REPRESENTATIONS OF HECKE ALGEBRA OF TYPE A

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Abstract

We give some new results about representations of the Hecke algebra $\mathscr{H}_{F,q}(\mathfrak{S}_n)$ of type A. In the first part we define the decomposition numbers $d_{\lambda\nu}$ to be the composition multiplicity of the irreducible module D^ν in the Specht module S^λ. Then we compute the decomposition numbers $d_{\lambda\nu}$ for all partitions of the form $\lambda = (a, c, 1^b)$ and ν 2–regular for the Hecke algebra $\mathscr{H}_{\mathbb{C},-1}(\mathfrak{S}_n)$. In the second part, we give some examples of decomposable Specht modules for the Hecke algebra $\mathscr{H}_{\mathbb{C},-1}(\mathfrak{S}_n)$. These modules are indexed by partitions of the form $(a, 3, 1^b)$, where a, b are even. Finally, we find a new family of decomposable Specht modules for $F\mathfrak{S}_n$ when $\mathrm{char}(F) = 2$.

Introduction

Let $\mathfrak{S}_n$ be the symmetric group on n letters and let F be an arbitrary field. James in [15] studied the representation theory of the symmetric group algebra $F\mathfrak{S}_n$. This is a specific example of the Hecke algebra $\mathscr{H}_{F,q}(\mathfrak{S}_n)$ introduced by Dipper and James in [5]. By setting $q = 1$ in the statement of results proved for representations of the Hecke algebra, we recover results in the representation theory for symmetric groups. Many of the theorems in the symmetric group have an analogue in the Hecke algebra. However, some theorems do not have an analogue, or the proof is considerably more complicated.

For each Hecke algebra $\mathscr{H}_{F,q}(\mathfrak{S}_n)$, we define a family of modules called Specht modules. If $\mathscr{H}_{F,q}(\mathfrak{S}_n)$ is semisimple, these form a complete set of pairwise non-isomorphic irreducible $\mathscr{H}_{F,q}(\mathfrak{S}_n)$–modules. They are indexed by the set of partitions of n. Moreover, if $\mathscr{H}_{F,q}(\mathfrak{S}_n)$ is not semisimple, the simple modules arise as the heads of the Specht modules and are labelled by a subset of the set of partitions of n. The decomposition number $d_{\lambda\mu}$ is defined to be the composition multiplicity of the irreducible module D^μ in the Specht module S^λ and the decomposition matrix is the matrix whose entries are the decomposition numbers $d_{\lambda\mu}$. For fixed n and quantum parameter e, the decomposition matrices of the Hecke algebras $\mathscr{H}_{F,q}(\mathfrak{S}_n)$ will have a similar structure. In particular, each matrix is lower unitriangular if the partitions are sorted according to a specific partial order. In general it is not known how to compute decomposition numbers for Hecke algebras over fields of positive characteristic. However, Lascoux, Leclerc and Thibon in [20] described an iterative algorithm to compute the decomposition numbers of the Iwahori-Hecke algebras $\mathscr{H}_{\mathbb{C},q}(\mathfrak{S}_n)$ of the symmetric group in characteristic zero. This algorithm, now known as the LLT algorithm, was proved later by Ariki in [2]. Knowing these decomposition numbers in $\mathscr{H}_{\mathbb{C},q}(\mathfrak{S}_n)$ provides information about the decomposition numbers for an arbitrary Hecke algebra. The decomposition matrices of the Hecke algebras $\mathscr{H}_{F,q}(\mathfrak{S}_n)$ can be obtained from the decomposition matrix of the Hecke algebras $\mathscr{H}_{\mathbb{C},w}(\mathfrak{S}_n)$ by multiplying by an adjustment matrix. So the problem of computing the decomposition matrices of the Hecke algebras $\mathscr{H}_{F,q}(\mathfrak{S}_n)$ reduces to finding an adjustment matrix, although these are not generally known. In [16] James gave a conjecture which suggested that for $n<pe$, the adjustment matrix should be the identity matrix and his conjecture is closely related to the celebrated conjecture of Lusztig [21]. Although, James's Conjecture was proved to be true in certain cases given in [9], [8] and [11], counterexamples to the conjecture and Lusztig's Conjecture were found this year by Williamson [33].

Peel in [29] studied the decomposition numbers of Specht modules corresponding to hook partitions for symmetric group algebras $F\mathfrak{S}_n$ in odd characteristic p. This study is continued by James in [9] and James and Mathas in [17] for the Hecke algebra $\mathscr{H}_{\mathbb{C},q}(\mathfrak{S}_n)$. Chuang, Miyachi and Tan in [4] described the decomposition numbers corresponding to rows labelled by hook partitions with $e \geq 2$. We continue this work in this thesis by studying Specht modules indexed by partitions of the form $\lambda = (a, c, 1^b)$.

For the symmetric group $\mathfrak{S}_n$, Peel in [30] showed that for characteristic $p \neq 2$ the Specht module S^λ indexed by a partition λ is indecomposable. Furthermore, James in [15] and Dipper and James in [5] show that if $e \neq 2$ or if λ is 2–regular then the Specht module S^λ is indecomposable. Moreover, James in [14] shows that the $F_2\mathfrak{S}_n$–module S^λ is a decomposable module for the partition $\lambda = (5, 1^2)$. Murphy in [27] continued this case and analysed the Specht modules labelled by hook partitions by computing the endomorphism ring of every such Specht module and determining when the Specht module is decomposable. Dodge and Fayers in [7] presented a new family of decomposable Specht modules for the symmetric group algebra $F_2\mathfrak{S}_n$ and these Specht modules are labelled by partitions of the form $(a, 3, 1^b)$, where a, b are even. However, we are far from knowing which Specht modules are decomposable for $e = 2$.

Now we give a brief description of the contents of each chapter in this thesis. In Chapter 1 we start by introducing some the background theory and results of the symmetric group and its Iwahori-Hecke algebra that we are going to use regularly throughout this thesis. We conclude by giving definitions and results of the LLT algorithm and adjustment matrix that we will need in the second Chapter. In Chapter 2 we extend results of Chuang, Miyachi and Tan. We compute the decomposition numbers $d_{\lambda\mu}$ for the Hecke algebra $\mathscr{H}_{\mathbb{C},-1}(\mathfrak{S}_n)$ for Specht modules S^λ for all $\lambda = (a, c, 1^b)$. Our main results are Theorem A and Theorem B which are given at the start of Section 1 and Theorem C which appears in Section 3. In Chapter 3 we give analogues of some of the results given in Dodge and Fayers in [7]. We provide some cases of decomposable Specht modules for the Hecke algebra $\mathscr{H}_{\mathbb{C},-1}(\mathfrak{S}_n)$ which are indexed by partitions of the form $(a, 3, 1^b)$, where a, b are even. Our main result is Theorem D which is given at the start of Section 3. In Chapter 4 we present a new family of decomposable Specht modules for the symmetric group algebra $F_2\mathfrak{S}_n$. These Specht modules are labelled by partitions of the form $(a, 5, 1^b)$, $(a, 7, 1^b)$ and $(a, c, 1^b)$ where a, b are even. Our main results are Theorems I, II and III which appear in Section 1.

Chapter 1

Preliminaries

1.1 The symmetric groups

We start by fixing $n \geq 1$ and let $\mathfrak{S}_n$ be the symmetric group acting on the set $\{1, 2, \ldots, n\}$.

Definition 1.1.1. For $1 \leq i \leq n-1$ suppose s_i is the basic transposition such that $\mathsf{s}_i = (i,\ i+1)$ and let $\mathsf{S} = \{\mathsf{s}_1, \ldots, \mathsf{s}_{n-1}\}$. Then as a Coxeter group, $\mathfrak{S}_n$ is generated by $\{\mathsf{s}_i \mid 1 \leq i \leq n-1\}$ with the relations:

$$\begin{aligned} \mathsf{s}_i^2 &= 1, && 1 \leq i \leq n-1, \\ \mathsf{s}_i\mathsf{s}_j &= \mathsf{s}_j\mathsf{s}_i, && 1 \leq i < j-1 \leq n-2, \\ \mathsf{s}_i\mathsf{s}_{i+1}\mathsf{s}_i &= \mathsf{s}_{i+1}\mathsf{s}_i\mathsf{s}_{i+1}, && 1 \leq i \leq n-2. \end{aligned}$$

Every permutation can be written as a product of basic transpositions.

Definition 1.1.2. For the permutation $w \in \mathfrak{S}_n$ we write $w = \mathsf{s}_{i_1} \cdots \mathsf{s}_{i_k}$, where $\mathsf{s}_{i_1}, \ldots, \mathsf{s}_{i_k} \in \mathsf{S}$ and if k is minimal we say that w has length k and write $\ell(w) = k$. Then $\mathsf{s}_{i_1} \cdots \mathsf{s}_{i_k}$ is called a reduced expression for w.

Definition 1.1.3. An odd permutation is defined to be a permutation that can be written as a product of an odd number of transpositions. Similarly, if a permutation can be written as a product of an even number of transpositions, then it is called an even permutation. The sign of a permutation is defined as follows:

$$\operatorname{sgn}(\sigma) = \begin{cases} +1, & \text{if } \sigma \text{ is even,} \\ -1, & \text{if } \sigma \text{ is odd} \end{cases}$$

where, $\sigma \in \mathfrak{S}_n$ and $\operatorname{sgn} : \mathfrak{S}_n \longrightarrow \{+1, -1\}$.

Theorem 1.1.4. *(Matsumoto)[26, Theorem 1.8] Let* $\mathsf{s}_{i_1}, \ldots, \mathsf{s}_{i_k}$ *and* $\mathsf{s}_{j_1}, \ldots, \mathsf{s}_{j_k}$ *be elements of* $\mathsf{S} = \{\mathsf{s}_1, \ldots, \mathsf{s}_{n-1}\}$ *such that* $\mathsf{s}_{i_1} \ldots \mathsf{s}_{i_k}$ *and* $\mathsf{s}_{j_1} \ldots \mathsf{s}_{j_k}$ *are two reduced expressions in* $\mathfrak{S}_n$. *Then write* $(i_1, \ldots, i_k) \sim_b (j_1, \ldots, j_k)$ *if one expression can be be transformed into the other using only the braid relations*

$(\mathsf{s}_i \mathsf{s}_j = \mathsf{s}_j \mathsf{s}_i,$ *for* $1 \le i < j-1 \le n-2$ *and* $\mathsf{s}_i \mathsf{s}_{i+1} \mathsf{s}_i = \mathsf{s}_{i+1} \mathsf{s}_i \mathsf{s}_{i+1},$ *for* $i = 1, \ldots, n-2)$.

Then,

$$(i_1, \ldots, i_k) \sim_b (j_1, \ldots, j_k) \Leftrightarrow \mathsf{s}_{i_1} \ldots \mathsf{s}_{i_k} = \mathsf{s}_{j_1} \ldots \mathsf{s}_{j_k}.$$

1.2 The Hecke algebra of the symmetric groups

Definition 1.2.1. Suppose R is a commutative domain with 1 and that $q \neq 0$ is an arbitrary element of R. Then the Iwahori-Hecke algebra $\mathscr{H} = \mathscr{H}_{R,q}(\mathfrak{S}_n)$ of $\mathfrak{S}_n$ is defined to be the unital associative R-algebra with generators $T_1, T_2, \ldots, T_{n-1}$ and relations:

$$\begin{aligned} (T_i - q)(T_i + 1) &= 0, && 1 \le i \le n-1, \\ T_i T_j &= T_j T_i, && 1 \le i < j-1 \le n-2, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, && 1 \le i \le n-2. \end{aligned}$$

Note that: if $q = 1$ in the first relation in the above Definition 3.1 we get $T_i^2 = 1$, which is the same as the defining relations for the symmetric group ring $R\mathfrak{S}_n$. Thus $\mathscr{H}$ is isomorphic to the group ring $R\mathfrak{S}_n$ of $\mathfrak{S}_n$ [26].

Now we introduce a basis of $\mathscr{H}$.

Definition 1.2.2. Let w be an element of $\mathfrak{S}_n$ and let $s_{i_1} \ldots s_{i_k}$ be a reduced expression for w. By Matsumoto's Theorem 1.1.4, we can define:

$$T_w = T_{i_1} \ldots T_{i_k}.$$

Remark 1.2.3. By Matsumoto's Theorem 1.1.4 the element T_w is independent of the choice of reduced expression for w and thus is well defined. If w is the identity element of $\mathfrak{S}_n$, we identify T_w with $1 = 1_R$ the identity element of R.

The next lemma gives us the right-hand multiplicative relation for $\mathscr{H}$:

Lemma 1.2.4. *[26, Lemma 1.12] Suppose* s *is a transposition in* S *and that* $w \in \mathfrak{S}_n$*. Then*

$$T_w T_{\mathsf{s}} = \begin{cases} T_{w\mathsf{s}}, & \text{if } \ell(w\mathsf{s}) > \ell(w), \\ qT_{w\mathsf{s}} + (q-1)T_w, & \text{if } \ell(w\mathsf{s}) < \ell(w) \end{cases}$$

Example 1.2.5. Take $w = (1,\ 3,\ 2)$, $\mathsf{s} = (1,\ 2)$. Then $w\mathsf{s} = (1,\ 3)$ and $\ell(w\mathsf{s}) = 3 > \ell(w) = 2$. So

$$T_{(1,\ 3,\ 2)} \times T_{(1,\ 2)} = T_{(1,\ 3)}.$$

If $w = (1,\ 2)(2,\ 3)$ and $\mathsf{s} = (2,\ 3)$, then $w\mathsf{s} = (1,\ 2)$ and $\ell(w\mathsf{s}) = 1 < \ell(w) = 2$. So

$$T_{(1,\ 2)(2,\ 3)} \times T_{(2,\ 3)} = qT_{(1\ 2)} + (q-1)T_{(1,\ 3,\ 2)}.$$

Lemma 1.2.6. *[5, Lemma 2.1] Suppose* $w \in \mathfrak{S}_n$*. Then* T_w *is invertible in* $\mathscr{H}$ *with inverse* $T_w^{-1} = T_{\mathsf{s}_k}^{-1} \dots T_{\mathsf{s}_2}^{-1} T_{\mathsf{s}_1}^{-1}$*, where* $w = \mathsf{s}_1 \mathsf{s}_2 \dots \mathsf{s}_k$ *is a reduced expression for* w*, and*

$$T_{\mathsf{s}}^{-1} = -(1 - (1/q)) + (1/q)T_{\mathsf{s}}$$

for s *is a transposition in* S*.*

Lemma 1.2.4 shows that $\mathscr{H}$ is spanned by the elements $\{T_w \mid w \in \mathfrak{S}_n\}$ when it is taken as an R-module. The following theorem gives us a basis of $\mathscr{H}$.

Theorem 1.2.7. *[26, Theorem 1.13] The Iwahori-Hecke algebra* $\mathscr{H}$ *of the symmetric group* $\mathfrak{S}_n$ *is free as an* R*–module with basis* $\{T_w \mid w \in \mathfrak{S}_n\}$*.*

Definition 1.2.8. We define e to be the smallest positive integer such that $1 + q + \dots + q^{e-1} = 0$. Let $e = \infty$ if no such integer exists. In other words, either $q = 1$ and e is equal to the characteristic of R or $q \neq 1$ and q is a primitive

e^{th} root of unity. e plays the same role as the characteristic of a field in the representation theory of finite groups.

Definition 1.2.9. Suppose $\mathcal{Z} = \mathbb{Z}[\hat{q}, \hat{q}^{-1}]$, where $\hat{q}$ is an indeterminate over $\mathbb{Z}$. Then $\mathscr{H}_{\mathcal{Z}} = \mathscr{H}_{\mathcal{Z},\hat{q}}(\mathfrak{S}_n)$ is called the generic Iwahori-Hecke algebra of $\mathfrak{S}_n$.

Lemma 1.2.10. *[26, Corollary 1.17] The algebra $\mathscr{H}_{\mathcal{Z}}$ is semisimple.*

Theorem 1.2.11. *[26, Corollary 1.15] Let F be a field and $q \in F\backslash\{0\}$. Define $\varphi : \mathcal{Z} \longrightarrow F$ to be the ring homomorphism determined by $\hat{q} \longmapsto q$. Then*

$$\mathscr{H}_{F,q}(\mathfrak{S}_n) \cong \mathscr{H}_{\mathcal{Z}} \otimes_{\mathcal{Z}} F$$

as $F-$algebras.

1.3 Combinatorics

Definition 1.3.1. A composition of a positive integer n is a sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that $\sum_{i=1}^{k} \lambda_i = n$.

Definition 1.3.2. We say $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a partition of n if the following conditions hold:

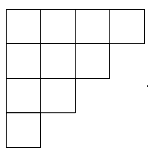
1. $\lambda_1, \lambda_2, \ldots, \lambda_k$ are positive integers and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$.
2. $\sum_{i=1}^{k} \lambda_i = n$

Note that: $\lambda \vdash n$ denotes λ is a partition of n.

Definition 1.3.3. Suppose λ is a partition of n. Then the diagram $[\lambda]$ is defined by :

$$[\lambda] := \{(i,j) \mid i,j \in \mathbb{Z}, 1 \leq i, 1 \leq j \leq \lambda_i\}.$$

Example 1.3.4. The partition $\lambda = (4,3,2,1)$ has a diagram

$[\lambda] =$  .

Definition 1.3.5. Suppose λ and μ are partitions of n. Then λ dominates μ and we write $\lambda \trianglerighteq \mu$ if the following condition holds

$$\sum_{i=1}^{j} \lambda_i \geq \sum_{i=1}^{j} \mu_i \quad \text{for all } j.$$

Note that: We write $\lambda \triangleright \mu$, if $\lambda \trianglerighteq \mu$ and $\lambda \neq \mu$.

Example 1.3.6. The dominance relation on the set of partitions when $n = 5$ is given by the diagram:

$$\begin{tikzcd}
& (5) \arrow[d, dash] & \\
& (4,1) \arrow[d, dash] & \\
& (3,2) \arrow[dl, dash] \arrow[dr, dash] & \\
(2,2,1) \arrow[dr, dash] & & (3,1,1) \arrow[dl, dash] \\
& (2,1,1,1) \arrow[d, dash] & \\
& (1^5) &
\end{tikzcd}$$

Definition 1.3.7. Suppose $[\lambda]$ is a diagram. Then the conjugate diagram $[\lambda']$ is obtained by interchanging the rows and columns in diagram $[\lambda]$. We say $\lambda' \vdash n$ is conjugate to λ.

Example 1.3.8. Take $\lambda = (4,2,1)$. Then the diagram of λ is $[\lambda] = \ydiagram{4,2,1}$

and the conjugate diagram $[\lambda'] = \ydiagram{3,2,1,1}$ and so $\lambda' = (3,2,1^2)$.

Definition 1.3.9. Suppose $[\lambda]$ is a diagram of a partition λ. Then a λ–tableau $\mathfrak{t}$ is obtained by replacing each node in $[\lambda]$ by one of the integers $1,2,3,\ldots,n$ without repeating.

Note that: λ–tableau can be defined as a bijection from $[\lambda] \longrightarrow \{1, 2, ..., n\}$.

Example 1.3.10. Let $\lambda = (2, 1)$, so we have the diagram of λ as $[\lambda] = \begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$.
Then all possible tableaux are:

$$\mathfrak{t}_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \mathfrak{t}_2 = \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \mathfrak{t}_3 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \mathfrak{t}_4 = \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \mathfrak{t}_5 = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \mathfrak{t}_6 = \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}.$$

Definition 1.3.11. Let $\mathfrak{t}$ be a tableau. Then we define its row-stabilizer $R_\mathfrak{t}$ as follows:

$$R_\mathfrak{t} = \{\sigma \in \mathfrak{S}_n \mid \text{ for all } i,\ i \text{ and } \sigma(i) \text{ belong to the same row of } \mathfrak{t}\}$$

similarly we can define the column-stabilizer $C_\mathfrak{t}$ as:

$$C_\mathfrak{t} = \{\sigma \in \mathfrak{S}_n \mid \text{ for all } j,\ j \text{ and } \sigma(j) \text{ belong to the same column of } \mathfrak{t}\}.$$

Example 1.3.12. If $\mathfrak{t} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2} 6 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$, then:

$$R_\mathfrak{t} = \mathfrak{S}_{\{1,2,3\}} \times \mathfrak{S}_{\{4,5\}} \times \mathfrak{S}_{\{6\}}$$

$$C_\mathfrak{t} = \mathfrak{S}_{\{1,4,6\}} \times \mathfrak{S}_{\{2,5\}} \times \mathfrak{S}_{\{3\}}.$$

Also,

$$|R_\mathfrak{t}| = 3!\ 2!\ 1!.$$

Now we can introduce the tabloid.

Definition 1.3.13. Define an equivalence relation on the set of λ–tableaux by $\mathfrak{t}_1 \sim \mathfrak{t}_2$ if and only if

$$\mathfrak{t}_1 \sigma = \mathfrak{t}_2 \quad \text{for some } \sigma \in R_{\mathfrak{t}_1}$$

where the symmetric group $\mathfrak{S}_n$ acts on tableaux by permuting entries of the given tableau. Now the tabloid $\{\mathfrak{t}\}$ is the equivalence class of $\mathfrak{t}$ under this equivalence relation.

Example 1.3.14. If $\lambda = (2,1)$, then the different λ–tabloids are

$$\{\mathfrak{t}_1\} = \begin{array}{c}\hline 1 \quad 2 \\ \hline 3 \\ \hline \end{array}, \quad \{\mathfrak{t}_3\} = \begin{array}{c}\hline 1 \quad 3 \\ \hline 2 \\ \hline \end{array}, \quad \{\mathfrak{t}_5\} = \begin{array}{c}\hline 2 \quad 3 \\ \hline 1 \\ \hline \end{array}$$

Note that: $\begin{array}{c}\hline 2 \quad 3 \\ \hline 1 \\ \hline \end{array} = \begin{array}{c}\hline 3 \quad 2 \\ \hline 1 \\ \hline \end{array}$

Lemma 1.3.15. *Suppose* $\mathfrak{t}$ *is a tableau and* σ *is a permutation. Then:*

$$R_{\mathfrak{t}\sigma} = \sigma^{-1} R_{\mathfrak{t}} \sigma.$$

Proof. Suppose that $\pi \in R_{\mathfrak{t}\sigma}$. Then we have the following:

$$\begin{aligned} \pi \in R_{\mathfrak{t}\sigma} &\iff \{\mathfrak{t}\sigma\}\pi = \{\mathfrak{t}\sigma\} \\ &\iff \{\mathfrak{t}\}\sigma\pi\sigma^{-1} = \{\mathfrak{t}\} \\ &\iff \sigma\pi\sigma^{-1} \in R_{\mathfrak{t}} \\ &\iff \pi \in \sigma^{-1} R_{\mathfrak{t}} \sigma \end{aligned}$$

□

Lemma 1.3.16. $\mathfrak{S}_n$ *acts on the set of* λ*–tabloids by* $\{\mathfrak{t}\}\sigma = \{\mathfrak{t}\sigma\}$. *This action is well defined.*

Proof. Let $\{\mathfrak{t}_1\} = \{\mathfrak{t}_2\}$, so that $\mathfrak{t}_1\pi = \mathfrak{t}_2$ for some $\pi \in R_{\mathfrak{t}_1}$. Then, $\sigma^{-1}\pi\sigma \in \sigma^{-1} R_{\mathfrak{t}_1} \sigma = R_{\mathfrak{t}_1\sigma}$ by Lemma 1.3.15. So, $\{\mathfrak{t}_1\sigma\} = \{\mathfrak{t}_1\pi\sigma\} = \{\mathfrak{t}_2\sigma\}$. □

Definition 1.3.17. Let $\mathfrak{t}$ be a λ–tableau. Then $\mathfrak{t}$ is row standard if the entries increase along the rows. The tableau $\mathfrak{t}$ is a standard tableau if the rows and columns of $\mathfrak{t}$ are increasing sequences. The tabloid $\{\mathfrak{t}\}$ is standard if there is a standard tableau in the equivalence class $\{\mathfrak{t}\}$.

Definition 1.3.18. We define a λ–tableau of type μ to be a tableau of shape λ with μ_i entries equal to i, for each i. A λ–tableau T of type μ is said to be row standard if its entries increase along the rows and is said to be semistandard if the entries increase along the rows and strictly increase down the columns. We denote the set of row standard λ–tableaux of type μ by $\mathcal{T}(\lambda,\mu)$ and we denote the set of semistandard λ–tableaux of type μ by $\mathcal{T}_0(\lambda,\mu)$.

Definition 1.3.19. Let λ be a partition of n. Define $\mathfrak{t}^{\lambda}$ to be the row standard λ–tableau with $1, 2, \ldots, n$ entered in order along its rows. Define $\mathfrak{t}_{\lambda}$ to be the row standard λ–tableau with $1, 2, \ldots, n$ entered in order down its columns. We denote the permutation that sends $\mathfrak{t}^{\lambda}$ to $\mathfrak{t}_{\lambda}$ by w_{λ}.

Definition 1.3.20. Suppose λ is a partition of n. The row-stabilizer of $\mathfrak{t}^{\lambda}$ is called the standard Young subgroup of $\mathfrak{S}_n$ with respect to a partition λ and is denoted by $\mathfrak{S}_{\lambda}$.

Example 1.3.21. Let $\lambda = (3, 1)$ and that $w_{\lambda} \in \mathfrak{S}_n$ be the permutation that sends $\mathfrak{t}^{\lambda}$ to $\mathfrak{t}_{\lambda}$. Then

$$\mathfrak{t}^{\lambda} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}, \quad \mathfrak{t}_{\lambda} = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}$$

where $w_{\lambda} = (2, 3, 4)$.

Now we let $e \geq 2$.

Definition 1.3.22. Let λ be a partition of n. Then λ is e–singular if for some i we have

$$\lambda_{i+1} = \lambda_{i+2} = \cdots = \lambda_{i+e} > 0, \text{ otherwise, it is } e\text{–regular.}$$

Definition 1.3.23. Suppose λ is a partition of n and that $[\lambda]$ is the Young diagram of λ. Then for each node $(i, j) \in [\lambda]$ define the e–residue of (i, j) by:

$$\operatorname{res}((i, j)) = (j - i) \mod e.$$

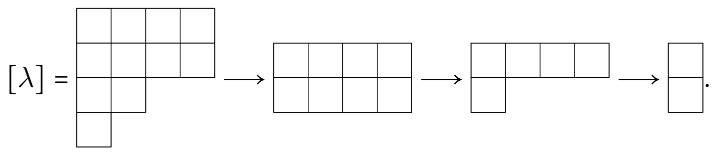
If the e–residue of a node $(i,\ j)$ is r, we say $(i,\ j)$ is an r–node. The e–residue diagram of a partition λ is defined to be the diagram obtained by replacing each node by $(j - i) \mod e$.

Example 1.3.24. Take $\lambda = (4, 2)$, $e = 3$. Then the e–residue diagram is

$$\begin{array}{|c|c|c|c|}\hline 0 & 1 & 2 & 0 \\ \hline 2 & 0 \\ \cline{1-2}\end{array}.$$

Definition 1.3.25. The rim of a partition λ is defined to be the set of all nodes $(i, j) \in [\lambda]$ such that $(i+1, j+1) \notin [\lambda]$. An e–rim hook of a partition λ is defined to be a connected subset of the rim of λ containing exactly e nodes which can be removed from $[\lambda]$ to leave a new Young diagram. The e–core of a partition λ is defined to be the partition formed by repeatedly removing e–rim hooks until no more e–rim hooks can be removed. The e–weight of a partition λ is the number of e–rim hooks which must be removed from $[\lambda]$ to get the e–core.

Example 1.3.26. Let $\lambda = (4^2, 2, 1)$, $e = 3$ then



Then the 3–core of λ is (1^2) and it has weight 3.

Definition 1.3.27. Let λ be a partition of n. For $l \geq 0$, define the *lth* ladder to be the set of nodes of the form $\{(i,j) \in \mathbb{N}^2 \mid j - i + (i-1)e = l\}$. The e–ladder diagram of a partition λ is defined to be the diagram obtained by replacing each node $(i,\ j)$ in $[\lambda]$ by the number $j - i + (i-1)e$.

Example 1.3.28. Let $\lambda = (4,3,2)$ and $e = 2$. Then the e–ladder diagram of the partition λ is

$$\begin{array}{|c|c|c|c|}\hline 0 & 1 & 2 & 3 \\ \hline 1 & 2 & 3 \\ \cline{1-3} 2 & 3 \\ \cline{1-2}\end{array}\ .$$

Definition 1.3.29. The regularization of λ is the partition λ^R whose Young diagram is obtained by moving the nodes in $[\lambda]$ as high as possible within their e–ladders. The partition λ^R is always e–regular.

Example 1.3.30. If $\lambda = (6,2,1^4)$ and $e = 2$, then $\lambda^R = (6,5,1)$ and their ladder diagrams are:

$$[\lambda] = \begin{array}{|c|c|c|c|c|c|}\hline 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 2 \\ \cline{1-2} 2 \\ \cline{1-1} 3 \\ \cline{1-1} 4 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}\ , \qquad \lambda^R = \begin{array}{|c|c|c|c|c|c|}\hline 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 2 & 3 & 4 & 5 \\ \cline{1-5} 2 \\ \cline{1-1}\end{array}\ .$$

Now we describe the Specht modules for $\mathfrak{S}_n$.

1.4 Specht modules

Before studying the Specht modules we need to describe the permutation module M^λ of $\mathfrak{S}_n$ on the Young subgroup $\mathfrak{S}_\lambda$. Let F be an arbitrary field of characteristic $p \geq 0$. So, we define the permutation module as follows:

Definition 1.4.1. Suppose λ is a partition of n. Let M^λ be the vector space over F whose basis elements are $\{\mathfrak{t}_1\}, \ldots, \{\mathfrak{t}_k\}$, where $\{\mathfrak{t}_1\}, \ldots, \{\mathfrak{t}_k\}$ are λ-tabloids, with the action

$$\{\mathfrak{t}\}\sigma = \{\mathfrak{t}\sigma\}, \text{ for } \sigma \in \mathfrak{S}_n.$$

Then the M^λ is called the permutation module corresponding to λ.

Example 1.4.2. From Example 1.3.14 the set $\{\{\mathfrak{t}_1\}, \{\mathfrak{t}_3\}, \{\mathfrak{t}_5\}\}$ is a basis of $M^{(2,1)}$.

Remark 1.4.3. Let λ be a partition of n. Then the permutation module M^λ of $\mathfrak{S}_n$ is a cyclic $F\mathfrak{S}_n$-module generated by any given λ–tabloid.

Definition 1.4.4. Let $\mathfrak{t}$ be a tableau. Then the signed column sum $K_\mathfrak{t}$ is defined as the element of the group algebra $F\mathfrak{S}_n$ which is obtained by summing the elements in the column stabilizer of $\mathfrak{t}$ and attaching the signature to each permutation. i.e:

$$K_\mathfrak{t} = \sum_{\sigma \in C_\mathfrak{t}} \operatorname{sgn}(\sigma)\sigma.$$

Now we can define the polytabloid as follows.

Definition 1.4.5. Let $\mathfrak{t}$ be a tableau. Then the polytabloid $e_\mathfrak{t}$ associated with this tableau is defined as

$$e_\mathfrak{t} = \{\mathfrak{t}\}K_\mathfrak{t}.$$

Remark 1.4.6.

1. If $\mathfrak{t}$ has columns $C_1, ..., C_k$, then $K_\mathfrak{t}$ factors as

$$K_\mathfrak{t} = K_{C_1} .... K_{C_k}.$$

2. The polytabloid $e_\mathfrak{t}$ is said to be standard if the tableau $\mathfrak{t}$ is standard.

Example 1.4.7. Let $\lambda = (3,2)$ and suppose $\mathfrak{t} = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}$. Then

$$K_{\mathfrak{t}} = (1 - (1,3))(1 - (2,4))$$

and

$$e_{\mathfrak{t}} = \begin{array}{ccc}\hline 1 & 2 & 5 \\ \hline 3 & 4 & \\ \cline{1-2}\end{array} - \begin{array}{ccc}\hline 2 & 3 & 5 \\ \hline 1 & 4 & \\ \cline{1-2}\end{array} - \begin{array}{ccc}\hline 1 & 4 & 5 \\ \hline 2 & 3 & \\ \cline{1-2}\end{array} + \begin{array}{ccc}\hline 3 & 4 & 5 \\ \hline 1 & 2 & \\ \cline{1-2}\end{array}.$$

Remark 1.4.8. A polytabloid depends on the tableau $\mathfrak{t}$ as well as the tabloid $\{\mathfrak{t}\}$. In addition, all tabloids involved in $e_{\mathfrak{t}}$ have coefficient ± 1.

Note that: If $\lambda = (n)$, then $e_{12\ldots n} = \begin{array}{cccc}\hline 1 & 2 & \cdots & n \\ \hline\end{array}$ is the only polytabloid and M^{λ} is the trivial $F\mathfrak{S}_n$–module. Previously, we constructed representations M^{λ} of $\mathfrak{S}_n$ known as permutation modules. Now we consider the Specht modules that corresponds uniquely to λ.

Definition 1.4.9. For any partition λ of n, the Specht module S^{λ} is the submodule of M^{λ} spanned by the polytabloids $e_{\mathfrak{t}}$.

Example 1.4.10. Consider $\lambda = (n)$. Then there is only one polytabloid, which is

$$\begin{array}{cccc}\hline 1 & 2 & \cdots & n \\ \hline\end{array}$$

Since this polytabloid is fixed by $\mathfrak{S}_n$, we see that S^{λ} is the one-dimensional trivial representation.

The basis for Specht modules S^{λ} is given by next theorem

Theorem 1.4.11. *[15, Theorem 8.4] The set* $\{e_{\mathfrak{t}} \mid \mathfrak{t}$ *is a standard* λ*–tableau*$\}$ *is a basis for the Specht module* S^{λ}.

Example 1.4.12. If $\lambda = (2,1)$, then standard λ–tableaux are

$$\mathfrak{t}_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \quad \mathfrak{t}_2 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$$

and

$$e_{\mathfrak{t}_1} = \begin{array}{cc}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array} - \begin{array}{cc}\hline 2 & 3 \\ \hline 1 & \\ \cline{1-1}\end{array}$$

$$e_{t_2} = \frac{\overline{\begin{array}{cc}1 & 3\end{array}}}{\underline{2}} - \frac{\overline{\begin{array}{cc}2 & 3\end{array}}}{\underline{1}}$$

Thus the set $\{e_{t_1}, e_{t_2}\}$ is a basis of $S^{(2,1)}$.

We now will introduce the irreducible modules [15]. All the results and these theorems are true over an arbitrary field F of characteristic p.

Definition 1.4.13. Suppose $\{t_1\}$ and $\{t_2\}$ are λ–tabloids. Define the inner product $\langle , \rangle$ to be the unique bilinear form on M^λ such that

$$\langle \{t_1\}, \{t_2\} \rangle = \begin{cases} 1 & \text{if } \{t_1\} = \{t_2\}; \\ 0 & \text{if } \{t_1\} \neq \{t_2\}. \end{cases}$$

We are now in a position to state the submodule theorem.

Theorem 1.4.14. *[15, Theorem 4.9] The module $S^\lambda/(S^\lambda \bigcap S^{\lambda^\perp})$ is zero or absolutely irreducible, so if this not zero then $S^\lambda \cap S^{\lambda^\perp}$ is the unique maximal submodule of S^λ and $S^\lambda/(S^\lambda \cap S^{\lambda^\perp})$ is self dual.*

Theorem 1.4.15. *[31, Theorem 2.4.6] The Specht modules S^λ give a complete list of irreducible $\mathfrak{S}_n$–modules over $\mathbb{C}$.*

Theorem 1.4.16. *[15, Theorem 11.1] Let S^λ be a Specht module over a field F of characteristic $p > 0$. Then $S^\lambda/(S^\lambda \cap S^{\lambda^\perp})$ is non zero if and only if λ is p–regular.*

Definition 1.4.17. Let $p > 0$ be prime. Suppose λ is a partition of n and that λ is p–regular. Then, we define

$$D^\lambda = S^\lambda/(S^\lambda \cap S^{\lambda^\perp}).$$

The next theorem is an analogue of Theorem 1.4.15.

Theorem 1.4.18. *[13, Theorem 6] The set $\{D^\lambda \mid \lambda\ p\text{–regular}\}$ gives a complete list of irreducible $\mathfrak{S}_n$–modules over a field of characteristic $p > 0$.*

Theorem 1.4.19. *[15, Corollary 12.2] Suppose $p > 0$ is prime. Let λ be p–regular. Then S^λ has a unique top composition factor $D^\lambda = S^\lambda/(S^\lambda \cap S^{\lambda^\perp})$. If D is a composition factor of $S^\lambda \cap S^{\lambda^\perp}$ then D is isomorphic to D^μ for $\mu \rhd \lambda$. If λ is p–singular, then all composition factors of S^λ have the form D^μ with $\mu \rhd \lambda$.*

Definition 1.4.20. Suppose F is a field of characteristic $p > 0$. Let λ and μ be partitions of n with μ is p–regular. Define $d_{\lambda\mu} = [S^\lambda : D^\mu]$ to be multiplicity of D^μ as a composition factors of S^λ. The matrix $D = (d_{\lambda\mu})$ is called the decomposition matrix of $\mathfrak{S}_n$.

1.5 Representation theory of the Hecke algebra

Now suppose that F is a field of characteristic $p \geq 0$ and that $q \in F\backslash\{0\}$. Take $\mathscr{H} = \mathscr{H}_{F,q}(\mathfrak{S}_n)$. Recall that e is the smallest positive integer such that $1 + q + ... + q^{e-1} = 0$. Let $e = \infty$ if no such integer exists. In this section we use the definitions of Dipper and James [5]. However, Murphy [28] has shown that the Iwahori-Hecke algebra $\mathscr{H}$ is a cellular algebra in the sense of Graham and Lehrer [12].

1.5.1 Permutation Modules

Definition 1.5.1. Let μ be a composition of n. Then define

$$\mathcal{D}_\mu = \{d \in \mathfrak{S}_n \mid \mathfrak{t}^\mu d \text{ is row standard}\}$$

which is a complete set of right coset representatives of the Young subgroup $\mathfrak{S}_\mu = \mathfrak{S}_{\mu_1} \times \cdots \times \mathfrak{S}_{\mu_k}$ in $\mathfrak{S}_n$. Moreover, the set $\mathcal{D}_\mu$ consists the unique element of minimal length from each coset. Furthermore, if $w \in \mathfrak{S}_\mu$ and $d \in \mathcal{D}_\mu$ then $\ell(wd) = \ell(w) + \ell(d)$ and $T_v = T_w T_d$ for $v = wd \in \mathfrak{S}_n$. Hence, each row standard μ-tableau corresponds to a right coset of $\mathfrak{S}_\mu$ in $\mathfrak{S}_n$. for each such row standard tableau $\mathfrak{t}$ we define $d(\mathfrak{t})$ to be the unique element of $\mathfrak{S}_n$ such that $\mathfrak{t} = \mathfrak{t}^\mu d(\mathfrak{t})$. These elements of $\mathfrak{S}_n$ are called distinguished coset representatives.

Example 1.5.2. Let $\mu = (2,2)$. Then a complete list of representatives are

$$\mathfrak{t}_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}, \mathfrak{t}_2 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}, \mathfrak{t}_3 = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 3 \\ \hline\end{array}, \mathfrak{t}_4 = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & 4 \\ \hline\end{array}, \mathfrak{t}_5 = \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline\end{array}, \mathfrak{t}_6 = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline\end{array}$$

and

$$d(\mathfrak{t}_1) = 1, d(\mathfrak{t}_2) = (2,3), d(\mathfrak{t}_3) = (2,3)(3,4), d(\mathfrak{t}_4) = (2,3)(1,2),$$

$$d(\mathfrak{t}_5) = (2,3)(1,2)(3,4), d(\mathfrak{t}_6) = (2,3)(1,2)(3,4)(2,3).$$

We come to state the main facts and results in this section. First, we define an analogue of the permutation modules M^μ of $\mathscr{H}$ as follows.

Definition 1.5.3. Suppose λ is a composition of n. Define

$$\begin{aligned} x_\lambda &= \sum_{w\in\mathfrak{S}_\lambda} T_w, \\ y_\lambda &= \sum_{w\in\mathfrak{S}_\lambda} (-q)^{-\ell(w)} T_w. \end{aligned}$$

We define $M^\lambda = x_\lambda\mathscr{H}$ to be the right $\mathscr{H}$–module generated by x_λ.

Example 1.5.4. Let $\lambda = (2,1)$. Then

$$x_\lambda = 1 + T_1, \text{and } y_\lambda = 1 - q^{-1}T_1.$$

Lemma 1.5.5. *[5, Lemma 3.2] Suppose λ is a composition of n. Then M^λ is a free R-module with basis $\{x_\lambda T_d \mid d \in \mathcal{D}_\lambda\}$. Moreover, if $d \in \mathcal{D}_\lambda$ and $\mathsf{s}_i = (i, i+1)$ for some $1 \le i < n$, then:*

$$x_\lambda T_d T_{\mathsf{s}_i} = \begin{cases} q x_\lambda T_d, & \text{if } i,\ i+1 \text{ belong to same row of } \mathfrak{t}^\lambda d, \\ x_\lambda T_{d\mathsf{s}_i}, & \text{if the row index of } i \text{ in } \mathfrak{t}^\lambda d \text{ is less than} \\ & \text{the row index of } i+1, \\ q x_\lambda T_{ds_i} + (q-1) x_\lambda T_d, & \text{otherwise.} \end{cases}$$

1.5.2 The Specht module of $\mathscr{H}$

The Specht module is contained in the permutation module $M^\lambda = x_\lambda\mathscr{H}$.

Definition 1.5.6. Suppose λ is partition of n. Define $c_\lambda \in \mathscr{H}$ by

$$c_\lambda = x_\lambda T_{w_\lambda} y_{\lambda'} = \sum_{u\in\mathfrak{S}_{\lambda'}} (-q)^{-\ell(u)} x_\lambda T_{w_\lambda u}.$$

Define the Specht module S^λ by $S^\lambda = c_\lambda\mathscr{H}$.

Example 1.5.7. Take $x_\lambda = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$, then $x_\lambda T_{w_\lambda} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$. Then

$$c_\lambda = x_\lambda T_{w_\lambda}(1 - q^{-1}T_1) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} - q^{-1}\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}.$$

Now we can write the basis of S^λ.

Lemma 1.5.8. *[5, Theorem 5.6] The Specht module S^λ is a F–free module with a basis*

$$\{c_\lambda T_d \mid \mathfrak{t}^\lambda w_\lambda d \text{ is standard}\}.$$

This basis is called the standard basis of S^λ.

Now we define the inner product.

Definition 1.5.9. Define the inner product $\langle , \rangle$ to be the unique bilinear form on M^λ such that:

$$\langle x_\lambda T_d, x_\lambda T_u \rangle = \begin{cases} q^{\ell(d)} & \text{if } d = u; \\ 0 & \text{otherwise} \end{cases}$$

where $d, u \in \mathcal{D}_\lambda$. This inner product extends linearly to M^λ, so $\langle , \rangle$ is a symmetric and non-degenerate bilinear form on M^λ.

Theorem 1.5.10. *[5, Lemma 4.9] Suppose λ is a partition of n. Then $S^\lambda/(S^\lambda \cap S^{\lambda^\perp})$ is zero or an absolutely irreducible self-dual $\mathscr{H}$–module.*

Definition 1.5.11. For each partition λ of n, define D^λ to be the right $\mathscr{H}$–module $S^\lambda/(S^\lambda \cap S^{\lambda^\perp})$.

From Theorem 1.5.10 we can state the following corollary

Corollary 1.5.12. *Suppose λ is a partition of n. Then either $D^\lambda = 0$ or D^λ is an absolutely irreducible and self-dual $\mathscr{H}$–module.*

Theorem 1.5.13. *[5, Theorem 7.7] Suppose λ is a partition of n and that $\mathscr{H}$ is semisimple. Then*

$$\{S^\mu \mid \ \mu \text{ is a partition of } n\}$$

is a complete set of non-isomorphic irreducible $\mathscr{H}$–modules.

Lemma 1.5.14. *Suppose λ is e–regular. Then $D^\lambda \neq 0$.*

Theorem 1.5.15. *[5, Theorem 7.6] Let F be a field. Then*

$$\{D^\lambda \mid \lambda \text{ a partition of } n \text{ and } \lambda \text{ is } e\text{–regular}\}$$

is a complete set of inequivalent irreducible $\mathscr{H}$-modules. All these irreducible $\mathscr{H}$–modules are self-dual. Moreover, if λ and μ are partitions of n with λ e–regular and D^λ is a composition factor of S^μ, then $\lambda \trianglerighteq \mu$, and D^λ occurs in S^λ with multiplicity 1.

Definition 1.5.16. Let λ and μ be partitions of n with μ e–regular. Then, we define $d_{\lambda\mu} = [\mathrm{S}^\lambda : D^\mu]$ to be the composition multiplicity of D^μ in S^λ. The matrix $D = (d_{\lambda\mu})$ is called the decomposition matrix of $\mathscr{H}$ and this matrix is upper unitriangular matrix and has the form

$$\begin{array}{cc} & D^\mu \\ S^\lambda & \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ & * & & \\ & & & \end{pmatrix} \end{array}$$

Note that: if $e = \infty$ then $\{S^\mu \mid \mu \text{ is a partition of } n\}$ are irreducible and the decomposition matrix is identify matrix.

Definition 1.5.17. Suppose $\mathscr{H} = B_1 \oplus \cdots \oplus B_s$, where each B_i is an indecomposable two-sided ideal. Then $B_1, \ldots, B_s$ are called the blocks of $\mathscr{H}$.

Lemma 1.5.18. *[26, Corollary 2.22] Every irreducible $\mathscr{H}$–module is a composition factor of exactly one block. Moreover, all composition factors of the Specht module S^λ lie in the same block. So we can say that two Specht modules lie in the block if their composition factors lie in the same block.*

Corollary 1.5.19. *Suppose λ and μ are partitions of n with μ e–regular. Then the Specht module S^λ can have a composition factor D^μ only if S^λ and S^μ lie in the same block.*

Recall that the definition of an e–core was given in Definition 1.3.25. Then

Theorem 1.5.20. *(The Nakayama conjecture). Let λ and μ be partitions of n. Then the $\mathscr{H}$–modules S^λ and S^μ belong to the same block of $\mathscr{H}$ if and only if λ and μ have the same e–core.*

Now we introduce some useful results about decomposition matrices and decomposition numbers. Recall Definition 1.3.29 for regularization, then we can state the following theorem.

Theorem 1.5.21. *[16, Theorem 6.21] Let λ and μ be partitions of n, with μ is an e–regular partition. Then*

- $[S^\lambda : D^{\lambda^R}] = 1$*;*
- $[S^\lambda : D^\mu] = 0$ *if* $\mu \not\trianglerighteq \lambda^R$.

Now we give theorems for computing decomposition matrices indexed by partitions of e–weight 0 and 1.

Theorem 1.5.22. *[16, Theorem 6.4]*

Let λ be a partition of n and λ be an e–core. Then $d_{\lambda\mu} = 0$ for every e–regular partition μ which is distinct from λ.

Theorem 1.5.23. *[16, Theorem 6.5]*

Suppose $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(e)}$ are partitions of weight 1 with e–core ν and that $\lambda^{(1)} \triangleright \lambda^{(2)} \triangleright \cdots \triangleright \lambda^{(e)}$. Then, for $1 \leq i < e$ $d_{\lambda^{(i+1)}\lambda^{(i)}} = 1$ and $d_{\lambda^{(i)}\lambda^{(j)}} = 0$ for $i \neq j, j-1$.

1.5.3 The LLT algorithm

Let q be a primitive e^{th} root of unity in $\mathbb{C}$ where $e \geq 2$. Then there exists a recursive algorithm that determines the decomposition matrices of the Iwahori-Hecke algebra $\mathscr{H}_{\mathbb{C},q}(\mathfrak{S}_n)$. The algorithm was first published in 1996 [20] where Lascoux, Leclerc and Thibon claimed to solve the decomposition matrices of $\mathscr{H}_{\mathbb{C},q}(\mathfrak{S}_n)$ but this claim was proved since in [2] by Ariki. This recursive algorithm is called now the LLT algorithm. This section shows how to calculate the decomposition matrices of $\mathscr{H}_{\mathbb{C},q}(\mathfrak{S}_n)$ using the LLT algorithm. Recall the Specht modules of $\mathscr{H}$ are indexed by partitions λ of n, also recall the irreducible modules D^μ, where μ is an e–regular partitions of n. Let $d_{\lambda\mu} = [S^\lambda : D^\mu]$ be the composition multiplicity of D^μ in S^λ and $D = (d_{\lambda\mu})$ be the decomposition matrix of $\mathscr{H}_{\mathbb{C},q}(\mathfrak{S}_n)$. Now we describe the LLT algorithm.

Definition 1.5.24. Let λ be a partition. A node x is said to be addable if $x \notin [\lambda]$ and $[\lambda \cup x]$ is the diagram of a partition and a node $x \in [\lambda]$ is said to be removable if $[\lambda \backslash x]$ is the diagram of a partition.

Now let v be an indeterminate over $\mathbb{C}$ and let $\mathcal{F}_v$ denote the $\mathbb{C}(v)$–vector space with basis the partitions of n, for all $n \geq 0$. Let λ, ν be partitions. If the e–residue diagram of ν is formed by adding s nodes to the e–residue diagram of λ, all of residue r, then we write $\lambda \xrightarrow{s:r} \nu$. Define

$$(\lambda)\uparrow_s^r = \sum_{\lambda \xrightarrow{s:r} \nu} v^{N_r(\lambda,\nu)}\nu$$

where

$$N_r(\lambda,\nu) = \sum_{\gamma \in [\nu]\setminus[\lambda]} (\#\{\gamma' : \gamma' \text{ is an addable } r\text{–node of } \nu \text{ above } \gamma\} - \\ \#\{\gamma' : \gamma' \text{ is a removable } r\text{–node of } \lambda \text{ above } \gamma\}).$$

We extend this definition linearly in order to define $B \uparrow_r^s$ for $B \in \mathcal{F}_v$. The algorithm works by calculating the crystallized decomposition matrix of$\mathscr{H}_{\mathbb{C},q}(\mathfrak{S}_n)$ which is a lower unitriangular matrix defined in [20] with the same structure as the decomposition matrix of $\mathscr{H}_{\mathbb{C},q}(\mathfrak{S}_n)$, but whose lower triangular entries are elements of $v\mathbb{N}[v]$. By Ariki's Theorem [2], the decomposition matrix of $\mathscr{H}_{\mathbb{C},q}(\mathfrak{S}_n)$ is then obtained by setting $v = 1$. Let $d_{\lambda\nu}(v)$ be the entry of the crystallized decomposition matrix in the row indexed by the partition λ and the column indexed by the e–regular partition ν. If ν is an e–regular partition of n, define $[B_c(\nu)] = \sum_{\lambda \vdash n} d_{\lambda\nu}(v)\lambda \in \mathcal{F}_v$, where we associate this with the column of the crystallized decomposition matrix indexed by ν. Since the LLT algorithm is recursive, we assume that we know $[B_c(\tau)]$ where τ is a partition of m and either $m < n$ or $m = n$ and $\nu \rhd \tau$. This is reasonable, since if $n = 1$ the crystallized decomposition matrix is simply the identity matrix. Now $[B_c(\nu)]$ is found by the LLT algorithm as follows.

1. Write the e–ladder diagram of ν and construct the partition τ by removing all nodes of maximal ladder number. Suppose there are s such nodes and they have common e-residue r.

2. Assume we know $[B_c(\tau)]$ and set $C_\nu = [B_c(\tau)]\uparrow_s^r$. Then C_ν is of the form

$$C_\nu = \sum_{\nu \unrhd \mu} c_{\mu\nu}(v)\mu = [B_c(\nu)] + \sum_{\nu \rhd \mu} \alpha_{\mu\nu}(v)[B_c(\mu)] \tag{1.1}$$

where $\alpha_{\mu\nu}(v) \in \mathbb{N}[v+v^{-1}]$ and $c_{\mu\nu}(v) \in \mathbb{N}[v,v^{-1}]$.

3. Find the largest partition, μ_0, such that $c_{\mu_0\nu}(v) \notin v\mathbb{N}[v]$ and $\nu \neq \mu_0$. If no such partition exists then $C_\nu = [B_c(\nu)]$ and we are done. Otherwise, $\alpha_{\mu_0\nu}(v)$ is the unique polynomial in $v+v^{-1}$ such that the coefficient of v^i in $\alpha_{\mu_0\nu}(v)$ is equal to the coefficient of v^i in $c_{\mu_0\nu}(v)$, for all $i \leq 0$. Replace C_ν by the element $C_\nu - \alpha_{\mu_0\nu}(v)[B_c(\mu_0)]$ and repeat step (3) until all the coefficients $c_{\mu\nu}(v)$ belong to $v\mathbb{N}[v]$ for all $\nu \rhd \mu$.

Example 1.5.25. Let $\nu = (4,2)$ and $e = 2$. We want to find $[B_c(\nu)]$ so

1. Find τ. The e–ladder diagram of ν is

$$\begin{ytableau} 0 & 1 & 2 & 3 \\ 1 & 2 \end{ytableau}$$

and the e–residue diagram of ν is

$$\begin{ytableau} 0 & 1 & 0 & 1 \\ 1 & 0 \end{ytableau}.$$

Then τ is obtained by removing one node of residue 1. Thus $\tau = (3,2)$.

2. We find that

$$[B_c(\tau)] = (3,2) + v(3,1^2) + v^2(2^2,1).$$

So we calculate $C_\nu = [B_c(\tau)] \uparrow_1^1$

$$\begin{ytableau} 0 & 1 & 0 \\ 1 & 0 \end{ytableau} + v\,\begin{ytableau} 0 & 1 & 0 \\ 1 \\ 0 \end{ytableau} + v^2\,\begin{ytableau} 0 & 1 \\ 1 & 0 \\ 0 \end{ytableau} \xrightarrow{1:1} \begin{ytableau} 0 & 1 & 0 & 1 \\ 1 & 0 \end{ytableau} + v\,\begin{ytableau} 0 & 1 & 0 \\ 1 & 0 & 1 \end{ytableau}$$
$$+ v\,\begin{ytableau} 0 & 1 & 0 & 1 \\ 1 \\ 0 \end{ytableau} + v^2\,\begin{ytableau} 0 & 1 & 0 \\ 1 \\ 0 \\ 1 \end{ytableau} + v^2\,\begin{ytableau} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{ytableau} + v^3\,\begin{ytableau} 0 & 1 \\ 1 & 0 \\ 0 \\ 1 \end{ytableau}.$$

Hence, C_ν has the right form so

$$[B_c(\nu)] = (4,2) + v(3^2) + v(4,1^2) + v^2(3,1^3) + v^2(2^3) + v^3(2^2,1^2).$$

Example 1.5.26. Let $\nu = (7)$ and $e = 2$. We want to find $[B_c(\nu)]$ so

1. Find τ. The e–ladder diagram of ν is

$$\begin{array}{|c|c|c|c|c|c|c|}\hline 0&1&2&3&4&5&6\\ \hline\end{array}$$

and the e–residue diagram of ν is

$$\begin{array}{|c|c|c|c|c|c|c|}\hline 0&1&0&1&0&1&0\\ \hline\end{array}.$$

Then τ is obtained by removing one node of residue 0. Thus $\tau = (6)$.

2. We find that

$$[B_c(\tau)] = (6) + v(5,1) + v(4,1^2) + v^2(3,1^3) + v^2(2,1^4) + v^3(1^6).$$

So we calculate $C_\nu = [B_c(\tau)] \uparrow_1^0$

$$\begin{array}{|c|c|c|c|c|c|}\hline 0&1&0&1&0&1\\ \hline\end{array} + v \begin{array}{|c|c|c|c|c|}\hline 0&1&0&1&0\\ \hline \multicolumn{1}{|c|}{1}&\multicolumn{4}{c}{}\\ \cline{1-1}\end{array} + v \begin{array}{|c|c|c|c|}\hline 0&1&0&1\\ \hline \multicolumn{1}{|c|}{1}&\multicolumn{3}{c}{}\\ \cline{1-1} \multicolumn{1}{|c|}{0}&\multicolumn{3}{c}{}\\ \cline{1-1}\end{array}$$

$$+ v^2 \begin{array}{|c|c|c|}\hline 0&1&0\\ \hline \multicolumn{1}{|c|}{1}&\multicolumn{2}{c}{}\\ \cline{1-1} \multicolumn{1}{|c|}{0}&\multicolumn{2}{c}{}\\ \cline{1-1} \multicolumn{1}{|c|}{1}&\multicolumn{2}{c}{}\\ \cline{1-1}\end{array} + v^2 \begin{array}{|c|c|}\hline 0&1\\ \hline \multicolumn{1}{|c|}{1}&\\ \cline{1-1} \multicolumn{1}{|c|}{0}&\\ \cline{1-1} \multicolumn{1}{|c|}{1}&\\ \cline{1-1} \multicolumn{1}{|c|}{0}&\\ \cline{1-1}\end{array} + v^3 \begin{array}{|c|}\hline 0\\ \hline 1\\ \hline 0\\ \hline 1\\ \hline 0\\ \hline 1\\ \hline\end{array}$$

$$\xrightarrow{1:0} \begin{array}{|c|c|c|c|c|c|c|}\hline 0&1&0&1&0&1&0\\ \hline\end{array} + \begin{array}{|c|c|c|c|c|}\hline 0&1&0&1&0\\ \hline \multicolumn{1}{|c|}{1}&\multicolumn{1}{c|}{0}&\multicolumn{3}{c}{}\\ \cline{1-2}\end{array} + v \begin{array}{|c|c|c|c|c|}\hline 0&1&0&1&0\\ \hline \multicolumn{1}{|c|}{1}&\multicolumn{4}{c}{}\\ \cline{1-1} \multicolumn{1}{|c|}{0}&\multicolumn{4}{c}{}\\ \cline{1-1}\end{array}$$

$$+ v \begin{array}{|c|c|c|c|c|}\hline 0&1&0&1&0\\ \hline \multicolumn{1}{|c|}{1}&\multicolumn{4}{c}{}\\ \cline{1-1} \multicolumn{1}{|c|}{0}&\multicolumn{4}{c}{}\\ \cline{1-1}\end{array} + v^2 \begin{array}{|c|c|c|c|}\hline 0&1&0&1\\ \hline \multicolumn{1}{|c|}{1}&\multicolumn{1}{c|}{0}&\multicolumn{2}{c}{}\\ \cline{1-2} \multicolumn{1}{|c|}{0}&\multicolumn{3}{c}{}\\ \cline{1-1}\end{array} + v \begin{array}{|c|c|c|}\hline 0&1&0\\ \hline \multicolumn{1}{|c|}{1}&\multicolumn{1}{c|}{0}&\\ \cline{1-2} \multicolumn{1}{|c|}{0}&\multicolumn{2}{c}{}\\ \cline{1-1} \multicolumn{1}{|c|}{1}&\multicolumn{2}{c}{}\\ \cline{1-1}\end{array}$$

$$+ v^2 \begin{array}{|c|c|c|}\hline 0&1&0\\ \hline \multicolumn{1}{|c|}{1}&\multicolumn{2}{c}{}\\ \cline{1-1} \multicolumn{1}{|c|}{0}&\multicolumn{2}{c}{}\\ \cline{1-1} \multicolumn{1}{|c|}{1}&\multicolumn{2}{c}{}\\ \cline{1-1} \multicolumn{1}{|c|}{0}&\multicolumn{2}{c}{}\\ \cline{1-1}\end{array} + v^2 \begin{array}{|c|c|c|}\hline 0&1&0\\ \hline \multicolumn{1}{|c|}{1}&\multicolumn{2}{c}{}\\ \cline{1-1} \multicolumn{1}{|c|}{0}&\multicolumn{2}{c}{}\\ \cline{1-1} \multicolumn{1}{|c|}{1}&\multicolumn{2}{c}{}\\ \cline{1-1} \multicolumn{1}{|c|}{0}&\multicolumn{2}{c}{}\\ \cline{1-1}\end{array} + v^3 \begin{array}{|c|c|}\hline 0&1\\ \hline 1&0\\ \hline \multicolumn{1}{|c|}{0}&\\ \cline{1-1} \multicolumn{1}{|c|}{1}&\\ \cline{1-1} \multicolumn{1}{|c|}{0}&\\ \cline{1-1}\end{array} + v^3 \begin{array}{|c|}\hline 0\\ \hline 1\\ \hline 0\\ \hline 1\\ \hline 0\\ \hline 1\\ \hline 0\\ \hline\end{array}.$$

Hence, C_ν is not of the right form because the partition $\mu = (5,2)$ has the coefficient $1 \notin v\mathbb{N}[v]$. Thus, $\alpha_{\mu\nu}(v) = 1$. So, we subtract $[B_c(5,2)]$ from C_ν and we get

$$\begin{aligned}&(7) + (5,2) + 2v(5,1^2) + v^2(4,2,1) + v(3,2,1^2) + 2v^2(3,1^4) + v^3(2^2,1^3) + v^3(1^7)\\&- [(5,2) + v(5,1^2) + v^2(4,2,1) + v(3,2,1^2) + v^2(3,1^4) + v^3(2^2,1^3)]\\&= (7) + v(5,1^2) + v^2(3,1^4) + v^3(1^7).\end{aligned}$$

Now this is of the correct form. Hence,

$$[B_c(\nu)] = (7) + v(5,1^2) + v^2(3,1^4) + v^3(1^7).$$

In Chapter 2, we assume $e = 2$ and use a slight variation of the LLT algorithm to compute decomposition numbers. If τ is an e–regular partition of m, then if $0 \le r < e$ and $s \ge 1$ we have

$$[B_c(\tau)] \uparrow_s^r = \sum_{\mu \vdash m+s} \alpha_{\mu\nu}(v)[B_c(\mu)]$$

where $\alpha_{\mu\nu}(v) \in \mathbb{N}[v + v^{-1}]$.

Lemma 1.5.27. *Suppose ν is an e–regular partition of m and that there exist $1 \le s$ and $0 \le r < e$ such that rows $1, 2, \dots, s$ of $[\nu]$ contain a removable r–node and the partition τ formed by removing these s nodes is e–regular. Then*

$$[B_c(\tau)] \uparrow_s^r = \sum_{\nu \trianglerighteq \mu} c_{\mu\nu}(v)\mu = [B_c(\nu)] + \sum_{\nu \triangleright \mu} \alpha_{\mu\nu}(v)[B_c(\mu)] \qquad (1.2)$$

where $\alpha_{\mu\nu}(v) \in \mathbb{N}[v + v^{-1}]$.

Proof. Since $[B_c(\tau)] \uparrow_s^r = \sum_{\mu \vdash m+s} \alpha_{\mu\nu}(v)[B_c(\mu)]$ it is sufficient to show that if $[B_c(\tau)] \uparrow_s^r = \sum_{\mu \vdash m+s} c_{\mu\nu}(v)\mu$, then $c_{\nu\nu}(v) = 1$ and $c_{\mu\nu}(v) = 0$ if $\nu \ntrianglerighteq \mu$. We have

$$[B_c(\tau)] = \sum_{\tau \triangleright \sigma} d_{\sigma\tau}(v)\tau,$$

where $d_{\tau\tau} = 1$. Then

$$\tau \uparrow_s^r = \nu + \sum_{\nu \triangleright \mu} \beta_{\mu\nu}(v)\mu$$

and if $\tau \triangleright \sigma$ then since ν is formed by adding nodes to the first s rows of τ then any partition $\tilde{\mu}$ formed by adding s nodes to σ will have the property that $\nu \triangleright \tilde{\mu}$. □

1.5.4 Adjustment matrices

Let F be a field of positive characteristic and recall $e \geq 2$ is the smallest positive integer such that $1 + q + ... + q^{e-1} = 0$ with $e = \infty$ if no such integer exists. In particular, if $e = \infty$ then the decomposition matrix is simply the identity matrix. Thus, suppose e is finite and assume $\mathscr{H}_0 = \mathscr{H}_{\mathbb{C},q}(\mathfrak{S}_n)$ where q is a primitive e^{th} root of unity in $\mathbb{C}$. Let D be the decomposition matrix of $\mathscr{H}_{F,q}(\mathfrak{S}_n)$ and D_0 be the decomposition matrix of $\mathscr{H}_0$ which can be computed by using the LLT algorithm [22].

Theorem 1.5.28. *There exists a square lower uni-triangular matrix A whose entries are non-negative integers such that*

$$D = D_0 A.$$

Then A is called an adjustment matrix.

Remark 1.5.29. Let μ be an e–regular partition and suppose $B(\mu)$ is the column of the matrix D and $B_0(\mu)$ is the column of the matrix D_0 both of them indexed by μ. If λ is an e–regular partition, then

$$B(\lambda) = B_0(\lambda) + \sum_{\lambda \rhd \nu} \alpha_{\nu\lambda} B_0(\nu)$$

where $\alpha_{\nu\lambda}(v) \in \mathbb{Z}_{\geq 0}$.

Theorem 1.5.30 ([3],Corollary 6.3)**.** *The decomposition matrix D depends only on e and char(F), not on the choice of q.*

In general the adjustment matrices are not known. However if part of the decomposition matrix of $\mathscr{H}$ is known, we can use it to find part of the adjustment matrix. The next example comes from [16, Appendix 1].

Example 1.5.31. Consider $e = 2$ and $n = 5$. Assume we known D_0 by the

LLT algorithm which is

$$\begin{array}{c c} & \begin{array}{ccc} (5) & (4,1) & (3,2) \end{array} \\ \begin{array}{l} (5) \\ (4,1) \\ (3,2) \\ (3,1^2) \\ (2^2,1) \\ (2,1^3) \\ (1^5) \end{array} & \left(\begin{array}{ccc} 1 & . & . \\ . & 1 & . \\ . & . & 1 \\ 1 & . & 1 \\ . & . & 1 \\ . & 1 & . \\ 1 & . & . \end{array}\right). \end{array}$$

The adjustment matrix A is

$$\begin{array}{c c} \begin{array}{l} (5) \\ (4,1) \\ (3,2) \end{array} & \left(\begin{array}{ccc} 1 & . & . \\ . & 1 & . \\ 1 & . & 1 \end{array}\right). \end{array}$$

The decomposition matrix $D = D_0 A$ is

$$\begin{array}{c c} & \begin{array}{ccc} (5) & (4,1) & (3,2) \end{array} \\ \begin{array}{l} (5) \\ (4,1) \\ (3,2) \\ (3,1^2) \\ (2^2,1) \\ (2,1^3) \\ (1^5) \end{array} & \left(\begin{array}{ccc} 1 & . & . \\ . & 1 & . \\ 1 & . & 1 \\ 2 & . & 1 \\ 1 & . & 1 \\ . & 1 & . \\ 1 & . & \end{array}\right). \end{array}$$

Chapter 2

Some results for decomposition numbers for $\mathscr{H}_{\mathbb{C},-1}(\mathfrak{S}_n)$

2.1 The decomposition numbers for all partitions of the form $\lambda = (a, c, 1^b)$.

2.1.1 Notation

Throughout this section we assume that $e = 2$ and that $F = \mathbb{C}$, that is $\mathscr{H} = \mathscr{H}_{\mathbb{C},-1}(\mathfrak{S}_n)$. The work in this chapter will appear in [1].

Definition 2.1.1. Define Γ to be the set:

$$\Gamma := \{(a, c, 1^b) \mid (a, c, 1^b) \text{ is a partition of some integer } n\ \}$$

where a, b, c are positive integers.

In this section, we write $a \equiv c$ to denote $a \equiv c \bmod 2$ for all $a,\ c \geq 0$.

Definition 2.1.2. Let λ be a partition of n. Then $\ell(\lambda)$ denotes the number of non–zero parts of λ. We define the set:

$$\Gamma^R = \{\nu \text{ is a partition} \mid \nu \text{ is 2–regular and either } \ell(\nu) = 2 \text{ or } \ell(\nu) = 3\}.$$

The Specht modules that we consider in this section will be labelled by partitions λ such that $\lambda \in \Gamma$.

Definition 2.1.3. If ν is 2–regular, then we define $[B_c(\nu)]_\Gamma = \sum_{\substack{\lambda \vdash n \\ \lambda \in \Gamma}} d_{\lambda\nu}(v)\lambda$ which we associate with the column of the crystallized decomposition matrix indexed by ν corresponding to only the rows indexed by partitions in Γ.

Lemma 2.1.4. *Suppose ν is a partition of n with ν 2–regular and that $\ell(\nu) \geq 4$. Then $[B_c(\nu)]_\Gamma = 0$.*

Proof. Let $\lambda \in \Gamma$ and suppose σ is a 2–regular partition of n. Now from Theorem 1.5.21 the decomposition numbers $d_{\lambda\sigma} = [S^\lambda : D^\sigma]$ can only be non–zero when $\sigma \trianglerighteq \lambda^R$. Since λ^R has the form either $\lambda^R = (x', y', z')$, $\lambda^R = (x', y')$ or $\lambda^R = (x')$ for some x', y', z' then if $\sigma \trianglerighteq \lambda^R$ then σ has the form $\sigma = (x)$ or $\sigma = (x, y)$ or $\sigma = (x, y, z)$ for some x, y, z. $\square$

2.1.2 Statement of main theorems

Our main results are the following theorems:

Theorem A. *Suppose $\nu = (x, y, z) \in \Gamma^R$. Then*

1. *Suppose $x \equiv y \not\equiv z$. Then*

$$[B_c(x,y,z)]_\Gamma = \sum_{f=0}^{\frac{y-z-1}{2}} \sum_{k=0}^{\frac{x-y-2}{2}} v^{\lfloor \frac{y-2f+2k+1}{2} \rfloor}(x-2k-1, z+2+2f, 1^{y-2f+2k-1}) + \sum_{f=0}^{\frac{y-z-1}{2}} \sum_{k=0}^{x-y-1} \alpha_k (x-k, z+1+2f, 1^{y-2f+k-1})$$

$$\text{where } \alpha_k = \begin{cases} v^{\lfloor \frac{y-2f+k-1}{2} \rfloor}, & \text{if } k \text{ even,} \\ v^{\lfloor \frac{y-2f+k+3}{2} \rfloor}, & \text{if } k \text{ odd.} \end{cases}$$

2. *Suppose $x \not\equiv y \not\equiv z$. Then*

$$[B_c(x,y,z)]_\Gamma = \sum_{f=0}^{\frac{y-z-1}{2}} \sum_{k=0}^{\frac{x-y-1}{2}} v^{\lfloor \frac{y-2f+2k-1}{2} \rfloor}(x-2k, z+1+2f, 1^{y-2f+2k-1}).$$

3. *Suppose $x \not\equiv y \equiv z$. Then*

$$[B_c(x,y,z)]_\Gamma = \sum_{f=0}^{\frac{y-z-2}{2}} \sum_{k=0}^{x-y} \alpha_k (x-k, z+2+2f, 1^{y-2f+k-2})$$
$$+ \sum_{f=0}^{\frac{y-z-2}{2}} \sum_{k=0}^{\frac{x-y-1}{2}} v^{\lfloor \frac{y-2f+2k+1}{2} \rfloor}(x-2k, z+1+2f, 1^{y-2f+2k-1})$$

$$\text{where } \alpha_k = \begin{cases} v^{\lfloor \frac{y-2f+k-2}{2} \rfloor}, & \text{if } k \text{ even,} \\ v^{\lfloor \frac{y-2f+k+2}{2} \rfloor}, & \text{if } k \text{ odd.} \end{cases}$$

4. *Suppose $x \equiv y \equiv z$. Then*

$$[B_c(x,y,z)]_\Gamma = \sum_{f=0}^{\frac{y-z-2}{2}} \sum_{k=0}^{\frac{x-y-2}{2}} v^{\lfloor \frac{y-2f+2k-2}{2} \rfloor}(x-2k, z+2+2f, 1^{y-2f+2k-2})$$
$$+ \sum_{f=0}^{\frac{y-z-2}{2}} \sum_{k=0}^{\frac{x-y-2}{2}} v^{\lfloor \frac{y-2f+2k+4}{2} \rfloor}(x-2k-2, z+2+2f, 1^{y-2f+2k})$$
$$+ \sum_{f=0}^{\frac{y-z-4}{2}} \sum_{k=0}^{x-y} v^{\lfloor \frac{y-2f+k-1}{2} \rfloor}(x-k, z+3+2f, 1^{y-2f+k-3})$$
$$+ \sum_{f=0}^{\frac{y-z-2}{2}} \sum_{k=0}^{x-y-2} v^{\lfloor \frac{y-2f+k}{2} \rfloor}(x-k-1, z+3+2f, 1^{y-2f+k-2})$$
$$+ \sum_{k=0}^{x-y-1} v^{\lfloor \frac{y+k+1}{2} \rfloor}(x-k, z+1, 1^{y-1+k}).$$

Theorem A describes $[B_c(\nu)]_\Gamma$, where $\ell(\nu) = 2$ or $\ell(\nu) = 3$. The case where $\ell(\nu) = 1$ was considered by James and Mathas [5, Theorem 3.2], [26, p112].

Theorem 2.1.5. *[5, Theorem 3.2] Suppose $e = 2$ and that ν is 2–regular and let $\nu = (x)$. Then*

- *If x is odd, then*

$$[B_c(x)]_\Gamma = \sum_{k=0}^{\frac{x-1}{2}} v^k (x-2k, 1^{2k}).$$

- *If x is even, then*

$$[B_c(x)]_\Gamma = \sum_{k=0}^{x-1} v^{\lfloor \frac{k+1}{2} \rfloor}(x-k, 1^k).$$

Definition 2.1.6. Let λ be a partition. We say that λ is a *hook* partition if λ has the form $(n-j, 1^j)$ where $0 \le j < n$.

The next theorem describes the decomposition numbers corresponding to rows labelled by hook partitions, where $e = 2$.

Theorem 2.1.7. *[4, Theorem 1] Suppose $e = 2$ and $\alpha_j^n = (n-j, 1^j)$ is a hook partition. Let $(\alpha_i^n)^R = (n-i, i)$ be the regularization of α_i^n. Then, for $0 \le j \le n-1$*

$$d_{\alpha_j^n (\alpha_i^n)^R}(v) = \begin{cases} v^{\lfloor \frac{j}{2} \rfloor}, & \text{if } i \le j < n-i \text{ and } j \equiv i, \\ v^{\frac{j+1}{2}}, & \text{if } i < j < n-i \text{ and } j \not\equiv i \equiv n \equiv 0, \\ v^{\frac{j}{2}+1}, & \text{if } i < j < n-i \text{ and } j \not\equiv i \not\equiv n \equiv 0. \end{cases}$$

Furthermore, $d_{\alpha_j^n \nu}(v) = 0$ for all other 2–regular partitions ν.

Combining Lemma 2.1.4, Theorem A and Theorem 2.1.5 we obtain the second result which gives the composition factors of S^λ for $\lambda \in \Gamma$.

Definition 2.1.8. If λ is a partition, define $B_r(\lambda) = \sum_{\nu \text{ 2-regular}} d_{\lambda\nu}(v)\nu$ which we associate with the row of the crystallized decomposition matrix corresponding to S^λ.

Theorem B. *Suppose $\lambda = (a, c, 1^b) \in \Gamma$ and λ is not a hook partition. Then*

1. *Suppose $a \equiv b \not\equiv c$. Then*

$$B_r(\lambda) = \sum_{\substack{y=c \\ y \equiv c}}^{a-1} \sum_{\substack{x=max\{a, a+b-y+1\} \\ x \not\equiv c}}^{n-y} v^{\lfloor \frac{b}{2} \rfloor}(x, y, n-x-y).$$

2. *Suppose $a \equiv b \equiv c$. Then*

$$B_r(\lambda) = \sum_{\substack{y=c-1\\ y\not\equiv c}}^{a-1} \sum_{\substack{x=max\{a+1,a+b-y+2\}\\ x\not\equiv c}}^{n-y} v^{\lfloor \frac{b+2}{2} \rfloor}(x,y,n-x-y)$$
$$+ \sum_{\substack{y=c+1\\ y\not\equiv c}}^{a-1} \sum_{\substack{x=max\{a,a+b-y+1\}\\ x\equiv c}}^{n-y} v^{\lfloor \frac{b+2}{2} \rfloor}(x,y,n-x-y)$$
$$+ \sum_{\substack{y=c\\ y\equiv c}}^{a-2} \sum_{\substack{x=max\{a,a+b-y+2\}\\ x\equiv c}}^{n-y} v^{\lfloor \frac{b}{2} \rfloor}(x,y,n-x-y)$$
$$+ \sum_{\substack{y=c\\ y\equiv c}}^{a} \sum_{\substack{x=max\{a+2,a+b-y+2\}\\ x\equiv c}}^{n-y} v^{\lfloor \frac{b+4}{2} \rfloor}(x,y,n-x-y).$$

3. *Suppose $a \not\equiv b$. Then*

$$B_r(\lambda) = \sum_{\substack{y=c\\ y\equiv c}}^{a-1} \sum_{\substack{x=max\{a,a+b-y+1\}\\ x\equiv c}}^{n-y} \alpha_b(x,y,n-x-y)$$
$$+ \sum_{\substack{y=c\\ y\equiv c}}^{a} \sum_{\substack{x=max\{a,a+b-y+2\}\\ x\not\equiv c}}^{n-y} \beta_b(x,y,n-x-y)$$
$$+ \sum_{\substack{y=c+1\\ y\not\equiv c}}^{a} \sum_{\substack{x=max\{a,a+b-y+3,y+1\}\\ x\not\equiv c}}^{n-y} v^{\lfloor \frac{b+2}{2} \rfloor}(x,y,n-x-y)$$
$$+ \sum_{\substack{y=c-1\\ y\not\equiv c}}^{a-1} \sum_{\substack{x=max\{a+1,a+b-y+3\}\\ x\not\equiv c}}^{n-y} v^{\lfloor \frac{b+2}{2} \rfloor}(x,y,n-x-y)$$
$$+ \sum_{\substack{y=c+1\\ y\not\equiv c}}^{min\{a-1,b+1\}} v^{\lfloor \frac{b+2}{2} \rfloor}(a+b-y+1,y,c-1)$$

$$\textit{where } \alpha_b = \begin{cases} v^{\lfloor \frac{b+4}{2} \rfloor}, & \textit{if } c \not\equiv a, \\ v^{\lfloor \frac{b}{2} \rfloor}, & \textit{if } c \equiv a, \end{cases}$$

$$\textit{where } \beta_b = \begin{cases} v^{\lfloor \frac{b}{2} \rfloor}, & \textit{if } c \not\equiv a, \\ v^{\lfloor \frac{b+4}{2} \rfloor}, & \textit{if } c \equiv a. \end{cases}$$

Combining Theorem B and Theorem 2.1.7, we obtain the decomposition numbers for Specht modules S^λ where $\lambda \in \Gamma$.

2.2 Proof of Main theorems

2.2.1 Proof of Theorem A

Before proving Theorem A, we state some relevant results.

Theorem 2.2.1. *(Row and column removal)[5, Theorem 1.2] Suppose $\lambda = (\lambda_1, ..., \lambda_i)$ and $\nu = (\nu_1, ..., \nu_j)$ are two partitions of n and ν is 2–regular.*

1. *If $\lambda_1 = \nu_1$ then $d_{\lambda\nu}(v) = d_{(\lambda_2,.....,\lambda_i)(\nu_2,....,\nu_j)}(v)$.*
2. *If $i = j$ then $d_{\lambda\nu}(v) = d_{(\lambda_1-1,.....,\lambda_i-1)(\nu_1-1,....,\nu_j-1)}(v)$.*

Let $[B_c(\nu)]_3 = \sum_{\ell(\lambda)\leq 3} d_{\lambda\nu}(v)\lambda \in \mathcal{F}_v$ which we identify with the portion of the column of the crystallized decomposition matrix indexed by ν corresponding to only the rows containing partitions with at most 3 parts.

Proposition 2.2.2. *[23, Theorem 3.1]*

Suppose $e = 2$ and that $\nu = (\nu_1, \nu_2)$ is a two part 2–regular partition of n. If ν_1 is odd and ν_2 is even, then for $\nu_2 \geq 2$

$$[B_c(\nu)]_3 = (\nu_1, \nu_2) + v(\nu_1, \nu_2 - 1, 1) + v^2(\nu_1 - 1, \nu_2, 1).$$

Proof of Theorem A We prove this theorem by induction on n and on the dominance order $\trianglerighteq$. Theorem A is trivially true for $n = 0, 1$. So suppose $\nu \in \Gamma^R$ is a partition of n where $n \geq 2$ and that Theorem A holds for all partitions $\sigma \in \Gamma^R$ where $\sigma \vdash m<n$ or $\sigma \vdash n$ and $\nu \triangleright \sigma$. Suppose that $0 \leq r<e$ and that for some $s \geq 1$, the first s rows of $[\nu]$ have a removable r–node. Let τ be the partition whose diagram is formed by removing these s nodes from $[\nu]$. By the induction hypothesis, we know $[B_c(\tau)]_\Gamma$. Note that if $\lambda \in \Gamma$ and $\lambda_0 \xrightarrow{s:r} \lambda$ then $\lambda_0 \in \Gamma$. Hence to find $[B_c(\nu)]_\Gamma$ we first want to consider $[B_c(\tau)]_\Gamma \uparrow_s^r$.

Definition 2.2.3. Suppose ν is a partition of n. Define $\Xi = \Xi(\nu)$ to be the set

$$\{\mu \vdash n \mid \nu \triangleright \mu,\ \mu \text{ is 2–regular},\ \mu \text{ lies in the same block as } \nu,\ \ell(\mu) \leq 3\}.$$

From equation 1.2, we have

$$[B_c(\tau)]_\Gamma \uparrow_s^r = [B_c(\nu)]_\Gamma + \sum_{\nu \rhd \mu} \alpha_{\mu\nu}(v)[B_c(\mu)]_\Gamma$$

where $\alpha_{\mu\nu}(v) \in \mathbb{N}[v + v^{-1}]$. We look for the coefficients $c_{\mu\nu}(v)$ of μ in $[B_c(\tau)]_\Gamma \uparrow_s^r$, where $\mu \in \Xi$. If $c_{\mu\nu}(v) \in v\mathbb{N}[v]$ for all $\mu \in \Xi$, then $[B_c(\nu)]_\Gamma = [B_c(\tau)]_\Gamma \uparrow_s^r$. Otherwise, we find the largest partition $\mu_0 \in \Xi$ such that $c_{\mu_0\nu}(v) \notin v\mathbb{N}[v]$ and replace $\sum_{\nu \unrhd \mu} c_{\mu\nu}(v)\mu$ by $\sum_{\nu \unrhd \mu} c_{\mu\nu}(v)\mu - \alpha_{\mu_0\nu}(v)[B_c(\mu_0)]_\Gamma$. We repeat until all coefficients $c_{\mu\nu}(v)\mu \in v\mathbb{N}[v]$ for all $\mu \in \Xi$. Now, we have 4 cases to consider. Throughout we assume $\nu = (x, y, z) \in \Gamma^R$, that is ν is 2–regular and $\ell(\nu) = 2$ or $\ell(\nu) = 3$.

Case 1: $x \equiv y \not\equiv z$

The Young diagram of ν is:

$$[\nu] = \begin{array}{|c|c|c|} \hline 0 & \cdots\cdots\cdots\cdots\cdots\cdots & i \\ \hline 1 & \cdots\cdots\cdots\cdots & j \\ \hline 0 & \cdots\cdots & j \\ \hline \end{array}$$

$$\text{where } i = \begin{cases} 0, & \text{if } z \text{ even,} \\ 1, & \text{if } z \text{ odd,} \end{cases}$$

$$\text{where } j = \begin{cases} 1, & \text{if } z \text{ even,} \\ 0, & \text{if } z \text{ odd.} \end{cases}$$

Let τ be the partition obtained by removing the highest i–node from ν. Hence τ is obtained by removing one node of residue i. So

$$\tau = (x - 1, y, z).$$

By the induction hypothesis

$$[B_c(x-1, y, z)]_\Gamma = \sum_{\tilde{f}=0}^{\frac{y-z-1}{2}} \sum_{\tilde{k}=0}^{\frac{x-y-2}{2}} v^{\lfloor \frac{y-2\tilde{f}+2\tilde{k}-1}{2} \rfloor}(x - 2\tilde{k} - 1, z + 1 + 2\tilde{f}, 1^{y-2\tilde{f}+2\tilde{k}-1}).$$

Now we find $[B_c(\tau)]_\Gamma \uparrow_1^i$. To do this, we add one node of residue i to each partition $(x-2\tilde{k}-1, z+1+2\tilde{f}, 1^{y-2\tilde{f}+2\tilde{k}-1})$ in all ways such that we obtain a partition which appears in $[B_c(\nu)]_\Gamma$.

$$\begin{array}{|c|c|c|c|}\hline 0 & 1 & \cdots\cdots & j \\ \hline 1 & \cdots & j & \\ \hline \vdots & & & \\ \hline j & & & \\ \hline \end{array} \xrightarrow{1:i} v^0 \begin{array}{|c|c|c|c|c|}\hline 0 & 1 & \cdots\cdots & j & i \\ \hline 1 & \cdots & j & & \\ \hline \vdots & & & & \\ \hline j & & & & \\ \hline \end{array} + v \begin{array}{|c|c|c|c|c|}\hline 0 & 1 & \cdots\cdots & & j \\ \hline 1 & \cdots & j & i & \\ \hline \vdots & & & & \\ \hline j & & & & \\ \hline \end{array}$$

$$+ v^{2+i} \begin{array}{|c|c|c|c|}\hline 0 & 1 & \cdots\cdots & j \\ \hline 1 & \cdots & j & \\ \hline \vdots & & & \\ \hline j & & & \\ \hline i & & & \\ \hline \end{array} .$$

Hence

$$\begin{aligned}
[B_c(\tau)]_\Gamma \uparrow_1^i &= \sum_{\tilde{f}=0}^{\frac{y-z-1}{2}} \sum_{\tilde{k}=0}^{\frac{x-y-2}{2}} v^{\lfloor \frac{y-2\tilde{f}+2\tilde{k}-1}{2} \rfloor} (x-2\tilde{k}, z+1+2\tilde{f}, 1^{y-2\tilde{f}+2\tilde{k}-1}) \\
&\quad + \sum_{\tilde{f}=0}^{\frac{y-z-1}{2}} \sum_{\tilde{k}=0}^{\frac{x-y-2}{2}} v^{\lfloor \frac{y-2\tilde{f}+2\tilde{k}+1}{2} \rfloor} (x-2\tilde{k}-1, z+2+2\tilde{f}, 1^{y-2\tilde{f}+2\tilde{k}-1}) \\
&\quad + \sum_{\tilde{f}=0}^{\frac{y-z-1}{2}} \sum_{\tilde{k}=0}^{\frac{x-y-2}{2}} v^{\lfloor \frac{y-2\tilde{f}+2\tilde{k}+3+2i}{2} \rfloor} (x-2\tilde{k}-1, z+1+2\tilde{f}, 1^{y-2\tilde{f}+2\tilde{k}}) \\
&= \sum_{\tilde{f}=0}^{\frac{y-z-1}{2}} \sum_{\tilde{k}=0}^{\frac{x-y-2}{2}} v^{\lfloor \frac{y-2\tilde{f}+2\tilde{k}+1}{2} \rfloor} (x-2\tilde{k}-1, z+2+2\tilde{f}, 1^{y-2\tilde{f}+2\tilde{k}-1}) \\
&\quad + \sum_{\tilde{f}=0}^{\frac{y-z-1}{2}} \sum_{\tilde{k}=0}^{x-y-1} \alpha_{\tilde{k}} (x-\tilde{k}, z+1+2\tilde{f}, 1^{y-2\tilde{f}+\tilde{k}-1})
\end{aligned}$$

$$\text{where } \alpha_{\tilde{k}} = \begin{cases} v^{\lfloor \frac{y-2\tilde{f}+\tilde{k}-1}{2} \rfloor}, & \text{if } \tilde{k} \text{ even}, \\ v^{\lfloor \frac{y-2\tilde{f}+\tilde{k}+3}{2} \rfloor}, & \text{if } \tilde{k} \text{ odd}. \end{cases}$$

Note that if $l = 2\tilde{k}+1$, then

$$\lfloor \frac{y-2\tilde{f}+2\tilde{k}+3+2i}{2} \rfloor = \lfloor \frac{y-2\tilde{f}+l+3}{2} \rfloor$$

since,

$$i = \begin{cases} 0, & \text{if } y \text{ odd,} \\ 1, & \text{if } y \text{ even.} \end{cases}$$

Now we show that in this case $[B_c(\nu)]_\Gamma = [B_c(\tau)]_\Gamma \uparrow_1^i$. Let $\mu \in \Xi$. We look for the coefficient of μ in $[B_c(\tau)] \uparrow_1^i$. If $\mu \in \Xi$, then μ has one of the following three forms:

$$[\mu_1] = \begin{array}{|l|l|l|l|} \hline 0 & \cdots\cdots\cdots\cdots\cdots\cdots & j & i \\ \hline 1 & \cdots\cdots\cdots\cdots & j & \\ \hline 0 & \cdots\cdots & j & \\ \hline \end{array} \quad \text{or} \quad [\mu_2] = \begin{array}{|l|l|l|l|} \hline 0 & \cdots\cdots\cdots\cdots\cdots\cdots\cdots & j & \\ \hline 1 & \cdots\cdots\cdots\cdots & j & i \\ \hline 0 & \cdots\cdots & j & \\ \hline \end{array}$$

$$\text{or} \quad [\mu_3] = \begin{array}{|l|l|l|l|} \hline 0 & \cdots\cdots\cdots\cdots\cdots\cdots\cdots & j & \\ \hline 1 & \cdots\cdots\cdots\cdots\cdots & j & \\ \hline 0 & \cdots\cdots & j & i \\ \hline \end{array} \quad .$$

For each μ_k, where $1 \le k \le 3$, there is a unique partition σ_k such that σ_k is obtained by removing an i–node from μ_k.

Each partition μ_1 comes from a partition σ_1 such that:

$$[\sigma_1] = \begin{array}{|l|l|l|} \hline 0 & \cdots\cdots\cdots\cdots\cdots\cdots & j \\ \hline 1 & \cdots\cdots\cdots\cdots & j \\ \hline 0 & \cdots\cdots & j \\ \hline \end{array} \xrightarrow{1:i} v^0 \begin{array}{|l|l|l|l|} \hline 0 & \cdots\cdots\cdots\cdots\cdots\cdots & j & i \\ \hline 1 & \cdots\cdots\cdots\cdots & j & \\ \hline 0 & \cdots\cdots & j & \\ \hline \end{array} \, .$$

Thus the coefficient of μ_1 in $[B_c(\tau)] \uparrow_1^i$ is the same as the coefficient of σ_1 in $[B_c(\tau)]$ which is in $v\mathbb{N}[v]$. Each partition μ_2 comes from a partition σ_2 such that:

$$[\sigma_2] = \begin{array}{|l|l|l|} \hline 0 & \cdots\cdots\cdots\cdots\cdots\cdots & j \\ \hline 1 & \cdots\cdots\cdots\cdots & j \\ \hline 0 & \cdots\cdots & j \\ \hline \end{array} \xrightarrow{1:i} v \begin{array}{|l|l|l|l|} \hline 0 & \cdots\cdots\cdots\cdots\cdots\cdots\cdots & j & \\ \hline 1 & \cdots\cdots\cdots\cdots & j & i \\ \hline 0 & \cdots\cdots & j & \\ \hline \end{array} \, .$$

Thus the coefficient of μ in $[B_c(\tau)] \uparrow_1^i$ is the coefficient of σ_2 in $[B_c(\tau)]$ multiplied by v which is in $v\mathbb{N}[v]$. Each partition μ_3 comes from a partition σ_3 such that:

$$[\sigma_3] = \begin{array}{|l|l|l|} \hline 0 & \cdots\cdots\cdots\cdots\cdots\cdots & j \\ \hline 1 & \cdots\cdots\cdots\cdots & j \\ \hline 0 & \cdots\cdots & j \\ \hline \end{array} \xrightarrow{1:i} v^2 \begin{array}{|l|l|l|l|} \hline 0 & \cdots\cdots\cdots\cdots\cdots\cdots\cdots & j & \\ \hline 1 & \cdots\cdots\cdots\cdots\cdots & j & \\ \hline 0 & \cdots\cdots & j & i \\ \hline \end{array} \, .$$

Thus the coefficient of μ_3 in $[B_c(\tau)] \uparrow_1^i$ is the coefficient of σ_3 in $[B_c(\tau)]$

multiplied by v^2 which is in $v\mathbb{N}[v]$. Since $c_{\mu\nu}(v) \in v\mathbb{N}[v]$, for all $\mu \in \Xi$, we have $[B_c(\nu)]_\Gamma = [B_c(\tau)]_\Gamma \uparrow_1^i$ as required. Hence, we have shown that Theorem A holds for all partitions described in Case 1.

Case 2: $x \not\equiv y \not\equiv z$

First, suppose that $z > 0$. The Young diagram of ν is:

$$[\nu] = \begin{array}{|c|l|c|} \hline 0 & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & i \\ \hline 1 & \cdots\cdots\cdots\cdots\cdots & i \\ \hline 0 & \cdots\cdots & i \\ \hline \end{array}$$

$$\text{where } i = \begin{cases} 1, & \text{if } z \text{ even,} \\ 0, & \text{if } z \text{ odd.} \end{cases}$$

Let τ be the partition obtained by removing three i–nodes from ν. So

$$\tau = (x-1, y-1, z-1).$$

By the induction hypothesis

$$[B_c(x-1,y-1,z-1)]_\Gamma = \sum_{\tilde{f}=0}^{\frac{y-z-1}{2}} \sum_{\tilde{k}=0}^{\frac{x-y-1}{2}} v^{\lfloor \frac{y-2\tilde{f}+2\tilde{k}-2}{2} \rfloor}(x-2\tilde{k}-1, z+2\tilde{f}, 1^{y-2\tilde{f}+2\tilde{k}-2}).$$

Now we find $[B_c(\tau)]_\Gamma \uparrow_3^i$. To do this, we add three nodes of residue i to each partition $(x-2\tilde{k}-1, z+2\tilde{f}, 1^{y-2\tilde{f}+2\tilde{k}-2})$ in all ways such that we obtain a partition which appears in $[B_c(\nu)]_\Gamma$.

$$\begin{array}{|c|c|l|c|} \hline 0 & 1 & \cdots\cdots\cdots & j \\ \hline 1 & \multicolumn{1}{l|}{\cdots\cdots} & j \\ \cline{1-3} \vdots \\ \cline{1-1} j \\ \cline{1-1} \end{array} \xrightarrow{3:i} v^i \begin{array}{|c|c|l|c|c|} \hline 0 & 1 & \cdots\cdots\cdots\cdots & j & i \\ \hline 1 & \multicolumn{1}{l|}{\cdots\cdots} & j & i \\ \cline{1-4} \vdots \\ \cline{1-1} j \\ \cline{1-1} i \\ \cline{1-1} \end{array}$$

$$\text{where } j = \begin{cases} 0, & \text{if } z \text{ even,} \\ 1, & \text{if } z \text{ odd.} \end{cases}$$

Hence

$$\begin{aligned}
[B_c(\tau)]_\Gamma \uparrow_3^i &= \sum_{\tilde{f}=0}^{\frac{y-z-1}{2}} \sum_{\tilde{k}=0}^{\frac{x-y-1}{2}} v^{\lfloor \frac{y-2\tilde{f}+2\tilde{k}-2+2i}{2} \rfloor} (x-2\tilde{k}, z+1+2\tilde{f}, 1^{y-2\tilde{f}+2\tilde{k}-1}) \\
&= \sum_{\tilde{f}=0}^{\frac{y-z-1}{2}} \sum_{\tilde{k}=0}^{\frac{x-y-1}{2}} v^{\lfloor \frac{y-2\tilde{f}+2\tilde{k}-1}{2} \rfloor} (x-2\tilde{k}, z+1+2\tilde{f}, 1^{y-2\tilde{f}+2\tilde{k}-1})
\end{aligned}$$

since

$$i = \begin{cases} 0, & \text{if } y \text{ even}, \\ 1, & \text{if } y \text{ odd}. \end{cases}$$

Now we show that in this case $[B_c(\nu)]_\Gamma = [B_c(\tau)]_\Gamma \uparrow_3^i$. Let $\mu \in \Xi$. We look for the coefficient of μ in $[B_c(\tau)] \uparrow_3^i$. If $\mu \in \Xi$, then μ has the form:

$$\begin{array}{|c|c|c|c|}
\hline
0 & \cdots\cdots\cdots\cdots\cdots\cdots & j & i \\
\hline
1 & \cdots\cdots\cdots\cdots & j & i \\
\hline
0 & \cdots & j & i \\
\hline
\end{array}.$$

Now we see that, for $\mu \in \Xi$ there is a unique σ such that σ is obtained by removing three i-nodes from μ. The partition μ comes from a partition σ such that

$$[\sigma] = \begin{array}{|c|c|c|}
\hline
0 & \cdots\cdots\cdots\cdots\cdots & j \\
\hline
1 & \cdots\cdots\cdots & j \\
\hline
0 & \cdots & j \\
\hline
\end{array} \xrightarrow{3:i} v^0 \begin{array}{|c|c|c|c|}
\hline
0 & \cdots\cdots\cdots\cdots\cdots\cdots & j & i \\
\hline
1 & \cdots\cdots\cdots\cdots & j & i \\
\hline
0 & \cdots & j & i \\
\hline
\end{array}.$$

Thus the coefficient of μ in $[B_c(\tau)] \uparrow_3^i$ is the same as the coefficient of σ in $[B_c(\tau)]$ which is in $v\mathbb{N}[v]$. Since $c_{\mu\nu}(v) \in v\mathbb{N}[v]$, for all $\mu \in \Xi$, we have $[B_c(\nu)]_\Gamma = [B_c(\tau)]_\Gamma \uparrow_3^i$.

Secondly, suppose $z = 0$. This implies x is even and y is odd. We separate this case into two cases. First, when $y > 1$. In this case the Young diagram of ν is:

$$[\nu] = \begin{array}{|c|c|c|}
\hline
0 & \cdots\cdots\cdots\cdots\cdots & 1 \\
\hline
1 & \cdots\cdots\cdots & 1 \\
\hline
\end{array}.$$

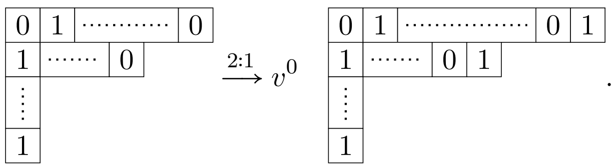
Let τ be the partition obtained by removing two highest 1–nodes from ν. So

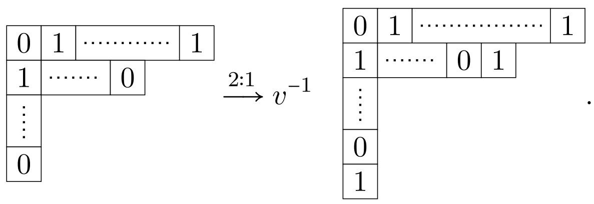
$$\tau = (x-1, y-1).$$

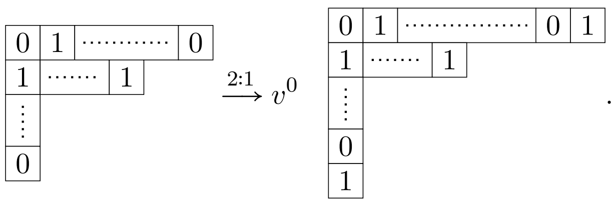
By the induction hypothesis

$$[B_c(\tau)]_\Gamma = \sum_{\tilde f=0}^{\frac{y-3}{2}} \sum_{\tilde k=0}^{\frac{x-y-1}{2}} v^{\lfloor \frac{y-2\tilde f+2\tilde k-3}{2} \rfloor}(x-2\tilde k-1, 2+2\tilde f, 1^{y-2\tilde f+2\tilde k-3})$$
$$+ \sum_{\tilde f=0}^{\frac{y-3}{2}} \sum_{\tilde k=0}^{\frac{x-y-1}{2}} v^{\lfloor \frac{y-2\tilde f+2\tilde k+2}{2} \rfloor}(x-2\tilde k-2, 2+2\tilde f, 1^{y-2\tilde f+2\tilde k-2})$$
$$+ \sum_{\tilde f=0}^{\frac{y-3}{2}} \sum_{\tilde k=0}^{\frac{x-y-1}{2}} v^{\lfloor \frac{y-2\tilde f+2\tilde k}{2} \rfloor}(x-2\tilde k-1, 1+2\tilde f, 1^{y-2\tilde f+2\tilde k-2}).$$

Now we find $[B_c(\tau)]_\Gamma \uparrow_2^1$. To do this, we add two nodes of residue 1 to each partition $(x-2\tilde k-1, 2+2\tilde f, 1^{y-2\tilde f+2\tilde k-3})$, $(x-2\tilde k-2, 2+2\tilde f, 1^{y-2\tilde f+2\tilde k-2})$ and $(x-2\tilde k-1, 1+2\tilde f, 1^{y-2\tilde f+2\tilde k-2})$ in all ways such that we obtain a partition which appears in $[B_c(\nu)]_\Gamma$.







Note that: in the second case, that is when

$$(x-2\tilde k-2, 2+2\tilde f, 1^{y-2\tilde f+2\tilde k-2}) \xrightarrow{2:1} (x-2\tilde k-2, 3+2\tilde f, 1^{y-2\tilde f+2\tilde k-1})$$

we do not get a partition when $\tilde k = \frac{x-y-1}{2}$ and $\tilde f = \frac{y-3}{2}$. We split this formula into two cases to obtain

$$
\begin{aligned}
[B_c(\tau)]_\Gamma\uparrow_2^1 = &\sum_{\tilde f=0}^{\frac{y-3}{2}}\sum_{\tilde k=0}^{\frac{x-y-1}{2}} v^{\lfloor\frac{y-2\tilde f+2\tilde k-3}{2}\rfloor}(x-2\tilde k, 3+2\tilde f, 1^{y-2\tilde f+2\tilde k-3})\\
&+\sum_{\tilde f=0}^{\frac{y-3}{2}}\sum_{\tilde k=0}^{\frac{x-y-3}{2}} v^{\lfloor\frac{y-2\tilde f+2\tilde k}{2}\rfloor}(x-2\tilde k-2, 3+2\tilde f, 1^{y-2\tilde f+2\tilde k-1})\\
&+\sum_{\tilde f=0}^{\frac{y-5}{2}} v^{\lfloor\frac{x-2\tilde f-1}{2}\rfloor}(y-1, 3+2\tilde f, 1^{x-2\tilde f-2})\\
&+\sum_{\tilde f=0}^{\frac{y-3}{2}}\sum_{\tilde k=0}^{\frac{x-y-1}{2}} v^{\lfloor\frac{y-2\tilde f+2\tilde k}{2}\rfloor}(x-2\tilde k, 1+2\tilde f, 1^{y-2\tilde f+2\tilde k-1}).
\end{aligned}
\tag{2.1}
$$

Now we can see that:

$$
\begin{aligned}
&\sum_{\tilde f=0}^{\frac{y-3}{2}}\sum_{\tilde k=0}^{\frac{x-y-1}{2}} v^{\lfloor\frac{y-2\tilde f+2\tilde k-3}{2}\rfloor}(x-2\tilde k, 3+2\tilde f, 1^{y-2\tilde f+2\tilde k-3})\\
&=\sum_{\tilde f=1}^{\frac{y-1}{2}}\sum_{\tilde k=0}^{\frac{x-y-1}{2}} v^{\lfloor\frac{y-2\tilde f+2\tilde k-1}{2}\rfloor}(x-2\tilde k, 1+2\tilde f, 1^{y-2\tilde f+2\tilde k-1}).
\end{aligned}
\tag{2.2}
$$

Also

$$
\begin{aligned}
&\sum_{\tilde f=0}^{\frac{y-5}{2}} v^{\lfloor\frac{x-2\tilde f-1}{2}\rfloor}(y-1, 3+2\tilde f, 1^{x-2\tilde f-2})\\
&=\sum_{\tilde f=0}^{\frac{y-5}{2}}\sum_{\tilde k=\frac{x-y+1}{2}}^{\frac{x-y+1}{2}} v^{\lfloor\frac{y-2\tilde f+2\tilde k-2}{2}\rfloor}(x-2k, 3+2\tilde f, 1^{y-2\tilde f+2\tilde k-3}).
\end{aligned}
\tag{2.3}
$$

and

$$
\begin{aligned}
&\sum_{\tilde f=0}^{\frac{y-3}{2}}\sum_{\tilde k=0}^{\frac{x-y-1}{2}} v^{\lfloor\frac{y-2\tilde f+2\tilde k}{2}\rfloor}(x-2\tilde k, 1+2\tilde f, 1^{y-2\tilde f+2\tilde k-1})\\
&=\sum_{\tilde f=0}^{\frac{y-5}{2}}\sum_{\tilde k=0}^{\frac{x-y-1}{2}} v^{\lfloor\frac{y-2\tilde f+2\tilde k-2}{2}\rfloor}(x-2\tilde k, 3+2\tilde f, 1^{y-2\tilde f+2\tilde k-3})\\
&\quad+\sum_{\tilde k=0}^{\frac{x-y-1}{2}} v^{\lfloor\frac{y+2\tilde k}{2}\rfloor}(x-2\tilde k, 1, 1^{y+2\tilde k-1}).
\end{aligned}
\tag{2.4}
$$

Now if we calculate (2.3)+(2.4), then we get the formula:

$$\sum_{\tilde{f}=0}^{\frac{y-5}{2}} \sum_{\tilde{k}=0}^{\frac{x-y+1}{2}} v^{\lfloor \frac{y-2\tilde{f}+2\tilde{k}-2}{2} \rfloor}(x-2\tilde{k}, 3+2\tilde{f}, 1^{y-2\tilde{f}+2\tilde{k}-3}) + \sum_{\tilde{k}=0}^{\frac{x-y-1}{2}} v^{\lfloor \frac{y+2\tilde{k}}{2} \rfloor}(x-2\tilde{k}, 1, 1^{y+2\tilde{k}-1}). \tag{2.5}$$

Finally, note that

$$\begin{aligned}
&\sum_{\tilde{f}=1}^{\frac{y-1}{2}} \sum_{\tilde{k}=0}^{\frac{x-y-1}{2}} v^{\lfloor \frac{y-2\tilde{f}+2\tilde{k}-1}{2} \rfloor}(x-2\tilde{k}, 1+2\tilde{f}, 1^{y-2\tilde{f}+2\tilde{k}-1}) \\
&+ \sum_{\tilde{k}=0}^{\frac{x-y-1}{2}} v^{\lfloor \frac{y+2\tilde{k}}{2} \rfloor}(x-2\tilde{k}, 1, 1^{y+2\tilde{k}-1}) \\
&= \sum_{\tilde{f}=0}^{\frac{y-1}{2}} \sum_{\tilde{k}=0}^{\frac{x-y-1}{2}} v^{\lfloor \frac{y-2\tilde{f}+2\tilde{k}-1}{2} \rfloor}(x-2\tilde{k}, 1+2\tilde{f}, 1^{y-2\tilde{f}+2\tilde{k}-1})
\end{aligned}$$

since y is odd. Hence, we can write the formula (2.1) as

$$\begin{aligned}
[B_c(\tau)]_\Gamma \uparrow_2^1 &= \sum_{\tilde{f}=0}^{\frac{y-3}{2}} \sum_{\tilde{k}=0}^{\frac{x-y-3}{2}} v^{\lfloor \frac{y-2\tilde{f}+2\tilde{k}}{2} \rfloor}(x-2\tilde{k}-2, 3+2\tilde{f}, 1^{y-2\tilde{f}+2\tilde{k}-1}) + (2.5) + (2.2) \\
&= \sum_{\tilde{f}=0}^{\frac{y-1}{2}} \sum_{\tilde{k}=0}^{\frac{x-y-1}{2}} v^{\lfloor \frac{y-2\tilde{f}+2\tilde{k}-1}{2} \rfloor}(x-2\tilde{k}, 1+2\tilde{f}, 1^{y-2\tilde{f}+2\tilde{k}-1}) \\
&\quad + \sum_{\tilde{f}=0}^{\frac{y-3}{2}} \sum_{\tilde{k}=0}^{\frac{x-y-3}{2}} v^{\lfloor \frac{y-2\tilde{f}+2\tilde{k}}{2} \rfloor}(x-2\tilde{k}-2, 3+2\tilde{f}, 1^{y-2\tilde{f}+2\tilde{k}-1}) \\
&\quad + \sum_{\tilde{f}=0}^{\frac{y-5}{2}} \sum_{\tilde{k}=0}^{\frac{x-y+1}{2}} v^{\lfloor \frac{y-2\tilde{f}+2\tilde{k}-2}{2} \rfloor}(x-2k, 3+2\tilde{f}, 1^{y-2\tilde{f}+2\tilde{k}-3}).
\end{aligned}$$

Now in this case $[B_c(\tau)]_\Gamma \uparrow_2^1 = [B_c(\nu)]_\Gamma + \sum_{\nu \rhd \mu} \alpha_{\mu\nu}(v)[B_c(\mu)]_\Gamma$, where $\alpha_{\mu\nu}(v) \neq 0$ for some $\mu \in \Xi$.

Lemma 2.2.4. *Let $\nu = (x, y)$ where x is even and y is odd and suppose that $\mu \in \Xi$. Then μ has the form*

$$[\mu] = \begin{array}{|c|c|c|c|c|} \hline 0 & \cdots\cdots\cdots\cdots & & 0 & 1 \\ \hline 1 & \cdots\cdots & 0 & 1 \\ \cline{1-4} \end{array} = (x-2j, y+2j), \quad 0<j$$

or

$$[\mu] = \begin{array}{l} \boxed{0}\cdots\cdots\cdots\cdots\cdots\boxed{0}\boxed{1} \\ \boxed{1}\cdots\cdots\cdots\boxed{0}\boxed{1} \\ \boxed{0}\cdots\boxed{1} \end{array} = (x-2j, y+2j-2m, 2m), j \geq 0,\ m > 0.$$

Now recall that $\tau = (x-1, y-1)$. Let $\mu \in \Xi$. We want to look for the coefficient of μ in $[B_c(\tau)] \uparrow_2^1$. Firstly, take the first case for μ which is

$$\mu \in \{(x-2j, y+2j)\}.$$

For each μ, there is a unique partition σ such that σ obtained from μ by removing two nodes of residue 1. Now μ come from the partition $\sigma = (x-2j-1, y+2j-1)$ such that

$$[\sigma] = \begin{array}{l} \boxed{0}\cdots\cdots\cdots\boxed{0} \\ \boxed{1}\cdots\cdots\boxed{0} \end{array} \xrightarrow{2:1} v^0 \quad \begin{array}{l} \boxed{0}\cdots\cdots\cdots\cdots\boxed{0}\boxed{1} \\ \boxed{1}\cdots\cdots\boxed{0}\boxed{1} \end{array}.$$

So the coefficient of μ in $B_c(\tau)] \uparrow_2^1$ is the same as the coefficient of σ in $[B_c(\tau)]$ which is in $v\mathbb{N}[v]$. Secondly, let $\mu \in \{(x-2j, y+2j-2m, 2m) | j \geq 0,\ m > 0\}$. There are three possible partitions σ such that σ is obtained from μ by removing two nodes of residue 1.

$$[\sigma_1] = \begin{array}{l} \boxed{0}\cdots\cdots\cdots\cdots\boxed{0} \\ \boxed{1}\cdots\cdots\cdots\boxed{0} \\ \boxed{0}\cdots\boxed{1} \end{array} \xrightarrow{2:1} v^0 \quad \begin{array}{l} \boxed{0}\cdots\cdots\cdots\cdots\cdots\boxed{0}\boxed{1} \\ \boxed{1}\cdots\cdots\cdots\boxed{0}\boxed{1} \\ \boxed{0}\cdots\boxed{1} \end{array}.$$

So the coefficient of μ in $B_c(\tau)] \uparrow_2^1$ is the same as the coefficient of σ_1 in $[B_c(\tau)]$ which is in $v\mathbb{N}[v]$.

$$[\sigma_2] = \begin{array}{l} \boxed{0}\cdots\cdots\cdots\cdots\boxed{0} \\ \boxed{1}\cdots\cdots\cdots\boxed{1} \\ \boxed{0}\cdots\boxed{0} \end{array} \xrightarrow{2:1} v^{-1} \quad \begin{array}{l} \boxed{0}\cdots\cdots\cdots\cdots\cdots\boxed{0}\boxed{1} \\ \boxed{1}\cdots\cdots\cdots\boxed{0}\boxed{1} \\ \boxed{0}\cdots\boxed{1} \end{array}.$$

So the coefficient $c_{\mu\nu}(v)$ of μ in $B_c(\tau)] \uparrow_2^1$ might not lie in $v\mathbb{N}[v]$.

$$[\sigma_3] = \begin{array}{l} \boxed{0}\cdots\cdots\cdots\cdots\boxed{1} \\ \boxed{1}\cdots\cdots\cdots\boxed{0} \\ \boxed{0}\cdots\boxed{0} \end{array} \xrightarrow{2:1} v^{-2} \quad \begin{array}{l} \boxed{0}\cdots\cdots\cdots\cdots\cdots\boxed{0}\boxed{1} \\ \boxed{1}\cdots\cdots\cdots\boxed{0}\boxed{1} \\ \boxed{0}\cdots\boxed{1} \end{array}.$$

So the coefficient $c_{\mu\nu}(v)$ of μ in $B_c(\tau)] \uparrow_2^1$ might not lie in $v\mathbb{N}[v]$.

Now we look at the coefficients of $\sigma_2 = (x-2j-1, y+2j-2m, 2m-1)$ and $\sigma_3 = (x-2j, y+2j-2m-1, 2m-1)$ in $[B_c(\tau)]$. By using Proposition

4.2.2 we see that:

$$\text{the coefficient of } \sigma_2 = \begin{cases} v, & \text{if } j = 0 \text{ and } m = 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$\text{the coefficient of } \sigma_3 = \begin{cases} v^2, & \text{if } j = 1 \text{ and } m = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, if $\mu = (x, y-2, 2)$ or $\mu = (x-2, y, 2)$ then $\alpha_{\mu\nu}(v) = 1$ and $\alpha_{\eta\nu}(v) = 0$ for all other $\eta \in \Xi$. Note that if $x = y+1$ then $(x-2, y, 2)$ is not a partition. In this case $\alpha_{\mu\nu}(v) = 1$ for $\mu = (x, y-2, 2)$ and $\alpha_{\eta\nu}(v) = 0$ for all other $\eta \in \Xi$. So if $x \neq y+1$ then

$$[B_c(\nu)]_\Gamma = [B_c(\tau)]_\Gamma \uparrow_2^1 - [B_c(x, y-2, 2)]_\Gamma - [B_c(x-2, y, 2)]_\Gamma.$$

And if $x = y+1$ then

$$[B_c(\nu)]_\Gamma = [B_c(\tau)]_\Gamma \uparrow_2^1 - [B_c(x, y-2, 2)]_\Gamma.$$

Now we find $[B_c(x, y-2, 2)]_\Gamma$ and $[B_c(x-2, y, 2)]_\Gamma$ by using the induction hypothesis:

$$[B_c(x, y-2, 2)]_\Gamma = \sum_{\tilde{f}=0}^{\frac{y-5}{2}} \sum_{\tilde{k}=0}^{\frac{x-y+1}{2}} v^{\lfloor \frac{y-2\tilde{f}+2\tilde{k}-3}{2} \rfloor} (x-2k, 3+2\tilde{f}, 1^{y-2\tilde{f}+2\tilde{k}-3}). \quad (2.6)$$

And

$$[B_c(x-2, y, 2)]_\Gamma = \sum_{\tilde{f}=0}^{\frac{y-3}{2}} \sum_{\tilde{k}=0}^{\frac{x-y-3}{2}} v^{\lfloor \frac{y-2\tilde{f}+2\tilde{k}-1}{2} \rfloor} (x-2\tilde{k}-2, 3+2\tilde{f}, 1^{y-2\tilde{f}+2\tilde{k}-1}). \quad (2.7)$$

Hence, if $x \neq y+1$ we subtract (2.6) and (2.7) from $[B_c(\tau)]_\Gamma \uparrow_2^1$ we get:

$$\begin{aligned} [B_c(\nu)]_\Gamma &= [B_c(\tau)]_\Gamma \uparrow_2^1 - [B_c(x, y-2, 2)]_\Gamma - [B_c(x-2, y, 2)]_\Gamma \\ &= \sum_{\tilde{f}=0}^{\frac{y-1}{2}} \sum_{\tilde{k}=0}^{\frac{x-y-1}{2}} v^{\lfloor \frac{y-2\tilde{f}+2\tilde{k}-1}{2} \rfloor} (x-2\tilde{k}, 1+2\tilde{f}, 1^{y-2f+2\tilde{k}-1}). \end{aligned}$$

And if $x = y + 1$ then

$$\begin{aligned}[B_c(\nu)]_\Gamma &= [B_c(\tau)]_\Gamma \uparrow_2^1 - [B_c(x, y-2, 2)]_\Gamma \\ &= \sum_{\tilde{f}=0}^{\frac{y-1}{2}} \sum_{\tilde{k}=0}^{\frac{x-y-1}{2}} v^{\lfloor \frac{y-2\tilde{f}+2\tilde{k}-1}{2} \rfloor} (x - 2\tilde{k}, 1 + 2\tilde{f}, 1^{y-2f+2\tilde{k}-1}).\end{aligned}$$

Second, if $y = 1$.

In this case the Young diagram of ν is:

$$[\nu] = \begin{array}{|c|c|c|}\hline 0 & \cdots\cdots\cdots & 1 \\ \hline 1 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}.$$

Let τ be the partition obtained by removing the two 1–nodes from ν. So

$$\tau = (x - 1)$$

and this is not covered by the induction hypothesis in Theorem A. Thus, we use Theorem 2.1.5 to find $[B_c(\tau)]_\Gamma$.

$$[B_c(\tau)]_\Gamma = \sum_{\tilde{k}=0}^{\frac{x-2}{2}} v^{\tilde{k}} (x - 2\tilde{k} - 1, 1^{2\tilde{k}}).$$

Now we find $[B_c(\tau)]_\Gamma \uparrow_2^1$. To do this, we add two nodes of residue 1 to each partition $(x - 2\tilde{k} - 1, 1^{2\tilde{k}})$ in all ways such that we obtain a partition which appears in $[B_c(\nu)]_\Gamma$.

$$\begin{array}{|c|c|c|c|}\hline 0 & 1 & \cdots\cdots\cdots & 0 \\ \hline 1 & \multicolumn{3}{c}{} \\ \cline{1-1} \vdots & \multicolumn{3}{c}{} \\ \cline{1-1} 0 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array} \xrightarrow{2:1} v^0 \begin{array}{|c|c|c|c|c|}\hline 0 & 1 & \cdots\cdots\cdots & 0 & 1 \\ \hline 1 & \multicolumn{4}{c}{} \\ \cline{1-1} \vdots & \multicolumn{4}{c}{} \\ \cline{1-1} 0 & \multicolumn{4}{c}{} \\ \cline{1-1} 1 & \multicolumn{4}{c}{} \\ \cline{1-1}\end{array}.$$

Hence,

$$[B_c(\tau)]_\Gamma \uparrow_2^1 = \sum_{\tilde{k}=0}^{\frac{x-2}{2}} v^{\tilde{k}} (x - 2\tilde{k}, 1^{2\tilde{k}+1}).$$

Now we show that in this case $[B_c(\nu)]_\Gamma = [B_c(\tau)]_\Gamma \uparrow_2^1$. Let $\mu \in \Xi$. We look for the coefficient of μ in $[B_c(\tau)]\uparrow_2^1$. If $\mu \in \Xi$, then μ has the form:

$$\begin{array}{|c|c|c|c|}\hline 0 & \cdots\cdots\cdots & 0 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}\,.$$

Now we see that, for $\mu \in \Xi$ there is a unique σ such that σ is obtained by removing two nodes of residue 1 from μ. The partition μ comes from a partition σ such that

$$[\sigma] = \begin{array}{|c|c|c|}\hline 0 & \cdots\cdots\cdots & 0 \\ \hline\end{array} \xrightarrow{2:1} v^0 \quad \begin{array}{|c|c|c|c|}\hline 0 & \cdots\cdots\cdots & 0 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}\,.$$

Thus the coefficient of μ in $[B_c(\tau)]\uparrow_2^1$ is the same as the coefficient of σ in $[B_c(\tau)]$ which is in $v\mathbb{N}[v]$. Since $c_{\mu\nu}(v) \in v\mathbb{N}[v]$, for all $\mu \in \Xi$, we have $[B_c(\nu)]_\Gamma = [B_c(\tau)]_\Gamma\uparrow_2^1$.

Hence, we have shown that Theorem A holds for all partitions described in Case 2.

Case 3: $x \not\equiv y \equiv z$

The Young diagram of ν is:

$$[\nu] = \begin{array}{|l}\hline 0 \ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \ i \\ \hline 1 \ \cdots\cdots\cdots\cdots\cdots \ i \\ \hline 0 \ \cdots\cdots \ j \\ \hline\end{array}$$

$$\text{where } i = \begin{cases} 0, & \text{if } z \text{ even}, \\ 1, & \text{if } z \text{ odd}, \end{cases}$$

$$\text{where } j = \begin{cases} 1, & \text{if } z \text{ even}, \\ 0, & \text{if } z \text{ odd}. \end{cases}$$

Let τ be the partition obtained by removing the highest two i–nodes from ν. So

$$\tau = (x-1, y-1, z).$$

By the induction hypothesis

$$[B_c(x-1,y-1,z)]_\Gamma = \sum_{\tilde{f}=0}^{\frac{y-z-2}{2}} \sum_{\tilde{k}=0}^{\frac{x-y-1}{2}} v^{\lfloor \frac{y-2\tilde{f}+2\tilde{k}-2}{2} \rfloor}(x-2\tilde{k}-1, z+1+2\tilde{f}, 1^{y-2\tilde{f}+2\tilde{k}-2}).$$

Now we find $[B_c(\tau)]_\Gamma \uparrow_2^i$. To do this, we add two nodes of residue i to each partition $(x-2\tilde{k}-1, z+1+2\tilde{f}, 1^{y-2\tilde{f}+2\tilde{k}-2})$ in all ways such that we obtain a partition which appears in $[B_c(\nu)]_\Gamma$.

$$\begin{array}{|c|c|c|c|}\hline 0 & 1 & \cdots\cdots & j \\ \hline 1 & \cdots & j & \\ \cline{1-3} \vdots & & & \\ \cline{1-1} j & & & \\ \cline{1-1} \end{array} \xrightarrow{2:i} v^0 \begin{array}{|c|c|c|c|c|}\hline 0 & 1 & \cdots\cdots & j & i \\ \hline 1 & \cdots & j & i & \\ \cline{1-4} \vdots & & & & \\ \cline{1-1} j & & & & \\ \cline{1-1} \end{array} + v^{1+i} \begin{array}{|c|c|c|c|c|}\hline 0 & 1 & \cdots\cdots & j & i \\ \hline 1 & \cdots & j & & \\ \cline{1-3} \vdots & & & & \\ \cline{1-1} j & & & & \\ \cline{1-1} i & & & & \\ \cline{1-1} \end{array}$$

$$+ v^{2+i} \begin{array}{|c|c|c|c|c|}\hline 0 & 1 & \cdots\cdots & j & \\ \hline 1 & \cdots & j & i & \\ \cline{1-4} \vdots & & & & \\ \cline{1-1} j & & & & \\ \cline{1-1} i & & & & \\ \cline{1-1} \end{array} .$$

Hence,

$$\begin{aligned}
[B_c(\tau)]_\Gamma \uparrow_2^i &= \sum_{\tilde{f}=0}^{\frac{y-z-2}{2}} \sum_{\tilde{k}=0}^{\frac{x-y-1}{2}} v^{\lfloor \frac{y-2\tilde{f}+2\tilde{k}-2}{2} \rfloor}(x-2\tilde{k}, z+2+2\tilde{f}, 1^{y-2\tilde{f}+2\tilde{k}-2}) \\
&\quad + \sum_{\tilde{f}=0}^{\frac{y-z-2}{2}} \sum_{\tilde{k}=0}^{\frac{x-y-1}{2}} v^{\lfloor \frac{y-2\tilde{f}+2\tilde{k}+2i}{2} \rfloor}(x-2\tilde{k}, z+1+2\tilde{f}, 1^{y-2\tilde{f}+2\tilde{k}-1}) \\
&\quad + \sum_{\tilde{f}=0}^{\frac{y-z-2}{2}} \sum_{\tilde{k}=0}^{\frac{x-y-1}{2}} v^{\lfloor \frac{y-2\tilde{f}+2\tilde{k}+2+2i}{2} \rfloor}(x-2\tilde{k}-1, z+2+2\tilde{f}, 1^{y-2\tilde{f}+2\tilde{k}-1}) \\
&= \sum_{\tilde{f}=0}^{\frac{y-z-2}{2}} \sum_{\tilde{k}=0}^{x-y} \alpha_k (x-\tilde{k}, z+2+2\tilde{f}, 1^{y-2\tilde{f}+\tilde{k}-2}) \\
&\quad + \sum_{\tilde{f}=0}^{\frac{y-z-2}{2}} \sum_{\tilde{k}=0}^{\frac{x-y-1}{2}} v^{\lfloor \frac{y-2\tilde{f}+2\tilde{k}+1}{2} \rfloor}(x-2\tilde{k}, z+1+2\tilde{f}, 1^{y-2\tilde{f}+2\tilde{k}-1})
\end{aligned}$$

$$\text{where } \alpha_{\tilde{k}} = \begin{cases} v^{\lfloor \frac{y-2\tilde{f}+\tilde{k}-2}{2} \rfloor}, & \text{if } \tilde{k} \text{ even,} \\ v^{\lfloor \frac{y-2\tilde{f}+\tilde{k}+2}{2} \rfloor}, & \text{if } \tilde{k} \text{ odd.} \end{cases}$$

Note that: if

$$i = \begin{cases} 0, & \text{if } y \text{ even,} \\ 1, & \text{if } y \text{ odd,} \end{cases}$$

then $\lfloor \frac{y-2\tilde{f}+2\tilde{k}+2i}{2} \rfloor = \lfloor \frac{y-2\tilde{f}+2\tilde{k}+1}{2} \rfloor$. Moreover, if $l = 2\tilde{k}+1$, then $\lfloor \frac{y-2\tilde{f}+2\tilde{k}+2+2i}{2} \rfloor = \lfloor \frac{y-2\tilde{f}+l+2}{2} \rfloor$.

Now we show that in this case $[B_c(\nu)]_\Gamma = [B_c(\tau)]_\Gamma \uparrow_2^i$. Let $\mu \in \Xi$. We look for the coefficient of μ in $[B_c(\tau)] \uparrow_2^i$. If $\mu \in \Xi$ then μ has one of the following forms:

$$[\mu_1] = \begin{array}{|l|} \hline 0 \;\cdots\cdots\cdots\cdots\cdots\; j \; i \\ \hline 1 \;\cdots\cdots\cdots\; j \; i \\ \hline 0 \;\cdots\; j \\ \hline \end{array} \quad \text{or} \quad [\mu_2] = \begin{array}{|l|} \hline 0 \;\cdots\cdots\cdots\cdots\cdots\; j \; i \\ \hline 1 \;\cdots\cdots\cdots\; j \\ \hline 0 \;\cdots\; j \; i \\ \hline \end{array}$$

$$\text{or} \quad [\mu_3] = \begin{array}{|l|} \hline 0 \;\cdots\cdots\cdots\cdots\cdots\; j \\ \hline 1 \;\cdots\cdots\cdots\; j \; i \\ \hline 0 \;\cdots\; j \; i \\ \hline \end{array} .$$

For each μ_k, where $1 \le k \le 3$, there is a unique partition σ_k such that σ_k is obtained by removing two i–nodes from μ_k.

Each partition μ_1 comes from a partition σ_1 such that

$$[\sigma_1] = \begin{array}{|l|} \hline 0 \;\cdots\cdots\cdots\cdots\; j \\ \hline 1 \;\cdots\cdots\cdots\; j \\ \hline 0 \;\cdots\; j \\ \hline \end{array} \xrightarrow{2:i} v^0 \begin{array}{|l|} \hline 0 \;\cdots\cdots\cdots\cdots\; j \; i \\ \hline 1 \;\cdots\cdots\cdots\; j \; i \\ \hline 0 \;\cdots\; j \\ \hline \end{array} .$$

So the coefficient of μ_1 in $[B_c(\tau)] \uparrow_2^i$ is the same as the coefficient of σ_1 in $[B_c(\tau)]$ which is in $v\mathbb{N}[v]$. Each partition μ_2 comes from partition σ_2 such that

$$[\sigma_2] = \begin{array}{|l|} \hline 0 \;\cdots\cdots\cdots\cdots\; j \\ \hline 1 \;\cdots\cdots\cdots\; j \\ \hline 0 \;\cdots\; j \\ \hline \end{array} \xrightarrow{2:i} v \begin{array}{|l|} \hline 0 \;\cdots\cdots\cdots\cdots\cdots\; j \; i \\ \hline 1 \;\cdots\cdots\cdots\; j \\ \hline 0 \;\cdots\; j \; i \\ \hline \end{array} .$$

Thus the coefficient of μ_2 in $[B_c(\tau)] \uparrow_2^i$ is the the coefficient of σ_2 in $[B_c(\tau)]$ multiplied by v which is in $v\mathbb{N}[v]$. Each μ_3 comes from a partition σ_3 such that

$$[\sigma_3] = \begin{array}{|c|l|}\hline 0 & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\; \boxed{j} \\ \hline 1 & \cdots\cdots\cdots\cdots\cdots\; \boxed{j} \\ \hline 0 & \cdots\cdots\; \boxed{j} \\ \hline \end{array} \xrightarrow{2:i} v^2 \begin{array}{|c|l|}\hline 0 & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\; \boxed{j} \\ \hline 1 & \cdots\cdots\cdots\cdots\cdots\cdots\; \boxed{j}\,\boxed{i} \\ \hline 0 & \cdots\cdots\; \boxed{j}\,\boxed{i} \\ \hline \end{array}.$$

Thus the coefficient of μ_3 in $[B_c(\tau)]\uparrow_2^i$ is the the coefficient of σ_3 in $[B_c(\tau)]\uparrow_2^i$ multiplied by v^2 which is in $v\mathbb{N}[v]$.

Since $c_{\mu\nu}(v) \in v\mathbb{N}[v]$, for all $\mu \in \Xi$, we have $[B_c(\nu)]_\Gamma = [B_c(\tau)]_\Gamma \uparrow_2^i$. Hence, we have shown that Theorem A holds for all partitions described in Case 3.

Case 4: $x \equiv y \equiv z$

The Young diagram of ν is:

$$[\nu] = \begin{array}{|c|l|}\hline 0 & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\; \boxed{i} \\ \hline 1 & \cdots\cdots\cdots\cdots\cdots\; \boxed{j} \\ \hline 0 & \cdots\cdots\; \boxed{i} \\ \hline \end{array}$$

$$\text{where } i = \begin{cases} 1, & \text{if } z \text{ even,} \\ 0, & \text{if } z \text{ odd,} \end{cases}$$

$$\text{where } j = \begin{cases} 0, & \text{if } z \text{ even,} \\ 1, & \text{if } z \text{ odd.} \end{cases}$$

Let τ be the partition obtained by removing the i–node from ν. So

$$\tau = (x-1, y, z).$$

By the induction hypothesis

$$
\begin{aligned}
[B_c(x-1,y,z)]_\Gamma = {} & \sum_{\tilde f=0}^{\frac{y-z-2}{2}} \sum_{\tilde k=0}^{\frac{x-y-2}{2}} v^{\lfloor \frac{y-2\tilde f+2\tilde k-2}{2} \rfloor} (x-2\tilde k-1, z+2+2\tilde f, 1^{y-2\tilde f+2k-2}) \\
& + \sum_{\tilde f=0}^{\frac{y-z-2}{2}} \sum_{\tilde k=0}^{\frac{x-y-2}{2}} v^{\lfloor \frac{y-2\tilde f+2\tilde k+3}{2} \rfloor} (x-2\tilde k-2, z+2+2\tilde f, 1^{y-2\tilde f+2k-1}) \\
& + \sum_{\tilde f=0}^{\frac{y-z-2}{2}} \sum_{k=0}^{\frac{x-y-2}{2}} v^{\lfloor \frac{y-2\tilde f+2\tilde k+1}{2} \rfloor} (x-2\tilde k-1, z+1+2\tilde f, 1^{y-2\tilde f+2\tilde k-1}).
\end{aligned}
$$

Now we find $[B_c(\tau)]_\Gamma \uparrow_1^i$. To do this, we add one node of residue i to each partition $(x-2\tilde k-1, z+2+2\tilde f, 1^{y-2\tilde f+2k-2})$, $(x-2\tilde k-2, z+2+2\tilde f, 1^{y-2\tilde f+2k-1})$ and $(x-2\tilde k-1, z+1+2\tilde f, 1^{y-2\tilde f+2\tilde k-1})$ in all ways such that we obtain a partition which appears in $[B_c(\nu)]_\Gamma$.

$$
\begin{array}{|c|c|c|c|}\hline 0 & 1 & \cdots & j \\ \hline 1 & \cdots & j \\ \cline{1-3} \vdots \\ \cline{1-1} i \\ \cline{1-1}\end{array}
\xrightarrow{1:i} v^0
\begin{array}{|c|c|c|c|c|}\hline 0 & 1 & \cdots & j & i \\ \hline 1 & \cdots & j \\ \cline{1-3} \vdots \\ \cline{1-1} i \\ \cline{1-1}\end{array}
+ v
\begin{array}{|c|c|c|c|}\hline 0 & 1 & \cdots & j \\ \hline 1 & \cdots & j & i \\ \hline \vdots \\ \cline{1-1} i \\ \cline{1-1}\end{array}
.
$$

$$
\begin{array}{|c|c|c|c|}\hline 0 & 1 & \cdots & i \\ \hline 1 & \cdots & j \\ \cline{1-3} \vdots \\ \cline{1-1} j \\ \cline{1-1}\end{array}
\xrightarrow{1:i} v^{-1}
\begin{array}{|c|c|c|c|}\hline 0 & 1 & \cdots & i \\ \hline 1 & \cdots & j & i \\ \hline \vdots \\ \cline{1-1} i \\ \cline{1-1}\end{array}
+ v^i
\begin{array}{|c|c|c|c|}\hline 0 & 1 & \cdots & i \\ \hline 1 & \cdots & j \\ \cline{1-3} \vdots \\ \cline{1-1} j \\ \cline{1-1} i \\ \cline{1-1}\end{array}
.
$$

Note that: in this case when we add one node of residue i such that

$$
(x-2\tilde k-2, z+2+2\tilde f, 1^{y-2\tilde f+2k-1}) \xrightarrow{1:i} (x-2\tilde k-2, z+2+2\tilde f+1, 1^{y-2\tilde f+2k-1})
$$

we do not get a partition when $\tilde k = \frac{x-y-2}{2}$ and $\tilde f = \frac{y-z-2}{2}$.

$$
\begin{array}{|c|c|c|c|}\hline 0 & 1 & \cdots & j \\ \hline 1 & \cdots & i \\ \cline{1-3} \vdots \\ \cline{1-1} j \\ \cline{1-1}\end{array}
\xrightarrow{1:i} v^0
\begin{array}{|c|c|c|c|c|}\hline 0 & 1 & \cdots & j & i \\ \hline 1 & \cdots & i \\ \cline{1-3} \vdots \\ \cline{1-1} i \\ \cline{1-1}\end{array}
+ v^i
\begin{array}{|c|c|c|c|}\hline 0 & 1 & \cdots & j \\ \hline 1 & \cdots & i \\ \cline{1-3} \vdots \\ \cline{1-1} j \\ \cline{1-1} i \\ \cline{1-1}\end{array}
.
$$

Hence,

$$
\begin{aligned}
[B_c(\tau)]_\Gamma \uparrow_1^1 = &\sum_{\tilde{f}=0}^{\frac{y-z-2}{2}} \sum_{\tilde{k}=0}^{\frac{x-y-2}{2}} v^{\lfloor \frac{y-2\tilde{f}+2\tilde{k}-2}{2} \rfloor} (x-2\tilde{k}, z+2+2\tilde{f}, 1^{y-2\tilde{f}+2\tilde{k}-2}) \\
&+ \sum_{\tilde{f}=0}^{\frac{y-z-2}{2}} \sum_{\tilde{k}=0}^{\frac{x-y-2}{2}} v^{\lfloor \frac{y-2\tilde{f}+2\tilde{k}}{2} \rfloor} (x-2\tilde{k}-1, z+3+2\tilde{f}, 1^{y-2\tilde{f}+2\tilde{k}-2}) \\
&+ \sum_{\tilde{f}=0}^{\frac{y-z-2}{2}} \sum_{\tilde{k}=0}^{\frac{x-y-4}{2}} v^{\lfloor \frac{y-2\tilde{f}+2\tilde{k}+1}{2} \rfloor} (x-2\tilde{k}-2, z+3+2\tilde{f}, 1^{y-2\tilde{f}+2\tilde{k}-1}) \\
&+ \sum_{\tilde{f}=0}^{\frac{y-z-4}{2}} v^{\lfloor \frac{x-2\tilde{f}-1}{2} \rfloor} (y, z+3+2\tilde{f}, 1^{x-2\tilde{f}-3}) \\
&+ \sum_{\tilde{f}=0}^{\frac{y-z-2}{2}} \sum_{\tilde{k}=0}^{\frac{x-y-2}{2}} v^{\lfloor \frac{y-2\tilde{f}+2\tilde{k}+3+2i}{2} \rfloor} (x-2\tilde{k}-2, z+2+2\tilde{f}, 1^{y-2\tilde{f}+2\tilde{k}}) \\
&+ \sum_{\tilde{f}=0}^{\frac{y-z-2}{2}} \sum_{\tilde{k}=0}^{\frac{x-y-2}{2}} v^{\lfloor \frac{y-2\tilde{f}+2\tilde{k}+1}{2} \rfloor} (x-2\tilde{k}, z+1+2\tilde{f}, 1^{y-2\tilde{f}+2\tilde{k}-1}) \\
&+ \sum_{\tilde{f}=0}^{\frac{y-z-2}{2}} \sum_{\tilde{k}=0}^{\frac{x-y-2}{2}} v^{\lfloor \frac{y-2\tilde{f}+2\tilde{k}+1+2i}{2} \rfloor} (x-2\tilde{k}-1, z+1+2\tilde{f}, 1^{y-2\tilde{f}+2\tilde{k}}) \\
= &\sum_{\tilde{f}=0}^{\frac{y-z-2}{2}} \sum_{\tilde{k}=0}^{\frac{x-y-2}{2}} v^{\lfloor \frac{y-2\tilde{f}+2\tilde{k}-2}{2} \rfloor} (x-2\tilde{k}, z+2+2\tilde{f}, 1^{y-2\tilde{f}+2\tilde{k}-2}) \\
&+ \sum_{\tilde{f}=0}^{\frac{y-z-2}{2}} \sum_{\tilde{k}=0}^{x-y-2} v^{\lfloor \frac{y-2f+\tilde{k}}{2} \rfloor} (x-\tilde{k}-1, z+3+2f, 1^{y-2f+\tilde{k}-2}) \\
&+ \sum_{\tilde{f}=0}^{\frac{y-z-2}{2}} \sum_{\tilde{k}=0}^{\frac{x-y-2}{2}} v^{\lfloor \frac{y-2\tilde{f}+2\tilde{k}+4}{2} \rfloor} (x-2\tilde{k}-2, z+2+2\tilde{f}, 1^{y-2\tilde{f}+2\tilde{k}}) \\
&+ \sum_{\tilde{f}=0}^{\frac{y-z-4}{2}} \sum_{\tilde{k}=0}^{x-y} v^{\lfloor \frac{y-2\tilde{f}+\tilde{k}-1}{2} \rfloor} (x-\tilde{k}, z+3+2\tilde{f}, 1^{y-2\tilde{f}+\tilde{k}-3}) \\
&+ \sum_{\tilde{k}=0}^{x-y-1} v^{\lfloor \frac{y+\tilde{k}+1}{2} \rfloor} (x-\tilde{k}, z+1, 1^{y-1+\tilde{k}}).
\end{aligned}
$$

Note that if

$$
i = \begin{cases} 1, & \text{if } y \text{ even,} \\ 0, & \text{if } y \text{ odd,} \end{cases}
$$

then

$$\lfloor\frac{y-2\tilde{f}+2\tilde{k}+3+2i}{2}\rfloor=\lfloor\frac{y-2\tilde{f}+2\tilde{k}+4}{2}\rfloor \text{ and } \lfloor\frac{y-2\tilde{f}+2\tilde{k}+1+2i}{2}\rfloor=\lfloor\frac{y-2\tilde{f}+2\tilde{k}+2}{2}\rfloor.$$

Now we come to show that in this case $[B_c(\nu)]_\Gamma=[B_c(\tau)]_\Gamma\uparrow_1^i$. Consider μ such that $\mu\in\Xi$. We look for the coefficient of μ in $[B_c(\tau)]_\Gamma\uparrow_1^i$. Recall $\tau=(x-1,y,z)$. Let $\tau'=(x-1-z,y-z)$. By using Proposition 4.2.2 we have

$$[B_c(\tau')]_3=(x-1-z,y-z)+v(x-1-z,y-z-1,1)+v^2(x-2-z,y-z,1).$$

Hence by Theorem 2.2.1,

$$\begin{aligned}[B_c(\tau)]_3\uparrow_1^i &= [(x-1,y,z)+v(x-1,y-1,1+z)+v^2(x-2,y,1+z)]\uparrow_1^i\\ &= v^0(x,y,z)+v(x-1,y+1,z)\\ &\quad+v(x,y-1,1+z)+v(x-1,y-1,2+z)\\ &\quad+v(x-2,y+1,1+z)+v^2(x-2,y,2+z).\end{aligned}$$

If $\mu\in\Xi$, then the coefficient of μ in $[B_c(\tau)]\uparrow_1^i$ lies in $v\mathbb{N}[v]$. Hence, we have shown that Theorem A holds for all partitions described in Case 4.

This completes the proof of Theorem A.

2.2.2 Proof of Theorem B.

Now we come to prove the second of our results.

Proof of Theorem B We prove this theorem by using Theorem A. From Theorem A, we may deduce the following lemmas.

Lemma 2.2.5. *Suppose $a\equiv b\not\equiv c$. Then $(a,c,1^b)$ appears in $[B_c(x,y,z)]_\Gamma$ if and only if $(a,c,1^b)=(x-2k,z+1+2f,1^{y-2f+2k-1})$ where $a\equiv x\not\equiv y\not\equiv z$ for some $0\le f\le\frac{y-z-1}{2}$, $0\le k\le\frac{x-y-1}{2}$ and in this case it occurs with coefficient $v^{\lfloor\frac{y-2f+2k-1}{2}\rfloor}=v^{\lfloor\frac{b}{2}\rfloor}$.*

Lemma 2.2.6. *Suppose $a\equiv b\equiv c$. Then $(a,c,1^b)$ appears in $[B_c(x,y,z)]_\Gamma$ if and only if $(a,c,1^b)$ has one of the three forms below:*

- *If* $(a,c,1^b) = (x-2k-1, z+2+2f, 1^{y-2f+2k-1})$ *where* $a \not\equiv x \equiv y \not\equiv z$ *for some* $0 \le f \le \frac{y-z-1}{2}$, $0 \le k \le \frac{x-y-2}{2}$ *and in this case it occurs with coefficient* $v^{\lfloor \frac{y-2f+2k+1}{2} \rfloor} = v^{\lfloor \frac{b+2}{2} \rfloor}$.

- *If* $(a,c,1^b) = (x-2k, z+1+2f, 1^{y-2f+2k-1})$ *where* $a \equiv x \not\equiv y \equiv z$ *for some* $0 \le f \le \frac{y-z-2}{2}$, $0 \le k \le \frac{x-y-1}{2}$ *and in this case it occurs with coefficient* $v^{\lfloor \frac{y-2f+2k+1}{2} \rfloor} = v^{\lfloor \frac{b+2}{2} \rfloor}$.

- *If* $(a,c,1^b) = (x-2l, z+2+2f, 1^{y-2f+2l-2})$ *where* $a \equiv x \equiv y \equiv z$ *for some* $0 \le f \le \frac{y-z-2}{2}$, $0 \le l \le \frac{x-y}{2}$ *and in this case it occurs with coefficient*

$$\begin{cases} v^{\lfloor \frac{y-2f+2l-2}{2} \rfloor} = v^{\lfloor \frac{b}{2} \rfloor}, & \textit{if } l = 0\ , \\ v^{\lfloor \frac{y-2f+2l+2}{2} \rfloor} = v^{\lfloor \frac{b+4}{2} \rfloor}, & \textit{if } l = \frac{x-y}{2}\ , \\ v^{\lfloor \frac{y-2f+2l-2}{2} \rfloor} + v^{\lfloor \frac{y-2f+2l+2}{2} \rfloor} = v^{\lfloor \frac{b}{2} \rfloor} + v^{\lfloor \frac{b+4}{2} \rfloor}, & \textit{otherwise.} \end{cases}$$

Lemma 2.2.7. *Suppose* $a \not\equiv b \not\equiv c$. *Then* $(a,c,1^b)$ *appears in* $[B_c(x,y,z)]_\Gamma$ *if and only if* $(a,c,1^b)$ *has one of the four forms below:*

- *If* $(a,c,1^b) = (x-2k, z+1+2f, 1^{y-2f+2k-1})$ *where* $a \equiv x \equiv y \not\equiv z$ *for some* $0 \le f \le \frac{y-z-1}{2}$, $0 \le k \le \frac{x-y-2}{2}$ *and in this case it occurs with coefficient* $v^{\lfloor \frac{y-2f+2k-1}{2} \rfloor} = v^{\lfloor \frac{b}{2} \rfloor}$.

- *If* $(a,c,1^b) = (x-2k-1, z+2+2f, 1^{y-2f+2k-1})$ *where* $a \not\equiv x \not\equiv y \equiv z$ *for some* $0 \le f \le \frac{y-z-2}{2}$, $0 \le k \le \frac{x-y-1}{2}$ *and in this case it occurs with coefficient* $v^{\lfloor \frac{y-2f+2k+3}{2} \rfloor} = v^{\lfloor \frac{b+4}{2} \rfloor}$.

- *If* $(a,c,1^b) = (x-2k-1, z+3+2f, 1^{y-2f+2k-2})$ *where* $a \not\equiv x \equiv y \equiv z$ *for some* $0 \le k \le \frac{x-y-2}{2}$ *and* $0 \le f \le \frac{y-z-2}{2}$ *and in this case it occurs with coefficient*

$$\begin{cases} v^{\lfloor \frac{y-2f+2k}{2} \rfloor} = v^{\lfloor \frac{b+2}{2} \rfloor}, & \textit{if } f = \frac{y-z-2}{2}\ , \\ 2v^{\lfloor \frac{y-2f+2k}{2} \rfloor} = 2v^{\lfloor \frac{b+2}{2} \rfloor}, & \textit{if } 0 \le f \le \frac{y-z-4}{2}. \end{cases}$$

- *If* $(a, c, 1^b) = (x - 2k - 1, z + 1, 1^{y+2k})$ *where* $a \not\equiv x \equiv y \equiv z$ *for some* $0 \leq k \leq \frac{x-y-2}{2}$ *and in this case it occurs with coefficient* $v^{\lfloor \frac{y+2k+2}{2} \rfloor} = v^{\lfloor \frac{b+2}{2} \rfloor}$.

Lemma 2.2.8. *Suppose* $a \not\equiv b \equiv c$. *Then* $(a, c, 1^b)$ *appears in* $[B_c(x, y, z)]_\Gamma$ *if and only if* $(a, c, 1^b)$ *has one of the four forms below:*

- *If* $(a, c, 1^b) = (x-2k-1, z+1+2f, 1^{y-2f+2k})$ *where* $a \not\equiv x \equiv y \not\equiv z$ *for some* $0 \leq f \leq \frac{y-z-1}{2}$, $0 \leq k \leq \frac{x-y-2}{2}$ *and in this case it occurs with coefficient* $v^{\lfloor \frac{y-2f+2k+4}{2} \rfloor} = v^{\lfloor \frac{b+4}{2} \rfloor}$.

- *If* $(a, c, 1^b) = (x-2k, z+2+2f, 1^{y-2f+2k-2})$ *where* $a \equiv x \not\equiv y \equiv z$ *for some* $0 \leq f \leq \frac{y-z-2}{2}$, $0 \leq k \leq \frac{x-y-1}{2}$ *and in this case it occurs with coefficient* $v^{\lfloor \frac{y-2f+2k-2}{2} \rfloor} = v^{\lfloor \frac{b}{2} \rfloor}$.

- *If* $(a, c, 1^b) = (x - 2l, z + 3 + 2g, 1^{y-2g+2l-3})$ *where* $a \equiv x \equiv y \equiv z$ *for some* $0 \leq g \leq \frac{y-z-4}{2}$, $0 \leq l \leq \frac{x-y}{2}$ *or* $g = \frac{y-z-2}{2}$ *and* $1 \leq l \leq \frac{x-y-2}{2}$ *and in this case it occurs with coefficient*

$$\begin{cases} v^{\lfloor \frac{y-2g+2l-1}{2} \rfloor} = v^{\lfloor \frac{b+2}{2} \rfloor}, & \text{if } l = 0 \text{ or } l = \frac{x-y}{2} \text{ or } g = \frac{x-y-2}{2}. \ , \\ 2v^{\lfloor \frac{y-2g+2l-1}{2} \rfloor} = 2v^{\lfloor \frac{b+2}{2} \rfloor}, & \text{otherwise} \ . \end{cases}$$

- *If* $(a, c, 1^b) = (x - 2k, z + 1, 1^{y+2k-1})$ *where* $a \equiv x \equiv y \equiv z$ *for some* $0 \leq k \leq \frac{x-y-2}{2}$ *and in this case it occurs with coefficient* $v^{\lfloor \frac{y+2k+1}{2} \rfloor} = v^{\lfloor \frac{b+2}{2} \rfloor}$.

Now we will prove the case (1) in Theorem B in detail.

Proposition 2.2.9. *If* $a \equiv b \not\equiv c$ *then:*

$$B_r(\lambda) = \sum_{\substack{y=c \\ y \equiv c}}^{a-1} \sum_{\substack{x=max\{a, a+b-y+1\} \\ x \not\equiv c}}^{n-y} v^{\lfloor \frac{b}{2} \rfloor}(x, y, n - x - y).$$

Proof. Suppose $a \equiv b \not\equiv c$. Then, from Lemma 2.2.5 the partitions $(a, c, 1^b)$ occur in $[B_c(x, y, z)]_\Gamma$ as $(x-2k, z+1+2f, 1^{y-2f+2k-1})$ for $0 \leq f \leq \frac{y-z-1}{2}$, $0 \leq k \leq \frac{x-y-1}{2}$ where $x > y > z \geq 0$ and $a \equiv x \not\equiv y \not\equiv z$. In this case the coefficient is $v^{\lfloor \frac{y-2f+2k-1}{2} \rfloor}$. Now we find all x, y such that there exist f, k where $0 \leq f \leq \frac{y-z-1}{2}$, $0 \leq k \leq \frac{x-y-1}{2}$ satisfying these equalities:

- $a = x - 2k$,
- $c = z + 1 + 2f$,
- $b = y - 2f + 2k - 1$,
- $a + c + b = x + y + z$.

Let $n = x + y + z$ so that $z = n - x - y$. Now $x = a + 2k$ and as $0 \leq k \leq \frac{x-y-1}{2}$ and $0 \leq f \leq \frac{2y+x-n-1}{2}$, then $a \leq x \leq a + x - y - 1$, that is $y + 1 \leq a \leq x$. Moreover, since $c = n - x - y + 1 + 2f$, then $n - x - y + 1 \leq c \leq y$ and this implies $n - c - y + 1 \leq x$ and $c \leq y$. Furthermore, since $b = y - 2f + 2k - 1$, then $n - x - y \leq b \leq x - 2$ implies $n - b - y \leq x$ and $b + 2 \leq x$. Also, $x > y > n - x - y \geq 0$ and this can be written as $y<x<n - y$ and $x > n - 2y$. Now we have:

$$c \leq y \leq a - 1 \quad \text{and } x \leq n - y, \text{ where } y \equiv c.$$

Also, x must satisfy the following inequalities:

- $x \geq a$.
- $x \geq n - c - y + 1$.
  By replacing $n = a + b + c$, this case can be written as $x \geq a + b - y + 1$.
- $x \geq b + 2$.
  Since $n = a + b + c$ we can write $b + 2 = n - c - a + 2$, but $y + 1 \leq a$ then $b + 2 = n - c - a + 2 \leq n - c - y + 1$. Thus, if $x \geq n - c - y + 1$, then $x \geq b + 2$.
- $x \geq y + 1$.
  If $x \geq a$, then $x \geq y + 1$, because $y + 1 \leq a$.
- $x > n - 2y$.
  Since $c \leq y$, then $n - 2y \leq n - 2c<n - 2c + 1 \leq n - c - y + 1$. So, if $x \geq n - c - y + 1$, then $x > n - 2y$.
- $x \geq n - b - y$.
  Since $n = a + b + c$, then $n - b - y = a + c - y$, but $c \leq y$. Thus $n - b - y = a + c - y \leq a$. So, if $x \geq a$, then $x \geq n - b - y$.

Hence all conditions above are satisfied if $x \geq \max\{a, a + b - y + 1\}$. Note that

$$v^{\lfloor \frac{y-2f+2k-1}{2} \rfloor} = v^{\lfloor \frac{b}{2} \rfloor}$$

where $b = y - 2f + 2k - 1$. Hence,

$$B_r(\lambda) = \sum_{\substack{y=c \\ y\equiv c}}^{a-1} \sum_{\substack{x=\max\{a,a+b-y+1\} \\ x\not\equiv c}}^{n-y} v^{\lfloor \frac{b}{2} \rfloor}(x, y, n-x-y).$$

□

Case (2) of Theorem B follows in the same way by Lemma 2.2.6.

Proposition 2.2.10. *If $a \equiv b \equiv c$ then:*

$$\begin{aligned} B_r(\lambda) = & \sum_{\substack{y=c-1 \\ y\not\equiv c}}^{a-1} \sum_{\substack{x=max\{a+1,a+b-y+2\} \\ x\not\equiv c}}^{n-y} v^{\lfloor \frac{b+2}{2} \rfloor}(x, y, n-x-y) \\ & + \sum_{\substack{y=c+1 \\ y\not\equiv c}}^{a-1} \sum_{\substack{x=max\{a,a+b-y+1\} \\ x\equiv c}}^{n-y} v^{\lfloor \frac{b+2}{2} \rfloor}(x, y, n-x-y) \\ & + \sum_{\substack{y=c \\ y\equiv c}}^{a-2} \sum_{\substack{x=max\{a,a+b-y+2\} \\ x\equiv c}}^{n-y} v^{\lfloor \frac{b}{2} \rfloor}(x, y, n-x-y) \\ & + \sum_{\substack{y=c \\ y\equiv c}}^{a} \sum_{\substack{x=max\{a+2,a+b-y+2\} \\ x\equiv c}}^{n-y} v^{\lfloor \frac{b+4}{2} \rfloor}(x, y, n-x-y). \end{aligned}$$

Proof. Let $a \equiv b \equiv c$. Then, from Lemma 2.2.6 the partitions $(a, c, 1^b)$ occur in $[B_c(x, y, z)]_\Gamma$ in three cases as follows:

1. First case, $(a, c, 1^b)$ occurs in $[B_c(x, y, z)]_\Gamma$ as $(x - 2k - 1, z + 2 + 2f, 1^{y-2f+2k-1})$ for $0 \le f \le \frac{y-z-1}{2}$, $0 \le k \le \frac{x-y-2}{2}$ where $x > y > z \ge 0$ and $a \not\equiv x \equiv y \not\equiv z$. In this case the coefficient is $v^{\lfloor \frac{y-2f+2k+1}{2} \rfloor} = v^{\lfloor \frac{b+2}{2} \rfloor}$. By finding all x, y such that there exist f, k where these properties, we see that
$$\sum_{\substack{y=c-1 \\ y\not\equiv c}}^{a-1} \sum_{\substack{x=\max\{a+1,a+b-y+2\} \\ x\not\equiv c}}^{n-y} v^{\lfloor \frac{b+2}{2} \rfloor}(x, y, n-x-y).$$

2. Second case, $(a, c, 1^b)$ occurs in $[B_c(x, y, z)]_\Gamma$ as $(x-2k, z+1+2f, 1^{y-2f+2k-1})$ for $0 \le f \le \frac{y-z-2}{2}$, $0 \le k \le \frac{x-y-1}{2}$ where $a \equiv x \not\equiv y \equiv z$. In this case the coefficient is $v^{\lfloor \frac{y-2f+2k+1}{2} \rfloor} = v^{\lfloor \frac{b+2}{2} \rfloor}$. By finding all x, y such that there

exist f, k where these properties, we see that

$$\sum_{\substack{y=c+1 \\ y\not\equiv c}}^{a-1} \sum_{\substack{x=\max\{a,a+b-y+1\} \\ x\equiv c}}^{n-y} v^{\lfloor \frac{b+2}{2} \rfloor}(x, y, n-x-y).$$

3. Third case, $(a, c, 1^b)$ occurs in $[B_c(x,y,z)]_\Gamma$ as $(x-2l, z+2+2f, 1^{y-2f+2l-2})$ where $a \equiv x \equiv y \equiv z$ for some $0 \le f \le \frac{y-z-2}{2}$, $0 \le l \le \frac{x-y}{2}$ and in this case it occurs with coefficient

$$\begin{cases} v^{\lfloor \frac{y-2f+2l-2}{2} \rfloor} = v^{\lfloor \frac{b}{2} \rfloor}, & \text{if } l = 0\ , \\ v^{\lfloor \frac{y-2f+2l+2}{2} \rfloor} = v^{\lfloor \frac{b+4}{2} \rfloor}, & \text{if } l = \frac{x-y}{2}\ , \\ v^{\lfloor \frac{y-2f+2l-2}{2} \rfloor} + v^{\lfloor \frac{y-2f+2l+2}{2} \rfloor} = v^{\lfloor \frac{b}{2} \rfloor} + v^{\lfloor \frac{b+4}{2} \rfloor}, & \text{otherwise.} \end{cases}$$

This case gives two cases:

(a) Firstly, when $(a, c, 1^b) = (x-2k, z+2+2f, 1^{y-2f+2k-2})$ for $0 \le f \le \frac{y-z-2}{2}$, $0 \le k \le \frac{x-y-2}{2}$ where $a \equiv x \equiv y \equiv z$. In this case the coefficient is $v^{\lfloor \frac{y-2f+2k-2}{2} \rfloor} = v^{\lfloor \frac{b}{2} \rfloor}$. By finding all x, y such that there exist f, k where these properties, we see that

$$\sum_{\substack{y=c \\ y\equiv c}}^{a-2} \sum_{\substack{x=\max\{a,a+b-y+2\} \\ x\equiv c}}^{n-y} v^{\lfloor \frac{b}{2} \rfloor}(x, y, n-x-y).$$

(b) Secondly, when $(a, c, 1^b) = (x-2k-2, z+2+2f, 1^{y-2f+2k})$ for $0 \le f \le \frac{y-z-2}{2}$, $0 \le k \le \frac{x-y-2}{2}$ where $a \equiv x \equiv y \equiv z$ and in this case the coefficient is $v^{\lfloor \frac{y-2f+2k+4}{2} \rfloor} = v^{\lfloor \frac{b+4}{2} \rfloor}$. By finding all x, y such that there exist f, k where these properties, we see that

$$\sum_{\substack{y=c \\ y\equiv c}}^{a} \sum_{\substack{x=\max\{a+2,a+b-y+2\} \\ x\equiv c}}^{n-y} v^{\lfloor \frac{b+4}{2} \rfloor}(x, y, n-x-y).$$

$\square$

Proposition 2.2.11. *Suppose $a \not\equiv b$. Then*

$$\begin{aligned}
B_r(\lambda) = &\sum_{\substack{y=c\\y\equiv c}}^{a-1} \sum_{\substack{x=max\{a,a+b-y+1\}\\x\equiv c}}^{n-y} \alpha_b(x,y,n-x-y)\\
&+ \sum_{\substack{y=c\\y\equiv c}}^{a} \sum_{\substack{x=max\{a,a+b-y+2\}\\x\not\equiv c}}^{n-y} \beta_b(x,y,n-x-y)\\
&+ \sum_{\substack{y=c+1\\y\not\equiv c}}^{a} \sum_{\substack{x=max\{a,a+b-y+3,y+1\}\\x\not\equiv c}}^{n-y} v^{\lfloor\frac{b+2}{2}\rfloor}(x,y,n-x-y)\\
&+ \sum_{\substack{y=c-1\\y\not\equiv c}}^{a-1} \sum_{\substack{x=max\{a+1,a+b-y+3\}\\x\not\equiv c}}^{n-y} v^{\lfloor\frac{b+2}{2}\rfloor}(x,y,n-x-y)\\
&+ \sum_{\substack{y=c+1\\y\not\equiv c}}^{min\{a-1,b+1\}} v^{\lfloor\frac{b+2}{2}\rfloor}(a+b-y+1,y,c-1)
\end{aligned}$$

$$\text{where } \alpha_b = \begin{cases} v^{\lfloor\frac{b+4}{2}\rfloor}, & \text{if } c \not\equiv a,\\ v^{\lfloor\frac{b}{2}\rfloor}, & \text{if } c \equiv a,\end{cases}$$

$$\text{where } \beta_b = \begin{cases} v^{\lfloor\frac{b}{2}\rfloor}, & \text{if } c \not\equiv a,\\ v^{\lfloor\frac{b+4}{2}\rfloor}, & \text{if } c \equiv a.\end{cases}$$

Proof.

First suppose that $a \not\equiv b \not\equiv c$. Then, from Lemma 2.2.7 the partitions $(a,c,1^b)$ occurs in $[B_c(x,y,z)]_\Gamma$ in four cases as follows:

1. First case, $(a,c,1^b)$ occurs in $[B_c(x,y,z)]_\Gamma$ as $(x-2k, z+1+2f, 1^{y-2f+2k-1})$ for $0 \le f \le \frac{y-z-1}{2}$, $0 \le k \le \frac{x-y-2}{2}$ where $x > y > z \ge 0$ and $a \equiv x \equiv y \not\equiv z$. In this case the coefficient is $v^{\lfloor\frac{y-2f+2k-1}{2}\rfloor} = v^{\lfloor\frac{b}{2}\rfloor}$. By finding all x, y such that there exist f, k where these properties, we see that

$$\sum_{\substack{y=c\\y\equiv c}}^{a-2} \sum_{\substack{x=\max\{a,a+b-y+1\}\\x\equiv c}}^{n-y} v^{\lfloor\frac{b}{2}\rfloor}(x,y,n-x-y).$$

2. Second case, $(a,c,1^b)$ occurs in $[B_c(x,y,z)]_\Gamma$ as $(x-2k-1, z+2+2f, 1^{y-2f+2k-1})$ for $0 \le f \le \frac{y-z-2}{2}$, $0 \le k \le \frac{x-y-1}{2}$ where $a \not\equiv x \not\equiv y \equiv z$. In this case the coefficient is $v^{\lfloor\frac{y-2f+2k+3}{2}\rfloor} = v^{\lfloor\frac{b+4}{2}\rfloor}$. By finding all x, y such

that there exist f,k where these properties, we see that

$$\sum_{\substack{y=c \\ y\equiv c}}^{a} \sum_{\substack{x=\max\{a+1,a+b-y+2\} \\ x\not\equiv c}}^{n-y} v^{\lfloor \frac{b+4}{2} \rfloor}(x,y,n-x-y).$$

3. Third case, $(a,c,1^b)$ occurs in $[B_c(x,y,z)]_\Gamma$ as $(x-2k-1,z+3+2f,1^{y-2f+2k-2})$ where $a \not\equiv x \equiv y \equiv z$ for some $0 \le k \le \frac{x-y-2}{2}$ and $0 \le f \le \frac{y-z-2}{2}$ and in this case it occurs with coefficient

$$\begin{cases} v^{\lfloor \frac{y-2f+2k}{2} \rfloor} = v^{\lfloor \frac{b+2}{2} \rfloor}, & \text{if } f = \frac{y-z-2}{2}, \\ 2v^{\lfloor \frac{y-2f+2k}{2} \rfloor} = 2v^{\lfloor \frac{b+2}{2} \rfloor}, & \text{if } 0 \le f \le \frac{y-z-4}{2}. \end{cases}$$

This case gives two cases

   (a) Firstly, when $(a,c,1^b) = (x-2k-1,z+3+2f,1^{y-2f+2k-2})$ for $0 \le f \le \frac{y-z-4}{2}$, $0 \le k \le \frac{x-y-2}{2}$ where $a \not\equiv x \equiv y \equiv z$. In this case the coefficient is $v^{\lfloor \frac{y-2f+2k}{2} \rfloor} = v^{\lfloor \frac{b+2}{2} \rfloor}$. By finding all x,y such that there exist f,k where these properties, we see that

$$\sum_{\substack{y=c+1 \\ y\not\equiv c}}^{a-1} \sum_{\substack{x=\max\{a+1,a+b-y+2,y+1\} \\ x\not\equiv c}}^{n-y} v^{\lfloor \frac{b+2}{2} \rfloor}(x,y,n-x-y).$$

   (b) Secondly, when $(a,c,1^b) = (x-2k-1,z+3+2f,1^{y-2f+2k-2})$ for $0 \le f \le \frac{y-z-2}{2}$, $0 \le k \le \frac{x-y-2}{2}$ where $a \not\equiv x \equiv y \equiv z$ and in this case the coefficient is $v^{\lfloor \frac{y-2f+2k}{2} \rfloor} = v^{\lfloor \frac{b+2}{2} \rfloor}$. By finding all x,y such that there exist f,k where these properties, we see that

$$\sum_{\substack{y=c-1 \\ y\not\equiv c}}^{a-1} \sum_{\substack{x=\max\{a+1,a+b-y+3\} \\ x\not\equiv c}}^{n-y} v^{\lfloor \frac{b+2}{2} \rfloor}(x,y,n-x-y).$$

4. fourth case, $(a,c,1^b)$ occurs in $[B_c(x,y,z)]_\Gamma$ as $(x-2k-1,z+1,1^{y+2k})$ where $a \not\equiv x \equiv y \equiv z$ for $0 \le k \le \frac{x-y-2}{2}$ and in this case the coefficient is $v^{\lfloor \frac{y+2k+2}{2} \rfloor} = v^{\lfloor \frac{b+2}{2} \rfloor}$. By finding all x,y such that there exist f,k where these properties, we see that

$$\sum_{\substack{y=c-1 \\ y\not\equiv c}}^{\min\{a-1,b\}} v^{\lfloor \frac{b+2}{2} \rfloor}(a+b-y+1,y,c-1).$$

New we summarize this case, if $a \not\equiv b \not\equiv c$, then

$$\begin{aligned} B_r(\lambda) = &\sum_{\substack{y=c\\ y\equiv c}}^{a-2} \sum_{\substack{x=\max\{a,a+b-y+1\}\\ x\equiv c}}^{n-y} v^{\lfloor \frac{b}{2} \rfloor}(x,y,n-x-y) \\ &+ \sum_{\substack{y=c\\ y\equiv c}}^{a} \sum_{\substack{x=\max\{a+1,a+b-y+2\}\\ x\not\equiv c}}^{n-y} v^{\lfloor \frac{b+4}{2} \rfloor}(x,y,n-x-y) \\ &+ \sum_{\substack{y=c+1\\ y\not\equiv c}}^{a-1} \sum_{\substack{x=\max\{a+1,a+b-y+2,y+1\}\\ x\not\equiv c}}^{n-y} v^{\lfloor \frac{b+2}{2} \rfloor}(x,y,n-x-y) \\ &+ \sum_{\substack{y=c-1\\ y\not\equiv c}}^{a-1} \sum_{\substack{x=\max\{a+1,a+b-y+3\}\\ x\not\equiv c}}^{n-y} v^{\lfloor \frac{b+2}{2} \rfloor}(x,y,n-x-y) \\ &+ \sum_{\substack{y=c-1\\ y\not\equiv c}}^{\min\{a-1,b\}} v^{\lfloor \frac{b+2}{2} \rfloor}(a+b-y+1,y,c-1). \end{aligned} \tag{2.8}$$

Now suppose $a \not\equiv b \equiv c$. Then, from Lemma 2.2.8 the partitions $(a,c,1^b)$ occur in $[B_c(x,y,z)]_\Gamma$ in four cases as follows:

1. First case, $(a,c,1^b)$ occur in $[B_c(x,y,z)]_\Gamma$ as $(x-2k-1, z+1+2f, 1^{y-2f+2k})$ for $0 \le f \le \frac{y-z-1}{2}$, $0 \le k \le \frac{x-y-2}{2}$ where $x > y > z \ge 0$ and $a \not\equiv x \equiv y \not\equiv z$. In this case the coefficient is $v^{\lfloor \frac{y-2f+2k+4}{2} \rfloor} = v^{\lfloor \frac{b+4}{2} \rfloor}$. By finding all x,y such that there exist f,k where these properties, we see that

$$\sum_{\substack{y=c\\ y\equiv c}}^{a-1} \sum_{\substack{x=\max\{a+1,a+b-y+1\}\\ x\equiv c}}^{n-y} v^{\lfloor \frac{b+4}{2} \rfloor}(x,y,n-x-y).$$

2. Second case, $(a,c,1^b)$ occurs in $[B_c(x,y,z)]_\Gamma$ as $(x-2k, z+2+2f, 1^{y-2f+2k-2})$ for $0 \le f \le \frac{y-z-2}{2}$, $0 \le k \le \frac{x-y-1}{2}$ where $a \equiv x \not\equiv y \equiv z$. In this case the coefficient is $v^{\lfloor \frac{y-2f+2k-2}{2} \rfloor} = v^{\lfloor \frac{b}{2} \rfloor}$. By finding all x,y such that there exist f,k where these properties, we see that

$$\sum_{\substack{y=c\\ y\equiv c}}^{a-1} \sum_{\substack{x=\max\{a,a+b-y+2\}\\ x\not\equiv c}}^{n-y} v^{\lfloor \frac{b}{2} \rfloor}(x,y,n-x-y).$$

3. Third case, $(a,c,1^b)$ occurs in $[B_c(x,y,z)]_\Gamma$ as $(x-2l, z+3+2g, 1^{y-2g+2l-3})$

where $a \equiv x \equiv y \equiv z$ for some $0 \le g \le \frac{y-z-4}{2}$, $0 \le l \le \frac{x-y}{2}$ or $g = \frac{y-z-2}{2}$ and $1 \le l \le \frac{x-y-2}{2}$ and in this case it occurs with coefficient

$$\begin{cases} v^{\lfloor \frac{y-2g+2l-1}{2} \rfloor} = v^{\lfloor \frac{b+2}{2} \rfloor}, & \text{if } l = 0 \text{ or } l = \frac{x-y}{2} \text{ or } g = \frac{x-y-2}{2}. \\ 2v^{\lfloor \frac{y-2g+2l-1}{2} \rfloor} = 2v^{\lfloor \frac{b+2}{2} \rfloor}, & \text{otherwise.} \end{cases},$$

This case gives two cases:

(a) Firstly, when $(a, c, 1^b) = (x - 2k, z + 3 + 2f, 1^{y-2f+2k-3})$ for $0 \le f \le \frac{y-z-4}{2}$, $0 \le k \le \frac{x-y}{2}$ where $a \equiv x \equiv y \equiv z$. In this case the coefficient is $v^{\lfloor \frac{y-2f+2k-1}{2} \rfloor} = v^{\lfloor \frac{b+2}{2} \rfloor}$. By finding all x, y such that there exist f, k where these properties, we see that

$$\sum_{\substack{y=c+1 \\ y \not\equiv c}}^{a} \sum_{\substack{x=\max\{a, a+b-y+3, y+1\} \\ x \not\equiv c}}^{n-y} v^{\lfloor \frac{b+2}{2} \rfloor}(x, y, n-x-y).$$

(b) Secondly, when $(a, c, 1^b) = (x - 2k - 2, z + 3 + 2f, 1^{y-2f+2k-1})$ for $0 \le f \le \frac{y-z-2}{2}$, $0 \le k \le \frac{x-y-2}{2}$ where $a \equiv x \equiv y \equiv z$ and in this case the coefficient is $v^{\lfloor \frac{y-2f+2k+1}{2} \rfloor} = v^{\lfloor \frac{b+2}{2} \rfloor}$. By finding all x, y such that there exist f, k where these properties, we see that

$$\sum_{\substack{y=c-1 \\ y \not\equiv c}}^{a} \sum_{\substack{x=\max\{a, a+b-y+3\} \\ x \not\equiv c}}^{n-y} v^{\lfloor \frac{b+2}{2} \rfloor}(x, y, n-x-y).$$

4. Fourth case $(a, c, 1^b)$ occurs in $[B_c(x, y, z)]_\Gamma$ as $(x - 2k, z + 1, 1^{y+2k-1})$ where $a \equiv x \equiv y \equiv z$ for $0 \le k \le \frac{x-y-2}{2}$ and in this case the coefficient is $v^{\lfloor \frac{y+2k+1}{2} \rfloor} = v^{\lfloor \frac{b+2}{2} \rfloor}$. By finding all x, y such that there exist f, k where these properties, we see that

$$\sum_{\substack{y=c-1 \\ y \not\equiv c}}^{\min\{a-1, b+1\}} v^{\lfloor \frac{b+2}{2} \rfloor}(a+b-y+1, y, c-1).$$

New we summarize this case, if $a \not\equiv b \equiv c$, then

$$
\begin{aligned}
B_r(\lambda) = &\sum_{\substack{y=c\\ y\equiv c}}^{a-1} \sum_{\substack{x=\max\{a+1,a+b-y+1\}\\ x\equiv c}}^{n-y} v^{\lfloor \frac{b+4}{2} \rfloor}(x,y,n-x-y)\\
&+ \sum_{\substack{y=c\\ y\equiv c}}^{a-1} \sum_{\substack{x=\max\{a,a+b-y+2\}\\ x\not\equiv c}}^{n-y} v^{\lfloor \frac{b}{2} \rfloor}(x,y,n-x-y)\\
&+ \sum_{\substack{y=c+1\\ y\not\equiv c}}^{a} \sum_{\substack{x=\max\{a,a+b-y+3,y+1\}\\ x\not\equiv c}}^{n-y} v^{\lfloor \frac{b+2}{2} \rfloor}(x,y,n-x-y) \qquad (2.9)\\
&+ \sum_{\substack{y=c-1\\ y\not\equiv c}}^{a} \sum_{\substack{x=\max\{a,a+b-y+3\}\\ x\not\equiv c}}^{n-y} v^{\lfloor \frac{b+2}{2} \rfloor}(x,y,n-x-y)\\
&+ \sum_{\substack{y=c-1\\ y\not\equiv c}}^{\min\{a-1,b+1\}} v^{\lfloor \frac{b+2}{2} \rfloor}(a+b-y+1,y,c-1).
\end{aligned}
$$

Hence, by combining equations 2.8 and 2.9 we get: If $a \not\equiv b$, then

$$
\begin{aligned}
B_r(\lambda) = &\sum_{\substack{y=c\\ y\equiv c}}^{a-1} \sum_{\substack{x=\max\{a,a+b-y+1\}\\ x\equiv c}}^{n-y} \alpha_b(x,y,n-x-y)\\
&+ \sum_{\substack{y=c\\ y\equiv c}}^{a} \sum_{\substack{x=\max\{a,a+b-y+2\}\\ x\not\equiv c}}^{n-y} \beta_b(x,y,n-x-y)\\
&+ \sum_{\substack{y=c+1\\ y\not\equiv c}}^{a} \sum_{\substack{x=\max\{a,a+b-y+3,y+1\}\\ x\not\equiv c}}^{n-y} v^{\lfloor \frac{b+2}{2} \rfloor}(x,y,n-x-y)\\
&+ \sum_{\substack{y=c-1\\ y\not\equiv c}}^{a-1} \sum_{\substack{x=\max\{a+1,a+b-y+3\}\\ x\not\equiv c}}^{n-y} v^{\lfloor \frac{b+2}{2} \rfloor}(x,y,n-x-y)\\
&+ \sum_{\substack{y=c+1\\ y\not\equiv c}}^{\min\{a-1,b+1\}} v^{\lfloor \frac{b+2}{2} \rfloor}(a+b-y+1,y,c-1)
\end{aligned}
$$

$$
\text{where } \alpha_b = \begin{cases} v^{\lfloor \frac{b+4}{2} \rfloor}, & \text{if } c \not\equiv a,\\ v^{\lfloor \frac{b}{2} \rfloor}, & \text{if } c \equiv a, \end{cases}
$$

$$
\text{where } \beta_b = \begin{cases} v^{\lfloor \frac{b}{2} \rfloor}, & \text{if } c \not\equiv a,\\ v^{\lfloor \frac{b+4}{2} \rfloor}, & \text{if } c \equiv a. \end{cases}
$$

$\square$

2.3 Decomposition numbers for $\mathscr{H}_{F,q}(\mathfrak{S}_n)$

Suppose that F is a field of an arbitrary characteristic $p \geq 0$ and let e be the smallest positive integer such that $1 + q + ... + q^{e-1} = 0$. If $e = \infty$, then the Hecke algebra $\mathscr{H}$ is semisimple, so we assume throughout that $e \geq 2$ is finite. Define $\mathcal{F}$ to be the vector space over $\mathbb{C}$ with basis the partitions of n for all $n \geq 0$. For a partition $\nu \vdash n$, let $[B_c(\nu)]_{F,q}$ denote the column of the decomposition matrix of $\mathscr{H}_{F,q}(\mathfrak{S}_n)$ indexed by the partition ν, that is

$$[B_c(\nu)]_{F,q} = \sum_{\lambda \vdash n} d_{\lambda\nu}\lambda.$$

If $s \geq 1$ and $0 \leq r<e$, define

$$(\lambda) \uparrow_s^r = \sum_{\lambda \xrightarrow{r:s} \nu} \nu$$

and extend linearly to define $B \uparrow_r^s$ for all $B \in \mathcal{F}$.

Proposition 2.3.1. *[26, P116] Suppose λ is a partition. Then*

$$[B_c(\lambda)]_{F,q} \uparrow_r^s = \sum_{\mu} \alpha_{\mu\lambda}[B_c(\mu)]_{F,q}$$

for some $\alpha_{\mu\lambda} \in \mathbb{Z}_{\geq 0}$.

Let $\mathscr{H} = \mathscr{H}_{F,q}(\mathfrak{S}_n)$ and that $\mathscr{H}_0 = \mathscr{H}_{\mathbb{C},\hat{q}}(\mathfrak{S}_n)$, where $\hat{q}$ is a primitive e^{th} root of unity in $\mathbb{C}$. Recall that the decomposition matrix of $\mathscr{H}_0$ can be computed by using the LLT algorithm.

Theorem 2.3.2. *[26, Theorem 6.35] Suppose D is the decomposition matrix of $\mathscr{H}$ and D_0 is the decomposition matrix of $\mathscr{H}_0$. Then there exists a square unitriangular matrix A where entries are non-negative integers such that*

$$D = D_0 A.$$

Corollary 2.3.3. *Let λ and ν be partitions of n with ν e–regular. Then*

$$[S^\lambda : D^\nu]_{\mathscr{H}_0} \leq [S^\lambda : D^\nu]_{\mathscr{H}}.$$

Now let $q = -1$. Suppose that $\nu = (x, y, z)$ and that $\ell(\nu) = 3$. Note that in the proof of Theorem A, in all cases we had

$$[B_c(\tau)]_\Gamma \uparrow_r^s = [B_c(\nu)]_\Gamma.$$

Hence, if $\lambda \in \Gamma$ and $\ell(\nu) = 3$, we have

$$[S^\lambda : D^\nu]_{\mathscr{H}_0} \geq [S^\lambda : D^\nu]_{\mathscr{H}}.$$

Combining this with Corollary 2.3.3, we have the following results.

Theorem C. *Suppose that* $\mathscr{H} = \mathscr{H}_{F,-1}(\mathfrak{S}_n)$, *that* $\lambda \in \Gamma$ *and that* $\nu = (x, y, z)$ *with* $\ell(\nu) = 3$. *Then*

$$[S^\lambda : D^\nu]_{\mathscr{H}_0} = [S^\lambda : D^\nu]_{\mathscr{H}}$$

and these decomposition numbers are given by Theorem B.

Chapter 3

Decomposable Specht modules for the Hecke algebra

In this Chapter, we find some cases of decomposable Specht modules for the Hecke algebra $\mathscr{H}_{\mathbb{C},-1}(\mathfrak{S}_n)$ which are indexed by partitions of the form $(a, 3, 1^b)$, where a, b are even.

3.1 Background

Recall that $\mathfrak{S}_n$ is the symmetric group on n letters and S^λ is the Specht module indexed by a partition λ. Now for any $e \neq 2$, the Specht modules S^λ for the Hecke algebra $\mathscr{H} = \mathscr{H}_{F,q}(\mathfrak{S}_n)$ is indecomposable [30],[15],[5]. If $e = p = 2$, Murphy in [27] shows which Specht modules labelled by hook partitions of the form $(n-a, 1^a)$ are decomposable. Moreover, Dodge and Fayers in [7] found a new family of decomposable Specht modules for the symmetric group algebra $F_2\mathfrak{S}_n$ and these decomposable Specht modules are labelled by partitions of the form $(a, 3, 1^b)$, where a, b are even. They found which Specht modules S^λ had a summand isomorphic to an irreducible Specht module S^μ by considering homomorphisms between Specht modules. They assumed that S^μ is irreducible and found when there are homomorphisms $\gamma : S^\mu \to S^\lambda$ and $\delta : S^\lambda \to S^{\mu'}$ such that $\delta \circ \gamma$ is non-zero. Recall the following result:

Lemma 3.1.1. *Suppose M and N are A–modules, for some algebra A. If*

M is irreducible and we have homomorphisms $\gamma : M \to N$ and $\delta : N \to M$ such that $\delta \circ \gamma = Id_M$, then M is a summand of N.

Proof. Let $C = \ker(\delta)$. We claim that $N \cong M \oplus C$. First note that $M/\ker(\gamma) \cong \mathrm{Im}(\gamma)$, so since M is irreducible, and $\gamma \neq 0$ we have $\mathrm{Im}(\gamma) \cong M$. Now suppose that $x \in \mathrm{Im}(\gamma) \cap \ker(\delta)$. Then $x = \gamma(a)$, some $a \in M$. Now $a = \delta(\gamma(a)) = \delta(x) = 0$ so $x = \gamma(a) = 0$. Hence, $\mathrm{Im}(\gamma) \cap \ker(\delta) = \{0\}$. By the rank-nullity Theorem,

$$\dim(N) = \dim(\mathrm{Im}(\gamma)) + \dim(\ker(\delta))$$

so $N = \mathrm{Im}(\gamma) + \ker(\delta)$. Hence, $N = \mathrm{Im}(\gamma) \oplus \ker(\delta) \cong M \oplus C$.

□

So the results of Dodge and Fayers show that S^μ occurs as a summand of S^λ, since $S^\mu \cong S^{\mu'}$ [7, Lemma 2.1] if $p = 2$ and S^μ is irreducible, so $\delta \circ \gamma$ is the identity on S^μ. They stated the following theorem:

Theorem 3.1.2. *[7, Theorem 3.1] Let $\lambda = (a, 3, 1^b)$ be a partition of n, where a, b are positive even integers with $a \geq 4$, and suppose μ is a partition of n such that S^μ is irreducible. Then S^λ has a direct summand isomorphic to S^μ if and only if one of the following holds.*

1. *μ or μ' equals (u, v), where $v \equiv 3 \mod 4$ and $\binom{u-v}{a-v}$ is odd.*
2. *μ or μ' equals $(u, v, 2)$, where $\binom{u-v}{a-v}$ is odd.*

Now we explain how Dodge and Fayers construct homomorphisms between Specht modules with more details. Let μ and λ be partitions of n and consider $\mathrm{Hom}_{F\mathfrak{S}_n}(S^\mu, S^\lambda)$. Assume $\mathscr{H} = F\mathfrak{S}_n$ where F is a field of characteristic $p \geq 2$. Recall that the Specht module S^λ is a submodule of M^λ, so any homomorphism from S^μ to S^λ can be written as a homomorphism from S^μ to M^λ. From Definition 1.3.18 recall that $\mathcal{T}(\mu, \lambda)$ is the set of row-standard μ–tableaux of type λ, and $\mathcal{T}_0(\mu, \lambda)$ is the set of semistandard μ–tableaux of type λ. For each $A \in \mathcal{T}(\mu, \lambda)$, James defines a homomorphism $\Theta_A : M^\mu \longrightarrow M^\lambda$ over any field and we do not need the exact definition here. Let $\hat{\Theta}_A$ denote the restriction of Θ_A to the Specht module S^μ.

Theorem 3.1.3. *[15, Lemma 13.11 and Theorem 13.13] The set*

$$\{\hat{\Theta}_A \mid A \in \mathcal{T}_0(\mu, \lambda)\}$$

is linearly independent. If either $e \neq 2$ or μ is 2–regular, then $\{\hat{\Theta}_A \mid A \in \mathcal{T}_0(\mu,\lambda)\}$ also spans $\mathrm{Hom}_{F\mathfrak{S}_n}(S^\mu, M^\lambda)$.

For any pair (d,t) such that $d \geq 1$ and $1 \leq t \leq \lambda_{d+1}$, there is a homomorphism $\psi_{d,t} : M^\lambda \longrightarrow M^\nu$, where the composition ν depends on λ, d, t. Now to check whether the image of a homomorphism $\theta : S^\mu \longmapsto M^\lambda$ lies in S^λ, we use the Kernel Intersection Theorem below.

Theorem 3.1.4. *[15, Corollary 17.18] Let λ be a partition of n. Then*

$$S^\lambda = \bigcap_{\substack{d\geq 1\\ 1\leq t\leq \lambda_{d+1}}} \mathrm{Ker}(\psi_{d,t}).$$

This provides a strategy for computing $\mathrm{Hom}_{F\mathfrak{S}_n}(S^\mu, M^\lambda)$ by finding all linear combinations θ of the homomorphisms $\hat{\Theta}_A$ such that $\psi_{d,t} \circ \theta = 0$ for every d, t.

Definition 3.1.5. Let X be a multiset of positive integers. We define X_i to be the number of is in X. If X and Y are multisets, we write $X \sqcup Y$ for the multiset with $(X \sqcup Y)_i = X_i + Y_i$ for all i. Moreover, if A is a row-standard tableau, we denote the multiset of entries in row j of A by A^j. In particular, we write A_i^j for the number of entries equal to i in row j of A.

The next theorems show how to compute the composition $\psi_{d,t} \circ \hat{\Theta}_A$ when $A \in \mathcal{T}(\mu,\lambda)$.

Theorem 3.1.6. *[10, Lemma 5] Let λ and μ be partitions of n, $A \in \mathcal{T}(\mu,\lambda)$, $d \geq 1$ and $1 \leq t \leq \lambda_{d+1}$. Suppose $\mathcal{S}$ is the set of all row-standard tableaux which can be obtained from A by replacing t of the entries equal to $d+1$ in A with ds. Then*

$$\psi_{d,t} \circ \Theta_A = \sum_{S\in\mathcal{S}} \prod_{j\geq 1} \binom{S_d^j}{A_d^j} \Theta_S.$$

The tableaux S in Theorem 3.1.6 are not necessarily semistandard, so it can be difficult to compute homomorphism spaces. The next theorem helps to express a tableau homomorphism in terms of semistandard homomorphisms.

Theorem 3.1.7. *[10, Lemma 7] Let μ be a partition of n and λ a composition of n, and suppose i, j, k are positive integers with $j \neq k$ and $\mu_j \geq \mu_k$. Consider $A \in \mathcal{T}(\mu,\lambda)$, and let $\mathcal{S}$ be the set of all $S \in \mathcal{T}(\mu,\lambda)$ such that:*

- $S_i^j = A_i^j + A_i^k$;
- $S_l^j \leq A_l^j$ *for every* $l \neq i$;
- $S^l = A^l$ *for all* $l \neq j, k$.

Then

$$\hat{\Theta}_A = (-1)^{A_i^k} \sum_{S \in \mathcal{S}} \prod_{l \geq 1} \binom{S_l^k}{A_l^k} \hat{\Theta}_S.$$

Example 3.1.8. Let $\lambda = (2,2,1)$ and $\mu = (3,2)$ and $p > 0$. We want to find $\mathrm{Hom}_{F\mathfrak{S}_n}(S^\mu, S^\lambda)$. So

$$\mathcal{T}_0(\mu,\lambda) = \left\{ A_1 = \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 3 \\ \cline{1-2} \end{array}, A_2 = \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 2 \\ \cline{1-2} \end{array} \right\}$$

now if $\Theta : S^\mu \to M^\lambda$ then $\Theta = \alpha\Theta_{A_1} + \beta\Theta_{A_2}$ for some $\alpha, \beta \in F$. If we identify a tableau A with the corresponding homomorphism $\hat{\Theta}_A$ then by using Theorem 3.1.6 we get

$$\psi_{1,1} \circ \hat{\Theta}_{A_1} = 3 \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 3 \\ \cline{1-2} \end{array} + \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 1 & 3 \\ \cline{1-2} \end{array}.$$

Now we use Theorem 3.1.7 to move 1 from row 2 to row 1 in $\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 1 & 3 \\ \cline{1-2} \end{array}$ then we get

$$\psi_{1,1} \circ \hat{\Theta}_{A_1} = 3 \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 3 \\ \cline{1-2} \end{array} - \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 3 \\ \cline{1-2} \end{array} = 2 \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 3 \\ \cline{1-2} \end{array}.$$

Similarly by applying Theorem 3.1.6 we get

$$\psi_{1,1} \circ \hat{\Theta}_{A_2} = \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 1 & 2 \\ \cline{1-2} \end{array},$$

then we use Theorem 3.1.7 to move 1 from row 2 to row 1 and we get

$$\psi_{1,1} \circ \hat{\Theta}_{A_2} = - \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 3 \\ \cline{1-2} \end{array}$$

hence if $\psi_{1,1} \circ \hat{\Theta} = 0$ then $2\alpha - \beta = 0$. Also, by applying Theorem 3.1.6 we get

$$\psi_{2,1} \circ \hat{\Theta}_{A_1} = 2 \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 2 \\ \cline{1-2} \end{array}$$

and

$$\psi_{2,1} \circ \hat{\Theta}_{A_2} = \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 2 \\ \cline{1-2}\end{array}$$

hence $2\alpha + \beta = 0$. Now if $p \neq 2$, then $\beta = 0$ and $2\alpha = 0$, so $\alpha = 0$. Thus, the only solution is $\alpha = \beta = 0$. If $p = 2$, then $\beta = 0$ and $2\alpha = 0$, which is it true for any α. So the homomorphism space is one dimensional, spanned by $\Theta(\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 3 \\ \cline{1-2}\end{array})$.

Definition 3.1.9. Let μ be a partition, and suppose that S and A are row-standard μ–tableaux of the same type. We say that S dominates A if we can obtain A from S by repeatedly swapping an entry of S with a larger entry in a lower row and re-ordering within each row. We write $S \trianglerighteq A$.

The following theorem gives that a linear combination of row-standard homomorphisms is non-zero without needing to go through the full process of expressing it.

Theorem 3.1.10. *[7, Lemma 4.6] Let μ be a partition of n and λ a composition of n, and $A \in \mathcal{T}(\mu, \lambda)$. If*

$$\hat{\Theta}_A = \sum_{S \in \mathcal{T}_0(\mu,\lambda)} a_S \hat{\Theta}_S,$$

then $a_S \neq 0$ only if $S \trianglerighteq A$.

We now show how to compute the composition of homomorphisms between two Specht modules.

Definition 3.1.11. Let $x_1, x_2, \dots, x_m$ be non-negative integers such that $\sum_{i=1}^{m} x_i = x$. We say $\binom{x}{x_1,x_2,\dots,x_m}$ for the corresponding multinomial coefficient which is defined to be $\binom{x}{x_1,x_2,\dots,x_m} = \frac{x!}{x_1! \quad x_2! \quad \dots \quad x_m!}$.

Theorem 3.1.12. *[7, Proposition 4.7] Let λ, μ, ν be compositions of n, S be a λ–tableau of type μ and let A be a μ–tableau of type ν. Consider $\mathcal{X}$ be the set of all collections $X = (X^{ij})_{i,j \geq 1}$ of multisets such that*

$$|X^{ij}| = S_i^j \quad \textit{for each } i,\ j, \qquad \bigsqcup_{j \geq 1} X^{ij} = A^i \quad \textit{for each } i.$$

For $X \in \mathcal{X}$, let U_X denote the row-standard λ-tableau with $(U_X)^j = \bigsqcup_{i \geq 1} X^{ij}$.

Then

$$\Theta_A \circ \Theta_S = \sum_{X \in \mathcal{X}} \prod_{i,j \geq 1} \binom{X_i^{1j} + X_i^{2j} + X_i^{3j} + \dots}{X_i^{1j}, X_i^{2j}, X_i^{3j}, \dots} \Theta_{U_X}.$$

By using Theorem 3.1.12 we can compute composition of homomorphisms between two Specht modules. Hence, we may use the technique of Dodge and Fayers to describe certain decomposable Specht modules.

Example 3.1.13. Take $\lambda = (4,3,1^2)$ and $\mu = (6,3)$. Consider a homomorphism $\sigma : S^{(4,3,1^2)} \longrightarrow S^{(2^3,1^3)}$. We construct this homomorphism in the case where $v = 3$. Suppose $\mathcal{U}$ is the set of λ-tableaux having the form:

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & \star \\ \hline 1 & \star & \star \\ \cline{1-3} \star \\ \cline{1-1} \star \\ \cline{1-1} \end{array}$$

where the $\star$s represent the numbers $2,3,4,5,6$. Define

$$\sigma = \sum_{T \in \mathcal{U}} \hat{\Theta}_T$$

and $\gamma : S^{(6,3)} \longrightarrow S^{(4,3,1^2)}$ by $\gamma = \hat{\Theta}_A + \hat{\Theta}_B$, where $A, B \in \mathcal{T}(\mu, \lambda)$ are given by:

$$A = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 2 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 \\ \cline{1-3} \end{array};$$

$$B = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 3 & 4 \\ \hline 1 & 2 & 2 \\ \cline{1-3} \end{array}.$$

Dodge and Fayers showed the following:

- $\sigma : S^{\lambda} \longrightarrow S^{\mu'}$ and $\sigma \neq 0$;
- $\gamma : S^{\mu} \longrightarrow S^{\lambda}$ and $\gamma \neq 0$;
- $\sigma \circ \gamma \neq 0$.

Hence, S^{λ} has a summand isomorphic to S^{μ}.

3.2 Decomposable Specht modules for Hecke algebra $\mathscr{H}_{\mathbb{C},-1}(\mathfrak{S}_n)$

In this section, we present some results on the representations of the Hecke algebra $\mathscr{H} = \mathscr{H}_{\mathbb{C},-1}(\mathfrak{S}_n)$. We first give analogues of some of the results in section 3.1.

Definition 3.2.1. Let $A \in \mathcal{T}(\mu, \lambda)$. We define 1_A to be permutation formed by taking $\mathfrak{t}^\lambda 1_A$ to be the row standard λ–tableau for which i belongs to row r when the place occupied by i in $\mathfrak{t}^\mu$ is occupied by r in A.

Definition 3.2.2. Define the relation $\sim_r$ on $\mathcal{T}(\mu, \lambda)$ by setting $A \sim_r B$ if row i of A has the same numbers as row i of B for all i, where $A, B \in \mathcal{T}(\mu, \lambda)$.

Definition 3.2.3. Suppose $A \in \mathcal{T}(\mu, \lambda)$. We define the homomorphism

$$\Theta_A : M^\mu \to M^\lambda.$$

by

$$\Theta_A(x_\mu h) = \left(x_\lambda \sum_{A' \sim_r A} T_{1_{A'}} \right) h$$

for all $h \in \mathscr{H}$.

Suppose λ and μ are partitions and that $\Theta : M^\mu \to M^\lambda$. Let $\hat{\Theta}$ denote the restriction of Θ to S^μ.

Theorem 3.2.4. *[6, Corollary 8.7] Let μ be a partition of n and λ be a composition of n. Then $\{\hat{\Theta}_A \mid A \in \mathcal{T}_0(\lambda, \mu)\}$ is a linearly independent subset of $\mathrm{Hom}_{\mathscr{H}}(S^\mu, M^\lambda)$. If μ is 2–regular then $\{\hat{\Theta}_A \mid A \in \mathcal{T}_0(\mu, \lambda)\}$ is a basis of $\mathrm{Hom}_{\mathscr{H}}(S^\mu, M^\lambda)$.*

Definition 3.2.5. Suppose μ is a partition. Let $d \geq 1$ and consider t such that $1 \leq t < \mu_{d+1}$. Suppose $\nu^{d,t}$ is the composition defined by

$$\nu_i^{d,t} = \begin{cases} \mu_i + t & \text{if } i = d, \\ \mu_i - t & \text{if } i = d+1, \\ \mu_i & \text{otherwise.} \end{cases}$$

Let A be the row standard μ–tableau of type $\nu^{d,t}$ with all entries in row i equal to i, except for $i = d+1$, when there are t entries equal to d

and $\mu_{d+1} - t$ entries equal to $d+1$. We write $\psi_{d,t}$ for the homomorphism $\Theta_A : M^\mu \to M^{\nu^{d,t}}$. Then we have the following theorem.

Theorem 3.2.6. *[5, Theorem 7.5] If μ is a partition of n, then*

$$S^\mu = \bigcap_{d \geq 1} \bigcap_{t=1}^{\mu_{d+1}} \mathrm{Ker}(\psi_{d,t}).$$

Now we give the generalization of Theorem 3.1.6 and Theorem 3.1.7. We define the Gaussian polynomials $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$.

Definition 3.2.7. Let $\alpha \geq 0$. Define

$$[\alpha] = \begin{cases} 1 + q + q^2 + \ldots + q^{\alpha-1} & \text{if } \alpha > 0; \\ 0 & \text{if } \alpha = 0. \end{cases}$$

We set

$$[\alpha]! = \begin{cases} [1][2]\ldots[\alpha] & \text{if } \alpha > 0; \\ 1 & \text{if } \alpha = 0. \end{cases}$$

If $\alpha \geq \beta \geq 0$, define

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{[\alpha]!}{[\beta]![\alpha - \beta]!}.$$

Remark 3.2.8. In general we have

$$\begin{bmatrix} \alpha + 1 \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + q^{\alpha-\beta+1} \begin{bmatrix} \alpha \\ \beta - 1 \end{bmatrix}.$$

So since $q = -1$ we have $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \in \mathbb{Z}$ for all $\alpha \geq \beta \geq 0$.

Theorem 3.2.9. *[24, Proposition 2.14] Suppose λ and μ are partitions of n and consider d and t with $d \geq 1$ and $0 \leq t < \mu_{d+1}$. Let $\nu = \nu^{d,t}$. Suppose $A \in \mathcal{T}(\lambda, \mu)$ is a row standard tableau. Let $\mathcal{S} \subseteq \mathcal{T}(\lambda, \nu)$ be the set of row standard tableaux obtained by replacing t entries of $d+1$ in A with d. For $S \in \mathcal{S}$ and $i \geq 1$, suppose that β_i entries were replaced in row i. Define $b_S \in F$ by*

$$b_S = \prod_{i \geq 1} q^{x_i \beta_i} \begin{bmatrix} y_i \\ \beta_i \end{bmatrix}$$

where x_i is the cardinality of the set $\{(k,j) \mid k > i \text{ and } A(k,j) = d\}$ and y_i

is the cardinality of the set $\{j \mid S(i,j) = d\}$*. Then*

$$\psi_{d,t} \circ \Theta_A = \sum_{S \in \mathcal{S}} b_S \Theta_S.$$

Definition 3.2.10. Let S be a tableau. Recall that S_i^j is the number of entries in row j of S which is equal to i. We generalize this by setting $S^r_{\leq x} = \sum_{i=j}^{x} S^r_j$, similarly for other definitions.

Definition 3.2.11. Let $\lambda = (\lambda_1, ..., \lambda_a)$ be a partition of n and $\nu = (\nu_1, ..., \nu_b)$ be a composition of n. Suppose $S \in \mathcal{T}(\lambda, \nu)$ and $r_1 \neq r_2$ with $1 \leq r_1, r_2 \leq a$ and $\lambda_{r_1} \geq \lambda_{r_2}$ and d with $1 \leq d \leq b$. Let

$$\mathcal{G} = \left\{ g = (g_1, g_2, \dots, g_b) \mid g_d = 0, \ \overline{g} = S_d^{r_2}, \ \text{ and } g_i \leq S_i^{r_1} \text{ for } 1 \leq i \leq b \right\}.$$

For $g \in \mathcal{G}$, define U_g to be the row-standard tableau obtained by moving all entries equal to d from row r_2 to row r_1 and for $i \neq d$ moving g_i entries equal to i from row r_1 to row r_2.

Theorem 3.2.12. *[25, Theorem 2.7] Suppose* $\lambda = (\lambda_1, \dots, \lambda_a)$ *is a partition of* n *and* $\nu = (\nu_1, \dots, \nu_b)$ *is a composition of* n*. Let* $S \in \mathcal{T}(\lambda, \nu)$ *and that* r_1, r_2 *satisfy* $1 \leq r_1 \leq a$ *and* $r_2 = r_1 + 1$*. Suppose that* $r = r_1$*. Consider* $1 \leq d \leq b$*. Then*

$$\Theta_S = (-1)^{S_d^{r+1}} q^{-\binom{S_d^{r+1}+1}{2}} q^{-S_d^{r+1} S_{<d}^{r+1}} \sum_{g \in \mathcal{G}} q^{\overline{g}_{d-1}} \prod_{i=1}^{b} q^{g_i S_{<i}^{r+1}} \begin{bmatrix} S_i^{r+1} + g_i \\ g_i \end{bmatrix} \Theta_{U_g}.$$

Definition 3.2.13. Let λ be a partition of n and let (a,b) be a node in the diagram of λ. The $(a,b)^{th}$ hook length is defined to be

$$h_{ab} = \lambda_a + \lambda_b' - a - b + 1.$$

The following results provide the classification of the irreducible Specht module for the Iwahori-Hecke algebras $\mathscr{H}$ in the case where λ is e–regular.

Definition 3.2.14. Let F be a field of characteristic p. Define: $\nu_{e,p} : \mathbb{N} \longrightarrow \mathbb{Z}$

$$\nu_{e,p}(k) = \begin{cases} \nu_p(k/e) + 1 & \text{if } e \text{ divides } k; \\ 0 & \text{otherwise.} \end{cases}$$

where $\nu_p(k)$ is maximal such that $p^{\nu_p(k)}|k$ for $p \geq 2$. If $p = 0$ then set $\nu_p(k) = 0$, for all non-negative integers k.

Theorem 3.2.15. *[18, Theorem 4.15] Let λ be a partition of n. Then λ is e–regular and S^λ is irreducible if and only if $\nu_{e,p}(h^\lambda_{ac}) = \nu_{e,p}(h^\lambda_{bc})$ for all nodes (a, c), $(b, c) \in [\lambda]$.*

Example 3.2.16. Let $e = 2$ and $p = 0$. If $\lambda = (6, 3)$. For all nodes $(a, c), (b, c) \in [\lambda]$ the hook length diagram is

$$\begin{array}{|c|c|c|c|c|c|}\hline 7 & 6 & 5 & 3 & 2 & 1 \\ \hline 3 & 2 & 1 \\ \cline{1-3}\end{array}$$

Now if replace each node (a, c) with the integer $\nu_{e,p}(h^\lambda_{ac})$ then we obtain the diagram

$$\begin{array}{|c|c|c|c|c|c|}\hline 0 & 1 & 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 \\ \cline{1-3}\end{array}$$

and we see that $\nu_{2,0}(h^\lambda_{ac}) = \nu_{2,0}(h^\lambda_{bc})$ for all nodes $(a, c), (b, c) \in [\lambda]$. By applying Theorem 3.2.15 $S^{(6,3)}$ is irreducible.

Let $\lambda = (a, 3, 1^b)$ then the 2–regularisation of λ is given by the next Lemma.

Lemma 3.2.17. *[7, Lemma 2.4] Suppose $a \geq 4$ and $b \geq 2$. Then*

$$(a, 3, 1^b)^R = \begin{cases} (a, b + 1, 2) & (a > b) \\ (b + 2, a - 1, 2) & (a \leq b). \end{cases}$$

Recall the definition of the generic Iwahori-Hecke algebra of $\mathfrak{S}_n$. Let $\mathcal{Z} = \mathbb{Z}[\hat{q}, \hat{q}^{-1}]$, where $\hat{q}$ is an indeterminate over $\mathbb{Z}$. Then $\mathscr{H}_{\mathcal{Z}} = \mathscr{H}_{\mathcal{Z},\hat{q}}(\mathfrak{S}_n)$ is semisimple which implies all Specht modules are irreducible and the set $\{S^\lambda \mid \lambda \vdash n\}$ is a complete set of non-isomorphic irreducible modules. If F is a field and $q \in F\backslash\{0\}$, define $\varphi : \mathcal{Z} \longrightarrow F$ to be the ring homomorphisms determined by $\hat{q} \longmapsto q$. Then

$$\mathscr{H}_{F,q}(\mathfrak{S}_n) \cong \mathscr{H}_{\mathcal{Z}} \otimes_{\mathcal{Z}} F$$

as F–algebras. We state the following lemma:

Lemma 3.2.18. *1. Let* $\hat{\theta} = \sum_{U \in \mathcal{T}(\alpha,\beta)} a_U \hat{\theta}_U$ *be a homomorphism* $\hat{\theta} : S^\alpha \longrightarrow M^\beta$ *in* $\mathscr{H}_{\mathcal{Z}}$ *so that it can be written* $\hat{\theta} = \sum_{R \in \mathcal{T}_0(\alpha,\beta)} b_R \hat{\theta}_R$ *so that* $\hat{\theta} \neq 0$ *if and only if* $b_R \neq 0$ *for some* R. *Define* $\hat{\theta}^F$ *to be the map in* $\mathscr{H}_{F,q}(\mathfrak{S}_n)$ *such that* $\hat{\theta}^F = \sum_{U \in \mathcal{T}(\alpha,\beta)} \overline{a}_U \hat{\theta}^F_U$, *where* $\overline{a}_U = \varphi(a_U)$. *Then* $\hat{\theta}^F = \sum_{R \in \mathcal{T}_0(\alpha,\beta)} \overline{b}_R \hat{\theta}^F_R$ *and* $\hat{\theta}^F \neq 0$ *if and only if* $\overline{b}_R \neq 0$ *for some* R.

2. Let $\phi = \sum_{U \in \mathcal{T}(\alpha,\beta)} a_U \hat{\theta}_U$ *be a homomorphism* $\phi : S^\alpha \longrightarrow M^\beta$ *and* $\varpi = \sum_{S \in \mathcal{T}(\beta,\gamma)} b_S \hat{\theta}_S$ *be a homomorphism* $\varpi : M^\beta \longrightarrow M^\gamma$ *in* $\mathscr{H}_{\mathcal{Z}}$. *Then the composition can be written as*

$$\sum_{T \in \mathcal{T}_0(\alpha,\gamma)} c_T \hat{\theta}_T \quad \text{for some } c_T \in \mathcal{Z}\ .$$

If we have the corresponding maps in $\mathscr{H}_{F,q}(\mathfrak{S}_n)$, *then the composition can be written as*

$$\sum_{T \in \mathcal{T}_0(\alpha,\gamma)} \overline{c}_T \hat{\theta}^F_T.$$

in the same way, that is, the coefficients are $\overline{c}_T$.

3. Suppose q *is a primitive* e^{th} *root of unity of* $\mathbb{C}$ *and* w *is a primitive* e^{th} *root of unity of* F_p. *Define* $\varphi_{\mathbb{C}} : \mathcal{Z} \longrightarrow \mathbb{C}$ *to be the homomorphism defined by setting* $\varphi_{\mathbb{C}}(\hat{q}) = q$ *and* $\varphi_F : \mathcal{Z} \longrightarrow F_p$ *to be the homomorphism defined by setting* $\varphi_F(\hat{q}) = w$. *Suppose* $z \in \mathcal{Z}$. *If* $\varphi_{\mathbb{C}}(z) = 0$ *then* $\varphi_{F_p}(z) = 0$.

Proof. 1. Working in $\mathscr{H}_{\mathcal{Z}}$, from Definition 3.2.3 we have

$$\begin{aligned}\hat{\theta}(c_\alpha) &= \big(x_\beta \sum_{U \in \mathcal{T}(\alpha,\beta)} \sum_{U' \sim_r U} a_U T_{1_{U'}}\big) T_{w_\alpha} y_{\alpha'} \\ &= \big(x_\beta \sum_{R \in \mathcal{T}_0(\alpha,\beta)} \sum_{R' \sim_r R} b_R T_{1_{R'}}\big) T_{w_\alpha} y_{\alpha'} \\ &= \sum_{w \in \mathfrak{S}_n} d_w T_w\end{aligned}$$

for some $d_w \in \mathcal{Z}$. Now working in $\mathscr{H}_{F,q}(\mathfrak{S}_n)$, from Definition 3.2.3 we have

$$\begin{aligned}\hat{\theta}^F(c_\alpha) &= \big(x_\beta \sum_{U\in\mathcal{T}(\alpha,\beta)} \sum_{U'\sim_r U} \varphi(a_u)T_{1_{U'}}\big)T_{w_\alpha}y_{\alpha'}\\ &= \sum_{w\in\mathfrak{S}_n} \varphi(d_w)T_w\\ &= \big(x_\beta \sum_{R\in\mathcal{T}_0(\alpha,\beta)} \sum_{R'\sim_r R} \varphi(b_R)T_{1_{R'}}\big)T_{w_\alpha}y_{\alpha'}\\ &= \sum_{R\in\mathcal{T}_0(\alpha,\beta)} \varphi(b_R)\hat{\Theta}^F_R(c_\alpha).\end{aligned}$$

So, $\hat{\theta}^F = \sum_{R\in\mathcal{T}_0(\alpha,\beta)} \overline{b}_R\hat{\theta}^F_R$.

2. Similarly, working in $\mathscr{H}_{\mathcal{Z}}$, from Definition 3.2.3 we have

$$\phi(c_\alpha) = \big(x_\beta \sum_{U\in\mathcal{T}(\alpha,\beta)} \sum_{U'\sim_r U} a_U T_{1_{U'}}\big)T_{w_\alpha}y_{\alpha'}.$$

$$\varpi(x_\beta) = x_\gamma \sum_{S\in\mathcal{T}(\beta,\gamma)} \sum_{S'\sim_r S} b_S T_{1_{S'}}.$$

Now the composition can be written as

$$\begin{aligned}\varpi(\phi(c_\alpha) &= \varpi\Big(\big(x_\beta \sum_{U\in\mathcal{T}(\alpha,\beta)} \sum_{U'\sim_r U} a_U T_{1_{U'}}\big)T_{w_\alpha}y_{\alpha'}\Big)\\ &= x_\gamma\big(\sum_{S\in\mathcal{T}(\beta,\gamma)} \sum_{S'\sim_r S} b_S T_{1_{S'}} \sum_{U\in\mathcal{T}(\alpha,\beta)} \sum_{U'\sim_r U} a_U T_{1_{U'}}\big)T_{w_\alpha}y_{\alpha'}\\ &= \sum_{w\in\mathfrak{S}_n} d_w T_w\\ &= \sum_{T\in\mathcal{T}_0(\alpha,\gamma)} c_T\hat{\theta}_T(c_\alpha)\\ &= x_\gamma\big(\sum_{T\in\mathcal{T}_0(\alpha,\gamma)} \sum_{T'\sim_r T} c_T T_{1_{T'}}\big)T_{w_\alpha}y_{\alpha'}\end{aligned}$$

for some $d_w \in \mathcal{Z}$. Now working in $\mathscr{H}_{F,q}(\mathfrak{S}_n)$, from Definition 3.2.3 we have

$$\begin{aligned}\varpi^F(\phi^F(c_\alpha)) &= x_\gamma\big(\sum_{S\in\mathcal{T}(\beta,\gamma)} \sum_{S'\sim_r S} \varphi(b_S)T_{1_{S'}} \sum_{U\in\mathcal{T}(\alpha,\beta)} \sum_{U'\sim_r U} \varphi(a_U)T_{1_{U'}}\big)T_{w_\alpha}y_{\alpha'}\\ &= \sum_{w\in\mathfrak{S}_n} \varphi(d_w)T_w.\end{aligned}$$

And

$$\sum_{T\in\mathcal{T}_0(\alpha,\gamma)} c_T\hat{\theta}^F_T(c_\alpha) = x_\gamma\Big(\sum_{T\in\mathcal{T}_0(\alpha,\gamma)}\sum_{T'\sim_r T}\varphi(c_T)T_{1_{T'}}\Big)T_{w_\alpha}y_{\alpha'}$$
$$= \sum_{w\in\mathfrak{S}_n}\varphi(d_w)T_w.$$

So, $\varpi^F\circ\phi^F = \sum_{T\in\mathcal{T}_0(\alpha,\gamma)}\overline{c}_T\hat{\theta}^F_T$.

3. Let $\Phi_e(x)$ be the e^{th} cyclotomic polynomial [32]. Then $\Phi_e(x)$ is the minimum polynomial for q in $\mathbb{C}$. Furthermore, $\varphi_{\mathbb{C}}(\Phi_e(\hat{q})) = 0 \in \mathbb{C}$ and $\varphi_F(\Phi_e(\hat{q})) = 0 \in F_p$ so $\varphi_{\mathbb{C}}$ and φ_F both factor through $\mathcal{Z}/(\Phi_e(\hat{q}))$. Hence, there are ring homomorphisms $\overline{\varphi_{\mathbb{C}}}$ and $\overline{\varphi_F}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{Z} \arrow[rr, "\varphi_F"] \arrow[dr, "\pi"] \arrow[dd, "\varphi_{\mathbb{C}}"'] & & F_p \\
& \mathcal{Z}/(\Phi_e(\hat{q})) \arrow[ur, "\overline{\varphi_F}"'] \arrow[dl, "\overline{\varphi_{\mathbb{C}}}"] & \\
\mathbb{C} & &
\end{tikzcd}$$

where $\pi : \mathcal{Z} \longrightarrow \mathcal{Z}/(\Phi_e(\hat{q}))$ is the natural projection. Since $\Phi_e(x)$ is the minimum polynomial for $\hat{q}$, note that $\overline{\varphi_{\mathbb{C}}}$ is injective. Now suppose that $\varphi_{\mathbb{C}}(z) = 0$ for some $z \in \mathcal{Z}$. Since $\overline{\varphi_{\mathbb{C}}}$ is injective, $z \in \Phi_e(\hat{q})$, which implies that $\varphi_F(z) = \overline{\varphi_F}(\pi(z)) = 0 \in F$.

□

Corollary 3.2.19. *Suppose that $\hat{\Theta}_{\mathcal{Z}} : S^\alpha_{\mathcal{Z}} \longrightarrow M^\beta_{\mathcal{Z}}$ is a $\mathcal{Z}$–homomorphism given by*

$$\hat{\Theta}_{\mathcal{Z}} = \sum_{R\in\mathcal{T}(\alpha,\beta)} a_R\hat{\Theta}^{\mathcal{Z}}_R.$$

Define the $F_2\mathfrak{S}_n$–homomorphism $\hat{\Theta}_{F_2} : S^\alpha_{F_2} \longrightarrow M^\beta_{F_2}$ by

$$\hat{\Theta}_{F_2} = \sum_{R\in\mathcal{T}(\alpha,\beta)}\varphi_{F_2}(a_R)\hat{\Theta}^{F_2}_R$$

and the $\mathscr{H}_{\mathbb{C},-1}(\mathfrak{S}_n)$*–homomorphism* $\hat{\Theta}_{\mathbb{C}} : S^{\alpha}_{\mathbb{C}} \longrightarrow M^{\beta}_{\mathbb{C}}$ *by*

$$\hat{\Theta}_{\mathbb{C}} = \sum_{R \in \mathcal{T}(\alpha,\beta)} \varphi_{\mathbb{C}}(a_R)\hat{\Theta}^{\mathbb{C}}_R.$$

If $\hat{\Theta}_{F_2} \neq 0$ *then* $\hat{\Theta}_{\mathbb{C}} \neq 0$*.*

Lemma 3.2.20. *Suppose* $e = 2$*. Then, for* $m > 0$*:*

- $\begin{bmatrix} 2m \\ 1 \end{bmatrix} = 0.$
- $\begin{bmatrix} 2m \\ 2 \end{bmatrix} = m.$
- $\begin{bmatrix} 2m+1 \\ 1 \end{bmatrix} = 1.$
- $\begin{bmatrix} 2m+1 \\ 2 \end{bmatrix} = m.$
- $\begin{bmatrix} 2m \\ 3 \end{bmatrix} = 0$

Proof. By using the definition of the Gaussian polynomial. We get

$$\begin{aligned}\begin{bmatrix} 2m \\ 1 \end{bmatrix} &= \frac{[2m]!}{[1]![2m-1]!} \\ &= 1 + q + q^2 + \ldots + q^{2m-2} + q^{2m-1} \\ &= (1+q)(1 + q^2 + q^4 + \ldots + q^{2m-2}) \\ &= 0.\end{aligned}$$

Similarly,

$$\begin{aligned}\begin{bmatrix} 2m \\ 2 \end{bmatrix} &= \frac{[2m][2m-1]}{[2]} \\ &= \frac{(1 + q + q^2 + \ldots + q^{2m-2} + q^{2m-1})(1 + q + \ldots + q^{2m-2})}{1+q} \\ &= \frac{(1+q)(1 + q^2 + \ldots + q^{2m-2})(1 + q + \ldots + q^{2m-2})}{1+q} \\ &= m.\end{aligned}$$

Similarly

$$\begin{aligned}\begin{bmatrix} 2m+1 \\ 1 \end{bmatrix} &= 1 + q + q^2 + \ldots + q^{2m-2} + q^{2m-1} + q^{2m} \\ &= (1+q)(1 + q^2 + q^4 + \ldots + q^{2m-2}) + q^{2m} = q^{2m} = 1.\end{aligned}$$

Also,

$$\begin{aligned}\begin{bmatrix}2m+1\\2\end{bmatrix} &= \frac{[2m+1][2m]}{[2]}\\ &= 1+q^2+\ldots+q^{2m-2} = m.\end{aligned}$$

Finally,

$$\begin{aligned}\begin{bmatrix}2m\\3\end{bmatrix} &= \frac{[2m][2m-1][2m-2]}{[3][2][1]}\\ &= \frac{(1+q+q^2+\ldots+q^{2m-2}+q^{2m-1})(1+q+\ldots+q^{2m-2})(1+q+\ldots+q^{2m-3})}{(1+q+q^2)(1+q)}\\ &= \frac{(1+q)^2(1+q^2+\ldots+q^{2m-2})(1+q+\ldots+q^{2m-2})(1+q^2+\ldots+q^{2m-4})}{(1+q+q^2)(1+q)}\\ &= \frac{(1+q)(1+q^2+\ldots+q^{2m-2})(1+q+\ldots+q^{2m-2})(1+q^2+\ldots+q^{2m-4})}{(1+q+q^2)}\\ &= 0.\end{aligned}$$

□

3.3 The main results

Recall that the Hecke algebra $\mathscr{H} = \mathscr{H}_{\mathbb{C},-1}(\mathfrak{S}_n)$. In this section we state the main theorem which describes some Specht modules $S^{(a,3,1^b)}$ which have a summand isomorphic to an irreducible Specht module of the form either $S^{(u,v)}$ or $S^{(u,v,2)}$, where u is even and v is odd. We assume that the field has characteristic zero and $e = 2$. Now we state the main theorem as follows

Theorem D. *Suppose $\lambda = (a,3,1^b)$ is a partition of n, where a,b are positive even integers with $a \geq 4$ and let μ be a partition of n such that S^μ is irreducible. If one of the following occurs:*

1. *If μ or μ' equals (u,v), where u is even and $u > v$ with $v \equiv 3 \bmod 4$ and $\begin{bmatrix}u-v\\a-v\end{bmatrix} \neq 0$,*
2. *If μ or μ' equals $(u,v,2)$, where u is even and v is odd with $u > v$ and $\begin{bmatrix}u-v\\a-v\end{bmatrix} \neq 0$,*

then S^λ has a direct summand isomorphic to S^μ.

Lemma 3.3.1. *Suppose $\mathscr{H} = \mathscr{H}_{F,-1}(\mathfrak{S}_n)$. Let μ be a partition of n and $e = 2$. Suppose S^μ is irreducible. Then $S^\mu \cong S^{\mu'}$.*

Proof. We have $S^\mu \cong D^{\mu^R}$ and since $\mu^R = {\mu'}^R$ we have $D^{\mu^R} = D^{{\mu'}^R} \cong S^{\mu'}$. Thus, $S^\mu \cong S^{\mu'}$. □

Theorem D is an analogue of Theorem 3.1.2. In order to use some of Dodge and Fayers' results, we use the generic Hecke algebra $\mathscr{H}_{\mathcal{Z}}$. We find there are homomorphisms $\gamma : S^\mu \to S^\lambda$ and $\delta : S^\lambda \to S^{\mu'}$ such that $\delta \circ \gamma$ is non-zero and S^μ irreducible. That is enough to show that S^μ occurs as a summand of S^λ, since $S^\mu \cong S^{\mu'}$ if $e = 2$ and S^μ irreducible, so $\delta \circ \gamma$ is the identity on S^μ.

3.3.1 Irreducible summands of the form $S^{(u,v)}$

In this section, we assume $\lambda = (a, 3, 1^b)$ and $\mu = (u, v)$, where a, b, u, v are positive integers with a, b, u even, $a \geq 4$, $u > v$, $n = a + b + 3 = u + v$ and $v \leq \min\{a + 1, b + 3\}$. Throughout our examples, we identify a tableau $T \in \mathcal{T}(\alpha, \beta)$ with the corresponding homomorphism $\hat{\Theta}_T : S^\alpha \to M^\beta$.

Homomorphism $\sigma : S^\lambda \longrightarrow S^{\mu'}$

Consider homomorphisms from S^λ to $S^{\mu'}$, where μ' is the conjugate of μ. We begin by constructing such a homomorphism in the case where $3 \leq v \leq a-1$. Suppose $\mathcal{U}$ is the set of λ-tableaux having the form:

$$\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
1 & 2 & 3 & \cdots\cdots & v & \star & \cdots\cdots & \star \\
\hline
1 & \star & \star \\
\cline{1-3}
\star \\
\cline{1-1}
\vdots \\
\cline{1-1}
\star \\
\cline{1-1}
\end{array}$$

where the $\star$s represent the numbers from 2 to u, and the entries are strictly increasing along each row and weakly increasing down each column.

Example 3.3.2. Let $\lambda = (4, 3, 1^2)$, $\mu = (6, 3)$. Then

$$\mathcal{U} = \{T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8, T_9, T_{10}, T_{11}, T_{12}, T_{13}, T_{14}, T_{15}, T_{16}, T_{17}, T_{18}\}.$$

where

$$T_1 = \begin{ytableau} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 \\ 5 \\ 6 \end{ytableau}, T_2 = \begin{ytableau} 1 & 2 & 3 & 4 \\ 1 & 2 & 5 \\ 3 \\ 6 \end{ytableau}, T_3 = \begin{ytableau} 1 & 2 & 3 & 4 \\ 1 & 3 & 5 \\ 2 \\ 6 \end{ytableau}, T_4 = \begin{ytableau} 1 & 2 & 3 & 5 \\ 1 & 2 & 3 \\ 4 \\ 6 \end{ytableau},$$

$$T_5 = \begin{ytableau} 1 & 2 & 3 & 5 \\ 1 & 2 & 4 \\ 3 \\ 6 \end{ytableau}, T_6 = \begin{ytableau} 1 & 2 & 3 & 5 \\ 1 & 3 & 4 \\ 2 \\ 6 \end{ytableau}, T_7 = \begin{ytableau} 1 & 2 & 3 & 6 \\ 1 & 2 & 3 \\ 4 \\ 5 \end{ytableau}, T_8 = \begin{ytableau} 1 & 2 & 3 & 6 \\ 1 & 2 & 4 \\ 3 \\ 5 \end{ytableau},$$

$$T_9 = \begin{ytableau} 1 & 2 & 3 & 6 \\ 1 & 3 & 4 \\ 2 \\ 5 \end{ytableau}, T_{10} = \begin{ytableau} 1 & 2 & 3 & 4 \\ 1 & 3 & 6 \\ 2 \\ 5 \end{ytableau}, T_{11} = \begin{ytableau} 1 & 2 & 3 & 4 \\ 1 & 2 & 6 \\ 3 \\ 5 \end{ytableau}, T_{12} = \begin{ytableau} 1 & 2 & 3 & 5 \\ 1 & 2 & 6 \\ 3 \\ 4 \end{ytableau},$$

$$T_{13} = \begin{ytableau} 1 & 2 & 3 & 5 \\ 1 & 3 & 6 \\ 2 \\ 4 \end{ytableau}, T_{14} = \begin{ytableau} 1 & 2 & 3 & 6 \\ 1 & 2 & 5 \\ 3 \\ 4 \end{ytableau}, T_{15} = \begin{ytableau} 1 & 2 & 3 & 6 \\ 1 & 3 & 5 \\ 2 \\ 4 \end{ytableau}, T_{16} = \begin{ytableau} 1 & 2 & 3 & 6 \\ 1 & 4 & 5 \\ 2 \\ 3 \end{ytableau},$$

$$T_{17} = \begin{ytableau} 1 & 2 & 3 & 5 \\ 1 & 4 & 6 \\ 2 \\ 3 \end{ytableau}, T_{18} = \begin{ytableau} 1 & 2 & 3 & 4 \\ 1 & 5 & 6 \\ 2 \\ 3 \end{ytableau}.$$

Now define

$$\sigma = \sum_{T \in \mathcal{U}} \hat{\Theta}_T.$$

Proposition 3.3.3. *We have $\psi_{d,t} \circ \sigma = 0$ for each d, t.*

Proof. First take $v<d<u$ and $t = 1$. If $T \in \mathcal{U}$, then T contains a single d and a single $d+1$. If these lie in the same row of T, then by Theorem 3.2.9 $\psi_{d,1} \circ \hat{\Theta}_T = (1+q)\hat{\Theta}_U$, where U is obtained from T by changing $d+1$ into d and hence $\psi_{d,1} \circ \hat{\Theta}_T = 0$. If these lie in the same column of T, then by Theorem 3.2.9 $\psi_{d,1} \circ \hat{\Theta}_T = \hat{\Theta}_U$, where U has row r and row $r+1$ both equal to $\boxed{d}$, so by using Theorem 3.2.12 we have $\psi_{d,1} \circ \hat{\Theta}_T = 0$. Otherwise, there is another tableau $T' \in \mathcal{U}$ obtained by interchanging the d and the $d+1$. Then, by using Theorem 3.2.9 we have $\psi_{d,1} \circ (\hat{\Theta}_T + \hat{\Theta}_{T'}) = (1+q)\hat{\Theta}_S = 0$, where S is the tableau obtained by replacing one entry of $d+1$ in T with d . Hence, $\psi_{d,1} \circ \sigma = 0$.

Example 3.3.4. Take $\mathcal{U}$ as in example 3.3.2 and assume $d = 4$ and $t = 1$ Then: by using Theorem 3.2.9 we get

$$\psi_{4,1} \circ \hat{\Theta}_{T_{16}} = (1+q) \begin{array}{|c|c|c|c|}\hline 1&2&3&6\\ \hline 1&4&4 \\ \cline{1-3} 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array} = 0$$

where d and $d+1$ lie in the same row of T_{16}. Similarly

$$\psi_{4,1} \circ \hat{\Theta}_{T_7} = \begin{array}{|c|c|c|c|}\hline 1&2&3&6\\ \hline 1&2&3 \\ \cline{1-3} 4 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$$

where d and $d+1$ lie in the same column of T_7. Applying Theorem 3.2.12 we get

$$\psi_{4,1} \circ \hat{\Theta}_{T_7} = 0.$$

For the case where a single d and a single $d+1$ lie in the different row and column, consider

$$\psi_{4,1} \circ \hat{\Theta}_{T_1} = q^0 \begin{bmatrix}1\\0\end{bmatrix} q^0 \begin{bmatrix}0\\0\end{bmatrix} q^0 \begin{bmatrix}1\\1\end{bmatrix} q^0 \begin{bmatrix}0\\0\end{bmatrix} \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\ \hline 1&2&3 \\ \cline{1-3} 4 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\ \hline 1&2&3 \\ \cline{1-3} 4 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array}$$

$$\psi_{4,1} \circ \hat{\Theta}_{T_4} = q \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\ \hline 1&2&3 \\ \cline{1-3} 4 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array}$$

and so

$$\psi_{4,1} \circ (\hat{\Theta}_{T_1} + \hat{\Theta}_{T_4}) = (1+q) \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\ \hline 1&2&3 \\ \cline{1-3} 4 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array} = 0.$$

Continuing in this way, we get $\psi_{4,1} \circ \sigma = 0$.

Second take $d = v$ and $t = 1$. If $T \in \mathcal{U}$, then T contains either a single v or a single v and a single $v+1$ below the first row. If there are a single v and a single $v+1$ below the first row and they occur in the same row of T, then by Theorem 3.2.9 $\psi_{v,1} \circ \hat{\Theta}_T$ has a factor $1+q$ and hence $\psi_{v,1} \circ \hat{\Theta}_T = 0$. If these occur the same column of T, then by Theorem 3.2.9 we get a tableau which has the row r and row $r+1$ both equal to $\boxed{v}$ with coefficient q, then by using Theorem 3.2.12 $\psi_{v,1} \circ \hat{\Theta}_T = 0$. Now if a single v and a single $v+1$ below the first row occur in a different row and column then there is another tableau $T' \in \mathcal{U}$ obtained by interchanging the v and the $v+1$. Then, by using Theorem 3.2.9 we have $\psi_{v,1} \circ (\hat{\Theta}_T + \hat{\Theta}'_T) = (1+q)\hat{\Theta}_S = 0$, where S is the unique tableau obtained by replacing one entry of $v+1$ in T with v. In the case that there is only a single v below the first row then by Theorem 3.2.9 we get tableau with a coefficient has a factor $q(1+q)$ and thus $\psi_{v,1} \circ \sigma = 0$. Hence, $\psi_{v,1} \circ \hat{\Theta}_T = 0$ for all $T \in \mathcal{U}$.

Example 3.3.5. Take $\mathcal{U}$ as in example 3.3.2 and assume $d = 3$ and $t = 1$. Then, by using Theorem 3.2.9 we get for the case only a single v below the first row,

$$\psi_{3,1} \circ \hat{\Theta}_{T_1} = b_S \begin{array}{|c|c|c|c|}\hline 1&2&3&3\\\hline 1&2&3\\\cline{1-3} 5\\\cline{1-1} 6\\\cline{1-1}\end{array}$$

such that

$$\begin{aligned} b_S &= \prod_{i \geq 1} q^{x_i \beta_i} \begin{bmatrix} y_i \\ \beta_i \end{bmatrix} \\ &= q^1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} q^0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} q^0 \begin{bmatrix} 0 \\ 0 \end{bmatrix} q^0 \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &= q(1+q). \end{aligned}$$

Hence, $\psi_{3,1} \hat{\Theta}_{T_1} = q(1+q) \begin{array}{|c|c|c|c|}\hline 1&2&3&3\\\hline 1&2&3\\\cline{1-3} 5\\\cline{1-1} 6\\\cline{1-1}\end{array} = 0.$

In the case where a single v and a single $v+1$ are in the same row below the first row, then by Theorem 3.2.9 we get

$$\psi_{3,1} \circ \hat{\Theta}_{T_6} = q^0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} q^0 \begin{bmatrix} 2 \\ 1 \end{bmatrix} q^0 \begin{bmatrix} 0 \\ 0 \end{bmatrix} q^0 \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{array}{|c|c|c|c|}\hline 1&2&3&5\\\hline 1&3&3\\\cline{1-3} 2\\\cline{1-1} 6\\\cline{1-1}\end{array} = (1+q) \begin{array}{|c|c|c|c|}\hline 1&2&3&5\\\hline 1&3&3\\\cline{1-3} 2\\\cline{1-1} 6\\\cline{1-1}\end{array} = 0.$$

In the case that a single v and a single $v+1$ are in the same column below the first row, then by Theorem 3.2.9 we get

$$\psi_{3,1} \circ \hat{\Theta}_{T_{14}} = q \begin{ytableau} 1 & 2 & 3 & 6 \\ 1 & 2 & 5 \\ 3 \\ 3 \end{ytableau} .$$

Applying Theorem 3.2.12 we get $\psi_{3,1} \circ \hat{\Theta}_{T_{14}} = 0$.

In the case where a single v and a single $v+1$ in different rows and columns below the first row. Then by Theorem 3.2.9 we get

$$\psi_{3,1} \circ \hat{\Theta}_{T_4} = \begin{ytableau} 1 & 2 & 3 & 5 \\ 1 & 2 & 3 \\ 3 \\ 6 \end{ytableau} \quad \text{and } \psi_{3,1} \circ \hat{\Theta}_{T_5} = q \begin{ytableau} 1 & 2 & 3 & 5 \\ 1 & 2 & 3 \\ 3 \\ 6 \end{ytableau} .$$

Then

$$\psi_{3,1}(\hat{\Theta}_{T_4} + \hat{\Theta}_{T_5}) = \begin{ytableau} 1 & 2 & 3 & 5 \\ 1 & 2 & 3 \\ 3 \\ 6 \end{ytableau} + q \begin{ytableau} 1 & 2 & 3 & 5 \\ 1 & 2 & 3 \\ 3 \\ 6 \end{ytableau} = 0.$$

If $1 \le d < v$ and $t = 2$, then we have at least one factor of $1+q$ in $\psi_{d,2} \circ \hat{\Theta}_T$ and then $\psi_{d,2} \circ \hat{\Theta}_T = 0$ for each $T \in \mathcal{U}$ by using Theorem 3.2.9.

Example 3.3.6. Let $d = 1$ and $t = 2$. Then by using Theorem 3.2.9

$$\psi_{1,2} \circ \hat{\Theta}_{T_1} = q(1+q)(1+q) \begin{ytableau} 1 & 1 & 3 & 4 \\ 1 & 1 & 3 \\ 5 \\ 6 \end{ytableau} = 0$$

$$\psi_{1,2} \circ \hat{\Theta}_{T_3} = q(1+q) \begin{ytableau} 1 & 1 & 3 & 4 \\ 1 & 3 & 5 \\ 1 \\ 6 \end{ytableau} = 0.$$

If $d = 2$ and $t = 2$, then by using Theorem 3.2.9

$$\psi_{2,2} \circ \hat{\Theta}_{T_1} = q(1+q)(1+q) \begin{ytableau} 1 & 2 & 2 & 4 \\ 1 & 2 & 2 \\ 5 \\ 6 \end{ytableau} = 0$$

$$\psi_{2,2} \circ \hat{\Theta}_{T_5} = q(1+q)\begin{ytableau} 1 & 2 & 2 & 5 \\ 1 & 2 & 4 \\ 2 \\ 6 \end{ytableau} = 0.$$

Now take $2 \le d < v$ and $t = 1$, and consider a tableau $T \in \mathcal{U}$. There are a single d and a single $d+1$ below the first row. If these lie in the same row then by using Theorem 3.2.9 we have at least one factor of $1+q$ and then $\psi_{d,1} \circ \hat{\Theta}_T = 0$. If these lie in the same column, then by Theorem 3.2.9 we have two tableaux; the coefficient of one of them has a factor of $1+q$ and the second has rows r and $r+1$ both equal to $\boxed{d}$ and by Theorem 3.2.12 $\psi_{d,1} \circ \hat{\Theta}_T = 0$. Otherwise, let T' be the tableau obtained by interchanging the d and the $d+1$ below the first row. Each homomorphism $\hat{\Theta}_S$, where S is a unique tableau obtained by replacing two entries of $d+1$ in T with d, that appears when we apply Theorem 3.2.9 occurs with a coefficient that has a factor of $1+q$ and hence $\psi_{d,1} \circ (\hat{\Theta}_T + \hat{\Theta}'_T) = (1+q)\hat{\Theta}_S = 0$. Thus, $\psi_{d,2} \circ \hat{\Theta}_T = 0$ for all $T \in \mathcal{U}$. For example:

Example 3.3.7. Suppose $d = 2$, $t = 1$. Suppose there is a single d and a single $d+1$ below the first row and that they lie in the same row. Then by using Theorem 3.2.9

$$\psi_{2,1} \circ \hat{\Theta}_{T_1} = q(1+q)\begin{ytableau} 1 & 2 & 2 & 4 \\ 1 & 2 & 3 \\ 5 \\ 6 \end{ytableau} + (1+q)\begin{ytableau} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 \\ 5 \\ 6 \end{ytableau} = 0.$$

For a single d and a single $d+1$ below the first row and lying in the same column,

$$\psi_{2,1} \circ \hat{\Theta}_{T_{16}} = q(1+q)\begin{ytableau} 1 & 2 & 2 & 6 \\ 1 & 4 & 5 \\ 2 \\ 3 \end{ytableau} + \begin{ytableau} 1 & 2 & 3 & 6 \\ 1 & 4 & 5 \\ 2 \\ 2 \end{ytableau}.$$

Using Theorem 3.2.12, then $\psi_{2,1} \circ \hat{\Theta}_{T_{16}} = 0$. For a single d and a single $d+1$ below the first row and lying in a different row and column,

$$\psi_{2,1} \circ \hat{\Theta}_{T_2} = q(1+q)\begin{ytableau} 1 & 2 & 2 & 4 \\ 1 & 2 & 5 \\ 3 \\ 6 \end{ytableau} + \begin{ytableau} 1 & 2 & 3 & 4 \\ 1 & 2 & 5 \\ 2 \\ 6 \end{ytableau}$$

$$\psi_{2,1} \circ \hat{\Theta}_{T_3} = q(1+q)\begin{ytableau} 1 & 2 & 2 & 4 \\ 1 & 3 & 5 \\ 2 \\ 6 \end{ytableau} + q\begin{ytableau} 1 & 2 & 3 & 4 \\ 1 & 2 & 5 \\ 2 \\ 6 \end{ytableau}$$

such that

$$\psi_{2,1} \circ (\hat{\Theta}_{T_2} + \hat{\Theta}_{T_3}) = 0.$$

We are left with the case $d = t = 1$. Applying Theorem 3.2.9, we get that $\psi_{1,1} \circ \hat{\Theta}_T$ is the sum of homomorphisms labelled by tableaux

$$\begin{ytableau} 1 & 2 & 3 & \cdots & v & \star & \cdots & \star \\ 1 & \star & \star \\ 1 \\ \vdots \\ \star \end{ytableau}$$

where the $\star$s denote to the numbers from 3 to u, and the entries are strictly increasing along rows and weakly increasing down columns. Now we apply Theorem 3.2.12 to each of these homomorphisms to move the 1 from row 3 to row 2, and then to reorder rows $3, \ldots, b+2$. We obtain a sum of tableaux of the form

$$\begin{ytableau} 1 & 2 & 3 & \cdots & v & \star & \cdots & \star \\ 1 & 1 & \star \\ \star \\ \vdots \\ \star \end{ytableau}$$

but each tableau occurs $\frac{b}{2}$ times with coefficient -1 and $\frac{b}{2}$ times with coefficient 1. Hence, by summing $\psi_{1,1} \circ \hat{\Theta}_T$ for each $T \in \mathcal{U}$ we get zero.

Example 3.3.8. If $d = 1$, $t = 1$, then by using Theorem 3.2.9

$$\psi_{1,1} \circ \hat{\Theta}_{T_1} = q(1+q)\begin{ytableau} 1 & 1 & 3 & 4 \\ 1 & 2 & 3 \\ 5 \\ 6 \end{ytableau} + (1+q)\begin{ytableau} 1 & 2 & 3 & 4 \\ 1 & 1 & 3 \\ 5 \\ 6 \end{ytableau} = 0$$

$$\psi_{1,1} \circ \hat{\Theta}_{T_2} = q(1+q)\begin{ytableau} 1 & 1 & 3 & 4 \\ 1 & 2 & 5 \\ 3 \\ 6 \end{ytableau} + (1+q)\begin{ytableau} 1 & 2 & 3 & 4 \\ 1 & 1 & 5 \\ 3 \\ 6 \end{ytableau} = 0$$

$$\psi_{1,1} \circ \hat{\Theta}_{T_3} = q(1+q)\begin{ytableau} 1 & 1 & 3 & 4 \\ 1 & 3 & 5 \\ 2 \\ 6 \end{ytableau} + \begin{ytableau} 1 & 2 & 3 & 4 \\ 1 & 3 & 5 \\ 1 \\ 6 \end{ytableau} = \begin{ytableau} 1 & 2 & 3 & 4 \\ 1 & 3 & 5 \\ 1 \\ 6 \end{ytableau}$$

$$\psi_{1,1} \circ \hat{\Theta}_{T_4} = q(1+q)\begin{ytableau} 1 & 1 & 3 & 5 \\ 1 & 2 & 3 \\ 4 \\ 6 \end{ytableau} + (1+q)\begin{ytableau} 1 & 2 & 3 & 5 \\ 1 & 1 & 3 \\ 4 \\ 6 \end{ytableau} = 0$$

$$\psi_{1,1} \circ \hat{\Theta}_{T_5} = q(1+q)\begin{ytableau} 1 & 1 & 3 & 5 \\ 1 & 2 & 4 \\ 3 \\ 6 \end{ytableau} + (1+q)\begin{ytableau} 1 & 2 & 3 & 5 \\ 1 & 1 & 4 \\ 3 \\ 6 \end{ytableau} = 0$$

$$\psi_{1,1} \circ \hat{\Theta}_{T_6} = q(1+q)\begin{ytableau} 1 & 1 & 3 & 5 \\ 1 & 3 & 4 \\ 2 \\ 6 \end{ytableau} + \begin{ytableau} 1 & 2 & 3 & 5 \\ 1 & 3 & 4 \\ 1 \\ 6 \end{ytableau} = \begin{ytableau} 1 & 2 & 3 & 5 \\ 1 & 3 & 4 \\ 1 \\ 6 \end{ytableau}$$

$$\psi_{1,1} \circ \hat{\Theta}_{T_7} = q(1+q)\begin{ytableau} 1 & 1 & 3 & 6 \\ 1 & 2 & 3 \\ 4 \\ 5 \end{ytableau} + (1+q)\begin{ytableau} 1 & 2 & 3 & 6 \\ 1 & 1 & 3 \\ 4 \\ 5 \end{ytableau} = 0$$

$$\psi_{1,1} \circ \hat{\Theta}_{T_8} = q(1+q)\begin{ytableau} 1 & 1 & 3 & 6 \\ 1 & 2 & 4 \\ 3 \\ 5 \end{ytableau} + (1+q)\begin{ytableau} 1 & 2 & 3 & 6 \\ 1 & 1 & 4 \\ 3 \\ 5 \end{ytableau} = 0$$

$$\psi_{1,1} \circ \hat{\Theta}_{T_9} = q(1+q)\begin{ytableau} 1 & 1 & 3 & 6 \\ 1 & 3 & 4 \\ 2 \\ 5 \end{ytableau} + \begin{ytableau} 1 & 2 & 3 & 6 \\ 1 & 3 & 4 \\ 1 \\ 5 \end{ytableau} = \begin{ytableau} 1 & 2 & 3 & 6 \\ 1 & 3 & 4 \\ 1 \\ 5 \end{ytableau}$$

$$\psi_{1,1} \circ \hat{\Theta}_{T_{10}} = q(1+q)\begin{ytableau} 1 & 1 & 3 & 4 \\ 1 & 3 & 6 \\ 2 \\ 5 \end{ytableau} + \begin{ytableau} 1 & 2 & 3 & 4 \\ 1 & 3 & 6 \\ 1 \\ 5 \end{ytableau} = \begin{ytableau} 1 & 2 & 3 & 4 \\ 1 & 3 & 6 \\ 1 \\ 5 \end{ytableau}$$

$$\psi_{1,1} \circ \hat{\Theta}_{T_{11}} = q(1+q)\begin{ytableau} 1 & 1 & 3 & 4 \\ 1 & 2 & 6 \\ 3 \\ 5 \end{ytableau} + (1+q)\begin{ytableau} 1 & 2 & 3 & 4 \\ 1 & 1 & 6 \\ 3 \\ 5 \end{ytableau} = 0$$

$$\psi_{1,1}\circ\hat{\Theta}_{T_{12}} = q(1+q)\begin{ytableau} 1 & 1 & 3 & 5 \\ 1 & 2 & 6 \\ 3 \\ 4 \end{ytableau} + (1+q)\begin{ytableau} 1 & 2 & 3 & 5 \\ 1 & 1 & 6 \\ 3 \\ 4 \end{ytableau} = 0$$

$$\psi_{1,1}\circ\hat{\Theta}_{T_{13}} = q(1+q)\begin{ytableau} 1 & 1 & 3 & 5 \\ 1 & 3 & 6 \\ 2 \\ 4 \end{ytableau} + \begin{ytableau} 1 & 2 & 3 & 5 \\ 1 & 3 & 6 \\ 1 \\ 4 \end{ytableau} = \begin{ytableau} 1 & 2 & 3 & 5 \\ 1 & 3 & 6 \\ 1 \\ 4 \end{ytableau}$$

$$\psi_{1,1}\circ\hat{\Theta}_{T_{14}} = q(1+q)\begin{ytableau} 1 & 1 & 3 & 6 \\ 1 & 2 & 5 \\ 3 \\ 4 \end{ytableau} + (1+q)\begin{ytableau} 1 & 2 & 3 & 6 \\ 1 & 1 & 5 \\ 3 \\ 4 \end{ytableau} = 0$$

$$\psi_{1,1}\circ\hat{\Theta}_{T_{15}} = q(1+q)\begin{ytableau} 1 & 1 & 3 & 6 \\ 1 & 3 & 5 \\ 2 \\ 4 \end{ytableau} + \begin{ytableau} 1 & 2 & 3 & 6 \\ 1 & 3 & 5 \\ 1 \\ 4 \end{ytableau} = \begin{ytableau} 1 & 2 & 3 & 6 \\ 1 & 3 & 5 \\ 1 \\ 4 \end{ytableau}$$

$$\psi_{1,1}\circ\hat{\Theta}_{T_{16}} = q(1+q)\begin{ytableau} 1 & 1 & 3 & 6 \\ 1 & 4 & 5 \\ 2 \\ 3 \end{ytableau} + \begin{ytableau} 1 & 2 & 3 & 6 \\ 1 & 4 & 5 \\ 1 \\ 3 \end{ytableau} = \begin{ytableau} 1 & 2 & 3 & 6 \\ 1 & 4 & 5 \\ 1 \\ 3 \end{ytableau}$$

$$\psi_{1,1}\circ\hat{\Theta}_{T_{17}} = q(1+q)\begin{ytableau} 1 & 1 & 3 & 5 \\ 1 & 4 & 6 \\ 2 \\ 3 \end{ytableau} + \begin{ytableau} 1 & 2 & 3 & 5 \\ 1 & 4 & 6 \\ 1 \\ 3 \end{ytableau} = \begin{ytableau} 1 & 2 & 3 & 5 \\ 1 & 4 & 6 \\ 1 \\ 3 \end{ytableau}$$

$$\psi_{1,1}\circ\hat{\Theta}_{T_{18}} = q(1+q)\begin{ytableau} 1 & 1 & 3 & 4 \\ 1 & 5 & 6 \\ 2 \\ 3 \end{ytableau} + \begin{ytableau} 1 & 2 & 3 & 4 \\ 1 & 5 & 6 \\ 1 \\ 3 \end{ytableau} = \begin{ytableau} 1 & 2 & 3 & 4 \\ 1 & 5 & 6 \\ 1 \\ 3 \end{ytableau}.$$

Therefore we find that $\psi_{1,1}\circ\sigma$ is the sum of homomorphisms labelled by tableaux

$$\psi_{1,1}\circ\hat{\Theta}_{T_3} = \begin{ytableau} 1 & 2 & 3 & 4 \\ 1 & 3 & 5 \\ 1 \\ 6 \end{ytableau}, \psi_{1,1}\circ\hat{\Theta}_{T_6} = \begin{ytableau} 1 & 2 & 3 & 5 \\ 1 & 3 & 4 \\ 1 \\ 6 \end{ytableau}, \psi_{1,1}\circ\hat{\Theta}_{T_9} = \begin{ytableau} 1 & 2 & 3 & 6 \\ 1 & 3 & 4 \\ 1 \\ 5 \end{ytableau}$$

$$\psi_{1,1}\circ\hat{\Theta}_{T_{10}} = \begin{ytableau} 1 & 2 & 3 & 4 \\ 1 & 3 & 6 \\ 1 \\ 5 \end{ytableau}, \psi_{1,1}\circ\hat{\Theta}_{T_{13}} = \begin{ytableau} 1 & 2 & 3 & 5 \\ 1 & 3 & 6 \\ 1 \\ 4 \end{ytableau}, \psi_{1,1}\circ\hat{\Theta}_{T_{15}} = \begin{ytableau} 1 & 2 & 3 & 6 \\ 1 & 3 & 5 \\ 1 \\ 4 \end{ytableau}$$

$$\psi_{1,1}\circ\hat{\Theta}_{T_{16}} = \begin{array}{|c|c|c|c|}\hline 1&2&3&6\\\hline 1&4&5&\multicolumn{1}{c}{}\\\cline{1-3} 1&\multicolumn{3}{c}{}\\\cline{1-1} 3&\multicolumn{3}{c}{}\\\cline{1-1}\end{array}, \psi_{1,1}\circ\hat{\Theta}_{T_{17}} = \begin{array}{|c|c|c|c|}\hline 1&2&3&5\\\hline 1&4&6&\multicolumn{1}{c}{}\\\cline{1-3} 1&\multicolumn{3}{c}{}\\\cline{1-1} 3&\multicolumn{3}{c}{}\\\cline{1-1}\end{array}, \psi_{1,1}\circ\hat{\Theta}_{T_{18}} = \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 1&5&6&\multicolumn{1}{c}{}\\\cline{1-3} 1&\multicolumn{3}{c}{}\\\cline{1-1} 3&\multicolumn{3}{c}{}\\\cline{1-1}\end{array}.$$

Now we use Theorem 3.2.12

$$\psi_{1,1}\circ\hat{\Theta}_{T_3} = -\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 1&1&5&\multicolumn{1}{c}{}\\\cline{1-3} 3&\multicolumn{3}{c}{}\\\cline{1-1} 6&\multicolumn{3}{c}{}\\\cline{1-1}\end{array} - \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 1&1&3&\multicolumn{1}{c}{}\\\cline{1-3} 5&\multicolumn{3}{c}{}\\\cline{1-1} 6&\multicolumn{3}{c}{}\\\cline{1-1}\end{array}$$

$$\psi_{1,1}\circ\hat{\Theta}_{T_6} = -\begin{array}{|c|c|c|c|}\hline 1&2&3&5\\\hline 1&1&4&\multicolumn{1}{c}{}\\\cline{1-3} 3&\multicolumn{3}{c}{}\\\cline{1-1} 6&\multicolumn{3}{c}{}\\\cline{1-1}\end{array} - \begin{array}{|c|c|c|c|}\hline 1&2&3&5\\\hline 1&1&3&\multicolumn{1}{c}{}\\\cline{1-3} 4&\multicolumn{3}{c}{}\\\cline{1-1} 6&\multicolumn{3}{c}{}\\\cline{1-1}\end{array}$$

$$\psi_{1,1}\circ\hat{\Theta}_{T_9} = -\begin{array}{|c|c|c|c|}\hline 1&2&3&6\\\hline 1&1&4&\multicolumn{1}{c}{}\\\cline{1-3} 3&\multicolumn{3}{c}{}\\\cline{1-1} 5&\multicolumn{3}{c}{}\\\cline{1-1}\end{array} - \begin{array}{|c|c|c|c|}\hline 1&2&3&6\\\hline 1&1&3&\multicolumn{1}{c}{}\\\cline{1-3} 4&\multicolumn{3}{c}{}\\\cline{1-1} 5&\multicolumn{3}{c}{}\\\cline{1-1}\end{array}$$

$$\psi_{1,1}\circ\hat{\Theta}_{T_{10}} = -\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 1&1&6&\multicolumn{1}{c}{}\\\cline{1-3} 3&\multicolumn{3}{c}{}\\\cline{1-1} 5&\multicolumn{3}{c}{}\\\cline{1-1}\end{array} - \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 1&1&3&\multicolumn{1}{c}{}\\\cline{1-3} 6&\multicolumn{3}{c}{}\\\cline{1-1} 5&\multicolumn{3}{c}{}\\\cline{1-1}\end{array} = -\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 1&1&6&\multicolumn{1}{c}{}\\\cline{1-3} 3&\multicolumn{3}{c}{}\\\cline{1-1} 5&\multicolumn{3}{c}{}\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 1&1&3&\multicolumn{1}{c}{}\\\cline{1-3} 5&\multicolumn{3}{c}{}\\\cline{1-1} 6&\multicolumn{3}{c}{}\\\cline{1-1}\end{array}$$

$$\psi_{1,1}\circ\hat{\Theta}_{T_{13}} = -\begin{array}{|c|c|c|c|}\hline 1&2&3&5\\\hline 1&1&6&\multicolumn{1}{c}{}\\\cline{1-3} 3&\multicolumn{3}{c}{}\\\cline{1-1} 4&\multicolumn{3}{c}{}\\\cline{1-1}\end{array} - \begin{array}{|c|c|c|c|}\hline 1&2&3&5\\\hline 1&1&3&\multicolumn{1}{c}{}\\\cline{1-3} 6&\multicolumn{3}{c}{}\\\cline{1-1} 4&\multicolumn{3}{c}{}\\\cline{1-1}\end{array} = -\begin{array}{|c|c|c|c|}\hline 1&2&3&5\\\hline 1&1&6&\multicolumn{1}{c}{}\\\cline{1-3} 3&\multicolumn{3}{c}{}\\\cline{1-1} 4&\multicolumn{3}{c}{}\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|c|}\hline 1&2&3&5\\\hline 1&1&3&\multicolumn{1}{c}{}\\\cline{1-3} 4&\multicolumn{3}{c}{}\\\cline{1-1} 6&\multicolumn{3}{c}{}\\\cline{1-1}\end{array}$$

$$\psi_{1,1}\circ\hat{\Theta}_{T_{15}} = -\begin{array}{|c|c|c|c|}\hline 1&2&3&6\\\hline 1&1&5&\multicolumn{1}{c}{}\\\cline{1-3} 3&\multicolumn{3}{c}{}\\\cline{1-1} 4&\multicolumn{3}{c}{}\\\cline{1-1}\end{array} - \begin{array}{|c|c|c|c|}\hline 1&2&3&6\\\hline 1&1&3&\multicolumn{1}{c}{}\\\cline{1-3} 5&\multicolumn{3}{c}{}\\\cline{1-1} 4&\multicolumn{3}{c}{}\\\cline{1-1}\end{array} = -\begin{array}{|c|c|c|c|}\hline 1&2&3&6\\\hline 1&1&5&\multicolumn{1}{c}{}\\\cline{1-3} 3&\multicolumn{3}{c}{}\\\cline{1-1} 4&\multicolumn{3}{c}{}\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|c|}\hline 1&2&3&6\\\hline 1&1&3&\multicolumn{1}{c}{}\\\cline{1-3} 4&\multicolumn{3}{c}{}\\\cline{1-1} 5&\multicolumn{3}{c}{}\\\cline{1-1}\end{array}$$

$$\psi_{1,1}\circ\hat{\Theta}_{T_{16}} = -\begin{array}{|c|c|c|c|}\hline 1&2&3&6\\\hline 1&1&5&\multicolumn{1}{c}{}\\\cline{1-3} 4&\multicolumn{3}{c}{}\\\cline{1-1} 3&\multicolumn{3}{c}{}\\\cline{1-1}\end{array} - \begin{array}{|c|c|c|c|}\hline 1&2&3&6\\\hline 1&1&4&\multicolumn{1}{c}{}\\\cline{1-3} 5&\multicolumn{3}{c}{}\\\cline{1-1} 3&\multicolumn{3}{c}{}\\\cline{1-1}\end{array} = \begin{array}{|c|c|c|c|}\hline 1&2&3&6\\\hline 1&1&5&\multicolumn{1}{c}{}\\\cline{1-3} 3&\multicolumn{3}{c}{}\\\cline{1-1} 4&\multicolumn{3}{c}{}\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|c|}\hline 1&2&3&6\\\hline 1&1&4&\multicolumn{1}{c}{}\\\cline{1-3} 3&\multicolumn{3}{c}{}\\\cline{1-1} 5&\multicolumn{3}{c}{}\\\cline{1-1}\end{array}$$

$$\psi_{1,1}\circ\hat{\Theta}_{T_{17}} = -\begin{array}{|c|c|c|c|}\hline 1&2&3&5\\\hline 1&1&6&\multicolumn{1}{c}{}\\\cline{1-3} 4&\multicolumn{3}{c}{}\\\cline{1-1} 3&\multicolumn{3}{c}{}\\\cline{1-1}\end{array} - \begin{array}{|c|c|c|c|}\hline 1&2&3&5\\\hline 1&1&4&\multicolumn{1}{c}{}\\\cline{1-3} 6&\multicolumn{3}{c}{}\\\cline{1-1} 3&\multicolumn{3}{c}{}\\\cline{1-1}\end{array} = \begin{array}{|c|c|c|c|}\hline 1&2&3&5\\\hline 1&1&6&\multicolumn{1}{c}{}\\\cline{1-3} 3&\multicolumn{3}{c}{}\\\cline{1-1} 4&\multicolumn{3}{c}{}\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|c|}\hline 1&2&3&5\\\hline 1&1&4&\multicolumn{1}{c}{}\\\cline{1-3} 3&\multicolumn{3}{c}{}\\\cline{1-1} 6&\multicolumn{3}{c}{}\\\cline{1-1}\end{array}$$

$$\psi_{1,1} \circ \hat{\Theta}_{T_{18}} = -\begin{array}{llll} 1 & 2 & 3 & 4 \\ 1 & 1 & 6 & \\ 5 & & & \\ 3 & & & \end{array} - \begin{array}{llll} 1 & 2 & 3 & 4 \\ 1 & 1 & 5 & \\ 6 & & & \\ 3 & & & \end{array} = \begin{array}{llll} 1 & 2 & 3 & 4 \\ 1 & 1 & 6 & \\ 3 & & & \\ 5 & & & \end{array} + \begin{array}{llll} 1 & 2 & 3 & 4 \\ 1 & 1 & 5 & \\ 3 & & & \\ 6 & & & \end{array}.$$

Hence $\psi_{1,1} \circ \hat{\Theta}_T = 0$.

□

Now let us take the follows lemma.

Lemma 3.3.9. *[7, Proposition 5.2] Let F be a field of characteristic 2. Suppose $\lambda = (a, 3, 1^b)$ and $\mu = (u, v)$, where a, b, u, v are positive integers with a, b, u are even and let $a \geq 4$, $u > v$, $n = a+b+3 = u+v$ and $v \leq \min\{a+1, b+3\}$. Then, $\sum_{T \in \mathcal{U}} \hat{\Theta}_T^F \neq 0$ for each d, t.*

Proposition 3.3.10. *From Corollary 3.2.19 and Lemma 3.3.9, we get that $\sigma \neq 0$.*

Homomorphisms γ from S^μ to S^λ

Throughout this section we consider homomorphisms from S^μ to S^λ. Assume that $3 \leq v \leq a-1$. Define A, B to be the μ-tableaux of type λ as follows

$$A = \begin{array}{|l|l|l|l|l|l|l|l|l|l|} \hline 1 & \cdots\cdots & 1 & 2 & 2 & 2 & 3 & 4 & \cdots & b+2 \\ \hline 1 & \cdots & 1 & 1 & 1 \\ \cline{1-5} \end{array};$$

$$B = \begin{array}{|l|l|l|l|l|l|l|l|l|l|} \hline 1 & \cdots\cdots & 1 & 1 & 1 & 2 & 3 & 4 & \cdots & b+2 \\ \hline 1 & \cdots & 1 & 2 & 2 \\ \cline{1-5} \end{array}.$$

Lemma 3.3.11. *$\hat{\Theta}_A$ and $\hat{\Theta}_B$ are non-zero, and are linearly independent if $v \leq b+1$.*

Proof. By using Theorem 3.2.12, we express $\hat{\Theta}_A$ and $\hat{\Theta}_B$ as linear combinations of semistandard homomorphisms such that there is at least one semistandard tableau appearing in each case. Thus, the homomorphisms are non-zero. Moreover, if $v \leq b+1$, then in the expression for $\hat{\Theta}_A$ there is at least one semistandard tableau with two 2s in the first row while there is no such tableau appearing in the expression for $\hat{\Theta}_B$. Hence, $\hat{\Theta}_A, \hat{\Theta}_B$ are linearly independent. □

Example 3.3.12. Let $\lambda = (4,3,1^2)$ and $\nu = (6,3)$ then

$$A = \begin{array}{|c|c|c|c|c|c|}\hline 1&2&2&2&3&4\\ \hline 1&1&1 \\ \cline{1-3}\end{array}, \quad B = \begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&2&3&4\\ \hline 1&2&2 \\ \cline{1-3}\end{array}$$

By using Theorem 3.2.12

$$\hat{\Theta}_A = -q^3 \begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&1&3&4\\ \hline 2&2&2 \\ \cline{1-3}\end{array} - q^3 \begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&1&2&3\\ \hline 2&2&4 \\ \cline{1-3}\end{array} - q^3 \begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&1&2&4\\ \hline 2&2&3 \\ \cline{1-3}\end{array}$$
$$- q^3 \begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&1&2&2\\ \hline 2&3&4 \\ \cline{1-3}\end{array}.$$

$$\hat{\Theta}_B = -[3] \begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&1&3&4\\ \hline 2&2&2 \\ \cline{1-3}\end{array} - q^2 \begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&1&2&4\\ \hline 2&2&3 \\ \cline{1-3}\end{array} - q^2 \begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&1&2&3\\ \hline 2&2&4 \\ \cline{1-3}\end{array}.$$

Theorem 3.3.13. *Suppose $\lambda = (a,3,1^b)$ and $\mu = (u,v)$ with $a \geq 4$ and $u > v$, where a,b,u are positive integers. Then*

$$\dim \operatorname{Hom}(S^\mu, S^\lambda) \geq 1.$$

Proof. Suppose $\hat{\Theta} = \alpha\hat{\Theta}_A + \beta\hat{\Theta}_B$, where α and $\beta \in \mathbb{C}$. Now we find $\psi_{d,t} \circ \hat{\Theta}$ for all d,t. By Theorem 3.2.9 for all $d > 2$ we have $\psi_{d,1} \circ \hat{\Theta}_A = (1+q)\hat{\Theta}_S = 0$ and $\psi_{d,1} \circ \hat{\Theta}_B = (1+q)\hat{\Theta}_{S'} = 0$, where S (respectively S') is the unique tableau obtained by changing 1 of the entries equal to $d+1$ in A (respectively B) into d. Now consider $d = 2$, then by Theorem 3.2.9 we have that

$$\psi_{2,1} \circ \hat{\Theta}_A = [4] \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&\cdots\cdots&\cdots&\cdots&\cdots&1&2&2&2&2&4&\cdots&b{+}2\\ \hline 1&\cdots&1&1&1 \\ \cline{1-5}\end{array} = 0.$$

Similarly, $\psi_{2,1} \circ \hat{\Theta}_B = q^2(1+q)\hat{\Theta}_S = 0$, where S is the unique tableau obtained by changing 1 of the entries equal to 3 in B into 2. Consider $d = 1$ and let $t = 1,2,3$. Then, if $t = 1$, we get by Theorem 3.2.9 and Lemma 3.2.20 that $\psi_{1,1} \circ \hat{\Theta}_A = q^v\left[\begin{smallmatrix}(a-v)+1\\1\end{smallmatrix}\right]\hat{\Theta}_S = 0$ where S is the unique tableau obtained by changing 2 in first row in A with 1. Also $\psi_{1,1} \circ \hat{\Theta}_B = q^{v-2}\left[\begin{smallmatrix}a-v+3\\1\end{smallmatrix}\right]\hat{\Theta}_{S_1} + \left[\begin{smallmatrix}v-1\\1\end{smallmatrix}\right]\hat{\Theta}_{S_2} = 0$, where S_1 is the unique tableau obtained by changing 2 in first row in B with 1 and S_2 is the unique tableau obtained by changing 1 entry equal to 2 in second row in B with 1. If $t = 3$, then by Theorem 3.2.9 and

Lemma 3.2.20 we get

$$\begin{aligned}\psi_{1,3} \circ \hat{\Theta}_A &= q^{3v}\begin{bmatrix} a-v+3 \\ 3 \end{bmatrix}\hat{\Theta}_S \\ &= 0,\end{aligned}$$

where S is the unique tableau obtained by changing 3 of the entries equal to 2 in A with 1. Also,

$$\begin{aligned}\psi_{1,3} \circ \hat{\Theta}_B &= q^{v-2}\begin{bmatrix} a-v+3 \\ 1 \end{bmatrix}\begin{bmatrix} v \\ 1 \end{bmatrix}\hat{\Theta}_S \\ &= q^{v-2}(0)(1)\hat{\Theta}_S = 0,\end{aligned}$$

where S is the unique tableau obtained by changing 3 of the entries equal to 2 in B with 1.

If $t = 2$, then by Theorem 3.2.9 we get

$$\psi_{1,2} \circ \hat{\Theta}_A = q^{2v}\begin{bmatrix} a-v+2 \\ 2 \end{bmatrix}\hat{\Theta}_S \neq 0 \quad \text{since } \hat{\Theta}_S \neq 0$$

also,

$$\psi_{1,2} \circ \hat{\Theta}_B = \begin{bmatrix} v \\ 2 \end{bmatrix}\hat{\Theta}_S \neq 0$$

where

$$S = \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 1 & \cdots\cdots\cdots\cdots\cdots\cdots & 1 & 1 & 1 & 2 & 3 & 4 & \cdots\cdots & b+2 \\ \hline 1 & \cdots\cdots & 1 & 1 & 1 \\ \cline{1-5}\end{array}\ .$$

Now we look for $\psi_{1,2} \circ (\alpha\hat{\Theta}_A + \beta\hat{\Theta}_B)$. Then

$$\begin{aligned}\psi_{1,2} \circ (\alpha\hat{\Theta}_A + \beta\hat{\Theta}_B) &= (\alpha\begin{bmatrix} a-v+2 \\ 2 \end{bmatrix} + \beta\begin{bmatrix} v \\ 2 \end{bmatrix})\hat{\Theta}_S \\ &= (\alpha\begin{bmatrix} \frac{2(a-v+1)}{2}+1 \\ 2 \end{bmatrix} + \beta\begin{bmatrix} 2(\frac{v-1}{2})+1 \\ 2 \end{bmatrix})\hat{\Theta}_S \\ &= (\alpha\frac{a-v+1}{2} + \beta\frac{v-1}{2})\hat{\Theta}_S, \text{ from Lemma 3.2.20}\end{aligned}$$

So if we set $\alpha = v-1$ and $\beta = -(a-v+1)$ then, $\psi_{1,2} \circ \hat{\Theta} = 0$. □

After we define the maps γ and σ and show that $\psi_{d,t}\circ\sigma = 0$ and $\psi_{d,t}\circ\gamma = 0$ we come to prove the first case of Theorem D. Recall that

Theorem D(1). *Suppose $\lambda = (a, 3, 1^b)$ is a partition of n, where a, b are positive even integers with $a \geq 4$ and let μ be a partition of n such that S^μ is irreducible. If μ or μ' equals (u, v), where u is even and $u > v$ with $v \equiv 3 \bmod 4$ and $\begin{bmatrix} u-v \\ a-v \end{bmatrix} \neq 0$, then S^λ has a direct summand isomorphic to S^μ.*

Proof. Let S^μ be irreducible, where $\mu = (u, v)$ with $u + v = a + b + 3$ and suppose that $v \equiv 3 \pmod 4$ and $\begin{bmatrix} u-v \\ a-v \end{bmatrix} \neq 0$ and that $0 \leq a - v \leq u - v$ which give $v \leq \min\{a-1, b+3\}$. We want to show there are homomorphisms $S^\mu \xrightarrow{\vartheta} S^\lambda \xrightarrow{\varpi} S^{\mu'}$ such that $\varpi \circ \vartheta \neq 0$. Assume $3 \leq v \leq a-1$ and consider $\varpi = \sigma$. Suppose $\vartheta = (\frac{v-1}{2})\hat{\Theta}_A - (\frac{a-v+1}{2})\hat{\Theta}_B$.Now we use Lemma 3.2.18 to show that $\varpi \circ \vartheta \neq 0$. If η is any partition, define $S^\eta_{\mathcal{Z}}$ and $M^\eta_{\mathcal{Z}}$ to be the $\mathscr{H}_{\mathcal{Z}}$–modules defined in 1.2.9, so that

$$S^\eta \cong S^\eta_{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathbb{C} \quad \text{and} \quad M^\eta \cong M^\eta_{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathbb{C}.$$

Define $S^\eta_{F_2}$ and $M^\eta_{F_2}$ to be the $F_2\mathfrak{S}_n$–modules, so that

$$S^\eta_{F_2} \cong S^\eta_{\mathcal{Z}} \otimes_{\mathcal{Z}} F_2 \quad \text{and} \quad M^\eta_{F_2} \cong M^\eta_{\mathcal{Z}} \otimes_{\mathcal{Z}} F_2.$$

For a tableau $R \in \mathcal{T}(\alpha, \beta)$ let $\hat{\Theta}^{\mathcal{Z}}_R$ define the corresponding homomorphism

$$\hat{\Theta}^{\mathcal{Z}}_R : S^\alpha_{\mathcal{Z}} \to M^\beta_{\mathcal{Z}}$$

and define $\hat{\Theta}^{F_2}_R$ to be the corresponding homomorphism

$$\hat{\Theta}^{F_2}_R : S^\alpha_{F_2} \to M^\beta_{F_2}.$$

Set

$$\vartheta_{\mathcal{Z}} : M^\mu_{\mathcal{Z}} \to S^\lambda_{\mathcal{Z}} \qquad \text{by} \quad \vartheta_{\mathcal{Z}} = (\frac{v-1}{2})\hat{\Theta}^{\mathcal{Z}}_A - (\frac{a-v+1}{2})\hat{\Theta}^{\mathcal{Z}}_B$$

also,

$$\vartheta_{F_2} : M^\mu_{F_2} \to S^\lambda_{F_2} \qquad \text{by} \quad \vartheta_{F_2} = \begin{cases} \hat{\Theta}^{F_2}_A + \hat{\Theta}^{F_2}_B & \text{if } a \equiv 0 \pmod 4; \\ \hat{\Theta}^{F_2}_A & \text{if } a \equiv 2 \pmod 4). \end{cases}$$

Suppose $a \equiv 0 \pmod 4$. Suppose $v = 4c+3$ and $a = 4d$ for $c, d \geq 0$. Then

$$\begin{aligned}\vartheta_{\mathcal{Z}} &= \frac{4c+2}{2}\hat{\Theta}_A^{\mathcal{Z}} - \frac{4d-4c-2}{2}\hat{\Theta}_B^{\mathcal{Z}} \\ &= (2c+1)\hat{\Theta}_A - (2d-2c-1)\hat{\Theta}_B^{\mathcal{Z}}.\end{aligned}$$

So that $\vartheta = (2c+1)\Theta_A - (2d-2c-1)\hat{\Theta}_B$ and $\vartheta_{F_2} = \hat{\Theta}_A^{F_2} + \hat{\Theta}_B^{F_2}$. Define

$$\varpi_{\mathcal{Z}} : S_{\mathcal{Z}}^{\lambda} \to M_{\mathcal{Z}}^{\mu'} \qquad \text{by} \quad \varpi_{\mathcal{Z}} = \sum_{T \in \mu} \hat{\Theta}_T^{\mathcal{Z}}$$

and

$$\varpi_{F_2} : S_{\mathcal{Z}}^{\lambda} \to M_{F_2}^{\mu'} \qquad \text{by} \quad \varpi_{F_2} = \sum_{T \in \mu} \hat{\Theta}_T^{F_2}.$$

Then

$$\varpi_{\mathcal{Z}} \circ \vartheta_{\mathcal{Z}} = \sum_{R \in \mathcal{T}_0(\mu', \mu)} b_R \hat{\Theta}_R^{\mathcal{Z}} \quad \text{for some } b_R \in \mathcal{Z}.$$

So that $\varpi_{F_2} \circ \vartheta_{F_2} = \sum_{R \in \mathcal{T}_0(\mu', \mu)} \varphi_{F_2}(b_R)\hat{\Theta}_R^{F_2}$ and $\varpi \circ \vartheta = \sum_{R \in \mathcal{T}_0(\mu', \mu)} \varphi_{\mathbb{C}}(b_R)\hat{\Theta}_R$. It was shown by Dodge and Fayers [7] that $\varpi_{F_2} \circ \vartheta_{F_2} \neq 0$, so that $\varphi_{F_2}(b_R) \neq 0$ for some $R \in \mathcal{T}_0(\mu', \mu)$. By Lemma 3.2.18(3), $\varphi_{\mathbb{C}}(b_R) \neq 0$ and hence $\varpi \circ \vartheta \neq 0$. Suppose $a \equiv 2 \pmod 4$. Suppose $v = 4c+3$ and $a = 4d+2$ for $c, d \geq 0$. Then set

$$\vartheta_{\mathcal{Z}} = (2c+1)\hat{\Theta}_A - (2d-2c)\hat{\Theta}_B,$$

so that

$$\vartheta = (2c+1)\hat{\Theta}_A^{\mathcal{Z}} - (2d-2c)\hat{\Theta}_B^{\mathcal{Z}}$$

by reducing modulo 2 gives

$$\vartheta_{F_2} = \hat{\Theta}_A^{F_2}.$$

Again, Dodge and Fayers [7] proved that $\varpi_{F_2} \circ \vartheta_{F_2} \neq 0$. Hence $\varpi \circ \vartheta \neq 0$ as above. By the argument given it, this shows that S^{λ} has a summand isomorphic to S^{μ}.

□

3.3.2 Irreducible summands of the form $S^{(u,v,2)}$

In this section, we show some Specht modules $S^{(a,3,1^b)}$ which have a summand isomorphic to an irreducible Specht module of the form $S^{(u,v,2)}$, where u is even and v is odd. Throughout this section we assume that $\lambda = (a, 3, 1^b)$

and $\mu = (u, v, 2)$, where a, b, u, v are positive integers with a, b, u even, $a \geq 4$, $u > v > 2$, $n = a + b + 3 = u + v + 2$ and $v \leq \min\{a - 1, b + 1\}$.

Homomorphisms $\delta : S^\lambda \to S^{\mu'}$

Consider a homomorphism from S^λ to $S^{\mu'}$. We use non-semistandard tableaux to construct this homomorphism. Suppose $\mathcal{U}$ is the set of λ-tableaux having the form:

$$\begin{array}{cccccccc}
1 & 2 & 3 & \cdots\cdots & v & \star & \cdots\cdots & \star \\
1 & 1 & 2 & & & & & \\
2 & & & & & & & \\
3 & & & & & & & \\
\vdots & & & & & & & \\
v & & & & & & & \\
\star & & & & & & & \\
\vdots & & & & & & & \\
\star & & & & & & &
\end{array}$$

where the $\star$s denote to the numbers from $v + 1$ to u, and the entries are weakly increasing along the first row and down the first column.

Example 3.3.14. Let $\lambda = (4, 3, 1^2)$, $\mu = (4, 3, 2)$. Then $\mathcal{U} = \{T\}$ such that

$$T = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & 1 & 2 & \\
2 & & & \\
3 & & &
\end{array}.$$

Now define

$$\delta = \sum_{T \in \mathcal{U}} \hat{\Theta}_T.$$

Proposition 3.3.15. *Suppose $\lambda = (a, 3, 1^b)$ and $\mu = (u, v, 2)$, where a, b, u, v are positive integers with a, b, u are even and let $a \geq 4$, $u > v > 2$, $n = a+b+3 = u + v$ and $v \leq \min\{a - 1, b + 1\}$. Then, $\psi_{d,t} \circ \delta = 0$ for each d, t.*

Proof. First take $d \geq v$ and $t = 1$. If $T \in \mathcal{U}$, then T contains a single d and a single $d + 1$. If these lie in the same row of T, then by Theorem 3.2.9 each map Θ_S which occurs has a coefficient which has a factor of $1 + q$. Therefore $\psi_{d,1} \circ \hat{\Theta}_T = 0$. If these lie in the same column of T, then by Theorem 3.2.9 get a tableau which has the row r and row $r + 1$ both equal to $\boxed{d}$ with coefficient $b_S = 1$ then by using Theorem 3.2.12 $\psi_{d,1} \circ \hat{\Theta}_T = 0$. A similar argument applies in the cases where $2 \le d < v$. So we are left with the cases where $d = 1$ and $t \in \{1, 2, 3\}$. If $t = 3$ then by Theorem 3.2.9 each coefficient that occurs has a factor of $1 + q$ and hence $\psi_{1,3} \circ \hat{\Theta}_T = 0$. For example

Example 3.3.16. Take T as in Example 3.3.14. Then by using Theorem 3.2.9 we get

$$\begin{aligned}
\psi_{1,3} \circ \hat{\Theta}_T &= q^2 \begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} \begin{array}{|c|c|c|c|}\hline 1&1&3&4\\\hline 1&1&1&\multicolumn{1}{c}{}\\\cline{1-3} 1&\multicolumn{3}{c}{}\\\cline{1-1} 3&\multicolumn{3}{c}{}\\\cline{1-1}\end{array} \\
&= q^2(1+q)(1+q+q^2) \begin{array}{|c|c|c|c|}\hline 1&1&3&4\\\hline 1&1&1&\multicolumn{1}{c}{}\\\cline{1-3} 1&\multicolumn{3}{c}{}\\\cline{1-1} 3&\multicolumn{3}{c}{}\\\cline{1-1}\end{array} \\
&= 0.
\end{aligned}$$

If $t = 2$ then by Theorem 3.2.9 the homomorphism $\psi_{1,2} \circ \hat{\Theta}_T$ is labelled by three tableaux. The coefficient of two of them have a factor of $1 + q$ and the third term is

$$\begin{bmatrix}3\\1\end{bmatrix} \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&2&3&\cdots\cdots&v&\star&\cdots\cdots&\star\\\hline 1&1&1&\multicolumn{5}{c}{}\\\cline{1-3} 1&\multicolumn{7}{c}{}\\\cline{1-1} 3&\multicolumn{7}{c}{}\\\cline{1-1} \vdots&\multicolumn{7}{c}{}\\\cline{1-1} v&\multicolumn{7}{c}{}\\\cline{1-1} \star&\multicolumn{7}{c}{}\\\cline{1-1} \vdots&\multicolumn{7}{c}{}\\\cline{1-1} \star&\multicolumn{7}{c}{}\\\cline{1-1}\end{array}$$

and this is zero by Theorem 3.2.12. Hence, $\psi_{1,2} \circ \hat{\Theta}_T = 0$.

Example 3.3.17. Take $T = \begin{array}{llll} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & \\ 2 & & & \\ 3 & & & \end{array}$. Then, by Theorem 3.2.9

$$\psi_{1,2} \circ \hat{\Theta}_T = q^2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{array}{llll} 1 & 1 & 3 & 4 \\ 1 & 1 & 1 & \\ 2 & & & \\ 3 & & & \end{array} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{array}{llll} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & \\ 1 & & & \\ 3 & & & \end{array} + q^2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{array}{llll} 1 & 1 & 3 & 4 \\ 1 & 1 & 2 & \\ 1 & & & \\ 3 & & & \end{array} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{array}{llll} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & \\ 1 & & & \\ 3 & & & \end{array} .$$

By using Theorem 3.2.12 we get:

$$\psi_{1,2} \circ \hat{\Theta}_T = 0$$

If $t = 1$ then by Theorem 3.2.9 the homomorphism $\psi_{1,1} \circ \hat{\Theta}_T$ labelled by three tableaux. The coefficient of one of them has factor $q+1$ and the other two are

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{array}{llllllll} 1 & 2 & 3 & \cdots & v & \star & \cdots & \star \\ 1 & 1 & 1 & & & & & \\ 2 & & & & & & & \\ 3 & & & & & & & \\ \vdots & & & & & & & \\ v & & & & & & & \\ \star & & & & & & & \\ \vdots & & & & & & & \\ \star & & & & & & & \end{array} + \begin{array}{llllllll} 1 & 2 & 3 & \cdots & v & \star & \cdots & \star \\ 1 & 1 & 2 & & & & & \\ 1 & & & & & & & \\ 3 & & & & & & & \\ \vdots & & & & & & & \\ v & & & & & & & \\ \star & & & & & & & \\ \vdots & & & & & & & \\ \star & & & & & & & \end{array}$$

which by Theorem 3.2.12 is equal to

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{array}{llllllll} 1 & 2 & 3 & \cdots & v & \star & \cdots & \star \\ 1 & 1 & 1 & & & & & \\ 2 & & & & & & & \\ 3 & & & & & & & \\ \vdots & & & & & & & \\ v & & & & & & & \\ \star & & & & & & & \\ \vdots & & & & & & & \\ \star & & & & & & & \end{array} - \begin{array}{llllllll} 1 & 2 & 3 & \cdots & v & \star & \cdots & \star \\ 1 & 1 & 1 & & & & & \\ 2 & & & & & & & \\ 3 & & & & & & & \\ \vdots & & & & & & & \\ v & & & & & & & \\ \star & & & & & & & \\ \vdots & & & & & & & \\ \star & & & & & & & \end{array} = 0.$$

Hence, $\psi_{1,1} \circ \hat{\Theta}_T = 0$.

Example 3.3.18. Let $T = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & \\ 2 & & & \\ 3 & & & \end{array}$. Then, by Theorem 3.2.9

$$\begin{aligned}
\psi_{1,1} \circ \hat{\Theta}_T &= q^2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{array}{cccc} 1 & 1 & 3 & 4 \\ 1 & 1 & 2 & \\ 2 & & & \\ 3 & & & \end{array} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & \\ 2 & & & \\ 3 & & & \end{array} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & \\ 1 & & & \\ 3 & & & \end{array} \\
&= \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & \\ 2 & & & \\ 3 & & & \end{array} + \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & \\ 1 & & & \\ 3 & & & \end{array} .
\end{aligned}$$

By using Theorem 3.2.12 we get:

$$\psi_{1,1} \circ \hat{\Theta}_T = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & \\ 2 & & & \\ 3 & & & \end{array} - \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & \\ 2 & & & \\ 3 & & & \end{array} = 0.$$

□

Now let us take the following lemma.

Lemma 3.3.19. *[7, Proposition 6.2] Let F be a field of characteristic 2. Suppose $\lambda = (a, 3, 1^b)$ and $\mu = (u, v, 2)$, where a, b, u, v are positive integers with a, b, u are even and let $a \geq 4$, $u > v > 2$, $n = a + b + 3 = u + v + 2$ and $v \leq \min\{a + 1, b + 3\}$. Then, $\sum_{T \in \mathcal{U}} \hat{\Theta}_T^F \neq 0$ for each d, t.*

From Corollary 3.2.19 and Lemma 3.3.19, we deduce

Proposition 3.3.20. *Suppose $\lambda = (a, 3, 1^b)$ and $\mu = (u, v, 2)$ with $a \geq 4$ and $u > v > 2$. Let $n = a + b + 3 = u + v$ and $v \leq \min\{a - 1, b + 1\}$. Then, $\delta \neq 0$.*

Homomorphisms $\gamma : S^\mu \longrightarrow S^\lambda$

In this section we consider homomorphisms from S^μ to S^λ. Assume that D is μ-tableaux of type λ as follows:

$$D = \begin{array}{ccccccccc} 1 & 1 & \cdots\cdots & 1 & 2 & 3 & \cdots & b{+}2 \\ 1 & 1 & \cdots & 1 & & & & \\ 2 & 2 & & & & & & \end{array} .$$

Proposition 3.3.21. *We have* $\psi_{d,t} \circ \hat{\Theta}_D = 0$*, for all* d, t *and* $\hat{\Theta}_D \neq 0$.

Proof. By using Theorem 3.2.9, if $d \geq 2$ then there is at least a factor of $q+1$ and hence $\psi_{d,1} \circ \hat{\Theta}_D = 0$. If $d = 1$ and $t = 1, 2$, then by using Theorem 3.2.9 we get either the homomorphism is labelled by tableau with coefficient $\begin{bmatrix} 2m \\ 1 \end{bmatrix}$ which is zero by Lemma 3.2.20 or the homomorphism which labelled by a sum of tableaux with more than v 1s in rows 2 and 3, and therefore by Theorem 3.2.12 are zero. If $d = 1$ and $t = 3$, then by using Theorem 3.2.9 we get the homomorphism is labelled by tableau with coefficient $\begin{bmatrix} 2m \\ 1 \end{bmatrix}$ which is zero by Lemma 3.2.20. By using Theorem 3.2.12 we get the sum of tableaux and sum two of them we get at leat a factor of 1 and hence $\hat{\Theta}_D \neq 0$. For example, we get at least the next tableau:

$$D = \begin{array}{|c|c|c|c|c|c|c|}
\hline
1 & 1 & \cdots\cdots\cdots & 1 & v{+}2 & \cdots\cdots & b{+}2 \\
\hline
2 & \cdots\cdots & v{+}1 \\
\cline{1-3}
3 & 4 \\
\cline{1-2}
\end{array}.$$

$\square$

Now we prove the second case of Theorem D. Recall that

Theorem D(2). *Suppose* $\lambda = (a, 3, 1^b)$ *is a partition of* n*, where* a, b *are positive even integers with* $a \geq 4$ *and let* μ *be a partition of* n *such that* S^μ *is irreducible. If* μ *or* μ' *equals* $(u, v, 2)$*, where* u *is even and* v *is odd with* $u > v$ *and* $\begin{bmatrix} u-v \\ a-v \end{bmatrix} \neq 0$*, then* S^λ *has a direct summand isomorphic to* S^μ*.*

Proof. Let S^μ be irreducible, where $\mu = (u, v, 2)$ and suppose that $\begin{bmatrix} u-v \\ a-v \end{bmatrix} \neq 0$ and that $0 \leq a - v \leq u - v$ which give $v \leq \min\{a-1, b+1\}$. So we have

$$\gamma : S^\mu \longrightarrow S^\lambda \text{ and } \delta : S^\lambda \longrightarrow S^{\mu'}.$$

The argument that $\delta \circ \gamma \neq 0$ is identical to the argument given in proof of Theorem D(1). $\square$

Chapter 4

New family of decomposable Specht modules of $F\mathfrak{S}_n$

In this Chapter, we find a new family of decomposable Specht modules for the symmetric group in characteristic 2.

4.1 The main results

In this section we state the main theorems which describe some Specht modules S^λ which have a summand isomorphic to an irreducible Specht module of the form $S^{(u,v)}$, where u, v are positive integers with v odd and u even and $u > v$. We assume that $q = 1$ and that the field F has characteristic 2, so that we are working with the symmetric group algebra $F_2\mathfrak{S}_n$. We use the method that we describe in section 3.1.

Theorem I. *Let F be a field of characteristic 2 and let $q = 1$. Suppose $\lambda = (a, 5, 1^b)$ is a partition of n, where a, b are positive even integers and let μ be a partition of n such that S^μ is irreducible. Suppose μ or μ' equals (u, v), where $u > v$ and u is even and v is odd with $v \leq \min\{a-1, b+1\}$ and $\binom{u-v}{a-v}$ is odd. If one of the following condition holds:*

- *If $v \equiv 7 \mod 8$,*
- *If $v \equiv 5 \mod 8$ and $a - v \equiv 3 \mod 4$,*

then S^λ has a direct summand isomorphic to S^μ.

Theorem II. *Let F be a field of characteristic 2 and let $q = 1$. Suppose $\lambda = (a, 7, 1^b)$ is a partition of n, where a, b are positive even integers and let μ be a partition of n such that S^μ is irreducible. Suppose μ or μ' equals (u, v), where $u > v$ and u is even and v is odd with $v \le \min\{a-1, b+1\}$ and $\binom{u-v}{a-v}$ is odd and $v \equiv 7 \mod 8$. Then S^λ has a direct summand isomorphic to S^μ.*

Theorem III. *Let F be a field of characteristic 2 and let $q = 1$. Suppose $\lambda = (a, c, 1^b)$ is a partition of n, where a, b are positive even integers and c is odd. Let μ be a partition of n such that S^μ is irreducible. Let m be minimal such that $c \le 2^m$. If μ or μ' equals (u, v), where $u > v$ and u is even and v is odd with $v \le \min\{a-1, b+1\}$ and $\binom{u-v}{a-v}$ is odd and $v \equiv -1 \mod 2^m$ and $a - v \equiv -1 \mod 2^m$. Then S^λ has a direct summand isomorphic to S^μ.*

Let us state the classification of irreducible Specht modules. Let m be a non-negative integer we say $l(m)$ the smallest positive integer such that $2^{l(m)} > m$.

Theorem 4.1.1. *[19, Main Theorem] Let μ be a partition of n and F_2 be the field of characteristic 2. Then the $F_2\mathfrak{S}_n$–module S^μ is irreducible if and only if one of the following holds:*

- $\mu_i - \mu_{i+1} \equiv -1 \mod 2^{l(\mu_{i+1}-\mu_{i+2})}$ *for each* $i \ge 1$;
- $\mu'_i - \mu'_{i+1} \equiv -1 \mod 2^{l(\mu'_{i+1}-\mu'_{i+2})}$ *for each* $i \ge 1$;
- $\mu = (2^2)$.

Remark 4.1.2. If μ is 2–regular then Theorem 4.1.1 is a special case of Theorem 3.2.15.

Corollary 4.1.3. *Suppose $\mu = (u, v)$, where u is even and v is odd with $u > v$. Let m be minimal such that $c \le 2^m$. If S^μ is irreducible, then*

$$u - v \equiv -1 \mod 2^m.$$

Now we state some useful results on binomial coefficients modulo 2.

Lemma 4.1.4. *[15, Lemma 22.4] Suppose*

$$c = c_0 + 2c_1 + 2^2c_2 + \cdots + 2^m c_m \quad \text{where } 0 \le c_i < 2.$$
$$v = v_0 + 2v_1 + 2^2v_2 + \cdots + 2^m v_m \quad \text{where } 0 \le v_i < 2.$$

Then
$$\binom{c}{v} \equiv \binom{c_0}{v_0}\binom{c_1}{v_1}\cdots\binom{c_m}{v_m} \mod 2.$$
Hence, $\binom{c}{v}$ *is divisible by* 2 *if and only if* $v_i > c_i$ *for some* i.

Lemma 4.1.5. *[15, Corollary 22.5] Let* $c \geq v \geq 1$. *Then all the binomial coefficients*
$$\binom{c}{v}, \binom{c-1}{v-1}, \ldots, \binom{c-v+1}{1}$$
are divisible by 2 *if and only if* $c - v \equiv -1 \mod 2^{l(v)}$.

4.2 The Specht modules labelled by $(a, 5, 1^b)$

In this section, we find Specht modules $S^{(a,5,1^b)}$ which have a direct summand isomorphic to an irreducible Specht module $S^{(u,v)}$, where u is even and v is odd with $u > v$. We assume that a, b, u, v are positive even integers with $a \geq 6$. Let $n = a + b + 5 = u + v$. The regularisation of partitions $(a, 5, 1^b)$ is given by following lemma.

Lemma 4.2.1. *Let* $a \geq 6$ *and* $b \geq 2$. *Then*
$$(a,5,1^b)^R = \begin{cases} (a,5,2) & (a > b = 2) \\ (a, b+1, 4) & (a > b) \\ (b+2, a-1, 4) & (a \leq b > 2). \end{cases}$$

Now from Theorem 1.5.21 and Lemma 4.2.1, we see that $D^{(u,v)}$ appears as a composition factor of $S^{(a,5,1^b)}$ only if $(u,v) \trianglerighteq (a,5,1^b)^R$, so we need $u \geq \max\{a, b+2\}$ that is $v \leq \min\{a+3, b+5\}$. In this section we need $5 \leq v \leq a-1$ for maps to be defined and $v \leq b+1$ for independence. Assume that $5 \leq v \leq \min\{a-1, b+1\}$.

4.2.1 Homomorphisms from S^λ to $S^{\mu'}$

Consider homomorphisms $\sigma : S^\lambda \longrightarrow S^{\mu'}$. Suppose $\mathcal{U}$ is the set of λ–tableaux of type μ' which take the form:

$$\mathcal{U} = \begin{array}{cccccccc} 1 & 2 & 3 & \cdots\cdots\cdots & v & \star & \cdots & \star \\ 1 & \star & \star & \star & \star & & & \\ \star & & & & & & & \\ \vdots & & & & & & & \\ \star & & & & & & & \end{array}$$

such that the $\star$s are the numbers from 2 to u, and the entries are strictly increasing along each row and weakly increasing down each column. Now define

$$\sigma = \sum_{T \in \mathcal{U}} \hat{\Theta}_T.$$

Proposition 4.2.2. *We have $\psi_{d,t} \circ \sigma = 0$ for each d, t.*

Proof. First choose $v<d<u$ and $t = 1$. If $T \in \mathcal{U}$, then T contains a single d and a single $d+1$. If these lie in the same row of T, then by Theorem 3.2.9 and Theorem 3.2.12 $\psi_{d,1} \circ \hat{\Theta}_T = 0$. Similarly, if these lie in the same column of T then $\psi_{d,1} \circ \hat{\Theta}_T = 0$. Otherwise, there is another tableau $T' \in \mathcal{U}$ obtained by interchanging the d and the $d+1$. By Theorem 3.2.9 we have $\psi_{d,1} \circ (\hat{\Theta}_T + \hat{\Theta}_{T'}) = 0$. Hence, $\psi_{d,1} \circ \sigma = 0$.

Second take $d = v$ and $t = 1$. Then $T \in \mathcal{U}$ contains either a single v and a single $v+1$ in the first row or a single v and a single $v+1$ below the first row. Suppose there is a single v and a single $v+1$ below the first row. If these occur in the same row or the same column of T, then by Theorem 3.2.9 and Theorem 3.2.12 $\psi_{v,1} \circ \hat{\Theta}_T = 0$. Now if a single v and a single $v+1$ below the first row occur in the different row and column then there is another tableau $T' \in \mathcal{U}$ obtained by interchanging the v and the $v+1$. By Theorem 3.2.9 we have $\psi_{v,1} \circ (\hat{\Theta}_T + \hat{\Theta}'_T) = 0$. If there is a single v a single $v+1$ in the first row then by Theorem 3.2.9, $\psi_{v,1} \circ \hat{\Theta}_T = 0$. Hence, $\psi_{v,1} \circ \sigma = 0$. If $1 \leq d < v$ and $t = 2$, then we get $\psi_{d,2} \circ \hat{\Theta}_T = 0$ for each $T \in \mathcal{U}$ by applying Theorem 3.2.9.

Now take $2 \leq d < v$ and $t = 1$, and consider a tableau $T \in \mathcal{U}$. There are a single d and a single $d+1$ below the first row. If these occur in the same row or in the same column then by using Theorem 3.2.9 and Theorem 3.2.12 we have $\psi_{d,1} \circ \hat{\Theta}_T = 0$. Otherwise, there is a tableau $T' \in \mathcal{U}$ formed by interchanging the d and the $d+1$ below the first row. Then by Theorem 3.2.9 $\psi_{d,1} \circ (\hat{\Theta}_T + \hat{\Theta}_{T'}) = 0$. Hence, by $\psi_{d,2} \circ \sigma = 0$.

Finally, suppose $d = t = 1$. By applying Theorem 3.2.9, we have that

$\psi_{1,1} \circ \sigma$ is the sum of homomorphisms labelled by tableaux

$$\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
1 & 2 & 3 & \cdots\cdots\cdots\cdots & v & \star & \cdots\cdots & \star \\
\hline
1 & \star & \star & \star & \star \\
\cline{1-5}
1 \\
\cline{1-1}
\vdots \\
\cline{1-1}
\star \\
\cline{1-1}
\end{array}$$

where the $\star$s represent the numbers from 3 to u, and where the entries strictly increase along rows and weakly increase down columns. Now to move the 1 from row 3 to row 2 we apply Theorem 3.2.12 to each of these homomorphisms and then to reorder rows $3, \ldots, b+2$. We obtain a sum of homomorphisms indexed by tableaux of the form

$$\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
1 & 2 & 3 & \cdots\cdots\cdots\cdots & v & \star & \cdots\cdots & \star \\
\hline
1 & 1 & \star & \star & \star \\
\cline{1-5}
\star \\
\cline{1-1}
\vdots \\
\cline{1-1}
\star \\
\cline{1-1}
\end{array}.$$

Now each tableau occurs b times in this way, but b is even. Then $\psi_{1,1} \circ \sigma = 0$. □

Now we show that $\sigma \neq 0$. So we consider the following Proposition.

Proposition 4.2.3. *We have $\sigma \neq 0$.*

We need Theorem 3.1.10 to prove this proposition. Consider the semistandard tableau S such that:

$$S = \begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
1 & 1 & 2 & \cdots\cdots\cdots\cdots & v & b{+}7 & b{+}8 & \cdots\cdots & u \\
\hline
2 & b{+}3 & b{+}4 & b{+}5 & b{+}6 \\
\cline{1-5}
3 \\
\cline{1-1}
4 \\
\cline{1-1}
\vdots \\
\cline{1-1}
b{+}2 \\
\cline{1-1}
\end{array}.$$

Let $T \in \mathcal{U}$ and consider expressing $\hat{\Theta}_T$ as a linear combination of semistandard homomorphisms. By applying Theorem 3.1.10, T contributes to S only if $S \trianglerighteq T$. Therefore, we can ignore all $T \in \mathcal{U}$ for which $S \ntrianglerighteq T$. In particular, we consider only those tableaux in $\mathcal{U}$ which have $b+7, \ldots, u$ in the first row and $b+3, b+4, b+5, b+6$ in the top two rows. Now assume that $v \leq b+1$, then the tableaux $T \in \mathcal{U}$ that we need to consider are those of the following forms:

1. Suppose $v < i \leq b+2$. Then

$$T[i] = \begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
1 & 2 & \cdots\cdots\cdots & v & i & b{+}7 & b{+}8 & \cdots & u \\
\hline
1 & b{+}3 & b{+}4 & b{+}5 & b{+}6 \\
\cline{1-5}
2 \\
\cline{1-1}
\vdots \\
\cline{1-1}
v \\
\cline{1-1}
v{+}1 \\
\cline{1-1}
\vdots \\
\cline{1-1}
\hat{i} \\
\cline{1-1}
\vdots \\
\cline{1-1}
b{+}2 \\
\cline{1-1}
\end{array} .$$

2. Suppose $2 \leq i \leq b+2$ and $3 \leq k \leq 6$. Then

$$U[i,k] = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
1 & 2 & 3 & \cdots\cdots\cdots & v & b{+}k & b{+}7 & b{+}8 & \cdots & u \\
\hline
1 & i & b{+}3 & \cdots & \widehat{b{+}k} & \cdots & b{+}6 \\
\cline{1-7}
2 \\
\cline{1-1}
\vdots \\
\cline{1-1}
\hat{i} \\
\cline{1-1}
\vdots \\
\cline{1-1}
b{+}2 \\
\cline{1-1}
\end{array} .$$

Note that, the $\hat{i}$ in the first column means that i does not appear in that column and $b \hat{+} k$ in the second row means that $b+k$ does not appear in that row. First apply Theorem 3.2.12 on the tableau $T[i]$ to move the 1 from row 2 to row 1. We see that of the tableaux appearing in the resulting expression, the only ones dominated by S are

$$T'[i] = \begin{array}{ccccccccc}
1 & 1 & 2 & \cdots\cdots & v & b+7 & b+8 & \cdots & u \\
i & b+3 & b+4 & b+5 & b+6 & & & & \\
2 & & & & & & & & \\
\vdots & & & & & & & & \\
v & & & & & & & & \\
v+1 & & & & & & & & \\
\vdots & & & & & & & & \\
\hat{i} & & & & & & & & \\
\vdots & & & & & & & & \\
b+2 & & & & & & & &
\end{array}$$

and

$$T'[i,j] = \begin{array}{ccccccccccc}
1 & 1 & 2 & \cdots\cdots & j & \cdots & v & i & b+7 & b+8 & \cdots & u \\
j & b+3 & b+4 & b+5 & b+6 & & & & & & & \\
2 & & & & & & & & & & & \\
\vdots & & & & & & & & & & & \\
v & & & & & & & & & & & \\
v+1 & & & & & & & & & & & \\
\vdots & & & & & & & & & & & \\
\hat{i} & & & & & & & & & & & \\
\vdots & & & & & & & & & & & \\
b+2 & & & & & & & & & & &
\end{array}$$

where $2 \leq j \leq v$. Now we move the 2 from row 3 to row 2 in $T'[i]$ by using Theorem 3.2.12. We obtain five tableaux, but four of these are not dominated by S. We are left with the tableau

$$T''[i] = \begin{array}{ccccccccc}
1 & 1 & 2 & \cdots\cdots & v & b+7 & b+8 & \cdots & u \\
2 & b+3 & b+4 & b+5 & b+6 & & & & \\
i & & & & & & & & \\
3 & & & & & & & & \\
\vdots & & & & & & & & \\
v & & & & & & & & \\
v+1 & & & & & & & & \\
\vdots & & & & & & & & \\
\hat{i} & & & & & & & & \\
\vdots & & & & & & & & \\
b+2 & & & & & & & &
\end{array}$$

and $i-5$ more applications of Theorem 3.2.12 prove that $\hat{\Theta}_{T''[i]}$ is equal to $\hat{\Theta}_S$. Similarly, we apply Theorem 3.2.12 to $T^i[i,j]$, to move the 2 from row 3 to row 2. If $j = 2$, then the four tableaux obtained are not dominated by S. If $j > 2$, then four of the five tableaux obtained are not dominated by S; the fifth has two rows equal to $\boxed{j}$, so the homomorphism is zero by Theorem 3.2.12. Hence, we get that $\hat{\Theta}_{T[i]} = \hat{\Theta}_S$ plus a linear combination of homomorphisms indexed by tableaux not dominated by S.

Now we apply Theorem 3.2.12 to $U[i,k]$ to move the 1 from row 2 to row 1. The tableaux obtained that are dominated by S are $T'[i]$ and the tableaux

$$U'[i,j,k] = \begin{array}{llllllllllll} 1 & 1 & 2 & \cdots\cdots\cdots & \hat{j} & \cdots & v & b+k & b+7 & b+8 & \cdots & u \\ j & i & b+3 & \cdots & \widehat{b+k} & \cdots & b+6 \\ 2 \\ \vdots \\ \hat{i} \\ \vdots \\ b+2 \end{array}$$

where $2 \leq j \leq v$ and $i \neq j$. Note that if $i < j$ where i and j are in the second row, then we write i before j. If $i = j$ then the accompanying coefficient would be $\binom{2}{1} = 0$, so this case dose not occur. If $i = 2$, then $U'[i,j,k]$ is a semistandard tableau different from S. If $i > 2$ then we apply Theorem 3.2.12 to move the 2 from row 3 to row 2; by ignoring the tableau not dominated by S and the tableaux with two rows equal to $\boxed{j}$, then the only tableau we get is

$$U''[i,j,k] = \begin{array}{llllllllllll} 1 & 1 & 2 & \cdots\cdots\cdots & \hat{j} & \cdots & v & b+k & b+7 & b+8 & \cdots & u \\ 2 & j & b+3 & \cdots & \widehat{b+k} & \cdots & b+6 \\ i \\ 3 \\ \vdots \\ \hat{i} \\ \vdots \\ b+2 \end{array}$$

and $i-5$ more applications of Theorem 3.2.12 show that $\hat{\Theta}_{U''[i,j,k]}$ equals a semistandard homomorphism different from $\hat{\Theta}_S$.

We conclude that $\hat{\Theta}_{U[i,k]} = \hat{\Theta}_S$ plus a linear combination of homomorphisms indexed by tableaux which are either not dominated by S or semistandard and different from S. Now by combining the two cases together, we find that the coefficient of $\hat{\Theta}_S$ in σ is the total number of tableaux of the form $T[i]$ or $U[i,k]$, which is $(b+2-v)+4(b+1)$ which is not zero modulo 2. Hence, we shown that $\sigma \neq 0$ as required.

4.2.2 Homomorphisms from S^μ to S^λ

In this section we consider homomorphisms $\gamma : S^\mu \to S^\lambda$. Define A, B, C to be the μ–tableaux of type λ as follows:

$$A = \begin{array}{|c|c|c|c|c|c|}\hline 1^{a-v} & 2^5 & 3 & 4 & \dots & b+2 \\ \hline 1^v \\ \cline{1-1}\end{array}$$

$$B = \begin{array}{|c|c|c|c|c|c|}\hline 1^{a-v+2} & 2^3 & 3 & 4 & \dots & b+2 \\ \hline 1^{v-2} & 2^2 \\ \cline{1-2}\end{array}$$

$$C = \begin{array}{|c|c|c|c|c|c|}\hline 1^{a-v+4} & 2 & 3 & 4 & \dots & b+2 \\ \hline 1^{v-4} & 2^4 \\ \cline{1-2}\end{array}$$

Remark 4.2.4. The notation above means that the first row of A contains $a - v$ entries equal to 1, 5 entries equal to 2 and 1 entry equal to j for $3 \leq j \leq b+2$. The second row of A contains v entries equal to 1.

Lemma 4.2.5. *$\hat{\Theta}_A$, $\hat{\Theta}_B$ and $\hat{\Theta}_C$ are non-zero, and are linearly independent if $v \leq b+1$.*

Proof. By using Theorem 3.2.12, we express $\hat{\Theta}_A$, $\hat{\Theta}_B$ and $\hat{\Theta}_C$ as linear combinations of semistandard homomorphisms such that there is at least one semistandard tableau appearing in each case. Thus, the homomorphisms are non-zero. Moreover, if $v \leq b+1$, then in the expression for $\hat{\Theta}_A$ there is at least one semistandard tableau with four 2s in the first row which there is no such tableau appearing in the expression for $\hat{\Theta}_B$ and $\hat{\Theta}_C$. Also, in the expression for $\hat{\Theta}_B$ there is at least one semistandard tableau with two 2s in the first row which there is no such tableau appearing in the expression for $\hat{\Theta}_C$. Hence, $\hat{\Theta}_A$, $\hat{\Theta}_B$ and $\hat{\Theta}_C$ are linearly independent. □

Proposition 4.2.6. *1. If $a - v \equiv 7 \mod 8$, then $\psi_{d,t} \circ \hat{\Theta}_A = 0$ for all d, t.*

2. If $a - v \equiv 1 \mod 4$ and $v \equiv 1 \mod 4$, then $\psi_{d,t} \circ \hat{\Theta}_B = 0$ for all d, t.

3. If $v \equiv 3 \mod 8$, then $\psi_{d,t} \circ \hat{\Theta}_C = 0$ for all d, t.

4. If $a - v \equiv 3 \mod 8$ and $v \equiv 7 \mod 8$. Then $\psi_{d,t} \circ (\hat{\Theta}_A + \hat{\Theta}_C) = 0$ for all d, t

5. If $a - v \equiv 3 \mod 8$ and $v \equiv 1 \mod 8$. Then $\psi_{d,t} \circ (\hat{\Theta}_B + \hat{\Theta}_C) = 0$ for all d, t.

6. If $a - v \equiv 3 \mod 8$ and $v \equiv 5 \mod 8$. Then $\psi_{d,t} \circ (\hat{\Theta}_A + \hat{\Theta}_B + \hat{\Theta}_C) = 0$ for all d, t.

7. If $a - v \equiv 1 \mod 8$ and $v \equiv 7 \mod 8$. Then $\psi_{d,t} \circ (\hat{\Theta}_A + \hat{\Theta}_B + \hat{\Theta}_C) = 0$ for all d, t.

8. If $a - v \equiv 5 \mod 8$ and either $v \equiv 3 \mod 8$ or $v \equiv 7 \mod 8$. Then $\psi_{d,t} \circ (\hat{\Theta}_A + \hat{\Theta}_B) = 0$ for all d, t.

9. If $a - v \equiv 7 \mod 8$ and $v \equiv 1 \mod 8$. Then $\psi_{d,t} \circ (\hat{\Theta}_B + \hat{\Theta}_C) = 0$ for all d, t.

Proof. If $d \geq 2$, then from Theorem 3.2.9 we get $\psi_{d,1} \circ \hat{\Theta}_A = \psi_{d,1} \circ \hat{\Theta}_B = \psi_{d,1} \circ \hat{\Theta}_C = 0$. Now, if $d = 1$ then $\psi_{1,t} \circ \hat{\Theta}_A = \psi_{1,t} \circ \hat{\Theta}_B = \psi_{1,t} \circ \hat{\Theta}_C = 0$ for $t = 1, 3, 5$, because each A, B, C have an odd number of 1s in each row and use that by Lemma 4.1.4 $\binom{2m}{2j+1} \equiv 0$ for all $m, j \geq 0$. Finally, if $d = 1$ and $t = 2, 4$, then repeatedly using Lemma 4.1.4 then we have

1. Suppose $a - v \equiv 7 \mod 8$. Then

$$\psi_{1,2} \circ \hat{\Theta}_A = \binom{a-v+2}{2} \begin{array}{|c|c|c|c|c|c|}\hline 1^{a-v+2} & 2^3 & 3 & 4 & \dots & b+2 \\ \hline 1^v \\ \cline{1-1}\end{array} = 0$$

and

$$\psi_{1,4}\circ\hat{\Theta}_A = \binom{a-v+4}{4} \begin{array}{|c|c|c|c|c|c|}\hline 1^{a-v+4} & 2 & 3 & 4 & \dots & b+2 \\ \hline 1^{v} \\ \cline{1-1}\end{array} = 0$$

2. Suppose $a-v \equiv 1 \mod 4$ and $v \equiv 1 \mod 4$. Then

$$\begin{aligned}\psi_{1,2}\circ\hat{\Theta}_B = {} & \binom{a-v+4}{2} \begin{array}{|c|c|c|c|c|c|}\hline 1^{a-v+4} & 2 & 3 & 4 & \dots & b+2 \\ \hline 1^{v-2} & 2^{2} \\ \cline{1-2}\end{array} \\ & + \binom{v}{2} \begin{array}{|c|c|c|c|c|c|}\hline 1^{a-v+2} & 2^{3} & 3 & 4 & \dots & b+2 \\ \hline 1^{v} \\ \cline{1-1}\end{array} \\ & + \binom{a-v+3}{1}\binom{v-1}{1} \begin{array}{|c|c|c|c|c|c|}\hline 1^{a-v+3} & 2^{2} & 3 & 4 & \dots & b+2 \\ \hline 1^{v-1} & 2 \\ \cline{1-2}\end{array} = 0\end{aligned}$$

and

$$\begin{aligned}\psi_{1,4}\circ\hat{\Theta}_B = {} & \binom{a-v+4}{2}\binom{v}{2} \begin{array}{|c|c|c|c|c|c|}\hline 1^{a-v+4} & 2 & 3 & 4 & \dots & b+2 \\ \hline 1^{v} \\ \cline{1-1}\end{array} \\ & + \binom{a-v+5}{3}\binom{v-1}{1} \begin{array}{|c|c|c|c|c|}\hline 1^{a-v+5} & 3 & 4 & \dots & b+2 \\ \hline 1^{v-1} & 2 \\ \cline{1-2}\end{array} = 0\end{aligned}$$

3. Suppose $v \equiv 3 \mod 8$. Then

$$\psi_{1,2} \circ \hat{\Theta}_C = \binom{v-2}{2} \begin{array}{|c|c|c|c|c|c|} \hline 1^{a-v+4} & 2 & 3 & 4 & \dots & b+2 \\ \hline 1^{v-2} & 2^2 \\ \cline{1-2} \end{array}$$

$$+ \binom{a-v+5}{1}\binom{v-3}{1} \begin{array}{|c|c|c|c|c|} \hline 1^{a-v+5} & 3 & 4 & \dots & b+2 \\ \hline 1^{v-3} & 2^3 \\ \cline{1-2} \end{array} = 0$$

and

$$\psi_{1,4} \circ \hat{\Theta}_C = \binom{v}{4} \begin{array}{|c|c|c|c|c|c|} \hline 1^{a-v+4} & 2 & 3 & 4 & \dots & b+2 \\ \hline 1^{v} \\ \cline{1-1} \end{array}$$

$$+ \binom{a-v+5}{1}\binom{v-1}{3} \begin{array}{|c|c|c|c|c|} \hline 1^{a-v+5} & 3 & 4 & \dots & b+2 \\ \hline 1^{v-1} & 2 \\ \cline{1-2} \end{array} = 0$$

4. Suppose $a-v \equiv 3 \mod 8$ and $v \equiv 7 \mod 8$. Then by applying Theorem 3.2.9 on A and C we get

$$\psi_{1,2} \circ \hat{\Theta}_A = \binom{a-v+2}{2} \begin{array}{|c|c|c|c|c|c|} \hline 1^{a-v+2} & 2^3 & 3 & 4 & \dots & b+2 \\ \hline 1^{v} \\ \cline{1-1} \end{array} = 0$$

and

$$\psi_{1,4} \circ \hat{\Theta}_A = \binom{a-v+4}{4} \begin{array}{|c|c|c|c|c|c|} \hline 1^{a-v+4} & 2 & 3 & 4 & \dots & b+2 \\ \hline 1^{v} \\ \cline{1-1} \end{array} \neq 0.$$

Also,

$$
\psi_{1,2}\circ\hat{\Theta}_C = \binom{v-2}{2}\begin{array}{|c|c|c|c|c|c|}\hline 1^{a-v+4} & 2 & 3 & 4 & \dots & b+2 \\ \hline 1^{v-2} & 2^2 \\ \cline{1-2}\end{array}
$$

$$
+\binom{a-v+5}{1}\binom{v-3}{1}\begin{array}{|c|c|c|c|c|}\hline 1^{a-v+5} & 3 & 4 & \dots & b+2 \\ \hline 1^{v-3} & 2^3 \\ \cline{1-2}\end{array} = 0
$$

and

$$
\psi_{1,4}\circ\hat{\Theta}_C = \binom{v}{4}\begin{array}{|c|c|c|c|c|c|}\hline 1^{a-v+4} & 2 & 3 & 4 & \dots & b+2 \\ \hline 1^{v} \\ \cline{1-1}\end{array}
$$

$$
+\binom{a-v+5}{1}\binom{v-1}{3}\begin{array}{|c|c|c|c|c|}\hline 1^{a-v+5} & 3 & 4 & \dots & b+2 \\ \hline 1^{v-1} & 2 \\ \cline{1-2}\end{array} \neq 0.
$$

Hence,

$$
\psi_{1,4}\circ(\hat{\Theta}_A+\hat{\Theta}_C) = 0.
$$

Similarly, we can show all other cases.

□

Remark 4.2.7. From Proposition 4.2.6, we can write the homomorphisms in the following table.

	$v\equiv 1$	$v\equiv 3$	$v\equiv 5$	$v\equiv 7$
$a-v\equiv 1$	$\hat{\Theta}_B$	$\hat{\Theta}_C$	$\hat{\Theta}_B$	$\hat{\Theta}_A+\hat{\Theta}_B+\hat{\Theta}_C$
$a-v\equiv 3$	$\hat{\Theta}_B+\hat{\Theta}_C$	$\hat{\Theta}_C$	$\hat{\Theta}_A+\hat{\Theta}_B+\hat{\Theta}_C$	$\hat{\Theta}_A+\hat{\Theta}_C$
$a-v\equiv 5$	$\hat{\Theta}_B$	$\hat{\Theta}_A+\hat{\Theta}_B,\ \hat{\Theta}_C$	$\hat{\Theta}_B$	$\hat{\Theta}_A+\hat{\Theta}_B$
$a-v\equiv 7$	$\hat{\Theta}_A,\ \hat{\Theta}_B+\hat{\Theta}_C$	$\hat{\Theta}_A,\ \hat{\Theta}_C$	$\hat{\Theta}_A$	$\hat{\Theta}_A$

As a corollary of proposition 4.2.6 we consider

Corollary 4.2.8. *We have*

$$\dim_F \operatorname{Hom}_{F\mathfrak{S}_n}(S^\mu, S^\lambda) \geq \begin{cases} 2 & \text{if } v \equiv 3 \mod 8 \text{ and either } a-v \equiv 5 \text{ or } 7 \mod 8 \\ & \quad \text{or if } a-v \equiv 7 \mod 8 \text{ and } v \equiv 1 \mod 8, \\ 1 & \text{otherwise.} \end{cases}$$

4.2.3 Composing the homomorphisms

Now we find when S^μ is a summand of S^λ by composing the homomorphisms. Let D be the μ–tableau of type μ'

$$D = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & \cdots\cdots\cdots\cdots\cdots\cdots & u \\ \hline 1 & 2 & 3 & \cdots\cdots \quad v & \\ \hline \end{array}.$$

Then we have the following theorem.

Theorem 4.2.9. *Suppose $T \in \mathcal{U}$, and let x be the entry in the $(2,2)$–position of T and z be the entry in the $(2,4)$–position of T. Then*

$$\hat{\Theta}_T \circ \hat{\Theta}_A = \hat{\Theta}_D, \qquad \hat{\Theta}_T \circ \hat{\Theta}_B = \begin{cases} \hat{\Theta}_D & (x \leq v < z) \\ 0 & (v < x \text{ or } z \leq v). \end{cases} \qquad \hat{\Theta}_T \circ \hat{\Theta}_C = \begin{cases} \hat{\Theta}_D & (z \leq v) \\ 0 & (z > v). \end{cases}$$

Furthermore, $\hat{\Theta}_D \neq 0$.

Proof. From Theorem 3.1.7 we get $\hat{\Theta}_D \neq 0$. Now let $T \in \mathcal{U}$ we apply Theorem 3.1.12 with S equal to either A or B or C. Suppose $X \in \mathcal{X}$. Since each T^i is a proper set, each X^{ij} must be as well. This means that if some integer i appears in two sets X^{kj}, X^{lj}, then the multinomial coefficient $\binom{X_j^{1j}+X_j^{2j}+X_j^{3j}+\ldots}{X_j^{1j},X_j^{2j},X_j^{3j},\ldots}$ from Theorem 3.1.12 will include a factor $\binom{2}{1}$, which gives 0.

So in order to get a non-zero coefficient in Theorem 3.1.12, we must have $X^{1j}, X^{2j}, X^{3j}, \ldots$ pairwise disjoint for each j, which means that we will have

$$X^{11} \sqcup X^{21} \sqcup \cdots = \{1, \ldots, u\}, \qquad X^{12} \sqcup X^{22} = \{1, \ldots, v\}; \qquad (\ddagger)$$

so U_X will equal D.

If $S = A$, the only way to achieve this is to have

$$X^{11} = T^1 \backslash \{1, \ldots, v\}, \quad X^{12} = \{1, \ldots, v\}, \quad X^{i1} = T^i \text{ for } i \geq 2.$$

Thus we have $\hat{\Theta}_T \circ \hat{\Theta}_A = \hat{\Theta}_D$.

In the case $S = B$, let y be the $(2,3)$–entry of T and z be the $(2,4)$–entry of T and z' be the $(2,5)$–entry of T. If $x > v$ then we cannot possibly achieve (‡). So we get $\hat{\Theta}_T \circ \hat{\Theta}_B = 0$ in this case. If $x \leq v < y$, then the only way to achieve (‡) is to have $X^{22} = \{1, x\}$ and $X^{12} = \{2, \dots, \hat{x}, \dots, v\}$, and this yields $\hat{\Theta}_T \circ \hat{\Theta}_B = \hat{\Theta}_D$. If $y \leq v < z$, then there are three possible ways to achieve (‡), that is we can have $X^{22} = \{1, x\}$ or $X^{22} = \{1, y\}$ or $X^{22} = \{x, y\}$ and $X^{12} = \{2, \dots, \hat{x}, \dots, v\}$ or $X^{12} = \{2, \dots, \hat{y}, \dots, v\}$ or $X^{12} = \{1, \dots, \hat{x}, \hat{y}, \dots, v\}$; each of these gives a coefficient of 1, and again we have $\hat{\Theta}_T \circ \hat{\Theta}_B = \hat{\Theta}_D$. If $z \leq v < z'$, then X^{22} must contain either x or y or z then there are six possible ways to achieve (‡); each of these gives a coefficient of 1, hence we have $\hat{\Theta}_T \circ \hat{\Theta}_B = 0$. If $z' \leq v$, then X^{22} must contain either x or y or z or z' then there are ten possible ways to achieve (‡); each of these gives a coefficient of 1, hence we have $\hat{\Theta}_T \circ \hat{\Theta}_B = 0$.

In the case $S = C$, we have

$$|X^{11}| = a - v + 4, \quad |X^{12}| = v - 4, \quad |X^{22}| = 4, \quad |X^{21}| = 1, \quad X^{i1} = T^i \text{ for } i \geq 3.$$

X^{11} must contain either z or z', so if $z > v$ then we cannot possibly achieve (‡). So we get $\hat{\Theta}_T \circ \hat{\Theta}_C = 0$ in this case. If $z \leq v < z'$, then the only way to achieve (‡) is to have $X^{22} = \{1, \dots, z\}$ and $X^{21} = \{z'\}$, and this yields $\hat{\Theta}_T \circ \hat{\Theta}_C = \hat{\Theta}_D$. Finally, if $z' \leq v$, then there are five possible ways to achieve (‡); each of these gives a coefficient of 1, and again we have $\hat{\Theta}_T \circ \hat{\Theta}_C = \hat{\Theta}_D$. □

Lemma 4.2.10.

- *The number of tableaux in $\mathcal{U}$ is $\binom{u-v}{a-v}\binom{u+v-a-1}{4}$.*
- *The number of tableaux in $\mathcal{U}$ whose $(2,2)$-entry is less than or equal to v and $(2,3)$-entry is greater than v is $\binom{u-v}{a-v}\binom{u-a}{3}\binom{v-1}{1}$.*
- *The number of tableaux in $\mathcal{U}$ whose $(2,3)$-entry is less than or equal to v and $(2,4)$-entry is greater than v is $\binom{u-v}{a-v}\binom{u-a}{2}\binom{v-1}{2}$.*
- *The number of tableaux in $\mathcal{U}$ whose $(2,4)$-entry is less than or equal to v and $(2,5)$-entry is greater than v is $\binom{u-v}{a-v}\binom{u-a}{1}\binom{v-1}{3}$.*
- *The number of tableaux in $\mathcal{U}$ whose $(2,5)$-entry is less than or equal to v is $\binom{u-v}{a-v}\binom{v-1}{4}$.*

Proof.

$$\mathcal{U} = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 2 & 3 & \cdots\cdots\cdots\cdots & v & \star & \cdots\cdots & \star \\ \hline 1 & \star & \star & \star & \star \\ \cline{1-5} \star \\ \cline{1-1} \vdots \\ \cline{1-1} \star \\ \cline{1-1} \end{array}$$

where the $\star$s are the numbers from 2 to u. The number of tableaux $|\mathcal{U}|$ is found by choosing the $a-v$ entries represented by $\star$ in the first row from the set $\{v+1, v+2, \dots, u\}$ which can be done in $\binom{u-v}{a-v}$ ways and the choosing the four entries in the second row from the remaining entries in the set, which can be done in $\binom{u+v-a-1}{4}$ ways. Hence, we have $\binom{u-v}{a-v}\binom{u+v-a-1}{4}$ number of choice of tableaux. Similarly, we can show the other cases. $\square$

As corollary of Lemma 4.2.10

Corollary 4.2.11. *Let $\sigma = \sum_{T \in \mathcal{U}} \hat{\Theta}_T$ and $\theta = \alpha\hat{\Theta}_A + \beta\hat{\Theta}_B + \gamma\hat{\Theta}_C$ where all congruence are modulo* 2*. Then*

$$\sigma \circ \hat{\Theta}_A \equiv \binom{u-v}{a-v}\binom{u+v-a-1}{4}\hat{\Theta}_D.$$

$$\begin{aligned}\sigma \circ \hat{\Theta}_B &= \left(\binom{u-v}{a-v}\binom{u-a}{3}\binom{v-1}{1} + \binom{u-v}{a-v}\binom{u-a}{2}\binom{v-1}{2}\right)\hat{\Theta}_D \\ &\equiv \binom{u-v}{a-v}\binom{u-a}{2}\binom{v-1}{2}\hat{\Theta}_D.\end{aligned}$$

$$\sigma \circ \hat{\Theta}_C \equiv \binom{u-v}{a-v}\binom{v-1}{4}\hat{\Theta}_D.$$

Before we prove Theorem I we want to show when in the cases in Proposition 4.2.6 we have $\delta \circ \gamma \neq 0$ for homomorphisms $S^\mu \xrightarrow{\gamma} S^\lambda \xrightarrow{\delta} S^{\mu'}$. Assume S^μ is irreducible, where $\mu = (u, v)$ with $u+v = a+b+5$. Since S^μ is irreducible, then by Corollary 4.1.3, $u - v \equiv = -1 \mod 8$.

1. Suppose $a - v \equiv 1 \mod 8$. If $v \equiv 1 \mod 8$, then $a \equiv 0 \mod 8$ and $u \equiv 0 \mod 8$. Let $\gamma = \hat{\Theta}_B$.

$$\delta \circ \gamma = \binom{u-v}{a-v}\binom{u-a}{2}\binom{v-1}{2}\hat{\Theta}_D.$$

The first term is odd by assumption, the second binomial coefficient is even because $u-a \equiv 0 \mod 8$ and third binomial coefficient is odd. Hence $\delta \circ \gamma = 0$. If $v \equiv 3 \mod 8$, then $a \equiv 4 \mod 8$ and $u \equiv 2 \mod 8$. Let $\gamma = \hat{\Theta}_C$.

$$\delta \circ \gamma = \binom{u-v}{a-v}\binom{v-1}{4}\hat{\Theta}_D.$$

The first term is odd by assumption, the second binomial coefficient is even because $v-1 \equiv 2 \mod 8$. Hence $\delta \circ \gamma = 0$. Finally, $v \equiv 5 \mod 8$, then $a \equiv 6 \mod 8$ and $u \equiv 4 \mod 8$. Let $\gamma = \hat{\Theta}_B$.

$$\delta \circ \gamma = \binom{u-v}{a-v}\binom{u-a}{2}\binom{v-1}{2}\hat{\Theta}_D.$$

The first term is odd by assumption, the second binomial coefficient is odd because $u-a \equiv 6 \mod 8$ and third binomial coefficient is even. Hence $\delta \circ \gamma = 0$.

2. Suppose $a-v \equiv 3 \mod 8$. If $v \equiv 1 \mod 8$, then $a \equiv 4 \mod 8$ and $u \equiv 0 \mod 8$. Let $\gamma = \hat{\Theta}_B + \hat{\Theta}_C$.

$$\delta \circ \gamma = \left(\binom{u-v}{a-v}\binom{u-a}{2}\binom{v-1}{2} + \binom{u-v}{a-v}\binom{v-1}{4}\right)\hat{\Theta}_D.$$

The first term $\binom{u-v}{a-v}$ is odd by assumption, the term $\binom{u-a}{2}$ is even because $u-a \equiv 4 \mod 8$. The binomial coefficient of $\binom{v-1}{4}$ is even because $v-1 \equiv 0 \mod 8$. Hence $\delta \circ \gamma = 0$. If $v \equiv 3 \mod 8$, then $a \equiv 6 \mod 8$ and $u \equiv 2 \mod 8$. Let $\gamma = \hat{\Theta}_C$.

$$\delta \circ \gamma = \delta \circ \gamma = \binom{u-v}{a-v}\binom{v-1}{4}\hat{\Theta}_D.$$

The first term $\binom{u-v}{a-v}$ is odd by assumption. The binomial coefficient is of $\binom{v-1}{4}$ is even because $v-1 \equiv 2 \mod 8$. Hence $\delta \circ \gamma = 0$.

3. Suppose $a-v \equiv 5 \mod 8$. If $v \equiv 1 \mod 8$, then $a \equiv 6 \mod 8$ and $u \equiv 0 \mod 8$. Let $\gamma = \hat{\Theta}_B$.

$$\delta \circ \gamma = \binom{u-v}{a-v}\binom{u-a}{2}\binom{v-1}{2}\hat{\Theta}_D.$$

The first term $\binom{u-v}{a-v}$ is odd by assumption. The term $\binom{u-a}{2}$ is odd because $u-a \equiv 2 \mod 8$. The binomial coefficient of $\binom{v-1}{2}$ is even because $v-1 \equiv 4 \mod 8$. Hence $\delta \circ \gamma = 0$. If $v \equiv 3 \mod 8$. Let $\gamma = \hat{\Theta}_A + \hat{\Theta}_B$. Then $a \equiv 0 \mod 8$ and $u \equiv 2 \mod 8$. Then

$$\delta \circ \gamma = \left(\binom{u-v}{a-v}\binom{u+v-a-1}{4} + \binom{u-v}{a-v}\binom{u-a}{2}\binom{v-1}{2}\right)\hat{\Theta}_D.$$

The first term $\binom{u-v}{a-v}$ is odd by assumption, the term $\binom{u+v-a-1}{4}$ is odd because $u+v-a-1 \equiv 4 \mod 8$, the term $\binom{u-a}{2}$ is odd because $u-a \equiv 2 \mod 8$. Hence $\delta \circ \gamma = 0$. Now Let $\gamma = \hat{\Theta}_C$. Then

$$\delta \circ \gamma = \binom{u-v}{a-v}\binom{v-1}{4}\hat{\Theta}_D.$$

The term $\binom{v-1}{4}$ is even because $v-1 \equiv 2 \mod 8$. Hence $\delta \circ \gamma = 0$. If $v \equiv 5 \mod 8$, then $a \equiv 2 \mod 8$ and $u \equiv 4 \mod 8$. Let $\gamma = \hat{\Theta}_B$.

$$\delta \circ \gamma = \binom{u-v}{a-v}\binom{u-a}{2}\binom{v-1}{2}\hat{\Theta}_D.$$

The first term $\binom{u-v}{a-v}$ is odd by assumption. The term $\binom{u-a}{2}$ is odd because $u-a \equiv 2 \mod 8$. The binomial coefficient of $\binom{v-1}{2}$ is even because $v-1 \equiv 4 \mod 8$. Hence $\delta \circ \gamma = 0$.

4. If $a-v \equiv 7 \mod 8$. If $v \equiv 1 \mod 8$. Then $a \equiv 0 \mod 8$ and $u \equiv 0 \mod 8$. Let $\gamma = \hat{\Theta}_A$. From Corollary 4.2.11

$$\delta \circ \gamma = \binom{u-v}{a-v}\binom{u+v-a-1}{4}\hat{\Theta}_D.$$

The first term is odd by assumption, the second binomial coefficient is even because $u+v-a-1 \equiv 0 \mod 8$. Hence $\delta \circ \gamma = 0$. Now let $\gamma = \hat{\Theta}_B + \hat{\Theta}_C$. From Corollary 4.2.11

$$\delta \circ \gamma = \binom{u-v}{a-v}\left(\binom{u-a}{2}\binom{v-1}{2} + \binom{v-1}{4}\right)\hat{\Theta}_D.$$

The first term $\binom{u-v}{a-v}$ is odd by assumption, but the binomial coefficients $\binom{v-1}{2}$ and $\binom{v-1}{4}$ are even because $v-1 \equiv 0 \mod 8$. Hence $\delta \circ \gamma = 0$. If $v \equiv 3 \mod 8$, then $a \equiv 2 \mod 8$ and $u \equiv 2 \mod 8$. Let $\gamma = \hat{\Theta}_A$. From

Corollary 4.2.11

$$\delta \circ \gamma = \binom{u-v}{a-v}\binom{u+v-a-1}{4}\hat{\Theta}_D.$$

The first term is odd by assumption, the second binomial coefficient is even because $u+v-a-1 \equiv 2 \mod 8$. Hence $\delta \circ \gamma = 0$. Now let $\gamma = \hat{\Theta}_C$. From Corollary 4.2.11

$$\delta \circ \gamma = \binom{u-v}{a-v}\binom{v-1}{4}\hat{\Theta}_D.$$

The first term is odd by assumption, the second binomial coefficient is even because $v-1 \equiv 2 \mod 8$. Hence $\delta \circ \gamma = 0$.

Now we prove Theorem I. Recall that

Theorem I. *Let F be a field of characteristic 2 and let $q = 1$. Suppose $\lambda = (a, 5, 1^b)$ is a partition of n, where a, b are positive even integers and let μ be a partition of n such that S^μ is irreducible. Suppose μ or μ' equals (u, v), where $u > v$ and u is even and v is odd with $v \le \min\{a-1, b+1\}$ and $\binom{u-v}{a-v}$ is odd. If one of the following condition holds:*

- *If $v \equiv 7 \mod 8$,*
- *If $v \equiv 5 \mod 8$ and $a - v \equiv 3 \mod 4$,*

then S^λ has a direct summand isomorphic to S^μ.

Proof. Let S^μ be irreducible, where $\mu = (u, v)$ with $u+v = a+b+5$. Since S^μ is irreducible, then by Corollary 4.1.3, $u - v \equiv= -1 \mod 8$. We would like to show that in the cases above there are homomorphisms $S^\mu \xrightarrow{\gamma} S^\lambda \xrightarrow{\delta} S^{\mu'}$ such that $\delta \circ \gamma \ne 0$. We take $\delta = \sigma$.

1. Suppose $a - v \equiv 1 \mod 8$. If $v \equiv 7 \mod 8$, then $a \equiv 0 \mod 8$ and $u \equiv 6 \mod 8$. Let $\gamma = \hat{\Theta}_A + \hat{\Theta}_B + \hat{\Theta}_C$. Then

$$\delta\circ\gamma = \left(\binom{u-v}{a-v}\binom{u+v-a-1}{4} + \binom{u-v}{a-v}\binom{u-a}{2}\binom{v-1}{2} + \binom{u-v}{a-v}\binom{v-1}{4}\right)\hat{\Theta}_D.$$

The first term $\binom{u-v}{a-v}$ is odd by assumption, the term $\binom{u+v-a-1}{4}$ is odd because $u+v-a-1 \equiv 4 \mod 8$, the term $\binom{u-a}{2}$ is odd because $u-a \equiv 6 \mod 8$. The binomial coefficients of $\binom{v-1}{2}$ and $\binom{v-1}{4}$ are odd because $v - 1 \equiv 6 \mod 8$. Hence $\delta \circ \gamma = \hat{\Theta}_D$.

2. Suppose $a - v \equiv 3 \mod 8$. If $v \equiv 5 \mod 8$, then $a \equiv 0 \mod 8$ and $u \equiv 4 \mod 8$. Let $\gamma = \hat{\Theta}_A + \hat{\Theta}_B + \hat{\Theta}_C$. Then

$$\delta \circ \gamma = \left(\binom{u-v}{a-v}\binom{u+v-a-1}{4} + \binom{u-v}{a-v}\binom{u-a}{2}\binom{v-1}{2} + \binom{u-v}{a-v}\binom{v-1}{4} \right) \hat{\Theta}_D.$$

The first term $\binom{u-v}{a-v}$ is odd by assumption, the term $\binom{u+v-a-1}{4}$ is even because $u+v-a-1 \equiv 0 \mod 8$, the term $\binom{u-a}{2}$ is even because $u-a \equiv 4 \mod 8$. The binomial coefficient is of $\binom{v-1}{4}$ is odd because $v-1 \equiv 4 \mod 8$. Hence $\delta \circ \gamma = \hat{\Theta}_D$. If $v \equiv 7 \mod 8$, then $a \equiv 2 \mod 8$ and $u \equiv 6 \mod 8$. Let $\gamma = \hat{\Theta}_A + \hat{\Theta}_C$.

$$\delta \circ \gamma = \left(\binom{u-v}{a-v}\binom{u+v-a-1}{4} + \binom{u-v}{a-v}\binom{v-1}{4} \right) \hat{\Theta}_D.$$

The term $\binom{u-v}{a-v}$ is odd by assumption, the term $\binom{u+v-a-1}{4}$ is even because $u+v-a-1 \equiv 2 \mod 8$, the binomial coefficient of $\binom{v-1}{4}$ is odd because $v-1 \equiv 6 \mod 8$. Hence $\delta \circ \gamma = \binom{u-v}{a-v}\binom{u-1}{4}\hat{\Theta}_D = \hat{\Theta}_D$.

3. Suppose $a - v \equiv 5 \mod 8$. If $v \equiv 7 \mod 8$. Let $\gamma = \hat{\Theta}_A + \hat{\Theta}_B$. Then $a \equiv 4 \mod 8$ and $u \equiv 6 \mod 8$. Then

$$\delta \circ \gamma = \left(\binom{u-v}{a-v}\binom{u+v-a-1}{4} + \binom{u-v}{a-v}\binom{u-a}{2}\binom{v-1}{2} \right) \hat{\Theta}_D.$$

The first term $\binom{u-v}{a-v}$ is odd by assumption, the term $\binom{u+v-a-1}{4}$ is even because $u+v-a-1 \equiv 0 \mod 8$, the term $\binom{u-a}{2}$ is odd because $u-a \equiv 2 \mod 8$. Hence $\delta \circ \gamma = \hat{\Theta}_D$.

4. If $a - v \equiv 7 \mod 8$. If $v \equiv 5 \mod 8$, then $a \equiv 4 \mod 8$ and $u \equiv 4 \mod 8$. Let $\gamma = \hat{\Theta}_A$. From Corollary 4.2.11

$$\delta \circ \gamma = \binom{u-v}{a-v}\binom{u+v-a-1}{4}\hat{\Theta}_D.$$

The first term is odd by assumption, the second binomial coefficient is odd because $u+v-a-1 \equiv 4 \mod 8$. Hence $\delta \circ \gamma = \hat{\Theta}_D$. If $v \equiv 7 \mod 8$, then $a \equiv 6 \mod 8$ and $u \equiv 6 \mod 8$. Let $\gamma = \hat{\Theta}_A$. From Corollary 4.2.11

$$\delta \circ \gamma = \binom{u-v}{a-v}\binom{u+v-a-1}{4}\hat{\Theta}_D.$$

The first term is odd by assumption, the second binomial coefficient is even because $u+v-a-1 \equiv 6 \mod 8$. Hence $\delta \circ \gamma = \hat{\Theta}_D$.

Hence we have shown that if $v \equiv 7 \mod 8$ or $v \equiv 5 \mod 8$ and $a-v \equiv 3 \mod 4$, then $\delta \circ \gamma = \hat{\Theta}_D$. This completes the proof of Theorem I.

$\square$

4.3 The Specht modules labelled by $(a, 7, 1^b)$

In this section, we find Specht modules $S^{(a,7,1^b)}$ which have a direct summand isomorphic to an irreducible Specht module $S^{(u,v)}$. The argument is similar to that given in Section 4.2. So, we can assume that $\lambda = (a, 7, 1^b)$ and $\mu = (u, v)$, where a, b, u, v are positive integers with a, b, u even and v odd and let $u > v$, $n = a + b + 7 = u + v$ and $7 \leq v \leq \min\{a-1, b+1\}$.

4.3.1 Homomorphisms from S^λ to $S^{\mu'}$

Consider homomorphisms $\sigma : S^\lambda \longrightarrow S^{\mu'}$. We start constructing such a homomorphism. Suppose $\mathcal{U}$ is the set of λ-tableaux having the form:

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
1 & 2 & 3 & \multicolumn{4}{c|}{\cdots\cdots\cdots\cdots} & v & \star & \cdots\cdots & \star \\
\hline
1 & \star & \star & \star & \star & \star & \star & \multicolumn{4}{c}{} \\
\cline{1-7}
\star & \multicolumn{10}{c}{} \\
\cline{1-1}
\vdots & \multicolumn{10}{c}{} \\
\cline{1-1}
\star & \multicolumn{10}{c}{} \\
\cline{1-1}
\end{array}$$

such that the $\star$s are the numbers from 2 to u, and the entries are strictly increasing along each row and weakly increasing down each column. Now define

$$\sigma = \sum_{T \in \mathcal{U}} \hat{\Theta}_T.$$

Proposition 4.3.1. *We have $\psi_{d,t} \circ \sigma = 0$.*

Proof. Similar argument as in the proof of Proposition 4.2.2. $\square$

Now we check that $\sigma \neq 0$. So we state the following proposition.

Proposition 4.3.2. *We have $\sigma \neq 0$.*

Proof. We need Theorem 3.1.10 to prove this proposition. Consider the semistandard tableau S such that:

$$S = \begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
1 & 1 & 2 & \cdots\cdots & v & b{+}9 & b{+}10 & \cdots\cdots & u \\
\hline
2 & b{+}3 & b{+}4 & b{+}5 & b{+}6 & b{+}7 & b{+}8 \\
\cline{1-7}
3 \\
\cline{1-1}
4 \\
\cline{1-1}
\vdots \\
\cline{1-1}
b{+}2 \\
\cline{1-1}
\end{array}.$$

Take $T \in \mathcal{U}$ and consider expressing $\hat{\Theta}_T$ as a linear combination of semistandard homomorphisms. By Theorem 3.1.10, T contributes to S only if $S \trianglerighteq T$. Therefore, we can ignore all $T \in \mathcal{U}$ for which $S \ntrianglerighteq T$. In particular, we need only consider those tableaux in $\mathcal{U}$ which have $b+9, \ldots, u$ in the first row and $b+3, b+4, \ldots, b+8$ in the top two rows. Now we assume that $v \le b+1$, then the tableaux $T \in \mathcal{U}$ that we need to consider are those of the following forms:

1. Suppose $v < i \le b+2$. Then

$$T[i] = \begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
1 & 2 & \cdots\cdots & v & i & b{+}9 & b{+}10 & \cdots & u \\
\hline
1 & b{+}3 & b{+}4 & b{+}5 & b{+}6 & b{+}7 & b{+}8 \\
\cline{1-7}
2 \\
\cline{1-1}
\vdots \\
\cline{1-1}
v \\
\cline{1-1}
v{+}1 \\
\cline{1-1}
\vdots \\
\cline{1-1}
\hat{i} \\
\cline{1-1}
\vdots \\
\cline{1-1}
b{+}2 \\
\cline{1-1}
\end{array}.$$

2. Suppose $2 \leq i \leq b+2$ and $3 \leq k \leq 8$. Then

$$U[i,k] = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
1 & 2 & 3 & \multicolumn{4}{c|}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} & v & b{+}k & b{+}9 & b{+}10 & \cdots\cdots & u \\
\hline
1 & i & b{+}3 & \cdots\cdots & \widehat{b{+}k} & \cdots\cdots & b{+}8 \\
\cline{1-7}
2 \\
\cline{1-1}
\vdots \\
\cline{1-1}
\hat{i} \\
\cline{1-1}
\vdots \\
\cline{1-1}
b{+}2 \\
\cline{1-1}
\end{array} .$$

Now we do a similar argument as in the proof of Proposition 4.2.3 by applying Theorems 3.2.9 and 3.2.12. We get that $\hat{\Theta}_{T[i]} = \hat{\Theta}_S$ plus a linear combination of homomorphisms indexed by tableaux not dominated by S and that $\hat{\Theta}_{U[i,k]} = \hat{\Theta}_S$ plus a linear combination of homomorphisms indexed by tableaux which are either not dominated by S or semistandard and different from S. Now by combining the two cases together, we find that the coefficient of $\hat{\Theta}_S$ in σ is the total number of tableaux of the form $T[i]$ or $U[i,k]$, which is $(b+2-v)+6(b+1)$ which is not zero. Hence, $\sigma \neq 0$.

□

4.3.2 Homomorphisms from S^μ to S^λ

In this section we consider homomorphisms $\gamma : S^\mu$ to S^λ. Define A, B, C, D to be μ-tableaux of type λ as follows:

$$A = \begin{array}{|c|c|c|c|c|c|}
\hline
1^{a-v} & 2^7 & 3 & 4 & \dots & b+2 \\
\hline
1^v \\
\cline{1-1}
\end{array} .$$

$$B = \begin{array}{|c|c|c|c|c|c|}
\hline
1^{a-v+2} & 2^5 & 3 & 4 & \dots & b+2 \\
\hline
1^{v-2} & 2^2 \\
\cline{1-2}
\end{array}$$

$$C = \begin{array}{|c|c|c|c|c|c|}
\hline
1^{a-v+4} & 2^3 & 3 & 4 & \dots & b+2 \\
\hline
1^{v-4} & 2^4 \\
\cline{1-2}
\end{array}$$

$D =$	1^{a-v+6}	2	3	4	$\dots$	$b+2$
	1^{v-6}	2^6				

Lemma 4.3.3. *$\hat{\Theta}_A$, $\hat{\Theta}_B$, $\hat{\Theta}_C$ and $\hat{\Theta}_D$ are non-zero, and are linearly independent if $v \leq b+1$.*

Proof. By using Theorem 3.2.12, we express $\hat{\Theta}_A$, $\hat{\Theta}_B$, $\hat{\Theta}_C$ and $\hat{\Theta}_D$ as linear combinations of semistandard homomorphisms such that there is at least one semistandard tableau appearing in each case. Thus, the homomorphisms are non-zero. Moreover, if $v \leq b+1$, then in the expression for $\hat{\Theta}_A$ there is at least one semistandard tableau with six 2s in the first row which there is no such tableau appearing in the expression for $\hat{\Theta}_B$, $\hat{\Theta}_C$ and $\hat{\Theta}_D$. Also, in the expression for $\hat{\Theta}_B$ there is at least one semistandard tableau with four 2 in the first row which there is no such tableau appearing in the expression for $\hat{\Theta}_C$ and $\hat{\Theta}_D$. In the expression for $\hat{\Theta}_C$ there is at least one semistandard tableau with two 2 in the first row which there is no such tableau appearing in the expression for $\hat{\Theta}_D$. Hence, $\hat{\Theta}_A$, $\hat{\Theta}_B$, $\hat{\Theta}_C$ and $\hat{\Theta}_D$ are linearly independent. □

Proposition 4.3.4.

1. *If $a-v \equiv 7 \mod 8$, then $\psi_{d,t} \circ \hat{\Theta}_A = 0$ for all d,t.*
2. *If $v \equiv 5 \mod 8$, then $\psi_{d,t} \circ \hat{\Theta}_D = 0$ for all d,t.*
3. *If $v \equiv 3 \mod 8$ and either $a-v \equiv 3 \mod 8$ or $a-v \equiv 7 \mod 8$. Then $\psi_{d,t} \circ \hat{\Theta}_C = 0$ for all d,t.*
4. *If $a-v \equiv 5 \mod 8$ and either $v \equiv 1 \mod 8$ or $v \equiv 5 \mod 8$. Then $\psi_{d,t} \circ \hat{\Theta}_B = 0$ for all d,t.*
5. *If $a-v \equiv 1 \mod 8$ and $v \equiv 1 \mod 8$, then $\psi_{d,t} \circ (\hat{\Theta}_B + \hat{\Theta}_D) = 0$ for all d,t.*
6. *If $a-v \equiv 1 \mod 8$ and $v \equiv 3 \mod 8$, then $\psi_{d,t} \circ (\hat{\Theta}_C + \hat{\Theta}_D) = 0$ for all d,t.*
7. *If $a-v \equiv 1 \mod 8$ and $v \equiv 7 \mod 8$, then $\psi_{d,t} \circ (\hat{\Theta}_A + \hat{\Theta}_B + \hat{\Theta}_C + \hat{\Theta}_D) = 0$ for all d,t.*

8. *If* $a - v \equiv 3 \mod 8$ *and* $v \equiv 1 \mod 8$. *Then* $\psi_{d,t} \circ (\hat{\Theta}_B + \hat{\Theta}_C) = 0$ *for all* d, t

9. *If* $a - v \equiv 3 \mod 8$ *and* $v \equiv 7 \mod 8$. *Then* $\psi_{d,t} \circ (\hat{\Theta}_A + \hat{\Theta}_C) = 0$ *for all* d, t.

10. *If* $a - v \equiv 5 \mod 8$ *and* $v \equiv 3 \mod 8$. *Then* $\psi_{d,t} \circ (\hat{\Theta}_A + \hat{\Theta}_B + \hat{\Theta}_C + \hat{\Theta}_D) = 0$ *for all* d, t.

11. *If* $a - v \equiv 5 \mod 8$ *and* $v \equiv 7 \mod 8$. *Then* $\psi_{d,t} \circ (\hat{\Theta}_A + \hat{\Theta}_B) = 0$ *for all* d, t.

12. *If* $a - v \equiv 7 \mod 8$ *and* $v \equiv 1 \mod 8$. *Then* $\psi_{d,t} \circ (\hat{\Theta}_A + \hat{\Theta}_C + \hat{\Theta}_D) = 0$ *for all* d, t.

Proof. The proof follows by case-by-case analysis as in the proof of Proposition 4.2.6. ☐

Remark 4.3.5. From Proposition 4.3.3, we can write the homomorphisms in the following table.

	$v \equiv 1$	$v \equiv 3$	$v \equiv 5$	$v \equiv 7$
$a - v \equiv 1$	$\hat{\Theta}_B + \hat{\Theta}_D$	$\hat{\Theta}_C + \hat{\Theta}_D$	$\hat{\Theta}_D$	$\hat{\Theta}_A + \hat{\Theta}_B$ $+\hat{\Theta}_C + \hat{\Theta}_D$
$a - v \equiv 3$	$\hat{\Theta}_B + \hat{\Theta}_C$	$\hat{\Theta}_C$	$\hat{\Theta}_D$	$\hat{\Theta}_A + \hat{\Theta}_C$
$a - v \equiv 5$	$\hat{\Theta}_B$	$\hat{\Theta}_A + \hat{\Theta}_B$ $+\hat{\Theta}_C + \hat{\Theta}_D$	$\hat{\Theta}_B,\ \hat{\Theta}_D$	$\hat{\Theta}_A + \hat{\Theta}_B$
$a - v \equiv 7$	$\hat{\Theta}_A,\ \hat{\Theta}_A + \hat{\Theta}_C + \hat{\Theta}_D$	$\hat{\Theta}_A,\ \hat{\Theta}_C$	$\hat{\Theta}_A,\ \hat{\Theta}_D$	$\hat{\Theta}_A$

4.3.3 Composing the homomorphisms

Let Δ be the μ-tableau of type μ'

$$\Delta = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & \cdots\cdots\cdots\cdots\cdots\cdots & u \\ \hline 1 & 2 & 3 & \cdots\cdots \quad v & \\ \cline{1-4}\end{array}.$$

Then we have the following theorem.

Theorem 4.3.6. *Suppose* $T \in \mathcal{U}$, *and let* x_i *be the entry in position* $(2, i+1)$ *of* T, *where* $1 \le i \le 6$. *Then*

$$\hat{\Theta}_T \circ \hat{\Theta}_A = \hat{\Theta}_\Delta, \qquad \hat{\Theta}_T \circ \hat{\Theta}_B = \begin{cases} \hat{\Theta}_\Delta & \text{if } x_1 \le v < x_3 \text{ and } x_5 \le v; \\ 0 & \text{if } v < x_1 \text{ and } x_3 \le v < x_5. \end{cases}$$

$$\hat\Theta_T \circ \hat\Theta_C = \begin{cases} \hat\Theta_\Delta & \textit{if } x_3 \le v; \\ 0 & \textit{if } v < x_3. \end{cases} \qquad \hat\Theta_T \circ \hat\Theta_D = \begin{cases} \hat\Theta_\Delta & \textit{if } x_5 \le v; \\ 0 & \textit{if } x_5 > v. \end{cases}$$

Furthermore, $\hat\Theta_\Delta \neq 0$.

Proof. From Theorem 3.2.12 we get $\hat\Theta_\Delta \neq 0$. Now we apply Theorem 3.1.12 with $T \in \mathcal{U}$ and S equal to either A or B or C or D. Suppose $X \in \mathcal{X}$. Since each T^i is a proper set, each X^{ij} must be as well. This means that if some integer i appears in two sets X^{kj}, X^{lj}, then the multinomial coefficient $\binom{X^{1j}_j + X^{2j}_j + X^{3j}_j + \dots}{X^{1j}_j, X^{2j}_j, X^{3j}_j, \dots}$ from Theorem 3.1.12 will include a factor $\binom{2}{1}$, which gives 0.

So in order to get a non-zero coefficient in Theorem 3.1.12, we must have $X^{1j}, X^{2j}, X^{3j}, \dots$ pairwise disjoint for each j, which means that we will have

$$X^{11} \sqcup X^{21} \sqcup \dots = \{1, \dots, u\}, \qquad X^{12} \sqcup X^{22} = \{1, \dots, v\}; \tag{‡}$$

so U_X will equal Δ.

If $S = A$, the only way to achieve this is to have

$$X^{11} = T^1 \setminus \{1, \dots, v\}, \quad X^{12} = \{1, \dots, v\}, \quad X^{i1} = T^i \text{ for } i \ge 2.$$

Thus we have $\hat\Theta_T \circ \hat\Theta_A = \hat\Theta_\Delta$.

In the case $S = B$, let x_2 be the $(2,3)$-entry of T and x_3 be the $(2,4)$-entry of T and x_4 be the $(2,5)$-entry, x_5 be $(2,6)$-entry and x_6 be $(2,7)$-entry of T. If $x_1 > v$ then we cannot possibly achieve (‡). So we get $\hat\Theta_T \circ \hat\Theta_B = 0$ in this case. If $x_1 \le v < x_2$, then the only way to achieve (‡) is to have $X^{22} = \{1, x_1\}$ and $X^{12} = \{2, \dots, \hat{x_1}, \dots, v\}$, and this yields $\hat\Theta_T \circ \hat\Theta_B = \hat\Theta_\Delta$. If $x_2 \le v < x_3$, then there are three possible ways to achieve (‡) is to have $X^{22} = \{1, x_1\}$ or $X^{22} = \{1, x_2\}$ or $X^{22} = \{x_1, x_2\}$ and $X^{12} = \{2, \dots, \hat{x_1}, \dots, v\}$ or $X^{12} = \{2, \dots, \hat{x_2}, \dots, v\}$ or $X^{12} = \{1, \dots, \hat{x_1}, \hat{x_2}, \dots, v\}$; each of these gives a coefficient of 1, and again we have $\hat\Theta_T \circ \hat\Theta_B = \hat\Theta_\Delta$. If $x_3 \le v < x_4$, then X^{22} must contain either x_1 or x_2 or x_3 then there are six possible ways to achieve (‡); each of these gives a coefficient of 1, hence we have $\hat\Theta_T \circ \hat\Theta_B = 0$. If $x_4 \le v < x_5$, then X^{22} must contain either x_1 or x_2 or x_3 or x_4 then there are ten possible ways to achieve (‡); each of these gives a coefficient of 1, hence we have $\hat\Theta_T \circ \hat\Theta_B = 0$. If $x_5 \le v < x_6$, then there are fifteen possible ways to achieve (‡); each of these gives a coefficient of 1, and again we have

$\hat{\Theta}_T \circ \hat{\Theta}_B = \hat{\Theta}_\Delta$. If $x_6 \le v$, then there are twenty-one possible ways to achieve (‡); each of these gives a coefficient of 1, and again we have $\hat{\Theta}_T \circ \hat{\Theta}_B = \hat{\Theta}_\Delta$.

In the case $S = C$ we have

$$|X^{11}| = a - v + 4, \quad |X^{12}| = v - 4, \quad |X^{22}| = 4, \quad |X^{21}| = 3, \quad X^{i1} = T^i \text{ for } i \ge 3.$$

If $x_1, x_2, x_3 > v$ then we cannot possibly achieve (‡). So we get $\hat{\Theta}_T \circ \hat{\Theta}_C = 0$ in this case. If $x_3 \le v<x_4$, then the only way to achieve (‡) is to have $X^{22} = \{1, x_1, x_2, x_3\}$ and $X^{21} = \{x_4, x_5, x_6\}$, and this yields $\hat{\Theta}_T \circ \hat{\Theta}_C = \hat{\Theta}_\Delta$. If $x_4 \le v<x_5$, then there are five possible ways to achieve (‡); each of these gives a coefficient of 1, and again we have $\hat{\Theta}_T \circ \hat{\Theta}_C = \hat{\Theta}_\Delta$. If $x_5 \le v<x_6$, then there are 15 possible ways to achieve (‡); each of these gives a coefficient of 1, and again we have $\hat{\Theta}_T \circ \hat{\Theta}_C = 0$. If $x_6 \le v$, then there are 35 possible ways to achieve (‡); each of these gives a coefficient of 1, and again we have $\hat{\Theta}_T \circ \hat{\Theta}_C = \hat{\Theta}_\Delta$.

In the case $S = D$ we have

$$|X^{11}| = a - v + 6, \quad |X^{12}| = v - 6, \quad |X^{22}| = 6, \quad |X^{21}| = 1, \quad X^{i1} = T^i \text{ for } i \ge 3.$$

X^{11} must contain either x_5 or x_6, so if $x_5 > v$ then we cannot possibly achieve (‡). So we get $\hat{\Theta}_T \circ \hat{\Theta}_D = 0$ in this case. If $x_5 \le v < x_6$, then the only way to achieve (‡) is to have $X^{22} = \{1, \dots, x_5\}$ and $X^{21} = \{x_6\}$, and this yields $\hat{\Theta}_T \circ \hat{\Theta}_D = \hat{\Theta}_\Delta$. Finally, if $x_6 \le v$, then there are seven possible ways to achieve (‡); each of these gives a coefficient of 1, and again we have $\hat{\Theta}_T \circ \hat{\Theta}_D = \hat{\Theta}_\Delta$. □

The proof of this lemma is given by a counting argument similar to that given in the proof of Lemma 4.2.10.

Lemma 4.3.7. *Let x_i be $(2, i+1)$–entry of T, where $1 \le i \le 6$. Then*

- *The number of tableaux in $\mathcal{U}$ is $\binom{u-v}{a-v}\binom{u+v-a-1}{6}$.*
- *The number of tableaux in $\mathcal{U}$ with $x_1 \le v<x_2$ is $\binom{u-v}{a-v}\binom{u-a}{5}\binom{v-1}{1}$.*
- *The number of tableaux in $\mathcal{U}$ with $x_2 \le v<x_3$ is $\binom{u-v}{a-v}\binom{u-a}{4}\binom{v-1}{2}$.*
- *The number of tableaux in $\mathcal{U}$ with $x_3 \le v<x_4$ is $\binom{u-v}{a-v}\binom{u-a}{3}\binom{v-1}{3}$.*
- *The number of tableaux in $\mathcal{U}$ with $x_4 \le v<x_5$ is $\binom{u-v}{a-v}\binom{u-a}{2}\binom{v-1}{4}$.*

- *The number of tableaux in $\mathcal{U}$ with $x_5 \le v < x_6$ is* $\binom{u-v}{a-v}\binom{u-a}{1}\binom{v-1}{5}$.
- *The number of tableaux in $\mathcal{U}$ with $x_6 \le v$ is* $\binom{u-v}{a-v}\binom{v-1}{6}$.

As corollary of Lemma 4.3.7

Corollary 4.3.8. *Suppose all congruences are modulo* 2. *Then*

$$\sigma \circ \hat{\Theta}_A \equiv \binom{u-v}{a-v}\binom{u+v-a-1}{6}\hat{\Theta}_\Delta.$$

$$\begin{aligned}\sigma \circ \hat{\Theta}_B &= \binom{u-v}{a-v}\left(\binom{u-a}{5}\binom{v-1}{1} + \binom{u-a}{4}\binom{v-1}{2} + \binom{u-a}{1}\binom{v-1}{5} + \binom{v-1}{6}\right)\hat{\Theta}_\Delta \\ &\equiv \binom{u-v}{a-v}\left(\binom{u-a}{4}\binom{v-1}{2} + \binom{v-1}{6}\right)\hat{\Theta}_\Delta.\end{aligned}$$

$$\begin{aligned}\sigma \circ \hat{\Theta}_C &= \binom{u-v}{a-v}\left(\binom{u-a}{3}\binom{v-1}{3} + \binom{u-a}{2}\binom{v-1}{4} + \binom{v-1}{6}\right)\hat{\Theta}_\Delta \\ &\equiv \binom{u-v}{a-v}\left(\binom{u-a}{2}\binom{v-1}{4} + \binom{v-1}{6}\right)\hat{\Theta}_\Delta.\end{aligned}$$

$$\begin{aligned}\sigma \circ \hat{\Theta}_D &= \binom{u-v}{a-v}\left(\binom{u-a}{1}\binom{v-1}{5} + \binom{v-1}{6}\right)\hat{\Theta}_\Delta \\ &\equiv \binom{u-v}{a-v}\binom{v-1}{6}\hat{\Theta}_\Delta.\end{aligned}$$

Now we prove Theorem II. Recall that

Theorem II. *Let F be a field of characteristic 2 and let $q = 1$. Suppose $\lambda = (a, 7, 1^b)$ is a partition of n, where a, b are positive even integers and let μ be a partition of n such that S^μ is irreducible. Suppose μ or μ' equals (u, v), where $u > v$ and u is even and v is odd with $v \le \min\{a-1, b+1\}$ and $\binom{u-v}{a-v}$ is odd and $v \equiv 7 \mod 8$. Then S^λ has a direct summand isomorphic to S^μ.*

Proof. Suppose S^μ is irreducible, where $\mu = (u, v)$ with $u + v = a + b + 5$. Suppose that u, v satisfy the given condition $\binom{u-v}{a-v}$ is odd. We need to show that there are homomorphisms $S^{\mu} \xrightarrow{\gamma} S^{\lambda} \xrightarrow{\delta} S^{\mu'}$ such that $\delta \circ \gamma \neq 0$. Let $\delta = \sigma$.

1. Suppose $a - v \equiv 1 \mod 8$. If $v \equiv 7 \mod 8$. Let $\gamma = (\hat{\Theta}_A + \hat{\Theta}_B + \hat{\Theta}_C + \hat{\Theta}_D)$. From Corollary 4.3.8

$$\delta \circ \gamma = \binom{u-v}{a-v}\left(\binom{u+v-a-1}{6} + \binom{u-a}{4}\binom{v-1}{2} + \binom{u-a}{2}\binom{v-1}{4} + \binom{v-1}{6}\right)\hat{\Theta}_{\Delta}.$$

   The first term $\binom{u-v}{a-v}$ is odd by assumption. The terms $\binom{v-1}{4}$, $\binom{v-1}{2}$ and $\binom{v-1}{6}$ are odd because $v - 1 \equiv 6 \mod 8$. Also, the term $\binom{u+v-a-1}{6}$ is even because $u + v - a - 1 \equiv 4 \mod 8$, the terms $\binom{u-a}{4}$ and $\binom{u-a}{2}$ are odd because $u - a \equiv 6 \mod 8$. Hence $\delta \circ \gamma = \hat{\Theta}_{\Delta}$.

2. Suppose $a - v \equiv 3 \mod 8$. If $v \equiv 7 \mod 8$. Let $\gamma = \hat{\Theta}_A + \hat{\Theta}_C$. From Corollary 4.3.8

$$\delta \circ \gamma = \binom{u-v}{a-v}\left(\binom{u+v-a-1}{6} + \binom{u-a}{2}\binom{v-1}{4} + \binom{v-1}{6}\right)\hat{\Theta}_{\Delta}.$$

   The first term $\binom{u-v}{a-v}$ is odd by assumption, the term $\binom{u+v-a-1}{6}$ is even because $u + v - a - 1 \equiv 4 \mod 8$. The term $\binom{u-a}{2}$ is even because $u - a \equiv 4 \mod 8$ and the term $\binom{v-1}{6}$ is odd because $v - 1 \equiv 6 \mod 8$. Hence $\delta \circ \gamma = \hat{\Theta}_{\Delta}$.

3. Suppose $a - v \equiv 5 \mod 8$. If $v \equiv 7 \mod 8$. Let $\gamma = \hat{\Theta}_A + \hat{\Theta}_B$. From Corollary 4.3.8

$$\delta \circ \gamma = \binom{u-v}{a-v}\left(\binom{u+v-a-1}{6} + \binom{u-a}{4}\binom{v-1}{2} + \binom{v-1}{6}\right)\hat{\Theta}_{\Delta}.$$

   The first term $\binom{u-v}{a-v}$ is odd by assumption, the term $\binom{u+v-a-1}{6}$ is even because $u + v - a - 1 \equiv 0 \mod 8$. The term $\binom{u-a}{4}$ is even because $u - a \equiv 2 \mod 8$ and the term $\binom{v-1}{6}$ is odd because $v - 1 \equiv 6 \mod 8$. Hence $\delta \circ \gamma = \hat{\Theta}_{\Delta}$.

4. Suppose $a - v \equiv 7 \mod 8$. If $v \equiv 7 \mod 8$. Let $\gamma = \hat{\Theta}_A$. Then $a \equiv 6 \mod 8$ and $u \equiv 6 \mod 8$. From Corollary 4.3.8

$$\delta \circ \gamma = \binom{u-v}{a-v}\binom{u+v-a-1}{6}\hat{\Theta}_\Delta.$$

The first term is odd by assumption, the second binomial coefficient is even because $u + v - a - 1 \equiv 6 \mod 8$. Thus $\delta \circ \gamma = \hat{\Theta}_\Delta$. Hence we have shown that if $v \equiv 7 \mod 8$, then $\delta \circ \gamma = \hat{\Theta}_D$. This completes the proof of Theorem II.

$\square$

4.4 The Specht modules labelled by $(a, c, 1^b)$

In this section, we assume that $\lambda = (a, c, 1^b)$ and $\mu = (u, v)$, where a, b, u, v are positive integers with a, b, u even, $a > c$, $u > v$, $n = a + b + c = u + v$ and $c \leq v \leq \min\{a - 1, b + 1\}$.

4.4.1 Homomorphisms from S^λ to $S^{\mu'}$

Consider homomorphisms $\sigma : S^\lambda \longrightarrow S^{\mu'}$. We begin by constructing such a homomorphism in the case where $c \leq v \leq a - 1$. Suppose $\mathcal{U}$ is the set of λ–tableaux having the form:

$$\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
1 & 2 & 3 & \cdots\cdots & v & \star & \cdots & \star \\
\hline
1 & \star & \cdots & \star \\
\cline{1-4}
\star \\
\cline{1-1}
\vdots \\
\cline{1-1}
\star \\
\cline{1-1}
\end{array}$$

where the $\star$s are the numbers from 2 to u, and in which

- the entries along each row are strictly increasing,
- the entries down each column are weakly increasing.

Now define

$$\sigma = \sum_{T \in \mathcal{U}} \hat{\Theta}_T.$$

Proposition 4.4.1. *We have $\psi_{d,t} \circ \sigma = 0$ for each d, t.*

Proof. Similar argument as in the proof of Proposition 4.2.2 ☐

Proposition 4.4.2. *We have $\sigma \neq 0$.*

Proof. The proof is similar to that given in 4.2.3. Consider the semistandard tableau S such that:

$$S = \begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
1 & 1 & 2 & \cdots\cdots\cdots & v & b+f_2 & b+f_3 & \cdots\cdots & u \\
\hline
2 & b+3 & b+4 & \cdots & b+f_1 \\
\cline{1-5}
3 \\
\cline{1-1}
4 \\
\cline{1-1}
\vdots \\
\cline{1-1}
b+2 \\
\cline{1-1}
\end{array}$$

where $f_j = c + j$ for all j.

Take $T \in \mathcal{U}$ and consider expressing $\hat{\Theta}_T$ as a linear combination of semistandard homomorphisms. By Theorem 3.1.10, T contributes to S if $S \trianglerighteq T$. Therefore, we can ignore all $T \in \mathcal{U}$ for which $S \ntrianglerighteq T$. In particular, we need only consider those tableaux in $\mathcal{U}$ which have $b + (c + 2), \ldots, u$ in the first row and $b + 3, b + 4, \ldots, b + (c + 1)$ in the second row. The tableaux $T \in \mathcal{U}$ that we need to consider are those of the following forms:

1. Suppose $v < i \leq b + 2$. Then

$$T[i] = \begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
1 & 2 & \cdots\cdots\cdots & v & i & b+f_2 & b+f_3 & \cdots & u \\
\hline
1 & b+3 & b+4 & \cdots & b+f_1 \\
\cline{1-5}
2 \\
\cline{1-1}
\vdots \\
\cline{1-1}
v \\
\cline{1-1}
v+1 \\
\cline{1-1}
\vdots \\
\cline{1-1}
\hat{i} \\
\cline{1-1}
\vdots \\
\cline{1-1}
b+2 \\
\cline{1-1}
\end{array} .$$

2. Suppose $2 \le i \le b+2$ and $3 \le k \le c+1$. Then

$$U[i,k] = \begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
1 & 2 & 3 & \cdots\cdots\cdots & v & b{+}k & b{+}f_2 & \cdots & u \\
\hline
1 & i & b{+}3 & \cdots & \widehat{b{+}k} & \cdots & b{+}f_1 \\
\cline{1-7}
2 \\
\cline{1-1}
\vdots \\
\cline{1-1}
\hat{i} \\
\cline{1-1}
\vdots \\
\cline{1-1}
b{+}2 \\
\cline{1-1}
\end{array}.$$

Now we do a similar argument as in proof of Proposition 4.2.3 by applying Theorems 3.2.9 and 3.2.12. We get that $\hat{\Theta}_{T[i]} = \hat{\Theta}_S$ plus a linear combination of homomorphisms indexed by tableaux not dominated by S and that $\hat{\Theta}_{U[i,k]} = \hat{\Theta}_S$ plus a linear combination of homomorphisms indexed by tableaux which are either not dominated by S or semistandard and different from S. Now by combining the two cases together, we find that the coefficient of $\hat{\Theta}_S$ in σ is the total number of tableaux of the form $T[i]$ or $U[i,k]$, which is $(b+2-v)+(c-1)(b+1)$ which is odd. Hence, $\sigma \neq 0$.

□

4.4.2 Homomorphisms from S^μ to S^λ

In this section we consider homomorphisms $\gamma : S^\mu \longrightarrow S^\lambda$. Assume that $c \le v \le a-1$. Define A_i to be the μ–tableau of type λ as follows

$$A_i = \begin{array}{|c|c|c|c|c|c|}
\hline
1^{a-v+2i} & 2^{c-2i} & 3 & 4 & \ldots & b+2 \\
\hline
1^{v-2i} & 2^{2i} \\
\cline{1-2}
\end{array}$$

where $0 \le i \le \frac{c-1}{2}$.

Lemma 4.4.3. *For $0 \le i \le \frac{c-1}{2}$, $\hat{\Theta}_{A_i}$ are non-zero, and are linearly independent if $v \le b+1$.*

Proof. By using Theorem 3.2.12, we express $\hat{\Theta}_{A_i}$ for $0 \le i \le \frac{c-1}{2}$ as linear combinations of semistandard homomorphisms such that there is at least one semistandard tableau appearing in each case. Thus, the homomorphisms are non-zero. Moreover, if $v \le b+1$, in the expression for $\hat{\Theta}_{A_i}$ we have a semistandard tableau with $(c-2i-1)$ 2s in the first row and this tableau does not occur in the expression for $\hat{\Theta}_{A_j}$, where $i \le j$. Hence $\hat{\Theta}_{A_i}$ are are linearly independent. $\square$

Proposition 4.4.4. *Let m be minimal such that $c \le 2^m$. If $a - v \equiv -1 \mod 2^m$, then $\psi_{d,t} \circ \hat{\Theta}_{A_0} = 0$ for all d, t.*

Proof. If $d \ge 2$, then from Theorem 3.2.9 we get $\psi_{d,1} \circ \hat{\Theta}_{A_0} = 0$. Now, if $d = 1$ then $\psi_{1,t_1} \circ \hat{\Theta}_{A_0} = 0$ for $t_1 \equiv 1 \mod 2$, because each A_i have an odd number of 1s in each row and $\binom{2k}{2j+1} \equiv 0 \mod 2$ for all $k, j \ge 0$. Finally, suppose $d = 1$ and $t_2 \equiv 0 \mod 2$. Then we have $a - v \equiv -1 \mod 2^m$. Then

$$\psi_{1,t_2} \circ \hat{\Theta}_{A_0} = \binom{a-v+t_2}{t_2} \begin{array}{|c|c|c|c|c|c|} \hline 1^{a-v+t_2} & 2^{c-t_2} & 3 & 4 & \dots & b+2 \\ \hline 1^v \\ \cline{1-1} \end{array}$$

we apply Lemma 4.1.4, since $t_2 > a - v + t_2$ then the binomial coefficient $\binom{a-v+t_2}{t_2}$ is divisible by 2 which is zero modulo 2. Thus, $\psi_{1,t} \circ \hat{\Theta}_{A_0} = 0$ $\square$

4.4.3 Composing the homomorphisms

Now we analysis when S^μ is a summand of S^λ. Let Δ be the μ–tableau of type μ'

$$\Delta = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & \cdots\cdots\cdots\cdots\cdots\cdots & u \\ \hline 1 & 2 & 3 & \cdots\cdots \; v \\ \cline{1-4} \end{array}.$$

Then we have the following theorem.

Theorem 4.4.5. *Suppose $T \in \mathcal{U}$. Then*

$$\hat{\Theta}_T \circ \hat{\Theta}_{A_0} = \hat{\Theta}_\Delta$$

Furthermore, $\hat{\Theta}_\Delta \ne 0$.

Proof. From Theorem 3.2.12 we get $\hat{\Theta}_\Delta \ne 0$. Now apply Theorem 3.1.12 with $T \in \mathcal{U}$ and S equal to A_0. Suppose $X \in \mathcal{X}$. Since each T^i is a proper set, each X^{ij} must be as well. This means that if some integer i appears in two sets X^{kj}, X^{lj}, then the multinomial coefficient $\binom{X_j^{1j}+X_j^{2j}+X_j^{3j}+\dots}{X_j^{1j},X_j^{2j},X_j^{3j},\dots}$ from Theorem 3.1.12 will include a factor $\binom{2}{1}$, which gives 0.

So in order to get a non-zero coefficient in Theorem 3.1.12, we must have $X^{1j}, X^{2j}, X^{3j}, \dots$ pairwise disjoint for each j, which means that we will have

$$X^{11} \sqcup X^{21} \sqcup \dots = \{1, \dots, u\}, \qquad X^{12} \sqcup X^{22} = \{1, \dots, v\};$$

so U_X will equal Δ.

If $S = A_0$, the only way to achieve this is to have

$$X^{11} = T^1 \smallsetminus \{1, \dots, v\}, \quad X^{12} = \{1, \dots, v\}, \quad X^{i1} = T^i \text{ for } i \geq 2.$$

Thus we have $\hat{\Theta}_T \circ \hat{\Theta}_{A_0} = \hat{\Theta}_\Delta$. □

Lemma 4.4.6. *Let x_i be $(2, i)$–entry of T, where $2 \leq i \leq c$. Then the number of tableaux in $\mathcal{U}$ is $\binom{u-v}{a-v}\binom{u+v-a-1}{c-1}$.*

As corollary of Lemma 4.4.6

Corollary 4.4.7. *Take all congruences modulo 2. Then*

$$\sigma \circ \hat{\Theta}_{A_0} \equiv \binom{u-v}{a-v}\binom{u+v-a-1}{c-1}\hat{\Theta}_\Delta.$$

Now we prove Theorem III. Recall that

Theorem III. *Let F be a field of characteristic 2 and let $q = 1$. Suppose $\lambda = (a, c, 1^b)$ is a partition of n, where a, b are positive even integers and c is odd. Let μ be a partition of n such that S^μ is irreducible. Let m be minimal such that $c \leq 2^m$. If μ or μ' equals (u, v), where $u > v$ and u is even and v is odd with $v \leq \min\{a-1, b+1\}$ and $\binom{u-v}{a-v}$ is odd and $v \equiv -1 \mod 2^m$ and $a - v \equiv -1 \mod 2^m$. Then S^λ has a direct summand isomorphic to S^μ.*

Proof. Suppose $S^\mu = S^{(u,v)}$ is irreducible, with $u + v = a + b + c$. Suppose that u, v satisfy the given condition that $\binom{u-v}{a-v}$ is odd. We would like to show that there are homomorphisms $S^\mu \xrightarrow{\gamma} S^\lambda \xrightarrow{\delta} S^{\mu'}$ such that $\delta \circ \gamma \neq 0$. Let $\delta = \sigma$.

Suppose $a - v \equiv -1 \mod 2^m$. Let $\gamma = \hat{\Theta}_{A_0}$. γ is a homomorphism from S^μ to S^λ. Then from Corollary 4.4.7

$$\delta \circ \gamma = \binom{u-v}{a-v}\binom{u+v-a-1}{c-1}\hat{\Theta}_\Delta.$$

The first term is odd by assumption, the second binomial coefficient $\binom{u+v-a-1}{c-1} \not\equiv 0 \mod 2$ because that $S^{(u,v)}$ is irreducible, so from Corollary 4.1.3 $u - v \equiv -1$

mod 2^m, then $u \equiv v-1 \mod 2^m$. Since $a-v \equiv -1 \mod 2^m$, then $v-a-1 \equiv 0 \mod 2^m$. Thus, binomial coefficient $\binom{u+v-a-1}{c-1} \equiv \binom{v-1}{c-1}$. It is enough to show that $\binom{v-1}{c-1} \not\equiv 0 \mod 2$. Now by applying Lemma 4.1.4, we have $v-1 = 2v_1 + 4v_2 + \cdots + 2^m v_m + \cdots + 2^k v_k$ and $c-1 = 2c_1 + 4c_2 + \cdots + 2^m c_m$. If $v \equiv -1 \mod 2^m$ then $v = 1 + 2 + \cdots + 2^{m-1} + \ldots$, so $v_i = 1$ for $0 \leq i$. Hence for modulo 2 $\binom{v-1}{c-1} \equiv \binom{v_1}{c_1}\binom{v_2}{c_2}\ldots\binom{v_{m-1}}{c_{m-1}}\binom{v_m}{0}$, where $v_i, c_i \in \{0,1\}$. So if $v_i \geq c_i$ for all i, then $\binom{v-1}{c-1} \not\equiv 0 \mod 2$.

$\square$

This concludes the proof of our main results. We also have the following conjectures. Let F be a field of characteristic 2. Suppose $\lambda = (a, c, 1^b)$ and $\mu = (u, v)$, where a, b, c, u, v are positive integers with a, b, u even and c, v odd such that $a > c$, $u > v$, $n = a + b + c = u + v$ and $c \leq v \leq \min\{a-1, b+1\}$. Suppose $\mathcal{U}$ is the set of λ–tableaux having the form:

$$\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
1 & 2 & 3 & \cdots\cdots & v & \star & \cdots & \star \\
\hline
1 & \star & \cdots & \star \\
\cline{1-4}
\star \\
\cline{1-1}
\vdots \\
\cline{1-1}
\star \\
\cline{1-1}
\end{array}$$

such that the $\star$s are the numbers from 2 to u, and the entries are strictly increasing along each row and weakly increasing down each column. Define

$$\sigma = \sum_{T \in \mathcal{U}} \hat{\Theta}_T$$

and define A_i to be the μ–tableau of type λ as follows

$$A_i = \begin{array}{|c|c|c|c|c|c|}
\hline
1^{a-v+2i} & 2^{c-2i} & 3 & 4 & \ldots & b+2 \\
\hline
1^{v-2i} & 2^{2i} \\
\cline{1-2}
\end{array}$$

where $0 \leq i \leq \frac{c-1}{2}$. Set

$$\gamma = \sum_{i=0}^{\frac{c-1}{2}} \hat{\Theta}_{A_i}.$$

Then we have the following conjecture.

Conjecture 4.4.8. *Let m be minimal such that $c \leq 2^m$. Then if $a - v \equiv 1$*

mod 2^m *and* $v \equiv -1 \mod 2^m$, *then* $\psi_{d,t} \circ \gamma = 0$ *for all* d, t.

Conjecture 4.4.9. *If* $a - v \equiv 1 \mod 2^m$ *and* $v \equiv -1 \mod 2^m$ *then we have homomorphisms* $\gamma : S^\mu \longrightarrow S^\lambda$ *and* $\sigma : S^\lambda \longrightarrow S^{\mu'}$ *and* $\sigma \circ \gamma \neq 0$.

If Conjecture 4.4.9 is true, the next conjecture follows immediately.

Conjecture IV. *Let* F *be a field of characteristic* 2 *and let* $q = 1$. *Suppose* $\lambda = (a, c, 1^b)$ *is a partition of* n, *where* a, b *are positive even integers and* c *is odd. Let* μ *be a partition of* n *such that* S^μ *is irreducible. Let* m *be minimal such that* $c \leq 2^m$. *Suppose* μ *or* μ' *equals* (u, v), *where* $u > v$ *and* u *even and* v *odd with* $v \leq \min\{a - 1, b + 1\}$ *and* $\binom{u-v}{a-v}$ *is odd and* $v \equiv -1 \mod 2^m$ *and* $a - v \equiv 1 \mod 2^m$. *Then* S^λ *has a direct summand isomorphic to* S^μ.

We believe that similar techniques can be used to prove the following conjecture which we have shown holds for $c = 5, 7$, although we do not know how to express the homomorphism γ in terms of the homomorphisms $\hat{\Theta}_{A_i}$.

Conjecture V. *Let* F *be a field of characteristic* 2 *and let* $q = 1$. *Suppose* $\lambda = (a, c, 1^b)$ *is a partition of* n, *where* a, b *are positive even integers and* c *is odd. Let* μ *be a partition of* n *such that* S^μ *is irreducible. Let* m *be minimal such that* $c \leq 2^m$. *Suppose* μ *or* μ' *equals* (u, v), *where* $u > v$ *and* u *even and* v *odd with* $v \leq \min\{a - 1, b + 1\}$ *and* $\binom{u-v}{a-v}$ *is odd and* $v \equiv -1 \mod 2^m$. *Then* S^λ *has a direct summand isomorphic to* S^μ.

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