UNIVERSITY OF EAST ANGLIA

DOCTORAL THESIS

Modelling skewness in Financial data

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A thesis submitted in fulfillment of the requirements for the degree of Doctor of Philosophy

in the

Department of Economics
Faculty of Social Science

March 26, 2014

Declaration

This thesis is an account of research undertaken between October 2008 and May 2013 at The Department of Economics, Faculty of Social Science, The University of East Anglia, Norwich, United Kingdom.

Except where acknowledged in the customary manner, the material presented in this thesis is, to the best of my knowledge, original and has not been submitted in whole or part for a degree in any university.

Ann, W.Y. SHUM
March 26, 2014

Statement Of The Word Count

Number of words including in-text references and contents: 10,191

Ann, W.Y. SHUM
March 26, 2014

Abstract

The first systematic analysis of the skew-normal distribution in a scalar case is done by Azzalini (1985). Unlike most of the skewed distributions, the skew-normal distribution allows continuity of the passage from the normal distribution to the skew-normal distribution and is mathematically tractable. The skew-normal distribution and its extensions have been applied in lots of financial applications. This thesis contributes to the recent development of the skew-normal distribution by, firstly, analyzing the the properties of annualization and time-scaling of the skew-normal distribution under heteroskedasticity which, in turn allows us to model financial time series with the skew-normal distribution at different time scales; and, secondly, extending the Skew-Normal-GARCH(1,1) model of Arellano-Valle and Azzalini (2008) to allow for time-varying skewness.

Chapter one analyses the performance of the time scaling rules for computing volatility and skewness under the Skew-Normal-GARCH(1,1) model at multiple horizons by simulation and applies the simulation results to the Skew-Normal-Black-Scholes option pricing model introduced by Corns and Satchell (2007). Chapter two tests the Skew-Normal Black-Scholes model empirically. Chapter three extends the Skew-Normal-GARCH(1,1) model to allow for time-varying skewness. The time-varying-skewness adjusted model is then applied to test the relationship between heterogeneous beliefs, short-sale restrictions and market declines.

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Chapter 1

Annualization of skewness with application to the Skew Normal Black Scholes model: A Monte Carlo Study

1.1 Introduction

Skewness of the return distribution is generally acknowledged in the literature. The skew normal distributions, firstly documented by Azzalini (1985), has been seen as a natural choice for modelling skewness. The class of distributions not only has properties accords with the fundamental principles of the efficient market hypothesis but also derives useful theoretical outcomes for varies financial applications which, for example, includes the capital asset pricing model with skew normal distribution discussed in Adcock (2004), the skew-in-Mean GARCH model introduced by De Luca and Loperfido (2004) and the stochastic frontier analysis with skew-normality studied by Domìnguez-Molina and Ramos-Quiroga (2004). While financial applications assuming the skew normal distributions have gained more and more recognition, theories related to multi-period returns under the distributions are remain untested. The Skew-Normal-Black-Scholes option pricing model

(Corns and Satchell, 2007) is one of the theories which carries great significance in the related area. The model assumes that underlying stock prices follow skew Brownian motion and option pricing formula derived from the model extends the original Black-Scholes equation (Black and Scholes, 1973) to allow for the present of skewness. The Skew-Normal-Black-Scholes model nests the Black-Scholes model as a special case and accommodates skewness in the option pricing equation. Once the Skew-Normal-Black-Scholes equation is derived, the solution can be solved by standard build in functions in most of the computer software nowadays. It is tempting to test the theories empirically by converting daily volatility and daily skewness to annual volatility and annual skewness by applying the time scaling rules; that is, by applying the $\sqrt{250}$ rule to daily volatility to obtain annual volatility and the $1/\sqrt{250}$ rule to daily skewness to obtain annual skewness. However, unlike volatility, the properties of annualization and time-scaling of skewness under heteroskedasticity, one of the most prominent features of financial data, are far from clear. In this chapter, we address this question by analyzing the properties of skewness in the Skew-Normal model (Arellano-Valle and Azzalini, 2008) and the Skew-Normal-GARCH(1,1) model (Liseo and Loperfido, 2006). The resulting annual volatility and annual skewness estimators obtained from the simulation study are then applied to the Skew-Normal-Black-Scholes model to analysis the performance of the time scaling rules on option prices.

We note that the Skew-Normal distribution is not the only distribution to model skewness and the Skew-Normal-GARCH(1,1) model is not the only model that can model heteroskedasticity. However, computing option prices by plugging in the volatilities, mainly obtained from the GARCH type models, into the Black-Scholes formula is a widely used strategy among market participants (Knight and Satchell, 2002; Xekalaki and Degiannakis, 2010). Moreover, the Skew-Normal-Black-Scholes formula is no more complicated than the original formula and it nests the Black-Scholes model as a special case. Therefore the Skew-Normal-GARCH(1,1) model together with the Skew-Normal-Black-Scholes model allow us to extent the original model at almost no cost. The study of the properties of annualized skewness under the

Skew-Normal and the Skew-Normal-GARCH(1,1) models enable us not only to test the performance of the time scaling rules but, perhaps, also help us to to bring the Skew-Normal-Black-Scholes option pricing theory into more practical uses.

In section 2, we review the theoretical and empirical work that motivate our study. In section 3, we present our Skew-Normal model and the Skew-Normal-GARCH(1,1) model which help us to test the appropriateness of the time scaling rules. In section 4, we discuss our simulation analysis. In section 5, we apply the simulation results to the Skew-Normal-Black-Scholes option pricing model in order to analysis the performance of the time scaling rules on the option pricing model.

1.2 Literature Review

Converting 1-day to h-day volatility by scaling daily volatility with $\sqrt{h}$, i.e. the square root of time rule, is widely accepted by market practitioners. For example, it is not uncommon to calculate annualized volatility in the Black-Scholes equation by scaling daily conditional volatility of a univariate GARCH model with $\sqrt{250}$. The practice is more than just a convention; the Basel Committee on Banking Supervision (1996), a banking supervisor, recommends the use of the square root of time rule to get a 10-day VAR by rescaling daily VAR with $\sqrt{10}$. The square root of time rule is asserted again strongly as it is well known that it provides good unconditional h-day volatility approximations provided asset price follows a martingale, then its return is serially uncorrelated and unpredictable in mean. In addition, we assume that the asset market is under a non-speculative environment where the transversality condition should hold such that prices will never rise quicker than their discounts. Meucci (2010a), for example, provides an analytical proof exposing market invariant returns. Moreover, as can be seen in Diebold et al. (1997), when returns appear to be heteroskedastic, the square root of time rule provides correct unconditional h-day volatility on average although it magnifies conditional volatility fluctuations. Drost and Nijman (1993) has also demonstrated analytically that volatility fluctuation disappears and conditional volatility converges to unconditional volatility as $h \to \infty$. However, the simulation analysis carried by Diebold et al. (1997) assumes that returns follow a GARCH(1,1) process with normally distributed errors whereas Drost and Nijman (1993) mention nothing about skewness. Therefore, although we are able to show that the $\sqrt{h}$ rule provides correct h-day unconditional volatility on average, we know nothing about the properties of h-day unconditional skewness. Indeed, since skewness was not considered in the previous studies, we may not even know the properties of h-day unconditional volatility with the present of skewness.

Separated works about skewness have been done. Similar to the time scaling of volatility, Lau and Wingender (1989) and Meucci (2010b) shows that if the time series is invariant or, equivalently, independent and identically distributed across time, 1-day skewness can be converted to h-day skewness by applying the $1/\sqrt{h}$ time-scaling rule which indicates that skewness decays with time and vanishes as $h \to \infty$. However, as suggested by Meucci (2010b), the $1/\sqrt{h}$ rule does not hold under heteroskedasticity and there is no analytical formula available for calculating skewness at multiple horizons under heteroskedasticity. The closest topic has been discussed by Wong and So (2003). They calculate the third and forth moments of return under a Quadratic-GARCH (QGARCH) model. However, skewness in their model is induced by asymmetric volatility. If the asymmetric term in the QGARCH model is insignificant, the third moment will vanish and skewness will disappear. Although asymmetric GARCH models are important, asymmetric volatility is not the only source of skewness. For example, the rational bubble theory (Blanchard and Watson, 1983; Diba and Grossman, 1988) suggests that a sharp fall in price followed by a period of sustained stock price increase contributes to the overall negative skewness in the market and the heterogeneous-agent-based theory (Hong and Stein, 2003) suggests that negative skewness is greater when short selling is not allowed and heterogeneous beliefs is high enough. In other words, skewness could be induced by factors other than asymmetric volatility and can be present even without heteroskedasticity.

In light of the previous studies, we are interest in extending their work by analyzing the $\sqrt{h}$ and the $1/\sqrt{h}$ rule when the normality assumption is replaced by the skew-normality assumption under both homoskedasticity and heteroskedasticity using the Skew-Normal and the Skew-Normal-GARCH(1,1) models.

1.3 The Skew-Normal and the Skew-Normal-GARCH(1,1) models

1.3.1 The Skew-Normal model

In the centered parameterized Skew-Normal model, we consider the specification of returns in which

$$r_t = \mu + u_t \ , \quad u_t = \sigma\varepsilon_t \ , \tag{1.1}$$

$$\varepsilon_t \sim CSN(0, 1, \gamma) \tag{1.2}$$

where r_t is return at time t, μ is the unconditional mean of returns, u_t is the unexpected part of returns which is generally referred to as "news" in the markets, σ^2 is a homoscedastic variance parameter and γ is the unconditional skewness of returns. Following the centered parametrization used in Arellano-Valle and Azzalini (2008) and Liseo and Loperfido (2006), the centered skew normal innovation term ε_t with zero mean, unit variance and unconditional skewness γ is the standardized version of z_t given by

$$\varepsilon_t = \frac{z_t - \mu_z}{\sigma_z} \ , \tag{1.3}$$

$$z_t \sim SN(0, 1, \alpha) \tag{1.4}$$

where $\mu_z = b\delta$ and $\sigma_z^2 = 1 - \mu_z^2$ with $b = (2/\pi)^{1/2}$ and $\delta = \alpha(1+\alpha^2)^{-1/2}$ are the mean and variance of z_t which is a sequence of independent, identically distributed standard skew normal random variable with density function

$$f(z\,; \eta, \omega, \alpha) = 2\,\phi\left(\frac{z-\eta}{\omega}\right)\,\Phi\left(\alpha\frac{z-\eta}{\omega}\right) \ . \tag{1.5}$$

Note that when $\alpha = 0$ the skew normal density function is identical to the normal density function. Having set the unconditional mean μ, variance σ^2 and skewness γ as

$$\mu = \eta + \omega\mu_z \ , \tag{1.6}$$

$$\sigma^2 = \omega^2 \left(1 - \mu_z^2\right) \tag{1.7}$$

and

$$\gamma = \frac{4-\pi}{2} \frac{\mu_z^3}{\left(1-\mu_z^2\right)^{3/2}} \tag{1.8}$$

the centered parameterized Skew-Normal model is equivalent to

$$r_t = \eta + \omega z_t \ , \quad z_t \sim SN(0,1,\alpha) \tag{1.9}$$

where return at time r_t in the model is parameterized by using the standard skew normal random variable z_t directly with location parameter η, scale parameter ω and shape parameter α. We denote the parameter vector for the centered parameterized Skew-Normal model as

$$\text{SKEWN}(\mu, \sigma^2, \gamma) \tag{1.10}$$

with

$$\text{SKEWN}(\mu_{(1)}, \sigma^2_{(1)}, \gamma_{(1)}) \quad \text{and} \quad \text{SKEWN}(\mu_{(h)}, \sigma^2_{(h)}, \gamma_{(h)}) \tag{1.11}$$

represents its daily and h-day parameter vectors respectively; and the direct parameterized Skew-Normal model as

$$\text{SKEWN}(\eta, \omega^2, \alpha) \tag{1.12}$$

with

$$\text{SKEWN}(\eta_{(1)}, \omega^2_{(1)}, \alpha_{(1)}) \quad \text{and} \quad \text{SKEWN}(\eta_{(h)}, \omega^2_{(h)}, \alpha_{(h)}) \tag{1.13}$$

represents its daily and h-day parameter vectors respectively. The two parameterization can be used interchangeably. However model parameters has to be estimated by using the centered parameterization since Azzalini (1985) and Arellano-Valle and Azzalini (2008) has shown that the maximum likelihood estimation can be problematic if the direct parameterization is used. Note that returns under the Skew-Normal model are independent and identically distributed (i.i.d) across time, and thus, the Skew-Normal model is in accordance with the assumptions of the $\sqrt{h}$ and the $1/\sqrt{h}$ rules.

1.3.2 The Skew-Normal-GARCH(1,1) model

In the centered parameterized Skew-Normal-GARCH(1,1) model, we consider the specification of returns in which

$$r_t = \mu + \sigma_t \varepsilon_t \tag{1.14}$$

where r_t is daily return at day t, μ is the unconditional mean, ε_t is the innovation terms which follows the centered skew normal distribution, $\mathrm{CSN}(0, 1, \gamma)$, and σ_t^2 is the conditional variance of a GARCH(1,1) process

$$\sigma_t^2 = a_0 + a_1 \sigma_{t-1}^2 + a_2 u_{t-1}^2 \ . \tag{1.15}$$

We denote the parameter vector for Skew-Normal-GARCH(1,1) model as

$$\text{SKEWN-GARCH}(\mu, \sigma^2, \gamma) \tag{1.16}$$

with unconditional mean μ, unconditional variance

$$\sigma^2 = E(\sigma_t^2) = a_0/(1 - a_1 - a_2) \tag{1.17}$$

and unconditional skewness γ. The corresponding daily and h-day parameter vectors are

$$\text{SKEWN-GARCH}(\mu_{(1)}, \sigma_{(1)}^2, \gamma_{(1)}) \quad \text{and} \quad \text{SKEWN-GARCH}(\mu_{(250)}, \sigma_{(250)}^2, \gamma_{(250)}) \tag{1.18}$$

respectively. While returns under the Skew-Normal model are invariant, similar to the GARCH model with normally distributed errors, returns under the Skew-Normal-GARCH(1,1) model are still uncorrelated but no longer i.i.d across time.

1.4 Annualization and time scaling of volatility and skewness with Simulated Data

Testing the time scaling rule empirically is difficult if not impossible. A data set which includes daily data from 1950 to 2013 has around 60 yearly non-overlapping observations. The annual data set, even the largest possible data set that we can obtain, is pitifully small in terms of sample size. We can achieve a larger data set by using overlapping data. However, the overlapped data are highly dependent, and thus, are not very useful for any statistical tests. We can also test the scaling rules at a shorter horizon. However, we cannot grantee the short-horizon behavior can be inferred to long-horizon behavior. Fortunately, we can confirm the validity of the time scaling rules using simulation. For testing the problem of annualization, we generate $m = 1000$ time series of daily returns or Monte Carlo sample paths with daily sample size $n_1 = 250000$ under the Skew-Normal model with daily parameters $\mu_{(1)} = 0$, $\sigma^2_{(1)} = 0.04^2$ and $\gamma_{(1)} = -0.7, -0.3, -0.1, 0, 0.1, 0.3, 0.7$; and the Skew-Normal-GARCH(1,1) model with daily parameters $\mu_{(1)} = 0$, $\gamma_{(1)} =$ -0.7, -0.3, -0.1, 0, 0.1, 0.3, 0.7 and

$$\sigma^2 = E(\sigma_t^2) = \frac{a_0}{1 - a_1 - a_2} = \frac{0.0041}{1 - 0.8 - 0.1} = 0.04^2 \ . \tag{1.19}$$

Note that the two models have the same unconditional variance, i.e. 0.04^2, for ease of comparison. We calculate the "theoretical" annual volatility by multiplying 1-day volatility with $\sqrt{250}$; and the annual skewness by multiplying 1-day skewness with $1/\sqrt{250}$. We denote annual volatility and skewness obtained by using the time-scaling rules as $\sigma^S_{(250)}$ and $\gamma^S_{(250)}$ respectively. Daily returns are then aggregate to obtain non-overlapping annual returns with sample size $n_{(250)} = 1000$. The annual unconditional parameters

$$\mu_{(250)}^{(m)}, \sigma_{(250)}^{2,(m)}, \gamma_{(250)}^{(m)}, \tag{1.20}$$

for each Monte Carlo sample paths, m=1,...1000, under both the Skew-Normal and the Skew-Normal-GARCH(1,1) models are estimated by the maximum likelihood method assuming that conditional returns follow the centered parameterized Skew-Normal model. We regard the "actual" unconditional annual variance for the underlying data generating process as

$$\sigma_{(250)}^{2,A} = \frac{1}{m} \sum_{m} \hat{\sigma}_{(250)}^{2,(m)} \tag{1.21}$$

and the "actual" unconditional annual skewness for the underlying data generating process as

$$\gamma_{(250)}^{A} = \frac{1}{m} \sum_{m} \hat{\gamma}_{(250)}^{(m)} \tag{1.22}$$

where $\hat{\sigma}_{(250)}^{2,(m)}$ and $\hat{\gamma}_{(250)}^{(m)}$ are the annual variance and annual skewness estimators for the m^{th} Monte Carlo sample path. To look at the performance of the time-scaling rules. We compare the actual values with the values obtained by applying the time-scaling rules. In other words, we are concerned with the problem of testing the two null hypothesis

$$\text{Hypothesis I:} \qquad H_0 : \sigma_{(250)}^{2,A} = \sigma_{(250)}^{2,S}$$

$$\text{Hypothesis II:} \qquad H_0 : \gamma_{(250)}^{A} = \gamma_{(250)}^{S}$$

against the alternatives that the actual values obtained by using the simulation method are not the same as the theoretical values obtained by using the time scaling rules for annual volatility and skewness.

The Matlab simulation program,"mysn_sim" and "mysngarch_sim" for the centered parameterized Skew-Normal Model and Skew-Normal-GARCH Model are presented in the Appendix.

1.4.1 Simulation results

Tables 1 and 2 contain the simulation results for the parameters $\sigma^2_{(1)}$ and $\gamma_{(1)}$ respectively. The interpretation of the content of these tables is best explained with an example. In the very first line of Table 1, we see that when the simulation is carried out using SKEWN(0,0.042,-0.7), the mean value of the actual unconditional annual variance obtained over the 1000 replications is 0.4002, which compares very closely to the true value of this parameter, which is 0.4000. The t-statistic for testing this difference is 0.2739, resulting in an acceptance of the null hypothesis in this case.

In fact, we see that all of the rows in Table 1 contain acceptances of this null hypothesis. From this we may conclude that the $\sqrt{250}$ rule for converting 1-day volatility to 250-day volatility is correct for both the Skew-Normal model and the Skew-Normal GARCH(1,1) model. Figures 1 and 2 present graphical representations of the same information, and these also suggest that the unconditional annual variance estimators for both models are closely centred around the scaling value.

Table 1: RMSE, BIAS and Test results for Hypothesis I: $H_0 : \sigma^{2,T}_{(250)} = \sigma^{2,S}_{(250)}$

Model		Daily Specification	$\sigma^{2,S}_{(250)}$	$\sigma^{2,A}_{(250)}$	$RMSE$	$BIAS$	H	*t-stat*	*st.err.*	[95% Conf.	Interval]
SKEWN	1.	(0,0.04^2,-0.7)	0.4000	0.4002	0.0178	0.0002	0	0.2739	0.0178	0.3990	0.4013
	2.	(0,0.04^2,-0.3)	0.4000	0.4000	0.0181	0.0000	0	0.0270	0.0181	0.3989	0.4011
	3.	(0,0.04^2,-0.1)	0.4000	0.4001	0.0180	0.0001	0	0.1548	0.0180	0.3990	0.4012
	4.	(0,0.04^2,0.0)	0.4000	0.4005	0.0174	0.0005	0	0.9556	0.0174	0.3994	0.4016
	5.	(0,0.04^2,0.1)	0.4000	0.4010	0.0174	0.0010	0	1.8229	0.0173	0.3999	0.4021
	6.	(0,0.04^2,0.3)	0.4000	0.3994	0.0174	-0.0006	0	-1.1136	0.0174	0.3983	0.4005
	7.	(0,0.04^2,0.7)	0.4000	0.4010	0.0178	0.0010	0	1.8639	0.0178	0.3999	0.4022
SKEWN-GARCH	1.	(0,0.04^2,-0.7)	0.4000	0.4006	0.0187	0.0006	0	1.0011	0.0187	0.3994	0.4017
	2.	(0,0.04^2,-0.3)	0.4000	0.4007	0.0188	0.0007	0	1.2336	0.0188	0.3996	0.4019
	3.	(0,0.04^2,-0.1)	0.4000	0.4009	0.0187	0.0009	0	1.6015	0.0187	0.3998	0.4021
	4.	(0,0.04^2,0.0)	0.4000	0.4002	0.0187	0.0002	0	0.3816	0.0187	0.3991	0.4014
	5.	(0,0.04^2,0.1)	0.4000	0.4002	0.0181	0.0002	0	0.3734	0.0181	0.3991	0.4013
	6.	(0,0.04^2,0.3)	0.4000	0.4002	0.0177	0.0002	0	0.3963	0.0177	0.3991	0.4013
	7.	(0,0.04^2,0.7)	0.4000	0.4002	0.0179	0.0002	0	0.4397	0.0179	0.3991	0.4014

"Daily Specifications" shows the seven parameter vectors for the Skew-Normal and the Skew-Normal-GARCH(1,1) models; "$\sigma^{2,S}_{(250)}$*" is the unconditional annual variance obtained by using the scaling rule; "*$\sigma^{2,A}_{(250)}$*" is the actual unconditional annual variance obtained by Monte Carlos simulation; 'RMSE" for unconditional variance are defined as:* $[\frac{1}{m}\sum_m(\hat{\sigma}^{2,(m)}_{(250)} - \sigma^{2,S}_{(250)})^2 + BIAS^2]^{1/2}$ *where* $BIAS = \sigma^{2,T}_{(250)} - \sigma^{2,S}_{(250)}$*; H is equal to 1 if the null hypothesis is rejected and is equal to 0 otherwise; "t-stat", "st.err." and "95% Conf. Interval" are the t-statistics, standard errors and the 95% confidence interval for the hypothesis tests.*

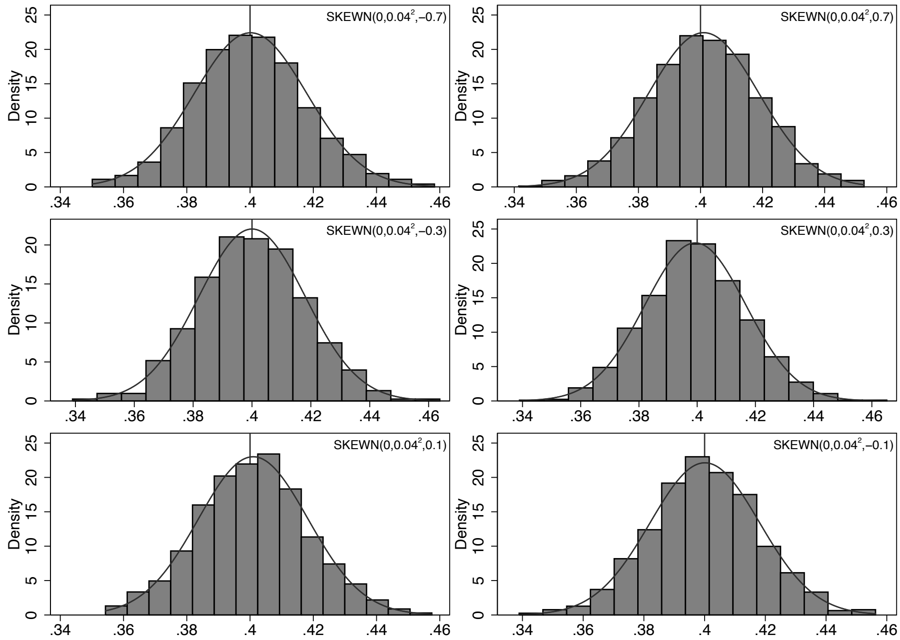


Figure 1: Estimated distributions of the unconditional annual variance estimators, $\hat{\sigma}^{2,(m)}_{(250)}$, for the Skew-Normal model.

Notes for Figure 1 to 5: The daily model parameter vectors for the Skew-Normal model and the Skew-Normal-GARCH model are displayed in the graphs as SKEWN($\mu_{(1)}$, $\sigma^2_{(1)}$, $\gamma_{(1)}$) and SKEWN-GARCH($\mu_{(1)}$, $\sigma^2_{(1)}$, $\gamma_{(1)}$) where $\mu_{(1)}$ is its daily unconditional mean, $\sigma^2_{(1)}$ is the daily unconditional variance and $\gamma_{(1)}$ is the daily unconditional skewness. The annual unconditional parameters, $\hat{\sigma}^{2,(m)}_{(250)}$ and $\hat{\gamma}^{(m)}_{(250)}$ for each Monte Carlo sample paths, m=1,...1000, are estimated by the maximum likelihood method assuming that conditional returns follow the centered parameterized Skew-Normal model. The vertical lines represent the scaling values $\sigma^{2,S}_{(250)}$ and $\gamma^{S}_{(250)}$.

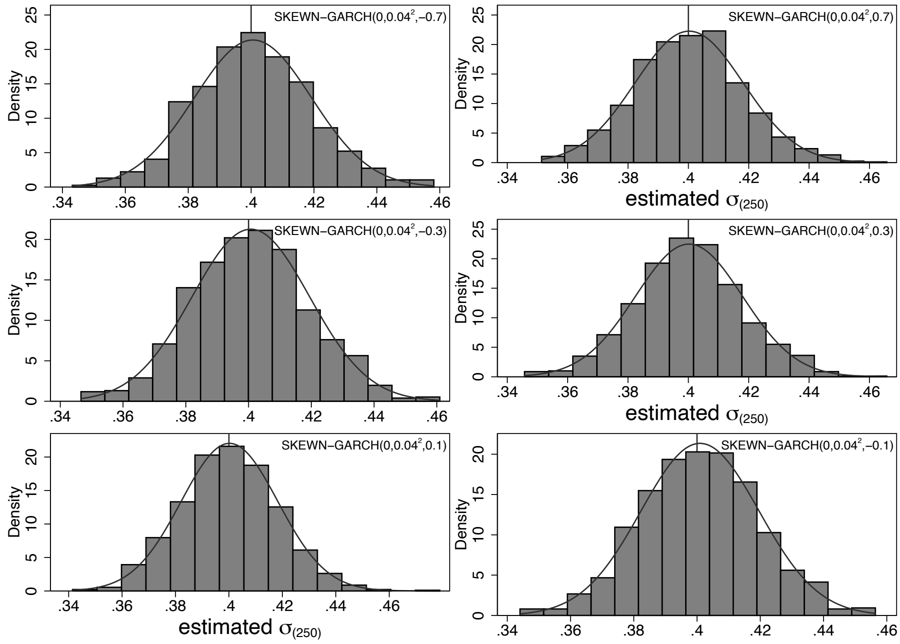


Figure 2: Estimated distributions of the unconditional annual variance estimators, $\hat{\sigma}^{2,(m)}_{(250)}$, for the Skew-Normal-GARCH model.

See notes under Figure 1

However, looking at Table 2 for the performance of the $1/\sqrt{250}$ rule, we conclude that the $1/\sqrt{250}$ rule for converting 1-day to 250-day skewness is appropriate only under the assumption of homoskedasticity while the rule is inappropriate for heteroskedastic returns. As, on one hand, we have evidence to show that the $1/\sqrt{250}$ rule under the Skew-Normal model provides correct h-day unconditional skewness estimates since we cannot reject the null hypothesis that the actual annual skewness is equal to the annual skewness obtained by using the scaling rule for all specifications under the Skew-Normal model, but on the other hand, we reject the null hypothesis and accept the alternative hypothesis that the actual annual skewness estimators obtained by using the simulation method are not the same as the theoretical values obtained by applying the $1/\sqrt{250}$ rule for almost all of the specifications under the Skew-Normal-GARCH(1,1) model. The only exception when the $1/\sqrt{250}$ rule under the Skew-Normal-GARCH(1,1) model provides

Table 2: $RMSE$, $BIAS$ and Test results for Hypothesis II: $H_0 : \gamma^T_{(250)} = \gamma^S_{(250)}$

Model	Daily Specification	$\gamma^S_{(250)}$	$\gamma^A_{(250)}$	$RMSE$	$BIAS$	H	*t-stat*	*st.err.*	[95% Conf.	Interval]
SKEWN	1. $(0,0.04^2,-0.7)$	-0.0443	-0.0446	0.0786	-0.0003	0	-0.1292	0.0786	-0.0495	-0.0397
	2. $(0,0.04^2,-0.3)$	-0.0190	-0.0200	0.0792	-0.0011	0	-0.4193	0.0793	-0.0249	-0.0151
	3. $(0,0.04^2,-0.1)$	-0.0063	-0.0075	0.0779	-0.0011	0	-0.4664	0.0779	-0.0123	-0.0026
	4. $(0,0.04^2,0.0)$	0.0000	-0.0004	0.0761	-0.0004	0	-0.1487	0.0761	-0.0051	0.0044
	5. $(0,0.04^2,0.1)$	0.0063	0.0069	0.0760	0.0005	0	0.2236	0.0760	0.0021	0.0116
	6. $(0,0.04^2,0.3)$	0.0190	0.0192	0.0761	0.0002	0	0.0932	0.0761	0.0145	0.0239
	7. $(0,0.04^2,0.7)$	0.0443	0.0429	0.0772	-0.0014	0	-0.5706	0.0772	0.0381	0.0477
SKEWN-GARCH	1. $(0,0.04^2,-0.7)$	-0.0443	-0.1714	0.1505	-0.1271	1	-49.8871	0.0806	-0.1764	-0.1664
	2. $(0,0.04^2,-0.3)$	-0.0190	-0.0753	0.1007	-0.0563	1	-21.3407	0.0835	-0.0805	-0.0701
	3. $(0,0.04^2,-0.1)$	-0.0063	-0.0261	0.0854	-0.0198	1	-7.5257	0.0831	-0.0313	-0.0209
	4. $(0,0.04^2, 0.0)$	0.0000	0.0003	0.0801	0.0003	0	0.1260	0.0802	-0.0047	0.0053
	5. $(0,0.04^2, 0.1)$	0.0063	0.0269	0.0823	0.0206	1	8.1791	0.0797	0.0220	0.0319
	6. $(0,0.04^2,0.3)$	0.0190	0.0761	0.0988	0.0571	1	22.4090	0.0806	0.0711	0.0811
	7. $(0,0.04^2,0.7)$	0.0443	0.1726	0.1511	0.1283	1	50.7547	0.0799	0.1676	0.1775

"Daily Specifications" shows the seven parameter vectors for the Skew-Normal and the Skew-Normal-GARCH(1,1) models; $\gamma^S_{(250)}$" is the unconditional annual skewness obtained by using the scaling rule; "$\gamma^A_{(250)}$"is the actual unconditional annual skewness obtained by Monte Carlos simulation; 'RMSE" for unconditional skewness are defined as: $[\frac{1}{m}\sum_m(\hat{\gamma}^{(m)}_{(250)} - \gamma^S_{(250)})^2 + BIAS^2]^{1/2}$ where $BIAS = \gamma^T_{(250)} - \gamma^S_{(250)}$; H is equal to 1 if the null hypothesis is rejected and is equal to 0 otherwise; "t-stat", "st.err." and "95% Conf. Interval" are the t-statistics, standard errors and the 95% confidence interval for the hypothesis tests.

a good approximation for the 250-day skewness is when the daily and, thus, the annual skewness parameters are equal to zero. Moreover, Figure 3 and Figure 4 indicate that annual skewness estimators under the Skew-Normal model are closely centered around the scaling values whereas the scaling values overestimate (underestimate) unconditional annual skewness when daily skewness is negative (positive) under the Skew-Normal-GARCH(1,1) model.

Note that the aim of the simulation is to show that controlling for both skewness and variance, i.e. given the same location, shape and scale parameters, the time scaling rule fail to provide good approximation for annual skewness under the assumption of heteroskedasticity. This is clearly shown in the results discussed above. However, it is difficult to say that when daily skewness is becoming more and more negative or positive, the precision of the time scaling rule will decay since skewness is affecting variance under the Skew-Normal and the Skew-Normal-GARCH models. Therefore, although the RMSE and the BIAS indicate that the time scaling rule provide less and less precise estimation for annual skewness when we have more and more negative or positive daily skewness, we cannot conclude that the degree of daily skewness will be affected the precision of the time scaling annual skewness because the lost in precision may be caused by increasing variance which is positively correlated with the severity of skewness.

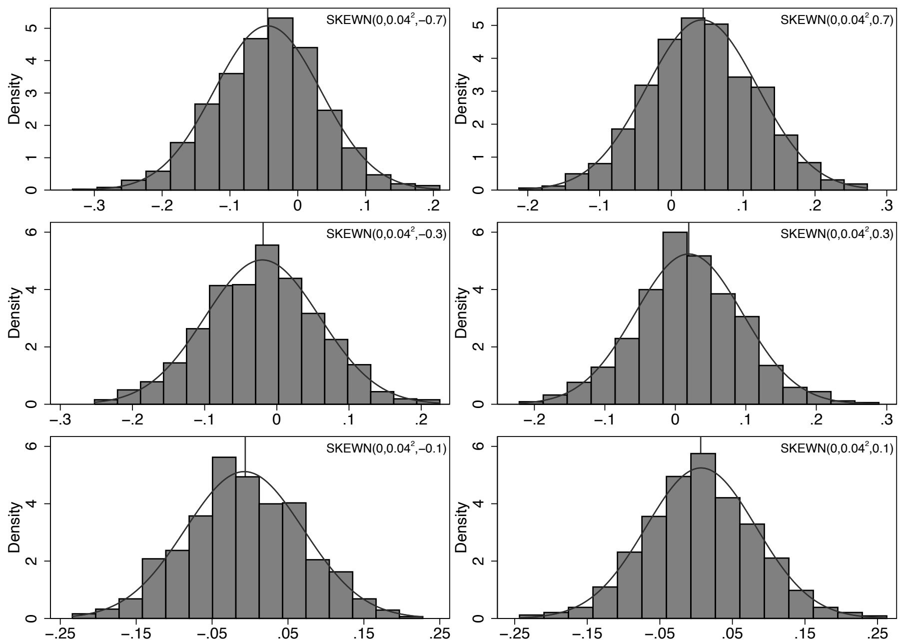


Figure 3: Estimated distributions of the unconditional annual skewness estimators, $\hat{\gamma}^{(m)}_{(250)}$, for the Skew-Normal model.

See notes under Figure 1

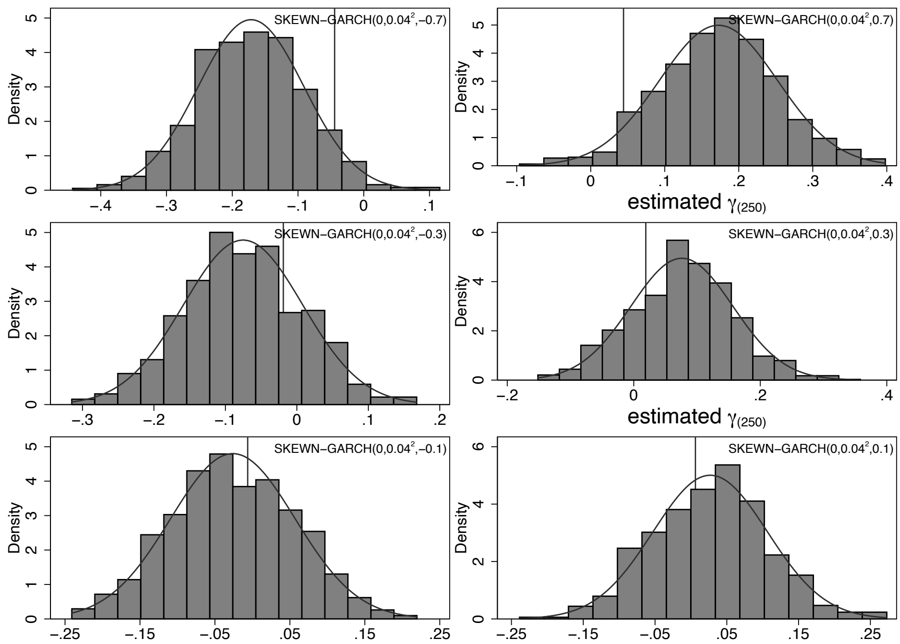


Figure 4: Estimated distributions of the unconditional annual skewness estimators, $\hat{\gamma}^{(m)}_{(250)}$*, for the Skew-Normal-GARCH model.*

See notes under Figure 1

The behavior of the scaling rules under the normality assumption can be seen from the fourth specification in Table 1 and 2 which display the hypothesis test results when the daily and annual skewness parameters are set equal to zero. When the skewness parameter is equal to zero, the Skew-Normal distribution collapses to the normal distribution. Since we cannot reject the null hypothesis that the actual annual variance and skewness estimators obtained by using the simulation method are the same as the theoretical values obtained by applying the $\sqrt{250}$ and the $1/\sqrt{250}$ rules when the daily and annual skewness parameters are equal to zero; and the actual unconditional variance and skewness estimators are centered closely around the scaling values as can been seen in Figure 5, we can conclude that the scaling rules work well under the normality assumption with or without the present of heteroskedasticity.

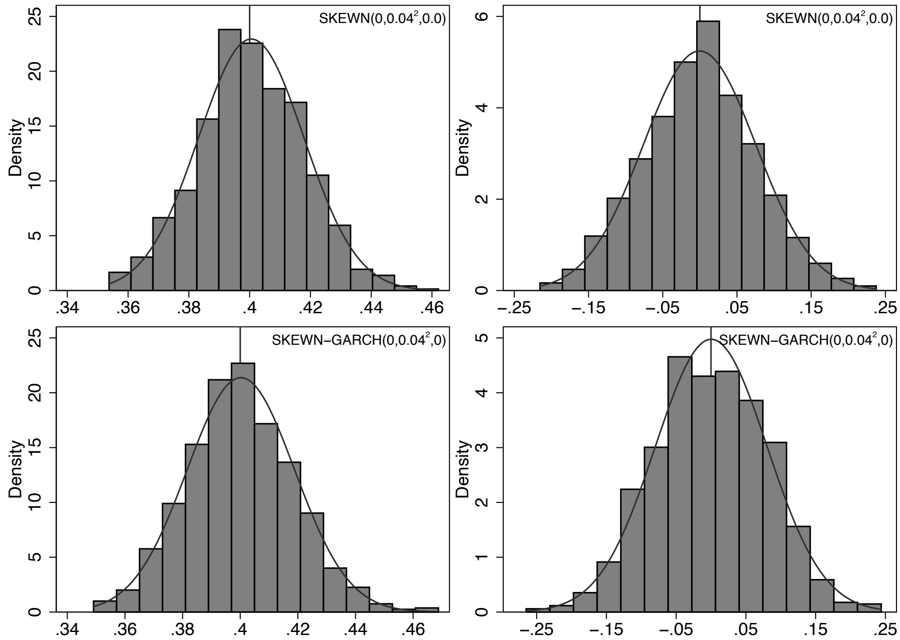


Figure 5: Estimated distributions of the unconditional annual variance, $\hat{\sigma}^{2,(m)}_{(250)}$, and unconditional annual skewness, $\hat{\gamma}^{(m)}_{(250)}$, estimators for the Skew-Normal and the Skew-Normal-GARCH models with daily skewness parameter $\gamma_{(1)} = 0$.

See notes under Figure 1

1.5 Application to the Skew-Normal-Black-Scholes option pricing model

The Skew-Normal-Black-Scholes Option pricing model introduced by Corns and Satchell (2007) assumes stock price follows skew Brownian motion. The European call option price with underlying stock price S, exercise price K, time to maturity τ and interest rate r derived from their model is:

$$CALL = \frac{1}{2\Phi(\delta_{(250)}\omega_{(250)}\sqrt{\tau})} S\Psi_1(\theta) - e^{-r\tau} K\Psi_2(\theta), \tag{1.23}$$

with

$$\Psi_1(\theta) = 2\int_{\theta}^{\infty}\int_{-\infty}^{s\alpha_{(250)}} \phi(s - \omega_{(250)}\sqrt{\tau})\phi(u)duds, \tag{1.24}$$

$$\Psi_2(\theta) = 2\int_{-\infty}^{-\theta}\int_{-\infty}^{-s\alpha_{(250)}} \phi(s)\phi(u)duds, \tag{1.25}$$

$$\theta = \frac{ln(K/S) - \{[r - (\omega^2_{(250)}/2)]\tau - ln2\Phi(\delta_{(250)}\omega_{(250)}\sqrt{\tau})\}}{\omega_{(250)}\sqrt{\tau}}, \tag{1.26}$$

where $\delta_{(250)} = \alpha_{(250)}(1+\alpha^2_{(250)})^{1/2}$, $\phi(\cdot)$ and $\Phi(\cdot)$ are the standard normal density and distribution functions. The Skew-Normal-Black-Scholes formula and the Black-Scholes differ only by the skewness parameters $\alpha_{(250)}$ which govern the degree of skewness of the underlying data. When $\alpha_{(250)} = 0$, the skew-Normal-Black-Scholes option pricing model reduces to the Black-Scholes model.

Empirically, the two parameters $\omega_{(250)}$ and $\alpha_{(250)}$ are not observable and have to be estimated. Since daily returns are almost surely heteroskedastic, in practice, one can estimate the center parameterized Skew-Normal-GARCH(1,1) model to obtain the daily centered parameters, $\sigma^2_{(1)}$ and $\gamma_{(1)}$, and then apply either the scaling rules or the simulation method to obtain the annual centered parameters, $\sigma^2_{(250)}$ and $\gamma_{(250)}$, which can be transformed into the annual direct parameters, $\omega^2_{(250)}$ and $\alpha_{(250)}$.

Consider a benchmark case with stock price $S = 100$, exercise price $K = 100$, annual risk free rate $r = 0.1$ and time to maturity $\tau = 0.25$. In order to study the performance of the time scaling rules on option pricing, we compare the European call option prices computed by the "actual" annual volatility and skewness estimators obtained by simulation with the prices computed by the scaling values. The centered annual parameters have been analyzed in the previous section and the parameter values are reported in Table 1 and Table 2. Since estimations haven been done by using the centered parameterization, the annual centered parameters obtained either by the simulation method or the scaling rules are transformed into direct annual parameters needed for the option pricing formula. By plugging in the transformed direct values into the Skew-Normal-Black-Scholes option pricing formula, we obtain the corresponding call option prices. Table 3 reports the European call option prices obtained by using the scaling parameters in panel A and the prices obtained by using the simulated parameters in panel B. As can be seen in

Table 3: European call option prices obtained by using the direct scaling parameters and the direct actual parameters

Panel A.		Daily Specifications	Centered Scaling Param.		Direct Scaling Param.			Scaling Call Option Prices
		$(\mu_{(1)},\sigma^2_{(1)},\gamma_{(1)})$	$[\sigma^{2,S}_{(250)}$	$\gamma^{S}_{(250)}]$	$[\omega^{2,S}_{(250)}$	$\alpha^{S}_{(250)}$	$\delta^{S}_{(250)}]$	CALL(S)
SKEWN-GARCH	1.	$(0,0.04^2,-0.7)$	0.4000	-0.0443	0.4880	-0.6287	-0.5323	13.6401
	2.	$(0,0.04^2,-0.3)$	0.4000	-0.0190	0.4501	-0.4601	-0.4180	13.6636
	3.	$(0,0.04^2,-0.1)$	0.4000	-0.0063	0.4240	-0.3123	-0.2981	13.6756
	4.	$(0,0.04^2,0.0)$	0.4000	0.0000	0.4000	0.0000	0.0000	13.6814
	5.	$(0,0.04^2,0.1)$	0.4000	0.0063	0.4240	0.3123	0.2981	13.6856
	6.	$(0,0.04^2,0.3)$	0.4000	0.0190	0.4501	0.4601	0.4180	13.6947
	7.	$(0,0.04^2,0.7)$	0.4000	0.0443	0.4880	0.6287	0.5323	13.7115
Panel B.		Daily Specifications	Centered simulated Param.		Direct simulated Param.			Simulated Call Option Prices
		$(\mu_{(1)},\sigma^2_{(1)},\gamma_{(1)})$	$[\sigma^{2,A}_{(250)}$	$\gamma^{A}_{(250)}]$	$[\omega^{2,A}_{(250)}$	$\alpha^{A}_{(250)}$	$\delta^{A}_{(250)}]$	CALL(A)
SKEWN-GARCH	1.	$(0,0.04^2,-0.7)$	0.4006	-0.1714	0.6169	-1.1107	-0.7432	13.5141
	2.	$(0,0.04^2,-0.3)$	0.4007	-0.0753	0.5254	-0.7743	-0.6122	13.6097
	3.	$(0,0.04^2,-0.1)$	0.4009	-0.0261	0.4619	-0.5162	-0.4587	13.6570
	4.	$(0,0.04^2,0.0)$	0.4002	0.0000	0.4000	0.0000	0	13.6814
	5.	$(0,0.04^2,0.1)$	0.4002	0.0269	0.4631	0.5219	0.4627	13.7001
	6.	$(0,0.04^2,0.3)$	0.4002	0.0761	0.5262	0.7776	0.6139	13.7324
	7.	$(0,0.04^2,0.7)$	0.4002	0.1726	0.6179	1.1145	0.7443	13.7955

The European call option prices shown in the table represent the call prices of the benchmark case with stock price $S = 100$, exercise price $K = 100$, annual risk free rate $r = 0.1$ and time to maturity $\tau = 0.25$ assuming that the underlying daily returns follow the SKKEW-Normal-GARCH(1,1) process with daily parameters specified under "Daily Specifications". In Panel A, the European call option prices are computed by the "actual" annual volatility and skewness estimators obtained by simulation, $\omega^{2,A}_{(250)}$ and $\delta^{A}_{(250)} = \alpha^{A}_{(250)}(1+\alpha^{2,A}_{(250)})^{-1/2}$. In Panel B, the call prices are computed by the simulated values $\omega^{2,A}_{(250)}$ and $\delta^{A}_{(250)}$.

the tables, the call option prices CALL(S) and CALL(A) are monotonically increasing in $\alpha^S_{(250)}$ and $\alpha^A_{(250)}$. By plotting CALL(S) and CALL(A) against their corresponding annual skewness parameters $\alpha^S_{(250)}$ and $\alpha^A_{(250)}$ in Figure 6, it can be easily seen that the scaling rule overestimates (underestimate) the skewness parameters as well as the call option prices when returns are negatively (positively) skewed.

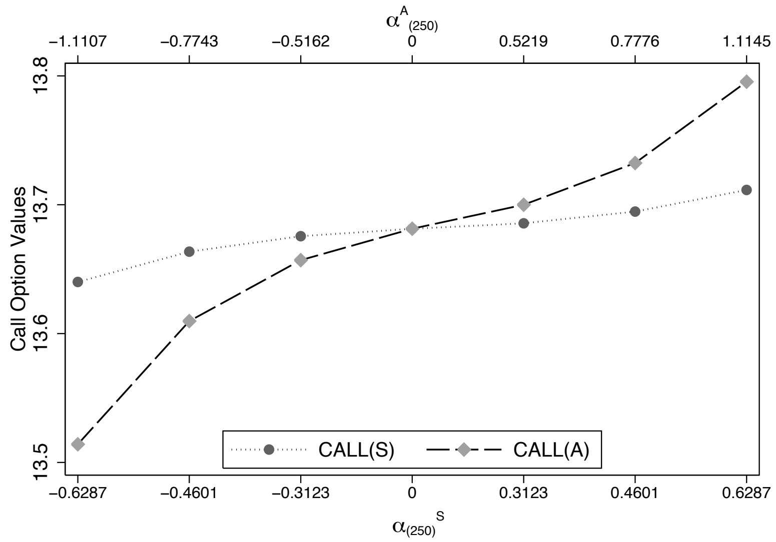


Figure 6: The relationship between Skew-Normal-Black-Scholes call option prices and the skewness parameters

To see how implied variance correlated with skewness when the annual skewness parameters are obtained by using the scaling values, consider the actual call option prices CALL(A) reported in Table 3 are observable with annual volatility not known. We substitute the call option prices with annual scaling skewness parameters $\alpha^S_{(250)}$ into the pricing formula and numerically solve for the variance rates. The resulting variance rate is the implied variance for the Skew-Normal-Black-Scholes model. The relationship between implied variance and the skewness parameter $\alpha^S_{(250)}$ are plot in Figure 7. Implied Variances in the figure are computed by numerically solving the Skew-Normal-Black-Scholes equation for the variance rate for each call prices CALL(A) and annual scaling skewness parameters $\alpha^S_{(250)}$ reported in Table 3. It is not surprising to see that implied variance are increasing with the skewness parameter $\alpha^S_{(250)}$ since the scaling values overestimate (underestimate) skewness as well as call option prices when returns are negatively (positively) skewed, the variance rates which are positively related to call prices have to be adjusted downward (upward) to account for the pricing bias. Therefore, we can also see that implied variances are lower (higher) than the actual value(40%) represented by the horizontal line in the diagram when returns are negatively (positively) skewed.

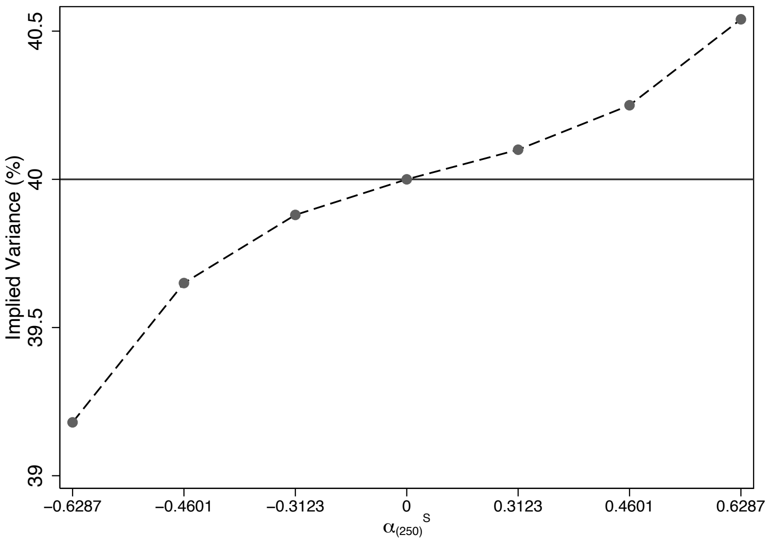
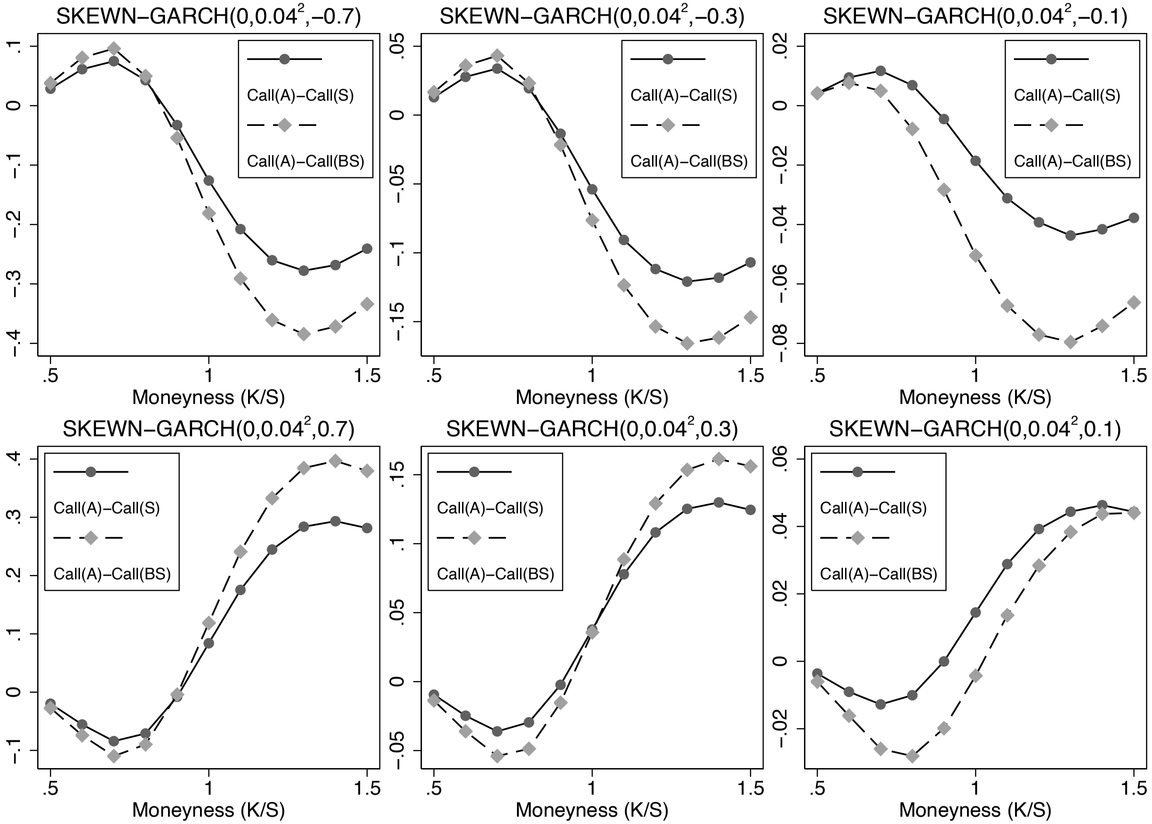


Figure 7: The relationship between the Skew-Normal-Black-Scholes's implied variance and skewness parameters

We now look at the potential pricing errors and misrepresentation of the relationship between implied variance and moneyness. Looking at Figure 8, all pictures represent call prices for the benchmark case with stock price $S = 100$ and moneyness defined as K/S. All figures show that both the the Skew-Normal-Black-Scholes call prices obtained by using the scaling annual parameters and the original Black-Scholes call prices computed by using annual variance obtained by historical variance overestimate (underestimate) in-the-money calls and underestimate (overestimate) at-the-money and out-of-the-money calls when returns are negatively(positively) skewed. This leads to what we can see in Figure 9 which shows that implied variance for the Skew-Normal-Black-Scholes model with scaling annual parameters and the original Black-Scholes model are monotonically decreasing (increasing) with moneyness when returns are negatively (positively) skewed. However, we observe the same implied variance across different moneyness if the actual annual skewness parameters obtained by simulation are used. Therefore, we can conclude that the relationship between implied variance and moneyness is misrepresented when the present of skewness is ignored or biasly estimated by the scaling rule $1/\sqrt{250}$ when returns are actually heteroskedastic.

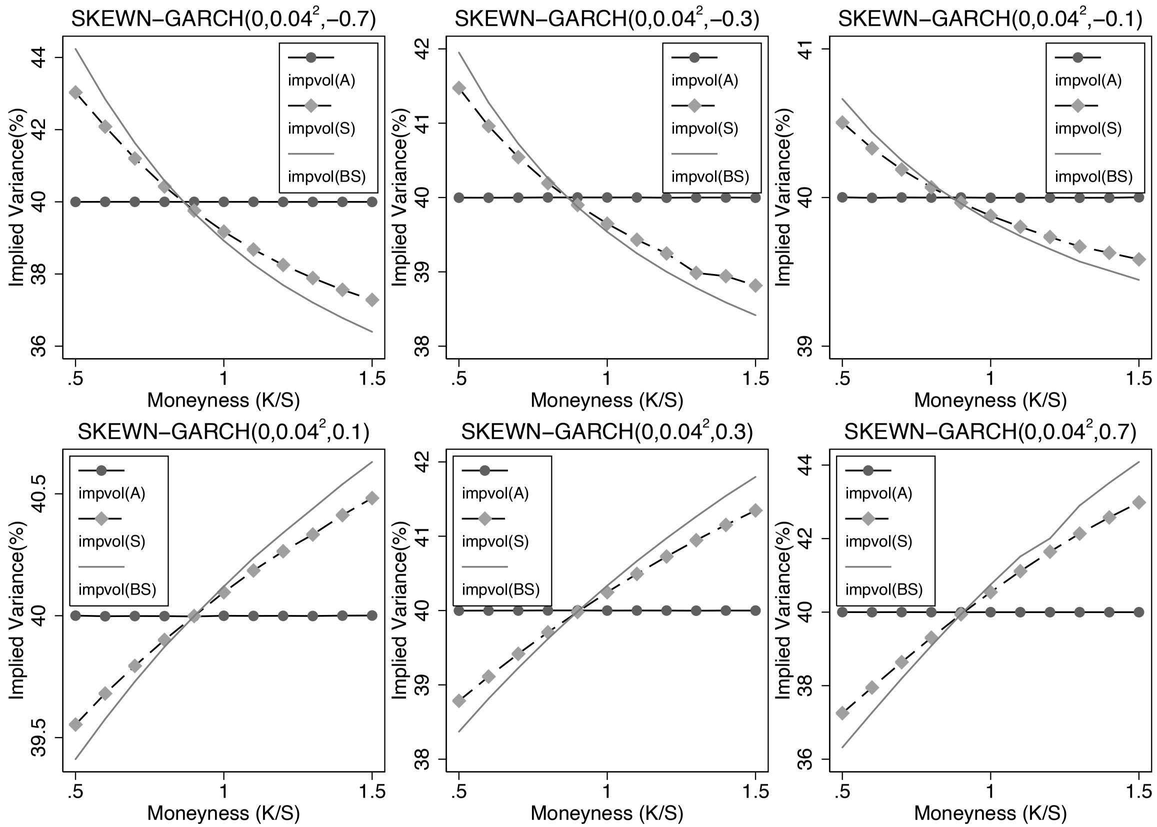
In this chapter, the pricing errors and the misrepresented relationship between implied variance and moneyness are generated purely by either ignoring the present of skewness or biasly estimated annual skewness using the scaling rule $1/\sqrt{250}$ when returns are actually heteroskedastic. However, the Skew-Normal-Black-Scholes model is computed by skew Brownian motion with constant variance. The hybrid procedure of estimating volatility and skewness from the discrete Skew-Normal-GARCH(1,1) but using the Skew-Normal-Black-Scholes model to price options have to be tested empirically.

Figure 8: The relationship between the Skew-Normal-Black-Scholes call option prices and moneyness



All pictures represent call prices for the benchmark case with stock price $S = 100$, annual risk free rate $r = 0.1$, time to maturity $\tau = 0.25$ and moneyness K/S where $CALL(T)$ represent the actual Skew-Normal-Black-Scholes call prices obtained by using the simulation parameter, $CALL(S)$ represent Skew-Normal-Black-Scholes call prices obtained by using the scaling annual parameters and $CALL(BS)$ represent the original Black-Scholes call prices computed by using historical annual variance.

Figure 9: The relationship between the Skew-Normal-Black-Scholes's implied variance and moneyness



Implied variances, impvol(A) and impvol(S), in the figures are computed by numerically solving the Skew-Normal-Black-Scholes equation for the variance rate for each call prices CALL(A) across different moneyness as shown in Figure 8 with annual scaling skewness parameters $\alpha^{A}_{(250)}$ *and* $\alpha^{S}_{(250)}$ *where implied variances, impvol(BS), is the original Black-Scholes implied volatility.*

Chapter 2

Testing the Skew Normal Black Scholes Model

2.1 Introduction

Fat-tailed and skewness of the return distributions have important implications for option pricing. Since the publication of the Black and Scholes (1973)'s option pricing theory, their model has been the cornerstone of the option pricing theory. The model assumes stock price follows geometric Brownian motion and has a closed form solution which is a function of the underlying share price of the option, the risk free rate, the exercise price, the volatility of the share and the option's time to maturity. Concerning geometric Brownian motion implies constant volatility and symmetric return distributions, the Black Scholes model has been criticized for its incapability of capturing time-varying volatility and negative skewness; the most prominent features of financial time series. To capture both time-varying volatility and skewness, in this chapter, we use the Skew-Normal-GARCH model introduced by Liseo and Loperfido (2006) to model volatility and skewness and use the Skew-Normal-Black-Scholes model developed by Corns and Satchell (2007) to predict the European call option prices in the Hang Seng Index options market in Hong Kong. Section 2 of the chapter reviews the theoretical and empirical work that motivate our study. Section 3 presents the Skew-Normal-Black-Scholes model. Section 4 review the Skew-Normal-GARCH(1,1) model which help us to estimate daily volatility and skewness. Section 5 describes the empirical data. Section 6 investigates the behavior of volatility and skewness in the data. Section 7 presents our empirical results. Section 8 concludes.

2.2 Literature Review

Numerous attempts have been made to relax the constant volatility assumption of the Black-Scholes model including the jump diffusion model discussed in Merton (1976) which assumes the dynamic of stock prices incorporates small diffusive movements with the presence of large jumps; the stochastic volatility model firstly introduced by Hull and White (1987) treating volatility as a random process; the stochastic volatility jump diffusion model of Bates (1996) which incorporate both the jump diffusion as well as the stochastic volatility processes in the option pricing models; the ARCH option pricing model of Engle and Mustafa (1992) with stock returns follow a ARCH process and the GARCH option pricing model of Duan (1995) which assumes stock returns follow a GARCH process. The list here is far from exhaustive and, theoretically, can be endless since new option pricing models can be derived once new compatible volatility processes are developed. Nevertheless, the time varying volatility adjusted option pricing models, including those not listed here, help providing extensive evidence to show that time varying volatility is capable of explaining the systematic errors between observed option prices and the Black-Scholes prices.

Pricing error depends not only on time varying volatility but depends also on skewness. The option pricing model has been extended to include skewness in the expense of assuming more complicated distribution functions. The Jarrow and Rudd (1982)'s skewness adjusted model is one of the option pricing models which have been applied in early empirical option pricing tests to incorporate the presence of skewness. The Jarrow-Rudd model different from the original Black-Scholes model by having an additive term which depends on the cumulants of the log-normal distribution and an unknown distribution. Corrado and Su (1997) empirically tested the model developed by Jarrow and Rudd (1982). They find significant negative skewness and positive excess kurtosis in the option-implied distribution of S&P 500 index prices. Moreover, they show that adding skewness- and kurtosis-adjustment terms in the Black-Scholes model yield significant improvement for pricing European options. Their findings suggest that skewness and kurtosis are factors affecting option prices. Instead of letting option prices depending on an unknown distribution, Eberlein et al. (1998) develop a closed form option pricing formula based on the hyperbolic Levy motion. Although not as plain and simple as the Black-Scholes price, the hyperbolic price can be computed by employing fast Fourier transformation and numerical integration. They find that employing the hyperbolic model help reducing volatility smile and improving the pricing accuracy. The skewness adjusted models mentioned above pay no attention to the behavior of time varying volatility. To capture both skewness and time varying volatility in financial time series, Menn and Rachev (2005) developed an option pricing model where stock returns follow a GARCH process with α-stable innovations. Their time varying volatility and skewness adjusted model reveals the unneglectable linkage between time varying volatility, skewness and option prices empirically.

Despite all these criticisms, with little doubt, the Black-Scholes model is still the standard option pricing model in the finance industry. On the theoretical side, the Black-Scholes model is constructed base on the assumption that the underlying stock price follows a geometric Brownian motion with constant variance. On the other hand, the GARCH(1,1) model assumes that the underlying stock price is a discrete process with time varying variance. Therefore, in principle, we cannot use the GARCH(1,1) variance in the Black-Scholes model. However, in practice, estimating volatility from the discrete GARCH model and suing the continuous Black-Scholes model to price options is widely used option pricing strategy among market participants and it is generally believe that the volatility adjusted Black-Scholes price is a good approximation of the actual price (Satchell and Knight, 2011). The assumptions behind the Black-Scholes model are overly simplified. Yet, if the model is based on more complicated stochastic processes or distributions, solutions of the model would have to be relied on more complicated numerical methods, algorithm development or computer simulations. This becomes one of the biggest obstacles impeding the application of a more realistic and accurate but complicated option pricing theory in everyday option trading operations.

The Skew-Normal-Black-Scholes option pricing model allows us to incorporate skewness in the original Black-Scholes formula with almost no extra cost. The model assumes that underlying stock prices follow skew Brownian motion and option pricing formula derived from the model nests the Black-Scholes model as a special case and extends the original Black-Scholes equation to allow for the present of skewness with an additional parameter. Unlike other option pricing models which involve skewness, once the Skew-Normal-Black-Scholes equation is derived, the solution can be solved by standard build in functions in most of the statistical or mathematical software nowadays within approximately one-tenth of a second. Indeed, the most complicated part of the formula is to evaluate a double integral numerically. Above all, the model has not been verified empirically. It is worth carrying out empirical tests to compare the performance of the Skew-Normal-Black-Scholes (SNBS) model and the original Black-Scholes (BS) model.

2.3 A Brief Review of The Skew-Normal-Black-Scholes Model

Under the SNBS framework of Corns and Satchell (2007), the stock price at expiry is

$$S_T = S_{t+\tau} = S_t \exp(\eta_\tau + \omega\sqrt{\tau} Z_t) \tag{2.1}$$

where

$$Z_t = \frac{1}{\sqrt{1+\alpha^2}} W_{1,t} + \frac{\alpha}{\sqrt{1+\alpha^2}} \mid W_{2,t} \mid \tag{2.2}$$

with t the current date, T the expiry date, $\tau = T - t$ the time to expiry, S_t the current stock price, S_T the stock price at expiry, $W_{1,t}$ and $W_{2,t}$ the independent standard Brownian motions. It follows that

$$\ln \frac{S_T}{S_t} = \eta\tau + \omega\sqrt{\tau} Z_t \tag{2.3}$$

is skew normally distributed with density

$$f\left(\ln \frac{S_T}{S_t}\right) = \frac{2}{\omega\sqrt{\tau}} \phi\left(\frac{ln\frac{S_T}{S_t} - \eta\tau}{\omega\sqrt{\tau}}\right) \Phi\left[\alpha\left(\frac{ln\frac{S_T}{S_t} - \eta\tau}{\omega\sqrt{\tau}}\right)\right] \tag{2.4}$$

where η is the location parameter, ω is the scale parameter, α is the shape parameter, $\phi(\cdot)$ and $\Phi(\cdot)$ are the standard normal density and distribution functions. From the moment generating function of $\ln S_T$

$$M_{\ln S_T(\beta)} = 2\exp(\ln S_t + \eta\tau + \frac{1}{2}\beta^2\omega^2\tau)\Phi(\beta\delta\omega\sqrt{\tau}) \tag{2.5}$$

we have

$$E\left(\frac{S_T}{S_t}\right) = 2\exp(\eta\tau + \frac{1}{2}\omega^2\tau)\Phi(\delta\omega\sqrt{\tau}) \ . \tag{2.6}$$

Moreover,

$$E\left(\frac{S_T}{S_t}\right) = \exp(r\tau) \tag{2.7}$$

since the average return of stocks equals to the risk-free return r. This implies that, assuming risk-neutrality, the stock price is

$$S_T = S_t \exp(r\tau - \frac{1}{2}\omega^2\tau - \ln 2\Phi(\delta\omega\sqrt{\tau})) \ . \tag{2.8}$$

Recall that a call option pays $\max(0, S_T - K)$ τ periods in the future, therefore, the current value of a call option is

$$CALL = e^{-r\tau} \int_K^\infty f(S_T \mid S_t) dS_T \ . \tag{2.9}$$

Using the properties of the log-skew-normal distribution discussed in Corns and Satchell (2007), we can evaluate the above integral. It follows that the European call option price with underlying stock price S, exercise price K, time to maturity τ and risk-free rate r is:

$$CALL = \frac{1}{2\Phi(\delta\omega\sqrt{\tau})} S_t \Psi_1(\theta) - e^{-r\tau} K \Psi_2(\theta), \tag{2.10}$$

with

$$\Psi_1(\theta) = 2 \int_{\theta}^{\infty} \int_{-\infty}^{s\alpha} \phi(s - \omega\sqrt{\tau}) \phi(u) du ds, \tag{2.11}$$

$$\Psi_2(\theta) = 2 \int_{-\infty}^{-\theta} \int_{-\infty}^{-s\alpha} \phi(s) \phi(u) du ds, \tag{2.12}$$

$$\theta = \frac{\ln(K/S) - \{[r - (\omega^2/2)]\tau - \ln 2\Phi(\delta\omega\sqrt{\tau})\}}{\omega\sqrt{\tau}} . \tag{2.13}$$

The SNBS formula and the BS differ only by the skewness parameters α which govern the degree of skewness of the underlying data. When $\alpha = 0$, the SNBS option pricing model reduces to the BS model.

2.4 Modeling Volatility and Skewness

Empirically, the two parameters ω and α in the SNBS formula are not observable and have to be estimated. Since daily returns are almost surely heteroskedastic, we obtain the parameters by estimating the centered Skew-Normal-GARCH(1,1) model with specification of returns in which

$$r_t = \mu + u_t \ , \quad u_t = \sigma_t \varepsilon_t \ , \tag{2.14}$$

where r_t is return at time t, μ is the unconditional mean of returns, u_t is the unexpected part of returns which is generally referred to as “news” in the markets and σ_t^2 is the conditional variance of a GARCH(1,1) process (Bollerslev, 1986)

$$\sigma_t^2 = a_0 + a_1 \sigma_{t-1}^2 + a_2 u_{t-1}^2 \tag{2.15}$$

with unconditional variance

$$\sigma^2 = E(\sigma_t^2) = a_0/(1 - a_1 - a_2) \ . \tag{2.16}$$

Following Arellano-Valle and Azzalini (2008), the centered skew normal innovation term ε_t with zero mean, unit variance and unconditional skewness γ is the standardized version of z_t given by

$$\varepsilon_t = \frac{z_t - \mu_z}{\sigma_z} \ , \tag{2.17}$$

where $\mu_z = b\delta$ and $\sigma_z^2 = 1 - \mu_z^2$ with $b = (2/\pi)^{1/2}$ and $\delta = \alpha(1+\alpha^2)^{-1/2}$ are the mean and variance of z_t which is a sequence of independent, identically distributed standard skew normal random variable with skew normal density function

$$f(z_t) = 2\,\phi\left(\frac{z_t - \eta}{\omega}\right)\,\Phi\left(\alpha\frac{z_t - \eta}{\omega}\right) \ . \tag{2.18}$$

Note that when $\alpha = 0$ the skew normal density function is identical to the normal density function. Having set the unconditional mean μ, variance σ^2 and skewness γ as

$$\mu = \eta + \omega\mu_z \ ,$$

$$\sigma^2 = \omega^2\left(1 - \mu_z^2\right)$$

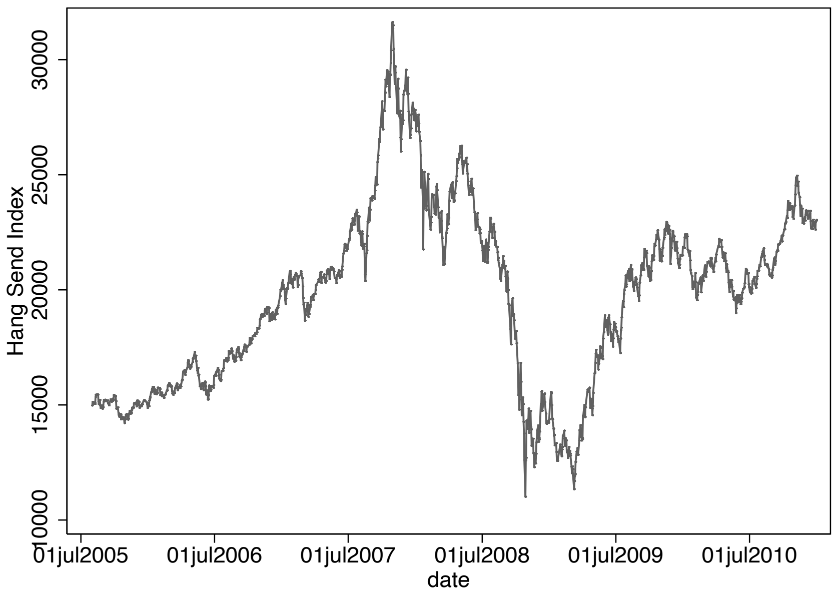
and

$$\gamma = \frac{4-\pi}{2}\frac{\mu_z^3}{(1-\mu_z^2)^{3/2}} \tag{2.19}$$

the centered parameters (μ, σ, γ) can be transformed into the direct parameters (η, ω, α) where the scale parameter ω and the shape parameter α are the two unknown parameters in the SNBS model. The two parameterization can be used interchangeably. However model parameters have to be estimated by using the centered parameterization since Azzalini (1985) and Arellano-Valle and Azzalini (2008) has shown that the maximum likelihood estimation can be problematic if the direct parameterization is used. We denote the Skew-Normal-GARCH(1,1) model as SNGARCH(1,1) model in this chapter with the emphasis of the number of lags used in modelling conditional variance. Estimation for the centered SNGARCH(1,1) model has been performed using the numerical optimization program in Matlab; the code is available from the authors on request.

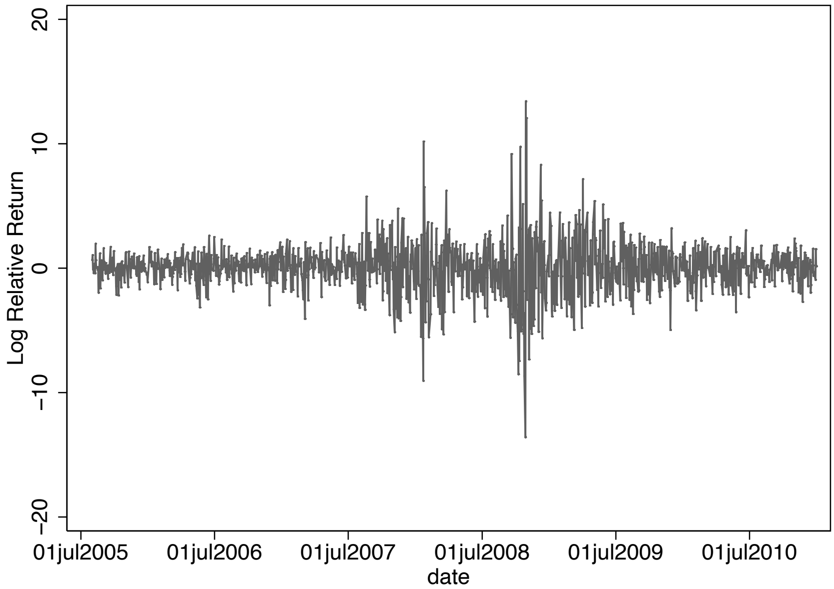
2.5 Data Description

Figure 10: Hang Seng Index



Hang Seng Index call options which are of European style are used to examine the performance of the Skew-Normal-Black-Scholes European option pricing model. Data used in this chapter are obtained from several sources. The Hang Seng Index call option prices for the sample period 1 August 2005 to 31 December 2010 are purchased from the Hong Kong Exchanges and Clearing Limited. The Hang Send Index historical prices for the sample period 17 July 2001 to 31 December 2010 are purchased from the Hang Seng Indexes Company Limited. Returns of the Hang Send Index time series are defined as $\ln(p_t) - \ln(p_{t-1})$ where p_t is the Hang Seng Index adjusted closing price at day t. The Hang Seng Index and its corresponding log relative returns series are shown in Figure 10 and Figure 11 respectively. The overnight Hong Kong Interbank Offered Rate (HIBOR) which is used as a proxy of the risk-free rate is download from the Hong Kong Monetary Authority's monthly statistical bulletin. To avoid thin trading, any option with less than 10 transactions or with less than 20 options traded are eliminated. Since by

Figure 11: Return



definition the bid price is always below the ask price, options which cannot satisfied the condition: $0 <$ bid price $\leq$ ask price, are excluded. Moreover, options prices which violate the basic no-arbitrage constraint, i.e. call prices should be greater than the intrinsic values defined as the difference between the underlying stock prices and exercise prices, are removed. Moneyness is defined as $[\ln(K/S) - r\tau]/\sqrt{\tau}$. At-the-Money corresponds to 0 moneyness, while in-the-money corresponds to negative moneyness and out-of-the-money corresponds to positive moneyness. We sort moneyness into three deciles, the in-the-money decile (ITM), the at-the-money decile (ATM) and the out-of-the-money decile (OTM). We first sort options into 2 groups, namely, positive moneyness and negative moneyness. We then assign one-third of the data closest to zero in both groups to the ITM decile and the remaining observations in the negative (positive) moneyness group to the ITM (OTM) decile. The summary statistics for moneyness and the Hang Seng Index call option data are reported in Table 4 and Table 5 respectively.

Table 4: Summary statistics for moneyness

	Moneyness Deciles			
	In-the-money (ITM)	At-the-Money (ATM)	Out-of-the-money (OTM)	Total
A. Percentile				
1th	-0.8773	0.0030	0.4666	-0.0951
5th	-2.0236	-0.2449	0.1737	-1.6385
10th	-1.5439	-0.2026	0.1876	-1.2541
50th	-1.3843	-0.1559	0.2033	-1.0398
90th	-0.8547	0.0145	0.3713	0.0418
95th	-0.3662	0.1389	0.8238	0.5289
99th	-0.3058	0.1541	1.0556	0.7239
B. Other Statistics				
Mean	-0.2632	0.1661	1.7883	1.2778
Min	-4.2777	-0.2542	0.1696	-4.2777
Max	-0.2544	0.1696	6.2028	6.2028
No. of Obs.	8661	9529	10398	28588

Moneyness is defined as $[\ln(K/S) - r\tau]/\sqrt{\tau}$. *At-the-Money corresponds to 0 moneyness, while in-the-money corresponds to negative moneyness and out-of-the-money corresponds to positive moneyness.*

Table 5: Summary statistics for the Hang Seng Index call option data

Moneyness Deciles	Settlement Price	Bid Price	Ask Price	Bid-Ask Spread	Volume	Deals
ITM	0.2819	0.2287	0.3030	0.0743	3477	834
ATM	0.3601	0.2992	0.3892	0.0900	3795	1176
OTM	0.0563	0.0461	0.0720	0.0258	2830	977
Total	0.2259	0.1858	0.2477	0.0619	10102	2987

The Settlement Price, the Bid and the Ask Prices reported in the table are the average prices for different moneyness category. The Bid-Ask Spread is calculated as the average difference between the bid and ask price. Volume is the total contract volume and Deals is the total number of transaction. All prices are displayed in thousands of Hong Kong dollars whereas volume and deals are displayed in thousands of options traded. All values are daily closing figures.

Table 6: Daily parameters estimation results for the centered SNGARCH(1,1) and the GARCH(1,1) Models using daily log relative returns of the Hand Seng Index

		Model	
	Parameter	SNGARCH(1,1)	GARCH(1,1)
Mean Eq.	μ	0.0296	0.0287
		(0.0322)	(0.0322)
Variance Eq.	con	0.0213*	0.0225*
		(0.0099)	(0.0101)
	σ^2_{t-1}	0.0652**	0.0682**
		(0.0109)	(0.0112)
	u^2_{t-1}	0.9215**	0.9181**
		(0.0125)	(0.0127)
Skewness Eq.	γ	-0.1413*	
		(0.0578)	
Log Likelihood		437.8533	435.7501
N		1367	1367

Standard errors in parentheses ($p < 0.05$, **$p < 0.01$). The parameter values reported in the table are estimated using the first 1000 observations from the sample period 17 July 2001 up to 1 August 2005.*

2.6 Volatility and skewness in Hang Seng Index

To obtain daily volatility and skewness in the returns of the Hang Seng Index time series, we first estimate the centered Skew-Normal-GARCH(1,1) model by using observations from the sample period 17 July 2001 up to 1 August 2005, then estimate the model again using observations up to 2 August 2005, and so on, finishing the estimation procedure with a total of 1376 estimated coefficient vectors by using all the observations up to 31 December 2010. Unconditional volatility and skewness on 1 August 2005 is estimated by using data up to 1 August 2005, and so on. We do the same to obtain the parameters for the GARCH(1,1) model with the shape parameter restricted to be zero. The parameter values obtained by using the first 1000 observations of our sample for both models are reported in table 6. The daily centered unconditional variance estimators for the SNGARCH(1,1) and the GARCH(1,1) models are plotted against time in Figure 12 whereas the time series of the daily centered unconditional skewness estimators for the SNGARCH(1,1) model is presented in Figure 13. Individual parameters except the unconditional mean of returns for both models are significantly different from zero at a 5% level on each day throughout the sample period for both models. As discussed in Adcock (2004), in the skew normal case, the hypothesis of $\alpha = 0$ against the alternative is the most powerful invariant test for testing normality; thus, we have evidence to reject the null hypothesis of normality in favor of the alternative of skew-normality since the null hypothesis of $\gamma = 0$, and thus $\alpha = 0$, in the centered SNGARCH(1,1) model has been consistently rejected at a 5% level on each day throughout the entire rolling sample as shown in Figure 14.

The annualized volatility for the Black-Scholes model is the daily unconditional volatility of the GARCH(1,1) scaled with $\sqrt{250}$. The square root of time rule is widely accepted by market practitioners and is asserted again strongly as it is well known that it provides good unconditional h-day volatility approximations. As can be seen in the simulation study of Diebold et al. (1997), even when returns appear to be heteroskedastic with returns

Figure 12: Daily Unconditional Variance Estimators $\hat{\sigma}^2_{(1)}$ for the Skew-Normal-GARCH(1,1) and the GARCH(1,1) models

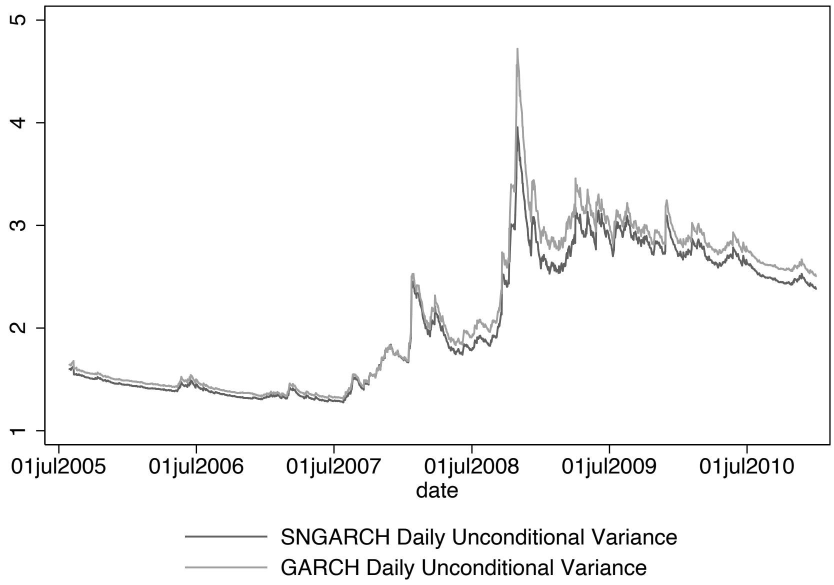


Figure 13: Daily Unconditional Skewness Estimator $\gamma_{(1)}$ for the Skew-Normal-GARCH(1,1) Model

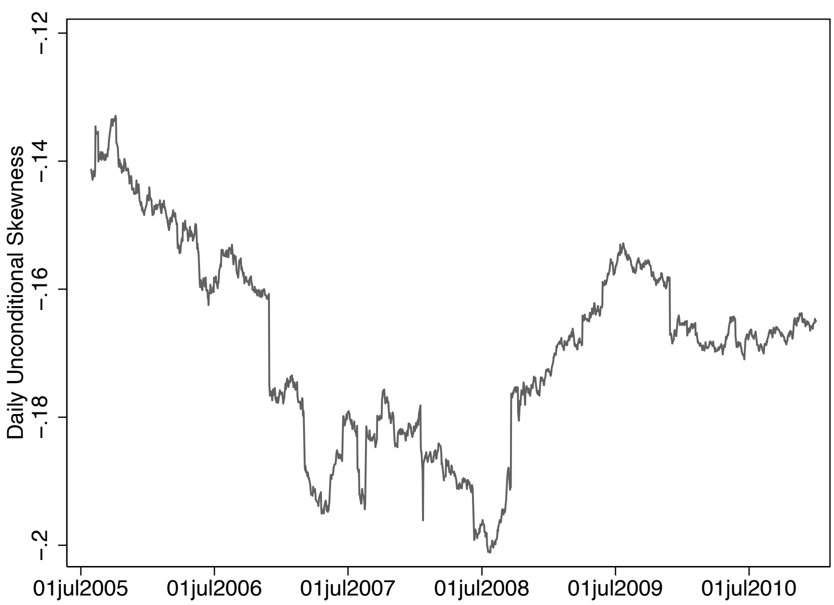
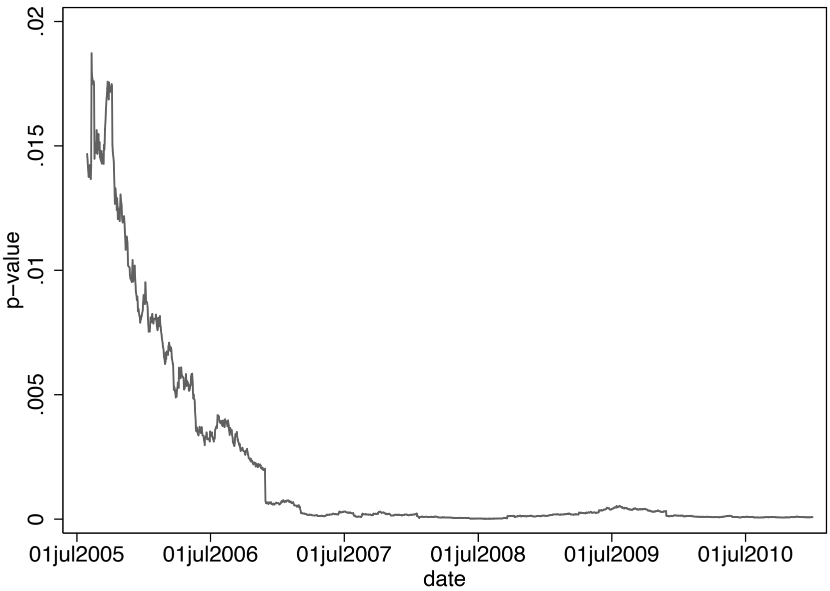
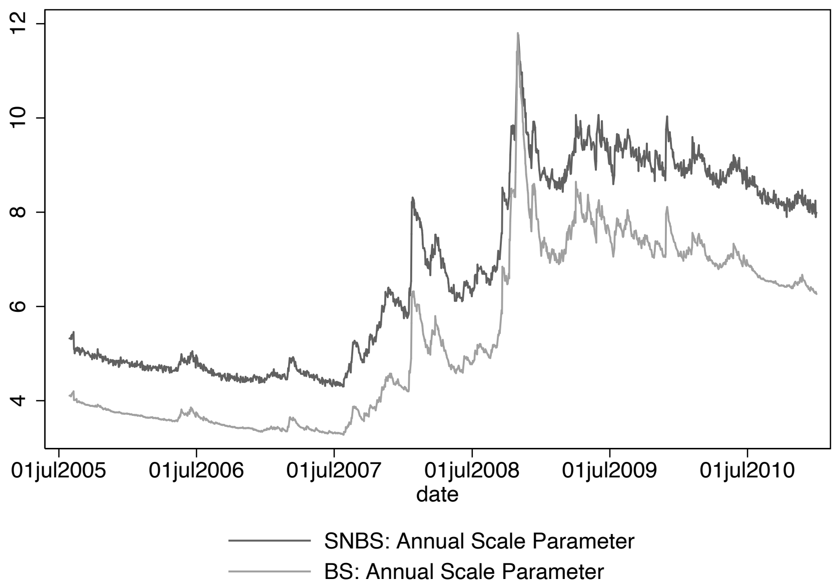


Figure 14: The p-values test for the daily unconditional skewness estimator $\gamma_{(1)}$ differences from zero for the SNGARCH(1,1) Model



follow the GARCH(1,1) process, the square root of time rule provides correct unconditional h-day volatility on average although it magnifies conditional volatility fluctuations. Drost and Nijman (1993) has also demonstrated analytically that volatility fluctuation disappears and conditional volatility converges to unconditional volatility as $h \to \infty$. Converting daily skewness to annual skewness, however, require the use of simulation since there is no analytical formula available for calculating skewness at multiple horizons under heteroskedasticity. This is a universal problem for all option pricing models which involve skewness. For converting daily skewness to annual skewness at day t, We generate $m = 1000$ time series of daily returns with daily sample size $n_{(1)} = 250000$ under the SNGARCH(1,1) model calibrated by daily parameters estimated at day t. Daily returns are then aggregate to obtain non-overlapping annual returns with sample size $n_{(250)} = 1000$. The annual unconditional parameters for each Monte Carlo sample paths, m=1,...1000, under the SNGARCH(1,1) models are estimated by the maximum likelihood method assuming that conditional returns follow the skew normal distribution function in equation 2.18. We regard the unconditional annual variance and unconditional skewness at day t for the underlying process as the average values obtained from the simulation process. These values are then transformed into annual scale parameters $\omega_{(250)}$ and annual location parameters $\alpha_{(1)}$ for the SNBS model. Note that annual scale parameters are the same as their corresponding unconditional variance values under the GARCH(1,1) model since centered parameters are the same as direct parameters in the absence of skewness. The rolling annual shape and scale parameters for the SNBS and the BS models are shown in Figure 15 and 16.

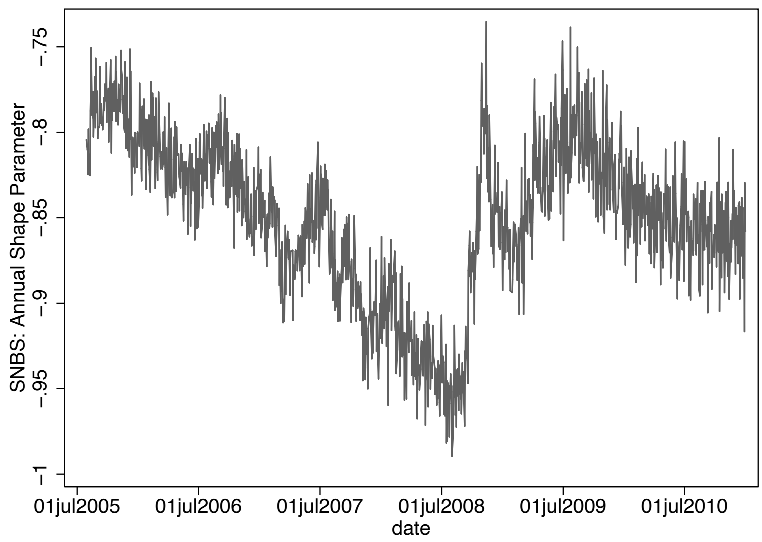
Figure 15: Annual Scale Estimators $\hat{\omega}^2_{(250)}$ for the SNBS and the BS models



2.7 The empirical performance of the SNBS and the BS models

Many criteria could be used to evaluate alternative option pricing models; four measurements are going to be used in this chapter. They are defined as follow:

Figure 16: Annual Shape Estimators $\hat{\alpha}_{(250)}$ for the SNBS models



1. The root mean squared valuation error (RMSVE) is the square root of the mean squared deviations of the observed option settlement prices from the theoretical prices. The RMSVE measures by how much the observed prices deviate from the theoretical prices.

2. The relative RMSVE (Rel.RMSVE) is the % difference between the SNBS root mean squared valuation error and the BS root mean squared valuation error. The relative RMSVE compares the performance of the SNBS and the BS option pricing models using the root mean squared valuation error. A negative figure implies the SNBS model outperforms the BS model, and vice versa.

3. The average absolute error (MAE) is the average absolute valuation error outside the bid-ask spread. The MAE measure how well the models fir within the bid and ask prices.

4. The relative MAE (Rel.MAE) is the % difference between the SNBS average absolute error and the BS average absolute error. The relative MAE compares the performance of the SNBS and the BS option pricing models using the average absolute error. A negative figure implies the

SNBS model outperforms the BS model, and vice versa.

Table 7: In sample goodness of fit statistics

	RMSVE			MAE		
Moneyness	SNBS	BS	Rel.RMSVE	SNBS	BS	Rel.MAE
ITM	0.0343	0.0337	1.7768	0.0765	0.0784	-2.4861
ATM	0.0375	0.0396	-5.3220	0.1078	0.1158	-6.8373
OTM	0.0068	0.0071	-4.7688	0.0373	0.0416	-10.3916
Total	0.0254	0.0260	-2.4783	0.0727	0.0775	-6.1969

2.7.1 In-sample Performance of the SNBS and the BS models

Table 7 reports the in-sample goodness of fit statistics described above. Both the root mean squared valuation error and the average absolute error rank the SNBS model the best model using the full sample since the RMSVE is 0.0254 for the SNBS model and is 0.0260 for the BS model, indicating that the RMSVE for the SNBS model is 2.5% less than that of the BS model; whereas the MAE is 0.0727 for the SNBS model and 0.0775 for the BS model; indicating that the MAE for the SNBS model is 6.2% less than that of the BS model. We can also see clearly from the relative root mean squared valuation error and the relative average absolute error that the SNBS model perform better when options are at-the-money or out-of-the-money. On one hand, the Rel.RMSVE for the SNBS model is 5.3% less than that of the BS model in the ATM decile and is 4.8% less than that of the BS model in the OTM decile. On the other hand, the Rel.MAE for the SNBS model is 6.8% less than that of the BS model in the ATM decile and is 10.4% less than that of the BS model in the OTM decile. However, the SNBS model showed a slight deterioration in performance in the ITM decile. Indeed there is insufficient evidence to indicate that the SNBS model prefers better than the BS model.

Yet, there is no evidence supporting the superiority of the BS model either. Although the Rel.RMSVE for the SNBS model is 1.8% larger than the BS model, the percentage difference is very small. Moreover, the Rel.MAE for the SNBS model is still 2.5% less than that of the BS model although the 2.5% difference is far less than that of the 10.4% in the OTM decile. We conclude that the SNBS model outperform the BS model since the SNBS perform better in both the ATM and the OTM deciles while the two models have similar performance in the ITM decile.

2.7.2 Out-of-the-sample Performance of the SNBS and the BS models

The question of which of SNBS and BS is a superior model is addressed in terms of out-of-sample forecasting performance. Clearly, SNBS is destined to perform better than BS in terms of in-sample forecasting performance, as a consequence of the fact that the former contains an additional parameter and is therefore more flexible. However, out-of-sample forecasting performance only improves if the additional parameter represents a genuine improvement to the model; addition of an unnecessary parameter would result in a worsening of out-of-sample forecasting performance. For this reason, we may use out-of-sample forecasting performance, without any adjustment for the number of parameters, as a valid criterion for judging which model is superior.

Table 8: Out of sample goodness of fit statistics

Panel A. One day ahead goodness of fit statistics

	RMSVE			MAE		
Moneyness	SNBS	BS	Rel.RMSVE	SNBS	BS	Rel.MAE
ITM	0.0362	0.0364	-0.7055	0.0807	0.0836	-3.5533
ATM	0.0380	0.0402	-5.4666	0.1086	0.1165	-6.8081
OTM	0.0069	0.0072	-4.5975	0.0374	0.0417	-10.2511
Total	0.0261	0.0270	-3.4391	0.0742	0.0793	-6.4268

Panel B. Five day ahead goodness of fit statistics

	RMSVE			MAE		
Moneyness	SNBS	BS	Rel.RMSVE	SNBS	BS	Rel.MAE
ITM	0.0416	0.0440	-5.4290	0.0951	0.1015	-6.2881
ATM	0.0397	0.0421	-5.6256	0.1113	0.1190	-6.4475
OTM	0.0071	0.0075	-4.6174	0.0379	0.0420	-9.8294
Total	0.0284	0.0301	-5.4473	0.0797	0.0857	-6.9931

Table 8 reports the out-of-the-sample goodness of fit statistics. Unlike the in-sample analysis, the relative root mean squared valuation error and the relative average absolute error for the SNBS models are less than those for the BS model in all moneyness deciles. However, looking at the one day ahead statistics, the Rel.RMSVE for the ITM deciles is -0.71%. This indicates that the one day ahead forecast performance of the two models are not very different for the ITM decile. Nevertheless, looking at all the other figures in the table, we can conclude that the SNBS model outperform the BS model in our out-of-the sample analysis. The Relative root mean squared valuation errors indicate that the SNBS model perform 5% better than the BS model on average excluding the ITM decile while the Relative average absolute errors indicate that the SNBS model perform at least 3.6% better than the BS model and can be up to 10.3% depending on the moneyness deciles.

2.8 Concluding Remarks

In this chapter, we tested the Skew-Normal-Black-Scholes model in the Hang Seng Index options market in Hong Kong. Our empirical evidence indicated that regardless of the performance yardsticks, taking into account of skewness improves both the in-sample and the out-of-the-sample pricing performance. The pricing procedure used in this chapter is the standard procedure used in the finance industry. We use the Skew-Normal-GARCH(1,1) model to predict daily volatility and skewness and use the SNBS model to price options. The only complication of the procedure is to simulate annual skewness. This complication can be solved by having an analytical formula for converting 1-day skewness to h-day skewness for the Skew-Normal-GARCH(1,1) model just like the Drost and Nijman (1993)'s formula which convert 1-day volatility to h-day volatility for the GARCH(1,1) model.

Chapter 3

Modeling Conditional Skewness: Heterogeneous Beliefs, Short-sale restrictions and Market Declines

3.1 Introduction

Skewness of the conditional return distribution has been widely recognized as a common phenomenon in financial markets. Some important aspects of skewness in returns have been studied. Firstly, a substantial body of literature documents that negative unconditional skewness is caused by asymmetric volatility (Nelson, 1990; Engle and Ng, 1993; Glosten et al., 1993). Secondly, many authors note that financial time series behave differently during market declines and periods of market growth. Officer (1973), Schwert (1989a,b), and Campbell and Lettau (1999) indicate that volatility is higher when price slumps than when price rises while Perez-Quiros and Timmermann (2001) identify more pronounced negative conditional skewness during late expansions and early recessions. Thirdly, starting from Hansen (1994), several empirical studies document that conditional skewness in market returns is time varying and predictable. Harvey and Siddique (1999), Jondeau and Rockinger (2003), Brooks et al. (2005), Leon et al. (2005) and Lanne and Saikkonen (2007) find significant presence of time varying conditional skewness in market returns. These studies view conditional skewness as analogous to heteroskedasticity and model conditional skewness as a function of lagged skewness.

However, more recent studies argue that time varying conditional skewness is not stable over time and thus cannot be explained by lagged skewness alone (Boyer et al., 2010). Motivated by the theory proposed by Hong and Stein (2003) which predicts negative skewness that is more pronounced when investors disagree more and short selling is restricted, a number of empirical studies address this question by analyzing the relationship between skewness, short sale constraints and heterogeneous beliefs. The validity of the theory, however, is answered with conflicting findings. Daouk and Charoenrook (2005), Chang et al. (2007) and Hueng and McDonald (2005) have found either insignificant or positive relationship between short sale restriction, heterogeneous beliefs and skewness whereas Chen et al. (2001) and Boyer et al. (2010) find a negative relationship.

The empirical analyses of skewness mentioned above pay no attention to the behavior of skewness under different market conditions. This paper argues that the relationship between short sale restrictions, heterogeneous beliefs and conditional skewness behaves differently during periods of market decline and periods of market growth. Time series models which ignore this difference are highly likely to be misspecified if sample skewness measured during the slump periods behaves differently from that in the expansion periods. Several theoretical and empirical papers motivate our empirical investigation on the effect of heterogeneous beliefs, short sale restrictions and market direction on conditional skewness. Section 2 reviews the theoretical and empirical work that motivate our study. Section 3 presents our conditionals skewness model which helps us to test our idea. Section 4 discusses our empirical results. Section 5 concludes.

3.2 Literature Review

A number of theories have been proposed to explain the existence of skewness. One of the most influential theories of skewness, firstly documented by Black (1976), attributes the negative relationship between current stock prices and future volatility to leverage. An increase in financial leverage followed by a period of price decline increases future volatility and thus introduces negative skewness while the debt level is fixed. Although the empirical effect of leverage on volatility has been proved to be statistically significant (Christie, 1982), the effect is not sufficiently large to account for all asymmetries in stock prices (Schwert, 1989b; Figlewski and Wang, 2000).

Second is the rational bubble theory (Blanchard and Watson, 1983; Diba and Grossman, 1988). A sharp fall in price followed by a period of sustained stock price increase contributes to the overall negative skewness in the market. The rational bubble theory, however, cannot help us to model skewness. Just like the bubble theory itself cannot predict when the bubble will burst, the model tells us nothing about when the distribution of returns will become more negatively skewed.

Third is the volatility feedback model (French et al., 1987; Pindyck, 1983) which assumes that both good news and bad news generate uncertainty and, hence, volatility shocks with respect to future prices. Risk averse investors will, therefore, require a higher rate of return and consequently a lower current price to compensate for a higher risk level regardless of the nature of the news. This volatility feedback effect strengthens the effect of the negative impact of bad news but moderates the effect of the positive impact of good news. As a result, on average, magnitude of the effect of negative events are larger than that of positive events, contributing to negative skewness in equity returns.

The above theories are representative-agent-based and assume rationality. The heterogeneous-agent-based theory proposed by Hong and Stein (2003), however, suggests that a mild assumption of investor irrationality together with some institutional frictions may offer us some insights into the abnormal behavior of daily skewness. They assume that at least some of the investors are overconfident and thus believe in their own private signals, which in turn generates differences of opinion. When differences of opinion are large and short selling is not allowed, the market price reflects only the valuation of the optimists since short sale constraints prevent negative information from being revealed in the market. The hidden information of those aggressive investors is more likely to flush into the market when prices fall than when prices rise. Therefore, the negative skewness that we expect as a result of a price fall is greater when short selling is not allowed.

One implication of this theory is that negative asymmetries are positively related to the degree of heterogeneous beliefs. Chen et al. (2001) develop a series of cross sectional analyses in an attempt to test this idea. In their analysis, they find that higher detrended turnover, a proxy of the degree of heterogeneous beliefs, can predict more negative skewness of daily returns measured. Thus, they find evidence to confirm the theory proposed by Hong and Stein (2003). Through the use of a similar methodology, Boyer et al. (2010) find similar results by showing that firms which have high turnover have more negatively skewed returns.

Not all evidence points to the same conclusion. There is empirical evidence against Hong and Stein (2003)'s theory. Charoenrook and Daouk (2004) find that higher detrended turnover predicts more negative unconditional skewness in countries where short selling is allowed than in countries where short selling is not allowed. Chang et al. (2007) find that skewness of unconditional returns increases when stocks are not allowed to be sold short and decreases when stocks are allowed to be sold short. Blau and Pinegar (2009) who approximate short sale constraints by using relative short interest show that there are positive relationships between turnover, relative short interest and unconditional skewness. Hueng and McDonald (2005) test the behavior of time-varying conditional skewness by assuming that conditional returns have a skewed-t distribution which allows for time varying conditional skewness and kurtosis. They show that a larger variance today is positively related to contemporaneous skewness in the market level.

None of the above empirical analyses of skewness pay attention to the possibility that the behavior of skewness may depend on market conditions.

We believe that a general market decline over time implies a stream of sequentially revealed bad news which was previously hidden during market expansion. Therefore the theory proposed by Hong and Stein (2003) predicts more pronounced negative skewness during market declines than during expansions and thus negative asymmetries are more positively related to the degree of heterogeneous beliefs and short sale restrictions during periods of general market declines.

The aim of this paper is therefore to look at the effects of short sale restrictions, heterogeneous beliefs and market direction on time varying conditional skewness. In particular, we would like to see how the relationship between heterogeneous beliefs, short sale restrictions and conditional skewness changes under different market conditions. In the next section, we present a skew normal generalized autoregressive conditional heteroskedasticity model which help us to test the idea.

3.3 Modelling Time-Varying Conditional Skewness

The Time-varying Conditional Skew-normal GARCH (TVSN-GARCH) model extends the GARCH model to allow for time varying conditional skewness by assuming that conditional returns follow a skew normal distribution. To model conditional skewness, we consider the specification of returns in which

$$r_t = \mu + u_t\,, \qquad u_t = \sigma_t \varepsilon_t$$

where r_t is daily return at day t, μ is the conditional mean, u_t is the unexpected part of returns which is generally referred to as "news" in the markets. Since arbitrage forces unexpected returns to have zero mean, we are going to assume that the innovation term ε_t is a sequence of independent, identically distributed random variates from the general skew normal distribution, $SN(\kappa, \tau^2, \alpha)$. Similarly to Liseo and Loperfido (2006), we assume that the location parameter is

$$\kappa = -\sqrt{\frac{2}{\pi}}\frac{\tau^2\alpha}{\sqrt{1+\alpha^2}}$$

and that the scale parameter is

$$\tau^2 = \left(1 - \frac{2\alpha^2}{\pi(1+\alpha^2)}\right)^{-1}$$

such that the innovation term ε_t has zero mean as required by the arbitrage free condition and the scale parameter σ_t^2 is retained to be the conditional variance of the model. We assume that conditional variance follow either the GJR-GARCH(1,1) process of Glosten et al. (1993) or the Q-GARCH(1,1) process of Sentana (1995). According to the GJR-GARCH model, the conditional variance follows an asymmetric GARCH process as follows:

$$\sigma_t^2 = a_0 + a_1\sigma_{t-1}^2 + a_2u_{t-1}^2 + a_3u_{t-1}^2 I^+$$

where $I^+ = 1$ if $u_{t-1} > 0$ and $I^+ = 0$ if $u_{t-1} \leq 0$. We expect the coefficient on the asymmetric term to be negative so that bad news has a larger impact than good news on the conditional variance of the return. In the Q-GARCH model[1], the variance equation is specified as:

$$\sigma_t^2 = a_0 + a_1\sigma_{t-1}^2 + a_2u_{t-1}^2 + a_3u_{t-1}.$$

Unlike Liseo and Loperfido (2006), we allow the shape parameter α and thus skewness in the Q-GARCH model to be time varying

$$\alpha_t = b_0 + b_1\alpha_{t-1} + b_2\varepsilon_{t-1}^2 + b_3\varepsilon_{t-1} + \mathcal{S}_t.$$

Note that when $b_3 > 0$, good news has a positive impact whereas bad news has a negative impact on the skewness of the returns. The use of ε_{t-1} instead

[1]The principal restriction of this model is that σ_t^2 should be positive although this constraint is not explicitly applied in our estimation process as the constraint is not binding in any part of the sample period.

of u_{t-1} in the skewness equation is to prevent conditional variance from having an effect on skewness. $\mathcal{S}_t$ represents all relevant skewness factors and can be defined as follows:

$$\begin{aligned}\mathcal{S}_t &= c_1 \cdot \mathrm{DTO}_t + c_2 \cdot \mathrm{DSI}_t + c_3 \cdot \mathrm{DTO}_t \cdot \Im + c_4 \cdot \mathrm{DSI}_t \cdot \Im \\ &\quad + c_5 \cdot \mathrm{DTO}_t \cdot \mathrm{DSI}_t + c_6 \cdot \mathrm{DTO}_t \cdot \mathrm{DSI}_t \cdot \Im + c_7 \cdot \Im \,.\end{aligned}$$

The model has three main skewness factors. DTO_t is detrended turnover, which is a proxy for heterogeneous beliefs; DSI_t is detrended short interest, which is a proxy of short sale restrictions. (On these uses of proxies, see Miller (1977), Epps and Epps (1976), Figlewski (1981), Jones and Lamont (2002), and others.) The third skewness factor is the market direction indicator, $\Im$, which is equal to one during periods of general market declines and zero otherwise. We expect the signs of the coefficients on "$\mathrm{DTO}_t \cdot \Im$" and "$\mathrm{DSI}_t \cdot \Im$" to be negative since the Hong and Stein (2003) theory predicts that negative asymmetries are more positively related to the degree of heterogeneous beliefs and short sale restriction during periods of general market declines. The other variables are there to control for any interaction effects between the three variables. We refer to the model which uses the GJR-GARCH conditional variance as TVSN-GJR-GARCH model and refer the model which uses the Q-GARCH conditional variance as TVSN-Q-GARCH model.

Estimation has been performed using the "optimize" routine in Mata; the code is available from the authors on request.

3.4 Empirical Tests

In this paper, we carry out tests of the effects of short sale restrictions, heterogeneous beliefs, market direction on conditional skewness by exploiting the unique short-sale restrictions present in the Hong Kong stock market. We start off with the general background of the Hong Kong borrowing market followed by a formal description of the data and then we present our empirical results.

3.4.1 Short Sales on the Hong Kong Stock Market

In Hong Kong, short selling was prohibited before January 1994. After that, 17 stocks under the pilot program listed on the "Designated Securities Eligible for Short Selling List" could be sold short. In March 1996, the list was expanded. Since than, the stocks that constitute the list are revised quarterly and all components of the Hang Seng Index are allowed to be sold short. In practice, short selling is done through the "Automatic Order Matching and Execution System" where brokers can identify potential lenders and short sellers, place trading requests and make short selling transactions. Short selling data are recorded on a daily basis and daily data from 1999 onward are available to the public on the Hong Kong Stock Exchange web site under the "Statistics and Research" section with usually one day delay. Full details of short selling regulations in Hong Kong can be found in the "Regulated Short Selling" page on the Hong Kong Stock Exchange web site.

Compared to other markets, the borrowing market in Hong Kong is more transparent, better regulated, and has a more complete and accessible database which covers the 12-year period of 1999-2012. Hence the Hong Kong stock market provides us with a unique opportunity to test the effect of short sale restrictions, heterogeneous beliefs and market direction on time varying conditional skewness.

3.4.2 Data

The stocks analyzed in this paper consist of all the components of the Hang Seng index (HSI). Our data, including short interest, trading volume and total shares outstanding of individual stocks which constitute the Hang Seng index over the period 4th Jan, 1999 to 31st May, 2011 were purchased from the Hong Kong Stock Exchange web site (www.hkex.com.hk). The daily return series for individual stocks are calculated as $r_{i,t} = ln(p_{i,t}) - ln(p_{i,t-1})$, where daily closing prices for individual stocks, $p_{i,t}$, are obtained from the Reuters EcoWin Pro database. Figure 17 plots the market returns of the HSI against time. Market return, turnover and short interest are capitalization-weighted and consist of all HSI components. Historical changes to the list of

HSI constituents can be downloaded from the HSI website (www.hsi.com.hk). The turnover and short interest series are measured in number of shares traded per day. Following the methodology used by Chen et al. (2001), the

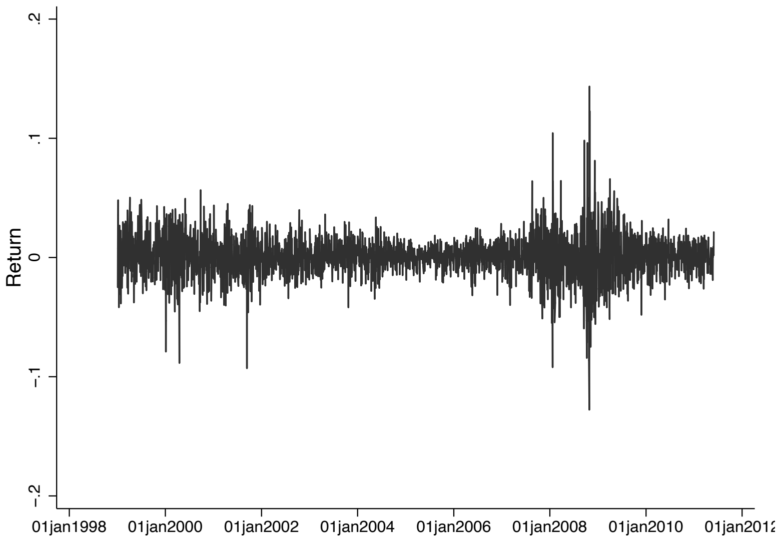


Figure 17: Return

normal level of heterogeneous beliefs and short sale restrictions in this paper are approximated by a centered moving average of market turnover and short interest over a 120-day window where the centered moving average with i-day window is defined as $\bar{x}_{t,i} = \frac{1}{2i+1}\sum_{j=-i}^{i} x_{t+j}$. Both the turnover and short interest series are detrended by first taking its natural log and then subtracting the moving average trends from the logged series. The degree of heterogeneous beliefs and short sale restrictions are the highest (lowest) when the level of turnover or short interest has the highest positive (negative) deviation from the normal levels. We use a centered instead of a backward moving average since we are interested in detrending, not forecasting. We also use 5, 20, 250 and 750-day windows to detrend the series since there is no solid rule to determine the size of the window for a moving average filter. We refer to the series thus obtained as (respectively) very short-term, short-term, long-term, and very long-term detrended turnover and detrended short interest. We label them as DTO_i and DSI_i, with i equal to 5, 20, 250 or 750. When we specify no particular value for i, we refer DTO and DSI as medium term detrended turnover and short interest which are detrended by a moving average trend indicator with a 120-day window. Figure 18 and 19 plot the medium term detrended turnover and detrended short interest series against time.

Figure 18: Detrended Turnover

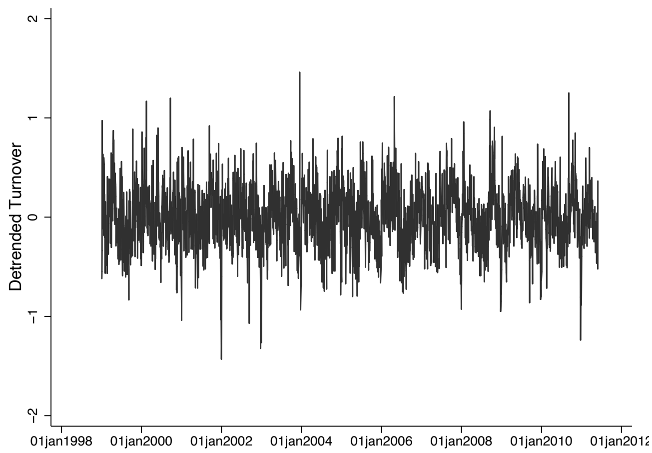
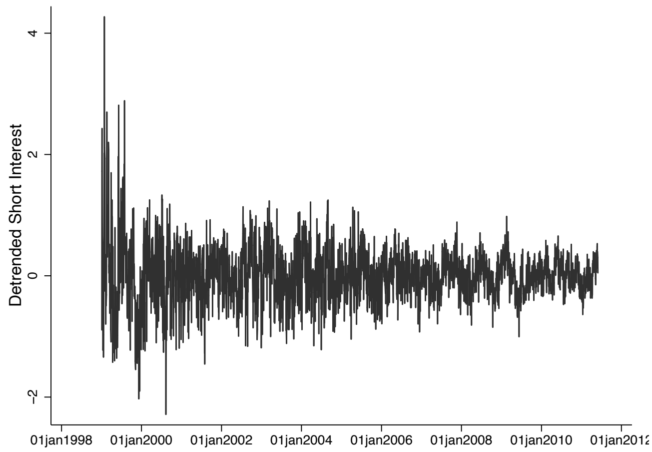


Figure 19: Detrended Short Interest



We define four different market direction indicators, namely, the crisis indicator $\Im_{crisis}$, the yearly market direction indicator $\Im_{year}$, the quarterly market direction indicator $\Im_{qtr}$ and the weekly market direction indicator $\Im_{week}$. We now look at the crisis indicator. There were two major financial crises in Hong Kong over the period 1999-2012. Firstly, shortly after recovery from the financial crisis of 1997-1998, Hong Kong's economy was hit by the global economic downturn in 2001 followed by the outbreak of Severe Acute Respiratory Syndrome (SARS) in 2003. Second is the global financial crisis that happened in 2008. We define the first crisis period to be 28th March, 2000 to 25th April, 2003 and the second crisis period to be 30th October, 2007 to 9th March, 2009. Figure 20 shows the HSI series along with the start and end dates of the crises. Since the purpose of the crisis indicator is to indicate general market declines, we pick up the starting and ending dates by using ex-post data. Since the crisis indicator is somewhat arbitrary, we also test other market direction indicators. The yearly market direction indicator $\Im_{year,t}$ is equal to 1 at day t for the year $y-1$ if yearly price difference calculated as $P_y - P_{y-1}$ is negative, where the variable y records the number of years from the start of 4th Jan, 1999 to 31st May, 2011 and P_y is the last observation of the daily price series at year y. The quarterly market direction indicator $\Im_{qtr,t}$ is equal to 1 at day t for the quarter $q-1$ if quarterly price difference calculated as $P_q - P_{q-1}$ is negative, where the variable q records the number of quarters from the start of 4th Jan, 1999 to 31st May, 2011 and P_q is the last observation of the daily price series at quarter q. The weekly market direction indicator $\Im_{week,t}$ is equal to 1 at day t for week w if weekly price difference calculated as $P_w - P_{w-1}$ is negative, where the variable w records the number of weeks from the start of 4th Jan, 1999 to 31st May, 2011 and P_w is the last observation of the daily price series at week w. Figure 21, 22 and 23 show the yearly, quarterly and weekly price difference indicators.

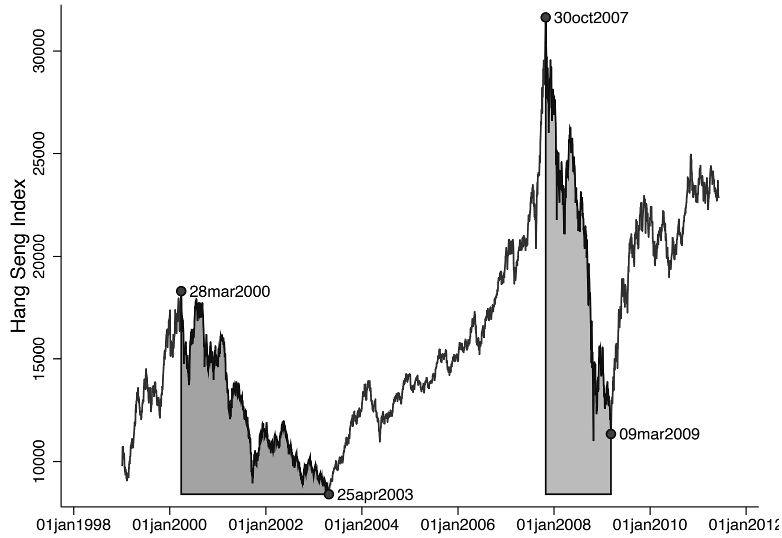


Figure 20: Hang Seng Index With Starting and Ending Dates For The Crises

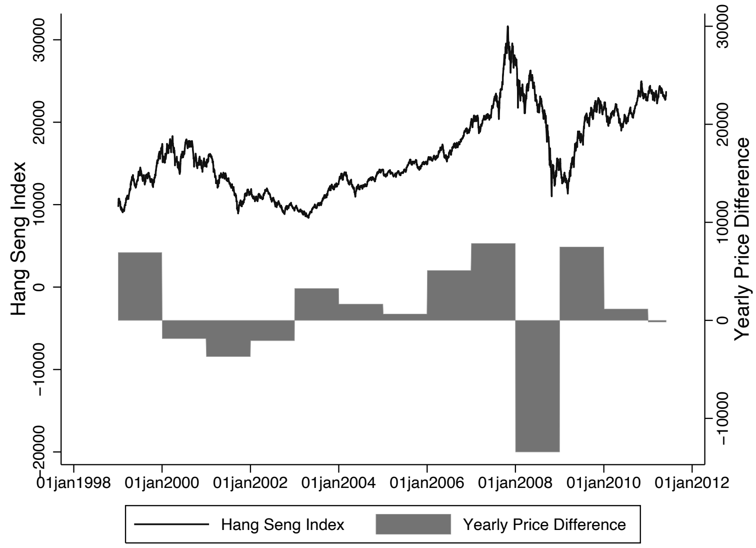


Figure 21: Yearly Price Difference

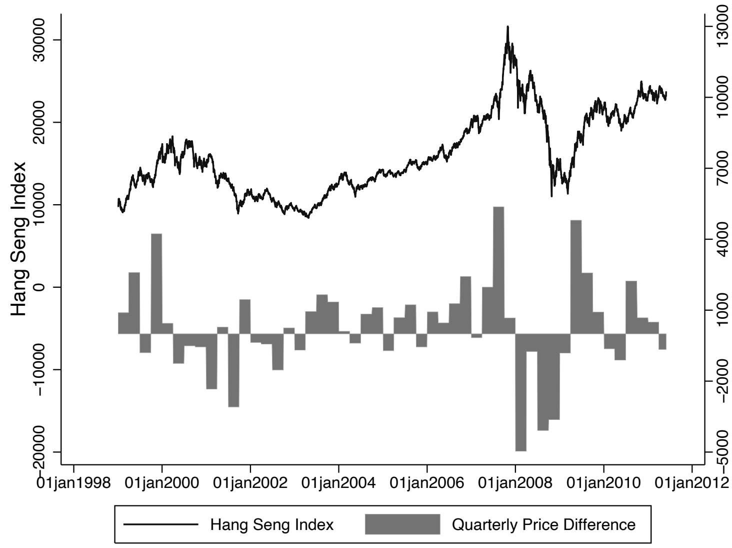


Figure 22: Quarterly Price Difference

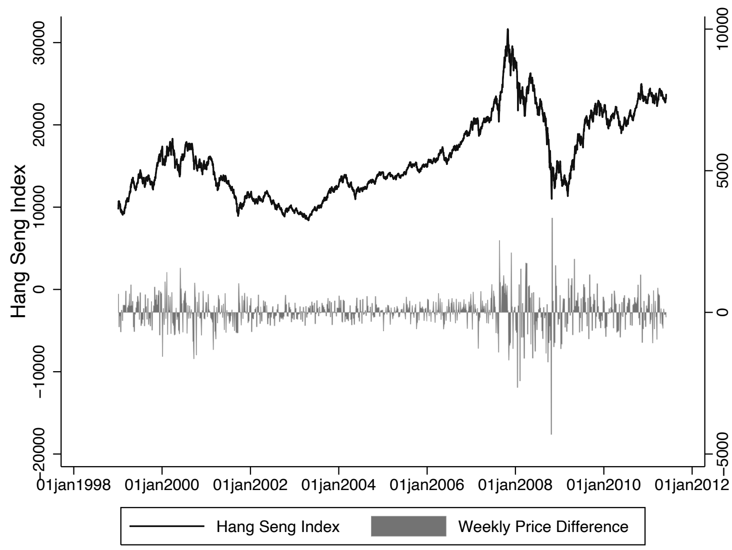


Figure 23: Weekly Price Difference

3.4.3 Results

Table 9 shows the estimation results and various specification tests for the TVSN-GJR-GARCH and TVSN-Q-GARCH models presented in the pre-

Table 9: TVSN-GJR-GARCH and TVSN-Q-GARCH Estimation Results

	TVSN-GJR-GARCH			TVSN-Q-GARCH		
	(1)	(2)	(3)	(4)	(5)	(6)
Variance Equation						
h_{t-1}	0.924716***	0.925830***	0.919784***	0.921695***	0.922646***	0.930184***
	(0.0075)	(0.0072)	(0.0076)	(0.0075)	(0.0071)	(0.0068)
u^2_{t-1}	0.093426***	0.093019***	0.095635***	0.067327***	0.066482***	0.062880***
	(0.0096)	(0.0091)	(0.0088)	(0.0069)	(0.0065)	(0.0066)
$u^2_{t-1}I^+$	-0.054642***	-0.054685***	-0.050366***			
	(0.0104)	(0.0098)	(0.0099)			
u_{t-1}				-0.000625***	-0.000644***	-0.000656***
				(0.0001)	(0.0001)	(0.0001)
_cons	0.000002***	0.000002***	0.000001***	0.000002***	0.000002***	0.000002***
	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
Skewness Equation						
s_{t-1}	-0.399160***	-0.400692***	-0.460596***	-0.430430***	-0.448076***	-0.328663***
	(0.1103)	(0.1213)	(0.1494)	(0.1052)	(0.1117)	(0.1098)
u^2_{t-1}	0.134970**	0.131701**	0.043110	0.149620***	0.130374***	0.106321
	(0.0580)	(0.0537)	(0.0504)	(0.0551)	(0.0492)	(0.0719)
u_{t-1}	0.458544***	0.410136***	0.208406**	0.440773***	0.396245***	0.440645***
	(0.1297)	(0.1194)	(0.0945)	(0.1250)	(0.1121)	(0.1021)
_cons	0.323589	0.837111***	-2.102074***	0.318898	0.955872***	1.345092***
	(0.4852)	(0.2949)	(0.3170)	(0.4731)	(0.2889)	(0.2968)
Skewness Factors						
DTO	0.748852		1.457434***	0.758511		-2.606873***
	(0.5700)		(0.3346)	(0.5649)		(0.3130)
DSI	-0.165695		-0.205270	0.390758		-0.705571***
	(0.3945)		(0.1252)	(0.4208)		(0.1832)
DTO$\times\Im_{Crisis}$	-1.742325**	-1.235309**		-1.936706***	-1.543076***	
	(0.7447)	(0.5073)		(0.7376)	(0.5372)	
DSI$\times\Im_{Crisis}$	-1.815924***	-1.672560***		-2.460995***	-1.705960***	
	(0.5183)	(0.3253)		(0.5395)	(0.3134)	
DTO×DSI	-0.098086			-0.792320		
	(0.7768)			(0.8731)		
DTO×DSI	1.069672			1.939779		
$\times\Im_{Crisis}$	(1.2503)			(1.2918)		
$\Im_{Crisis}$	-1.278767**	-1.657665***		-1.161699**	-1.563355***	
	(0.5584)	(0.4192)		(0.5314)	(0.4077)	
Mean Equation						
_cons	0.000447	0.000196	0.002260***	0.000358	0.000009	-0.000626**
	(0.0003)	(0.0003)	(0.0003)	(0.0003)	(0.0004)	(0.0003)
N	3061	3061	3061	3061	3061	3061
LL	8844.3516	8842.1595	8825.3523	8844.3096	8842.3898	8828.8507

Standard errors in parentheses

* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

vious section. In the variance equations of both models, all the signs are consistent with our expectation. Coefficients on the asymmetric terms in the variance equations are negative and strongly signification at the 1% level for various specifications, implying that negative shocks tend to cause higher volatility than positive shocks. This result is consistent with Nelson (1990), Engle and Ng (1993), Glosten et al. (1993) and other empirical asymmetric GARCH studies. In the skewness equations of both models, coefficients are generally significant at the 5% level indicating that conditional skewness is time varying which is consistent with Harvey and Siddique (1999), Jondeau and Rockinger (2003), Brooks et al. (2005), Leon et al. (2005) and Lanne and Saikkonen (2007). The coefficients on u_{t-1} in the skewness equations of both models are positive, implying that good news has a positive impact while bad news has a negative impact on the skewness of the returns. This shows that our asymmetric terms in the variance equations and skewness equations are consistent with each other. The asymmetric terms in the variance equations tell us that the distribution of return is negatively skewed for the estimation period whereas the asymmetric terms in the skewness equations tell us that today's return is more negatively skewed when the market received bad news yesterday.

We now turn to the skewness factors equation. The coefficients on $\mathrm{DTO}_t \cdot \Im_{Crisis}$ and $\mathrm{DSI}_t \cdot \Im_{Crisis}$ are negative and strongly significant for various specifications in both models, indicating that both turnover and market short interest have statistically significant power in predicting conditional skewness during the two crisis periods. Specifically, during the two crisis periods, negative conditional skewness is more pronounced when people disagree about the market more or the short sale constraints are more and more binding. However, heterogeneous beliefs and short sale restrictions do not seem important to the determination of market conditional skewness during market growth since the coefficients on DTO and DSI in our baseline specifications, specification (1) for the TVSN-GJR-GARCH model and specification (4) for the TVSN-Q-GARCH model, are not statistically significant. However, this does not necessarily imply a falsification of the Hong and Stein (2003)'s theory since the theory predicts more pronounced negative skewness when the short sale constraint is binding and the differences in opinions are high. Our results might indicate that the short sale constraint is binding and the differences in opinions are high enough during the two crisis periods but not during the periods of market growth. Note that in specification (3) and (6), when we omit the crisis indicator and its related terms, heterogeneous beliefs and short sale restrictions have either positive, negative or no impacts on conditional skewness. It may shed light on why the effect of heterogeneous beliefs and short sale restrictions on skewness has been answered with conflicting findings.

Second, in table 10 and 11, we estimate the model with different terms of detrended turnover and detrended short interest. In our model, detrended turnover and detrended short interest are proxies for the degree of heterogeneous beliefs and short sale constraints. We use a centered moving average over a 120-day window to detrend both series. However, the size of the moving average window is subject to debate. Therefore, we also use 5, 20, 250 and 750-day moving average windows to detrend the series. Our results are consistent with various terms of detrended turnover and short interest. Coefficients on "$\mathrm{DTO}_i \cdot \Im_{Crisis}$" and "$\mathrm{DSI}_i \cdot \Im_{Crisis}$" are negative and significant except for i equal to 20 in both models. Therefore, we have further evidence that market turnover and short interst have negative effect on conditional skewness during the two crisis periods. Coefficients on "DTO_i" and "DSI_i" are either positive or insignificant except for i equal to 20 in the TVSN-Q-GARCH model. This finding is consistent with Hueng and McDonald (2005) who have found either insignificant or positive relationship between short sale restriction, heterogeneous beliefs and skewness. Our results, instead of rejecting the Hong and Stein (2003)'s model or any previous empirical tests, suggest that the relationship between heterogeneous beliefs, short sale restrictions and skewness behave differently under different market states.

Third, in table 12, we test the TVSN-GJR-GARCH and TVSN-Q-GARCH models with different market direction indicators. As mentioned in the previous section, the crisis indicator is somewhat arbitrary, as a robustness check, we test also the yearly, quarterly and weekly market direction indicators. As can be seen form figure 20 and 21, the yearly market indicator spots

Table 10: TVSN-GJR-GARCH Estimation Results with Different Terms of Detrended Turnover and Detrended Short Interest

	Terms of Detrended Turnover and Detrended Short Interest				
	Very Short (i=5)	Short (i=20)	Medium (i=120)	Long (i=250)	Very Long (i=750)
Variance Equation					
_cons	0.000002***	0.000002***	0.000002***	0.000002***	0.000001***
	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
h_{t-1}	0.924641***	0.928194***	0.924716***	0.925026***	0.923910***
	(0.0075)	(0.0073)	(0.0075)	(0.0074)	(0.0076)
u_{t-1}^2	0.095164***	0.089566***	0.093426***	0.092972***	0.087010***
	(0.0098)	(0.0090)	(0.0096)	(0.0094)	(0.0085)
u_{t-1}	-0.056851***	-0.051924***	-0.054642***	-0.054171***	-0.044262***
	(0.0103)	(0.0099)	(0.0104)	(0.0103)	(0.0105)
Skewness Equation					
_cons	0.461767	0.578186	0.323589	0.346870	-1.987924***
	(0.3837)	(0.3692)	(0.4852)	(0.4839)	(0.3161)
s_{t-1}	-0.368912***	-0.383245***	-0.399160***	-0.429248***	-0.460735***
	(0.0896)	(0.0897)	(0.1103)	(0.1072)	(0.1211)
u_{t-1}^2	0.165522**	0.191689***	0.134970**	0.142773***	0.084871
	(0.0687)	(0.0694)	(0.0580)	(0.0544)	(0.0588)
u_{t-1}	0.480123***	0.460506***	0.458544***	0.437505***	0.269005**
	(0.1304)	(0.1151)	(0.1297)	(0.1263)	(0.1108)
Skewness Factors					
DTO_i	0.351370	-0.802228	0.748852	0.727349	1.418962***
	(0.5701)	(0.5067)	(0.5700)	(0.5249)	(0.3430)
DSI_i	1.089533**	1.345161***	-0.165695	-0.139840	0.220347
	(0.5274)	(0.4423)	(0.3945)	(0.3210)	(0.1672)
$\text{DTO}_i \times \Im_{Crisis}$	-2.057438**	-0.081393	-1.742325**	-1.897282***	-0.972849*
	(0.8230)	(0.7270)	(0.7447)	(0.7185)	(0.5403)
$\text{DSI}_i \times \Im_{Crisis}$	-3.712871***	-3.682146***	-1.815924***	-1.938751***	-1.979838***
	(0.7906)	(0.5703)	(0.5183)	(0.4471)	(0.5155)
$\text{DTO}_i \times \text{DSI}_i$	-3.370001**	-4.542003***	-0.098086	0.011697	0.056073
	(1.5268)	(1.1738)	(0.7768)	(0.5608)	(0.3130)
$\text{DTO}_i \times \text{DSI}_i \times \Im_{Crisis}$	6.840392***	6.711280***	1.069672	1.176087	-0.200208
	(2.2309)	(1.6373)	(1.2503)	(1.1277)	(0.8631)
$\Im_{Crisis}$	-1.058154**	-1.628250***	-1.278767**	-1.160213**	-0.159771
	(0.4896)	(0.4605)	(0.5584)	(0.5491)	(0.4022)
Mean Equation					
_cons	0.000289	0.000376	0.000447	0.000416	0.002351***
	(0.0003)	(0.0003)	(0.0003)	(0.0003)	(0.0003)
N	3061	3061	3061	3061	3061
LL	8847.1701	8846.8856	8844.3516	8844.2970	8845.0959

Standard errors in parentheses
* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Table 11: TVSN-Q-GARCH Estimation Results with Different Terms of Detrended Turnover and Detrended Short Interest

	Terms of Detrended Turnover and Detrended Short Interest				
	Very Short (i=5)	Short (i=20)	Medium (i=120)	Long (i=250)	Very Long (i=750)
Variance Equation					
_cons	0.000002***	0.000002***	0.000002***	0.000002***	0.000002***
	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
h_{t-1}	0.920874***	0.926413***	0.921695***	0.921641***	0.919847***
	(0.0076)	(0.0072)	(0.0075)	(0.0074)	(0.0077)
u_{t-1}^2	0.068125***	0.064648***	0.067327***	0.067391***	0.068674***
	(0.0070)	(0.0067)	(0.0069)	(0.0069)	(0.0070)
u_{t-1}	-0.000662***	-0.000624***	-0.000625***	-0.000623***	-0.000622***
	(0.0001)	(0.0001)	(0.0001)	(0.0001)	(0.0001)
Skewness Equation					
_cons	0.481314	1.543825***	0.318898	0.333259	0.218796
	(0.3829)	(0.3105)	(0.4731)	(0.4714)	(0.4215)
s_{t-1}	-0.371087***	-0.363995***	-0.430430***	-0.435419***	-0.380221***
	(0.0892)	(0.0941)	(0.1052)	(0.1044)	(0.1222)
u_{t-1}^2	0.165315**	0.187735***	0.149620***	0.157144***	0.176111***
	(0.0671)	(0.0690)	(0.0551)	(0.0522)	(0.0528)
u_{t-1}	0.476801***	0.370677***	0.440773***	0.428017***	0.414066***
	(0.1289)	(0.1056)	(0.1250)	(0.1231)	(0.1210)
Skewness Factors					
DTO_i	0.419213	-2.671636***	0.758511	0.700311	0.818530*
	(0.5719)	(0.4582)	(0.5649)	(0.5308)	(0.4726)
DSI_i	1.068138**	0.020256	0.390758	0.414690	0.661859
	(0.5419)	(0.4208)	(0.4208)	(0.4183)	(0.4736)
$\text{DTO}_i \times \Im_{Crisis}$	-2.194376***	1.316030*	-1.936706***	-1.953506***	-1.488445**
	(0.8174)	(0.6963)	(0.7376)	(0.7154)	(0.6133)
$\text{DSI}_i \times \Im_{Crisis}$	-3.615895***	-2.313101***	-2.460995***	-2.511521***	-2.155910***
	(0.7922)	(0.5357)	(0.5395)	(0.5312)	(0.6408)
$\text{DTO}_i \times \text{DSI}_i$	-3.191515**	-1.795606	-0.792320	-0.607398	-1.001791
	(1.5820)	(1.1174)	(0.8731)	(0.7100)	(0.7434)
$\text{DTO}_i \times \text{DSI}_i \times \Im_{Crisis}$	6.357902***	3.646588**	1.939779	1.801792	0.578859
	(2.1972)	(1.5243)	(1.2918)	(1.1919)	(1.1992)
$\Im_{Crisis}$	-1.028812**	-2.198506***	-1.161699**	-1.098822**	-1.183944**
	(0.4850)	(0.4015)	(0.5314)	(0.5273)	(0.4995)
Mean Equation					
_cons	0.000202	-0.000304	0.000358	0.000351	0.000498
	(0.0003)	(0.0003)	(0.0003)	(0.0003)	(0.0003)
N	3061	3061	3061	3061	3061
LL	8847.0001	8850.6517	8844.3096	8844.2785	8845.5999

Standard errors in parentheses

* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Table 12: Estimation Results with Different Market Direction Indicators

	TVSN-GJR-GARCH			TVSN-Q-GARCH		
Indicator ($\Im$) =	$\Im_{Year}$	$\Im_{Qtr}$	$\Im_{Week}$	$\Im_{Year}$	$\Im_{Qtr}$	$\Im_{week}$
Variance Equation						
h_{t-1}	0.921061***	0.920933***	0.918887***	0.924689***	0.924108***	0.921815***
	(0.0075)	(0.0074)	(0.0082)	(0.0076)	(0.0074)	(0.0082)
u_{t-1}^2	0.068102***	0.069428***	0.066425***	0.094890***	0.093352***	0.095022***
	(0.0070)	(0.0070)	(0.0070)	(0.0097)	(0.0094)	(0.0098)
$u_{t-1}^2 I^+$	-0.000647***	-0.000588***	-0.000577***			
	(0.0001)	(0.0001)	(0.0001)			
u_{t-1}				-0.057155***	-0.050765***	-0.058381***
				(0.0106)	(0.0108)	(0.0103)
_cons	0.000002***	0.000002***	0.000002***	0.000002***	0.000002***	0.000001***
	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
Skewness Equation						
s_{t-1}	-0.395437***	-0.335266***	0.180489***	-0.393245***	-0.318248***	0.188818***
	(0.1067)	(0.0905)	(0.0549)	(0.1094)	(0.0938)	(0.0535)
u_{t-1}^2	0.190641***	0.171903***	0.086169	0.185546***	0.167586***	0.086943
	(0.0651)	(0.0566)	(0.0579)	(0.0689)	(0.0578)	(0.0587)
u_{t-1}	0.449871***	0.504000***	-0.113099	0.435419***	0.497151***	-0.142707
	(0.1296)	(0.1214)	(0.1054)	(0.1279)	(0.1252)	(0.1019)
_cons	0.201394	0.398088	1.630147***	0.182494	0.391532	1.634555***
	(0.4893)	(0.4326)	(0.1748)	(0.4891)	(0.4424)	(0.1724)
Skewness Factors						
DTO	0.914384	0.807952	-0.138865	0.938103	0.797533	-0.164152
	(0.5817)	(0.5607)	(0.3103)	(0.5739)	(0.5487)	(0.3137)
DSI	-0.167603	0.265931	-0.430184*	-0.169346	-0.221363	-0.154579
	(0.3310)	(0.3445)	(0.2305)	(0.3174)	(0.3217)	(0.2012)
DTO×$\Im$	-2.075701***	-2.029631***	-0.852966*	-2.056578***	-1.898429***	-0.917132*
	(0.7184)	(0.6706)	(0.5149)	(0.7188)	(0.6612)	(0.5335)
DSI×$\Im$	-1.822093***	-2.395390***	0.062384	-1.794744***	-1.821227***	-0.244383
	(0.4740)	(0.4554)	(0.4508)	(0.4642)	(0.4345)	(0.4366)
DTO×DSI	-0.125198	-0.715096	1.173728	-0.127797	-0.093430	0.478808
	(0.6540)	(0.7492)	(0.9001)	(0.6251)	(0.6507)	(0.7259)
DTO×DSI	1.441900	2.881235***	-0.508792	1.437034	2.246968**	0.326687
×$\Im$	(1.0968)	(1.0884)	(1.2879)	(1.0996)	(1.0026)	(1.2084)
$\Im$	-0.951439*	-0.766636*	-3.334832***	-0.992285*	-0.864254*	-3.348793***
	(0.5381)	(0.4539)	(0.2619)	(0.5457)	(0.4693)	(0.2599)
Mean Equation						
_cons	0.000359	0.000268	0.000252	0.000431	0.000355	0.000270
	(0.0003)	(0.0003)	(0.0003)	(0.0003)	(0.0003)	(0.0003)
N	3061	3061	3061	3061	3061	3061
LL	8841.2225	8843.4227	9030.5384	8841.9794	8843.9140	9033.3730

Standard errors in parentheses
* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

similar turning points in the markets whereas the quarterly market indicator spots more frequent changes in market direction compare with the crisis indicator. The crisis and yearly indicator help us to test how conditional skewness responses to general market decline for a relative long period of time. The quarterly market indicator, which can be seen from figure 22, help us to test how conditional skewness responses to a more frequent changes in market direction and the weekly market direction indicator, which can be seen from figure 23, help us to test how conditional skewness responses to very short term changes in market direction. The estimation results are presented in table 12. All coefficients on "$\mathrm{DTO}_i \cdot \Im$" and "$\mathrm{DSI}_i \cdot \Im$" are negative and strongly significant when the yearly and quarterly market indicators are used. However, we find that market turnover only has mild statistical power in forecasting conditioanl skewness while market short interest is not important in the determination of the skewness of market return even during periods of general market declines when the weekly market direction indicator is used. This may imply that the degree of heterogeneous beliefs and short sale restrictions in the market has an effect on conditional skewness when the downward trend in the market persists more than a week. It is also possible that the weekly market direction indicator that we have used to obtain the above results may be a noisy proxy for indicating real changes of market direction.

Finally, in table 13, we test the effect of past returns on conditional skewness by including cumulative return in our model. Both Hong and Stein (2003) and Hueng and McDonald (2005) find that past returns as far back as 36 months are negatively related to conditional skewness. Therefore, following Hong and Stein (2003), we define cumulative return as $\mathrm{RET}_t = \prod_{\tau=1}^{\tau=750} (1 + r_{t-\tau}) - 1$ and include it as one of our skewness factors. For $t < 750$, we set τ as the number of maximum possible days that we can use. We drop the first five observations such that our shortest cumulation period is five days. For each specifications, we estimate the TVSN-GJR-GARCH and TVSN-Q-GARCH models with RET_t as an additional skewness factor measured in percentage. Unlike Hong and Stein (2003) and Hueng and McDonald (2005), we find that past returns has no predictive power on conditional skewness when skewness factors are included. However, when we omit the skewness factors, the coefficients on cumulative return are negative and strongly significant. This result is similar to the results shown in Hong and Stein (2003) and Hueng and McDonald (2005).

3.5 Conclusion

In this chapter, we have analyzed the relationship between short sale restrictions, heterogeneous beliefs and conditional skewness by using a skew normal generalized autoregressive conditional heteroskedasticity model. Unlike previous studies, our paper considers the relationship under different market conditions. We show that negative conditional skewness is more pronounced when people disagree about the market more or the short sale constraints are more and more binding during general market declines but the effect is undetermined during periods of market growth. We demonstrated the importance of market conditions on condition skewness and reconciled conflicting evidence in recent empirical studies on the relationship between heterogeneous beliefs, short sale restrictions and conditional skewness.

Table 13: Estimation Results With Cumulative Return In the Skewness Equation

	TVSN-GJR-GARCH		TVSN-Q-GARCH	
	(1)	(2)	(3)	(4)
Variance Equation				
h_{t-1}	0.925946***	0.923723***	0.922395***	0.921480***
	(0.0075)	(0.0076)	(0.0075)	(0.0073)
u_{t-1}^2	0.090851***	0.098176***	0.066697***	0.068671***
	(0.0095)	(0.0092)	(0.0069)	(0.0067)
$u_{t-1}^2 I^+$	-0.052000***	-0.060109***		
	(0.0103)	(0.0094)		
u_{t-1}			-0.000601***	-0.000659***
			(0.0001)	(0.0001)
_cons	0.000002***	0.000002***	0.000002***	0.000002***
	(0.0000)	(0.0000)	(0.0000)	(0.0000)
Skewness Equation				
s_{t-1}	-0.449809***	0.193720	-0.455548***	0.241970
	(0.1034)	(0.1619)	(0.1024)	(0.1594)
u_{t-1}^2	0.157330***	0.232349***	0.164145***	0.291517***
	(0.0584)	(0.0752)	(0.0579)	(0.0579)
u_{t-1}	0.407628***	0.545841***	0.422381***	0.462237***
	(0.1213)	(0.1206)	(0.1224)	(0.1225)
_cons	0.247934	-0.075924	0.264295	-0.036921
	(0.4766)	(0.1867)	(0.4743)	(0.1529)
Skewness Factors				
DTO	0.898929		0.891090	
	(0.5598)		(0.5572)	
DSI	0.491968		0.481711	
	(0.4946)		(0.4885)	
DTO$\times\Im_{Crisis}$	-2.091444***		-2.155092***	
	(0.7377)		(0.7311)	
DSI$\times\Im_{Crisis}$	-2.573345***		-2.551889***	
	(0.6011)		(0.6007)	
DTO×DSI	-0.987776		-0.969622	
	(0.9728)		(0.9595)	
DTO×DSI$\times\Im_{Crisis}$	2.285931*		2.236892*	
	(1.3760)		(1.3503)	
$\Im_{Crisis}$	-0.925596*		-0.876804*	
	(0.5385)		(0.5311)	
RET$_t$	-0.005536	-0.008793***	-0.005625	-0.009379***
	(0.0040)	(0.0024)	(0.0041)	(0.0024)
Mean Equation				
_cons	0.000495*	0.000607**	0.000416	0.000443
	(0.0003)	(0.0003)	(0.0003)	(0.0003)
N	3056	3056	3056	3056
LL	8833.8753	8813.5079	8833.8377	8813.3772

Standard errors in parentheses
* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

3.6 Appendix

The Matlab Simulation Program "mysn_sim" for the centered parameterized Skew-Normal Model

```
function [dr kr]=mysn_sim(mu,sigma2,gamma,obs,k)
% ***********************************************************************
%PURPOSE: 1/ Simulate a time series of SNGARCH(1,1) daily return
% under the assumption that residuals are skew normal distributed
% 2/ Calculate multi period K-day return from daily return
%Remark: if K is specified as 1, daily return = K-day return
%***********************************************************************
%INPUT: mu,sigma2,gamma – SN(mu,sigma2,gamma) parameters
% mu: A constant for the mean equation
% sigma2: A constant for the variance equation
% gamma: A constant for the skewness equation
% obs – number of observation for centered_dr
% seed – set seed for normrnd
% K – set the numnber of day for the multi period
% K-day return
% ***********************************************************************
% RETURN: centered_dr – daily return ; centered_kr – K-day return
% ***********************************************************************
   K = k + 1;
   sigma = sqrt(sigma2);
   obs = obs+1;
% Transforming CP to DP using the cp2dp function
   [eta w alpha] = cp2dp(mu,sigma,gamma);
   w2 = w^2;
% Creating panel index
if K ~= 1
```

```
    j = [0;kron((1:(obs/k))',ones(k,1))];
    idk = find((j(2:end)-j(1:end-1))==1);
    clear A B j
end % Creating daily and K-day return, centered_dr and centered_kr
    c1 = 1./sqrt(1+((alpha).^2));
    c2 = alpha./sqrt(1+((alpha).^2));
    z = (c1.*normrnd(0,1,obs,1))+(c2.*abs(normrnd(0,1,obs,1)));
    delta = alpha/sqrt(1+((alpha)^2));
    ez = sqrt(2/pi)*delta;
    sdz = sqrt(1-((sqrt(2/pi)*delta)^2));
% centered skew normal variate
    zo = (z-ez)/sdz;
    y = exp(mu+(sqrt(sigma2).*zo));
% this is log price. price is log skew normal
    p = cumsum(log(y));
    dp = [p(2:end);p(end)];
    dr = dp-p;
if k ~=1
    kp = [p(K:end);zeros((K-1),1)];
    kr1 = kp-p;
    kr = kr1(idk,:);
elseif k==1
    kr = dr;
end
```

The Matlab Simulation Program "mysngarch_sim" for the centered parameterized Skew-Normal-GARCH(1,1) Model

```
function [dr kr]=mysngarch_sim(mu,sigma2,gamma,obs,k)
% *******************************************************************
```

```
%PURPOSE: 1/ Simulate a time series of SNGARCH(1,1) daily return
% under the assumption that residuals are skew normal distributed
% 2/ Calculate multi period K-day return from daily return
%Remark: if K is specified as 1, daily return = K-day return
%*********************************************************************
%INPUT: mu,sigma2,gamma – SN(mu,sigma2,gamma) parameters
% mu: A constant for the mean equation
% sigma2: A constant for the variance equation
% gamma: A constant for the skewness equation
% obs – number of observation for centered_dr
% seed – set seed for normrnd
% K – set the numnber of day for the multi period
% K-day return
% *********************************************************************
% RETURN: centered_dr – daily return ; centered_kr – K-day return
% *********************************************************************
    K = k + 1;
    sigma = sqrt(sigma2);
    obs = obs+1;
% Transforming CP to DP using the cp2dp function
    [eta w alpha] = cp2dp(mu,sigma,gamma);
    w2 = w^2;
% Creating panel index
if K ~= 1
    j = [0;kron((1:(obs/k))',ones(k,1))];
    idk = find((j(2:end)-j(1:end-1))==1);
    clear A B j
end % Creating daily and K-day return, centered_dr and centered_kr
    c1 = 1./sqrt(1+((alpha).^2));
    c2 = alpha./sqrt(1+((alpha).^2));
    z = (c1.*normrnd(0,1,obs,1))+(c2.*abs(normrnd(0,1,obs,1)));
    delta = alpha/sqrt(1+((alpha)^2));
    ez = sqrt(2/pi)*delta;
```

```
    sdz = sqrt(1-((sqrt(2/pi)*delta)^2));
% centered skew normal variate
    zo = (z-ez)/sdz;

    sigma2t = zeros(obs,1);
    sigma2t(1) = 0.1;
    ut2 = zeros(obs,1);
    ut2(1) = 0.1;

    for j = 2:obs
% conditional variance process
        sigma2t(j)= b1 + (b2*ut2(j-1)) + (b3*sigma2t(j-1));
        ut2(j) = (sqrt(sigma2t(j))*zo(j))^2 ;
    end

    y = exp(mu+(sqrt(sigma2).*zo));
% this is log price. price is log skew normal
    p = cumsum(log(y));
    dp = [p(2:end);p(end)];
    dr = dp-p;
if k ~=1
    kp = [p(K:end);zeros((K-1),1)];
    kr1 = kp-p;
    kr = kr1(idk,:);
elseif k==1
    kr = dr;
end
```

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