Incidence homology for the hyperoctahedral group

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By

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Abstract

The incidence structure of the cross-polytope gives rise to certain modular representations for the hyperoctahedral group. In this thesis we introduce and begin the study of these natural representations. In particular we show that they satisfy a branching rule. This branching rule is used to extract information about the representations and underlying combinatorial objects. Amongst the information extracted is a formula for the dimensions of the representations. This has applications in calculating the p-rank of incidence matrices arising from the cross-polytope. We also construct explicit generators for the representations and identify cases where the representations are irreducible.

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Chapter 1

Introduction

The n-cross-polytope is the n-dimensional analogue of the octahedron and the dual of the n-cube. Attached to the cross-polytope are many ranked posets. In this thesis we are interested in the combinatorics of these ranked posets and associated modular representations. The posets arise by picking certain types of subsets of the vertex set of the cross-polytope. In the case where these subsets are the faces of the cross-polytope we recover the usual simplicial complex of (the boundary of) the cross-polytope. This is an important special case. If $n = 2$ it is a generalized quadrangle of order $(1, 1)$. More generally it underlies the structure of a polar space of rank n.

The incidence structure of these posets naturally gives rise to incidence matrices which encode the covering relation of the poset in 0's and 1's. The elements of a poset can be viewed as the basis of a vector space or more generally a free module over a ring. The rank function extends to a grading of this module – given an integer k the k-th graded component is the submodule generated by the elements of rank k in the poset. With this viewpoint the incidence matrices can be regarded as linear maps between graded components or as homogeneous maps on the whole graded module. These maps are known as *incidence maps*. This is a common idea. However, this construction can also be viewed from a homological point of view and this is one of the main objectives of this thesis. We will describe the construction explicitly shortly. For now we merely note that the homology modules arising from this construction are known as *incidence homology modules*. We are interested mainly in the case where the coefficient domain has positive characteristic. This homology dates back to the 1940s paper [22] of Mayer, but remains relatively underdeveloped. Incidence homology has the advantage that it can be attached to any finite ranked poset and it is defined purely by the combinatorics of the poset.

The symmetry group of the n-cross-polytope is the well-known *hyperoctahedral group* B_n. All of our constructions are compatible with the group action of B_n on the cross-polytope. This has the side-effect that chain modules of chain complexes become modules for B_n, incidence maps commute with B_n and the homology modules become modules for B_n. This leads to questions invisible from the purely combinatorial viewpoint: How do these modules look on restriction to a subgroup? Is there a branching rule for the restriction to B_{n-1}? Under what conditions are the modules irreducible? More generally, what are the composition factors? Additionally, the homological approach lends itself to the calculation of dimension formulae in the form of alternating sums. This can be applied to calculate the modular rank or p-rank of the incidence matrices.

The ranked posets we study can be roughly described as follows. Let $\{e_1, \dots, e_n\}$ be the standard basis in $\mathbb{R}^n$. A concrete realisation of the n-cross-polytope is given by taking the convex hull of the vertex set

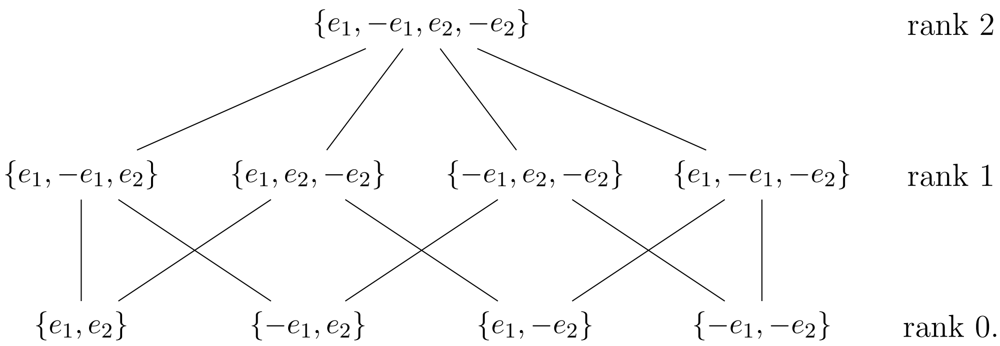
$$\Delta = \{\pm e_1, \dots, \pm e_n\}.$$

We partition the subsets of Δ into types as follows. Two subsets are considered of the same type if and only if they

(a) contain an equal number of subsets of the form $\{e_j, -e_j\}$ and
(b) have an equal number of e_j appearing, disregarding whether they appear with a "+" or a "−".

Thus the subsets of Δ fall into a 2-dimensional array of types. The posets we study arise by fixing one coordinate of this array and taking the union of all types obtained by varying the other coordinate. In each case, a rank function is naturally provided by this other coordinate.

Let us describe these ranked posets a little more explicitly so we can set up notation for the corresponding homological sequences and homology modules. By the previous paragraph we have a 2-dimensional array of types. Let x be a subset of Δ. Consider the number in (a) for x, that is the number of subsets of x of the form $\{e_j, -e_j\}$. We call subsets of this form *doubles* of x. For d an integer we set $L_{*,d}$ to be the set of all subsets of Δ which contain precisely d doubles. We think of $L_{*,d}$ as a row in the 2-dimensional array of types. Notice that $L_{*,d}$ is partially ordered by subset inclusion. To define a rank function on $L_{*,d}$ we use the other number, the number defined in (b). That is for $x \in L_{*,d}$ the rank of x is given by the number of e_j appearing in x, disregarding whether they appear with a "+" or a "−". We call this number the *unsigned size* of x. We define a similar notation for the columns: For $u \in \mathbb{Z}$ set $L_{u,*}$ to be the set of subsets of Δ of unsigned size u. Then $L_{u,*}$ is also partially ordered by subset inclusion. A rank function is provided by the number of doubles. Explicitly, for $x \in L_{u,*}$ define the rank of x to be the number of doubles of x. Note that the rank functions are defined differently in $L_{*,d}$ and $L_{u,*}$. The following picture shows the column $L_{2,*}$ in the case $n = 2$:



Next we show how incidence homology can be defined for any finite ranked poset. We outline the general construction as follows. Afterwards we will apply this to the posets defined above. Let $\mathcal{P}$ be a finite ranked poset and let F be a field. Denote by M the F-vector space with basis $\mathcal{P}$. This space is naturally graded by the rank function of $\mathcal{P}$. For k an integer let $\mathcal{P}_k$ denote the set of elements of rank k in $\mathcal{P}$. Then the k-th graded part of M is $M_k = F\mathcal{P}_k$ with $M_k = 0$ whenever $\mathcal{P}_k = \emptyset$. The crucial point where incidence homology diverges from the classical is in the definition of the differential. The partial order on $\mathcal{P}$ gives rise to a linear *incidence map* on M. This incidence map is defined as the linear map ∂ that takes each element of $\mathcal{P}$ to the sum of the elements that it covers. Consider the sequence

$$\ldots \xleftarrow{\partial} M_1 \xleftarrow{\partial} M_2 \xleftarrow{\partial} M_3 \xleftarrow{\partial} M_4 \xleftarrow{\partial} M_5 \xleftarrow{\partial} \ldots . \qquad (1.0.1)$$

Since $\mathcal{P}$ is finite this sequence is terminated by an infinite string of zeros at each end. In particular, ∂ is nilpotent. Let m be a positive integer such that $\partial^m = 0$. This means that any composition of m arrows in (1.0.1) is zero. We call a sequence with this property an *m-complex*. For instance a 2-complex is an ordinary chain complex or homological sequence. Fixing k and an integer i with $0 < i < m$ gives rise to the sequence

$$\mathcal{M}^{\mathcal{P}}_{k,i}: \quad \ldots \xleftarrow{\partial^i} M_{k-m} \xleftarrow{\partial^{m-i}} M_{k-i} \xleftarrow{\partial^i} M_k \xleftarrow{\partial^{m-i}} M_{k+m-i} \xleftarrow{\partial^i} M_{k+m} \xleftarrow{\partial^{m-i}} \ldots.$$

Since $\partial^m = 0$ this sequence is homological, that is the composition of any two arrows is zero. We thus can form homology modules in the usual way. We denote the homology of the sequence at position k by

$$H_{k,i} = H^{\mathcal{P}}_{k,i} = (\ker \partial^i \cap M_k)/\partial^{m-i}(M_{k+m-i}).$$

This module is known as an *incidence homology module.*

We now apply this general construction to our posets. Let F be a field of characteristic $p > 0$. Since each row $L_{*,d}$ and each column $L_{u,*}$ is a ranked poset, we can apply the construction above to get homological sequences and incidence homology modules. The resulting incidence map for the row $L_{*,d}$ is the linear map which takes each subset $x \in L_{*,d}$ to the sum of its subsets that can be obtained by removing precisely one element which does not lie in a double of x. The resulting incidence map for the column $L_{u,*}$ is the linear map which takes each subset $x \in L_{u,*}$ to the sum of its subsets that can be obtained by removing precisely one element which does lie in a double of x. It can be shown, either by Theorem 3.3.5 or by counting chains, that the incidence maps of $L_{*,d}$ and $L_{u,*}$ are both nilpotent of degree p. Let $\mathcal{P} = L_{*,d}$ and $\mathcal{Q} = L_{u,*}$. For each integer i with $0 < i < p$ we set the notation

$$\dot{\mathcal{M}}^n_{u,d,i} = \mathcal{M}^{\mathcal{P}}_{u,i}$$

with homology

$$\dot{H}^n_{u,d,i} = H^{\mathcal{P}}_{u,i}$$

and

$$\ddot{\mathcal{M}}^n_{u,d,i} = \mathcal{M}^{\mathcal{Q}}_{d,i}$$

with homology

$$\ddot{H}^n_{u,d,i} = H^{\mathcal{Q}}_{d,i}.$$

So far we have not mentioned how the symmetry group B_n is involved. We rectify this now. Observe that the permutation action of B_n on the vertices of the cross-polytope extends to a permutation action on the subsets of Δ. What we have been calling types are actually the orbits of B_n on the set of subsets of Δ. Therefore B_n acts as a group of rank-preserving automorphisms on the posets $L_{*,d}$ and $L_{u,*}$. Because of this the vector spaces involved in $\dot{\mathcal{M}}^n_{u,d,i}$ and $\ddot{\mathcal{M}}^n_{u,d,i}$ become permutation FB_n-modules and the incidence maps become FB_n-maps. Thus the kernels, images and homology modules also become FB_n-modules. In other words these are p-modular representations for B_n. So, as stated above, we are naturally led to representation theoretic questions about these sequences and modules which are not immediately apparent from the purely combinatorial viewpoint. We should emphasize that the representation theory of B_n is well-understood, for example by the paper [11] of Dipper and James. The combinatorics of the cross-polytope are reasonably well-understood. The purpose of this thesis is to isolate parts of the representation theory that are relevant to the combinatorics and to highlight aspects of the combinatorics which are only accessible via the representation theory.

1.1 Overview

In Chapter 2 we set out the preliminaries and background material required to comprehend the rest of the thesis. Most of this material can be found in standard texts with the exception of Section 2.4 which introduces m-complexes more thoroughly. In that section, amongst other things, we define the notion of a "double m-complex" which will be central in later chapters. Although this is a natural idea, which can already be seen in the paper [21] of Kapranov and the dissertation [23] of Mirmohades, it would appear that this thesis is the first place where it is explicitly given a name. The notion is useful for us because of the 2-dimensional grid structure of the orbits described in the previous section. Section 2.8 provides an overview of the known results for the simplex case (also known as the Type A, Boolean algebra, or symmetric group case) with special emphasis on the results we will leverage for our own purposes. There is quite an extensive literature on this case but the results we need are mostly found in the paper [3] of Bell, Jones and Siemons.

From Chapter 3 onwards the work is mostly original. In Chapter 3 we describe in detail the specific setup for the hyperoctahedral case. More specifically, we define the 2-dimensional grid structure above and go into more depth about the incidence maps, resulting sequences and homology modules. We end the chapter with our first main result, Theorem 3.6.1, which relates all of these sequences back down to the boundary of the cross-polytope. This is of great help later as it enables us to reduce many arguments down to this case.

Chapter 4 is devoted to the proof of some branching rules which describe the restriction of the sequences and homology modules to B_{n-1}. The strongest versions of these results are Theorem 4.2.3 and Theorem 4.2.4 which can be stated as follows.

Theorem 4.2.3. *Let $1 \leq n$ and $0 < i < p$. We have an isomorphism of chain complexes of FB_{n-1}-modules*

$$\dot{\mathcal{M}}^n_{u,d,i} \cong \dot{\mathcal{M}}^{n-1}_{u,d,i+1} \oplus \dot{\mathcal{M}}^{n-1}_{u-1,d,i} \oplus \dot{\mathcal{M}}^{n-1}_{u-1,d,i-1} \oplus \dot{\mathcal{M}}^{n-1}_{u-1,d-1,i}.$$

Theorem 4.2.4. *Let $1 \leq n$ and $0 < i < p$. We have an isomorphism of chain complexes of FB_{n-1}-modules*

$$\ddot{\mathcal{M}}^n_{u,d,i} \cong \ddot{\mathcal{M}}^{n-1}_{u,d,i} \oplus \ddot{\mathcal{M}}^{n-1}_{u-1,d,i} \oplus \ddot{\mathcal{M}}^{n-1}_{u-1,d,i+1} \oplus \ddot{\mathcal{M}}^{n-1}_{u-1,d-1,i-1}.$$

These branching rules are powerful tools since they enable many results to be proved by induction on n. We demonstrate this in Chapter 5. Note that the two theorems look very similar. We comment more on this observation later.

Chapter 5 is really an exercise in squeezing as much out of the branching rules as possible. It is well known that the representation theory of B_n in even characteristic is much simpler than in odd characteristic. This fact is echoed in Section 5.1 where we completely determine all homology modules in the even characteristic case. They turn out to be isomorphic to permutation modules. We also prove a duality result in the even characteristic case which we believe is true regardless of the characteristic. This is related to the similarity between Theorem 4.2.3 and Theorem 4.2.4.

The remainder of the thesis constitutes the beginning of a study of the more interesting case where the characteristic is odd. The first step on this path is the derivation of necessary and sufficient combinatorial conditions for the incidence homology modules to be non-zero, namely we prove the following.

Theorem 5.2.1. *Let $p > 2$. Let $0 \leq d \leq u \leq n$ be integers. Let $0 < i < p$. The module $\dot{H}^n_{u,d,i}$ is non-zero if and only if $0 < 2u + p - i - n - d$.*

Theorem 5.2.2. *Let $p > 2$. Let $0 \leq d \leq u \leq n$ be integers. Let $0 < i < p$. The module $\ddot{H}^n_{u,d,i}$ is non-zero if and only if $0 < u - 2d + i$.*

Section 5.3 goes a bit further: We prove a dimension formula for the homology modules, in the form of an alternating sum of ranks of certain chain modules. The result of the previous section is crucial here as it enables us to swiftly identify branches of the branching rule that lead to a zero module. The end result is Theorem 5.3.2, shown below. In the theorem $H^{n-d-m}_{u-d-m,i}$ is the incidence homology module for the symmetric group S_{n-d-m} studied in [3]. The notation $f_{u'}$ in the theorem is defined as follows. Suppose $0 \leq d \leq u \leq n$ and $0 < i < p$ are fixed. Set $\ell = 2u - d + p - i - n$. For $u' \in \mathbb{Z}$ define

$$f_{u'} = \sum_{b=\ell+d-p+1}^{\ell+d-1} \binom{n}{u'}\binom{u'}{d}\binom{u'-d}{b-d}.$$

Theorem 5.3.2. *Let $p < 2$. Let $0 \leq d \leq u \leq n$ and $0 < i < p$. Set $\ell = 2u + p - i - n - d$. Then*

$$\dim \dot{H}^n_{u,d,i} = \binom{n}{d} \sum_{m=0}^{\ell-1} \binom{n-d}{m} \dim H^{n-d-m}_{u-d-m,i} = \sum_{t\in\mathbb{Z}} f_{u-pt} - f_{u-i-pt}.$$

This result depends on a similar result for the modules $H^{n-d-m}_{u-d-m,i}$ which appears in [3]. In the text we show that the $f_{u'}$ are the ranks of chain modules in a generalised version of the defining chain complex for the incidence homology module $H^n_{u,i}$ for the symmetric group. Thus Theorem 5.3.2 gives the dimension as an alternating sum of ranks of chain modules from a chain complex, a common trend in Algebraic Topology. In the simplest case where $\ell = 1$ and $d = 0$ Theorem 5.3.2 reduces to the statement $\dim \dot{H}^n_{u,0,i} = \dim H^n_{u,i}$. This together with Lemma 5.6.3 shows that these two modules are isomorphic as FS_n-modules, where S_n is identified with the subgroup of B_n which preserves $\{e_1, \ldots, e_n\}$.

The *p-rank* of an integer matrix is the rank of the matrix when regarded as a matrix over a field of characteristic p. Theorem 5.3.2 finds an application in the calculation of p-ranks of incidence matrices arising from the boundary of the cross-polytope. Explicitly, recall the row $L_{*,0}$ consists of all subsets of the vertex set of the cross-polytope which contain no doubles. These subsets are just the faces of the usual simplicial complex of the boundary of the cross-polytope. In the following, by a $(k-1)$-simplex we will mean such a subset of size k. For integers $0 \leq s \leq t \leq n$ we define the incidence matrix $W_{s,t}$ to be the $\{0,1\}$-matrix with rows indexed by the $(s-1)$-simplices x and columns indexed by the $(t-1)$-simplices y with xy-entry equal to 1 if and only if x is a face of y. To state our formula for the p-rank of $W_{s,t}$

(valid under certain conditions on s and t) we need two counting functions defined as follows. For any integer u define

$$c_u = 2^u \binom{n}{u}.$$

For integers u and ℓ define

$$f_u^\ell = \sum_{b=\ell-p+1}^{\ell-1} \binom{n}{u}\binom{u}{b}.$$

These functions count certain subsets of $L_{*,0}$ which are related to the action of S_n on $L_{*,0}$. We obtain the following formula for the p-rank of $W_{s,t}$.

Theorem 5.4.6. *Let $p > 2$. Let $0 \leq s \leq t \leq n$ and suppose $0 < t - s < p$. Set $\ell = s + t + p - n$. Then the p-rank of $W_{s,t}$ is*

$$\operatorname{rk}_p W_{s,t} = \sum_{k\in\mathbb{Z}} \sum_{m\in\mathbb{Z}_{>0}} \left(f_{t-kp}^{\ell-mp} - f_{s-kp}^{\ell-mp} \right) + \sum_{k'\in\mathbb{Z}_{\geq 0}} \left(c_{s-k'p} - c_{t-(k'+1)p} \right).$$

Although this formula looks rather unwieldy, it can easily be evaluated by a computer. It would appear this case has gone somewhat untouched in the literature. In Section 5.5 we exhibit generating sets for the homology modules. The results are Theorem 5.5.3 and Theorem 5.5.12. These are used in Section 5.6 to identify irreducible homology modules in terms of the standard modular representation theory of the hyperoctahedral group. Explicitly, we prove the following two theorems.

Theorem 5.6.1. *Let $p > 2$. Let $0 \leq d \leq u \leq n$ and i be integers with $0 < i < p$. Suppose $H = \dot{H}^n_{u,d,i}$ is non-zero. Then H is irreducible if any of the following conditions hold:*

(a) $0 = d = u$, *in which case* $H \cong 1_{B_n} \cong D^{((n),0)}$;
(b) $0 = d < u$ *and* $2u + p - i - n = 1$ *in which case* $H \cong D^{(\lambda,0)}$ *where* λ *is the partition of* n *into two parts* u *and* $n - u$;
(c) $d = u = n$, *in which case* $H \cong 1_{B_n} \cong D^{((n),0)}$;
(d) $d < u = n$ *and* $i = 1$, *in which case* $H \cong D^{((d),(n-d))}$.

Theorem 5.6.2. *Let $p > 2$. Let $0 \leq d \leq u \leq n$ and i be integers with $0 < i < p$. Suppose $H = \ddot{H}^n_{u,d,i}$ is non-zero. Then H is irreducible if any of the following conditions hold:*

(a) $d = u = n$, *in which case* $H \cong 1_{B_n} \cong D^{((n),0)}$;

(b) $d < u = n$ *and* $u - 2d + i = 1$ *in which case* $H \cong D^{(\lambda,0)}$ *where* λ *is the partition of* n *into two parts* d *and* $n - d$*;*

(c) $0 = d = u$*, in which case* $H \cong 1_{B_n} \cong D^{((n),0)}$*;*

(d) $0 = d < u$ *and* $i = p - 1$*, in which case* $H \cong D^{((n-u),(u))}$*.*

By comparing these two results, we see that in each of the cases (a)–(d) we have $\dot{H}^n_{u,d,i} \cong \ddot{H}^n_{n-d,n-u,p-i}$. This is the duality proved in Section 5.1 for the even characteristic case. The conjecture is that this duality holds in odd characteristic also.

In Chapter 6 we state the aforementioned duality conjecture which seems to resonate throughout the thesis. We also outline some steps towards a proof, but so far it is out of reach.

Conjecture 6.0.1. *Let* $0 \leq d \leq u \leq n$ *and* i *be integers with* $0 < i < p$*. Then we have an isomorphism of* FB_n*-modules*

$$\dot{H}^n_{u,d,i} \cong \ddot{H}^n_{n-d,n-u,p-i}.$$

This is related to Poincaré duality for the p-complexes. Finally the appendices contain some explicit descriptions of homology modules for small n. Appendix A shows bases for the kernels and images while Appendix B contains tables of Brauer character decompositions.

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Chapter 2

Preliminaries

The purpose of this chapter is to introduce the basic terminology and some general results that will be used throughout the rest of the text. Throughout the thesis F will be a field with characteristic $p \geq 0$. Often we will assume $p > 0$, but this will be stated. The integers will be denoted by $\mathbb{Z}$. The non-negative integers will be denoted by $\mathbb{N}$ or $\mathbb{Z}_{\geq 0}$. The positive integers will be denoted by $\mathbb{Z}_{>0}$.

2.1 Ranked posets

Let $\mathcal{P}$ be a set. We say that $\mathcal{P}$ is a *partially ordered set*, or *poset* for short, if there is a binary relation $\leq$ on $\mathcal{P}$ such that the following hold for all $x, y, z \in \mathcal{P}$

(a) $x \leq x$ (reflexivity),
(b) $x \leq y$ and $y \leq z$ implies $x \leq z$ (transitivity) and
(c) $x \leq y$ and $y \leq x$ implies $x = y$ (antisymmetry).

The relation $\leq$ is called a *partial order*. We say that $\mathcal{P}$ is *partially ordered by* $\leq$. Sometimes, for example if we already know we are talking about a partial order, we drop the "partially" and just say $\mathcal{P}$ is *ordered by* $\leq$. For elements x and y in $\mathcal{P}$ we write $x < y$ to mean that $x \leq y$ and $x \neq y$. An element $x \in \mathcal{P}$ such that $x \leq y$ for all $y \in \mathcal{P}$ is called a *minimum* element of $\mathcal{P}$. It is easy to see that if a minimum element exists then it is unique so we can say *the minimum element*. A *chain* is a poset in which all elements x and y satisfy either $x \leq y$ or $y \leq x$. Note that later on we will also use the word "chain" to mean something completely different, but the meaning will be clear from the context. An *automorphism* of $\mathcal{P}$ is a bijection $g : \mathcal{P} \to \mathcal{P}$ such that $x \leq y$ implies $g(x) \leq g(y)$ for all $x, y \in \mathcal{P}$.

Example 2.1.1 (A poset)**.** Let X be a set. Then any collection of subsets of X is a poset partially ordered by subset inclusion $\subseteq$. The notation $x \subset y$ means $x \subseteq y$ and $x \neq y$. Most posets in this thesis will be of this form.

Let $x, y \in \mathcal{P}$. We say that y *covers* x if $x \leq y$ and there is no $z \in \mathcal{P}$ such that $x < z < y$.

Definition 2.1.2 (Ranked poset)**.** A *ranked poset* (sometimes called a *graded poset*) is a poset $\mathcal{P}$ with a function $r : \mathcal{P} \to \mathbb{N}$ such that for all $x, y \in \mathcal{P}$ we have

(a) $x \leq y$ implies $r(x) \leq r(y)$,
(b) if y covers x then $r(y) = r(x) + 1$ and
(c) if $\mathcal{P}$ has minimum element $\hat{0}$ then $r(\hat{0}) = 0$.

Note that definitions of ranked poset vary between authors. In particular, condition (c) is often omitted but it is convenient for our purposes. The function r in the definition is called the *rank function* of $\mathcal{P}$. The number $r(x)$ is the *rank* of x (with respect to r). Given a ranked poset $\mathcal{P}$ and $k \in \mathbb{N}$ we write $\mathcal{P}_k$ for the set of elements of rank k.

Example 2.1.3 (A ranked poset)**.** Let X be a finite set. The set of all subsets of X is a ranked poset. As in Example 2.1.1 a partial order is given by subset inclusion. A rank function compatible with this partial order is given by the cardinality function $| - |$. The empty set is the unique element of rank 0.

Note that a poset can easily fail to be ranked, even if it is contained in a poset which is ranked. The following example demonstrates this.

Example 2.1.4 (A poset which is not ranked)**.** Let $X = \{1, 2, 3\}$. Consider the set of subsets $\{\emptyset, \{1\}, \{3\}, \{1, 2\}, \{1, 2, 3\}\}$ ordered by inclusion as usual. The following diagram illustrates the poset structure

$$
\begin{tikzcd}[column sep=small]
 & \{1,2,3\} \arrow[dl, dash] \arrow[ddr, dash] & \\
\{1,2\} \arrow[d, dash] & & \\
\{1\} \arrow[dr, dash] & & \{3\} \arrow[dl, dash] \\
 & \emptyset &
\end{tikzcd}
$$

We claim that this poset is not ranked. It cannot be ranked by cardinality since $\{1,2,3\}$ covers $\{3\}$ but $|\{1,2,3\}| = 3 \neq 2 = |\{3\}| + 1$. Suppose it is ranked by a different rank function r. Then by the left hand part of the diagram we have $r(\{1,2,3\}) = r(\{1,2\}) + 1 = r(\{1\}) + 2 = r(\emptyset) + 3$. Similarly, by the right hand part of the diagram we have $r(\{1,2,3\}) = r(\emptyset) + 2$, a contradiction.

2.2 Graded modules

Throughout this section R denotes a ring. A *graded R-module* is an R-module M together with a decomposition of M as a direct sum of R-modules

$$M = \bigoplus_{k\in\mathbb{N}} M_k.$$

We call the decomposition a *grading* of M. The module M_k is called the *k-th graded component* or *k-th graded part* of M. For convenience we also define $M_k = 0$ for $k \in \mathbb{Z}$ with $k < 0$.

A *bigraded R-module* is an R-module M together with a decomposition of M as a direct sum of R-modules

$$M = \bigoplus_{a,b\in\mathbb{N}} M_{a,b}.$$

We call the decomposition a *bigrading* of M. The module $M_{a,b}$ is called the *(a,b)-th graded component* or *(a,b)-th graded part* of M. For convenience we also define $M_{a,b} = 0$ for $a, b \in \mathbb{Z}$ with $a < 0$ or $b < 0$.

Of course every R-module can be graded. The purpose of the definition is to provide a shorthand for setting up a module with components accessed by subscripts. For example instead of writing "Let $M = \oplus_{k\in\mathbb{Z}} M_k$ be a direct sum of R-modules with $M_k = 0$ for $k < 0$" we can just write "Let M be a graded R-module." Similar comments apply for bigraded modules.

Let M be a graded R-module. Let $\varphi : M \to M$ be an R-homomorphism. If there exists $j \in \mathbb{Z}$ with $\varphi(M_k) \subseteq M_{k+j}$ for all $k \in \mathbb{N}$ then φ is called *homogeneous of degree* j. If there exists $j \in \mathbb{Z}$ such that φ is homogeneous of degree j then we call φ *homogeneous*.

Similarly, let M be a bigraded R-module. Let $\varphi : M \to M$ be an R-homomorphism. If there exist $j_1, j_2 \in \mathbb{Z}$ with $\varphi(M_{k_1,k_2}) \subseteq M_{k_1+j_1,k_2+j_2}$ for all $k_1, k_2 \in \mathbb{N}$ then φ is called *homogeneous of degree* (j_1, j_2). If there exist $j_1, j_2 \in \mathbb{Z}$ such that φ is homogeneous of degree (j_1, j_2) then we call φ *homogeneous*.

A common theme throughout the text will be the construction of graded modules from ranked posets. We outline this construction now. Suppose $\mathcal{P}$ is a ranked poset. Let M be the free R-module with basis $\mathcal{P}$. For $k \in \mathbb{N}$ let M_k be the free R-module with basis $\mathcal{P}_k$. This makes M into a graded R-module. An important case is if R is a field, F say. Then M is just the F-vector space with basis $\mathcal{P}$ and the k-th graded part of M is the subspace spanned by $\mathcal{P}_k$. If G is a group of automorphisms of $\mathcal{P}$ whose action restricts to each $\mathcal{P}_k$ then each M_k becomes an FG-module. Hence M is a graded FG-module. We attach to this construction a homogeneous map of degree -1 defined by taking each element in $\mathcal{P}$ to the sum of the elements it covers. This is an example of an *incidence map*. We will discuss this in more detail in Section 2.5.

2.3 Simplicial complexes

Simplicial complexes are roughly triangulations of a topological space. They have a combinatorial nature which eases computation of topological invariants such as homology groups. Although we do not necessarily use the idea of a simplicial complex directly, we feel it is important background.

Definition 2.3.1. An *abstract simplicial complex* is a set of finite sets closed under taking subsets[1].

A set of size $k + 1$ in an abstract simplicial complex is called a *k-simplex*. The

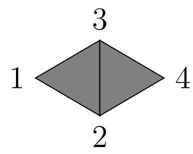
[1]Some authors, for example Munkres [25], require that the sets be non-empty and that the complex should be closed under taking non-empty subsets.

union of all the simplices is called the *vertex set* of the complex. The elements of the vertex set are the *vertices*. We usually identify each vertex with the singleton (0-simplex) containing it.

Example 2.3.2. Let

$$X = \{\{1,2,3\},\{2,3,4\},\{1,2\},\{1,3\},\{2,3\},\{2,4\},\{3,4\},\{1\},\{2\},\{3\},\{4\}\}.$$

Then X is an abstract simplicial complex. Clearly the notation is a bit verbose and not very visual so we want a better way to think of an abstract simplicial complex. We draw X as



.

The 0-simplices are just the vertices $1, 2, 3, 4$. The 1-simplices are the edges. The 2-simplices are the shaded triangles. If we hadn't shaded the triangle 123 this would indicate that $\{1, 2, 3\}$ was not a simplex of X.

The drawing of an abstract simplicial complex as in the example is known as the *geometric realisation* of the abstract simplicial complex. It is an example of a *geometric simplicial complex*. This can be made rigorous [25, Theorem 3.1] but it is not necessary for our purposes. We regard it as simply a visual aid when thinking about an abstract simplicial complex. From now on, by a *simplicial complex* we will always mean an abstract simplicial complex, unless stated otherwise.

Attached to each simplicial complex is a sequence of abelian groups, or in our case vector spaces, called simplicial homology groups or homology modules. The remainder of this section is devoted to defining these. First we need the notion of "orientation". An *orientation* of a simplex is an equivalence class of orderings of its vertices under the equivalence that two orderings are equivalent if and only if they differ by an even permutation. Thus to any simplex there are precisely two orientations of that simplex (except 0-simplices which have only one orientation). If $(\sigma_0, \sigma_1, \ldots, \sigma_k)$ is an ordering of the vertices of a simplex σ then we write $[\sigma_0, \sigma_1, \ldots, \sigma_k]$ for the orientation corresponding to this ordering.

Let X be a simplicial complex. Fix an orientation of each simplex in X. Recall F is a field. We define $C_k(X)$ to be the F-vector space with basis the k-simplices of X with this orientation. The oppositely orientated simplices also lie in $C_k(X)$: we regard them as the negations of their counterparts in the basis. For example $[\sigma_0, \sigma_1] = -[\sigma_1, \sigma_0]$ for any 1-simplex $\{\sigma_0, \sigma_1\}$. The elements of $C_k(X)$ are called the *k-chains* of X. We call $C_k(X)$ a *chain module*. We define the *differential* or *boundary map* $\partial : C_k(X) \to C_{k-1}(X)$ by

$$\partial([\sigma_0, \sigma_1, \dots, \sigma_k]) = \sum_{i=0}^{k} (-1)^i [\sigma_0, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots, \sigma_k]$$

for each oriented k-simplex $[\sigma_0, \sigma_1, \dots, \sigma_k]$. The key fact is the following.

Lemma 2.3.3 ([25, Lemma 5.3])**.** $\partial^2 = 0$.

Thus from X we obtain the sequence

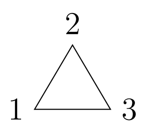
$$\dots \leftarrow C_{k-2}(X) \xleftarrow{\partial} C_{k-1}(X) \xleftarrow{\partial} C_k(X) \xleftarrow{\partial} C_{k+1}(X) \xleftarrow{\partial} C_{k+2}(X) \leftarrow \dots$$

where going down twice always gives you zero. A sequence with this property is called a *chain complex* or a *homological sequence*. Because $\partial^2 = 0$ we have $\partial(C_{k+1}(X)) \subseteq \ker \partial \cap C_k(X)$. We define $B_k(X) = \partial(C_{k+1}(X))$ and $Z_k(X) = \ker \partial \cap C_k(X)$. The elements of $B_k(X)$ are called *k-boundaries* and the elements of $Z_k(X)$ are called *k-cycles*. The reason for these names should become clear after seeing some examples. Since $B_k(X) \subseteq Z_k(X)$ we can take the quotient space. We define

$$H_k(X) = Z_k(X)/B_k(X).$$

This is called the *k-th simplicial homology module* (or group) of X.

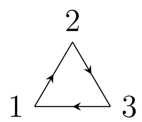
Example 2.3.4. Let X be the simplicial complex defined by the picture



We have

$$
\begin{aligned}
&\vdots \\
C_{-2}(X) &= 0 \\
C_{-1}(X) &= 0 \\
C_0(X) &= \langle [1], [2], [3] \rangle_F \\
C_1(X) &= \langle [1,2], [2,3], [3,1] \rangle_F \\
C_2(X) &= 0 \\
C_3(X) &= 0 \\
&\vdots
\end{aligned}
$$

We compute the homology $H_1(X)$. First, since $C_2(X) = 0$ we immediately have $B_1(X) = 0$. So $H_1(X) \cong Z_1(X)$. To compute $Z_1(X)$ notice $\partial([1,2]) = [2] - [1]$, $\partial([2,3]) = [3] - [2]$ and $\partial([3,1]) = [1] - [3]$. Suppose $x = x_1[1,2] + x_2[2,3] + x_3[3,1]$ is a 1-chain with $\partial(x) = 0$. Then $x_1 = x_2 = x_3$. So $Z_1(X) = \langle [1,2] + [2,3] + [3,1] \rangle_F$. In particular $H_1(X)$ is 1-dimensional. We picture this generator $[1,2] + [2,3] + [3,1]$ of $H_1(X)$ as the cycle in the following picture



We take the view that a simplicial complex is a special kind of ranked poset. The cardinality function provides a rank function. The vast majority of ranked posets however are not simplicial complexes. For a general ranked poset Lemma 2.3.3 can easily fail. So for these posets we do not get homology modules directly by the above method. In the next two sections we describe a different kind of homology known as *incidence homology* which can be defined for any finite ranked poset.

2.4 A different kind of complex

Let m be a natural number. In this section we introduce a generalisation of a chain complex called an m-chain complex. In all of our applications m will be prime but this is not necessary for the definitions. We assume basic knowledge of homological algebra, a good reference is Weibel's book [30].

Let R be a ring.

Definition 2.4.1. Let M be a graded R-module. Suppose M is equipped with a homogeneous map ∂ of degree -1 such that $\partial^m = 0$. Then we call the pair (M, ∂) an *m-complex* (of R-modules).

Often we will just say that M is an m-complex, rather than (M, ∂). The map ∂ is called the *differential* of the m-complex. We can think of an m-complex M as a sequence

$$\cdots \xleftarrow{\partial} M_{k-1} \xleftarrow{\partial} M_k \xleftarrow{\partial} M_{k+1} \xleftarrow{\partial} \cdots$$

where any composition of m consecutive arrows is zero. We have already seen m-complexes for $m = 1$ and $m = 2$: A 1-complex is simply a graded module and a 2-complex is a chain complex in the usual sense.

The idea of an m-complex seems to appear first in a paper [22] of Mayer. For p prime he describes a method for assigning a p-complex to a simplicial complex and a theory of p-homology is developed in a similar vein to "standard" simplicial homology (the case $p = 2$). Recently there has been activity in the category theoretic properties of m-complexes (often called N-complexes in the literature) [21, 23]. There is also motivation from physics and quantum groups [13]. In these papers an m-complex is defined as a $\mathbb{Z}$-graded module with differential as in our definition. Our definition differs slightly as our graded modules M have by definition $M_k = 0$ for $k < 0$. The reason for this is that this thesis is more concerned with the study of explicit combinatorial examples in detail along with attached modular representations.

The m-complexes of R-modules are the objects of a category. We do not prove this. The morphisms in this category are defined as follows. Let (M, ∂_M) and (N, ∂_N) be m-complexes of R-modules. An *m-chain map* or a *map of m-complexes* from M to N is an R-homomorphism $f : M \to N$ which restricts to maps $f_k : M_k \to N_k$ such that $\partial_N f_k = f_{k-1} \partial_M$ for all $k \in \mathbb{Z}$, that is such that the diagram

$$\begin{array}{ccccccccc}
\cdots & \xleftarrow{\partial_M} & M_{k-1} & \xleftarrow{\partial_M} & M_k & \xleftarrow{\partial_M} & M_{k+1} & \xleftarrow{\partial_M} & \cdots \\
 & & \downarrow f_{k-1} & & \downarrow f_k & & \downarrow f_{k+1} & & \\
\cdots & \xleftarrow{\partial_N} & N_{k-1} & \xleftarrow{\partial_N} & N_k & \xleftarrow{\partial_N} & N_{k+1} & \xleftarrow{\partial_N} & \cdots
\end{array}$$

commutes. A bijective map of m-complexes is called an *isomorphism* of m-complexes.

Proposition 2.4.2. *Let (M, ∂_M) and (N, ∂_N) be m-complexes of R-modules. Let $f : M \to N$ be a bijective map of m-complexes. Then the inverse $f^{-1} : N \to M$ is a map of m-complexes $N \to M$.*

Proof. Let $x \in N_k$. Then $f\partial_M f^{-1}(x) = \partial_N f f^{-1}(x) = \partial_N(x)$. Applying f^{-1} gives $\partial_M f^{-1}(x) = f^{-1}\partial_N(x)$. $\square$

Note that isomorphism is extremely strong since it preserves all properties of an m-complex. We will use the following lemma later.

Lemma 2.4.3. *Let I be some index set. Suppose for each $\alpha \in I$ we have a chain complex (that is a 2-complex) $(C^{(\alpha)}, \partial^{(\alpha)})$. Let $(C, \partial) = \oplus_{\alpha \in I}(C^{(\alpha)}, \partial^{(\alpha)})$ be the direct sum chain complex, that is $C = \oplus_{\alpha \in I} C^{(\alpha)}$ and $\partial = \oplus_{\alpha \in I}\partial^{(\alpha)}$. Then we have*

$$\begin{aligned}
\ker \partial \cap C_k &= \bigoplus_{\alpha \in I}(\ker \partial^{(\alpha)} \cap C_k^{(\alpha)}), \\
\partial(C_{k+1}) &= \bigoplus_{\alpha \in I} \partial^{(\alpha)}(C_{k+1}^{(\alpha)}) \text{ and} \\
(\ker \partial \cap C_k)/\partial(C_{k+1}) &\cong \bigoplus_{\alpha \in I}(\ker \partial^{(\alpha)} \cap C_k^{(\alpha)})/\partial^{(\alpha)}(C_{k+1}^{(\alpha)}).
\end{aligned}$$

Proof. Let $x \in C_k$. Then there exist unique $x^{(\alpha)} \in C_k^{(\alpha)}$ such that $x = \sum_{\alpha \in I} x^{(\alpha)}$. Since $\partial(x) = \oplus_{\alpha \in I}\partial^{(\alpha)}(x^{(\alpha)})$ we have $\partial(x) = 0$ if and only if $\partial^{(\alpha)}(x^{(\alpha)}) = 0$ for each $\alpha \in I$. That is if and only if $x \in \oplus_{\alpha \in I}(\ker \partial^{(\alpha)} \cap C_k^{(\alpha)})$. Similarly, $\partial(C_{k+1}) = \oplus_{\alpha \in I}\partial^{(\alpha)}(C_{k+1}^{(\alpha)})$. Since direct sums commute with quotients we are done. $\square$

Weaker notions of equivalence which only preserve certain desired properties can also be useful. We will introduce one of these weaker notions later, namely "quasi-isomorphism".

We say that M is an *m-subcomplex* of N if each M_k is an R-submodule of N_k and the differential ∂_M is the restriction of ∂_N to M. If M is an m-subcomplex of N then N/M is an m-complex with k-th graded part N_k/M_k and differential $\partial_{N/M}$ defined by

$$\partial_{N/M}(x + M) = \partial_N(x) + M$$

for each $x \in N$. Such an m-complex is called a *quotient m-complex*.

Now we will move on to the construction of homology modules from m-complexes. These will be the main objects of study later. Let (M, ∂) be an m-complex. For any pair of integers k, i with $0 < i < m$ we may extract from M the sequence

$$\mathcal{M}_{k,i}: \quad \cdots \xleftarrow{\partial^*} M_{k-i} \xleftarrow{\partial^*} M_k \xleftarrow{\partial^*} M_{k+m-i} \xleftarrow{\partial^*} \cdots$$

where ∂^* alternates between the compositions ∂^i and ∂^{m-i} as appropriate. The identity $\partial^m = 0$ implies that $\mathcal{M}_{k,i}$ is a homological sequence (in other words a 2-complex). So we may form the homology module

$$H_{k,i}(M) := (\ker \partial^i \cap M_k)/\partial^{m-i}(M_{k+m-i}).$$

The dimensions of the homology modules are known as the *Betti numbers* of the sequence.

Remark 2.4.4. The assignments of the complex $\mathcal{M}_{k,i}$ and the homology $H_{k,i}(M)$ to the m-complex M are functorial. Concretely, let (M, ∂_M) and (N, ∂_N) be m-complexes. Let $f : M \to N$ be a map of m-complexes. Then we get a map of chain complexes $\mathcal{M}_{k,i} \to \mathcal{N}_{k,i}$ by restriction to those $M_{k'}$ occurring in $\mathcal{M}_{k,i}$. We also get a map f_* of homology modules $H_{k,i}(M) \to H_{k,i}(N)$ defined by

$$f_*(z + \partial_M^{m-i}(M_{k+m-i})) = f(z) + \partial_N^{m-i}(N_{k+m-i})$$

for each $z \in \ker \partial_M^i \cap M_k$. This map is called *the map induced on homology.* The image of two composed maps is the composition of their two images.

We say that f as in the remark is a *quasi-isomorphism* if for each for each pair of integers k and i with $0 < i < m$ the map $f_* : H_{k,i}(M) \to H_{k,i}(N)$ is an isomorphism of R-modules.

A frequently occurring object in homological algebra is the double complex (or bicomplex). We will need an m-complex analogue. First recall the definition of a double complex [30, Example 1.2.4] as follows. Let M be a bigraded R-module. Let ∂ and δ be R-endomorphisms of M homogeneous of degrees $(-1, 0)$ and $(0, -1)$ such that $(\partial+\delta)^2 = 0$. Then the triple (M, ∂, δ) is called a *double complex*. The double complex M can be visualised as a grid

$$\begin{tikzcd}
 & \vdots \arrow[d, "\delta"] & \vdots \arrow[d, "\delta"] & \vdots \arrow[d, "\delta"] & \\
\cdots & M_{-1,1} \arrow[l, "\partial"'] \arrow[d, "\delta"] & M_{0,1} \arrow[l, "\partial"'] \arrow[d, "\delta"] & M_{1,1} \arrow[l, "\partial"'] \arrow[d, "\delta"] & \cdots \arrow[l, "\partial"'] \\
\cdots & M_{-1,0} \arrow[l, "\partial"'] \arrow[d, "\delta"] & M_{0,0} \arrow[l, "\partial"'] \arrow[d, "\delta"] & M_{1,0} \arrow[l, "\partial"'] \arrow[d, "\delta"] & \cdots \arrow[l, "\partial"'] \\
\cdots & M_{-1,-1} \arrow[l, "\partial"'] \arrow[d, "\delta"] & M_{0,-1} \arrow[l, "\partial"'] \arrow[d, "\delta"] & M_{1,-1} \arrow[l, "\partial"'] \arrow[d, "\delta"] & \cdots \arrow[l, "\partial"'] \\
 & \vdots & \vdots & \vdots & .
\end{tikzcd}$$

The condition $(\partial+\delta)^2 = 0$ can be expanded to $\partial^2 + \partial\delta + \delta\partial + \delta^2 = 0$. Now ∂^2, $\partial\delta+\delta\partial$ and δ^2 are homogeneous of degrees $(-2, 0)$, $(-1, -1)$ and $(0, -2)$ respectively. Therefore they must each be zero for the condition to hold. So (M, ∂, δ) is a double complex if and only if each of its rows and columns are chain complexes and each square in its diagram anticommutes.

We introduce what seems to be the natural generalisation of a double complex for m-complexes as follows.

Definition 2.4.5. Let M be a bigraded module. Suppose M has two R-endomorphisms ∂ and δ homogeneous of degrees $(-1, 0)$ and $(0, -1)$ respectively such that

$$(\partial+\delta)^m = 0. \tag{2.4.6}$$

Then the triple (M, ∂, δ) is called a *double m-complex.*

Setting $m = 2$ gives the usual definition of a double complex as we wanted. Condition (2.4.6) can be interpreted diagrammatically as follows. If we think of M as the (not necessarily commuting) diagram

$$\begin{tikzcd}
 & \vdots \arrow[d, "\delta"] & \vdots \arrow[d, "\delta"] & \vdots \arrow[d, "\delta"] & \\
\cdots & M_{-1,1} \arrow[l, "\partial"'] \arrow[d, "\delta"] & M_{0,1} \arrow[l, "\partial"'] \arrow[d, "\delta"] & M_{1,1} \arrow[l, "\partial"'] \arrow[d, "\delta"] & \cdots \arrow[l, "\partial"'] \\
\cdots & M_{-1,0} \arrow[l, "\partial"'] \arrow[d, "\delta"] & M_{0,0} \arrow[l, "\partial"'] \arrow[d, "\delta"] & M_{1,0} \arrow[l, "\partial"'] \arrow[d, "\delta"] & \cdots \arrow[l, "\partial"'] \\
\cdots & M_{-1,-1} \arrow[l, "\partial"'] \arrow[d, "\delta"] & M_{0,-1} \arrow[l, "\partial"'] \arrow[d, "\delta"] & M_{1,-1} \arrow[l, "\partial"'] \arrow[d, "\delta"] & \cdots \arrow[l, "\partial"'] \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

then (2.4.6) is equivalent to the statement that the sum of all length m compositions of arrows between any two objects in the diagram is zero.

Example 2.4.7. Let $m = 3$. Then

$$\begin{aligned}(\partial + \delta)^3 = \partial^3 + \partial^2\delta + \partial\delta\partial + \delta\partial^2 \\ + \partial\delta^2 + \delta\partial\delta + \delta^2\partial + \delta^3.\end{aligned} \tag{2.4.8}$$

Evaluating (2.4.8) on an element $x \in M_{a,b}$ amounts to calculating $\sum f(x)$ as f runs over the possible length 3 compositions of arrows in the truncated diagram

$$\begin{tikzcd}
M_{a-3,b} & M_{a-2,b} \arrow[l, "\partial"'] \arrow[d, "\delta"] & M_{a-1,b} \arrow[l, "\partial"'] \arrow[d, "\delta"] & M_{a,b} \arrow[l, "\partial"'] \arrow[d, "\delta"] \\
 & M_{a-2,b-1} & M_{a-1,b-1} \arrow[l, "\partial"'] \arrow[d, "\delta"] & M_{a,b-1} \arrow[l, "\partial"'] \arrow[d, "\delta"] \\
 & & M_{a-1,b-2} & M_{a,b-2} \arrow[l, "\partial"'] \arrow[d, "\delta"] \\
 & & & M_{a,b-3}
\end{tikzcd}$$

The condition $(\partial + \delta)^3 = 0$ implies that

$$\partial^3 = \partial^2\delta + \partial\delta\partial + \delta\partial^2 = \partial\delta^2 + \delta\partial\delta + \delta^2\partial = \delta^3 = 0.$$

Let M be a double m-complex with differentials ∂ and δ. Then we define the *total* *m-complex* of M, denoted $\mathrm{Tot}(M)$, by

$$\mathrm{Tot}(M)_k = \bigoplus_{a+b=k} M_{a,b}$$

with differential $d : \mathrm{Tot}(M) \to \mathrm{Tot}(M)$ defined by

$$d = \delta + \partial.$$

The following is then immediate from the definitions.

Proposition 2.4.9. *The total m-complex of a double m-complex is an m-complex.*

2.5 Incidence homology

Let $\mathcal{P}$ be a finite ranked poset. Let F be a field. Let G act as a group of automorphisms of $\mathcal{P}$ which preserves each $\mathcal{P}_k$ for $k \in \mathbb{N}$. In this section we describe the general method of constructing from $\mathcal{P}$ an m-complex of FG-modules. Note that m depends on both $\mathcal{P}$ and F. The resulting homology modules will be FG-modules and they are called "incidence homology modules."

Recall the construction from the end of Section 2.2. That is, let M be the F-vector space with basis $\mathcal{P}$. For $k \in \mathbb{N}$ let M_k be the F-vector space with basis $\mathcal{P}_k$. This makes M into a graded F-module in other words a graded F-vector space. Since G acts on each $\mathcal{P}_k$ we get that each M_k is an FG-module and hence M is a graded FG-module.

We define a linear map $\partial : M \to M$ as follows. For $x \in \mathcal{P}$ we set $\partial(x) = \sum_y y$ where the sum runs over all $y \in \mathcal{P}$ that are covered by x. This map ∂ is called the *incidence map* of $\mathcal{P}$ (over F). Clearly ∂ restricts to a map $M_k \to M_{k-1}$ for each $k \in \mathbb{N}$. Also observe that since G acts on $\mathcal{P}$ we have that $g(y)$ is covered by $g(x)$ if and only if y is covered by x. Therefore ∂ commutes with G. Now since $\mathcal{P}$ is finite, there exists $m \in \mathbb{N}$ such that $\partial^m = 0$. Therefore (M, ∂) is an m-complex of FG-modules.

For integers k and i with $0 < i < m$ we therefore obtain, by Section 2.4, the sequence

$$\mathcal{M}_{k,i} : \quad \dots \xleftarrow{\partial^i} M_{k-m} \xleftarrow{\partial^{m-i}} M_{k-i} \xleftarrow{\partial^i} M_k \xleftarrow{\partial^{m-i}} M_{k+m-i} \xleftarrow{\partial^i} M_{k+m} \xleftarrow{\partial^{m-i}} \dots$$

of FG-modules and FG-maps. The kernel $\ker \partial^i \cap M_k$ and the image $\partial^{m-i}(M_{k+m-i})$

are therefore FG-modules. Hence the homology module

$$H_{k,i}(M) = (\ker \partial^i \cap M_k)/\partial^{m-i}(M_{k+m-i})$$

is also an FG-module. This is called an *incidence homology module* for $\mathcal{P}$.

2.6 The hyperoctahedral group

Many of the objects studied in this thesis are closely tied to the hyperoctahedral group, as the title should suggest. In this section we define this well-known group. We will also illustrate several nice concrete realisations of the group. No attempt will be made to prove these realisations are isomorphic although we will indicate the isomorphisms.

Let $n \in \mathbb{N}$. We will define an n-dimensional analogue of the octahedron. Let $\{e_1, \ldots, e_n\}$ be the standard basis in $\mathbb{R}^n$. Consider the set of points

$$\Delta = \{\pm e_1, \ldots, \pm e_n\}.$$

The convex hull[2] of Δ is called the *n-cross-polytope*. Setting $n = 3$ we obtain the usual octahedron. The symmetry group of the n-cross-polytope is the well-known *hyperoctahedral group*, denoted B_n.

Observe that the subsets of Δ of the form $\{e_j, -e_j\}$ with $1 \leq j \leq n$ are preserved by B_n. Thus [12, Exercise 2.6.2] B_n embeds in the wreath product

$$S_2 \wr S_n.$$

Let t be the linear map that sends e_1 to $-e_1$ whilst fixing all the other basis vectors. For $1 \leq i \leq n-1$ let s_i be the linear map that swaps e_i and e_{i+1} whilst fixing all other basis vectors. Then t and s_i are symmetries of the n-cross-polytope, that is they are elements of B_n. It is not difficult to see that these elements generate B_n. The matrices of these generators with respect to the basis $\{e_1, \ldots, e_n\}$ are given (by

[2]Recall the convex hull of a set of points is the shape you would get if you tried to vacuum pack those points.

abuse of notation) by

$$t = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}, \qquad s_i = \begin{bmatrix} I_{i-1} & & & \\ & & 1 & \\ & 1 & & \\ & & & I_{n-i-1} \end{bmatrix}$$

where for each integer m the notation I_m denotes the $m \times m$ identity matrix and where empty spaces denote zeros. The elements of this group are precisely the $n \times n$ monomial matrices where we allow each non-zero entry to be ± 1. Hence the order of B_n is

$$|B_n| = 2^n n!.$$

This matches the order of the wreath product so we have

$$B_n \cong S_2 \wr S_n.$$

If we regard the matrices above as matrices over $\mathbb{C}$ then the group they generate is a *complex reflection group*. In the Shephard–Todd classification [28] this group is labelled as $G(2, 1, n)$. So we have $B_n \cong G(2, 1, n)$.

Our final description of B_n is as a Coxeter group. By abuse of notation B_n is the Coxeter group of type B_n. It is also the Weyl group of type B_n. This group is the Coxeter group with Dynkin diagram

In other words B_n is the group with generators $t, s_1, \ldots, s_{n-1}$ subject to the relations

(a) $t^2 = 1$,
(b) $s_i^2 = 1$ for $1 \leq i \leq n-1$,
(c) $ts_1ts_1 = s_1ts_1t$,
(d) $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$ for $1 \leq i \leq n-2$,
(e) $ts_i = s_it$ for $i > 1$ and
(f) $s_is_j = s_js_i$ for $|i - j| > 1$.

Sending these generators to the generators above with the same names produces an isomorphism. For more information on Coxeter groups we refer the reader to [18].

Two important subgroups of B_n are the following. The *base group* of B_n is the subgroup generated by all conjugates of t. The base group is isomorphic to the direct product S_2^n of n copies the symmetric group of order 2. The *top group* of B_n is the subgroup generated by the s_i with $1 \leq i \leq n-1$. The top group is isomorphic to the symmetric group S_n by mapping s_i to the transposition $(i, i+1)$ in S_n. In the matrix description of B_n, the base group consists of the diagonal matrices and the top group consists of the permutation matrices.

2.7 The representation theory of the hyperoctahedral group

Let F be a field of characteristic $p \geq 0$. Fix $n \in \mathbb{N}$. In this section we describe the simple FB_n-modules. Let $G = B_n$. As G is isomorphic to the wreath product $S_2 \wr S_n$, the simple FG-modules can be built up, by Clifford Theory, from the simple FS_k-modules with $k \leq n$. This process is easily generalised to the setting where G is the wreath product of an arbitrary finite group with any subgroup of S_n, provided F is large enough (any F is large enough for B_n). For details see [20, Chapter 4]. Our approach is fairly elementary requiring only basic Clifford Theory. For an alternative approach via Hecke algebras of type B we refer the reader to [11].

Some important facts are the following:

(a) If F is a splitting field for a finite group then every irreducible representation of the group over any extension field of F can be realised over F itself,
(b) Every field is a splitting field for B_n.

We start by defining some important subgroups of G. Let $H_0 = \langle t \rangle$ be the subgroup generated by t. Let H be the base group of G, that is the subgroup generated by all conjugates of t. Let T be the top group of G, that is the subgroup $\langle s_1, \ldots, s_{n-1} \rangle \cong S_n$. In terms of the matrix description of G, given in Section 2.6, H consists of the diagonal matrices in G and T consists of the permutation matrices in G. We have $H_0 \cong S_2$ and $H \cong H_0^n \cong S_2^n$. Also $T \cong S_n$ and $G = HT$ with $H \trianglelefteq G$.

Let D be a simple FH-module. For $g \in G$ we define the *conjugate* FH-module D^g

to be the FH-module with the same underlying vector space as D, but with action of $h \in H$ given by

$$hx := g^{-1}hgx$$

for all $x \in D$. The conjugate D^g is an FH-module since H is normal in G. Furthermore, D^g is simple. Thus G acts on the simple FH-modules by conjugation. The stabilizer of D under this action is called the *inertia group* of D and is denoted by G_D. The subgroup $T_D = T \cap G_D$ is called the *inertia factor*[3] *of* D.

Clifford Theory says that the simple FG-modules are parametrised by the pairs (D_1, D_2) where D_1 runs through the non-conjugate simple FH-modules and, while D_1 is fixed, D_2 runs through the simple FT_{D_1}-modules; see [20, Theorem 4.3.34] or Clifford's original paper [8].

All of these simple modules are labelled by certain combinatorial objects defined as follows. A *composition of* n is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of non-negative integers whose sum, denoted $|\lambda|$, is n. We call the λ_i the *parts* of λ. We identify each composition λ with the tuple obtained by removing all parts that are zero. For example $(1, 3, 0, 5, 5, 0, 0, \ldots) = (1, 3, 5, 5)$. We also indicate repeated parts by an exponent. So the previous example becomes $(1, 3, 5, 5) = (1, 3, 5^2)$. The unique composition $(0, 0, \ldots)$ of 0 will be denoted by 0. A composition λ of n with $\lambda_i \geq \lambda_{i+1}$ for each $i \in \mathbb{N}$ is called a *partition* of n. It is well-known that for $p = 0$ the simple FS_n-modules are parametrised by the partitions of n. We write S^λ for the simple module corresponding to the partition λ. Moreover these modules S^λ can be constructed in a characteristic free fashion, that is over $\mathbb{Z}$. We write S^λ for the corresponding FS_n-module in any characteristic. In the case $p > 0$ the module S^λ is often not simple. However it has an S_n-invariant bilinear form. We set $D^\lambda = S^\lambda / \operatorname{rad} S^\lambda$. Then it can be shown [19, Theorem 4.9] that D^λ is either zero or absolutely irreducible. The partitions λ with $D^\lambda \neq 0$ are those with the following combinatorial property. Let λ be a partition of n. For $p > 0$ we say that λ is *p-regular* if there is no $i \in \mathbb{N}$ with $\lambda_{i+1} = \lambda_{i+p}$. In other words a partition is p-regular if none of its parts are repeated p or more times. All partitions are defined to be 0-*regular*.

Theorem 2.7.1 ([19, Theorem 11.5])**.** *As λ varies over p-regular partitions of n,*

[3] The reason for the name "factor" is that, in general Clifford Theory, G can be any finite group with arbitrary normal subgroup H and the inertia factor is defined to be the factor group G_D/H which may not be a subgroup of G. However in our case $G_D/H \cong T_D$ and the subgroup T_D is easier to work with.

D^λ varies over a complete set of inequivalent irreducible FS_n-modules. Each D^λ is self-dual and absolutely irreducible. Every field is a splitting field of S_n.

For B_n we also need the following notion. A *bipartition* of n is a pair of partitions (λ, μ) such that $|\lambda| + |\mu| = n$. A bipartition is *p-regular* if each of its constituent partitions is p-regular.

We are now ready to construct the simple FB_n-modules. The case $p = 2$ is rather special since in this case there is only one simple FH-module: the trivial module. So suppose $p = 2$. Let D_1 be the trivial FH-module. The inertia factor T_{D_1} is then the whole of $T \cong S_n$. The simple FG-modules are precisely the modules

$$D^{(\lambda)}$$

(to be defined), where λ is a 2-regular partition of n. To construct $D^{(\lambda)}$, we start with the simple FT-module D^λ (recall $T \cong S_n$). Then $D^{(\lambda)}$ is obtained from D^λ by letting H act as the identity on D^λ.

We assume now that $p \neq 2$. This case is more interesting. The simple FG-modules are parametrised by the set of p-regular bipartitions (λ, μ) of n. We will denote the simple module (to be constructed) corresponding to the bipartition (λ, μ) by

$$D^{(\lambda,\mu)}.$$

We describe the construction of $D^{(\lambda,\mu)}$. Let (λ, μ) be a bipartition of n in which each of λ and μ is p-regular. There are precisely two simple FH_0-modules (upto isomorphism): the trivial module, $D^{(2)}$, and the sign module, $D^{(1,1)}$. Since F is a splitting field for H_0, the outer tensor product modules

$$\underbrace{D^{(2)} \# \cdots \# D^{(2)}}_{a \text{ copies}} \# \underbrace{D^{(1,1)} \# \cdots \# D^{(1,1)}}_{n-a \text{ copies}}, \tag{2.7.2}$$

as a runs over the integers $0 \leq a \leq n$, form a complete set of non-isomorphic non-conjugate simple FH-modules [9, Theorem 10.33]. Note in particular that these are all 1-dimensional. Let $D_1^{(\lambda,\mu)}$ be the FH-module given by (2.7.2) with $a = |\lambda|$. Set $a = |\lambda|$. The inertia group and inertia factor of $D_1^{(\lambda,\mu)}$ are

$$G_{D_1^{(\lambda,\mu)}} = B_a \times B_{n-a}$$

and

$$T_{D_1^{(\lambda,\mu)}} = S_a \times S_{n-a}$$

where we regard $S_a \times S_{n-a}$ as the subgroup of T generated by s_i with $1 \leq i \leq n$ and $i \neq a$ and we regard $B_a \times B_{n-a}$ as the subgroup of B_n generated by H and $S_a \times S_{n-a}$. We make $D_1^{(\lambda,\mu)}$ into a module for the inertia group by letting $S_a \times S_{n-a}$ act trivially[4]. Define $D_2^{(\lambda,\mu)}$ to be the $F[B_a \times B_{n-a}]$-module obtained from the simple $F[S_a \times S_{n-a}]$-module $D^\lambda \# D^\mu$ by letting H act trivially. Then $D_2^{(\lambda,\mu)}$ is a simple $F[B_a \times B_{n-a}]$-module. Finally, we set

$$D^{(\lambda,\mu)} = \mathrm{Ind}_{B_a \times B_{n-a}}^{G} \left(D_1^{(\lambda,\mu)} \otimes D_2^{(\lambda,\mu)} \right).$$

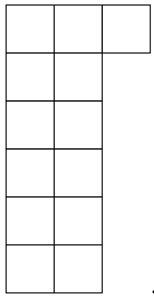
Example 2.7.3 (The trivial module)**.** We claim $D^{((n),0)}$ is the trivial FB_n-module. Indeed $D_1^{((n),0)}$ is obtained by taking the outer tensor product of n copies of the trivial module for H_0 and letting S_n act trivially. This is the trivial $F[B_n \times B_0]$-module. The module $D_2^{((n),0)}$ is obtained by taking the trivial module $D^{(n)} \# D^0$ for $S_n \times S_0$ and letting H act trivially. This is also the trivial $[B_n \times B_0]$-module. Finally, tensoring these two modules and inducing up to B_n does nothing (since the inertia group $B_n \times B_0$ is already B_n).

Example 2.7.4 (The determinant representation)**.** Consider the interpretation of B_n as matrices in Section 2.6. Then the determinant map provides a 1-dimensional representation of B_n. We claim for $p > n$ the module $D^{(0,(1^n))}$ is the 1-dimensional module affording this representation. First $D_1^{(0,(1^n))}$ is the outer tensor product of n copies of the sign representation of H_0 with $S_0 \times S_n$ acting as the identity. This is the 1-dimensional module where each conjugate of t acts as -1. In terms of the matrix description of B_n these elements are the diagonal matrices with one entry equal to -1 and the other $n-1$ entries equal to 1. The module $D_2^{(0,(1^n))}$ is the sign representation of $S_0 \times S_n$ with H acting trivially. Thus, the inner tensor product of these two modules affords the determinant representation as required.

For $p \leq n$ the sign representation of S_n is not labelled by (1^n) but rather by the image of (n) under the Mullineux map[5], see for example [4]. This image is the partition whose diagram is obtained by placing boxes underneath each other to a maximum depth of $p-1$ boxes, then repeating starting in the first row of the next column to the right until n boxes are laid out. For example the 7-modular sign representation of

[4]For other wreath products $A \wr S_n$ this step is non-trivial.
[5]For $p > n$ this image is just (1^n).

S_{13} is labelled by the partition $(3, 2^5)$ whose diagram has two full columns of height 6 followed by a single column with only one box:



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2.8 Incidence homology from the Boolean algebra

In the paper [24] of Mnukhin and Siemons incidence homology modules for the symmetric group are defined and studied. These are studied further in the paper [3] of Bell, Jones and Siemons. In this chapter we give an overview with particular emphasis on those results which will be useful for us later. It is worth noting that all of these results have counterparts in the case the Boolean algebra is replaced by a finite projective space over a field of characteristic coprime to p and the symmetric group is replaced by a general linear group. This case is recent work [29] of Siemons and Smith. The results here can all be obtained by "setting $q = 1$" in these projective space analogues.

Fix $n \in \mathbb{N}$. Let $[n] = \{1, \ldots, n\}$. We denote by L the set of all subsets of $[n]$. Notice that L is partially ordered by subset inclusion. It is also ranked by cardinality. So L is a ranked poset.

As L is a ranked poset we can apply the constructions of Section 2.5 to obtain incidence homology modules for the symmetric group S_n as follows. Let k be an integer. Then L_k is the set of k element subsets of $[n]$. Let M be the F-vector space with basis L. Let M_k be the F-vector space with basis L_k. The incidence map is the linear map $\partial : M \to M$ given on L by taking each subset of $[n]$ to the sum of its subsets which can be obtained by removing precisely one element. It can be shown that $\partial^p = 0$. Thus (M, ∂) is a p-complex, visualised as the sequence

$$\ldots \xleftarrow{\partial} M_1 \xleftarrow{\partial} M_2 \xleftarrow{\partial} M_3 \xleftarrow{\partial} M_4 \xleftarrow{\partial} M_5 \xleftarrow{\partial} \ldots .$$

For $0 \leq i \leq p$ we obtain the chain complex

$$\mathcal{M}_{k,i}: \quad \dots \xleftarrow{\partial^i} M_{k-p} \xleftarrow{\partial^{p-i}} M_{k-i} \xleftarrow{\partial^i} M_k \xleftarrow{\partial^{p-i}} M_{k+p-i} \xleftarrow{\partial^i} M_{k+p} \xleftarrow{\partial^{p-i}} \dots$$

Set $K^n_{k,i} = \ker \partial^i \cap M_k$ and $I^n_{k,i} = \partial^{p-i}(M_{k+p-i})$. We obtain the incidence homology module

$$H^n_{k,i} = K^n_{k,i}/I^n_{k,i}.$$

The defining action of S_n on $[n]$ extends to a rank-preserving action on L. Thus by the arguments in Section 2.5 $H^n_{k,i}$ is an FS_n-module.

The case $p = 2$ turns out to be not so interesting. In all of the following we assume $p > 2$. Trivially $H^n_{k,i} = 0$ if any of the following hold

(a) $k < 0$,
(b) $k > n$,
(c) $i = 0$,
(d) $i = p$.

Theorem 2.8.1 ([3, Theorem 3.2])**.** *Let $p > 2$. The module $H^n_{k,i}$ is non-zero if and only if $0 \leq k \leq n$ and $0 < i < p$ satisfy*

$$n < 2k + p - i < n + p. \tag{2.8.2}$$

In particular, Theorem 2.8.1 says that each chain complex $\mathcal{M}_{k,i}$ when viewed as a sequence is *almost exact* in the sense that there is at most one position where the sequence is not exact, equivalently the sequence has at most one non-zero homology module. Given n and p, a value of (k, i) satisfying (2.8.2) is called a *middle term* (for n and p).

Theorem 2.8.3 ([3, Theorem 6.2])**.** *Let $p > 2$. On restriction to S_{n-1} we have*

$$H^n_{k,i} \cong H^{n-1}_{k,i+1} \oplus H^{n-1}_{k-1,i-1}.$$

Theorem 2.8.3 is known as a *branching rule*. The proof of this branching rule is constructive in the sense that an explicit isomorphism is given. This means the proof is self-contained. This and the rescursive nature of the result allows many of the other results to be proved by induction (on n) using only the branching rule.

Theorem 2.8.4 ([3, Theorem 6.4])**.** *Let $p > 2$. Suppose that $0 \leq k \leq n$ and $0 < i < p$ satisfy $2k + p - i = n + p - 1$. Then $H^n_{k,i}$ is irreducible. Furthermore, if (k', i') is another pair of positive integers satisfying the above conditions then $H^n_{k,i} \ncong H^n_{k',i'}$.*

Theorem 2.8.5 ([3, Theorem 6.5]). *Let $p > 2$. Suppose $i \leq k$ and $0 < i < p$ satisfy $2k + p - i = n + p - 1$. Then*

$$H^n_{k,i} \cong D^\lambda$$

where D^λ is the Specht module S^λ modulo its radical and λ is the partition of n into two parts of size k and $n - k$. In particular $H^n_{k,i}$ is irreducible.

Lemma 2.8.6 ([3, Lemma 6.7]). *Let $p > 2$. If $2k + p - i = n + a$ then $H^n_{k,i} \cong H^n_{k,a}$.*

Lemma 2.8.7 ([3, Lemma 6.8]). *Let $p > 2$. Then $H^n_{k,i} \cong H^n_{n-k,p-i}$.*

Theorem 2.8.8 ([3, Theorem 6.9]). *Let $p > 2$. Let $2k+p-i = n+a$ and $0 < a < p/2$. Then the composition factors of $H^n_{k,i}$ each have multiplicity one and are given by*

(a) $\{ H^n_{k-j,i+a-1-2j} : j = 0, \ldots, a-1 \}$ *if $a \leq i \leq p - a$,*
(b) $\{ H^n_{k-j,i+a-1-2j} : j = 0, \ldots, i-1 \}$ *if $i < a$ and*
(c) $\{ H^n_{k-j,i+a-1-2j} : j = i - (p-a), \ldots, a-1 \}$ *if $i > p - a$.*

These results together show that each of the composition factors of a given $H^n_{k,i}$ is isomorphic to $H^n_{k',i'}$ for some integers k', i'. We record a particular special case which will be useful to us later.

Corollary 2.8.9. *Let $p > 2$. Suppose $0 \leq k \leq n$ and $0 < i < p$ satisfy $2k+p-i-n = 1$. Then*

$$H^n_{k,i} \cong D^\lambda$$

where λ is the partition of n into two parts of size k and $n-k$ or the partition (n) in the case $k = 0$.

Proof. We have $n = 2k + p - i - 1$. Lemma 2.8.7 gives $H^n_{k,i} \cong H^n_{k',i'}$ where $k' = n - k$ and $i' = p - i$. Now $2k' + p - i' = n + p - 1$. If $k \neq 0$ then $i' \leq k'$ so Theorem 2.8.5 gives $H^n_{k',i'} \cong D^\lambda$ where λ is the partition of n into two parts of size $k' = n - k$ and $n - k' = k$. On the other hand if $k = 0$ then $M^n_{k'} \cong 1_{S_n} \cong D^{(n)}$. So $H^n_{k',i'} \neq 0$ implies $H^n_{k',i'} \cong D^{(n)}$. □

Since the sequences $\mathcal{M}_{k,i}$ are almost exact the Hopf trace formula [25, Theorem 22.1] can give very strong results. As an example a formula for $\dim H^n_{k,i}$ can be found:

Theorem 2.8.10 ([3, Theorem 4.5]). *Let $p > 2$. Suppose $0 < i < p$ and $0 < 2k + p - i - n < p$. Then*

$$\dim H^n_{k,i} = \sum_{t \in \mathbb{Z}} \binom{n}{k - pt} - \binom{n}{k - i - pt}.$$

Chapter 3

Incidence homology for the hyperoctahedral group

This chapter serves as an introduction to the main objects of study and notation. In Section 3.1 we introduce the natural partially ordered set associated to the hyperoctahedral group B_n. It is a Boolean algebra with extra structure. The automorphism group of this structure is isomorphic to B_n. In Section 3.2 the associated permutation modules are introduced over a field F of characteristic $p > 0$. These modules cannot help falling into a grid which is shown in 3.2.2. The grid has a rich combinatorial structure and many symmetries. With the addition of natural incidence maps between these modules, the grid becomes a double p-complex. Homology modules thus arise, giving representations of B_n. These are introduced in Section 3.4. The combinatorics and symmetry give information about these representations but we believe they are also interesting in their own right. They are discussed in later chapters.

3.1 The sets $L^n_{u,d}$

Throughout n will be some fixed natural number. We consider zero to be the first natural number. Let $[n]$ denote the set of integers $\{1, \ldots, n\}$. Let $[n\overline{n}]$ denote the set of integers and *barred integers* $\{1, \overline{1}, \ldots, n, \overline{n}\}$. Let L^n denote the powerset of $[n\overline{n}]$. On $[n\overline{n}]$ we define an involutory map $^-$ by $\alpha \mapsto \overline{\alpha}$ for each $\alpha \in [n\overline{n}]$ with the rule that

$\overline{\overline{\alpha}} := \alpha$. So, for example, $1 \mapsto \overline{1}$ and $\overline{2} \mapsto 2$. Naturally, this operation extends to L^n by setting

$$\overline{\{\alpha_1, \dots, \alpha_k\}} := \{\overline{\alpha_1}, \dots, \overline{\alpha_k}\}$$

for all $\{\alpha_1, \dots, \alpha_k\}$ in L^n. For example $\overline{\{1, \overline{2}\}} = \{\overline{1}, 2\}$. We are interested in certain collections of subsets of $[n\overline{n}]$ as follows. Suppose $x \subseteq [n\overline{n}]$. Let $\alpha \in x$. Call $\{\alpha\}$ a *single* of x if $|\{\alpha, \overline{\alpha}\} \cap x| = 1$. Call $\{\alpha, \overline{\alpha}\}$ a *double* of x if $|\{\alpha, \overline{\alpha}\} \cap x| = 2$. Let $\dot{x}$ be the union of all singles of x. Let $\ddot{x}$ be the union of all doubles of x. This is the same thing as $\dot{x} = \{\, \alpha \in x \,:\, \overline{\alpha} \notin x \,\}$ and $\ddot{x} = \{\, \alpha \in x \,:\, \overline{\alpha} \in x \,\}$. We can write x as the disjoint union

$$x = \dot{x} \cup \ddot{x}.$$

This will come into play later. The collections we are interested in are

$$L^n_{u,d} := \left\{\, x \in L^n \,:\, |\dot{x}| + \tfrac{1}{2}|\ddot{x}| = u \text{ and } \tfrac{1}{2}|\ddot{x}| = d \,\right\}$$

as u and d vary over the integers. The mnemonics here are u for *unsigned size* and d for *number of doubles*. These sets $L^n_{u,d}$ form a partition of L^n. This partition provides a structure within the powerset of $[n\overline{n}]$ for the group B_n which will be defined shortly. The case $n = 2$ is illustrated in Table 3.1. Notice that $L^n_{u,d} \neq \emptyset$ if and only if $0 \leq d \leq u \leq n$.

Example 3.1.1. The partition of the powerset of $[2\overline{2}]$ into the collections $L^2_{u,d}$ is shown in Table 3.1. For example the sets in $L^2_{1,0}$ are shown in the bottom row of the second column. Thus $L^2_{1,0} = \big\{\{1\}, \{\overline{1}\}, \{2\}, \{\overline{2}\}\big\}$ consists of those subsets of $[2\overline{2}]$ which contain exactly one single and no doubles.

d \ u	0	1	2
2			$\{1, \overline{1}, 2, \overline{2}\}$
1		$\{1, \overline{1}\}, \{2, \overline{2}\}$	$\{1, 2, \overline{2}\}, \{\overline{1}, 2, \overline{2}\}, \{1, \overline{1}, 2\}, \{1, \overline{1}, \overline{2}\}$
0	$\emptyset$	$\{1\}, \{\overline{1}\}, \{2\}, \{\overline{2}\}$	$\{1, 2\}, \{\overline{1}, 2\}, \{1, \overline{2}\}, \{\overline{1}, \overline{2}\}$

Table 3.1: The collections $L^2_{u,d}$ as u and d vary.

Example 3.1.2 (The hyperoctahedron or cross-polytope)**.** Let $n \geq 1$. Let V be an n-dimensional Euclidean space with basis $e_1, \dots, e_n$. The *hyperoctahedron* or more commonly the *n-cross-polytope* is defined to be the convex hull of the points $\pm e_1, \dots, \pm e_n$. For $n = 3$ this is the standard octahedron. How this relates to our setup is as follows. For $1 \leq j \leq n$ identify e_j with j and $-e_j$ with $\overline{j}$. Let $1 \leq u \leq n$. For $x \in L^n_{u,0}$ let s_x be the convex hull of the elements of x under this identification. It turns out this gives the boundary of the cross-polytope the structure of an $(n-1)$-dimensional simplicial complex. The $(u-1)$-simplices are precisely the s_x with $x \in L^n_{u,0}$. Thus the union of those $L^n_{u,d}$ with $d = 0$ is an important subposet of L^n. In fact many arguments throughout the thesis can be reduced to the case $d = 0$. This is facilitated by the results of Section 3.6. Note that this subposet is the bottom row in Table 3.1. We will try to adhere to this orientation throughout the thesis.

Example 3.1.3 (The hypercube)**.** The dual of the cross-polytope is the hypercube. This also occurs as a subposet in L^n as follows. Let $e_1, \dots, e_n$ be the standard basis in $\mathbb{R}^n$. The vertices of the cube are $\pm e_1 \pm e_2 \dots \pm e_n$ in the sense that each assignment of signs gives one vertex and every vertex occurs in this way. For each integer i with $1 \leq i \leq n$ we identify e_i with i and $-e_i$ with $\bar{i}$. Then the vertices correspond to the elements of $L^n_{n,0}$. For example $\{1, \overline{2}, 3\} \in L^3_{3,0}$ corresponds to the vertex $e_1 - e_2 + e_3$. Now an edge between vertices can be identified with the union of the sets corresponding to the vertices. For example the edge between $\{1, \overline{2}, 3\}$ and $\{\overline{1}, \overline{2}, 3\}$ is identified with $\{1, \overline{1}, \overline{2}, 3\}$. Two vertices lie on the same edge if and only if their signs differ only in one position. Thus edges are the elements of $L^n_{n,1}$. Continuing in this way we identify the set of d-dimensional faces with $L^n_{n,d}$ for each $0 \leq d \leq n$. Note that here the whole hypercube is identified with the set $\{1, \overline{1}, 2, \overline{2}, \dots, n, \overline{n}\}$. This differs from Example 3.1.2 in that here we include a set representing the entire hypercube and no set representing the -1-dimensional face. In Example 3.1.2 we have the -1-dimensional face $\emptyset$ but no set representing the entire cross-polytope. This fits with the idea that these two objects should be dual to each other. Note also that the faces here are actually the faces of the cube in the usual sense, there is no triangulation involved. In particular, the subposet consisting of the union of all $L^n_{n,d}$ with $0 \leq d \leq n$ is not a simplicial complex. This subposet is the righthand column in Table 3.1.

In the future, to save space when writing elements of L^n, we will omit commas in the set notation. For example we will write $\{1\overline{2}3\overline{3}\}$ instead of $\{1, \overline{2}, 3, \overline{3}\}$.

Definition 3.1.4 (Another form of the group B_n)**.** A *signed permutation of* $[n\overline{n}]$ is a permutation g of $[n\overline{n}]$ such that $g(\overline{\alpha}) = \overline{g(\alpha)}$ for all $\alpha \in [n\overline{n}]$. Denote the group of all such permutations by B_n.

Note that this group is isomorphic to all of the previous forms of the hyperoctahedral group which we saw in Section 2.6. An isomorphism is given by identifying the signed permutation $(1,\overline{1})$ with the generator t and the signed permutation $(i,i+1)(\overline{i},\overline{i+1})$ with the generator s_i for $1 \leq i \leq n-1$. An *automorphism of* L^n is defined to be a permutation g of L^n such that for each $x, y \in L^n$ we have

(a) $\overline{g(x)} = g(\overline{x})$, and
(b) $x \subseteq y$ implies $g(x) \subseteq g(y)$.

Denote the group of all automorphisms of the above form by $\mathrm{Aut}(L^n)$. We define the action of B_n on L^n by

$$g(\{\alpha_1 \dots \alpha_k\}) = \{g(\alpha_1) \dots g(\alpha_k)\}$$

for all $g \in B_n$ and all $\{\alpha_1 \dots \alpha_k\} \in L^n$. This action gives an embedding of B_n in $\mathrm{Aut}(L^n)$. In fact, it is not difficult to show that this embedding is surjective. So $\mathrm{Aut}(L^n) \cong B_n$. In the future we will make no distinction between $\mathrm{Aut}(L^n)$ and B_n. Some important classes of elements in B_n are the following.

(a) The bar operation is an element of B_n. In cycle notation it is the element $(1,\overline{1}) \dots (n,\overline{n})$.
(b) For each $\alpha \in [n]$ there is the signed permutation $(\alpha,\overline{\alpha})$. The subgroup generated by all such signed permutations is known as the *base group* of B_n. It is an elementary abelian group of order 2^n. The bar operation lies in this subgroup as it is the product of all these generators.
(c) Any permutation g of $[n]$ yields a signed permutation g' of $[n\overline{n}]$ by setting $g'(\alpha) = g(\alpha)$ and $g'(\overline{\alpha}) = \overline{g(\alpha)}$ for each $\alpha \in [n]$. We will abuse notation and write simply g when strictly we mean g'. For example, we write $(1,2)$ for the signed permutation $(1,2)(\overline{1},\overline{2})$.

Proposition 3.1.5. *The orbits of B_n on L^n are precisely the collections $L^n_{u,d}$ as u and d range over the integers $0 \leq d \leq u \leq n$. The size of $L^n_{u,d}$ is* $|L^n_{u,d}| = \binom{n}{u}\binom{u}{d}2^{u-d}$.

Proof. Let $x \in L^n$. Then there exist integers u and d with $0 \leq d \leq u \leq n$ such that $x \in L^n_{u,d}$. Let $g \in B_n$. Set $y = g(x)$. Suppose $\{\alpha\overline{\alpha}\}$ is a double of x. Then $g(\{\alpha\overline{\alpha}\})$ is a double of y. Similarly, if $\{\beta\overline{\beta}\}$ is a double of y then $g^{-1}(\{\beta\overline{\beta}\})$ is a double of x. Hence $|\ddot{x}| = |\ddot{y}|$. But since g is a bijection on $[n\overline{n}]$ we also have $|x| = |y|$. Hence $|\dot{x}| = |\dot{y}|$. So B_n preserves $L^n_{u,d}$. It is not difficult to see that B_n is also transitive on $L^n_{u,d}$. This shows that $L^n_{u,d}$ is the orbit of x.

To count $L^n_{u,d}$ suppose we want to choose an element $x \in L^n_{u,d}$. First choose the *unsigned support of* x, by which we mean the set of elements $\alpha \in [n]$ such that $\{\alpha\overline{\alpha}\} \cap x \neq \emptyset$. There are $\binom{n}{u}$ ways to do this. Then choose which of these are to become the doubles of x, that is which α should have $\{\alpha\overline{\alpha}\} \subseteq x$. There are $\binom{u}{d}$ choices. Finally we must decide for each remaining α whether it is $\{\alpha\}$ or $\{\overline{\alpha}\}$ which is to appear as a single of x. There are 2^{u-d} choices for this. Multiplying these three numbers together gives the result. $\square$

Since we have group actions, we can define permutation modules. This is described in the next section.

3.2 The modules $M^n_{u,d}$

Let F be a field. Let Ω be a set. We write $F\Omega$ for the F-vector space with basis Ω. If $x_1, \dots, x_t$ are already elements of some F-vector space, then we write $\langle x_1, \dots, x_t\rangle_F$ for the subspace spanned by $x_1, \dots, x_t$.

With this in mind, define

$$M^n := FL^n.$$

For any integers u and d set

$$M^n_{u,d} := FL^n_{u,d}.$$

Define a multiplication $\cdot$ on M^n by setting

$$x \cdot y = x \cup y$$

for $x, y \in L^n$ and extending linearly. For example

$$\begin{aligned}\big(5\{12\} + \{\overline{3}\}\big) \cdot \{1\overline{2}3\} &= 5\{12\} \cdot \{1\overline{2}3\} + \{\overline{3}\} \cdot \{1\overline{2}3\} \\ &= 5\{12\overline{2}3\} + \{1\overline{2}3\overline{3}\}.\end{aligned}$$

Note in this example 5 is a scalar in our field. The multiplication interacts well with certain incidence maps which we define in the next section.

If G is a group acting on Ω then $F\Omega$ is the associated *permutation module*. It has the structure of a (left) FG-module by setting

$$\left(\sum_{g\in G}\lambda_g g\right)\left(\sum_{\omega\in\Omega}\mu_\omega\omega\right)=\sum_{g\in G}\sum_{\omega\in\Omega}\lambda_g\mu_\omega g\omega$$

where each λ_g and each μ_ω are elements of F. We call this procedure (and similar procedures) *extending the action of G linearly.* Let Δ be an orbit of G on Ω. The G-action restricts to Δ. Extending linearly gives $F\Delta$ the structure of a permutation module for G. Furthermore, it is clear that $F\Omega$ decomposes as the direct sum of these submodules as Δ runs over all orbits, that is

$$F\Omega=\bigoplus_{\Delta\subseteq\Omega \text{ a } G\text{-orbit}} F\Delta.$$

Recall the action of B_n on L^n described in Section 3.1. Extending this action linearly thus gives M^n the structure of a permutation module for B_n. Proposition 3.1.5 tells us the orbits of B_n on L^n are the $L^n_{u,d}$ as u and d range over the integers $0\le d\le u\le n$. Restricting the action of B_n to the orbit $L^n_{u,d}$ and extending linearly yields the permutation module

$$M^n_{u,d}:=FL^n_{u,d}.$$

By the above, M^n decomposes as the direct sum of these submodules

$$M^n=\bigoplus_{0\le d\le u\le n} M^n_{u,d}. \tag{3.2.1}$$

Note that we have $L^n_{u,d}=\emptyset$ unless $0\le d\le u\le n$. For convenience we also define $M^n_{u,d}:=FL^n_{u,d}=F\emptyset=0$ in these cases. So, in fact we have

$$M^n=\bigoplus_{u,d\in\mathbb{Z}} M^n_{u,d}.$$

This gives M^n the structure of a bigraded FB_n-module. The parameters u and d can be thought of as coordinates in a 2-dimensional grid. Then the decomposition (3.2.1) is visualised as a grid with the submodule $M^n_{u,d}$ at the (u,d)-coordinate. The non-zero submodules occupy the triangular region $0\le d\le u\le n$ in this grid. This is shown in Figure 3.2.2.

$$
\begin{array}{ccccccc}
 & & & & & 0 & \\
 & & & & 0 & M^n_{n,n} & 0 \\
 & & & 0 & M^n_{n-1,n-1} & M^n_{n,n-1} & 0 \\
 & & \iddots & \iddots & \vdots & \vdots & \\
 & 0 & M^n_{1,1} & \cdots & M^n_{n-1,1} & M^n_{n,1} & 0 \\
0 & M^n_{0,0} & M^n_{1,0} & \cdots & M^n_{n-1,0} & M^n_{n,0} & 0 \\
 & 0 & 0 & & 0 & 0 &
\end{array}
$$

Figure 3.2.2: The decomposition of M^n into submodules $M^n_{u,d}$.

3.3 Incidence maps

The elements of L^n (in other words the subsets of $[n\overline{n}]$) are partially ordered by inclusion. This can be used to compare the permutation modules $M^n_{u,d}$ of the previous section for different values of the parameters u and d. Define an incidence relation as follows. Let $x, y \in L^n$. We say that x and y are *incident* and write $x \sim y$ if $x \neq y$ and either $x \subseteq y$ or $y \subseteq x$. For any two pairs (u, d) and (u', d') there is a canonical *incidence map* $M^n_{u,d} \to M^n_{u',d'}$ derived from this relation. It is defined to be the linear map that sends each basis element $x \in L^n_{u,d}$ to the sum (in the vector space) of those $y \in L^n_{u',d'}$ that are incident with x. Each of these incidence maps commutes with the action of B_n.

We find it useful to take a slightly different viewpoint. Forget about the source and destination coordinates (u, d) and (u', d') and retain only their difference $(a, b) = (u' - u, d' - d)$. Define a linear map on the whole of M^n as follows. Let $x \in L^n$ be a basis element. Then there exist u and d such that $x \in L^n_{u,d}$. Map x to the sum of those $y \in L^n_{u+a,d+b}$ that are incident with x. This map again commutes with B_n. For each

$u, d \in \mathbb{Z}$ it restricts to a map $M^n_{u,d} \to M^n_{u+a,d+b}$, that is it is homogeneous of degree (a, b). The restriction is precisely the incidence map defined in the previous paragraph. We will refer to this "larger" map as an *incidence map* also. This viewpoint has the advantage that an incidence map can be composed with itself. This will be important in the next section.

Two incidence maps are of special importance as they control much of the structure of M^n. We describe them here.

Definition 3.3.1 (The singles and doubles maps)**.** We define the *singles map* $\dot{\partial}$ as the linear map $\dot{\partial} : M^n \to M^n$ that takes each $x \in L^n_{u,d}$ to

$$\dot{\partial}(x) := \sum_y y \in M^n_{u-1,d}$$

where the sum runs over those $y \in L^n_{u-1,d}$ that are contained in x. The *doubles map* $\ddot{\partial}$ is the linear map $\ddot{\partial} : M^n \to M^n$ that takes each $x \in L^n_{u,d}$ to

$$\ddot{\partial}(x) := \sum_y y \in M^n_{u,d-1}$$

where the sum runs over those $y \in L^n_{u,d-1}$ that are contained in x.

These maps maps are homogeneous of degrees $(-1, 0)$ and $(0, -1)$ respectively. Intuitively, the singles map is the map that takes x to the sum of those y obtained from x by removing one element from the singles part $\dot{x}$ of x in all possible ways. Similarly, the doubles map takes x to the sum of those y obtained by removing one element from the doubles part $\ddot{x}$ of x in all possible ways. Note that since every element of x lies in either a single or a double of x, the map $\partial = \dot{\partial} + \ddot{\partial}$ is the map which takes x to the sum of all of its subsets of size $|x| - 1$. Note that the notation ∂ will denote various maps throughout the thesis whereas the notations $\dot{\partial}$ and $\ddot{\partial}$ will always denote the singles and doubles maps.

Example 3.3.2. Let $x = \{123\overline{3}\}$. Then $\dot{\partial}(x) = \{13\overline{3}\} + \{23\overline{3}\}$, $\ddot{\partial}(x) = \{123\} + \{12\overline{3}\}$ and $\partial(x) = \{13\overline{3}\} + \{23\overline{3}\} + \{123\} + \{12\overline{3}\}$.

The singles and doubles maps can be superimposed on Figure 3.2.2. This is shown in Figure 3.3.3. It is an important picture which we will often refer back to. For reasons which will become clear we call it the *homology grid*.

$$
\begin{tikzcd}
 & & & & & 0 \arrow[d, "\ddot{\partial}"] & \\
 & & & & 0 & M^n_{n,n} \arrow[l, "\dot{\partial}"'] \arrow[d, "\ddot{\partial}"] & 0 \arrow[l, "\dot{\partial}"'] \\
 & & & 0 & M^n_{n-1,n-1} \arrow[l, "\dot{\partial}"'] \arrow[d, "\ddot{\partial}"] & M^n_{n,n-1} \arrow[l, "\dot{\partial}"'] \arrow[d, "\ddot{\partial}"] & 0 \arrow[l, "\dot{\partial}"'] \\
 & \iddots & & \iddots & \vdots \arrow[d, "\ddot{\partial}"] & \vdots \arrow[d, "\ddot{\partial}"] & \\
 & 0 & M^n_{1,1} \arrow[l, "\dot{\partial}"'] \arrow[d, "\ddot{\partial}"] & \cdots \arrow[l, "\dot{\partial}"'] & M^n_{n-1,1} \arrow[l, "\dot{\partial}"'] \arrow[d, "\ddot{\partial}"] & M^n_{n,1} \arrow[l, "\dot{\partial}"'] \arrow[d, "\ddot{\partial}"] & 0 \arrow[l, "\dot{\partial}"'] \\
0 & M^n_{0,0} \arrow[l, "\dot{\partial}"'] \arrow[d, "\ddot{\partial}"] & M^n_{1,0} \arrow[l, "\dot{\partial}"'] \arrow[d, "\ddot{\partial}"] & \cdots \arrow[l, "\dot{\partial}"'] & M^n_{n-1,0} \arrow[l, "\dot{\partial}"'] \arrow[d, "\ddot{\partial}"] & M^n_{n,0} \arrow[l, "\dot{\partial}"'] \arrow[d, "\ddot{\partial}"] & 0 \arrow[l, "\dot{\partial}"'] \\
 & 0 & 0 & & 0 & 0 &
\end{tikzcd}
$$

Figure 3.3.3: Decomposition of M^n showing incidence maps, the homology grid.

From now on we specify F to be a field of characteristic $p > 0$. The reason for this choice is that we are interested in homological sequences and methods. In positive characteristic M^n acquires properties which invite the use of such methods. Recall from Definition 2.4.5 that a *double p-complex* is a bigraded module M with two maps δ and ∂ homogeneous of degrees $(-1, 0)$ and $(0, -1)$ respectively such that

$$(\delta + \partial)^p = 0. \tag{3.3.4}$$

Our first main structural result is the following.

Theorem 3.3.5. *Let $n \in \mathbb{N}$. Then $(M^n, \dot{\partial}, \ddot{\partial})$ is a double p-complex of FB_n-modules.*

Proof. We know $\dot{\partial}$ and $\ddot{\partial}$ are homogeneous linear maps of degrees $(-1, 0)$ and $(0, -1)$ respectively. We also know they both commute with B_n. Let $\partial = \dot{\partial} + \ddot{\partial}$. It remains to show $\partial^p = 0$. But ∂ is the map that sends each $x \in L^n_{u,d}$ to the sum of *all* of its subsets of size $|x| - 1$. By identifying $[n\overline{n}]$ with $[2n]$ we see that ∂ is the map from the Boolean algebra case discussed in Section 2.8 (for S_{2n} rather than S_n). Thus $\partial^p = 0$ is immediate. More directly, we have $\partial^p(x) = p! \sum_y y$ where y runs over all the subsets of x of size $|x| - p$. The factor $p!$ counts the number of maximal chains between a fixed y and x. Since p is the characteristic of the field, this is zero. $\square$

Since $(M^n, \dot{\partial}, \ddot{\partial})$ is a double p-complex, homological methods can be used to yield more information. We conclude this section with a slightly more detailed look at M^n. Let $M = M^n$. Consider the total complex (T, ∂) of $(M, \dot{\partial}, \ddot{\partial})$. Recall, this is the p-complex with k-th homogeneous component

$$T_k = \bigoplus_{u+d=k} M_{u,d}$$

together with the differential $\partial = \dot{\partial} + \ddot{\partial}$. This is illustrated in Figure 3.3.6. The dot at the (u, d)-coordinate represents the submodule $M^n_{u,d}$. The grey lines $u + d = k$ as k varies represent T_k. The arrows represent the map ∂.

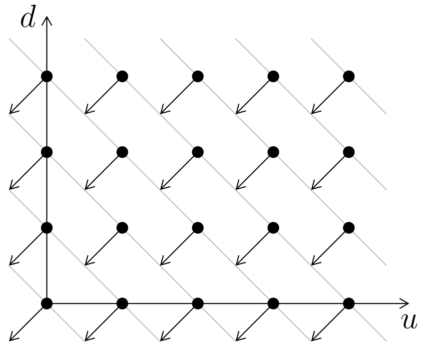


Figure 3.3.6: The total complex of M^n.

For $x \in L^n_{u,d}$ we have $|x| = u + d$. Hence T_k has basis consisting of all subsets of $[n\overline{n}]$ of size k. The map ∂ takes x to the sum of all its subsets of size $|x|-1$. On identifying $[n\overline{n}]$ with $[2n]$ it becomes clear that (T, ∂) is actually the p-complex for S_{2n} arising from the Boolean algebra of rank $2n$, as discussed in Section 2.8.

The p-complex for S_n arising from the Boolean algebra of rank n also occurs, and in many ways. We illustrate two essentially different ways. For $d \in \mathbb{Z}$ define $C_d = M^n_{d,d}$. Set $C = \bigoplus_d C_d$. Then C has basis the subsets of $[n\overline{n}]$ containing no singles, purely doubles. A typical element of this basis looks like $x = \{\alpha_1\overline{\alpha_1} \dots \alpha_d\overline{\alpha_d}\}$ for some $\alpha_1, \dots, \alpha_d$ in $[n]$. Define $\partial : C \to C$ by taking x to the sum of its subsets which are obtained by removing an entire double from x and extending linearly.

Example 3.3.7. $\partial(\{1\overline{1}2\overline{2}3\overline{3}\}) = \{1\overline{1}2\overline{2}\} + \{1\overline{1}3\overline{3}\} + \{2\overline{2}3\overline{3}\}$.

Identifying $\{\alpha_1\overline{\alpha_1} \dots \alpha_d\overline{\alpha_d}\}$ with $\{\alpha_1 \dots \alpha_d\}$ gives us our isomorphism of (C, ∂) with the p-complex of Section 2.8. A note of caution: on C, this map ∂ that removes a double in all possible ways coincides with the (unique) incidence map $\partial_{(-1,-1)}$ homogeneous of degree $(-1,-1)$. This is not true outside of C, for example $\partial_{(-1,-1)}(\{1\bar{1}2\}) = \{1\} + \{\bar{1}\} + \{2\}$ whereas $\partial(\{1\bar{1}2\}) = \{2\}$. The p-complex (C, ∂) is shaded in Figure 3.3.8.

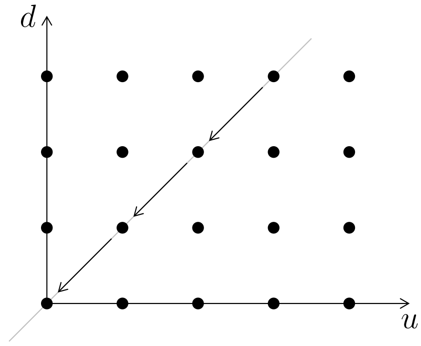


Figure 3.3.8: First copy of the S_n p-complex in M^n.

Our second copy of the S_n p-complex is simply the p-complex itself, as defined in Section 2.8. We recall the definition here. For $k \in \mathbb{Z}$ let L_k be the set of k-subsets of $[n]$. We have $L_k \subseteq L^n_{k,0}$. Let $C_k := FL_k$. Define $C = \oplus_k C_k$. The differential $\partial : C \to C$ is defined as the linear map that takes each $x \in L_k$ to the sum of its subsets in L_{k-1}. Thus (C, ∂) is a p-subcomplex of FS_n-modules in the bottom row of the homology grid, Figure 3.3.3. This bottom row is highlighted in Figure 3.3.9.

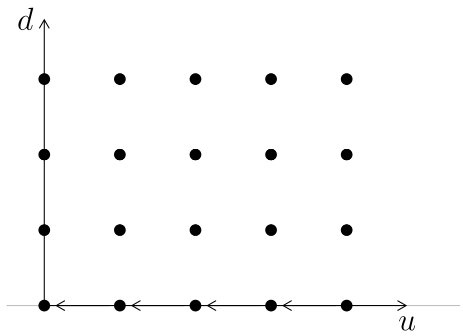


Figure 3.3.9: The bottom row of M^n.

We now define the two main classes of p-complexes we are interested in. The double p-complex structure of M^n implies each row and each column is a p-complex. For $d \in \mathbb{Z}$ we denote the d-th row by $M^n_{*,d}$. Then $(M^n_{*,d}, \dot{\partial})$ is a p-complex. This is highlighted as the grey line in Figure 3.3.10.

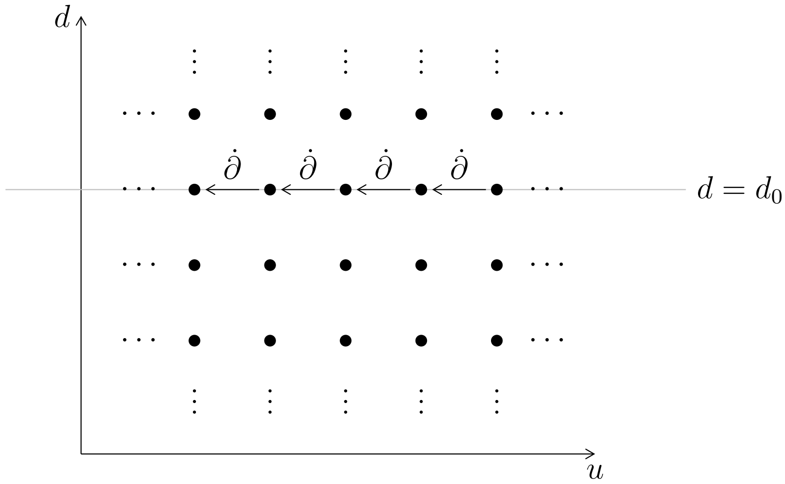


Figure 3.3.10: The p-complex $(M^n_{*,d_0}, \dot{\partial})$.

Similarly, for $u \in \mathbb{Z}$ we denote the u-th column of M^n by $M^n_{u,*}$. Then $(M^n_{u,*}, \ddot{\partial})$ is a p-complex. It is highlighted as the grey line in Figure 3.3.11.

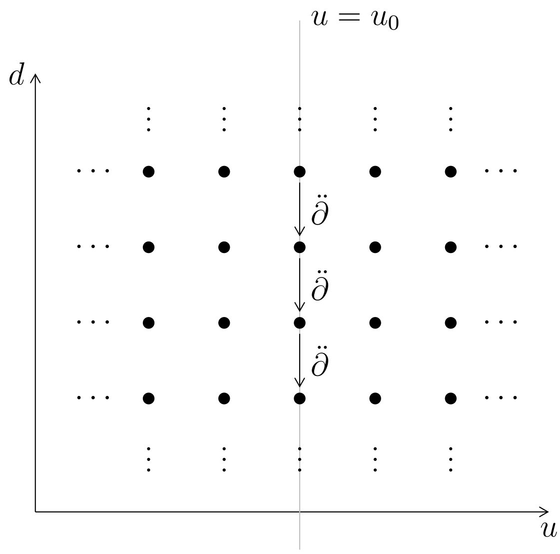


Figure 3.3.11: The p-complex $(M^n_{u_0,*}, \ddot{\partial})$.

In the next section we introduce the homology of these last two p-complexes.

3.4 The homology modules $\dot{H}^n_{u,d,i}$ and $\ddot{H}^n_{u,d,i}$

We have seen that $(M^n, \dot{\partial}, \ddot{\partial})$ is a double p-complex. In particular, this means that certain subsequences have homological properties. Amongst the most important subsequences to understand are the rows and the columns of M^n, as pictured in the homology grid, Figure 3.3.3. This section serves to set up notation for these sequences and their homology modules which are central throughout the remainder of the thesis.

Let $n, d \in \mathbb{Z}$ with $n \geq 0$. Consider the row $(M^n_{*,d}, \dot{\partial})$ of the homology grid shown in Figure 3.3.10. For any $u, i \in \mathbb{Z}$ with $0 \leq i \leq p$ define $\dot{\mathcal{M}}^n_{u,d,i}$ to be the sequence

$$\dot{\mathcal{M}}^n_{u,d,i}: \qquad \ldots \leftarrow M^n_{u-p,d} \xleftarrow{\dot{\partial}^{p-i}} M^n_{u-i,d} \xleftarrow{\dot{\partial}^{i}} M^n_{u,d} \xleftarrow{\dot{\partial}^{p-i}} M^n_{u+p-i,d} \xleftarrow{\dot{\partial}^{i}} M^n_{u+p,d} \leftarrow \ldots .$$

The way to think of this sequence is as the horizontal sequence passing through $M^n_{u,d}$ with the map $\dot{\partial}^i$ leaving $M^n_{u,d}$ and where the differentials alternate between $\dot{\partial}^i$ and $\dot{\partial}^{p-i}$. Since $\dot{\partial}^p = 0$ we see that $\dot{\mathcal{M}}^n_{u,d,i}$ is a homological sequence. Explicitly, $\dot{\partial}^i\dot{\partial}^{p-i} = \dot{\partial}^p = 0$ and $\dot{\partial}^{p-i}\dot{\partial}^i = \dot{\partial}^p = 0$.

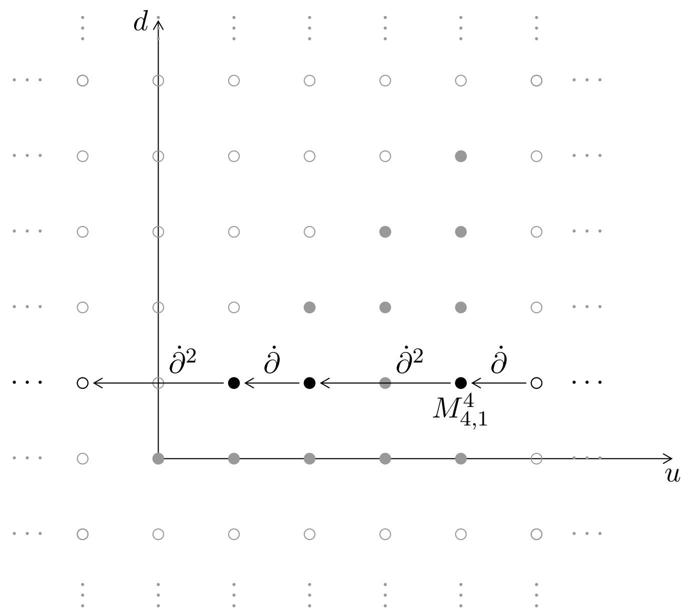


Figure 3.4.1: The homological sequence $\dot{\mathcal{M}}^4_{4,1,2}$ in the case $p = 3$.

Example 3.4.2. Let $p = 3$. Then $\dot{\mathcal{M}}^4_{4,1,2}$ is the horizontal sequence passing through $M^4_{4,1}$ with differential $\dot{\partial}^2$ leaving $M^4_{4,1}$ and differentials alternating between $\dot{\partial}^i = \dot{\partial}^2$ and $\dot{\partial}^{p-i} = \dot{\partial}$. This is highlighted in black in the homology grid in Figure 3.4.1. The rest of M^4 is faded to grey. Note the filled dots represent non-zero modules while the non-filled dots represent modules that are zero. For example $M^4_{2,4} = 0$ since $L^4_{2,4} = \emptyset$ as any subset of $[4\overline{4}]$ containing 4 doubles must have unsigned size at least $4 > 2$.

Since $\dot{\mathcal{M}}^n_{u,d,i}$ is homological, we can take the homology in the usual sense. We set $\dot{K}^n_{u,d,i} := \ker \dot{\partial}^i \cap M^n_{u,d}$ and $\dot{I}^n_{u,d,i} := \dot{\partial}^{p-i}\left(M^n_{u+p-i,d}\right)$.

Definition 3.4.3 (Singles homology modules)**.** The homology module

$$\dot{H}^n_{u,d,i} := \dot{K}^n_{u,d,i}/\dot{I}^n_{u,d,i}$$

is called the *singles homology module* for the index u, d, i.

The reason for the name is that the module arises from a sequence involving only the singles map. Since the singles map is linear and commutes with B_n, the singles homology modules are FB_n-modules. Our aim is to study these modules and the corresponding modules arising from the doubles map $\ddot{\partial}$, defined below. Before proceeding any further it is worth noting that there are two trivial cases (this exact argument applies to the doubles map case also). For $i = 0$ we have $\dot{\partial}^i = \mathrm{Id}$. This means $\dot{K}^n_{u,d,0} = 0$ and so $\dot{H}^n_{u,d,0} = 0$. For $i = p$ we have $\dot{\partial}^i = 0$ and $\dot{\partial}^{p-i} = \mathrm{Id}$. So $\dot{K}^n_{u,d,p} = M^n_{u,d} = \dot{I}^n_{u,d,p}$. Thus $\dot{H}^n_{u,d,p} = 0$. For this reason it is useful to think of the cases $i = 0$ and $i = p$ as boundary conditions where the homology modules are zero. Results and examples will often only be concerned with the case $0 < i < p$ for this reason. Let us compute an example.

Example 3.4.4. Let $(n, u, d) = (1, 1, 0)$. We wish to calculate the homology at the middle position of the sequence

$$M^1_{1-i,0} \xleftarrow{\dot{\partial}^i} M^1_{1,0} \xleftarrow{\dot{\partial}^{p-i}} M^1_{1+p-i,0}.$$

If $i > 1$ then $\dot{\partial}^i$ maps everything in $M^1_{1,0}$ to zero since $M^1_{1-i,0} = 0$. So $\dot{K}^1_{1,0,i} = M^1_{1,0}$ for $i > 1$. If $i = 1$ then $M^1_{1-i,0} = M^1_{0,0} = F\{\emptyset\}$. We have $M^1_{1,0} = F\{\{1\}, \{\overline{1}\}\}$. Let $x \in M^1_{1,0}$. Then $x = \lambda\{1\} + \mu\{\overline{1}\}$ with $\lambda, \mu \in F$. We have $\dot{\partial}^i(x) = \dot{\partial}(x) = \lambda\emptyset + \mu\emptyset = (\lambda + \mu)\emptyset$. So $x \in \dot{K}^1_{1,0,1}$ if and only if $\lambda = -\mu$. Thus $\dot{K}^1_{1,0,1} = \langle \{1\} - \{\overline{1}\} \rangle_F$. Finally $\dot{I}^1_{1,0,i} = \dot{\partial}^{p-i}\left(M^1_{1+p-i,0}\right) = \dot{\partial}^{p-i}(0) = 0$ since $1 + p - i > 1 = n$. To summarize, we have

$$\dot{H}^1_{1,0,i} \cong \dot{K}^1_{1,0,i} = \begin{cases} \left\langle \{1\} - \{\overline{1}\} \right\rangle_F & \text{if } i = 1 \text{ (1-dimensional)}, \\ F\{\{1\}, \{\overline{1}\}\} & \text{if } 2 \le i \le p-1 \text{ (2-dimensional)}. \end{cases}$$

Appendix A contains similar descriptions of all homology modules for $n \le 2$.

Now we set up the notation for the columns. Define $\ddot{\mathcal{M}}^n_{u,d,i}$ to be the sequence

$$
\begin{tikzcd}[row sep=large]
 & \vdots \arrow[d] \\
 & M^n_{u,d+p-i} \arrow[d, "\ddot{\partial}^{p-i}"] \\
\ddot{\mathcal{M}}^n_{u,d,i} : & M^n_{u,d} \arrow[d, "\ddot{\partial}^{i}"] \\
 & M^n_{u,d-i} \arrow[d] \\
 & \vdots
\end{tikzcd}
$$

where the differentials alternate between $\ddot{\partial}^i$ and $\ddot{\partial}^{p-i}$ in both directions. This sequence is also homological for the same reasons as above.

Example 3.4.5. Let $p = 3$. Then $\ddot{\mathcal{M}}^4_{4,1,2}$ is shown in Figure 3.4.6.

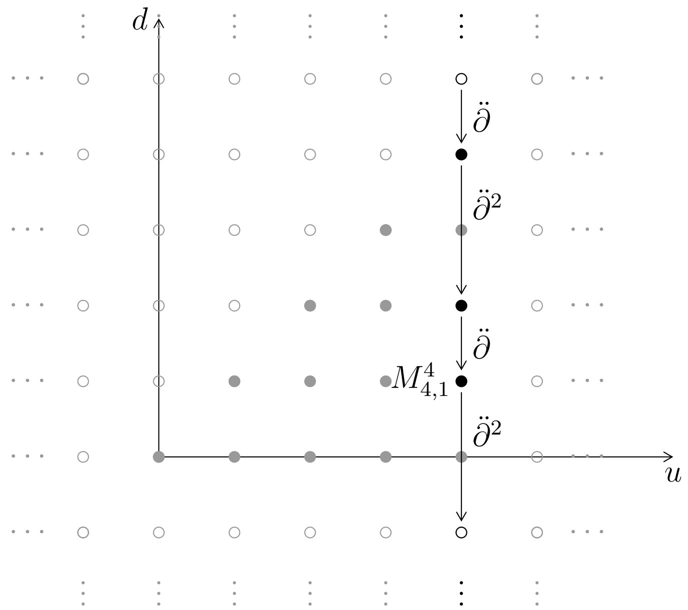


Figure 3.4.6: The 2-complex $\ddot{\mathcal{M}}^4_{4,1,2}$ in the case $p = 3$.

Definition 3.4.7 (Doubles homology modules)**.** As in the singles case, we define $\ddot{K}^n_{u,d,i} := \ker \ddot{\partial}^i \cap M^n_{u,d}$ and $\ddot{I}^n_{u,d,i} := \ddot{\partial}^{p-i}\left(M^n_{u,d+p-i}\right)$. The homology module

$$\ddot{H}^n_{u,d,i} := \ddot{K}^n_{u,d,i}/\ddot{I}^n_{u,d,i}$$

is called the *doubles homology module* for the index u, d, i.

The reason for the name is that the module arises from a sequence involving only the doubles map $\ddot{\partial}$. The doubles homology modules are FB_n-modules for the same reasons given above for the singles homology modules.

3.5 Basic properties of the incidence maps

In this section we derive some basic properties of the incidence maps $\dot{\partial}$ and $\ddot{\partial}$. Let $x \in L^n$. Recall from Section 3.1 that $x = \dot{x} \cdot \ddot{x}$ where we have used the notation $\dot{x} \cdot \ddot{x} := \dot{x} \cup \ddot{x}$ introduced in Section 3.2, that is x is the union of its singles part $\dot{x}$ and its doubles part $\ddot{x}$. This decomposition of x interacts with the singles and doubles maps in the following way, which may be viewed as an alternative definition of the maps.

Lemma 3.5.1. *For $x \in L^n$ we have*

$$\dot{\partial}(x) = \dot{\partial}(\dot{x}) \cdot \ddot{x}$$

and

$$\ddot{\partial}(x) = \dot{x} \cdot \ddot{\partial}(\ddot{x}).$$

Proof. The sets occurring with non-zero coefficient in $\dot{\partial}(x)$ are those y obtained by removing precisely one single from x. For such y we have $\ddot{y} = \ddot{x}$ so we may take out the factor $\ddot{x}$. The proof for the doubles map $\ddot{\partial}$ is similar. □

Lemma 3.5.1 is perhaps most clearly expressed by example. Take $x = \{123\bar{3}4\bar{4}\}$. Then $\dot{x} = \{12\}$ and $\ddot{x} = \{3\bar{3}4\bar{4}\}$. We have

$$\begin{aligned} \dot{\partial}(x) &= \dot{\partial}(\{123\bar{3}4\bar{4}\}) = \{23\bar{3}4\bar{4}\} + \{13\bar{3}4\bar{4}\} \\ &= (\{1\} + \{2\}) \cdot \{3\bar{3}4\bar{4}\} = \dot{\partial}(\{12\}) \cdot \{3\bar{3}4\bar{4}\} \\ &= \dot{\partial}(\dot{x}) \cdot \ddot{x} \end{aligned}$$

and
$$\begin{aligned}\ddot{\partial}(x) &= \ddot{\partial}(\{123\overline{3}4\overline{4}\}) = \{12\overline{3}4\overline{4}\} + \{1234\overline{4}\} + \{123\overline{3}\overline{4}\} + \{123\overline{3}4\} \\ &= \{12\} \cdot (\{\overline{3}4\overline{4}\} + \{34\overline{4}\} + \{3\overline{3}\overline{4}\} + \{3\overline{3}4\}) = \{12\} \cdot \ddot{\partial}(\{3\overline{3}4\overline{4}\}) \\ &= \dot{x} \cdot \ddot{\partial}(\ddot{x}).\end{aligned}$$

Lemma 3.5.2. *Let $\partial \in \{\dot{\partial}, \ddot{\partial}\}$. If $x, y \in L^n$ with $\{\alpha\overline{\alpha}\} \cap y = \emptyset$ for all $\alpha \in x$ then*
$$\partial(x \cdot y) = \partial(x) \cdot y + x \cdot \partial(y).$$

Proof. Let $\partial = \dot{\partial}$. Let $x, y \in L^n$. Suppose $x \cdot y \in L^n_{u,d}$. Then we have
$$\partial(x \cdot y) = \sum z$$
where z runs over the set $\{\, z \in L^n_{u-1,d} : z \subseteq x \cdot y \,\}$. Each such z is obtained by removing a single of $x \cdot y$. But the condition $\{\alpha\overline{\alpha}\} \cap y = \emptyset$ for all $\alpha \in x$ means that the singles part of $x \cdot y$ is the disjoint union of the singles parts of x and y and the doubles part of $x \cdot y$ is the disjoint union of the doubles parts of x and y. Hence
$$\partial(x \cdot y) = \partial(x) \cdot y + x \cdot \partial(y)$$
where the term $\partial(x) \cdot y$ is the sum of those z obtained from $x \cdot y$ by removing a single of x and $x \cdot \partial(y)$ is the sum of those z obtained from $x \cdot y$ by removing a single of y. The proof for $\partial = \ddot{\partial}$ is similar. □

3.6 Identifying induced modules

The aim of this section is to prove a result which identifies many of our homology modules roughly as induced modules from homology modules for subgroups B_m with $m < n$. The precise statement is Theorem 3.6.1. In particular, the only singles homology modules not of this form are the singles homology modules with $d = 0$, that is those arising from the bottom row of M^n, see Figure 3.3.9. The only doubles homology modules not of this form are those with $u = n$, that is those arising from the rightmost column of M^n. The feel is similar to the idea of Harish-Chandra induction and cuspidal representations of linear groups.

Before stating the theorem, we need the notion of the inflation of a module, which we now define. Suppose we have two groups G and H. Then any FG-module M can be made into an $F[G \times H]$-module by setting

$$(g, h)m = gm$$

for all $g \in G$, all $h \in H$ and all $m \in M$, then extending linearly. We call the resulting $F[G \times H]$-module the *inflation of M to $G \times H$* and denote it by $\mathrm{Infl}_G^{G \times H}(M)$. We say that M has been inflated to $G \times H$ *by letting H act trivially*. Suppose N is another FG-module. Let $f : M \to N$ be an FG-map. Then f is also an $F[G \times H]$-map $\mathrm{Infl}_G^{G\times H}(M) \to \mathrm{Infl}_G^{G\times H}(N)$. Setting $\mathrm{Infl}_G^{G\times H}(f) = f$ makes $\mathrm{Infl}_G^{G\times H}$ a functor from the category of FG-modules to the category of $F[G \times H]$-modules. Note there are more general notions of inflation but this is the only one we will use.

For the B_n group we identify certain Young-type subgroups as follows. Let m be an integer with $0 \leq m \leq n$. Then the direct product $B_m \times B_{n-m}$ has a canonical embedding in B_n. We identify it as the setwise stabilizer in B_n of $\{1\overline{1} \ldots m\overline{m}\}$. This is the product in B_n of the subgroup which only moves the elements of $\{1\overline{1} \ldots m\overline{m}\}$ and the subgroup which only moves the elements of $\{(m+1)\overline{(m+1)} \ldots n\overline{n}\}$. Occasionally we will refer to similar more general combinations of subgroups. For example $S_3 \times B_4 \times B_2 \subseteq B_9$ is the product of the symmetric group on $\{123\}$ (embedded in B_3), the signed symmetric group on $\{4\overline{4}5\overline{5}6\overline{6}7\overline{7}\}$ and the signed symmetric group on $\{8\overline{8}9\overline{9}\}$.

This gives us a way of going from FB_m-modules up to FB_n-modules which produces smaller modules than we would get by inducing directly from B_m to B_n. Suppose M is an FB_m-module. Then we may inflate M to an $F[B_m \times B_{n-m}]$ module by letting B_{n-m} act trivially. Then by the identification of $B_m \times B_{n-m}$ as the specific subgroup of B_n described in the previous paragraph we may induce from $B_m \times B_{n-m}$ up to B_n. We will suppress the inflation stage from the notation by writing $\mathrm{Ind}_{B_m \times B_{n-m}}^{B_n}(M)$ for $\mathrm{Ind}_{B_m \times B_{n-m}}^{B_n} \mathrm{Infl}_{B_m}^{B_m \times B_{n-m}}(M)$.

Since inflation and the usual induction are functors, it makes sense to apply them to sequences of modules and maps. The following result makes use of this fact.

Theorem 3.6.1. *Let $0 \leq d \leq n$. There is an isomorphism of FB_n-sequences*

$$\left(M^n_{*,d}, \dot{\partial}\right) \cong \mathrm{Ind}_{B_{n-d} \times B_d}^{B_n} \left(M^{n-d}_{*,0}, \dot{\partial}\right).$$

For each $u \in \mathbb{Z}$ it gives an isomorphism $M^n_{u,d} \cong \mathrm{Ind}_{B_{n-d} \times B_d}^{B_n} \left(M^{n-d}_{u-d,0}\right)$.

Dually, let $0 \le u \le n$. There is an isomorphism of FB_n-sequences

$$\left(M^n_{u,*}, \ddot{\partial}\right) \cong \operatorname{Ind}^{B_n}_{B_u \times B_{n-u}} \left(M^u_{u,*}, \ddot{\partial}\right).$$

For each $d \in \mathbb{Z}$ it gives an isomorphism $M^n_{u,d} \cong \operatorname{Ind}^{B_n}_{B_u \times B_{n-u}} \left(M^u_{u,d}\right)$.

Notice we have used the word "dually". This result is the first indication of the existence of a certain duality between the singles and doubles maps. We will comment more on this later. For now we merely remark that for every result involving the singles map (or related objects) there will be a corresponding, similar-looking result for the doubles map.

Proof. We will prove the singles map case first. Let $u \in \mathbb{Z}$. First we show $M^n_{u,d} \cong \operatorname{Ind}^{B_n}_{B_{n-d} \times B_d} \left(M^{n-d}_{u-d,0}\right)$. Explicitly writing the inflation stage, we need to show

$$M^n_{u,d} \cong \operatorname{Ind}^{B_n}_{B_{n-d} \times B_d} \operatorname{Infl}^{B_{n-d} \times B_d}_{B_{n-d}} \left(M^{n-d}_{u-d,0}\right).$$

Fix $v = \{n-d+1\overline{n-d+1} \dots n\overline{n}\} \in L^n_{d,d}$. Consider the subspace $M^{n-d}_{u-d,0} \cdot v$ of $M^n_{u,d}$ with basis the set of $x \in L^n_{u,d}$ such that $\ddot{x} = v$. This is an $F[B_{n-d} \times B_d]$-submodule of $M^n_{u,d}$. Clearly it is isomorphic to the inflation of $M^{n-d}_{u-d,0}$ from B_{n-d} to $B_{n-d} \times B_d$. Recall in terms of tensor products

$$\operatorname{Ind}^{B_n}_{B_{n-d} \times B_d}(M^{n-d}_{u-d,0} \cdot v) := FB_n \otimes_{F[B_{n-d} \times B_d]} (M^{n-d}_{u-d,0} \cdot v). \tag{3.6.2}$$

This has a basis consisting of elements of the form $g \otimes x$ where g runs over a complete set of left coset representatives of $B_{n-d} \times B_d$ in B_n and x runs over those $x \in L^n_{u,d}$ with $\ddot{x} = v$. Observe that two elements $g, h \in B_n$ lie in the same coset of $B_{n-d} \times B_d$ if and only if $g(v) = h(v)$. Thus the cosets are parameterised by the elements of $L^n_{d,d}$: For each $y \in L^n_{d,d}$ fix a group element g_y with $g_y(v) = y$. Then $\{\, g_y : y \in L^n_{d,d} \,\}$ is a complete system of (left) coset representatives of $B_{n-d} \times B_d$ in B_n. Define a linear map $\varphi : M^n_{u,d} \to \operatorname{Ind}^{B_n}_{B_{n-d} \times B_d}(M^{n-d}_{u-d,0} \cdot v)$ as follows. For each $x \in L^n_{u,d}$ we have $\ddot{x} \in L^n_{d,d}$. Set

$$\varphi(x) = g_{\ddot{x}} \otimes g^{-1}_{\ddot{x}}(x) = g_{\ddot{x}} \otimes g^{-1}_{\ddot{x}}(\dot{x}) \cdot v.$$

For each element $x \in L^n_{u,d}$ we have $\ddot{x} \in L^n_{d,d}$ and $g^{-1}_{\ddot{x}}(\dot{x}) \in L^{n-d}_{u-d,0}$ with

$$x = \dot{x} \cdot \ddot{x} = g_{\ddot{x}}(g^{-1}_{\ddot{x}}(\dot{x})) \cdot g_{\ddot{x}}(v) = g_{\ddot{x}}(g^{-1}_{\ddot{x}}(\dot{x}) \cdot v) \in g_{\ddot{x}} \left(M^{n-d}_{u-d,0} \cdot v\right).$$

Thus as a vector space $M^n_{u,d}$ decomposes as

$$M^n_{u,d} = \bigoplus_{y \in L^n_{d,d}} g_y \left(M^{n-d}_{u-d,0} \cdot v\right).$$

This implies φ is an isomorphism, see Curtis and Reiner's book [9, Proposition 10.5].

It remains to show the differential of $(M^n_{*,d}, \dot\partial)$ coincides with that induced up from $(M^{n-d}_{*,0}, \dot\partial)$. This amounts to showing the following diagram commutes.

$$\begin{tikzcd} M^n_{u-1,d} \arrow[d, "\varphi"] & & M^n_{u,d} \arrow[ll, "\dot\partial"'] \arrow[d, "\varphi"] \\ \mathrm{Ind}^{B_n}_{B_{n-d}\times B_d}\left(M^{n-d}_{u-d-1,0}\cdot v\right) & & \mathrm{Ind}^{B_n}_{B_{n-d}\times B_d}\left(M^{n-d}_{u-d,0}\cdot v\right) \arrow[ll, "1\otimes\dot\partial"'] \end{tikzcd}$$

Let $x \in L^n_{u,d}$. We have $\varphi\dot\partial(x) = \varphi\left(\sum_y y\right) = \sum_y \varphi(y)$ where the sum runs over those $y \in L^n_{u-1,d}$ such that $y \subseteq x$. Note that for such y we have $\ddot y = \ddot x$ and $\dot y \subseteq \dot x$. Hence $\varphi(y) = g_{\ddot x} \otimes g_{\ddot x}^{-1}(\dot y)\cdot v$. So

$$\varphi\dot\partial(x) = \sum_y g_{\ddot x} \otimes g_{\ddot x}^{-1}(\dot y)\cdot v. \tag{3.6.3}$$

Going the other way around the diagram we have

$$\begin{aligned}(1\otimes\dot\partial)\varphi(x) &= (1\otimes\dot\partial)(g_{\ddot x}\otimes g_{\ddot x}^{-1}(\dot x)\cdot v \\ &= g_{\ddot x}\otimes\dot\partial(g_{\ddot x}^{-1}(\dot x)\cdot v) \\ &= g_{\ddot x}\otimes\dot\partial(g_{\ddot x}^{-1}(\dot x))\cdot v \\ &= g_{\ddot x}\otimes\sum_{\dot y} g_{\ddot x}^{-1}(\dot y)\cdot v\end{aligned}$$

where the sum runs over those $\dot y \subseteq \dot x$ with $|\dot y| = |\dot x| - 1$. Clearly this is the same as (3.6.3). This completes the proof for the singles map case.

The proof for the doubles map case is very similar, even more transparent perhaps. Define the isomorphism $\varphi : M^n_{u,d} \to \mathrm{Ind}^{B_n}_{B_u\times B_{n-u}}\left(M^u_{u,d}\right)$ as follows. The cosets of $B_u \times B_{n-u}$ in B_n are parameterised by $L^n_{u,u}$: Let $v = \{1\overline{1}\ldots u\overline{u}\} \in L^n_{u,u}$. For each $y \in L^n_{u,u}$ fix some $g_y \in B_n$ with $g_y(v) = y$. Then $\{\, g_y : y \in L^n_{u,u} \,\}$ is a complete set of left coset representatives of $B_u \times B_{n-u}$ in B_n. Let $x \in L^n_{u,d}$. Then $g^{-1}_{x\cdot\overline{x}}(x) \in L^u_{u,d}$. Define $\varphi(x) = g_{x\cdot\overline{x}} \otimes g^{-1}_{x\cdot\overline{x}}(x)$ and extend linearly.

Now we must show that the following diagram commutes.

$$\begin{tikzcd}
M^n_{u,d} \arrow[r, "\varphi"] \arrow[d, "\ddot{\partial}"] & \operatorname{Ind}^{B_n}_{B_u \times B_{n-u}} \left(M^u_{u,d}\right) \arrow[d, "1 \otimes \ddot{\partial}"] \\
M^n_{u,d-1} \arrow[r, "\varphi"] & \operatorname{Ind}^{B_n}_{B_u \times B_{n-u}} \left(M^u_{u,d-1}\right)
\end{tikzcd}$$

Let $x \in L^n_{u,d}$. Then $\varphi\ddot{\partial}(x) = \varphi\left(\sum_y y\right) = \sum_y \varphi(y)$ where y runs over the set $\{\, y \in L^n_{u,d-1} : y \subseteq x \,\}$. Now $\varphi(y) = g_{y\cdot\overline{y}} \otimes g^{-1}_{y\cdot\overline{y}}(y) = g_{x\cdot\overline{x}} \otimes g^{-1}_{x\cdot\overline{x}}(y)$ since $y \cdot \overline{y} = x \cdot \overline{x}$. So

$$\varphi\ddot{\partial}(x) = \sum_y g_{x\cdot\overline{x}} \otimes g^{-1}_{x\cdot\overline{x}}(y).$$

On the other hand

$$\begin{aligned}
(1 \otimes \ddot{\partial})\varphi(x) &= (1 \otimes \ddot{\partial})\left(g_{x\cdot\overline{x}} \otimes g^{-1}_{x\cdot\overline{x}}(x)\right) \\
&= g_{x\cdot\overline{x}} \otimes \ddot{\partial}\left(g^{-1}_{x\cdot\overline{x}}(x)\right) \\
&= g_{x\cdot\overline{x}} \otimes g^{-1}_{x\cdot\overline{x}}\ddot{\partial}(x) \text{ since } \ddot{\partial} \text{ commutes with } B_n \\
&= \sum_y g_{x\cdot\overline{x}} \otimes g^{-1}_{x\cdot\overline{x}}(y)
\end{aligned}$$

where the sum runs over the same y as above. Hence $\varphi\ddot{\partial}(x) = (1 \otimes \ddot{\partial})\varphi(x)$ as required. □

We also give a less direct proof that $M^n_{u,d} \cong \operatorname{Ind}^{B_n}_{B_{n-d}\times B_d}(M^{n-d}_{u-d,0})$.

Alternative proof that $M^n_{u,d} \cong \operatorname{Ind}^{B_n}_{B_{n-d}\times B_d}(M^{n-d}_{u-d,0})$. The idea of the proof is encapsulated by the following diagram.

$$\begin{tikzcd}
 & M^n_{u,d} \\
M^{n-d}_{u-d,0} \arrow[r, mapsto, "\operatorname{Infl}^{B_{n-d}\times B_d}"] & M^{n-d}_{u-d,0} \arrow[u, mapsto, "\operatorname{Ind}^{B_n}"] \\
1_{S_{u-d}\times B_{n-u}} \arrow[u, mapsto, "\operatorname{Ind}^{B_{n-d}}"] \arrow[r, mapsto, "\operatorname{Infl}^{S_{u-d}\times B_{n-u}\times B_d}"] & 1_{S_{u-d}\times B_{n-u}\times B_d} \arrow[u, mapsto, "\operatorname{Ind}^{B_{n-d}\times B_d}"] \arrow[uu, bend right, mapsto, "\operatorname{Ind}^{B_n}"']
\end{tikzcd}$$

Since $M^n_{u,d}$ is a transitive permutation module we know it is isomorphic to $\mathrm{Ind}_H^{B_n}(1_H)$ where H is the stabilizer in B_n of an element in $L^n_{u,d}$. Let

$$x = \{1 \dots u-d\} \cdot \{n-d+1\overline{n-d+1} \dots n\overline{n}\} \in L^n_{u,d}.$$

Then the stabilizer of x in B_n is $\mathrm{Stab}_{B_n}(x) = S_{u-d} \times B_{n-u} \times B_d$. Thus $M^n_{u,d} \cong \mathrm{Ind}^{B_n}_{S_{u-d}\times B_{n-u}\times B_d}(1_{S_{u-d}\times B_{n-u}\times B_d})$. Similarly $M^{n-d}_{u-d,0} \cong \mathrm{Ind}^{B_{n-d}}_{S_{u-d}\times B_{n-u}}(1_{S_{u-d}\times B_{n-u}})$. By transitivity of induction [9, Proposition 10.6(ii)], we have

$$M^n_{u,d} \cong \mathrm{Ind}^{B_n}_{B_{n-d}\times B_d} \mathrm{Ind}^{B_{n-d}\times B_d}_{S_{u-d}\times B_{n-u}\times B_d}(1_{S_{u-d}\times B_{n-u}\times B_d}).$$

We may rewrite the trivial module as the inflation of the trivial module for $S_{u-d}\times B_{n-u}$ to get

$$M^n_{u,d} \cong \mathrm{Ind}^{B_n}_{B_{n-d}\times B_d} \mathrm{Ind}^{B_{n-d}\times B_d}_{S_{u-d}\times B_{n-u}\times B_d} \mathrm{Infl}^{S_{u-d}\times B_{n-u}\times B_d}_{S_{u-d}\times B_{n-u}}(1_{S_{u-d}\times B_{n-u}}).$$

This inflation can be viewed as taking the outer tensor product with the trivial module for B_d. By the the standard fact that outer tensor products commute with induction [9, Lemma 10.17], we may swap the first induction and inflation stages to get

$$M^n_{u,d} \cong \mathrm{Ind}^{B_n}_{B_{n-d}\times B_d} \mathrm{Infl}^{B_{n-d}\times B_d}_{B_{n-d}} \mathrm{Ind}^{B_{n-d}}_{S_{u-d}\times B_{n-u}}(1_{S_{u-d}\times B_{n-u}}).$$

Now by our earlier remark that $M^{n-d}_{u-d,0} \cong \mathrm{Ind}^{B_{n-d}}_{S_{u-d}\times B_{n-u}}(1_{S_{u-d}\times B_{n-u}})$ we have finally

$$M^n_{u,d} \cong \mathrm{Ind}^{B_n}_{B_{n-d}\times B_d} \mathrm{Infl}^{B_{n-d}\times B_d}_{B_{n-d}}(M^{n-d}_{u-d,0}),$$

as required. □

The isomorphisms of Theorem 3.6.1, as they live in the world of p-complexes, are very strong and allow isomorphisms of objects derived from these p-complexes to be immediately read off. One such statement is Theorem 3.6.6 which describes how the homology modules arise as induced modules. The key machinery for this is the following.

Lemma 3.6.4 (Exactness of induction [9, Section 10, Exercise 20])**.** *Let $H \subseteq G$ be finite groups. Let A and B be FH-modules with $\alpha : A \to B$ an FH-map. Consider the induced map $1 \otimes \alpha : \mathrm{Ind}_H^G(A) \to \mathrm{Ind}_H^G(B)$. We have*

(a) $\ker(1\otimes\alpha) = \mathrm{Ind}_H^G(\ker\alpha)$ *and*
(b) $\mathrm{im}(1\otimes\alpha) = \mathrm{Ind}_H^G(\mathrm{im}\,\alpha)$.

Proof. Let $G = g_1H \cup \dots \cup g_sH$ be a decomposition of G into left cosets of H. Let $a_1, \dots, a_t$ be a basis of A. Then $\mathrm{Ind}_H^G(A)$ has basis $\{\, g_i \otimes a_j \;:\; 1 \le i \le s \text{ and } 1 \le j \le t \,\}$.

Let $x \in \ker(1 \otimes \alpha)$. Write $x = \sum_{i,j} \lambda_{ij}(g_i \otimes a_j)$ with $\lambda_{ij} \in F$. Then

$$\begin{aligned} 0 = (1 \otimes \alpha)(x) &= (1 \otimes \alpha)\left(\sum_{i,j} \lambda_{ij}(g_i \otimes a_j)\right) \\ &= \sum_i g_i \otimes \alpha\left(\sum_j \lambda_{ij} a_j\right). \end{aligned}$$

This implies $\alpha\left(\sum_j \lambda_{ij} a_j\right) = 0$ for each i. In other words $\sum_j \lambda_{ij} a_j \in \ker \alpha$. So $x \in \mathrm{Ind}_H^G(\ker \alpha)$. So $\ker(1 \otimes \alpha) \subseteq \mathrm{Ind}_H^G(\ker \alpha)$. On the other hand suppose $x \in \mathrm{Ind}_H^G(\ker \alpha)$. Then $x = \sum_i g_i \otimes z_i$ with $z_i \in \ker \alpha$. We have

$$(1 \otimes \alpha)(x) = (1 \otimes \alpha)\left(\sum_i g_i \otimes z_i\right) = \sum_i g_i \otimes \alpha(z_i) = \sum_i g_i \otimes 0 = 0.$$

So $\mathrm{Ind}_H^G(\ker \alpha) \subseteq \ker(1 \otimes \alpha)$. So $\mathrm{Ind}_H^G(\ker \alpha) = \ker(1 \otimes \alpha)$. This completes the proof of (a).

For (b) let $x \in \mathrm{im}(1 \otimes \alpha)$. Then $x = (1 \otimes \alpha)(y)$ for some $y \in \mathrm{Ind}_H^G(A)$. Write $y = \sum_{i,j} \lambda_{ij}(g_i \otimes a_j)$ with each λ_{ij} in F. Then

$$\begin{aligned} x = (1 \otimes \alpha)\left(\sum_{i,j} \lambda_{ij}(g_i \otimes a_j)\right) &= \sum_{i,j} \lambda_{ij} g_i \otimes \alpha(a_j) \\ &= \sum_i g_i \otimes \left(\sum_j \lambda_{ij} \alpha(a_j)\right) \\ &= \sum_i g_i \otimes \alpha\left(\sum_j \lambda_{ij} a_j\right) \in \mathrm{Ind}_H^G(\mathrm{im}\, \alpha). \end{aligned}$$

So $\mathrm{im}(1 \otimes \alpha) \subseteq \mathrm{Ind}_H^G(\mathrm{im}\, \alpha)$. Suppose on the other hand $x \in \mathrm{Ind}_H^G(\mathrm{im}\, \alpha)$. Then $x = \sum_i g_i \otimes x_i$ with $x_i \in \mathrm{im}\, \alpha$. Write $x_i = \alpha(y_i)$ for each i. Then $x = \sum_i g_i \otimes \alpha(y_i) = (1 \otimes \alpha)\left(\sum_i g_i \otimes y_i\right) \in \mathrm{im}(1 \otimes \alpha)$. So $\mathrm{Ind}_H^G(\mathrm{im}\, \alpha) \subseteq \mathrm{im}(1 \otimes \alpha)$. So (b) holds. $\square$

Note Lemma 3.6.4 implies that Ind_H^G is an *exact functor* which is by definition a functor which preserves exactness of sequences. We don't go into detail about exact functors merely we remark that Lemma 3.6.4 holds in general, replacing Ind_H^G with any exact functor between abelian categories and replacing equality with isomorphism in (a) and (b). The following corollary also holds in general for any exact functor.

Corollary 3.6.5. *Let $H \subseteq G$ be finite groups. Suppose we have a homological se-*

quence of FH-modules

$$\ldots \leftarrow A_0 \overset{\partial}{\leftarrow} A_1 \overset{\partial}{\leftarrow} A_2 \leftarrow \ldots.$$

Let $k \in \mathbb{Z}$. Using the standard homology notation we have

$$H_k\left(\mathrm{Ind}_H^G(A)\right) \cong \mathrm{Ind}_H^G\left(H_k(A)\right),$$

that is

$$\frac{\ker(1 \otimes \partial) \cap \mathrm{Ind}_H^G(A_k)}{(1 \otimes \partial)\left(\mathrm{Ind}_H^G(A_{k+1})\right)} \cong \mathrm{Ind}_H^G\left(\frac{\ker \partial \cap A_k}{\partial(A_{k+1})}\right).$$

Proof. Consider the short exact sequence

$$0 \to \partial(A_{k+1}) \xrightarrow{\mathrm{Id}} \ker \partial \cap A_k \xrightarrow{\pi} \frac{\ker \partial \cap A_k}{\partial(A_{k+1})} \to 0$$

where π is the natural projection. Apply Ind_H^G to get

$$0 \to \mathrm{Ind}_H^G\left(\partial(A_{k+1})\right) \xrightarrow{1\otimes\mathrm{Id}} \mathrm{Ind}_H^G\left(\ker \partial \cap A_k\right) \xrightarrow{1\otimes\pi} \mathrm{Ind}_H^G\left(\frac{\ker \partial \cap A_k}{\partial(A_{k+1})}\right) \to 0.$$

By Lemma 3.6.4 this sequence is also exact. This implies

$$\mathrm{Ind}_H^G\left(\frac{\ker \partial \cap A_k}{\partial(A_{k+1})}\right) \cong \frac{\mathrm{Ind}_H^G\left(\ker \partial \cap A_k\right)}{\mathrm{Ind}_H^G\left(\partial(A_{k+1})\right)}.$$

But by Lemma 3.6.4 we have

$$\mathrm{Ind}_H^G\left(\ker \partial \cap A_k\right) = \ker(1 \otimes \partial) \cap \mathrm{Ind}_H^G(A_k)$$

and

$$\mathrm{Ind}_H^G\left(\partial(A_{k+1})\right) = (1 \otimes \partial)\left(\mathrm{Ind}_H^G(A_{k+1})\right).$$

Hence

$$\mathrm{Ind}_H^G\left(\frac{\ker \partial \cap A_k}{\partial(A_{k+1})}\right) \cong \frac{\ker(1 \otimes \partial) \cap \mathrm{Ind}_H^G(A_k)}{(1 \otimes \partial)\left(\mathrm{Ind}_H^G(A_{k+1})\right)}$$

as required. □

Theorem 3.6.6. *We have*

$$\dot{H}^n_{u,d,i} \cong \mathrm{Ind}_{B_{n-d}\times B_d}^{B_n}\left(\dot{H}^{n-d}_{u-d,0,i}\right)$$

and

$$\ddot{H}^n_{u,d,i} \cong \mathrm{Ind}_{B_u\times B_{n-u}}^{B_n}\left(\ddot{H}^u_{u,d,i}\right).$$

Proof. We give a proof for the singles map case. A similar proof can be used for the doubles map case. By Theorem 3.6.1 we have an isomorphism of p-complexes of FB_n-modules

$$\left(M^n_{*,d}, \dot{\partial}\right) \cong \mathrm{Ind}_{B_{n-d}\times B_d}^{B_n}\left(M^{n-d}_{*,0}, \dot{\partial}\right).$$

For each $u \in \mathbb{Z}$ this isomorphism induces an isomorphism of homological sequences of

FB_n-modules
$$\dot{\mathcal{M}}^n_{u,d,i} \cong \mathrm{Ind}^{B_n}_{B_{n-d}\times B_d}\left(\dot{\mathcal{M}}^{n-d}_{u-d,0,i}\right).$$

By Corollary 3.6.5 we have an isomorphism in homology
$$\dot{H}^n_{u,d,i} \cong \mathrm{Ind}^{B_n}_{B_{n-d}\times B_d}\left(\dot{H}^{n-d}_{u-d,0,i}\right). \qquad \square$$

Theorem 3.6.6 should be viewed as a reduction result. It allows many problems about the singles (respectively doubles) homology modules to be reduced to the case $d = 0$ (respectively $u = n$). For example, the module $\dot{H}^n_{u,d,i}$ is non-zero if and only if the module $\dot{H}^{n-d}_{u-d,0,i}$ is non-zero. This fact is utilised in Section 5.2 to streamline the proof that certain combinatorial conditions determine whether these modules are non-zero. The reduction result illustrates the importance of the bottom row of the homology grid, 3.3.3. This should not be surprising as the sets $L^n_{u,0}$ involved in the bottom row correspond to simplices of the cross-polytope (see Example 3.1.2) which is a very important object for B_n. The reduction in the doubles map case is to the righthand column $u = n$. The sets occurring in $L^n_{n,d}$ are simply the complements in $[n\overline{n}]$ of the sets occurring in $L^n_{n-d,0}$. Thus they also correspond to the simplices of the cross-polytope.

Chapter 4

Branching rules

Suppose we have a recursively defined family of groups, for example the hyperoctahedral groups $B_0 \subseteq B_1 \subseteq B_2 \subseteq \ldots$. If a group in the family acts on some object then since the groups lower down in the family are subgroups they also act on the same object by restriction. Thus we can look at how our object "decomposes" into objects for a group lower down in the family. There are often ways to index objects so that these decompositions obey a combinatorial rule. Such a rule is known as a *branching rule*. Branching rules play an important role in the representation theory of groups with this recursive structure. They provide a means for proving results by induction on the position of your group in the family. They are especially good for extracting combinatorial information, for example dimension formulae. Perhaps the most well-known branching rule is that for the complex irreducible representations of the symmetric groups, as follows.

Theorem 4.0.1 ([26])**.** *Let n be a natural number. Let λ be a partition of n. Let S^λ denote the Specht module corresponding to λ. Then as $\mathbb{C}S_{n-1}$-modules we have*

$$S^\lambda \cong \bigoplus_\mu S^\mu$$

where the sum runs over the partitions μ of $n-1$ whose Ferrers diagram is obtained by removing a node from the Ferrers diagram of λ.

Proof. For a proof see Sagan's book [27, Theorem 2.8.3]. $\square$

The aim of this chapter is to obtain some branching rules for the homological sequences defined in Chapter 3 and their homology modules. In the next chapter we will use these rules to prove various results about the sequences and their homology modules.

4.1 A branching rule for the permutation modules

In this section we discuss a branching rule for the module $M^n_{u,d}$. We noted in Proposition 3.1.5 that $|L^n_{u,d}| = \binom{n}{u}\binom{u}{d}2^{u-d}$. Recall for integers $0 \leq k \leq n$ the binomial coefficients satisfy the recurrence

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}. \tag{4.1.1}$$

This recurrence can be obtained by partitioning the set of all k-subsets of $\{\alpha_1, \ldots, \alpha_n\}$ into two parts, for example those subsets that contain α_n and those subsets that do not. The term $\binom{n-1}{k-1}$ counts the subsets containing α_n and the term $\binom{n-1}{k}$ counts the subsets not containing α_n. Using (4.1.1) we may produce a similar recurrence for $|L^n_{u,d}|$ as follows.

Proposition 4.1.2. *For integers $1 \leq n$ and $0 \leq d \leq u \leq n$ we have*

$$\left|L^n_{u,d}\right| = \left|L^{n-1}_{u,d}\right| + 2\left|L^{n-1}_{u-1,d}\right| + \left|L^{n-1}_{u-1,d-1}\right|.$$

Proof. We have

$$\begin{aligned}
\left|L^n_{u,d}\right| &= \binom{n}{u}\binom{u}{d}2^{u-d} \\
&= \left(\binom{n-1}{u} + \binom{n-1}{u-1}\right)\left(\binom{u-1}{d} + \binom{u-1}{d-1}\right)2^{u-d} \\
&= \binom{n-1}{u}\binom{u}{d}2^{u-d} + 2\binom{n-1}{u-1}\binom{u-1}{d}2^{u-d-1} \\
&\quad + \binom{n-1}{u-1}\binom{u-1}{d-1}2^{u-d} \\
&= |L^{n-1}_{u,d}| + 2|L^{n-1}_{u-1,d}| + |L^{n-1}_{u-1,d-1}|.
\end{aligned}$$

□

This proof is not very enlightening. Alternatively, Proposition 4.1.2 can be obtained directly by partitioning $L^n_{u,d}$ into those sets intersecting $\{n\overline{n}\}$ in each of $\emptyset$,

$\{n\}$, $\{\overline{n}\}$ and $\{n\overline{n}\}$. We notice what we are really doing is partitioning $L^n_{u,d}$ into B_{n-1}-orbits. So we can reformulate Proposition 4.1.2 as a branching rule.

Theorem 4.1.3 (Branching rule for $M^n_{u,d}$)**.** *As FB_{n-1}-modules we have a decomposition*

$$M^n_{u,d} \cong M^{n-1}_{u,d} \oplus M^{n-1}_{u-1,d} \oplus M^{n-1}_{u-1,d} \oplus M^{n-1}_{u-1,d-1}.$$

Proof. As stated above, this amounts to partitioning the basis $L^n_{u,d}$ of $M^n_{u,d}$ into B_{n-1}-orbits. It is easy to see that elements x and y in $L^n_{u,d}$ lie in the same B_{n-1}-orbit if and only if

$$x \cap \{n\overline{n}\} = y \cap \{n\overline{n}\}.$$

Hence there are at most four B_{n-1}-orbits. These are given by the non-empty sets among the four sets

$$\begin{aligned} L^{n-1}_{u,d} &= \{\, x \in L^n_{u,d} \,:\, x \cap \{n\overline{n}\} = \emptyset \,\}, \\ L^{n-1}_{u-1,d} \cdot \{n\} &= \{\, x \in L^n_{u,d} \,:\, x \cap \{n\overline{n}\} = \{n\} \,\}, \\ L^{n-1}_{u-1,d} \cdot \{\overline{n}\} &= \{\, x \in L^n_{u,d} \,:\, x \cap \{n\overline{n}\} = \{\overline{n}\} \,\} \text{ and} \\ L^{n-1}_{u-1,d-1} \cdot \{n\overline{n}\} &= \{\, x \in L^n_{u,d} \,:\, x \cap \{n\overline{n}\} = \{n\overline{n}\} \,\}. \end{aligned}$$

□

Theorem 4.1.3 allows us to write any x in $M^n_{u,d}$ uniquely as

$$x = x_\emptyset + x_{\{n\}} \cdot \{n\} + x_{\{\overline{n}\}} \cdot \{\overline{n}\} + x_{\{n\overline{n}\}} \cdot \{n\overline{n}\}$$

with $(x_\emptyset, x_{\{n\}}, x_{\{\overline{n}\}}, x_{\{n\overline{n}\}}) \in M^{n-1}_{u,d} \oplus M^{n-1}_{u-1,d} \oplus M^{n-1}_{u-1,d} \oplus M^{n-1}_{u-1,d-1}$. This subscript notation will be standard.

4.2 Branching rules for chain complexes

None of the results of the previous section should be surprising. What is perhaps surprising is that the form of the branching rule, Theorem 4.1.3, continues through to branching rules for chain complexes. We will prove the precise statements in Theorem 4.2.3 and Theorem 4.2.4. First we will prove two lemmas about how the differentials interact with the branching rule.

Lemma 4.2.1. *Let $x \in M^n$. Then*

$$\dot{\partial}^i(x) = \dot{\partial}^i(x_\emptyset) + i\dot{\partial}^{i-1}(x_{\{n\}} + x_{\{\overline{n}\}}) + \dot{\partial}^i(x_{\{n\}}) \cdot \{n\} + \dot{\partial}^i(x_{\{\overline{n}\}}) \cdot \{\overline{n}\} + \dot{\partial}^i(x_{\{n\overline{n}\}}) \cdot \{n\overline{n}\}.$$

Lemma 4.2.2. *Let $x \in M^n$. Then*

$$\ddot{\partial}^i(x) = \ddot{\partial}^i(x_\emptyset) + \left(\ddot{\partial}^i(x_{\{n\}}) + i\ddot{\partial}^{i-1}(x_{\{n\overline{n}\}})\right) \cdot \{n\} + \left(\ddot{\partial}^i(x_{\{\overline{n}\}}) + i\ddot{\partial}^{i-1}(x_{\{n\overline{n}\}})\right) \cdot \{\overline{n}\} + \ddot{\partial}^i(x_{\{n\overline{n}\}}) \cdot \{n\overline{n}\}.$$

Proof of Lemmas 4.2.1 and 4.2.2. Let $\partial \in \{\dot{\partial}, \ddot{\partial}\}$. By linearity of the projections $x \mapsto x_\emptyset, x \mapsto x_{\{n\}}, x \mapsto x_{\{\overline{n}\}}, x \mapsto x_{\{n\overline{n}\}}$ and the map ∂, it suffices to consider the case x is an element of the basis $L^n_{u,d}$. Notice that the pairs $(x_{\{n\}}, \{n\}), (x_{\{\overline{n}\}}, \{\overline{n}\})$ and $(x_{\{n\overline{n}\}}, \{n\overline{n}\})$ each satisfy the condition on (x, y) in Lemma 3.5.2. Therefore we can expand $\partial^i(x \cdot y)$ using the standard binomial formula from calculus

$$\partial^i(x \cdot y) = \sum_{j=0}^{i} \binom{i}{j} \partial^{i-j}(x) \cdot \partial^j(y).$$

Now note that for $y \in \{\{n\}, \{\overline{n}\}, \{n\overline{n}\}\}$ we have $\partial^j(y) = 0$ for $j \geq 2$. This gives all of the expressions in the lemmas (noting that $\ddot{\partial}(\{n\overline{n}\}) = \{n\} + \{\overline{n}\}$). □

The branching rules for the chain complexes are as follows. Note that these are generalisations of the earlier branching rule Theorem 4.1.3.

Theorem 4.2.3 (Branching rule for the singles map chain complexes)**.** *Let $1 \leq n$ and $0 < i < p$. We have an isomorphism of chain complexes of FB_{n-1}-modules*

$$\dot{\mathcal{M}}^n_{u,d,i} \cong \dot{\mathcal{M}}^{n-1}_{u,d,i+1} \oplus \dot{\mathcal{M}}^{n-1}_{u-1,d,i} \oplus \dot{\mathcal{M}}^{n-1}_{u-1,d,i-1} \oplus \dot{\mathcal{M}}^{n-1}_{u-1,d-1,i}.$$

Theorem 4.2.4 (Branching rule for the doubles map chain complexes)**.** *Let $1 \leq n$ and $0 < i < p$. We have an isomorphism of chain complexes of FB_{n-1}-modules*

$$\ddot{\mathcal{M}}^n_{u,d,i} \cong \ddot{\mathcal{M}}^{n-1}_{u,d,i} \oplus \ddot{\mathcal{M}}^{n-1}_{u-1,d,i} \oplus \ddot{\mathcal{M}}^{n-1}_{u-1,d,i+1} \oplus \ddot{\mathcal{M}}^{n-1}_{u-1,d-1,i-1}.$$

As with Theorem 3.6.1, there is a striking similarity between Theorems 4.2.3 and 4.2.4 which further develops the picture of a certain duality between the singles and doubles maps.

Chain complex isomorphism is a very strong property, see Section 2.4. Thus Theorems 4.2.3 and 4.2.4 are very strong. A consequence of this strength is that they allow us to immediately deduce branching rules for objects derived from the chain complexes. For example, the following branching rules about kernels, images and homology modules follow immediately, see Section 2.4 for details.

Theorem 4.2.5. *Let f be the isomorphism in the proof of Theorem 4.2.3. Then f induces the following isomorphisms of FB_{n-1}-modules*

$$\begin{aligned}
\dot{K}^n_{u,d,i} &\cong \dot{K}^{n-1}_{u,d,i+1} \oplus \dot{K}^{n-1}_{u-1,d,i} \oplus \dot{K}^{n-1}_{u-1,d,i-1} \oplus \dot{K}^{n-1}_{u-1,d-1,i}, \\
\dot{I}^n_{u,d,i} &\cong \dot{I}^{n-1}_{u,d,i+1} \oplus \dot{I}^{n-1}_{u-1,d,i} \oplus \dot{I}^{n-1}_{u-1,d,i-1} \oplus \dot{I}^{n-1}_{u-1,d-1,i}, \text{ and} \\
\dot{H}^n_{u,d,i} &\cong \dot{H}^{n-1}_{u,d,i+1} \oplus \dot{H}^{n-1}_{u-1,d,i} \oplus \dot{H}^{n-1}_{u-1,d,i-1} \oplus \dot{H}^{n-1}_{u-1,d-1,i}.
\end{aligned}$$

Theorem 4.2.6. *Let f be the isomorphism in the proof of Theorem 4.2.4. Then f induces the following isomorphisms of FB_{n-1}-modules*

$$\begin{aligned}
\ddot{K}^n_{u,d,i} &\cong \ddot{K}^{n-1}_{u,d,i} \oplus \ddot{K}^{n-1}_{u-1,d,i} \oplus \ddot{K}^{n-1}_{u-1,d,i+1} \oplus \ddot{K}^{n-1}_{u-1,d-1,i-1}, \\
\ddot{I}^n_{u,d,i} &\cong \ddot{I}^{n-1}_{u,d,i} \oplus \ddot{I}^{n-1}_{u-1,d,i} \oplus \ddot{I}^{n-1}_{u-1,d,i+1} \oplus \ddot{I}^{n-1}_{u-1,d-1,i-1}, \text{ and} \\
\ddot{H}^n_{u,d,i} &\cong \ddot{H}^{n-1}_{u,d,i} \oplus \ddot{H}^{n-1}_{u-1,d,i} \oplus \ddot{H}^{n-1}_{u-1,d,i+1} \oplus \ddot{H}^{n-1}_{u-1,d-1,i-1}.
\end{aligned}$$

Before proceeding with the proof of Theorem 4.2.3, let us have a brief discussion as to why the branching rule looks as it does. Let $\partial = \dot{\partial}$ and let $\mathcal{M}^n_{u,d,i} = \dot{\mathcal{M}}^n_{u,d,i}$. Also, set $J = (u + p\mathbb{Z}) \cup (u - i + p\mathbb{Z})$ to be the set of integers congruent either to u (mod p) or to $u - i$ (mod p). Recall Theorem 4.1.3 told us that as FB_{n-1}-modules we have the rather natural decomposition

$$M^n_{u,d} = M^{n-1}_{u,d} \oplus M^{n-1}_{u-1,d} \cdot \{n\} \oplus M^{n-1}_{u-1,d} \cdot \{\overline{n}\} \oplus M^{n-1}_{u-1,d-1} \cdot \{n\overline{n}\}.$$

Hence each $x \in M^n_{u,d}$ can be written uniquely as

$$x = x_\emptyset + x_{\{n\}} \cdot \{n\} + x_{\{\overline{n}\}} \cdot \{\overline{n}\} + x_{\{n\overline{n}\}} \cdot \{n\overline{n}\}$$

with

$$(x_\emptyset, x_{\{n\}}, x_{\{\overline{n}\}}, x_{\{n\overline{n}\}}) \in M^{n-1}_{u,d} \oplus M^{n-1}_{u-1,d} \oplus M^{n-1}_{u-1,d} \oplus M^{n-1}_{u-1,d-1}.$$

Let $\varphi_{u,d}$ be the associated isomorphism defined for each $x \in M_{u,d}$ by

$$x \mapsto (x_\emptyset, x_{\{n\}}, x_{\{\overline{n}\}}, x_{\{n\overline{n}\}}).$$

We might expect that the sequence $(\varphi_{j,d})_{j \in J}$ of maps associated to the terms

$$\ldots \leftarrow M^n_{u-p,d} \leftarrow M^n_{u-i,d} \leftarrow M^n_{u,d} \leftarrow M^n_{u+p-i,d} \leftarrow M^n_{u+p,d} \leftarrow \ldots$$

of $\mathcal{M}^n_{u,d,i}$ would give an isomorphism of chain complexes. However, this is not the case since the diagram

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
M^n_{u+p-i,d} \arrow[r, "\varphi_{u+p-i,d}"] \arrow[d, "\partial^{p-i}"] & M^{n-1}_{u+p-i,d} \oplus M^{n-1}_{u+p-i-1,d} \oplus M^{n-1}_{u+p-i-1,d} \oplus M^{n-1}_{u+p-i-1,d-1} \arrow[d, "\partial^{p-i} \oplus \partial^{p-i} \oplus \partial^{p-i} \oplus \partial^{p-i}"] \\
M^n_{u,d} \arrow[r, "\varphi_{u,d}"] \arrow[d, "\partial^i"] & M^{n-1}_{u,d} \oplus M^{n-1}_{u-1,d} \oplus M^{n-1}_{u-1,d} \oplus M^{n-1}_{u-1,d-1} \arrow[d, "\partial^i \oplus \partial^i \oplus \partial^i \oplus \partial^i"] \\
M^n_{u-i,d} \arrow[r, "\varphi_{u-i,d}"] \arrow[d] & M^{n-1}_{u-i,d} \oplus M^{n-1}_{u-i-1,d} \oplus M^{n-1}_{u-i-1,d} \oplus M^{n-1}_{u-i-1,d-1} \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

does not commute (so the map is not a chain map, let alone an isomorphism of chain complexes). The idea is to deform this map slightly to obtain a chain map.

The problem is that for $x \in M^n_{u,d}$ there is no way to recover $\partial^i(x_\emptyset)$ (the first component of the value obtained by tracing x across and then down in the diagram) from $\partial^i(x)$ (the value obtained by tracing x down in the diagram). However, writing x' for $\partial^i(x)$, we have

$$\partial^{i+1}(x_\emptyset) = \partial(x'_\emptyset) - i(x'_{\{n\}} + x'_{\{\overline{n}\}}).$$

So although we cannot recover $\partial^i(x_\emptyset)$ from x', we can recover $\partial^{i+1}(x_\emptyset)$ from x'. This gives a clue as to why there is an $i+1$ appearing in the branching rule. Once we have an $i+1$ somewhere, it is then clear we must have an $i-1$ somewhere else to compensate. Furthermore, it is clear that there is no room for an $i+2$ or higher term (or an $i-2$ or lower term).

Proof of Theorem 4.2.3. Recall the local notation $\partial = \dot{\partial}$ and $\mathcal{M}^n_{u,d,i} = \dot{\mathcal{M}}^n_{u,d,i}$. Also, set $J = (u+p\mathbb{Z}) \cup (u-i+p\mathbb{Z})$ to be the set of integers congruent either to $u \pmod p$ or to $u-i \pmod p$. We construct an isomorphism of chain complexes of FB_{n-1}-modules

$$f : \mathcal{M}^n_{u,d,i} \to \mathcal{M}^{n-1}_{u,d,i+1} \oplus \mathcal{M}^{n-1}_{u-1,d,i} \oplus \mathcal{M}^{n-1}_{u-1,d,i-1} \oplus \mathcal{M}^{n-1}_{u-1,d-1,i}.$$

Such an isomorphism will be a sequence $(f_j)_{j\in J}$ of maps. We define such a sequence as follows. For $j \equiv u \pmod p$ we define

$$f_j : M^n_{j,d} \to M^{n-1}_{j,d} \oplus M^{n-1}_{j-1,d} \oplus M^{n-1}_{j-1,d} \oplus M^{n-1}_{j-1,d-1}$$

by

$$f_j(x) = (-i^{-1}x_\emptyset,\, x_{\{\overline{n}\}},\, x_{\{n\}} + x_{\{\overline{n}\}} + i^{-1}\partial(x_\emptyset),\, x_{\{n\overline{n}\}}). \tag{4.2.7}$$

For $j \equiv u - i \pmod p$ we define

$$f_j : M^n_{j,d} \to M^{n-1}_{j-1,d} \oplus M^{n-1}_{j-1,d} \oplus M^{n-1}_{j,d} \oplus M^{n-1}_{j-1,d-1}$$

by

$$f_j(x) = (x_{\{n\}} + x_{\{\overline{n}\}} - i^{-1}\partial(x_\emptyset),\, x_{\{\overline{n}\}},\, i^{-1}x_\emptyset,\, x_{\{n\overline{n}\}}). \tag{4.2.8}$$

Notice the symmetry between (4.2.7) and (4.2.8): Suppose we start with (4.2.7). If we interchange the first and third components of $f_j(x)$ and negate each of the two terms involving i^{-1} then we obtain the definition of $f_j(x)$ in (4.2.8). The same procedure maps (4.2.8) to (4.2.7).

We claim that the sequence of maps $f = (f_j)_{j\in J}$ is an isomorphism of chain complexes of FB_{n-1}-modules. To verify this we need to show two things:

(a) f is a chain map of complexes of FB_{n-1}-modules, that is
   (i) each f_j is an FB_{n-1}-map, and

(ii) the diagram

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
M^n_{u+p-i,d} \arrow[r, "f_{u+p-i}"] \arrow[d, "\partial^{p-i}"] & M^{n-1}_{u+p-i-1,d} \oplus M^{n-1}_{u+p-i-1,d} \oplus M^{n-1}_{u+p-i,d} \oplus M^{n-1}_{u+p-i-1,d-1} \arrow[d, "\partial^{p-i-1} \oplus \partial^{p-i} \oplus \partial^{p-i+1} \oplus \partial^{p-i}"] \\
M^n_{u,d} \arrow[r, "f_u"] \arrow[d, "\partial^i"] & M^{n-1}_{u,d} \oplus M^{n-1}_{u-1,d} \oplus M^{n-1}_{u-1,d} \oplus M^{n-1}_{u-1,d-1} \arrow[d, "\partial^{i+1} \oplus \partial^i \oplus \partial^{i-1} \oplus \partial^i"] \\
M^n_{u-i,d} \arrow[r, "f_{u-i}"] \arrow[d] & M^{n-1}_{u-i-1,d} \oplus M^{n-1}_{u-i-1,d} \oplus M^{n-1}_{u-i,d} \oplus M^{n-1}_{u-i-1,d-1} \arrow[d] \\
\vdots & \vdots
\end{tikzcd} \tag{4.2.9}$$

commutes;

(b) each f_j has an inverse.

Note that these two properties automatically imply that each f_j^{-1} is an FB_{n-1}-map and that $f^{-1} = (f_j^{-1})_{j \in J}$ is the inverse chain map of f.

First we prove (a). The maps $x \mapsto x_\emptyset, x \mapsto x_{\{n\}}, x \mapsto x_{\{\overline{n}\}}, x \mapsto x_{\{n\overline{n}\}}$ and $x \mapsto \partial(x)$ are clearly each FB_{n-1}-maps. Hence, we have that each f_j is an FB_{n-1}-map. To verify that the diagram (4.2.9) commutes it suffices to show that each square

$$\begin{tikzcd}
M^n_{j,d} \arrow[r, "f_j"] \arrow[d, "\partial^{i^*}"] & f_j(M^n_{j,d}) \arrow[d] \\
M^n_{j-i^*,d} \arrow[r, "f_{j-i^*}"] & f_{j-i^*}(M^n_{j-i^*,d})
\end{tikzcd} \tag{4.2.10}$$

in (4.2.9) commutes, where $i^* = i$ or $p - i$ dependent on whether $j \equiv u$ or $u - i$

(mod p) respectively. To see this, we use Lemma 4.2.1:

$$\partial^i(x) = \partial^i(x_\emptyset) + i\partial^{i-1}(x_{\{n\}} + x_{\{\overline{n}\}}) + \partial^i(x_{\{n\}}) \cdot \{n\} + \partial^i(x_{\{\overline{n}\}}) \cdot \{\overline{n}\} + \partial^i(x_{\{n\overline{n}\}}) \cdot \{n\overline{n}\}.$$

First consider the case $j \equiv u \pmod p$. Then the corresponding square is

$$\begin{tikzcd} M^n_{j,d} \arrow[r, "f_j"] \arrow[d, "\partial^i"] & M^{n-1}_{j,d} \oplus M^{n-1}_{j-1,d} \oplus M^{n-1}_{j-1,d} \oplus M^{n-1}_{j-1,d-1} \arrow[d, "\partial^{i+1} \oplus \partial^i \oplus \partial^{i-1} \oplus \partial^i"] \\ M^n_{j-i,d} \arrow[r, "f_{j-i}"] & M^{n-1}_{j-i-1,d} \oplus M^{n-1}_{j-i-1,d} \oplus M^{n-1}_{j-i,d} \oplus M^{n-1}_{j-i-1,d-1}. \end{tikzcd}$$

This commutes since for $x \in M^n_{j,d}$ we have

$$\begin{aligned}
&(\partial^{i+1} \oplus \partial^i \oplus \partial^{i-1} \oplus \partial^i) f_j(x) \\
&\quad = (\partial^{i+1} \oplus \partial^i \oplus \partial^{i-1} \oplus \partial^i)(-i^{-1}x_\emptyset, x_{\{\overline{n}\}}, x_{\{n\}} + x_{\{\overline{n}\}} + i^{-1}\partial(x_\emptyset), x_{\{n\overline{n}\}}) \\
&\quad = (-i^{-1}\partial^{i+1}(x_\emptyset), \partial^i(x_{\{\overline{n}\}}), \partial^{i-1}(x_{\{n\}} + x_{\{\overline{n}\}}) + i^{-1}\partial^i(x_\emptyset), \partial^i(x_{\{n\overline{n}\}})) \\
&\quad = (\partial^i(x_{\{n\}}) + \partial^i(x_{\{\overline{n}\}}) - i^{-1}\partial(\partial^i(x_\emptyset) + i\partial^{i-1}(x_{\{n\}} + x_{\{\overline{n}\}})), \partial^i(x_{\{\overline{n}\}}), \\
&\qquad\quad i^{-1}(\partial^i(x_\emptyset) + i\partial^{i-1}(x_{\{n\}} + x_{\{\overline{n}\}})), \partial^i(x_{\{n\overline{n}\}})) \\
&\quad = f_{j-i}(\partial^i(x_\emptyset) + i\partial^{i-1}(x_{\{n\}} + x_{\{\overline{n}\}}) + \partial^i(x_{\{n\}}) \cdot \{n\} + \partial^i(x_{\{\overline{n}\}}) \cdot \{\overline{n}\} \\
&\qquad\qquad + \partial^i(x_{\{n\overline{n}\}}) \cdot \{n\overline{n}\}) \\
&\quad = f_{j-i}\partial^i(x).
\end{aligned}$$

Now suppose $j \equiv u - i \pmod p$. The corresponding square is

$$\begin{tikzcd} M^n_{j,d} \arrow[r, "f_j"] \arrow[d, "\partial^{p-i}"] & M^{n-1}_{j-1,d} \oplus M^{n-1}_{j-1,d} \oplus M^{n-1}_{j,d} \oplus M^{n-1}_{j-1,d-1} \arrow[d, "\partial^{p-i-1} \oplus \partial^{p-i} \oplus \partial^{p-i+1} \oplus \partial^{p-i}"] \\ M^n_{j-p+i,d} \arrow[r, "f_{j-p+i}"] & M^{n-1}_{j-p+i,d} \oplus M^{n-1}_{j-p+i-1,d} \oplus M^{n-1}_{j-p+i-1,d} \oplus M^{n-1}_{j-p+i-1,d-1}. \end{tikzcd}$$

This commutes since for $x \in M^n_{j,d}$ we have

$$\begin{aligned}
&(\partial^{p-i-1} \oplus \partial^{p-i} \oplus \partial^{p-i+1} \oplus \partial^{p-i}) f_j(x) \\
&\quad = (\partial^{p-i-1} \oplus \partial^{p-i} \oplus \partial^{p-i+1} \oplus \partial^{p-i})(x_{\{n\}} + x_{\{\overline{n}\}} - i^{-1}\partial(x_\emptyset), x_{\{\overline{n}\}}, i^{-1}x_\emptyset, x_{\{n\overline{n}\}}) \\
&\quad = (\partial^{p-i-1}(x_{\{n\}} + x_{\{\overline{n}\}}) - i^{-1}\partial^{p-i}(x_\emptyset), \partial^{p-i}(x_{\{\overline{n}\}}), i^{-1}\partial^{p-i+1}(x_\emptyset), \partial^{p-i}(x_{\{n\overline{n}\}})) \\
&\quad = (-i^{-1}(\partial^{p-i}(x_\emptyset) - i\partial^{p-i-1}(x_{\{n\}} + x_{\{\overline{n}\}})), \partial^{p-i}(x_{\{\overline{n}\}}), \\
&\qquad \partial^{p-i}(x_{\{n\}}) + \partial^{p-i}(x_{\{\overline{n}\}}) + i^{-1}\partial(\partial^{p-i}(x_\emptyset) - i\partial^{p-i-1}(x_{\{n\}} + x_{\{\overline{n}\}})), \\
&\qquad \partial^{p-i}(x_{\{n\overline{n}\}})) \\
&\quad = f_{j-p+i}(\partial^{p-i}(x_\emptyset) + (p-i)\partial^{p-i-1}(x_{\{n\}} + x_{\{\overline{n}\}}) \\
&\qquad\qquad + \partial^{p-i}(x_{\{n\}}) \cdot \{n\} + \partial^{p-i}(x_{\{\overline{n}\}}) \cdot \{\overline{n}\} + \partial^{p-i}(x_{\{n\overline{n}\}}) \cdot \{n\overline{n}\}) \\
&\quad = f_{j-p+i}\partial^{p-i}(x).
\end{aligned}$$

So we have established (a), that f is a chain map.

It remains to show (b), that each f_j has an inverse. In the case $j \equiv u \pmod p$ we claim f_j^{-1} is given by

$$f_j^{-1}(x_1, x_2, x_3, x_4) = -ix_1 + \partial(x_1) \cdot \{n\} + x_2 \cdot (\{\overline{n}\} - \{n\}) + x_3 \cdot \{n\} + x_4 \cdot \{n\overline{n}\} \quad (4.2.11)$$

for all $(x_1, x_2, x_3, x_4) \in M^{n-1}_{j,d} \oplus M^{n-1}_{j-1,d} \oplus M^{n-1}_{j-1,d} \oplus M^{n-1}_{j-1,d-1}$. The proof of the claim is straightforward: For $x \in M^n_{j,d}$ we have

$$\begin{aligned}
f_j^{-1} f_j(x) &= -i(-i^{-1}x_\emptyset) + \partial(-i^{-1}x_\emptyset) \cdot \{n\} + x_{\{\overline{n}\}} \cdot (\{\overline{n}\} - \{n\}) \\
&\qquad + (x_{\{n\}} + x_{\{\overline{n}\}} + i^{-1}\partial(x_\emptyset)) \cdot \{n\} + x_{\{n\overline{n}\}} \cdot \{n\overline{n}\} \\
&= x_\emptyset + x_{\{n\}} \cdot \{n\} + x_{\{\overline{n}\}} \cdot \{\overline{n}\} + x_{\{\overline{n}\}} \cdot \{n\overline{n}\} \\
&= x.
\end{aligned}$$

This shows that we have defined a left inverse for f_j. Hence f_j is an embedding. But, by dimension counting, f_j must be an isomorphism. So what we have defined is also a right inverse. So f_j^{-1} is indeed defined by (4.2.11).

Now suppose $j \equiv u - i \pmod p$. Let $(x_1, x_2, x_3, x_4) \in M^{n-1}_{j-1,d} \oplus M^{n-1}_{j-1,d} \oplus M^{n-1}_{j,d} \oplus M^{n-1}_{j-1,d-1}$. We have

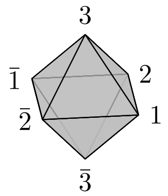
$$f_j^{-1}(x_1, x_2, x_3, x_4) = x_1 \cdot \{n\} + x_2 \cdot (\{\overline{n}\} - \{n\}) + ix_3 + \partial(x_3) \cdot \{n\} + x_4 \cdot \{n\overline{n}\}.$$

This follows from the symmetry discussed after defining f_j. Incase we don't trust the symmetry, let $x \in M^n_{j,d}$. We have

$$\begin{aligned} f_j^{-1} f_j(x) &= (x_{\{n\}} + x_{\{\overline{n}\}} - i^{-1}\partial(x_\emptyset)) \cdot \{n\} + x_{\{\overline{n}\}} \cdot (\{\overline{n}\} - \{n\}) + i(i^{-1}x_\emptyset) \\ &\qquad + \partial(i^{-1}x_\emptyset) \cdot \{n\} + x_{\{n\overline{n}\}} \cdot \{n\overline{n}\} \\ &= x_\emptyset + x_{\{n\}} \cdot \{n\} + x_{\{\overline{n}\}} \cdot \{\overline{n}\} + x_{\{n\overline{n}\}} \cdot \{n\overline{n}\} \\ &= x. \end{aligned}$$

By the same argument as in the case $j \equiv u \pmod p$ we are done. This completes the proof for $\partial = \dot{\partial}$ and $\mathcal{M}^n_{u,d,i} = \dot{\mathcal{M}}^n_{u,d,i}$. □

Example 4.2.12. Let $p = 2$. Let $n = 3$ and $d = 0 \le u \le n$. In this case for $u \neq 0$ the sets in $L^3_{u,0}$ may be viewed as the $(u-1)$-simplices of the octahedron, namely the 2-simplices are the faces, the 1-simplices are the edges and the 0-simplices are the vertices. Let X be the octahedron



.

The homology $\dot{H}^3_{u,0,i}$ is zero unless $i = 1$. In characteristic 2 the map $\dot{\partial}$ is the same as the usual boundary map from simplicial homology with coefficients in F [25, Page 28]. Thus $\dot{H}^3_{u,0,1}$ is just the usual reduced homology of the octahedron with coefficients in F, that is $\dot{H}^3_{u,0,1} = \tilde{H}_{u-1}(X; F)$ using standard notation. Note that the reduced homology is the same as the non-reduced homology in all degrees except zero. The inclusion of the "-1-simplex" $\emptyset$ is what makes our homology the reduced homology rather than the non-reduced. Observe X is homeomorphic to a 2-sphere. Hence by a standard result [25, Theorem 31.8], we have

$$\tilde{H}_{u-1}(X; F) = \begin{cases} F & \text{if } u = 3 \\ 0 & \text{if } u \neq 3. \end{cases}$$

The branching rule provides an alternative proof of this fact. It gives

$$\begin{aligned} \dot{H}^3_{u,0,1} &\cong \dot{H}^2_{u,0,2} \oplus \dot{H}^2_{u-1,0,1} \oplus \dot{H}^2_{u-1,0,0} \oplus \dot{H}^2_{u-1,-1,1} \\ &= \dot{H}^2_{u-1,0,1} \cong \dot{H}^1_{u-2,0,1} \cong \dot{H}^0_{u-3,0,1}. \end{aligned}$$

Now clearly

$$\dot{H}^0_{u-3,0,1} = \begin{cases} F & \text{if } u = 3 \\ 0 & \text{if } u \neq 3. \end{cases}$$

We can go further. Let f be the isomorphism constructed in the proof of the branching rule, Theorem 4.2.3. We can use f to find a generator of $\dot{H}^3_{3,0,1}$. We know that $\dot{H}^0_{0,0,1} \cong \langle \emptyset \rangle_F$. Now we just apply f^{-1} three times to $\emptyset$ to get

$$\dot{H}^3_{3,0,1} \cong \big\langle (\{1\} - \{\overline{1}\}) \cdot (\{2\} - \{\overline{2}\}) \cdot (\{3\} - \{\overline{3}\}) \big\rangle_F.$$

In Subsection 5.5.1 we will use this method to find generators of $\dot{H}^n_{u,0,i}$ in general.

The proof of Theorem 4.2.4 (the branching rule for the doubles map) is similar to the singles map case.

Proof of Theorem 4.2.4. Let $J = (d + p\mathbb{Z}) \cup (d - i + p\mathbb{Z})$. We will define a chain map $f = (f_j)_{j \in J} : \ddot{\mathcal{M}}^n_{u,d,i} \to \ddot{\mathcal{M}}^{n-1}_{u,d,i} \oplus \ddot{\mathcal{M}}^{n-1}_{u-1,d,i} \oplus \ddot{\mathcal{M}}^{n-1}_{u-1,d,i+1} \oplus \ddot{\mathcal{M}}^{n-1}_{u-1,d-1,i-1}$ and its inverse. The proof that f is an isomorphism of chain complexes of FB_{n-1}-modules will be omitted as it is similar to the proof provided for the singles map case, Theorem 4.2.3.

For $j \equiv d \pmod p$ define $f_j : M^n_{u,j} \to M^{n-1}_{u,j} \oplus M^{n-1}_{u-1,j} \oplus M^{n-1}_{u-1,j} \oplus M^{n-1}_{u-1,j-1}$ by

$$f_j(x) = \Big(x_\emptyset, x_{\{n\}} - x_{\{\overline{n}\}}, -i^{-1} x_{\{\overline{n}\}}, i^{-1} \ddot{\partial}(x_{\{\overline{n}\}}) + x_{\{n\overline{n}\}} \Big)$$

for all $x \in M^n_{u,j}$. For $j \equiv d - i \pmod p$ define $f_j : M^n_{u,j} \to M^{n-1}_{u,j} \oplus M^{n-1}_{u-1,j} \oplus M^{n-1}_{u-1,j-1} \oplus M^{n-1}_{u-1,j}$ by

$$f_j(x) = \Big(x_\emptyset, x_{\{n\}} - x_{\{\overline{n}\}}, -i^{-1} \ddot{\partial}(x_{\{\overline{n}\}}) + x_{\{n\overline{n}\}}, i^{-1} x_{\{\overline{n}\}} \Big)$$

for all $x \in M^n_{u,j}$.

Now we define f^{-1}. For $j \equiv d \pmod p$ define $f_j^{-1} : M^{n-1}_{u,j} \oplus M^{n-1}_{u-1,j} \oplus M^{n-1}_{u-1,j} \oplus M^{n-1}_{u-1,j-1} \to M^n_{u,j}$ by

$$f_j^{-1}(x_1, x_2, x_3, x_4) = x_1 + x_2 \cdot \{n\} + \ddot{\partial}(x_3) \cdot \{n\overline{n}\} - i x_3 \cdot (\{n\} + \{\overline{n}\}) + x_4 \cdot \{n\overline{n}\}$$

for all $(x_1, x_2, x_3, x_4) \in M^{n-1}_{u,j} \oplus M^{n-1}_{u-1,j} \oplus M^{n-1}_{u-1,j} \oplus M^{n-1}_{u-1,j-1}$. For $j \equiv d - i \pmod p$ define $f_j^{-1} : M^{n-1}_{u,j} \oplus M^{n-1}_{u-1,j} \oplus M^{n-1}_{u-1,j-1} \oplus M^{n-1}_{u-1,j} \to M^n_{u,j}$ by

$$f_j^{-1}(x_1, x_2, x_3, x_4) = x_1 + x_2 \cdot \{n\} + x_3 \cdot \{n\overline{n}\} + \ddot{\partial}(x_4) \cdot \{n\overline{n}\} + i x_4 \cdot (\{n\} + \{\overline{n}\})$$

for all $(x_1, x_2, x_3, x_4) \in M^{n-1}_{u,j} \oplus M^{n-1}_{u-1,j} \oplus M^{n-1}_{u-1,j-1} \oplus M^{n-1}_{u-1,j}$. $\square$

Chapter 5

Consequences of the branching rules

In this chapter we use the branching rules of Chapter 4 to extract as much information as we can about the homology modules. This information includes combinatorial conditions for the modules to be non-zero, conditions for irreducibility, dimension formulae and construction of explicit generators. The dimension formulae allow us to compute the p-rank of certain incidence matrices. It is worth noting that up until now our results have only required that we be in positive characteristic. From now on we will see a separation of the even and odd characteristic cases. A complete account of the even characteristic case is possible and this is provided by Section 5.1. This really has a distinct flavour from the odd characteristic case which is much less transparent and is explored in the remaining sections.

5.1 Even characteristic

As we saw in Section 2.7, the representation theory of B_n in even characteristic is really distinct from that in odd characteristics. The former involves partitions, whereas the latter involves bipartitions. This is due to the fact that the base group 2^n has only one irreducible representation in even characteristic. In even characteristic, the irreducible representations of B_n resemble those of S_n but they are extended by letting the base group act trivially. It should therefore be no surprise that the behaviour of the homology modules in even characteristic is rather different from that in odd characteristics. Furthermore, in even characteristic our boundary map coincides with the usual boundary map of algebraic topology (see Example 4.2.12) so the even characteristic case resembles the more classical situation. All of this contributes to the fact that the even characteristic case is much more straightforward to study and describe. This section covers the even characteristic case. Our main interest is the odd characteristic case which is studied in the later sections.

Throughout this section we assume $p = 2$. The first thing to notice is that the only integer i satisfying $0 < i < p$ is $i = 1$. We recall some concepts from Section 2.7. Let $0 \leq d \leq n$ be integers. Let $\lambda = (d, n-d)$ be a composition of n into two parts. We denote by M^λ the permutation module over F obtained from the action of S_n on the cosets of the Young subgroup $S_d \times S_{n-d}$. We make M^λ into a B_n-module by putting the base group C_2^n in the kernel of the action. Note $M^{(d,n-d)} \cong M^{(n-d,d)}$. If we order the parts of λ in non-increasing order then λ becomes a partition of n and the resulting module can be thought of as the permutation module with basis the λ-tabloids.

The structure of $\dot{H}^n_{u,d,1}$ is described by the following theorem. In particular, it says that each $\dot{H}^n_{u,d,1}$ is a permutation module.

Theorem 5.1.1. *Let $p = 2$. Let $0 \leq d \leq u \leq n$ be integers. Then*

$$\dot{H}^n_{u,d,1} \cong \begin{cases} M^{(d,n-d)} & \text{if } u = n \\ 0 & \text{if } u \neq n. \end{cases}$$

In particular, $\dot{H}^n_{u,d,1} \cong \dot{H}^n_{u,n-d,1}$ and $\dot{H}^n_{n,d,1}$ is irreducible if and only if $d = 0$ or $d = n$, in which case it is the trivial module.

Note that the result $\dot{H}^n_{u,d,1} \cong \dot{H}^n_{u,n-d,1}$ is a kind of duality. This has a partial generalisation to the odd characteristic case, see Theorem 5.6.1(c) and (d). The duality can be explained as follows. First note that $\dot{H}^n_{n,d,1} \cong \dot{K}^n_{n,d,1}$ since $\dot{I}^n_{n,d,1} = 0$. It is not difficult to show directly that $\dot{K}^n_{n,d,1}$ has a basis consisting of the orbit sums of the base group on $L^n_{n,d}$. Two elements lie in the same orbit if and only if their doubles parts are equal. Thus as an S_n-module $\dot{K}^n_{n,d,1}$ is isomorphic to the permutation module on the d-subsets of $[n]$. It is well-known that the module for d-subsets is isomorphic to that for $(n-d)$-subsets. The base group acts trivially in both cases so we have our isomorphism of B_n-modules.

Note also that $\dot{H}^n_{u,0,1} = 0$ unless $u = n$ in which case it is the trivial module. This fact can be obtained alternatively by classical algebraic topology: The elements of $L^n_{u,0}$ correspond to the $(u-1)$-simplices of a simplicial complex associated to the boundary of the cross-polytope (see Example 3.1.2). Since we are in characteristic 2 the incidence map is the same as the usual boundary map (involving alternating sums) from algebraic topology. Then $\dot{H}^n_{u,0,1}$ is the reduced homology of the boundary of the cross-polytope with coefficients in F (see Example 4.2.12 for the case $n = 3$). Since the boundary of the cross-polytope is homeomorphic to a sphere we have that its reduced homology vanishes everywhere except at the top ($u = n$) where it is 1-dimensional.

We now prove Theorem 5.1.1. We split off two parts of the proof into lemmas.

Lemma 5.1.2. *Let $p = 2$. Let $0 \leq u \leq n$ be integers. Then $\dot{H}^n_{u,0,1} = 0$ unless $u = n$.*

Proof. Clearly $\dot{H}^0_{u,0,1} = 0$ unless $u = 0$. The branching rule Theorem 4.2.3 gives

$$\begin{aligned}\dot{H}^n_{u,0,1} &\cong \dot{H}^{n-1}_{u,0,2} \oplus \dot{H}^{n-1}_{u-1,0,1} \oplus \dot{H}^{n-1}_{u-1,0,0} \oplus \dot{H}^{n-1}_{u-1,-1,1} \\ &= \dot{H}^{n-1}_{u-1,0,1}\end{aligned}$$

since the other three terms vanish. By induction $\dot{H}^{n-1}_{u-1,0,1} = 0$ unless $u - 1 = n - 1$, that is $u = n$. □

Lemma 5.1.3. *Let $p = 2$. Let $0 \leq n$ be integers. Then $\dot{H}^n_{n,0,1}$ is the trivial FB_n-module.*

Proof. We have $\dot{H}^0_{0,0,1} \cong \langle \emptyset \rangle_F$. Tracing this back up through the isomorphism f in the proof of Theorem 4.2.3 we obtain (after n steps)

$$\dot{H}^n_{n,0,1} \cong \left\langle \prod_{k=1}^{n} (\{k\} - \{\bar{k}\}) \right\rangle_F .$$

Since we are in characteristic 2 this becomes

$$\dot{H}^n_{n,0,1} \cong \left\langle \prod_{k=1}^{n} (\{k\} + \{\bar{k}\}) \right\rangle_F .$$

Thus B_n acts trivially as required. □

Proof of Theorem 5.1.1. By Theorem 3.6.6 we have
$$\dot{H}^n_{u,d,1} \cong \mathrm{Ind}^{B_n}_{B_{n-d}\times B_d}\left(\dot{H}^{n-d}_{u-d,0,1}\right).$$
Note that for any two part composition (a,b) of n the module $M^{(a,b)}$ is the induced module
$$M^{(a,b)} \cong \mathrm{Ind}^{B_n}_{B_a\times B_b}(1_{B_a\times B_b}) \cong \mathrm{Ind}^{B_n}_{B_b\times B_a}(1_{B_b\times B_a}).$$
Thus it suffices to show that $\dot{H}^{n-d}_{u-d,0,1}$ is the trivial module for $u=n$ and zero otherwise. Lemma 5.1.2 says that $\dot{H}^{n-d}_{u-d,0,1}$ is zero unless $u-d=n-d$, that is $u=n$. Lemma 5.1.3 says that $\dot{H}^{n-d}_{n-d,0,1}$ is the trivial FB_{n-d}-module. □

The analogue of Theorem 5.1.1 for the doubles map is the following.

Theorem 5.1.4. *Let $p=2$. Let $0 \le d \le u \le n$ be integers. Then*
$$\ddot{H}^n_{u,d,1} \cong \begin{cases} M^{(u,n-u)} & \text{if } d=0 \\ 0 & \text{if } d\neq 0. \end{cases}$$
In particular, $\ddot{H}^n_{u,d,1} \cong \ddot{H}^n_{n-u,d,1}$ and $\ddot{H}^n_{u,0,1}$ is irreducible if and only if $u=0$ or $u=n$, in which case it is the trivial module.

As before, we break the proof into two lemmas.

Lemma 5.1.5. *Let $p=2$. Let $0 \le d \le n$ be integers. Then $\ddot{H}^n_{n,d,1}=0$ unless $d=0$.*

Proof. Clearly $\ddot{H}^0_{0,d,1}=0$ unless $d=0$. The branching rule, Theorem 4.2.6, gives
$$\begin{aligned}\ddot{H}^n_{n,d,1} &\cong \ddot{H}^{n-1}_{n,d,1} \oplus \ddot{H}^{n-1}_{n-1,d,1} \oplus \ddot{H}^{n-1}_{n-1,d,2} \oplus \ddot{H}^{n-1}_{n-1,d-1,0} \\ &= \ddot{H}^{n-1}_{n-1,d,1}\end{aligned}$$
since the other three terms are zero. By induction, $\ddot{H}^{n-1}_{n-1,d,1}=0$ unless $d=0$. □

Lemma 5.1.6. *Let $p=2$. Let $0 \le n$ be an integer. Then $\ddot{H}^n_{n,0,1}$ is the trivial FB_n-module.*

Proof. We have $\ddot{H}^0_{0,0,1} \cong \langle\emptyset\rangle_F$ which is the trivial module. By the branching rule Theorem 4.2.6 we have $\ddot{H}^n_{n,0,1} \cong \ddot{H}^{n-1}_{n-1,0,1}$. An inverse isomorphism is given by $[z] \mapsto [\{n\}\cdot z]$ for $z \in \ddot{K}^{n-1}_{n-1,0,1}$ where square brackets denote the coset in the homology module, see the proof of Theorem 4.2.4. So $\ddot{H}^n_{n,0,1} \cong \langle\{1\dots n\} + \ddot{I}^n_{n,0,1}\rangle_F$. This is the trivial module. To see this we first notice S_n acts trivially. Then since the module is 1-dimensional and we are in characteristic 2, the base group must act trivially also. □

Proof of Theorem 5.1.4. By Theorem 3.6.6 we have

$$\ddot{H}^n_{u,d,1} \cong \mathrm{Ind}^{B_n}_{B_u \times B_{n-u}} \left(\ddot{H}^u_{u,d,1} \right).$$

Theorem 5.3.6 says this is zero unless $d = 0$. Finally, Lemma 5.3.7 says $\ddot{H}^u_{u,0,1}$ is the trivial module. □

We end this section on the characteristic 2 case with a duality result.

Theorem 5.1.7 (Duality in even characteristic)**.** *Let $p = 2$. We have*

$$\dot{H}^n_{u,d,1} \cong \ddot{H}^n_{n-d,n-u,1}.$$

Proof. This is a one line proof using Theorem 5.1.1 and Theorem 5.1.4. □

Some additional remarks about this duality are in order as the proof is not particularly enlightening. Suppose we wish to find an explicit isomorphism $\dot{H}^n_{u,d,1} \to \ddot{H}^n_{n-d,n-u,1}$. By Theorem 3.6.6 it suffices to consider the case $d = 0$. In this case we are seeking an isomorphism $\dot{H}^n_{u,0,1} \to \ddot{H}^n_{n,n-u,1}$. By Lemma 5.1.2 we have $\dot{H}^n_{u,0,1} = 0$ unless $u = n$. Similarly, by Lemma 5.1.5 we have $\ddot{H}^n_{n,n-u,1} = 0$ unless $u = n$. So it suffices to find an isomorphism $\dot{H}^n_{n,0,1} \to \ddot{H}^n_{n,0,1}$. By the proof of Lemma 5.1.3 we have $\dot{H}^n_{n,0,1} \cong \langle (\{1\} + \{\overline{1}\}) \cdots (\{n\} + \{\overline{n}\}) \rangle_F \cong 1_{B_n}$. By the proof of Lemma 5.1.6 we have $\ddot{H}^n_{n,0,1} \cong \langle \{1 \ldots n\} + \ddot{I}^n_{n,0,1} \rangle_F \cong 1_{B_n}$. So of course we can just define an isomorphism by taking the class of $(\{1\} + \{\overline{1}\}) \cdots (\{n\} + \{\overline{n}\})$ in $\dot{H}^n_{n,0,1}$ to the class of $\{1 \ldots n\}$ in $\ddot{H}^n_{n,0,1}$. But suppose we want to define a map of kernels $\dot{K}^n_{n,0,1} \to \ddot{K}^n_{n,0,1}$ which commutes with B_n and induces isomorphism in homology. Suppose f is such a map. Let $z = (\{1\} + \{\overline{1}\}) \cdots (\{n\} + \{\overline{n}\})$ be the generator of $\dot{K}^n_{n,0,1}$. Then since z is fixed by B_n we must have that $f(z)$ is fixed by B_n also. Thus $f(z) = \lambda \sum_x x$ where the sum runs over all $x \in L^n_{n,0}$. For each $x \in L^n_{n,0}$ write $[x]$ for $x + \ddot{I}^n_{n,0,1}$. Then $[x] = [y]$ for all $x, y \in L^n_{n,0}$. Hence $[f(z)] = [0]$ since $|L^n_{n,0}| = 2^n = 0 \pmod 2$. This means f cannot possibly induce a surjective map on homology, a contradiction. So no such f exists. But all is not lost as we can still try to find a map going the other way, that is an FB_n-map $\ddot{K}^n_{n,0,1} \to \dot{K}^n_{n,0,1}$ which induces an isomorphism in homology. The map

g defined by $g([x]) = z + \dot{I}^n_{n,0,1}$ for all $x \in L^n_{n,0}$ is clearly such a map. This leads to a slightly stronger version of Theorem 5.1.7 as follows.

Theorem 5.1.8 (Stronger duality in even characteristic)**.** *Let $p = 2$. Then there exists $f \in \mathrm{Hom}_{FB_n}(\ddot{K}^n_{n-d,n-u,1}, \dot{K}^n_{u,d,1})$ such that f induces an isomorphism in homology $\ddot{H}^n_{n-d,n-u,1} \cong \dot{H}^n_{u,d,1}$.*

Before the proof we need a general lemma. Note, in the lemma we have opted to stick with the standard notation from Algebraic Topology. Unfortunately this includes the notation "B_k" which denotes the image of the appropriate differential. This should not be confused with our usual notation where it denotes the hyperoctahedral group.

Lemma 5.1.9. *Let $H \subseteq G$ be finite groups. Let K be a field. Let C and D be chain complexes of KH-modules. Fix $k \in \mathbb{Z}$. Suppose we have a KH-map $f : Z_k(C) \to Z_k(D)$ which restricts to a map of boundaries $B_k(C) \to B_k(D)$. Suppose the map f_* induced by f on homology is an isomorphism. Then $(1 \otimes f)_* : H_k(\mathrm{Ind}^G_H C) \to H_k(\mathrm{Ind}^G_H D)$ is also an isomorphism. In particular Ind^G_H preserves quasi-isomorphisms.*

Proof. Since f_* is an isomorphism $H_k(C) \to H_k(D)$ exactness of induction implies $1 \otimes f_* : \mathrm{Ind}^G_H H_k(C) \to \mathrm{Ind}^G_H H_k(D)$ is an isomorphism. By Corollary 3.6.5 we know that $H_k \mathrm{Ind}^G_H(C) \cong \mathrm{Ind}^G_H H_k(C)$ and $H_k \mathrm{Ind}^G_H(D) \cong \mathrm{Ind}^G_H H_k(D)$. An explicit isomorphism $H_k(\mathrm{Ind}^G_H(C)) \to \mathrm{Ind}^G_H(H_k(C))$ is given as follows. Let $g_1, \ldots, g_t$ be a complete set of coset representatives for H in G. Then each element of $\mathrm{Ind}^G_H(Z_k(C))$ can be written uniquely as $\sum_{i=1}^t g_i \otimes z_i$ with $z_i \in Z_k(C)$. The image of this element in $H_k(\mathrm{Ind}^G_H(C))$ is the coset $\sum_{i=1}^t g_i \otimes z_i + \mathrm{Ind}^G_H(B_k(C))$. The image in $\mathrm{Ind}^G_H(H_k(C))$ is $\sum_{i=1}^t g_i \otimes (z_i + B_k(C))$. The isomorphism is given simply by sending the first image to the second image, that is

$$\sum_{i=1}^t g_i \otimes z_i + \mathrm{Ind}^G_H(B_k(C)) \mapsto \sum_{i=1}^t g_i \otimes (z_i + B_k(C)).$$

Denote this isomorphism by φ_C. An explicit isomorphism between $H_k(\mathrm{Ind}^G_H(D))$ and $\mathrm{Ind}^G_H(H_k(D))$ is given similarly. Denote this by φ_D. Consider the square

$$\begin{tikzcd}
H_k \operatorname{Ind}_H^G(C) \arrow[rr, "\varphi_C"] \arrow[d, "(1\otimes f)_*"] & & \operatorname{Ind}_H^G H_k(C) \arrow[d, "1\otimes f_*"] \\
H_k \operatorname{Ind}_H^G(D) \arrow[rr, "\varphi_D"] & & \operatorname{Ind}_H^G H_k(D).
\end{tikzcd}$$

By the above comments we know φ_C, φ_D and $1 \otimes f_*$ are isomorphisms. Hence to show that $(1 \otimes f)_*$ is an isomorphism it suffices to show that the square commutes, that is that $(1 \otimes f_*)\varphi_C = \varphi_D(1 \otimes f)_*$. Let $\sum_i g_i \otimes z_i + \operatorname{Ind}_H^G B_k(C)$ be an arbitrary element of $H_k \operatorname{Ind}_H^G(C)$. Then

$$\begin{aligned}
(1 \otimes f_*)\varphi_C\left(\sum_i g_i \otimes z_i + \operatorname{Ind}_H^G B_k(C)\right) &= (1 \otimes f_*)\left(\sum_i g_i \otimes (z_i + B_k(C))\right) \\
&= \sum_i g_i \otimes (f(z_i) + B_k(D))
\end{aligned}$$

and

$$\begin{aligned}
\varphi_D(1 \otimes f)_*\left(\sum_i g_i \otimes z_i + \operatorname{Ind}_H^G B_k(C)\right) &= \varphi_D\left(\sum_i g_i \otimes f(z_i) + \operatorname{Ind}_H^G B_k(D)\right) \\
&= \sum_i g_i \otimes (f(z_i) + B_k(D)).
\end{aligned}$$

These two values agree so we are done. □

Proof of Theorem 5.1.8. By the comments before the statement of Theorem 5.1.8, the result holds for $d = 0$. For $d > 0$ it follows by Lemma 5.1.9 and the results of Section 3.6. □

5.2 Combinatorial conditions for non-zero homologies

From now on we assume $p > 2$. Our first application of the branching rule, Theorem 4.2.3, in this case is to find combinatorial conditions for the homology modules $\dot{H}^n_{u,d,i}$ to be non-zero. We saw in Theorem 5.1.1 that in the even characteristic case, the singles homology modules coming from a given row of the homology grid are all zero except for the rightmost one where $u = n$. Already, the odd characteristic case is more complicated and we will see that roughly speaking the leftmost half of the homology modules from a given row are zero while the rightmost half are non-zero. The precise statement is Theorem 5.2.1 which is the aim of this section. Other than the branching rule, no preliminaries are needed.

Note that there are two trivial necessary conditions for $\dot{H}^n_{u,d,i}$ to be non-zero. The first comes from the fact that $\dot{H}^n_{u,d,i}$ is a subquotient of $M^n_{u,d}$. This gives the condition $M^n_{u,d} \neq 0$. This is clearly equivalent to the condition $0 \leq d \leq u \leq n$. The other trivial condition is $0 < i < p$.

Theorem 5.2.1. *Let $p > 2$. Let $0 \leq d \leq u \leq n$ be integers. Let $0 < i < p$. The module $\dot{H}^n_{u,d,i}$ is non-zero if and only if $0 < 2u + p - i - n - d$.*

Proof. First we reduce to the case $d = 0$ by Theorem 3.6.6. This usage of an external result may be avoided but the resulting branching rule and case analysis are more complicated. Theorem 3.6.6 states that

$$\dot{H}^n_{u,d,i} \cong \operatorname{Ind}^{B_n}_{B_{n-d} \times B_d} \left(\dot{H}^{n-d}_{u-d,0,i} \right).$$

Thus $\dot{H}^n_{u,d,i} \neq 0$ if and only if $\dot{H}^{n-d}_{u-d,0,i} \neq 0$. Note that $0 \leq 0 \leq u-d \leq n-d$ so $\dot{H}^{n-d}_{u-d,0,i}$ satisfies the hypothesis. Thus if the result holds for $d = 0$ then $\dot{H}^{n-d}_{u-d,0,i} \neq 0$ if and only if $0 < 2(u-d) + p - i - (n-d) - 0 = 2u + p - i - n - d$. Thus $\dot{H}^n_{u,d,i} \neq 0$ if and only if $0 < 2u + p - i - n - d$. That is, the result for $d = 0$ implies the result for arbitrary d.

We now prove the result in the case $d = 0$. The proof is by induction on n via the branching rule. The cases $n = 0$ and $n = 1$ are easily verified by direct calculation or by consulting Appendix A. Suppose $n > 1$. By the branching rule, Theorem 4.2.3, we have

$$\begin{aligned} \dot{H}^n_{u,0,i} &\cong \dot{H}^{n-1}_{u,0,i+1} \oplus \dot{H}^{n-1}_{u-1,0,i} \oplus \dot{H}^{n-1}_{u-1,0,i-1} \oplus \dot{H}^{n-1}_{u-1,-1,i} \\ &= \dot{H}^{n-1}_{u,0,i+1} \oplus \dot{H}^{n-1}_{u-1,0,i} \oplus \dot{H}^{n-1}_{u-1,0,i-1} \end{aligned}$$

since the rightmost term is zero. Now we can determine conditions under which each of the three terms of the branching rule vanishes as follows. We have $\dot{H}^{n-1}_{u,0,i+1} \neq 0$ unless $u = n$ or $i = p-1$ or by induction $2u+p-(i+1)-(n-1) = 2u+p-i-n \leq 0$. We have $\dot{H}^{n-1}_{u-1,0,i} \neq 0$ unless $u = 0$ or by induction $2(u-1) + p - i - (n-1) \leq 0$, that is $2u + p - i - n \leq 1$. Finally, we have $\dot{H}^{n-1}_{u-1,0,i-1} \neq 0$ unless $u = 0$ or $i = 1$ or by induction $2(u-1) + p - (i-1) - (n-1) = 2u + p - i - n \leq 0$. So clearly if $2u + p - i - n \leq 0$ then all three terms of the branching rule vanish and $\dot{H}^{n}_{u,0,i} = 0$. Suppose $0 < 2u + p - i - n$ and for a contradiction $\dot{H}^{n}_{u,0,i} = 0$. Then $\dot{H}^{n-1}_{u-1,0,i} = 0$ implies $u = 0$ or $2u + p - i - n = 1$. Suppose $2u + p - i - n = 1$. Then $\dot{H}^{n-1}_{u,0,i+1} = 0$ implies $u = n$ or $i = p - 1$. If $u = n$ then $\dot{H}^{n-1}_{u-1,0,i-1} = 0$ implies $i = 1$ which implies $2u + p - i - n = n + p - 1 = 1$, a contradiction since $n > 1$ and $p > 2$. On the other hand if $i = p - 1$ then $\dot{H}^{n-1}_{u-1,0,i-1} = 0$ and $p > 2$ imply $u = 0$. But then $2u + p - i - n = p - (p-1) - n = 1 - n < 0$, another contradiction. The only remaining possibility is $1 < 2u + p - i - n$ and $u = 0$. But then $\dot{H}^{n-1}_{u,0,i+1} = 0$ and $n > 1$ implies $i = p - 1$. This implies $2u + p - i - n = p - (p-1) - n = 1 - n < 0$ since $n > 1$. This is also a contradiction. $\square$

We can interpret Theorem 5.2.1 in terms of the homology grid as follows. Suppose p and n are fixed. Fix $0 < i < p$. We can rearrange the condition in Theorem 5.2.1 as $\frac{n+d+i-p}{2} < u$. Thus the interpretation is that a module $\dot{H}^{n}_{u,d,i}$ with $0 \leq d \leq u \leq n$ is non-zero if and only if it comes from the right of the line $u = \frac{n+d+i-p}{2}$. Of course the modules coming from outside the triangular region $0 \leq d \leq u \leq n$ are all zero.

A result for the doubles map can be proved similarly.

Theorem 5.2.2. *Let $p > 2$. Let $0 \leq d \leq u \leq n$ be integers. Let $0 < i < p$. The module $\ddot{H}^{n}_{u,d,i}$ is non-zero if and only if $0 < u - 2d + i$.*

We will see later, by Theorem 5.3.18, that $\dim \ddot{H}^{n}_{u,d,i} = \dim \dot{H}^{n}_{n-d,n-u,p-i}$. The proof of Theorem 5.3.18 does not depend on Theorem 5.2.2. Therefore Theorem 5.2.2 becomes equivalent to Theorem 5.2.1.

5.3 Dimension formulae

For small n the dimensions of the homology modules are readily computed, see Appendix A for a list of all homology modules with $n \leq 2$ for example. Thus for any given n we can theoretically compute the dimension of the corresponding homology modules recursively directly from the branching rule, that is provided we have enough resources. As n grows large however this recursive method can quickly become infeasible in practice since to compute the dimension of a given homology module requires knowledge of the dimensions of all summands from the branching rule. A closed formula is therefore desirable. The main purpose of this section is to prove such a formula, Theorem 5.3.2. The proof will be via the branching rule. In Section 5.4 we will apply this formula to calculate the p-ranks of certain incidence matrices. We will also show that the homology modules arising from the doubles map are isomorphic to the cohomology modules arising from the singles map and vice versa. It is a general fact that the homology and cohomology modules have the same dimension, if finite. Therefore for the purposes of calculating dimensions, it suffices to consider the singles map case alone.

We begin by recalling the analogous results for the Boolean algebra case. This will be useful later. Let $H^n_{k,i}$ be the incidence homology module for the symmetric group defined in Section 2.8. Recall, by Theorem 2.8.10, the dimension of $H^n_{k,i}$ is given by the alternating sum

$$\dim H^n_{k,i} = \sum_{t\in\mathbb{Z}} \binom{n}{k-pt} - \binom{n}{k-i-pt}$$

provided $0 < i < p$ and $0 < 2k + p - i - n < p$. Note that this is a closed formula since the binomial terms $\binom{n}{k'}$ vanish unless $0 \leq k' \leq n$. This dimension formula was obtained by Bell, Jones and Siemons [3] by an application of the Hopf trace formula from algebraic topology, see [25, Theorem 22.1]. We recall the setup from Section 2.8. For k an integer let M_k be the permutation FS_n-module with basis the k-subsets of $[n]$ and the natural S_n-action. Let ∂ be the map $M_k \to M_{k-1}$ defined on bases by taking each k-subset to the sum of the $(k-1)$-subsets it contains. The key property that allows the dimension of the homology modules to be readily computed from the Hopf trace formula is that the sequences

$$\ldots \leftarrow M_{k-i} \xleftarrow{\partial^i} M_k \xleftarrow{\partial^{p-i}} M_{k+p-i} \leftarrow \ldots \tag{5.3.1}$$

are almost exact. In other words, at most one of the homology modules $H^n_{k,i}$ arising from a given sequence of the form (5.3.1) is non-zero. This follows directly from Theorem 2.8.1. The modules we are considering for the hyperoctahedral group do not share this almost exactness property, but it turns out that the branching rule is very well-behaved in its interactions with the branching rule for the symmetric group modules. This enables us to express the dimensions of our modules in terms of the dimensions of the symmetric group modules. The resulting closed formula for $\dim \dot{H}^n_{u,d,i}$ turns out to be an alternating sum of sizes of certain subcollections of the bases $L^n_{u',d}$ involved in the sequence $\dot{\mathcal{M}}^n_{u,d,i}$. In this sense it is very similar to the formula for the symmetric group case.

Before stating Theorem 5.3.2 we need to define some new notation. Suppose $0 \le d \le u \le n$ and $0 < i < p$ are fixed. Set $\ell = 2u - d + p - i - n$. For $u' \in \mathbb{Z}$ define

$$f_{u'} = \sum_{b=\ell+d-p+1}^{\ell+d-1} \binom{n}{u'} \binom{u'}{d} \binom{u'-d}{b-d}.$$

This number counts a certain subset of $L^n_{u',d}$ which we will describe shortly. The main result of this section is the following.

Theorem 5.3.2. *Let $2 < p$. Let $0 \le d \le u \le n$ and $0 < i < p$. Set $\ell = 2u - d + p - i - n$. Then*

$$\dim \dot{H}^n_{u,d,i} = \binom{n}{d} \sum_{m=0}^{\ell-1} \binom{n-d}{m} \dim H^{n-d-m}_{u-d-m,i} = \sum_{t \in \mathbb{Z}} f_{u-pt} - f_{u-i-pt}.$$

Recall that $H^n_{k,i}$ is the incidence homology module for the symmetric group. The infinite sum notation is just a convenience. The sum actually only involves a finite number of non-zero terms since $f_{u'} = 0$ unless $d \le u' \le n$.

It should be emphasised that the formula in Theorem 5.3.2 is an alternating sum

$$\cdots + f_{u-p} - f_{u-i} + f_u - f_{u+p-i} + f_{u+p} - \cdots .$$

This suggests that the terms may be the ranks of chain modules from a chain complex. This is indeed the case. Although this isn't needed for the proof of Theorem 5.3.2, it is important enough to warrant a brief diversion. Let $0 \le d \le u \le n$. An element of $L^n_{u,d}$ may contain anywhere between d and u barred elements or equivalently between 0 and $u - d$ barred singles. Up until now we have not cared about this number of barred elements. One reason for this is that we have been considering B_n-symmetry and the number of barred elements is not a property preserved by this symmetry. For the purposes of computing dimensions however the symmetry is not so important. Indeed, if we were to compute the dimension of a homology module recursively via the branching rule then by the n-th step we would have thrown away all symmetry and would be left only with a vector space decomposition of the FB_n-module. It turns out that the $f_{u'}$ in our dimension formula, Theorem 5.3.2, are sums of orbit lengths of S_n on $L^n_{u',d}$. This suggests we are actually only throwing away the base group symmetry. With this in mind we introduce notation for the orbits as follows. For b an integer define the subcollection $L^{n,b}_{u',d}$ of $L^n_{u',d}$ by

$$L^{n,b}_{u',d} := \left\{ x \in L^n_{u',d} \,:\, x \text{ has exactly } b \text{ barred elements} \right\}.$$

Then it is not difficult to see that the $L^{n,b}_{u',d}$, as b varies over $d \leq b \leq u'$, are precisely the orbits of S_n on $L^n_{u',d}$. It will be convenient to have a shorthand for certain unions of these orbits as follows. For ℓ any integer define

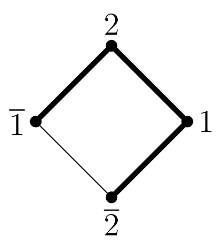
$$L^{n,[\ell]}_{u',d} := \bigcup_{b=\ell+d-p+1}^{\ell+d-1} L^{n,b}_{u',d}.$$

Thus $L^{n,[\ell]}_{u',d}$ contains precisely those sets in $L^n_{u',d}$ which have strictly between $\ell - p$ and ℓ barred singles. This is what $f_{u'}$ counts. More precisely suppose $0 \leq d \leq u \leq n$ and $0 < i < p$ are fixed. Set $\ell = 2u + p - i - n - d$. Then for any integer u' we have

$$f_{u'} = \left| L^{n,[\ell]}_{u',d} \right|.$$

Note that ℓ depends on u, not u'.

Example 5.3.3 (The cross-polytope)**.** Suppose $d = 0$. Then $L^n_{u,d}$ may be viewed as the set of $(u-1)$-simplices of the boundary of the n-cross-polytope, see Example 3.1.2 for the details. Suppose $n = 2$. The boundary of the 2-cross-polytope is the square. We identify the vertices with $\{1\}$, $\{\overline{1}\}$, $\{2\}$ and $\{\overline{2}\}$. The collection $L^{2,[2]}_{1,0}$ consists of all 1-subsets of $[2\overline{2}]$ which contain strictly between $2 - p$ and 2 barred singles. Since $p > 2$ this means either 0 or 1 barred singles. Well this is just all 1-subsets. Thus $L^{2,[2]}_{1,0} = L^2_{1,0}$ consists of all vertices of the square. The collection $L^{2,[2]}_{2,0}$ on the other hand consists of the edges which contain either 0 or 1 barred singles. The only disallowed edge is therefore $\{\overline{1}\overline{2}\}$. So $L^{2,[2]}_{1,0}$ together with $L^{2,[2]}_{2,0}$ forms a simplicial subcomplex of the square which is highlighted in bold in the following diagram.



Recall we wish to describe the $f_{u'}$, as u' varies, as ranks of chain modules in a chain complex. The most obvious candidate for such a chain complex would be that with chain modules $FL^{n,[\ell]}_{u',d}$ and differential obtained by restriction of the appropriate power of the singles map. However this does not work: Recall $L^{n,[\ell]}_{u',d}$ consists of the subsets in $L^n_{u',d}$ which contain strictly between $\ell - p$ and ℓ barred singles. This lower bound $\ell - p$ is a problem since the singles map can decrease the number of barred singles. Thus the singles map does not restrict to a map as we hoped. This problem is easily rectified though as follows. For each positive integer k define $\tilde{L}^k_{u'}$ to be the set of subsets in $L^n_{u',d}$ which contain at most k barred singles. Then

$$L^{n,[\ell]}_{u',d} = \tilde{L}^{\ell-1}_{u'} \setminus \tilde{L}^{\ell-p}_{u'}.$$

Define $\tilde{M}^k_{u'} = F\tilde{L}^k_{u'}$. Then $\tilde{M}^k_{u'}$ is an FS_n-module. Clearly removing elements from subsets cannot increase the number of barred singles. Therefore the singles map $\dot{\partial}$ restricts to a map $\tilde{M}^k_{u'} \to \tilde{M}^k_{u'-1}$. We thus obtain the p-complex

$$\tilde{M}^k : \quad \ldots \overset{\dot{\partial}}{\leftarrow} \tilde{M}^k_{u-2} \overset{\dot{\partial}}{\leftarrow} \tilde{M}^k_{u-1} \overset{\dot{\partial}}{\leftarrow} \tilde{M}^k_{u} \overset{\dot{\partial}}{\leftarrow} \tilde{M}^k_{u+1} \overset{\dot{\partial}}{\leftarrow} \tilde{M}^k_{u+2} \overset{\dot{\partial}}{\leftarrow} \ldots.$$

Since each $\tilde{L}^{\ell-p}_{u'}$ is a subset of $\tilde{L}^{\ell-1}_{u'}$ (preserved by S_n) we have that $\tilde{M}^{\ell-p}_{u'}$ is a submodule of $\tilde{M}^{\ell-1}_{u'}$. Since the differential of $\tilde{M}^{\ell-p}$ is just the restriction of that on $\tilde{M}^{\ell-1}$ we have that $\tilde{M}^{\ell-p}$ is a p-subcomplex of $\tilde{M}^{\ell-1}$. The quotient p-complex $\tilde{M}^{\ell-1}/\tilde{M}^{\ell-p}$ has the property that its u'-th chain module has dimension $|\tilde{L}^{\ell-1}_{u'}| - |\tilde{L}^{\ell-p}_{u'}| = |L^{n,[\ell]}_{u',d}| = f_{u'}$ as required.

Conjecture 5.3.4. *The quotients*

$$0 = \tilde{M}^{\ell-p}/\tilde{M}^{\ell-p} \leq \tilde{M}^{\ell-p+1}/\tilde{M}^{\ell-p} \leq \ldots \leq \tilde{M}^{\ell-1}/\tilde{M}^{\ell-p}$$

give rise to a filtration of $\dot{H}^n_{u,d,i}$ as an FS_n-module.

We now begin to tackle the proof of the dimension formula, Theorem 5.3.2. To begin with let us consider the case $d = 0$. The general case will follow by an application of Theorem 3.6.6. The result essentially comes from close examination of a particular graph, the "branching graph", which we now define. Consider the branching rule, Theorem 4.2.3

$$\dot{H}^n_{u,d,i} \cong \dot{H}^{n-1}_{u,d,i+1} \oplus \dot{H}^{n-1}_{u-1,d,i} \oplus \dot{H}^{n-1}_{u-1,d,i-1} \oplus \dot{H}^{n-1}_{u-1,d-1,i}.$$

Since $d = 0$ this reduces to

$$\dot{H}^n_{u,0,i} \cong \dot{H}^{n-1}_{u,0,i+1} \oplus \dot{H}^{n-1}_{u-1,0,i} \oplus \dot{H}^{n-1}_{u-1,0,i-1} \oplus 0, \tag{5.3.5}$$

by Theorem 5.2.1. To make this neater, let $H(n,u,i) = \dot{H}^n_{u,0,i}$. Then (5.3.5) becomes

$$H(n,u,i) \cong H(n-1,u,i+1) \oplus H(n-1,u-1,i) \oplus H(n-1,u-1,i-1).$$

We define three functions $b_j : \mathbb{N}^3 \to \mathbb{N}^3$ which capture the nature of the branching rule by

$$\begin{aligned} b_1(n,u,i) &= (n-1,u,i+1), \\ b_2(n,u,i) &= (n-1,u-1,i), \\ b_3(n,u,i) &= (n-1,u-1,i-1). \end{aligned}$$

The *branching graph* Γ is the directed graph with vertex set

$$V(\Gamma) = \mathbb{N}^3$$

and edge set

$$E(\Gamma) = \{\, (n,u,i) \to b_j(n,u,i) \,:\, (n,u,i) \in V(\Gamma) \text{ and } j \in \{1,2,3\} \,\}.$$

We call an edge $(n,u,i) \to b_j(n,u,i)$ a *type* j edge (or *branch*). Let $\Gamma_{\neq 0}$ be the induced subgraph of Γ on the vertices (n,u,i) such that $H(n,u,i) \neq 0$. We also let $\Gamma^0_{\neq 0}$ be the induced subgraph of $\Gamma_{\neq 0}$ on those vertices (n,u,i) with $n = 0$. Then $\Gamma^0_{\neq 0}$ has no edges and has $p-1$ vertices, namely $(0,0,i)$ for $0 < i < p$.

By Theorem 5.2.1, the vertices of $\Gamma_{\neq 0}$ are those (n,u,i) such that

(a) $0 \leq u \leq n$,
(b) $0 < i < p$, and
(c) $0 < 2u + p - i - n$.

We now define some functions $\mathbb{N}^3 \to \mathbb{N}$ which will help us to track these conditions as follows

$$\begin{aligned} c_1(n,u,i) &= u, \\ c_2(n,u,i) &= n-u, \\ c_3(n,u,i) &= i, \text{ and} \\ \ell(n,u,i) &= 2u+p-i-n. \end{aligned}$$

We call $\ell(n,u,i)$ the *level* of (n,u,i). Although we are only considering the $d = 0$ case at the moment, it is worth mentioning that we define the level generally for $d \geq 0$ by $\ell = 2u + p - i - n - d$.

Theorem 5.3.6. *Let $1 \leq n$. Then* $\dim H(n,u,i)$ *is the number of directed paths in $\Gamma_{\neq 0}$ from (n,u,i) to $\Gamma^0_{\neq 0}$, that is with last vertex in $\Gamma^0_{\neq 0}$.*

Proof. First suppose $H(n,u,i)=0$. Then $(n,u,i)\notin V(\Gamma_{\neq 0})$. So the number of paths in $\Gamma_{\neq 0}$ from (n,u,i) to $\Gamma^0_{\neq 0}$ is zero. So we're done.

Now suppose $H(n,u,i)\neq 0$. Proceed by induction on n. If $2\leq n$ then by the branching rule

$$\dim H(n,u,i)=\sum_{j=1}^{3}\dim H(b_j(n,u,i)).$$

Then, by induction, $\dim H(b_j(n,u,i))$ is the number of directed paths in $\Gamma_{\neq 0}$ from $b_j(n,u,i)$ to $\Gamma^0_{\neq 0}$. It remains to check the base case. Suppose $n=1$. Then by the branching rule

$$\dim H(1,u,i)=\sum_{j=1}^{3}\dim H(b_j(1,u,i)).$$

We have that $b_j(1,u,i)=(0,u',i')$ for some u' and i'. Now, by Appendix A, we have that $\dim H(0,u',i')\in\{0,1\}$. So $\dim H(1,u,i)$ is simply the number of $j\in\{1,2,3\}$ such that $H(b_j(1,u,i))\neq 0$. Well this is precisely the number of directed paths from $(1,u,i)$ to $\Gamma^0_{\neq 0}$ in $\Gamma_{\neq 0}$. □

Lemma 5.3.7. *Let $1\leq n$ and let $(n,u,i)\in V(\Gamma_{\neq 0})$. Then the number of directed paths in $\Gamma_{\neq 0}$ from (n,u,i) to $\Gamma^0_{\neq 0}$ involving only edges of types 1 and 3 is $\dim H^n_{u,i}$ where $H^n_{u,i}$ is the incidence homology module for the symmetric group from Section 2.8.*

Proof. First note that level is preserved across edges of types 1 and 3, that is $\ell(b_j(n,u,i))=\ell(n,u,i)$ for $j\in\{1,3\}$. Hence if $\ell(n,u,i)\geq p$ and we have a directed path in $\Gamma_{\neq 0}$ from (n,u,i) to $(0,0,i')$ involving only edges of types 1 and 3, then $\ell(0,0,i')\geq p$. But $\ell(0,0,i')=2\cdot 0+p-i'-0=p-i'$. Hence $i'\leq 0$ which implies $H(0,0,i')=0$. So $(0,0,i')\notin V(\Gamma_{\neq 0})$, contrary to our assumption. In other words, no such path can exist, so the number of such paths is zero. Now, by Theorem 2.8.1, we have $\dim H^n_{u,i}=0$. So in the case $\ell(n,u,i)\geq p$ the dimension matches the number of paths, as required.

On the other hand, if $\ell(n,u,i)\leq 0$ then $H(n,u,i)=0$ by Theorem 5.2.1. So the number of paths is zero again. By Theorem 2.8.1, $H^n_{u,i}$ is also zero.

It remains to consider the case $0<\ell(n,u,i)<p$. In this case we have $H(n,u,i)\neq 0$ if and only if $0\leq u\leq n$ and $0<i<p$, by Theorem 5.2.1. The same is true of $H^n_{u,i}$, by Theorem 2.8.1 (in the symmetric group language, this is the case (u,i) is a middle index). So if $(n,u,i)\notin V(\Gamma_{\neq 0})$ then there is nothing to prove. Suppose $(n,u,i)\in$

$V(\Gamma_{\neq 0})$. In this case we have $\ell(n',u',i') = \ell(n,u,i)$ for all vertices (n',u',i') occurring in paths from (n,u,i) involving only edges of types 1 and 3. Hence $H(n',u',i') \neq 0$ if and only if $0 \leq u' \leq n'$ and $0 < i' < p$. But this is exactly the condition for $H^{n'}_{u',i'}$ to be non-zero in Theorem 2.8.1. Also, $\dim H^0_{0,i'} = 1$ if $0 < i' < p$ and $\dim H^0_{0,i'} = 0$ otherwise. The same is true of $\dim H(0,0,i')$. Finally, the branching rule, Theorem 2.8.3, for $H^n_{u,i}$ is precisely our branching rule for $H(n,u,i)$ when ignoring type 2 branches. So we are done. $\square$

Lemma 5.3.8. *Let $P = (n_0,u_0,i_0) \to \ldots \to (n_m,u_m,i_m)$ be a directed path in Γ. Then*

(a) $c_1(n_0,u_0,i_0) \geq c_1(n_m,u_m,i_m)$,

(b) $c_2(n_0,u_0,i_0) \geq c_2(n_m,u_m,i_m)$,

(c) $c_3(b_j(n,u,i)) = c_3(n,u,i) + \epsilon_j$ *where* $\epsilon_1 = 1, \epsilon_2 = 0$ *and* $\epsilon_3 = -1$. *In particular, if $(n_0,u_0,i_0) \in V(\Gamma_{\neq 0})$ then the path P lies entirely in the induced subgraph Δ of Γ on the vertices (n,u,i) with $0 < c_3(n,u,i) < p$ if and only if the path $P_{\setminus 2}$ obtained from P by truncating all type 2 edges in P to their initial vertex while preserving the types of the other edges lies in Δ.*

(d) $\ell(b_j(n,u,i)) = \ell(n,u,i) + \delta_j$ *where* $\delta_1 = \delta_3 = 0$ *and* $\delta_2 = -1$. *In particular, if $(n_0,u_0,i_0) \in V(\Gamma_{\neq 0})$ then P lies in the induced subgraph of Γ on the vertices (n,u,i) with $\ell(n,u,i) > 0$ if and only if the number of type 2 edges in P is less than $\ell(n_0,u_0,i_0)$.*

Proof. Easy exercise. $\square$

Lemma 5.3.9. *Let $1 \leq n$ and $(n,u,i) \in V(\Gamma_{\neq 0})$. Then the number of directed paths P from (n,u,i) to $\Gamma^0_{\neq 0}$ with type 2 edges in precisely the positions $t_1 < \ldots < t_m$ amongst the edges of P depends only on the number m and is equal to $\dim H^{n-m}_{u-m,i}$, where $H^{n-m}_{u-m,i}$ is the symmetric group module from Section 2.8.*

Proof. First we count the number of paths P as in the hypothesis in which the type 2 edges are precisely the first m edges. Clearly, this is just the number of paths from $(n-m,u-m,i)$ to $\Gamma^0_{\neq 0}$ involving only edges of types 1 and 3. By Lemma 5.3.7, this number is $\dim H^{n-m}_{u-m,i}$.

It remains to show that the same is true no matter where the m type 2 edges occur in our paths. We will introduce some new notation to make the proof less cumbersome.

Suppose $(n,u,i) \in V(\Gamma_{\neq 0})$. For a tuple $(j_1,\ldots,j_t) \in \{1,2,3\}^t$ let $P(j_1,\ldots,j_t)$ denote the path

$$(n,u,i) \xrightarrow{b_{j_1}} b_{j_1}(n,u,i) \xrightarrow{b_{j_2}} b_{j_2}b_{j_1}(n,u,i) \to \ldots \xrightarrow{b_{j_t}} b_{j_t}\cdots b_{j_1}(n,u,i)$$

in Γ. For example, let $(n,u,i) = (3,3,2)$. Then $P(2,3,1)$ is the path

$$(3,3,2) \xrightarrow{b_2} (2,2,2) \xrightarrow{b_3} (1,1,1) \xrightarrow{b_1} (0,1,2).$$

Thus S_t acts on the paths $P(j_1,\ldots,j_t)$ by

$$gP(j_1,\ldots,j_t) = P(j_{g(1)},\ldots,j_{g(t)})$$

for all $g \in S_t$.

Suppose P is a path in $\Gamma_{\neq 0}$ from (n,u,i) to $(0,0,i_n)$ with type 2 branches in precisely the first m positions, that is $P = P(\underbrace{2,2,\ldots,2}_{m\text{ times}}, j_{m+1},\ldots,j_n)$ for some $j_{m+1},\ldots,j_n \in \{1,3\}$. Now apply a permutation in S_n which shuffles the type 2 edges around amongst the other edges whilst leaving the order of the other edges intact. We claim the resulting path Q also lies in $\Gamma_{\neq 0}$. Suppose not. Let (n',u',i') be the first vertex of Q not in $V(\Gamma_{\neq 0})$. Then at least one of the following must hold

(a) $c_1(n',u',i') < 0$,
(b) $c_2(n',u',i') < 0$,
(c) $c_3(n',u',i') \leq 0$,
(d) $c_3(n',u',i') \geq p$, or
(e) $\ell(n',u',i') \leq 0$.

Since $c_1(n,u,i) \geq 0$, Lemma 5.3.8(a) says $c_1(n',u',i') \geq 0$. Similarly, since $c_2(n,u,i) \geq 0$, Lemma 5.3.8(b) says that $c_2(n',u',i') \geq 0$. Lemma 5.3.8(c) implies that (c) and (d) do not hold. Finally Lemma 5.3.8(d) shows that (e) does not hold. □

We are now ready to prove Theorem 5.3.2 in the case $d = 0$. For convenience we state this special case here.

Theorem 5.3.10 (Theorem 5.3.2 in the case $d = 0$)**.** *Let $p > 2$. Let $0 \leq u \leq n$ and let $d = 0$. Suppose $0 < i < p$. Let $\ell = 2u + p - i - n$. Then we have*

$$\dim \dot{H}^n_{u,0,i} = \sum_{m=0}^{\ell-1} \binom{n}{m} \dim H^{n-m}_{u-m,i} = \sum_{t\in\mathbb{Z}} f_{u-pt} - f_{u-i-pt}.$$

Note in the case $d = 0$ we have
$$f_{u'} = \sum_{b=\ell-p+1}^{\ell-1} \binom{n}{u'}\binom{u'}{b}.$$

Proof. By Theorem 5.3.6 we must count the directed paths in $\Gamma_{\neq 0}$ from (n,u,i) to $\Gamma^0_{\neq 0}$. Let S be the set of directed paths in $\Gamma_{\neq 0}$ from (n,u,i) to $\Gamma^0_{\neq 0}$. Then S can be partitioned according to where in these paths the type 2 edges occur. By Lemma 5.3.8(d) such paths can have at most $\ell(n,u,i)-1 = \ell-1$ type 2 edges. Our starting partition is as follows
$$S = \bigcup_{m=0}^{\ell-1} \{\text{paths with precisely } m \text{ type 2 edges}\}.$$
We refine this partition to
$$S = \bigcup_{m=0}^{\ell-1} \bigcup_{\substack{T\subseteq[n]\\|T|=m}} \{\text{paths with type 2 edges in precisely the positions } T\}$$
By Lemma 5.3.9, the size of the set
$$\{\text{paths with type 2 edges in precisely the positions } T\}$$
depends only on $|T| = m$ and is equal to $\dim H^{n-m}_{u-m,i}$. So we get
$$\begin{aligned}|S| &= \sum_{m=0}^{\ell-1} \sum_{\substack{T\subseteq[n]\\|T|=m}} |\{\text{paths with type 2 edges in precisely the positions } T\}| \\ &= \sum_{m=0}^{\ell-1} \sum_{\substack{T\subseteq[n]\\|T|=m}} \dim H^{n-m}_{u-m,i} \\ &= \sum_{m=0}^{\ell-1} \binom{n}{m} \dim H^{n-m}_{u-m,i}.\end{aligned}$$
This completes the proof of the first equality.

It remains to show
$$\sum_{m=0}^{\ell-1} \binom{n}{m} \dim H^{n-m}_{u-m,i} = \sum_{t\in\mathbb{Z}} f_{u-pt} - f_{u-i-pt}. \tag{5.3.11}$$
We wish to expand $\dim H^{n-m}_{u-m,i}$ in the above. Observe $2(u-m)+p-i-(n-m) = \ell-m$. Thus if $\ell - m \geq p$ then $\dim H^{n-m}_{u-m,i} = 0$ by Theorem 2.8.1. So we only have to sum

over those m with $0 < \ell - m < p$. For such values Theorem 2.8.10 applies and gives

$$\dim H^{n-m}_{u-m,i} = \sum_{t\in\mathbb{Z}} \binom{n-m}{u-m-pt} - \binom{n-m}{u-m-i-pt}.$$

Thus the left hand side of (5.3.11) becomes

$$\sum_{m=\ell-p+1}^{\ell-1} \binom{n}{m} \sum_{t\in\mathbb{Z}} \binom{n-m}{u-m-pt} - \binom{n-m}{u-m-i-pt}.$$

Now observe that for any $u' \in \mathbb{Z}$ we have $\binom{n}{m}\binom{n-m}{u'-m} = \binom{n}{u'}\binom{u'}{m}$. Therefore

$$\sum_{m=\ell-p+1}^{\ell-1} \binom{n}{m}\binom{n-m}{u'-m} = f_{u'}. \qquad \square$$

We are now ready to drop the assumption $d = 0$. The full version of Theorem 5.3.2 is an easy corollary of the $d = 0$ case, Theorem 5.3.10. For the proof we need a notation for $f_{u'}$ with different "ambient parameters" to n, u, d. Suppose $0 \le d \le u \le n$ and $0 < i < p$ are fixed. Set $\ell = 2u - d + p - i - n$. Define

$$f_{u',0} = \left| L^{n-d,[\ell]}_{u',0} \right| = \sum_{b=\ell-p+1}^{\ell-1} \binom{n-d}{u'}\binom{u'}{b}.$$

Note that $\ell = 2(u-d) - 0 + p - i - (n-d)$. So $f_{u',0}$ is just $f_{u'}$ but with ambient parameters $n-d, u-d, 0$ rather than n, u, d.

Proof of Theorem 5.3.2. Recall we have $0 \le d \le u \le n$ and $0 < i < p$ with $\ell = 2u - d + p - i - n$. The first equality

$$\dim \dot{H}^n_{u,d,i} = \binom{n}{d} \sum_{m=0}^{\ell-1} \binom{n-d}{m} \dim H^{n-d-m}_{u-d-m,i}$$

follows from Theorem 5.3.10 as follows. By Theorem 3.6.6, we have

$$\begin{aligned}\dim \dot{H}^n_{u,d,i} &= [B_n : B_{n-d} \times B_d] \dim \dot{H}^{n-d}_{u-d,0,i} \\ &= \binom{n}{d} \dim \dot{H}^{n-d}_{u-d,0,i}.\end{aligned}$$

By Theorem 5.3.10, we have

$$\dim \dot{H}^{n-d}_{u-d,0,i} = \sum_{m=0}^{2(u-d)+p-i-(n-d)-1} \binom{n-d}{m} \dim H^{n-d-m}_{u-d-m,i}$$
$$= \sum_{m=0}^{\ell-1} \binom{n-d}{m} \dim H^{n-d-m}_{u-d-m,i}$$

as required.

It remains to show the second equality

$$\dim \dot{H}^{n}_{u,d,i} = \sum_{t\in\mathbb{Z}} f_{u-pt} - f_{u-i-pt}.$$

As before, we have

$$\dim \dot{H}^{n}_{u,d,i} = \binom{n}{d} \dim \dot{H}^{n-d}_{u-d,0,i}. \tag{5.3.12}$$

Note that $2(u-d)+p-i-(n-d) = 2u+p-i-n-d = \ell$. Thus by Theorem 5.3.10, we have

$$\binom{n}{d} \dim \dot{H}^{n-d}_{u-d,0,i} = \binom{n}{d} \sum_{t\in\mathbb{Z}} f_{u-d-pt,0} - f_{u-d-i-pt,0}. \tag{5.3.13}$$

We claim

$$f_{u'} = \binom{n}{d} f_{u'-d,0}$$

for each $u' \in \mathbb{Z}$. This is easily checked:

$$f_{u'} = \sum_{b=\ell+d-p+1}^{\ell+d-1} \binom{n}{u'}\binom{u'}{d}\binom{u'-d}{b-d}$$
$$= \sum_{b=\ell-p+1}^{\ell-1} \binom{n}{u'}\binom{u'}{d}\binom{u'-d}{b}$$
$$= \sum_{b=\ell-p+1}^{\ell-1} \binom{n}{d}\binom{n-d}{u'-d}\binom{u'-d}{b}$$
$$= \binom{n}{d} \sum_{b=\ell-p+1}^{\ell-1} \binom{n-d}{u'-d}\binom{u'-d}{b}$$
$$= \binom{n}{d} f_{u'-d,0}.$$

□

Note that the equality $f_{u'} = \binom{n}{d} f_{u'-d,0}$ above is a manifestation of the more structural fact that $FL^{n,[\ell]}_{u',d} = \mathrm{Ind}^{S_n}_{S_{n-d}\times S_d}(FL^{n-d,[\ell]}_{u'-d,0})$.

Example 5.3.14. Suppose $p = 3$ and $i = 1$. With a little effort (see below) it can be

shown that
$$\dim \dot{H}^n_{u,0,1} = \binom{n+1}{2(n-u)} = \binom{n+1}{2u-n+1}.$$
This is sequence A098157 in The On-Line Encyclopedia of Integer Sequences [2]. For the proof we first claim the following. Let k be any integer. If k is even then
$$\sum_{t\in\mathbb{Z}} \binom{k}{\frac{k}{2}+j+3t} = \begin{cases} \frac{2^k+2}{3} & \text{if } j \equiv 0 \pmod 3 \\ \frac{2^k-1}{3} & \text{otherwise.} \end{cases}$$
If k is odd then
$$\sum_{t\in\mathbb{Z}} \binom{k}{\frac{k+1}{2}+j+3t} = \begin{cases} \frac{2^k-2}{3} & \text{if } j \equiv 1 \pmod 3 \\ \frac{2^k+1}{3} & \text{otherwise.} \end{cases}$$
For small k this is easily verified. Suppose $k > 1$. In the case k is even we have
$$\sum_{t\in\mathbb{Z}} \binom{k}{\frac{k}{2}+3t} = \sum_{t\in\mathbb{Z}} \binom{k-1}{\frac{k}{2}+3t} + \binom{k-1}{\frac{k}{2}-1+3t}.$$
By induction this is
$$\frac{2^{k-1}+1}{3} + \frac{2^{k-1}+1}{3} = \frac{2^k+2}{3}$$
as required. By the symmetry of the binomial coefficients we have
$$\sum_{t\in\mathbb{Z}} \binom{k}{\frac{k}{2}+1+3t} = \sum_{t\in\mathbb{Z}} \binom{k}{\frac{k}{2}+2+3t} = \frac{1}{2}\left(2^k - \frac{2^k+2}{3}\right) = \frac{2^k-1}{3}$$
as required. The proof for the case k is odd is similar. Now by Theorem 5.3.2 we have
$$\begin{aligned}
\dim \dot{H}^n_{u,0,1} &= \sum_{m=2u-n}^{2u-n+1} \binom{n}{m} \sum_{t\in\mathbb{Z}} \left(\binom{n-m}{u-m-3t} - \binom{n-m}{u-m-1-3t} \right) \\
&= \sum_{t\in\mathbb{Z}} \binom{n}{2u-n} \left(\binom{2(n-u)}{n-u-3t} - \binom{2(n-u)}{n-u-1-3t} \right) \\
&\quad + \binom{n}{2u-n+1} \left(\binom{2(n-u)-1}{n-u-1-3t} - \binom{2(n-u)-1}{n-u-2-3t} \right) \\
&= \binom{n}{2u-n} \left(\frac{2^{2(n-u)}+2}{3} - \frac{2^{2(n-u)}-1}{3} \right) \\
&\quad + \binom{n}{2u-n+1} \left(\frac{2^{2(n-u)-1}+1}{3} - \frac{2^{2(n-u)-1}-2}{3} \right) \\
&= \binom{n}{2u-n} \cdot 1 + \binom{n}{2u-n+1} \cdot 1 \\
&= \binom{n+1}{2u-n+1}
\end{aligned}$$
as required.

Example 5.3.15. As in Example 5.3.14 we set $p = 3$. But this time consider $i = 2$. In this case we have
$$\dim \dot{H}^n_{u,0,2} = \binom{n+1}{2u-n}.$$
This is sequence A119900 in The On-Line Encyclopedia of Integer Sequences [10]. This formula can be obtained in a similar manner to the formula in Example 5.3.14. However, notice the task is simplified if we instead try to prove both formulas simultaneously since then we can simply use induction via the branching rule.

We end this section with a result which identifies the homology modules arising from the doubles map as cohomology modules arising from the singles map. This resembles one half of the classical Poincaré duality, see Munkres [25] for an exposition of the classical result. To this end we define a symmetric bilinear form b on M^n by
$$b(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$
for all $x, y \in L^n$. This form is non-degenerate. For $0 \leq d \leq u \leq n$ its restriction to $M^n_{u,d}$ is also non-degenerate. We define the *cosingles map* $\dot{\varepsilon}$ to be the transpose of $\dot{\partial}$ with respect to b, that is the unique map such that
$$b(x, \dot{\varepsilon}(y)) = b\left(\dot{\partial}(x), y\right)$$
for all $x, y \in M^n$. Thus $\dot{\varepsilon}$ is the linear map which takes each subset of $[n\overline{n}]$ to the sum of the subsets obtained by adding a single element whilst not increasing the number of doubles, that is for $x \in L^n_{u,d}$ we have
$$\dot{\varepsilon}(x) = \sum_y y$$
where the sum runs over those $y \in L^n_{u+1,d}$ such that $y \supseteq x$. Let c be the linear map on M^n which takes each subset of $[n\overline{n}]$ to its complement in $[n\overline{n}]$. We call c the *complement map*. This may be thought of in terms of the homology grid as reflection in the line $u + d = n$.

Theorem 5.3.16 (One half of a Poincaré-type duality)**.** *Let $p \geq 2$. Let n, u and d be integers. The complement map gives an isomorphism of p-complexes of FB_n-modules*
$$(M^n_{*,d}, \dot{\varepsilon}) \cong (M^n_{n-d,*}, \ddot{\partial})$$
which maps $M^n_{u,d}$ to $M^n_{n-d,n-u}$. In particular for $0 \leq i \leq p$ it induces an isomorphism

of homology modules

$$\frac{\ker \dot{\varepsilon}^{p-i} \cap M^n_{u,d}}{\dot{\varepsilon}^i(M^n_{u-i,d})} \cong \frac{\ker \ddot{\partial}^{p-i} \cap M^n_{n-d,n-u}}{\ddot{\partial}^i(M^n_{n-d,n-u+i})} = \ddot{H}^n_{n-d,n-u,p-i}.$$

Proof. The complement map satisfies $c^2 = \mathrm{Id}$. Therefore it is invertible. It remains to show that the complement map commutes with the boundary maps, that is that the following diagram commutes

$$\begin{tikzcd}
 & & M^n_{n-d,n-u} \arrow[d, "\ddot{\partial}"] \\
 & & M^n_{n-d,n-u-1} \\
M^n_{u,d} \arrow[r, "\dot{\varepsilon}"] \arrow[uurr, "c"] & M^n_{u+1,d}. \arrow[ur, "c"] &
\end{tikzcd}$$

Let $x \in L^n_{u,d}$ be a basis element. Then $\dot{\varepsilon}(x)$ is the sum of all subsets of $[n\overline{n}]$ that can be obtained by adding a single element to x without increasing its number of doubles. Let $\alpha \in [n\overline{n}]$. Then $x \cup \{\alpha\}$ is a subset of the above form if and only if α lies in the doubles part of the complement of x. Suppose α lies in the doubles part of the complement of x. Then we have

$$c\,(x \cup \{\alpha\}) = c\,(x) \setminus \{\alpha\}.$$

Summing over all such α we thus obtain $\ddot{\partial}(c\,(x))$. This shows that the diagram commutes. $\square$

Remark 5.3.17. Temporarily, let $(-)^*$ denote the dualising functor $\mathrm{Hom}_F(-, F)$. The *cohomology* of the chain complex $\dot{\mathcal{M}}^n_{u,d,i}$ at position (u, d) is the homology of the chain complex $(\dot{\mathcal{M}}^n_{u,d,i})^*$:

$$\ldots \to (M^n_{u-i,d})^* \xrightarrow{(\dot{\partial}^i)^*} (M^n_{u,d})^* \xrightarrow{(\dot{\partial}^{p-i})^*} (M^n_{u+p-i,d})^* \to \ldots$$

at position (u, d). Since each $M^n_{u',d}$ is finite dimensional it is isomorphic to its dual. This together with other standard facts from linear algebra imply that the above sequence is isomorphic to the sequence

$$\ldots \to M^n_{u-i,d} \xrightarrow{\dot\varepsilon^i} M^n_{u,d} \xrightarrow{\dot\varepsilon^{p-i}} M^n_{u+p-i,d} \to \ldots,$$

see for example Greub's book [16, Chapter 2, Section 5]. It is not difficult to see that this is an isomorphism of sequences of FB_n-modules. Therefore Theorem 5.3.16 may be interpreted as saying that the cohomology of $\dot{\mathcal{M}}^n_{u,d,i}$ at the module $(M^n_{u,d})^*$ is isomorphic to the homology of $\ddot{\mathcal{M}}^n_{n-d,n-u,p-i}$ at the module $M^n_{n-d,n-u}$.

The following result allows computation of the dimensions of the doubles homology modules from that of the singles homology modules.

Theorem 5.3.18. *Let $p \geq 2$. Let n, u, d and i be integers. Let $0 \leq i \leq p$. Then we have*

$$\dim \dot H^n_{u,d,i} = \dim \ddot H^n_{n-d,n-u,p-i}.$$

Proof. By Theorem 5.3.16 we have

$$\ddot H^n_{n-d,n-u,p-i} \cong \frac{\ker \dot\varepsilon^{p-i} \cap M^n_{u,d}}{\dot\varepsilon^i(M^n_{u-i,d})}.$$

For u' any integer write $M_{u'}$ for $M^n_{u',d}$. Then

$$\begin{aligned}\dim \dot H^n_{u,d,i} &= \dim \ker(\dot\partial^i) \cap M_u - \dim \dot\partial^{p-i}(M_{u+p-i}) \\ &= \dim M_u - \operatorname{rk} \dot\partial^i|_{M_u} - \operatorname{rk} \dot\partial^{p-i}|_{M_{u+p-i}}.\end{aligned}$$

Also

$$\begin{aligned}\dim\left(\frac{\ker \dot\varepsilon^{p-i} \cap M^n_{u,d}}{\dot\varepsilon^i(M^n_{u-i,d})}\right) &= \dim \ker(\dot\varepsilon^{p-i}) \cap M_u - \dim \dot\varepsilon^i(M_{u-i}) \\ &= \dim M_u - \operatorname{rk} \dot\varepsilon^{p-i}|_{M_u} - \operatorname{rk} \dot\varepsilon^i|_{M_{u-i}}.\end{aligned}$$

Now since $\dot\varepsilon$ is the transpose of $\dot\partial$ we have $\operatorname{rk} \dot\varepsilon^{p-i}|_{M_u} = \operatorname{rk} \dot\partial^{p-i}|_{M_{u+p-i}}$ and $\operatorname{rk} \dot\varepsilon^i|_{M_{u-i}} = \operatorname{rk} \dot\partial^i|_{M_u}$. □

This is another manifestation of the duality which we wanted to prove. Note that we have not used anything specific about our chain complexes in this proof. In fact given any chain complex of finite dimensional vector spaces there is a vector space isomorphism between the homology and the cohomology at each position in the complex. This follows from the (far more general) Universal Coefficient Theorem for Cohomology, see Weibel's book [30, Theorem 3.6.5].

5.4 p-rank of incidence matrices

An important application of Theorem 5.3.2 is the derivation of a formula for the p-rank of certain incidence matrices arising from the n-cross-polytope. There does not seem to be any existing work on this particular problem. Incidence matrices appear in many diverse areas of mathematics, usually when studying a relation between two sets. The report [15] provides a good overview of the current state of knowledge on invariants of incidence matrices such as the p-rank and the Smith Normal Form. The survey [32] is also a good reference. We highlight some important examples, but first let us define what we mean by "p-rank".

Definition 5.4.1 (The p-rank of an integer matrix)**.** Suppose M is a matrix with integer entries. Then M may be regarded as a matrix over any field. The *p-rank of M* is the rank of M when regarded as a matrix over a field of characteristic p.

Example 5.4.2 (The Boolean algebra, or the simplex)**.** Let $0 \leq t \leq k \leq n$ be integers. Denote by W_{tk} the $\{0, 1\}$-matrix with rows indexed by the t-subsets x of $[n]$ and columns indexed by the k-subsets y of $[n]$ with xy-entry equal to 1 if and only if $x \subseteq y$. The p-rank of W_{tk} was determined by Wilson [31].

Example 5.4.3 (Subspaces of a finite vector space)**.** Let $0 \leq t \leq k \leq n$ be integers. Let V be a finite vector space of dimension n over a field of characteristic $p > 0$. Similarly to the previous example, set W_{tk} to be the $\{0, 1\}$-matrix with rows indexed by the t-dimensional subspaces x of V and columns indexed by the k-dimensional subspaces y of V with xy-entry equal to 1 if and only if $x \subseteq y$. For ℓ prime with $\ell \neq p$ the ℓ-rank of W_{tk} is known by the work of Frumkin and Yakir [14]. The p-rank of W_{tk} has long been known for $t = 1$. This result, known as the Hamada Formula, dates back to a 1960s paper of Hamada [17]. Determining the p-rank of W_{tk} for general t and k remains an open problem.

Example 5.4.4 (Subspaces of a formed space)**.** Again suppose we have a vector space, but now suppose it comes equipped with a non-degenerate alternating bilinear, non-degenerate Hermitian or non-singular quadratic form. Then we can consider the restriction of the incidence relation from Example 5.4.3 to particular classes of subspaces, for example the totally isotropic subspaces or totally singular subspaces. This resembles how the bottom row in the homology grid is obtained by restricting the inclusion map from the Boolean algebra case (the map which takes any subset of

$[n\overline{n}]$ to the sum of all subsets obtained by removing a single element) to those subsets which have no doubles. Representation theoretic methods have been successful in determining the p-ranks of incidence matrices between points and hyperplanes in unitary and orthogonal spaces and between points and flats of any fixed dimension in the symplectic case. The unitary and orthogonal cases are treated in the paper [1] of Arslan and Sin. For the symplectic case see the papers [6, 7] of Chandler, Sin and Xiang.

Let X be the boundary of the n-cross-polytope with the usual simplicial complex structure as in Example 3.1.2. For $0 \leq s \leq t \leq n$ define $W_{s,t}$ to be the $\{0,1\}$-matrix with rows indexed by the $(s-1)$-simplices x of X and columns indexed by the $(t-1)$-simplices y of X with xy-entry equal to 1 if and only if $x \subseteq y$. The matrix $W_{s,t}$ is known as the *incidence matrix* or *inclusion matrix* of $(s-1)$-simplices versus $(t-1)$-simplices (recall in this case a $(k-1)$-simplex is just a set of size k). As far as we know the p-rank of $W_{s,t}$ does not appear in the literature. For simplicity from now on we will regard $W_{s,t}$ as a matrix over F, our field of characteristic $p > 0$. Let $0 \leq s \leq t \leq n$ and set $i = t - s$. If $0 < i < p$ then $i!$ is invertible in F and $W_{s,t}$ is the matrix of $\frac{1}{i!}\dot{\partial}^i|_{M^n_{t,0}}$ with respect to the bases $L^n_{s,0}$ and $L^n_{t,0}$. Thus $\operatorname{rk} W_{s,t} = \operatorname{rk} \dot{\partial}^i|_{M^n_{t,0}}$.

The main tool to transition from results about dimensions of homology modules to results about ranks of incidence matrices is the following.

Lemma 5.4.5. *Suppose (C, d) is a positive*[1] *chain complex of vector spaces with differential d. For $k \in \mathbb{Z}$ let $r_k = \operatorname{rk} d_k$ where d_k is the restriction of d to C_k and let $\beta_k = \dim H_k(M)$ be the k-th Betti number of C. Then*

$$r_k = \sum_{j=0}^{k-1} (-1)^{k-j}(\beta_j - \dim C_j).$$

Proof. For any $k \in \mathbb{Z}$ we have by the Rank-nullity theorem that

$$\beta_k = (\dim C_k - r_k) - r_{k+1}.$$

Rearranging gives the recurrence

$$r_{k+1} = \dim C_k - r_k - \beta_k.$$

But since C is positive we have $r_0 = 0$. The result now follows by induction on k. $\square$

[1]This just means $C_k = 0$ for all $k < 0$.

Suppose n is fixed. For integers u and ℓ define

$$f_u^\ell = \sum_{b=\ell-p+1}^{\ell-1} \binom{n}{u}\binom{u}{b} = \left|L_{u,0}^{n,[\ell]}\right|$$

and

$$c_u = 2^u \binom{n}{u} = \left|L_{u,0}^{n}\right|$$

where, as in Section 5.3, $L_{u,0}^{n,[\ell]}$ is the set of all subsets of $[n\overline{n}]$ of size u which contain no doubles and strictly between $\ell - p$ and ℓ barred elements. Let r_{st} denote the p-rank of $W_{s,t}$.

Theorem 5.4.6. *Let $p > 2$. Let $0 \leq s \leq t \leq n$ and suppose $0 < t - s < p$. Set $\ell = s + t + p - n$. Then the p-rank of $W_{s,t}$ is*

$$r_{st} = \sum_{k\in\mathbb{Z}} \sum_{m\in\mathbb{Z}_{>0}} \left(f_{t-kp}^{\ell-mp} - f_{s-kp}^{\ell-mp}\right) + \sum_{k'\in\mathbb{Z}_{\geq 0}} \left(c_{s-k'p} - c_{t-(k'+1)p}\right).$$

Note if p is large enough then the formula becomes independent of p and gives the rank of $W_{s,t}$ in characteristic zero. Explicitly, if p is large enough we obtain

$$r_{st} = f_t^{\ell-p} - f_s^{\ell-p} + c_s = c_s + \sum_{b=0}^{s+t-n-1} \binom{n}{t}\binom{t}{b} - \binom{n}{s}\binom{s}{b}.$$

If, in addition, $s + t \leq n$ then the sum vanishes and we are left with $r_{st} = c_s$, that is $W_{s,t}$ is of full rank (in particular, $c_s \leq c_t$, which can also be verified directly).

Proof of Theorem 5.4.6. Recall $\dot{\mathcal{M}}_{t,0,t-s}^n$ is the chain complex through $M_{t,0}^n$ with $\dot{\partial}^{t-s}$ exiting $M_{t,0}^n$:

$$\ldots \leftarrow M_{t-p,0}^n \xleftarrow{\dot{\partial}^{p-t+s}} M_{s,0}^n \xleftarrow{\dot{\partial}^{t-s}} M_{t,0}^n \xleftarrow{\dot{\partial}^{p-t+s}} M_{s+p,0}^n \xleftarrow{\dot{\partial}^{t-s}} M_{t+p,0}^n \leftarrow \ldots.$$

By a suitable choice of indices $\dot{\mathcal{M}}_{t,0,t-s}^n$ is a positive complex. Thus we can apply Lemma 5.4.5 to calculate the rank of $\dot{\partial}^{t-s}$ on $M_{t,0}^n$. We obtain

$$r_{st} = \sum_{j\in\mathbb{Z}_{\geq 0}} \left(\dim M_{s-jp,0}^n - \dim \dot{H}_{s-jp,0,p-t+s}^n - \dim M_{t-(j+1)p,0}^n + \dim \dot{H}_{t-(j+1)p,0,t-s}^n\right).$$

Let $\ell = s + t + p - n$. Noting that $\dim M_{u,0}^n = c_u$ for each $u \in \mathbb{Z}$ and expanding the dimensions of the homology modules by the dimension formula Theorem 5.3.2 yields

$$r_{st} = \sum_{j\in\mathbb{Z}_{\geq 0}} \left(c_{s-jp} - \sum_{r\in\mathbb{Z}} \left(f^{\ell-(2j+1)p}_{s-(j+r)p} - f^{\ell-(2j+1)p}_{t-(j+r+1)p} \right) \right.$$
$$\left. - c_{t-(j+1)p} + \sum_{r\in\mathbb{Z}} \left(f^{\ell-(2j+2)p}_{t-(j+r+1)p} - f^{\ell-(2j+2)p}_{s-(j+r+1)p} \right) \right). \tag{5.4.7}$$

Now let $k \in \mathbb{Z}$. Consider the subsets of $L^n_{t-kp,0}$ which contribute to the sum (5.4.7). Suppose $j \in \mathbb{Z}_{\geq 0}$. Set $r = k - j - 1$. Then we have

$$L^{n,[\ell-(2j+2)p]}_{t-(j+r+1)p,0} \subseteq L^n_{t-kp,0}$$

and

$$L^{n,[\ell-(2j+1)p]}_{t-(j+r+1)p,0} \subseteq L^n_{t-kp,0}.$$

As j runs over all non-negative integers we see that $2j + 2$ covers all even integers starting at 2 and $2j + 1$ covers all odd integers starting at 1. Thus $f^{\ell-mp}_{t-kp}$ occurs with coefficient $+1$ for all $m \in \mathbb{Z}_{>0}$. The only other way in which subsets of $L^n_{t-kp,0}$ contribute is if $k \geq 1$. Then c_{t-kp} itself occurs as $c_{t-(j+1)p}$ for $j = k - 1$. The coefficient of c_{t-kp} in this case is -1. Similar arguments apply to the subsets of $L^n_{s-kp,0}$ contributing to (5.4.7). Thus we obtain

$$r_{st} = \sum_{k\in\mathbb{Z}} \sum_{m\in\mathbb{Z}_{>0}} \left(f^{\ell-mp}_{t-kp} - f^{\ell-mp}_{s-kp} \right) + \sum_{k'\in\mathbb{Z}_{\geq 0}} \left(c_{s-k'p} - c_{t-(k'+1)p} \right). \qquad \square$$

Example 5.4.8. Let $n = 3$. We use Theorem 5.4.6 to calculate the p-ranks of the incidence matrices arising from the boundary of the octahedron for any odd prime p. There are three such matrices: $W_{1,2}$ for vertices and edges, $W_{1,3}$ for vertices and faces and $W_{2,3}$ for edges and faces. Suppose $1 \leq s < t \leq 3$. Set $\ell = s + t + p - 3$. By Theorem 5.4.6 we have

$$\operatorname{rk} W_{s,t} = \sum_{k\in\mathbb{Z}} \sum_{m\in\mathbb{Z}_{>0}} \left(f^{\ell-mp}_{t-kp} - f^{\ell-mp}_{s-kp} \right) + \sum_{k'\in\mathbb{Z}_{\geq 0}} \left(c_{s-k'p} - c_{t-(k'+1)p} \right).$$

Now since $p \geq 3$ most of these terms will vanish, at least all terms with $k \neq 0$ or $k' \neq 0$ with the possible exception of $f^{\ell-mp}_{t-kp}$ in the case $t = p = 3$ with $k = 1$. Additionally with $k' = 0$ the term $c_{t-(k'+1)p}$ vanishes unless $t = p = 3$. We treat these two terms separately now. They cancel as follows. Suppose $t = p = 3$ with $k = 1$ and $k' = 0$. We have $\ell = s + 3 + 3 - 3 = s + 3 \in \{4, 5\}$ since $s \in \{1, 2\}$. Thus $f^{\ell-mp}_0 \neq 0$ only if $m = 1$ in which case we have $f^{\ell-p}_0 = \binom{3}{0}\binom{0}{0} = 1$ for both $\ell = 4$ and $\ell = 5$. The other term we need to consider is $c_{t-(k'+1)p} = c_0 = 1$. Therefore the two terms cancel as claimed. So, returning to the general case, we in fact have

$$r_{st} = c_s + \sum_{m \in \mathbb{Z}_{>0}} \left(f_t^{\ell - mp} - f_s^{\ell - mp} \right).$$

Now

$$f_t^{\ell - mp} = \sum_{b = \ell - mp - p + 1}^{\ell - mp - 1} \binom{3}{t} \binom{t}{b}.$$

This is zero if $m > 1$ since then $\ell - mp - 1 = s + t + p - 3 - mp - 1 \leq 1 + (1 - m)p \leq 1 - p < 0$. So

$$r_{st} = c_s + f_t^{\ell - p} - f_s^{\ell - p} = c_s + f_t^{s+t-3} - f_s^{s+t-3}.$$

We can now evaluate this. We obtain

$$\begin{aligned}
r_{12} &= c_1 + f_2^0 - f_1^0 = \binom{3}{1} 2^1 + 0 - 0 = 6, \\
r_{13} &= c_1 + f_3^1 - f_1^1 = \binom{3}{1} 2^1 + \binom{3}{3}\binom{3}{0} - \binom{3}{1}\binom{1}{0} = 4 \text{ and} \\
r_{23} &= c_2 + f_3^2 - f_2^2 = \binom{3}{2} 2^2 + \binom{3}{3}\left(\binom{3}{0} + \binom{3}{1}\right) - \binom{3}{2}\left(\binom{2}{0} + \binom{2}{1}\right) = 7.
\end{aligned}$$

In particular, these p-ranks do not depend on $p \geq 3$. For $p = 2$ however we have $\operatorname{rk} W_{1,2} = 5 \neq 6$ (by GAP). Note that the octahedron has $c_1 = 6$ vertices, $c_2 = 12$ edges and $c_3 = 8$ faces. Therefore only $W_{1,2}$ has full rank here. This agrees with our earlier observation (immediately after Theorem 5.4.6) that for large enough p the matrix $W_{s,t}$ has full rank if $s + t \leq n$.

The more general problems of finding the rank of powers of the singles map on modules with $d > 0$ and finding the rank of powers of the doubles map can be reduced down to this case as follows. First the case of the singles map with $d > 0$. We can apply Theorem 3.6.1 to find the p-rank of the incidence matrix W of $L^n_{s,d}$ and $L^n_{t,d}$ provided $0 < t - s < p$ as follows. Theorem 3.6.1 says that

$$\dot{K}^n_{t,d,t-s} = \operatorname{Ind}^{B_n}_{B_{n-d} \times B_d} \left(\dot{K}^{n-d}_{t-d,0,t-s} \right).$$

Therefore the p-rank of W is given by

$$\operatorname{rk}_p W = \dim M^n_{t,d} - \binom{n}{d} \left(\dim M^{n-d}_{t-d,0} - \operatorname{rk}_p W^{n-d}_{s-d,t-d} \right).$$

where $W^{n-d}_{s-d,t-d}$ denotes $W_{s-d,t-d}$ but with n replaced by $n - d$. Theorem 5.4.6 gives a closed formula for $\operatorname{rk}_p W^{n-d}_{s-d,t-d}$ so we have a closed formula for $\operatorname{rk}_p W$.

We can also calculate the ranks for the doubles map by using the fact that

$$\operatorname{rk} \ddot{\partial}^i|_{M^n_{u,d}} = \operatorname{rk} \dot{\varepsilon}^i|_{M^n_{n-d,n-u}} = \operatorname{rk} \dot{\partial}^i|_{M^n_{n-d+i,n-u}}.$$

These equalities follow from Theorem 5.3.16 and the fact that $\dot{\varepsilon}$ is the transpose of $\dot{\partial}$.

We end this section with some remarks on possible future study in this area. Our formula for the rank of the incidence matrix $W_{s,t}$ is only valid for $0 < t - s < p$. For $t - s$ outside this range it is possible homological methods can still be applied but instead of the singles map $M^n_{t,0} \to M^n_{t-1,0}$ we would have to consider a more general incidence map $M^n_{t,0} \to M^n_{t-j,0}$ for some $j > 1$. It is also likely similar methods can be employed for the calculation of p-ranks of incidence matrices arising from a polar space since the boundary of the cross-polytope lies at the heart of the structure of these spaces. More explicitly, the maximal totally isotropic or totally singular subspaces all have the same dimension n say. Then the space is a direct sum of n hyperbolic lines and an anisotropic space. For reference we refer the reader to Cameron's notes [5]. Denote these hyperbolic lines by $H_1, \ldots, H_n$. For $1 \leq \alpha \leq n$ fix a basis $v_\alpha, v_{\overline{\alpha}}$ of H_α. Then the subspaces spanned by the subsets of $\{v_1, v_{\overline{1}}, \ldots, v_n, v_{\overline{n}}\}$ correspond to the subsets of $[n\overline{n}]$ in the obvious way. The totally isotropic or totally singular subspaces correspond to the subsets which contain no doubles, that is the subsets which form the bases in the bottom row of the homology grid. We have seen in Example 3.1.2 that these subsets naturally label the simplices of the boundary of the n-cross-polytope.

5.5 Kernel generators

In this section we exhibit generators for the kernels of powers of the singles and doubles maps restricted to $M^n_{u,d}$. We also show that the singles and doubles homology modules are cyclic FB_n-modules. By the usual reduction arguments afforded by Theorem 3.6.1 it suffices to consider the bottom row of the homology grid in the singles case and the rightmost column of the homology grid in the doubles case.

5.5.1 Generators for the kernel of the singles map

We start with the singles map case. Let ∂ be the singles map. Let n, u and i be integers with $0 \le u \le n$ and $0 < i < p$. Let L_u consist of the u-subsets of $[n\overline{n}]$ containing no doubles, that is $L_u = L^n_{u,0}$. Let M_u be the permutation module with basis L_u. Let $K^n_{u,i} = \dot{K}^n_{u,0,i}$ be the kernel of ∂^i on M_u and $I^n_{u,i} = \dot{I}^n_{u,0,i}$ be the image $\partial^{p-i}(M_{u+p-i})$. As usual, let $H^n_{u,i} = K^n_{u,i}/I^n_{u,i} = \dot{H}^n_{u,0,i}$. We are interested in finding generators of $K^n_{u,i}$ and consequently of $H^n_{u,i}$. If $H^n_{u,i} = 0$ then $K^n_{u,i} = I^n_{u,i} = \partial^{p-i}(M_{u+p-i})$. So $\partial^{p-i}(L_{u+p-i})$ is an efficient set of generators. So we may assume $H^n_{u,i} \ne 0$. By Theorem 5.2.1 this is the case if and only if $0 < 2u + p - i - n$. There is one further straightforward reduction: If $u < i$ then $K^n_{u,i} = M_u$ and L_u is a set of generators for $K^n_{u,i}$. It remains to consider the case $i \le u$.

Let $\tau = (\alpha_1, \ldots, \alpha_{2u-i+1})$ be a tuple of distinct elements of $[n\overline{n}]$. Consider the element

$$z_\tau = \{\alpha_1 \ldots \alpha_{i-1}\} \cdot \prod_{j=i}^{u} (\{\alpha_j\} - \{\alpha_{j+u-i+1}\}) .$$

Note that z_τ may lie outside M_u since it may involve subsets containing doubles. We will show that $K^n_{u,i}$ is spanned by the image $I^n_{u,i}$ together with the elements z_τ that do lie in M_u. First observe that z_τ lies in M_u if and only if $\alpha_k = \overline{\alpha_m}$ with $k < m$ implies $i \le k$ and $m = k + u - i + 1$, in other words if and only if $(\{\alpha_k\} - \{\alpha_m\})$ is one of the factors occurring in the definition of z_τ. We set T to be the set of all tuples $\tau = (\alpha_1, \ldots, \alpha_{2u-i+1})$ of distinct elements of $[n\overline{n}]$ which satisfy $z_\tau \in M_u$. Let $\dot{Z}^n_{u,i}$ be the set

$$\dot{Z}^n_{u,i} = \{\, z_\tau \,:\, \tau \in T \,\}.$$

For example, the element

$$\{1\overline{2}3\} \cdot (\{4\} - \{5\}) \cdot \big(\{6\} - \{\overline{6}\}\big) = \{1\overline{2}346\} - \{1\overline{2}34\overline{6}\} - \{1\overline{2}356\} + \{1\overline{2}35\overline{6}\}$$

lies in $\dot{Z}^n_{5,4}$ for any $n \ge 6$.

Remark 5.5.1. The elements of $\dot{Z}^n_{u,i}$ can be identified with polytabloids in the following way. We refer the reader to James' book [19] for a more thorough explanation of the terms used here. Let $\tau = (\alpha_1, \ldots, \alpha_s, \beta_1, \ldots, \beta_t, \gamma_1, \ldots, \gamma_t) \in T$ where $s = i - 1$ and $t = u - i + 1$. We associate with τ the $(u, u - i + 1)$-tableau[2]

$$\underline{\tau} = \begin{array}{|c|c|c|c|c|c|} \hline \beta_1 & \cdots & \beta_t & \alpha_1 & \cdots & \alpha_s \\ \hline \gamma_1 & \cdots & \gamma_t \\ \cline{1-3} \end{array} \quad .$$

[2] Note that this "tableau" is not on the alphabet $1, \ldots, 2u - i + 1$.

Let B_τ be the group of signed permutations of the elements of τ and their barred counterparts. The signed column stabilizer of $\underline{\tau}$ in FB_τ is

$$k_{\underline{\tau}} = (1 - (\beta_1, \gamma_1)) \cdots (1 - (\beta_t, \gamma_t)).$$

Disregarding the order of the elements in each row of $\underline{\tau}$ gives a tabloid $\overline{\underline{\tau}}$. The linear combination $k_{\underline{\tau}}\overline{\underline{\tau}}$ of tabloids is called the polytabloid associated to $\underline{\tau}$. It is clear that the map $\tau \mapsto k_{\underline{\tau}}\overline{\underline{\tau}}$ is an isomorphism between $\langle \dot{Z}^n_{u,i} \rangle_F$ and the FB_n-module generated by the set of polytabloids $\{\, k_{\underline{\tau}}\overline{\underline{\tau}} \,:\, \tau \in T \,\}$. For each $\tau \in T$ the FB_τ-submodule generated by z_τ has a similar feel to a Specht module.

We now prove our claim that $\dot{Z}^n_{u,i}$ together with $I^n_{u,i}$ generates $K^n_{u,i}$. The next lemma handles the case $i = u$.

Lemma 5.5.2. *Let $p > 2$. Let $0 \le u \le n$ with $0 < i < p$ and $0 < 2u + p - i - n$. Then $\dot{Z}^n_{u,u}$ spans $K^n_{u,u}$.*

Proof. For $z \in K^n_{u,u}$ the sum of the coefficients of elements of L_u occurring in z must be zero. Hence we have that $K^n_{u,u}$ is spanned by $\{\, x - y \,:\, x, y \in L_u \,\}$. We claim each such $x - y$ lies in the span of $\dot{Z}^n_{u,u}$. The proof is by induction on $|x \setminus y|$. If $|x \setminus y| = 0$ then $x = y$ so $x - y = 0 \in \langle \dot{Z}^n_{u,u} \rangle_F$. If $|x \setminus y| = 1$ then without loss of generality $x = \{\alpha_1 \dots \alpha_{u-1}\alpha_u\}$ and $y = \{\alpha_1 \dots \alpha_{u-1}\beta_u\}$ with $\beta_u \ne \alpha_u$. Hence $x - y = \{\alpha_1 \dots \alpha_{u-1}\} \cdot (\{\alpha_u\} - \{\beta_u\}) \in \dot{Z}^n_{u,u}$. Now suppose $|x \setminus y| = m \ge 2$. Then without loss of generality $x = \{\alpha_1 \dots \alpha_u\}$ and $y = \{\alpha_1 \dots \alpha_{u-m}\beta_{u-m+1} \dots \beta_u\}$ with $\{\alpha_{u-m+1} \dots \alpha_u\} \cap \{\beta_{u-m+1} \dots \beta_u\} = \emptyset$. Now since $m \ge 2$ we may arrange that $\overline{\alpha_{u-m+1}} \notin \{\beta_{u-m+2} \dots \beta_u\}$. Set $v = \{\alpha_1 \dots \alpha_{u-m}\alpha_{u-m+1}\beta_{u-m+2} \dots \beta_u\}$. Then (by our special arrangement) $v \in L_u$ and $x - y = x - v + v - y$. We have $|x \setminus v| = m - 1$ and $|v \setminus y| = 1$ so by induction we are done. $\square$

Theorem 5.5.3. *Let $p > 2$. Let n, u and i be integers with $0 \le i \le p$. If $i \le u$ then $K^n_{u,i} = \langle \dot{Z}^n_{u,i}, I^n_{u,i} \rangle_F$.*

Proof. If $H^n_{u,i} = 0$ then $K^n_{u,i} = I^n_{u,i}$ and the conclusion holds in this case. So we may assume $H^n_{u,i} \ne 0$. By Theorem 5.2.1 we have

(a) $0 < i < p$,

(b) $0 \le u \le n$ and

(c) $0 < 2u + p - i - n$.

The result is easily seen to hold for $n = 1$ (using Appendix A). Suppose $n > 1$. Since (a) holds we may apply the branching rule, Theorem 4.2.3. This gives an isomorphism

$$K^n_{u,i} \overset{f}{\cong} K^{n-1}_{u,i+1} \oplus K^{n-1}_{u-1,i} \oplus K^{n-1}_{u-1,i-1}. \tag{5.5.4}$$

The inverse of f is given by

$$f^{-1}(x_1, x_2, x_3) = -ix_1 + \partial(x_1) \cdot \{n\} + x_2 \cdot (\{\overline{n}\} - \{n\}) + x_3 \cdot \{n\}.$$

We need to show that the inverse image of each of the summands in (5.5.4) is contained in $\langle \dot{Z}^n_{u,i}, I^n_{u,i} \rangle_F$. By Lemma 5.5.2 we may assume that $i < u$ so that for each $(u', i') \in \{(u, i+1), (u-1, i), (u-1, i-1)\}$ we have $i' \leq u'$. Then by induction $K^{n-1}_{u',i'} = \langle \dot{Z}^{n-1}_{u',i'}, I^{n-1}_{u',i'} \rangle_F$ for $(u', i') \in \{(u, i+1), (u-1, i), (u-1, i-1)\}$. So it suffices to show that the inverse images of the copies of $\dot{Z}^{n-1}_{u',i'}$ and $I^{n-1}_{u',i'}$ lie in $\langle \dot{Z}^n_{u,i}, I^n_{u,i} \rangle_F$. But f restricts to an isomorphism

$$I^n_{u,i} \overset{f}{\cong} I^{n-1}_{u,i+1} \oplus I^{n-1}_{u-1,i} \oplus I^{n-1}_{u-1,i-1}$$

so we know that the inverse images of the $I^{n-1}_{u',i'}$ are contained in $I^n_{u,i} \subseteq \langle \dot{Z}^n_{u,i}, I^n_{u,i} \rangle_F$. All that remains is to check the inverse images of the $\dot{Z}^{n-1}_{u',i'}$. A typical element of $\dot{Z}^{n-1}_{u,i+1}$ is

$$\{\alpha_1 \ldots \alpha_i\} \cdot \prod_{j=i+1}^{u} (\{\alpha_j\} - \{\beta_j\}).$$

We have

$$\begin{aligned}
&f^{-1}\left(\{\alpha_1 \ldots \alpha_i\} \cdot \prod_{j=i+1}^{u} (\{\alpha_j\} - \{\beta_j\}), 0, 0\right) \\
&= -i\{\alpha_1 \ldots \alpha_i\} \cdot \prod_{j=i+1}^{u} (\{\alpha_j\} - \{\beta_j\}) + \{n\} \cdot \partial\left(\{\alpha_1 \ldots \alpha_i\} \cdot \prod_{j=i+1}^{u} (\{\alpha_j\} - \{\beta_j\})\right) \\
&= -i\{\alpha_1 \ldots \alpha_i\} \cdot \prod_{j=i+1}^{u} (\{\alpha_j\} - \{\beta_j\}) + \{n\} \cdot \partial(\{\alpha_1 \ldots \alpha_i\}) \cdot \prod_{j=i+1}^{u} (\{\alpha_j\} - \{\beta_j\}) \\
&= \sum_{k=1}^{i} (\{n\} \cdot \{\alpha_1 \ldots \alpha_{k-1}\widehat{\alpha_k}\alpha_{k+1} \ldots \alpha_i\} - \{\alpha_1 \ldots \alpha_i\}) \cdot \prod_{j=i+1}^{u} (\{\alpha_j\} - \{\beta_j\}) \\
&= \sum_{k=1}^{i} (\{n\} - \{\alpha_k\}) \cdot (\{\alpha_1 \ldots \alpha_{k-1}\widehat{\alpha_k}\alpha_{k+1} \ldots \alpha_i\} \cdot \prod_{j=i+1}^{u} (\{\alpha_j\} - \{\beta_j\})
\end{aligned}$$

where the notation $\widehat{\alpha_k}$ means omit α_k. But each term of this sum is an element of $\dot{Z}^n_{u,i}$. So the sum lies in $\langle \dot{Z}^n_{u,i} \rangle_F$ as required.

A typical element of $\dot{Z}^{n-1}_{u-1,i}$ is

$$\{\alpha_1 \dots \alpha_{i-1}\} \cdot \prod_{j=i}^{u-1} (\{\alpha_j\} - \{\beta_j\}) .$$

We have

$$f^{-1}\left(0, \{\alpha_1 \dots \alpha_{i-1}\} \cdot \prod_{j=i}^{u-1} (\{\alpha_j\} - \{\beta_j\}), 0\right)$$
$$= \{\alpha_1 \dots \alpha_{i-1}\} \cdot (\{\overline{n}\} - \{n\}) \cdot \prod_{j=i}^{u-1} (\{\alpha_j\} - \{\beta_j\}) \in \dot{Z}^n_{u,i}.$$

Finally, a typical element of $\dot{Z}^{n-1}_{u-1,i-1}$ is

$$\{\alpha_1 \dots \alpha_{i-2}\} \cdot \prod_{j=i-1}^{u-1} (\{\alpha_j\} - \{\beta_j\}) .$$

We have

$$f^{-1}\left(0, 0, \{\alpha_1 \dots \alpha_{i-2}\} \cdot \prod_{j=i-1}^{u-1} (\{\alpha_j\} - \{\beta_j\})\right)$$
$$= \{\alpha_1 \dots \alpha_{i-2}\} \cdot \{n\} \cdot \prod_{j=i-1}^{u-1} (\{\alpha_j\} - \{\beta_j\}) \in \dot{Z}^n_{u,i}. \qquad \square$$

By using Theorem 3.6.1, the case $d > 0$ can be derived from Theorem 5.5.3 as follows.

Corollary 5.5.5. *Let $p > 2$. Let $0 \le d \le u \le n$ and i be integers with $0 \le i \le p$. If $i \le u-d$ then $\dot{K}^n_{u,d,i}$ is generated by $\dot{I}^n_{u,d,i}$ and the elements $z \cdot \{n-d+1\overline{n-d+1} \dots n\overline{n}\}$ with $z \in \dot{Z}^{n-d}_{u-d,i}$.*

Proof. By Theorem 3.6.1 we have

$$\dot{K}^n_{u,d,i} = \operatorname{Ind}^{B_n}_{B_{n-d} \times B_d} \left(\dot{K}^{n-d}_{u-d,0,i}\right).$$

Since $u - d \le i$ Theorem 5.5.3 implies that $\dot{K}^{n-d}_{u-d,0,i}$ is generated by $\dot{Z}^{n-d}_{u-d,i}$ and $\dot{I}^{n-d}_{u-d,0,i}$. Theorem 3.6.1 says that $\operatorname{Ind}^{B_n}_{B_{n-d} \times B_d} \left(\dot{I}^{n-d}_{u-d,0,i}\right) = \dot{I}^n_{u,d,i}$. Therefore $\dot{K}^n_{u,d,i}$ is generated by $\dot{I}^n_{u,d,i}$ and $z \cdot \{n-d+1\overline{n-d+1} \dots n\overline{n}\}$ with $z \in \dot{Z}^{n-d}_{u-d,i}$. $\square$

There is alot of redundancy (linear dependence) in the generators $\dot{Z}^n_{u,i}$, for example

$z \in \dot{Z}^n_{u,i}$ implies $-z \in \dot{Z}^n_{u,i}$ (for $i \leq u$). The next result shows that $\langle \dot{Z}^n_{u,i} \rangle_F$ is a cyclic FB_n-module. For this we need a definition.

Definition 5.5.6. Recall M^n is the F-vector space with basis the subsets of $[n\overline{n}]$. Also recall the bilinear form b defined for $x, y \in L^n$ by $b(x,y) = \delta_{xy}$ where δ is the Kronecker delta. Let $x \in M^n$. The *unsigned support size of* x is the cardinality of the set

$$\{\, \alpha \in [n] \;:\; \text{there exists } y \in L^n \text{ with } b(x,y) \neq 0 \text{ and } \{\alpha\overline{\alpha}\} \cap y \neq \emptyset \,\}.$$

We denote the unsigned support size of x by $\tilde{\mathrm{u}}(x)$.

Intuitively, the unsigned support size of an element is just the number of symbols from $[n]$ that we see when writing out the element, regardless of whether they appear with a bar or without. For example the unsigned support size of the element $\{12\overline{3}\} - \{\overline{1}5\overline{5}\}$ is four because we see the four symbols 1, 2, 3 and 5.

Lemma 5.5.7. *Let $p > 2$. Let n, u and i be integers with $0 \leq u \leq n$ and $0 < i < p$. Suppose $i \leq u$. Then the set of orbits of B_n on $\dot{Z}^n_{u,i}$ is*

$$\Big\{ \{\, z \in \dot{Z}^n_{u,i} \;:\; \tilde{\mathrm{u}}(z) = t \,\} \;:\; u \leq t \leq \min\{2u - i + 1, n\} \Big\}.$$

In particular, the maximum unsigned support size of an element in $\dot{Z}^n_{u,i}$ is $\min\{2u - i + 1, n\}$.

Proof. It is clear that B_n preserves the unsigned support size of elements in $\dot{Z}^n_{u,i}$. It is also clear that B_n is transitive on each set $\{\, z \in \dot{Z}^n_{u,i} \;:\; \tilde{\mathrm{u}}(z) = t \,\}$. Suppose $z \in \dot{Z}^n_{u,i}$. Since $z \in M_u$ we have $u \leq \tilde{\mathrm{u}}(z)$. The element

$$z = \{1 \dots i-1\} \cdot (\{i\} - \{\overline{i}\}) \cdot (\{i+1\} - \{\overline{i+1}\}) \cdot \dots \cdot (\{u\} - \{\overline{u}\})$$

attains this lower bound. If $u < n$ then we may replace $\{\overline{u}\}$ in z with $\{u+1\}$ to obtain an element of unsigned support size $u+1$. More generally for each integer a with $0 \leq a \leq u - i$ if $u + a < n$ then we can replace the factor $(\{u-a\} - \{\overline{u-a}\})$ by $(\{u-a\} - \{u+a+1\})$. Each replacement increases the unsigned support size by 1. Therefore by performing replacements we can continually increment the unsigned support size of z up to a maximum of $u + \min\{u - i + 1, n - u\} = \min\{2u - i + 1, n\}$. This is clearly an upper bound on the unsigned support size among elements of $\dot{Z}^n_{u,i}$ since each element involves only $2u - i + 1$ symbols from $[n\overline{n}]$. $\square$

Lemma 5.5.8. *Let $p > 2$. Let n, u and i be integers with $0 \leq u \leq n$ and $0 < i <$*

p. Let Z' be the set of elements of maximum unsigned support size in $\dot{Z}^n_{u,i}$. Then $\langle \dot{Z}^n_{u,i} \rangle_F = \langle Z' \rangle_F$.

Proof. For $n \in \{1, 2\}$ the result is easily checked. Suppose $n > 2$. Suppose $z \in \dot{Z}^n_{u,i}$ is not of maximum unsigned support size. Then $\tilde{\mathrm{u}}(z) < n$. Without loss of generality we may assume $z = z_0 \cdot (\{n\} - \{\overline{n}\})$ with $z_0 \in \dot{Z}^{n-1}_{u-1,i}$. Note there is a potential problem here if $i = u$ since then $i > u - 1$ but the definition of $\dot{Z}^{n-1}_{u-1,i}$ still makes sense in this case and gives $\dot{Z}^{n-1}_{u-1,i} = L_{u-1}$. The unsigned support size of z_0 is less than $n - 1$. So we can assume without loss of generality that z_0 lies in $\dot{Z}^{n-2}_{u-1,i}$. But then

$$z = z_0 \cdot (\{n\} - \{\overline{n}\}) = z_0 \cdot (\{n\} - \{n-1\}) - z_0 \cdot (\{\overline{n}\} - \{n-1\}) .$$

These two terms lie in $\dot{Z}^n_{u,i}$. Furthermore, they satisfy

$$\tilde{\mathrm{u}}(z_0 \cdot (\{n\} - \{n-1\})) = \tilde{\mathrm{u}}(z_0 \cdot (\{\overline{n}\} - \{n-1\})) = \tilde{\mathrm{u}}(z) + 1 > \tilde{\mathrm{u}}(z).$$

Continuing in this fashion completes the proof. □

Corollary 5.5.9. *Let $p > 2$. The FB_n-module $\langle \dot{Z}^n_{u,i} \rangle_F$ is cyclic, generated by any element $z \in \dot{Z}^n_{u,i}$ of maximum unsigned support size.*

Proof. By Lemma 5.5.8, the elements $z \in \dot{Z}^n_{u,i}$ with maximum unsigned support size span $\langle \dot{Z}^n_{u,i} \rangle_F$. But B_n is transitive on the set of such elements. □

Corollary 5.5.10. *Let $p > 2$. The homology module $\dot{H}^n_{u,0,i}$ is cyclic.*

Proof. Since $\dot{H}^n_{u,0,i} = \dot{K}^n_{u,0,i}/\dot{I}^n_{u,0,i}$, we have that $\dot{H}^n_{u,0,i}$ is generated by the classes $[z] = z + \dot{I}^n_{u,0,i}$ as z runs over $\dot{K}^n_{u,0,i}$. If $i > u$ then $\dot{K}^n_{u,0,i} = M^n_{u,0}$ and this module is cyclic. Otherwise $i \leq u$ and by Theorem 5.5.3 we have that $\dot{H}^n_{u,0,i}$ is spanned by the classes $[z]$ with $z \in \dot{Z}^n_{u,i}$. But, by Corollary 5.5.9, the module $\langle \dot{Z}^n_{u,i} \rangle_F$ is cyclic. □

5.5.2 Generators for the kernel of the doubles map

Now we will do the same for the doubles map. Redefine the local notation as follows:

$$\begin{aligned} K^n_{d,i} &:= \ddot{K}^n_{n,d,i}, \\ I^n_{d,i} &:= \ddot{I}^n_{n,d,i}, \text{ and} \\ H^n_{d,i} &:= \ddot{H}^n_{n,d,i}. \end{aligned}$$

For $\alpha, \beta \in [n\overline{n}]$ let $X_{\alpha,\beta}$ be the element

$$X_{\alpha,\beta} = (\{\alpha\} + \{\overline{\alpha}\}) \cdot \{\beta\overline{\beta}\} - \{\alpha\overline{\alpha}\} \cdot \big(\{\beta\} + \{\overline{\beta}\}\big)$$

of $M^n_{2,1}$. If $\{\alpha\overline{\alpha}\} \cap \{\beta\overline{\beta}\} = \emptyset$ we have

$$\ddot{\partial}\,(X_{\alpha,\beta}) = (\{\alpha\} + \{\overline{\alpha}\}) \cdot \big(\{\beta\} + \{\overline{\beta}\}\big) - (\{\alpha\} + \{\overline{\alpha}\}) \cdot \big(\{\beta\} + \{\overline{\beta}\}\big) = 0.$$

So $X_{\alpha,\beta}$ is an element of $\ker \ddot{\partial}$. In a way made precise in Theorem 5.5.12, the elements $X_{\alpha,\beta}$ are really all that is needed to build arbitrary non-trivial elements of the kernel $\ddot{K}^n_{n,d,i}$. Let T_n be the set of tuples $(\alpha_1, \ldots, \alpha_n)$ of distinct elements of $[n\overline{n}]$ such that the set $\{\alpha_1 \ldots \alpha_n\}$ contains no doubles. For $\tau = (\alpha_1, \ldots, \alpha_n) \in T_n$ define the element

$$z_\tau = \prod_{j=1}^{d-i+1} X_{\alpha_{2j-1},\alpha_{2j}} \cdot \prod_{k=1}^{i-1} \{\beta_k\overline{\beta_k}\} \cdot \prod_{\ell=1}^{n-2d+i-1} \{\gamma_\ell\}$$

where $\beta_k = \alpha_{2(d-i+1)+k}$ and $\gamma_\ell = \alpha_{2(d-i+1)+i-1+\ell} = \alpha_{2d-i+1+\ell}$.

Let

$$\ddot{Z}^n_{d,i} = \{\, z_\tau \;:\; \tau \in T_n \,\}.$$

As an example, take $(n, d, i) = (6, 3, 2)$. Let $\tau = (1, 2, 3, 4, 5, 6)$. Then

$$\begin{aligned} z_\tau &= X_{1,2} \cdot X_{3,4} \cdot \{5\overline{5}\} \cdot \{6\} \\ &= \Big(\big(\{1\} + \{\overline{1}\}\big) \cdot \{2\overline{2}\} - \{1\overline{1}\} \cdot \big(\{2\} + \{\overline{2}\}\big)\Big) \\ &\quad \cdot \Big(\big(\{3\} + \{\overline{3}\}\big) \cdot \{4\overline{4}\} - \{3\overline{3}\} \cdot \big(\{4\} + \{\overline{4}\}\big)\Big) \cdot \{5\overline{5}6\} \end{aligned}$$

is an element of $\ddot{Z}^6_{3,2}$. All other elements of $\ddot{Z}^6_{3,2}$ are obtained by applying signed permutations to z_τ.

Computing the images of elements of $\ddot{Z}^n_{d,i}$ under powers of $\ddot{\partial}$ is straightforward and summarized by the following lemma.

Lemma 5.5.11. *Let $p > 2$. Let n, d and i be integers. Let*

$$(\alpha_1, \ldots, \alpha_{2(d-i+1)}, \beta_1, \ldots, \beta_{i-1}, \gamma_1, \ldots, \gamma_{n-2d+i-1}) \in T_n.$$

Let t be a non-negative integer. Then

$$\ddot{\partial}^t(z_\tau) = \left(\prod_{j=1}^{d-i+1} X_{\alpha_{2j-1},\alpha_{2j}}\right) \cdot \ddot{\partial}^t \left(\prod_{k=1}^{i-1} \{\beta_k\overline{\beta_k}\}\right) \cdot \left(\prod_{\ell=1}^{n-2d+i-1} \{\gamma_\ell\}\right).$$

In particular, if $i \leq t$ then $\ddot{\partial}^t(z_\tau) = 0$, that is $\ddot{Z}^n_{d,i} \subseteq K^n_{d,i}$.

Proof. Notice that $\ddot{\partial}\left(X_{\alpha_{2j-1},\alpha_{2j}}\right) = 0$ and $\ddot{\partial}\,(\{\gamma_\ell\}) = 0$. The result now follows by multiple applications of Lemma 3.5.2. □

Theorem 5.5.12. *Let $p > 2$. Let n, d and i be integers with $0 \leq i \leq p$ and $i \leq d$. We have $K^n_{d,i} = \langle \ddot{Z}^n_{d,i}, I^n_{d,i} \rangle_F$.*

Proof. The inclusion $K^n_{d,i} \supseteq I^n_{d,i}$ is clear. The inclusion $K^n_{d,i} \supseteq \ddot{Z}^n_{d,i}$ holds by Lemma 5.5.11. So $K^n_{d,i} \supseteq \langle \ddot{Z}^n_{d,i}, I^n_{d,i} \rangle_F$.

It remains to show the reverse inclusion $K^n_{d,i} \subseteq \langle \ddot{Z}^n_{d,i}, I^n_{d,i} \rangle_F$. The proof is by induction on n via the branching rule. The case $n = 1$ is easily verified or checked in Appendix A. If $H^n_{d,i} = 0$ then $K^n_{d,i} = I^n_{d,i}$ so there is nothing to prove in this case. Suppose $n > 1$ and $H^n_{d,i} \neq 0$. Then in particular $0 < i < p$. The branching rule, Theorem 4.2.4, applies and yields an isomorphism of FB_{n-1}-modules

$$K^n_{d,i} \overset{f}{\cong} 0 \oplus K^{n-1}_{d,i} \oplus K^{n-1}_{d,i+1} \oplus K^{n-1}_{d-1,i-1}$$

(see the proof of Theorem 4.2.4 for the construction of the explicit f we will be using). To complete the proof we must show that the inverse images under f of the summands $K^{n-1}_{d,i}$, $K^{n-1}_{d,i+1}$ and $K^{n-1}_{d-1,i-1}$ are contained in $\langle \ddot{Z}^n_{d,i}, I^n_{d,i} \rangle_F$. By induction $K^{n-1}_{d,i} = \langle \ddot{Z}^{n-1}_{d,i}, I^{n-1}_{d,i} \rangle_F$ and $K^{n-1}_{d-1,i-1} = \langle \ddot{Z}^{n-1}_{d-1,i-1}, I^{n-1}_{d-1,i-1} \rangle_F$. Also by induction $K^{n-1}_{d,i+1} = \langle \ddot{Z}^{n-1}_{d,i+1}, I^{n-1}_{d,i+1} \rangle_F$ unless $i = d$. If $i = d$ then $K^{n-1}_{d,i+1} = M^{n-1}_{n-1,d} = \langle L^{n-1}_{n-1,d} \rangle_F$. So the proof reduces to checking the inverse images of $\ddot{Z}^{n-1}_{d,i+1}$ and $I^{n-1}_{d,i+1}$ if $i < d$, the inverse image of $L^{n-1}_{n-1,d}$ if $i = d$ and the inverse images of $\ddot{Z}^{n-1}_{d,i}, I^{n-1}_{d,i}, \ddot{Z}^{n-1}_{d-1,i-1}$ and $I^{n-1}_{d-1,i-1}$ in both cases. Note that f restricts to an isomorphism

$$I^n_{d,i} \overset{f}{\cong} 0 \oplus I^{n-1}_{d,i} \oplus I^{n-1}_{d,i+1} \oplus I^{n-1}_{d-1,i-1}.$$

So we know the inverse images of $I^{n-1}_{d,i}$, $I^{n-1}_{d,i+1}$ and $I^{n-1}_{d-1,i-1}$ are contained in $I^n_{d,i} \subseteq \langle \ddot{Z}^n_{d,i}, I^n_{d,i} \rangle_F$. The elements of $\ddot{Z}^{n-1}_{d,i}$ are of the form

$$z = \left(\prod_{j=1}^{d-i+1} X_{\alpha_{2j-1},\alpha_{2j}} \right) \cdot \left(\prod_{k=1}^{i-1} \{\beta_k \overline{\beta_k}\} \right) \cdot \left(\prod_{\ell=1}^{n-1-2d+i-1} \{\gamma_\ell\} \right)$$

with $(\alpha_1, \ldots, \alpha_{2(d-i+1)}, \beta_1, \ldots, \beta_{i-1}, \gamma_1, \ldots, \gamma_{n-2d+i-2}) \in T_{n-1}$. We have

$$f^{-1}(0, z, 0, 0) = z \cdot \{n\}$$

(see the proof of Theorem 4.2.4 for the construction of f^{-1}). Thus $f^{-1}(0, z, 0, 0)$ lies in $\ddot{Z}^n_{d,i} \subseteq \langle \ddot{Z}^n_{d,i}, I^n_{d,i} \rangle_F$, as required. The elements of $\ddot{Z}^{n-1}_{d-1,i-1}$ are of the form

$$z = \left(\prod_{j=1}^{d-i+1} X_{\alpha_{2j-1},\alpha_{2j}} \right) \cdot \left(\prod_{k=1}^{i-2} \{\beta_k \overline{\beta_k}\} \right) \cdot \left(\prod_{\ell=1}^{n-1-2d+i} \{\gamma_\ell\} \right)$$

with $(\alpha_1, \ldots, \alpha_{2(d-i+1)}, \beta_1, \ldots, \beta_{i-2}, \gamma_1, \ldots, \gamma_{n-1-2d+i}) \in T_{n-1}$. We have

$$f^{-1}(0,0,0,z) = z \cdot \{n\overline{n}\}$$

which lies in $\ddot{Z}^n_{d,i} \subseteq \langle \ddot{Z}^n_{d,i}, I^n_{d,i} \rangle_F$, as required. Now suppose $i < d$. The elements of $\ddot{Z}^{n-1}_{d,i+1}$ are of the form

$$z = \left(\prod_{j=1}^{d-i} X_{\alpha_{2j-1},\alpha_{2j}} \right) \cdot \left(\prod_{k=1}^{i} \{\beta_k \overline{\beta_k}\} \right) \cdot \left(\prod_{\ell=1}^{n-1-2d+i} \{\gamma_\ell\} \right)$$

with $(\alpha_1, \ldots, \alpha_{2(d-i)}, \beta_1, \ldots, \beta_i, \gamma_1, \ldots, \gamma_{n-1-2d+i}) \in T_{n-1}$. We have

$$f^{-1}(0,0,z,0) = \ddot{\partial}(z) \cdot \{n\overline{n}\} - iz \cdot (\{n\} + \{\overline{n}\}) \, .$$

But by Lemma 5.5.11

$$\begin{aligned}
\ddot{\partial}(z) &= \left(\prod_{j=1}^{d-i} X_{\alpha_{2j-1},\alpha_{2j}} \right) \cdot \ddot{\partial} \left(\prod_{k=1}^{i} \{\beta_k \overline{\beta_k}\} \right) \cdot \left(\prod_{\ell=1}^{n-1-2d+i} \{\gamma_\ell\} \right) \\
&= \left(\prod_{j=1}^{d-i} X_{\alpha_{2j-1},\alpha_{2j}} \right) \cdot \left(\sum_{m=1}^{i} \ddot{\partial} \left(\{\beta_m \overline{\beta_m}\} \right) \cdot \prod_{k \in [i] \setminus \{m\}} \{\beta_k \overline{\beta_k}\} \right) \cdot \left(\prod_{\ell=1}^{n-1-2d+i} \{\gamma_\ell\} \right) \\
&= \sum_{m=1}^{i} \left(\prod_{j=1}^{d-i} X_{\alpha_{2j-1},\alpha_{2j}} \right) \cdot \left(\{\beta_m\} + \{\overline{\beta_m}\} \right) \cdot \left(\prod_{k \in [i] \setminus \{m\}} \{\beta_k \overline{\beta_k}\} \right) \cdot \left(\prod_{\ell=1}^{n-1-2d+i} \{\gamma_\ell\} \right).
\end{aligned}$$

So

$$\begin{aligned}
&\ddot{\partial}(z) \cdot \{n\overline{n}\} - iz \cdot (\{n\} + \{\overline{n}\}) \\
&= \sum_{m=1}^{i} \left(\left(\prod_{j=1}^{d-i} X_{\alpha_{2j-1},\alpha_{2j}} \right) \cdot X_{\beta_m,n} \cdot \left(\prod_{k \in [i] \setminus \{m\}} \{\beta_k \overline{\beta_k}\} \right) \cdot \left(\prod_{\ell=1}^{n-1-2d+i} \{\gamma_\ell\} \right) \right) .
\end{aligned}$$

Now each term of this sum is an element of $\ddot{Z}^n_{d,i}$ so the sum lies in $\langle \ddot{Z}^n_{d,i} \rangle_F \subseteq \langle \ddot{Z}^n_{d,i}, I^n_{d,i} \rangle_F$ as required. Finally, suppose $i = d$. Then $K^{n-1}_{d,i+1} = M^{n-1}_{n-1,d}$. Let $x \in L^{n-1}_{n-1,d}$. Then

$$f^{-1}(0,0,x,0) = \ddot{\partial}(x) \cdot \{n\overline{n}\} - dx \cdot (\{n\} + \{\overline{n}\}) \, .$$

Write $x = \left(\prod_{k=1}^{d}\{\beta_k\overline{\beta_k}\}\right) \cdot \left(\prod_{\ell=1}^{n-1-d}\{\gamma_\ell\}\right)$ with $\bigcup_{k,\ell}\{\beta_k\gamma_\ell\} \in L_{n-1,0}^{n-1}$. Then

$$\begin{aligned}
&\ddot{\partial}(x) \cdot \{n\overline{n}\} - dx \cdot (\{n\} + \{\overline{n}\}) \\
&= \sum_{m=1}^{d} \left(\left(\{\beta_m\} + \{\overline{\beta_m}\}\right) \cdot \left(\prod_{k\in[d]\setminus\{m\}} \{\beta_k\overline{\beta_k}\}\right) \cdot \{n\overline{n}\} \cdot \left(\prod_{\ell=1}^{n-1-d} \{\gamma_\ell\}\right) \right. \\
&\qquad\qquad \left. - x \cdot \left(\{n\} + \{\overline{n}\}\right)\right) \\
&= \sum_{m=1}^{d} \left(X_{\beta_m,n} \cdot \left(\prod_{k\in[d]\setminus\{m\}} \{\beta_k\overline{\beta_k}\}\right) \cdot \left(\prod_{\ell=1}^{n-1-d} \{\gamma_\ell\}\right)\right).
\end{aligned}$$

Each term of this sum is an element of $\ddot{Z}_{d,d}^{n}$ so the sum lies in $\langle \ddot{Z}_{d,d}^{n}\rangle_F \subseteq \langle \ddot{Z}_{d,d}^{n}, I_{d,d}^{n}\rangle_F$ as required. $\square$

Corollary 5.5.13. *Let $p > 2$. Let $0 \leq d \leq u \leq n$ and i be integers with $0 \leq i \leq p$. If $i \leq d$ then $\ddot{K}_{u,d,i}^{n}$ is generated by $\ddot{I}_{u,d,i}^{n}$ and $\ddot{Z}_{d,i}^{u}$.*

Proof. Theorem 3.6.1 tells us

$$\ddot{K}_{u,d,i}^{n} = \mathrm{Ind}_{B_u\times B_{n-u}}^{B_n}\left(\ddot{K}_{u,d,i}^{u}\right).$$

Since $i \leq d$ Theorem 5.5.12 says that $\ddot{K}_{u,d,i}^{u}$ is generated by $\ddot{I}_{u,d,i}^{u}$ and $\ddot{Z}_{d,i}^{u}$. Theorem 3.6.1 again implies $\mathrm{Ind}_{B_u\times B_{n-u}}^{B_n}\left(\ddot{I}_{u,d,i}^{u}\right) = \ddot{I}_{u,d,i}^{n}$. Therefore $\ddot{K}_{u,d,i}^{n}$ is generated by $\ddot{I}_{u,d,i}^{n}$ and $\ddot{Z}_{d,i}^{u}$. $\square$

We remark that there seems to be a discrepancy between the singles and doubles cases in that for the singles case our natural set of generators broke up into multiple B_n-orbits and a little work was required to show that a single special orbit would suffice. In the doubles case our generators already fall into a single orbit.

Lemma 5.5.14. *Let $p > 2$. Let n, d and i be integers with $0 \leq i \leq p$. The module $H_{d,i}^{n}$ is cyclic.*

Proof. If $i > d$ then $K_{d,i}^{n} = M_{n,d}^{n}$. This module is a transitive permutation module so it is cyclic. Therefore the quotient $H_{d,i}^{n}$ is cyclic. Otherwise $i \leq d$. Then by Theorem 5.5.12 we have that $K_{d,i}^{n}$ is generated by the elements z_τ with $\tau \in T_n$. But B_n is easily seen to be transitive on the set of such elements. Hence the kernel $K_{d,i}^{n}$ is again cyclic. So $H_{d,i}^{n}$ is cyclic. $\square$

5.6 Irreducibility

Throughout this section we assume $p > 2$. Let $0 \leq d \leq u \leq n$ and i be integers with $0 < i < p$. Set $H = \dot{H}^n_{u,d,i}$. We give sufficient conditions for H to be irreducible. Under these conditions we identify H in terms of the standard representation theory, that is we identify the bipartition (λ, μ) of n for which the given irreducible homology module H is isomorphic to $D^{(\lambda,\mu)}$ (see Section 2.7 for the definition of $D^{(\lambda,\mu)}$). In short, we prove the following.

Theorem 5.6.1. *Let $p > 2$. Let $0 \leq d \leq u \leq n$ and i be integers with $0 < i < p$. Suppose $H = \dot{H}^n_{u,d,i}$ is non-zero. Then H is irreducible if any of the following conditions hold:*

- *(a) $0 = d = u$, in which case $H \cong 1_{B_n} \cong D^{((n),0)}$;*
- *(b) $0 = d < u$ and $2u + p - i - n = 1$ in which case $H \cong D^{(\lambda,0)}$ where λ is the partition of n into two parts u and $n - u$;*
- *(c) $d = u = n$, in which case $H \cong 1_{B_n} \cong D^{((n),0)}$;*
- *(d) $d < u = n$ and $i = 1$, in which case $H \cong D^{((d),(n-d))}$.*

Case (d) is the only case where the base group acts non-trivially, as indicated by the non-empty second constituent $(n-d)$ of the bipartition $((d),(n-d))$. Note that (c) and (d) give $\dot{H}^n_{n,d,1} \cong \dot{H}^n_{n,n-d,1}$ as FS_n-modules. This is the appropriate generalisation of the small duality result in Theorem 5.1.1 from the even characteristic case.

We also prove the corresponding result for the doubles homology modules:

Theorem 5.6.2. *Let $p > 2$. Let $0 \leq d \leq u \leq n$ and i be integers with $0 < i < p$. Suppose $H = \ddot{H}^n_{u,d,i}$ is non-zero. Then H is irreducible if any of the following conditions hold:*

- *(a) $d = u = n$, in which case $H \cong 1_{B_n} \cong D^{((n),0)}$;*
- *(b) $d < u = n$ and $u - 2d + i = 1$ in which case $H \cong D^{(\lambda,0)}$ where λ is the partition of n into two parts d and $n - d$;*
- *(c) $0 = d = u$, in which case $H \cong 1_{B_n} \cong D^{((n),0)}$;*
- *(d) $0 = d < u$ and $i = p - 1$, in which case $H \cong D^{((n-u),(u))}$.*

Again, case (d) is the only case where the base group acts non-trivially. Note that as usual these results looks very similar, hinting at the possible duality.

The S_n-structure of the homology modules will prove invaluable when proving irreducibility in some cases. Recall the setup for the Boolean algebra case in Section 2.8 as follows. Let k be an integer. Let L_k denote the k element subsets of $[n]$. Set $M_k = FL_k$. Define the map $\partial : M_k \to M_{k-1}$ by taking each subset of $[n]$ to the sum of its subsets which can be obtained by removing precisely one element. For $0 \leq i \leq p$ set $K^n_{k,i} = \ker \partial^i \cap M_k$ and $I^n_{k,i} = \partial^{p-i}(M_{k+p-i})$ with homology module $H^n_{k,i} = K^n_{k,i}/I^n_{k,i}$.

Lemma 5.6.3. *Let $p > 2$. Let n, u and i be integers with $0 \leq i \leq p$. Let $H^n_{u,i}$ be the incidence homology module for the symmetric group defined in Section 2.8. Then there is an FS_n-embedding*

$$H^n_{u,i} \hookrightarrow \dot{H}^n_{u,0,i}.$$

Proof. Define a map $\varphi : H^n_{u,i} \to \dot{H}^n_{u,0,i}$ by

$$\varphi\left(z + I^n_{u,i}\right) = z + \dot{I}^n_{u,0,i}$$

for all $z \in K^n_{u,i}$. We claim φ is an FS_n-embedding. First we check φ is well-defined. Suppose $x, y \in K^n_{u,i}$ with $x - y \in I^n_{u,i}$. Then $x - y \in \dot{I}^n_{u,0,i}$, since $I^n_{u,i} \subseteq \dot{I}^n_{u,0,i}$. Clearly φ is linear and commutes with S_n. It remains to show that φ is injective. Suppose $z \in K^n_{u,i}$ with $\varphi(z + I^n_{u,i}) = 0$. Then $z \in \dot{I}^n_{u,0,i}$. We need to show that $z \in I^n_{u,i}$. Let $y \in M^n_{u+p-i,0}$ be such that $\partial^i(y) = z$. For any integer k denote by M_k the space with basis the k element subsets of $[n]$ as above. Define an F-linear map $\pi : M^n_{u+p-i,0} \to M_{u+p-i}$ by

$$x \mapsto (x \cdot \overline{x}) \cap [n]$$

for each $x \in L^n_{u+p-i,0}$. The map π simply removes all bars from elements. For example

$$\pi\left(\{1\overline{3}4\overline{5}\} + \{2\overline{4}56\}\right) = \{1345\} + \{2456\}.$$

We claim $\partial^{p-i}(\pi(y)) = \partial^{p-i}(y)$ so that $z \in I^n_{u,i}$, as required. This is clear since $\partial^{p-i}(y) \in M_u$. Note that we did not use that $\partial^i(z) = 0$ so in fact we have shown the slightly more general:

$$\dot{I}^n_{u,0,i} \cap M_u = I^n_{u,i}.$$

□

Proof of Theorem 5.6.1. Throughout, H is assumed to be non-zero. Let $\partial = \dot{\partial}$. In the case $0 = d = u$ we want to show that H is the trivial FB_n-module $D^{((n),0)}$. The sequence to consider is

$$0 = M^n_{-i,0} \xleftarrow{\partial^i} M^n_{0,0} = F\{\emptyset\} \xleftarrow{\partial^{p-i}} M^n_{p-i,0}.$$

So clearly we have $\dot{K}^n_{0,0,i} = \ker \partial^i \cap M^n_{0,0} = F\{\emptyset\}$, which is the trivial module. Since

$H \neq 0$ we must have $\dot{I}^n_{0,0,i} = 0$ and $H \cong \dot{K}^n_{0,0,i}$. Thus H is the trivial module. It is clear from the definition in Section 2.7 that $D^{((n),0)}$ is the trivial module. This shows (a).

The case $d = u = n$ is similar. The sequence to consider is

$$0 = M^n_{n-i,n} \xleftarrow{\partial^i} M^n_{n,n} = F\{\{1\bar{1}\dots n\bar{n}\}\} \xleftarrow{\partial^{p-i}} M^n_{n+p-i,n} = 0.$$

So $\dot{K}^n_{n,n,i} = \ker \partial^i \cap M^n_{n,n} = M^n_{n,n} = F\{\{1\bar{1}\dots n\bar{n}\}\}$ is the trivial module. The same argument as in case (a) gives $H \cong D^{((n),0)}$ is the trivial module. Thus (c) is true.

Suppose $0 = d < u$ and $2u+p-i-n = 1$. Lemma 5.6.3 tells us the symmetric group module $H^n_{u,i}$ embeds in H. On the other hand Theorem 5.3.2 gives $\dim H = \dim H^n_{u,i}$ since $\ell = 2u + p - i - n = 1$. Thus $H \cong H^n_{u,i}$ as FS_n-modules. Now Corollary 2.8.9 says $H^n_{u,i} \cong D^\lambda$ is the irreducible module for the symmetric group corresponding to the partition λ of n into two parts of size u and $n-u$. It remains to show that the base group acts trivially. Since $H \neq 0$ the branching rule Theorem 4.2.3 gives

$$H = \dot{H}^n_{u,0,i} \cong \dot{H}^{n-1}_{u,0,i+1} \oplus \dot{H}^{n-1}_{u-1,0,i} \oplus \dot{H}^{n-1}_{u-1,0,i-1} \oplus \dot{H}^{n-1}_{u-1,-1,i}.$$

The rightmost summand is clearly zero. Using the notation of Section 5.3 we have $\ell(n,u,i) = 2u + p - i - n = 1$. Lemma 5.3.8 says $\ell(n-1, u-1, i) = 0$. Therefore Theorem 5.2.1 says the summand $\dot{H}^{n-1}_{u-1,0,i}$ is zero. Since $0 \neq H \cong \dot{H}^{n-1}_{u,0,i+1} \oplus \dot{H}^{n-1}_{u-1,0,i-1}$ at most one of these summands is zero. If $\dot{H}^{n-1}_{u,0,i+1}$ is non-zero then it satisfies condition (b) so the element $(1,\bar{1})$ acts as the identity by induction. If $\dot{H}^{n-1}_{u-1,0,i-1}$ is non-zero we have two cases. Either it satisfies (b) and $(1,\bar{1})$ acts as the identity by induction or it satisfies (a) and $(1,\bar{1})$ acts as the identity since we have already proved (a). Therefore the element $(1,\bar{1})$ acts as the identity on H. Now the conjugates of $(1,\bar{1})$ under S_n must all act as the identity. Since these generate the base group this proves (b).

The remaining case is $d < u = n$ and $i = 1$. We want to show $H \cong D^{((d),(n-d))}$. By Theorem 3.6.6 we have

$$H = \dot{H}^n_{n,d,1} \cong \mathrm{Ind}^{B_n}_{B_{n-d}\times B_d}\left(\dot{H}^{n-d}_{n-d,0,1}\right).$$

The module $\dot{H}^{n-d}_{n-d,0,1}$ also satisfies (d). So the proof reduces to two steps:

(i) Show that $\dot{H}^n_{n,0,1} \cong D^{(0,(n))}$.
(ii) Show that $\mathrm{Ind}^{B_n}_{B_{n-d}\times B_d}\left(D^{(0,(n-d))}\right) \cong D^{((d),(n-d))}$.

For (i) note that $\dot{I}^n_{n,0,1} = \partial^{p-1}(M^n_{n+p-1,0}) = \partial^{p-1}(0) = 0$. Hence $\dot{H}^n_{n,0,1} \cong \dot{K}^n_{n,0,1}$. By

Theorem 5.5.3 we have

$$\dot{H}^n_{n,0,1} \cong \big\langle (\{1\} - \{\overline{1}\}) \cdot \dots \cdot (\{n\} - \{\overline{n}\}) \big\rangle_F. \tag{5.6.4}$$

From (5.6.4) we see that $\dot{H}^n_{n,0,1}$ is 1-dimensional. We can also read off the action of the group elements. The generators $(\alpha, \overline{\alpha})$ for $1 \leq \alpha \leq n$ of the base group act as -1. The top group S_n acts as the identity. Hence $\dot{H}^n_{n,0,1} \cong D^{(0,(n))}$, as required. Now we prove (ii). Following Section 2.7, the module $D^{((d),(n-d))}$ is obtained as follows. For the representation of the base group, we take the outer tensor product V say of d copies of the trivial S_2 module and $n-d$ copies of the sign module. Enlarging this to a module for the inertia group $B_d \times B_{n-d}$ of V yields a module isomorphic to the outer tensor product

$$D^{((d),0)} \# D^{(0,(n-d))}. \tag{5.6.5}$$

Call this D_1. This is the first of two modules for the inertia group which we will eventually tensor together before inducing to B_n to obtain $D^{((d),(n-d))}$. The inertia factor of V is $S_d \times S_{n-d}$. The second module of the inertia group is the module obtained by starting with the outer tensor product $S^{(d)} \# S^{(n-d)}$. Letting the base group act trivially yields the module for the inertia group. Call this D_2. Note that D_2 is just the trivial module $1_{B_d \times B_{n-d}}$ of the inertia group. Finally

$$\begin{aligned} D^{((d),(n-d))} &:= \operatorname{Ind}^{B_n}_{B_d \times B_{n-d}} (D_1 \otimes D_2) \\ &\cong \operatorname{Ind}^{B_n}_{B_d \times B_{n-d}} (D_1) \\ &\cong \operatorname{Ind}^{B_n}_{B_d \times B_{n-d}} \left(D^{((d),0)} \# D^{(0,(n-d))} \right). \end{aligned}$$

We need to check this agrees with $\operatorname{Ind}^{B_n}_{B_{n-d} \times B_d} \left(D^{(0,(n-d))} \right)$. Explicitly writing out the inflation stage we have

$$\begin{aligned} \operatorname{Ind}^{B_n}_{B_{n-d} \times B_d} \left(D^{(0,(n-d))} \right) &= \operatorname{Ind}^{B_n}_{B_{n-d} \times B_d} \operatorname{Infl}^{B_{n-d} \times B_d}_{B_{n-d}} \left(D^{(0,(n-d))} \right) \\ &\cong \operatorname{Ind}^{B_n}_{B_{n-d} \times B_d} \left(D^{(0,(n-d))} \# D^{((d),0)} \right). \end{aligned}$$

Unfortunately the $n-d$ and d are back to front but the modules are nonetheless isomorphic. □

We now turn to the doubles case. An analogue of Lemma 5.6.3 for the doubles case is the following.

Lemma 5.6.6. *Let $p > 2$. Let n, d and i be integers with $0 \leq i \leq p$. Let $H^n_{d,i}$ be the incidence homology module for the symmetric group defined in Section 2.8. Then there is an FS_n-embedding*

$$H^n_{d,i} \hookrightarrow \ddot{H}^n_{n,d,i}.$$

Proof. Recall the setup of Section 2.8: For k an integer define the following. Let L_k be the set of k element subsets of $[n]$. Set $M_k = FL_k$. Let ∂ be incidence map which takes any subset of $[n]$ to the sum of its subsets which can be obtained by removing precisely one element. This restricts to a linear map $M_k \to M_{k-1}$ for each k. We define a linear map $\varphi_k : M_k \to M^n_{n,k}$ by

$$\varphi_k(x) = x \cdot \overline{x} \cdot \prod_{\alpha \in [n] \setminus x} (\{\alpha\} + \{\overline{\alpha}\})$$

for each $x \in L_k$. Set $\varphi = \oplus_k \varphi_k$. We claim that φ is a map of p-complexes of FS_n-modules, that is φ commutes with S_n and the following diagram commutes

$$\begin{tikzcd}
\dots & M_{d-1} \arrow[l, "\partial"'] \arrow[d, "\varphi"] & M_d \arrow[l, "\partial"'] \arrow[d, "\varphi"] & M_{d+1} \arrow[l, "\partial"'] \arrow[d, "\varphi"] & \dots \arrow[l, "\partial"'] \\
\dots & M^n_{n,d-1} \arrow[l, "\ddot{\partial}"'] & M^n_{n,d} \arrow[l, "\ddot{\partial}"'] & M^n_{n,d+1} \arrow[l, "\ddot{\partial}"'] & \dots . \arrow[l, "\ddot{\partial}"']
\end{tikzcd}$$

Clearly φ commutes with S_n. To see that the diagram commutes let $x \in L_k$. Then

$$\begin{aligned}
\ddot{\partial}\varphi(x) &= \ddot{\partial} \left(x \cdot \overline{x} \cdot \prod_{\alpha \in [n] \setminus x} (\{\alpha\} + \{\overline{\alpha}\}) \right) \\
&= \ddot{\partial} \left(x \cdot \overline{x} \right) \cdot \prod_{\alpha \in [n] \setminus x} (\{\alpha\} + \{\overline{\alpha}\}) \text{ since } \ddot{\partial}(\{\alpha\} + \{\overline{\alpha}\}) = 0 \text{ for each } \alpha \in [n] \\
&= \sum_{\beta \in x \cdot \overline{x}} \left(x \cdot \overline{x} \setminus \{\beta\} \right) \cdot \prod_{\alpha \in [n] \setminus x} (\{\alpha\} + \{\overline{\alpha}\}) . \qquad (5.6.7)
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
\varphi\partial(x) &= \varphi \left(\sum_{\gamma \in x} x \setminus \{\gamma\} \right) \\
&= \sum_{\gamma \in x} (x \setminus \{\gamma\}) \cdot \overline{(x \setminus \{\gamma\})} \cdot (\{\gamma\} + \{\overline{\gamma}\}) \cdot \prod_{\delta \in [n] \setminus x} (\{\delta\} + \{\overline{\delta}\}).
\end{aligned}$$

This agrees with (5.6.7). So φ commutes with the boundary maps. Hence φ induces a map φ_* on homology $H^n_{d,i} \to \ddot{H}^n_{n,d,i}$. We claim this map is injective. First we will show that φ itself is injective. For each integer k define a map $\psi_k : M^n_{n,k} \to M_k$ by

$$\psi(y) = 2^{k-n} \left(\ddot{y} \cap [n] \right)$$

where $\ddot{y}$ denotes the doubles part of y, see Section 3.1. Note $p \neq 2$ so 2^{k-n} exists in

F. Then for $x \in L_k$ we have

$$\psi_k\varphi_k(x) = \psi_k\left(x \cdot \overline{x} \cdot \prod_{\alpha\in[n]\setminus x}(\{\alpha\} + \{\overline{\alpha}\})\right).$$

Since $x \in L_k$ the expression inside the brackets is a sum of 2^{n-k} signed sets. The doubles part of each of these signed sets is $x \cdot \overline{x}$. Therefore $\psi_k\varphi_k(x) = 2^{k-n}2^{n-k}x = x$.

It remains to show that φ_* is injective. Let $z \in \ker\partial^i \cap M_d$ with $\varphi(z) \in \ddot{\partial}^{p-i}(M^n_{n,d+p-i})$. Pick $y \in M^n_{n,d+p-i}$ so that $\ddot{\partial}^{p-i}(y) = \varphi(z)$. Notice that $\varphi(z)$ is fixed by the base group H of B_n. Therefore

$$\sum_{h\in H} h\ddot{\partial}^{p-i}(y) = |H|\,\ddot{\partial}^{p-i}(y) = 2^n\ddot{\partial}^{p-i}(y).$$

Since $p > 2$ we have that 2^n is invertible in F. Therefore

$$\ddot{\partial}^{p-i}\left(2^{-n}\sum_{h\in H} hy\right) = 2^{-n}\sum_{h\in H} h\ddot{\partial}^{p-i}(y) = \ddot{\partial}^{p-i}(y).$$

The element $y' = 2^{-n}\sum_{h\in H} hy$ is fixed by the base group. Therefore y' is a linear combination of orbit sums of the base group on $L^n_{n,d}$. We claim that φ is surjective onto the space spanned by these orbit sums. This is clear since the orbit sum of any element $v \in L^n_{n,d+p-i}$ is given by $\varphi(\ddot{v}\cap[n])$. So there exists $x \in M_{d+p-i}$ with $\varphi(x) = y'$. Then

$$\varphi\partial^{p-i}(x) = \ddot{\partial}^{p-i}\varphi(x) = \ddot{\partial}^{p-i}(y') = \ddot{\partial}^{p-i}(y) = \varphi(z).$$

Since φ is injective we must have $\partial^{p-i}(x) = z$. This completes the proof that φ_* is injective. □

Proof of Theorem 5.6.2. This proof will be very similar to the proof of the singles map analogue Theorem 5.6.1. In some cases we will rely on work already done in that case. Let $H = \ddot{H}^n_{u,d,i}$. In cases (a) and (c) the module $M^n_{u,d}$ is the trivial module. Therefore since $H \neq 0$ and H is a subquotient of $M^n_{u,d}$ we must have that H is the trivial module $D^{((n),0)}$.

Suppose we are in case (b), that is $d < u = n$ and $u - 2d + i = 1$. By Lemma 5.6.6 the symmetric group module $H^n_{d,i}$ embeds in H. Note that $2d - i = n - 1$ and $d < n$ implies $i \leq d$. Hence Theorem 2.8.5 applies and gives $H^n_{d,i} \cong D^\lambda$ the irreducible FS_n-module indexed by the partition λ of n into two parts of size d and $n - d$. Theorem 5.3.18 says that $\dim H = \dim \dot{H}^n_{n-d,0,p-i}$. But $\ell(n, n-d, p-i) = 2(n -$

$d) + p - (p - i) - n = n - 2d + i = 1$. Therefore $\dot{H}^n_{n-d,0,p-i}$ satisfies the conditions of Theorem 5.6.1(b). Hence $\dot{H}^n_{n-d,0,p-i} \cong D^\mu$ where μ is the partition of n into two parts of size $n - d$ and $n - (n - d) = d$. Since $\lambda = \mu$ we have $\dot{H}^n_{n-d,0,p-i} \cong H^n_{d,i}$ as S_n-modules. Hence $\dim H = \dim H^n_{d,i}$ and $H \cong H^n_{d,i} \cong D^\lambda$ as an S_n-module. It remains to show that the base group acts as the identity on H. Since $H \neq 0$ the branching rule Theorem 4.2.4 applies and gives

$$H \cong \ddot{H}^{n-1}_{n,d,i} \oplus \ddot{H}^{n-1}_{n-1,d,i} \oplus \ddot{H}^{n-1}_{n-1,d,i+1} \oplus \ddot{H}^{n-1}_{n-1,d-1,i-1}$$

as an FB_{n-1}-module. Clearly the first summand vanishes since $n - 1 < n$. Since $n-1-2d+i = 0$ the second summand also vanishes by Theorem 5.2.2. We check the action of $(1, \overline{1})$ on the two remaining summands. Note that because H is non-zero at most one of these two summands may vanish. If the third summand is non-zero, we have two cases. If $n-1 = d$ then this summand satisfies condition (a) so $(1, \overline{1})$ acts as the identity. Otherwise, since we have $(n-1)-2d+(i+1) = 1$, this summand satisfies condition (b) and again $(1, \overline{1})$ acts as the identity, this time by induction. We have $(n - 1) - 2(d - 1) + (i - 1) = 1$. Therefore the fourth summand, if non-zero, satisfies (b). So $(1, \overline{1})$ acts as the identity by induction. Thus $(1, \overline{1})$ acts as the identity on H. Therefore all conjugates of $(1, \overline{1})$ act as the identity also. Since the conjugates of $(1, \overline{1})$ generate the base group (b) is proved.

It remains to show (d). In this case we have $0 = d < u$ and $i = p - 1$. We need to show $H \cong D^{((n-u),(u))}$. By Theorem 3.6.6 we have

$$\ddot{H}^n_{u,0,p-1} \cong \mathrm{Ind}^{B_n}_{B_u \times B_{n-u}} \left(\ddot{H}^u_{u,0,p-1} \right).$$

Thus we need to show

(i) $\ddot{H}^n_{n,0,p-1} \cong D^{(0,(n))}$ and
(ii) $\mathrm{Ind}^{B_n}_{B_u \times B_{n-u}} \left(D^{(0,(u))} \right) \cong D^{((n-u),(u))}$.

We have already shown (ii) in the proof of Theorem 5.6.1(d). To prove (i) we use induction via the branching rule, Theorem 4.2.4. The cases $n \leq 2$ can be checked in Appendix A, noting that the module $D^{(0,(n))}$ is the 1-dimensional module where $(1, \overline{1})$ acts as -1 and S_n acts trivially. Suppose $n > 2$. The branching rule gives

$$\begin{aligned} \ddot{H}^n_{n,0,p-1} &\cong \ddot{H}^{n-1}_{n,0,p-1} \oplus \ddot{H}^{n-1}_{n-1,0,p-1} \oplus \ddot{H}^{n-1}_{n-1,0,p} \oplus \ddot{H}^{n-1}_{n-1,-1,p-2} \\ &\cong \ddot{H}^{n-1}_{n-1,0,p-1}. \end{aligned}$$

By induction this module is 1-dimensional. Also by induction the element $(1, \overline{1})$ acts as -1 and the element $(1,2)$ acts as the identity. By conjugating $(1,2)$ by suitable elements we see that the whole of S_n acts trivially. Thus (d) is proved. $\square$

Chapter 6

A conjecture

Throughout the thesis we have taken note that for every result about the singles map there is a corresponding, similar looking, result about the doubles map. In this chapter we formulate this more precisely as a conjecture. As usual let $p > 0$ be the characteristic of our field F. Let $0 \leq d \leq u \leq n$ and i be integers. Let $0 < i < p$. We have seen in Theorem 5.3.18 that there is an isomorphism of vector spaces $\dot{H}^n_{u,d,i} \cong \ddot{H}^n_{n-d,n-u,p-i}$. The question of whether there is such an isomorphism which commutes with the group action seems to be much more difficult. I conjecture there is such a map.

Conjecture 6.0.1 (Duality)**.** *Let $0 \leq d \leq u \leq n$ and i be integers. Let $0 < i < p$. Then we have an isomorphism of FB_n-modules*

$$\dot{H}^n_{u,d,i} \cong \ddot{H}^n_{n-d,n-u,p-i}.$$

In the even characteristic case this is Theorem 5.1.7. Comparing the irreducibility results, Theorem 5.6.1 and Theorem 5.6.2, we see that the conjecture holds in the cases (a) $0 = d = u$, (b) $0 = d < u$ and $2u + p - i - n = 1$, (c) $d = u = n$ and (d) $d < u = n$ and $i = 1$.

The complement map produces an isomorphism of permutation modules $M^n_{u,d} \cong M^n_{n-d,n-u}$. So it would be tempting to think that the conjecture arises from an isomorphism of chain complexes. However the conjecture is strictly about homology modules. The reason for this is that an isomorphism of chain complexes would induce isomorphisms in kernels and images but there are straightforward counterexamples to this. For example $\dot{K}^n_{0,0,i} = M^n_{0,0} \neq 0$ but $\ddot{K}^n_{n,n,p-i} = 0$ for $p - i \leq n$. Note that the bottom row ($d = 0$) of the homology grid corresponds to the cross-polytope. The dual of this (according to the conjecture) is the right hand column ($u = n$) which corresponds to the hypercube, the dual polytope. Thus for $d = 0$ the conjecture becomes "for $0 \leq u \leq n$ the u-th incidence homology of the cross-polytope, with parameter i, is the $(n - u)$-th incidence homology of the hypercube, with parameter $p - i$". If we replace one of these homologies with cohomology then we obtain a duality resembling one half of Poincaré duality, see Munkres [25] for an exposition of the usual Poincaré duality. We have already stated and proved this modified duality as Theorem 5.3.16. In the proof we saw that this version does come from a chain complex isomorphism, the one induced by the complement map.

6.1 Reduction to the case $d = 0$

As usual we can reduce to the case $d = 0$.

Lemma 6.1.1. *Suppose Conjecture 6.0.1 holds for $d = 0$. Then Conjecture 6.0.1 holds for all d.*

Proof. Suppose Conjecture 6.0.1 holds for $d = 0$. This means that for all $n \geq 0$ and all $0 \leq u \leq n$ with $0 < i < p$ we have

$$\dot{H}^n_{u,0,i} \cong \ddot{H}^n_{n,n-u,p-i}.$$

Suppose $d > 0$. Then

$$\begin{aligned}
\dot{H}^n_{u,d,i} &\cong \mathrm{Ind}^{B_n}_{B_{n-d} \times B_d} \left(\dot{H}^{n-d}_{u-d,0,i} \right) && \text{by Theorem 3.6.6} \\
&\cong \mathrm{Ind}^{B_n}_{B_{n-d} \times B_d} \left(\ddot{H}^{n-d}_{n-d,n-d-(u-d),p-i} \right) && \text{by Conjecture 6.0.1 with } d = 0 \\
&= \mathrm{Ind}^{B_n}_{B_{n-d} \times B_d} \left(\ddot{H}^{n-d}_{n-d,n-u,p-i} \right) \\
&\cong \ddot{H}^n_{n-d,n-u,p-i} && \text{by Theorem 3.6.6.}
\end{aligned}$$

□

With this in mind computer evidence shows that Conjecture 6.0.1 is true for $p \leq 13$ with $n \leq 4$ and also for $p \leq 5$ with $n = 5$.

6.2 The case $i \leq u$ and $p - i > n - u$

Recall the notation from the end of Section 5.3. Specifically $b(-,-)$ is our bilinear form, c is the complement map and $\dot{\varepsilon}$ is the transpose of the singles map $\dot{\partial}$.

Lemma 6.2.1. *We have $\dot{K}^n_{u,0,i} = (\dot{\varepsilon}^i(M^n_{u-i,0}))^\perp$ with respect to the standard bilinear form on $M^n_{u,0}$.*

Proof. Let $x \in M^n_{u,0}$. Then $\dot{\partial}^i(x) = 0$ if and only if $b\left(\dot{\partial}^i(x), y\right) = 0$ for all $y \in L^n_{u-i,0}$ since the bilinear form on $M^n_{u-i,0}$ is non-degenerate. This holds if and only if $b(x, \dot{\varepsilon}^i(y)) = 0$ for all $y \in L^n_{u-i,0}$. But this is the same thing as $x \in (\dot{\varepsilon}(L^n_{u-i,0}))^\perp = (\dot{\varepsilon}(M^n_{u-i,0}))^\perp$. □

Conjecture 6.2.2. *Suppose $\dot{H}^n_{u,0,i} \neq 0$. Let $p - i > n - u$. Then the identity map of $M^n_{u,0}$ induces an isomorphism in homology*

$$\dot{H}^n_{u,0,i} \cong c\left(\ddot{H}^n_{n,n-u,p-i}\right) = M^n_{u,0}/\dot{\varepsilon}^i(M^n_{u-i,0}).$$

Notes:

(a) $p-i > n-u$ means we can't use our result about generators of doubles homology.

(b) In the $i = 1$ case we already know there exists an isomorphism by the irreducibility theorems.

Steps towards a proof:

Lemma 6.2.3. *Suppose $\dot{H}^n_{u,0,i} \neq 0$. Let $p - i > n - u$. Then the identity map of $M^n_{u,0}$ induces an isomorphism in homology*

$$\dot{H}^n_{u,0,i} \cong c\left(\ddot{H}^n_{n,n-u,p-i}\right) = M^n_{u,0}/\dot{\varepsilon}^i(M^n_{u-i,0})$$

if and only if $\dot{\varepsilon}^i(M^n_{u-i,0})$ is non-degenerate.

Proof. Since $p - i > n - u$ we have $\dot{I}^n_{u,0,i} = \dot{\partial}^{p-i}(M^n_{u+p-i,0}) = \dot{\partial}^{p-i}(0) = 0$. So the identity maps images to images. Also since $p - i > n - u$ we have $c\left(\ddot{K}^n_{n,n-u,p-i}\right) = \ker \dot{\varepsilon}^{p-i} \cap M^n_{u,0} = M^n_{u,0}$. So the identity also maps kernels to kernels. So it induces a map on homology, Id_* say. By the standard dimension counting arguments it suffices to show that Id_* is either injective or surjective. The map is injective if and only if for all $z \in \dot{K}^n_{u,0,i}$ we have $\mathrm{Id}(z) \in \dot{\varepsilon}^i(M^n_{u-i,0})$ implies $z \in \dot{I}^n_{u,0,i} = 0$. That is if and only if $\dot{K}^n_{u,0,i} \cap \dot{\varepsilon}^i(M^n_{u-i,0}) = 0$. By Lemma 6.2.1 $\dot{K}^n_{u,0,i} = (\dot{\varepsilon}^i(M^n_{u-i,0}))^{\perp}$. So injectivity of Id_* is equivalent to the condition that $\dot{\varepsilon}^i(M^n_{u-i,0})$ is non-degenerate[1]. $\square$

Conjecture 6.2.4. *Let $0 \leq u \leq n$. Suppose $p - i > n - u$. Then $\dot{\varepsilon}^i(M^n_{u-i,0})$ is non-degenerate.*

This has been checked with GAP for $n \leq 8$ and $p \leq 17$. It seems that this could be brute-forced by hand but requires some very involved linear equations and combinatorics.

6.3 The case $u = n$ and $p > 2$

Let $p > 2$. In the case $u = n$ we have a decomposition of $M^n_{u,0}$. We use this to prove Conjecture 6.0.1 in the case $u = n$. I am relatively confident this method can be extended to a full proof of Conjecture 6.0.1, at least in the semisimple case (that is $p > n$).

For $0 \leq j \leq n$ define the module

$$V^n_j = \mathrm{Ind}^{B_n}_{B_j \times B_{n-j}} \left(\langle \prod_{a=1}^{j} (\{a\} + \{\overline{a}\}) \cdot \prod_{b=j+1}^{n} (\{b\} - \{\overline{b}\}) \rangle_F \right).$$

This is an FB_n-module. Clearly we have $V^n_j \cong D^{((j),(n-j))}$.

Lemma 6.3.1. *Let $p > 2$. Let $n \geq 1$. We have*

$$M^n_{n,0} = \bigoplus_{j=0}^{n} V^n_j.$$

Proof. First note that $\dim V^n_j = \binom{n}{j}$. Hence

$$\sum_{j=0}^{n} \dim V^n_j = \sum_{j=0}^{n} \binom{n}{j} = 2^n = \binom{n}{n} 2^n = \dim M^n_{n,0}.$$

It remains to show that $\sum_{j=0}^{n} V^n_j$ spans $M^n_{n,0}$. For this we can use induction on n. Let

[1]Non-degenerate means there is no non-zero v in $\dot{\varepsilon}^i(M^n_{u-i,0})$ with $b\,(v, y) = 0$ for all y in $\dot{\varepsilon}^i(M^n_{u-i,0})$.

$x = \{1 \dots n\}$. By induction we can write

$$x = \left(\sum_{j=0}^{n-1} v_j \right) \cdot \{n\} = \sum_{j=0}^{n-1} v_j \cdot \{n\}$$

with each $v_j \in V_j^{n-1}$. But

$$v_j \cdot \{n\} = \frac{1}{2} \left(v_j \cdot (\{n\} + \{\overline{n}\}) + v_j \cdot (\{n\} - \{\overline{n}\}) \right).$$

Clearly this element lies in $\sum_{i=0}^{n} V_i^n$. Since x generates $M_{n,0}^n$ we are done. □

Lemma 6.3.2. *Let $p > 2$. Let $n \geq 1$. Suppose $0 < i < p$. We have*

$$\dot{K}_{n,0,i}^n = \bigoplus_{j=0}^{i-1} V_j^n$$

and

$$\ddot{I}_{n,0,p-i}^n = \bigoplus_{j=i}^{n} V_j^n.$$

Proof. First we will show the statement about the singles kernel $\dot{K}_{n,0,i}^n$. By Lemma 6.3.1 we have $M_{n,0}^n = \bigoplus_{j=0}^n V_j^n$. Let $0 \leq j \leq n$. Consider the element

$$x = (\{1\} + \{\overline{1}\}) \cdots (\{j\} + \{\overline{j}\}) \cdot (\{j+1\} - \{\overline{j+1}\}) \cdots (\{n\} - \{\overline{n}\})$$

which generates V_j^n. We can write x as

$$x = \left(\sum_{y \in M_{j,0}^j} y \right) \cdot (\{j+1\} - \{\overline{j+1}\}) \cdots (\{n\} - \{\overline{n}\}).$$

With this notation for x it is not difficult to see that

$$\dot{\partial}^i(x) = 2^i i! \left(\sum_{y \in M_{j-i,0}^j} y \right) \cdot (\{j+1\} - \{\overline{j+1}\}) \cdots (\{n\} - \{\overline{n}\}).$$

This is zero if and only if $j < i$. Since each V_j^n is irreducible we have $\partial^i(V_j^n) \cong V_j^n$ for $j \geq i$. Thus $\dot{K}_{n,0,i}^n = \oplus_{j=0}^{i-1} V_j^n$ as required.

Now if $\partial^i(x) \neq 0$ then $\dot{\varepsilon}^i \partial^i(x) = (2^i i!)^2 x \neq 0$. Thus $\dot{\varepsilon}^i \partial^i(V_j^n) = V_j^n$. By dimension counting we have $\dot{\varepsilon}^i(M_{n-i,0}^n) = \bigoplus_{j=i}^n V_j^n$. Since the complement map c is an element of B_n and $\ddot{I}_{n,0,p-i}^n = c\left(\dot{\varepsilon}^i(M_{n-i,0}^n)\right)$ we are done. □

Lemma 6.3.3. *Let $p > 2$. Let $n \geq 1$. Suppose $0 < i < p$. Then*

$$\dot{H}_{n,0,i}^n \cong \ddot{H}_{n,0,p-i}^n.$$

Proof. We have

$$\dot{H}^n_{n,0,i} \cong \dot{K}^n_{n,0,i} = \bigoplus_{j=0}^{i-1} V^n_j$$

by Lemma 6.3.2. We also have

$$\ddot{H}^n_{n,0,p-i} = M^n_{n,0}/\ddot{I}^n_{n,0,p-i} = M^n_{n,0}/\bigoplus_{j=i}^{n} V^n_j$$

by Lemma 6.3.2. But by Lemma 6.3.1 this quotient is isomorphic to $\oplus_{j=0}^{i-1} V^n_j$. □

Appendix A

Incidence homology modules for B_n with $n \leq 2$

In this appendix we list for reference all homology modules $\dot{H}^n_{u,d,i}$ and $\ddot{H}^n_{u,d,i}$ for $n \leq 2$. Explicit bases of the kernels $\dot{K}^n_{u,d,i}$, $\ddot{K}^n_{u,d,i}$ and images $\dot{I}^n_{u,d,i}$, $\ddot{I}^n_{u,d,i}$ are given. In particular, dimensions can be read off as the number of presented basis elements. To avoid clutter and since $\dot{H}^n_{u,d,i} = \ddot{H}^n_{u,d,i} = 0$ in the cases $i = 0$ or $i = p$, the assumption $0 < i < p$ is implicit throughout.

A.1 $n = 0$

The case is $n = 0$ is trivial. We have

$$M^0_{0,0} = F\{\emptyset\}.$$

The images are all zero, that is

$$\dot{I}^0_{0,0,i} = 0 = \ddot{I}^0_{0,0,i}.$$

This means the homology modules are isomorphic to the kernels. Futhermore, the kernels are all the entire permutation module. In other words

$$\dot{H}^0_{0,0,i} \cong \dot{K}^0_{0,0,i} = F\{\emptyset\} = \ddot{K}^0_{0,0,i} \cong \ddot{H}^0_{0,0,i}.$$

A.2 $n = 1$

For $n = 1$ there are three pairs of integers (u, d) with $0 \leq d \leq u \leq n$. We give each its own subsection.

A.2.1 $(u, d) = (0, 0)$

$$M^1_{0,0} = F\{\emptyset\}$$

$$\dot{K}^1_{0,0,i} = F\{\emptyset\}$$

$$\dot{I}^1_{0,0,i} = \begin{cases} 0 & \text{if } i \leq p-2 \\ F\{\emptyset\} & \text{if } i = p-1 \end{cases}$$

$$\dot{H}^1_{0,0,i} = \begin{cases} F\{\emptyset\} & \text{if } i \leq p-2 \\ 0 & \text{if } i = p-1 \end{cases}$$

$$\ddot{K}^1_{0,0,i} = F\{\emptyset\}$$

$$\ddot{I}^1_{0,0,i} = 0$$

$$\ddot{H}^1_{0,0,i} = F\{\emptyset\}$$

A.2.2 $(u, d) = (1, 0)$

$$M^1_{1,0} = F\{\{1\}, \{\bar{1}\}\}$$

$$\dot{K}^1_{1,0,i} = \begin{cases} F\{\{1\} - \{\bar{1}\}\} & \text{if } i = 1 \\ F\{\{1\}, \{\bar{1}\}\} & \text{if } 2 \leq i \end{cases}$$

$$\dot{I}^1_{1,0,i} = 0$$

$$\dot{H}^1_{1,0,i} = \begin{cases} F\{\{1\} - \{\bar{1}\}\} & \text{if } i = 1 \\ F\{\{1\}, \{\bar{1}\}\} & \text{if } 2 \leq i \end{cases}$$

$$\ddot{K}^1_{1,0,i} = F\{\{1\}, \{\bar{1}\}\}$$

$$\ddot{I}^1_{1,0,i} = \begin{cases} 0 & \text{if } i \leq p-2 \\ F\{\{1\} + \{\bar{1}\}\} & \text{if } i = p-1 \end{cases}$$

$$\ddot{H}^1_{1,0,i} = \begin{cases} F\{\{1\}, \{\bar{1}\}\} & \text{if } i \leq p-2 \\ \dfrac{F\{\{1\}, \{\bar{1}\}\}}{F\{\{1\} + \{\bar{1}\}\}} & \text{if } i = p-1 \end{cases}$$

A.2.3 $(u, d) = (1, 1)$

$$M^1_{1,1} = F\{\{1\bar{1}\}\}$$

$$\dot{K}^1_{1,1,i} = F\{\{1\bar{1}\}\}$$
$$\dot{I}^1_{1,1,i} = 0$$
$$\dot{H}^1_{1,1,i} = F\{\{1\bar{1}\}\}$$

$$\ddot{K}^1_{1,1,i} = \begin{cases} 0 & \text{if } i = 1 \\ F\{\{1\bar{1}\}\} & \text{if } 2 \leq i \end{cases}$$
$$\ddot{I}^1_{1,1,i} = 0$$
$$\ddot{H}^1_{1,1,i} = \begin{cases} 0 & \text{if } i = 1 \\ F\{\{1\bar{1}\}\} & \text{if } 2 \leq i \end{cases}$$

A.3 $n = 2$

For $n = 2$ there are six pairs of integers (u, d) with $0 \leq d \leq u \leq n$. We give each its own subsection.

A.3.1 $(u, d) = (0, 0)$

$$M^2_{0,0} = F\{\emptyset\}$$

$$\dot{K}^2_{0,0,i} = F\{\emptyset\}$$
$$\dot{I}^2_{0,0,i} = \begin{cases} 0 & \text{if } i \leq p-3 \\ F\{\emptyset\} & \text{if } p-2 \leq i \end{cases}$$
$$\dot{H}^2_{0,0,i} = \begin{cases} F\{\emptyset\} & \text{if } i \leq p-3 \\ 0 & \text{if } p-2 \leq i \end{cases}$$

$$\ddot{K}^2_{0,0,i} = F\{\emptyset\}$$
$$\ddot{I}^2_{0,0,i} = 0$$
$$\ddot{H}^2_{0,0,i} = F\{\emptyset\}$$

A.3.2 $(u, d) = (1, 0)$

$$M^2_{1,0} = F\{\{1\}, \{\bar{1}\}, \{2\}, \{\bar{2}\}\}$$

$$\dot{K}^2_{1,0,i} = \begin{cases} F\{\{1\}-\{\bar{1}\},\{\bar{1}\}-\{2\},\{2\}-\{\bar{2}\}\} & \text{if } i=1 \\ F\{\{1\},\{\bar{1}\},\{2\},\{\bar{2}\}\} & \text{if } 2 \le i \end{cases}$$

$$\dot{I}^2_{1,0,i} = \begin{cases} 0 & \text{if } i \le p-2 \\ F\{\{1\}+\{2\},\{1\}+\{\bar{2}\},\{\bar{1}\}+\{2\}\} & \text{if } i=p-1 \end{cases}$$

$$\dot{H}^2_{1,0,i} = \begin{cases} F\{\{1\}-\{\bar{1}\},\{\bar{1}\}-\{2\},\{2\}-\{\bar{2}\}\} & \text{if } i=1 \le p-2 \\ 0 & \text{if } i=1=p-1 \\ F\{\{1\},\{\bar{1}\},\{2\},\{\bar{2}\}\} & \text{if } 2 \le i \le p-2 \\ \dfrac{F\{\{1\},\{\bar{1}\},\{2\},\{\bar{2}\}\}}{F\{\{1\}+\{2\},\{1\}+\{\bar{2}\},\{\bar{1}\}+\{2\}\}} & \text{if } 2 \le i = p-1 \end{cases}$$

$$\ddot{K}^2_{1,0,i} = F\{\{1\},\{\bar{1}\},\{2\},\{\bar{2}\}\}$$

$$\ddot{I}^2_{1,0,i} = \begin{cases} 0 & \text{if } i \le p-2 \\ F\{\{1\}+\{\bar{1}\},\{2\}+\{\bar{2}\}\} & \text{if } i=p-1 \end{cases}$$

$$\ddot{H}^2_{1,0,i} = \begin{cases} F\{\{1\},\{\bar{1}\},\{2\},\{\bar{2}\}\} & \text{if } i \le p-2 \\ \dfrac{F\{\{1\},\{\bar{1}\},\{2\},\{\bar{2}\}\}}{F\{\{1\}+\{\bar{1}\},\{2\}+\{\bar{2}\}\}} & \text{if } i=p-1 \end{cases}$$

A.3.3 $(u, d) = (2, 0)$

$$M^2_{2,0} = F\{\{12\},\{1\bar{2}\},\{\bar{1}2\},\{\overline{12}\}\}$$

$$\dot{K}^2_{2,0,i} = \begin{cases} F\{\{12\}-\{1\bar{2}\}+\{\overline{12}\}-\{\bar{1}2\}\} & \text{if } i=1 \\ F\{\{12\}-\{1\bar{2}\},\{1\bar{2}\}-\{\bar{1}2\},\{\bar{1}2\}-\{\overline{12}\}\} & \text{if } i=2 \\ F\{\{12\},\{1\bar{2}\},\{\bar{1}2\},\{\overline{12}\}\} & \text{if } 3 \le i \end{cases}$$

$$\dot{I}^2_{2,0,i} = 0$$

$$\dot{H}^2_{2,0,i} = \begin{cases} F\{\{12\}-\{1\bar{2}\}+\{\overline{12}\}-\{\bar{1}2\}\} & \text{if } i=1 \\ F\{\{12\}-\{1\bar{2}\},\{1\bar{2}\}-\{\bar{1}2\},\{\bar{1}2\}-\{\overline{12}\}\} & \text{if } i=2 \\ F\{\{12\},\{1\bar{2}\},\{\bar{1}2\},\{\overline{12}\}\} & \text{if } 3 \le i \end{cases}$$

$$\ddot{K}^2_{2,0,i} = F\{\{12\},\{1\overline{2}\},\{\overline{1}2\},\{\overline{12}\}\}$$

$$\ddot{I}^2_{2,0,i} = \begin{cases} 0 & \text{if } i \le p-3 \\ F\{\{12\}+\{1\overline{2}\}+\{\overline{1}2\}+\{\overline{12}\}\} & \text{if } i = p-2 \\ F\{\{12\}+\{1\overline{2}\},\{\overline{1}2\}+\{\overline{12}\},\{12\}+\{\overline{1}2\}\} & \text{if } i = p-1 \end{cases}$$

$$\ddot{H}^2_{2,0,i} = \begin{cases} F\{\{12\},\{1\overline{2}\},\{\overline{1}2\},\{\overline{12}\}\} & \text{if } i \le p-3 \\ \dfrac{F\{\{12\},\{1\overline{2}\},\{\overline{1}2\},\{\overline{12}\}\}}{F\{\{12\}+\{1\overline{2}\}+\{\overline{1}2\}+\{\overline{12}\}\}} & \text{if } i = p-2 \\ \dfrac{F\{\{12\},\{1\overline{2}\},\{\overline{1}2\},\{\overline{12}\}\}}{F\{\{12\}+\{1\overline{2}\},\{\overline{1}2\}+\{\overline{12}\},\{12\}+\{\overline{1}2\}\}} & \text{if } i = p-1 \end{cases}$$

A.3.4 $(u, d) = (1, 1)$

$$M^2_{1,1} = F\{\{1\overline{1}\},\{2\overline{2}\}\}$$

$$\dot{K}^2_{1,1,i} = F\{\{1\overline{1}\},\{2\overline{2}\}\}$$

$$\dot{I}^2_{1,1,i} = \begin{cases} 0 & \text{if } i \le p-2 \\ F\{\{1\overline{1}\},\{2\overline{2}\}\} & \text{if } i = p-1 \end{cases}$$

$$\dot{H}^2_{1,1,i} = \begin{cases} F\{\{1\overline{1}\},\{2\overline{2}\}\} & \text{if } i \le p-2 \\ 0 & \text{if } i = p-1 \end{cases}$$

$$\ddot{K}^2_{1,1,i} = \begin{cases} 0 & \text{if } i = 1 \\ F\{\{1\overline{1}\},\{2\overline{2}\}\} & \text{if } 2 \le i \end{cases}$$

$$\ddot{I}^2_{1,1,i} = 0$$

$$\ddot{H}^2_{1,1,i} = \begin{cases} 0 & \text{if } i = 1 \\ F\{\{1\overline{1}\},\{2\overline{2}\}\} & \text{if } 2 \le i \end{cases}$$

A.3.5 $(u, d) = (2, 1)$

$$M^2_{2,1} = F\{\{12\overline{2}\},\{\overline{1}2\overline{2}\},\{1\overline{1}2\},\{1\overline{1}\overline{2}\}\}$$

$$\dot{K}^2_{2,1,i} = \begin{cases} F\{\{12\overline{2}\} - \{\overline{1}2\overline{2}\}, \{1\overline{1}2\} - \{1\overline{1}\overline{2}\}\} & \text{if } i = 1 \\ F\{\{12\overline{2}\}, \{\overline{1}2\overline{2}\}, \{1\overline{1}2\}, \{1\overline{1}\overline{2}\}\} & \text{if } 2 \leq i \end{cases}$$

$$\dot{I}^2_{2,1,i} = 0$$

$$\dot{H}^2_{2,1,i} = \begin{cases} F\{\{12\overline{2}\} - \{\overline{1}2\overline{2}\}, \{1\overline{1}2\} - \{1\overline{1}\overline{2}\}\} & \text{if } i = 1 \\ F\{\{12\overline{2}\}, \{\overline{1}2\overline{2}\}, \{1\overline{1}2\}, \{1\overline{1}\overline{2}\}\} & \text{if } 2 \leq i \end{cases}$$

$$\ddot{K}^2_{2,1,i} = \begin{cases} F\{\{12\overline{2}\} - \{1\overline{1}2\} + \{\overline{1}2\overline{2}\} - \{1\overline{1}\overline{2}\}\} & \text{if } i = 1 \\ F\{\{12\overline{2}\}, \{\overline{1}2\overline{2}\}, \{1\overline{1}2\}, \{1\overline{1}\overline{2}\}\} & \text{if } 2 \leq i \end{cases}$$

$$\ddot{I}^2_{2,1,i} = \begin{cases} 0 & \text{if } i \leq p-2 \\ F\{\{\overline{1}2\overline{2}\} + \{12\overline{2}\} + \{1\overline{1}\overline{2}\} + \{1\overline{1}2\}\} & \text{if } i = p-1 \end{cases}$$

$$\ddot{H}^2_{2,1,i} = \begin{cases} F\{\{12\overline{2}\} - \{1\overline{1}2\} + \{\overline{1}2\overline{2}\} - \{1\overline{1}\overline{2}\}\} & \text{if } i = 1 \leq p-2 \\ 0 & \text{if } i = 1 = p-1 \\ F\{\{12\overline{2}\}, \{\overline{1}2\overline{2}\}, \{1\overline{1}2\}, \{1\overline{1}\overline{2}\}\} & \text{if } 2 \leq i \leq p-2 \\ \dfrac{F\{\{12\overline{2}\}, \{\overline{1}2\overline{2}\}, \{1\overline{1}2\}, \{1\overline{1}\overline{2}\}\}}{F\{\{\overline{1}2\overline{2}\} + \{12\overline{2}\} + \{1\overline{1}\overline{2}\} + \{1\overline{1}2\}\}} & \text{if } 2 \leq i = p-1 \end{cases}$$

A.3.6 $(u, d) = (2, 2)$

$$M^2_{2,2} = F\{\{1\overline{1}2\overline{2}\}\}$$

$$\dot{K}^2_{2,2,i} = F\{\{1\overline{1}2\overline{2}\}\}$$

$$\dot{I}^2_{2,2,i} = 0$$

$$\dot{H}^2_{2,2,i} = F\{\{1\overline{1}2\overline{2}\}\}$$

$$\ddot{K}^2_{2,2,i} = \begin{cases} 0 & \text{if } i \leq 2 \\ F\{\{1\overline{1}2\overline{2}\}\} & \text{if } 3 \leq i \end{cases}$$

$$\ddot{I}^2_{2,2,i} = 0$$

$$\ddot{H}^2_{2,2,i} = \begin{cases} 0 & \text{if } i \leq 2 \\ F\{\{1\overline{1}2\overline{2}\}\} & \text{if } 3 \leq i \end{cases}$$

Appendix B

Tables of Brauer character decompositions

This appendix contains decompositions of the homology modules' Brauer characters as sums of irreducible Brauer characters for $n \in \{1, 2, 3, 4\}$ and $p \in \{3, 5, 7\}$. By Section 2.7 the irreducible Brauer characters of B_n are parameterised by the p-regular bipartitions of n. In these tables we denote each Brauer character by its corresponding bipartition. We also do not write any brackets or commas except for the comma that separates the two constituent partitions of a bipartition. For example for $p > 2$ the irreducible Brauer character of B_{11} corresponding to the bipartition $((4, 2, 1^2), (3))$ is denoted by $421^2, 3$ in the tables. The partition $(0, 0, \dots)$ is denoted by 0. A "." denotes the zero character.

Evidence for the conjectured duality, Conjecture 6.0.1, can be seen: The singles and doubles tables for fixed parameters p and n are interchanged by flipping horizontally followed by turning upside down. Note however this only tells you the composition factors of $\dot{H}^n_{u,d,i}$ and $\ddot{H}^n_{n-d,n-u,p-i}$ are the same, the conjecture is stronger.

Another observation is that for $p > n$ the tables become independent of p in the following sense. Suppose $p' > p > n$. Then each column in the table for p' and n contains at least $p' - p$ repeated entries. Removing $p' - p$ of these from each column gives us the table for p and n.

i, u	0	1
1	$1, 0$	$0, 1$
2	.	$1, 0 + 0, 1$

Table B.1: Brauer characters of $\dot{H}^1_{u,0,i}$ for $p = 3$.

i, u	0	1
1	$1, 0$	$0, 1$
2	$1, 0$	$1, 0 + 0, 1$
3	$1, 0$	$1, 0 + 0, 1$
4	.	$1, 0 + 0, 1$

Table B.2: Brauer characters of $\dot{H}^1_{u,0,i}$ for $p = 5$.

i, u	0	1
1	$1, 0$	$0, 1$
2	$1, 0$	$1, 0 + 0, 1$
3	$1, 0$	$1, 0 + 0, 1$
4	$1, 0$	$1, 0 + 0, 1$
5	$1, 0$	$1, 0 + 0, 1$
6	.	$1, 0 + 0, 1$

Table B.3: Brauer characters of $\dot{H}^1_{u,0,i}$ for $p = 7$.

i, u	0	1	2
1	.	$1^2, 0 + 1, 1$	$0, 2$
2	.	$1^2, 0$	$1, 1 + 0, 2$

Table B.4: Brauer characters of $\dot{H}^2_{u,0,i}$ for $p = 3$.

i, u	0	1	2
1	$2, 0$	$1^2, 0 + 1, 1$	$0, 2$
2	$2, 0$	$1^2, 0 + 1, 1 + 2, 0$	$1, 1 + 0, 2$
3	.	$1^2, 0 + 1, 1 + 2, 0$	$1, 1 + 2, 0 + 0, 2$
4	.	$1^2, 0$	$1, 1 + 2, 0 + 0, 2$

Table B.5: Brauer characters of $\dot{H}^2_{u,0,i}$ for $p = 5$.

i, u	0	1	2
1	$2, 0$	$1^2, 0 + 1, 1$	$0, 2$
2	$2, 0$	$1^2, 0 + 1, 1 + 2, 0$	$1, 1 + 0, 2$
3	$2, 0$	$1^2, 0 + 1, 1 + 2, 0$	$1, 1 + 2, 0 + 0, 2$
4	$2, 0$	$1^2, 0 + 1, 1 + 2, 0$	$1, 1 + 2, 0 + 0, 2$
5	.	$1^2, 0 + 1, 1 + 2, 0$	$1, 1 + 2, 0 + 0, 2$
6	.	$1^2, 0$	$1, 1 + 2, 0 + 0, 2$

Table B.6: Brauer characters of $\dot{H}^2_{u,0,i}$ for $p = 7$.

i, u	0	1	2	3
1	.	$21, 0$	$1^2, 1 + 1, 2$	$0, 3$
2	.	.	$21, 0 + 1^2, 1$	$0, 3 + 1, 2$

Table B.7: Brauer characters of $\dot{H}^3_{u,0,i}$ for $p = 3$.

i, u	0	1	2	3
1	$3, 0$	$21, 0 + 2, 1$	$1^2, 1 + 1, 2$	$0, 3$
2	.	$21, 0 + 2, 1 + 3, 0$	$1^2, 1 + 21, 0 + 1, 2 + 2, 1$	$1, 2 + 0, 3$
3	.	$21, 0$	$1^2, 1 + 21, 0 + 1, 2 + 2, 1 + 3, 0$	$1, 2 + 2, 1 + 0, 3$
4	.	.	$1^2, 1 + 21, 0$	$1, 2 + 2, 1 + 3, 0 + 0, 3$

Table B.8: Brauer characters of $\dot{H}^3_{u,0,i}$ for $p = 5$.

i, u	0	1	2	3
1	$3, 0$	$21, 0 + 2, 1$	$1^2, 1 + 1, 2$	$0, 3$
2	$3, 0$	$21, 0 + 2, 1 + 3, 0$	$1^2, 1 + 21, 0 + 1, 2 + 2, 1$	$1, 2 + 0, 3$
3	$3, 0$	$21, 0 + 2, 1 + 3, 0$	$1^2, 1 + 21, 0 + 1, 2 + 2, 1 + 3, 0$	$1, 2 + 2, 1 + 0, 3$
4	.	$21, 0 + 2, 1 + 3, 0$	$1^2, 1 + 21, 0 + 1, 2 + 2, 1 + 3, 0$	$1, 2 + 2, 1 + 3, 0 + 0, 3$
5	.	$21, 0$	$1^2, 1 + 21, 0 + 1, 2 + 2, 1 + 3, 0$	$1, 2 + 2, 1 + 3, 0 + 0, 3$
6	.	.	$1^2, 1 + 21, 0$	$1, 2 + 2, 1 + 3, 0 + 0, 3$

Table B.9: Brauer characters of $\dot{H}^3_{u,0,i}$ for $p = 7$.

i,u	0	1	2	3	4
1	.	.	$2^2,0+21,1$	$1,3+1^2,2$	$0,4$
2	.	.	$2^2,0$	$21,1+1^2,2$	$0,4+1,3$

Table B.10: Brauer characters of $\dot{H}^4_{u,0,i}$ for $p=3$.

i,u	0	1	2	3	4
1	.	$31,0+3,1$	$21,1+2^2,0+2,2$	$1^2,2+1,3$	$0,4$
2	.	$31,0$	$21,1+2^2,0+2,2+31,0+3,1$	$1^2,2+21,1+2,2+1,3$	$1,3+0,4$
3	.	.	$21,1+2^2,0+31,0$	$1^2,2+21,1+2,2+31,0+1,3+3,1$	$2,2+1,3+0,4$
4	.	.	$2^2,0$	$1^2,2+21,1+31,0$	$2,2+1,3+3,1+0,4$

Table B.11: Brauer characters of $\dot{H}^4_{u,0,i}$ for $p=5$.

i,u	0	1	2	3	4
1	$4,0$	$31,0+3,1$	$21,1+2^2,0+2,2$	$1^2,2+1,3$	$0,4$
2	$4,0$	$31,0+3,1+4,0$	$21,1+2^2,0+2,2+31,0+3,1$	$1^2,2+21,1+2,2+1,3$	$1,3+0,4$
3	.	$31,0+3,1+4,0$	$21,1+2^2,0+2,2+31,0+3,1+4,0$	$1^2,2+21,1+2,2+31,0+1,3+3,1$	$2,2+1,3+0,4$
4	.	$31,0$	$21,1+2^2,0+2,2+31,0+3,1+4,0$	$1^2,2+21,1+2,2+31,0+1,3+3,1+4,0$	$2,2+1,3+3,1+0,4$
5	.	.	$21,1+2^2,0+31,0$	$1^2,2+21,1+2,2+31,0+1,3+3,1+4,0$	$2,2+1,3+3,1+4,0+0,4$
6	.	.	$2^2,0$	$1^2,2+21,1+31,0$	$2,2+1,3+3,1+4,0+0,4$

Table B.12: Brauer characters of $\dot{H}^4_{u,0,i}$ for $p=7$.

i,d	0	1
1	$1,0+0,1$	.
2	$0,1$	$1,0$

Table B.13: Brauer characters of $\ddot{H}^1_{1,d,i}$ for $p=3$.

i,d	0	1
1	$1,0+0,1$	.
2	$1,0+0,1$	$1,0$
3	$1,0+0,1$	$1,0$
4	$0,1$	$1,0$

Table B.14: Brauer characters of $\ddot{H}^1_{1,d,i}$ for $p=5$.

i,d	0	1
1	$1,0+0,1$	.
2	$1,0+0,1$	$1,0$
3	$1,0+0,1$	$1,0$
4	$1,0+0,1$	$1,0$
5	$1,0+0,1$	$1,0$
6	$0,1$	$1,0$

Table B.15: Brauer characters of $\ddot{H}^1_{1,d,i}$ for $p=7$.

i, d	0	1	2
1	$1,1+0,2$	$1^2,0$	.
2	$0,2$	$1^2,0+1,1$	.

Table B.16: Brauer characters of $\ddot{H}^2_{2,d,i}$ for $p=3$.

i, d	0	1	2
1	$1,1+2,0+0,2$	$1^2,0$	.
2	$1,1+2,0+0,2$	$1^2,0+1,1+2,0$	.
3	$1,1+0,2$	$1^2,0+1,1+2,0$	$2,0$
4	$0,2$	$1^2,0+1,1$	$2,0$

Table B.17: Brauer characters of $\ddot{H}^2_{2,d,i}$ for $p=5$.

i, d	0	1	2
1	$1,1+2,0+0,2$	$1^2,0$	.
2	$1,1+2,0+0,2$	$1^2,0+1,1+2,0$	.
3	$1,1+2,0+0,2$	$1^2,0+1,1+2,0$	$2,0$
4	$1,1+2,0+0,2$	$1^2,0+1,1+2,0$	$2,0$
5	$1,1+0,2$	$1^2,0+1,1+2,0$	$2,0$
6	$0,2$	$1^2,0+1,1$	$2,0$

Table B.18: Brauer characters of $\ddot{H}^2_{2,d,i}$ for $p=7$.

i,d	0	1	2	3
1	$0,3+1,2$	$21,0+1^2,1$	.	.
2	$0,3$	$1^2,1+1,2$	$21,0$	.

Table B.19: Brauer characters of $\ddot{H}^3_{3,d,i}$ for $p=3$.

i,d	0	1	2	3
1	$1,2+2,1+3,0+0,3$	$1^2,1+21,0$	.	.
2	$1,2+2,1+0,3$	$1^2,1+21,0+1,2+2,1+3,0$	$21,0$	.
3	$1,2+0,3$	$1^2,1+21,0+1,2+2,1$	$21,0+2,1+3,0$	.
4	$0,3$	$1^2,1+1,2$	$21,0+2,1$	$3,0$

Table B.20: Brauer characters of $\ddot{H}^3_{3,d,i}$ for $p=5$.

i,d	0	1	2	3
1	$1,2+2,1+3,0+0,3$	$1^2,1+21,0$	.	.
2	$1,2+2,1+3,0+0,3$	$1^2,1+21,0+1,2+2,1+3,0$	$21,0$	.
3	$1,2+2,1+3,0+0,3$	$1^2,1+21,0+1,2+2,1+3,0$	$21,0+2,1+3,0$	.
4	$1,2+2,1+0,3$	$1^2,1+21,0+1,2+2,1+3,0$	$21,0+2,1+3,0$	$3,0$
5	$1,2+0,3$	$1^2,1+21,0+1,2+2,1$	$21,0+2,1+3,0$	$3,0$
6	$0,3$	$1^2,1+1,2$	$21,0+2,1$	$3,0$

Table B.21: Brauer characters of $\ddot{H}^3_{3,d,i}$ for $p=7$.

i,d	0	1	2	3	4
1	$0,4+1,3$	$21,1+1^2,2$	$2^2,0$	.	.
2	$0,4$	$1,3+1^2,2$	$2^2,0+21,1$	.	.

Table B.22: Brauer characters of $\ddot{H}^4_{4,d,i}$ for $p=3$.

i,d	0	1	2	3	4
1	$2,2+1,3+3,1+0,4$	$1^2,2+21,1+31,0$	$2^2,0$	.	.
2	$2,2+1,3+0,4$	$1^2,2+21,1+2,2+31,0+1,3+3,1$	$21,1+2^2,0+31,0$	.	.
3	$1,3+0,4$	$1^2,2+21,1+2,2+1,3$	$21,1+2^2,0+2,2+31,0+3,1$	$31,0$	.
4	$0,4$	$1^2,2+1,3$	$21,1+2^2,0+2,2$	$31,0+3,1$	.

Table B.23: Brauer characters of $\ddot{H}^4_{4,d,i}$ for $p=5$.

i,d	0	1	2	3	4
1	$2,2+1,3+3,1+4,0+0,4$	$1^2,2+21,1+31,0$	$2^2,0$	.	.
2	$2,2+1,3+3,1+4,0+0,4$	$1^2,2+21,1+2,2+31,0+1,3+3,1+4,0$	$21,1+2^2,0+31,0$	.	.
3	$2,2+1,3+3,1+0,4$	$1^2,2+21,1+2,2+31,0+1,3+3,1+4,0$	$21,1+2^2,0+2,2+31,0+3,1+4,0$	$31,0$	.
4	$2,2+1,3+0,4$	$1^2,2+21,1+2,2+31,0+1,3+3,1$	$21,1+2^2,0+2,2+31,0+3,1+4,0$	$31,0+3,1+4,0$	.
5	$1,3+0,4$	$1^2,2+21,1+2,2+1,3$	$21,1+2^2,0+2,2+31,0+3,1$	$31,0+3,1+4,0$	$4,0$
6	$0,4$	$1^2,2+1,3$	$21,1+2^2,0+2,2$	$31,0+3,1$	$4,0$

Table B.24: Brauer characters of $\ddot{H}^4_{4,d,i}$ for $p=7$.

Appendix C

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