SECONDARY STUDENTS' PROOF SCHEMES DURING THE FIRST ENCOUNTERS WITH FORMAL MATHEMATICAL REASONING:

APPRECIATION, FLUENCY AND READINESS

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An analysis approach by means of Harel and Sowder's proof schemes taxonomy which reflects multiplicity of proof schemes, proof appreciation, proof fluency, proof-readiness

ABSTRACT

The topic of the thesis is proof. At Year 9 Greek students encounter proof for the first time in Algebra and Geometry. Thus the principal research question of the thesis is: How do students' perceive proof when they first encounter it? The analysis tool in order to obtain an image of students' perception of proof, the Harel and Sowder's taxonomy, is itself a research question in what concerns its applicability under Greek conditions. Its applicability, of which there is strong evidence, provides the space to shape an image of students' proof fluency, proof appreciation, proof readiness etc.

In order to collect data with regard to answering the research questions in collaboration principally with the class teacher I constructed the two tests on proof that are presented in this thesis. The first test was administered to the students of Year 9 at the beginning of the school year 2010-2011 before the teaching of proof. The second was administered after the teaching of proof of the same school year. Students' answers were analyzed and provided strong evidence that the Harel and Sowder's taxonomy is applicable on them. Thus every answer was characterized in terms of the taxonomy. As a result every individual student but also the whole sample is depicted by proof schemes.

The major findings of the analysis are the two following:

- Students' proof fluency is higher in simple proof issues. Although they face difficulties when the issues are more demanding, they show high proof appreciation.
- There is strong evidence of the applicability of the Harel and Sowder's taxonomy in a completely different socio-cultural and educational environment in comparison to that of its original invention and application. In the same vein the research proposes the mixture of proof schemes within one proof as theoretical and methodological contribution.

Finally from the findings emerge new research questions as e.g.

- How teaching and curriculum affect students' proof schemes?
- What is the origin of mixed proof schemes?

CONTENTS

To my loving wife Eva,

and to my beloved children Lena and Ili

who always support and encourage me

and in memory of my beloved mother Eleni Kanellou

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CHAPTER 1: INTRODUCTION

Pompey: … but what mystery there should be in hanging, if I should be hanged, I cannot imagine.

Abhorson: Sir, it is a mystery.

Pompey: Proof?

Abhorson: Every true man's apparel fits your thief. If it be too little for your thief, your true man thinks it big enough; if it be too big for your thief, your thief thinks it little enough: so every true man's apparel fits your thief.

(William Shakespeare, Measure for measure)

In this thesis I report on a research project on Greek secondary education students' first encounters with proof as predicated and described in the mathematical curriculum for Year 9 secondary education in Greece. The project was conceived in collaboration with my supervisors Elena Nardi and Irene Biza in the context of Doctorate in Education (EdD) studies in the School of Education and Lifelong Learning in UEA and it began in 2008. The three of us discussed the project extensively during the biannual Conference of the Greek Association of Researchers in Mathematics Education (ENEDIM) in Rhodes in October-November 2009. I then carried out the data collection for the project in the 2010-2011 school year which in Greece begins mid-September and ends mid-May. With my present professional engagement as a secondary school advisor responsible for teaching mathematics in the prefectures of Heraklion and Lassithi in Crete, I was in the privileged position of being able to implement and conduct the project.

To reach the professional status of secondary school advisor I first studied mathematics from October 1972 to May 1977 at the Aristotle University of

Thessaloniki, Greece. After graduating with a bachelor's degree, I worked in the private sector as a mathematics teacher. Parallel to this in the summer of 1977 I applied for a position teaching mathematics in Greece's secondary state schools which are under the supervision of the Greek Ministry of Education. In Greece, at the time, with a bachelor's degree in mathematics one was allowed to teach mathematics in secondary state schools and the equivalent private schools. I was appointed as a state school teacher in 1982 and from then until March 2003, with a break between 1999 and 2001, I taught mathematics in lower and upper secondary schools.

Lower secondary Greek education includes Years 7, 8 and 9 with students' aged 13, 14 and 15 years respectively; Upper secondary education includes the Years 10, 11 and 12 with students' aged 16, 17 and 18 years respectively. At the end of Year 12, students take the university entrance examinations. The Greek secondary education system has remained almost unaltered for the last forty years and I went through it myself before passing my university entrance examinations.

At the end of 1998, while working as a mathematics teacher, I passed the examinations of the Education Department at the University of Crete and was granted a study leave by the Ministry of Education Greek to study the teaching of mathematics in the Department at Masters level from March 1999 to December 2001. In December 2001 I obtained my Master of Science degree in the Didactic of Mathematics.

In mid-2002 I applied for a position as secondary school advisor responsible for teaching mathematics. I was appointed in March 2003 and still remain in this position after having been assessed two more times, in July 2007, and in October 2011. According to the Greek law school advisors have to be assessed every four years in order to remain in their position.

Mathematics, and especially mathematical proof, plays an important role in Greek education. Indeed the role of mathematics in Greek education, with mathematical proof at its core, has special weight as for a large number of Year12 students, one of the six subjects examined for university entrance is mathematics.

More specifically, the mathematics in this examination is about calculus, elementary analysis and complex numbers. The examination has typically four questions and almost all four require proof and this requirement for proof is at a rather high standard. Consider a problem from an example from Inglis and Mejia-Ramos (2008) (referred in Tall and Mejia-Ramos (2006)) set a second year university student: "Prove that the derivative of a differentiable even function is an odd function" to study the university students' perceptions of the mathematical notions in question as well as their proof behaviour. The same question, had it been set in the aforementioned Greek secondary education examination would have been perceived as one of the simplest. Consequently, in order to reach this level of mathematical thinking and to be able to understand proof and carry out proof processes, students must become acquainted with proof relatively early in their education. It is in this spirit that proof has been a significant issue in the Greek curriculum for decades.

From 2007 new text-books of Mathematics were introduced in the Lower Secondary Education. In this text-books proof is introduced in Year 9. This was valid while I was conducting the research. Proof appears in two forms in this context: algebraic and geometric. Algebraic proof mainly includes proof identities such as $(a+b)^2 = a^2 + 2ab + b^2$ and other algebraic relations as inequalities using the laws of algebraic operations. Geometric proof appears in the form of applying the triangle congruency criteria which then are used to prove various properties of geometric figures such as that every point of the bisector of any angle is equidistant from the sides of the angle.

Some of the main issues in teaching proof are the degree to which proof is taught successfully; how this success, if any, is obtained; how students perceive proof; and, consequently, how they attempt to engage with the proving process. As I mentioned above Greek students' first encounters with proof are of paramount importance at least with regard to the effect this has on their subsequent engagement with proof, their success in the aforementioned final examinations and, ultimately, influences strongly their choice of university studies. In other words mathematics and proof are of decisive importance in students' lives. Thus, research into how the first encounters with proof take place in typical secondary classrooms is crucial; and it is the key idea underlying the conception of my research project.

Choosing this kind of research has practical use because it studies learning in real life school situations; it is useful because it can have direct implications for practice and, to a school advisor and experienced mathematics teacher, it is also an attractive task. Beyond this personal and local interest however, there is a rather broader interest in a project like this: although its success in international comparison such as Trends in International Mathematics and Science Studies (TIMSS) and Program for International Student Assessment (PISA) is questionable, Greece is one of the countries that associates high school standards in mathematics with an emphasis on proof in the school curriculum. In the last decade several other countries introduced proof into the school curriculum and it makes sense that the Greek experience on this matter is likely to be of international relevance. Mariotti comments:

> 'Reasoning and proof are not special activities reserved for special times or special topics in the curriculum but should be a natural, ongoing part of classroom discussions, no matter what topic is being studied'.(NCTM, 2000, p. 342)

> I wonder whether these words would have been possible only a few years ago, and still now the idea of "proof for all" claimed in this quotation is not a view that most teachers hold, even in countries where there is a longstanding tradition of including proof in the curriculum. (Mariotti, 2006, p. 173)

I find Mariotti's thoughts on this to be well founded:

> Why does a pupil learn to speak his mother tongue but not mathematics? In the mother tongue he is living the whole day, may be in his dreams too. Mathematics can only claim 4 or 5 hours a week. What is learned unrelatedly does not last long. Is it not the disappointment familiar to every teacher that subjects taught a few weeks ago seem to have disappeared out of the pupils' minds, with no trace left unless they have been retrained in the meantime? (Freudenthal, 1973, p. 77)

Freudenthal speaks of mathematics in general as a curriculum subject, but if mathematical notions disappear from students' minds, as he argues, then the same is even truer for proof and proving, because without these mathematical notions and knowledge no proof can be understood not to speak of carrying proof out. And further:

> Till now, education in Western Europe has been élite education, that is to say education of an élite or at least for an élite. This tendency alas has been reinforced by most of the innovation movements. As for mathematics I am afraid that its educational programmes and methods are influenced by a belief which is natural for every mathematician, that mathematical education is education to become a mathematician-those who cannot keep pace are left behind. And for those who were left behind or who never even embarked, they serve up as a second infusion of this mathematics for the élite. (ibid., p. 62)

Freudenthal warns us here not to accept the idea that mathematics is for just a few students, which directly implies that proof is not for all. In a way, he anticipates the 'proof for all' movement before it was given birth. At the same time, Mariotti not only highlights the tendency to introduce proof into the school mathematical curriculum but also proposes:

> The evolution of a mathematical culture in the classroom is a long-term process, requiring specific strategies of intervention that begin very early and develop over a long period. In this respect, investigation cannot be detached from classroom reality and, generally speaking, from the school environment: classroom investigations are of great value, and, although

> they raise difficult methodological problems, they should be promoted both in the form of comparison between different cultural experiences and in the form of teaching experiments. (Mariotti, 2006, p. 199)

Although Mariotti does not explicitly refer to proof in the above quotations it fits perfectly as well in the case of teaching proof because proof is at the heart of mathematical culture. At the same time she offers a strong argument in favour of research like the one I present here which is based on the experience of the natural learning environment, the classroom, with typical learners.

Some researchers although did not work on proof explicitly, focussed on aspects of mathematics and produced research results which are of great importance in relation to proof and proving in school. For instance the Van Hieles (1984) developed the theory of cognitive levels in geometry. Fuys, Geddes and Tischler successfully took on the task of translating the doctoral thesis of Dina van Hiele-Geldof and other works of the van Hieles from the Dutch into English. The Van Hiele cognitive levels may not refer directly to proof, but presumably remaining at low cognitive levels does not help to develop competency at proof. Brousseau (1997) developed the theory of teaching situations[1] and, although also not directly referring to proof, analyses the didactical value of Euclidean geometry (2000) which leads directly to the question of proving ability, because due to its origin, Euclidean geometry has the logical structure that bears proof as mode of existence.

Many researchers have studied the teaching and learning of proof explicitly. Balacheff offers a proof taxonomy in at least one of his works (Balacheff, 1987). Healey and Hoyles (2000) work on students' conceptions of proof. Harel and Sowder observe a taxonomy in the ways that students attempt to prove propositions in various fields of mathematics such as geometry, linear algebra etc., and formulate their

[1] The terms are known in French as situations didactiques

conclusions of corresponding teaching experiments in their theory of students' proof schemes (Harel & Sowder, 1998, 2007). The whole of the 19th International Commission on Mathematics Instruction (ICMI) conference, held in Taiwan in 2009, was dedicated to proof. In 2010 Hanna and others edited a collection of works on proof (Hanna, Jahnke, & Pulte, 2010). Thus many researchers underline the significance of the learning and teaching of proof. I return to this research on proof in the literature review but the fleeting references to this research here serve the purpose of highlighting that there is substantial and influential work in this area, particularly in the form of classroom based investigations and also theoretical analysis. The richness of this field may suggest that the tendency to study proof processes in secondary education will become even stronger in the future. The project reported in this thesis aspires to make a contribution in this respect.

In addition to this support from tendencies in the international literature I have accumulated substantial professional and personal experience of the difficulties involved in the teaching and learning of proof and proving. I still remember vividly the teacher trying to teach the following theorem, to the Year 9 class- and me among them, at a school in Athens in 1970:

> "If the external bisector of a triangle is parallel to the opposite side thereof, then the triangle is an isosceles one, and vice versa."

I also remember that the proof given by the teacher was, to me, somewhat vague and not easily understandable. I could not see the proof process clearly and could not do it correctly. I cannot recall the exact issue I was struggling with, but the sense of hardship I experienced as a learner is still with me today. Thus the research project stems from my commitment to observe the endeavours of today's students, analyse their difficulties and find ways of helping them to overcome these. This is an exciting prospect. Apart from this personal and professional commitment the merits of such an effort are multiple. The professional development of teachers can only gain from discussion of what a study like this finds. I have found so from personal experience. As a newly-appointed teacher I learned to appreciate research in the field of teaching through reading and engaging with it. I remember demonstrating triangle inequality empirically to Year 8 and 9 students using the Castelnuovo's triangle after reading an article in Euclid 3[2] (Valtas, 1983).

Later in my carrier as a teacher, while reading articles such as Anna Sfard's "On reform movement and the limits of mathematical discourse" (2000), I arrived at the conclusion that had I had the chance to read such works as a newcomer to the teaching profession, it would have provided me with a guide to teaching mathematics satisfyingly and to clarifying what reforming teaching actually means.

In this context and in order to gain insight into the world of students' proof perceptions the need of an analytical tool is necessary. In my research project this tool is the Harel and Sowder taxonomy. The choice of the analytical tool will be explained in a more detailed manner in Chapter 2: Literature Review. For the time-being I name the taxonomy only for the sake of making clear what I refer to in the formulation of the research questions of my study as they emerge at this point. Namely the purpose of the study is to find answers to the following research questions:

- **a)** What are students' pre-proof perceptions?
- **b)** What are students' perceptions of proof when they first encounter it?
- **c)** How, if at all, is the Harel and Sowder taxonomy applicable to the Greek secondary educational contexts?
- **d)** How, if at all, can the Harel and Sowder taxonomy be used to elucidate students' competence in proving as well as how they value proof within the Greek secondary educational contexts?

[2] A journal published by the Greek Mathematical Society which can be useful for mathematics teachers looking for teaching models.

By the term pre-proof perceptions of the students I mean the perceptions of proof the students have before the teaching of proof whereas by proof perceptions I mean the perceptions the students develop during the teaching of proof as well as after having been taught proof.

Beyond reading about research in the field of mathematics education, engaging with it in collaboration with colleagues is the other great source of insight that I have found. Therefore I want to emphasise the creative collaboration with my colleagues that underlies the carrying out of this study. Just for historical reasons I want to name the works of Marton and Pang (Marton & Pang, 2006; Pang, 2006) that I came across while shaping my ideas on collaboration with my colleagues from a methodological point of view.

I give now a brief description of the application of the research project which was as follows. An appropriate school was chosen for the project: the teachers had already developed a high-quality professional relationship with me, and in particular I enjoyed excellent professional collaboration with the Year 9 class teacher. The principal, also a mathematics teacher and the other mathematics teachers were informed about the project already before the beginning of 2010-2011 school year and in May 2010 all agreed to help in any way they could. At the beginning of the 2010-2011 school year the students and their parents were informed about the project and their reception of the idea was remarkably warm.

The Year 9 teacher and I began to implement the project in September 2010 discussing the creation of a test to collect information on how the students who had not yet been formally introduced to proof and the proving process, would perform and work with problems that involved of elementary proof. A 60-minute test was designed mainly by the class teacher and myself and with a further colleague taking part in relevant discussions. We were concerned about how the students would understand the very word ‘proof’ in a given problem. The class teacher informed us that in the first lessons of the school year, she had introduced not only the word but also an overview of the notion of proof in mathematics while teaching material that had been left uncovered in Year 8. We wondered what kinds of problems would fit our purposes of investigating ideas of proof. These could be called pre-proof in the sense that the official introduction and teaching of proof would be applied later in the school year. We ended up with six problems of elementary geometric proof and we decided that at this stage algebra did not lend itself to our purpose. This test was intended to provide information on the research question a): What are students’ pre-proof perceptions? On the other hand it provides the first elements concerning the research question c): How, if at all, is the Harel and Sowder’s taxonomy applicable to the Greek secondary educational contexts?

After administering the test to 90 students in the four Year 9 classes at the end of September 2010, at the end of October, I began to follow the teaching of the four Year 9 classes of the school, audio recording the lessons and taking extensive notes during every lesson in which proof was taught. This class observation lasted until March 2011 and, during this period, the class teacher and I had many discussions before and after lessons on matters of teaching proof and more general issues having to do with teaching mathematics. We regularly discussed our perceptions of the students’ reactions to the new knowledge. Our discussions were audio recorded and some of them video recorded.

In March 2011 we set a new test of approximately 30 minutes with proof problems in geometry. We gave to half of the four classes a problem mainly created by class teacher and to the other half a problem mainly created by me, although both were products of our discussions. The results of this second test do not appear in this work as I explain in the methodology chapter. Data collection was completed in May 2011. The final phase for the study reported here was to give to the students a general 90-minutes test on proof, which this time included problems in both geometry and algebra. This final test was voluntary and was taken by 85 of the 92 Year 9 students.

Both tests were intended to collect data on students' proof perceptions. Thus they provide information concerning the research questions b): What are students' perceptions of proof when they first encounter it? As well as c): How, if at all, is the Harel and Sowder's taxonomy applicable to the Greek secondary educational contexts? Finally especially T3 provides information on research question d): How, if at all, can the Harel and Sowder's taxonomy be used to elucidate students' competence in proving as well as how they value proof within the Greek secondary educational contexts?

Results of the pre-proof Year 9 test have already been presented in the poster section (Kanellos, Nardi, & Biza, 2011b) of the 35th conference of the International Group for the Psychology of Mathematics Education (PME 35) held in Ankara-Turkey in July 2011. Further at the beginning of the 2010-2011 school year another test, created also collectively, was given to three Year 10 classes to investigate proof ideas in geometry that they had been taught in Year 9. The results of the Year 10 test combined with the results of the Year 9 pre-proof test were accepted and presented as a research report (Kanellos, Nardi, & Biza, 2011a) in the 14th European Association for Research in Learning and Instruction (EARLI) conference held in Exeter in the UK in August-September 2011. Finally in PME 37 held in Kiel Germany results of algebra questions of the May 2011 test were presented (Kanellos, Nardi, & Biza, 2013) as a short oral report.

In this thesis I present the project focusing on the analysis of the two tests, administered in September 2010, in May 2011, before and after teaching students about proof. The presentation is as follows:

Chapter 2 offers a literature review; Chapter 3 explains the methodology; Chapters 4 and 5 present the data analysis and findings, and Chapter 6 my conclusions.

In the literature review I discuss studies relevant to proof that have influenced this study. I describe and justify my decision to use the taxonomy of Harel and Sowder's to analyse the students' perceptions of proof.

In the methodology chapter I describe and reflect upon the creation of the data collection tools. As mentioned I collaborated with a number of my colleagues, but mainly with the Year 9 teacher teaching the classes on which data collection focused.

In the analysis and conclusion chapters I present the analysis of the data from the two tests as well as the findings of this analysis and its implications for theory, practice and further research on how to better understand how Year 9 students perceive proof.

CHAPTER 2: LITERATURE REVIEW

2.0 Introductory remarks

In this chapter I describe the review of the literature on mathematics education I conducted in order to find the appropriate analytical tools for my research project. I note that I have conducted this search for analytical tools in full awareness that the bulk of research in this area is in cultural and educational milieus that are substantially different to the one in which this study was conducted.

As I explained in Chapter 1 my intention was to investigate how secondary school students in Greece perceive proof when they first encounter it, but this aim did not at all exclude research work on other educational levels such as tertiary or primary education from my review. In fact, the tool of analysis that I finally chose is the Harel and Sowder’s taxonomy of proof schemes (Harel & Sowder, 1998, 2007) that comes from a very different cultural, educational and cognitive context, namely, a study conducted at the US tertiary educational context. In the Greek educational system proof is introduced at the secondary level whereas other systems seem to do so at the tertiary level. I can add here, without pre-empting the final findings of my research, that I used also various theoretical constructs in my research besides Harel and Sowder’s taxonomy that also describe as well university students’ proof behaviours.

In the sections that follow I present my perception of a small part of the plethora of theoretical constructs concerning the learning and teaching of proof. Also, I explain how the polyphony of theoretical constructs indicates the progress of research on the one hand and the divergent currents inherent in this progress on the other hand. This diversity also reveals the lack of a general educational theory of proof and what follows is presented in full awareness of this absence. First, in section 2.1, I refer to some works on proof in mathematics education. Then, in section 2.2 I discuss and justify my choice of the tool of Harel and Sowder's taxonomy of students' proof schemes as an analytical tool and briefly describe it. Finally, in section 2.3, I summarise the present chapter.

2.1 Research projects related to the teaching and learning of proof

Hanna (2002) judging by the many research papers on proof in at least the last two decades, considers proof a prominent issue in mathematics education. Although proof is a controversial issue, it deserves the attention of mathematics educators regarding its role in teaching. Proof in the classroom is important for mathematical understanding. Among many other issues there is discussion about whether dynamic geometry software (DGS) can help with problems of teaching about proof and whether is more appropriate to teach proof following the line of mathematical rigor or not. The DGS question remains open. In what regards mathematical formality, it has become rather apparent that it does not necessarily result in the understanding of proof (Hanna, 2002) especially when we speak of the secondary education. Hanna (2006) also believes that on the bottom line proof may be the engine driving the development of individuals' analytic thinking in general, but as well, and more importantly, it is the engine by which mathematics can be developed further through understanding it. In a paper on proof in mathematics, Hanna and Barbeau (2006) see proof as a result of historical evolution. Hana and Barbeau's (ibid.) explanation of the logic of proof from a teaching point of view indicates that proof is really a very dense field of human knowledge requiring repeated efforts to understand it and it must be comprehensibly taught. All these aforementioned considerations lead, among others, also to the questions where and how to start teaching and learning of proof, what path to follow to keep proof coherence intact throughout the educational levels, facilitating at the same time students' learning of it and which didactical problems would probably emerge and what constructs and frameworks are rather appropriate to either describe or to solve them. Below I review a number of research works referring to primary, secondary and tertiary levels of education investigating the teaching and learning of proof.

2.1.1 Examples of research works regarding proof in primary education

At the elementary school level it seems that little research has focused on the issue of characterising and understanding proof. In that regard Stylianides (2007b) considers four features of an argument: foundation, formulation, representation and social dimension. These features are examined within the theoretical framework of two principles: (i) the intellectual-honesty principle meaning that proof should be conceptualised in a manner that both student and mathematics are served and (ii) the continuum principle which states that proof is coherently conceptualised through the different grade levels. The examination results in the acceptance or rejection of an argument concerning its counting as proof. Stylianides (2007c) conceptualise proof with the aim to offer a framework of teaching proof in school mathematics on the elementary level and not only. In the same spirit Stylianides (2007a) underlines the notion of assumption in two directions: that of the primary school students and that of teachers offering to both parties grounds to develop activities reach in mathematical content. Bartolini (2009) experiments with students of the second grade and on, in a primary school and suggests tasks that are manageable by students and teachers on this level which can promote logical thinking and reasoning. In a nutshell the above works on the one hand support the idea of teaching proof in primary level but at the same time shed light on the students' and teachers' difficulties with proof and propose ways of overcoming them.

Thus the question that naturally arises therefore is whether secondary school students in any educational context are being taught proof well enough. The answer is probably not. This answer seems to be indicated by a substantial number of research projects and empirical studies some of which I review below.

2.1.2 Examples of research works regarding proof in secondary education

In the UK Healy and Hoyles' (2000) and Hoyles and Healey's (2006) longitudinal studies of algebra and geometry, respectively, employ relatively large samples of secondary school students. They find that high attaining students seem to think mostly empirically when it comes to proof and problem solving, although a tendency has also been observed in them to produce semantic proofs (Healy & Hoyles, 2000). Hoyles and Healy (2006) warn us not to expect easy solutions such as the change of the curriculum to improve students' performance. Believing that the problem of teaching proof is mainly a curriculum matter is misleading (ibid.). It is accepted that progress in mathematical thinking and consequently in proof thinking is painstakingly slow (Küchemann & Hoyles, 2006). This is to be expected, since even the most elementary mathematical constructs, such as the if-then implication, constitute a difficulty for students (Hoyles & Küchemann, 2002). In connection to this and the comment of Hoyles and Healy, not to expect easy solutions on the issue of teaching proof as the change of the curriculum, questions are often raised about whether the intentions of reforms aiming to improve students' mathematical performance produce substantial results. For example reforms aiming to the teaching of mathematics through problem solving is one issue that seems to benefit students' of lower social and economic status, but questions must be answered concerning whether it is a means for learning other mathematical concepts and skills (Lubienski, 2000). Another example of a rather unsuccessful reform is the New Math reform movement. The New Math movement in the US and Europe called for a curriculum oriented to formal proof, guided by the idea that a coherent logical system can attract the attention of the average student. But the problem lies exactly in the fact that in a formal proof the questions do not emerge in a natural way for the students and they become uninterested about the answers. Thus reforming curriculum has not always proved to be the best way to obtain better results in teaching proof. What is needed with or without reform movements is that deductive proof should be the final step in the long mathematical process of learning about proof. Proofs, of whatever nature, should be invoked only where the students are convinced they are required. Proof is meaningful when it answers the students' doubts and proves what is not obvious. It is thus natural to conclude, taking in account students' proof difficulties, that the ability to prove depends on forms of knowledge to which most students are rarely, if ever, exposed (Dreyfus, 1999).

Research goes on, however, and for the researchers it is natural to study, investigate the students' difficulties and propose methods for overcoming them. Thus some researchers as Bieda (2009) propose the adoption of certain mathematical activities rich in proof tasks in the context of an appropriate curriculum.

Chinnappan, Ekanayake and Brown (2011), in a study of Sri Lankan 10th, 11th graders' construction of proof, invent predictive indices concerning the students' knowledge and skills which influence the successful production of proof in geometry. The study's main conclusion underlines the need for robust geometrical knowledge combined with guided problem solving and reasoning skills.

Students in Germany first encounter proof in Year 8. Their problems with it according to Heinze (2004) may be explained by students' insufficient knowledge of concepts, their deficits in methodological knowledge about mathematical proofs, and the lack of knowledge about how to develop and implement a proof strategy. In an earlier work Heinze and Kwak (2002) use the theoretical construct of "declarative" and "methodological" knowledge in an experiment on the ability of students to articulate and produce proofs. By the term "declarative knowledge" the authors mean the knowledge on geometrical axioms, definitions and theorems. By the term "methodological knowledge" the authors mean knowledge of the principles of mathematical proofs. The deficit of both declarative and methodological knowledge seems to play a decisive part in difficulties with proof.

Stylianides and Al-Murani (2010) investigate students' conceptions about proofs and refutations examining whether a proof can coexist in students understanding with a counter example. The whole idea of the research has a strong association to Lakatos' work Proofs and refutations (1976). Although the survey data offered evidence for the presence of the misconception that a proof and a counter example to it, can coexist the followed-up interviews did not point to the same direction as strongly. Under these conditions they propose measures to be taken to avoid ambiguity in future researches.

In the US, the two-column proof is part of the tradition of teaching proof in geometry and used to be considered a successful model. Revising the two-column method under modern reform terms has led to the view that its application was at the expenses of students' initiative and participation and thus of their conquering new ideas (Herbst, 2002b). Herbst (Herbst, 2002a) concludes that emphasis must be put not on procedural methods but on the deepening of knowledge.

Heuristics is a solving problem approach by which a solver uses experienced based ideas, both informal and formal, on a problem to reach its solution. Sometimes heuristics are simple actions focussed on obtaining a certain result, as the decomposition of an integer in prime factors in order to obtain or count all its divisors, whereas sometimes are of strategic character as in the case of decomposing a difficult problem in smaller parts easier to handle. Heuristics is probably an indication that although mathematics has its own language and formality, it cannot replace all fields of rational thinking and needs a 'bridge' to this wider world of human logic. This may be an indication that mathematical ideas cannot always be fertile without an accompanying nebula of non-formal 'heuristic' ideas, at least when one has to solve mathematical problems. Taking advantage of this consideration Koichu, Berman and Moore (2007) propose the theoretical construct of heuristic literacy as a descriptive instrument of students' richness in heuristic ideas. By the term "heuristic literacy" the authors mean a solver's capacity to use heuristic vocabulary and to approach the solution of mathematical problems by a multitude of heuristic ideas. Thus the progress of mathematical thinking for Koichu, Berman and Moore (2007), who experimented with students taking intensive classes and thus high-attaining students, is proportional to higher degree of heuristic literacy. One kind of heuristics is the deliberate and purposeful organisation of knowledge and information. In an experiment described by Marton and Booth (1997) the participants had to memorise a list of personalities. The most successful strategy proved to be the creation of a net connecting information on these personalities in comparison to simple memorising without any structure. It is thus not surprising that an analogous strategy described by the theoretical construct called knowledge connectedness, plays an important role in students' mathematical efforts. Indeed high-achievers seem to be able to retrieve more information than low-achievers, as reflected in the better results of those with higher knowledge connectedness (Lawson & Chinnappan, 2000). Examples of the absence of mathematical knowledge connectedness can be found in Monaghan (2000).

Mariotti (2001) builds a theoretical construct of cognitive unity. For Mariotti conjecture and proof are bonded together when substantial cognitive obstacles do not decisively affect the final result, which, of course, is proof. But even if non-negligible cognitive obstacles are present, the same theoretical construct might serve for the description of the situation in terms of these very obstacles. The theoretical construct of cognitive unity, owes its origin, from the historical and teaching perspectives, to Euclid's Elements where the 'what is to be said' should be said in a certain order. The rupture of that certain order reveals an absence of cognitive unity. In the same way, students with the necessary cognitive unity can find their way and prove after formulating an appropriate conjecture. On the contrary students who lack this cognitive unity, experience problems in their progress and face stagnation regarding proof (Mariotti, 2001, 2006). Antonini and Mariotti (2008) study indirect proof and come to the conclusion that intertwining the teaching of mathematical logic with the teaching of proof in mathematics is important for achieving satisfying teaching results.

Seen as a dynamic evolution the learning of proof could be interpreted as a continuous process of liberation from the chains of the empirical thinking towards the freedom of the ideal formal thinking. Arzarello, Domingo and Sabena (2009b) experiment with 10th-grade students on early calculus. The researchers introduce the terms 'semi-empirical' and 'semi-theoretical' to describe the proof behaviour of students which indicative for the students' thinking. The terms 'semi-empirical' refer to the Lakatos' view of mathematics as a semi-empirical science whereas the terms 'semi-theoretical' refer to methods developed by the students influenced by the experiment's software to cope in paper and pencil environment with limits and ratios. Barrier, Durand-Guerrier and Blossier (2009) see empirical facts as a tool to gradual abstraction towards a deductive thinking. Students, for their part, reduce the abstraction of problems when they feel unable to grasp the ideas or notions that are connected to these problems. Reducing abstraction is a process used in attempting to solve mathematical problems, and probably represents the need to perceive the mathematical objects involved empirically. If a given problem is difficult to handle there is a tendency to simplify it by reducing its abstraction (Hazzan & Zazkis, 2005).

Miyazaki (2000) studies the level of proof in algebra in Japanese schools. He proposes a model of the levels of proof observed and an ordering of the steps to be taken by the pupils under their teachers' guidance. These steps also correspond to an ascension from the empirical to the formal thinking along a smooth pathway, although he admits that his model is only appropriate for algebra. Another of Miyazaki's models interprets and describes the structure of the empirical proof schemes of students emerging from measurement in Geometry (Miyazaki, 2008).

Kospentaris, Spyrou and Lappas (2011) study the perceptions of 12 grade students and students in their first year of university studies regarding area congruency. To address such problems the students must develop deductive thinking. The authors observe that when the students cannot find a way to solve a problem they seek help in empirical evidence as a substitute for deductive thinking.

Lin, Yang and Chen (2004), inspired by Healy and Hoyles (2000) research project in the UK, present a corresponding research project with 7th, 8th and 9th graders in Taiwan that investigates their choice of proof for their own and for the best mark. They scrutinise the students' reasoning, proving and understanding of proof using certain models of counting in geometrical patterns. The researchers discuss the students' difficulties with proof under this scope and suggest counting in geometrical patterns as a mean of developing deductive algebraic thinking.

Battie (2009) studies the difficulties that students experience with proof while solving number theoretical problems of congruencies *modulo n* in their transition from the secondary to tertiary education. She analyses their attempts to solve relevant problems through the lens of the organising and operative dimensions. The organising dimension in proof requires the ability to create a plan, which must be practically implemented; this is where the operative dimension is needed and must come into action. The two dimensions are complementary and any loss of balance creates obstacles in the proving process.

The efficacy of DGS in students' understanding is rather controversial, which is again natural given that DGS is a relatively new element in the teaching and learning of mathematics. DGS is being introduced slowly because on the one hand it demands certain infrastructure, and on the other it must of course be combined with the guidance of trained teachers in order to benefit students. However, research has produced interesting findings although not always compatible with one another. DGS keep researchers busy considering the probable and possible consequences of the role of proof in a digitalised world. Borwein (2009) finds that DGS not only challenges proof but also provides it with opportunities and Hanna (2000) believes that the role of proof will remain intact. Certainly a number of researchers believe in DGS's didactical potential to support deductive reasoning (Jones, 2000; Laborde, 2000). Researchers' use of DGS environments combined with questions provoking students' surprise of the unexpected is another way to make them to feel the need of proof (Hadas, Hershkowitz, & Schwarz, 2000). Marrades and Gurierrez (2000) use of examples in DGS environments leads to the division of students' justifications of various assertions into two main categories: those that are deductive and those that are empirical. The researchers assert that appropriate use of some DGS might improve students' attitudes towards to proof. For Wares (2007), investigating by means of DGS conjectures on difficult geometry problems not encountered during mathematic teaching would stimulate students to provide proof. For Chazan (1993), instead, computer software is suspected of contributing to empirical perceptions. In his research evidence and proof sometimes seem to be mixed up in students' ideas, making the issue important. Aiming for more general enhancement of mathematical understanding, Kordaki (2003) chooses the mathematical issue of area and uses DGS environments to help 9th-grade students understand the issue better. Bloch (2003) uses technology to ameliorate students' perceptions of mathematical objects such as functions.

2.1.3 Examples of research works regarding proof in tertiary education

Proof at the tertiary level appears to be difficult both in terms of teaching and learning. Researchers have raised various aspects of this problem.

Epp (2003) gives a very clear picture of the problems of which she has become conscious since the late 1970's and after. Having presented students' difficulties with proof and formal logic in her work proposes courses which, for instance logic and geometry are interwoven to make the logic vivid and applicable on the one hand and facilitate learning about proof on the other. Thus the combined teaching of logic and geometry and in general of proof and logic is indispensable. Epp does not miss the social factor; she discusses the possibility of more instructors per student, although she accepts the difficulty of such a solution. For Alibert and Thomas (2002), proof in the text-books and in the research jargon is algorithmic, linear and opaque to students, for whom it should be structured and provide main ideas. The main issue seems to be the necessity for communicating scientific results in a productive way in order to detect and solve problems of understanding. Durand-Guerrier (2003) observes that students of tertiary education seem to experience as many problems as those in secondary education with implications which are “at the very heart of deductive reasoning” (ibid., p. 11). Edwards and Ward (2004) used the theoretical construct of concept image and concept definition developed by Tall and Vinner (1981) to analyse the phenomenon misuse of definitions and not only. They think that for the misuse of definition etc. the teaching is among the contributing factors when it becomes stereotypical. Based on their observations and experiences with students learning abstract algebra, they propose some teaching measures in order to help students understand the different meanings of words. Sowder and Harel (2003) see proof understanding, production, and appreciation (PUPA) as “important parts of a mathematician's repertoire” (ibid., p. 2). Finding that students in US universities demonstrate a deficiency in proving abilities, Sowder and Harel seek the reasons for this in their aforementioned work. According to their results, students need to see that proof is concrete, convincing and essential implying that these decisive elements are not always present in the tertiary teaching of proof. Moore (1994) thinks that students’ main difficulties are in understanding concept, mathematical language and notation, and getting started on a proof. Recio and Godino’s (2001) research project looks at students’ difficulties with deductive reasoning and formulates the conjecture that they may be due to different institutional meanings of proof. Selden and Selden’s (2003) study of the proof perceptions of students finds that their limited ability to distinguish proof from ‘fake’ proof shows their poor understanding of logical structure due to the stagnation on superficial features of proof. Stylianides, Sylianides and Philippou (2004) studying students’ understanding of the contraposition equivalence rule, reveal the complexity of the factors that influence students’ logical thinking. For Tall (2005) students’ difficulties with formal proof have their origin in the earlier ‘mathematical life’. Weber (2001, 2003) speaks of students’ lack of strategic knowledge in the face proof questions which makes them incapable of putting the proof process within a relevant theoretical context in order to apply the theory required to reach a proof. Furinghetti, Morselli and Antonini (2011) asked university students to produce examples in analysis to study the dialectic of visual versus symbolic and find analogies to the dialectic of the formal versus the informal. Uhlig (2002) proposes an alternative way of introducing students to proof in linear algebra, based on an analysis of educational and historical dimensions. The central idea is to avoid the high degree of formality that traditionally characterises courses in linear algebra and to appeal to a more natural way of understanding that is closer to students’ the ability to grasp such ideas. Dorier, Robert and Rogalski (2002) concur Uhlig’s view. Iannone, Inglis, Mejia-Ramos, Simpson and Weber (2011) explore whether the generation of examples is actually predictive for successful handling of proof tasks. Their conclusion about the method’s effectiveness remains ambiguous without rejecting it. Weber and Alcock (2004) study proof productions and develop the theoretical construct of semantic and syntactic proof production:

> We define a syntactic proof production as one which is written solely by manipulating correctly stated definitions and other relevant facts in a logically permissible way. In a syntactic proof production, the prover does not make use of diagrams or other intuitive and non-formal representations of mathematical concepts. In the mathematics community, a syntactic proof production can be colloquially defined as a proof in which all one does is ‘unwrap the definitions’ and ‘push symbols’.
>
> We define a semantic proof production to be a proof of a statement in which the prover uses instantiation(s) of the mathematical object(s) to which the statement applies to suggest and guide the formal inferences that he or she draws. By an instantiation, we refer to a systematically repeatable way that an individual thinks about a mathematical object, which is internally meaningful to that individual. (ibid., p. 210)

The authors make a very interesting analogy of semantic and syntactic proof productions to Skemp’s (1976) relational and instrumental understanding respectively concluding therefore that semantic proof production is likely to lead to proofs more efficiently. Alcock and Simpson (2004, 2005) study the role of visualisation and suggest that the fidelity of visualisation to the formal definition contributes positively to proof production. This use, creative or not, of visualisation and the parallel correct or incorrect use of definitions and theorems is connected, according to the authors, to the Tall and Vinner's (1981) notions of concept image and concept definition. Problem solving is comparable to proof production, and Stylianou, Chae and Blanton (2006) study the parallel between the two activities' interrelation and interaction. The notion of isomorphism in its educational meaning, that is of problems or problem solving situations essentially similar, is analysed by Harel and Greer (1998) who review a number of research papers on the subject and support the idea that research must be carried out in an appropriate context. Mamona-Downs (2001) works on proposals regarding a more effective understanding of limits. She supports the idea that imagery potential can be helpful in proofs relevant to limits. Mamona-Downs and Downs (2004) made a teaching experiment with volunteers from a class in which proof was taught, on problem solving connected with bijections used for enumeration. In their course they stressed the basic logic, mathematical language and practice in doing simple proofs.

Harel (1998, 2001, 2007, 2008a, 2008b) develops the idea of the DNR system, which aims to clarify what mathematics should be taught at school and university and how it should be taught. Although the DNR system is basically inspired by tertiary level teaching experiences and the project PUPA it embraces and permeates, as a teaching philosophy, all educational levels. DNR stands for the duality principle (=D), the necessity principle (=N) and the repeated reasoning principle (=R).

> Mathematics is a union of two sets: The first set is a collection, or structure, of structures consisting of particular axioms, definitions,

> theorems, proofs, problems, and solutions. This subset consists of all the institutionalized ways of understanding in mathematics throughout history. The second set consists of all the ways of thinking that are characteristics of the mental acts whose products comprise the first set. (Harel, 2008a, p. 490)

Out of this thesis arise the Duality Principle, a product of interaction between ways of understanding and ways of thinking:

> The Necessity Principle has its roots in the Piagetian theory of learning and is consistent with the current theory of Problematique put forth by French mathematics educators. […] for example, [...] pupils' learning depends on their recognition and re-construction of problems as being their own... A problem is a problem for a student only if she or he takes the responsibility for the validity of its solution. This transfer of the responsibility for truth from teacher to pupils' must occur in order to allow the construction of meaning. (Harel, 1998, p. 259).

The third cornerstone of the DNR system is the Repeated Reasoning Principle which means that students must practice reasoning in order to internalize desirable ways of understanding and ways of thinking. DNR embraces all mathematical teaching but puts emphasis on reasoning with at least the Repeated Reasoning Principle and thus on proof.

In some works the researchers are interested in deductive thinking, not necessarily in the narrow educational context of teaching and learning proof but also in a broader sense. Ayalon and Even (2008) discuss how people professionally engaged in mathematical activities perceive deductive thinking. Akin to this work are, Raman (2003) on key ideas and Inglis and Mejia-Ramos (2009) investigation of the persuasive power of visual arguments when accompanied or not by a text supporting them.

2.1.4 Examples of research works regarding proof focusing mainly on teachers

There are researchers who focus on teachers, exploring various aspects of the influence the teacher factor on teaching proof and problem solving in both primary and secondary education. Brousseau and Gibel (2005) analyse a classroom situation where 5th graders are attempting to solve a problem whose solutions must be supported by logical arguments and accentuate the teacher not being able to enhance their reasoning because he could not process the students' reasoning appropriately. Biza, Nardi and Zachariades (2009) explore the relationship between beliefs about the sufficiency and persuasiveness of a visual argument and personal images about tangent lines of secondary education teachers. It turns out that some teachers accept incorrect arguments because they are carried away by visual 'evidence'. Bjuland (2004) works on a teaching experiment with future teachers. Through their efforts to solve Geometry problems the future teachers begin to understand among other things the role of the simplification of related problems when confronted with students' difficulties in solving a problem. Dekker and Elshout-Mohr (2004) study teachers' interventions focused on mathematical content and students' interactions and find indications of how teachers should organise their interventions for better teaching results. Harel, Fuller and Rabin (2008) warn us not to risk teaching mathematics in a way that could generate in our students the perception that mathematics is a procedural routine with irrelevant and arbitrary elements. Knuth (2002), taking advantage of the fact that some schools are and others are not following the 'proof for all' reform in the US distinguishes the pedagogical problems that the teachers themselves seem to have with proof per se or as a teaching material. Barbé, Bosch, Espinoza and Gascón (2005) analyse the teaching of *limits* in Spanish schools from a praxeological point of view, which distinguishes the teaching in didactical moments. For the authors, types of problems, techniques, technologies and theories in the field of mathematics form what it is called mathematical praxeology. On the other hand to teach a mathematical praxeology the teaching has to be organised in didactical moments. They conclude that the teaching they observed suffered in organisation because it lacked some of the necessary to a successful teaching didactical moments in question. Martin, McCrone, Bower and Dindyal (2005) having observed a geometry class for a significant length of time, emphasise the importance of interplay between teacher and students, the teacher guiding the students to act for themselves on matters of formal proof, although even in such cases one cannot be sure whether the students have indeed internalised the axiomatic method. Schoenfeld (1988) presents observations of a 10th-grade geometry class over a long period of time. The class is a typical achieving class where curriculum material is taught and state-administered tests show off students' satisfying achievements. The researcher is concerned that aspects of the class do not involve the development of mathematical thinking, and from this emanates the paradox of good teaching with bad results. Schoenfeld makes the important point that the bad results including those in problem solving and proof are not necessarily a consequence of teachers' inefficiency and inadequacy. Instead they are the complex product of tight attachment of the teaching to curricular premises that are predicated on performing well on state-designed and -administered tests. Stylianides and Stylianides (2009) study perspective elementary teachers' perceptions of proof and find that even perspective mathematics teachers are not always able to recognise what an empirical argument is and explain what a proof is, although a number of the research project participants were aware that an empirical argument is not a proof. Van-Schalkwijk, Bergen and Van Rooij (2000) experiment with students interested in mathematics in a double-bind study: on the one hand the students learn to investigate and on the other the teachers learn to coach this investigation. Thus, it is very important to find a balance between mere concentration on guiding the process of the students' investigations and active intervention in the learning process of proving.

2.1.5 Examples of research works regarding proof based on non mathematical theoretical constructs

Sometimes researchers use analytical models framed outside mathematics education research and mathematical sciences. Toulmin (2008) and Habermas (2003) offer two impressive examples of such models.

Toulmin's model of argumentation (2008) has influenced a number of researchers. Briefly according to Toulmin an argument is constituted basically of the data, the warrant, the backing of the warrant, the qualifier, the claim or conclusion and the rebuttal. When a person builds an argument, she appeals to data by using the warrant, which is supported by the backing. Consequently she can assert using a qualifier that the conclusion is valid unless there is a rebuttal negating this conclusion. Knipping (2008) uses the Toulmin's model to analyse students' thinking about proof. Inglis, Mejia-Ramos and Simpson (2007) analyse high-attaining post-graduate mathematics students' arguments and conclude that instruction should offer students the ability to match modal qualifiers to warrant types. Krummheuer (2003) applies Toulmin's model to primary students' thinking and asserts that it allows the reconstruction and thus the study of the argumentative character of their thinking in retrospect. Arzarello, Domingo and Sabena (2009a) claimed that results are not always in favour of the Toulmin's model as an instrument of analysis and they criticised Toulmin's model for not being able to explain all argumentative phenomena.

Habermas' theoretical construct of rational behaviour presented in his book 'Truth and Justification' (2003) inspired Morselli and Boero (2011) to use it as an instrument to analyse students' handling of algebraic issues and algebraic proof. They consider that their observation and subsequent analysis can be used by curriculum developers to production appropriate teaching material.

2.1.6 Taxonomic theoretical constructs related to students' proof perceptions

There are research works which offer taxonomic theoretical constructs either on students' knowledge or proving ability. The most widely known work on geometry perceptions is the work of the Van Hiele couple. More specifically It is known that in the late 1950's the Van Hieles developed a theory of geometrical knowledge levels (Van-Hiele-Geldof & Van-Hiele, 1984):

> According to the Van Hieles, the learner, assisted by appropriate instructional experiences, passes through the following five levels, where the learner cannot achieve one level of thinking without having passed through the previous levels.
> Level 0: The student identifies names, compares and operates on geometric Figures (e.g., triangles, angles, intersecting or parallel lines) according to their appearance.
> Level 1: The student analyses Figures in terms of their components and relationships among components and discovers properties/rules of a class of shapes empirically (e.g., by folding, measuring, using a grid or diagram).
> Level 2: The student logically interrelates previously discovered properties/rules by giving or following informal arguments.
> Level 3: The student proves theorems deductively and establishes interrelationships among networks of theorems.
> Level 4: The student establishes theorems in different postulational systems and analyses/compares the systems." (Fuys, Geddes, & Tischler, 1988, p. 5)

Senk (1989) uses the Van Hiele model on students' level of geometric knowledge. The model is indeed general, since it offers a taxonomy of the students' geometrical knowledge levels. It is confined to geometry and treats the students' proof behaviour in a predictive manner. Doubt is even being cast upon the predictive element according to Senk's research.

Balacheff's taxonomy (1987) on the other hand is more general than Van Hieles' regarding students' proof behaviour because it does not confine itself in geometry only. It takes as its starting point the proof behaviour of the students of Class 4 (students 13-14 years old). Here is how Balacheff sees proof:

> We call proof an explication accepted by a given community at a given moment. This decision may be the object of debate the significance thereof being the demand to determine a system of validation common to the interlocutors[1]. (Balacheff, 1987 p. 148)

On the basis of the previous definition this is what Balacheff's general view:

> The carrying out of a decision or the realization of the content of an assertion permits what we will call pragmatic validation of the decision or pragmatic proof if they are carried out by the student himself in order to establish the validity of a proposition. If this access to realization is not possible then the validations are necessarily conceptual. The production of these conceptual proofs demand indeed the language formulation of these objects to which they refer and the relations of these objects.[2]" (Balacheff, 1987 p. 157)

Having given the above definition Balacheff continues his classification, saying that from pragmatic proofs (preuves pragmatiques) to conceptual proofs (preuves intellectuelles) one can distinguish various types of proofs as follows:

> Naïve empiricism is the first type of proof that we encounter in this hierarchy. It consists of concluding the truth of an assertion from the observation of a small number of cases.
> The crucial experiment is processes of validation of an assertion where the individual explicitly poses the problem of generalization and resolves it, betting on a case which he recognise the less particular as possible.[3]." (ibid., p. 163)
> "The generic example involves making explicit the reasons for the validity of an assertion by means of the realisation of operations or transformations of an object that is not present itself but is a

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[1] Nous appelons preuve une explication acceptée par une communauté donnée à un moment donné. Cette décision peut être l'objet d'un débat dont la signification est l'exigence de déterminer un système de validation commun aux interlocuteurs. (Original text in French, my translation; this also applies to all texts translated here from the French original)

[2] La mise à exécution d'une décision, ou la réalisation du contenu d'une affirmation, permet ce que nous appellerons des validations pragmatiques de la décision, ou des preuves pragmatiques lorsqu'elles sont effectuées par l' élève lui-même pour établir la validité d'une proposition. Lorsque cet accès a la réalisation n'est pas possible alors les validations sont nécessairement intellectuelles. La production de ces preuves intellectuelles requièrt notamment l'expression langagière des objets sur lesquelles elles portent et de leurs relations.

[3] L'empirisme naïf est dans cette hiérarchie le premier type de preuve que nous rencontrons. II consiste à tirer de l'observation d'un petit nombre de cas la certitude de la vérité d'une assertion (…) L'expérience cruciale est un procédé de validation d'une assertion dans lequel l'individu pose le problème de la généralisation et le rèsoud en pariant sur la réalisation d'un cas qu'il reconnaisse pour aussi peu particulier que possible.

> characteristic representative of a class of individuals[4]. (ibid., pp. 164-165)
> "The mental experiment appeals to the action interiorising it and detaching it from its realisation by a particular representative. It remains marked by anecdotal temporality, but the operations and founding relations of the proof are differently designed by the result of their carrying out as is the case for the generic example.[5] (ibid., p. 165)

Naïve empiricism (l'empirisme naïf), crucial experiment (l'expérience cruciale) and generic example (l'exemple générique) belong to the general class of pragmatic proofs whereas mental experiment (l' expérience mentale) belongs to the class of conceptual proofs.

2.1.7 Research questions emerging from the literature review

From the literature review so far two main questions emerge in a natural way regarding the students' proof perceptions. Do students possess proof perceptions before being taught proof? I shall call these perceptions pre-proof perceptions because they are, if they exist in any form, perceptions about proof before the relevant teaching of proof. Let it be noted that there are research works investigating this question even on primary level (Stylianides, 2007b). On the other hand it follows logically to ask, what are the students' proof perceptions during and after the first teaching of proof. In concise formulation the research questions are:

a) What are students' pre-proof perceptions?

b) What are students' perceptions of proof when they first encounter it?

To answer the questions previously cited a tool of analysis is definitely needed. I close the discussion of the literature with the next section 2.2 which is devoted to the

[4] L'exemple générique consiste en 1'explicitation des raisons de la validité d'une assertion par la réalisation d'opérations ou de transformations sur un objet présent non pour lui-même, mais en tant que représentant caractéristique d'une classe d'individus.

[5] L'expérience mentale invoque l'action en l'intériorisant et en la détachant de sa réalisation sur un représentant particulier. Elle reste marquée par la temporalité anecdotique, mais les opérations et les relations fondatrices de la preuve sont désignées autrement que par le résultat de leur mise en oeuvre; ce qui était le cas pour l' exemple générique.

chosen tool of my research analysis, the Harel and Sowder's (1998, 2007) proof scheme taxonomy which describes the proof behaviour of students also from a taxonomic point of view. Being the chosen tool of analysis it merits a distinguished presentation. To this end in 2.2.1 section I give first the philosophy underlying and supporting the taxonomy and in 2.2.2 section a detailed presentation of the taxonomy in the context of other proof related works – and explain why I chose to use it to analyse how Greek students perceive proof when they first encounter it.

2.2 The conceptual framework of this study: Harel and Sowder's taxonomy

2.2.1 The philosophy underlying the Harel and Sowder taxonomy

The central concept of the Harel and Sowder taxonomy, as is natural, is the concept of proof. But proof, as clear as it might be as a concept in the minds of the mathematicians, it is not at all clear for many students.

> Overall the performance of students at secondary and under graduate levels in proof, is weak as the findings reported in this paper will show. Whether the cause lies in the curriculum, the textbooks, the instruction, the teachers' background, or the students themselves, it is clear that the status quo needs, and has needed improvement (…)This chapter argues for "comprehensive perspectives" on proof learning and teaching and provides an example of such a perspective. A comprehensive perspective on the learning and teaching of proofs is one that incorporates a broad range of factors: mathematical, historical-epistemological, cognitive, sociological, and instructional. A unifying and organizing element of our perspective is the construct of "proof scheme." (Harel & Sowder, 2007, p. 2)

I call the broad range of all these factors, to which Harel and Sowder make reference to, the "philosophy" underlying the Harel and Sowder's taxonomy because it is the base to answer the question "what is proof?" and thus in brevity represents the ontological nature of the question as well as the answer that Harel and Sowder give to it. In order to find and formulate a satisfying answer to this question one has to ineluctably indulge in the historical development of the proof concept and study it thoroughly. Harel and Sowder studied works on the historical development of the proof concept and parallel to the studying they conducted instructional experiments with students. As a result of this combined effort they shaped ideas and formulated an answer to the question “what is proof?” I summarise my perception of their work (Harel & Sowder, 1998, 2007) on this matter.

It is widely accepted today that the pre - Greek mathematics is proof free. A number of mathematical truths were known to civilisations prior to the Greek civilisation, as the Babylonian and the Egyptian. These were truths concerning geometrical objects or of arithmetical character as e.g. operations on fractions. But all these mathematical truths were not explained, not supported and thus not justified by corresponding arguments but were seen as rules of algorithmic and computational character for practical usage in certain cases which called for or needed such handling. The rise of the Greek mathematics benchmarks a new era in the human thought. In this new era nothing is allowed to be left unexplained and unjustified especially in mathematics. According to Sfard (1991) the birth of new abstract ideas from previously mainly operational and procedural ideas in various mathematical topics is not a product of chance. She believes, on the contrary, that a certain development of procedural, algorithmic and computational ideas reaches a quantitative limit up and that is the crucial and critical moment where a qualitative leap forward generalises these ideas and produces the abstraction thereof igniting the mechanism of progress in the various mathematical fields. Under this light she sees the eruption of new mathematical ideas in the sixteenth century. She is in resonance in this respect with the explanation Harel and Sowder (1998) attribute the genesis of the proof concept by the Greeks to a number of factors among of which is the resolving of contradictory computational results obtained by earlier civilisations. There are probably other things as more general factors of social character which influence the concept proof as e.g. the political system of democracy demanding argumentation to support decisions and choices concerning the economic development or various social measures which help a society function “better”. Limiting ourselves to the inner developmental reasons of mathematics itself as described by Harel and Sowder and Sfard, in other words limiting our interest in observing how and why the mathematical ideas mature, is of paramount importance for our contemporary understanding of students difficulties with proof (Harel & Sowder, 2007). At the same time our acceptance of the proof value of Greek mathematics emphasises even more intensely the question “what is proof?” Indeed, we take as an exemplary exposition of the Greek concept of proof the work of Euclid in the Elements. What is salient and exceptional in the work of Euclid is the logical structure which precedes any engagement in argumentation, proof and proving of any proposition. The acceptance of some fundamental and undefined truths as a base for further argumentation and justification makes the Elements a monumental work which offered to human thought the paradigm of an axiomatic system. However the mathematical developed further even if the evolution was slow and painstaking. Passing through the sixteenth to nineteenth centuries, where considerable progress was recorded, some two thousand years later the mathematical thought in its development found itself in the need of a new fresh reconsideration of the axiomatic structure after having understood that the geometry of Greeks based on the axiomatic ideas of Euclid was not the unique answer to questions regarding the notion of parallelism. The final consequence of this new revolution in mathematics was the development of the axiomatic system of Hilbert which not only answered questions but spontaneously put new ones. Now, if Hilbert’s axiomatic system is more complete than the one developed by Euclid, does proof in the sense and under the assumptions of Hilbert represent the same thing as proof in the sense and under the assumptions of Euclid? Harel and Sowder propose the distinction of the one system from the other by calling that of Euclid the Greek axiomatic system or “Greek axiomatic proof scheme” and that of Hilbert the modern mathematical axiomatic system or “modern axiomatic proof scheme” (Harel & Sowder, 2007, p. 9). Their proposal leads obviously to the result that both Greek proof and modern proof are accepted as representing the concept of proof. In doing so Harel and Sowder are totally aware that such a point of view could not by many be logically accepted since obviously there are contradictory elements in the two systems. For example the Greek axiomatic system idealises the geometrical objects but does not free itself from the “material” substance of these objects and the impression they exert on us. Thus in the Elements a point is defined as having no parts, definition which idealises what we sketch as a point in a geometrical figure. In the same vein, in proposition I.32 (Heath, 1956, pp. 316-317) the parallel from a triangle’s vertex to the side opposite to the vertex is considered as belonging totally to the external angle of the triangle with the same vertex because our experience and empirical perception of these objects lead us to this conclusion. In radical revision of such ideas which encounters in a number of cases in the Elements the modern axiomatic deprives its objects of any so called real world interpretation making them void meaningless variables. The contradiction is resolved in the following manner. The primacy of the modern axiomatic system is clearly and beyond any doubt recognised. Consequently proof teaching has as educational goal and ultimate aim to make for our students possible to understand modern axiomatic proof and use it productively and fruitfully. In a way it can be said that the “objectivity” of the modern axiomatic system is acknowledged. On the other hand to the concept of proof, which is not developed within the realm of the modern axiomatic, is attributed the property of “subjectivity” by the following definition: Proof is an argument that a person or a community uses to convince others of the validity of a certain assertion or the rejection thereof. By virtue of such a consideration one is led to accept the historical as well the social nature of the proof itself. The historical aspect perceives proof in its development in time. The social aspect sees proof as a collective human activity. Indeed the way an ancient Greek mathematician differs from the ones of the sixteenth century. Similarly the mathematician of the sixteenth century differs from the ones in the nineteenth century. And finally the mathematicians of the nineteenth century differ from the mathematicians of the twentieth century and so on.

At this point I want to underline that Harel and Sowder idea of the “subjectivity” of proof, whether consciously or unconsciously, spontaneously or not, is an accepted notion within the context of education. Indeed, for centuries or at least the recent several decades the teaching of mathematics does not begin by presenting the students with its modern axiomatic foundation. There is more to that if we consider that some attempts to proceed in this manner in the secondary education led to the complete failure of the teaching regarding proof and not only. Thus in the world of education e.g. the empirical element is taken into account and is being used as a first means to approach the concept of proof. For example the superposition of triangles in order to check their equality is accepted, as in the times of Euclid, as a valid criterion. The axiomatisation of this empirical process is left for a later time, mostly during the tertiary education. From this observation angle the Harel and Sowder’s conception of proof summarises what is already being practised in the classrooms for decades at least. In other words beside the existence of the modern axiomatic system we accept at least for instructional, cognitive and psychological reasons the coexistence of the

Greek axiomatic system or even more primitive pre – proof ideas without any endorsing axiomatic systems.

Returning to the Harel and Sowder taxonomy's philosophy we find that the subjectivity of proof leads to the study of certain fundamental aspects or functions of proof which in their turn guide to the concept of proof scheme. These are the following: Conjecture versus fact, proving, ascertaining versus persuading.

Conjecture is an assertion formulated by a person or by a community which is not automatically true. Thus it can imply that the person making the conjecture might not be sure of the validity of the conjecture's truth. If the person believes in the truth of the spoken out conjecture then the latter becomes, for the person's point of view, a fact.

Proving is the process which removes doubts or just the contrary consolidates doubts about an assertion expressed by a person or a community.

Ascertaining and persuading are sub processes of proving. Ascertaining removes a person's or a community's doubts or consolidates them with regard to an assertion. In a way it has to do with introvert actions of a person or a community. Persuading is the extrovert action taken by a person or a community to persuade others of the validity or the invalidity of an assertion.

Thus term proof scheme is used instead of the term proof in order to put an emphasis in the subjectivity of the proof either seen historically or as an individual action. I repeat here that the acceptance of this mode of thinking towards proof does not imply that proof is never "objective". Far from any such ideas the modern axiomatic system is the objective deductive system to prove mathematical propositions and is literally the learning objective of mathematical education. Especially in education the concept of proof scheme makes the teaching and learning of proof student - centred shifting the focus for us as teachers not solely on the proof for itself but at the same time at the students perceptions of proof.

On this basis I proceed to the next section.

2.2.2 The description of Harel and Sowder taxonomy

Below I describe the Harel and Sowder's taxonomy (2007) in which they also address the relationship of their taxonomy to other taxonomies and the functions of proof in mathematics. Harel and Sowder present the complete taxonomy in their work Students' Proof Schemes (1998). In my research I used the names of proof schemes presented in Harel and Sowder (2007). According to the authors the taxonomy of proof schemes comprises of three classes: the *external conviction proof scheme*, the *empirical proof scheme*, and the *deductive proof scheme*. The authors give the following description of the first class of proof schemes:

> External conviction proof schemes. Proving within the external conviction proof schemes class depends (a) on an authority such as a teacher or a book, (b) on strictly the appearance of the argument (for example, proofs in geometry must have a two-column format), or (c) on symbol manipulations, with the symbols or the manipulations having no potential coherent system of referents (e.g., quantitative, spatial, etc.) in the eyes of the student (e.g., $\frac{(a+b)}{(c+b)} = \frac{(a+\not{b})}{(c+\not{b})} = \frac{a}{c}$). (Harel & Sowder, 2007, p. 7)

According to the above description, three cases can be distinguished within the class of external conviction proof schemes. If an authority such as the teacher or a book is appealed in order to support a proof argument the proof scheme is an *authoritarian proof scheme*. If an argument is judged logically adequate due to its appearance but not because of its actual logical validity is a *ritual proof scheme*. If a proof scheme is based on arbitrary manipulations of any kind, is a *non-referential symbolic proof scheme*. Thus in brief the external conviction proof scheme class has the following structure:

- External conviction proof schemes class
  - Authoritarian proof scheme
  - Ritual proof scheme
  - Non-referential symbolic proof scheme

The second class of proof schemes is *empirical proof schemes* which Harel and Sowder (2007) describe as follows:

> Empirical proof schemes**.** Schemes in the empirical proof scheme class are marked by their reliance on either (a) evidence from examples (sometimes just one example) of direct measurements of quantities, substitutions of specific numbers in algebraic expressions, and so forth, or (b) perceptions. (ibid., p. 7)

According to this description there are two types of empirical proof schemes. If a proof scheme is based on the use of examples and sometimes only on one example or on the direct measurement of quantities such as lengths, angles etc., or on the substitution of variables by certain numbers it is an *inductive proof scheme*. If on the other hand the argumentation of a proof scheme is based on perceptions, it is a *perceptual proof scheme*. Harel and Sowder (1998) explain what a perception is by means of an example where a student perceives two non-congruent line segments as congruent and a rectangle as a square (ibid., pp. 256-258). Thus in brief the empirical proof scheme class has the following structure:

- Empirical proof schemes class
  - Inductive proof schemes
  - Perceptual proof schemes

The third class of proof schemes is *deductive proof schemes*. As their name indicates, they are proof schemes where the arguments are of deductive character. This class has two kinds of proof schemes: *transformational* and *axiomatic*. Transformational proof schemes use *common generality, operational thought* and *logical inference*. In other words the arguments of a transformational proof scheme seek to be valid for all cases and not just for isolated ones, with exceptions not generally accepted. Operational thought is present in the manner that a proof is organised in appropriate steps to reach the final goal that completes the proof. Logical inference is made apparent in the way the individual offering a proof justifies his use of the data given in the partial steps of the proof and their connection. Transformational proof schemes differ from the previous classes in the fact that they provide elaborate demonstrations. On this last point the authors of the taxonomy give the following example taken from Harel (2001):

> Consider the following two responses … to the problem:
> Prove that for all positive integers n,
> $\log(a_1 \cdot a_2 \cdot ... a_n) = \log a_1 +_1 \log a_2 + \cdots + \log a_n$.
> Response I
> $\log(4 \cdot 3 \cdot 7) = \log 84 = 1.924 \qquad \log(4 \cdot 3 \cdot 6) = \log 72 = 1.857$
> $\log 4 + \log 3 + \log 7 = 1.924 \qquad \log 4 + \log 3 + \log 6 = 1.857$
> Since these work, then $\log(a_1 \cdot a_2 \cdot ... a_n) = \log a_1 +_1 \log a_2 + \cdots + \log a_n$
> A probe into the reasoning of the students who provide responses of this kind reveals that their conviction stems from the fact that the proposition is shown to be true in a few instances, each with numbers that are randomly chosen —a behaviour that is a manifestation of the empirical proof scheme.
> Response 2
> (1) $\log(a_1 a_2) = \log a_1 + \log a_2$ by definition
> (2) $\log(a_1 a_2 a_3) = \log a_1 + \log a_2\, a_3$. Similar to $\log(ax)$ as in step (1), where this time $x = a_2 a_3$.
> Then
> $$\log(a_1 \cdot a_2 \cdot a_3) = \log a_1 + \log a_2 + \log a_3$$
> (3) We can see from step (2) any $\log(a_1 \cdot a_2 \cdot ... a_n)$ can be repeatedly broken down to
> $\log a_1 + \log a_2 + \cdots + \log a_n$
> It is important to point out that in Response 2 the student recognizes that the process employed in the first and second cases constitutes a pattern that recursively applies to the entire sequence of propositions, $\log(a_1 \cdot a_2 \cdot ... a_n) = \log a_1 + \log a_2 + \log a_3$, n=1, 2, 3,...
> In both responses the generalizations are made from two cases. This may suggest, therefore, that both are empirical. As is explained in Harel (2001), this is not so: response 2, unlike response 1, is an expression of the transformational proof scheme. To see why, one needs to examine the two responses against the definitions of the two schemes. While both responses share the first characteristic—i.e., in both the students respond

> to the "for all" condition in the log-identity problem statement—they differ in the latter two: whereas the mental operations in Response 1 are incapable of anticipating possible subsequent outcomes in the sequence and are devoid of general principles in the evidencing process, the mental operations in Response 2 correctly predict, on the basis of the general rule, $\log(ax) = \log a + \log x$, that the same outcome will be obtained in each step of the sequence. Further, in Response 1 the inference rule that governs the evidencing process is empirical; namely, $(\exists r \in R)(P(r)) \Rightarrow (\forall r \in R)(P(r))$. In Response 2, on the other hand, it is deductive; namely, it is based on the inference rule $(\forall r \in R)(P(r)) \wedge (w \in R) \Rightarrow P(w)$. (Here r is any pair of real numbers a and x, R is the set of all pairs of real numbers, P(r) is the statement $\log(ax) = \log a + \log x$," and w in step n is a pair of real numbers $a_1 a_2 \ldots a_{n-1}$ and a_n .) (Harel & Sowder, 2007, pp. 8,9)

The axiomatic proof scheme also has the three characteristics that define the transformational proof scheme. The transformational and the axiomatic proof schemes differ in the following sense: an axiomatic proof scheme is given by an individual who has acquired the more general knowledge of the fact that mathematics in whatever field of its development starts from accepted principles that is from axioms. In this research for reasons explained in the methodology chapter I have not used the axiomatic proof scheme. Summarising in brief the deductive proof scheme class has the following structure:

- ➢ Deductive proof schemes class
  - ✓ Transformational proof schemes
  - ✓ Axiomatic proof schemes

Throughout the present work the following abbreviations are used for the proof schemes above:

- The external conviction proof schemes class **(**=EC.**)** comprising the ritual proof scheme **(**=EC.R.); the authoritarian proof scheme **(**=EC.A.**)**; and the non-referential symbolic proof scheme **(**=EC.NRS.**)**.
- The empirical proof schemes class **(**=E.) comprising the inductive proof scheme **(**=E.I.**)**; and the perceptual proof scheme **(**=E.P.).

- The deductive proof scheme class (=D.) comprising the transformational proof scheme (=D.T.); and the axiomatic proof scheme (=D.A.).

Harel and Sowder (2007) call the major classes “classes” and the ‘subclasses’ sometimes “subschemes” and sometimes “subcategories”. I prefer simply to use the term “proof scheme”: thus for example I speak of the external conviction (=EC.) proof scheme and of the external conviction non referential symbolic (=EC.NRS.) proof scheme. The deductive axiomatic proof scheme (=D.A.) and consequently finer sub-subclasses of it does not appear in my analysis as mentioned; using it would have constituted a methodological error because I speak of the first encounter with proof whereas the D.A. proof scheme, according to Harel and Sowder obviously refers to situations that occur only after systematic work on proof and the gathering of substantial amounts of proof experience. Finer sub-classes of D.T. proof scheme found in Harel and Sowder (1998) first work on proof schemes do not appear neither in my analysis (see methodology chapter).

2.2.3 The choice of Harel and Sowder taxonomy as analytical tool of the research

In what follows I explain the choice for Harel and Sowder’s taxonomy as an analytical tool for my research project by reviewing constructs, methods and ideas that were presented in section 2.1 as well aspects of the philosophy underlying the taxonomy and its structure presented in sections 2.2.1 and 2.2.3.

A plethora of ideas developed in a scientific field does not necessarily imply controversy, contradiction or conflict but makes their appearance possible as well as probable. Every science worthy of its name is alive because of controversy, contradiction and conflict. Thus the progress of ideas is normally accompanied by a divergence in ideas. In relation to this Balacheff (2008) goes as far as to underline how different notions or perceptions of proof in research work could even constitute an obstacle to further progress. Sometimes even the mathematical terminology is not universally agreed upon, as in the case of “indirect proof” (Antonini & Mariotti, 2008). Bergsten’s (2008) work indicates how difficult is to analyse even straightforward problems with which all agree that students have difficulties, even where interpretation of the difficulties diverges substantially, although they could be taken as part of a bigger interpretation embracing and entailing the partial interpretations. Bartolini-Bussi (2005) points out the difficulties of communicating the results of certain research experiments. Goldin expresses doubts about the quality of research and sets the following criterion for attaining it:

> Our knowledge bases in mathematics and the natural sciences should ‘fit’ easily with and augment the knowledge bases deriving from educational research in these domains. (Goldin, 2003, p. 198)

Lester (2005) engages in an analysis, with political features, considering the factors that seem to affect and influence mathematics research and believes that combining different perspectives would profit mathematics research. Reacting to Lester’s paper Harel (2006) supports the idea that, it is the mathematics and its unique constructs, history and epistemology that makes mathematics education a discipline in its own right. Anna Sfard (2000) looking critically at some popular ideas about teaching mathematics asks how far one may go in re-negotiating and relaxing the rules of mathematical discourse before seriously affecting its learnability. According to Sfard this also applies to proof:

> I was trying to show, the idea of a negative number cannot be fully understood within a discourse which is regarded as describing the physical world, since there is nothing in this world, as it is known to the student, which would dictate the rule “minus times minus is plus.” Similarly, the request for rigorous definitions which may count as “truly mathematical” cannot sound convincing without its being related to the idea of mathematical proof; and the mathematical rules of proving, in their turn, cannot be understood without the agreement that the ultimate criterion of a proper argumentation is the logical bond between

> propositions, and not relations between these propositions and physical reality. (Sfard, 2000, pp. 28-29)

There are also cases too where it is difficult for the newcomer to distinguish between theoretical constructs and terminology. For example what Brousseau calls the "cognitive obstacle" (1997) bears, to my eyes, a strong resemblance or connection to Tall and Vinner's (1981) notions of concept image and concept definition discrepancy of which constitutes a cognitive obstacle.

In a nutshell, I can summarise the situation as follows:

There are studies such as those of Healy and Hoyles (2000) and Hoyles and Healy (2006) which have captured moments in development of mathematical thinking and proof behaviour of students using a classical model for the assessment of their texts, assigning marks on a scale decided by the researchers. Specifically in the Healy and Hoyles studies reference is made, in what regards proof, to the taxonomy of Harel and Sowder and develop notions such as proof production and appreciation by testing students' perceptions of arguments accepted as proofs.

There are also studies that refer to qualitative model for assessing knowledge or proof behaviour such as:

a. The Van Hiele model of assessing geometrical knowledge
b. Balacheff's proof behaviour taxonomy

Besides these, there is a plethora of qualitative theoretical constructs that could be used to analyse students' proof behaviour including cognitive unity, proof production in comparison to proof appreciation, semantic and syntactic proof productions, etc. as I explained in the previous section.

On the other hand DGS and generally ICT technology offer ideas and contribute to the research regarding proof and proving as reflected in various research projects examples of which I have already mentioned.

Finally a number of research projects implement theoretical considerations outside mathematics or mathematics education such as those of Habermas and of Toulmin. As already noted in 2.1 Habermas' theories use by Morselli and Boero (2011) and Toulmin's by Inglis et al (2007), in Knipping (2008) and Krummheuer (2003).

First of all I wanted to investigate students' proof behaviour when they encounter proof for the first time and needed an analytic tool to help me understand the perception of proof and proving behaviour. I did not want to use the traditional method of texts assessment or any other assessment model and classify various aspects of proving performance. I was seeking a qualitative approach to how students think when proving. Choosing Harel and Sowder's taxonomy I wanted, in a smaller scale than above mentioned studies of Healy and Hoyles, to capture moments in the development of mathematical thinking and proof behaviour of students through a qualitative lens.

Although the relevant literature provides many creative ideas, I excluded using DGS from the beginning because it does not correspond whatsoever with the reality of the first encounter with proof in Greek classrooms.

I also wanted to understand how the students perceive proof in both geometry and algebra, so a model like Van Hiele's, although very important and influential, investigates only issues of geometry and, further, it investigates proof behaviour only tangentially in the broader context of growth of geometrical knowledge.

The theoretical constructs of syntactic and semantic proof productions are very interesting points of view from which to analyse proof behaviour. Although I did not use them as a general analytical tool, there are cases, as the analysis of the students' texts shows, where they clarify some aspects of my observations. The same is valid for important constructs as the relational and instrumental understanding (Skemp, 1976), concept image and concept definition (Tall & Vinner, 1981) etc.

Toulmin's argumentation model is the product of an admirable and very seminal work. Indeed, Toulmin warns the logicians and consequently the mathematicians not to turn a blind eye to the complexity of the real world in favour of mathematical eternal truths. However, this work analyses an argument in details and goes deeper into the structure of the argument itself. I, on the other hand, wanted instead to go in the opposite direction, understanding what 'family' the argument belongs to and thus classifying it, thus the choice of Harel and Sowder's taxonomy. However, I find it very attractive to use Toulmin's model in the future for the analysis of students' arguments.

Habermas theory might as well be a choice as analytical tool in the future research if my knowledge of it permits me to adapt to it.

Balacheff's taxonomy is a very important taxonomic proposal and has influenced many researchers although there are also cases where his taxonomy has been seen with a critical eye as in Varghese (2011). Harel and Sowder (2007) refer to him and his work and find parallels of their work to his. However, Harel and Sowder's taxonomy, in comparison to Balacheff's taxonomy appeared to me closer to my experience in the classroom and in students' texts throughout the years I have been teaching mathematics, and to my experience as a school advisor. I refer to this point in more detailed fashion below where I explain the choice of the analytical tool.

There is a number of reasons that led me to the choice of the Harel end Sowder's taxonomy as an analytical tool of the present research. I can divide these reasons in the two main following categories:

a) Reasons related to the philosophy underlying the Harel and Sowder's taxonomy as I perceive and interpret this philosophy.

b) Reasons related to my experience as a teacher which influenced the understanding of the Harel and Sowder's taxonomy as a potential applicable tool of proof behaviour analysis.

In what concerns the first category of reasons, I explained, in relatively extended manner in section 2.2.1, what is to be understood under the terms "philosophy underlying the Harel and Sowder's taxonomy" from my point of view. The whole of section 2.2.1 constitutes an argument in favour of the use of the Harel and Sowder's taxonomy in this respect. I only repeat here in brevity that the Harel and Sowder's taxonomy has, among others, the following features:

- It sets for the students as learning objective the understanding and the practicing of mathematical proof as it is considered and seen by the modern axiomatic system.
- Although the final purpose of teaching proof is the learning the practising of modern axiomatic proof the term proof is being replaced by the term proof scheme in order to embrace proofs that cannot necessarily be characterised as a deductive axiomatic proof scheme. Thus proof in this sense is characterised by a kind of subjectivity either of the individual or of the community that is giving a proof to an assertion. This other looking at matters regarding proof is a result of the study of proof's historical evolution on one hand as well of the observation of students' attempts to formulate and give proofs.
- The taxonomy shifts the focus of educators, researchers and teachers on students without forgetting or neglecting the concept of proof as it has been modulated by the modern axiomatic system. By doing so offers an important

pedagogical service because deeper understanding of how students think is of crucial importance in the teaching and learning of proof.

Summarising, the taxonomy's theoretical background (Harel & Sowder, 1998, 2007) is concrete, productive and philosophically strong. I find that this mode of thinking formulates and expresses proof in mathematics as an educational task. I believe also that Harel & Sowder's taxonomy follows the transformation from the empirical to the deductive in a sufficiently trustworthy and reliable manner, shedding light on an important evolutionary element in what regards students' reasoning.

The second category of the reasons for my choice is intertwined with my experience as a teacher. For example, I mentioned earlier that Harel and Sowder's taxonomy seemed to me closer than the Balacheff's one. For me the categories of proof schemes are in Harel and Sowder work very good understandable and very strongly related to what I had as well encountered as students' proof behaviour. This could be the result of the refinement Harel and Sowder have made presenting the taxonomy's proof schemes. On the other hand Balacheff's proof categories appeared to my eyes less relevant to what I had encountered as students' proof behaviour making for me more difficult to apply it. Furthermore in the very early stages of my research I experimented by applying the Harel and Sowder taxonomy in a small amount of data collected for the needs of a different study (Kanellos & Nardi, 2009). From this application I gained the feeling that the taxonomy might be a useful tool. At the same time, although I tried, I found it difficult to apply the taxonomy of Balacheff to the same amount of data. In comparison to the Harel and Sowder's taxonomy, it was much more difficult to allocate student's proof to Balacheff's categories. Additionally I found the Harel and Sowder's taxonomy implemented by others as e.g. Housman and Porter (2003). These authors offer a rigorous implementation of the taxonomy in question as an instrument of analysis, where the researchers analyse above-average students' proof schemes. This aspect of this work made me ask the question: why should be used only for above-average students? Why not implement it with a sample of normal students in a normal school?

I conclude this section with the consequence of my choice. Namely, the choice of Harel and Sowder's taxonomy leads unavoidably to the two following research questions:

c) How, if at all, is the Harel and Sowder taxonomy applicable to the Greek secondary educational contexts?

d) How, if at all, can the Harel and Sowder taxonomy be used to elucidate students' competence in proving as well as how they value proof within the Greek secondary educational contexts?

2.3 Summary

In this chapter I have described my investigation of the literature on the various currents of research, theoretical analysis and constructs. I have reviewed a number of works concerning primary, secondary and a tertiary education, and others that do not necessarily refer to an educational level. Although they represent a tiny fraction of the vast field of relevant literature, these studies gave me the chance to think about various theoretical and practical problems and they helped me to understand my own orientation with regard to the epistemology mathematics education. I have also explained why I have chosen Harel and Sowder's taxonomy as an instrument of analysis for my research project and presented briefly. In the methodology chapter I explain how I implemented this taxonomy as an analytical tool in my research project.

CHAPTER 3: METHODOLOGY

3.0 Introduction

This is a qualitative study with quantitative elements. Specifically, the method is a mixture of qualitative data analysis with some descriptive statistics.

While this is not a grounded theory study, I was deeply impressed by the spirit of grounded theory, as in Glaser and Strauss (1967), who offer a proposition on how theory can be grounded in data collected in real life conditions. I collected such data in a typical Greek school, focusing on mathematics lessons in which proof was being taught. I did not produce a theory grounded in the data that I collected and analysed as such but I deployed a variation of the Harel and Sowder's taxonomy that I generated through the analysis of my data. The taxonomy itself was produced in the following manner:

> The system of proof schemes reported in this paper has undergone numerous revisions dictated by the results from our qualitative analysis data, cross-checked through interviews with mathematics majors at a separate institution. *The current version of this system's structure and components seems to have reached a stable stage. By this we mean in completing the analysis of about 50% of the data, we discovered no additional categories of proof schemes and none of the existent categories has been altered* (Harel & Sowder, 1998, p. 238; my italics)

The passage above is strongly reminiscent of the emergence of categories and the stabilisation thereof after certain levels of analysis as in Glaser and Strauss (1967). Dey (1999), to underline the impact of grounded theory, speaks of armchair analysis which is based on abstract deductive thought , contrasting it to grounded theory, when the latter first appeared. In the same spirit I did not accept the taxonomy in an axiomatic deductive manner as armchair analysis does, but I tested the taxonomy initially on small amounts of data (Kanellos & Nardi, 2009), was convinced it is a viable and meaningful way to analyse my data and then proceeded to using it across the bulk of my data.

In order to collect my data I collaborated with my colleagues, especially the Year 9 mathematics teacher of the school that agreed to participate in my study. Collaboration of this kind also has a touch of grounded theory because I followed the teaching of proof in the classrooms for an extended period of time. In my effort to find a theoretical context for the collaboration I came across the ideas of the lesson study, lesson design, learning study and learning awareness in the works of Pang and Marton (Marton & Pang, 2006; Pang, 2006; Pang & Marton, 2003), Miyakawa and Winslow (2009), Marton and Booth (1997), Marton & Tsui (2004) and Pang (2008). Although my research project, in terms of the collaboration with teachers, has been influenced by the spirit of these works, I cannot say that my study is a learning study project.

In what follows in section 3.1 of this chapter I describe how my research was conducted. In section 3.2 I briefly outline the Greek educational context in which the study was carried out. As the data that I collected substantially exceeds the data that I present in this thesis, in section 3.3 I describe the data I collected and what part of it I finally analysed for the purpose of completing this thesis. In section 3.4 I describe how I analysed these data. Section 3.5 is dedicated to ethical issues. The concluding section 3.6 summarises the chapter.

3.1 How the study was conducted

In Greece mechanisms that bring teachers together to work on the planning, implementation, evaluation, revision and dissemination of a research lesson do not officially exist. In my research project I functioned as a mechanism of this kind being a member of the team that planned the lessons taught by my colleague J (anonymised thereafter as J) and simultaneously acted as a school advisor and researcher. The teachers' team consisted almost exclusively of two members, namely my colleague J and I me. Occasionally other colleagues at the same school took part in our discussions, as I explain later.

J and I agreed on experimenting with the teaching of proof which became our object of learning. So we aimed to pool our experience in one or a series of research lessons to improve teaching and learning of proof and proving. My aim as a researcher was to observe and qualitatively describe how students perceive proof and proving when they first encounter it using Harel and Sowder's taxonomy (1998, 2007) as a theoretical tool of analysing data collected in this process. Choosing this taxonomy added a new element to the research process. Analysing the students' perceptions of proof and proving tests the applicability of the tool of analysis itself because Greece's cultural and secondary educational environment is very different to that of tertiary education in the US, where the Harel and Sowder's taxonomy was constructed. To teach proof a teacher has to think about how to teach effectively and plan and implement lessons in the classroom that will solve students' problems with the objects of learning in question. Indeed, J and I thought about how to teach congruency criteria of triangles and algebraic identities: in fact, we agreed upon a method to teach triangles' congruency criteria. We did so by asking the students to construct a triangle (for each criterion) of which the corresponding elements were given, say the three sides, and then letting them compare their individual constructions by superimposing them. Our objective was to give students the chance to understand that each criterion describes not only two but a class of equal triangles, making any two necessarily congruent. Another important problem for students we identified in our discussions is the confusion between data and the unknown in a mathematics problem. We had both empirically observed that students engaging in the proof procedure of a theorem, an exercise etc., sometimes appeal to properties that are invited to prove as already valid (Mariotti, 2000). This confusion is in essence an inability to distinguish the hypothesis and the conclusion. We decided that the students would be taught from the beginning, before proving anything, to write down clearly what the data are and what the unknown is. Polya in *How to solve it* (1990), among others, underlines the importance of the distinction data-conclusion. With regard to algebraic identities and relations connected to them our main concern was to assist students to understand the significance of, on the one hand, the sequence in which operations can be carried out and on the other, the ability to distinguish between what is a sum and what a product in an algebraic expression.

Proof as an object of learning and proving as a capability in the present study are both confined to the Year 9 curriculum, i.e. algebraic identities and triangle congruency criteria, and that is what the students encounter here as proof. Thus proof under the previous consideration is an object of learning that has two facets: (i) capability to prove in the context of the Year 9 curriculum and (ii) appreciation of proof.

The second step of the study was the ascertaining of students' pre-understanding of the object of learning. To scrutinise the students' pre-understanding of proof and proving in my study I created a test (hereafter T1 or pre-proof test) in September 2010 in collaboration with my colleagues J, and N (anonymised thereafter as N). While preparing the test we discussed ideas about appropriate questions that we would set for the students starting Year 9. The final form of the test was mainly created by J, N and myself and included only geometry questions, which we thought most appropriate for testing pre-proof understanding, given what the students had been taught in Years

7 and 8. The test was administered at the end of September 2010 on a normal school day and helped to create a picture of what the students could achieve in what we saw as simple proof problems. T1 was created in the spirit of the pre-understanding of students' perceptions and its results are analysed in Chapter 4.

The third step was the designing and implementation of lesson plans. In Greece in Year 9 mathematics is taught for four lessons per week. About 20 hours were allocated to teaching proof in the 2010-2011 school year and there were four Year 9 classes in the school, so that J and I collaborated on the planning intensively from October 25, 2010 until March 11, 2011.

The fourth step was the evaluation of the whole process, e.g. through tests that focusing on the object of learning. Parallel to our very frequent discussions on the performance of the students we administered an intermediate test (hereafter T2) to the students. This test is not analysed in the thesis for reasons that I explain in section 3.3 (data collection) of this chapter.

In the fourth step of the study, just before the beginning of the official May-June school examinations we administered another test (T3) to students who volunteered for it. This test asked for proof in algebra as well as geometry, both of which they had been taught between October 2010 to March 2011. J and I created the test after discussion on what should be expected of the students at the end of the school year. T3 is analysed in Chapter 5 and provides information on the conceptions of proof that the students developed during teaching in the classroom. The students' perceptions are described in the terms of the proof scheme taxonomy by Harel and Sowder.

The fifth step of the study consisted of reporting and disseminating a number of results of the study. Up to now, on various occasions, in the context of my activities as a school advisor, small parts of the study have been presented to my colleagues with the aim of making them more aware of the problems that the students first encounter with proof. Full realisation of this step is an on-going process. Furthermore in the context of the fifth step Elena Nardi, Irene Biza and I have presented preliminary findings of the study at the following conferences: at PME 35 in Ankara (Turkey) in July 2011 (poster entitled “Tendencies towards deductive reasoning in secondary students’ pre-proof ideas: A Greek case” (2011b)); at the 14th biennial EARLI Conference in Exeter (UK) in August-September 2011 as a research report entitled “Greek secondary students’ early encounters with mathematical proof in algebra and Euclidean geometry” (2011a); at the fourth ENEDIM conference in December 2011 in Ioannina, Greece, as a research report entitled “Tendencies towards deductive thinking in students’ pre-proof conceptions’” (2011c); at PME 37 in Kiel (Germany) in July-August 2012 a short oral presentation entitled “The interplay between fluency and appreciation in secondary students’ first encounter with proof” (2013). This thesis is also intended as a means of disseminating the results of the study.

3.2 Context of the study

In the first chapter I introduced Greek secondary education in brief. Below I give a more detailed picture of the mathematics curriculum, with emphasis on Years 7, 8 and 9 and introduce the participating teachers, students and school. From here onwards, I use the word curriculum to mean mathematics curriculum.

Greek education is compulsory for 10 years. A preschool year is followed by six years of primary education and three years of lower secondary education, or Gymnasium, followed by non-compulsory upper secondary education or Lyceum. Lower secondary education includes Year 7 (age 13), 8 and 9.

Upper secondary education includes Years 10 (age 15-16), 11 and 12. The students graduate at 17-18 years of age. There are two types of upper secondary education: general lyceum and vocational lyceum. This thesis is only concerned with the general lyceum's curriculum.

Mathematics is taught for four lessons per week in lower secondary schools in Greece. The curriculum is divided into Algebra and Geometry.

The curriculum prescribes the following topics[1] for Year 7 (in brackets: recommended number of lessons)

Arithmetic-Algebra:

- natural numbers, ordering, rounding (1 hour)
- addition, subtraction and multiplication (2 hours)
- powers of numbers (2 hours)
- Euclidean division, divisibility, divisibility criteria, greatest common divisor, lowest common multiple, prime factorisation of a natural number (3 hours)
- the notion of fraction (2 hours)
- congruent (equivalent) fractions (1 hour)
- comparing fractions (1 hour)
- addition and subtraction of fractions (2 hours)
- multiplication and division of fractions (4 hours)
- decimal fractions, decimal numbers, ordering decimal numbers, rounding decimal numbers (2 hours)
- operations with decimal numbers. powers of decimal numbers (4 hours)
- scientific notation (standard form) of big numbers (1 hour)

[1] Retrievable in Greek from http://www.pi-schools.gr/programs/depps/ as well as from the National Printing House at http://www.et.gr/ in the form of the Official Journal of the Hellenic Republic under the name: ΦΕΚ 303-Β'/13.03.03 (Ministry, 2003).

- units of measurement (2 hours)
- the notion of equation, equations of the form a+x=b, x-a=b, a-x=b, ax=b, a:x=b, x:a=b (2 hours)
- solving problems (3 hours)
- percentages (3 hours)
- Cartesian coordinates of points in two dimensions (1 hour)
- ratio of two numbers, proportion (2 hours)
- proportional quantities, properties of proportional quantities (2 hour)
- graphic representation of proportion (1 hour)
- problems which can be solved using proportions (2 hours)
- inversely proportional quantities (2 hours)
- positive and negative rational numbers, the rational line, point's abscissa, the absolute value of rational number, opposite rationals, comparing rationals (3 hours)
- addition and subtraction of rational numbers (3 hours)
- multiplication of rational numbers (2 hours)
- division of rational numbers (2 hours)
- decimal form of rational numbers (1 hour)
- powers of rational numbers with integer exponent, scientific notation (standard form) of big and small numbers (4 hours)

Geometry:

- plane, point, line segment, straight line, ray, half plane (2 hours)
- measurement of line segments, comparison of line segments, congruency of line segments, distance between points, middle point of line segment (2 hours)
- addition and subtraction of line segments (1 hour)

- measurement of angles, comparison of angles, angle bisector, congruency of linear shapes (2 hours)
- types of angles, perpendicular straight lines (2 hours)
- adjacent angles, sum of angles (2 hours)
- supplementary angles, complementary angles, vertical angles (2 hours)
- positions of straight lines on the plane (2 hours)
- distance from a point to a straight line, distance between parallel straight lines (1 hour)
- the circle, elements of the circle (1 hour)
- central angle, relation of central angle to corresponding arc, arc measurement (2 hours)
- relative positions of straight lines and circles (2 hours)
- axial symmetry, axis of symmetry (3 hours)
- perpendicular bisector of line segment (2 hours)
- central symmetry, centre of symmetry (3 hours)
- parallel straight lines cut by a transversal straight line (2 hours)
- elements of triangle, sum of angles of a triangle, types of triangles, properties of the isosceles triangles (4 hours)
- parallelogram, rectangle, rhombus, square, trapezoid, isosceles trapezoid and the properties thereof (4 hours)

For Year 8:

Algebra:

- the notion of variable (1 hour)
- equations of first degree in one unknown, resolving formulas (4 hours)
- solving problems using equations (4 hours)

- inequalities of first degree in one unknown (4 hours)
- Pythagorean theorem (2 hours)
- square root of a positive number (3 hours)
- irrational numbers (2 hours)
- the notion of function (2 hours)
- Cartesian coordinates, graphic representation of functions (3 hours)
- the functions y=ax (3 hours)
- the function y=ax+b (3 hours)
- the function y=a/x, the hyperbola (2 hours)
- fundamental notion of statistics, population, sample (2 hours)
- graphical representations, pictographs, bar graphs pie charts, time charts (3 hours)
- frequency and relative frequency distribution (2 hours)
- grouping data (3 hours)
- mean value, median, variance (5 hours).

Geometry:

- sines, cosines of acute angles (5 hours)
- tangent of acute angles (2 hours)
- the notion of the vector, norm of a vector (1 hour)
- sum and difference of vectors, analysis of a vector in two mutually perpendicular components (3 hours)
- area of plane figure (2 hours)
- measurement units of plane figures (3 hours)
- area of various plane figures (6 hours)
- central and inscribed angles (2 hours)

- regular polygons (3 hours)
- length of a circle's circumference, length of an arc of a circle (4 hours)
- area of a circle, area of a circular sector (4 hours)
- relative positions of straight lines and planes, straight line perpendicular to plane, distance of a point from a plane, distance between parallel planes (2 hours)
- prism, cylinder and elements thereof, surface area of prism and cylinder, volume measurement units, volume of prism and cylinder (3 hours)
- pyramids, cone and elements thereof, surface area of pyramid and cone, volume of pyramid and cone (4 hours)
- the sphere and its elements, measurement of the sphere (4 hours)

For Year 9:

Algebra:

- real numbers and operations (5 hours)
- monomial and polynomials, operations with monomials, addition and subtraction of polynomials (4 hours)
- multiplication of polynomials (2 hours)
- basic algebraic identities (5 hours)
- factorisation of algebraic expressions, greatest common divisor and lowest common multiple of algebraic expressions (8 hours)
- division of polynomials (3 hours)
- rational algebraic expressions (5 hours)
- the equation $ax+b=0$ (1 hour)
- second degree equations in one unknown, problems leading to second degree equations in one unknown (7 hours)

- rational equations (3 hours)
- inequalities, inequalities in one unknown (4 hours)
- the notion of a linear equation, the notion of a linear system and its graphical solution, algebraic solution of a linear system (7 hours)
- the function $y=ax^2$ (5 hours)
- sets (3 hours)
- sample space, events (3 hours)
- the notion of probability (3 hours)

Geometry:

- triangle congruency (5 hours)
- ratio of line segments (2 hours)
- Thales' theorem (2 hours)
- homotheticity, similarity (6 hours)
- area of similar plane figures (2 hours)
- trigonometric numbers of angle φ with $0° \leq \varphi \leq 180°$ (2 hours)
- trigonometric numbers of supplementary angles (2 hours)
- relations between the trigonometric numbers of an angle (4 hours)
- law of sines, law of cosines (5 hours)

This list of mathematics topics gives general direction on what should be taught. Every bullet on the list is a topic for teaching and brief instructions are given as to how it should be taught.

The curriculum is reflected in the content of the state-approved textbooks. Only one textbook for each year, from Year 1 in elementary education through to Year 12 of secondary education is approved by the state and they cover the above list of curriculum topics and other instructions included in the official state document containing the curriculum (Ministry, 2003). The students receive the approved textbook gratis at the beginning of each school year.

Almost every year the Ministry of Education sends additional instructions, new topics and how they should be taught and minor or major changes to the amount of material to be taught. For 2010-2011, in which the present study was conducted, the Ministry of Education sent an instructions' document titled 114368/Γ2/15-09-2010 prescribing the following for Year 9 referring to the textbook by Argyrakis, Vourganas, Mentis, Tsikopoulou and Chryssovergis (2010):

Part 1

Chapter 1: Algebraic expressions (hours 29 in total)

1.1 Real numbers (repetition of Year 8)

A. Real number operations (2 hours)

B. Powers of real numbers (1 hours)

C. Square root of real numbers (2 hours)

1.2 Monomials – operations with monomials

A. Algebraic expressions—monomials (1 hour)

B. Operations with monomials (1 hour)

1.3 Polynomials – addition and subtraction (2 hours)

1.4 Multiplication of polynomials (2 hours)

1.5 Fundamental identities without sum and difference of cubes (6 hours)

1.6 Factorisation of algebraic expressions without sum and difference of cubes and without factorisation of trinomial of the form $x^2+(a+b)x+ab$ (6 hours)

1.8 Greatest common divisor, least common multiple of algebraic expressions (1 hour)

1.9 Rational algebraic expressions (2 hours)

1.10 Operations with rational algebraic expressions

A. Multiplication – division of rational expressions (1 hour)

B. Addition – subtraction of rational expressions (2 hour)

Chapter: 2 Equations – Inequalities (13 hours in aggregate)

2.2 Second degree equations

A. Solution of second degree equations by factorisation (2 hours)

B. Solution of second degree equations using formula (3 hours)

2.3 Problems leading to second degree equations (2 hours)

2.4 Rational equations (3 hours)

2.5 Inequalities – inequalities in one unknown (3 hours)

A. Order of real numbers

B. Properties of real number ordering

C. First degree inequalities in one unknown

Chapter 3: Systems of linear equations (7 hours in aggregate)

3.1 The notion of the linear equation (2 hours)

3.2 The notion of the linear system and its graphic solution (2 hours)

3.3 Algebraic solution of a linear system (3 hours)

Chapter 4: Functions (4 hours)

4.1 The function $y=ax^2$ with $a\neq0$ (2 hours)

4.2 The function $y=ax^2+bx+c$ with $a\neq0$ (2 hours)

Chapter 5: Probabilities (6 hours)

5.1 Sets (without operations with sets) (2 hours)

5.2 Sample space – events (without operations with events) (2 hours)

5.3 The notion of probability (without basic rules of probabilities' calculus) (2 hours)

<u>Part 2</u>

Chapter 1: Geometry (17 hours in total)

1.1 Congruency of triangles (5 hours)

1.2 Ratio of line segments (2 hours)

1.5 Similarity

A. Similar polygons (2 hours)

B. Similar triangles (2 hours)

1.6 Ratio of areas of similar plane figures (2 hours)

Chapter 2: Trigonometry (12 hours in aggregate)

2.1 Trigonometric numbers of an angle ω with $0° \leq ω \leq 180°$ (2 hours)

2.2 Trigonometric numbers of supplementary angles (2 hours)

2.3 Relations between the trigonometric numbers of an angle (4 hours)

2.4 Law of sines – law of cosines (4 hours)

The above stipulations are accompanied by instructions on teaching each item and what exercises to solve in the lessons or allocate as homework.

I now give concisely the general contour of the upper secondary education curriculum [2] or otherwise stated the curriculum for the general Lyceum (Years 10, 11 and 12). In Years 10 and 11 the students are taught a course of Euclidean Geometry in the spirit of Euclid's *Elements*. In Year 10 the Geometry course includes fundamental notions, basic plane figures, triangles, parallel lines, parallelograms and trapezoids and plane figures inscribed in circles. In Year 10 Algebra includes an introduction to probabilities, real numbers, equations, inequalities, progressions, basic notions of functions and the study of linear and quadratic functions.

In Years 11 and 12 the school offers three different study options, all three including obligatory Geometry and Algebra in Year 11. The course in geometry

[2] I make no particular reference to secondary vocational education where in general are being taught the same topics from almost the same books but under different time table.

includes proportions, similarity of plane figures, numerical properties of figures, areas, circle measurement and lines and planes in space. The algebra course includes linear and non-linear systems of equations, properties of functions, trigonometry, polynomials and polynomial equations, and exponential and logarithmic functions. Two of the options offer the same intensive mathematics course in analytic geometry which includes vectors, straight lines on the plane, conic sections and mathematical induction.

In Year 12 two study options offer a similarly intensive course in mathematics: an introduction to complex numbers and Calculus with elements of Analysis. All three options include a common obligatory course on Statistics and elementary Analysis.

Below I give some more detailed information about my colleagues who collaborated on the project and about the school in which the class observations took place. I note that in Greece there are two types of schools: state schools and the private schools. Both are controlled by the Ministry of Education. This means that private schools are obliged to follow the same curriculum as state schools. In state schools the teachers are civil servants with open or fixed term contracts. I myself am a civil servant with a permanent job as a teacher, and I am currently also a school advisor having been assessed and appointed to this job every four years since 2003 (2003, 2007 and 2011).

The principal of the school, anonymised thereafter as V, is a mathematician. We have collaborated on many projects in the past and, among other things, we experimented with simultaneous teaching in the same classroom on proof and proving in the 2009-2010 school year. We have had many long discussions on educational and philosophical aspects of mathematics. Although he contributed a little to the research project, because he was too busy with the school's administration most of the time, as the school principal, he embraced the study whole-heartedly.

My main collaborator on the research project, teacher J, was appointed as a civil servant with a permanent position in 2002. So by the time the study took place she had about 8 years of experience. She is a very highly educated mathematician and has a doctorate in pure mathematics from a French university. We have collaborated many times.

My colleague N is also a civil servant with a permanent position who, at the time of the study had been in service for 16 years as a civil servant. He also has a doctorate in pure mathematics from the University of Crete.

My colleague A (anonymised thereafter as A), who participated in our discussions to a limited extent, is also a civil servant with a permanent position and an MSc degree in mathematics from the University of Crete and had been in service as a civil servant with a permanent position for about eight years at the time of the study.

The school where the study took place was a typical secondary school. Its typicality can be seen in the data on the overall performance and the performance in mathematics in Years 7, 8 and 9 of the students taking part in the research project (see Appendix III). Typicality can also be seen in the occupations of the parents (see Appendix II). However, it has a strong reputation as a progressive school – while non-selective and inclusive of all student abilities – it attracts highly qualified teachers (see also above) and parental expectations are high.

3.3 Data collection

The data I collected include the following:

- Audio-recorded discussions with my colleagues, mainly with J about teaching of the four Year 9 classes.

- 22 hours audio-recordings of teaching from each of the four Year 9 classes. In total 88 hours audio-recordings.
- Handwritten notes kept during the audio-recording in which I described what was going on in the classroom. Thus every audio-recorded hour is accompanied by handwritten notes.
- Students' written answers to the pre proof (T1) test.
- Students' written answers to the intermediate (T2) test.
- Students' written answers to the test at the end of the school year just before the official school examinations (T3).
- The students' answers to the official examination covering proof and other subjects at the end of the school year (T4).

I initially also planned to observe the teaching of proof in Year 10 but abandoned this at the beginning of the school year 2010-2011 because it proved impossible to combine observations of Year 9 with that Year 10 due to timetables clashes.

In this thesis I present only the analyses of the Year 9 T1 and T3 tests. These analyses offer important information on students' perceptions of proof before and after their first encounter with it. There are two main reasons why I chose not to extend the present thesis beyond the analysis of tests T1 and T3.

The first reason is of practical character. The presentation of the analysis of test T2, let alone to include the analysis of test T4, would have made the thesis disproportionally lengthy which was not desirable.

The second reason is of methodological character. The students were not under pressure to obtain a good mark in these two tests. T1 was administered on a normal school day but the students were not obliged to answer its questions; they had the right to leave the classroom giving no answer at all, without any consequences on what mark they would obtain. Participation in T3 was also completely voluntarily – additionally because it was administered on a non-school day a few days before the beginning of the official school examinations. Students could choose not to come to school that day also with no consequences. In this sense T1 and T3 (formative assessment) are different to T2 and T4 (summative assessment). Further the voluntary character is lost in what regards the tests T2 and T4 because students were competing for better grades writing the tests in question and thus were under psychological pressure.

Below I proceed to present T1 in section 3.4.1 and T3 in section 3.4.2.

3.3.1 T1: The pre-proof test[3]

The purpose of T1 was to collect data about the students' perceptions of proof before teaching the relevant material prescribed in the Year 9 curriculum. This is why I call these perceptions pre-proof ideas.

T1 was created as a result of two meetings, between J, N and me on two different days, at the end of September 2010. However, I am exclusively responsible for the final formulation of the questions and the printed form administered to the students. The discussion during the meetings in question lasted about three hours and was audio-recorded. The object of the discussion was to determine appropriate questions answers to which would provide information about the students' pre-proof ideas and their emergent ability to prove. The translation of these questions in English follows. For every question a figure accompanied the Greek text. The original T1 in Greek can be found in Appendix I.

[3] The tests questions are named as follows: first the name of the test then follows the number of the question followed by a or b depending on the part of the question being referred to. Thus T13a means test T1, question 3, part a.

Question T11: *In a triangle ΑΒΓ angle* $\hat{A}$ *is 85° and angle* $\hat{B}$ *is 75°. Prove that angle* $\hat{\Gamma}$ *is 20° (see Figure 3.4.1.1).*

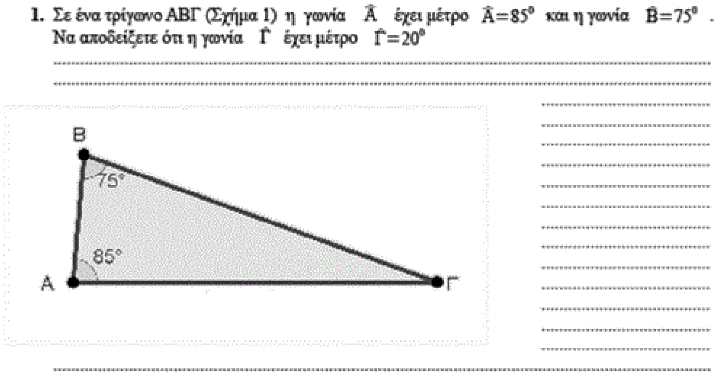

Figure 3.4.1. 1 Question T11

An adequate answer would be based on the theorem that $\hat{A} + \hat{B} + \hat{\Gamma}$ *=180°* (1). Substituting in (1) $\hat{A}$ *=85° a*nd $\hat{B}$ *=75°* and solving it for $\hat{\Gamma}$ the result $\hat{\Gamma}$ *=20° c*an be obtained. As can be verified in the curriculum of Year 7 and, as I have cited in section 3.3, the sum of the angles of a triangle is taught in Year 7. In the Year 7 textbook (Vandoulakis, Kalligas, Markakis, & Ferentinos, 2010, p. 221) a mathematical activity is proposed on this question. The students are asked to measure the angles of various triangles and then to find their sum. After this empirical approach they are asked to develop logical arguments to justify that the sum of the angles stays the same independently of the shape of the triangle. To this end it is proposed that they consider a parallel line from a vertex of a triangle to the opposite side. Then they are invited to note which angles the angles formed by the parallel and the sides of the triangle are equal to. Finally the students are prompted to see which angles are adjacent to the vertex of the triangle from which the parallel was drawn. The conclusion is that each one of the three adjacent angles is equal to the respective angles of the triangle and their geometric sum is an angle of *180°*. Additionally a number of activities and exercises are given that can be solved only by using the sum of the angles of a triangle

(Vandoulakis et al., 2010, pp. 221-224). In Year 8 the students are taught the inscribed angles in a circle. The sum of the angles of the triangle is used again in various exercises to calculate angles (Vlamos, Droutsas, Presvis, & Rekoumis, 2010, pp. 176-179). Consequently students can be considered sufficiently familiar with the sum of the angles of a triangle.

Question T12: *In a triangle ΑΒΓ the angle* $\widehat{BA\Gamma}$ *is 78° and the angle* $\widehat{AB\Gamma}$ *is 66°. The segments AI and BI are bisectors of the angles* $\widehat{BA\Gamma}$ *and* $\widehat{AB\Gamma}$ *respectively. Prove that the angle* $\widehat{AIB}$ *is 108° (see Figure 3.4.1 2).*

2. Σε ένα τρίγωνο ΑΒΓ (Σχήμα 2) η γωνία $\widehat{BA\Gamma}$ έχει μέτρο $\widehat{BA\Gamma}=78^0$ και η γωνία $\widehat{AB\Gamma}=66^0$. Οι ΑΙ και ΒΙ είναι διχοτόμοι των γωνιών $\widehat{BA\Gamma}$ και $\widehat{AB\Gamma}$ αντίστοιχα. Να αποδείξετε ότι η γωνία $\widehat{AIB}$ έχει μέτρο $\widehat{AIB}=108^0$ ....

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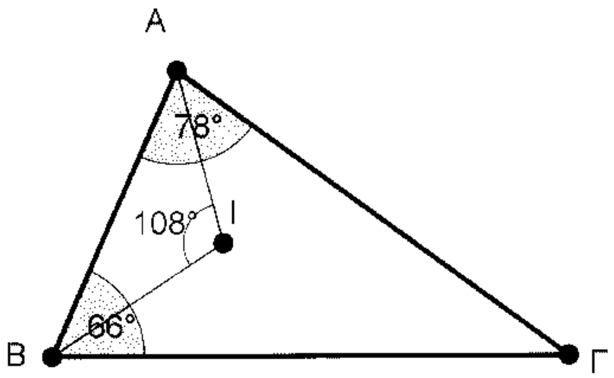


Figure 3.4.1. 2 Question T12

An adequate answer would be based on the fact that AIB is a triangle with angles $\widehat{BAI}$ and $\widehat{ABI}$ which are 39° and 33° due to the fact that AI and BI are on the bisectors of the angles $\widehat{BA\Gamma}$ and $\widehat{AB\Gamma}$ respectively. The rest of the proof should be a calculation of the angle $\widehat{BIA}$ as in Question T1 referring to the triangle AIB. Question T12 combines the property of the bisector of an angle and the sum of the angles in a triangle in a more complicated context than Question T1, allowing testing proof ability on a more difficult scale. The bisector of an angle is defined in the Year 7 textbook followed by various activities and exercises (Vandoulakis et al., 2010, pp.

167-168). The same textbook also gives the definition of the bisector of an angle belonging to a triangle (ibid., p. 219).

Question T13: *The point M on the line segment AB is at the midpoint of AB (MA=MB). The line (ε) is the perpendicular bisector of the line segment AB. Let Σ be a point on the perpendicular bisector (ε). Let us draw the line segments ΣA and ΣB. Prove that the triangle ΣAB is an isosceles triangle (see Figure 3.4.1.3).*

3. Ενός ευθυγράμμου τμήματος ΑΒ το σημείο Μ είναι το μέσο του (ΜΑ=ΜΒ). Η ευθεία (ε) είναι η μεσοκάθετος του τμήματος ΑΒ (Σχήμα 3). Έστω Σ ένα σημείο της μεσοκαθέτου (ε). Φέρουμε τα τμήματα ΣΑ και ΣΒ. Να αποδείξετε ότι το τρίγωνο ΣΑΒ είναι ισοσκελές ....

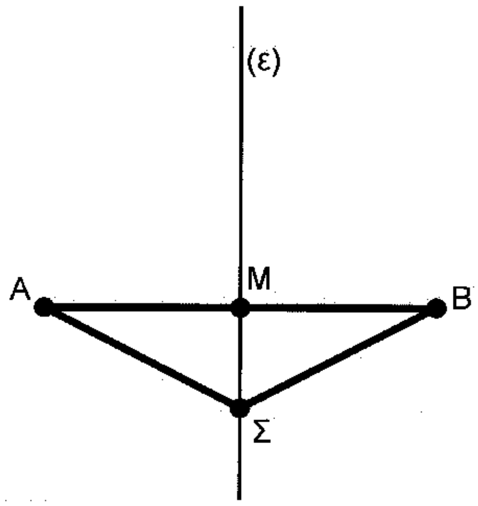


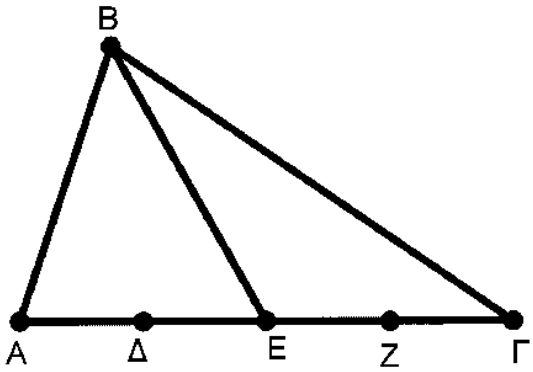
Figure 3.4.1. 3 T13

An adequate answer to this question can be based on the property of the perpendicular bisector. In other words one could assert that ΣΑ=ΣΒ because of the property of every point on the perpendicular bisector. Thus the triangle ΣΑΒ has two equal sides and is consequently an isosceles triangle. The definition and property of all points of the perpendicular bisector of a line segment and activities and exercises relevant to this material can be found in the Year 7 text book of (Vandoulakis et al., 2010, pp. 206-209). In Year 8 the property of the perpendicular bisector appears indirectly in various problems involving the isosceles and equilateral triangles and the rhombus.

This question explores how the students treat a problem concerning the implication of a property known to be valid.

<u>Question T14</u>: *In a triangle ΑΒΓ the side ΑΓ is divided into four equal parts by means of the points Δ, Ε, Ζ (that is ΒΔ=ΔΕ=ΕΖ=ΖΓ). Prove: (a) ΑΕ=ΕΓ (b) The line segment ΒΕ is the median of the triangle from the vertex Β, which corresponds to the side ΒΓ (see Figure 3.4.1.4).*

4. Σε ένα τρίγωνο ΑΒΓ (Σχήμα 4) διαιρούμε την πλευρά ΑΓ σε τέσσερα ίσα μέρη με τα σημεία Δ, Ε και Ζ (δηλαδή ΑΔ=ΔΕ=ΕΖ=ΖΓ). Να αποδείξετε ότι



a) ΑΕ=ΕΓ ....

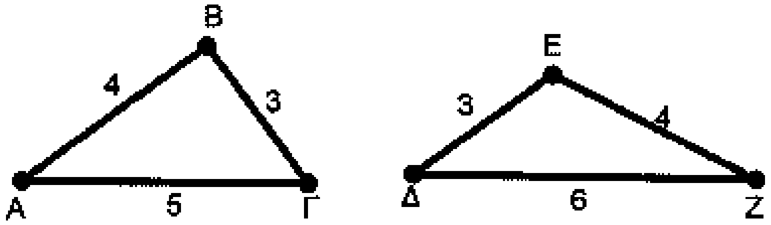
b) Το ευθύγραμμο τμήμα ΒΕ είναι η διάμεσος του τριγώνου από την κορυφή Β που αντιστοιχεί στην πλευρά ΑΓ..........................

Figure 3.4.1. 4 Question T14

An adequate answer to part (a) of T14 can be based on the fact that since ΒΔ=ΔΕ=ΕΖ=ΖΓ=x then ΑΕ=2x and ΕΓ=2x. Consequently ΑΕ=ΕΓ. An adequate answer to part (b) could comprise the description of the line segment ΒΕ. The line segment connects the vertex Β with the midpoint Ε of the side ΑΓ. Thus ΒΕ is by definition the median of the triangle ABC from the vertex B corresponding to the side BC. The Year 7 textbook gives the definitions of the midpoint (Vandoulakis et al., 2010, p. 160) and the median (ibid., p. 219). This question investigates whether the students knew what a midpoint and what a median are and whether they could manipulate a situation where the given data can be used to reach conclusions emerging out of it on the basis of definitions of such objects as the midpoint of a line segment and the median of a triangle.

<u>Question T15</u>: *In the figure you see the triangles ΑΒΓ and ΔΕΖ. (a) In the triangle ΑΒΓ the lengths of the sides are ΑΓ=5, ΓΒ=3 and ΒΑ=4. Prove that triangle ΑΒΓ is a right-angled triangle. (b) In the triangle ΔΕΖ the lengths of the sides are ΔΖ=6, ΖΕ=4 and ΕΔ=3. Prove that the triangle ΔΕΖ is not a right-angled triangle (see Figure 3.4.1.5).*

5. Στο Σχήμα 5 βλέπετε τα τρίγωνα ΑΒΓ και ΔΕΖ.



a) Στο τρίγωνο ΑΓΒ τα μήκη πλευρών είναι ΑΓ=5, ΓΒ=3 και ΒΑ=4. Να αποδείξετε ότι το τρίγωνο ΑΓΒ είναι ορθογώνιο...

...

...

...

...

b) Στο τρίγωνο ΔΖΕ τα μήκη πλευρών είναι ΔΖ=6, ΖΕ=4 και ΕΔ=3. Να αποδείξετε ότι το τρίγωνο ΔΖΕ <u>δεν</u> είναι ορθογώνιο...

...

Figure 3.4.1. 5 Question T15

An adequate answer to part (a) can draw on the converse Pythagorean Theorem. In other words to test whether 5^2, which is the square of the length of the biggest side ΑΓ, is equal to $3^2 + 4^2$ which is the sum of the squares of the lengths of the two remaining sides. After the necessary calculations it turns out that $5^2 = 3^2 + 4^2$. Thus, by virtue of the converse of the Pythagorean Theorem, triangle ΑΒΓ is a right-angled triangle with side ΑΓ=5 as the hypotenuse, or in other words with $\widehat{AB\Gamma}$ equal to a right angle. An adequate answer to part (b) would again compare 6^2, that is the square of the length of the biggest side ΔZ, to $3^2 + 4^2$ which is the sum of the squares of the lengths of the two remaining sides. Upon verifying that $6^2 \neq 3^2 + 4^2$ one could appeal to the Pythagorean Theorem and obtain a contradiction concluding that since $6^2 \neq 3^2 + 4^2$ is valid the triangle cannot be right-angled one because, if it is, then necessarily $6^2 = 3^2 + 4^2$ would be true. Thus the triangle ΑΒΓ is not a right-angled triangle. In the Year 8 text book we find the formulation of the Pythagorean Theorem, its converse and a number of activities and exercises (Vlamos et al., 2010, pp. 127-131). The Question T15 was intended to gather information on whether the students could handle this problem even though they had only latently been introduced to proof in Year 8.

<u>Question T16</u>: *In the figure 6 an isosceles triangle ΑΒΓ has angles $\widehat{AB\Gamma}$ and $\widehat{A\Gamma B}$ as equal and both 44°. Calculate the measure of the angle $\widehat{BA\Gamma}$ (see Figure 3.4.1.6).*

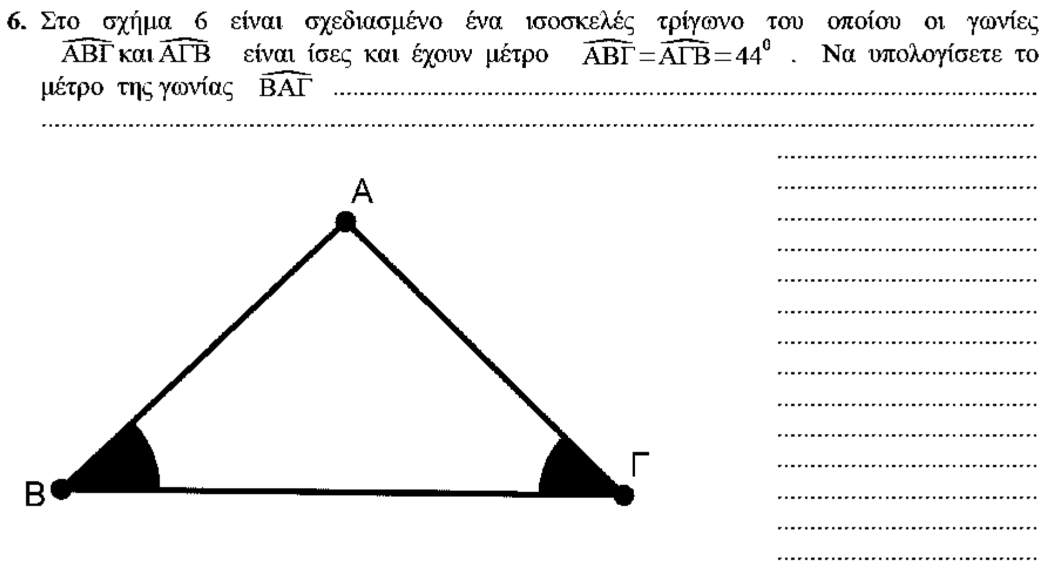

6. Στο σχήμα 6 είναι σχεδιασμένο ένα ισοσκελές τρίγωνο του οποίου οι γωνίες $\widehat{AB\Gamma}$ και $\widehat{A\Gamma B}$ είναι ίσες και έχουν μέτρο $\widehat{AB\Gamma}=\widehat{A\Gamma B}=44^0$. Να υπολογίσετε το μέτρο της γωνίας $\widehat{BA\Gamma}$ ....

Figure 3.4.1. 6 Question T15

An adequate answer would resemble the one given to Question T11. Thus all is needed is to subtract from 180° the sum of the measures of given angles, 88°, from 180° to find that $\widehat{BA\Gamma}$ is 92°. At first sight Question T15 is identical to Question T11 in this respect. But there are two underlying purposes in it: (a) the first tested whether the slight change of context in comparison to Question T11 would provoke different answers, and to what extent. The words 'proof' or 'prove' are not used and the triangle is an isosceles triangle. (b) The second purpose sought to detect whether the students are misled by the figure and perceive the triangle in the figure as a right-angled triangle[4].

The students had 45 minutes to complete the test and if a student asked for more time s/he could be granted an additional 15 minutes time. No student did.

3.3.2 The test T3

This test consists of two sections: a first section containing three algebra questions and a second section containing three geometry questions where the proof has to do with Geometry. J and I created the test: I proposed about fourteen questions, on the base of what had been taught in the previous months, and we discussed which of them we would use. Our discussion lasted about 45 minutes and was audio-recorded. Our intention in choosing the questions was to give the students questions that tended to be slightly demanding and avoid questions that were too easy. Questions T34 and T35 in the second section were accompanied by figures. Only Question T34 required the students to draw their own figure. The original T3 in Greek can be found in Appendix I. The description of the questions follows.

Question T31: *For the real numbers a and b the following relation is valid:* $a^2+b^2=5^2$. *Prove that* $\left(a\sqrt{3}+b\sqrt{2}\right)^2+\left(a\sqrt{2}-b\sqrt{3}\right)^2=125$.

An adequate answer could use of identities $(A\pm B)^2=A^2\pm 2AB+B^2$. After the expansions and all necessary calculations the left side of the equality takes the form $5a^2+5b^2=5(a^2+b^2)=5\cdot 5^2=125$. Question T31 intended to check to what

[4] The idea comes from Harel and Sowder (1998, p. 257) in whose example a student is carried away by the figure drawn and perceives a parallelogram as a square.

degree the students learned, during the school year, how to use elementary algebraic identities and symbols, such as that of the square root, to obtain a certain result. The underlying parallel purpose was to check whether Empirical Inductive (E.I.) proof schemes were present. Indeed, some students seeing the relation $a^2 + b^2 = 5^2$ think that the numbers a, b involved have the values 3 and 4. Such a perception is probably due to the fact that the triangle with sides 3, 4, 5 is a right-angled triangle and additionally the relation $a^2 + b^2 = 5^2$ is strongly reminiscent of the Pythagorean Theorem. According to Harel and Sowder the substitution of certain values for the variables, under whatever perceptions, reveals the presence of an E.I. proof scheme.

Question T32: *If the difference of the squares of two unequal natural numbers κ and λ (κ>λ) is equal to the sum of the two natural numbers (a) prove that the difference of the two natural numbers is equal to 1 and (b) prove that* 5556^2-5555^2*=11111.*

An adequate answer to (a) would begin setting κ^2-λ^2=κ+λ which implies (κ-λ)(κ+λ)=κ+λ leading to the conclusion κ-λ=1 by observing that κ+λ>0 and either by dividing both members by κ+λ or by transferring all the quantities to the right member and factorising to obtain (κ+λ)[(κ-λ)-1]=0. Part (b) can be answered by implementing the identity $A^2 - B^2$=(A−B)(A+B) setting A=5556 and B=5555. The question was intended to check whether the identity A^2—B^2 =(A—B)(A+B) could be used by the students in proof processes. On the other hand regarding in part (a) students might be tempted to substitute for κ, λ numerical values. Thus the question could detect the presence of E.I. proof schemes as well. Finally, if κ^2-λ^2=κ+λ then κ-λ=1. The converse would be: if κ-λ=1 then κ^2-λ^2=κ+λ. In part (b) 5556-5555=1. The underlying purpose was to test whether the students understood this difference. If in solving part (b) invoked part (a) that could imply they did not.

Question T33: *Two of your peers are wondering how to prove: $(\alpha—\beta)(\alpha+\beta)=\alpha^2—\beta^2$. One of them proposes to give the variables numerical values (e.g. α=2 and β=1) and to calculate the left and the right parts and see if the calculated values are equal. They experiment with some values of α and β and verify that the numerical results on the right and on the left are equal. After that they think they have proved the relation. (a) Overhearing the conversation, do you agree with them? If not, what would you suggest to them? (b) Do you think the teacher would agree with them?*

An adequate answer to part (a) would be to propose application of the distributive law on the left member of the given relation to arrive, after the necessary simplifications at the right member. Part (a) is an indirect question about what constitutes proof and what the verification of an algebraic relation. If verification is taken for proof, one could assert to have detected E.I. proof scheme. Part (b) of the question searches for EC.A.proof schemes. Whether this proof scheme is present depends on the type of answer[5].

Before describing Questions T34, T35, and T36 I briefly cite the congruency criteria of triangles and right-angled triangles as they appear in Year 9 the textbook (Argyrakis et al., 2010).

The first congruency criterion the textbook gives the following: if two triangles have two sides equal one by one and equal the angles included by the equal sides then the triangles are congruent (ibid., p. 188). In English this triangle congruency criterion is called Side-Angle-Side (SAS).

[5] Question T33ab is in the spirit of Healy and Hoyles (2000), who gave proofs to students and asked them to assess what proof would be judged the best by the teacher and what proof the students themselves would have given.

The second criterion is as follows: if two triangles have a side equal and the adjacent angles to the side equal one by one they are congruent (ibid., p. 189). This is called Angle-Side-Angle (ASA).

The third criterion is: if two triangles have their sides equal one by one they are congruent (ibid., p. 189). This is called Side-Side-Side (SSS).

The textbook notes specifically for the right-angled triangles:

Two right-angled triangles are congruent when they have

- two corresponding sides equal one by one
- one corresponding side and one corresponding acute angle equal (ibid., p. 190).

When the term “corresponding” refers to sides, it means that either both are perpendicular or both hypotenuse. According to the textbook the term can be applied to acute angles as well.

<u>*Question T34*</u>*: A non-rectangular parallelogram ΑΒΓΔ is given. From the vertex A we draw a perpendicular line (α) to ΔΓ. Line (a) intersects line ΔΓ at the point E. From vertex Γ we draw line (β) perpendicular to the line AB. Line (b) intersects the side AB at the point Z.*

a. Draw the figure.

b. Prove that triangle ΑΔΕ is equal to triangle ΓΒΖ (see Figure 3.4.2.1).

Figure 3.4.2.1 is a possible adequate construction according to the instructions in the text. As for T34b, referring to Figure 3.4.2.1 an adequate answer would be the following: compare triangles ΑΕΔ and ΓΒΖ. These are both right-angled triangles having equal their hypotenuses ΔΑ=ΒΓ as opposite sides of the parallelogram ΑΒΓΔ.

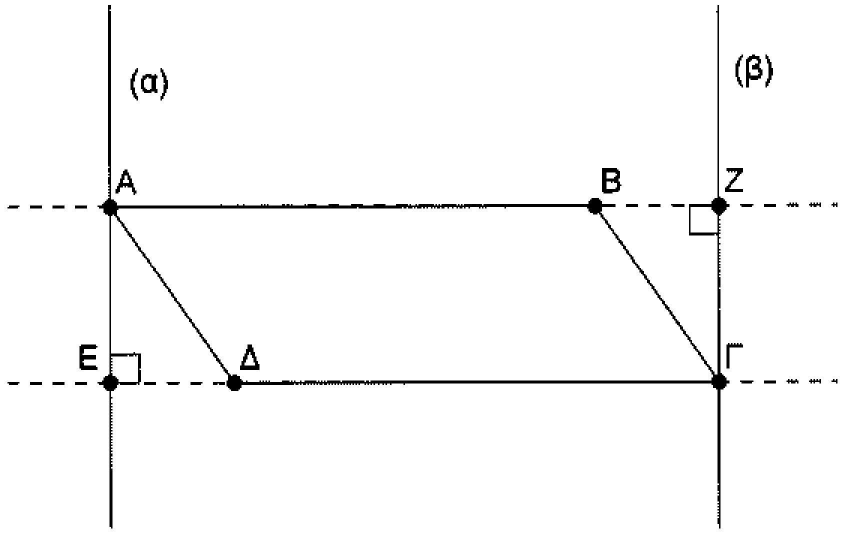


Figure 3.4.2. 1 Question 34ab (a possible drawing)

Additionally the two triangles have AE=ΓZ both distances between the parallel lines AB and ΓΔ. Thus they are equal according to the congruency criterion for right-angled triangles referring to two equal corresponding sides. Part a. of the question was meant to collect information on the students' efficiency at drawing a figure to the given instructions. If they managed to do this, then part b of the question can be proved using the corresponding congruency criterion. By attempting this proof the students would provide information on their proof schemes.

Question T35: *In the figure (see Figure 3.4.2.2) the following are valid: Line ζ passing through points A and B is the perpendicular bisector of the line segment ΓΔ. Prove that the triangles ABΓ and ABΔ are congruent.*

An adequate answer can be based on any of the three congruency criteria for triangles. For example the criterion SSS is valid. Indeed, since ζ is the perpendicular bisector of ΓΔ it follows immediately that BΓ=BΔ and AΓ=AΔ. On the other hand AB is a common side of both triangles. Thus the triangles in question are in fact congruent. Let it be noted that all three criteria of triangle congruency could have been invoked each with its own justification. In this question the triangles are deliberately

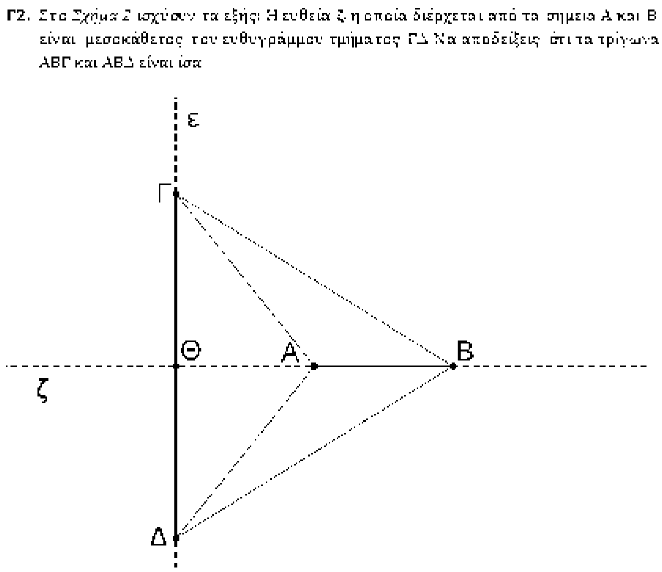


Figure 3.4.2. 2 Question T35

positioned to check whether their unusual position would make proving difficult, but, the main purpose was to observe which proof schemes would be present in the student responses.

<u>QuestionT36</u>: *In figure 3 (see Figure 3.4.2.3) the triangles ΑΓΒ and ΕΒΔ have ΑΓ=ΕΒ, ΑΒ=ΕΔ and ΓΒ=ΒΔ. Points Α, Β and Δ lie on the same line a. (a) Prove that the triangles ΑΓΒ and ΕΒΔ are equal. (b) Prove that the lines ΒΓ and ΕΔ are parallel.* An adequate answer could be based on the fact that triangles ΑΓΒ and ΕΒΔ have three pairs of equal sides and thus according to the SSS criterion they are necessarily congruent. From the congruency of the triangles it follows that the angles $\widehat{\Gamma B A}$ and $\widehat{B\Delta E}$ are equal. Consequently the lines ΒΓ and ΔΕ are parallel since they form corresponding equal angles with line ΑΒΔ. The first part of the question, on the congruency of the triangles ΑΓΒ and ΕΒΔ is the simplest part: here the students have to make use of the rest respective equal elements that are implied by the triangles' congruency, namely they must choose the appropriate corresponding angles to prove

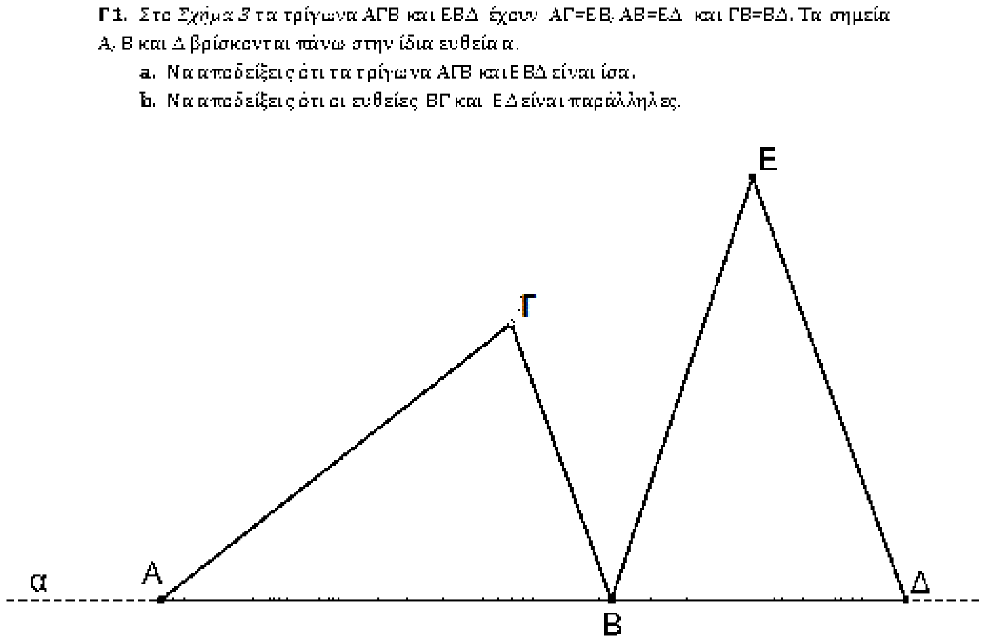


Figure 3.4.2. 3 Question T36

the parallelism of lines ΒΓ and ΕΔ. The students' efforts especially to prove the second part of the question which is relatively difficult, should reveal interesting elements of their proof schemes.

Characterisation of selected answers to questions of T1 and T3 according to Harel and Sowder's taxonomy follows in the next section.

3.4 Data analysis

In this section I describe the analysis of the students' scripts using Harel and Sowder's taxonomy as presented in Chapter 2.

Harel and Sowder (2007) divide students' proof schemes into three classes: external conviction proof scheme, empirical proof scheme, and the deductive proof scheme. Each class is divided in its turn into sub-schemes or subcategories[6]. As noted in Chapter 2, I prefer to simply use the term "proof scheme".

The external conviction proof scheme class includes:

- the authoritarian proof scheme,

[6] In Harel and Sowder (2007) is made use of both terms.

- the ritual proof scheme,
- the non-referential symbolic proof scheme.

The empirical proof scheme class includes:

- the empirical inductive proof scheme,
- the empirical perceptual proof scheme.

The deductive proof scheme class includes:

- the deductive transformational proof scheme,
- the deductive axiomatic proof scheme.

For individual and simultaneously a general picture of the students' proof schemes, as they emerge through the characterisation of their scripts, an EXCEL spreadsheet was created for each test. Each row on the spreadsheet corresponds to the participants and columns corresponding to each question or part of a question. Thus every cell of the spreadsheet contains the proof scheme used for the particular question or part of a question. Abbreviation for each class and its subcategories were presented in section 2.3. I repeat them here in the following table 3.5.1.Finally an extra abbreviation has been added, not belonging to the taxonomy when there is no response to a question or part of a question: NS stands for *No Solution*.

I commented in section 2.3 that evidence of the D.A. proof scheme could not be expected to appear in the data. Thus, of the two schemes in the D class, I used exclusively use the characterisation D.T.

Harel and Sowder (1998) call the deductive proof scheme analytical. Additionally the analytical transformational, or in Harel and Sowders (2007) terms, the deductive transformational subclass, is further divided into internalised, interiorised and restrictive. The analytical transformational restrictive scheme is divided into contextual, generic and constructive and the contextual includes the spatial. None of

Proof scheme class	Abbreviation	Proof schemes in the class	Abbreviation
External Conviction proof scheme	EC.	Authoritarian	EC.A.
		Ritual	EC.R.
		Non Referential Symbolic	EC.NRS.
Empirical proof scheme	E.	Inductive	E.I.
		Perceptual	E.P.
Deductive proof scheme	D.	Transformational	D.T.
		Axiomatic	D.A.

Table 3.5. 1 Proof schemes abbreviations

these subdivisions appear in my analysis. My main preoccupation is whether the major proof schemes that appear in the Table 3.5.1 are present in the student scripts. Harel and Sowder's (2007) deductive axiomatic scheme in Harel and Sowder (1998) is called analytical axiomatic and is subdivided into intuitive axiomatic, axiomatising and structural. From the moment that I decided to use only the deductive transformational characterisation, I did not include these distinctions in my analysis either.

Early on in the characterisation of the students' scripts evidence of two proof schemes started emerging. I decided that both proof schemes would be attached to the script and that a combination of the two abbreviations would be entered in the cell. Examples of this follow later in this section.

In what follows I introduce examples of the analysis emerging from the spreadsheets. I symbolise each participant with a capital P and their number in brackets. Thus the symbol P[56] means participant number 56.

I give examples of D.T., E.P., E.I., EC.NRS., EC.A., EC.R. proof schemes and further examples of mixed schemes.

P[01] gives the following answer to T13 (see Figure 3.5.1):

The perpendicular bisector of a line segment AB is the line that all its points have equal distance from the two endpoints of AB, that is the points A, B. Since the point Σ is a point on the perpendicular bisector, AΣ and BΣ are equal.

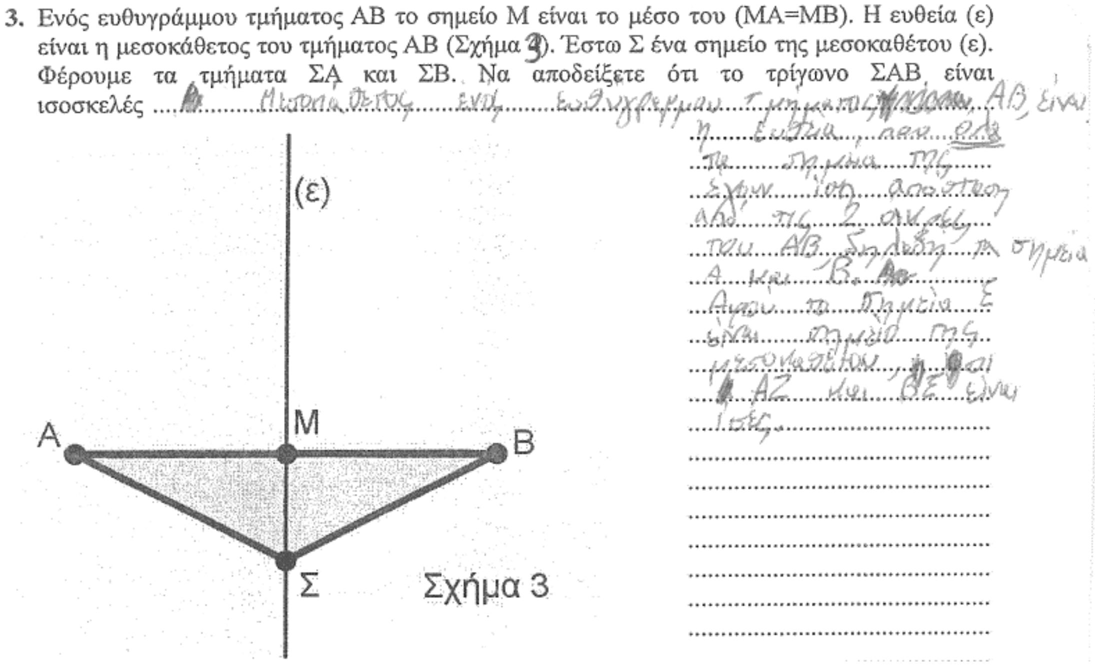


Figure 3.5. 1 Participants' [01] response to Question T13

I characterised this proof as D.T. because P[01] invokes the fundamental property of the perpendicular bisector to justify his assertion that AΣ and BΣ are equal as in the adequate answer I have given for this question. I note that there is a slight deficiency in the conclusion of the participant's argumentation: namely one could observe that the explicit conclusion "the triangle is thus isosceles" after the sentence "AΣ and BΣ are equal", is missing. Taking into account that the students had not been taught proof the lack of rigor can be considered negligible.

Let us now see the same participant's answer to the T36a (see Figure 3.5.2).

Hypothesis AΓ=EB, AB=EΔ, ΓB=BΔ. Conclusion (i) triangle AΓB=triangle EBΔ, (ii) BΓ// EΔ. We compare the triangles AΓB and EBΔ and observe that AΓ=EB from the hypothesis, AB=EB from the hypothesis and ΓB=BΔ from the hypothesis. Thus from

SSS it is valid that triangle ΑΓΒ=triangle ΕΒΔ. Thus since the triangles are equal they have the rest of their respective elements equal, thus $\hat{A}=\hat{E}$, $\hat{\Gamma}=\widehat{B_1}$, $\widehat{B_2}=\hat{\Delta}$ *(i) end.*

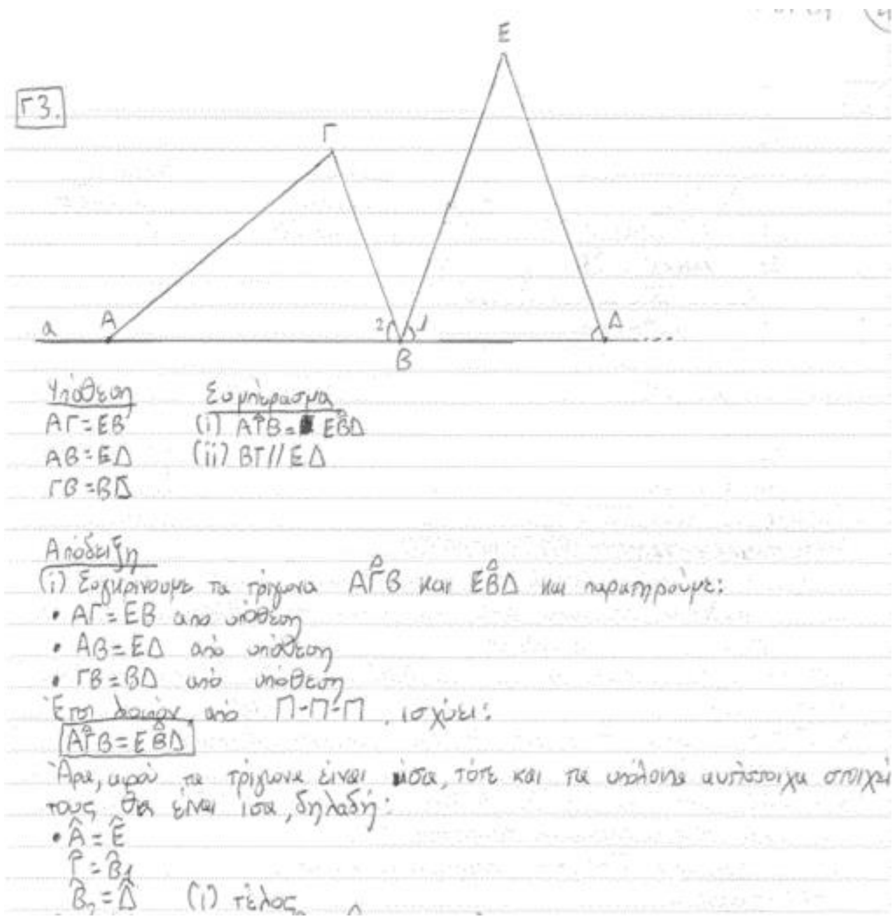


Figure 3.5. 2 Participant's [01] response to Question T36a

I characterise this proof as D.T. because P[01] writes down orderly why the congruency criterion SSS is valid, justifying correctly the elements that he asserts are equal by referring to the data given. His answer is thus in accordance with the adequate answer I gave in section 3.4.2. It is noteworthy that after having proved the congruency, P[01] writes down the rest equal elements, which in this case are all angles, without having been asked to. The odd sentence "(i) end" means "this is the end of part (i) of the question.

P[05] gives T13 the following answer (see Figure 3.5.3):

It is isosceles because the perpendicular bisector cuts it in the middle and two congruent right triangles are shaped.

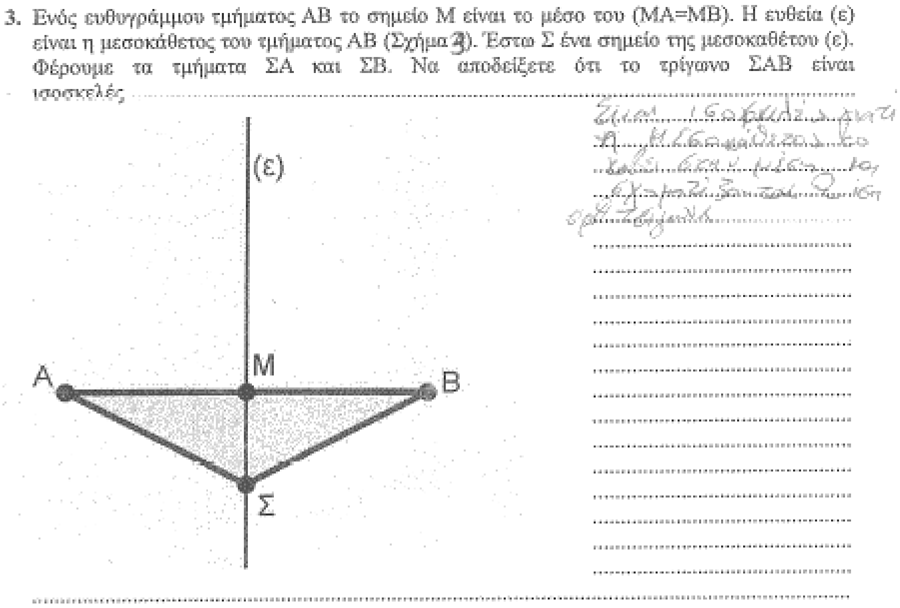


Figure 3.5. 3 Participant's [05] response to Question T13

I characterise this proof as E.P. because P[05] argument is not supported by a logical justification as in this case the fundamental property of the perpendicular bisector but by judging by eye the figure. Indeed P[05] sees the perpendicular bisector 'cutting' the figure in congruent triangles. Probably P[05] wants to say that if the triangles are congruent then the 'whole' triangle is isosceles perceiving thus visually the property to be proved. I ignore that there is no mention which triangles exactly are congruent and concentrate to the fact that judging properties of plane figures by eye and without any logical support is evidence of E.I. proof scheme.

Let us now take a look how the same participant handled Question T35. He writes (see Figures 3.5.4 and 3.5.5):

AB common side, A_1*=*A_2*,* B_1*=*B_2*.*

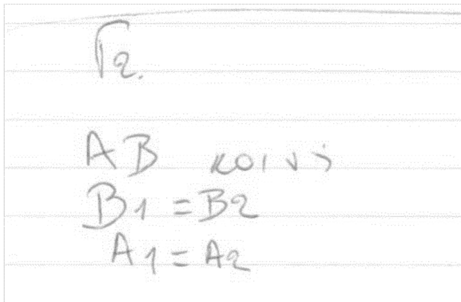


Figure 3.5. 4 Participant's [05] response to T35

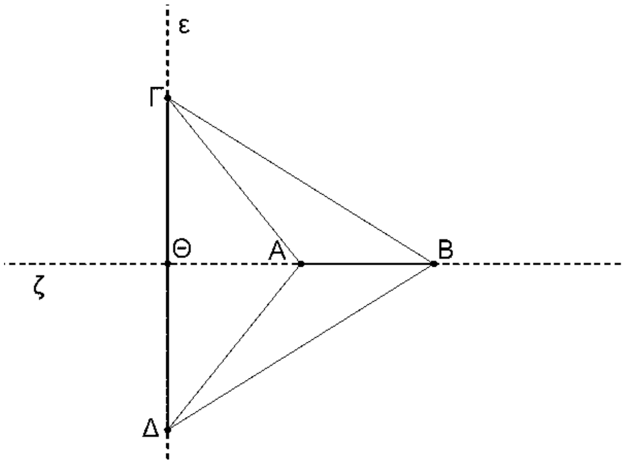


Figure 3.5. 5 Question T35

I characterise this proof as E.P because P[05] cites elements that are supposed to be equal without any supportive argument. This means that he sees the equality of these elements by eye in the figure. He has not noted any information on the ready-made figure (Figure 3.5.5) included on the test paper he was given. Consequently it is not clear what he means by B_1=B_2 and A_1=A_2. Probably he means the angles of the triangles ABΓ, ABΔ having vertices at A and B. In this case he means, the angles $\widehat{BA\Gamma}$ and $\widehat{\Gamma BA}$ of the triangle ABΓ and $\widehat{\Delta AB}$ and $\widehat{AB\Delta}$ of the triangle ABΔ without explicit formulation which is when he uses the symbolism B_1=B_2 and A_1=A_2. It is true that AB is the only element which is correctly described as common side. P[05] apparently tries to support the validity of the ASA criterion. He fails finally because the equality of the pairs of angles he refers to is not justified logically but by mere assertion which seems to be based on judging by eye thus providing evidence of E.P. proof scheme.

Participant P[87] gives to the Question T13 the following answer (see Figure 3.5.6):

Since ΣA is 5cm and the other ΣB is 5cm and AM is 4,5cm and MB is 4,5cm then the triangle is equal because its sides are equal thus the triangle is an isosceles triangle.

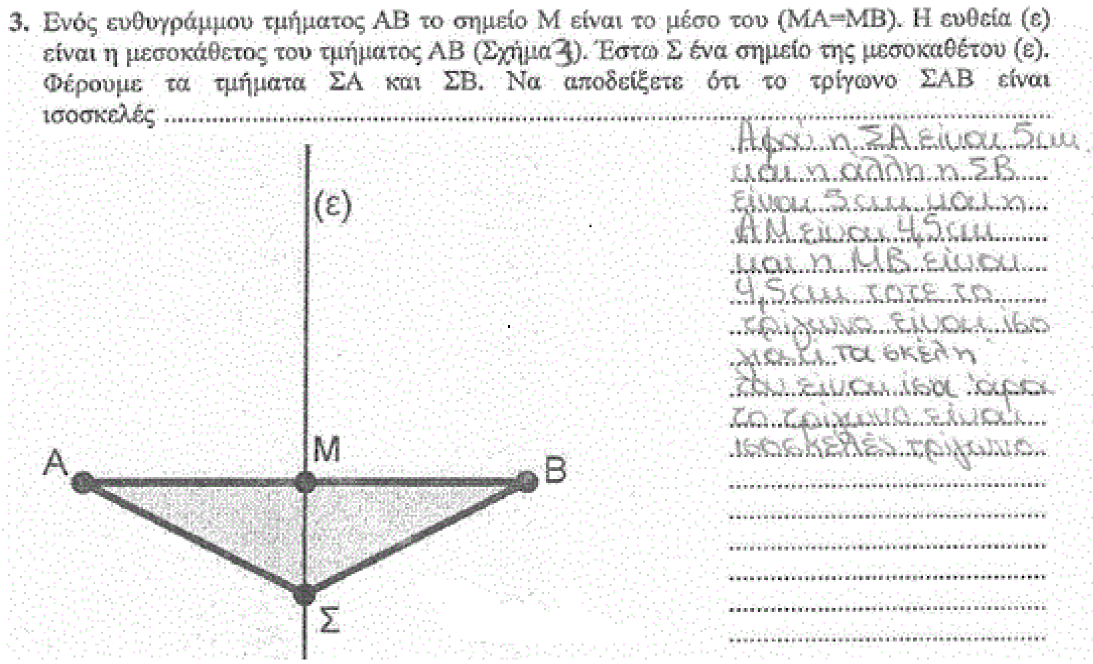


Figure 3.5. 6 Participant's [87] response to Question T13

I characterise this proof as E.I. because P[87] feels the need to assign numeric values to the lengths of the line segments ΣA, ΣB in order to prove that the triangle ΣAB is isosceles. P[87] assigned as well numeric values to the lengths of MA and MB probably wanting to support the idea that indeed M is the midpoint of AB. Assigning specific numeric values to lengths of various geometrical magnitudes or substituting numerical values for variables and using these assignments as an argument to prove assertions concerning properties of figures or of quantities is evidence of E.I. proof scheme. Admittedly there is a slight nuance of an EC.NRS. proof scheme at the point where the student asserts that “the triangle is equal” since it has to do with inappropriate use of terminology. By this expression P[87] wants to state that the triangle is isosceles. The EC.NRS. proof scheme element has been ignored in this case as not especially important or decisive.

P[40] answers Question T13 as follows (see Figure 3.5.7):

The triangle we find and knew is an isosceles and we can say that this triangle has equal angles, equal sides, equal perpendicular lines and all the rest are equal with each other and so we see one and the same figure which is the correct. Thus the figure we see is an isosceles.

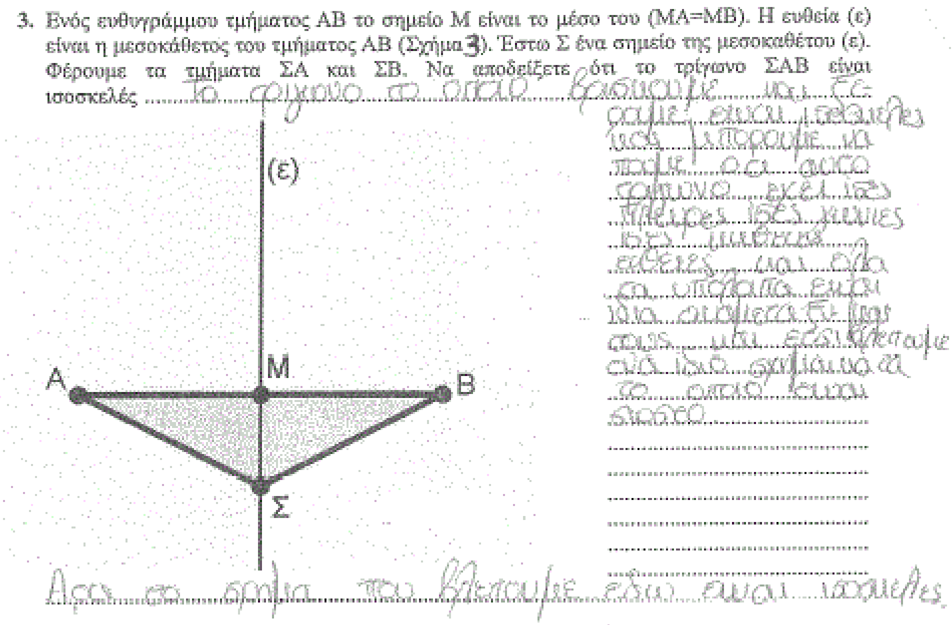


**Figure 3.5. 7	Participant’s [40] response to Question T13**

I characterise this proof as EC.NRS. and explain in what follows why. Harel and Sowder’s (1998, 2007) examples of the EC.NRS. proof scheme are examples of the misuse of algebraic symbols without any logical coherence. P[40]’s answer is not of algebraic character but geometric and symbols are not involved in the text of the question or are they necessary to articulate an answer: for examples P[40] begins by asserting that the triangle is an isosceles triangle which means that it has among other things equal "perpendicular lines". But no element of any triangle can be said to be "perpendicular lines". Thus the notion of triangle's element has been misused. P[40] continues to assert "…all the rest are equal we see one and the same figure which is the correct" which is a sentence without logical coherence and thus lacks meaning. Such inappropriate use of notions and lack of logical meaning are evidence of an EC.NRS. proof scheme.

The same participant gives Question T31 the following answer (see Figure 3.5.8):

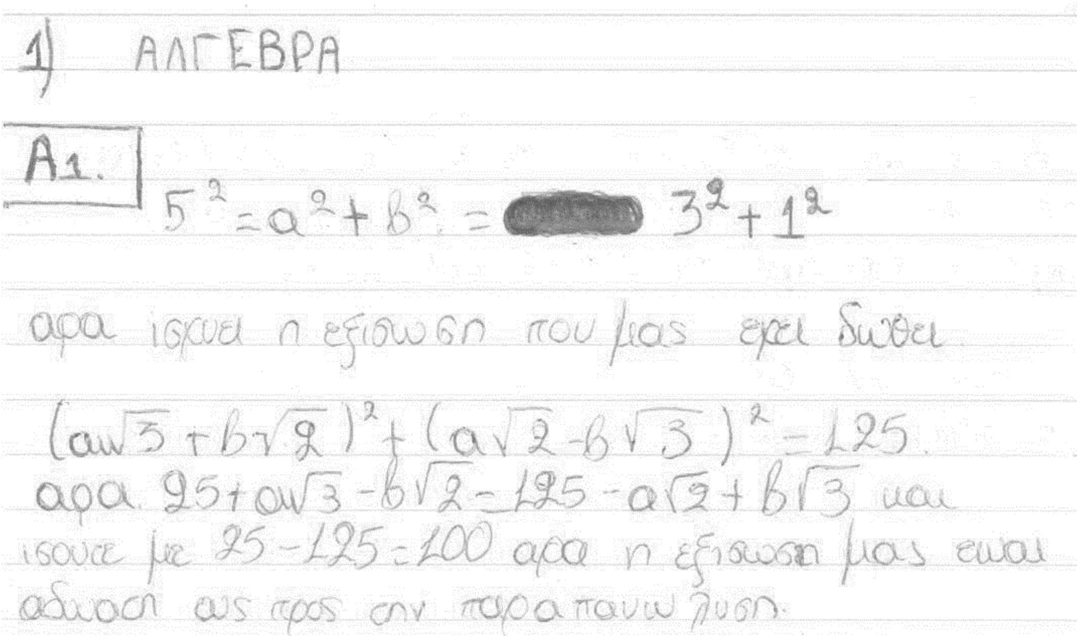


Figure 3.5. 8 Participant's [40] response to Question T31

$5^2 = \alpha^2 + \beta^2 = (sco^7) = 3^2 + 1^2$

thus the equation we have been given is valid

$\left(a\sqrt{3} + b\sqrt{2}\right)^2 + \left(a\sqrt{2} - b\sqrt{3}\right)^2 = 125$

thus $25 + a\sqrt{3} - b\sqrt{2} = 125 - a\sqrt{2} + b\sqrt{3}$ *and*

is equal with 25-125=100 consequently our equation

is impossible regarding the above solution

---

[7] sco stands for 'something crossed out' henceforward

I characterise this proof as EC.NRS. proof scheme as the following analysis explains. P[40] writes $5^2 = \alpha^2 + \beta^2 = 3^2 + 1^2$ (1) . It seems probable that he has substituted the variables as $\alpha = 3$ and $\beta = 1$ or vice versa. After that he comments that "…the equation we have been given is valid". P[40] asserts that the given "equation" is valid, but the assertion does not comply with the result $25 = 10$ yielded after calculation of the number powers on the left and right member of relation (1). In terms of proof scheme characterisation, this mistake is a first sign of EC.NRS. proof scheme. In the next step the participant manipulates the identity to be proved since after writing $\left(\alpha\sqrt{3} + \beta\sqrt{2}\right)^2 + \left(\alpha\sqrt{2} - \beta\sqrt{3}\right)^2 = 125$ (2) he adds the word "thus" and writes: $25 + \alpha\sqrt{3} - \beta\sqrt{2} = 125 - \alpha\sqrt{2} + \beta\sqrt{3}$ (3). There is no expansion of both parentheses, each of which should had been raised to the second power. The number 25 appears on the left member of (3), the plus sign before $\beta\sqrt{2}$ in the first parenthesis is changed to minus and the content of the second parenthesis on the left of (2) is transferred to the right member in (3) whereby the signs are changed. As none of these manipulations are in accordance to operation laws of real numbers they can be taken as a sign also of EC.NRS. proof scheme. Concluding his manipulations on (3) P[40] writes: $25 - 125 = 100$ (4). Again there is no law of the real numbers allowing such a conclusion as that presented by (4) as consequence of (3). The relation (4) itself is not correct since it states that: $-100 = 100$. P[40] seems to understand this because he writes "…consequently our equation is impossible regarding the above solution". Here is what Harel and Sowder (1998, p. 251) say about such cases: "Symbolic reasoning is a habit of mind students acquire during their school years-from elementary school to secondary and post secondary school-a habit that is very persistent and extremely difficult to relinquish". P[40] seems to fit this description. His symbolic reasoning is characterised by mistakes which emerge from inappropriate use of symbols and rules about number operations. Thus these considerations explain why P[40]'s answer provides evidence of the EC.NRS. proof scheme.

P[09] gives the following answer to T33b (see Figure 3.5.9):

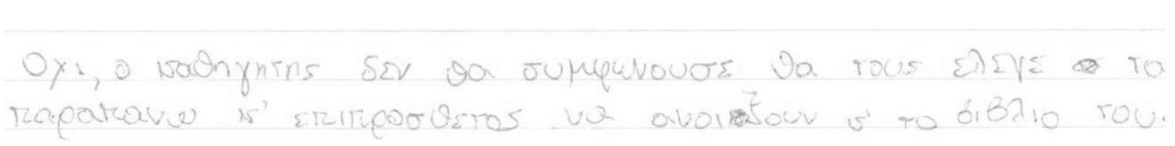


Figure 3.5. 9 Participant's [09] response to Question T33

No the teacher would not agree he would say to them the above and additionally to even open their books

I characterise this answer as an EC.A. proof scheme. Let us follow P[09]'s argument. P[09] thinks that the teacher would not agree with her peers. She depicts a teacher who, instead of explaining the procedure for a type of proof, would only confirm that the identity is written like this because it is written like this and its validity is due to "the law of identities" (the word "above" refers to the answer to part (a) of the question where P[09] speaks of a "law of identities"). Furthermore, according to P[09], the teacher would urge the peers to open their books. Thus P[09] imagines that an authority, the teacher, would suggest to the peers to attend another authority, the book. Appealing to the opinion of an authority and expecting the confirmation of truth by authorities of mathematical propositions without seeking any logical justification is evidence of EC.A. proof scheme.

P [39] writes (see Figure 3.5.10):

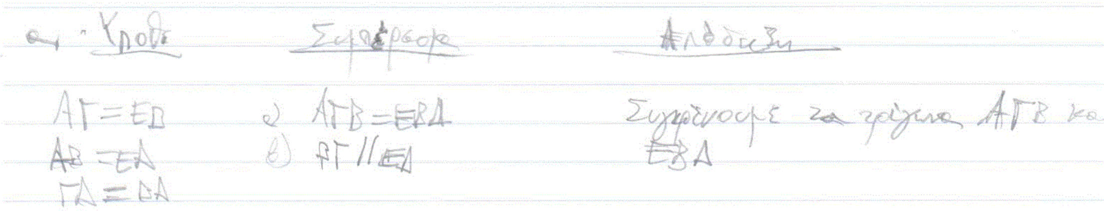


Figure 3.5. 10 Participant's [39] response to Question T36a

<u>*Hypoth*</u>	<u>*Conclusion*</u>	<u>*Proof*</u>
ΑΓ=ΕΒ	*α) ΑΓΒ=ΕΒΔ*	*We compare the triangles ΑΓΒ and*
ΑΒ=ΕΔ	*β) ΒΓ//ΕΔ*	*ΕΒΔ*
ΓΔ=ΒΑ		

I characterise this answer as an EC.R. proof scheme. Participant [39] gives the hypothesis, the conclusion, and the point from which to start to carry out a proof. Nevertheless, he offers no proof as he appeals to none of the congruency criteria. Thus the principal element of his answer is the ritual character of writing the data without justification of any kind. In this respect the answer of P[39] provides evidence of EC.R. proof scheme.

Below are some examples of combination of proof schemes. I start with a case where I observed a mixture of D.T. and EC.NRS. proof schemes.

Participant P[21] answering T11 writes (see Figure 3.5.11):

In the triangle $A\hat{B}\Gamma$ *the angle*

$\hat{\Gamma}$ *=20°, because the sum of the triangle from what*

we know is 180°. Thus $\hat{A}$*=85°+*$\hat{B}$*=75°=160*

Then in order

to have in the triangle

sum 180°

the other side that is

Γ must be 20°.

Thus $\hat{A}+\hat{B}+\hat{\Gamma}$ *they will*

do 180°

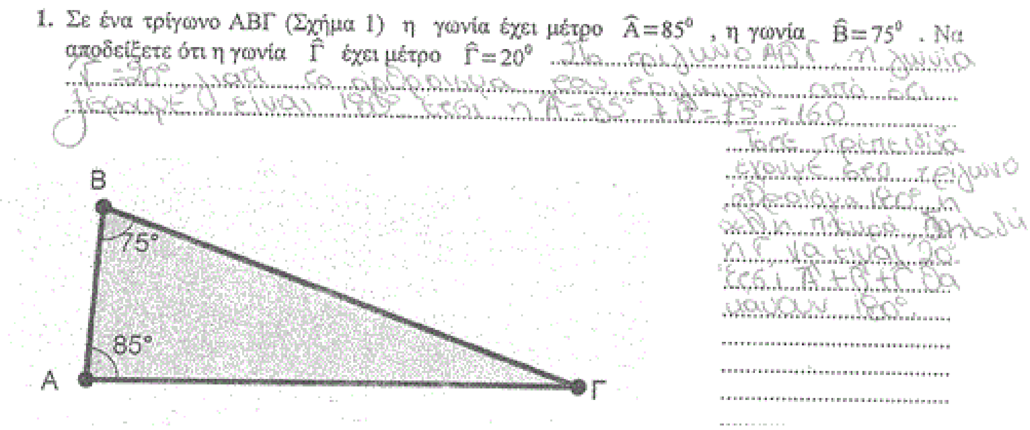


Figure 3.5. 11 Participant's [21] response to Question T11

I characterise this proof as a mixture of D.T. and EC.NRS. proof schemes. It is obvious that P[21] is aware of the theorem of the sum of the angles of a triangle. From this point of view the response offers evidence of a D.T proof scheme, considering that on the basis of this knowledge P[21] finds the correct value for angle $\hat{\Gamma}$. However, there is a series of non appropriate use of symbols. The first is when P[21] writes "In the triangle $A\hat{B}\Gamma$". Here the symbol of an angle is used for a triangle. More weighty is the case where P[21] writes: "Thus $\hat{A}=85°+\hat{B}=75°=160$". Her obvious intention was to write: "$\hat{A}+\hat{B}=85°+75°=160°$". She failed, however, to reach this end because it seems that she did not notice the inaccuracy which is implied by what she wrote. Indeed what she wrote implies for example, among other things, that $75°=160°$ which cannot be true in real number system. Then P[21] writes the sentence "…the other side that is Γ must be 20°" that is instead of writing the word "angle" she writes the word "side". The combination of knowledge of the theorem on the sum of the angles of a triangle and the correct calculation of the angle $\hat{\Gamma}$ on the one hand and the not appropriate use of symbols and terminology on the other provides evidence of both D.T. proof scheme and EC.NRS. proof scheme.

Participant [62] answering T12 (see Figure 3.5.12) writes:

Since AI is

the bisector of $B\hat{A}\Gamma$

then 78:2

and BI is

the bisector of $A\hat{B}\Gamma$

then 66:2.

78:2=36

66:2=33

then (36+33)-180

180°-69°=180°

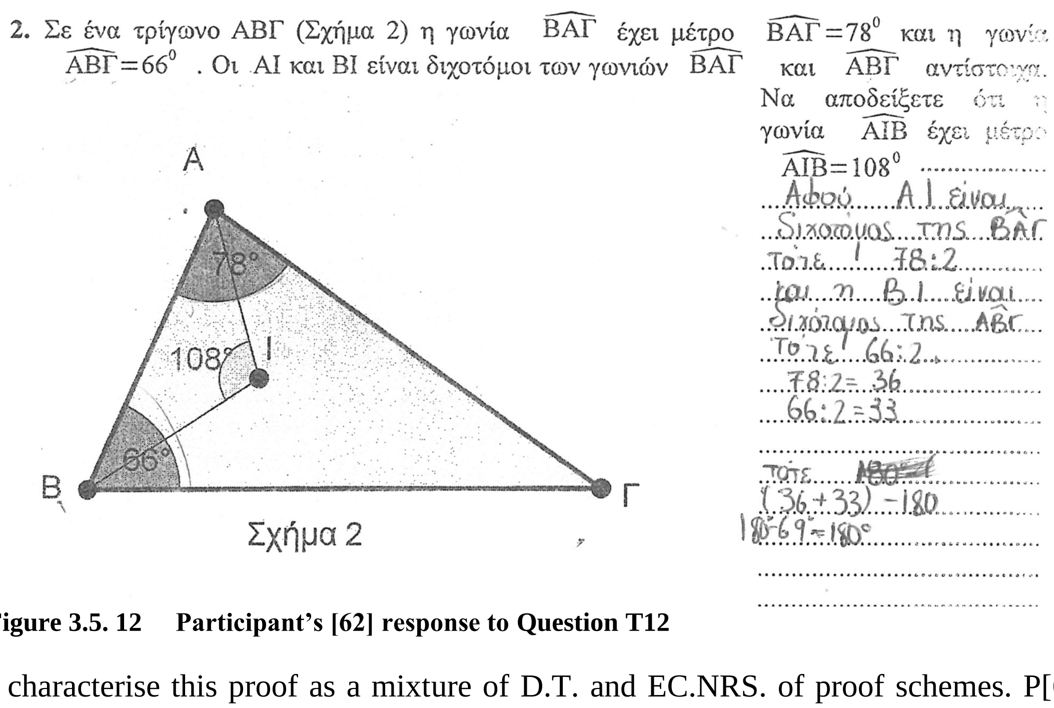


Figure 3.5. 12 Participant's [62] response to Question T12

I characterise this proof as a mixture of D.T. and EC.NRS. of proof schemes. P[62] understands what is to be done. First of all she halves the measures of the angles $B\hat{A}\Gamma$ and $A\hat{B}\Gamma$ and then proceeds to add the half measures and subtracts their sum from 180°. In this respect her line of thinking is a D.T. proof scheme But in halving 78 she calculates a value of 36, which is a mistake because 78:2=39. Of course, the inconsistent use of the degrees symbol does not escape the attention and should normally be considered as a mistake too even if a negligible one. Then P[62] writes "(36+33)-180". This difference, the false value 36 aside, is also not correct. P[62]'s intention was to find the difference 180°-(39°+33°) but she failed. As a conclusion P[62] writes "180°-69°=180°". Again the subtraction gives a false result, because even setting aside the false value 36°, the value that should have been found is 111°. The P[62]'s knowledge of the theorem of the sum of the angles of a triangle is recognisable and clearly offers evidence of a D.T. proof scheme. . On the other hand, she makes many mistakes operating with integers providing as well evidence of EC.NRS. proof scheme. Thus the characterisation given to P[62]'s answer.

P[62] also provides the next example of a mixture of proof schemes. Attempting to answer T13 (see Figure 3.5.13) she writes:

Since AM=MB then if we suppose that AM=1 and MB=1 and the triangle $\Sigma\hat{B}M$ *is a right triangle then the height* $\sqrt{2}$ *and the triangle is isosceles.*

I characterise this proof as a mixture of E.I. and EC.NRS. proof schemes. P[62] begins the proof with the valid equality AM=MB but then immediately supposes that both the equal line segments are of unitary length. But assigning numerical values to various magnitudes or variables provides evidence of E.I. proof scheme. The next step is the assertion that the height of the triangle "$\Sigma\hat{B}M$ is $\sqrt{2}$". Obviously there is a non appropriate use of symbols concerning the triangle in question because the symbolism $\Sigma\hat{B}M$ symbolise an angle and not a triangle. On the other hand, what is meant by the height of the triangle is not clear. A triangle with perpendicular sides of unitary length has its hypotenuse equal to $\sqrt{2}$, but P[62] makes no reference to such a

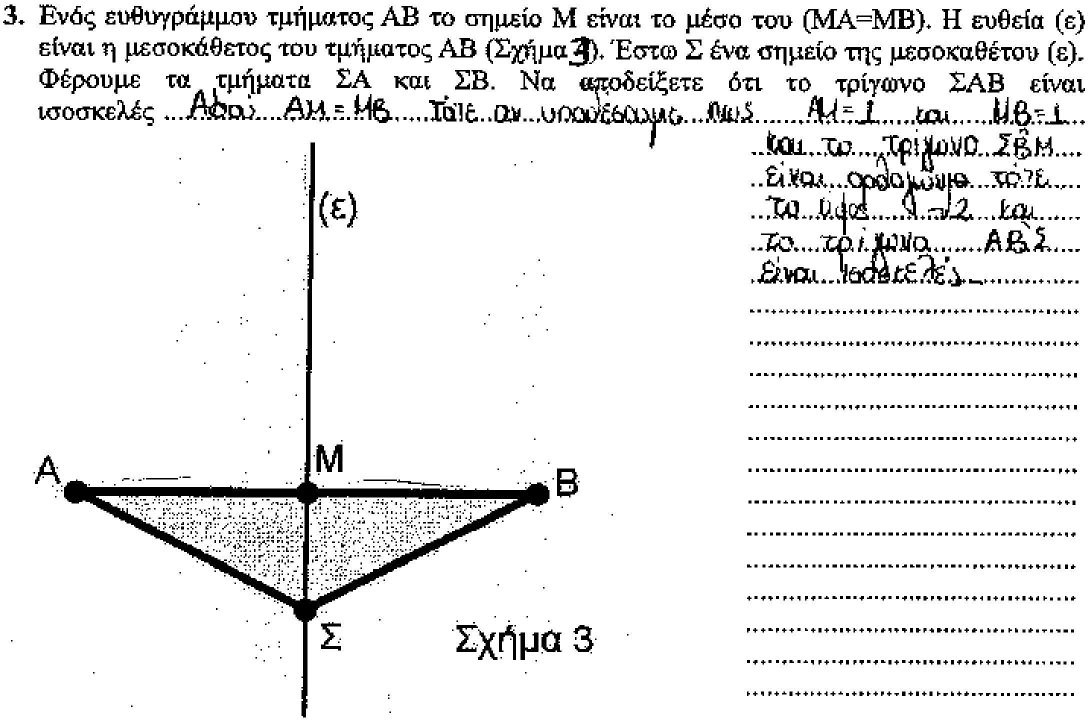


Figure 3.5. 13 Participant's [62] response to Question T13

triangle. Thus, how the triangle's height is calculated remains not clear. But let's set aside this unclear point. Let us accept for a moment that there is a height and has indeed length $\sqrt{2}$. Why the conclusion implicated by this fact should be that the triangle $A\hat{B}\Sigma$, in the participant's symbolism, is an isosceles is again unclear. To recapitulate: around the axis of an idea asserting AM=MB=1, thus empirical inductive in Harel and Sowder's (2007) taxonomy, an argument of EC.NRS. character is developed which includes another idea that a 'height' is of the length $\sqrt{2}$. Under these circumstances the answer offers evidence of an E.I. proof scheme as well as an EC.NRS. proof scheme.

Participant P[02] answers T32a (see Figure 3.5.14) as follows:

I know that $\kappa^2 - \lambda^2 = \kappa + \lambda$

α) I want to prove that $\kappa - \lambda = 1$

$$\kappa^2 - \lambda^2 = \kappa + \lambda \Rightarrow (\kappa - \lambda)(\kappa + \lambda) = \kappa + \lambda \Rightarrow \kappa + \lambda - [(\kappa + \lambda)(\kappa - \lambda)] = 0 \Rightarrow$$

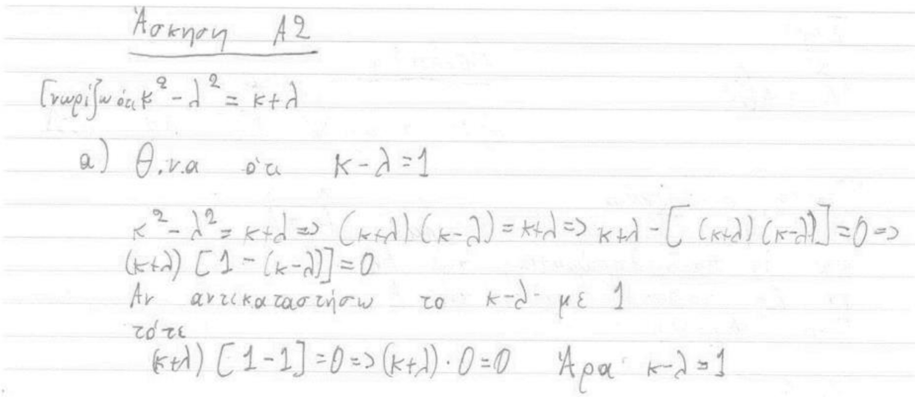


Figure 3.5. 14 Participant's [02] response to Question T32a

$(\kappa+\lambda)[1-(\kappa-\lambda))]=0$

If I substitute $\kappa-\lambda$ *by 1*

then

$(\kappa+\lambda)[1-1]=0 \Rightarrow (\kappa+\lambda)\cdot 0=0$ *thus* $\kappa-\lambda=1$

I characterise this proof as a mixture of D.T. and E.I. proof scheme. The answer offers evidence of D.T. proof scheme. The participant correctly manipulates the original equality till the point where he writes that $(\kappa+\lambda)[1-(\kappa-\lambda))]=0$. From this point the expected next step would be to observe first that $\kappa+\lambda$ is not zero due to the fact that κ and λ are both natural numbers and are not equal to each other. Thus, even if one of them were equal to *0*, the other could not be. On the other hand the product $(\kappa+\lambda)[1-(\kappa-\lambda))]=0$ can be equal to zero only if the factor $[1-(\kappa-\lambda))]$ is equal to zero. This leads to the conclusion that $\kappa-\lambda=1$. Instead of following the previous line of argument P[02] chooses to substitute the value 1 for $\kappa-\lambda$. But choosing a convenient value for a variable and substituting it in order to achieve a desired result when substitution is not needed to reach a conclusion offers evidence of E.I. proof scheme. Consequently P[02] provides with his answer evidence of a mixture of D.T. and E.I. proof schemes.

3.5 Ethics

This research project was carried out according to the proposal for a study of Year 9 students' first encounter with proof that I submitted as part of the third assignment for my EdD studies in April 2010 to the School of Education and Lifelong Learning at UEA. The proposal was approved by the EDU Ethics Committee.

To prepare for the implementation of the research proposal, in May 2010, I paid a preparatory visit, to the school where the project would take place. I informed the school principal and the mathematics teachers of my intentions to carry out a research project on the teaching of proof in mathematics in the next school year. Among the mathematics teachers was V, my colleague with whom I had piloted some ideas in February and March 2010, by teaching Year 9 classes various questions of geometrical proof. This is a normal task and part of my professional work as school advisor, but it nevertheless shows the sense of trust between us and mutual respect. At the beginning of the school year 2010-2011 V, knowing and supporting my plans for a research project, was appointed school principal at the school in question. In addition to this favourable fact, another welcome coincidence occurred at the beginning of that school year. My colleague J, was appointed that year as a teacher of mathematics at the school and was allocated the teaching of the four Year 9 classes.

In these circumstances, at the beginning of the year 2010-2011, I had preparatory meetings with V the school principal, and J, N and A mathematics teachers at the school at which I discussed my intentions with them again and described research project asking for their contribution. J agreed to allow me to sit in the classes and to collaborate with me on the implementation of the project. There has been no kind of problem whatsoever with the audio-recording of the lessons and our discussions with my taking notes during the observations. However I provided V, as a principal, and J and my other colleagues with consent forms (for consent forms and information sheets see Appendix IV). All accepted to help and they all consented to my carrying out the project.

The teacher J would become my main collaborator because she taught the four Year 9 classes. She kindly prepared the students for my visits to their classes telling them about my future presence in the classrooms. After J’s preparatory explanations, I visited the four classes and told the students what I intended to do. I gave them information sheets for their parents and for themselves and a consent form for their parents.

The school year in Greece ends by June 30th and begins September the first. I attended the Union of Guardians and Parents meeting that is traditionally held every year at every school. It took place just a few days after my visit to each of the four Year 9 classes. I was introduced to the parents and guardians by V, the principal, not only as a researcher but also as the school advisor responsible for matters concerned with the teaching of mathematics to their children. This raised their confidence in my plan because they understood that my research had to do with the broader aim of improving Greek students’ learning processes and that their children would have nothing to lose by my presence in their classes. Thus the parents and guardians approved my research project and the students’ participation in it. There was not one withdrawal or any objection to my audio-recording lessons.

The students almost immediately became accustomed to my presence, and from the beginning none were in any way embarrassed or disturbed by my being in their classroom or expressed any kind of discontent. On the contrary there have been occasions, toward the end of the class observations, when students asked why I had not been with them in the classroom.

To summarise: The principal, my colleagues, and other school personnel, the students and their parents gave me a warm reception and their consent. As a result the data collection proceeded smoothly from beginning to end.

3.6 Summary

In this chapter I presented in section 3.1 a brief summary on the study was conducted.

In 3.2 I presented the general background to Greek secondary education with particular emphasis on the curriculum for Years 7,8 and 9. I and described the participating school, teachers and students.

In section 3.3 I presented the tests intended to collect written answers from the students and explained their creation.

In section 3.4 I gave examples of students' scripts and their characterisation to show how I worked with the students' answers using the Harel and Sowder's taxonomy as a lens through which I investigated the details of their written thinking.

In 3.5 I have briefly laid out how I covered all the necessary steps to get the prescribed ethical approval and the acceptance of my proposal by the UEA Ethics Committee.

In the next chapter I present the analysis of the students' answers to tests T1 and T3.

CHAPTER 4: PRE-PROOF TEST DATA ANALYSIS

4.0 Introduction

In this chapter I analyse the pre-proof test (T1), which was administered to in September 2010 to collect data on the participants' pre-proof perceptions[1].

The T1 data analysis is laid out as follows: first each Question is presented with a brief adequate answer, followed by selected examples of the students' answers covering different proof schemes according to Harel and Sowder's taxonomy (Harel & Sowder, 2007) that appeared by the analysis of the corresponding Question. The concluding section offers general comments on the participants' answers to the Question and a table grouping the answers under the different proof schemes.

4.1 Analysis of responses to Question T11

The participants' answers to Question T11 can be divided into five groups: four of which are of various proof schemes. The first group includes D.T.; the second group D.T.-EC.NRS; the third E.P.-EC.NRS. ; and the fourth, one EC.NRS. proof scheme. The fifth group is the NS group.

First I give examples of particular D.T. proof schemes selected to illustrate the variety of answers that can be characterised as D.T. although they differ in various respects.

[1] For practical reasons I repeat the meaning of the abbreviations here. T14b for instance has the following meaning: T stands for the word test; the first number after T is the number of the test; the second number is the number of the test Question, and a or b refer to the sub-Question.

P[01] writes (see Figure 4.1.1):

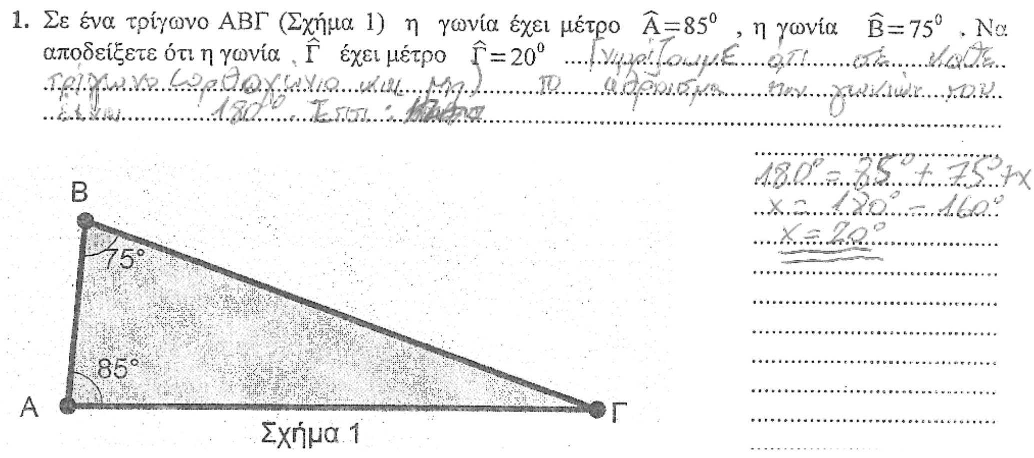


Figure 4.1. 1 Participant's [01] response to Question T11

We know that in every triangle (right or not)

the sum of its angles is 180°. Thus : (something crossed out[2]*)*

180°=85°+75°+x

x=180°−160°

<u>*x=20°*</u>

P[01] begins the proof by invoking the theorem that the sum of the angles of a triangle is equal to *180°* and then writes an equation in which the sought-for angle is represented by the symbol x, no explanation concerning the connection between the symbol x and the angle $\hat{\Gamma}$. Then P[01] proceeds to solve the equation for x by calculating the correct value x=20°. The final result is doubly underlined by the participant as if to announce: "Thus I have proved the desired result and it is indeed 20°". This answer is adequate and additionally shows the participant's tendency to use algebraic knowledge creatively to solve the problem. Thus the answer provides evidence of the D.T. proof scheme and has been characterised respectively.

[2] From now on I use the abbreviation 'sco' standing for 'something crossed out'

P[04] writes (see Figure 4.1.2):

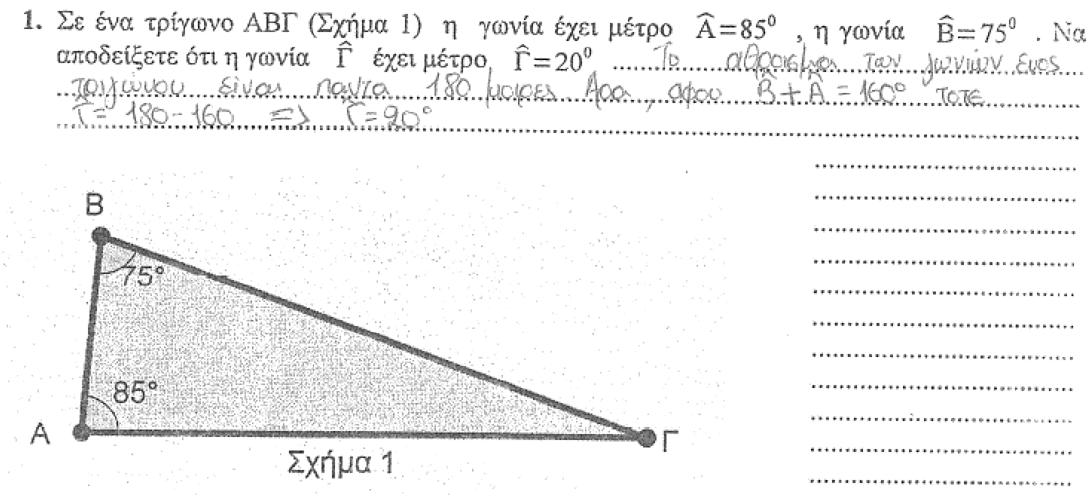


Figure 4.1. 2 Participant's [04] answer to Question T11

The sum of the angles of a

triangle is always 180 degrees. Thus , since $\hat{B}+\hat{A}$*=160° then*

$\hat{\Gamma}$ *=180−160⇒* $\hat{\Gamma}$ *=20°*

This answer is adequate. P[04] does not solve the problem by means of an equation; she finds the sum of the two given angles $\hat{B}+\hat{A}$*=160°* and subtracts the sum $\hat{B}+\hat{A}$*=160°* from *180* degrees to find angle $\hat{\Gamma}$. The alternative use of the symbol for degrees or the Greek word meaning degrees when the symbol is not used is noteworthy. However, when P[04] writes $\hat{\Gamma}$ *=180−160* in the third line of her answer she forgets the degrees symbol which I see a negligible mistake that does not substantially reduce the adequacy of the answer. The answer, being adequate provides evidence of D.T. proof scheme and has been characterised respectively.

P[05] writes (see Figure 4.1.3):

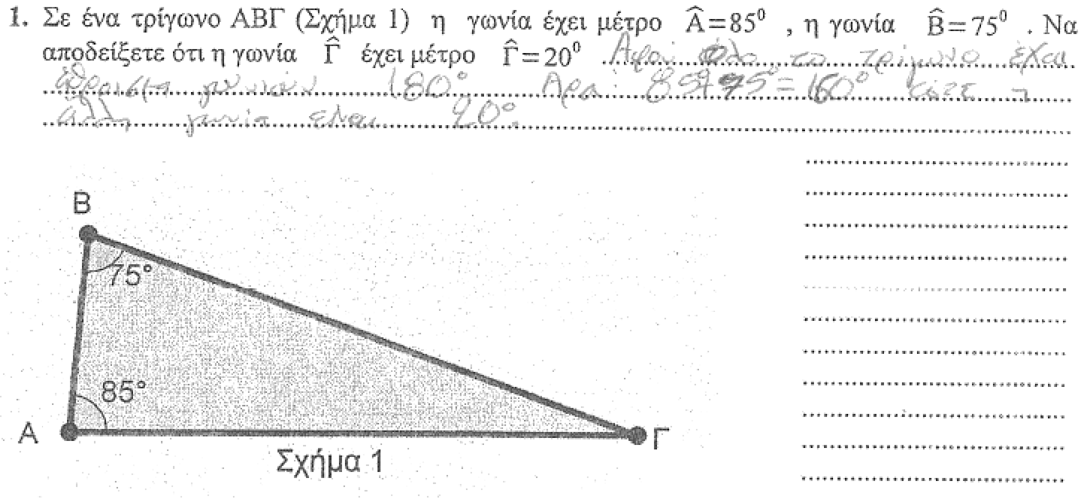


Figure 4.1. 3 Participant's [05] response to Question T11

Since all the triangle has

sum of angles 180° Thus 85°+75°=160° then the

other angle is 20°.

P [05]'s answer is adequate. He invokes the law regarding the sum of the angles of a triangle in the first step. In the second step he calculates the sum of the given angles *85°+75°=160°*; in the third step he calculates the correct measure of the angle $\hat{\Gamma}$, presumably mentally since there is no sign of written calculation. The calculation of the measure of the angle $\hat{\Gamma}$ is, indeed, very easy and obvious and thus can acceptably be computed mentally. No symbol of any angle or of the triangle is used throughout the whole proof, only the given measures of angles and the number *180°* as the measure of the sum of all the triangle's angles. The fact that the answer is adequate provides evidence of a D.T. proof scheme and thus has been characterised respectively.

P[72] writes (see Figure 4.1.4):

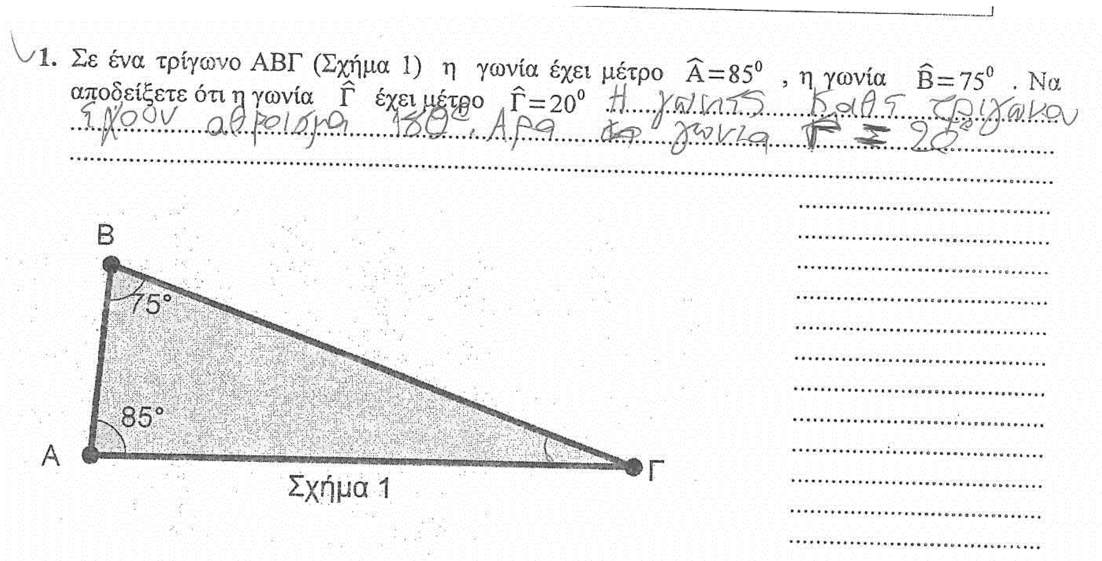


Figure 4.1. 4 Participant's [72] response to Question T11

The (incorrect Greek spelling of 'the') angles of every triangle

have sum 180°. Thus the angle $\hat{\Gamma}$ *=20°*

P[72] correctly invokes the sum of the angles of a triangle and immediately gives the correct measure of angle $\hat{\Gamma}$, which he appears to have computed mentally. There is no use of symbols apart from that for angle $\hat{\Gamma}$ nor is there explicit reference to the given measures of the angles. However, the answer has to be accepted as a D.T. proof scheme from the point at which P[72] is aware of the sum of the angles of a triangle and his calculation of the measure of $\hat{\Gamma}$ is correct and easily carried out mentally.

The second group consists of answers characterised as mixed D.T. and EC.NRS. proof schemes, some examples of which I present below.

P[09] writes (see Figure 4.1.5):

ΑΒΓ=180° Since *A=85°*

B=75° then (sco) 180=A+B+Γ Γ=(sco) 180(sco)-(A+B)

Γ=180-85+75 Γ=180−160 <u>*Γ=20°*</u>.

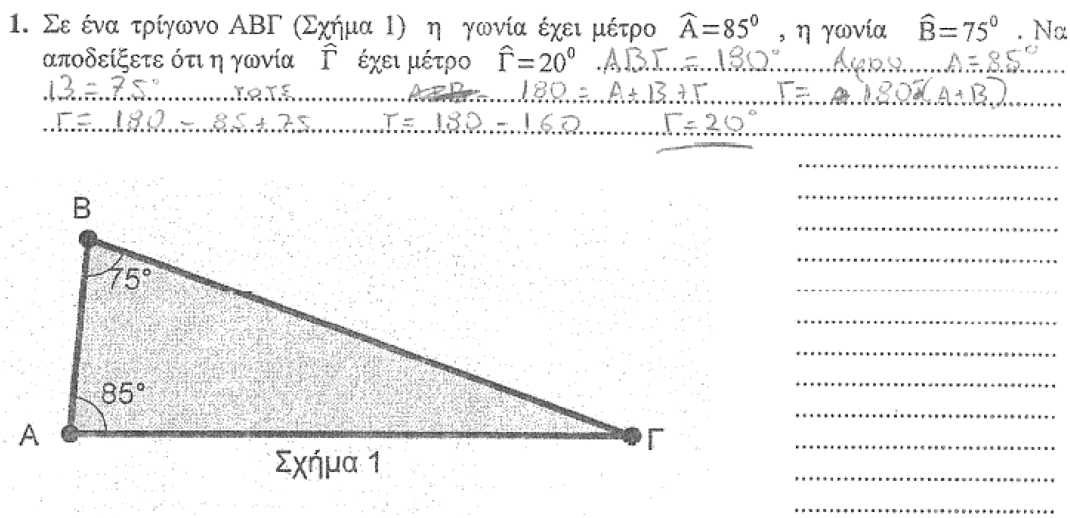


Figure 4.1. 5 Participant's [09] response to Question T11

P[09] begins the proof by referring to the law of the sum of a triangle's angles and then substitutes the given measures of the angles $\hat{A}$ and $\hat{B}$ in the sum of all the angles of the triangle and solves the resulting relation for angle $\hat{\Gamma}$. Finally he finds the measure of $\hat{\Gamma}$ to be *20*°. In this effort P[09] writes the sum of the angles of a triangle using the arbitrary symbol *ΑΒΓ=180°*, the probable origin of which is the formulation "the sum of the angles of a triangle" which we use in writing as well orally. However, whatever the reason behind its use the symbol itself is no less arbitrary. There is no use of the angle symbol over the letters symbolising angles, although this could be seen of little importance. Then next arbitrary use of symbols is the false removal of the parentheses when P[09] writes 180*(sco)-(A+B)* and then *Γ=180-85+75*. Thus P[09]'s answer demonstrates knowledge of the law on the sum of the angles of a triangle and the procedure by which the unknown angle is calculated, on the one hand: on the other P[09] uses an arbitrary symbol for the sum of the angles and, more importantly, mistakenly removes the parentheses in what regards the signs of the quantities involved. In this sense the answer is not completely adequate and thus provides evidence of a D.T. proof scheme but also of the EC.NRS. proof scheme, and thus the answer has been characterised as a mixture of the two.

P[45] writes (4.1.6):

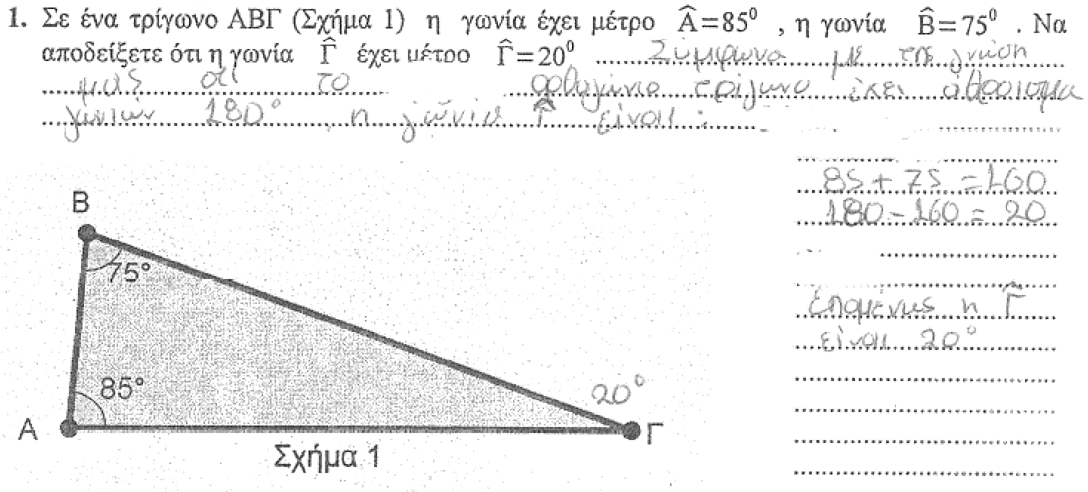


Figure 4.1. 6 Participant's [45] response to Question T11

According to our knowledge

that the right triangle has

sum of angles 180° angle $\hat{\Gamma}$ is:

85+75=160

180−160=20

Thus $\hat{\Gamma}$

is 20°

Participant P[45] invokes the theorem on the sum of the angles of a triangle but restricts it to right-angled triangles, and then calculates the sum of the measures of the given angles and subtracts the result from *180* to find *20* and concludes that angle $\hat{\Gamma}$ is *20°*. This answer is not completely adequate. Although P[45] calculates the correct measure of angle $\hat{\Gamma}$ this is done on the basis of a misuse of the law on the angles of a triangle. I reject the diagnosis of an E.P. proof scheme because P[45] considers no angle as a right one but uses the given measures which directly indicate that the triangle is a right-angled one. Thus the notion of right-angled triangle is not applied on the triangle or any of its angles and seems only of arbitrary character. Thus P[45]'s answer provides evidence of D.T. proof scheme combined with arbitrary formulations which also categorises it as an EC.NRS. proof scheme. Consequently it is a D.T.-EC.NRS. mixture of proof schemes.

Analogous answers where such mixed occurrences of expected mathematical manipulations and formulations with misuse of algebraic or arithmetical symbols and arbitrary use of terminology are similarly characterised as D.T.-EC.NRS.

There are two examples of mixed E.P. and EC.NRS. proof schemes. In one of them (see Figure 4.1.7) P[89] writes:

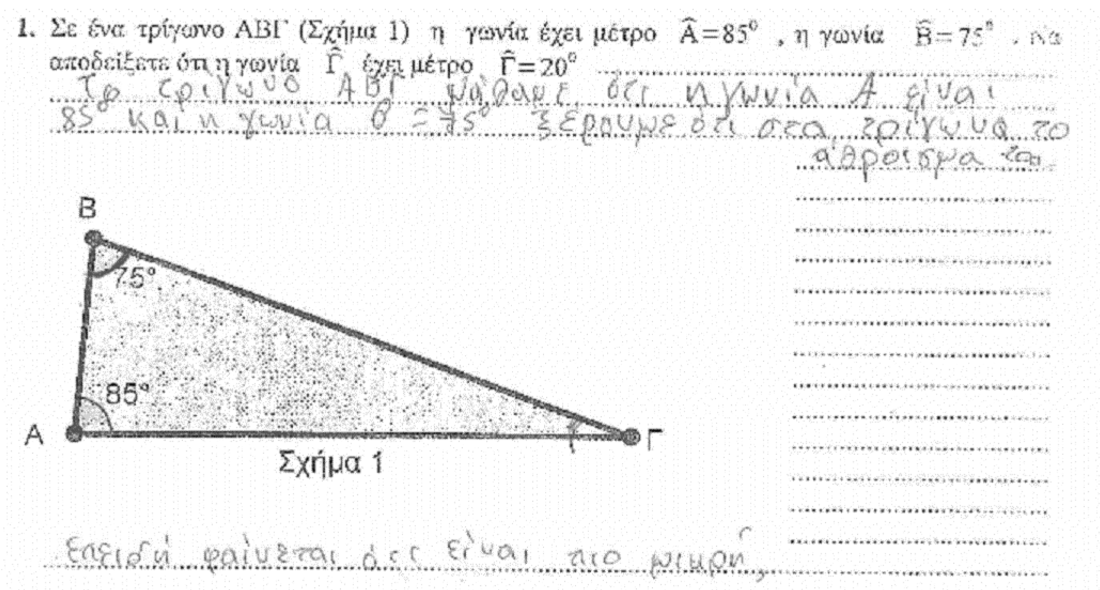


Figure 4.1. 7 Participant's [89] response to Question T11

The triangle ΑΒΓ we learned that angle A is

85° and angle B=75° we know that in triangles

its sum (something not clearly legible)

because it seems the smallest.

P[89] first refers to the measures of the given angles and then asserts something about the sum of the angles using an illegible symbol. There is no evidence of calculations of any kind. The written answer is barely comprehensible. The formulation is incoherent because it is incomplete. Indeed P[89] begins with the angles of the triangle and then abruptly turns to the sum of the angles, but the actual sum is, is not to given. Consequently this answer is inadequate. The first part of the answer provides evidence of arbitrariness and thus of the EC.NRS. proof scheme, and the rest of the next sentence is not comprehensibly connected to the previous ones. Probably P[89] regards angle $\hat{\Gamma}$ as the smallest in the given triangle and for this reason we can take this as providing evidence of the E.P. proof scheme. Based on these considerations this answer is a mixture of E.P. and EC.NRS. proof schemes.

Finally I present the unique EC.NRS. proof scheme example. Participant P[39] writes (see Figure 4.1.8):

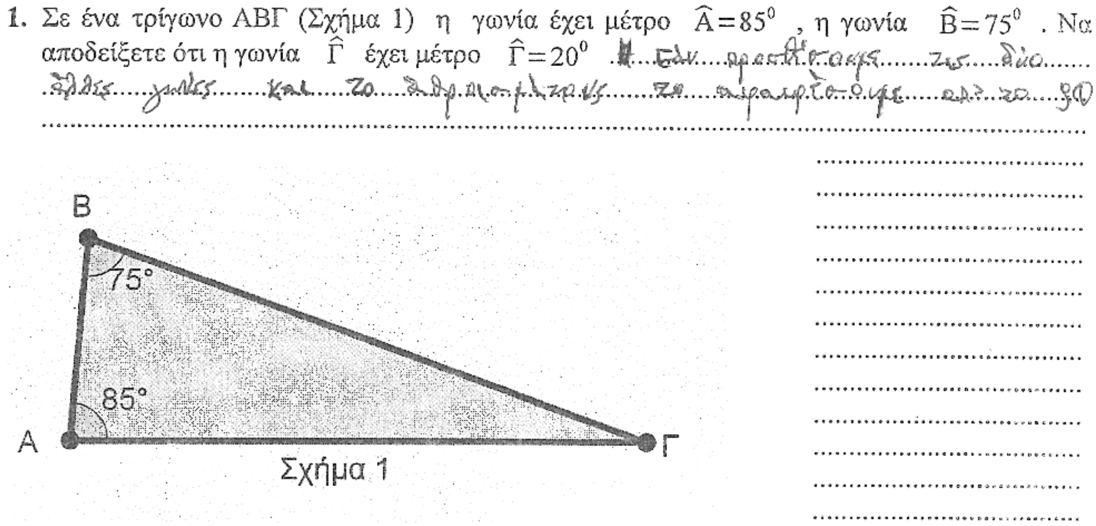

1. Σε ένα τρίγωνο ΑΒΓ (Σχήμα 1) η γωνία έχει μέτρο $\hat{A}=85^0$, η γωνία $\hat{B}=75^0$. Να αποδείξετε ότι η γωνία $\hat{\Gamma}$ έχει μέτρο $\hat{\Gamma}=20^0$. Εάν προσθέσουμε τις δύο άλλες γωνίες και το άθροισμά τους το αφαιρέσουμε από το 90

Figure 4.1. 8 Participant's [39] response to Question T11

"*If we add the two*

other angles and their sum we subtract from 90"

P[39] believes that to solve the problem it suffices to add the two angles and subtract their sum from *90.* The sum of the known angles to which P[39] refers is *85°+75°=160°*. Obviously the sum should be subtracted from *180°*, but P[39] suggests that it should be subtracted from *90.* Even in a right-angled, triangle finding the difference of the sum of the two acute angles from *90* degrees would lead to a zero degrees result. Apart from the fact that no degree symbol appears beside the number *90*, it seems that the fact that subtracting *160°* from *90°* would result in negative number has escaped P[39]'s attention. These two facts, difference zero and difference negative, are symptoms of misuse of the theorem on the sum of the angles of a triangle while attempting to find the measure of angle $\hat{\Gamma}$ of triangle *ΑΒΓ*. Thus the answer provides adequate evidence to characterise the P[39]'s proof scheme as EC.NRS.

Table 4.1.1 below summarizes the results of the script analysis and shows that 61 (67.78%) answers are classed as D.T. proof schemes, forming the biggest group. This result is in accordance with the fact that the sum of the angles of a triangle is sufficiently known to the participants, as explained. I justifiably accepted a wide range of answers as D.T. proof schemes since they fulfilled the conditions of an adequate answer. That makes those answers no less D.T. but it is a symptom connected with the nature of the question. Indeed the question leaves plenty of room for different answers ranging from algebraic ones to simple arithmetical calculations. The second biggest group of 22 answers (24.44%) D.T.-EC.NRS. answers, all refer to the sum of the angles of a triangle as being *180°*. With the D.T. answers these makes up a total of 83 (92.22%) answers reinforcing the 'popularity' of the theorem on the sum of the angles of a triangle among the participants. At the same time it signals the difficulty the students encountered even before the official teaching of proof when they attempted to use mathematical symbols and to formulate mathematical thoughts. Indeed, the very sum of the angles of a triangle is expressed in a number of answers using arbitrary symbols such as $AB\Gamma = 180°$ instead of $\hat{A} + \hat{B} + \hat{\Gamma} = 180°$. In other cases, on this same issue, instead of referring to the given triangle the participants

PROOF SCHEMES OBSERVED IN THE RESPONSES TO QUESTION T11				
PROOF SCHEME	***FREQUENCY***	***CUMULATIVE FREQUENCY***	***RELATIVE FREQUENCY (%)***	***CUMULATIVE RELATIVE FREQUENCY (%)***
D.T.	*61*	*61*	*67.78*	*67.78*
D.T.-EC.NRS.	*22*	*83*	*24.44*	*92.22*
E.P.-EC.NRS.	*2*	*85*	*2.22*	*94.44*
EC.NRS.	*1*	*86*	*1.11*	*95.55*
N.S.	*4*	***90***	*4.44*	***99.99***
SUM	***90***		***99.99***	

Table 4.1. 1 Summary of Question T11 proof schemes

speak of a right-angled triangle. There are also instances of miscalculation, as for example *160−180=20*. These examples exemplify the rising problems relevant to the development of the various proof schemes at this level of symbolism and formulation.

The next two groups, one with two (2.22%) E.P.-EC.NRS. answers and one with just one (1.11%) EC.NRS. answer are almost of negligible size, as also are the four (4.44%) NS. The latter is another indication of the widespread knowledge of the sum of the angles of a triangle, and the former two groups can be counted among the answers revealing the students' difficulties with proof.

4.2 Analysis of responses to Question T12

Question T12 combines the property of the bisector of an angle and the sum of the angles of a triangle. This makes its context more complicated than that of Question

T11, allowing testing proof ability on a more difficult scale. Consequently a student's performance can be expected to be inferior to their performance answering T11

Analysis of all the answers produced five groups of answers: four groups of different proof schemes and one group of NS. The first is the D.T. group; the second the D.T.-EC.NRS. proof scheme; the third is E.P.-EC.NRS. proof scheme, with only one answer; and the fourth group is the EC.NRS. proof scheme. The fifth group is NS.

Some members of the D.T. and some of the D.T.-EC.NRS. proof scheme are very close to the line setting the two groups apart. Some answers characteristic of the D.T. proof scheme would belong to the D.T.-EC.NRS. proof scheme group if their arbitrariness in using symbols or the arbitrariness in language formulation had gone beyond a certain limit. Similarly some answers classed as D.T.-EC.NRS. proof schemes could have been characterized D.T. if the arbitrariness had been proved unimportant.

Finally the D.T. proof scheme answers are not identical but vary from brief answers to very detailed ones, as the examples that follow illustrate.

P[18] (see Figure 4.2.1) writes:

Since AI and BI

bisectors of the angles

$\widehat{BA\Gamma}$ *and* $\widehat{AB\Gamma}$ *the*

angle $B\widehat{A}I = \frac{1}{2}\widehat{BA\Gamma}$ *and*

the $A\widehat{B}I = \frac{1}{2}\widehat{AB\Gamma}$ *thus*

$B\widehat{A}I = 39°$ *and*

$A\widehat{B}I = 33°$

The sum of the angles

in a triangle is 180°

therefore $\widehat{AIB} = 180° - (\widehat{BAI}$

$+\widehat{ABI}) = 180° - 72° =$

$108°$

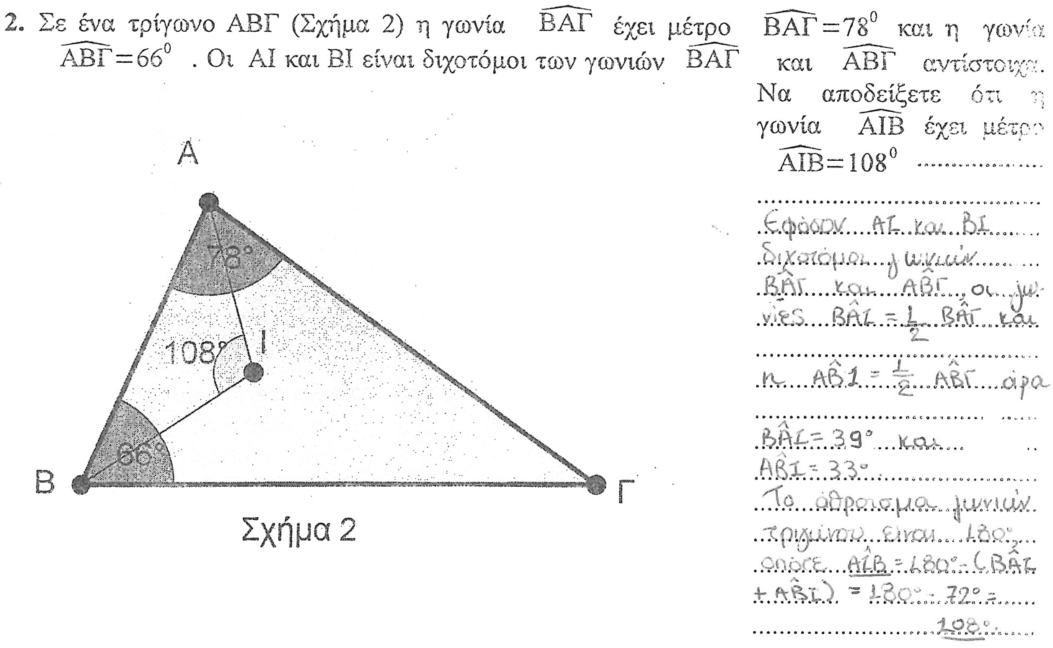


Figure 4.2. 1 Participant's [18] response to Question T12

P[18] calculates the measures of $\widehat{BAI} = 39$ and $\widehat{ABI} = 33°$ first by appealing to the property of an angle's bisector. Then she invokes the theorem on the sum of the angles of a triangle to calculate $\widehat{AIB} = 108°$ correctly without explicitly referring to triangle *ABI*: however it is obvious that the calculation took place in direct connection to this triangle. The answer has been characterised a D.T. proof scheme because it uses the definition of an angle bisector and the theorem on the sum of the angles of a triangle appropriately.

P[88] writes (see Figure 4.2.2):

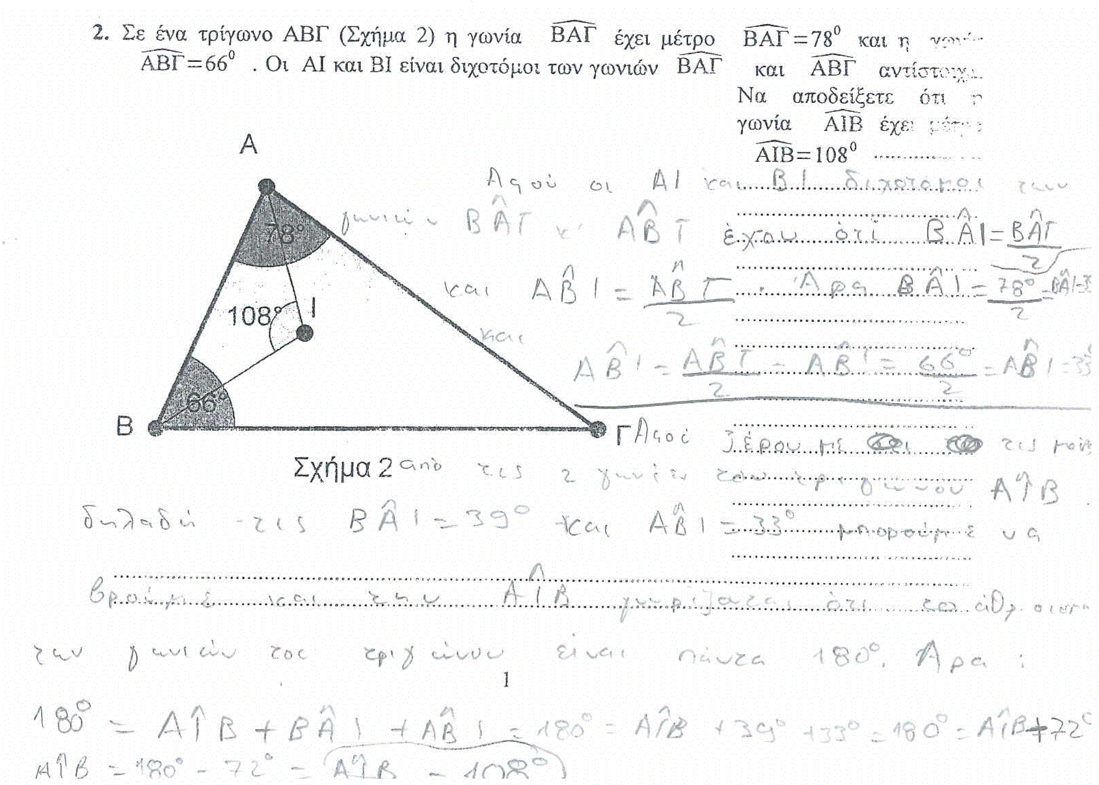


Figure 4.2. 2 Participant's [88] response to Question T12

Since AI and BI bisectors of

the angles $\widehat{BA\Gamma}$ $\widehat{AB\Gamma}$ *we have* [or I have] $\hat{BAI} = \frac{\widehat{BA\Gamma}}{2}$

and $\hat{ABI} = \frac{\widehat{AB\Gamma}}{2}$*. Hence* $\hat{BAI} = \frac{78^\circ}{2} = \hat{BAI} = 39°$

and

$\hat{ABI} = \frac{\widehat{AB\Gamma}}{2} = \hat{ABI} = \frac{66^\circ}{2} = \hat{ABI} = 33°$

Since we know the degrees

from the 2 angles of the triangle AIB (*ambiguous symbol over* AIB)

that is $\hat{BAI} = 39°$ and $\hat{ABI} = 33°$ *we can*

find also the $\widehat{AIB}$ *knowing that the sum* *of the angles in a triangle is always 180°. Thus:*

$180° = \widehat{AIB} + \widehat{BAI} + \widehat{ABI} = 180° = \widehat{AIB} + 39° + 33° = 180° = \widehat{AIB} + 72°$

$\widehat{AIB} = 180° - 72° = \widehat{AIB} = 108°$

P[88] starts the proof by invoking the property of an angle bisector and expresses symbolically the angles $\widehat{BAI}$ and $\widehat{ABI}$ as $\widehat{BAI} = \frac{\widehat{BA\Gamma}}{2}$ as $\widehat{ABI} = \frac{\widehat{AB\Gamma}}{2}$ first, and then computes their measures respectively. He then makes explicit reference to triangle AIB to which he applies the theorem on the sum of the angles of a triangle to give him the measure of $\widehat{AIB}$. It is true that his application of the symbol of equality is rather peculiar, but does not misuse it unacceptably. There is also some ambiguity as to whether the symbol over the triangle AIB is the symbol of a triangle or that of an angle. Even if it is the latter I did not to take it into in account which would have led characterising the proof scheme both D.T. and EC.NRS., and I considered the answer as providing evidence of a D.T. proof scheme.

P[13] (see Figure 4.2.3) writes:

Bisector of an angle

is called a straight line

that divides

the angle in two

equal parts.

Thus : $B\hat{A}I = \frac{B\hat{A}\Gamma}{2}$

$\frac{B\hat{A}\Gamma}{2} = \frac{78}{2}$

$\widehat{BA\Gamma} = 39$

The same is valid also

for the angle $A\hat{B}I$. *Thus* $A\hat{B}I = \dfrac{A\hat{B}\Gamma}{2} \Rightarrow \dfrac{66}{2} = 33$ *.(sco) **

(sco)

** Since the angles of the triangle have sum* 180° *then*

$$A\hat{B}I + B\hat{A}I + A\hat{I}B = 180 \text{. Thus } I = 180 - (B\hat{A}I - A\hat{B}I)$$

$$I = 180 - (39 + 33)$$

$$I = 180 - 72 \Rightarrow 108$$

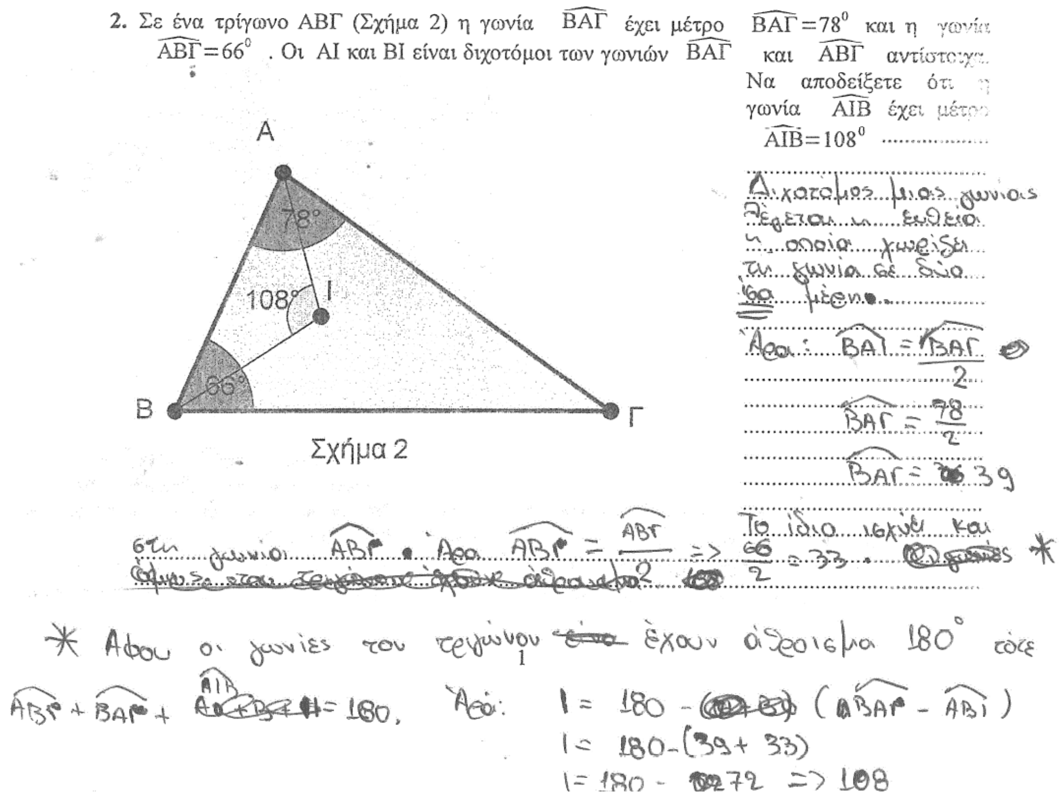


Figure 4.2. 3 Participant's [13] response to Question T12

From line 7 to line 9 P[13] writes $B\hat{A}\Gamma$ instead of $B\hat{A}I$. This is an obvious mistake, which she corrects in line 10. Indeed she wanted to write $B\hat{A}I = 39$ but failed, replacing the capital *I* (iota) with capital *Γ* (gamma). One can make out in line 9, that she has crossed out the upper horizontal of the *Γ* to create the letter *I.* Apparently the mistake in line 7 escaped her attention and remains uncorrected. There is also an inaccuracy regarding the symbol of implication when she writes $I = 180 - 72 \Rightarrow 108$. Finally the symbol for the degree, e.g. *180°* is used in an inconsistently appearing once in line 10 and nowhere else. However, I have opted to categorise this as a D.T. proof scheme, as that these mistakes do not significantly overshadow the elements of an adequate answer.

P[24] (see Figure 4.2.4) writes:

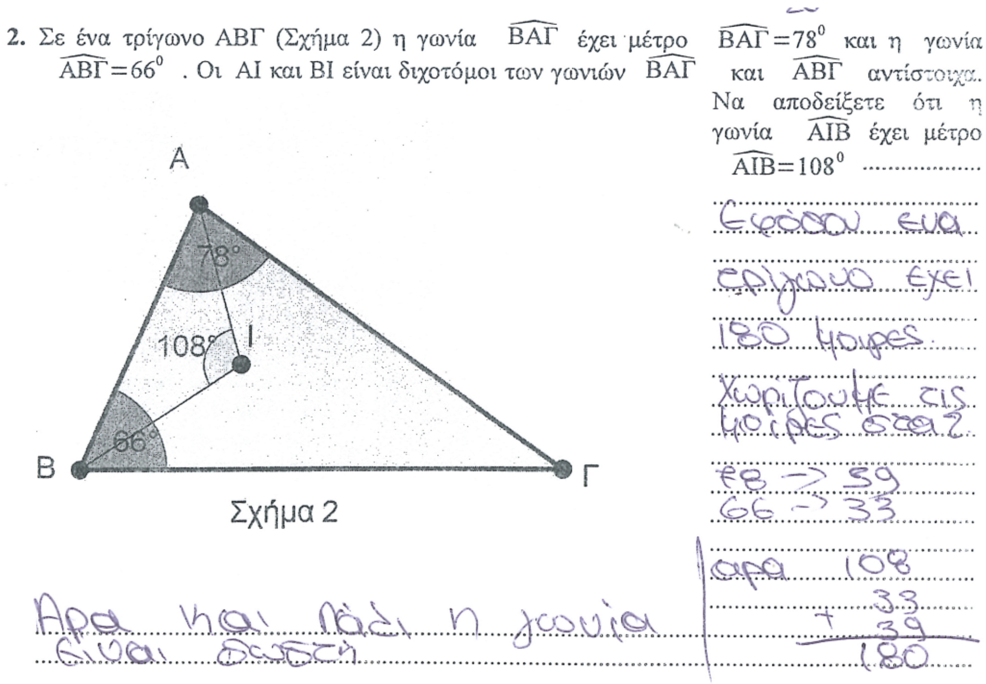


Figure 4.2. 4 Participant's [24] response to Question T12

Since a

triangle has

180 degrees

we divide the

degrees in 2

78→39

66→33

thus 108

Thus again the angle	*33*
is the correct one	*+ 39*
	180

P[24] invokes the theorem on the sum of the angles in a triangle, making only implicit reference to the triangle to which the theorem is applied. Then he finds the measures of the halves of the angles required for proof and verifies that these add up to *180* degrees. The odd sentence "*Thus again the angle is the correct on"*" has in all likelihood the following meaning: Questions T11 and T12 ask for proof that an angle's measure has a certain value. Thus this odd sentence means that P[24] found again, in other words as in T11, the sought for value for the angle's measure. Under these considerations this answer provides evidence of a D.T. proof scheme and has been characterised accordingly.

Next I present answers that include both D.T. and EC.NRS. proof schemes.

Participant P[03] writes (see Figure 4.2.5):

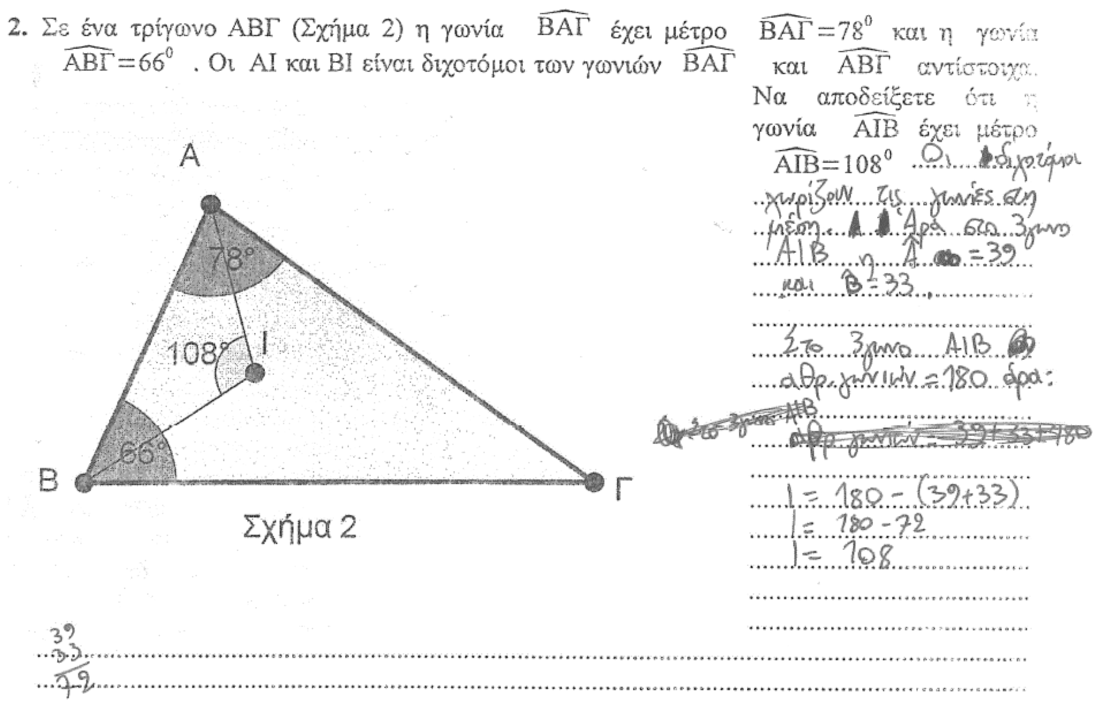


Figure 4.2. 5 Participant's [03] response to T12

The bisectors

divide the angles in

the middle. (sco) Thus in the 3angle

AIB the $\hat{A}$ *(sco)=39*

and $\hat{B}$ *=33.*

In the 3angle AIB (sco)

Sum angles=180 thus:

(sco)

I=180-(39+33)

I=180-72

I=108

There is also the addition of *39* and *33* in the lower far left corner. In his answer the participant begins by referring to the property of the angle bisector, but uses the words "*divide in the middle*" instead of "*divide in equal parts*". This is frequently encountered in the Greek students' writing and speech. The word "*middle"*, however, even if not accurate, is used here to mean "*equal parts*". There are some other inaccuracies in the use of the angle and degree symbols. P[03]'s answer belongs to those in which there is confusion when symbolising the angles of triangle *AIB*. Indeed P[03] symbolises angles $\widehat{ABI}$ and $\widehat{BAI}$ as $\hat{A}$ and $\hat{B}$ and uses the symbol '3angle' instead of the word 'triangle'. I think this is as a stylistic aspect of the participant's writing. Summarising I consider that the previous inaccuracies constitute elements of the EC.NRS. proof scheme. As the answer also contains all necessary information to be characterised as a D.T. proof scheme, this is a mixture of D.T. and EC.NRS. proof schemes.

P[26] writes (see Figure 4.2.6):

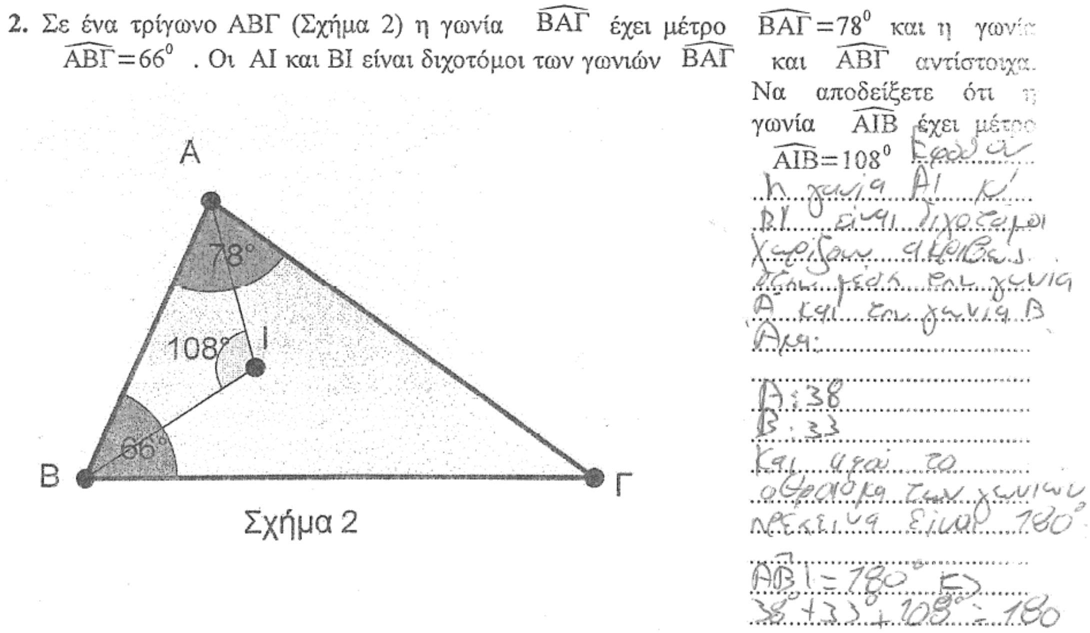


Figure 4.2. 6 Participant's [26] response to Question T12

Since

The angle AI &

BI are bisectors

cut exactly

in the middle the angle

A and the angle B

Thus:

A:38

B:33

And since the

Sum of the angles

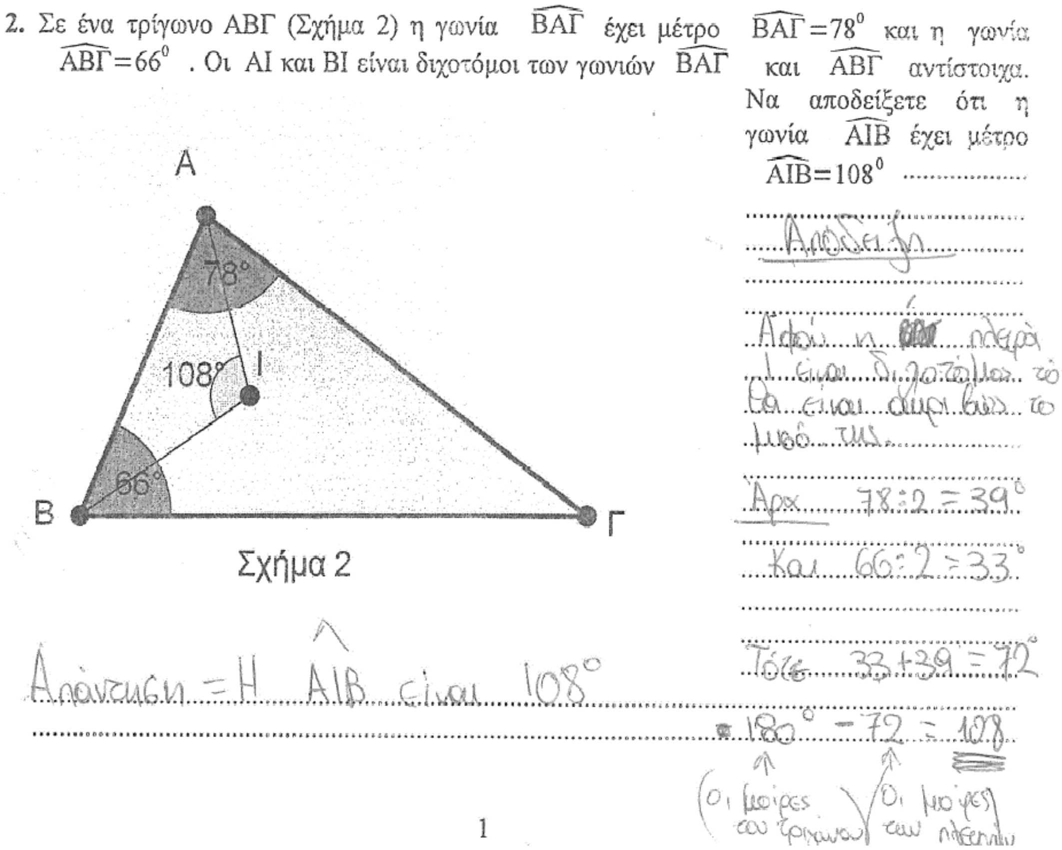
has to be 180°

$\widehat{ABI} = 180° \Rightarrow$

$38°+33°+108°=180°$

The participant misuses the words angle and bisector in the second and third lines of his script: "*The angle AI and BI are bisectors*". Then he writes *A:38* and *B:33*. By this symbolism obviously means the measures of the angles $\widehat{BAI} = 38°$ and $\widehat{ABI} = 33°$, but the symbols are arbitrary and ambiguous although interpretable: arbitrary because customarily this would be written *A=38*; and ambiguous because, on the one hand in mathematics *A:38* literally means a fraction with nominator *A* and denominator *38*, and on the other it is not clear what angle is referred to since there are, at least, three with the same vertex. Additionally the measure of angle $\widehat{BAI}$ is miscalculated as *38°* , whereas correct calculation yields *39°*. In the penultimate line of his script P[26] writes that $\widehat{ABI} = 180°$. The symbol $\widehat{ABI}$ stands for the sum of the angles of triangle *ABI* as we understand by reading the next and last lines of his script. However, the symbol is again arbitrary. Indeed there is no such symbol for the angles of a triangle in aggregate. The miscalculation of the measure of angle $\widehat{BAI}$ is repeated and extended to a new miscalculation. In fact the sum *38°+33°+108°* does not yield *180°* but *179°*. These comments notwithstanding participant P[26] has substantial knowledge of the bisector of an angle and the theorem on the sum of the angles of a triangle. He also knows he needs to use the theorem to verify the sought-for measure of the angle $\widehat{BAI}$. If we ignore his relevant mistakes the answer can be classed as mixture of D.T. and EC.NRS. proof schemes.

P[42] writes (see Figure 4.2.7):



2. Σε ένα τρίγωνο ΑΒΓ (Σχήμα 2) η γωνία $\widehat{ΒΑΓ}$ έχει μέτρο $\widehat{ΒΑΓ}=78^0$ και η γωνία $\widehat{ΑΒΓ}=66^0$. Οι ΑΙ και ΒΙ είναι διχοτόμοι των γωνιών $\widehat{ΒΑΓ}$ και $\widehat{ΑΒΓ}$ αντίστοιχα. Να αποδείξετε ότι η γωνία $\widehat{ΑΙΒ}$ έχει μέτρο $\widehat{ΑΙΒ}=108^0$ ....

Figure 4.2. 7 Participant's [42] response to Question T12

<u>*Proof*</u>

Since the side

I is bisector then

it will be exactly its

half.

Thus 78:2=39°

and 66:2=33°.

Then 33+39=72°

Answer= $\widehat{AIB}$ *is 108°*

$$\underset{\substack{\uparrow \\ \left(\text{the degrees of the triangle}\right)}}{180^\circ} - \underset{\substack{\uparrow \\ \left(\text{the degrees of the sides}\right)}}{72} = 108$$

Participant [42] begins the proof by invoking the property of the bisector. In doing so he writes "*since the side I is bisector then it will be exactly its half*". He apparently intended to write about *AI* and *BI* as bisectors of the angles and thus divide each of the corresponding angles of the triangle *ABΓ* into two equal parts each, but failed ending up with inaccurate and ambiguous formulation. Concluding the proof, the participant wants the reader to understand that the number *72* is "the degrees of the sides", but again the sentence is vague and arbitrary from the point of view of mathematical terminology. P[42] probably wanted to say that in triangle *ABI* the sides *AI* and *BI* form, with *AB*, angles the sum of which is 72 degrees. The ambiguous and arbitrary formulation is evidence of the EC.NRS. proof scheme. On the other hand P[42] is aware of the property of an angle bisector, the theorem on the sum of the angles of a triangle and to which triangle the theorem in question must be applied. Additionally she managed to put these properties together to solve the problem correctly. Thus her answer is characterised as a mixture of D.T. and EC.NRS. proof schemes.

P[59] (see Figure 4.2.8) writes:

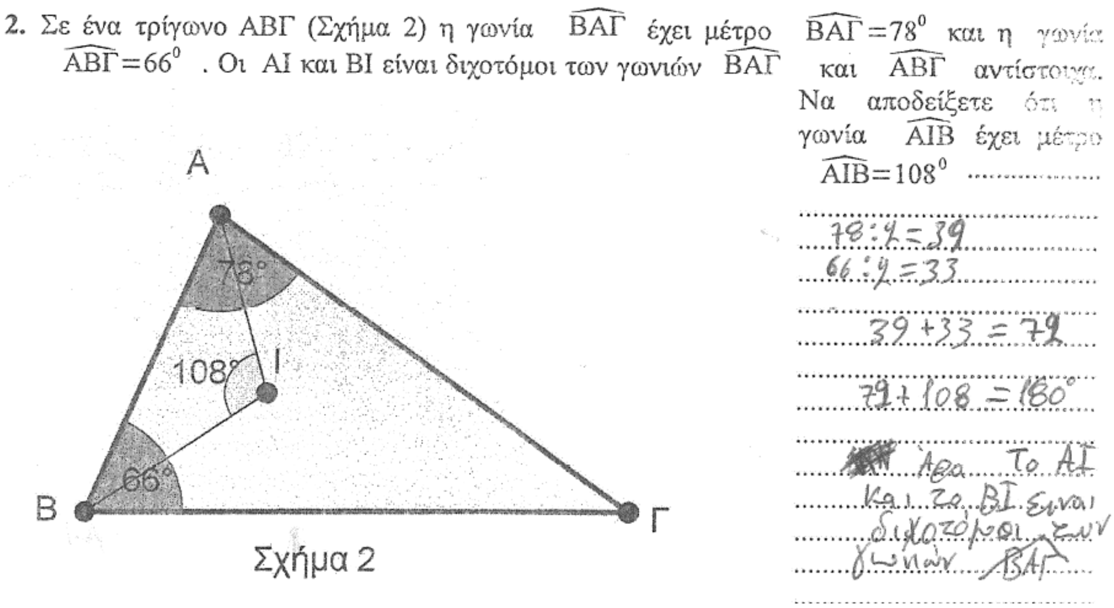


Figure 4.2. 8 Participant's [59] response to Question T12

78:2=39

66:2=33

39+33=72

72+108= 108°

Thus the AI

and BI are

bisectors of the

angles $B\hat{A}\Gamma$

P[59] verifies that angle $\widehat{BIA}$ indeed measures *108* degrees. At this point, apart from the fact that he offers no explanation whatsoever, one could accept the answer as a D.T. proof scheme. But then the participant concludes that *AI* and *BI* are bisectors of the angles of triangle *ABI*. Parallel to this the symbol for the triangle, if that was indeed the intention, is used instead the symbol of an angle. In this respect P[59] provides evidence of the EC.NRS. proof scheme with the misuse of the symbols and the conclusion that *AI* and *BI* are bisectors whereas this is data given. Thus the answer is characterised as a mixture of D.T. and EC.NRS. proof schemes.

P[62] (see Figure 4.2.9) writes:

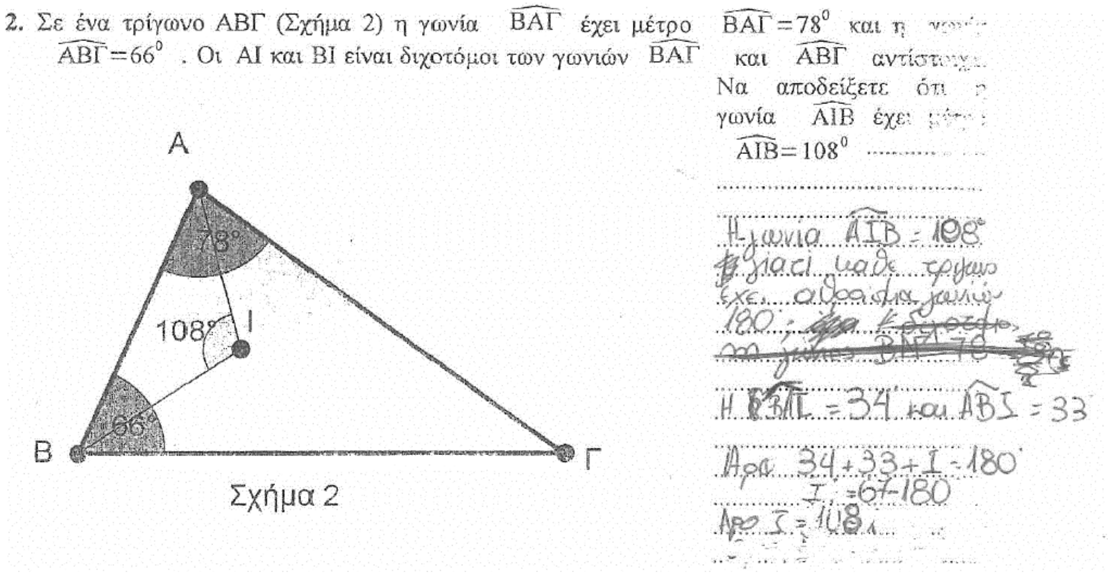


Figure 4.2. 9 Participant's [62] response to Question T12

The angle $\widehat{AIB}$ *=108°*

because every triangle

has sum of angles

180 (sco)

(sco)

$\widehat{BAI}$ *=34 and* $\widehat{ABI}$ *=33*

Thus 34+33+I=180

I=67-180

Thus I=108

P[62] invokes the theorem on the sum of the angles of a triangle at the beginning to calculate that $\widehat{I}$ *=108°* and then gives the measures of angles $\widehat{BAI}$ and $\widehat{ABI}$. By this effort he miscalculates the measure of the former angle as *34°* and asserts that *34+33+I=180* from which concludes that *I=67-180* thus solving for *I* not correctly.

The next step is arbitrary. Indeed, P[62] asserts that *I=108* whereas the equation *34+33+I=180* even if correctly solved does not yield this value but instead yields *I=113*. In all of these manipulations there is inconsistent use of the degree and angles symbols. At the same time, miscalculations and arbitrariness notwithstanding, the participant clearly shows knowledge of the property of an angle bisector, and of the theorem on the sum the angles in a triangle and its use to calculate the requirred angle. Under these circumstances the answer is characterised as a mixture of D.T. and EC.NRS. proof schemes.

Only one answer is classed as E.P.-EC.NRS. Participant P[87] (see Figure 4.2.10) writes:

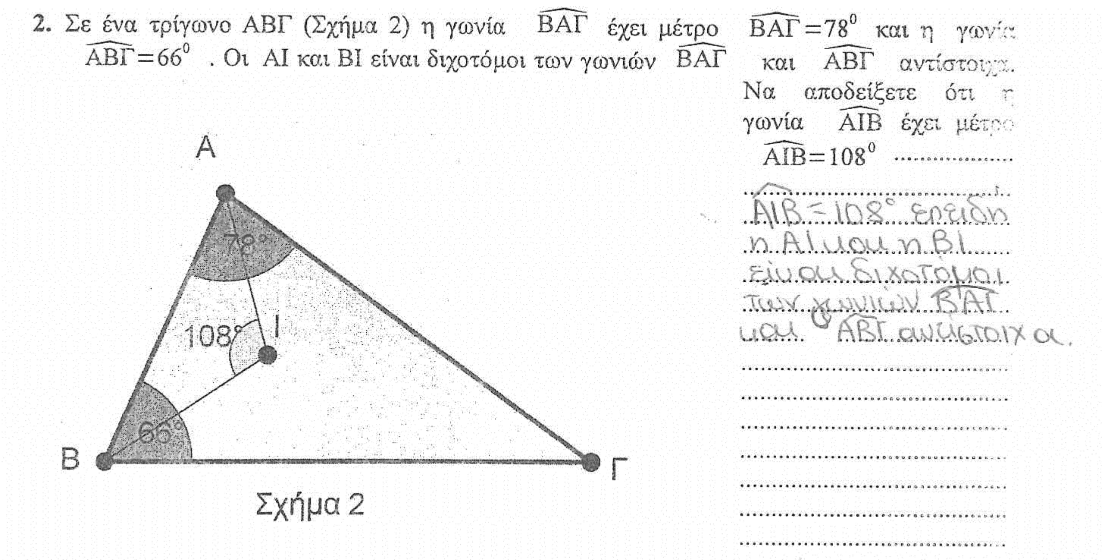


Figure 4.2. 10 Participant’s [87] response to Question T12

$\widehat{AIB}$ *=108°* because

AI and BI

are bisectors

of the angles $\widehat{BA\Gamma}$

and $\widehat{AB\Gamma}$ *respectively*”

P[87] asserts that $\widehat{AIB} = 108°$ because AI and BI are bisectors of angles $\widehat{BA\Gamma}$ and $\widehat{AB\Gamma}$. This assertion is arbitrary because these two facts do not have any cause and result relation. AI and BI can be bisectors of angles $\widehat{BA\Gamma}$ and $\widehat{AB\Gamma}$ without this fact forcing their angle to be *108°*. In the particular context of Question T12 indeed $\widehat{AIB} = 108°$ because *BI* and *AI* are bisectors of angles of given measures and simultaneously form with *AB* triangle *AIB*. However, the same formulation seems to be saying that "$\widehat{AIB} = 108°$ because I see it in the Figure". In this sense the answer contains a mixture of E.P. and EC.NRS proof schemes.

To end this presentation of examples of answers, I present one that is classed as EC.NRS. proof scheme.

P[16] writes (see Figure 4.2.11):

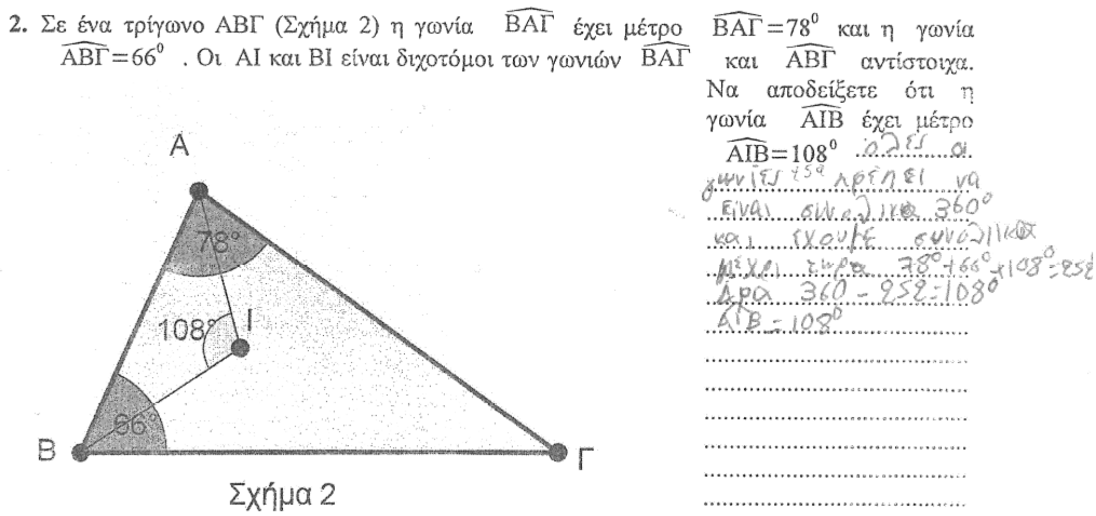


Figure 4.2. 11 Participant's [16] response to the Question T12

All the

angles 252 must be

in aggregate 360°

and we have in aggregate

up to now 78°+66°+108°=252°

Thus 360-252=108°

$\widehat{AIB} = 108°$

Participant [16] begins his proof by asserting that "*all the angles* 252 *must be in aggregate* 360°". The meaning of his sentence is explained in the next lines of his script: first he adds the measures *66°+78°+108°* to find *252°*. Then, arbitrarily considering that the sum of the measures should have been *360°*, he subtracts *252°* from *360°* to find *108°*. Of course there is no reason at all why the angles used as summands should add up to *360°* as they add up to *252°*. Thus from this point of view the answer is an EC.NRS. proof scheme because it is based on irrelevant assumptions and conditions that have nothing to do with the property of the angle bisector, the theorem on the sum of the angles of a triangle and its application or any other known valid assumption or assertion.

Table 4.2.1 gives the quantitative whole picture of the proof schemes that appeared in the participants' answers to Question T12. The first thing to observe in the table is the drastically reduction of the number of answers characterised as D.T. proof schemes (28, or 31.11%) in comparison to the corresponding number of D.T. answers given to Question T11 (61, or 67.78%). The 30 (33.33%) D.T.-EC.NRS. proof schemes and the number 28 (31.11%) D.T. proof schemes encountered in the analysis of the answers to Question T12 add up to 58 (64.44%) which is near to number 61 (67.78%) D.T. proof schemes encountered among the answers to Question. This smaller number of D.T. answers is expected but the reduction of 33 (61-28=33) is substantial. The change of context in Question T12 in comparison to T11 made the field more difficult for a number of participants. To answering Question T11, the students had either to verify that the sum of the angles *75°*, *85°* and *20°* is

PROOF SCHEMES OBSERVED IN THE RESPONSES TO QUESTION T12				
PROOF SCHEME	***FREQUENCY***	***CUMULATIVE FREQUENCY***	***RELATIVE FREQUENCY (%)***	***CUMULATIVE RELATIVE FREQUENCY (%)***
D.T.	*28*	*28*	*31.11*	*31.11*
D.T.-EC.NRS.	*30*	*58*	*33.33*	*64.44*
E.P.-EC.NRS.	*1*	*59*	*1.11*	*65.55*
EC.NRS.	*8*	*67*	*8.89*	*74.44*
N.S.	*23*	***90***	*25.56*	***100.00***
SUM	***90***		***100.00***	

Table 4.2. 1 Summary of Question T12 proof schemes

$180°$ or to form an equation of the form $75° + 85° + \hat{\Gamma} = 180°$ and solve it for the unknown $\hat{\Gamma}$ to find $\hat{\Gamma}$ =*20°*. They had only to invoke and apply the theorem on the sum of the angles of a triangle. There was no need to write much or to justifying their actions. However, in Question T12 they had to invoke the property of an angle bisector and the theorem on the sum of the angles of a triangle in connection not with the original triangle, *ABΓ*, but with the triangle formed by the bisectors *BI*, *AI* and side *AB*. To do this they had to formulate some thoughts concerning the measures of angles $\widehat{ABI}$ and $\widehat{BAI}$. Then they had to explain that they were referring to triangle *ABI*. At the same time they had to deliver some calculations. All these efforts led to various mistakes in calculations, correct language formulation and even acceptable use of symbols. This explains why the number of EC.NRS. proof scheme grew from 25 (27.77%) in Question T11[3] to 39 (43.33%) in Question T12. Another indicator of the participants' difficulties is the number of NS answers, which rose from 4 (4.44%) in T11 to 23 (25.56%) in T12.

The relatively high number of D.T. answers is a sign that the students can build logical, proof-like arguments although they cannot always use the language and the notation properly. Their performance underlines their problems with proof at this stage before they have been officially taught it. Their answers to Question T12 fell largely into two categories: NS and EC.NRS. proof scheme.

The lack of answers involving proof schemes such as E.I. is due to the nature of Question T12 which does not include variable quantities that can be measured or substituted with numbers. Proof schemes as EC.A. or EC.R. are also seldom elicited by Questions such as T12.

4.3 Analysis of responses to Question T13

The participants, regarding mathematics, are at a turning point in their school life at which they must be able to clearly and successfully formulate properties, hypotheses, and conclusions. This is probably difficult exactly because of its perceived simplicity, not in the sense of it being an easy task but in the sense of logical steps. Sometimes the points at which to begin and end the argument are not obvious to them. Questions T13 and , T11 and T12 are similar because they demand the use of a fundamental geometric property, but they are also different because T11and T12's frames of reference have to do with a fundamental property expressible in simple arithmetic or even algebraic terms, whereas the property needed to answer T13 correctly is of a logical nature. I decided to characterise as D.T. answers with the

[3] See table 4.1.1

basic aspects of the above adequate answer; i.e. (i) the property of the perpendicular bisector (ii) the argument that point Σ has this property and (iii) the conclusion concerning triangle ΣAB. Any other answer using convincing argumentation would be acceptable. As I show some answers diverge from the adequate example above and yet their mathematical content and logical structure are correct.

The curriculum stipulates the teaching of the perpendicular bisector. The definition and the property of all points on the perpendicular bisector of a line segment and activities and exercises relevant to this can be found in the Year 7 textbook (Vandoulakis, Kalligas, Markakis, & Ferentinos, 2010, pp. 206-209). In Year 8 the property of the perpendicular bisector appears indirectly on a number of occasions in various problems concerning isosceles and equilateral triangles and the rhombus. Question T13 intended to gather information on how the students treated a problem concerning the logical laying of thoughts when a hypothesis is given and it and one is asked to reach to a certain conclusion from it.

Analysis of the participants' answers to Question T13 revealed a wider scattering of proof schemes than in answers to questions T11 and T12. The answers fall into eight different groups of various proof schemes or mixtures of proof schemes: D.T, D.T.-E.P., D.T.-EC.NRS., E.I., E.I.-EC.NRS., E.P., E.P.-EC.NRS., EC.NRS., and one group NS. Below I present examples.

P[33] writes (see Figure 4.3.1):

Isosceles is called the triangle in which the two
sides are equal.
We know from the
properties of

the perpendicular bisector, that every

point situated

on the perpendicular bisector

is equidistant from the

endpoints of the line

segment. Thus:

ΑΣ=ΣΒ and since

the two sides of

the triangle are equal

we say that the

triangle ΑΣΒ

is isosceles.

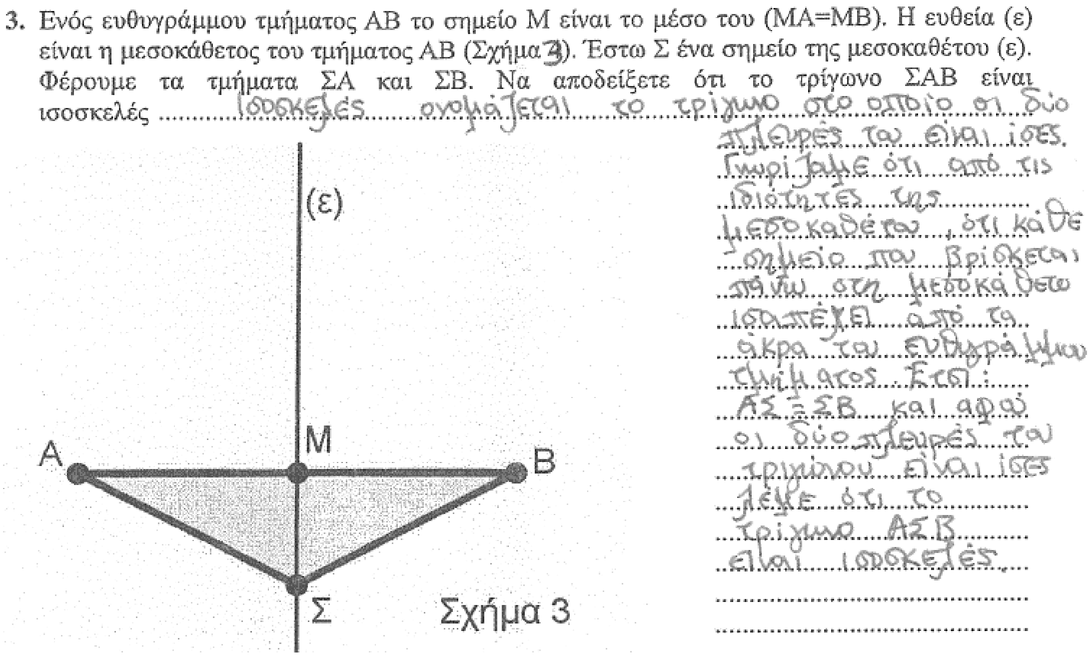


Figure 4.3. 1 Partticipant's [33] response to Question T13

P[33] gives the definition of an isosceles triangle and then the property of the perpendicular bisector of a line segment. Based on this he asserts that *ΑΣ=ΣΒ*. Finally by virtue of the equality *ΑΣ=ΣΒ* he concludes that triangle *ΑΣΒ* answers to the definition of an isosceles triangle and thus completes the proof. The answer satisfies the conditions of an adequate answer, conclusively providing evidence of a D.T. proof scheme.

P[06] (see Figure 4.3.2) writes:

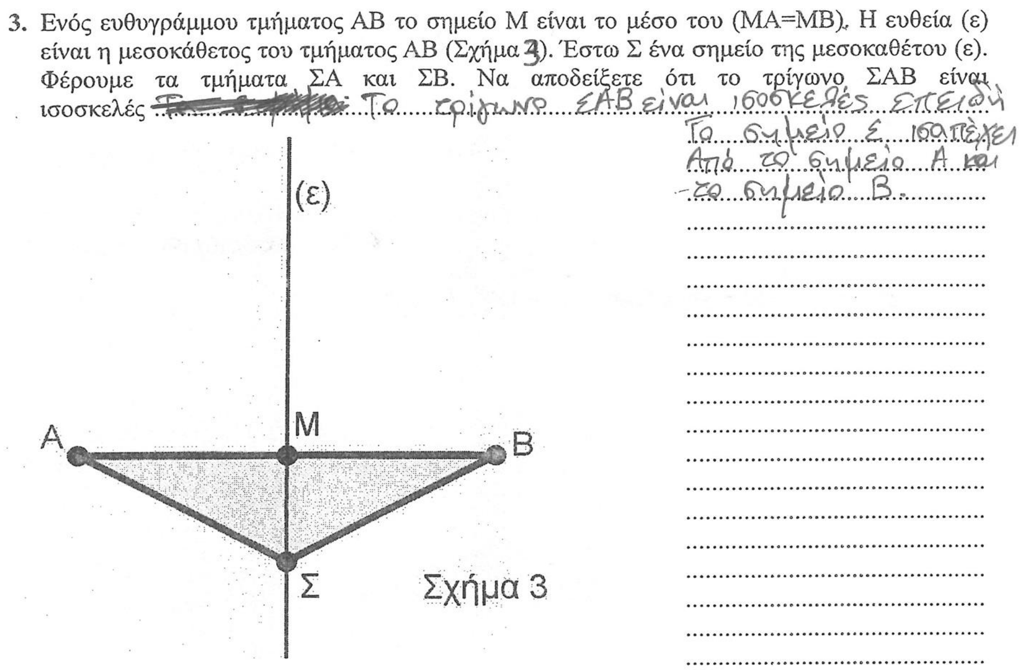


Figure 4.3. 2 Participant's [06] response to Question T13

The triangle ΣAB is isosceles because

the point Σ is equidistant

from the point A and

the point B.

P[06]'s proof asserts that triangle *ΣAB* is an isosceles triangle because point *Σ* is equidistant from points A and B. This is true under the condition that point *Σ* is a point on the perpendicular bisector of the segment *AB*. The last assertion is true because every point on the perpendicular bisector is equidistant from end points *A* and *B*. But P[06] does not feel the need to invoke the property of the perpendicular bisector because he sees it in the figure. Thus where the definition of the isosceles triangle is concerned his answer provides evidence of D.T. proof scheme. At the same time the answer also provides evidence of E.P. characteristics and is therefore classed as a mixture of D.T. and E.P. proof schemes.

P[64] (see Figure 4.3.3) writes:

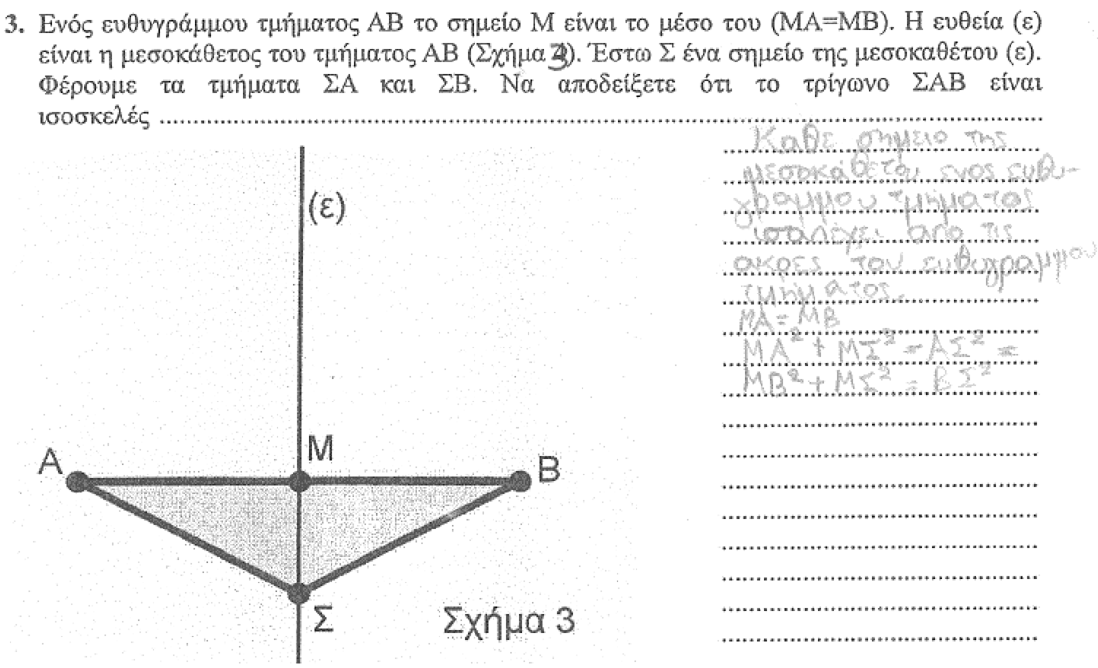

3. Ενός ευθυγράμμου τμήματος ΑΒ το σημείο Μ είναι το μέσο του (ΜΑ=ΜΒ). Η ευθεία (ε) είναι η μεσοκάθετος του τμήματος ΑΒ (Σχήμα 3). Έστω Σ ένα σημείο της μεσοκαθέτου (ε). Φέρουμε τα τμήματα ΣΑ και ΣΒ. Να αποδείξετε ότι το τρίγωνο ΣΑΒ είναι ισοσκελές

Figure 4.3. 3 Participant's [64] response to Question T13

Every point of

the perpendicular bisector of a

straight line segment

is equidistant from

the edges of the straight line

segment

$MA=MB$

$MA^2 + M\Sigma^2 = A\Sigma^2 =$

$MB^2 + M\Sigma^2 = B\Sigma^2$

P[64] begins her proof by stating the property of the perpendicular bisector, but does not take the expected next step, *AΣ=BΣ*. Instead she takes an unexpected turn: rather than declaring the triangle *ABΣ* an isosceles triangle and concluding the proof, she writes $MA=MB$ which is true. Her last step is to write $MA^2 + M\Sigma^2 = A\Sigma^2 = MB^2 + M\Sigma^2 = B\Sigma^2$ (1). The relation $MA^2 + M\Sigma^2 = A\Sigma^2$ is the valid Pythagorean theorem for the right triangle $MA\Sigma$ with M vertex of the right angle, since the straight line (ε) is perpendicular to line segment AB at M. Similarly $MB^2 + M\Sigma^2 = B\Sigma^2$ is valid in triangle $MB\Sigma$ with M vertex of the right angle. P[64] ends the proof after presenting relation (1), probably because P[64] considers that from (1) comes up $A\Sigma^2 = B\Sigma^2$ and thus $A\Sigma=B\Sigma$. However, there is no need to resort to (1) for this purpose because after having stated the property of the perpendicular bisector of a line segment the direct conclusion is $A\Sigma=B\Sigma$ and consequently that the triangle $AB\Sigma$ is an isosceles triangle. To summarise: what P[64] writes is mathematically correct and thus there is evidence of D.T. proof scheme. On the other hand the incompleteness of her arguments, caused by the failure to state clearly and explicitly how the final conclusion can be reached, makes her proof scheme also an EC.NRS. Accordingly her answer is classed as mixture of D.T. and EC.NRS. proof schemes

P [87] writes (see Figure 4.3.4):

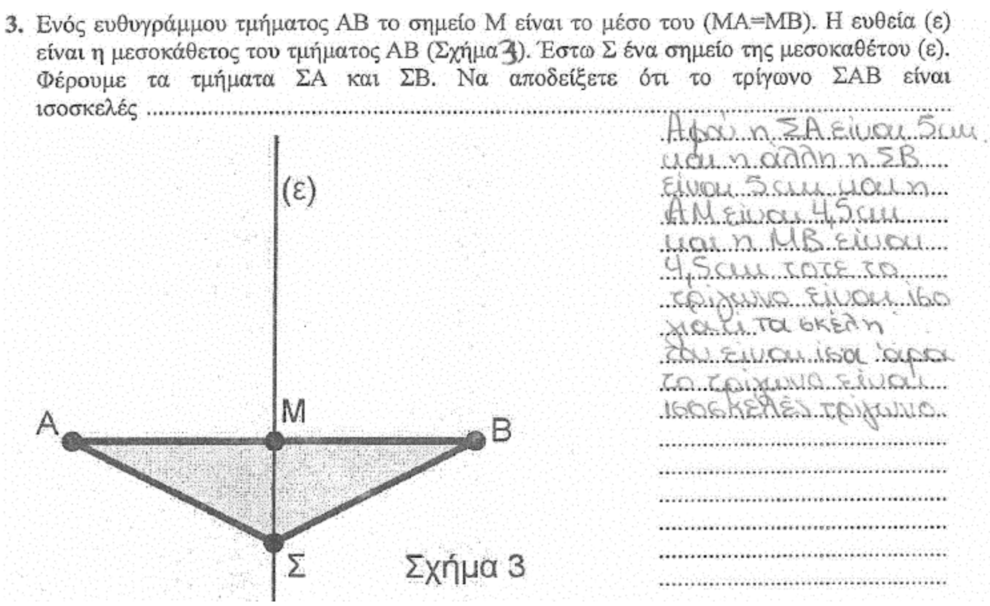


Figure 4.3. 4 Participant's [87] response to Question T13

Since ΣA is 5cm

and the other ΣB

is 5cm and

AM is 4.5cm

and MB is

4.5cm then

the triangle is equal

because its legs

are equal thus

the triangle is

isosceles triangle

P [87] start her proof by asserting that $\Sigma A=\Sigma B=5cm$ and $AM = MB =4.5cm$. It is not clear how she assigned these numerical values to the corresponding line segments, but the need to assign values to various quantities with no given numerical magnitude in order to articulate a proof is considered, in the taxonomy of Harel and Sowder's (2007, p. 7) taxonomy, evidence of an E.I. proof scheme. Thus P[87]'s need for numerical substitution in order to formulate a proof classes this answer as an E.I. proof scheme.

P[62] (see Figure 4.3.5) writes:

Since AM=MB then if we assume that AM=1 and MB=1

and the triangle $\Sigma\hat{B}M$

is a right one then

the height $\sqrt{2}$ and

the triangle $A\hat{B}\Sigma$

is isosceles.

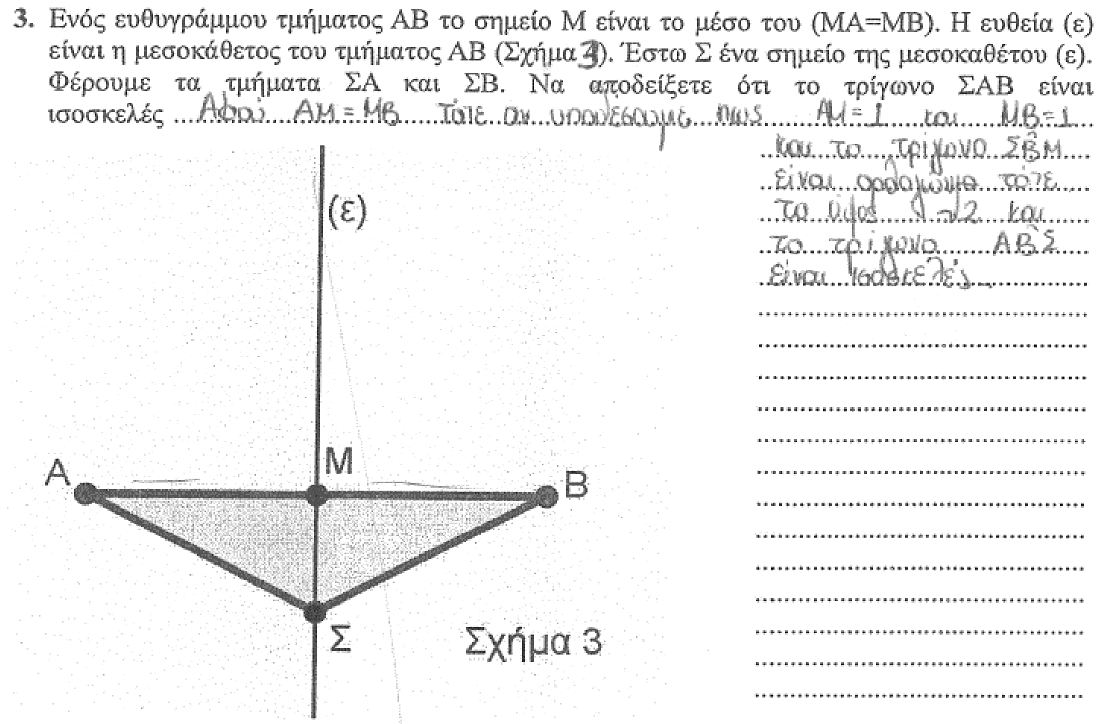


Figure 4.3. 5 Participant's [62] response to Question T13

Participant [62] begins her argument with the assumption that AM= MB= 1. She thus assigns number values to the lengths of AM and MB. In the fourth line she assigns the number $\sqrt{2}$ to the length of the height of the triangle *ΣBM*. Triangle *ΣBM* is a right-angled triangle with the vertex of the right angle at *M*. Thus the sides *MΣ* and *MB* are both heights. The third height, from point *M* to side *ΣB* is not drawn in the figure. There is no information or any other evidence to indicate which height P[62] is speaking of and nor is there any implication as to whether the new numerical assignment is the product of a calculation or of an arbitrary action. The conclusion, that triangle *ABΣ* is an isosceles triangle is plainly arbitrary because it is not based on the existence of two equal sides or any other plausible argument. As previously seen, the need to assign inconsistently numerical values to various quantities in a proof demonstrates what Harel and Sowder define as E.I. proof scheme. On the other hand, the obscurity of what P[62] means by the word "height" and its rather arbitrary way of its appearance is a sign of logical incoherence and thus of the EC.NRS. proof scheme.

The conclusion is of the same quality. Why a triangle should be an isosceles triangle when it has height of length $\sqrt{2}$ is not at all clear and it is not justified by argument. It may be that the participant has a memory of a right-angled isosceles triangle with equal sides of length 1 which has indeed a hypotenuse of length $\sqrt{2}$. However there is no height of length $\sqrt{2}$ in this case. Thus the answer provides evidence of both E.I. and EC.NRS. proof schemes and it is characterised as a mixture of E.I. and EC.NRS. proof schemes.

P[38] (see Figure 4.3.6) writes:

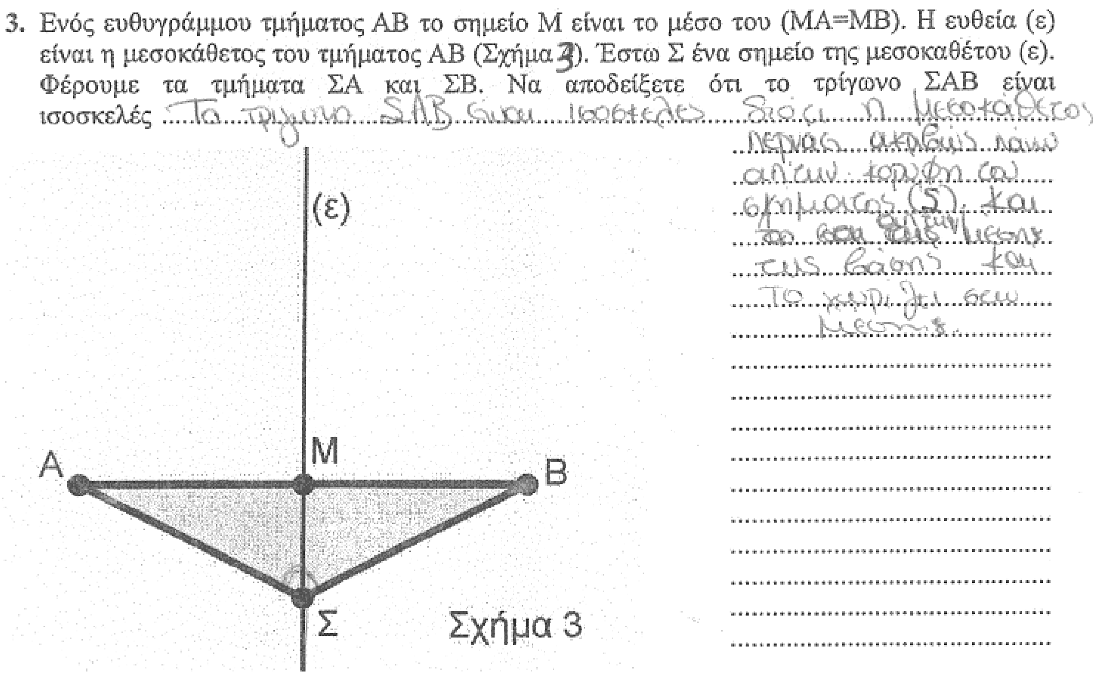


Figure 4.3. 6 Participant's [38] response to Question T13

The triangle ΣAB is isosceles because the perpendicular bisector

passes exactly over

the vertex of

the figure (Σ) and

(sco) the middle

of the base and

divides it

in the middle

Participant [38] makes no reference to the fundamental property of the perpendicular bisector and neither does she refer to any property of the point *Σ.* She confines herself only to an empirical description of the position of the perpendicular bisector, which she perceives as a line that divides in the 'middle'. This perception probably has to do with activities in Year 7 and possibly also at primary school, where folding a piece of paper along an axis of symmetry along an axis of symmetry in a drawing is a way of showing that one half will fit the other. I class this answer as an E.P. proof scheme because P[38] perceives but and does not prove the validity of the property to be proved.

P[42] (see Figure 4.3.7) writes:

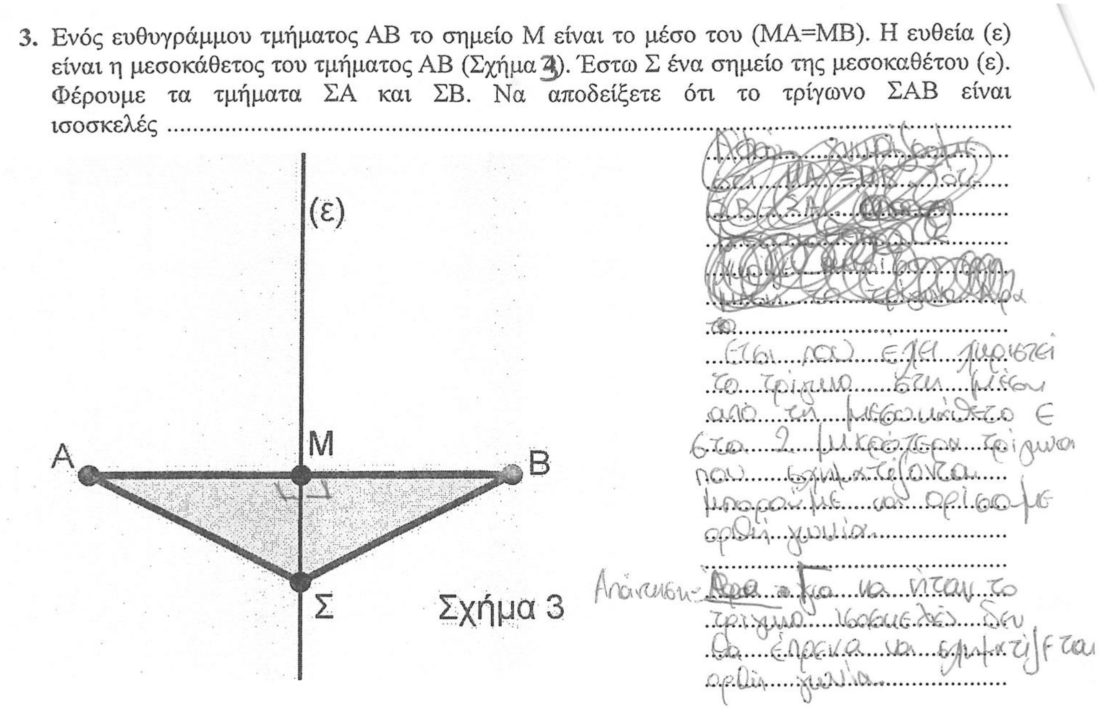


Figure 4.3. 7 Participant's [42] response to Question T13

In the way that the triangle has been divided

in the middle

by the perpendicular bisector ε

into 2 smaller triangles

that have been created

we can define

a right angle.

Answer: Thus for the triangle

to be isosceles

it ought not be formed

a right angle."

In the first 7 lines P[42] describes her perception of the figure, in which she sees in the perpendicular bisector and two "smaller" triangles. According to her formulation the "smaller" triangles are what allow us to "define" a right angle. Of course, there is no need to turn to the smaller triangles to define a right angle or the particular right angle in this case. The right angle or angles are there, in this case, because the straight line (ε) is indeed perpendicular to line segment AB at point M. Then P[42] writes the word "answer". The answer itself, in the text that follows is an argument in which P[42] justifies the opposite of what is asked. She concludes that the triangle is not an isosceles because a right angle is formed. The argument is arbitrary because she gives no plausible reason to justify the impossibility of the existence of an isosceles triangle due to the formation of a right angle. The first 7 lines of P[42]'s script have a perceptual character and are thus an E.P. proof scheme, while the second part after the word "answer" consists of arbitrary and unjustified reasoning and is thus an EC.NRS. proof scheme. Consequently the answer is classed as mixture of E.P. and EC.NRS. proof schemes.

P[69] (see Figure 4.3.8) writes:

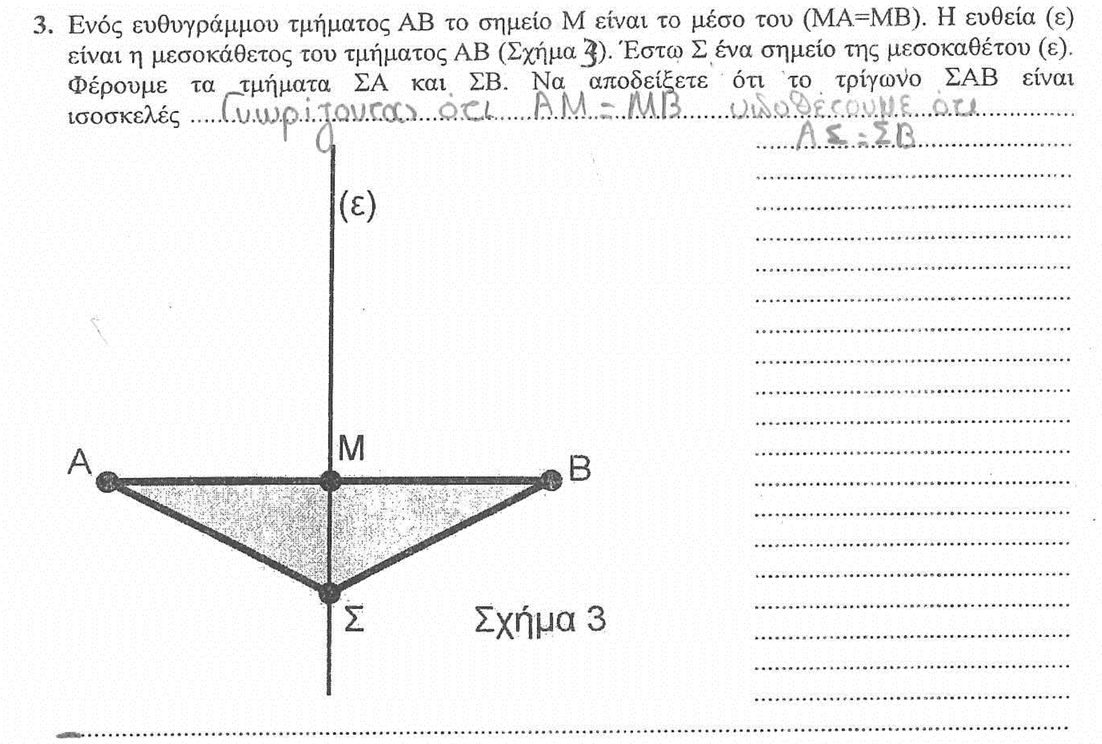


Figure 4.3. 8 Participant's [69] response to Question T13

Knowing that MA=MB we suppose that

ΑΣ=ΣΒ

This laconic formulation is an example of an arbitrary and irrelevant argument. The connecting element between *MA=MB* and *ΑΣ=ΣΒ* is the presence of the perpendicular bisector, but no reference is made to this fact. Of course, in whichever triangle not every median is drawn between equal sides as *ΣΜ* in Question T13. Thus not appealing to any valid property P[69] arbitrarily asserts that the conclusion is valid as a direct consequence of the hypothesis. This answer provides evidence of an EC.NRS. proof scheme and is classed accordingly.

Table 4.3.1 below, presents the general picture of the participants' answers showing that the number of D.T. answers has declined further in T13 to 14 (15.56%) in comparison to questions T11 and T12 which had 61 (67.78%) and 28 (31.11%) such answers respectively. This is to be expected because each of the questions T11, T12, T13 is more complicated than the previous one. T11 and T12 require the use of a combination of properties and calculations whereas Question T13 requires logical thinking, and the participants are facing such issues for the first time in their school life. Mathematics research provides evidence of the difficulty of proof when it has been taught, and it is all the more difficult when it has not been taught as in the case of this study.

Another aspect of the general picture is the appearance of the E.P. proof schemes in bigger numbers than in T11 and T12. In fact the biggest group, after NS, proves to be that of E.P. proof schemes at 21 (23.33%). Unlike Question T11 and T12, T13 has no arithmetical data and thus, all the proof steps should be based on properties and logic. In their efforts to articulate an adequate answer, the participants seek support from perceptions without justifying them logically, leading to substantial augmentation of the number of E.P. proof scheme. Parallel to the reinforcement of the E.P. numbers is the appearance of a small number of E.I. proof scheme. Any appearance of numerical substitution representing line segment lengths etc. is expected to be connected with such E.I. proof scheme. In other words the appearance of the various proof schemes, particularly this of the empirical class, is not independent of the structure of the question.

PROOF SCHEMES OBSEREVED IN THE RESPONSES TO QUESTION T13				
PROOF SCHEME	***FREQUENCY***	***CUMULATIVE FREQUENCY***	***RELATIVE FREQUENCY (%)***	***CUMULATIVE RELATIVE FREQUENCY (%)***
D.T.	*14*	*14*	*15.56*	*15.56*
D.T.-E.P.	*4*	*18*	*4.44*	*20.00*
D.T.-EC.NRS.	*6*	*24*	*6.67*	*26.67*
E.I.	*1*	*25*	*1.11*	*27.78*
E.I.-EC.NRS	*2*	*27*	*2.22*	*30.00*
E.P.	*21*	*48*	*23.33*	*53.33*
E.P.-EC.NRS.	*12*	*60*	*13.33*	*66.66*
EC.NRS.	*4*	*64*	*4.44*	*71.10*
NS	*26*	***90***	*28.89*	***99.99***
SUM	***90***		***99.99***	

Table 4.3. 1 Summary of Question T13 proof schemes

4.4 Analysis of responses to Question T14a

The curriculum specifies the definition of the elements of a triangle, i.e. its sides, angles, heights, angle bisectors and medians. The textbook of Year 7 gives the definitions of the midpoint (Vandoulakis et al., 2010, p. 160) and that of the median (ibid., p. 219).

This question was intended to collect information on whether the students knew what a midpoint and a median are. Whether they could manipulate a situation in which given data can be used to reach conclusions emerging from it on the basis of the definition of objects as the midpoint of a line segment and the median of a triangle.

In this section I present the analysis of answers to T14a, which fall into seven groups. Six groups of D.T., D.T.-EC.NRS., E.I., E.P., E.P.-EC.NRS., EC.NRS. proof schemes and seventh group that of NS. Below I present one example of each group in the order listed above.

P[59] (see Figure 4.4.1) writes:

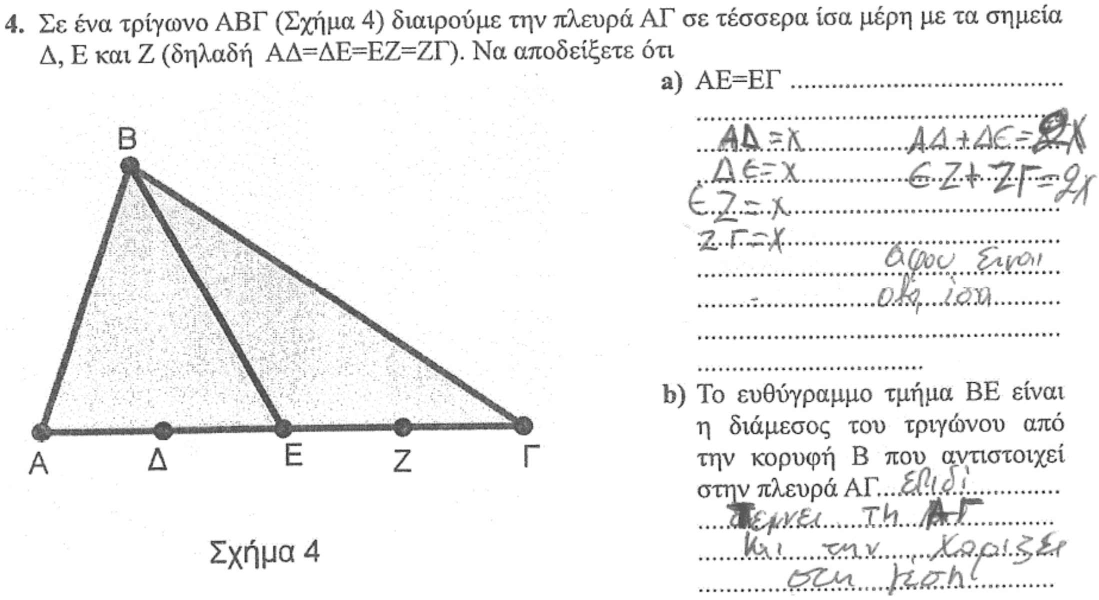


Figure 4.4. 1 Participant's [59] response to Question T14a

AΔ=x *AΔ+ΔE=2x*

ΔE=x *EZ+ZΓ=2x*

EZ=x

ZΓ=x

since they are

all equal

P [59] gives an algebraic answer similar to the proposed adequate answer, using the symbol x to express the common length of the line segments *AΔ=ΔE=EZ=ZΓ=x*.. Then the sums *AΔ+ΔE=2x* and *EZ+ZΓ=2x* become expressions of x. These two last equalities prove that the sums are equal. Without stating it explicitly, P[59] in writing *AΔ+ΔE=2x* and *EZ+ZΓ=2x* intends to express the respective line segments AE and EΓ in the form of the sum of the line segments and to show that these last two are equal. The sentence "*since they are all equal*", meaning *AΔ=ΔE=EZ=ZΓ=x*, is written to this purpose. Summing up, the answer is, although elliptical, adequate. Thus the answer provides evidence of D.T. proof scheme is classed accordingly.

P[45] (see Figure 4.4.2) writes:

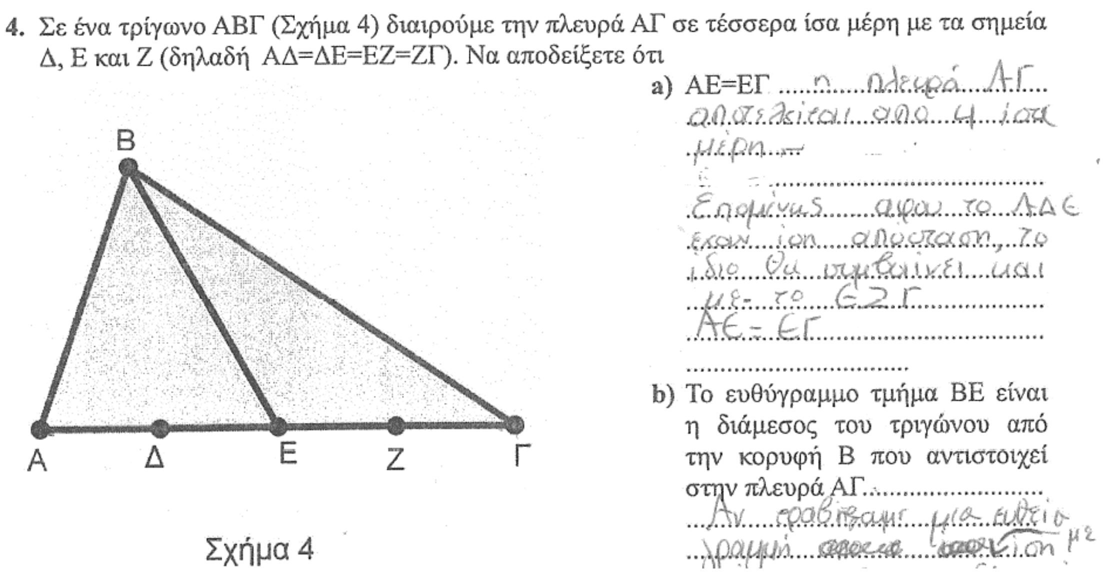


Figure 4.4. 2 Participant's [45] response to Question T14a

Side AΓ

is constituted of 4 equal

parts.

Thus since AΔE

have equal distance the

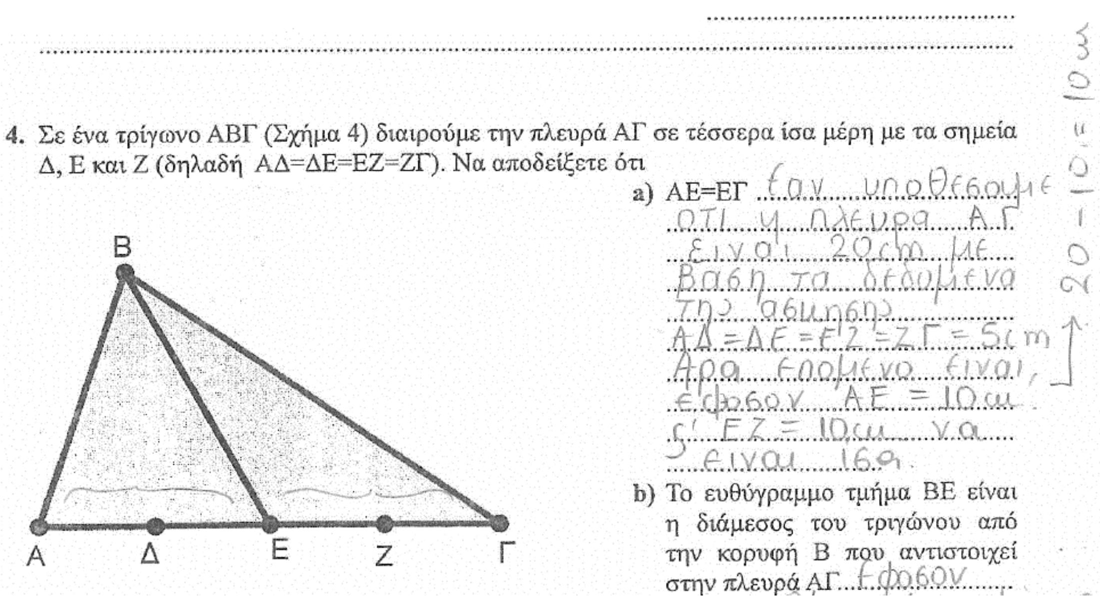
same would occur even

with EZΓ

AE=EΓ"

P[45] begins the proof with a description of the structure of line segment ΑΓ and continues "…*since AΔE have the same distance*…" obviously meaning, that the three points *A, Δ, E* and in that order are equidistant with each other i.e. *AΔ=ΔE* . The symbols *AΔE* used for this purpose is indeed ambiguous and arbitrary. The same is true of the next formulation '*the same would occur even with EZΓ*'. The conclusion *AE=EΓ* follows. Thus in writing *AΔE* and *EZΓ* the participant means *AE* and *EΓ* respectively. On the other hand, the essence of the thinking is correct and thus P[45]'s understanding of the conditions of the problem provides evidence of a D.T. proof scheme. At the same time the vagueness in the use of symbols provides evidence of the EC.NRS. proof scheme, so the Thus answer is categorised as a mixture of D.T. and EC.NRS. proof schemes.

P[77] (see Figure 4.4.3) writes:


4. Σε ένα τρίγωνο ΑΒΓ (Σχήμα 4) διαιρούμε την πλευρά ΑΓ σε τέσσερα ίσα μέρη με τα σημεία Δ, Ε και Ζ (δηλαδή ΑΔ=ΔΕ=ΕΖ=ΖΓ). Να αποδείξετε ότι

a) ΑΕ=ΕΓ εαν υποθεσουμε οτι η πλευρα ΑΓ ειναι 20cm με βαση τα δεδομενα της ασκησης ΑΔ=ΔΕ=ΕΖ=ΖΓ=5cm Αρα επομενο ειναι, εφοσον ΑΕ=10cm &' ΕΖ=10cm να ειναι ισα.

20-10=10cm

b) Το ευθύγραμμο τμήμα ΒΕ είναι η διάμεσος του τριγώνου από την κορυφή Β που αντιστοιχεί στην πλευρά ΑΓ εφοσον

Figure 4.4. 3 Participant's [77] response to Question T14a

If we assume

that side ΑΓ

is 20cm on the

basis of the data

of the exercise

ΑΔ=ΔΕ=ΕΖ= ΖΓ=5cm

Thus it is to conclude

since AE=10cm

& EZ=10cm[4] *that*

they are equal

20-10=10cm

Participant [77] assigns a numerical value to *ΑΓ=20cm* serves the purpose of illustrating the solution. The reasons for using an argument based on numbers are not clear, and nor is it clear why the participant does not make the next step, that is to generalise and thus offer a D.T. proof, as in the D.T. answers discussed previously. If

[4] Slight mistake. The correct equality is AE=EΓ

P[77] had taken it a step further and asserted that the same procedure would be valid independently of the numeric values for the lengths of the lines segments his answer could have seen as a kind of a generic proof in the spirit of Harel and Sowder (1998, 2007). In such a case the answer would have been characterised as a D.T. proof scheme, but the step in question is not taken. P[77] has not yet decisively freed himself from the assignment of arithmetic values, in contrast to participants that employ a generalised argument. However, the substitution of numerical values to variable magnitudes in order to achieve a solution or a proof is evidence of an E.I. proof scheme. Thus I characterised this answer as an E.I. proof scheme.

P[01] (see Figure 4.4.4) writes:

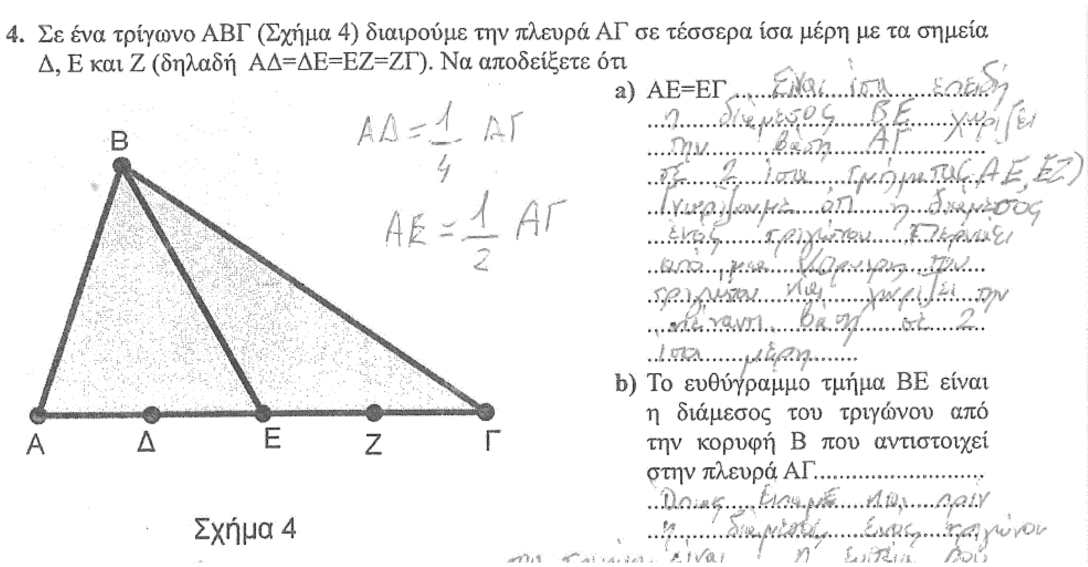


Figure 4.4. 4 Participant [01] response to Question T14a

They are equal because

$A\Delta = \frac{1}{4} A\Gamma$ *the median BE cuts*

$AE = \frac{1}{2} A\Gamma$ *the base AΓ*

in 2 equal segments (AE,EZ).

We know that the median

of a triangle passes

through a vertex of

the triangle and cuts

the base opposite the vertex in 2

equal parts

P[01] has noted beside the figure the equalities $A\Delta=\frac{1}{4}A\Gamma$ and $AE=\frac{1}{2}A\Gamma$ which are both true but unjustified, thus they have an empirical perceptual character. Furthermore in line 4 of his script P[01] refers in the parentheses to line segments *AE*,*EZ*. I assume that these line segments are not correctly written. Indeed the equal segments are not the written ones but *AE* and *EΓ*. Probably the participant intended to write these down but failed. Let's ignore this apparently minor mistake. The participant argues that *E* is the middle point of *AΓ* because *BE* is the median corresponding to it. Like a number of other participants P[01] has been carried away by the power of the figure. Consequently, without noticing it, P[01] accepts he property that is to be proved for the line segment *BE* or alternatively for the point *E*, in advance and does not use the given data at all. Any proof based on the perception or perceptions about the properties in a figure constitute substantial evidence of the E.P. proof scheme. Consequently the proof scheme in question has been characterised as E.P. proof scheme.

P [91] (see Figure 4.4.5) writes:

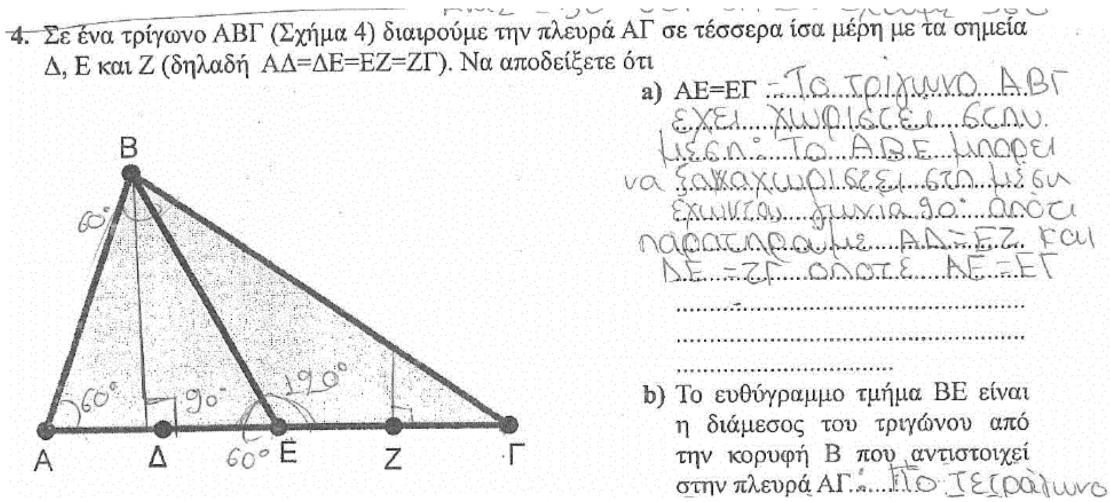


Figure 4.4. 5 Participant's [91] response to Question T14a

Triangle ABΓ

has been cut down

the middle: ABE can

be cut down the middle anew

having angle 90° so

we observe AΔ=EZ and

ΔE=ZΓ so AE=EΓ

P[91] begins his answer by asserting that triangle *ABΓ* "has been cut down the middle" and asserts that triangle *ABE* can also be "cut down the middle" presumably in the same spirit. This cut down the middle is connected to the as-yet unproved fact that E is the midpoint of A*Γ*. As one can see from the notation by the figure, the participant 'sees' triangle *ABE* as equilateral and *AΔ* as its height, and consequently as its median. The way he tries to sketch the height of triangle *ABE* from *B* and make it pass through *Δ*, although it does not necessarily do this, is interesting. These assertions and attempts, and the notation, reinforce the evidence of the answer's E.P. character. The participant concludes that *AΔ=EZ* and *ΔE=ZΓ* which is not relevant to the previous perception that triangle *ABE* is equilateral. However, the assertion that

ΑΔ=ΕΖ and *ΔΕ=ΖΓ* have been proved is an EC.NRS. quality, simply because the equality of these segments is given from the beginning and there is nothing to prove here. Thus the answer provides evidence of both E.P. and EC.NRS. proof schemes and has been characterised accordingly.

P [26] (see Figure 4.4.6) writes:

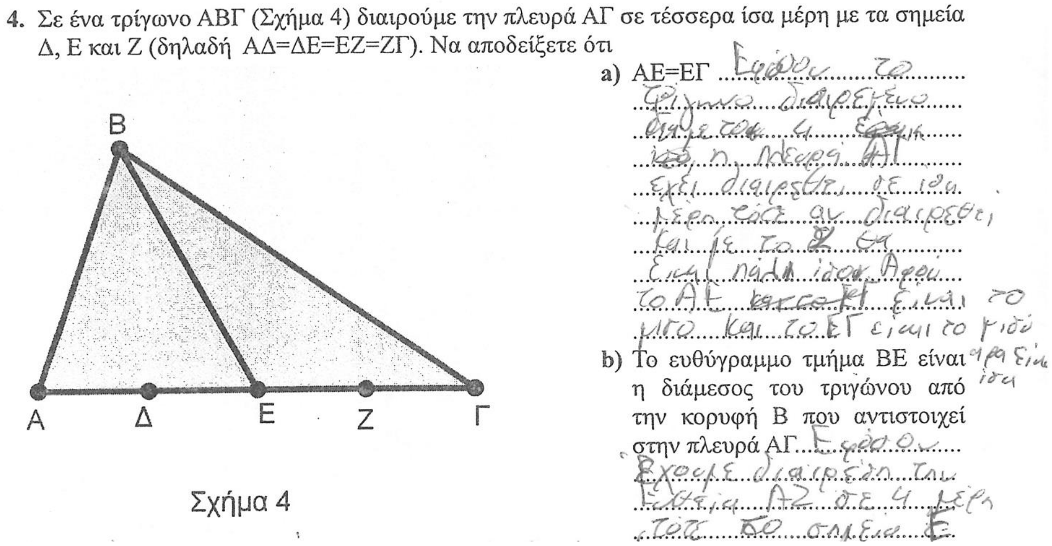


Figure 4.4. 6 Participant's [26] response to Question T14a

Since the
triangle is divided
by the number 4 (sco)
and side ΑΓ
has been divided into
equal parts then if it is divided
and by number 2 it will
have again equal. Since
ΑΕ (sco) is the
half and ΕΓ is the half
thus they are equal

P [26]'s first sentence of the proof is the phrase "…the triangle is divided by the number 4…". This sentence is an arbitrary distortion of the given situation. Indeed, side *ΑΓ*, and not the triangle, is divided into four equal parts by points *Δ, Ε, Ζ*. P[26] continues in the same vein. The implication he has to prove is as follows: if the side *ΑΓ* is indeed divided into four equal parts *ΑΔ=ΔΕ=ΕΖ=ΖΓ* by points *Δ, Ε, Ζ* then *Ε* is the midpoint of *ΑΓ*. But his formulation of the implication sounds as "…and side *ΑΓ* has been divided into equal parts; then if it is divided and by number 2 it will have gain equal". The arbitrary distortion is again clear. His conclusion is no less arbitrary. He writes "Since *ΑΕ* is the half and *ΕΓ* is the half thus they are equal'. But that is exactly the question: Why is *ΑΕ=ΕΓ*? The question is never answered. The comparison of P[26]'s formulations to the data and formulation of the problem lead me to the conclusion that the proof scheme here is an EC.NRS. one.

Table 4.4.1 illustrates the general picture of the proof schemes. The number of D.T. proof schemes has risen slightly to 22 (24.44%) answers 8 more than in T13.
On the other hand the empirical proof schemes persist. In fact, there are only 2 (2.22%) answers classed as E.I., but answers classed as E.P. stand at 24 (26.67%); and EC.NRS. appears in a total[5] of 32 (35.56%) answers. The problems with proof in the answers to Question T13, generally remains unchanged in the answers to T14a. Indeed, the two questions are similar in quality. There is no need for algebraic or arithmetic calculations instead demand logical steps from the data and the hypothesis to the conclusion. Questions T13 and T14a thus emphasise the students' problems

[5] Alone or in a mixture

PROOF SCHEMES OBSERVED IN THE RESPONSES TO QUESTION T14a				
PROOF SCHEME	***FREQUENCY***	***CUMULATIVE FREQUENCY***	***RELATIVE FREQUENCY (%)***	***CUMULATIVE RELATIVE FREQUENCY (%)***
D.T.	*22*	*22*	*24.44*	*24.44*
D.T.-EC.NRS.	*6*	*28*	*6.67*	*31.11*
E.I.	*2*	*30*	*2.22*	*33.33*
E.P.	*14*	*44*	*15.56*	*48.89*
E.P.-EC.NRS.	*10*	*54*	*11.11*	*60.00*
EC.NRS.	*16*	*70*	*17.78*	*77.78*
N.S.	*20*	*90*	*22.22*	***100.00***
		90		
SUM	***90***		***100.00***	

Table 4.4. 1 Summary of Question T14a proof schemes

with logical steps. However, almost one fourth of the participants managed to give an adequate answer and thus to deliver a D.T. proof scheme, a fact that must be seen as indicating readiness to deal with proof successfully.

4.5 Analysis of responses to Question T14b

The participants' answers fell into seven groups: D.T., D.T.-E.P., D.T.-EC.NRS., E.P., E.P.-EC.NRS., EC.NRS. and NS.

I present one example from each of the groups of various proof schemes in the above order.

P [02] (see Figure 4.5.1) writes:

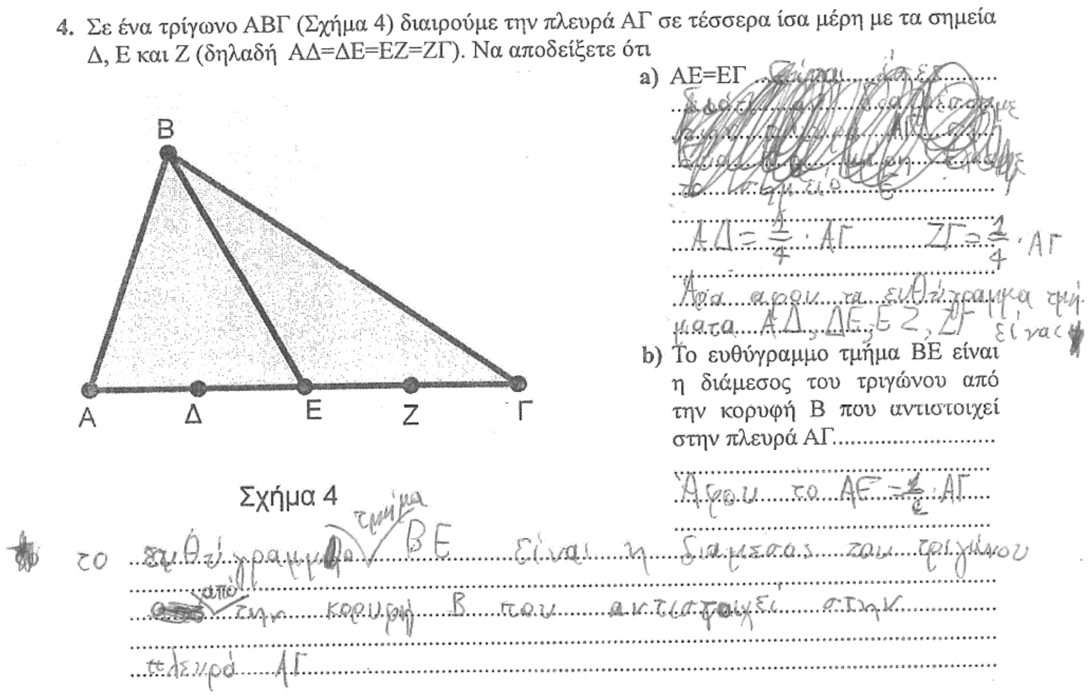


Figure 4.5. 1 Participant's [02] response to Question T14b

Since $AE = \frac{1}{2} A\Gamma$

(sco) the straight line segment BE is the median of the triangle

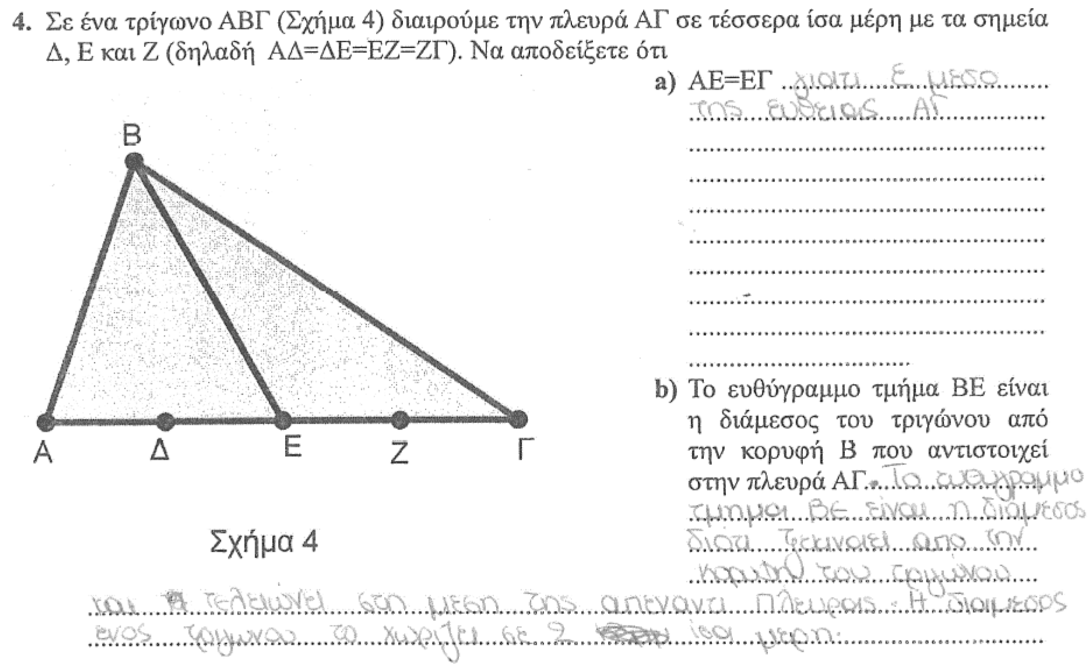
(sco) from the vertex B which corresponds to

the side ΑΓ"

By writing the equality $AE = \frac{1}{2} A\Gamma$ P[02] writes indirectly the fact that E is at the midpoint of ΑΓ. On this basis his conclusion about BE can be accepted as the definition of the median of a triangle and his answer is accepted as adequate and can be considered as a D.T. proof scheme. P[02]'s answer to T14a also falls under D.T.

The characterisation of the answers to T14a is independent of those of T14b. The answer to each sub-question has been categorised according to whether it can be accepted as adequate or not. However, for each participant quoted here I repeat the proof scheme in which fell the participant's answer to T14a.

P[23] (see Figure 4.5.2) writes:


4. Σε ένα τρίγωνο ΑΒΓ (Σχήμα 4) διαιρούμε την πλευρά ΑΓ σε τέσσερα ίσα μέρη με τα σημεία Δ, Ε και Ζ (δηλαδή ΑΔ=ΔΕ=ΕΖ=ΖΓ). Να αποδείξετε ότι

a) ΑΕ=ΕΓ διότι Ε μέσο της ευθείας ΑΓ

b) Το ευθύγραμμο τμήμα ΒΕ είναι η διάμεσος του τριγώνου από την κορυφή Β που αντιστοιχεί στην πλευρά ΑΓ. Το ευθύγραμμο τμήμα ΒΕ είναι η διάμεσος διότι ξεκινάει από την κορυφή του τριγώνου και τελειώνει στο μέσο της απέναντι πλευράς. Η διάμεσος ενός τριγώνου το χωρίζει σε 2 ίσα μέρη.

Figure 4.5. 2 Participant's [23] response to Question T14b

The straight line

segment BE is the median

because it starts from the

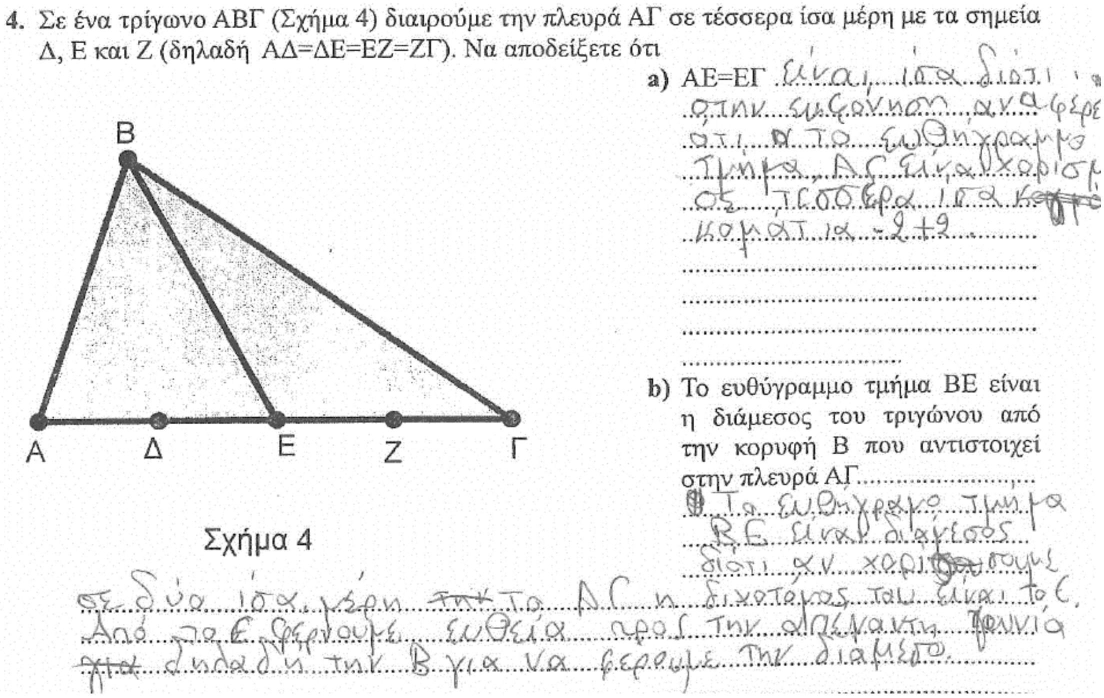
vertex of the triangle

and ends at the midpoint of the opposite side . The median

of a triangle divides it into 2 equal parts.

Participant's [23] proof is in two parts: The first part, from line one to line five, is an adequate answer regarding the definition of a median. Accordingly the assertion that *BE* is the median follows out of the fact that it connects a vertex of the triangle to the midpoint of the opposite side provides evidence of D.T. proof scheme. The second part of the proof begins in line five and ends in line six. The participant adds a comment on an alleged property of the median, namely that of dividing the triangle into two equal parts. However, this is not valid or at least it is only valid in the sense that triangles *ΒΑΕ* and *ΕΓΒ* are of equal areas. This property is not common knowledge at the beginning of Year 9 and the probability that P[23] is referring to it is unlikely. His comment is rather a false perception often encountered among in Years 7, 8 and 9 students when they try to describe a median of a triangle. In such cases an interesting change of formulation takes place: from the fact that *E* divides *ΑΓ* into two equal parts, students pass to the formulation "*ΒΕ* divides *ΑΓ* into two equal parts" and finally to "*ΒΕ* divides triangle *ΑΒΓ* into two equal parts". Thus the fact that *E* is the midpoint of *ΑΓ* is distorted to the perception that *ΒΕ* "divides" the triangle *ΑΒΓ* into two equal parts. Thus the participant's answer to T14b is classed as a mixture of D.T. and E.P. proof schemes.

P [78] (see Figure 4.5.3) writes:


4. Σε ένα τρίγωνο ΑΒΓ (Σχήμα 4) διαιρούμε την πλευρά ΑΓ σε τέσσερα ίσα μέρη με τα σημεία Δ, Ε και Ζ (δηλαδή ΑΔ=ΔΕ=ΕΖ=ΖΓ). Να αποδείξετε ότι

a) ΑΕ=ΕΓ

b) Το ευθύγραμμο τμήμα ΒΕ είναι η διάμεσος του τριγώνου από την κορυφή Β που αντιστοιχεί στην πλευρά ΑΓ

Figure 4.5. 3 Participant's [78] response to Question T14b

The line segment

BE is a median

because if we cut

ΑΓ in two equal parts its bisector is E.

From E we draw a line to the angle opposite to it

that is B to draw the median

P[78] tries to define segment *BE*. The meaning of his script is in the spirit of an adequate answer, that is, one should connect the midpoint of *ΑΓ*, which is *E*, with vertex *B*. Under this consideration the answer provides evidence of a D.T. proof scheme. However, the participant uses the word "bisector" for the point E, instead of "midpoint" for point E, and then writes "From *E* we draw a line to the angle opposite it" instead of that *BE* connects *E* with the vertex *B* opposite *ΑΓ*. In other words his formulation contains an arbitrary use of terms which is an element of the EC.NRS.

proof scheme. Thus on aggregate P[78]'s answer is classed as a mixture of D.T. and EC.NRS. proof schemes.

P[01] (see Figure 4.5.4) writes:

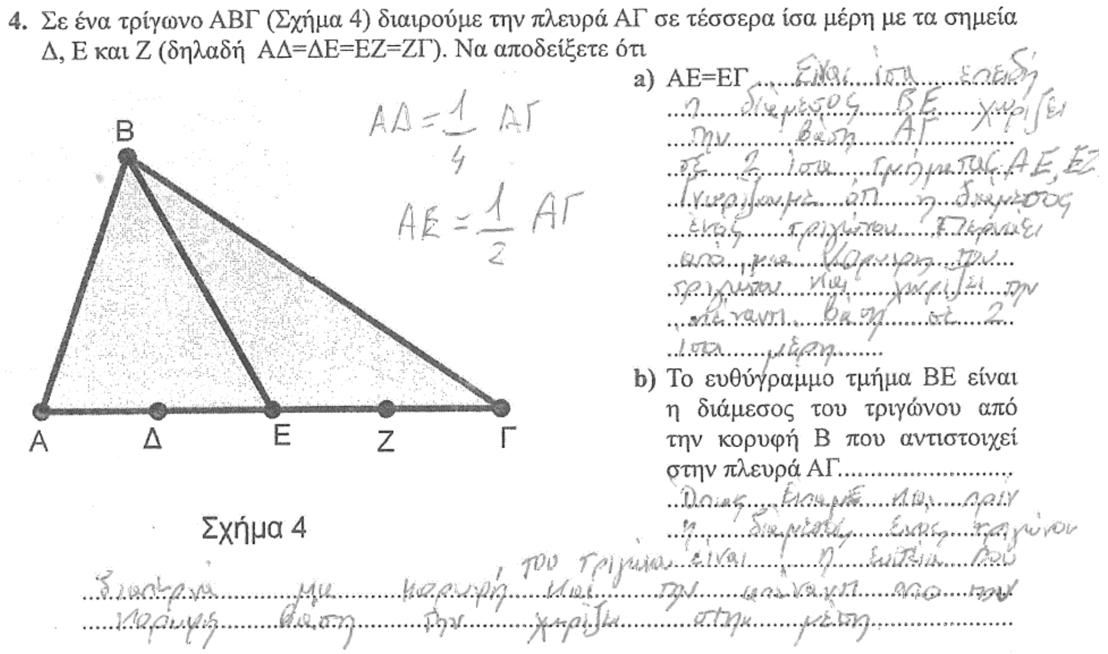


Figure 4.5. 4 Participant's [01] response to Question T14b

As we have said as well previously

the median of a triangle

is the straight line that

passes through a vertex of the triangle and cuts the opposite

to the vertex base in the middle

Participant [01] refers to his answer to Question T14a which, however, belongs to the E.P. proof schemes. In other words from the beginning this participant has seen the line segment *BE* as the median dividing the side to which it corresponds into two equal parts. His perception inverts the fact that first, *E* is the midpoint of *AΓ*, and then that *BE* is the median. Probably he does not adequately understand that he has to prove that *E* is the midpoint of *AΓ* in Question T14a and has already perception of *BE* as a median. Consequently in T14b when he is asked to prove that *BE* is the median he repeats the justification he gave in T14a. This argument that *BE* is the median because we see it as a median provides evidence of an E.P. proof scheme and is classed accordingly.

P[91] (see Figure 4.5.5) writes:

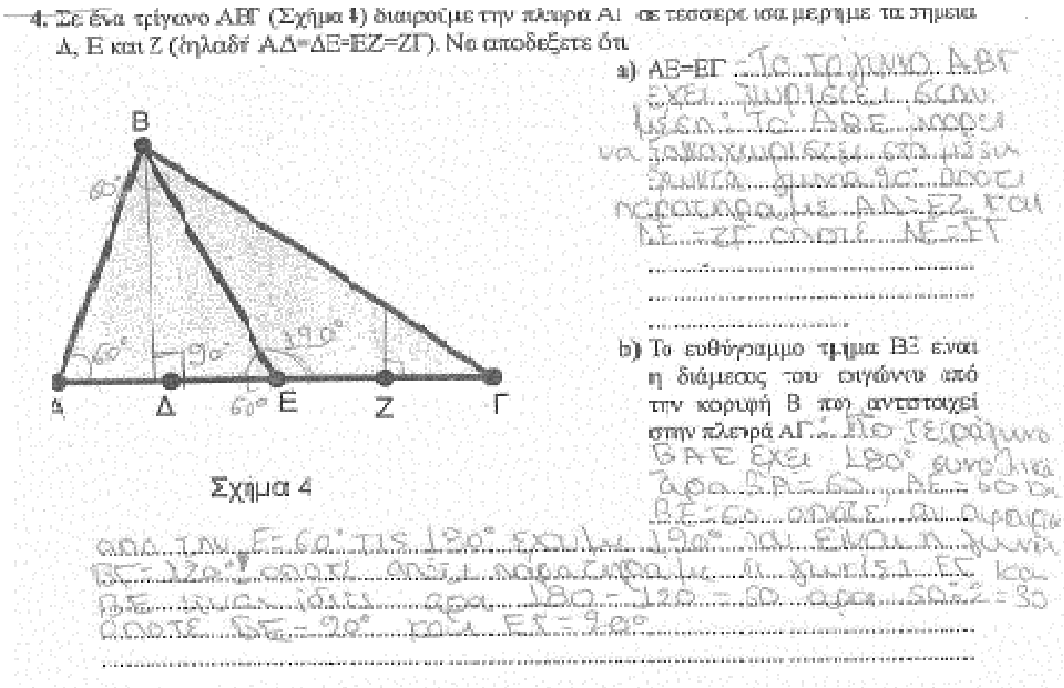


Figure 4.5. 5 Participant's [91] response to T14b

The square

BAE has 180° in aggregate

thus BA=60 and

BE=60 thus if I subtract

from E=60° the 180° we have 120° and that is the angle

BΓ=120° thus we observe the angles EΓ and

BE are the same thus 180-120=60 thus 60:2=30

thus BE=20° and EΓ=20°

P[91] perceives the triangle *ABE* as equilateral, but rather than calling it a triangle she calls it a square. She then asserts that *BA=60* and *BE=60*. Both equalities demonstrate an arbitrary use of the angle symbol and at the same time the meaning is ambiguous.

The arbitrary and ambiguous use of the angle symbol continues when the participant refers to angles *ΕΓ* and *ΒΕ* which, she claims, are "same", probably meaning "equal". In the last two lines of her script there are various arbitrary calculations without any validation. However, they seem to refer to angles *ΕΒΓ* and *ΒΓΕ*. Even if this were true it has nothing to do with the definition of the median. Summarising: P[91] perceives properties of the figure that are not given and could in no way be concluded from the data given. From this point of view her proof scheme is E.P.. At the same time she misuses symbols and terminology and makes arbitrary calculations, providing evidence of the EC.NRS. proof scheme. Consequently the answer of P[91] is classed as mixture of E.P. and EC.NRS. proof schemes. Let it be noted that the same mixture of proof schemes characterises her answer to T14a.

P[82] (see Figure 4.5.6) writes:

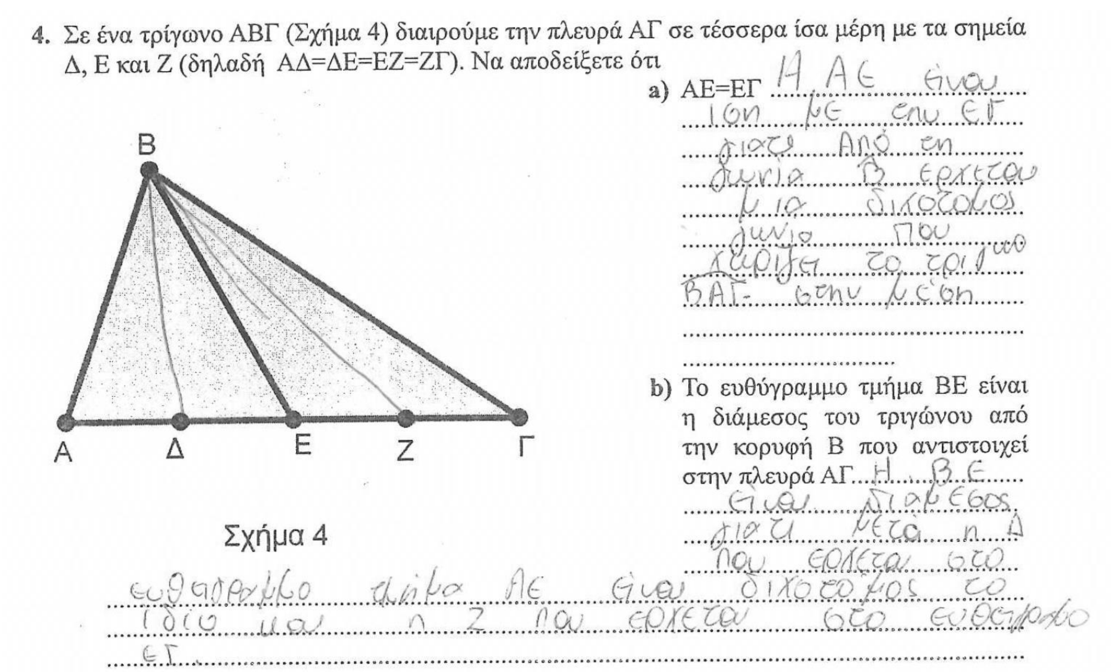


Figure 4.5. 6 Participant's [82] response to Question T14b

BE

is median

because after Δ

which comes to

the straight line segment AE is bisector the

same as well Z which comes to the straight line

EΓ.

P [82]'s answer is difficult to interpret which is why the English translation also appears random and is syntactically incoherent. Basically P[82] asserts that point *E* is between points *Δ* and *Z* which are midpoints of the straight line segments *AE* and *EΓ* respectively. Thus, according to P[82], *BE* is the median. Generally speaking, if *Δ* is the midpoint of *AE* and *Z* the midpoint of *EΓ* it does not follow that E is the midpoint of *AΓ*. In T14 it has been given that *AΔ=ΔE=EZ=ZΓ*. Only under this assumption is *E* in fact the midpoint of *AΓ*. On the other hand this proof had to be provided in part (a). P[82] confuses the words "median" and "bisector" using both to mean median. Thus his proof is of EC.NRS. character since there is no readily discernible meaning in what he writes and he misuses the mathematical terminology.

Table 4.5.1 illustrates the general picture of the answers to Question T14b.

PROOF SCHEMES OBSERVED IN THE RESPONSES TO QUESTION T14b				
PROOF SCHEME	***FREQUENCY***	***CUMULATIVE FREQUENCY***	***RELATIVE FREQUENCY (%)***	***CUMULATIVE RELATIVE FREQUENCY (%)***
D.T.	*17*	*17*	*18.89*	*18.89*
D.T.-E.P.	*2*	*19*	*2.22*	*21.11*
D.T.-EC.NRS.	*15*	*34*	*16.67*	*37.78*
E.P.	*7*	*41*	*7.78*	*45.56*
E.P.-EC.NRS.	*5*	*46*	*5.56*	*51.12*
EC.NRS.	*9*	*55*	*10.00*	*61.12*
N.S.	*35*	***90***	*38.89*	***100.01***
SUM	***90***		***100.01***	

Table 4.5. 1 Summary of Question T14b proof schemes

NS is the biggest group here, and rises considerably from 20 (22.22%) in T14a to 35 (38.89%) in T14b. The D.T. group follows in size, with 17 (18.89%) cases. Overall D.T. proof scheme appears the most 34 occurrences (37.78%) among the various proof schemes. The EC.NRS. proof scheme follows, at 29 (32.23%), and finally the E.P. proof scheme appears in total 14 (15.56%) times. The E.I. proof scheme is not present, a normal consequence of the structure of Question T14b which does not lend itself to empirical trial. The general picture is as expected because, as in T13 and T14a the participants have to provide proofs based on logical assumptions and definition without having been taught to do so, and thus a number of problems arise regarding the use of mathematical definitions, the properties of mathematical objects and mathematical terminology. Many participants fail to formulate their thoughts properly because of these problems, combined with their main difficulty in distinguishing between the data and the conclusion. However the presence of D.T. proof scheme either alone or in combination with others proof schemes is encouraging.

4.6 Analysis of responses to Question T15a

The Year 8 curriculum stipulates teaching the Pythagorean theorem and its converse, which are formulated in the textbook as follows:

> In every right-angled triangle the sum of the squares of the two perpendicular sides is equal to the square of the hypotenuse […]
> If in a triangle the square of the biggest side is equal to the sum of the squares of the two other sides the angle opposite to the biggest side is right.(Vlamos, Droutsas, Presvis, & Rekoumis, 2010, p. 127)

A number of activities and exercises using these two theorems can also be found in the Year 8 textbook (ibid., pp. 127-131). Question T15 was intended to collect information on whether the students could handle this unique case when proof had only been taught to them officially in Year 8.

Question T15 revealed a problem with characterising the participants' answers according to Harel and Sowder's taxonomy. This problem emerged in 25 answers to T14a where students compared 3^2+4^2 to 5^2, correctly found that $3^2+4^2=5^2$, and concluded that the triangle is a right-angled but made no reference to the converse Pythagorean theorem. There are also 26 such answers to T14b, in which 3^2+4^2 is compared to 6^2 to arrive at the conclusion that the triangle is not right-angled, but no reference is made to the Pythagorean theorem as an argument justifying this conclusion. I decided to accept the parts of these answers with correct calculation of 3^2+4^2 and correct comparison of the aforementioned sum to 5^2 or to 6^2 as evidence of a D.T. proof scheme. Where correct reference to the respective theorem is missing this was taken as evidence of the EC.NRS. proof scheme. I made this decision to retain consistency of the criteria used to classify the answers to previous Questions.

Under this assumption the analysis of the answers revealed six groups. Five were following proof schemes: D.T., D.T.-EC.NRS., E.P., E.P.-EC.NRS., EC.NRS. The sixth group is NS.

Below I present examples of answers belonging to various proof schemes in the order given above.

P[14] (see Figure 4.6.1) writes:

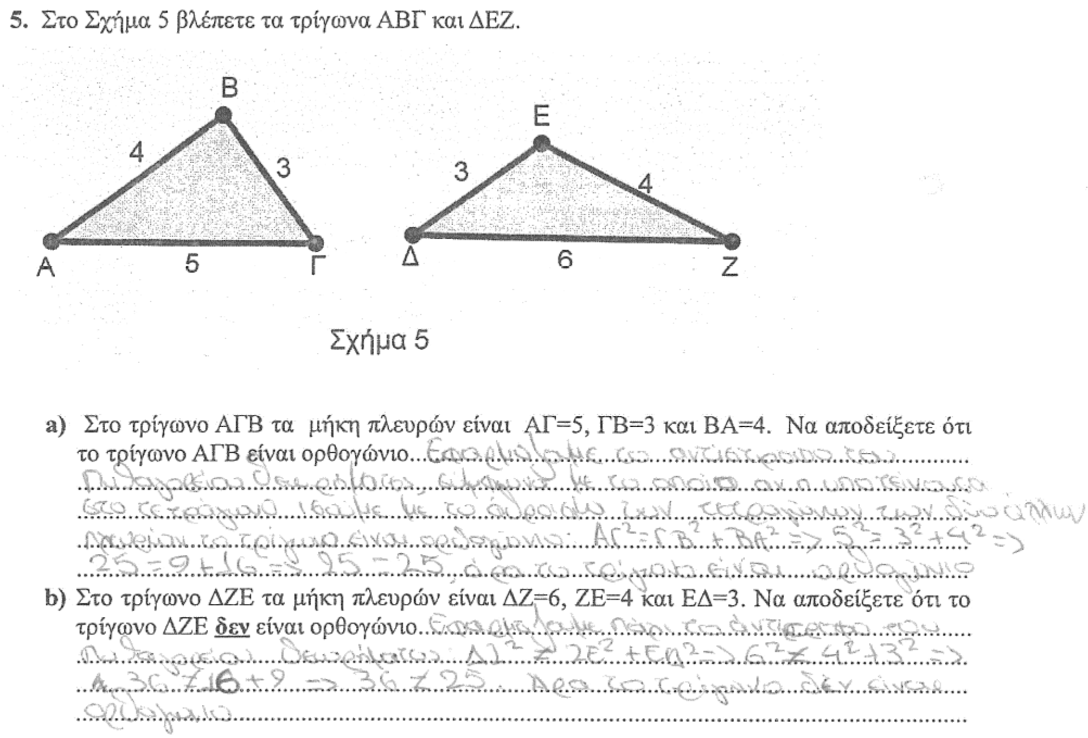


Figure 4.6. 1 Participant's [14] response to Question T15a

We apply the converse of the

Pythagorean theorem according to which if the hypotenuse *raised to the second power is equal to the sum of the squares of the two other sides the triangle is a right-angled one:* $A\Gamma^2 = \Gamma B^2 + BA^2 \Rightarrow$ $5^2=3^2+4^2 \Rightarrow 25=9+16$ *thus the triangle is a right-angled one*

Participant [14] invokes the appropriate theorem and verifies its validity. In doing so she calls the biggest side the hypotenuse. In terminology of mathematics books, including Greek mathematics textbooks, the word "hypotenuse" is used for the side of a right-angled triangle opposite to the vertex of its right angle of the triangle. In this respect the word, before having proved the existence of a right angle, is a slight inaccuracy which I deliberately ignored characterising the answer as a D.T. proof scheme.

P [75] (see Figure 4.6.2) writes:

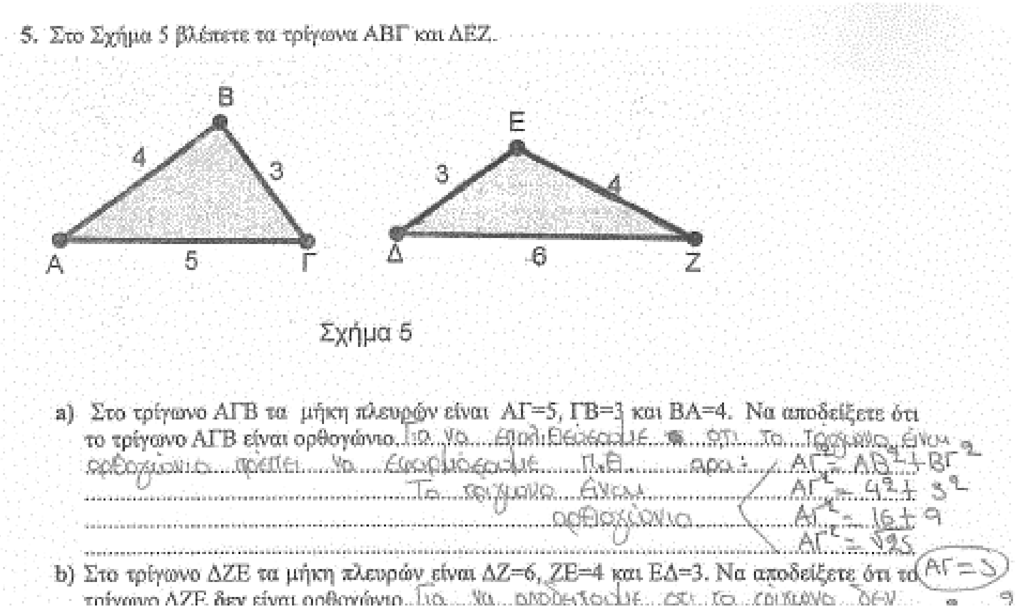


Figure 4.6. 2 Participant's [75] response to Question T15a

To verify that the triangle is a right-angled one we must apply the P.T. thus: the triangle is a right-angled one

$A\Gamma^2 = AB^2 + B\Gamma^2$

$A\Gamma^2 = 4^2+3^2$

$A\Gamma^2 = 16+9$

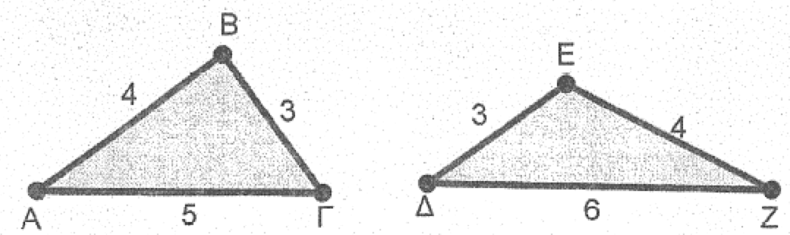
$$A\Gamma^2 = \sqrt{25}$$

$$A\Gamma = 5$$

P[75] verifies that triangle ABΓ is right-angled by applying the Pythagorean theorem (symbolised as P.T.). In the penultimate line P[75] writes “ $A\Gamma^2 = \sqrt{25}$ ”. I see these two points in his proof as evidence of the EC.NRS. proof scheme In fact the theorem to be applied is the converse of the Pythagorean theorem and in the equality $A\Gamma^2 = \sqrt{25}$ the symbol of the second power and of the square root are both misused since the equality should be in the form $A\Gamma = \sqrt{25}$. On the other hand P[75] knows how to check whether triangle ABΓ is a right-angled. Thus his proof also provides evidence of a D.T. proof scheme and his answer is characterised a mixture of D.T. and EC.NRS. proof schemes.

P[48] (see Figure 4.6.3) writes:

5. Στο Σχήμα 5 βλέπετε τα τρίγωνα ΑΒΓ και ΔΕΖ.



Σχήμα 5

a) Στο τρίγωνο ΑΓΒ τα μήκη πλευρών είναι ΑΓ=5, ΓΒ=3 και ΒΑ=4. Να αποδείξετε ότι το τρίγωνο ΑΓΒ είναι ορθογώνιο........ Η γωνια Β ειναι 90°

b) Στο τρίγωνο ΔΖΕ τα μήκη πλευρών είναι ΔΖ=6, ΖΕ=4 και ΕΔ=3. Να αποδείξετε ότι το τρίγωνο ΔΖΕ **δεν** είναι ορθογώνιο.... Καμια γωνια δεν ειναι ορθη - 90°

Figure 4.6. 3 Participant’s [48] response to Question T15a

(sco) *Angle B is 90°*

Participant [48] perceives the right angle in triangle *ΑΒΓ*, as the angle with the vertex at point *B* by just looking at the figure. All that we have here is evidence that the student knows what a right-angled triangle is: a triangle with one right angle. But P[48] answers the question asking which of the three angles of the triangle *ΑΒΓ* is the right-angled one by naming the angle $\hat{B}$, without any logical justification. In this respect one can reasonably claim evidence of an E.P. proof scheme, the main aspect of which is the perception of properties of plane figures from the shape they visually seem to have without logical justification.

P [91] (see Figures 4.6.4 & 4.6.5) writes:

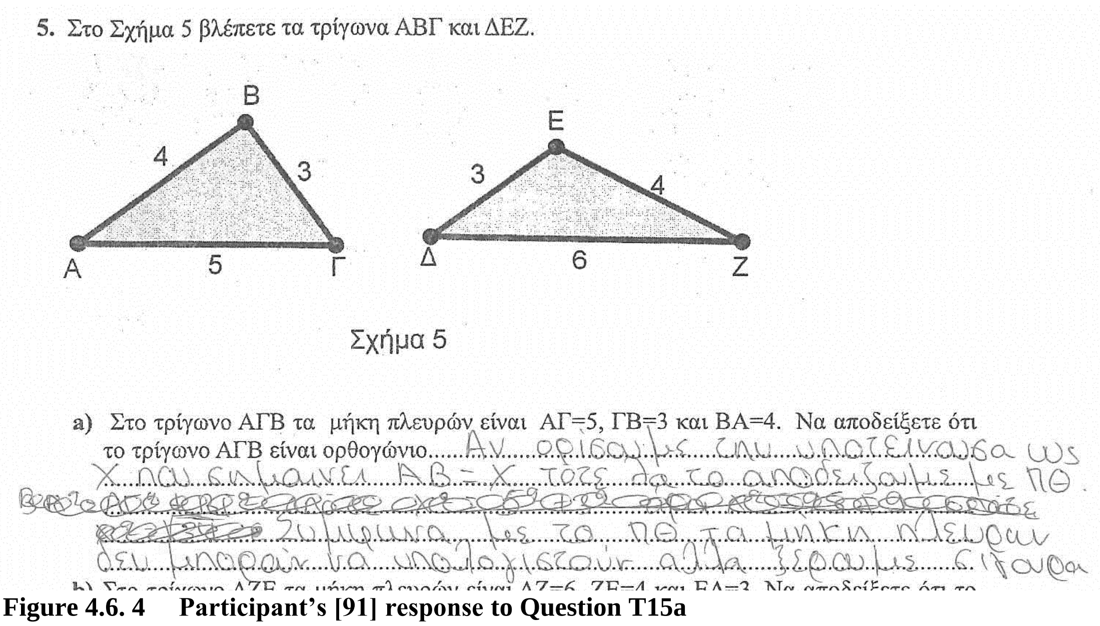


Figure 4.6. 4 Participant's [91] response to Question T15a

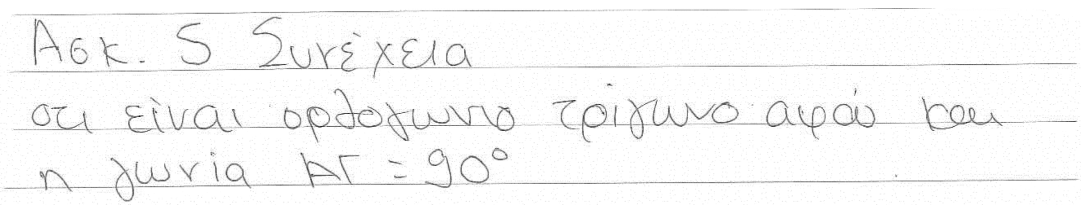


Figure 4.6. 5 Participant's [91] response to Question T15a

[Figure 4.6.4] If we name the hypotenuse

X the meaning of which is AB=X then we will prove it with PT.

According to PT the length of the sides

cannot be calculated but we know for sure

[Figure 4.6.5] that the triangle is a right one since

the angle AΓ=90°

Participant [91] gives to *AB* the symbolic name X writing $AB = X$, but in what follows use anywhere the symbol *X*. Thus remains incomprehensible the symbol's X meaning and seems to be just an arbitrary action. P[91] then asserts the impossibility of calculating the lengths of the sides of triangle *ABΓ*; however, these are given in the figure and consequently there is no need to calculate them. These two points in her answer, both of which are arbitrary, are evidence of an EC.NRS. proof scheme. Finally P[91] declares the triangle as right-angled because "*AΓ=90°*", thus continuing to develop an EC.NRS. proof scheme with this last arbitrary angle symbol comprising two letters. Probably she means angle $\widehat{AB\Gamma}$ but fails. While she sees that the angle is a right angle she does not feel any need to justify this with logical arguments. And thus this is an E.P. proof scheme, and in aggregate the answer is characterised by a mixture of E.P. and EC.NRS. proof schemes.

P[50] (see Figure 4.6.6) writes:

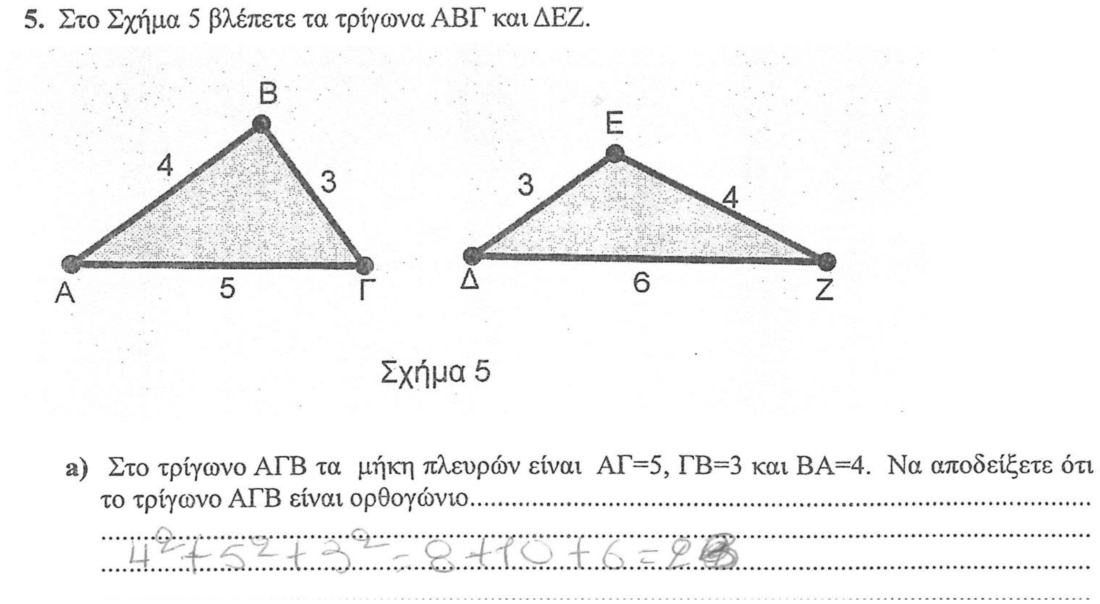


Figure 4.6. 6 Participant's [50] response to Question T15a

$4^2+5^2+3^2=8+10+6=25$

Participant [50] gives a very abbreviated EC.NRS. proof scheme. First, there is no word of explanation regarding the purpose of the calculation made; second, every power is wrongly calculated; and third, the sum of the three numbers is ambiguously written. I think these three points offer enough evidence to justify this single-line proof as an EC.NRS. proof scheme

Table 4.6.1 illustrates the general picture of the answers given to Question T15a.

The disproportionality of the 58 (64.44%) answers in the D.T.-EC.NRS. group of proof schemes compared to numbers in the other groups is due to the fact that some answers reveal practical knowledge of how to check whether a triangle is a right-angled one but do not clearly refer to the converse Pythagorean theorem, thus providing evidence of a D.T. proof scheme on the one hand and an EC.NRS. proof scheme on the other. Only five (6.67%) answers invoked the converse Pythagorean theorem and provided generally correct calculations, thus qualifying as D.T. proof schemes.

PROOF SCHEMES OBSERVED IN THE RESPONSES TO QUESTION T15a				
PROOF SCHEME	***FREQUENCY***	***CUMULATIVE FREQUENCY***	***RELATIVE FREQUENCY (%)***	***CUMULATIVE RELATIVE FREQUENCY (%)***
D.T.	*5*	*5*	*5.56*	*5.56*
D.T.-EC.NRS.	*58*	*63*	*64.44*	*70.00*
E.P.	*2*	*65*	*2.22*	*72.22*
E.P.-EC.NRS.	*1*	*66*	*1.11*	*73.33*
EC.NRS.	*5*	*71*	*5.56*	*78.89*
N.S.	*19*	***90***	*21.11*	***100.00***
SUM	***90***		***100.00***	

Table 4.6. 1 Summary of Question T15a proof schemes

4.7 Analysis of responses to Question T15b

I explained the problem emerged when categorising the answers to T15a and T15b in section 4.6. I repeat here only that any answer not appealing explicitly to the Pythagorean theorem is accepted as D.T.-EC.NRS. if it contains comparison of 6^2 to 3^2+4^2 correct calculations and the conclusion that the triangle is not a right-angled one.

Under these assumptions as in the case of T15a, the answers fell into six groups presented here: D.T., D.T.-EC.NRS., E.P., E.P.-EC.NRS., EC.NRS; the sixth group is that of NS.

I present examples of the answers in the same order.

P[53] (see Figure 4.7.1) writes:

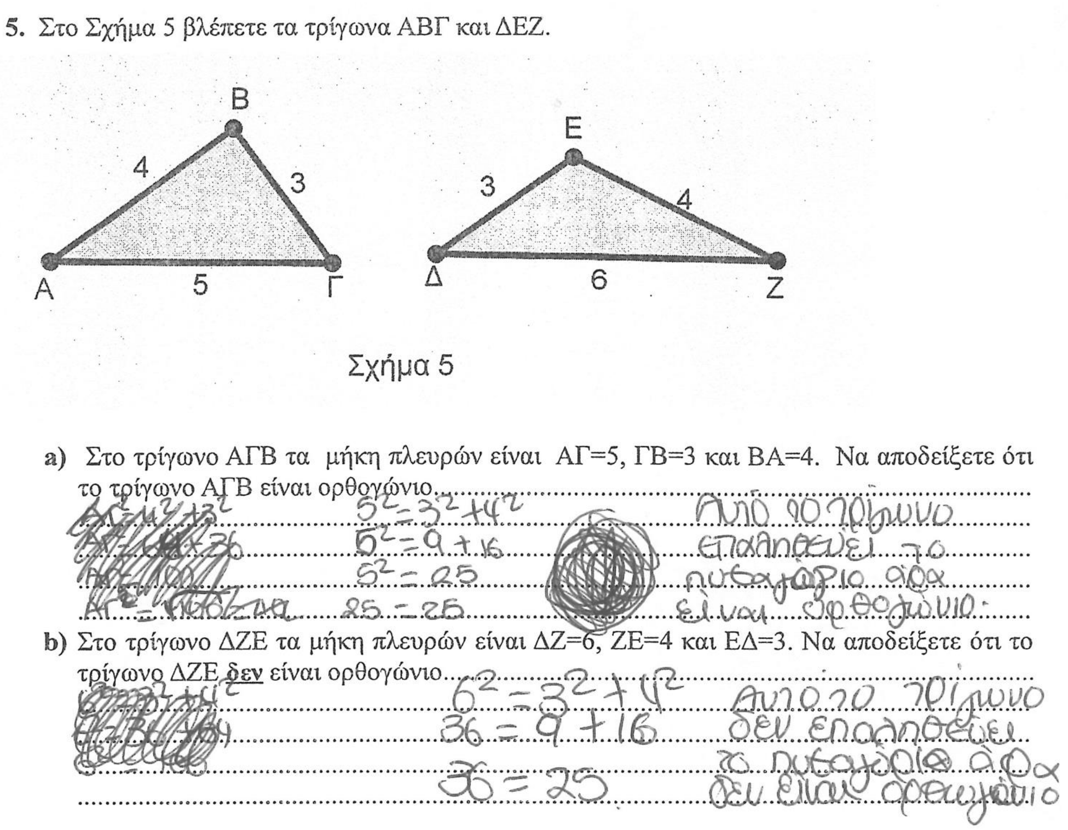


Figure 4.7. 1 Participant's [53] response to Question T15b

(sco) $6^2=3^2+4^2$ *this triangle*

(sco) $36=9+16$ *does not verify*

the Pythagorean theorem thus

36=25 *it is not a right-angled triangle*

Participant [53] checks if $6^2 = 3^2 + 4^2$ is valid and finds it is not; she thus concludes that the triangle is not a right-angled one since it does not satisfy the Pythagorean theorem. Her answer is adequate and is characterised as a D.T. proof scheme. It is worth noting that she uses the same reasoning when answering T15a. This illustrates the problem that arose in the categorisation of T15 answers. Many participants turn to the Pythagorean Theorem whether they have to check equalities as $5^2 = 3^2 + 4^2$ or as $6^2 = 3^2 + 4^2$. They have not understood that checking if $5^2 = 3^2 + 4^2$ means that the converse Pythagorean is applied. Neither have they understood that checking if $6^2 = 3^2 + 4^2$ which is not valid is equivalent to arguing that the triangle is not right-angled because otherwise the Pythagorean theorem would be valid which is not. According to the convention I have used throughout to classify the responses this answer to T15b is adequate and thus is a D.T. proof scheme.

P [72] (see Figure 4.7.2) writes:

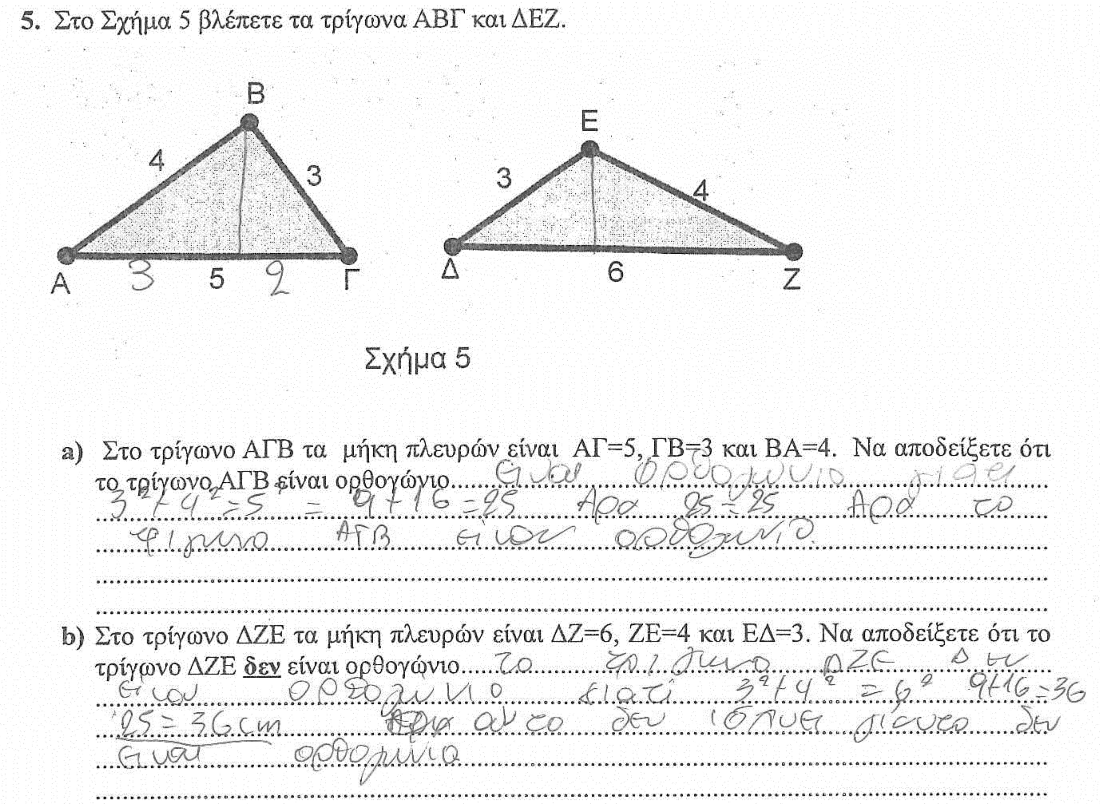


Figure 4.7. 2 Participant's [72] response to Question T15b

the triangle ΔZE is not

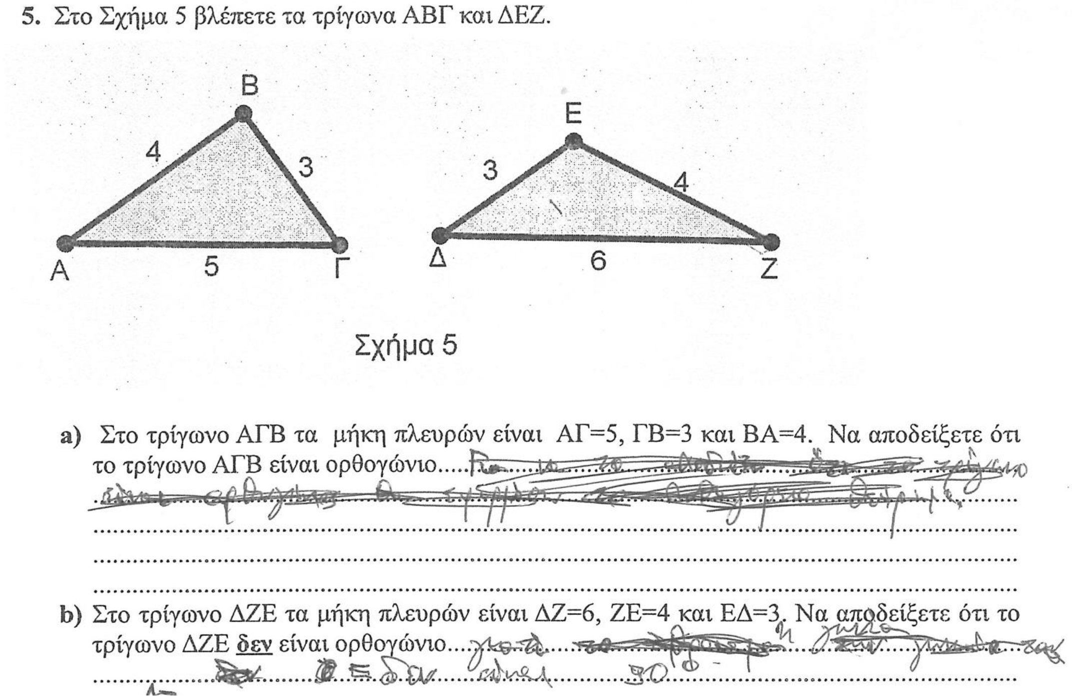
a right-angled one because $3^2+4^2=6^2$ *9+16=36*

<u>25=36cm</u> (sco) this is not valid thus it is not

a right-angled one

Participant [72] asserts correctly that triangle *ΔZE* is not a right-angled but does not refer to any theorem, just as in his answer to T15a, as well which is interesting. Thus both P[72] and P[53] illustrate the problem of the categorisation of proof schemes: few participants answered both T15a and T15b with reference to the correct theorem. Thus as defined earlier P[72]'s answer is a D.T.-EC.NRS. proof scheme because on the one hand he knows what to do and on the other he does not have a clear of which theorem is applicable.

P[39] (see Figure 4.7.3) writes:



5. Στο Σχήμα 5 βλέπετε τα τρίγωνα ΑΒΓ και ΔΕΖ.

a) Στο τρίγωνο ΑΓΒ τα μήκη πλευρών είναι ΑΓ=5, ΓΒ=3 και ΒΑ=4. Να αποδείξετε ότι το τρίγωνο ΑΓΒ είναι ορθογώνιο.....

b) Στο τρίγωνο ΔΖΕ τα μήκη πλευρών είναι ΔΖ=6, ΖΕ=4 και ΕΔ=3. Να αποδείξετε ότι το τρίγωνο ΔΖΕ **<u>δεν</u>** είναι ορθογώνιο....

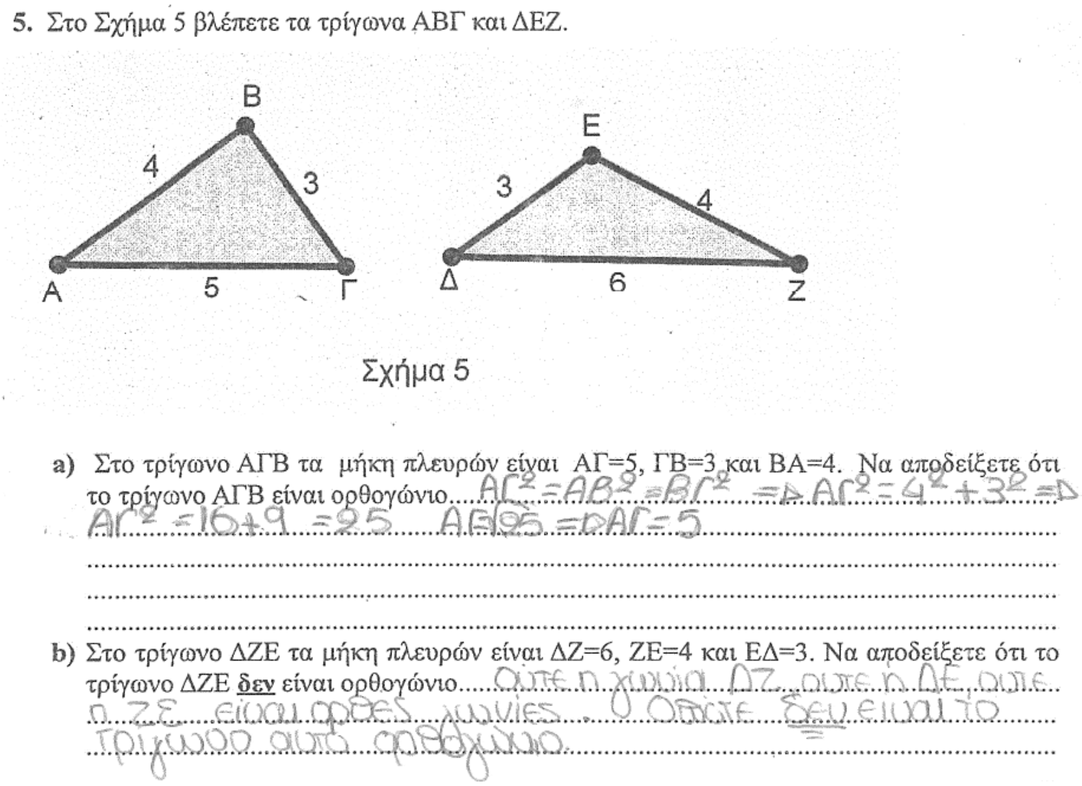
Figure 4.7. 3 Participant's [39] response to T15b

because (sco) the angle (sco)

(sco) E is not 90°

Participant [39] perceives angle $\hat{E}$ as not a right angle without feeling any need to logically justify his perception. Perceptions of properties of a figure that are not justified or not given as data indicate an E.P. proof scheme. P[39] does not refer to the other angles in the figure. Triangle *ΔEZ* could have been a right-angled triangle, for example, with vertex at *Δ* or *Z*. Probably P[39] perceives these angles as acute. It is worth noting that P[39] does not answer T15a at all. The crossed-out sentence in T15a as far as I can make out, read: "to prove whether the triangle is a right one I will apply the Pythagorean theorem". However, no application of the Pythagorean theorem is to be seen. In summary, the answer is classed as an E.P. proof scheme.

P[56] (see Figure 4.7.4) writes:


5. Στο Σχήμα 5 βλέπετε τα τρίγωνα ΑΒΓ και ΔΕΖ.

a) Στο τρίγωνο ΑΓΒ τα μήκη πλευρών είναι ΑΓ=5, ΓΒ=3 και ΒΑ=4. Να αποδείξετε ότι το τρίγωνο ΑΓΒ είναι ορθογώνιο..... $ΑΓ^2 = ΑΒ^2 + ΒΓ^2 \Rightarrow ΑΓ^2 = 4^2 + 3^2 \Rightarrow ΑΓ^2 = 16+9 = 25$ $ΑΓ = \sqrt{25} \Rightarrow ΑΓ = 5$

b) Στο τρίγωνο ΔΖΕ τα μήκη πλευρών είναι ΔΖ=6, ΖΕ=4 και ΕΔ=3. Να αποδείξετε ότι το τρίγωνο ΔΖΕ **δεν** είναι ορθογώνιο..... Ούτε η γωνία ΔΖ ούτε η ΔΕ ούτε η ΖΕ είναι ορθές γωνίες. Οπότε δεν είναι το τρίγωνο αυτό ορθογώνιο.

Figure 4.7. 4 Participant's [56] response to T15b

Neither angle ΔZ nor ΔE nor

ZE are right angles. Thus this

triangle is not a right-angled one

Participant [56] is more consistent than P[39] in her perception that triangle *ΔEZ* is not a right-angled. She refers to all the angles of triangle *ΔEZ* perceiving none of them as right angles. In doing so she misuses the angle symbol and symbolises them with two capital letters. Thus on the one hand her proof scheme is E.P. because she does not feel any need to justify, logically or by reference to the data given her statement that the angles of triangle *ΔEZ* are not right angles; on the other hand the misuse of symbols provides evidence of the EC.NRS. proof scheme. Thus her answer provides evidence of a mixture of the two. Her answer to T15a checks whether the triangle is a right-angled one by calculating 3^2+4^2 and finding it equal to 5^2 but again without explanation.

P[38] (see Figure 4.7.5) writes:

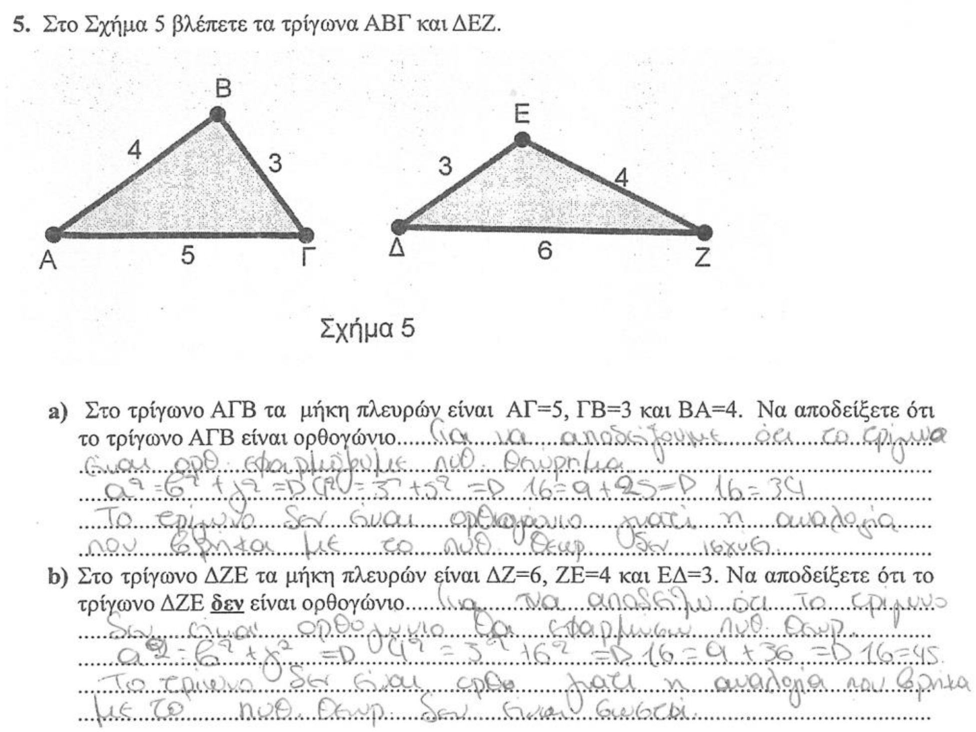


Figure 4.7. 5 Participant's [38] response to Question T15b

To prove that the triangle

is not a right-angled one I will apply the Pythagorean theorem

$\alpha^2 = \beta^2 + \gamma^2 \Rightarrow 4^2=3^2+6^2 \Rightarrow 16=9+36 \Rightarrow 16=45$

the triangle is not right-angled because the analogy I found

by the Pythagorean theorem is not correct.

P [38] first announces that she will apply the Pythagorean theorem to prove that the triangle is not a right-angled one, but in doing so she writes $4^2=3^2+6^2$ to arrive at $16=45$. She has not understood that when checking whether a triangle is a right-angled triangle, in all cases, the square of the length of the longest side is compared to the sum of squares of the lengths of the two remaining sides, because the biggest angle is to be found opposite the longest side of a triangle. What P[38] does reminds of relational and instrumental understanding. In the Harel and Sowder's taxonomy misuse of the criteria for judging whether a triangle is right-angled or not is evidence of EC.NRS. proof scheme. P[38] is also confused in her answer to T15a regarding of which side the square should be computed and compared to the sum of the squares of the remaining sides. This is a strong indication of systematic misuse of the criteria in question.

Table 4.7.1 illustrates the general picture of proof schemes observed in the answers to T15b. Table 4.7.1 shows that there are 26 (28.89%) D.T. answers to Question T15b compared to 5 (5.56%) for T15a. This indicates that the participants do not have a clear idea of when the Pythagorean theorem and when its converse is the correct argument to use. Furthermore only P[57] clearly appeals to the converse Pythagorean theorem in answering to T15a as well as to the Pythagorean theorem in answering T15b. The same participant demonstrates D.T. proof schemes in her answers to questions T11, T13, T15a, and T15b but does not answer Questions T12, T14, T16. Another element of the answers to T15b is the lack of E.I. proof schemes, because the nature of the question leaves little space for a proof scheme of this kind, while there are five instances of the E.P. proof scheme. Finally the EC.NRS. proof scheme appears in total more in T15a and in T15b than in the other questions because I had to distinguish the answers appealing to the appropriate theorem from those that did not. As a result the total number of EC.NRS. raised higher since every answer not appealing to a theorem is considered as EC.NRS.

PROOF SCHEMES OBSERVED IN THE RESPONSES TO QUESTION T15b				
PROOF SCHEME	***FREQUENCY***	***CUMULATIVE FREQUENCY***	***RELATIVE FREQUENCY (%)***	***CUMULATIVE RELATIVE FREQUENCY (%)***
D.T.	*26*	*26*	*28.89*	*28.89*
D.T.-EC.NRS.	*37*	*63*	*41.11*	*70.00*
E.P.	*4*	*67*	*4.44*	*74.44*
E.P.-EC.NRS.	*1*	*68*	*1.11*	*75.55*
EC.NRS.	*4*	*72*	*4.44*	*79.99*
N.S.	*18*	***90***	*20.00*	***99.99***
SUM	***90***		***99.99***	

Table 4.7. 1 Summary of Question T15b proof schemes

4.8 Analysis of responses to Question T16

At first sight the Question T16 is identical to Question T11 in this respect. There were two reasons for giving the participants a question that is almost identical to T11: (a) to test whether the slight change of context in comparison to Question T11 would provoke different answers and to what extent; the words proof or prove are not used and the triangle is an isosceles one; (b) to detect whether the students would be misled by the figure and perceive the triangle as a right-angled, because although it is not it bears a strong resemblance to a right-angled triangle. This idea, which I have mentioned in some occasions earlier, comes from Harel and Sowder (1998, p. 257) in whose example a student perceives a parallelogram as a square.

These answers fell into five groups: the four proof schemes: D.T., D.T.-EC.NRS., EC.NRS., E.P.-EC.NRS. , and the NS group.

Below I present examples of various proof schemes in the order given above.

P[72] (see Figure 4.8.1) writes:

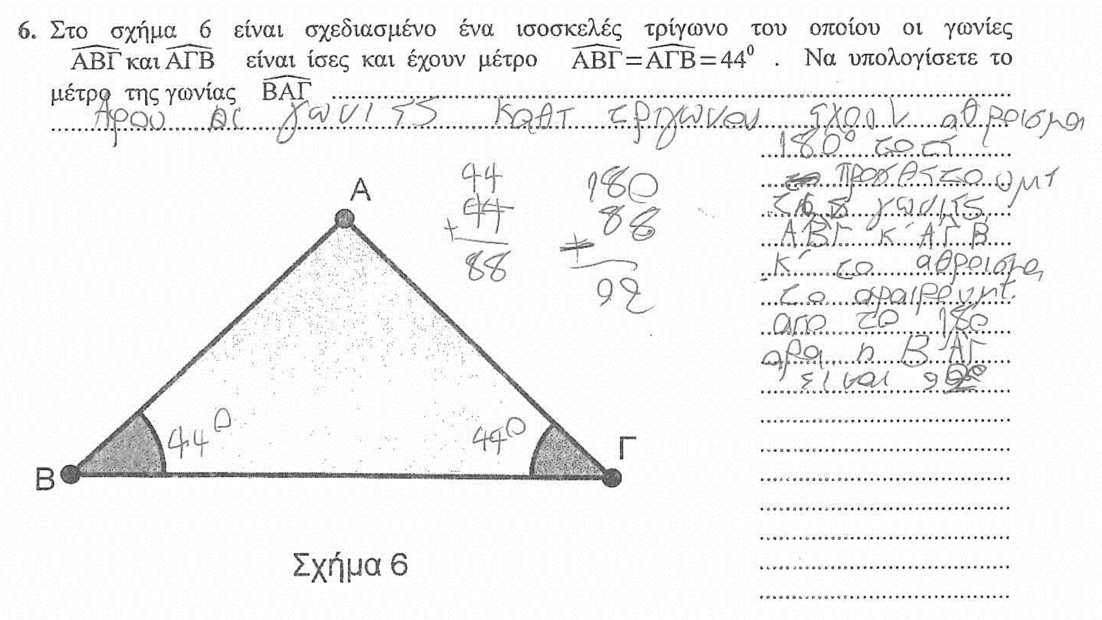


Figure 4.8. 1 Participant's [72] response to Question T16

Since the angles of every triangle have sum

180° then

(sco) we add

the angles

$\widehat{AB\Gamma}$ & $\widehat{A\Gamma B}$

& *the sum*

we subtract it

from 180

thus $\widehat{BA\Gamma}$

is 92°

P[72] gives an adequate answer, providing evidence of a D.T. proof scheme. Indeed P[72] appeals to the sum of the angles of a triangle. On this basis he subtracts the sum of angles $\widehat{AB\Gamma}$ and $\widehat{A\Gamma B}$ from *180* degrees to find *92°*. The calculations find the desired angle are visible beside the given figure of the triangle. P[72]'s response to Question T11 also included evidence of a D.T. proof scheme.

P [11] (see Figure 4.8.2) writes:

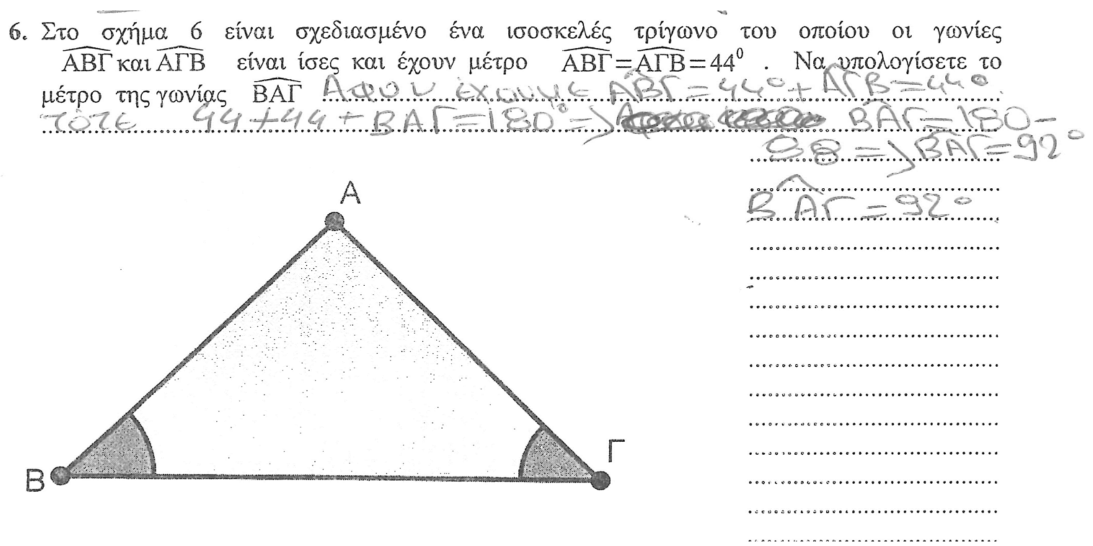


Figure 4.8. 2 Participant's [11] response to Question T16

Since we have $\widehat{AB\Gamma}$ *=44°+*$\widehat{A\Gamma B}$ *=44°*

then 44+44+ΒΑΓ=180°⇒ (sco) $\widehat{ΒΑΓ}$ *=180−*

88⇒ $\widehat{ΒΑΓ}$ *=92°*

$\widehat{ΒΑΓ}$ *=92°*

Participant [11] uses the theorem of the sum of the angles of a triangle to calculate the correct value of angle $\widehat{ΒΑΓ}$ *=92°*. In doing so, to write the sum of the given angles she uses the arbitrary symbolism “$\widehat{ΑΒΓ}$ *=44°+*$\widehat{ΑΓΒ}$ *=44°*”. Thus the answer provides evidence of a D.T. proof scheme, there is also evidence of an EC.NRS. proof scheme in the arbitrary symbolism for the sum of the given angles. Thus the answer is a mixture of D.T. and EC.NRS. proof scheme, as was her answer to T11, which also included arbitrary symbolism.

P [10] (see Figure 4.8.3) writes:

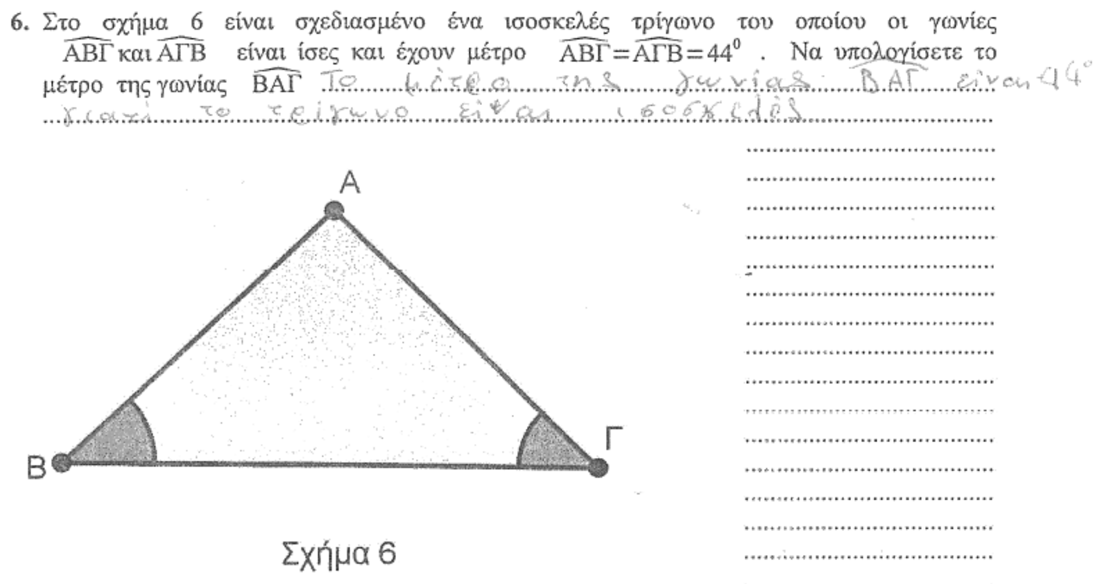

6. Στο σχήμα 6 είναι σχεδιασμένο ένα ισοσκελές τρίγωνο του οποίου οι γωνίες $\widehat{ΑΒΓ}$ και $\widehat{ΑΓΒ}$ είναι ίσες και έχουν μέτρο $\widehat{ΑΒΓ}=\widehat{ΑΓΒ}=44^0$. Να υπολογίσετε το μέτρο της γωνίας $\widehat{ΒΑΓ}$ Το μέτρο της γωνίας ΒΑΓ είναι 44° γιατί το τρίγωνο είναι ισοσκελές

Figure 4.8. 3 Participant’s [10] response to Question T16

The measure of angle $\widehat{ΒΑΓ}$ *is 44°*

because the triangle is isosceles

Participant’s [10] answer is not adequate. He asserts that “the measure of the angle $\widehat{ΒΑΓ}$ *is 44°* because the triangle is isosceles”. The triangle is in fact isosceles because, according to the data given it has already two equal angles $\widehat{ΑΒΓ}$ and $\widehat{ΑΓΒ}$ both measuring *44°*. Consequently if angle $\widehat{BA\Gamma}$ had been *44°* triangle *ΑΒΓ* would have been an equilateral triangle; but as an equilateral triangle, all three angles must be equal to *60°*. Thus the following facts escape P[10]'s attention: (i) a triangle with three equal angles cannot be an isosceles triangle but is equilateral; (ii) a triangle with three equal angles should have angles of *60°* ; and (iii) the sum of three angles measuring each one 44° is equal to 132° and not 180° as it should be. Thus P[10] misuses the theorem on the sum of the angles of a triangle as well as the terminology calling a triangle, necessarily equilateral according to his thoughts, isosceles. But arbitrary misuse of theorems and terminology is evidence of an EC.NRS. proof scheme.

P[92] writes (see Figure 4.8.4):

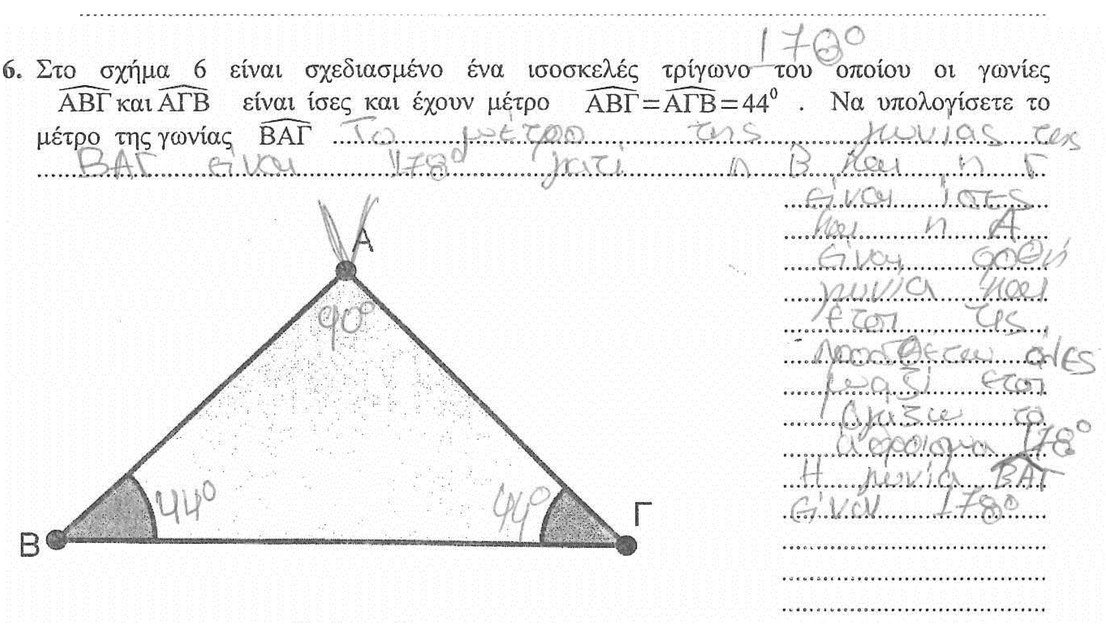


Figure 4.8. 4 Participant's [92] response to Question T16

The measure of the angle of

ΒΑΓ is 178° because B and Γ

are equal

and A

is a right

angle and

thus

I add them all together

I take the

sum 178°.

The angle $\widehat{BA\Gamma}$

is 178°

P[92]'s answer is not adequate. He thinks that he is being asked to find the sum of the angles of triangle ΑΒΓ, as this becomes obvious not only from his script but also from the numbers he has written on the figure. In the script he asserts that "…*B* and *Γ* are equal and *A* is a right angle…" and he has written *90°* in the figure in angle $\widehat{BA\Gamma}$. Additionally the symbol $\widehat{BA\Gamma}$ appears to mean all the angles of triangle *ΑΒΓ* to P[92], which is why he concludes his answer "Angle $\widehat{BA\Gamma}$ is *178°*". In fact *178°=44°+44°+90°*. Thus P[92] is the only participant who perceives angle $\widehat{BA\Gamma}$ as a right angle, in this respect offering evidence of an E.P. proof scheme. The rest of his proof is arbitrary and irrelevant: he adds up the angles of triangle *ΑΒΓ* to arrive at *178°*, which constitutes a misuse of the theorem on the sum of the angles of a triangle, which is always *180°*. Thus his arbitrary misuse of theorems and terminology provides evidence of an EC.NRS. proof scheme and his answer is characterised as a mixture of both EC.NRS. and E.P. proof schemes.

Table 4.8.1 illustrates the general picture regarding the various proof schemes given as answers to Question T16.

In general the theorem on the sum of the angles of a triangle is widely known and thus the 56 (62.22%) D.T. answers naturally result from this. However, some points regarding the D.T. answers given to both T11 and T16 are worth noting: 46 of the 61 (67.78%) participants who gave D.T. answers to T11 also gave a D.T. answer to T16. In other words, for various reasons 15 participants failed to articulate a D.T. proof answering T16: 5 gave an EC.NRS. answer and 10 gave an NS answer. If we reverse the direction of observation, of the 56 participants who gave T16 a D.T. answer, 46 participants also gave a D.T. answer to T11. The other 10 participants gave a D.T.-EC.NRS. answer for T11. The increased number in the NS answers to

PROOF SCHEMES OBSERVED IN THE RESPONSES TO QUESTION T16				
PROOF SCHEME	***FREQUENCY***	***CUMULATIVE FREQUENCY***	***RELATIVE FREQUENCY (%)***	***CUMULATIVE RELATIVE FREQUENCY (%)***
D.T.	*56*	*56*	*62.22*	*62.22*
D.T.-EC.NRS.	*2*	*58*	*2.22*	*64.44*
EC.NRS.	*11*	*69*	*12.22*	*76.66*
EC.NRS.-E.P.	*1*	*70*	*1.11*	*77.77*
N.S.	*20*	***90***	*22.22*	***99.99***
SUM	***90***		***99.99***	

Table 4.8. 1 Summary of Question T16 proof schemes

T16 is also important : there are 20 compared to 4 for T11. Of these, 2 participants answered neither T16 nor T11. Of the remaining 18, 10 gave D.T. and 8 D.T.-EC.NRS. answers to T11. The essence of these numbers is the instability that characterises the participants' attempts to articulate D.T. answers facing the definite questions. A slight change of context disoriented a number of participants. The E.I. and especially E.P. proof schemes are weakly represented: only one participant perceived the triangle T16 as right-angled.

4.9 Summary

This research project has scrutinised students' perceptions of proof at the beginning of Year 9 aimed by means of the T1pre-proof test. The small size of the research sample and small number of questions do not allow generalisation of the results. Below I briefly recapitulate some of my observations. Within the aforementioned methodological context I am going briefly to recapitulate some observations.

	QUESTIONS OF TEST T1							
PROOF SCHEMES	***T11***	***T12***	***T13***	***T14a***	***T14b***	***T15a***	***T15b***	***T16***
D.T.	*67.78*	*31.11*	*15.56*	*24.44*	*18.89*	*5.56*	*28.89*	*62.22*
D.T.-E.P.	*0.00*	*0.00*	*4.44*	*0.00*	*2.22*	*0.00*	*0.00*	*0.00*
D.T.-EC.NRS.	*24.44*	*33.33*	*6.67*	*6.67*	*16.67*	*64.44*	*41.11*	*2.22*
E.I.	*0.00*	*0.00*	*1.11*	*2.22*	*0.00*	*0.00*	*0.00*	*1.11*
E.I.-EC.NRS	*0.00*	*0.00*	*2.22*	*0.00*	*0.00*	*0.00*	*0.00*	*1.11*
E.P.	*0.00*	*0.00*	*23.33*	*15.56*	*7.78*	*2.22*	*4.44*	*0.00*
E.P.-EC.NRS.	*2.22*	*1.11*	*13.33*	*11.11*	*5.56*	*1.11*	*1.11*	*1.11*
EC.NRS.	*1.11*	*8.89*	*4.44*	*17.78*	*10.00*	*5.56*	*4.44*	*10.00*
NS	*4.44*	*25.56*	*28.89*	*22.22*	*38.89*	*21.11*	*20.00*	*22.22*
SUM	***99.99***	***100.00***	***99.99***	***100.00***	***100.01***	***100.00***	***99.99***	***99.99***

Table 4.9. 1 Percentages of proof schemes observed per Question of test T1

Table 4.9.1 illustrates the whole picture of test T1. In the first column each of the rows from 3 to 11 contain a name of a proof scheme or mixture of proof schemes observed by the analysis of the students' scripts. In the second row, each of the columns from 2 to 8 contains the name of a respective question of the T1 test. Each cell formed by the aforementioned rows and columns contains the percentage reached by the respective proof scheme or mixture of proof schemes in the respective question.

The most commonly-encountered proof schemes are the D.T. followed by D.T.-EC.NRS mix and then by NS. There is a rather weak presence of the rest of proof schemes and mixtures of proof schemes in the research sample. Charts 4.9.1 and 4.9.2 show the number of answers evidencing D.T. proof scheme per Question and total D.T. appearance per Question. In the bar of each chart one can read the corresponding percentage. The number of D.T. per Question is an indicator of the participants' readiness and preparedness to work with proof issues. Thus specifically in what regards the D.T. proof scheme (see charts 4.9.1 and 4.9.2) it can be said that:

- The number of D.T. answers is diminishing when the participants have to answer with logical arguments combining properties and given data in order to reach a conclusion. Nevertheless, the fact that proof is not yet taught to them underlines the importance of the fraction of them that managed to deliver proofs of this quality even so. Besides it had to be expected that students not yet taught proof should have the most problems dealing with questions demanding logical thinking.
- The number of D.T. answers raise when the participants have to answer questions where calculations based on widespread knowledge is needed, as the sum of the angles in a triangle.

- In what concerns T15a and T15b the ‘irregularity’ observed in D.T. answers is explicable as follows: The participants practically seem to ignore what the converse of the Pythagorean Theorem is. It seems that for the majority of participants if a relation has the form $a^2+b^2=c^2$ and either they have to apply it or test its validity, for them it is the Pythagorean theorem. Thus in T15b where indeed the Pythagorean theorem has to be invoked the numbers of D.T. are bigger than those of T15a where the converse of the Pythagorean theorem has to be invoked.
- Comparing T11 and T12 one can observe immediately that the change of context for applying the same theorem on the sum of the angles in a triangle, changes to a certain extent the D.T. number of answers.
- The same as in the previous comment is valid when one compares T11 and T16. This time, however, the reduction in D.T. answers is substantially smaller due in all likelihood to simpler and only change from a scalene triangle to an isosceles one. Anyway it is there signalling that even in a small number there are participants that cannot deal successfully with small context changes of the in respect with the applying of the same principle.

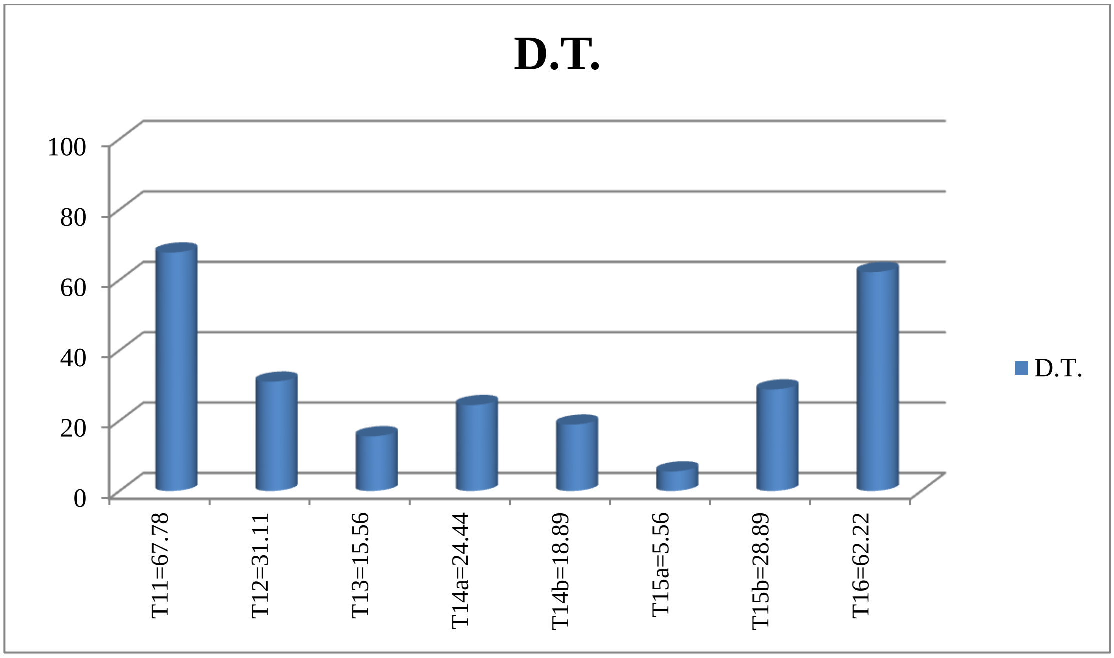


Chart 4.9. 1 D.T. proof scheme percentage per Question in bar chart form

Last but not least, the Harel and Sowder's taxonomy of proof schemes proves to be applicable even in Greek educational environment. The particular aspects of their presence need in the future to be further analysed but D.T., EC.NRS., E.I., E.P. proof schemes seem to characterize the answers of the participants even though E.I. and E.P. are of relatively low numbers. The various proof schemes that have been so far

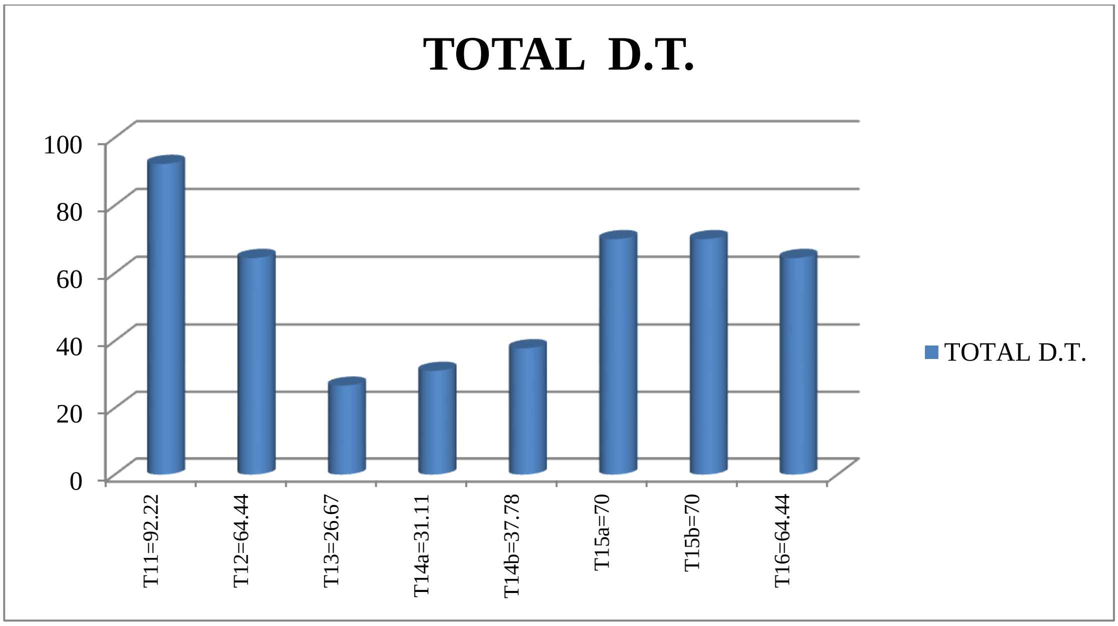


Chart 4.9. 2 Total D.T. proof scheme percentage per Question in bar chart form

encountered are functions of the Questions. In other words some proof schemes have not been encountered because the Questions did not leave much space for them. I speak here of EC.A. and EC.R. proof schemes because the D.A. proof schemes were excluded from the beginning.

I rest the case of further conclusions for the last chapter and I pass now to chapter five where I present the analysis of the T3 test.

CHAPTER 5: ANALYSIS OF T3 DATA

5.0 Introduction

In this chapter I present the analysis of the T3 test, which was taken by 85 of the 92 Year 9 students at the beginning of May 2011. T3 aimed to investigate the students' ability to prove algebraic relations such as identities and solve geometrical problems involving, for example, congruency of triangles theorems. The relevant material, including about 22 hours of lessons on proof, had been taught between the end of October 2010 and March 2011.

The presentation of T3 is as follows: first I present each Question and a brief adequate answer. This is followed by selected examples of answers evidencing different proof schemes according to the Harel and Sowder's taxonomy. The concluding section includes general comments on the participants' answers and a table containing the numerical data from the characterisations of the student answers, grouped by proof schemes (or combinations of). I have only used combinations of up to two proof schemes.

As mentioned before the use of the symbols of implication and logical equivalence are not taught systematically either before or during Year 9. Consequently I do not take their use into account when I classify a proof as containing evidence of the D.T. proof scheme group, if the answer is otherwise adequate.

5.1 Analysis of responses to Question T31

This algebra question was intended to explore how well the students had learned to use fundamental algebraic identities and symbols, such as the square root symbol. The underlying aim was to explore whether student answers – by trying specific values for *a*, and *b* – would contain evidence of the empirical inductive (E.I.) proof scheme. For example, some students, seeing the relation $a^2+b^2=5^2$, may think that numbers *a*, and *b* have the values *3* and *4* or *4* and *3*. Such a perception is probably due to the fact that a triangle with sides of lengths *3*, *4*, *5* is a right triangle and thus reminds students of $3^2+4^2=5^2$ of the Pythagorean Theorem.

Substituting the values 3, and 4 for variables *a*, and *b* would lead me to characterise an answer as E.I.

I found evidence of six proof schemes: D.T., D.T.-EC.NRS., E.I., E.I.-EC.NRS., and EC.NRS. and NS. In the following I present examples of the various proof schemes in that order.

P[08] (see Figure 5.1.1) writes:

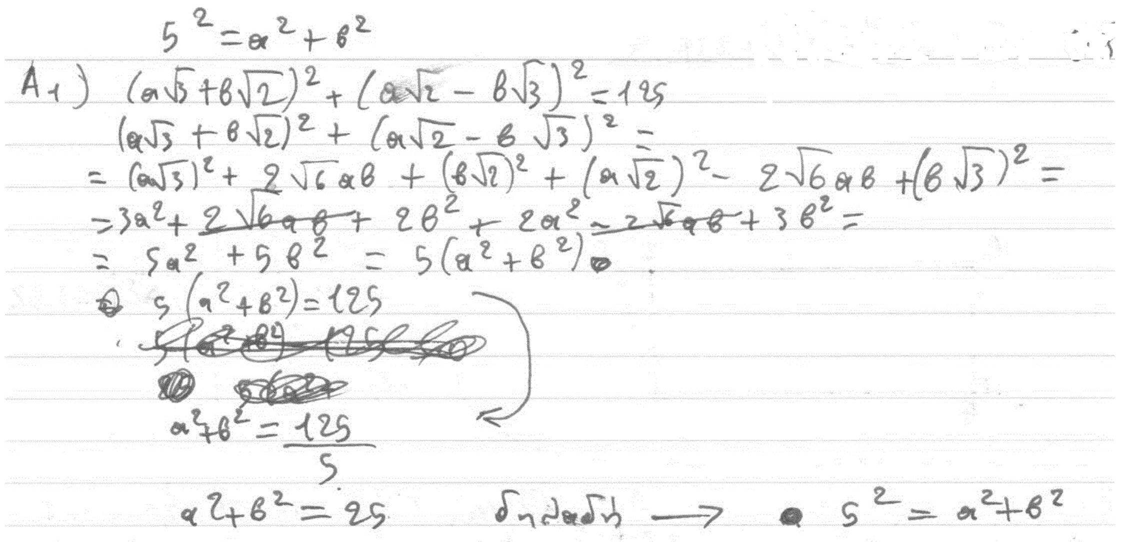
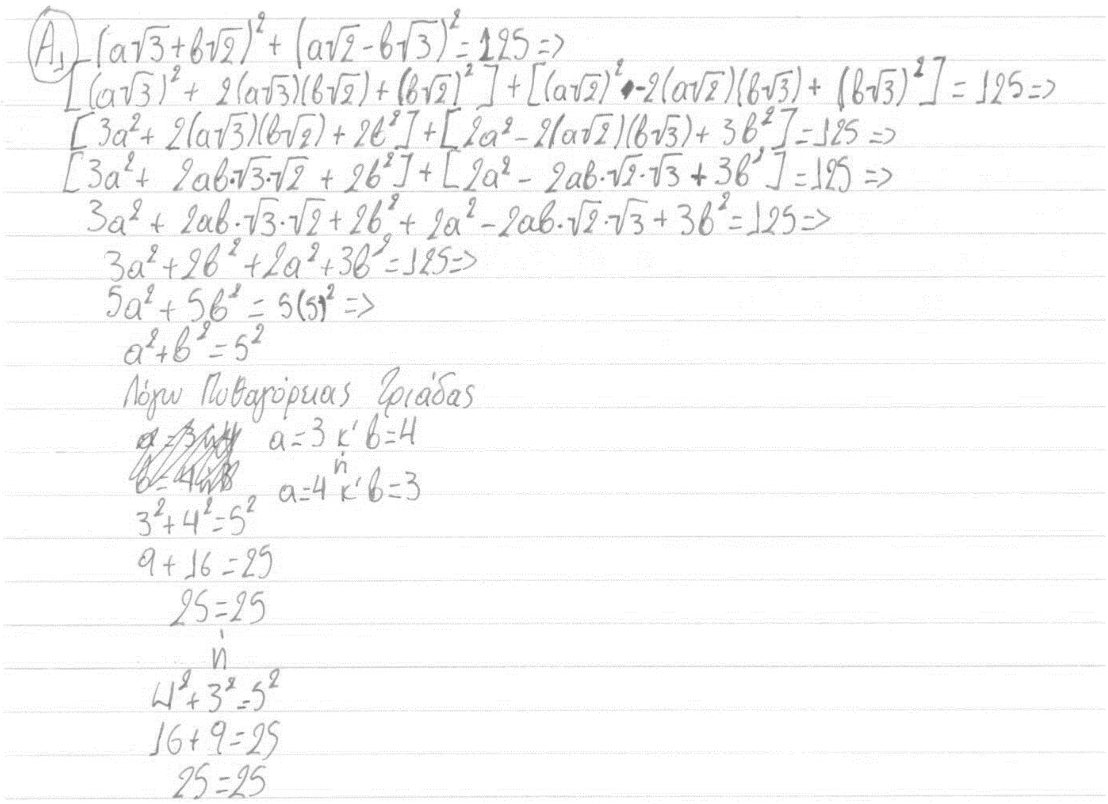


Figure 5.1. 1 Participant's [08] response to Question T31

Participant [08] gives an adequate answer to T31. First he gives the relations $5^2=a^2+\beta^2$ and $\left(\alpha\sqrt{3}+\beta\sqrt{2}\right)^2+\left(\alpha\sqrt{2}-\beta\sqrt{3}\right)^2=125$ and then takes the left side of the latter and expands the identities, making the proper reductions and finding $5(a^2+\beta^2)$ which is correct. Then he writes $5(a^2+\beta^2)=125$, by which he means that the left side, which has been transformed to $5(a^2+\beta^2)$, must now be equal to 125. From $5(a^2+\beta^2)=125$ he concludes that $5^2=a^2+\beta^2$. There he stops because this is the first given relation. Indeed there is a problem of logical equivalence which I put aside, because P[08] shows that he can use the symbol of square root correctly, knows how to expand the identities $(A\pm B)^2$ and operates flawlessly. In this respect this answer provides evidence of a D.T. proof scheme.

P[37] (see Figure 5.1.2) writes:


(A₁) $(a\sqrt{3}+b\sqrt{2})^2+(a\sqrt{2}-b\sqrt{3})^2=125 \Rightarrow$

$[(a\sqrt{3})^2+2(a\sqrt{3})(b\sqrt{2})+(b\sqrt{2})^2]+[(a\sqrt{2})^2-2(a\sqrt{2})(b\sqrt{3})+(b\sqrt{3})^2]=125 \Rightarrow$

$[3a^2+2(a\sqrt{3})(b\sqrt{2})+2b^2]+[2a^2-2(a\sqrt{2})(b\sqrt{3})+3b^2]=125 \Rightarrow$

$[3a^2+2ab\cdot\sqrt{3}\cdot\sqrt{2}+2b^2]+[2a^2-2ab\cdot\sqrt{2}\cdot\sqrt{3}+3b^2]=125 \Rightarrow$

$3a^2+2ab\cdot\sqrt{3}\cdot\sqrt{2}+2b^2+2a^2-2ab\cdot\sqrt{2}\cdot\sqrt{3}+3b^2=125 \Rightarrow$

$3a^2+2b^2+2a^2+3b^2=125 \Rightarrow$

$5a^2+5b^2=5(5)^2 \Rightarrow$

$a^2+b^2=5^2$

Λόγω Πυθαγόρειας Τριάδας

$a=3$ κ' $b=4$

ή

$a=4$ κ' $b=3$

$3^2+4^2=5^2$

$9+16=25$

$25=25$

ή

$4^2+3^2=5^2$

$16+9=25$

$25=25$

Figure 5.1. 2 Participant's [37] response to Question T31

The E.I proof scheme has a very strong presence in P [37]'s answer. Indeed, having given a proof to the Question T31, P[37] continues to assign values for *α* and *β*, (*α=3* and *β=4*; and *α=4* and *β=3)*. In line 9 of his script he writes "*Because of the Pythagorean Triad…*" aiming to justify what follows, and continues towards verifying that numbers *3* and *4* can be accepted as values for *α* and *β*. Probably he has been influenced by the Pythagorean Triad *3, 4, 5* and so he finds it natural to substitute definite values for ***a*** and ***b***. What seems to escape his attention is that numbers *α* and *β* are real according to Question T13. For example, one could have observed that $(\sqrt{2})^2+(\sqrt{23})^2=25$ where both values $\sqrt{2}$ and $\sqrt{23}$ are irrational, i.e. real numbers. The conjecture that P[37], and other participants who offered the same justification for the values of *α* and *β,* may think that numbers are integer or rational always and probably have not understood the existence of irrational numbers could be plausible. Thus P[37]'s need to substitute integer values for a and b provides evidence of an E.I. proof scheme. However he gives an adequate answer regarding the proof of the given relation which also provides evidence of the D.T. proof scheme. Under these considerations P[37]'s answer is classified as containing evidence of a mixture of E.I. and D.T. proof schemes.

P[85] (see Figure 5.1.3) writes:

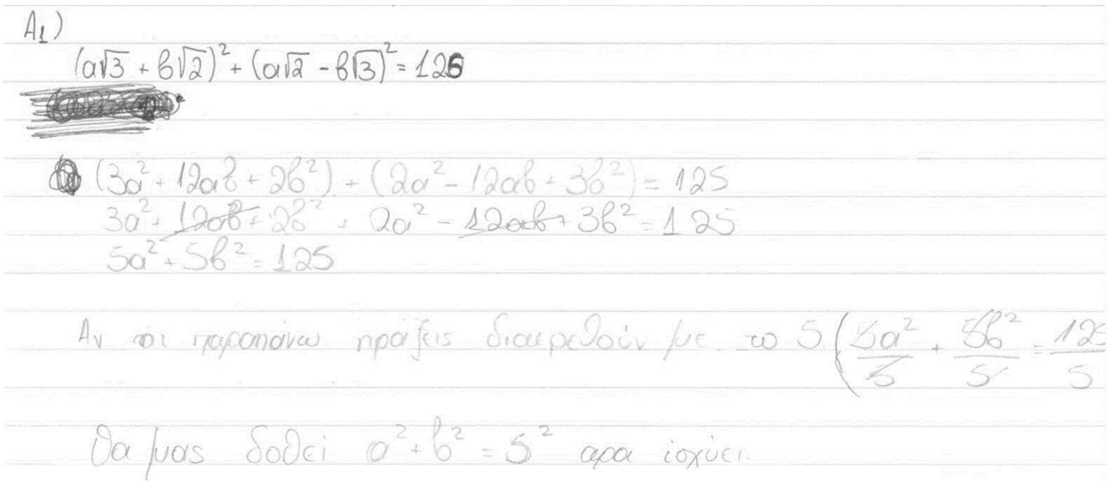


Figure 5.1. 3 Participant's [85] response to Question T31

Although P[85]'s answer seems to have characteristics of the D.T. proof scheme there are also signs of arbitrary use or misuse of symbols. P[85] expands the parentheses but fails to use the symbol for the product of the square roots correctly. Thus the term $2(a\sqrt{3})(b\sqrt{2})$ in the first parenthesis takes the irrelevant form *12αβ.* The mistake is repeated in the expansion of the second parenthesis giving $-12\alpha\beta$. Probably P[85] thinks that the product of two square roots leads to the elimination of the square root symbol, ignoring the fact that the radicands must be the same for the elimination to be valid as in the case of $\sqrt{2}\,\sqrt{2} = (\sqrt{2})^2 = 2$. Thus the misuse of the radicals by P[85] seems to relate to this perception of their properties . The opposite signs of the previous terms in question make their sum equal to zero; thus the final result is not affected by the mistakes. P[85]'s answer provides evidence that he understands what must be done to prove the validity of the given relation and his answer provides evidence of an EC.NRS. proof scheme in his use of symbols. Under these considerations the answer is categorised as a mixture of D.T. and EC.NRS. proof schemes.

P[05] (see Figure 5.1.4) writes:

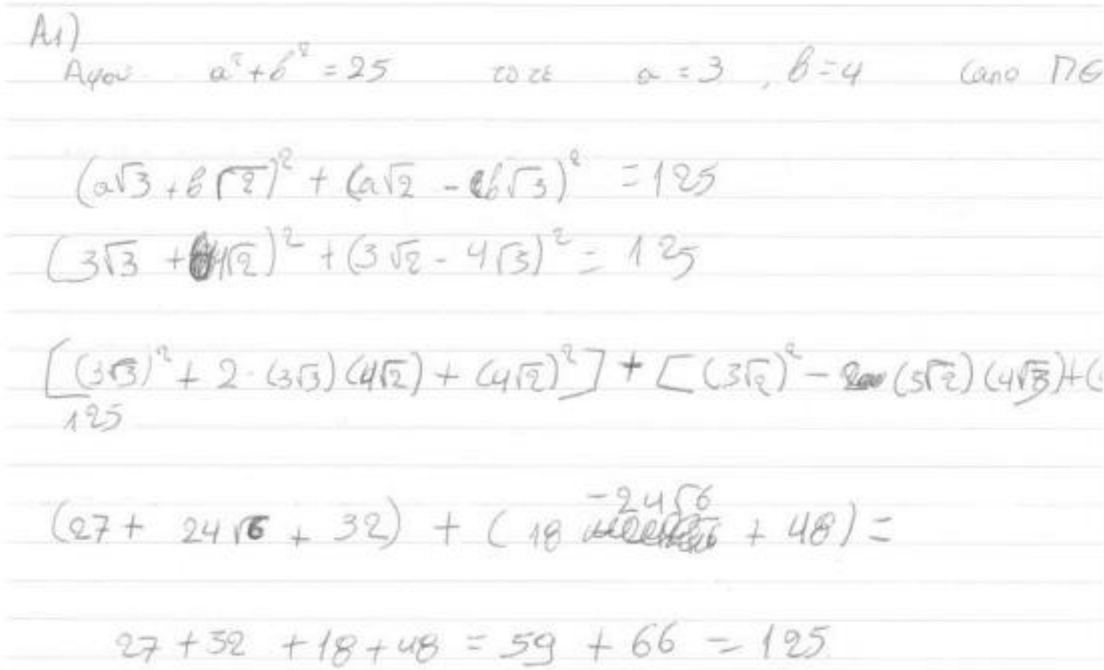


Figure 5.1. 4   Participant's [05] response to Question T31

P [05] gives an inadequate proof. He substitutes the values *α=3*, *β=4* before expanding the identities; i.e. the second line of his script reads: "Since $5^2=\alpha^2+\beta^2$ then *α=3*, *β=4* (from PT)". By the abbreviation "PT" P[05] means the Pythagorean theorem. In the rest of the proof the expansion of the identities is correct and thus the final result of the computations is indeed 125. However, the need to substitute specific numeric values for the variables provides evidence of an E.I. proof scheme and the answer has been classified accordingly.

P [52] (see Figure 5.1.5) writes:

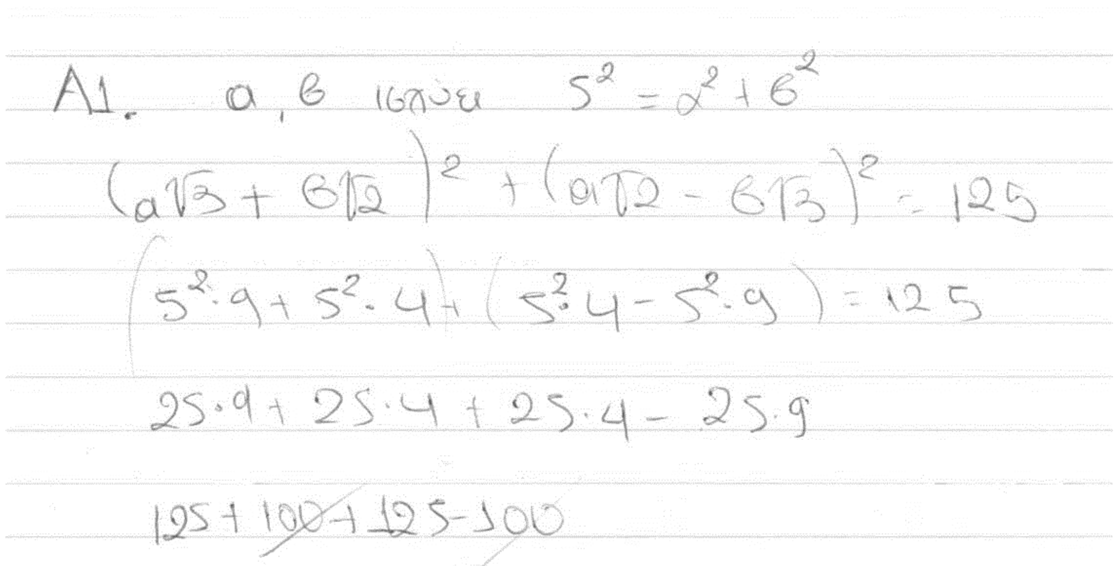


Figure 5.1. 5 Participant's [52] response to Question T31

P [52]'s answer is inadequate. In her proof she makes the substitution $\alpha=\beta=5$. This substitution of variables without logical justification is a sign of an E.I. proof scheme. At the same time this very substitution is evidence of an EC.NRS. proof scheme because there is no logical explanation for why α, β should be substituted by 5. On the other hand, if $\alpha=\beta=5$, then from $5^2=\alpha^2+\beta^2$ one would be led to $5^2=5^2+5^2$, which is not valid. Besides this, P[52] expands the parentheses correctly. Her expansion is a misuse of the identities $(A\pm B)^2$ again offering evidence of an EC.NRS. proof scheme. Thus P[52]'s answer provides evidence of a mixture of E.I. and EC.NRS. proof schemes and is characterised accordingly.

P[72] offers an inadequate answer (see figures 5.1.6 and 5.1.7). In figure 5.1.6 he manipulates the relation $5^2=\alpha^2+\beta^2$. This in itself is not a problem to start with, given that there are no arbitrary or absurd transformations. However in line 4 of his script he first misuses the parenthesis and as a result finds a false product in line 5, which leads him to see the expression at hand as a quadratic trinomial in one variable, although it is not in one variable. I understand this misperception as he calculates the alleged 'discriminant' of the alleged 'quadratic trinomial' and finds it "'$\Delta=-84$". There is no written comment or conclusion, and thus all these procedures remain unexplained. Then the participant leaps to a next page and another misuse of the identities resulting in the false relation $5\alpha+5\beta=125$ (Figure 5.1.7). This arbitrary misuse of symbols is evidence of an EC.NRS. proof scheme and P[72]'s answer is characterised accordingly.

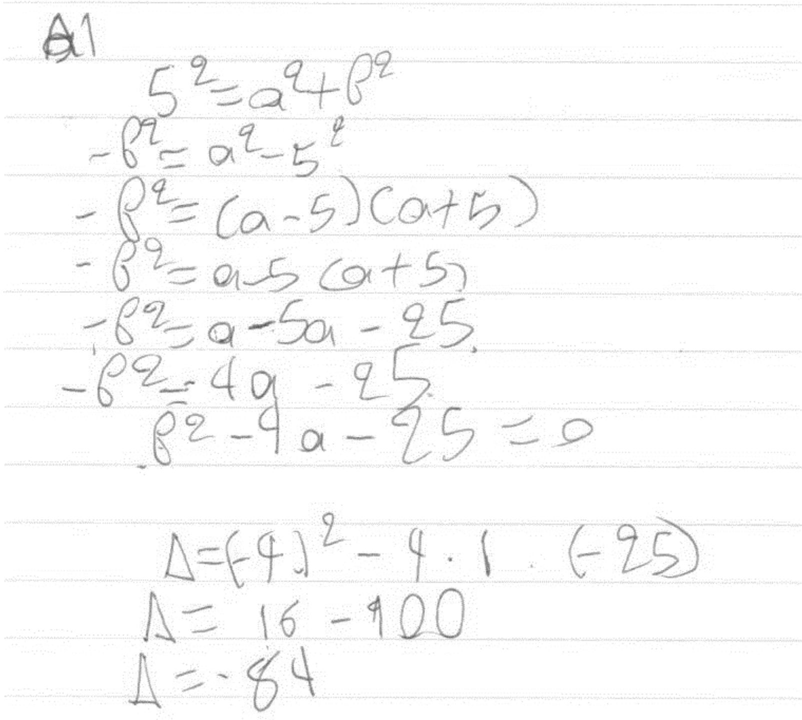


Figure 5.1. 6 Participant's [72] response to Question T31 (i)

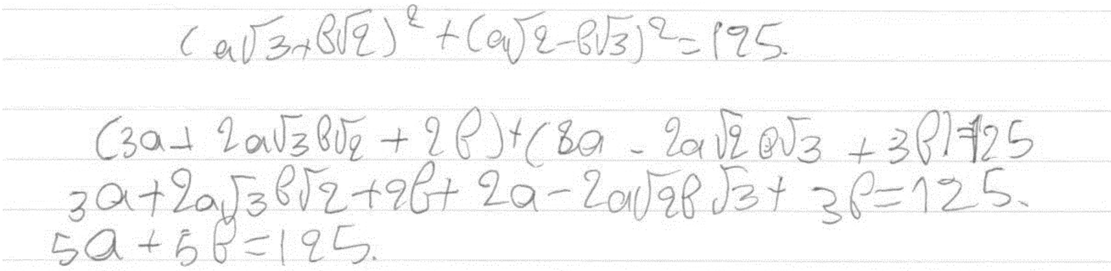


Figure 5.1. 7 Participant's [72] response to Question T31 (ii)

Table 5.1.1 illustrates the overview of answers given to Question T31.

The biggest group is that of NS indicating the participants' difficulty with the question. The expansion of the identities combined with the symbol of the square root and the use of the

PROOF SCHEMES OBSERVED IN THE RESPONSES TO QUESTION T31				
PROOF SCHEME	***FREQUENCY***	***CUMULATIVE FREQUENCY***	***RELATIVE FREQUENCY (%)***	***CUMULATIVE RELATIVE FREQUENCY (%)***
D.T.	*12*	*12*	*14.12*	*14.12*
D.T.-E.I.	*1*	*13*	*1.18*	*15.30*
D.T.-EC.NRS.	*8*	*21*	*9.41*	*24.71*
E.I.	*8*	*29*	*9.41*	*34.12*
E.I.-EC.NRS.	*15*	*44*	*17.65*	*51.77*
EC.NRS.	*12*	*56*	*14.12*	*65.89*
N.S.	*29*	***85***	*34.12*	***100.01***
SUM	***85***		***100.01***	

Table 5.1. 1 Summary of Question T31 proof schemes

relation $\alpha^2 + \beta^2 = 5^2$ to reach the final result seem to have been the difficult aspects of Question T31. Indeed the fact that there are only 12 (14.12%) answers characterised as D.T. is a strong indicator of these problems. There are 21 (24.71%) answers characterised as D.T. in total but only 12 (14.12%) are free of minor or major errors. This reflects the problems inherent in the transition from handling and mastering arithmetical operations to handling and mastering algebraic expressions experienced by a substantial number of participants.

The expected appearance of E.I. proof schemes indeed occurred in 24 (28.24%) answers in total. This is evident in the numerical substitution of the real variables α, β in the relation $\alpha^2+\beta^2=5^2$ with a variety of values. The tendency to make numerical substitutions is indicator of the still immature understanding of the role of the variable.

The highest number of answers, 35 (41.18%), are in the EC.NRS. group, lending evidence to the fact that students at this stage make arbitrary misuse of symbols.

5.2 Analysis of responses to Question T32a

In part *(a)* students might be tempted to substitute numerical values for κ, and λ. Thus some presence of E.I. proof schemes was expected.

The characterisations of the answers fell into seven groups; the proof schemes D.T., D.T.-E.I., D.T.-EC.NRS., E.I., E.I.-EC.NRS., EC.NRS. and NS. In the following I present examples of each in the above order.

P[74] (see Figure 5.2.1) writes:

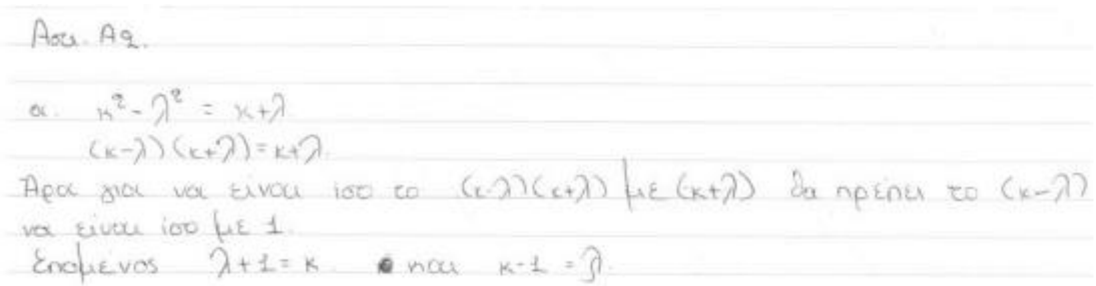


Figure 5.2. 1 Participant's [74] response to Question T32a

Participant [74] gives an adequate answer. In the fourth and fifth line of her script she asserts:

Thus for the $(\kappa-\lambda)(\kappa+\lambda)$ *to be equal to* $\kappa+\lambda$ *the* $(\kappa-\lambda)$

has to be equal to 1.

Thus $\lambda+1=\kappa$ *and* $\kappa-1=\lambda$.

P [74] gives another dimension of an adequate proof. The assertion that from the relation $(\kappa-\lambda)(\kappa+\lambda)=\kappa+\lambda$ it follows that $\kappa-\lambda=1$ is valid: it draws on the properties of number 1 as a neutral element of multiplication. This reminds me of the syntactic and semantic proof productions (Weber & Alcock, 2004) because P[74] does not proceed to solve for $\kappa-\lambda$ but correctly perceives the solution for $\kappa-\lambda$ which she logically proves to be equal to 1. The final, correct conclusion "$\lambda+1=\kappa$ and $\kappa-1=\lambda$" is neither necessary nor asked for but it appears that P[74] wanted to emphasise the fact that $\kappa - \lambda = 1$. Thus the answer is characterised as containing evidence of a D.T. proof scheme.

P[02] (see Figure 5.2.2) writes:

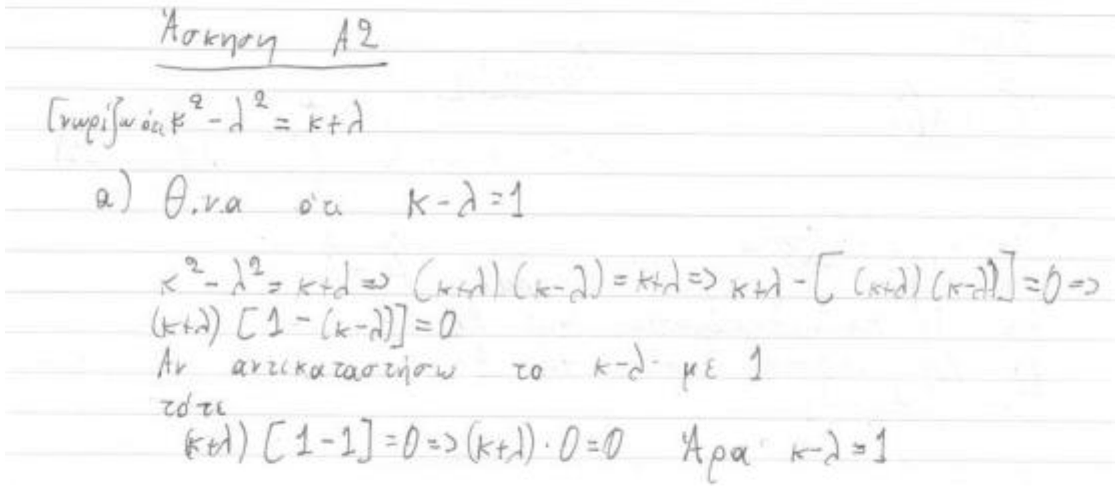


Figure 5.2. 2 Participant's [02] response to Question T32a

P[02] gives an adequate answer but with the following deficiency: before concluding that *κ-λ=1*, instead of proving, he proceeds to substitute for *κ-λ* the value 1. The proof begins in line 4. In line 5, P[02] writes:

If I substitute the κ-λ with 1

then

$(\kappa + \lambda)[1 - 1] = 0 \Rightarrow (\kappa + \lambda) \cdot 0 = 0$ *Thus κ-λ=1*

Up to the point where P[02] writes $(\kappa + \lambda)[1 - (\kappa - \lambda)] = 0$ the answer is adequate and thus can be classified as D.T.. From this point onwards the expected next step would be to observe that $\kappa + \lambda > 0$, and thus for the product $(\kappa + \lambda)[1 - (\kappa - \lambda)] = 0$ to be equal to zero the only remaining possibility is that $1 - (\kappa - \lambda) = 0$ which leads to $\kappa - \lambda = 1$. But in order to prove this fact, P[02] substitutes for $\kappa - \lambda$ the value 1. The substitution, I think, is evidence of an E.I. proof scheme because rather than the logical conclusion previously described, he prefers numerical validation to be sure that the product is zero. Thus P[02] is capable of manipulating efficiently the algebraic expressions. In this respect his answer provides evidence of a D.T. proof scheme. The numerical substitution κ-λ=1 instead of the logical proof that κ-λ=1, by application of real number properties on $(\kappa+\lambda)[1-(\kappa-\lambda)]=0$ is a sign of confusion between what constitutes a proof and what constitutes a verification. Thus the answer offers evidence of an E.I. proof scheme. Summarising, the answer of P[02] offers evidence of both D.T. and E.I. proof schemes and is classified accordingly.

P[13] (see Figure 5.2.3) writes:

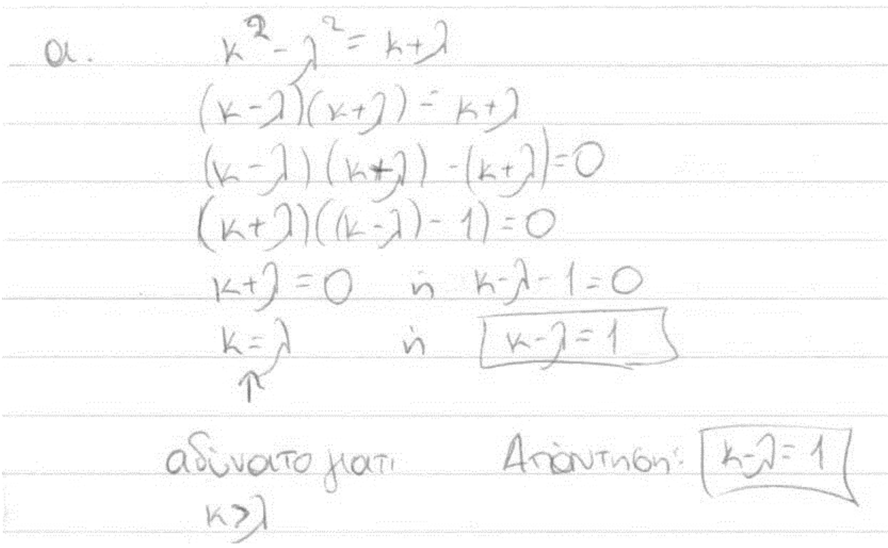


Figure 5.2. 3 Participant's [13] response to Question T32a

P [13] answer is adequate to a certain extent, but beyond a certain point it is deficient in handling the results obtained. Let's see in detail what happens. Up to line 4 the proof develops smoothly. Thus up to this point can be characterised as containing evidence of the D.T. proof scheme. In line 5 the problem begins when P[13] concludes that "$\kappa+\lambda=0$ or $\kappa-\lambda-1=0$". Even at this point P[13] could have rejected that $\kappa+\lambda=0$ as κ and λ are unequal natural numbers. Instead P[13] accepts the possibility that $\kappa+\lambda=0$ and continues, making another mistake by concluding that κ=λ which he considers impossible as κ>λ. Thus we see his logical effort to reject the case $\kappa+\lambda=0$. However, this effort is characterised by logical gaps and arbitrary assertions providing evidence of an EC.NRS. proof scheme. Believing he has correctly rejected the case $\kappa+\lambda=0$ he concludes that "Answer:

$\boxed{\kappa-\lambda=1}$" on the right of the seventh line of his script. P[13]'s provides evidence of both D.T. and EC.NRS. proof schemes and is characterised accordingly.

P[60] (see Figure 5.2.4) writes:

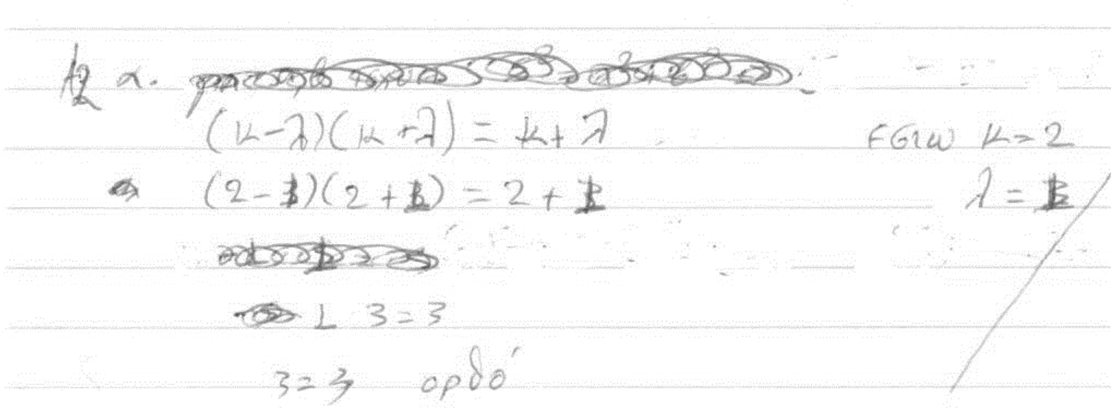


Figure 5.2. 4 Participants [60] response to Question T32a

Participant [60] gives an inadequate answer. Indeed the proof consists of her substituting the values 2 and 1 for κ and λ respectively, and then checking the validity of the expression. The first line of her script being sco, in line 2 of her script, on the right, she writes "Let κ=2" and directly under this substitution in line 3 writes "λ=1" although this is not clearly written. In line 3 the verification "(2-1)(2+1)=2+1" can be seen. The procedure of verification continuous in line 5 when the participant writes "1· 3=3" and in line 6 "3=3 correct". A clear general conclusion is nowhere to be found. Obviously the verification of the given relation for the aforementioned chosen values for the variables is 'seen as proof' enough. But the perception that any verification of an algebraic relation constitutes a general proof of its validity is evidence of an E.I. proof scheme.

P[88] (see Figure 5.2.5) writes:

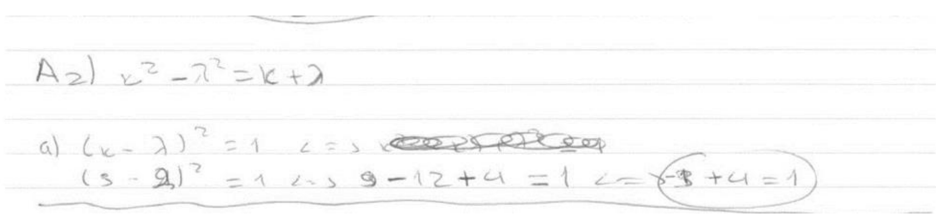


Figure 5.2. 5 Participant's [88] response to Question T32a

P[88] gives an inadequate answer. In his script in line 2 he writes $(\kappa-\lambda)^2=1$. No explanation is given as to the origin of this assertion. Probably he has the incorrect idea that the relation $(\kappa-\lambda)^2=\kappa^2-\lambda^2$ is valid. P[88] proceeds in line 3 with a numerical substitution of the variables κ and λ. It seems that κ takes on the value 3 and λ the value 2. No explanation is given for why these particular numbers were chosen. The most plausible explanation is that their difference is equal to one. The next step in line 3 is the expansion of the parenthesis $(3\text{-}2)^2$ which, is correct. Finally P[88] calculates the arithmetical expression 9-12+4 and verifies that its result is indeed equal to 1. No other explanation or comment is offered. P[88] may think that the proof is complete and so no further explanation is needed. This answer of P[88] contains the arbitrary relation $(\kappa-\lambda)^2=1$. Writing arbitrary relations without any logical justification of their validity is evidence of an EC.NRS. proof scheme. On the other hand, substituting numerical values for variables without giving a plausible reason for doing so, from one stand point, and believing that numerical verification of algebraic relations constitute proof from another, is evidence of an E.I. proof scheme. Thus this answer is classified as containing evidence of a mixture of E.I. and EC.NRS. proof schemes.

P[86] (see Figure 5.2.6) writes:

Figure 5.2. 6 Participant's [86] response to Question T32a

P[86]'s answer offers is inadequate. The goal of the proof is to show that κ-λ=1. P[86] transforms the hypothesis given $\kappa^2-\lambda^2=\kappa+\lambda$ to $(\kappa-\lambda)(\kappa+\lambda)=\kappa+\lambda$. And then writes:

"*as* $\kappa-\lambda=1\quad\kappa+\lambda=\kappa+\lambda$."

It seems that Participant [86] proves that $\kappa+\lambda=\kappa+\lambda$ using as supportive argument exactly that what was to be proved, namely that $\kappa-\lambda=1$. But using what is to be proved as data given and proving the obvious $\kappa + \lambda = \kappa + \lambda$ is evidence of arbitrary confusion of data, hypothesis, and conclusion and thus evidence an EC.NRS. proof scheme.

Table 5.2.1 illustrates the overview of answers to Question T32a. There were 49 (57.65%) NS to Question T32a even more than the 29 (34.12%) given to Question T31. The number of D.T. answers dropped from 12 (14.12%) in the latter to 7 (8.24%). Of the 12 participants who give a clear D.T. answer to Question T31, 6 six did the same with T32a; and of the 7 who gave a clear D.T. answer to Question T32a, 6 also did for T31 and 1 gave a D.T.-E.I. answer. These data indicate many participants' difficulty in handling Question T32a.

D.T. appears 11 times (12.94%). in the answers to Question T32a, E.I. 15 times (17.65%) and EC.NRS. 19 times 19 (22.35%). There is no appearance of E.P. proof scheme because Question T32a left almost no space for such schemes. Instead the nature of the data and probably the difficulty of T32a, led to the appearance of E.I. proof scheme.

Summarising the general handling of proof matters, the participants of the sample found many difficulties in dealing with proof in a context that was more complicated than one

PROOF SCHEMES OBSERVED IN THE RESPONSES TO QUESTION T32a				
PROOF SCHEME	***FREQUENCY***	***CUMULATIVE FREQUENCY***	***RELATIVE FREQUENCY (%)***	***CUMULATIVE RELATIVE FREQUENCY (%)***
D.T.	*7*	*7*	*8.24*	*8.24*
D.T.-E.I.	*1*	*8*	*1.18*	*9.42*
D.T.-EC.NRS.	*3*	*11*	*3.53*	*12.95*
E.I.	*9*	*20*	*10.59*	*23.54*
E.I.-EC.NRS.	*5*	*25*	*5.88*	*29.42*
EC.NRS.	*11*	*36*	*12.94*	*42.36*
N.S.	*49*	***85***	*57.65*	***100.01***
SUM	***85***		***100.01***	

Table 5.2. 1 Summary of Question T32a proof schemes

simply requiring expansion of identities and using algebraic expressions. The structure of T32a requires beginning from the data and the hypothesis via the appropriate steps to reach the conclusion. This is not yet a field in which many participants feel at ease.

And yet the seven answers who offered a D.T. proof scheme shows that even in small numbers there are very efficient students in what regards proof at the end of Year 9.

5.3 Analysis of responses to Question T32b

The underlying purpose of T32b was to test whether the students understood the converse of a proposition. If they invoked part *(a)* in solving part *(b)* then they did not understand the difference between the two. This underlying purpose was inspired by various works about students' underpinning problems with implications (Durand-Guerrier, 2003; Epp, 2003; Hoyles & Küchemann, 2002).

The answers fall into four groups: the proof schemes, D.T., D.T.-EC.NRS., EC.NRS. and NS. Eleven participants only calculated the difference 5556^2-5555^2 without applying the identity A^2-B^2=$(A$-$B)(A+B)$ and these responses are classified as D.T. proofs. Of the remaining participants, 19 used the identity, among whom 6 falsely invoked part (a). However, I consider these proof as D.T. as well because my purpose was only to investigate whether the participants would confuse a proposition and its converse is this way. On the bottom line, on the one hand, recognition of the converse is very difficult and on the other and from practical point of view since 5556-$5555=1$ independently of wrongly invoking part (a) they arrived at the correct result.

In the following I present the examples of these proof schemes in the above order.

P[81] (see Figure 5.3.1) writes:

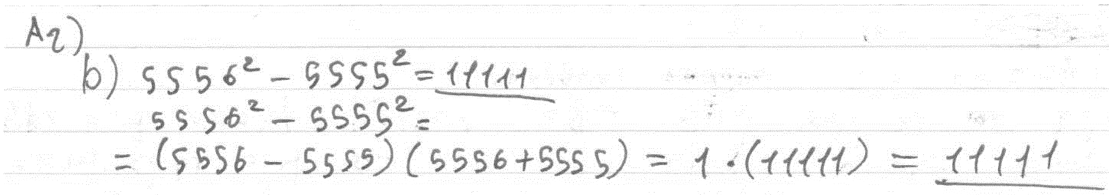


Figure 5.3. 1 Participant's [81] response to Question T32b

P [81] gives an adequate answer which applies the identity $(A+B)^2$ making the following transformation: $5556^2=(5555+1)^2$. In this respect this proof diverges creatively from what I have given as an adequate answer, but answers not identical with or in some cases even close to the answer proposed above are accepted if they offer an alternative adequate answer. There is a minor problem in the last line of the proof where P[81] writes $5556^2-5555=11111$ instead of $5556^2-5555^2=11111$: in other words mistakenly omits the exponent 2 of the second power of *5555*. I consider this lack negligible mistake and in any case non-systematic. Under these considerations the answer is classified as D.T..

P[08] (see Figure 5.3.2) writes:

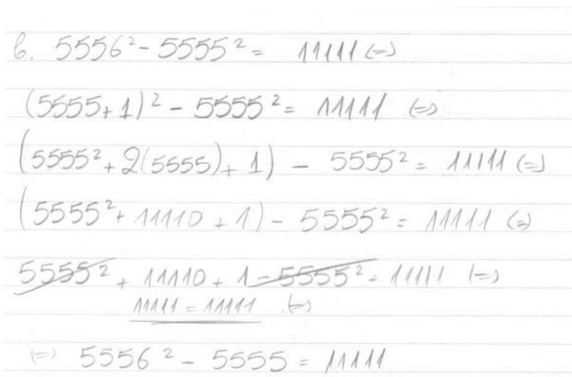


Figure 5.3. 2 Participant's [08] response to Question T32a

P[08]'s answer is adequate. P[08] uses the identity $A^2-B^2=(A-B)(A+B)$, putting $A=5556$ and $B=5555$ and proving, by application of the identity, that the relation is true. Thus P[08]'s answer is characterised as containing evidence of a D.T. proof scheme.

P[10] (see Figure 5.3.3) writes:

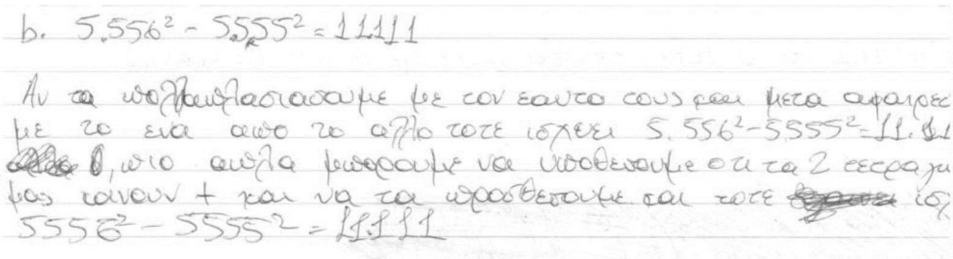


Figure 5.3. 3 Participant's [10] response to Question T32a

P[10] gives an adequate answer, preferring the direct computation *5556^2=30869136* and *5555^2=30858025* and then calculates the correct result *11111* subtracting the latter result from the former. It is true that omits the exponent '*2*' of the second power of *5555* but I put aside this mistake as the calculation is correct. Thus the answer has been categorised as a D.T. proof scheme.

P[67] (see Figure 5.3.4) writes:

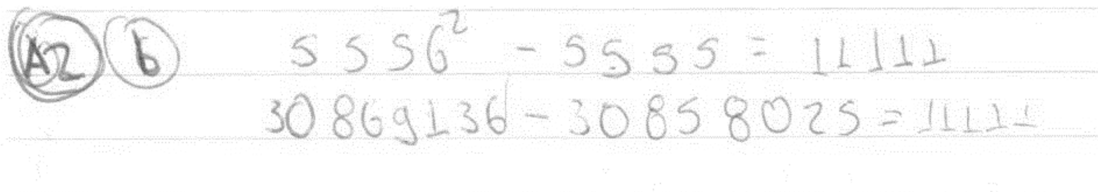


Figure 5.3. 4 Participant's [67] response to Question T32a

Participant [67] gives an ambiguous answer. She writes:

If we multiply them by themselves and then we subtract

from one another then it is valid that 5556^2-5555^2=11111

(sco) more simple we can suppose that the 2 squares

yield to us + and add with one another and then

5556^2-5555^2=11111

From a formal point of view, line 1 and line 2of the script give an adequate answer, describing what action has to be taken to prove what the questions asks to. From line 3 to line 5 the formulation is false because instead of subtraction P[67] proposes addition. I disregard this minor mistake because in these lines she simply asserts that the square of a non-zero number is a positive number and says little else. On the other hand there is no evidence that she calculated the squares as well as their difference and found them all correctly. From this viewpoint the answer is inadequate because it is an arbitrary assertion without any justification. Thus the answer is a D.T. proof scheme regarding what must be done, and it is, also, an EC.NRS. proof scheme because it contains unjustified assertions. Thus the answer is a mixture of D.T. and EC.NRS. proof schemes.

P[49] (see Figure 5.3.5) writes:

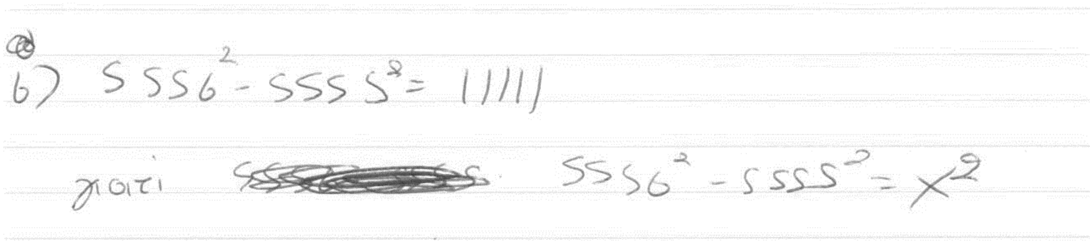


Figure 5.3. 5 Participant's [49] response to Question T32a

P[49] gives an inadequate answer which he considers complete in line 2, writing "*because* $5556^2 - 5555^2 = x^2$". He gives no information about what x is. Neither is there any explanation of why this undefined x and consequently x^2 has the power to prove the relation to be proved. The arbitrariness of the assertion is evidence enough to consider the answer an EC.NRS. proof scheme.

Table 5.3.1 illustrates the overview of answers to the Question T32b.

PROOF SCHEMES OBSERVED IN THE RESPONSES TO QUESTION T32b				
PROOF SCHEME	***FREQUENCY***	***CUMULATIVE FREQUENCY***	***RELATIVE FREQUENCY (%)***	***CUMULATIVE RELATIVE FREQUENCY (%)***
D.T.	*30*	*30*	*35.29*	*35.29*
D.T.-EC.NRS.	*1*	*31*	*1.18*	*36.47*
EC.NRS.	*10*	*41*	*11.76*	*48.23*
N.S.	*44*	***85***	*51.76*	***99.99***
SUM	***85***		***99.99***	

Table 5.3. 1 Summary of Question T32b proof schemes

The first thing to observe is that the number of NS is not much smaller than, those found in Question T32a, at 44 (51.76%) compared to the latter's 49 (57.65%). On the other hand there are considerably more D.T. answers at 30 (35.29%) compared to 7 (8.24%) for Question T32a. Thus, although a significant number of participants found Question T32b easier than T32a, for an equally significant number the question was hard to handle. Nineteen gave a D.T. answer using the identity A^2-B^2=$(A$-$B)(A+B)$ and eleven calculated the powers 5556^2 and 5555^2, and their difference, to find 11111. Thus the arithmetic nature of the question helped those who did not think of using the identity to give a D.T. answer.

Question T32b was about specific numbers and so gave no opportunity for E.I. evidence, which was not present in any answer.

There are 10 (11.76%) clear EC.NRS. answers compared to the 11 (12.94%) answers to T32a and a total of 12 (12.95%) compared to 19 (22.35%) for T32a.

5.4 Analysis of responses to Question T33ab

Part *(a)* of T33 is an indirect question about what constitutes the proof and what the verification of an algebraic relation. If verification is taken for proof then this would be characteristic of E.I. proof scheme. Part (a) is in the spirit of Healy and Hoyles (2000) who gave certain arguments to students and asked them to assess which the teacher would judge the best and what proof the students themselves would give.

In what regards part (b) from the view point any participant: what the participant thinks, what the participants' peers think, and what the teacher as a person with authority thinks. Parallel to proof appreciation, the question investigates whether the participants consider a persuasive argument to be a proof; it also looks for characteristics of EC.A. proof scheme, described by Harel and Sowder (1998, 2007). The EC.A. proof scheme refers to situations in which where the student seeks the validity of a proof by referring to an authority such as the teacher, a book etc.

The answers to T33a fall into: the proof schemes D.T., D.T.-E.I., D.T.-EC.NRS., E.I., EC.A., EC.NRS. and NS.

The answers to T33b fall into: the proof schemes D.T., D.T.-E.I., D.T.-EC.NRS., E.I., EC.A., EC.NRS. and NS.

In the following I present in the same previous order of T32a examples of answers and I insert examples of T33b if needed to cover all the cases of proof schemes.

P[02] (see Figure 5.4.1) writes:

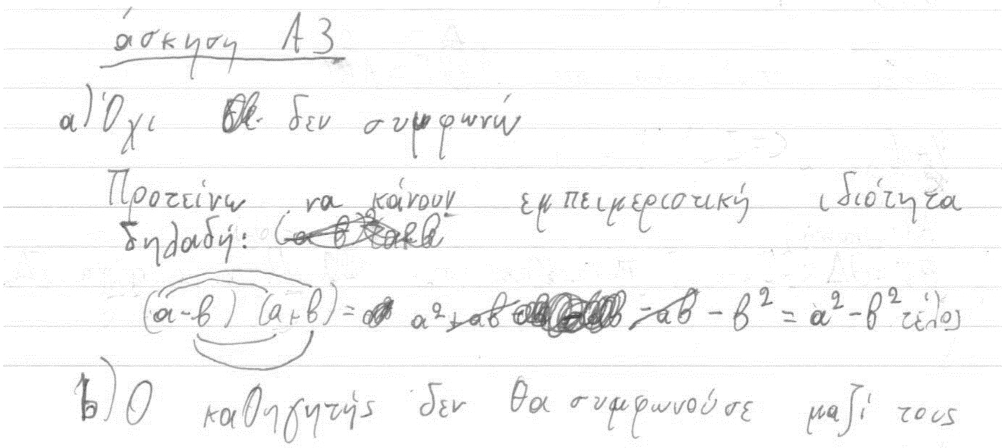


Figure 5.4. 1 Participant's [02] response to Question T33ab

<u>exercise A3</u>

a) No (sco) I do not agree

I suggest they applied the distributive property

that is (sco)

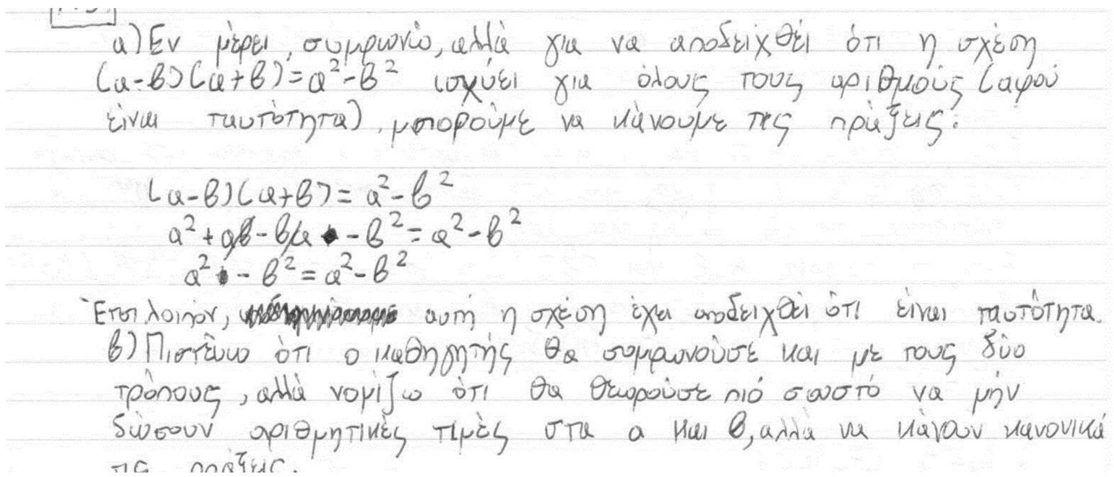
$(a-b)(a+b)$ =*(sco)* $a^2 + \not{ab}$ *(sco)* $-\not{ab} - b^2 = a^2 - b^2$ *end*

b) The teacher would not agree with them

P [02] gives adequate answers to both T33a and T33b (see figure 5.4.1). In his answer to T33a he disagrees with his peers. He thinks that proof is a procedure that justifies the validity of the identity in question in general and does not depend on the definite values of the variables involved in it. He explains his opinion by correctly applying the distributive law to the product *(a−b)(a+b).* He draws lines showing the multiplications that must be carried out according to the distributive law. Carries out the indicated multiplications and after simplification finds the correct final result $a^2 - b^2$. Consequently his answer has been characterised as containing evidence of a D.T. proof scheme.

P[02]'s answer to T33b is laconic. He just certifies that the teacher would not agree with his peers. Laconic answers are generally difficult to characterise but in the case of Question T33ab I have to accept an interdependence of answers. P[02]'s answer to T33a has already provided evidence of an adequate answer, and in a way has already answered both question by answering T33a because apparently the teacher would give the same explanation as P[02] did. Under these considerations his answer to T33b has been characterised as well as containing evidence of a D.T. proof scheme.

P[01] (see Figure 5.4.2) writes:


α) Εν μέρει, συμφωνώ, αλλά για να αποδειχθεί ότι η σχέση $(α-β)(α+β)=α^2-β^2$ ισχύει για όλους τους αριθμούς (αφού είναι ταυτότητα), μπορούμε να κάνουμε τις πράξεις:

$(α-β)(α+β)=α^2-β^2$

$α^2+αβ-βα-β^2=α^2-β^2$

$α^2-β^2=α^2-β^2$

Έτσι λοιπόν, [illegible] αυτή η σχέση έχει αποδειχθεί ότι είναι ταυτότητα.

β) Πιστεύω ότι ο καθηγητής θα συμφωνούσε και με τους δύο τρόπους, αλλά νομίζω ότι θα θεωρούσε πιό σωστό να μην δώσουν αριθμητικές τιμές στα α και β, αλλά να κάνουν κανονικά τις πράξεις.

Figure 5.4. 2 Participant's [01] response to Question T33ab

P[01] gives a partly adequate answer because for various reasons she does not completely reject the numerical value substitution. P[01] argues:

a) I agree partly, but to prove that the relation

$(a-b)(a+b)=a^2-b^2$ *is valid for all numbers (since*

it is an identity) we can do the computations:

$(a-b)(a+b)=a^2-b^2$

$a^2+a\not{b}-\not{b}a-b^2=a^2-b^2$

$a^2-b^2=a^2-b^2$

Thus, (sco) this relation has been proved that it is an identity.

b) I believe that the teacher would agree with both *ways, but I think that* [the teacher] *would consider more correct not to give numerical values to a and b, but to do normally the computations.*

P[01] understands that the general truth of the relation is established by the application of the distributive law to the product $(a-b)(a+b)$. Nevertheless he does not completely reject the use of numerical values. The answer to T33b reinforces this impression. Indeed, P[01] tells that the teacher would agree with both methods and would consider the application of distributive law as ‘more correct’. Probably he is influenced by the common practice of investigating before embarking on a full proof process. In this sense he leaves room for us to believe that, to him, the experimenting with numerical substitutions still has something of a proof and is not to be completely rejected in this respect. Sometimes numerical substitutions are used as examples in the classroom. However, this is not done with the aim of underpinning the role of an example to prove the validity of a relation but exactly the opposite, namely to show the insufficiency of resorting to examples as a general proof. Namely one can indeed prove that a relation *is not* generally valid if one finds at least one example of numerical substitution making the relation not valid. I have to accept that probably P[01] is taking a friendly approach towards his peers and consequently is lenient in his criticism of their numerical substitutions. However, his answer differs from those categorically rejecting the substitutions as a method of proof and so I decided to classify the answer as a mixture of D.T. and E.I proof schemes.

P[82] (see Figure 5.4.3) writes:

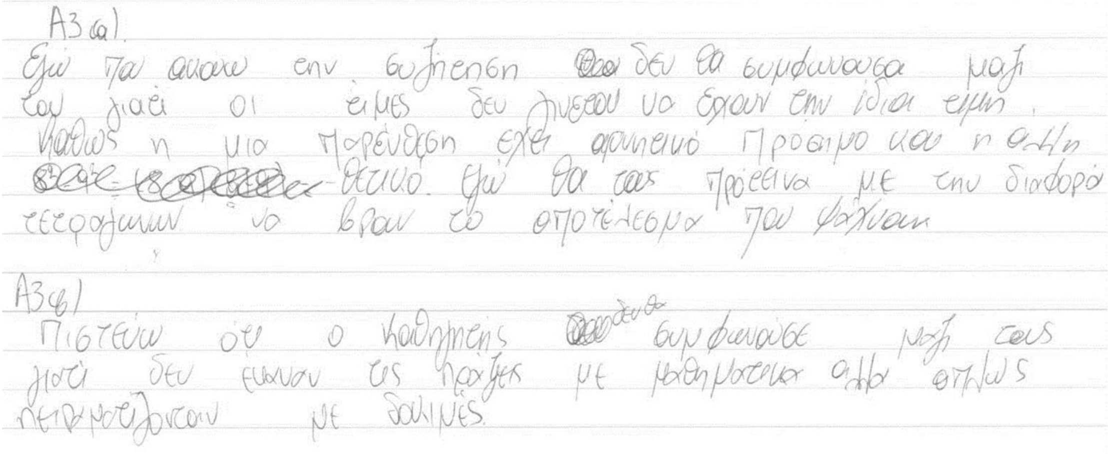


Figure 5.4. 3 Participant’s [82] response to Question T33ab

P[82] gives a partly adequate answer. P[82] argues:

“*A3a*

I who follow the discussion (sco) I would not agree with

them because the values cannot have the same value

since the first parenthesis has negative sign and the other

(sco) positive. I would suggest to them that they use the difference

of squares to find the result they are seeking

A3b

I believe that the teacher (sco) would not agree with them

because they did not carry out the operations with mathematics but simply

experimented with trials”

P[82]’s answer to both T33a and T33b has aspects of adequacy. P[82] does not agree with his peers and believes that it would have been better to use the identity of the difference of squares. He says that one does not prove an identity by substituting values in the identity. However, the formulation of his premise is flawed. For example, the meaning of the phrase “*because the values cannot have the same value since the first parenthesis has negative sign and the other (sco) positive*” is ambiguous. He seems to be saying that the values of the parentheses are different; this is not a valid argument against the use of values but a reality, exactly because the parentheses are not identical. Here an element of arbitrariness is to be found. P[82] next proposes to apply the difference of squares, but this is exactly the problem one has to prove its validity. P[82] proposes the application of the identity to be proved as a proof of the identity. This indicates a confusion of hypothesis and conclusion and is a sign of arbitrariness; thus the answer to T32a is characterised as containing evidence of a mixture of D.T. and EC.NRS. proof schemes. The same applies to T33b because again is vague when he writes "*they did not carry out the operations with mathematics*". But in the analysis of T33a the "*operations with mathematics*" has a controversial meaning. Altogether P[82]'s answer is without the clear meaning as it would have had if he had referred to for instance the application of distributive law etc.

P[25] (see Figure 5.4.4) writes:

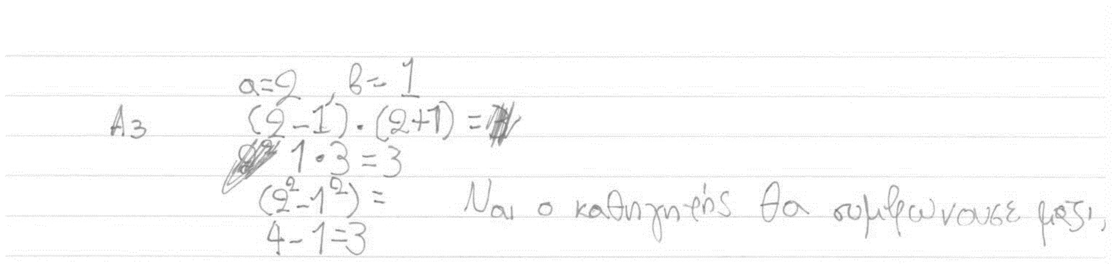


Figure 5.4. 4 Participant's [25] response to Question T33ab

P[25] gives an inadequate answer to both T33a and T33b. P[25] argues:

α=2, β=1

(2-1)(2+1)= (s.c.o.)

1·3=3

(2^2-1^2)= *Yes the teacher would agree with them*

4-1=3

Participant [25] gives an inadequate answer. P[25] does not distinguish clearly between parts (a) and (b). It seems that he uses numbers 2 and 1, to which the script refers, and verifies the identity without comment or explanation. However, it is obvious that this part of the answer is his own verification of the identity the proof of which is the issue in the discussion of the two peers. P[25] in the same vain with his makes numerical substitutions. Thus this part can be considered an answer to (a). The need to substitute numerical values for the variables in order to check the validity of an algebraic relation without justification and the generalisation of the validity beyond the concrete values is a sign of an E.I. proof scheme; in his answer to part (b) P[25] writes "Yes the teacher would agree with them" thus he is considering the substitution of numerical values as a method that even the teacher proposes and accepts. Consequently P[25]'s answer to part (b) provides also evidence of an E.I. proof scheme.

P[09] (see Figure 5.4.5) writes:

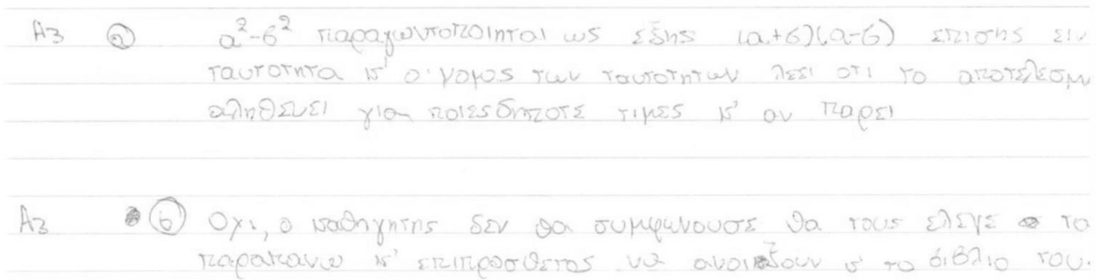


Figure 5.4. 5 Participant's [09] response to Question T33ab

A3 (*a*) *$a^2 - b^2$ is factorised as follows (a+b)(a-b) and also is*

an identity and the law of identities says that the result

is valid for whichever values it takes

A3 (*b*) *No the teacher would not agree he would say to them the*

above and to open their books

The answers to both parts of the question are inadequate. In the answer to *(a)* in the question the peers are wondering how to *prove* the identity and whether the substitution of values for the variables and the verification of the identity for these values is enough to achieve this. P[09] supports the idea that the identity is valid because it is factorized as follows: $a^2 - b^2 = (a+b)(a-b)$. But the problem is exactly whether this factorization is the logical result of some procedure. P[09] seems to think that there is no need to apply the distributive law on the product $(a-b)(a+b)$ to obtain after all the simplifications the result a^2-b^2. At this point is worth noting that the identity in the text has the order $(a+b)(a-b)= a^2-b^2$ whereas P[09] writes it as $a^2-b^2=(a+b)(a-b)$. This reinforces the thought that he sees the identity as formula prescribed by an authority and thus P[09] does not feel the need to prove the identity because he is convinced that this is the only way to write it and its validity is beyond doubt because of "*the law of the identities*". However, the only law that the proof is based on is the distributive law. The declarative character of the answer regarding the validity of the identity, and the inversion of the order of the text for the formula of the identity constitute evidence of an external conviction of the validity of the identity, which is characteristic of an EC.A. proof scheme.

In the answer to part (b) P[09] thinks that the teacher would repeat the argument in part (a) to his peers, so instead of explaining the procedure for some kind of proof the teacher would only confirm that the identity is written like this way because it is written this, and its validity is due to "*the law of identities*". Additionally, according to P[09], the teacher would urge the peers to open their books. Thus P[09] thinks the teacher would repeat to the peers similar arguments with P[09]'s with a new element, the strict order "open your books" which is a clear sign of seeking an authoritative opinion on the validity of the identity. Seeking the opinion of an authority and believing their confirmation of the validity of mathematical truths without any logical justification is evidence of the EC.A proof scheme putting both this and P[09]'s answer to T33b into that category proof schemes.

P[73] (see Figure 5.4.6) writes:

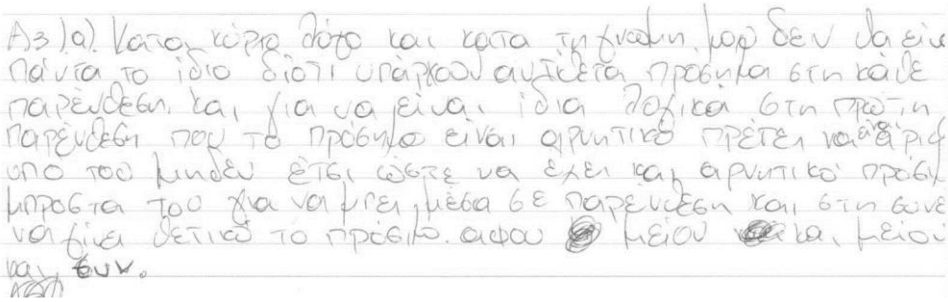


Figure 5.4. 6 Participant's [73] response to Question T33a

A3)a) Principally and in my opinion it will not be

always the same because there are opposite signs in every

parenthesis and for them to be equal logically in the first

parenthesis where the sign is negative it must be a number smaller

then zero in order to have the negative sign

in front of it so that it would need a parenthesis in which it will be written and

[consequently

to be transformed in positive since minus and minus

yields plus

The translation of P[73]'s script is difficult. Basically P[73] believes that the identity is not always valid, based on the difference between the signs in the two parentheses which, she argues, must be the same. Thus she believes that instead of *(a-b)(a+b)* one should have *(a+b)(a+b)* necessary for a valid identity. She unfolds her argument regarding this change by asserting that if *b* has a negative sign then this negative sign combined with the minus sign before *b* would give plus. I suspect that she has confused the given identity with the identity $(a+b)^2=a^2+2ab+b^2$. Under this assumption the meaning of the assertion "*in my opinion it will not be always the* same" is understandable. Thus she supports the arbitrary idea that the two parentheses have to be equal to each other. However, if that were possible we would have $(a+b)^2$ the left side leading to another arbitrary result, namely $(a+b)^2=a^2-b^2$. P[73]'s fails to realise that if her argument about *b* were correct, the minus sign would appear in the second parenthesis. Indeed if $b=-t$ then $(a-b)(a+b)=(a+t)(a-t)$. Thus the argument regarding the minus sign is arbitrary. The arbitrariness of the various assertions, the confusion of identities and even the ambiguous formulation constitute evidence that P[73]'s answer offers evidence of an EC.NRS. proof scheme.

Tables 5.4.1 and 5.4.2 illustrate the overview of answers to Question T33. Table 5.4.1 shows that the biggest group of answers to T33a are in the D.T. group, 38 (44.71%) and similarly Table 5.4.2 shows the corresponding number to be 36 (42.35%). There is a drastic improvement of student performance in Question T33 in comparison to T31 and T32 indicating that many participants found Question T33 easier to solve. The total number of answers in which some evidence of D.T. proof schemes was found is 51 (60.01%) for T33a and 43 (50.59%) for T33b; the highest number so far has been 31 (36.47%) for T32b.

PROOF SCHEMES OBSERVED IN THE RESPONSES TO QUESTION T33a

PROOF SCHEME	***FREQUENCY***	***CUMULATIVE FREQUENCY***	***RELATIVE FREQUENCY (%)***	***CUMULATIVE RELATIVE FREQUENCY (%)***
D.T.	*38*	*38*	*44.71*	*44.71*
D.T.-E.I.	*9*	*47*	*10.59*	*55.30*
D.T.-EC.NRS.	*4*	*51*	*4.71*	*60.01*
E.I.	*8*	*59*	*9.41*	*69.42*
EC.A.	*1*	*60*	*1.18*	*70.60*
EC.NRS.	*2*	*62*	*2.35*	*72.95*
N.S.	*23*	***85***	*27.06*	***100.01***
SUM	***85***		***100.01***	

Table 5.4. 1 Summary of Question T33a proof schemes

Therefore D.T, numbers are significantly high. Although the participants faced many difficulties in handling Questions T31 and T32ab adequately, they appear to recognise an acceptable proof when they are presented with one. In their research with prospective primary teachers Stylianides and Sylianides (2009) found similar results.

In conclusion even if there are considerable difficulties involved in producing a proof the appreciation of a proof is rather strong.

There are 8 (09.41%) answers to Question T33a that contain evidence of E.I. proof schemes and 9 (10.59%) for Question T33b. There is a total of 17 (20.00%) appearances of

PROOF SCHEMES OBSERVED IN THE RESPONSES TO QUESTION T33b

PROOF SCHEME	***FREQUENCY***	***CUMULATIVE FREQUENCY***	***RELATIVE FREQUENCY (%)***	***CUMULATIVE RELATIVE FREQUENCY (%)***
D.T.	*36*	*36*	*42.35*	*42.35*
D.T.-E.I.	*4*	*40*	*4.71*	*47.06*
D.T.-EC.A.	*1*	*41*	*1.18*	*48.24*
D.T.-EC.NRS.	*2*	*43*	*2.35*	*50.59*
E.I.	*9*	*52*	*10.59*	*61.18*
EC.A.	*3*	*55*	*3.53*	*64.71*
N.S.	*30*	***85***	*35.29*	***100.00***
SUM	***85***		***100.00***	

Table 5.4. 2 Summary of Question T33b proof schemes

the E.I. proof scheme, alone or with other proof schemes, in the answers to Question T33a and 13 (15.29%) for Question T33b. This is to be expected as these questions lend themselves easily to substitution of numerical values. For similar reasons the E.P. proof scheme is completely absent in the answers to both questions.

Other important findings here are that only 1 (01.18%) answer to Question T33a is classified as EC.A. and only 3 (03.53%) for Question T33b; and the total number of appearances of the E.I. proof scheme (4 or 4.71%) is higher only in T33b.

Finally the number of NS remains rather high at 23 (27.06%) for Question T33a and 30 (35.29%) for Question T33b.

5.5 Analysis of responses to Question T34ab

Part (a) of the question was intended to gather information on the students' efficiency at drawing a figure according to given instructions. If they managed this part (b) can be proved using the appropriate congruency criterion for right-angled triangles. In other words either a criterion which refers to two pairs of equal corresponding sides or one which refers to one pair of equal corresponding sides and one pair of corresponding angles. Thus Question T34b was open to the application of more than one congruency criterion for right-angled triangles. Attempting this proof the students would provide information on their proof schemes.

For T34a I decided to mark a figure as correct if it generally satisfied the following criteria: (i) the final result strongly resembles a parallelogram; (ii) the names of the vertices are in the right order; (iii) the perpendiculars resemble perpendiculars or the right-angle symbol is drawn in the right place. Judging by these criteria I found 55 (64.71%) of figures to be correct and 19 (22.35%) not correct; 11 (12.94%) participants neither drew a figure nor answered T34b, apart from one who gave an EC.NRS. answer. Among the 19 participants who drew incorrect figures 2 offered T34b answers characterised as D.T.-EC.NRS., 1 as E.P., 2 as E.P.- EC.NRS., 9 as EC.NRS., and 5 NS.

The answers to T34b fell into eight groups; the 7 proof schemes group D.T., D.T.-EC.NRS., D.T.-E.P., E.P., E.P.-EC.NRS., EC.NRS., EC.NRS.-EC.R. and one NS group.

While analysing the students' scripts I sometimes found it difficult to decide whether an answer was characteristic of the E.P. or the EC.NRS. proof scheme. If a participant, for instance, named congruent sides without justification I decided to characterise the situation as evidence of the E.P. proof scheme. If on the other hand a participant named for instance congruent sides with an invalid justification I saw it as characteristic of the EC.NRS. proof scheme.

When teaching the congruency of triangles, class teacher J underlined the distinction that must be made from the beginning between the hypothesis and the conclusion before engaging in the proof procedure, and taught the students to write hypothesis and conclusion explicitly. Additionally she taught them after accomplishing a proof to explicitly set out not only the final conclusion concerning the triangles' congruency but also the rest of the elements of congruent triangles that could be concluded from their congruency. This intended to give the students have a holistic idea of the congruency of triangles and to teach them to use these rest elements to prove something beyond the initial congruency of triangles. Writing data, hypothesis, and conclusion as well as the rest equal elements of triangles proved to be congruent might lead to the presence of a 'ritual' element in a proof and under certain conditions to an EC.R. proof scheme. The EC.R. proof scheme is one of Harel and Sowder's external conviction proof schemes which, as far as I understand they refer to the negative sense when students use a ritual form such as the traditional two-column proof habitual in US educational without productive results. Thus as I understand it we can categorise a proof as belonging to the EC.R. proof scheme if is not D.T. otherwise it has no meaning to speak of a D.T. ritual proof. From this viewpoint the 'ritual' element as taught by J is present to a greater or a lesser degree in 26 of the answers. Of these, 2 have been characterised as D.T., 4 as D.T.-E.P., 7 as D.T.-EC.NRS., 1 as E.P., 3 as E.P.-EC.NRS., 6 as EC.NRS. and 2 as EC.NRS.-EC.R. For all but the latter two I do not believe that the EC.R characterisation would offer anything further of importance to the analysis and, as I mentioned in section 5.0, I had decided not to make characterisations of more than two proof schemes at once.

In the following I present examples of the participants’ responses in the order given above.

P[14] (see Figure 5.5.1) writes:

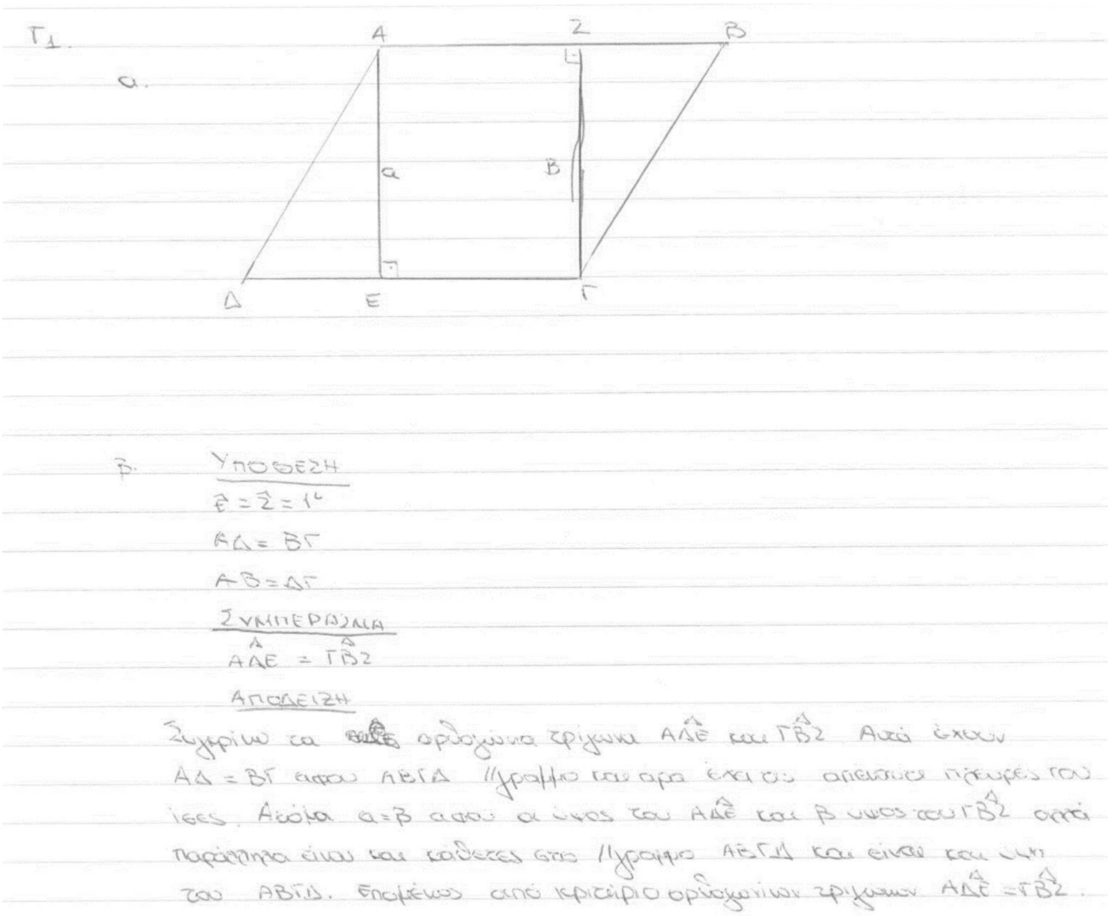


Figure 5.5. 1 Participant’s [14] response to T34ab

β. *<u>HYPOTHESIS</u>*

$\hat{E} = \hat{Z} = 1^{\llcorner}$

ΑΔ=ΒΓ

ΑΒ=ΔΓ

<u>CONCLUSION</u>

$\overset{\triangleleft}{A\Delta E} = \overset{\triangleleft}{\Gamma BZ}$

<u>PROOF</u>

I compare the (sco) right-angled triangles $\overset{\triangleleft}{A\Delta E}$ *and* $\overset{\triangleleft}{\Gamma BZ}$. *These have*

ΑΔ=ΒΓ since ΑΒΓΔ //gram and thus it has the opposite of its sides congruent. Also α=β since α height of $\overset{\triangleleft}{A\Delta E}$ *and β height of* $\overset{\triangleleft}{\Gamma BZ}$ *but at the same time are perpendicular in the //gram and they are heights of ΑΒΓΔ. Thus from the criterion for right-angled triangles* $\overset{\triangleleft}{A\Delta E} = \overset{\triangleleft}{\Gamma BZ}$.

In her proof P[14] uses the abbreviation " //gram" meaning parallelogram. Her answer is adequate: she invokes the criterion of congruency for right-angled triangles having two pairs of respective sides equal. She calls *ΑΔ=ΒΓ* a pair of congruent sides arguing that they are opposite sides of a parallelogram, and then names the pair of sides *α=β* (see Figure 5.5.2) and gives as her reason that they are the heights of the parallelogram *ΑΒΓΔ*. While her formulation is ambiguous, her final argument is that both segments are heights between the same parallel sides and so ignoring the ambiguity I have characterise the answer as a D.T. proof scheme.

P[10] (see Figure 5.5.2) writes:

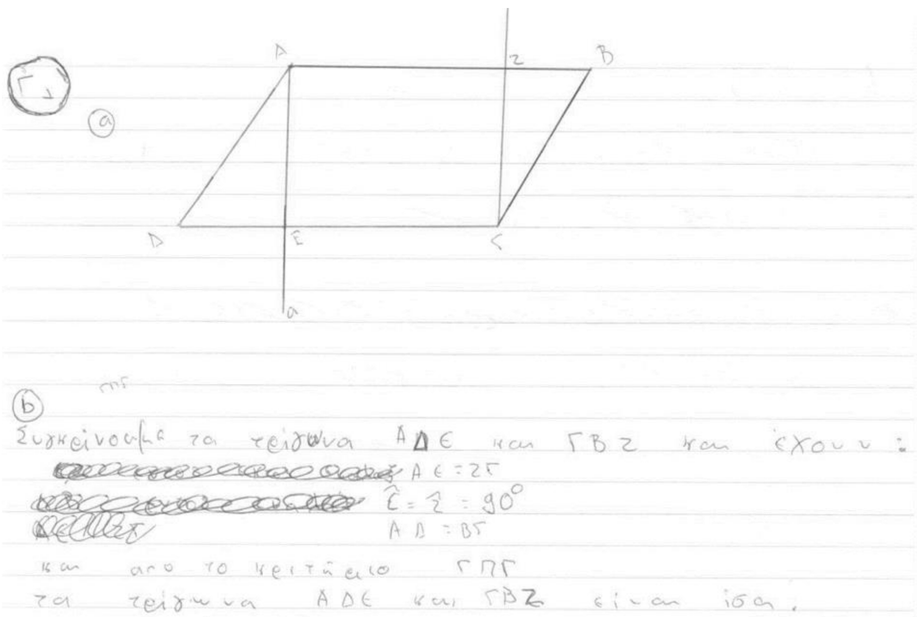


Figure 5.5. 2 Participant's [10] response to Question T34a

(b)

We compare the triangles $A\Delta E$ and ΓBZ and they have:

(sco) $AE=Z\Gamma$

We compare the triangles $A\Delta E$ and ΓBZ and they have:

(sco) $AE=Z\Gamma$

(sco) $\hat{E}=\hat{Z}=90°$

(sco) $A\Delta=B\Gamma$

and from the criterion ASA

the triangles $A\Delta E$ and ΓBZ are congruent.

This is an adequate answer to a certain extent. P[10] invokes the correct elements in order to prove the congruency, namely $AE=Z\Gamma$, $\hat{E}=\hat{Z}$, $A\Delta=B\Gamma$, but justifies this only by stating that the angles are equal as they are right angles. The congruency of the two pairs of sides is not justified by any argument. Thus invoking the congruency criterion is characteristic of D.T. proof scheme. Probably P[10] has not a clear idea what kind of congruency criterion she is using as she names the applied criterion Angle-Side-Angle. Practically what she writes is correct but we do not usually refer to the angle included between two sides if it is a right angle. Under these considerations the answer has been finally characterized as a mixture of D.T. and E.P. proof schemes.

P[66] (see Figure 5.5.3) writes:

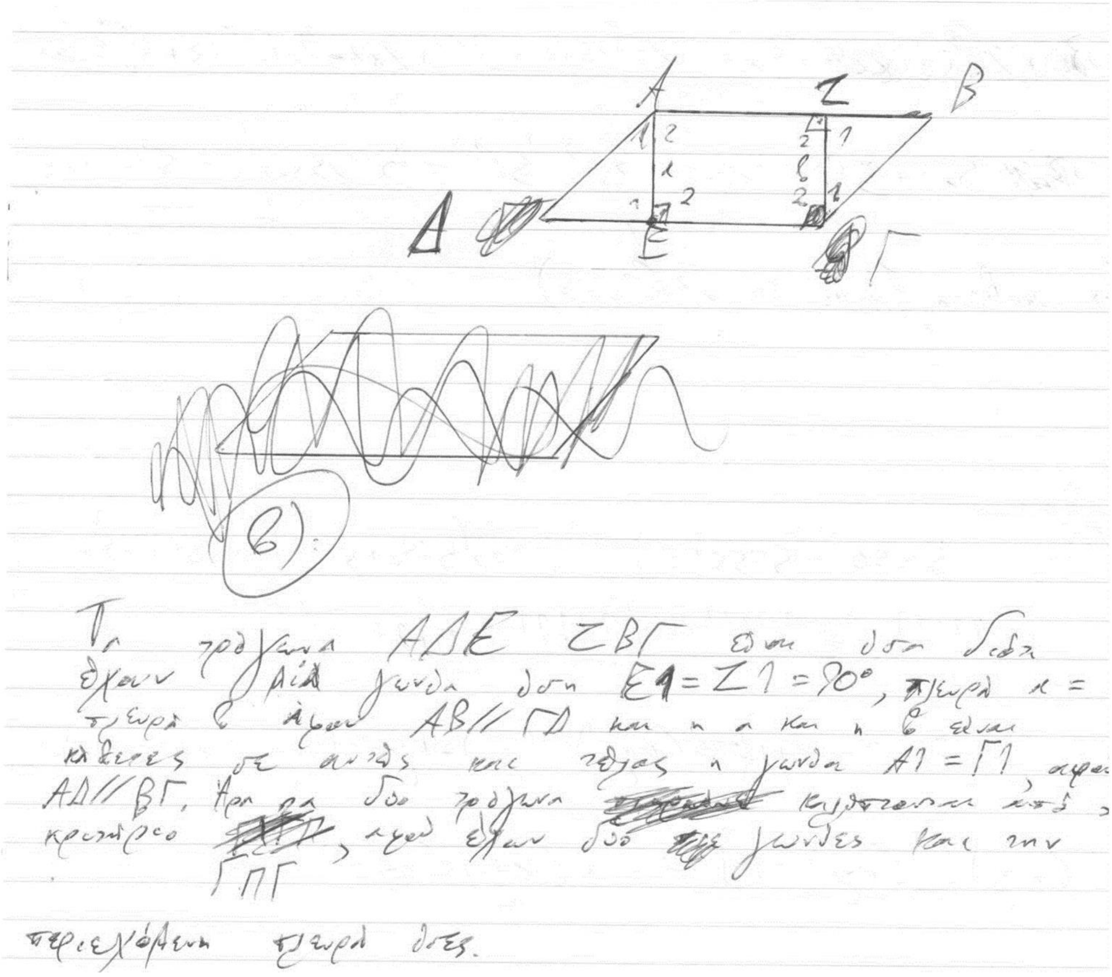


Figure 5.5. 3 Participant's [66] response to Question T34ab

Triangles ΑΔΕ ΒΖΓ are congruent because

they have one congruent angle $\widehat{E_1} = Z_1$ *=90°, side α=*

side β since ΑΒ//ΓΔ and α and β are

perpendicular to them and finally angle $A_1 = \Gamma_1$ *as*

ΑΔ//ΒΓ. Thus the two triangles (sco) (non readable) from the

criterion (sco) ASA since they have (sco) angles and the

included side congruent

P[66] gives a partially adequate answer. He refers to the fact that $\widehat{E_1} = Z_1 = 90°$, which means that the triangles are both right-angled, and then asserts that α=β. He justifies the last equality because the segments in question are perpendicular to the parallel lines *AB* and *ΓΔ*.

The formulation is vague but I accept it as valid. Finally he asserts that $A_1 = \Gamma_1$ because *ΑΔ//ΒΓ*. This part of the answer is not adequate because the assertion is arbitrary in the sense that it is not adequately justified. While it is true that angles with parallel sides are congruent or supplementary, Year 9 students do not yet know this, and even if they did, the argument is not complete because it has not excluded the case of supplementary angles. To this end it had sufficed to observe that both angles in question are complementary to the equal angles $\hat{B}$ and $\hat{\Gamma}$. Arbitrary and irrelevant justifications are taken as evidence of the EC.NRS. proof scheme. According to the above, this answer provides evidence of a mixture of D.T. and EC.NRS. proof schemes

P[44] (see Figure 5.5.4) writes:

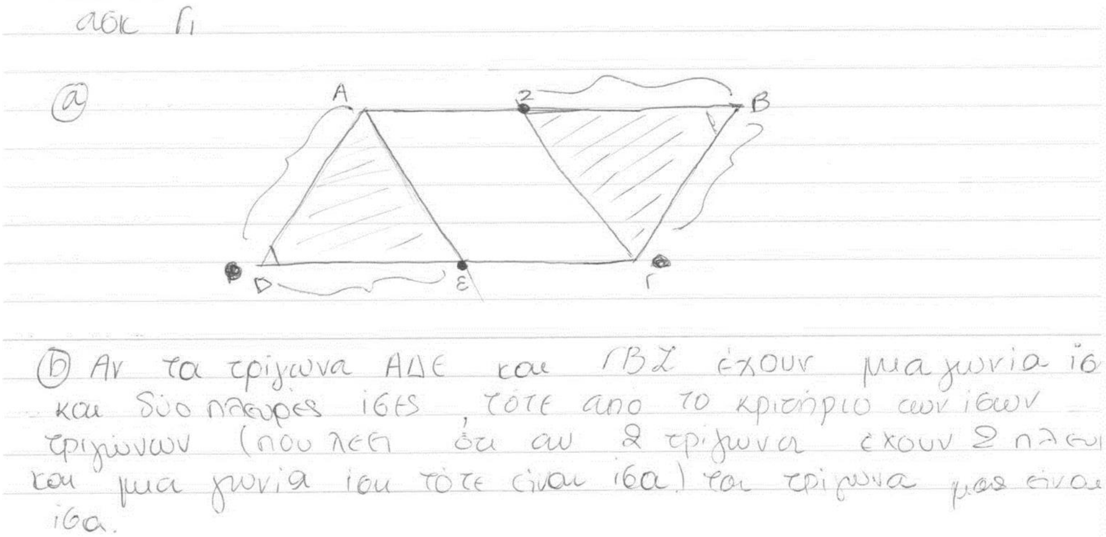


Figure 5.5. 4 Participant's [44] response to Question T34ab

(b) If the triangles ΑΔΕ and ΓΒΖ have an angle congruent

and two sides congruent then from the criterion of equal

triangles (which asserts that if 2 triangles have 2 sides

and an angle congruent then they are congruent) our triangles are

congruent.

P[44] gives an inadequate answer to T34b and draws an incorrect figure. He invokes a falsely formulated criterion, but the necessary congruent elements are all indicated in his figure. The criterion regards the pairs of sides *AΔ=BΓ*, *ΔE=ZB* and the angles $\hat{\Delta} = \hat{B}$; however, none of these congruencies are supported by logical justification. Not logically justifying properties that one asserts are valid because one sees them as valid in a figure is evidence of the E.P. proof scheme. Thus the answer of P[44] is characterised accordingly as an E.P. proof scheme.

P[11] (see Figure 5.5.5) writes:

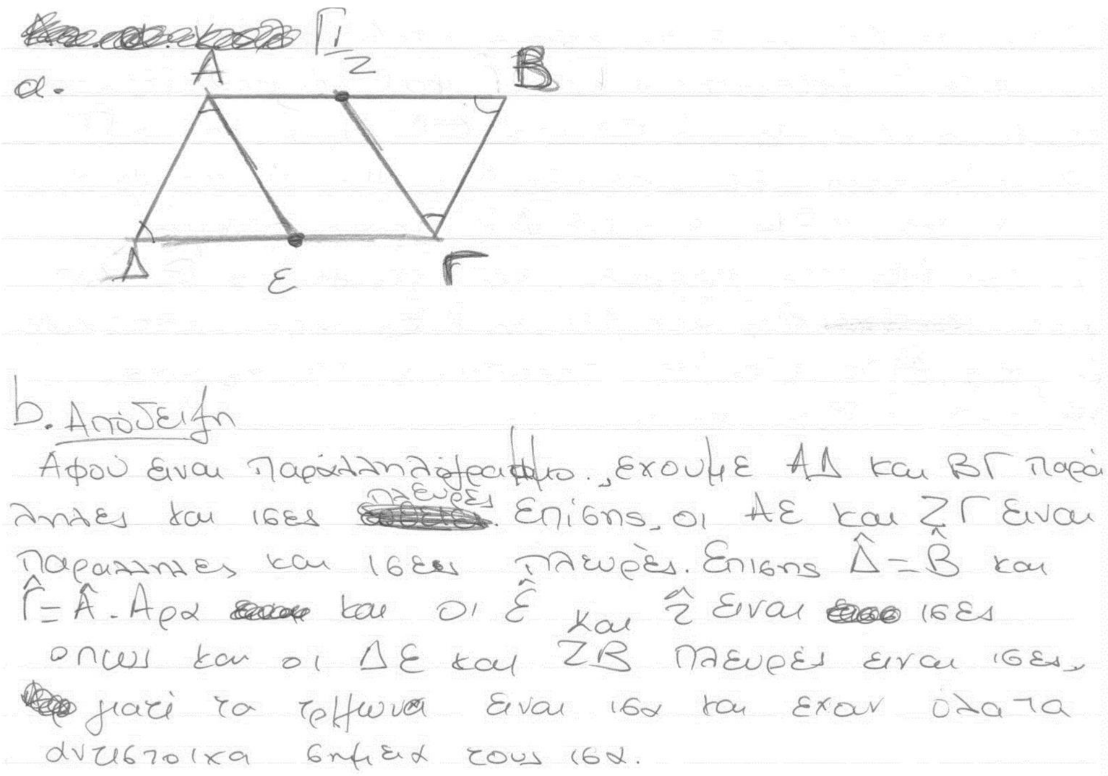


Figure 5.5. 5 Participant's [11] response to Question T34ab

b. <u>Proof</u>

Since it is a parallelogram, we have AΔ and BΓ parallel and congruent (sco) sides. Also AE and ZΓ are parallel and congruent sides. Also $\hat{\Delta} = \hat{B}$ and $\hat{\Gamma} = \hat{A}$. Thus (sco) also $\hat{E}$ and $\hat{Z}$ are (sco) congruent as well as ΔE and ZB sides are congruent *(sco) because the triangles are congruent and have all*

their corresponding points equal.

P[11] gives an inadequate answer. First she repeats the pairs of congruent sides of the parallelogram and then proceeds to assert that $\hat{\Delta} = \hat{B}$ which is correct, but she does not offer the justification of their being opposite angles of a parallelogram. She next asserts that $\hat{\Gamma} = \hat{A}$ but this time there is a strong suspicion that she is not referring to the corresponding angles of the parallelogram but to the angles $\widehat{\Delta AE}$ and $\widehat{Z\Gamma B}$. In any case the reference is ambiguous and not logically supported. She goes on to assert that angles $\hat{E}$ and $\hat{Z}$ are equal again without logical support. Up to this point P[11] sees properties in a figure as valid without logical support, which is evidence of the E.P. proof scheme. The last part of the proof justifies all the previous equalities in the name of the congruency of the triangles. The argument is cyclical and thus arbitrary. This is evidence of the EC.NRS. proof scheme. Thus this answer is categorised as a mixture of E.P. and EC.NRS. proof schemes.

P[54] (see Figure 5.5.6) writes:

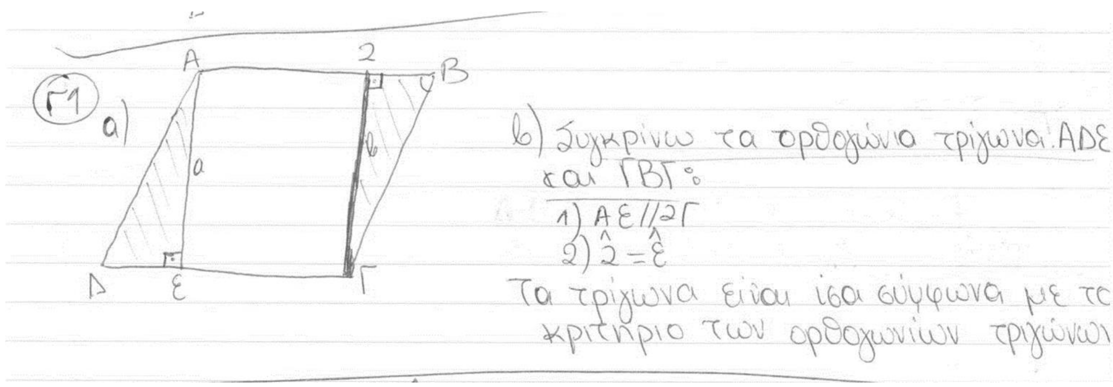


Figure 5.5. 6 Participant's [54] response to Question T34

β) I compare the right-angled triangles AΔE

<u>and ΓΒΓ</u>:

1) AE//ZΓ

2) $\hat{Z} = \hat{E}$

The triangles are congruent according to the *criterion of right-angled triangles.*

P[54] gives an inadequate answer. There is an obvious mistake when instead of writing ZBΓ he writes ΓBΓ P[54]. This minor mistake can be put aside, but the whole argument that follows is arbitrary. P[54] appeals to *AE//ZΓ* and $\hat{Z} = \hat{E}$ as congruency elements supporting the congruency of the triangles. But parallelism is not an element of congruency and the equality of angles does not suffice to support a criterion of congruency. Thus the whole argument is irrelevant regarding parallelism and, as a whole, arbitrary. Arbitrary and irrelevant assertions constitute evidence of the EC.NRS. proof scheme and thus P[54]'s is classified as such.

P[32] (see Figure 5.5.7) writes:

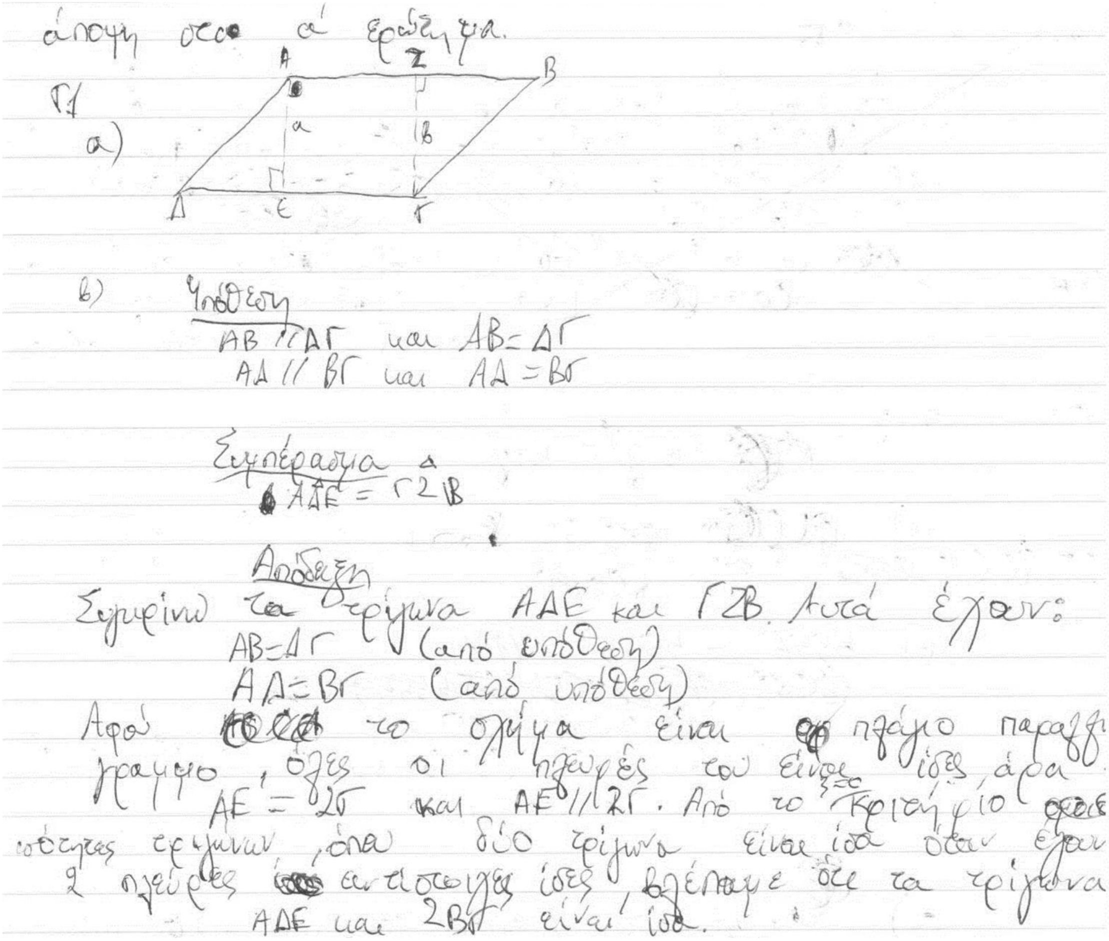


Figure 5.5. 7 Participant's [32] response to Question T34

β) Hypothesis

AB//ΓΔ   AB=ΔΓ

AΔ//BΓ   AΔ=BΓ

<u>Conclusion</u>

$\overset{\triangleleft}{A}\Delta E = \Gamma\overset{\triangleleft}{Z}B$

<u>Proof</u>

I compare triangles AΔE and ΓZB. These have:

AB=ΔΓ (from hypothesis)

AΔ=BΓ (from hypothesis)

As (sco) the figure is a not right-angled paral-

lelogram, all it sides are congruent thus

AE=ZΓ and AE//ZΓ. From the 3<u>o</u> criterion (sco)

of the congruency of triangles, where the two triangles are congruent when they have

2 corresponding (sco) sides congruent, we see that the triangles

AΔE and ZBΓ are congruent.

P[32] gives an inadequate answer. I want to emphasize her efforts to follow the ritual element in writing down the hypothesis, the conclusion and the proof procedure clearly and explicitly. In this respect I characterised the proof scheme as EC.R.. Where the assertions contained in hypothesis, conclusion, and proof are concerned: elements such as the perpendicular to the sides of the parallelogram from vertices A and Γ are lacking from hypothesis, but I do not think this particularly important. The conclusion is a repetition of Question T34b. In the proof, although she refers to triangles *AΔE* and *ΓZB* she appeals to the equality *AB=ΔΓ*, which is irrelevant to the triangles. Then she refers to the equality *AE=ZΓ* as a consequence of the congruency of the sides of the parallelogram, which is again irrelevant. Finally she appeals to the third criterion of congruency, asserting that it refers to two sides only, which is a distortion of whichever criterion she means. Deforming the formulation, and making arbitrary or irrelevant assertions are evidence of the EC.NRS. proof scheme. Under these considerations I classify P[32]'s answer as a mixture of the EC.NRS. and EC.R. proof schemes.

Table 5.5.1 illustrates the overview of answers to Question T34b. The table shows one of the lowest incidences of clear D.T. proof schemes in the whole test, namely 8 (9.41%). At the same time the D.T. proof schemes appears alone and in mixture with other proof schemes significantly in more answers than in Questions as in T31, T32a at 31 (36.47%). Thus, taking this evidence of D.T. presence as an indicator

PROOF SCHEMES OBSERVED IN THE RESPONSES TO QUESTION T34b				
PROOF SCHEME	***FREQUENCY***	***CUMULATIVE FREQUENCY***	***RELATIVE FREQUENCY (%)***	***CUMULATIVE RELATIVE FREQUENCY (%)***
D.T.	*8*	*8*	*9.41*	*9.41*
D.T.-EC.NRS.	*16*	*24*	*18.82*	*28.23*
D.T.-E.P.	*7*	*31*	*8.24*	*36.47*
E.P.	*5*	*36*	*5.88*	*42.35*
E.P.-EC.NRS.	*8*	*44*	*9.41*	*51.76*
EC.NRS.	*19*	*63*	*22.35*	*74.11*
EC.NRS.-EC.R.	*2*	*65*	*2.35*	*76.46*
N.S.	*20*	***85***	*23.53*	***99.99***
SUM	***85***		***99.99***	

Table 5.5. 1 Summary of Question T34b proof schemes

the performance of the participants is overall higher compared to T31, T32a. I take this to imply a readiness for proving which may become more technically fluent in the future.

There are 5 (5.88%) answers containing evidence of E.P. proof schemes and 20 (23.53%) answers where the proof scheme appears alone or in mixture with other proof schemes.

Given the nature of the question it is not surprising that there is no evidence of the E.I. proof scheme.

There are 19 (22.35%) answers containing evidence of the EC.NRS proof scheme, making this the second largest group. Overall there are 45 (52.93%) instances of EC.NRS. in the answers of this question.

Question T34b is the first question in T3 with some evidence of the EC.R. proof scheme.

Finally this question had the smallest number of NS is the whole of T3.

5.6 Analysis of responses to Question T35

In this question the position of the triangles is deliberately drawn to explore whether their relatively unusual position causes the students problems with proving. Of course, the main purpose was to explore what evidence of proof schemes would emerge.

I take answers appealing without justification to the equality of sides or angles, which are indeed congruent as evidence of an E.P. proof scheme. If an answer appeals to the equality of sides or angles and includes irrelevant or arbitrary justification this is evidence of an EC.NRS. proof scheme.

The answers to T35 fell into eight groups: proof schemes D.T., D.T.-E.P., D.T.-EC.NRS., E.I., E.P., E.P.-EC.A., E.P.-EC.NRS., EC.NRS., EC.NRS.-EC.R. and NS. During the analysis I found some answers which could be characterised by a mixture of three proof schemes; basically I speak again of the ritual element. There are 24 answers where the ritual element is present to a greater or lesser degree, and of these I have classified 10 as D.T., 3 as D.T.-E.P., 3 as D.T.-EC.NRS., 1 as E.P., 2 as E.P.-EC.NRS., 3 as EC.NRS., 2 as EC.NRS.-EC.R.. In the D.T. answers the ritual character does not have the negative connotation that it has in Harel and Sowder's taxonomy. There are also 1 E.P. and 3 EC.NRS. answers in which the EC.R.

element is present but not counted in the table of answers. Thus there are these two answers where I thought that the EC.R. element could be included in the characterisation as an indication of the presence of the respective proof scheme.

In the following I present examples of participants' responses in the order cited above.

P[37] (see Figure 5.6.1) writes:

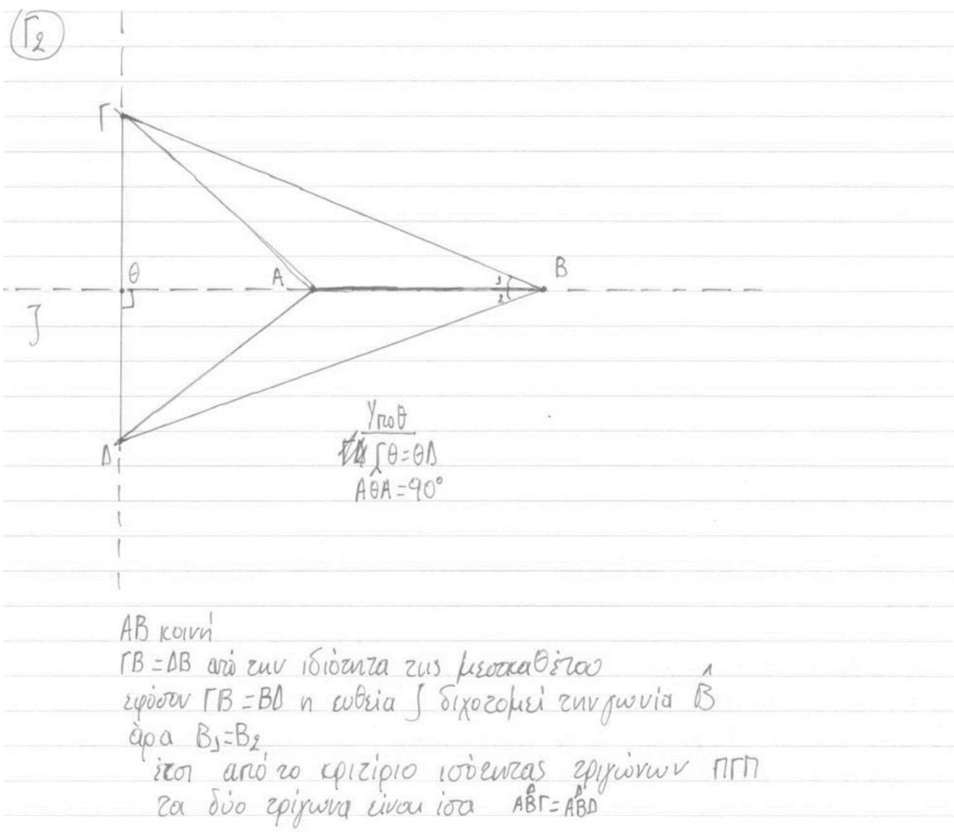


Figure 5.6. 1 Participant's [37] response to question T35

Hypoth

(sco) $\Gamma\Theta=\Theta\Delta$

$$\widehat{A\Theta A} = 90°$$

AB common

ΓΒ=ΔΒ from the property of the perpendicular bisector

since ΓΒ=ΒΔ the straight line ζ bisects angle B

thus $B_1=B_2$

and so from the congruency criterion SAS *the two triangles are congruent* $\overset{\triangle}{AB\Gamma} = \overset{\triangle}{AB\Delta}$

P[37] gives an adequate answer. He begins with the hypothesis of T35. There is an inaccuracy in $A\hat{\Theta}A = 90°$; probably he meant $A\hat{\Theta}\Delta = 90°$, but this inaccuracy is negligible. Then P[37] appeals to the fact that *AB* is a side common to the triangles $\overset{\triangle}{AB\Gamma}$ and $\overset{\triangle}{AB\Delta}$. While he does not mention the triangles it is clear in what follows that he is referring to them. He then appeals to the property of the perpendicular bisector in order to establish the relations $\Gamma B = \Delta B$ and $B_1 = B_2$. In the equality of the equality of the angles there is a minor inaccuracy in the absence of the angle symbol. The justification of this last equality is adequate, although slightly cryptic. Finally the invoked congruency elements indeed constitute the criterion SAS. Under these considerations the answer provides evidence of a D.T. proof scheme.

P[68] (see Figure 5.6.2) writes:

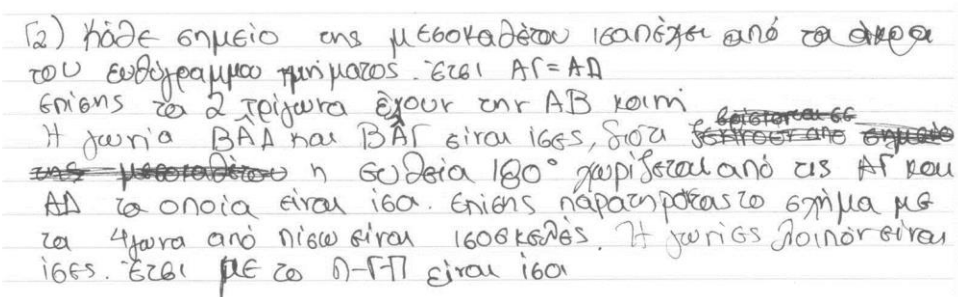


Figure 5.6. 2 Participant's [68] response to Question T35

Γ2) Every point of the perpendicular bisector is equidistant from the endpoints

of the line segment. Thus ΑΓ=ΑΔ

also the two triangles share AB

The angle $\widehat{BA\Delta}$ *and* $\widehat{BA\Gamma}$ *are equal, because (sco)*

(sco) the straight line 180° *is divided by ΑΓ and*

ΑΔ which are equal. Also observing the figure with

the 4angles from behind is isosceles. Thus the angles are

congruent. Thus by virtue of SAS are equal.

P[68] gives a partly adequate answer. Referring to the triangles $\overset{\triangleleft}{AB\Gamma}$ and $\overset{\triangleleft}{AB\Delta}$ she justifies the equality $A\Gamma=A\Delta$ by the property of the perpendicular bisector and adds that *AB* is shared by the two triangles. Then she has a problem, having chosen the previously mentioned equal sides, proving that angles $\widehat{BA\Delta}$ and $\widehat{BA\Gamma}$ included by the pairs *BA, AΔ* and *BA, AΓ* respectively, are equal. At this point, instead of justifying the equality she writes "*Also observing the figure with the 4angles from behind is isosceles*". The ambiguous "*4angles*" could probably refers to triangle *AΓΔ*. However, the justification of why *AΓΔ* is an isosceles triangle and why this fact leads to the equality of the angles in question is substituted by the verb "*observing*". Thus the angles' equality is based on a perception of properties judging from the figure and not by logical arguments. Here part of the answer provides evidence of a D.T. proof scheme and part of an E.P. proof scheme, thus the answer is classified as a mixture of the two.

P[79] (see Figure 5.6.3) writes:

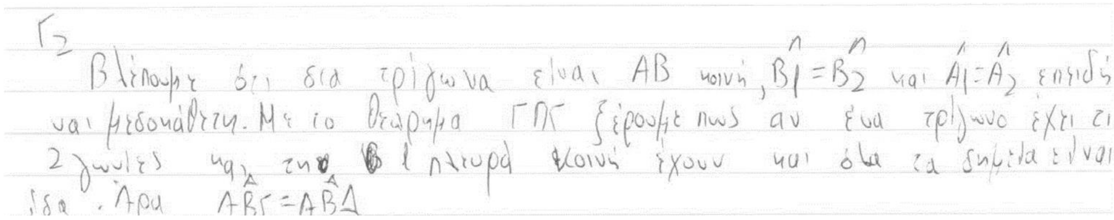


Figure 5.6. 3 Participant's [79] response to Question T35

Γ2

We see that for the triangles it is valid AB is common, $\hat{B}_1 = \hat{B}_2$ *and* $\hat{A}_1 = \hat{A}_2$ *because it is perpendicular bisector. By the theorem ASA we know that if a triangle has 2 angles and the (sco) 1 side common they have also all the points the same. Thus* $\overset{\triangleleft}{AB\Gamma}$ and $\overset{\triangleleft}{AB\Delta}$.

P[79]'s answer is partly adequate. He is clearly trying to prove that the two triangles share a common side, which is included between congruent pairs of sides. In this respect the answer provides evidence of D.T. proof scheme. Observing his notes in the figure (see Figure 5.6.4)

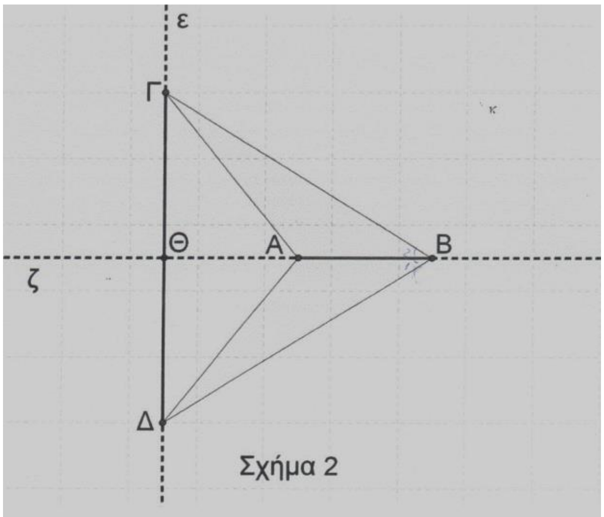


Figure 5.6. 4 Participant's [79] notes on figure of Question T35

we certify the existence of $\hat{B}_1$ and $\hat{B}_2$. But $\hat{A}_1$ and $\hat{A}_2$ are not noted, which gives the symbolism in the script an arbitrary character. His assertion that the pairs of angles in question are equal because of the perpendicular bisector is also cryptic. Even if we put aside these objections the formulation of the criterion is false, as it is not sufficient for two triangles to have two equal angles and a side but the <u>included</u> side to be congruent. Thus the answer provides evidence of both D.T. and an EC.NRS. proof scheme, the latter element due to the arbitrariness of the symbolism and mis-formulation of the appropriate congruency criterion.

P[17] (see Figure 5.6.5) writes:

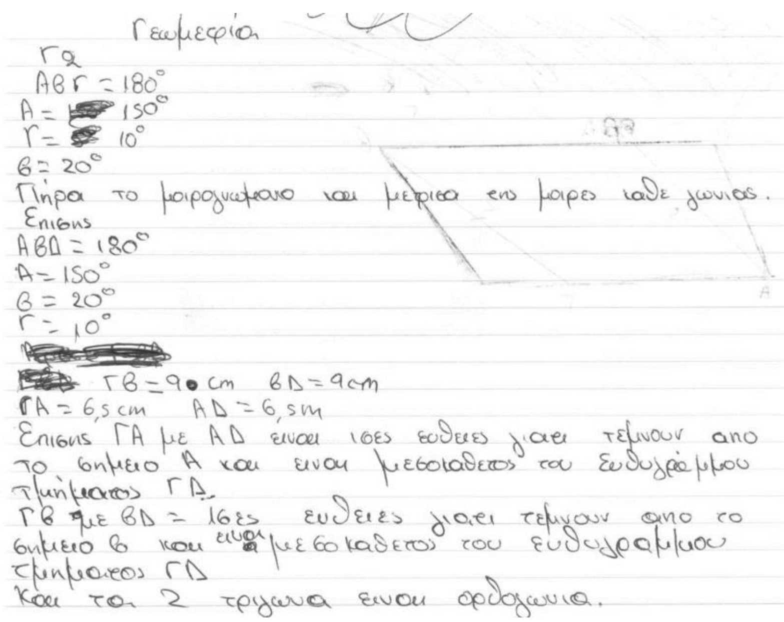


Figure 5.6. 5 Participant's [17] response to Question T35

Geometry

Γ2

ΑΒΓ=180°

A=150°

Γ=10°

B=20°

I took the protractor and measured the degrees of every angle.

Also

ΑΒΔ=180°

A=150°

B=20°

Γ=10°

(sco)

(sco) ΓΒ=9 *cm ΒΔ*=9 *cm*

ΓΑ=6,5 *cm ΑΔ*=6,5 *cm*

Also ΓΑ with ΑΔ are congruent lines because they cut from

the point A and it is perpendicular bisector of the line

segment ΓΔ.

ΓΒ and = congruent lines because they cut from

the point B and is perpendicular bisector of the line

segment ΓΔ

and the two triangles are right-angled.

P [17] gives an inadequate answer, the main aspect of which is the measurement of the lengths of the sides of the triangles whose congruency she is asked to prove. The need to find concrete numbers, either assigned or by measurement, representing variable magnitudes with which to formulate an argument or justify an assertion is evidence of E.I. proof scheme. There are also elements of an EC.NRS. proof scheme where P[17] attempts to formulate the property of the perpendicular bisector, but I neglected this element. P[17] speaks of right-angled triangles, probably meaning triangles *ΑΘΓ* and *ΑΘΔ* or *ΒΘΓ* and *ΒΘΔ*. In any case the assertion, valid or not, is irrelevant. Eventually I decided that the answer as provides evidence of proof scheme E.I. because this is the only answer in which there is measurement of the elements of the figure in accordance with Harel and Sowder's theoretical description.

Participant [05] writes (see figure 5.6.6):

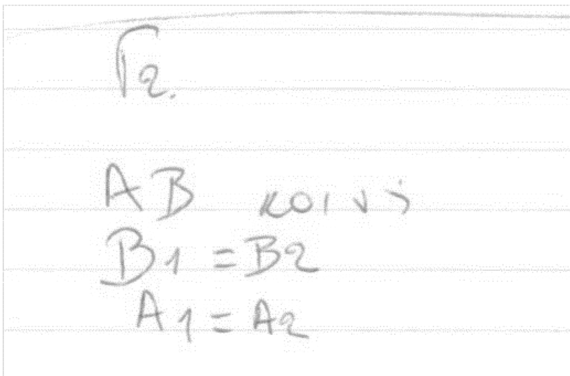


Figure 5.6. 6 Participant's [05] response to Question T35

Γ2.

AB common

$B_1=B_2$

$A_1=A_2$

P[05]'s answer is inadequate. P[05] cites three equalities of elements of the triangles in question which are needed to support the congruency of the triangles, without any explanation. Not even the congruency criterion is named. Thus P[05] sees the equal elements that lead to the congruency the triangles in the figure without any support and justification. Seeing properties in a figure, valid or not, without any justification is evidence of an E.P. proof scheme. Under these considerations the answer characterised respectively.

Participant [51] writes (see figure 5.6.7):

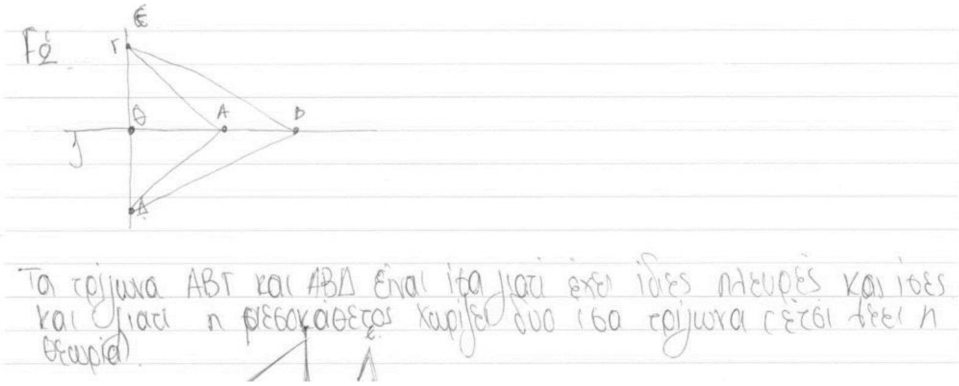


Figure 5.6. 7 Participant's [51] response to Question T35

Triangles ΑΒΓ and ΑΒΔ are congruent because it has same sides and congruent because the perpendicular bisector cuts in two congruent triangles (so says the theory).

P [51] answer is inadequate. The main body of the answer supports the congruency of the triangles as a consequence of an alleged property of the perpendicular bisector to bisect two congruent triangles. I put aside the touch of EC.NRS. in the answer and focus on the perception of congruency seen in the figure or in other words on the evidence of an E.P. proof scheme. P[51] concludes her argument by appealing to "the theory". Although it is not clear which 'theory' she is referring to, and whether from a teacher or book, the formulation is characteristic of an EC.A. proof scheme in which the truth of an assertion is supported by appealing to an authority. In other words one cannot consider an answer as D.T. because a D.T. proof scheme cites theory explicitly. This answer is a unique example in this respect and is classified as containing evidence of a mixture of E.P. and EC.A. proof schemes.

P[53] (see Figure 5.6.8) writes:

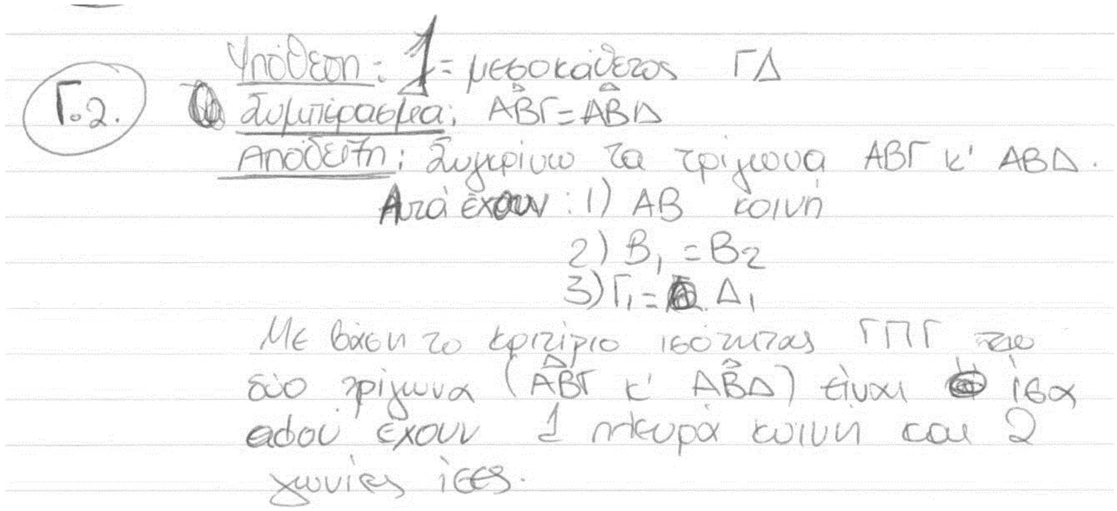


Figure 5.6. 8 Participant's [53] response to Question T35

Hypothesis : ζ=perpendicular bisector ΓΔ

(Γ.2.) (sco) Conclusion : $\overset{\triangleleft}{AB\Gamma} = \overset{\triangleleft}{AB\Delta}$

Proof : I compare triangles ΑΒΓ & ΑΒΔ

These have : 1) AB common

2) $B_1=B_2$

3) Γ_1*=(sco)* Δ_1

On the basis of the congruency criterion ASA the

two triangles ($\overset{\triangleleft}{AB\Gamma}$ & $\overset{\triangleleft}{AB\Delta}$*) are congruent*

since they have 1 side in common and 2

angles congruent.

P[53]'s answer is inadequate. It starts by citing congruent elements of the two triangles. The equality $B_1=B_2$, although correct, is perceived as valid only by looking at the figure, because there is no supportive argument, providing evidence of an E.P. proof scheme. The congruency $\Gamma_1= \Delta_1$ is again not logically supported, and it is not clear which angles P[53] is referring to. Even if it is accepted that she is referring to angles $\widehat{A\Gamma B}$ and $\widehat{A\Delta B}$ the asserted equality thereof is characteristic of an E.P. proof scheme, but on the other hand the whole argument about the triangles' congruency is arbitrary, which is evidence of an EC.NRS. proof scheme. The same can be said about the formulation of the ASA congruency criterion. Indeed the ASA is not correctly formulated by referring to two angles and one side of the respective triangles but included side. Under these considerations there is evidence of both E.P. and EC.NRS. proof schemes and the answer characteristic of a mixture thereof .

P[26] (see Figure 5.6.9) writes:

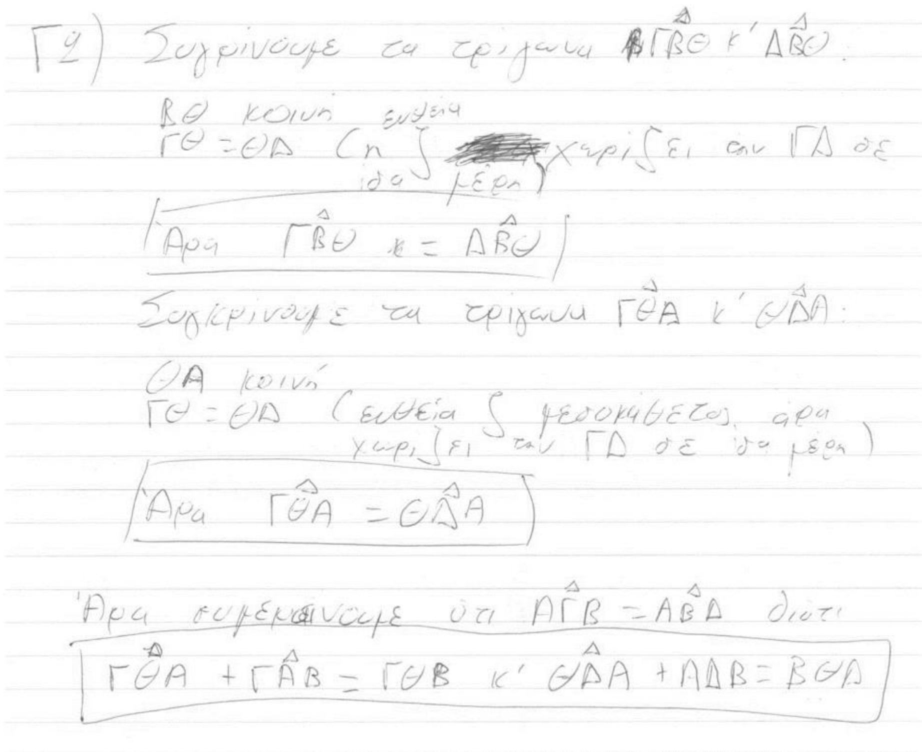

Γ2) Συγκρίνουμε τα τρίγωνα ΓΒ̂Θ κ' ΔΒ̂Θ:
ΒΘ κοινή ευθεία
ΓΘ=ΘΔ (η ζ χωρίζει την ΓΔ σε ίσα μέρη)
Άρα ΓΒ̂Θ = ΔΒ̂Θ
Συγκρίνουμε τα τρίγωνα ΓΘ̂Α κ' ΘΔ̂Α:
ΘΑ κοινή
ΓΘ=ΘΔ (ευθεία ζ μεσοκάθετος άρα χωρίζει την ΓΔ σε ίσα μέρη)
Άρα ΓΘ̂Α = ΘΔ̂Α
Άρα συμπεραίνουμε ότι ΑΓ̂Β = ΑΒ̂Δ διότι
ΓΘ̂Α + ΓÂΒ = ΓΘΒ κ' ΘΔ̂Α + ΑΔΒ = ΒΘΑ

Figure 5.6. 9 Participant's [26] response to Question T35

Γ2) We compare triangles $\Gamma\hat{B}\Theta$ & $\Delta\hat{B}\Theta$

ΒΘ common line

ΓΘ=ΘΔ (the line ζ cuts ΓΔ in equal parts)

Thus $\Gamma\hat{B}\Theta$ & $\Delta\hat{B}\Theta$

We compare triangles $\Gamma\hat{\Theta}A$ & $\Delta\hat{B}\Theta$

ΘΑ common line

ΓΘ=ΘΔ (the line ζ cuts ΓΔ in equal parts)

Thus $\Gamma\overset{\triangleleft}{\Theta}A$ & $\Theta\overset{\triangleleft}{\Delta}A$

Thus we conclude that $A\overset{\triangleleft}{\Gamma}B = A\overset{\triangleleft}{B}\Delta$ *because*

$\Gamma\overset{\triangleleft}{\Theta}B + \Gamma\overset{\triangleleft}{A}B = \Gamma\overset{\triangleleft}{\Theta}B$ & $\Theta\overset{\triangleleft}{\Delta}A + A\overset{\triangleleft}{\Delta}B = B\overset{\triangleleft}{\Theta}\Delta$

P[26] gives an inadequate answer, which he completes in three steps. The first refers to the congruency of triangles $\Gamma\overset{\triangleleft}{B}\Theta$ & $\Delta\overset{\triangleleft}{B}\Theta$. It misuses the criterion because two pairs of equal elements do not suffice to support the congruency unless they are right-angled triangles the equal sides are the appropriate, but the fact that the triangles are both right-angled is missing. Whether P[26] is aware of the latter is not clear. The same is true of step two. As for step three, P[26] supports an arbitrary idea of the sum of the triangles which is neither defined nor described. The arbitrary formulation and misuse of triangle congruency criteria and the arbitrary invention of a 'law' adding triangles provide evidence of an EC.NRS. proof scheme and the answer is characterised accordingly.

P[38] (see Figure 5.6.10) writes:

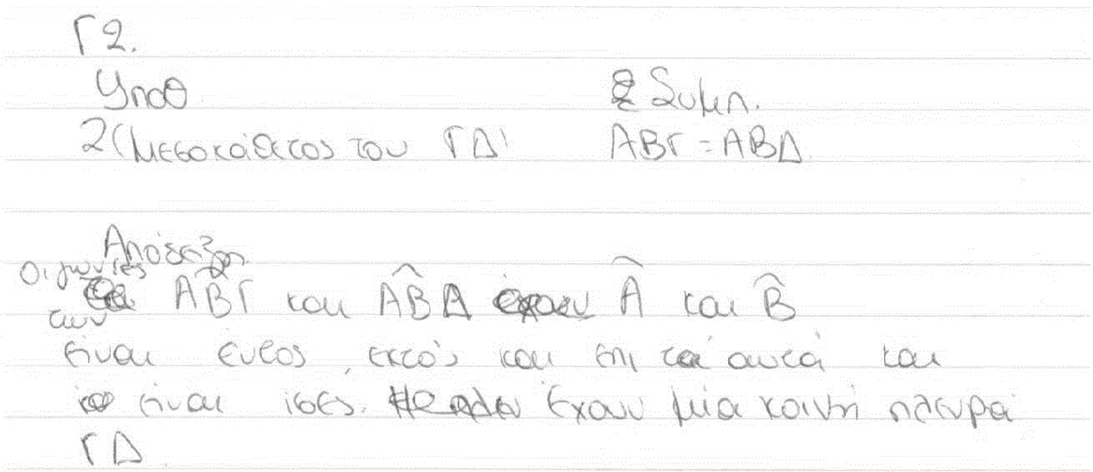


Figure 5.6. 10 Participant's [38] response to Question T35

Hypoth *Concl*

ζ (perpendicular bisector of ΓΔ) *ΑΒΓ=ΑΒΔ*

Proof

The angles of (sco) $\widehat{AB\Gamma}$ *and* $\widehat{AB\Delta}$ *(sco)* $\hat{A}$ *and* $\hat{B}$

are corresponding angles and

(sco) are congruent. (sco) They have a side in common

ΓΔ

P[38] answer is inadequate. At the beginning it seems from the formulation that she uses the angle symbol to symbolise triangles. Then she asserts that angles $\hat{A}$ and $\hat{B}$ are congruent because of parallel lines. This arbitrary assertion is not supported by any justification. Finally she asserts, again arbitrarily, that the triangles in question have in common the side *ΓΔ*, whereas none of the triangles have as a side the line segment *ΓΔ*. The arbitrariness of the assertions constitutes evidence of EC.NRS. proof scheme. At the same time we observe that the ritual element is present where P[38] writes the hypothesis and conclusion in an orderly way and announces the proof. From this point of view P[38]'s answer provides evidence of both EC.R. and EC.NRS. proof scheme and is characterised accordingly.

Table 5.6.1 illustrates the overview of answers to T35. The biggest group 22 (25.88%) of those who answered T35 includes evidence of D.T. proof schemes in their scripts. The total appearance of D.T. proof scheme, alone or in mixture with others, is even higher at 38 (44.70%).

There is only one 1 (1.18%) answer characterised as containing evidence of an E.I. proof scheme; three (3.53%) of E.P., and in total 17 (20.00%) appearances of E.P. proof scheme;

15 (17.65%)of EC.NRS. and in total 26 (30.59%) appearances of this proof scheme. See also my earlier comments on the EC.R. proof scheme. Finally 22 (25.88%) participants did not answer this question (NS).

Judging by the aforementioned numbers, and particularly from the D.T 22 (25.88%) and total D.T. 38 (44.70%), the participants' performance can be considered as impressive. Even those answers without D.T. elements sometimes include allusions to knowledge about the congruency criteria but they cannot yet correctly articulate a proof. I see this as rather natural

PROOF SCHEMES OBSERVED IN THE RESPONSES TO QUESTION T35				
PROOF SCHEME	***FREQUENCY***	***CUMULATIVE FREQUENCY***	***RELATIVE FREQUENCY (%)***	***CUMULATIVE RELATIVE FREQUENCY (%)***
D.T.	*22*	*22*	*25.88*	*25.88*
D.T.-E.P.	*10*	*32*	*11.76*	*37.64*
D.T.-EC.NRS.	*6*	*38*	*7.06*	*44.70*
E.I.	*1*	*39*	*1.18*	*45.88*
E.P.	*3*	*42*	*3.53*	*49.41*
E.P.-EC.A	*1*	*43*	*1.18*	*50.59*
E.P.-EC.NRS.	*3*	*46*	*3.53*	*54.12*
EC.NRS.	*15*	*61*	*17.65*	*71.77*
EC.NRS.-EC.R.	*2*	*63*	*2.35*	*74.12*
NS	*22*	***85***	*25.88*	***100.00***
SUM	***85***		***100.00***	

Table 5.6. 1 Summary of Question T35 proof schemes

for students taught proof for the first time.

5.7 Analysis of responses to Question T36a

Question T36a allows observation of how the change of context affects the participants' efficacy at formulating a proof. T35 is practically the same asT36a from the point of view of the triangle congruency criterion SSS; the difference lies in the form in which T36 is offered to the participants. In T35 the participants are given a common side and then have to recognise the congruency of two missing pairs of sides by invoking the property of the perpendicular bisector. Additionally, the position of the triangles, in the figure drawn for Question T35, is rather unusual. For Question T36a, three pairs of congruent sides are given directly and clearly from the beginning. Thus there is no need for any other justification apart from invoking the appropriate criterion SSS.

As before unjustified but valid assertions about congruent pairs of sides or angles are taken as evidence of E.P. proof scheme and arbitrary justified assertions are taken as evidence of an EC.NRS. proof scheme.

The answers fall into seven groups: those characteristic of the proof schemes D.T., D.T.-E.P., D.T.-EC.NRS., E.P., E.P.-EC.NRS., EC.NRS., and NS.

The 'ritual' element as discussed earlier is present as well in 22 answers to T36a to greater or lesser degrees. Of these 19 are characteristic of D.T. neutralising any negative aspect of the rituality; one is E.P.-EC.NRS., one is EC.NRS. and a student did not answer. This is most impressive NS of participant [39] (see Figure 5.7.1).

Participant [39] writes (see figure 5.7.1):

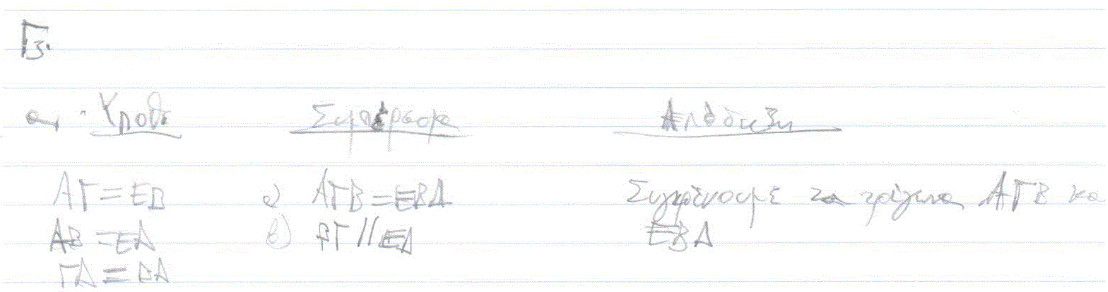


Figure 5.7. 1 Participant's [39] response to Question T36ab

Γ3.

Hypoth	*Conclusion*	*Proof*
ΑΓ=ΕΒ	*α) ΑΓΒ=ΕΒΔ*	*We compare the triangles ΑΓΒ and*
ΑΒ=ΕΔ	*β) ΒΓ//ΕΔ*	*ΕΒΔ*
ΓΔ=ΒΑ		

P [39] repeats the data given in the problem as taught by J in ritual manner. However, under 'proof' he only states which triangles are to be compared. The sentence should also include "these are congruent according to the SSS criterion". Thus we have an NS answer with all the rituality retained.

In the following I present examples of answers characteristic of the above proof schemes in the same order.

P[32] (see Figure 5.7.2) writes:

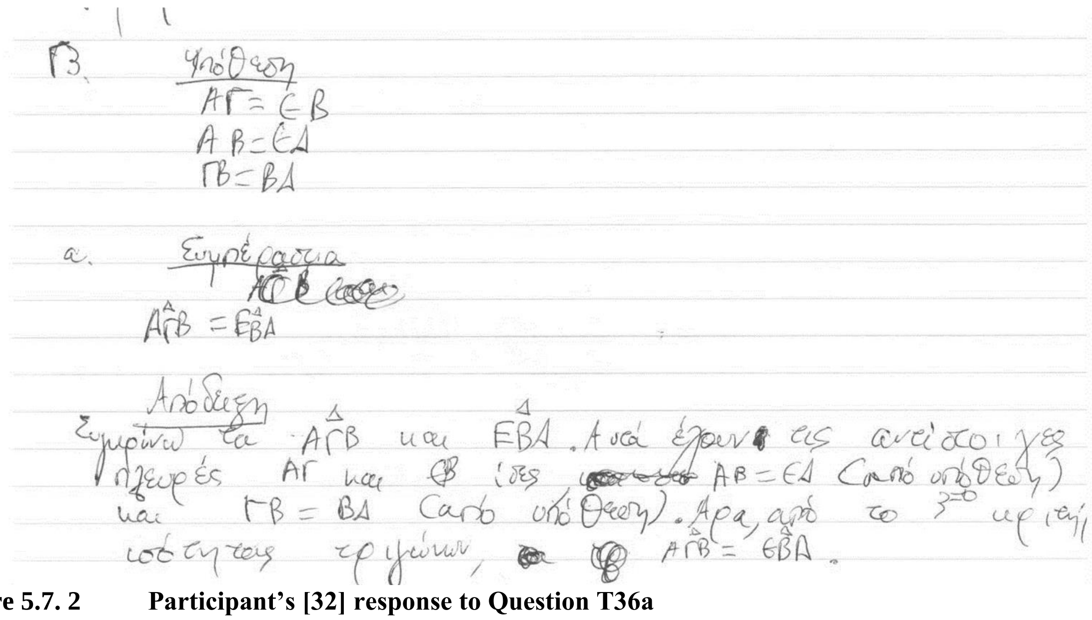


Figure 5.7. 2 Participant's [32] response to Question T36a

Γ3. Hypothesis

ΑΓ=ΕΒ

ΑΒ=ΕΔ

ΓΒ=ΒΔ

α. *Conclusion*

(sco)

$\overset{\triangleleft}{A\Gamma B} = \overset{\triangleleft}{EBA}$

Proof

I compare $\overset{\triangleleft}{A\Gamma B}$ *and* $\overset{\triangleleft}{EBA}$. *These have the respective sides ΑΓ and ΕΒ congruent (sco) ΑΒ=ΕΒ (from hypothesis) and ΓΒ=ΒΔ (from hypothesis). Thus from the 3rd criterion of triangles' congruency, (sco)* $\overset{\triangleleft}{A\Gamma B} = \overset{\triangleleft}{EBA}$

P [32] gives an adequate answer. She first sets out the data and the conclusion in an orderly way, and in the proof she invokes the hypothesis and the appropriate criterion to prove the triangles' congruency. A minor inaccuracy concerning the sides *ΑΓ* and *ΕΒ* where the hypothesis is not invoked is taken as negligible. Under these considerations the answer is characteristic of a D.T. proof scheme.

P[69] (see Figure 5.7.3) writes:

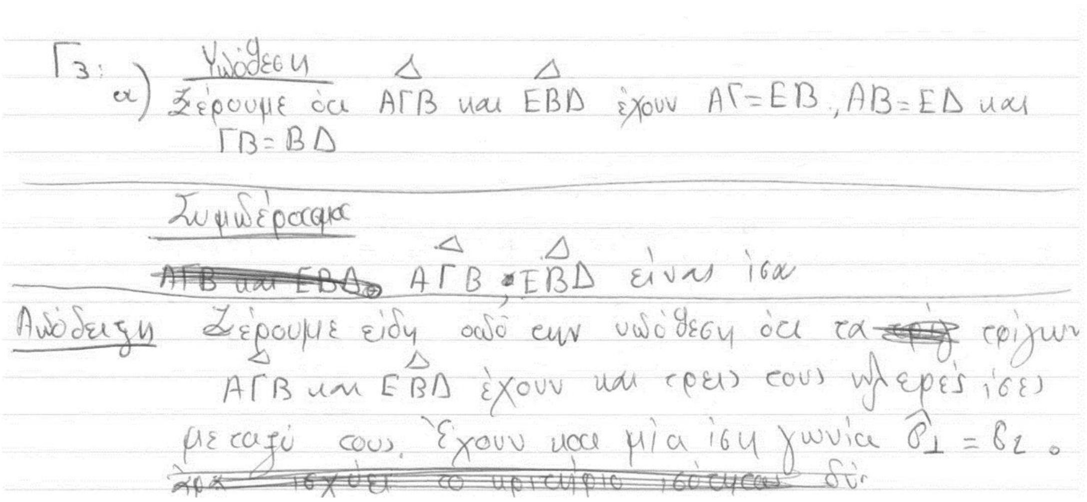


Figure 5.7. 3 Participant's [69] response to Question T36a

Γ3 *Hypothesis*

α) We know that $\overset{\triangleleft}{A\Gamma B}$ *and* $\overset{\triangleleft}{EB\Delta}$ *have ΑΓ=ΕΒ, ΑΒ=ΕΔ and*

ΓΒ=ΒΔ

Conclusion

(sco) $\overset{\triangleleft}{A\Gamma B}$, $\overset{\triangleleft}{EB\Delta}$ *are congruent*

Proof *We know already from the hypothesis that the (sco) triangles*

$\overset{\triangleleft}{A\Gamma B}$ *and* $\overset{\triangleleft}{EB\Delta}$ *have all three sides congruent*

to each other. They have as well an angle $\widehat{B_1} = \widehat{B_2}$ *.*

(sco)

P[69] gives a partly adequate answer. Writing the proof she appeals to the congruency of the three pairs of sides. But although this is sufficient to support the congruency of the triangles she feels the need to add that $\widehat{B_1} = \widehat{B_2}$, which does not justify logically. Thus the assertion is made from her judgment of what she perceives looking at the figure. Assertions of validity of properties of geometric objects only by looking at a figure offers evidence of an E.P. proof scheme. Under this point of view the answer of P[69] provides evidence of D.T. and E.P. proof schemes and has been characterized accordingly.

P[52] (see Figure 5.7.4) writes:

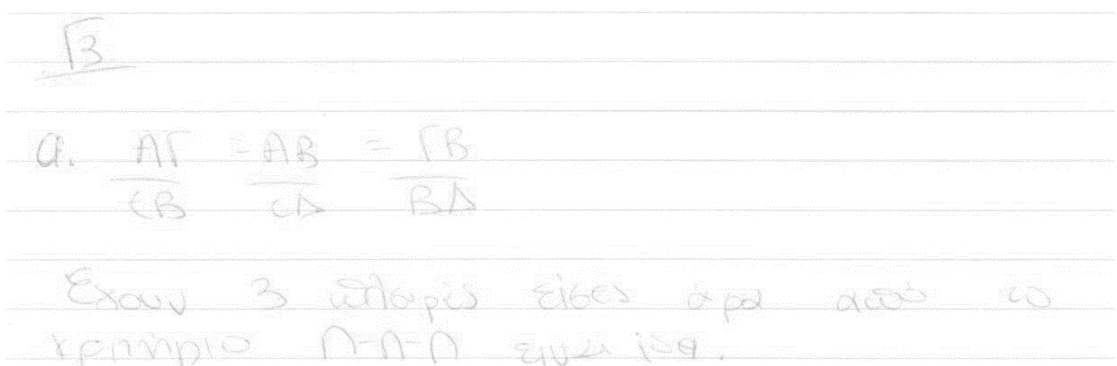


Figure 5.7. 4 Participant's [52] response to Question T36a

Γ3

α. *ΑΓ* = *ΑΒ* = *ΓΒ*

ΕΒ *ΕΔ* *ΒΔ*

They have 3 sides congruent thus from the

criterion SSS are congruent.

P[52] gives an adequate answer appealing to the correct criterion for triangles' congruency but uses an arbitrary symbol resembling the fraction symbol for the congruent sides. Thus the answer provides evidence of both D.T. and EC.NRS. proof schemes.

P[03] (see Figure 5.7.5) writes:

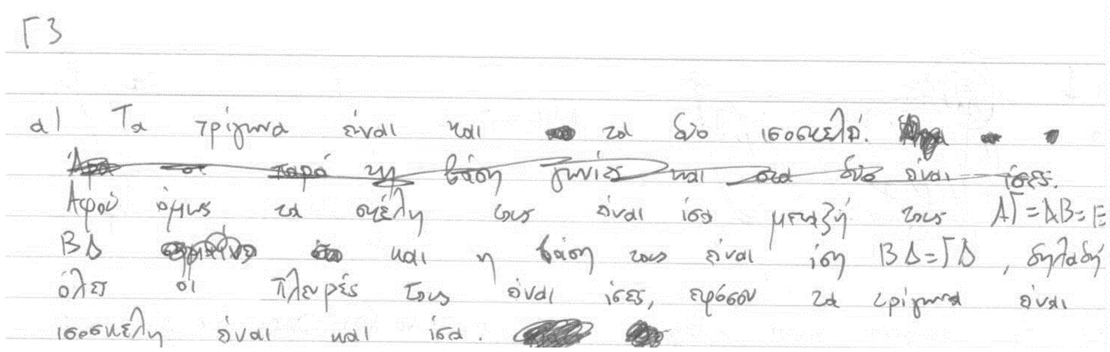


Figure 5.7. 5 Participant's [03] response to Question T36a

Γ3

α) The triangles are (sco) both isosceles. (sco)

(sco)

But since the equal sides of each are congruent to each other ΑΓ=ΑΒ=Ε

ΒΔ (sco) and their bases are congruent ΒΔ=ΓΔ thus

all their sides are congruent since the triangles are

isosceles they are congruent (sco).

P[03]'s answer is inadequate. The whole proof is based almost exclusively on the perception that the triangles are both isosceles triangles rather than on their three pairs of congruent sides. I put aside the inaccuracy in writing the sides that he sees "making" the triangles isosceles ones. Under these considerations P[03]'s answer has perceptual elements that overshadow any correct use of the data given, providing evidence for an E.P. proof scheme and thus the answer is characterized accordingly.

P[82] (see Figure 5.7.6) writes:

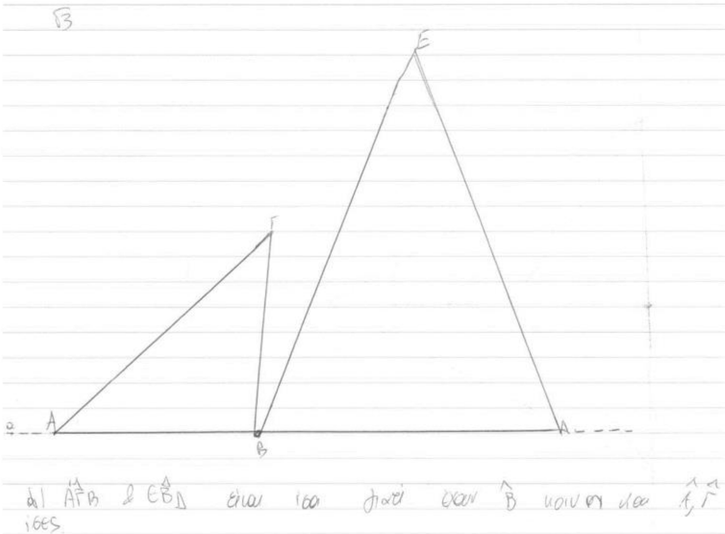


Figure 5.7. 6 Participant's [82] response to Question T36a

α) $\overset{\triangleleft}{A\Gamma B}$ & $\overset{\triangleleft}{EB\Delta}$ *are congruent because they have* $\hat{B}$ *in common and* $\hat{A}, \hat{\Gamma}$ *equal*

P[82]'s answer is inadequate. On the one hand, looking at the figure he sees properties but does not justify them, and in this respect the answer is characteristic of an E.P. proof scheme. However, even if the perceptions were both true they do not offer a basis from which to appeal to triangles' congruency criteria. P[82] arbitrarily asserts that his perceptions suffice to support the congruency of the triangles, thus misusing triangle congruency criteria. This second aspect is characteristic of an EC.NRS. proof scheme, and thus this answer of P[82] is classified as a mixture of E.P. and EC.NRS..

P[47] (see Figure 5.7.7) writes:

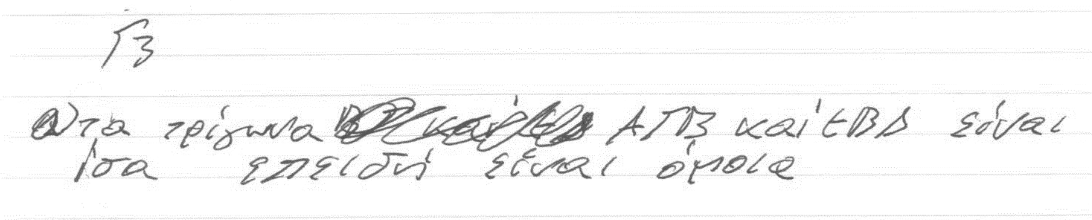


Figure 5.7. 7 Participant's [47] response to Question T36a

Γ3

α) The triangles (sco) ΑΓΒ and ΕΒΔ are

congruent because they are similar

P [47] answer is not adequate in that it asserts that triangles *ΑΓΒ* and *ΕΒΔ* are equal because they are similar. This constitutes a misuse of the criteria for the congruency of triangles and arbitrary invention of a criterion without logical support and is characteristic of an EC.NRS. proof scheme.

Table 5.7.1 illustrates the overview of answers to T36a. The biggest group shown in Table 5.7.1 is D.T., 42 (49.41%). The total number of D.T. appearances is 45 (52.94%); these can be compared to the results for Question T35, which are 22 (25.88%) and 38 (44.70%) respectively. This gives us a measure of the influence of the context on the difficulty of formulating a proof: in T35 the three pairs of congruent sides are not given directly and one has to arrive at the point of being able to use the SSS criterion by way of using the property of the perpendicular bisector. This task has proved to be complicated judging by the numbers of D.T. answers. When the context of a question calls for the application of the SSS criterion more directly, the criterion is recognised by many more participants. This is evident in the shift of D.T. number of answers in T35 and T36a: from the 22 in T35 to 42 in T36a, almost double.

PROOF SCHEMES OBSERVED IN THE RESPONSES TO QUESTION T36a				
PROOF SCHEME	***FREQUENCY***	***CUMULATIVE FREQUENCY***	***RELATIVE FREQUENCY (%)***	***CUMULATIVE RELATIVE FREQUENCY (%)***
D.T.	*42*	*42*	*49.41*	*49.41*
D.T.-E.P.	*1*	*43*	*1.18*	*50.59*
D.T.-EC.NRS.	*2*	*45*	*2.35*	*52.94*
E.P.	*2*	*47*	*2.35*	*55.29*
E.P.-EC.NRS.	*4*	*51*	*4.71*	*60.00*
EC.NRS.	*8*	*59*	*9.41*	*69.41*
N.S.	*26*	***85***	*30.59*	***100.00***
SUM	***85***		***100.00***	

Table 5.7. 1 Summary of Question T36a proof schemes

D.T. and N.S. answers apart, there are 17 answers distributed to the remaining groups of proof schemes. The biggest of these is the EC.NRS. group, 8 (9.41%).

There are 26 (30.59%) NS, compared to the 22 (25.88%) of T35.

These results were generally expected, particularly the numbers in the D.T. groups given the ease with which the congruency criteria are invoked by the formulation of the question. It seems that a rather big number of participants can handle the proof issues in this question well. What is less expected – and less easily interpreted – is the substantial N.S. number.

5.8 Analysis of responses to Question T36b

Students' efforts to prove the second part of the question were again expected to be interesting as they faced relatively complicated tasks. Their answers fell into six groups: those providing evidence of the proof schemes D.T., D.T.-EC.NRS., E.P., E.P.-EC.NRS., EC.NRS., and NS.

In the following I present examples of the answers using these proof schemes in the order given here.

P[81] (see Figure 5.8.1) writes:

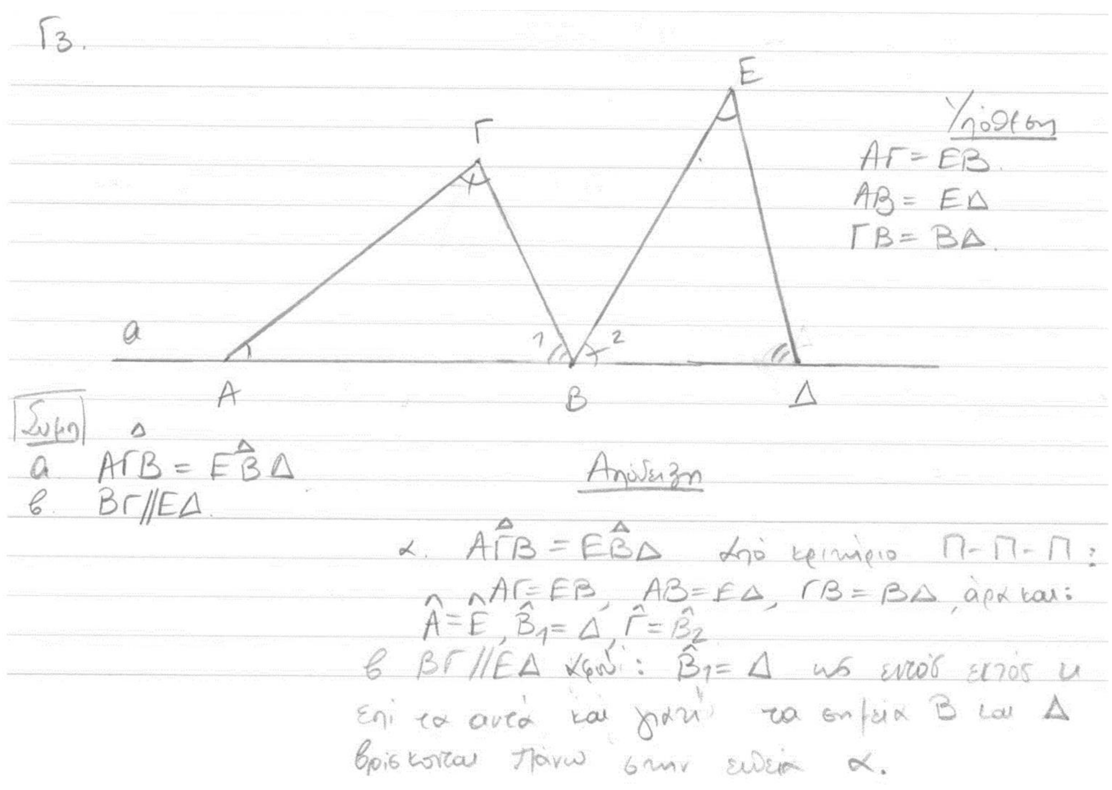


Figure 5.8. 1 Participant's [81] response to Question T36b

<u>*Hypothesis*</u>

ΑΓ=ΕΒ

ΑΒ=ΕΔ

ΓΒ=ΒΔ

α $\overset{\triangleleft}{A\Gamma B} = \overset{\triangleleft}{EB\Delta}$

β $B\Gamma // E\Delta$

<u>*Proof*</u>

α. $\overset{\triangleleft}{A\Gamma B} = \overset{\triangleleft}{EB\Delta}$ *from criterion SSS:*

$A\Gamma=EB$, $AB=E\Delta$, $\Gamma B=B\Delta$ thus as well:

$\hat{A} = \hat{E}, \widehat{B_1} = \hat{\Delta}, \hat{\Gamma} = \widehat{B_2}$

β *$B\Gamma//E\Delta$ since :* $\widehat{B_1} = \hat{\Delta}$ *as corresponding*

and because points B and Δ

lie on straight line α.

P[81] gives an adequate answer to T36b. She asserts that ΒΓ//ΕΔ because $\widehat{B_1} = \hat{\Delta}$. There is a minor ambiguity in the Greek formulation regarding the justification of parallelism which I cannot translate it into English and render absolutely clear. However I put aside this minor ambiguity because earlier P[81] writes "*$B\Gamma//E\Delta$ as :* $\widehat{B_1} = \hat{\Delta}$" which is unambiguous and correct. Under these considerations this answer provides evidence of D.T. proof scheme and has been characterised accordingly.

P[02] (see Figures 5.8.2 and 5.8.3) writes:

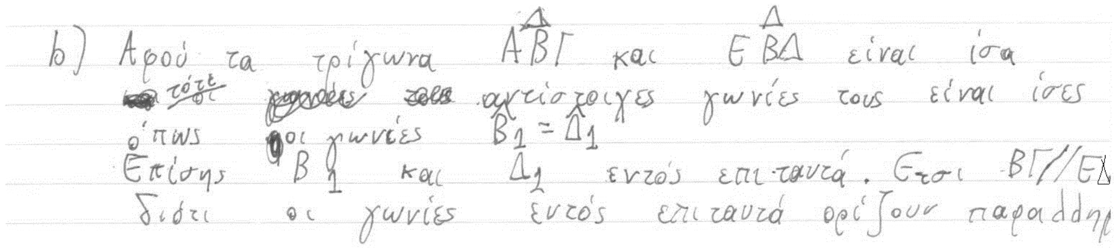

b) Αφού τα τρίγωνα $\overset{\triangle}{AB\Gamma}$ και $\overset{\triangle}{EB\Delta}$ είναι ίσα
τότε ~~γωνίες~~ ~~τους~~ αντίστοιχες γωνίες τους είναι ίσες
όπως οι γωνίες $B_1 = \Delta_1$
Επίσης B_1 και Δ_1 εντός επι ταυτά. Έτσι ΒΓ//ΕΔ
διότι οι γωνίες εντός επι ταυτά ορίζουν παράλληλ

Figure 5.8. 2 Participant's [02] response to Question T36b

b) Since the triangles $\overset{\triangleleft}{AB\Gamma}$ *and* $\overset{\triangleleft}{EB\Delta}$ *are congruent*

(sco) then (sco) their respective angles are congruent

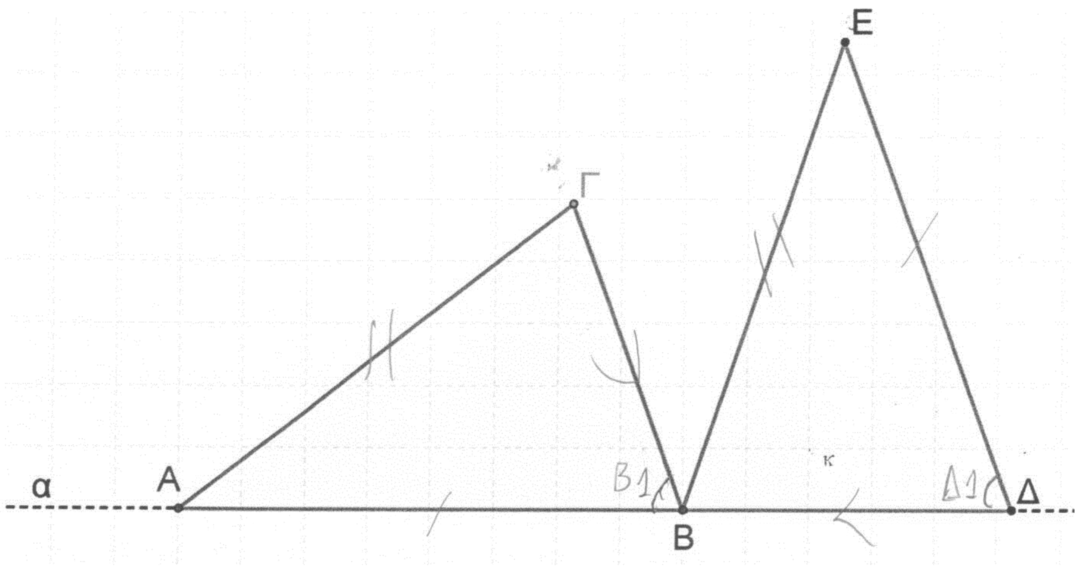


Figure 5.8. 3 **Participant's [02] notes on the figure of Question T36b**

as $\widehat{B_1} = \widehat{\Delta_1}$

Also $\widehat{B_1}$ *and* $\widehat{\Delta_1}$ *are interior alternating angles. Thus ΕΓ//ΒΔ*

because the interior alternating angles define parallels.

P[02] gives a partly adequate answer. From the congruency of the triangles he concludes that angles $\widehat{B_1}$ and $\widehat{\Delta_1}$ are congruent (see figure 5.8.3). Up to this point, even if elliptic in its justification, the proof is adequate. But then P[02] calls the angles "*interior alternating*" constituting a misuse of the terminology because the angles in question are corresponding and not alternating. This misuse of terminology is evidence of EC.NRS. proof scheme. Under these considerations this answer provides evidence of both D.T. and EC.NRS. proof schemes and is classified as a mixture of the two.

P[21] (see Figure 5.8.4) writes:

b) ΒΓ//ΕΔ

ΒΓ is parallel to ΕΔ because:

We draw straight line ΓΕ. Thus a parallelogram ΓΕΒΔ is formed.

Thus it will be ΓΕ//ΒΔ and ΓΒ//ΕΔ.

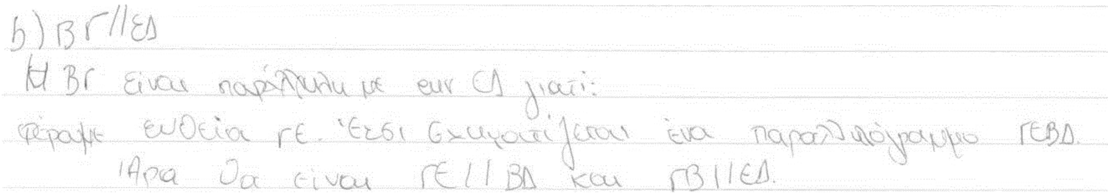


Figure 5.8. 4 Participant's [21] response to Question T36b

P[21]'s answer is not adequate. She refers to the given figure, thinking that if the line *ΓE* is drawn then a parallelogram is formed, of which the order of the vertices is falsely written, but that is minor mistake. Thus looking at the given figure P[21] perceives the existence of parallelogram *ΓEΔB*. Her perception is at the same time her justification of the parallelism of BΓ and ΔE. Perceiving properties in a plane figure without justifying them logically is evidence of an E.P. proof scheme.

P[60] (see Figure 5.8.5) writes:

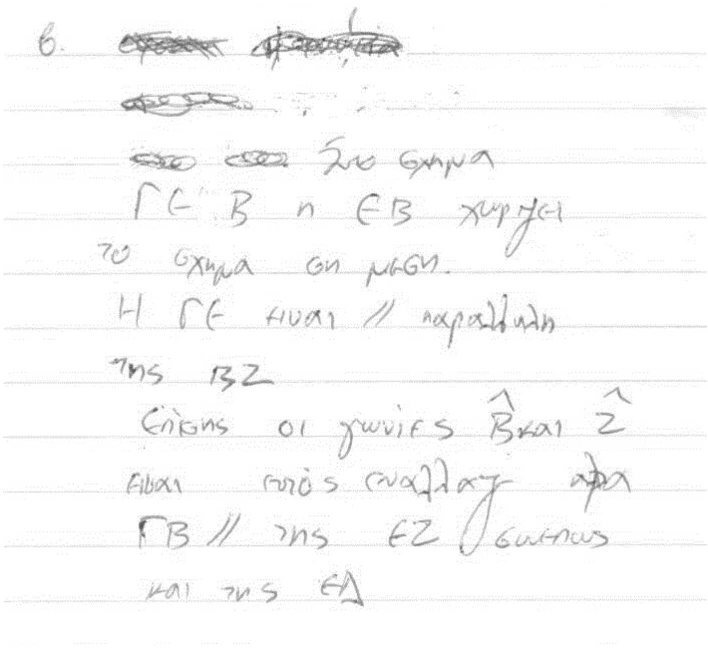


Figure 5.8. 5 Participant's [60] response to Question T36b

β (sco)

(sco)

(sco) In the figure

ΓEZB, EB divides

the figure in the middle.

ΓE is // parallel

to BZ

Also angles $\hat{B}$ *and* $\hat{Z}$

are interior alternating thus

ΓB // to EZ consequently

to EΔ

P [60] answer is not adequate. In her proof she asserts that "*EB divides the figure in the middle*", the meaning of which is not clear. However, whatever the meaning, perceiving a property in a figure without offering a logical argument to support offers evidence of an E.P. proof scheme. P[60] (see figure 5.8.6) goes on to assert that

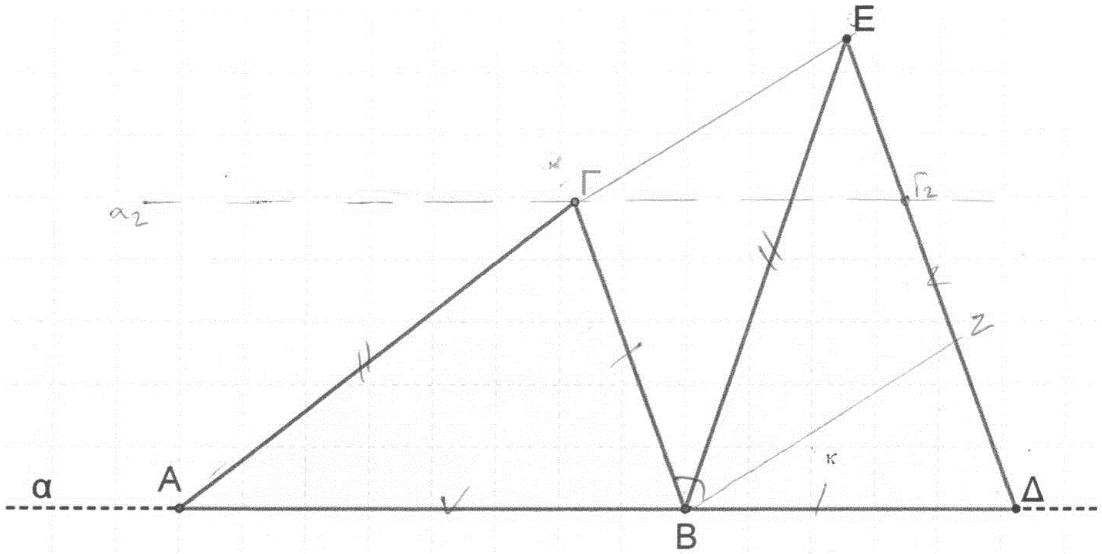


Figure 5.8. 6 Participant's [60] notes on the figure of Question T36b

ΓE//BZ, by which she probably means that *BZ* is drawn parallel to *ΓE*. She adds: "$\hat{B}$ *and* $\hat{Z}$ *are interior alternating*". Judging by the notation on $\widehat{\Gamma BZ}$ probably by $\hat{B}$ she means angle $\widehat{\Gamma BZ}$ and by $\hat{Z}$ she means angle $\widehat{BZ\Delta}$. Then from the fact that $\hat{B}$ and $\hat{Z}$ are interior alternating angles results that ΓB//EZ. This last assertion is arbitrary and a misuse of the notion of 'interior alternating angles' as when the interior alternating angles are congruent, the lines forming them with a transversal are parallel and vice versa. This instance of misuse has to do with the concept image and the concept definition (Tall & Vinner, 1981). In cases such as this there is clearly confusion about the notion of 'interior alternating angles' which is a property of position on the plane independent of the parallelism as we speak of 'interior alternating angles' even when there is no parallelism. Misuse of terminology and the confusion of a concept definition with an idiosyncratic concept image are evidence of an EC.NRS. proof scheme. This answer thus provides evidence of both E.P. and EC.NRS. proof schemes and is characterised accordingly.

P[69] (see Figure 5.8.7) writes:

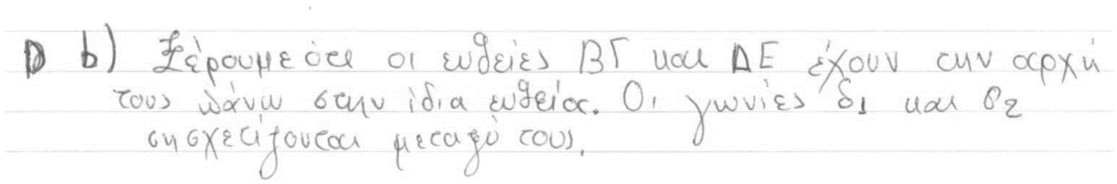


Figure 5.8. 7 Participant's [69] response to Question T36b

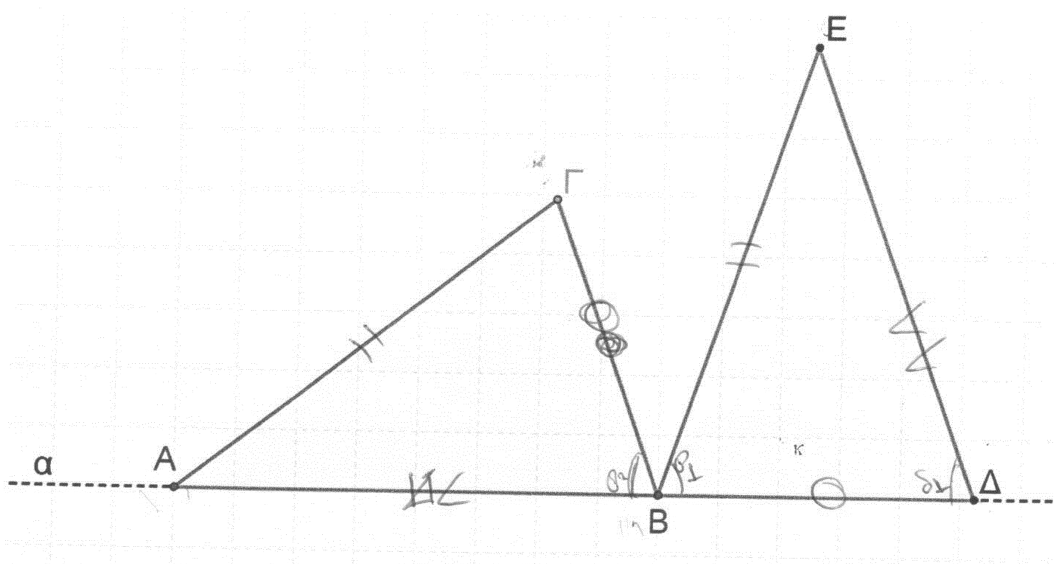


Figure 5.8. 8 Participant's [69] notes on the figure of Question T36b

b) We know that the straight lines ΒΓ and ΔΕ have their origin

on the same line. The angles δ_1 and β_2

are related to each other.

P [69]'s answer is not adequate. He asserts the obvious fact that the line segments *ΒΓ* and *ΔΕ* have one of their endpoints on line α (see figure 5.8.8) and then completes the proof by asserting that "*The angles δ_1 and β_2 are related to each other*". This arbitrary and ambiguous assertion provides evidence of an EC.NRS. proof scheme because it substitutes the usual terminology for congruent corresponding angles with an idiosyncratic formulation. From this point of view P[69]'s answer provides evidence of an EC.NRS. proof scheme and is classified accordingly.

Table 5.8.1 illustrates the overview of answers to T36b. In the table the large NS number, 43 (or 50.59%) is striking. It might be the case that the context of T36b impeded the participants' ability to reach an answer; they were perhaps unable to discern the parallel lines that would help reaching this answer. Two lines intersected by a transversal is one thing; embedding this in a more complicated context and diagram is quite another. Further, many

PROOF SCHEMES OBSERVED IN THE RESPONSES TO QUESTION T36b

PROOF SCHEME	***FREQUENCY***	***CUMULATIVE FREQUENCY***	***RELATIVE FREQUENCY (%)***	***CUMULATIVE RELATIVE FREQUENCY (%)***
D.T.	*4*	*4*	*4.71*	*4.71*
D.T.-EC.NRS.	*4*	*8*	*4.71*	*9.42*
E.P.	*4*	*12*	*4.71*	*14.13*
E.P.-EC.NRS.	*7*	*19*	*8.24*	*22.37*
EC.NRS.	*23*	*42*	*27.06*	*49.43*
N.S.	*43*	***85***	*50.59*	***100.02***
SUM	***85***		***100.02***	

Table 5.8. 1 Summary of Question T36b proof schemes

understood the description of the position of the angles as corresponding or interior alternating to provide the condition for parallelism. The relatively high number of EC.NRS. answers (23 or 27.06%) is also indicative of the difficulties the students faced.

5.9 Summary

In this chapter I have presented the analysis of the T3 test, as summarised in Table 5.9.1.

QUESTIONS OF TEST T3									
PROOF SCHEMES	***T31***	***T32a***	***T32b***	***T33a***	***T33b***	***T34b***	***T35***	***T36a***	***T36b***
D.T.	*14.12*	*8.24*	*35.29*	*44.71*	*42.35*	*9.41*	*25.88*	*49.41*	*4.71*
D.T.-E.I.	*1.18*	*1.18*	*0.00*	*10.59*	*4.71*	*0.00*	*0.00*	*0.00*	*0.00*
D.T.-E.P.	*0.00*	*0.00*	*0.00*	*0.00*	*0.00*	*8.24*	*11.76*	*1.18*	*0.00*
D.T.-EC.A.	*0.00*	*0.00*	*0.00*	*0.00*	*1.18*	*0.00*	*0.00*	*0.00*	*0.00*
D.T.-EC.NRS.	*9.41*	*3.53*	*1.18*	*4.71*	*2.35*	*18.82*	*7.06*	*2.35*	*4.71*
E.I.	*9.41*	*10.59*	*0.00*	*9.41*	*10.59*	*0.00*	*1.18*	*0.00*	*0.00*
E.I.-EC.NRS.	*17.65*	*5.88*	*0.00*	*0.00*	*0.00*	*0.00*	*0.00*	*0.00*	*0.00*
E.P.	*0.00*	*0.00*	*0.00*	*0.00*	*0.00*	*5.88*	*3.53*	*2.35*	*4.71*
E.P.-EC.A	*0.00*	*0.00*	*0.00*	*0.00*	*0.00*	*0.00*	*1.18*	*0.00*	*0.00*
E.P.-EC.NRS.	*0.00*	*0.00*	*0.00*	*0.00*	*0.00*	*9.41*	*3.53*	*4.71*	*8.24*
EC.A.	*0.00*	*0.00*	*0.00*	*1.18*	*3.53*	*0.00*	*0.00*	*0.00*	*0.00*
EC.NRS.	*14.12*	*12.94*	*11.76*	*2.35*	*0.00*	*22.35*	*17.65*	*9.41*	*27.06*
EC.NRS.-	*0.00*	*0.00*	*0.00*	*0.00*	*0.00*	*2.35*	*2.35*	*0.00*	*0.00*
N.S.	*34.12*	*57.65*	*51.76*	*27.06*	*35.29*	*23.53*	*25.88*	*30.59*	*50.59*
SUM	***100.01***	***100.01***	***99.99***	***100.01***	***100.00***	***99.99***	***100.00***	***100.00***	***100.02***

Table 5.9. 1 Proof schemes observed in test T3

Table 5.9.1 reveals the following characteristics of the answers to T3:

- A wide range of proof schemes, with some occurring in small or very small percentages. For example, T33a and T33b also elicited answers that contained evidence of the EC.A. and EC.R. proof schemes.
- The D.T. and EC.NRS proof schemes, occurred most frequently.
- The participants had the most difficulty answering questions T32a T36b.
- It is interesting that, although T35 and T36a could be proved using the congruency criterion Side-Side-Side, a number of participants did not do this. This might explain the rather high difference in percentages of D.T. proof schemes from 9.41% for T35 to 49.41% for T36a.
- The participants' technical difficulties with proof notwithstanding, the answers offer evidence of strong proof appreciation in algebra. The substantial percentages of D.T. proof scheme in the answers to T33a and T33b (44.71% and 42.35% respectively) offer evidence that the students seem to appreciate that proof of mathematical relations is based on certain laws of real numbers.
- The largest number in the N.S. group combined with the large number of D.T. proof schemes occurs in Question T36a where students overall either provided a complete answer (49.41% D.T.) or none at all (30.59% N.S.).

In the following, bar charts 5.9.1 and 5.9.2 focus on the students' fluency with proving with a presentation of the D.T. proof scheme percentages. Chart 5.9.1 depicts the percentages of answers classified as D.T. alone and Chart 5.9.2 adds to these the mixture with other proof schemes.

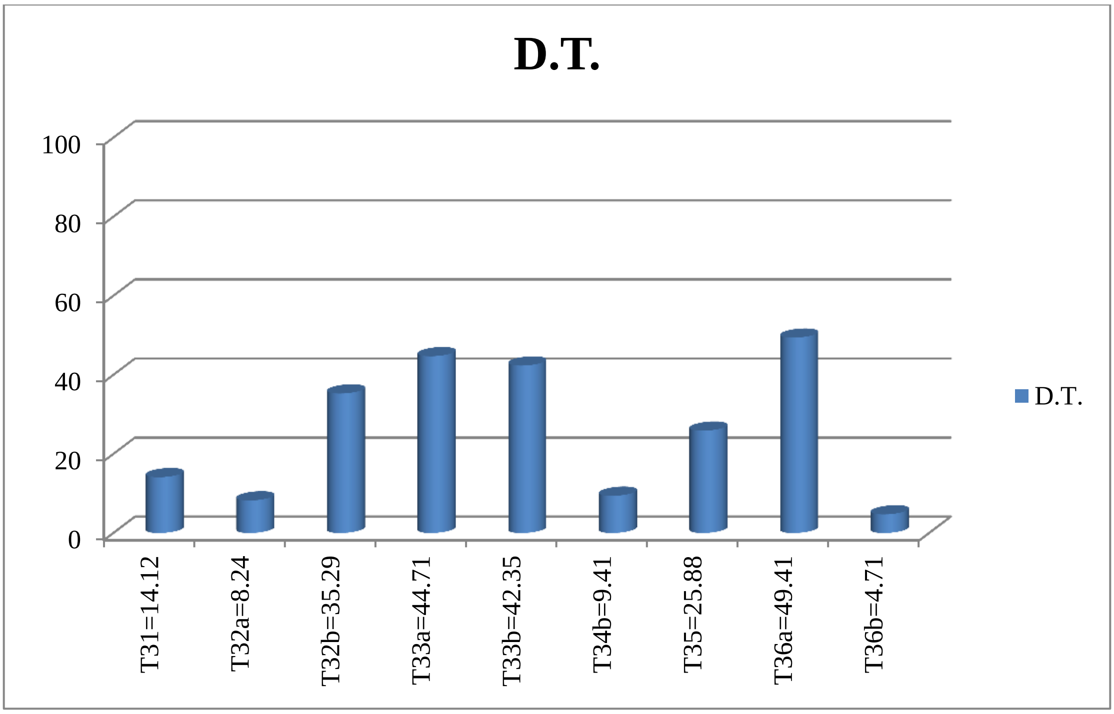


Chart 5.9. 1 D.T. proof scheme bar chart in percentages per Question of test T3

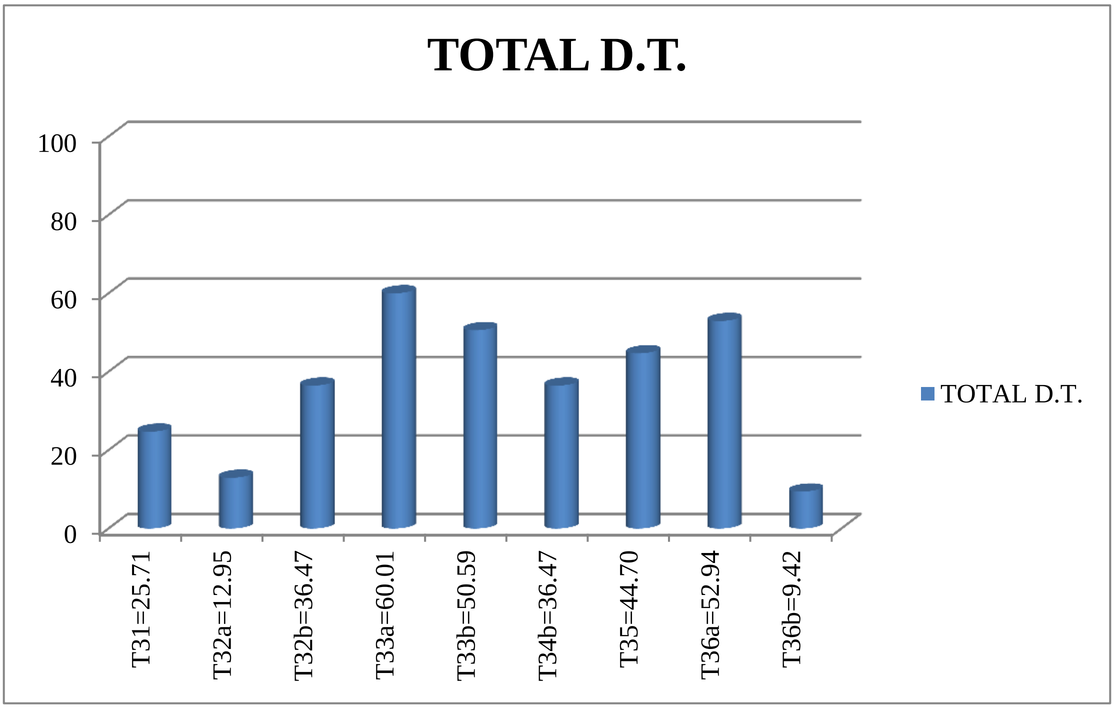


Chart 5.9. 2 Total D.T. proof scheme bar chart in percentages per Question of test T3

Charts 5.9.1 and 5.9.2 can be interpreted as follows:

- Proof fluency may oscillate due to the difficulty of the questions but it seems that teaching proof in Year 9 is productive, as a number of participants offer evidence of D.T. proofs in their answers even to the most difficult question.

- At this stage participants perform satisfactorily when called upon to prove simple algebraic and geometric propositions. This is an important first step, as the more difficult proof problems the students will be tackling soon are often composed of several more simple ones.

CHAPTER 6: CONCLUSION

6.0 Introduction

In this chapter I summarise, discuss and conclude the data analysis I presented in Chapters 4 and 5. In section 6.1 I present a general summary of the findings in relation to the reviewed literature on teaching and learning proof. The findings of the study and its implications provide answers to the research questions stated in chapters 1 and 2 which were the following:

a) What are students' pre-proof perceptions?

b) What are students' perceptions of proof when they first encounter it?

c) How, if at all, is the Harel and Sowder's taxonomy applicable to the Greek secondary educational contexts?

d) How, if at all, can the Harel and Sowder's taxonomy be used to elucidate students' competence in proving as well as how they value proof within the Greek secondary educational contexts?

In section 6.2 I discuss the contribution of the study and its implications from four different perspectives:

(1) the perspective of teaching and learning proof;

(2) the theoretical perspective regarding the application of Harel and Sowder's proof schemes taxonomy (1998, 2007) ;

(3) the methodological perspective of the analysis of students' answers;

(4) the classroom practice perspective in relation to educational policy, curriculum and pedagogy.

In section 6.3 I discuss the limitations of the study. Finally, in section 6.4., I discuss the wider project in which the present study is embedded and make suggestions for further research.

6.1 The findings of the study

The first two research questions of this study ask, what are the students' proof perceptions before and after being introduced to proof and proving. To record these perceptions the Harel and Sowder's taxonomy was chosen as an analytical tool which led to the third research question, whether this taxonomy is applicable within the context of the Greek secondary education. In students' responses in both T1 (pre-proof test) and T3 (post-proof test), I found strong evidence of the various proof schemes proposed in the Harel and Sowder's taxonomy (1998, 2007). I discuss in more details the proof scheme taxonomy, its background and its role in my study in section 6.2. Here I am only interested in the following point: Harel and Sowder (1998) developed the taxonomy mainly parallel to and after teaching students about proof. In their study, the students had already encountered and experienced proof previously in their secondary school education. My study indicates that even when secondary school students have not yet been introduced to the proof explicitly in advance, they appear to develop proof schemes corresponding to the taxonomy. Thus it can be said, judging by the answers given to T1 test, that the students' pre-proof perceptions are very well described by Harel and Sowder's taxonomy. This conclusion is also valid for T3 test which followed the teaching of proof.

According to Tall (2005), students' difficulties with formal proof have their origin in their earlier 'mathematical life'. This opinion can be applied to the participants of my research, who had not been taught proof when they sat T1 at the beginning of Year 9. They indeed demonstrated such difficulties. Based on the results of my study, if I transform Tall's argument, I can also say that students' efficiency has its origin in their earlier 'mathematical life'. Thus the emergence of the taxonomy's proof schemes in their answers can be presumed to be the result of how students perceived previous teaching of mathematics. In primary school and in Years 7 and 8 in the secondary school the participants have encountered various mathematical notions, which shape their concept images that evoke in students' effort to handle specific mathematical tasks (Tall & Vinner, 1981). For example what is a parallelogram and how to calculate its area or what is an equation of the first degree in one unknown and how to solve it etc.. In other words they have already accumulated some experience in the field of mathematics. The quality and content of the experience they have acquired, and how and what they have understood of the mathematical objects, affect and influence how they handle proof problems. Thus when the participants are invited to deal with proof questions they do not begin from scratch; they have already formed ways of understanding, ways of learning (Harel & Sowder, 2005) and ways of working which they demonstrate in their handling of proof problems in the form of proof schemes. Their methods of understanding and learning are not always adequate or productive. Specifically speaking of, students' ways of understanding of particular notions might be responsible for their underperformance in proof tasks. If, for instance, there is a misconception about the square root multiplication that includes incorrect statements such as $\sqrt{2}\sqrt{3}=6$, this misconception will pop up in a question related to the expansion of an algebraic expression with roots such as $\left(a\sqrt{2}+b\sqrt{3}\right)^2$. And if this expansion is embedded in a proof task, particular misunderstanding of the square root will lead to evidence of the EC.NRS. proof scheme. This can occur even after the teaching of proof, especially if the specific concept image of the square root diverges from its formal concept definition (Tall & Vinner, 1981). In the case where the square root has been semantically understood it will be treated correctly and if it is involved in a proof process then the use of it will support D.T. proof schemes (Weber & Alcock, 2004). I should make clear that I am not interpreting the genesis of proof schemes here; I am only following Harel and Sowder's (ibid.) line of thinking which is inspiring and gives ideas for a deeper investigation of the proof schemes genesis. Students' level of linguistic articulacy is potentially affecting their mathematical performance as indicated by PISA studies (Heinze & Kwak, 2002). This is sometimes evident in students' answers in the present research, but the decisive aspect is not so much how students formulate their thoughts in their native language as what mathematical content they include in this formulations. Herein lies the importance of cognitive aspects, which research into mathematics education has proposed and investigated. For example, in Skemp's (1976) work on relational and instrumental understanding, Tall and Vinner's (1981) concept image and concept definition, and research into the key ideas that students have developed or not yet developed (Raman, 2003), etc.. These theoretical constructs are evident in the participants' answers. The above example of the square root is a case of discrepancy between concept image and concept definition leading to cognitive obstacles in presenting proof. The way the participants treated, for example, Question T36b show that many of them do not yet understand the key ideas on possible parallels and their transversals, but a number who did understand them used them creatively showing heuristic literacy (Koichu, Berman, & Moore, 2007) and cognitive unity (Antonini & Mariotti, 2008). If an expression of the type $(a+b)^2$ is to be expanded, the student may demonstrate relational understanding by showing knowledge of the mechanism of expansion, or instrumental understanding if he/she simply tries to memorise the final expansion's result or misunderstands the expansion in a 'linear' manner as in $(a+b)^2 = a^2 + b^2$ and so on (Skemp, 1976).

Particularly T1 test requires both technical and theoretical knowledge. The participants have to be acquainted with the symbols denoting degrees as a measure of an angle. Probably many cannot tell the difference between the measure of an angle and the angle itself because the distinction is little, if ever, discussed in the lower secondary school classroom. They have to be acquainted with the symbols for angles as geometrical objects, using either three letters or one. They have to be able to recognise in a figure the angle they refer to, independently of the symbol used for it. Thus they have to be acquainted with the geometrical shape of an angle and how an angle is formed, which means that they have to understand that line segments with a common origin form an angle, and in fact they form more than one angle. Also, students need to understand that they should not judge the validity of any geometrical property from the figure and they need to use the geometrical definitions and properties and to distinguish between data and conclusion in order to ground their judgements (Mariotti, 2000). Students must also understand theoretically how many degrees there are in the sum of the angles of a triangle. Consequently they have to be able to manipulate the given measures arithmetically or algebraically, if they refer to the angles with their symbols, in order to obtain the measure of the unknown angle. They have to formulate correctly definitions as that of a perpendicular bisector of line segment, of a midpoint of a line segment, a median of a triangle and apply them appropriately if needed to. They have to know what the second power of a number is and to formulate and apply the Pythagorean Theorem. All the above pieces of knowledge are necessary for the formulation of coherent and accurate responses. As a result, the inappropriate use of any of these pieces can lead to at least an EC.NRS. proof scheme. Misunderstanding when and why a property of a geometrical figure is valid can lead to an E.P. or E.I. proof schemes depending on whether the participant perceives the properties by eye or by measuring the figure. However, regardless the high demand in knowledge I describe above, there are still participants who give adequate answers or in other words produce D.T. proof scheme. Thus previous students' mathematical knowledge, previous pre-proof mathematical experience at school, namely the development of habits as, using symbols appropriately or arbitrarily, formulating correctly or incorrectly definitions, understanding or misunderstanding given data, etc., might lead to the various proof schemes; all of these are reflected in T1 as evidence and indication of what pre-proof origins might the various proof schemes have their roots in.

The answers to T1 also show that even such knowledge, well known to the students, as the sum of the angles of a triangle cannot be exploited successfully when the context of the problem is not familiar to the students. It also indicates that the participants have more difficulty arranging their thoughts properly to reach conclusions from given properties and relations than through calculation. It can thus be said that T1 provides the contour of the problems that the participants were expected to have with proof issues at this early stage. In this respect the picture gained through T1 is successful and has yielded plenty of information.

The second picture from the participants' proof perception is shaped by their answers to T3. The test deliberately included some demanding questions because I wanted to avoid an oversimplified and overoptimistic picture of the participants' performance. Thus the overall results offer a useful indication of what should be expected at the end of Year 9 regarding proof. It should be noted that from the 95 hours teaching of mathematics, 10 hours were devoted to algebraic and 10 hours to geometric proof. In aggregate about 20 hours were devoted to proof and this was for the first time in the participants' school life.

In T3 test the algebraic questions allowed investigation of the students' fluency in and appreciation of proof. Healy and Hoyles (2000) gave students various proofs to evaluate them and found that they demonstrated strong proof appreciation in what they recognised correct proofs. Inspired by their example I included in T3 the Question T33 which tests the students' ideas on the proof of the identity $(a-b)(a+b)=a^2-b^2$. It proved to be productive as it revealed students' high proof appreciation with regard to how algebraic identities are justified although proof fluency is of significantly lower level in my participants' sample. Stylianides and Stylianides (2009) tested prospective elementary school teachers setting to them a construction-evaluation activity. Here is their opinion on that matter:

> Our focus on "construction–evaluation" tasks in the domain of proof revealed the interesting phenomenon of some prospective elementary teachers providing erroneous responses to mathematical tasks posed to them by the instructor while being aware that their responses were incorrect (see, e.g., Sherrill and Joan's responses). This phenomenon, which presumably is particular neither to prospective elementary teachers nor to proof, has received little attention in the literature.(ibid., p. 251)

Thus my study's relevant results meet the results of the aforementioned studies and in this manner it contributes to turn the researchers' attention to this matter. In the spirit of these two studies, and following Harel & Sowder (1998, 2007), proof fluency could be defined as the students' ability to articulate acceptable proofs within the Greek educational context of Year 9. Accordingly we could define proof appreciation as the students' ability to recognise, within the same context, acceptable proofs as defined above even as 'superior' to their own. On this basis the participants in the present research showed strong proof appreciation in algebraic matters as described before. This handling of the students' responses gives a satisfying answer to research question d): How, if at all, can the Harel and Sowder's taxonomy be used to elucidate students' competence in proving as well as how they value proof within the Greek secondary educational contexts? Indeed, using the notions of proof fluency and proof appreciation on the data collected a structure is bestowed upon them, revealing the internal relation of the proof schemes appeared in the analysis. The level of proof fluency is an indication of the difficulties that the participants encountered in their attempts to answer the test questions. There are participants who have learnt to deal with algebraic identities and relations to such a degree that they can successfully tackle problems involving the extensive use of mathematical symbolisation. The algebraic questions in T3 invited the participant to step beyond the simple application of identities. However, the very same application of identities that was necessary for an adequate response to these questions uncovered issues that are frequently observed in everyday teaching regarding the use of the three fundamental identities $(a \pm b)^2 = a^2 \pm 2ab + b^2$ and $(a-b)(a+b) = a^2 - b^2$. A part of the sample population provides evidence of problems manipulating algebraic expressions which in some cases are the main reason for failing to reach a proof, due to the accumulation of incorrect steps. In other words, more work and practice is needed before these participants will be able to handle relevant matters.

However, at first sight and 'contrary' to the evidence of the many difficulties the students encountered in producing proof, strong evidence was found of high proof appreciation and ability in fundamental issues such as how to explain and prove basic identities. The participants provide evidence of a very widespread persuasion that the truth of an identity results from certain rules and logical steps. The teaching approach of the mathematics teacher J has probably affected these students' responses. She insisted on the application of rules when she began teaching algebraic identities, emphasising the logic of algebraic manipulations and presenting it as a 'game' with rules rather than an arbitrary and incoherent process. Based on this strong proof appreciation and other factors, certain students have developed the ability to answer even the most difficult questions. The proofs the participants were asked to produce were not simple for them if we judge their simplicity by the objectivity of the numbers found when counting the adequate answers to the various questions. Thus the gap, between the strong proof appreciation and not analogously strong proof fluency, is not surprising.

The geometry-related questions in T3, which are mainly on the congruency criteria for triangles, also provided evidence of students' difficulties in producing a proof unless the question is simple and the necessary criterion is easily recognisable. However, it is natural for them to demonstrate these difficulties because this was the first time in their school life that they were asked to explicitly prove difficult geometric problems. Proof in this context is a completely different task to learning how to draw a figure and mastering the terminology. I have already commented on the geometric proof, which many of the participants appeared to find difficult, even in the pre-proof test, because it requires not arithmetic or algebraic calculations but a sequence of steps in which thoughts are put in the correct order using definitions and properties to deduce certain conclusions.

However, some participants delivered immaculate proofs even for the most difficult questions. Anne Watson (2010) speaks of the shifts of mathematical thinking in adolescence that is needed in order to develop mathematical efficiency. The participants, who had experienced such shifts and were able to offer difficult proofs, constitute, together with the wide spread phenomenon of proof appreciation across the sample, a very interesting aspect of the findings. It seems that a relative readiness of many participants to be taught proof is present. This readiness needs to be further investigated and promoted if we want proving ability to flourish in adolescence.

It should also be taken very much into account that many of the students in my sample appear to understand or appreciate proof at end of Year 9, at least in the relatively simple context of some the T1 and T3 questions. Quite a few though could not. The pace of evolution in proof efficiency has been observed to be laborious (Harel & Sowder, 1998; Healy & Hoyles, 2000). Thus the present research in its own way confirms the results of many studies of the difficulty of proof.

My findings cannot be seen as an assessment of any kind. The size of the sample does not allow generalisations and this was not in my intention. On the other hand and although my analysis uses a very new tool for the Greek education context, it provides indications of the kinds of problems that emerge when students face proof. I encountered the same mistakes and false perceptions in the participants' scripts that I have come across many times before in my professional life, but the new light in which I saw them is valuable because it opens up a new conception in the research analysis that draws attention to the factors that constitute mathematics teaching, in other words the curriculum, the teacher and the students in a dynamic interactive reality. I will discuss this in more detail in the next section.

6.2 Contribution of the study

In this section I discuss the contribution and implications of the study from four different perspectives:

(1) the perspective of teaching and learning proof;

(2) the theoretical perspective regarding the application of Harel and Sowder's proof schemes taxonomy (1998, 2007) ;

(3) the methodological perspective of the analysis of students' answers;

(4) the classroom practice perspective in relation to educational policy, curriculum and pedagogy.

6.2.1 Contribution to teaching and learning of proof

This study has investigated how Year 9 students perceive proof on encountering it for the first time. Teacher J and I discussed and exchanged thoughts how to teach algebraic identities and triangle congruency criteria and agreed on the following general points. First, algebraic operations with real numbers should be prioritised in proving. Second, in geometry, the congruency criteria should be taught as follows: first the students construct triangles from the elements of which the corresponding criterion makes use; then the students compare the constructing triangles empirically by superimposing them. We also decided to emphasise the writing down and understanding of data and conclusions prior to undertaking proof procedures as suggested by Polya (1990). The teaching under these considerations by no means covers all the knowledge pieces necessary for proof fluency. Heinze (2004) is of the opinion that when students lack knowledge of facts (=Faktenwissen) it is because they have not absorbed the teaching. Analysing the tests I found many cases of lack of knowledge of facts that could hinder students' completion of an adequate proof.

Without going into every detail, first of all, even before they are taught about proof the students have already shaped various ideas and perceptions of mathematical objects that define their understanding of proof and proving. These perceptions and ideas are not always desirable ones that will help them to develop a good relationship with proof procedures. Limited understanding of mathematical objects leads to a limited understanding of proof. Thus, as expected, previous knowledge plays an important role in the encounter with proof and proving, and teachers should take their students' previous knowledge very much into account, when starting to teach proof.

The process of examining students' efficiency in pre-proof mathematics poses problems beyond the scope of this study, such as, under what conditions, is proof teachable in a given classroom. Students in a class naturally perform in a range of ways. However if this range goes beyond certain extremes one cannot expect the class to be able to follow the teaching of not only proof but also any material as a team. Thus when I speak of the teacher's taking into account the previous mathematical understanding I do not mean to solve unsolvable problems. It is true that up to a quarter or a third of the participants did not provide an answer to at least one of the questions in T1. Questions about what this means and its consequences are posed spontaneously even in this research, but addressing this particular and very important problem is not one of my aims. In this research the only thing that can be said is that the number of failures to provide a proof underlines the difficulty of teaching proof and mathematics in general. This is a problem to be addressed via curriculum policy under the condition that this policy will be inspired by and based on further research.

Returning to my previous argument, it is important to be aware of students' previous knowledge. Even if a teacher is not able to resolve all of the students' difficulties she/he can address many of them if he/she is aware of students' background and understanding. A pre-proof test, or any kind of test, cannot embrace the whole gamut of potential problems, although testing the students at the beginning of the school year is indispensable. The results of any such test are only indicative, however, and can serve as the basis of a dynamic process and an on-going dialogue between teacher and students which allows the teacher to deepen his understanding of his students' mathematical thinking and provides a chance for mutual feed-back. Things observed in one school year may sometimes be exploited later, rather than directly and immediately. For example, with awareness of some students' tendency to 'see' in the relation $\alpha^2 + \beta^2 = 5^2$ e.g. the numbers $\alpha = 4$, $\beta = 3$, it could probably be prevented by citing other possible cases where a or b or both are allowed to also take on non-integer values such as $1^2 + \left(\sqrt{24}\right)^2 = 5^2$, $\left(\sqrt{3}\right)^2 + \left(\sqrt{22}\right)^2 = 5^2$ etc. Naturally the fact that the numbers in question such as $\sqrt{22}$ are irrationals and not integers has to be explained and stressed. Such examples allow the distinction of the 'Pythagorean triads' from the 'common' triads of numbers satisfying the relation $a^2 + b^2 = 5^2$. This approach will not eradicate mistakes once and for all, but the teacher's contract is always to find new ways of conveying ideas. It is important to spread many seeds if some of them are to blossom.

6.2.2 Contribution regarding the application of Harel and Sowder's taxonomy

I used Harel and Sowder's taxonomy of students' proof schemes to analyse the participants' answers. This taxonomy emerged out of a relevant study of the proof behaviour of university students taking part in programs in which linear algebra, Euclidean geometry and number theory were taught (Harel & Sowder, 1998). In their work on the taxonomy Harel and Sowder (1998, 2007) give examples of proof behaviour that refer not only to written texts but also, especially where they involve the EC.A. proof scheme, to oral answers. Harel and Sowder do not discuss whether the various proof schemes could make simultaneous appearances in the proof behaviour of students. Using the taxonomy in question Housman and Porter (2003) found that a student could provide evidence of different proof schemes answering different questions. The participants of this research were above average students.

The educational context of my study is in many respects different to that in which Harel and Sowder's research led to the theory of proof schemes as I am going to explain. First, their research was conducted at the tertiary level of education and consequently the participants had more experience of mathematics, and probably some at least had been taught proof at high school. Second, Harel and Sowder's research participants were being taught elementary number theory, college geometry, linear algebra and advanced linear algebra all of which are more advanced subjects than those that are taught in Year 9 mathematics classes in the US, in Greece or elsewhere. Third, the cultural context, and within that, the educational context in the US is different to that in Greece, as I indicated with the presentation of the syllabus and the description of the Greek context in Chapter 3. Additionally the sample in the present research was composed not of university or above average or high attaining students of any kind but of ordinary Year 9 students. All these factors constitute an important and weighty difference to the circumstances under which the taxonomy first emerged and was applied. Thus the verification of the applicability of the taxonomy by the present research reinforces its universal character and its independence of particular socio-cultural and educational conditions, and if this character of the taxonomy is scientifically accepted it can be used as a tool for broader analysis in longitudinal studies. From this point of view I once again emphasise that my research evidence answers positively the research question c): How, if at all, is the Harel and Sowder's taxonomy applicable to the Greek secondary educational contexts? I return to this point in section 6.4.

Analysis of the answers provided evidence of the various proof schemes foreseen and described by Harel and Sowder's taxonomy. Thus, the emergence of the taxonomy in my sample is independent of the specific educational environment and appears to characterise the proof behaviour of students regardless of their level of education. This strong evidence of the taxonomy's existence includes some very interesting aspects which appear to be present in my research sample.

First, the appearance of the E.I. and E.P. proof schemes, depends on the nature of the questions that the participants are called upon to prove. Harel and Sowder may insinuate this dependence through the examples they use to illustrate the theoretical descriptions of the proof schemes. However, they do not formulate this dependence explicitly. In the present research, evidence of the E.P. proof scheme appears in the answers to questions of geometric character and is rare in the algebraic answers if not impossible. The E.I. proof scheme is more likely to appear in the answers to algebraic questions involving variables. A very small number of students' answers provided evidence of E.I. proof scheme, however, because the corresponding participants, extraordinarily, measured the readymade geometric figures given to them. Analogously, the proof scheme EC.A. made few appearances in their answers because it principally has to do with an appeal to an authority when something is discussed, mainly, in the classroom, which was not an option under test conditions. In Harel and Sowder's description of the EC.R. proof scheme the ritual character overpowers any logical element, almost replacing it. In the present study it seems that this ritual element did not have such a strong character. Finally, I deliberately decided not to classify any answer as D.A. which means that the absence of this proof scheme is due only to methodological reasons. I have explained in methodology chapter why this choice was made. I repeat briefly that to classify an answer as D.A. means that the answer provides evidence of knowledge of the axiomatic structure of mathematics. This can only happen after a systematic study and accumulation of experience in proving, which was not yet the case for the participants of my study.

Second, as Housman and Porter (2003) observed when they used the Harel and Sowder's taxonomy, different proof schemes appear in different answers of the participants. In my research many of the answers provided evidence of a mixture of proof schemes. For example, students often did not succeed in their efforts to prove propositions completely, and various inefficiencies can be found, for example, in the misuse of concepts, or the validation of properties by judging the properties of a figure perceptually etc., yet their answers are sometimes accompanied by elements of deductive thinking. This led to my use of combinations of proof schemes. In this respect the taxonomy – and my refined use of it in the form of mixed proof schemes – provides new insight into how proof schemes develop.

Recapitulating, the main contribution of this research where Harel and Sowder's taxonomy is concerned is the strong indication of its applicability under different social, cultural and educational circumstances and conditions and this supports its theoretical generality. In this sense the taxonomy describes at satisfactory level how students perceive proof even before they have been taught it. These observations indicate that previous mathematical discourse, in the broader sense as it takes place in everyday school practice, contributes positively as well as negatively to students' understanding of proof. In this respect, used appropriately, the taxonomy could be a valuable tool to enrich our understanding of the consequences of mathematics teaching.

6.2.3 Contribution regarding the methodology used to analyse the students' answers

There are some methodological contributions of the study reported in this thesis. I briefly discuss them in this section.

To analyse the answers and keep them in order, every participant was assigned a code number. Then each of the proof schemes was codified with capital initials indicating the full spelling of the corresponding term. The external conviction proof schemes are coded EC. with the addition of authoritative proof , coded A., thus EC.A.; the ritual R., thus EC.R.; and the non-referential coded NRS., thus EC.NRS.. Similarly the empirical proof schemes take the code symbol E. plus the perceptual proof scheme, P. thus, E.P., and the inductive, I. thus, E.I.. I coded the deductive transformational proof scheme on its own as D.T.. The participants and their answers to each question, or sub-question were tracked on an EXCEL spreadsheet for each test. The complete spreadsheets create an overall picture of each test and can be used to create a table showing the absolute and relative frequencies of the proof schemes. All the previous steps are indispensable but procedural and are customary in research, as they represent the measures necessary for gathering all the relevant information in a form that is easy to read and study.

I now present some aspects of the methodology that can be regarded as a contribution to the research, given that the analytical tool used was Harel and Sowder's taxonomy. I am speaking only of the characterisation of the written responses to the question set. Before going into detail I want to emphasise that Harel and Sowder's taxonomy is not an assessment tool; it is a research tool. Every assessment puts the person making the assessment in a rather 'antagonistic' position towards the person being assessed. The assessor accepts what is 'correct', 'rational', 'acceptable' etc. and rejects what is 'incorrect', 'irrational', 'unacceptable' etc. The taxonomy in question, by contrast, demands from the researcher a deeper understanding of what the participant is doing or writing. In a way one has to take the place of the participants for a moment and go as deeply as possible into their mode of thinking. This is a consistent work based on criteria emerging from the taxonomy. In doing this, the researcher not only reflects on what the participants wrote but also aligns the investigation with any observable reasons why the various ideas of the participants were formed.

I now take each proof scheme separately and examine what criteria should be used, according to the taxonomy, to categorise the scripts. These criteria constitute the main methodological contribution to the literature because they solidify how the analysis was carried out and thus could be applied again to written answers in the future, not only in research but also as a practical tool for analysing texts in the school practice.

Evidence of an EC.A. proof scheme can be said to be present if a script, in support of an argument, includes elements of reference to an authority such as the textbook, the teacher or some other scientific authority without attention to whether the argument is logically valid or without proposing a justification which, independently of the reference to an authority, supports the argument content. I have already explained that such occurrences are not frequent in written responses, at least in the present study, although the possibility of their appearance cannot be absolutely excluded. An EC.A. proof scheme can theoretically occur independently of the question.

Evidence of an EC.R. proof scheme can be said to be present in a script that includes elements of a ritual exposition of ideas which, however, is not logical or valid. For example, there may be rituality in writing down the hypothesis and the conclusion followed by an attempt to proceed to a proof which partially or totally fails. Of course, the ritual writing down of the hypothesis and conclusion and proceeding to the proof cannot necessarily have a negative meaning (Herbst, 2002; Polya, 1990). The negative element lies in the fact that the proof that follows or is embedded in the ritual element is not based of logical arguments and justifications. This is why, as I believe, Harel and Sowder relate EC.R. proof scheme to the two-column proof, meaning the use of the two-column structure, as well as any other procedural structure, but void of logical content. An EC.R. proof scheme can theoretically occur independently of the nature of the question.

Evidence of EC.NRS. proof scheme can be said to be present where the definitions, or conclusions of theorems etc., or various symbols are misused, or where the participant makes arbitrary assertions or invents idiosyncratic laws in her or his answer. A discrepancy between concept image and concept definition (Tall & Vinner, 1981) may be responsible for such occurrences as well as the instrumental understanding (Skemp, 1976) of algebraic manipulations, the poor knowledge of mathematical objects (Heinze & Kwak, 2002) the insufficient previous knowledge (Chinnappan, Ekanayake, & Brown, 2011) etc.. All these considerations explain why the EC.NRS. proof scheme can occur independently of the nature of the question.

Evidence of an E.P. proof scheme can be said to be present in a script that includes assertions regarding geometrical properties which the participant has estimated by eye from a given or self-drawn figure, without any justification or supporting argument. The E.P. proof scheme, proved to be question-dependent in my research. It is closely connected to geometrical figures and perception of them.

Evidence of an E.I. proof scheme can be said to be present where a participant assigns values to variables and proves the question on this basis. There is no generalisation of the solution, for example by asserting that the same would be valid for any other value-assignment. The E.I. proof scheme is question dependent. Indeed, it usually appears when variables or possibility of measurement or both are involved in proof processes. In cases of E.I. proof scheme appearance, variables are substituted by assigned numerical values and various quantities, for example line segment lengths, are measured, without any generalisation of the proof thus obtained beyond the numbers assigned or found as measure. In my research sample it appeared in a limited number of responses to algebraic questions and in few responses to questions of geometry where the participant measured elements of the plane figure.

Finally the D.T proof scheme is particularly question-independent. It has a broad meaning in the present research and its presence was equated with evidence that the proof offered by a participant is logically adequate. A characteristic example is the proof of the relation $5556^2 - 5555^2 = 11111$. In this case the arithmetical computation in the left member of the relation to be proved is an acceptable D.T. proof, although it does not use the algebraic identity $a^2 - b^2 = (a-b)(a+b)$. If the arithmetical computation is correct it is also logically acceptable.

A major methodological contribution is the multiple characterisation of proof procedures to embrace the appearance of more than one proof scheme in a number of the participants' answers.

6.2.4 Contribution to classroom practice, curriculum policy, and teaching

In a recent seminar to mathematics teachers on the use of a DGS software in mathematical teaching, one of them, who was having difficulty in understanding how to use the software, exclaimed "I will never make any of my students repeat the mathematics examination again!". For her, being a student and learning the software made her realise the difficulty of being a student and trying to follow the teacher's instructions.

Sometimes in the history of human civilisation a reordering of priorities, the invention of a symbol as naïve as that for the 'number' zero etc., which in retrospect seems simple, has produced a great leap forward in the development of thought. Harel and Sowder's taxonomy is not only a technical instrument, and must not be understood instrumentally but relationally. It is not simply another research proposal regarding proof. There is a philosophy behind it which is reified by the taxonomy. This philosophy gives proof a socio-cultural face and makes it a product of the evolution of human thought. In this respect it allows us to see mathematical proof in its historical dimension, and not as a truth once and for all given. Today, after centuries of mathematical development mathematicians know that Euclid's proof is different from Hilbert's proof. Both however have in common the acceptance of axioms on which the proof of theorems is built. That is why Euclid's Elements is a work of paramount importance because it gave birth to the seminal idea of axiomatic foundation of mathematics. Efimov (1980, p. 18; my italics) comments on the arguments of the proof of Proposition 32 of Book I (Heath, 1956, pp. 316-322) as follows: "Thus the above arguments depend *heavily* on the visual evidence" (1980, p. 18; my italics). However, nobody dares to think that, because of his visual evidence in this and other cases, Euclid, is not among the greatest mathematicians of all times. Seeing the taxonomy from this point of view spurs a simple rearrangement in our minds and invites us to impose it creatively on the school reality. The students' proof behaviour must be understood in an evolutionary manner. The genetic principal is based on this philosophy (Freudenthal, 1973; Schubring, 1978; Wittenberg, 1963). Thus it must be understood as repeating in condensed form the progress and set-backs of the historical evolution of mathematical thinking. This angle of observation reorders our priorities from assessing to understanding, and from comparing the proof thinking of the students with abstract deductive thinking to the laborious birth of deductive thinking from various others forms such as empirical proof thinking.

Harel and Sowder's taxonomy provides teachers and educators with a particular insight into students' difficulties and thus offers a basis for rational compassion and empathy with them. The taxonomy forces one to understand their mode of thinking about such a complicated matter as proof. Since proof embraces all the mathematical knowledge of students, the various proof schemes analysed under the magnifying glass of the taxonomy illustrate how knowledge is constructed in the classroom as well as by each individual student in the class. Consequently analysis by means of the taxonomy makes the divergence from the wishful deductive reasoning clear and thus turns our attention to the probable reasons responsible for this.

The two factors that can account for students' proof behaviour, apart from the students themselves, are the teaching of mathematics, which is always personified by the teacher, and the curriculum, which institutionalises proof.

By observing the proof behaviour of the students and analysing it in terms of the taxonomy in question, the teacher can detect where the teaching has allowed or supported deviations from deductive thinking and adjust his/her teaching accordingly for better results.

Regarding the curriculum and its influence on the students' proof behaviour, the taxonomy is offered as a tool of analysis; however, longitudinal studies are needed to create credible results.

6.3 Reflections on some limitations of the study

The scope of this study was to investigate how the students in Greece perceive proof, just before being taught proof procedures at the beginning of Year 9, and at the end of Year 9, after having had a time interval in the school year of being taught about proof. The first limitation of the study is the small size of the participants' sample, compared to the population of students at this level across the country, although it comes from a school with ordinary students. Another is that this school has attracted teachers with particularly high qualifications in mathematics. Thus the findings of the research are indicative and need to be validated with further investigation.

The study offers evidence of the applicability of Harel and Sowder's taxonomy (positively answering research question c) but this evidence needs to be strengthened further by future, broader studies in the same vein. Obviously this evidence is limited to the students who participated in this study; and to the mathematical contexts of the questions in the two tests (e.g. applications of algebraic identities and geometrical proofs that involve, e.g., the use of the triangle congruency theorems). I do believe however that the significant number of students in whose answers substantial evidence of the D.T. proof scheme was found – and the overall quality of these answers – allows the emergence of the main findings reported in this study with some confidence. Of course this confidence needs to be strengthened further with larger and deeper investigations.

6.4 The larger study project this study is embedded in. Suggestions for further research

The present study is a result of collaboration mainly between teacher J and myself and, to some extent, with other school colleagues during the school year 2010-2011. Our common work has some exceptional features: Teachers and researchers work together towards agreeing on an object of learning and aim to teach it effectively. Usually the object of learning is one that demonstrates cognitive difficulties for the students, and the teaching is planned with the aim of overcoming these difficulties. This type of collaboration is not well-known in the Greek educational context and the larger project, which the study presented in this thesis is part of, can be considered a pilot project of this type of collaboration among teachers. Among other products of this collaboration are audio-taped meetings with my colleagues and audio-taped classroom sessions together with the notebook in which I took notes on the teaching. Additionally there are further written answers to test named T2 and to the final official examinations, T4. The audio-taped meetings and teaching observation can be analysed and exploited to aid understanding of how the enacted curriculum is applied to teaching about proof and the probable interrelationships among teachers, the teaching of proof, students' understanding as a case study but also more generally.

Therefore many possibilities for further study have emerged in the process. Here are some:

- How should students be taught mathematics before encountering proof, and how can they be prepared for a successful encounter with proof?
- On what grounds do the various proof schemes develop?
- How does proving ability, as seen through Harel and Sowder's taxonomy, evolve in the years beyond Year 9?
- Is the Year 9 students' performance with respect to the proof schemes they produced predictive of their future mathematical development?
- To what extent and in what manner does the teacher's teaching approach affect the production of the various proof schemes?
- To what extent and in what manner does the curriculum influence the proof scheme production and the distribution of the various proof schemes?
- To what extent and in what manner can the collaboration between researchers and practitioners in mathematics education curriculum influence the students' learning experiences in proof, and more widely, in mathematics?

I look forward to engaging with the analyses of the remaining bulk of data collected in the context of the larger study (but not included here) and I hope to have the opportunity to engage in further research projects investigating some of these and other relevant questions in the near future.

APPENDIX I
TEST T1 AND TEST T3 IN GREEK

ΔΙΑΓΝΩΣΤΙΚΟ ΤΕΣΤ (Τ1)

ΚΩΔΙΚΟΣ________________________________ ΗΜΕΡΟΜΗΝΙΑ____________

1. Σε ένα τρίγωνο ΑΒΓ (Σχήμα 1) η γωνία $\hat{A}$ έχει μέτρο $\hat{A} = 85°$ και η γωνία $\hat{B} = 75°$. Να αποδείξετε ότι η γωνία $\hat{\Gamma}$ έχει μέτρο $\hat{\Gamma} = 20°$

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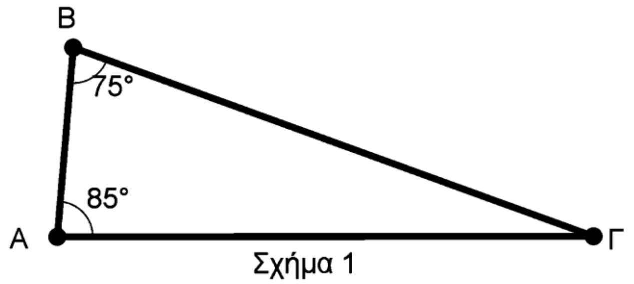

Σχήμα 1

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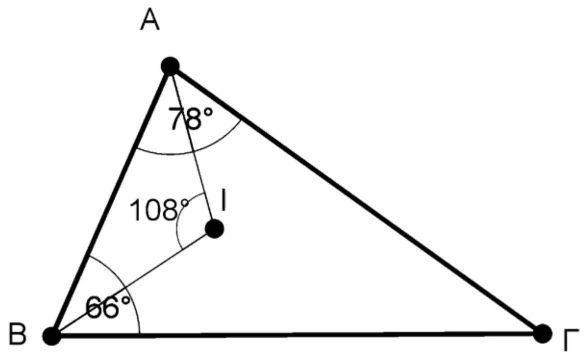
2. Σε ένα τρίγωνο ΑΒΓ (Σχήμα 2) η γωνία $\widehat{BA\Gamma}$ έχει μέτρο $\widehat{BA\Gamma} = 78°$ και η γωνία $\widehat{AB\Gamma} = 66°$. Οι ΑΙ και ΒΙ είναι διχοτόμοι των γωνιών $\widehat{BA\Gamma}$ και $\widehat{AB\Gamma}$ αντίστοιχα. Να αποδείξετε ότι η γωνία ΑΙΒ έχει μέτρο $\widehat{AIB} = 108°$



Σχήμα 2

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3. Ενός ευθυγράμμου τμήματος ΑΒ το σημείο Μ είναι το μέσο του (ΜΑ=ΜΒ). Η ευθεία (ε) είναι η μεσοκάθετος του τμήματος ΑΒ (Σχήμα 3). Έστω Σ ένα σημείο της μεσοκαθέτου (ε).Φέρουμε τα τμήματα ΣΑ και ΣΒ. Να αποδείξετε ότι το τρίγωνο ΣΑΒ είναι ισοσκελές...

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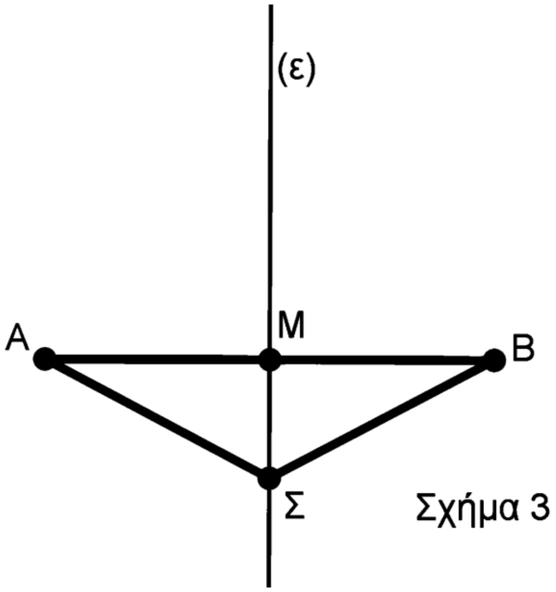

Σχήμα 3

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4. Σε ένα τρίγωνο ΑΒΓ (Σχήμα 4) διαιρούμε την πλευρά ΑΓ σε τέσσερα ίσα μέρη με τα σημεία Δ, Ε και Ζ (δηλαδή ΑΔ=ΔΕ=ΕΖ=ΖΓ). Να αποδείξετε ότι

a) ΑΕ=ΕΓ..

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b) Το ευθύγραμμο τμήμα ΒΕ είναι η διάμεσος του τριγώνου από την κορυφή Β που αντιστοιχεί στην πλευρά ΑΓ...

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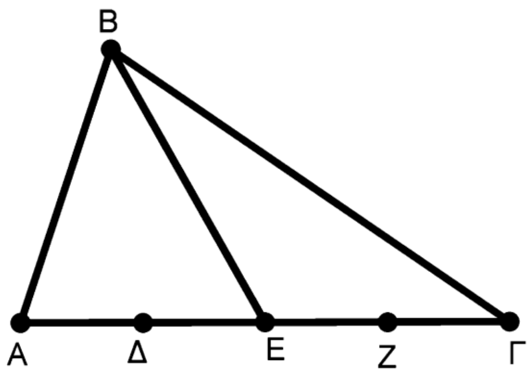

Σχήμα 4

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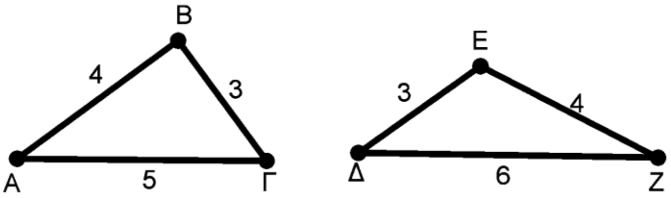
5. Στο Σχήμα 5 βλέπετε τα τρίγωνα ΑΒΓ και ΔΕΖ.



Σχήμα 5

a) Στο τρίγωνο ΑΓΒ τα μήκη πλευρών είναι ΑΓ=5, ΓΒ=3 και ΒΑ=4. Να αποδείξετε ότι το τρίγωνο ΑΓΒ είναι ορθογώνιο...

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....

b) Στο τρίγωνο ΔΖΕ τα μήκη πλευρών είναι ΔΖ=6, ΖΕ=4 και ΕΔ=3. Να αποδείξετε ότι το τρίγωνο ΔΖΕ **<u>δεν</u>** είναι ορθογώνιο……………………………………………..

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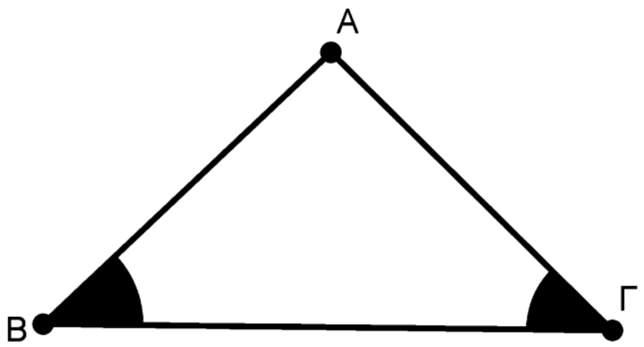
6. Στο σχήμα 6 είναι σχεδιασμένο ένα ισοσκελές τρίγωνο του οποίου οι γωνίες $\widehat{AB\Gamma}$ και $\widehat{A\Gamma B}$ είναι ίσες και έχουν μέτρο $\widehat{AB\Gamma} = \widehat{A\Gamma B} = 44°$. Να υπολογίσετε το μέτρο της γωνίας $\widehat{BA\Gamma}$ ....



Σχήμα 6

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ΤΕΣΤ ΑΠΟΔΕΙΚΤΙΚΩΝ ΘΕΜΑΤΩΝ
ΣΤΗΝ ΑΛΓΕΒΡΑ ΚΑΙ ΤΗ ΓΕΩΜΕΤΡΙΑ
ΤΗΣ Γ΄ ΓΥΜΝΑΣΙΟΥ (Τ3)

ΚΩΔΙΚΟΣ:_____________________________ ΗΜΕΡΟΜΗΝΙΑ:___________

1. ΑΛΓΕΒΡΑ

Α1. Δίνεται ότι για τους πραγματικούς αριθμούς α, β ισχύει $5^2 = \alpha^2 + \beta^2$. Να αποδείξεις ότι $\left(\alpha\sqrt{3}+\beta\sqrt{2}\right)^2 + \left(\alpha\sqrt{2}-\beta\sqrt{3}\right)^2 = 125$.

Α2. Αν η διαφορά των τετραγώνων δυο άνισων φυσικών αριθμών κ και λ (κ>λ) είναι ίση με το άθροισμα αυτών των φυσικών αριθμών τότε:

a. Να αποδείξεις ότι η διαφορά των δυο φυσικών αριθμών κ και λ είναι ίση με τη μονάδα.

b. Να αποδείξεις ότι $5556^2 - 5555^2 = 11111$.

Α3. Δυο συμμαθήτριες σου συζητούν πώς να αποδείξουν ότι $(\alpha-\beta)(\alpha+\beta) = \alpha^2 - \beta^2$. Η μια προτείνει να δώσουν αριθμητικές τιμές στα α και β (π.χ. $\alpha = 2$ και $\beta = 1$) και να κάνουν τις πράξεις για να διαπιστώσουν αν το αριστερό μέλος δίνει το ίδιο αριθμητικό αποτέλεσμα με το δεξιό μέλος. Πειραματίζονται με μερικές τιμές των α και β και διαπιστώνουν ότι το αριθμητικό αποτέλεσμα στο αριστερό και στο δεξιό μέλος είναι κάθε φορά το ίδιο. Μετά από αυτά πιστεύουν ότι η σχέση αποδείχθηκε.

a. Εσύ που παρακολουθείς τη συζήτηση συμφωνείς με την άποψη τους; Αν όχι τι έχεις να τους προτείνεις;

b. Πιστεύεις ότι ο καθηγητής τους θα συμφωνούσε μαζί τους;

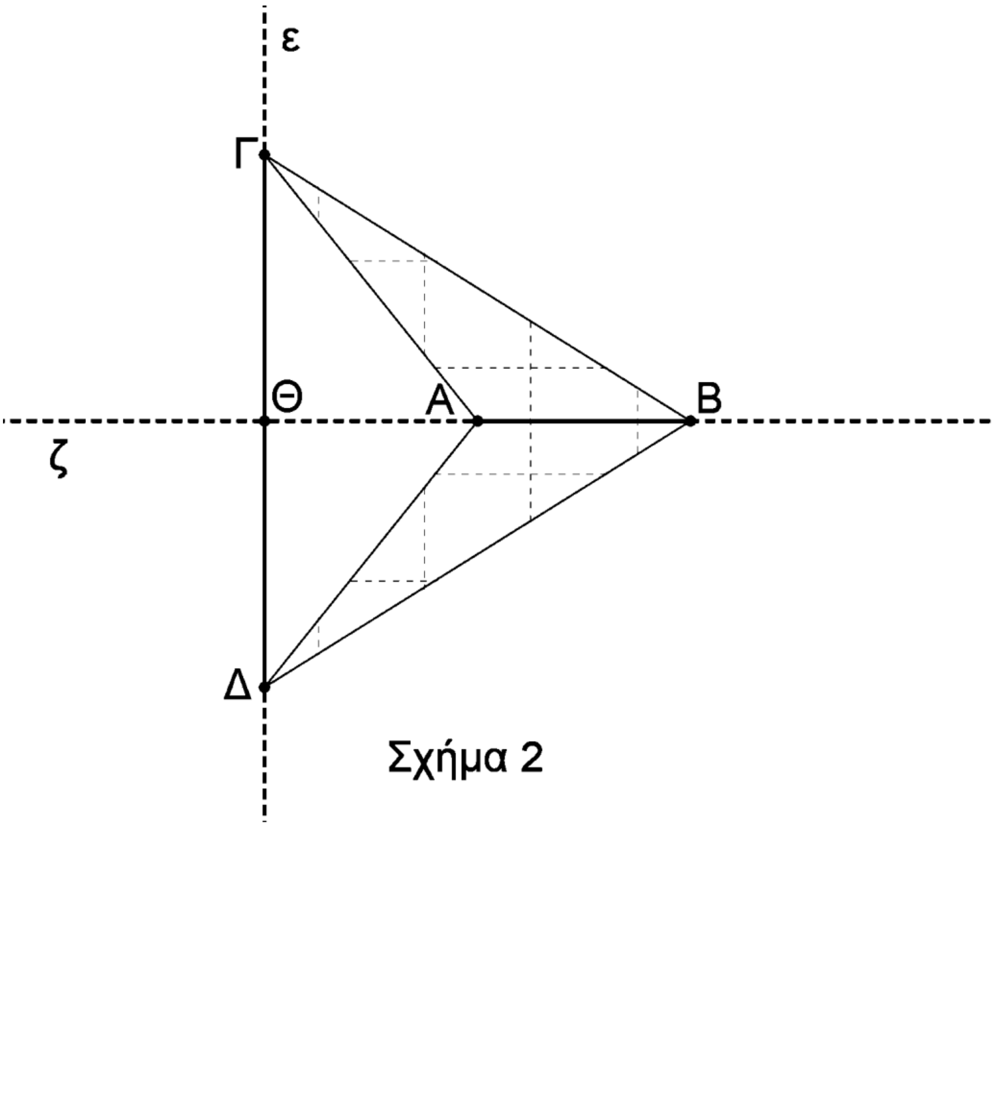
2. ΓΕΩΜΕΤΡΙΑ

Γ1. Δίνεται ένα πλάγιο παραλληλόγραμμο ΑΒΓΔ. Από την κορυφή Α φέρουμε την ευθεία (α) κάθετη στην ευθεία ΔΓ. Η ευθεία (α) τέμνει την ευθεία ΔΓ στο σημείο Ε. Από την κορυφή Γ φέρουμε την ευθεία (β) κάθετη στην ευθεία ΑΒ. Η ευθεία (β) τέμνει την ευθεία ΑΒ στο σημείο Ζ.

a. Να σχεδιάσεις το σχήμα.

b. Να αποδείξεις ότι το τρίγωνο ΑΔΕ είναι ίσο προς το τρίγωνο ΓΒΖ.

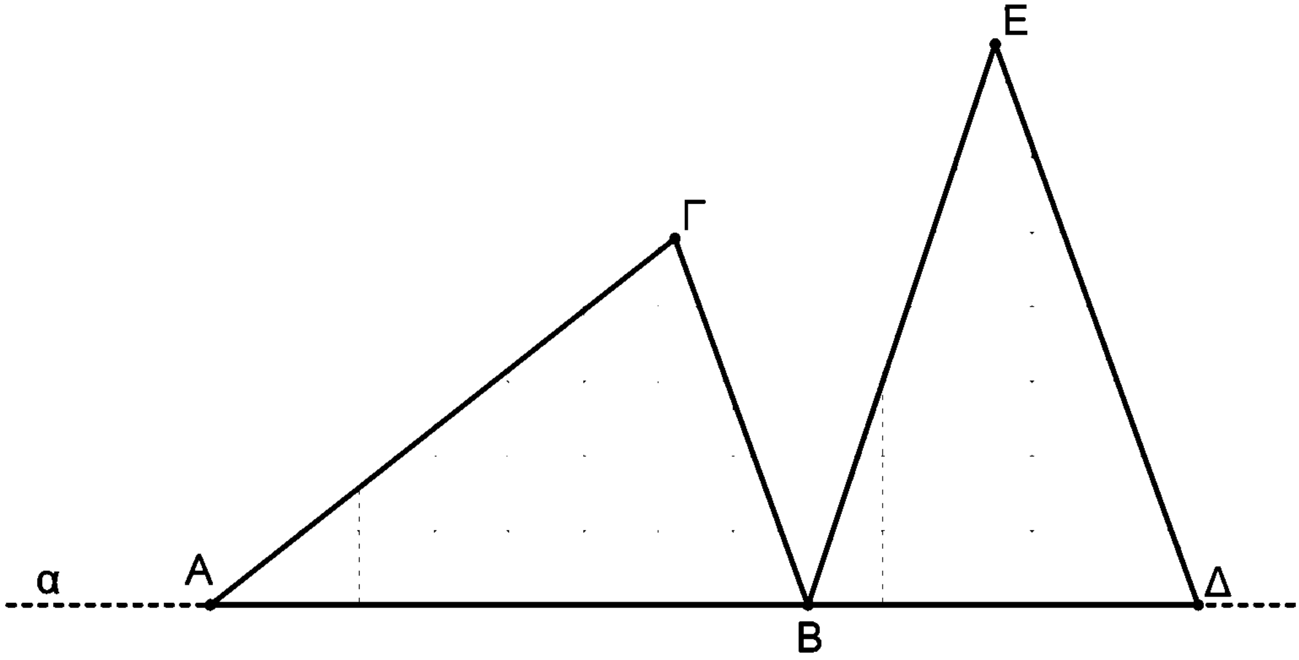
Γ2. Στο *Σχήμα 2* ισχύουν τα εξής: Η ευθεία ζ, η οποία διέρχεται από τα σημεία Α και Β, είναι μεσοκάθετος του ευθυγράμμου τμήματος ΓΔ. Να αποδείξεις ότι τα τρίγωνα ΑΒΓ και ΑΒΔ είναι ίσα.



Σχήμα 2

Γ3. Στο *Σχήμα 3* τα τρίγωνα ΑΓΒ και ΕΒΔ έχουν ΑΓ=ΕΒ, ΑΒ=ΕΔ και ΓΒ=ΒΔ. Τα σημεία Α, Β και Δ βρίσκονται πάνω στην ίδια ευθεία α.

- **a.** Να αποδείξεις ότι τα τρίγωνα ΑΓΒ και ΕΒΔ είναι ίσα.
- **b.** Να αποδείξεις ότι οι ευθείες ΒΓ και ΕΔ είναι παράλληλες.



Σχήμα 3

APPENDIX II

PARENTS AND GUARDIANS PROFESSIONS

CODE NUMBER	FATHER'S PROFESSION	MOTHER'S PROFESSION
01	PRIVATE SECTOR EMPLOYEE	HOUSEWIFE
02	UNIVERSITY PROFESSOR	CLOTHES MERCHANT
03	NOT DECLARED	NOT DECLARED
04	MERCHANT	AGRONOMIST
05	HOTEL DIRECTOR	SALESWOMAN
06	CIVIL SERVANT	CIVIL SERVANT
07	CLOTHES HANDICRAFT OWNER	CLOTHES HANDICRAFT OWNER
08	DENTAL TECHNICIAN	HOUSEWIFE
09	ELECTRICAL ENGINEER	HOUSEWIFE
10	NEUROSURGEON	GYNECOLOGIST
11	POLICEMAN	TEACHER
24	PRIVATE SECTOR EMPLOYEE	BUSINESSWOMAN
25	MARITIME BUSINESSMAN	ART HISTORY SPECIALIST
12	DRAPE HANDICRAFT	DRAPE HANDICRAFT
13	PHYSICIAN GENERAL PRACTIONER	SECONDARY SCHOOL TEACHER PHYSICIST
14	CHIEF POLICE INSPECTOR	SECONDARY SCHOOL TEACHER PHILOLOGIST
26	PENSIONER SEAMAN	UNEMPLOYED
15	TAXI DRIVER	BANK EMPLOYEE
27	TRAINER	NURSE

CODE NUMBER	FATHER'S PROFESSION	MOTHER'S PROFESSION
16	BUILDER	HOUSEWIFE
28	EARTH WORKS	TYPIST
17	PRIVATE SECTOR EMPLOYEE	SELF-EMPLOYED
29	SELF-EMPLOYED	PRIVATE SECTOR EMPLOYEE
18	NURSE	NURSE
30	BUSINESSMAN	HOUSEWIFE
19	SECONDARY SCHOOL TEACHER	SECONDARY SCHOOL TEACHER
31	INFORMATICS EMPLOYEE	SECONDARY SCHOOL TEACHER
32	WORKER	TEACHER OF GERMAN LANGUAGE
47	HOTEL DIRECTOR	HOTEL EMPLOYEE
20	CAR MECHANICHER	HOUSEWIFE
33	PATHOLOGIST	PATHOLOGIST
34	SALESMAN	SALESWOMAN
21	AUTOMOBILIST	PRIVATE SECTOR EMPLOYEE
22	NOT DECLARED	NOT DECLARED
48	POLICEMAN CAPTAIN SARGENT	HOUSEWIFE
35	BUILDER	PRIVATE SECTOR EMPLOYEE
36	ACCOUNTANT	ACCOUNTANT
49	PRIVATE SECTOR EMPLOYEE	PRIVATE SECTOR EMPLOYEE
37	CIVIL ENGINEER	CIVIL ENGINEER
50	CRAFTSMAN	DENTIST ASSISTANT

CODE NUMBER	FATHER'S PROFESSION	MOTHER'S PROFESSION
51	NOT DECLARED	NOT DECLARED
52	BUSINESSMAN	HOUSEWIFE
38	SELF-EMPLOYED	NURSE
53	NOT DECLARED	NOT DECLARED
54	ACCOUNTANT EMPLOYEE	SECONDARY SCHOOL TEACHER
39	MILITARY	SECONDARY SCHOOL
55	PORT EMPLOYEE	PRIVATE SECTOR EMPLOYEE
56	PLUMER	NURSE
40	MERCHANT	HOUSEWIFE
57	NOT DECLARED	NOT DECLARED
41	CONSULTANT	UNIVERSITY EMPLOYEE
58	CIVIL SERVANT	CIVIL SERVANT
59	IMPORT-EXPORT MERCHANT	NUTRITIONIST-DIETICIAN
69	REPAIR SHOP	PRIVATE SECTOR EMPLOYEE
42	CIVIL SERVANT	SELF-EMPLOYED
70	PRIVATE SECTOR EMPLOYEE	PRIVATE SECTOR EMPLOYEE
71	PRIVATE SECTOR EMPLOYEE	CIVIL SERVANT
72	MILITARY	BANK EMPLOYEE
43	TOUR GUIDE	PIANO TEACHER
73	PHYSICIAN	PHYSICIAN
74	PHYSICIAN	PHYSICIAN

CODE NUMBER	FATHER'S PROFESSION	MOTHER'S PROFESSION
75	AGRICULTURIST	ACCOUNTANT
76	CIVIL SERVANT	CIVIL SERVANT
23	SELF-EMPLOYED	NURSERY GOVERNESS
77	PHYSIOTHERAPIST	PHYSIOTHERAPIST
44	SALES REPRESENTATIVE	SELF-EMPLOYED
60	NURSE	NURSE
45	BUSINESSMAN	MERCHANT
46	DENTIST	MILITARY
78	MERCHANT	MERCHANT
79	ELECTRICAL ENGINEER	ENGLISH TEACHER
61	SELF-EMPLOYED	SELF-EMPLOYED
80	PEROPERTY DEVELOPMENT	CIVIL SERVANT
81	PHYSICIST	CIVIL SERVANT
82	CIVIL SERVANT	CIVIL SERVANT
83	SELF-EMPLOYED	CIVIL SERVANT
84	ELECTRICAL ENGINEER	EMPLOYEE IN THE GREEK ELECTRICAL COMPANY
62	NOT DECLARED	ACCOUNTANT
85	NOT DECLARED	ACCOUNTANT
86	ELECTRONIC	NURSE
87	CARPENTER	HOUSEWIFE
63	NOT DECLARED	NOT DECLARED

CODE NUMBER	FATHER'S PROFESSION	MOTHER'S PROFESSION
88	PRIVATE SECTOR EMPLOYEE	CIVIL SERVANT
64	ACCOUNTANT	RETIRED NURSE
89	BUILDER	HOUSEWIFE
65	COMPUTER PROGRAMMER	PRIMARY EDUCATION TEACHER & COSMETICIAN
66	NOT DECLARED	NOT DECLARED
90	PRIVATE SECTOR EMPLOYEE	PRIVATE SECTOR EMPLOYEE
67	CARDIOLOGIST	ECONOMIST
91	PRIVATE SECTOR EMPLOYEE	EMPLOYEE IN THE GREEK ELECTRICAL COMPANY
68	SECONDARY SCHOOL TEACHER	HEMATOLOGIST
92	BUTCHER	PRIVATE SECTOR EMPLOYEE

APPENDIX III

PARTICIPANTS' MATHEMATICS SCHOOL PERFORMANCE IN YEARS 7, 8, AND 9

PARTICIPANT'S CODE NUMBER	YEAR 7 MATHE-MATICS MARK OF THE FINAL OFFICIAL EXAM	YEAR 7 AVERAGE MATHE-MATICS MARK	YEAR 8 MATHE-MATICS MARK OF THE FINAL OFFICIAL EXAM	YEAR 8 AVERAGE MATHE-MATICS MARK	YEAR 9 MATHE-MATICS MARK OF THE FINAL OFFICIAL EXAM	YEAR 9 AVERAGE MATHE-MATICS MARK
01	19	19	20	20	20	20
02	19	19	20	19	19	20
03	20	19	20	20	20	20
04	18	19	19	18	19	19
05	18	19	20	18	14	17
06	20	19	18	19	20	20
07	17	16	20	19	18	20
08	10	15	14	16	18	18
09	19	19	18	18	18	18
10	18	18	18	17	14	16
11	14	17	20	18	18	19
12	8	13	16	12	19	17
13	20	19	20	20	20	20
14	20	19	20	20	20	20
15	10	13	8	10	15	14
16	11	14	10	12	12	17
17	2	9	9	10	4	12
18	20	20	20	20	19	20
19	17	19	18	18	19	20
20	16	18	20	19	20	20
22	2	8	5	9	3	12
23	15	18	18	19	19	20
24	8	16	10	16	16	17
25	11	13	13	12	17	16
26	14	17	19	19	18	19
27	12	16	5	14	18	18
28	13	14	12	13	10	14
29	11	11	13	11	4	12
30	17	18	19	19	18	19
31	9	15	4	9	16	17
32	12	14	20	16	17	18
33	20	19	20	20	18	20
35	4	9	9	10	6	11
36	12	17	18	19	15	18
37	16	18	18	19	18	19
38	17	18	14	16	16	17

PARTICIPANT'S CODE NUMBER	YEAR 7 MATHE-MATICS MARK OF THE FINAL OFFICIAL EXAM	YEAR 7 AVERAGE MATHE-MATICS MARK	YEAR 8 MATHE-MATICS MARK OF THE FINAL OFFICIAL EXAM	YEAR 8 AVERAGE MATHE-MATICS MARK	YEAR 9 MATHE-MATICS MARK OF THE FINAL OFFICIAL EXAM	YEAR 9 AVERAGE MATHE-MATICS MARK
39	13	15	9	12	10	14
40	4	13	4	11	3	11
41	19	19	18	18	17	19
42	16	18	9	14	14	16
43	13	13	12	14	16	17
44	14	17	15	16	10	16
45	14	16	17	17	13	16
46	8	10	7	10	9	13
47	12	14	14	14	11	16
48	14	16	13	17	14	18
49	9	14	4	11	6	12
50	15	16	14	17	17	17
51	5	10	9	9	6	11
52	11	15	13	14	14	15
53	9	11	9	12	15	17
54	19	17	15	17	14	17
55	2	9	2	9	6	11
56	8	12	16	15	16	16
57	16	17	17	18	17	19
58	15	14	14	13	12	14
59	17	16	15	15	19	18
60	19	18	17	17	17	18
61	14	14	15	14	14	14
62	10	14	17	18	19	20
63	11	13	11	12	10	14
64	19	19	20	19	19	20
65	14	17	18	18	13	16
66	20	19	17	18	19	19
68	20	19	19	19	18	19
69	11	15	14	15	14	16
70	8	10	8	10	6	10
71	14	16	13	15	13	15
72	13	15	10	15	16	18
73	14	15	10	14	15	16
74	20	20	20	20	19	20
75	13	16	17	14	18	18
76	9	13	9	10	9	14
77	11	15	18	19	19	19
78	18	19	19	19	18	19
79	15	15	13	15	11	16
80	10	13	10	13	10	13

PARTICIPANT'S CODE NUMBER	YEAR 7 MATHE-MATICS MARK OF THE FINAL OFFICIAL EXAM	YEAR 7 AVERAGE MATHE-MATICS MARK	YEAR 8 MATHEMATICS MARK OF THE FINAL OFFICIAL EXAM	YEAR 8 AVERAGE MATHEMATICS MARK	MATHEMATICS YEAR 9 MARK OF THE FINAL OFFICIAL EXAM	YEAR 9 AVERAGE MATHEMATICS MARK
81	19	19	19	18	20	19
82	18	18	16	17	12	16
83	12	16	19	18	18	18
84	15	18	14	16	16	17
85	12	16	17	18	16	17
86	16	17	17	14	13	17
87	10	11	6	10	11	14
88	19	19	19	19	18	19
89	3	8	8	10	4	11
90	18	19	17	17	14	17
92	14	15	10	14	17	17
21	11	15	16	15	16	18
34	19	17	15	17	15	18
67	15	16	16	14	14	14
91	9	15	16	16	14	18
AVE-RAGE	13 59/92	15 15/23	14 43/92	15 11/23	14 61/92	16 3/4
ST DE-VIATION	4 120/161	3 6/277	4 379/489	3 127/389	4 202/375	2 11/16
MAX	20	20	20	20	20	20
MIN	2	8	2	9	3	10
RANGE	18	12	18	11	17	10

Grades in scale 1-20

APPENDIX IV

Information sheets and consent forms
and their Greek translations

1. Model of information sheet for parents/guardians

Dear ________________________

My name is Ioannis Kanellos and I am supervisor of teaching Mathematics in the prefectures of Heraklion and Lassithi. Aiming at improving the quality of our mathematical education, enhancing the professional skills of my colleagues as well as mine and as a student of Doctorate in Education of the School of Education and Lifelong Learning of the University of East Anglia (UEA) I am conducting a research on the teaching of proof in our schools.

Title of Research Project: "The Learning Studies approach to explore and improve the learning experience of year 9 and 10 Greek students with regard to their first encounter with mathematical proof."
Researcher: Ioannis Kanellos
Supervisors: Elena Nardi, Irene Biza

I would like to invite your child to take part in my research. Before you decide you need to know why I am doing this research and what it will involve. Please take time to read this information carefully together with your child to help you decide whether or not to take part. Please ring me and ask if there is anything that is not clear or if you would like more information. Thank you for reading this.

What is this study about?

I am trying to explore and improve the teaching and learning of proof of secondary school children in Years 9 and 10. The reason for this study is that understanding proof is a decisive point in the development of mathematical thinking. Let it be noted that there is currently little research in this area in our country. I hope that through this research, I will be able to contribute to the design of more effective strategies that enhance deeper mathematical understanding among students.

How will my child be involved?

The research will be conducted within the context of the current curriculum so that your child will take part in lessons as usual. The lessons plans will be taught by the classroom teacher or me. But in order to evaluate, revise and investigate the lesson process I will tape-record and video-record the lessons. Parallel to this procedure in the classroom could be present also teachers members of the lessons' plan working group.

What are the potential benefits?

This is an opportunity for your child to get involved in research. Besides that, the result from this study will contribute to the design of activities that promote more effective strategies for students' deeper mathematical understanding.

Will it affect my child's Mathematics lesson?
No, your child's Mathematics lesson will not be affected in any way since as previous already mentioned there will be no divergence of the curriculum and the syllabus.

Can you change your mind?
Yes. You and your child have the right to withdraw at any time.

Who will have the access to the video-recordings (data)?
Data management will follow the current Data Protection Act valid in England. I will not keep information about your child that could identify it to someone else. Only I and my supervisors will have access to the data. The data will be only analysed for the scope of my final dissertation and this will be accessible only by me, my supervisors and the two other markers of my work. All the names of the children taking part in the research and the Schools will be anonymised to preserve confidentiality.

Who has reviewed the study?
The School of Education and Lifelong Learning, UEA and the UEA Ethics Committee have reviewed and approved the study.

Whom do I speak to if problems arise?
If there is a problem please let me know. You can contact me at the following address:
Ioannis Kanellos
Parodos Ikarou 28, 71601
Nea Alikarnassos
Or
If you would like to speak to someone else you can contact Elena Nardi
School of Education and Lifelong Learning
University of East Anglia
NORWICH NR4 7TJ
Tel: +4401603592631

OK, I want to take part – what do I do next?

You and your child need to fill in the consent form, both sign it and your child will take it back to the school.

Thank you for your time.

2. Student information sheet

Dear Student,
You are being invited to take part in a research study. Before you decide if you would like to take part, it is important that you understand why the research is being done and what taking part will involve. Please take some time to read this sheet carefully and discuss it with your parents.

Who is doing this research?

- The research is conducted by me, Ioannis Kanellos, supervisor for teaching Mathematics in the prefectures of Heraklion and Lassithi. I will be working under the supervision of Elena Nardi and Irene Biza

What I want to find out?

- I am trying to research and improve the teaching and learning of proof in Years 9 and 10. I hope that through completing this study, I will be able to design more effective strategies in enhancing a deeper mathematical understanding among students.

Why have you been chosen?

- You have been chosen as a participant in this research because your school is helping me out in conducting this research.

Do you have to take part?

- NO. You do not have to take part in this study.
- If you decide YES, it is still okay to change your mind later and say NO.
- You do not have to give a reason for your decision.

How will you be involved?

- While engaged in a mathematical task, you will be working as usual in the classroom.
- There will be video recordings of every lesson of the research project.

Can I change my mind?

- YES. You have the right to stop participating at any time.

Will information about me be kept private?

- YES. All recordings and information about you will not be revealed or shown to someone else.
- Only I and my supervisor will have access to these recordings.

What happens at the end?

- The results will be written as part of my final dissertation. Your identity will be protected.

Whom do I speak to if there is a problem?

- If there is a problem please let me know. You can contact me at the following address: Ioannis Kanellos, Parodos Ikarou 28, 716 01 Nea Alikarnassos

Or

- If you would like to speak to someone else you can contact Elena Nardi or Irene Biza School of Education and Lifelong Learning University of East Anglia NORWICH NR47TJ Tel; +4401603592631

OK, I want to take part – what do I do next?

- You need to fill in the consent form with your parent, both of you sign it and then take it back to school.

Thank you for your time**.**

3. Head Teacher/Teacher information sheet

DATE:

Ioannis Kanellos

The person this letter is going to

Dear Head Teacher/Teacher,

You know me as a supervisor for the teaching of Mathematics in the prefectures of Heraklion and Lassithi. In order to enhance my professional skills as well as those of my colleagues I am willing to conduct a research as a supervisor as well as a student of Doctorate in Education of the School of Education and Lifelong Learning at the University of East Anglia (UEA).

My research, entitled "The learning studies approach to explore and improve the learning experience of year 9 and 10 Greek students with regard to their first encounter with mathematical proof", will focus on the teaching and learning of proof within the context of the current curriculum.

I am to carry out my fieldwork in a school, particularly, in a classroom with students as participants engaged in Mathematics lessons.

My research requires the tape- and video-recording of the lessons. Besides that, if necessary, the study will employ video-stimulated recall interview to obtain further details on recordings made.

I would greatly appreciate your consent to my request. If you require any additional information, please do not hesitate to contact me. I can be reached at:

E-mail : I.Kanellos@uea.ac.uk

And my supervisors E.Nardi@uea.ac.uk, I.Biza@uea.ac.uk

Yours sincerely

4. Parent/guardian consent form

Dear Parent/Guardian,

I am writing to you about the research that I am conducting as a supervisor for teaching Mathematics and as a student of Doctorate in Education of the School of Education and Lifelong Learning at the University of East Anglia (UEA). I am interested in researching and improving teaching and learning of proof in Years 9 and 10.

I have approached the School your child attends, and have explained to them the purpose of the study, and they have kindly agreed to distribute these letters to you.

If you are not interested in allowing your child to take part in this research, please read together with your child the information sheet enclosed. If you are willing for your child to take part in this study, please sign the form enclosed, ask your child to sign it too and hand it in to the school where it will be passed on to me.

If you have any further questions about the research, please contact me.

Yours sincerely,
Ioannis Kanellos

I have read the information about the study and I am willing for my child to take part in the study

Name : …………………………………………….

School : …………………………………………….

Class : ……………………………………………..

Parent/Guardian Signature: …………………………………………..

Student Signature: …………………………………………..

Date : ……………………………………………..

5. Head Teacher/Teacher consent form

Dear ……………………………………

I am writing to you about the research that I am conducting as a supervisor for teaching Mathematics and as a student of Doctorate in Education of the School of Education and Lifelong Learning at the University of East Anglia (UEA). I am interested in researching and improving teaching and learning of proof in Years 9 and 10.

Please read the information sheet enclosed. If you are willing to support/take part in this study, please sign this form.

If you have any further questions about the research, please contact me.

Yours sincerely,
Ioannis Kanellos

I have read the information sheet about the study and I am willing to support/take part in the study

Name : …………………………………………….

Head Teacher/ Teacher Signature: …………………………………………..

Date : ……………………………………………..

Greek translation of information sheet for parents

Αγαπητέ/-ή ___________________

Ονομάζομαι Ιωάννης Κανέλλος και είμαι σχολικός σύμβουλος Μαθηματικών στους νομούς Ηρακλείου και Λασιθίου. Στοχεύοντας την βελτίωση της ποιότητας της μαθηματικής μας εκπαίδευσης, την ανάπτυξη της επαγγελματικής ικανότητας τόσο των συναδέλφων όσο και της δικής μου και ως φοιτητής του Διδακτορικού του School of Education and Lifelong Learning του Πανεπιστημίου East Anglia (UEA) διεξάγω έρευνα με θέμα την διδασκαλία της απόδειξης στα σχολεία μας.

Τίτλος της έρευνας: «Η προσέγγιση Learning Studies για την διερεύνηση και τη βελτίωση της γνωστικής εμπειρίας των μαθητών Γ' Γυμνασίου και Α' Λυκείου σε σχέση με την πρώτη τους συνάντηση με την μαθηματική απόδειξη.»
Ερευνητής: Ιωάννης Κανέλλος
Επιβλέπουσες Καθηγήτριες: Έλενα Ναρδή, Ειρήνη Μπιζά

Θα ήθελα να καλέσω το παιδί σας να λάβει μέρος στην έρευνα. Πριν αποφασίσετε χρειάζεται να γνωρίζετε γιατί κάνω αυτήν έρευνα και τι συμπεριλαμβάνει. Σας παρακαλώ να διαθέσετε λίγο χρόνο να διαβάσετε αυτές τις πληροφορίες μαζί με το παιδί σας για να βοηθηθείτε να αποφασίσετε αν θα λάβετε μέρος. Παρακαλώ επικοινωνήστε μαζί μου αν κάτι δεν σας είναι σαφές ή αν θέλετε περισσότερες πληροφορίες. Σας ευχαριστώ για τον κόπο σας να διαβάσετε το παρόν κείμενο.

Ποιο το αντικείμενο της έρευνας;

Προσπαθώ να διερευνήσω και να βελτιώσω τη διδασκαλία και τη μάθηση της απόδειξης των μαθητών της Γ' Γυμνασίου και της Α' Λυκείου. Η αιτία για την έρευνα αυτή είναι το ότι η κατανόηση της απόδειξης είναι ένα αποφασιστικό σημείο στην ανάπτυξη της μαθηματικής σκέψης. Να σημειωθεί ότι υπάρχει λίγη έρευνα στη χώρα μας σε αυτήν την περιοχή αυτή τη στιγμή. Ελπίζω μέσω αυτής της έρευνας να σταθώ ικανός να συμβάλλω στο σχεδιασμό αποτελεσματικότερων στρατηγικών που θα συμβάλλουν στην βαθύτερη μαθηματική κατανόηση των μαθητών.

Πώς θα εμπλακεί το παιδί μου;

Η έρευνα θα διεξαχθεί μέσα στα πλαίσια του ισχύοντος αναλυτικού προγράμματος πράγμα που σημαίνει ότι το παιδί σας θα λάβει μέρος στο μάθημα ως συνήθως. Τα σχέδια μαθήματος θα διδαχθούν στην τάξη από τον καθηγητή της τάξης ή από εμένα. Για να μπορέσω όμως να εκτιμήσω, να επανελέγξω και να διερευνήσω τη διαδικασία του μαθήματος θα βιντεοσκοπήσω και μαγνητοφωνήσω τις διδασκαλίες. Παράλληλα με αυτήν την διαδικασία στην τάξη μπορεί να είναι παρόντες και άλλοι καθηγητές/-τριες μέλη της ομάδας εργασίας.

Ποια μπορεί να είναι οφέλη;

Δίνεται στο παιδί σας η ευκαιρία να εμπλακεί στην έρευνα. Εκτός αυτού το αποτέλεσμα της έρευνας θα συμβάλει στο σχεδιασμό δραστηριοτήτων που προάγουν πιο αποτελεσματικές στρατηγικές για την βαθύτερη μαθηματική κατανόηση των μαθη-τών/-τριών.

Θα επηρεάσει τις ώρες των Μαθηματικών του παιδιού μου;
Όχι! Οι ώρες των Μαθηματικών του παιδιού σας δεν θα επηρεαστούν κατά κανένα τρόπο αφού όπως ειπώθηκε προηγουμένως δεν θα υπάρξει απόκλιση από το αναλυτικό και το ωρολόγιο πρόγραμμα.

Μπορείτε να αλλάξετε γνώμη;
Ασφαλώς. Εσείς και το παιδί σας μπορείτε να αποσυρθείτε από την έρευνα όποτε θελήσετε.

Ποιος θα έχει πρόσβαση στις βιντεοσκοπήσεις και τα δεδομένα;
Η διαχείριση των δεδομένων της έρευνας υπόκειται στον ισχύοντα νόμο περί Προστασίας Δεδομένων που ισχύει στην Αγγλία. Δεν θα διατηρήσω πληροφορίες που θα μπορούσαν να αποκαλύψουν την ταυτότητα του παιδιού σας σε τρίτα πρόσωπα. Μόνον εγώ και οι επιβλέπουσες καθηγήτριες θα έχουν πρόσβαση στα δεδομένα. Τα δεδομένα θα αναλυθούν από την σκοπιά της τελικής μου διατριβής και θα είναι προσβάσιμα μόνο από εμένα, τις επιβλέπουσες καθηγήτριες και τους δυο βαθμολογητές της τελικής μορφής της διατριβής. Ονόματα των παιδιών που λαμβάνουν μέρος στην έρευνα και τα σχολεία δεν θα αναφέρονται για να τηρηθεί ο εμπιστευτικός χαρακτήρας τους.

Ποιος έχει ελέγξει και εγκρίνει την έρευνα;
Το School of Education and Lifelong Learning και η Ethics Committee (Επιτροπή Προστασίας Προσωπικών Δεδομένων) του Πανεπιστημίου East Anglia (UEA)

Με ποιον μπορώ να μιλήσω αν προκύψουν προβλήματα;
Εάν υπάρξουν προβλήματα παρακαλώ ενημερώστε με. Μπορείτε να επικοινωνήσετε με εμένα στην διεύθυνση

Ιωάννης Κανέλλος
Πάροδος Ικάρου 28 , 76 01
Νέα Αλικαρνασσός

Ή

Αν επιθυμείτε να μιλήσετε σε κάποιον άλλον μπορείτε να απευθυνθείτε

Στην Κα Έλενα Ναρδή
School of Education and Lifelong Learning
University of East Anglia
NORWICH NR4 7TJ
Tel: +4401603592631

Σας ευχαριστώ για τον χρόνο που διαθέσατε

Greek translation of
student information sheet

Αγαπητέ μαθητή/αγαπητή μαθήτρια
Σε προσκαλώ να λάβεις μέρος σε μια έρευνα. Πριν αποφασίσεις αν θα ήθελες να λάβεις μέρος είναι σημαντικό να κατανοήσεις για ποιο λόγο διεξάγεται η έρευνα και τι σημαίνει να λαμβάνεις σε αυτήν μέρος. Σε παρακαλώ να διαθέσεις λίγο από το χρόνο σου να διαβάσεις προσεκτικά το ενημερωτικό σημείωμα και να το συζητήσεις με τους γονείς σου.

Ποιος κάνει την έρευνα;

- Ο υποφαινόμενος, Ιωάννης Κανέλλος, σχολικός σύμβουλος Μαθηματικών στους νομούς Ηρακλείου και Λασιθίου θα εργασθεί ερευνητικά υπό την εποπτεία των κκ. Έλενας Ναρδή και Ειρήνης Μπιζά.

Τι επιδιώκω να ανακαλύψω;

- Προσπαθώ να ερευνήσω και να βελτιώσω τη διδασκαλία και τη μάθηση της απόδειξης των μαθητών της Γ' Γυμνασίου και της Α' Λυκείου. Ελπίζω με την ολοκλήρωση της έρευνας θα είμαι σε θέση να σχεδιάζω αποτελεσματικότερες στρατηγικές διευρύνοντας την βαθύτερη μαθηματική κατανόηση των μαθητών.

Γιατί επιλέχθηκες;

- Επιλέχθηκες να λάβεις μέρος στην έρευνα επειδή το σχολείο σου με βοηθά να διεξάγω την έρευνα.

Είσαι υποχρεωμένος/-νη να λάβεις μέρος;

- Όχι. Δεν είσαι υποχρεωμένος/-νη να λάβεις μέρος σε αυτήν την έρευνα.
- Αν αποφασίσεις ότι θέλεις δεν υπάρχει πρόβλημα αν αλλάξεις αργότερα γνώμη και θες να αποχωρήσεις.
- Δεν έχεις υποχρέωση να εξηγήσεις τους λόγους της απόφασης σου.

Πως θα λάβεις μέρος;

- Θα λάβεις μέρος στο καθημερινό μάθημα της τάξης
- Κάθε ώρα διδασκαλίας του ερευνητικού προγράμματος θα βιντεοσκοπείται

Μπορώ να αλλάξω γνώμη;

- Ναι. Έχεις το δικαίωμα να σταματήσεις να συμμετέχεις όποια στιγμή θες.

Θα διαφυλαχθούν τα προσωπικά μου δεδομένα;

- Ναι. Όλες οι πληροφορίες που σε αφορούν δεν θα αποκαλυφθούν σε τρίτα πρόσωπα.
- Μόνο εγώ και οι επιβλέπουσες καθηγήτριες θα έχουν πρόσβαση στα δεδομένα της έρευνας.

Τι θα συμβεί στο τέλος;

- Τα αποτελέσματα θα αποτελέσουν μέρος της τελικής μου διατριβής. Η ταυτότητά σου θα προστατευθεί.

Με ποιον θα μιλήσω αν υπάρξει πρόβλημα;

- Αν υπάρξει πρόβλημα σε παρακαλώ να με ενημερώσεις. Μπορείς να επικοινωνήσεις μαζί μου στην διεύθυνση: Ιωάννης Κανέλλος, Πάροδος Ικάρου 28, 716 01, Νέα Αλικαρνασσός

ή

- Αν θες να μιλήσεις με κάποιον άλλο μπορείς να απευθυνθείς στην κ. Έλενα Ναρδή ή κ. Ειρήνη Μπιζά στη διεύθυνση School of Education and Lifelong Learning University of East Anglia NORWICH NR4 7TJ Tel: 01603

Θέλω να λάβω μέρος, τι κάνω;

- Χρειάζεται να συμπληρώσεις το φύλλο συγκατάθεσης μαζί με τους γονείς σου και να το φέρεις στο σχολείο.

Σε ευχαριστώ για το χρόνο σου.

Greek translation of
Head Teacher/Teacher information sheet

Ημερομηνία :________________________________

Ιωάννης Κανέλλος

Προς____

Αγαπητέ κ. Διευθυντή

Με γνωρίζετε ως σχολικό σύμβουλο των Μαθηματικών στους νομούς Ηρακλείου και Λασιθίου. Επιδιώκοντας να αναπτύξω την επαγγελματική μου ικανότητα καθώς και αυτή των συναδέλφων μου προτίθεμαι να διεξάγω έρευνα τόσο ως Σχολικός Σύμβουλος αλλά και ως υποψήφιος διδάκτωρ του προγράμματος Doctorate in Education του School of Education and Lifelong Learning του Πανεπιστημίου της East Anglia (UEA).

Η έρευνά μου που έχει τίτλο «Η προσέγγιση Learning Studies για τη διερεύνηση και βελτίωση των γνωστικών εμπειριών των μαθητών Γ' Γυμνασίου και Α' Λυκείου κατά την πρώτη τους επαφή με τη μαθηματική απόδειξη», θα εστιάσει στη διδασκαλία και τη μάθηση της απόδειξης στα πλαίσια του ισχύοντος αναλυτικού προγράμματος.

Πρόκειται να επιτελέσω την εργασία μου στο σχολείο, ιδιαίτερα στη τάξη με τους μαθητές/-τριες ως συμμετέχοντες/συμμετέχουσες στο μάθημα των Μαθηματικών.

Η έρευνά μου χρειάζεται την μαγνητοφώνηση και βιντεοσκόπηση των μαθημάτων. Πέραν αυτού αν κριθεί αναγκαίο μπορεί να εφαρμόσει συνέντευξη επανάληψης που πυροδοτείται από παρακολούθηση βιντεοσκοπημένου υλικού για την επίτευξη παραπέρα πληροφοριών επί των βιντεοσκοπημένων στιγμιοτύπων.

Θα εκτιμούσα ιδιαίτερα τη σύμφωνη γνώμη σας στο αίτημά μου. Αν χρειάζεστε πρόσθετες πληροφορίες σας παρακαλώ μη διστάσετε να επικοινωνήσετε μαζί μου στην ηλεκτρονική διεύθυνση

I.Kanellos@uea.ac.uk

Και με τις επιβλέπουσες καθηγήτριες στις ηλεκτρονικές διευθύνσεις

E.Nardi@uea.ac.uk, I.Biza@uea.ac.uk

Με εκτίμηση

Greek translation of the
parent/guardian consent form

Αγαπητοί Γονείς/Κηδεμόνες,

Σας ενημερώνω για την έρευνα που διεξάγω τόσο ως σύμβουλος Μαθηματικών όσο και ως υποψήφιος διδάκτωρ του προγράμματος Doctorate in Education του School of Education and Lifelong Learning του Πανεπιστημίου East Anglia (UEA). Ενδιαφέρομαι να ερευνήσω και να βελτιώσω την διδασκαλία και τη μάθηση της απόδειξης των μαθητών Γ’ Γυμνασίου και Α’ Λυκείου.

Επισκέφθηκα το σχολείο που παρακολουθεί το παιδί σας και εξήγησα στα παιδιά το σκοπό της έρευνας και είχαν την καλοσύνη να δεχθούν να σας επιδώσουν την παρούσα επιστολή.

Αν δεν ενδιαφέρεστε να επιτρέψετε στο παιδί σας να λάβει μέρος σε αυτήν την έρευνα σας παρακαλώ να διαβάσετε με το παιδί σας το φύλλο πληροφοριών. Αν επιθυμείτε τη συμμετοχή του παιδιού σας παρακαλώ υπογράψτε την αντίστοιχη φόρμα μαζί με το παιδί σας και στείλετε την με αυτό σε μένα μέσω του σχολείου.

Αν έχετε παραπέρα ερωτήσεις για την έρευνα επικοινωνήστε με εμένα στο τηλέφωνο ____

Με εκτίμηση
Ιωάννης Κανέλλος
Σχολικός Σύμβουλος Μαθηματικών

Διάβασα το φύλλο πληροφοριών της έρευνας και προτίθεμαι να επιτρέψω στο παιδί μου να λάβει μέρος στην έρευνα.

Όνομα : ____

Σχολείο: ____

Τάξη : ____

Υπογραφή γονέα/κηδεμόνα : ____

Υπογραφή μαθητή/μαθήτριας:

Ημερομηνία: ____

Greek translation of
Head Teacher/Teacher consent form

Αγαπητέ συνάδελφε ....

Σας ενημερώνω για την έρευνα που διεξάγω ως Σχολικός Σύμβουλος Μαθηματικών και ως υποψήφιος διδάκτωρ του προγράμματος Doctorate in Education του School of Education and Lifelong Learning του πανεπιστημίου East Anglia (UEA). Ενδιαφέρομαι να ερευνήσω και να βελτιώσω τη διδασκαλία και τη μάθηση της απόδειξης στην Γ΄ Γυμνασίου και Α΄ Λυκείου.

Σας παρακαλώ να διαβάσετε το φύλλο πληροφοριών της έρευνας. Αν προτίθεσθε να υποστηρίξετε/λάβετε μέρος στην έρευνα παρακαλώ υπόγραψε παρακάτω.

Αν έχετε πρόσθετες απορίες γύρω από την έρευνα παρακαλώ επικοινωνήστε μαζί μου.

Στη διάθεσή σας πάντοτε.

Διάβασα το φύλλο πληροφοριών γύρω από την έρευνα και προτίθεμαι να υποστηρίξω/λάβω μέρος στην έρευνα.

Ονοματεπώνυμο:..

Υπογραφή Διευθυντή/Καθηγητή:..

Ημερομηνία:..

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