_1}, ..., \bar{a}_{i_n}/A\bar{b}) \\
&\leq \mathrm{etd}(\bar{b}/A) + \sum_{k=1}^{n} \mathrm{etd}(\bar{a}_{i_k}/A\bar{b}) \\
&\leq \mathrm{etd}(\bar{b}/A) + \sum_{k=1}^{n} [\mathrm{etd}(\bar{a}_{i_k}/A) - 1] \\
&= \mathrm{etd}(\bar{b}/A) + n(d-1)
\end{aligned}$$

Therefore $\mathrm{etd}(\bar{b}/A\bar{a}_{i_1}, ..., \bar{a}_{i_n}) + nd \leq \mathrm{etd}(\bar{b}/A) + n(d-1)$ and so $\mathrm{etd}(\bar{b}/A\bar{a}_{i_1}, ..., \bar{a}_{i_n}) + n \leq \mathrm{etd}(\bar{b}/A)$, so n is bounded by $\mathrm{etd}(\bar{b}/A)$, and hence I is finite.

Define $J = \{i : \bar{a}_i \mathop{\not\perp\!\!\!\perp}_A \bar{b}\} \cap I^c$. We show that J is also finite. Since A is a strong ELA-subfield and $\mathrm{etd}(\bar{a}_i/A) = \mathrm{etd}(\bar{a}_i/A\bar{b})$ for all $i \in J$, by Lemma 3.5.5 we have $J = \{i : \bar{a}_i \mathop{\not\perp\!\!\!\perp}_A^{\mathbb{Q}\text{-lin}} \bar{b}\} \cap I^c$.

Suppose J is not finite. Treating the $\bar{a}_i$ as sets, $\bigcup_{i \in J} \bar{a}_i$ is a $\mathbb{Q}$-linearly independent set over A, where $\mathrm{ldim}_{\mathbb{Q}}(\bar{a}_i) = n$ for each $i \in J$. The $\lceil A\bar{a}_i \rceil$ are orthogonal as subspaces of $\mathcal{M}$ over A, that is $\mathrm{ldim}_{\mathbb{Q}}(\bar{a}_{i_1}, \bar{a}_{i_2}, ..., \bar{a}_{i_r}/A) = rn$ for any $i_1, ..., i_r \in J$. Suppose then that for each $i \in J$, there exists non-zero $u_i \in \lceil A\bar{b} \rceil \cap \lceil A\bar{a}_i \rceil$ for each $i \in J$. Setting $m = \mathrm{ldim}_{\mathbb{Q}}(\bar{b}/A)$, there exist $\mathbb{Q}$-linearly dependent $u_{i_1}, ..., u_{i_{m+1}} \in \lceil A\bar{b} \rceil$. But the $u_i \in \lceil A\bar{a}_i \rceil$ and the $\lceil A\bar{a}_i \rceil$

are orthogonal, which is a contradiction.

For any $i, j \notin J$ we have $\bar{a}_i \mathop{\smile\!\!\!\!\!\mid}_A \bar{b}$, $\bar{a}_j \mathop{\smile\!\!\!\!\!\mid}_A \bar{b}$ and $\mathrm{tp}^g(\bar{a}_i/A) = \mathrm{tp}^g(\bar{a}_j/A)$; therefore by stationarity over strong ELA-subfields from Proposition 3.5.7 we have $\mathrm{tp}^g(\bar{a}_i/A\bar{b}) = \mathrm{tp}^g(\bar{a}_j/A\bar{b})$, and so $\mathrm{tp}(\bar{a}_i/A\bar{b}) = \mathrm{tp}(\bar{a}_j/A\bar{b})$. If $\phi(\bar{x}, \bar{b}) \in \mathrm{tp}(\bar{a}_i/A\bar{b})$ for some $i \in J^c$, then $\phi(\bar{x}, \bar{b}) \in \mathrm{tp}(\bar{a}_i/A\bar{b})$ for all $i \in J^c$, that is $\mathcal{M} \models \phi(\bar{a}_i, \bar{b})$ for all $i \in J^c$. But then $\{i : \mathcal{M} \models \phi(\bar{a}_i, \bar{b})\} \supseteq J^c$, and in particular is cofinite. If $\phi(\bar{x}, \bar{b}) \notin \mathrm{tp}(\bar{a}_i/A\bar{b})$ for some $i \in J^c$ then $\neg\phi(\bar{x}, \bar{b}) \in \mathrm{tp}(\bar{a}_i/A\bar{b})$, and proceeding as before we see that $\{i : \mathcal{M} \models \neg\phi(\bar{a}_i, \bar{b})\}$ is cofinite. $\square$

Corollary 4.2.6. *Assume CIT, and let p be a complete type over a saturated model $\mathcal{M}$ that is orthogonal to the kernel and grounded at A. Then:*

(i) p is A-definable.

(ii) p is finitely satisfiable over A, that is, any finite partial type comprised of formulas from p is satisfiable in any elementary substructure of $\mathcal{M}$ containing A.

(iii) Any Morley sequence of p over A is totally indiscernible.

(iv) p is the unique non-forking extension of $p|A$.

Proof. By Theorem 4.2.5 p is A-invariant and generically stable. Then [21, Prop. 1(ii)-(iv)] gives us the above results. $\square$

We have seen that in **ECF**, assuming CIT, the exponential-field theoretic property of orthogonality to the kernel coincides with the model-theoretic property of generic stability. **ECF**-independence is therefore a useful notion of independence for this class, and could potentially lead to further results. It would be interesting to study other meaningful model-theoretic properties of types orthogonal to the kernel. We shall describe this and other possible future directions in the final section.

4.3 Final remarks

In this chapter we have proved that a model-theoretic property, generic stability, is equivalent to an exponential-algebraic property, orthogonality to the kernel. Assuming CIT means that **ECF** is the class of all models of a complete first order theory, which allows for many first-order model-theoretic concepts. We would like to know which of these concepts can also be understood in terms of exponential algebra. We provide an encouraging example.

Definition 4.3.1. [21, Definition 3] Let p be a non-algebraic complete type over a saturated model $\mathcal{M}$ in **ECF**. Then p is *invariant regular* if for some small A, it is A-invariant and for any superset $B \supseteq A$ in $\mathcal{M}$ and $\bar{a} \in \mathcal{M}$ realising $p|A$, either $\bar{a}$ realises $p|B$, or $p|B \vdash p|B\bar{a}$.

We say that p is *invariant strongly regular* if there exists a formula $\phi \in p$ and some small A such that p is A-invariant and for any superset $B \supseteq A$ in $\mathcal{M}$ and $\bar{a} \in \mathcal{M}$ such that $\mathcal{M} \models \phi(\bar{a})$, then either $\bar{a}$ realises $p|B$ or $p|B \vdash p|B\bar{a}$.

If p is invariant (strongly) regular and generically stable we say that it is *generically stable (strongly) regular.*

Proposition 4.3.2. *The exponentially transcendental complete type q over* *a model $\mathcal{M} \in$ **ECF** is generically stable strongly regular for the formula $x = x$.*

Proof. **Generic stability:** Let $\phi(x, \bar{y})$ be a formula and $\bar{b}$ a parameter set in the model $\mathcal{M}$. Let $(a_i)_{i<\omega}$ be a Morley sequence for q over A, so the a_i are exponentially transcendental over A and exponentially algebraically independent over A. Since $\mathrm{etd}(\bar{b}) \leq |\bar{b}|$, by the exchange property $|\{i : a_i \text{ exponentially algebraic over } \bar{b}\}| \leq |\bar{b}|$, and so either $\{i : \mathcal{M} \models \phi(a_i, \bar{b})\}$ or its complement will be finite.

Regularity: Let $A \subseteq B$ be subsets of $\mathcal{M}$. Let a be a realisation of $q|A$, that is a exponentially transcendental over A. If $a \not\models q|B$ then a is exponentially algebraic over B, so if $c \models q|B$ then $c \models q|Ba$. $\square$

The above proof that the exponentially transcendental type in **ECF** is generically stable is a special case of the proof of Theorem 4.2.5. Strong regularity of q follows from $\mathrm{etd}(-)$ being the dimension of a pregeometry on $\mathcal{M}$.

We would like to know what other types are generically stable regular in **ECF**.

Definition 4.3.3. [15, Definition 5.1] Let $V \subseteq G^n$ be an algebraic subvariety. We say that V is *perfectly rotund* iff it is irreducible, $\dim V = n$, and for every matrix $M \in \mathrm{Mat}_{n\times n}(\mathbb{Z})$ such that $0 < \mathrm{rk}M < n$,

$$\dim M \cdot V \geq \mathrm{rk}M + 1.$$

Assume CIT. Let p be a complete type over a model $\mathcal{M}$ orthogonal to the kernel, grounded at $A \subseteq \mathcal{M}$. Then p has a realisation $\bar{a}$ in some strong extension $\mathcal{N} \in \mathbf{ECF}$ of $\mathcal{M}$, and let $\bar{a}'$ be a grounding basis for $\lceil \bar{a}\mathcal{M} \rceil_{\mathcal{N}}$ over $\mathcal{M}$. With this setup, we make the following conjecture.

Conjecture 4.3.4. If $\mathrm{Loc}(\bar{a}', e^{\bar{a}'}/A, e^A)$ is perfectly rotund, then p is regular.

The converse does not always hold; let $\bar{a} = (a_1, ..., a_n) \in \mathbb{M}$ be a grounding basis for $\lceil \bar{a}\mathcal{M} \rceil$ over $\mathcal{M}$ such that $\mathrm{tp}(\bar{a}, e^{\bar{a}}/\mathcal{M})$ is regular, and $V = \mathrm{Loc}(\bar{a}, e^{\bar{a}}/\mathcal{M})$ is perfectly rotund. Setting $\bar{b} = \bar{a}e^{a_1}$ we have $W = \mathrm{Loc}(\bar{b}, e^{\bar{b}})$ not perfectly rotund. Similarly $U = \{(\bar{x}, \bar{x}, \bar{y}, \bar{y}) : (\bar{x}, \bar{y}) \in V\}$ is not perfectly rotund. However U, V, W all give rise to a regular type p.

A pertinent direction for future research could be to determine, assuming CIT, what other model-theoretic properties are equivalent to meaningful exponential algebraic properties. For instance, in **ECF** how can one describe a locally modular type, or a trivial type, in terms of exponential algebra? It is hoped that this thesis is an encouraging first step towards answering these sorts of questions, and that our independence relation $\mathop{\smile\!\!\!\!\!\!|}^{\mathbf{ECF}}$ may prove a useful tool in future research of **ECF**.

The assumption that CIT holds has been used at several points in this thesis to allow us to consider **ECF** as an elementary class, in particular so that we can apply first order tools such as compactness in the proof of Theorem 2.6.7, and assume that Galois and syntactic types coincide over sets. This application of CIT has quite an effect on later chapters; in particular our proof of Proposition 4.1.3 relies on the set of formulas defined in the proof of Theorem 2.6.7. Referencing this Proposition appears to be the only use of CIT in Theorem 4.2.5. It should be possible to remove the assumption of CIT from the thesis entirely, and certainly a good first step would be to find an alternative proof to Theorem 2.6.7 not assuming CIT.

We make a final observation, based on a suggestion of Kirby and Zilber in [16, Section 7]. Hitherto in this thesis the proofs of results for **ECF** have not explicitly used axiom (IIb), which states that $(Z; +, \cdot) \models \mathrm{Th}(\mathbb{Z}; +, \cdot)$. Without this axiom, by axiom (IIa) we still have $(Z; +, \cdot)$ an integral domain with $(Z; +) \equiv (\mathbb{Z}; +)$. Replacing axiom (IIb) with an axiom stating that $(Z; +, \cdot)$ is a model of the complete theory of any other integral domain whose additive group is a model of $\mathrm{Th}(\mathbb{Z}; +)$, it would be interesting to see if our conclusions also hold for this theory.

Appendix A

Classes of exponential fields

We provide a summary of the AECs of exponential fields studied in this thesis. The axiomatic definition of these classes are from Definition 2.1.3 and finitary/non-finitary results are from Proposition 2.2.19.

$\underline{(\mathbf{ECF_{SK,CCP}}, \subseteq^{cl})}$ is the class of all structures satisfying axioms (I), (II), (III), (IV), (V). It is a non-finitary AEC.

Definition 2.2.18 For $\mathcal{M} \subseteq \mathcal{N}$ in $\mathbf{ECF_{SK,CCP}}$, we say $\mathcal{M} \subseteq^{cl} \mathcal{N}$ if $\mathrm{ecl}_{\mathcal{N}}(\mathcal{M}) = \mathcal{M}$.

$\underline{(\mathbf{ECF_{SK}}, \triangleleft)}$ is the class of all structures satisfying axioms (I), (II), (III), (IV). It is a finitary AEC.

Definition 2.2.7 For $\mathcal{M} \subseteq \mathcal{N}$ in $\mathbf{ECF_{SK}}$, we say $\mathcal{M} \triangleleft \mathcal{N}$ if $\Delta(\bar{a}/\mathcal{M}) \geq 0$ for all $\bar{a} \in \mathcal{N}$.

$\underline{(\mathbf{ECF}, \leq)}$ is the class of all structures satisfying axioms (I), (IIa), (IIb), (III), (IV), (V). It is a finitary AEC.

Definition 2.2.12 For $\mathcal{M} \subseteq \mathcal{N}$ in $\mathbf{ECF}$ we say $\mathcal{M} \leq \mathcal{N}$ if $\mathcal{M} \prec \mathcal{N}$ and $Z(\mathcal{M}) \preccurlyeq Z(\mathcal{N})$.

Bibliography

[1] H Adler, *'Explanation of Independence,' Dissertation, 2005. arXiv:math/0511616*

[2] M Bays, M Gavilovich, M Hils, *'Some Definability Results in Abstract Kummer Theory,' International Mathematics Research Notices Vol. 2014 (14): 3975-4000, 2013. doi:10.1093/imrn/rnt057*

[3] S Buechler, *'Essential Stability Theory,' Berlin: Springer-Verlag, 1996.*

[4] R Grossberg, O Lessmann, *'Dependence Relations in Pregeometries,' Algebra Universalis 44 (3-4): 199-216, 2000. doi:10.1007/s000120050182*

[5] T Hyttinen, M Kesala, *'Independence in Finitary Abstract Elementary Classes,' Annals of Pure and Applied Logic 143 (1-3): 103-138, 2006. doi:10.1016/j.apal.2006.01.009*

[6] T Hyttinen, M Kesala, *'Categoricity Transfer in Finitary Abstract Elementary Classes,' Journal of Symbolic Logic 76 (3): 759-806, 2011. doi:10.2178/jsl/1309952520*

[7] W Hodges, *'Model Theory,' Cambridge University Press, 1993.*

[8] K Kangas, *'Finding Groups in Zariski-like Structures,' Licentiate's thesis, 2014. arXiv:1404.6811*

[9] B Kim, *'Forking in Simple Unstable Theories,' Journal of the London Mathematical Society Vol. 57 (2): 257-267, 1998. doi:10.1112/S0024610798005985*

[10] B Kim, *'Simplicity, and stability in there,' Journal of Symbolic Logic Vol. 66 (2): 822-836, 2001. doi:10.2307/2695047*

[11] B Kim, A Pillay, *'Simple Theories,' Annals of Pure and Applied Logic Vol. 88 (2-3): 149-164, 1997. doi:10.1016/S0168-0072(97)00019-5*

[12] J Kirby, *'Exponential Algebraicity in Exponential Fields,' Bulletin of the London Mathematical Society 42 (5): 879-890, 2010. doi:10.1112/blms/bdq044*

[13] J Kirby, *'On quasiminimal excellent classes,' Journal of Symbolic Logic Vol. 75 (2): 551-564, 2010. doi:10.2178/jsl/1268917494*

[14] J Kirby, *'A note on the axioms for Zilber's pseudo-exponential fields,' Notre Dame Journal of Formal Logic Vol. 54 (3-4): 509-520, 2013. doi:10.1215/00294527-2143844*

[15] J Kirby, *'Finitely Presented Exponential Fields,' Algebra and Number Theory Vol. 7(4): 943-980, 2013. doi:10.2140/ant.2013.7.943*

[16] J Kirby, B Zilber, *'Exponentially closed fields and the conjecture on intersections with tori,' Annals of Pure and Applied Logic, 2014. doi:10.1016/j.apal.2014.06.002*

[17] D Kueker, *'Abstract Elementary Classes and Infinitary Logics,' Annals of Pure and Applied Logic Vol. 156: 274-286, 2008. doi:10.1016/j.apal.2008.07.002*

[18] A Macintyre, *'Exponential Algebra,' Logic and Algebra, Lecture Notes in Pure and Applied Mathematics 180: 191-210, 1996.*

[19] D Marker, *'Model Theory: An Introduction,' Graduate Texts in Mathematics, Springer-Verlag, 2002.*

[20] D Marker, *'A remark on Zilber's pseudo-exponentiation,' Journal of Symbolic Logic Vol. 71: 791-798, 2006. doi:10.2178/jsl/1154698577*

[21] A Pillay, P Tanovic, *'Generic stability, regularity, quasi-minimality,' Models, Logics and Higher-dimensional categories: a tribute to the work of Mihaly Makkai, CRM Proceedings and Lecture Notes, Vol. 53, Journal of the American Mathematical Society, 2011.*

[22] P Ribenboim, *'My numbers, my friends: popular lectures on number theory,' Springer-Verlag, New York, 2000.*

[23] K Tent, M Ziegler, *'A Course in Model Theory,' Lecture Notes in Logic, Cambridge University Press, 2012.*

[24] A Wilkie, *'Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function,' Journal of the American Mathematics Society Vol. 9: 1051-1094, 1996. doi:10.1090/S0894-0347-96-00216-0*

[25] B Zilber, *'Exponential sums equations and the Schanuel conjecture,' Journal of the London Mathematical Society 65 (1): 27-44, 2002. doi:10.1112/S0024610701002861*

[26] B Zilber, *'Pseudo-exponentiation on algebraically closed fields of characteristic zero,' Annals of Pure and Applied Logic 132: 67-95, 2005. doi:10.1016/j.apal.2004.07.001*