CONCEPTUAL AND LEARNING ISSUES IN MATHEMATICS UNDERGRADUATES' FIRST ENCOUNTER WITH GROUP THEORY: A COMMOGNITIVE ANALYSIS

By

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Μεθ' ἡμῶν ὁ Θεός!

ABSTRACT

Group Theory is one of the mandatory courses taught usually in the second year of a Bachelor degree in Mathematics and is typically considered by students as one of the most challenging ones, mainly because of its abstract and rigorous nature. Often, after their first encounter students tend to avoid third-year or further courses in this area of Mathematics.

This study is a close examination of the conceptual and learning aspects of Year 2 Mathematics undergraduates' learning experience in Group Theory. The course was mandatory. The data consists of: observation notes and audio-recordings of lectures and seminars; lecture notes; student and staff interviews; and, marked coursework and examination scripts. For the interpretation of data I have used the Commognitive Theoretical Framework (Sfard, 2008), focusing on three general issues including: *the object-level and metalevel learning* and the conceptual difficulties that may occur; *the teaching and learning*, within the context of lecture, seminar and tutorial, as a form of communication; and the students' applied *study skills*.

Data analysis suggests that object-level and metalevel understanding are intertwined when learning a new mathematical discourse, and the discursive shift involved in object-level and metalevel learning is a complex procedure, especially within the abstract context of Group Theory. Two important milestones in the learning of Group Theory is the introduction of equivalence relations and normality. Regarding transition, this study suggests that it cannot be limited with respect to secondary-tertiary level Mathematics, but it rather involves a more complex shift, among different theories, and fields of undergraduate Mathematics. Such discursive shift requires adjustments in the students' study skills. Students have applied several techniques for the preparation of coursework and examination revision, with one, the *spiral revision model* being particularly prevalent. In addition, students have shown remarkable sensitivity on the effectiveness of communication in different contexts, e.g. mathematical conversations, or presentation of their reasoning.

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List of Abbreviations

APOS	Action, Process, Object, Schema
CS1E1	Coursework Problem Sheet 1 Exercise 1 (example of notation)
CTF	Commognitive Theoretical Framework
DAM	Data Analysis Map
EPM	Experienced Pure Mathematician
FEE4	Final Examination Exercise 4 (example of notation)
FIT	First Isomorphism Theorem
LCR	Lecturer
PST	Pedagogical Statement Table
SAA	Seminar Assistant A
SAB	Seminar Assistant B
SLA	Seminar Leader A
SLB	Seminar Leader B
TFA	Threefold Data Analysis Account
VA	Visual and Analytic

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Chapter 1 Introduction

This doctorate study is part of ongoing research in the field of Undergraduate Mathematics Education. In particular, I examine the conceptual and learning issues regarding second year undergraduate students' first encounter with Group Theory, using the interpretive lens of the Commognitive Theoretical Framework, introduced by Anna Sfard (2008).

My decision to focus on Group Theory was triggered by two facts: my previous personal encounter with this module as a student and the nature of this pure mathematical theory. The characteristics of Group Theory that make it is so distinct are its level of abstraction, the non-direct visual representation of its concepts, the rigour and precision that is required in the proof production, and the initially problematic student adjustment with the demands of this module as part of the secondary-tertiary transition (Gueudet, 2008).

Educational research in Group Theory, and Abstract Algebra in general, is significant for several reasons. Novice students consider Group Theory as one of the most demanding subjects in the undergraduate Mathematics syllabus (Dubinsky et al., 1994). One possible explanation is that "Abstract Algebra is the first course in which students must go beyond 'imitative behavior patterns' for mimicking the solution of a large number of variations on a small number of themes" (Dubinsky et al., 1994, p268). A typical first Abstract Algebra module, i.e. Group Theory and Ring Theory, requires a deep understanding of the abstract concepts involved as well as the ability to produce valid and rigorous proofs using them.

According to Gallian (1990), Abstract Algebra is very important for mathematically trained individuals because of its wide use in other parts of mathematics and other disciplines. However, it is a module that requires "deeper levels of insight and sophistication" (Barbeau, 1995, p139). One important aspect of students' difficulty with this module is that instructors do not always give adequate time to students to reflect on the new material (Clark et al., 1997).

An important element that causes students' difficulty with Abstract Algebra is its 'abstract' nature, which can be a serious setback, but also a key to a new world. The deductive way of teaching Abstract Algebra is unfamiliar to students and, in order to achieve mastery of the subject, it is necessary to "think selectively about its entities, paying attention to those aspects consistent with the context and ignoring those that are irrelevant" (Barbeau, 1995, p140). Similarly, Hazzan (1999) states that students' difficulty with the course can mainly be explained by the fact that for the first time they have to deal with concepts which are introduced abstractly, i.e. "defined and presented by their properties and by an examination of what facts can be determined just from their properties alone" (Hazzan, 1999, p73).

Gueudet (2008) suggests that many pedagogical issues emerging in undergraduate Mathematics Education are based on the transition from secondary to tertiary Mathematics. This transition is a process that can also occur beyond the period between the end of secondary school and the beginning of a university degree in Mathematics. Transition may occur in a second year module, such as Group Theory, in which students are required to develop different study skills while facing new challenges, such as adjusting to the new, higher level of abstraction and producing rigorous proofs. In fact, student difficulties in Group Theory may be an indication of problematic transition, mainly due to the particular nature of this mathematical theory.

As mentioned above, an intertwined element of transition from secondary to tertiary Mathematics is the production of rigorous, explicit and elegant proofs, especially in Pure Mathematics and Abstract Algebra in particular. Weber (2001) associates student difficulty with Group Theory partly with the difficulty to construct proofs: "when left to their own devices, students usually fail to acquire optimal strategies for completing mathematical tasks and often acquire deficient ones" (Weber, 2001, p116). Alcock et al (2009) similarly point out that learning Group Theory is challenging because of the abstract nature of its concepts and because it involves reading and writing proofs using various learning practices and beliefs.

The difficulty to directly visualise the objects of study is another issue that differentiates Group Theory from other mathematical fields. According to Zazkis et al. (1996), there is a strong relation between the visual and analytic thinking in learning Group Theory and therefore the interchange between the modes of thinking is necessary for constructing deep understanding of the abstract group-theoretic concepts. Other studies on visualisation, such as Ioannou and Nardi (2009a, 2009b, 2010) suggest that the ability or inability to visualise influences mathematical learning in Group Theory. Ioannou (2010) and Ioannou and Iannone (2011) link mathematical learning with the ability to visualise mentally certain concepts such as the coset.

Nowadays, research in Mathematics Education related to Group Theory, although not very extensive, can be divided into two strands: firstly, teaching methods and learning understanding, and, secondly, concept development, which sometimes involves affective and engagement issues. The first strand includes teaching strategies applied in the lecture theatre, assessment approaches and IT programmes supporting teaching. The second strand refers to studies concerned with student understanding of the abstract concepts of Group Theory, strategies applied by students to make concepts accessible, and studies that examine proof production and its difficulties. Additionally, it includes studies about student engagement and how it is linked to students' affective issues.

Research in Group Theory is one of the many strands of research within the general research field of Undergraduate Mathematics Education, a relatively new discipline, which has mainly been developed during the last three decades. Research in this field emerged as a result of the direct necessity to investigate teaching and learning of Mathematics at university level.

Sierpinska et al. (1993), define the specific objects of study in Mathematics Education as the teaching of Mathematics, the learning of Mathematics, teaching and learning situations, the relations between teaching, learning and mathematical knowledge, the reality of Mathematics classes, societal views of Mathematics and its teaching, and the system of education itself.

According to Schoenfeld (2000), there are two purposes of research in the field of Mathematics Education: one pure and one applied. The first purpose is to understand the nature of mathematical thinking, teaching and learning, and the second is to use this understanding in order to improve mathematical instruction. Sierpinska et al. (1993) identified two types of aims of Mathematics Education research: pragmatic aims, which refer to the development of teaching practice and students' understanding and performance, and scientific aims, which refer to the development of Mathematics Education as an academic field of research.

Moreover, the specific objects of research in Undergraduate Mathematics Education are the teaching and learning at university level, focusing on cognitive, metacognitive, socio-cultural, affective and other issues. Research in Mathematics Education, including Undergraduate Mathematics Education, is not disconnected from other disciplines. It has strong connections with disciplines such as psychology, sociology, and neuroscience (Adda, 1998).

Schoenfeld (2000) suggests that research in Mathematics Education in general, and research focusing on the tertiary level in particular, addresses, and contributes to several fundamental issues such as: the theoretical perspectives for understanding thinking, learning and teaching; aspects of cognition such as students' mathematical thinking and student understandings and misunderstandings of several mathematical concepts; existence proofs (evidence of students' encounter with problem solving, induction, Group Theory etc. and evidence of the viability of various kinds of instruction); and consequences of various forms of instruction.

Moreover, this doctorate study aims to investigate both conceptual and learning issues of undergraduate students' first encounter with Group Theory, and in particular to address the following general questions, which fall in the aforementioned two categories of issues. The general questions falling in the first category are the following: What are the specific conceptual difficulties that students face in their first encounter with Group Theory? How do these conceptual difficulties evolve as the module progresses? Are these difficulties connected with the object-level understanding of group-theoretic concepts or with the metalevel understanding related to the application of several routines, and how? The questions falling in the second category are the following: What are the pedagogical issues that arise from this study? How do students cope with the process of proof production? What skills do students apply for the preparation of the coursework and the revision for the final examination, and what are own perceptions about these two activities? What are the students' perceptions about the issue of communication between mathematicians?

Even though I considered including other issues regarding students' learning experience, such as affective issues and their connection with conceptual difficulties and effectiveness of learning, I eventually decided to limit the scope of the study to the above questions and to consider other issues as part of further research in the future.

Chapter 2 Literature Review

In this chapter I present a critical account of the literature related to the theoretical and substantive aims of this study. In the first section, 2.1, I present a comparative account of a number of well-established theoretical frameworks, such as Concept Image/Concept definition and APOS Theory, widely used in the context of Undergraduate Mathematics Education. Moreover, I present the Commognitive Theoretical Framework, which has been used for the purposes of this study and justify this decision.

In the second section, 2.2, I present an overview of the literature on Group Theory Education. I discuss issues related to the learning of the fundamental group-theoretic concepts, the use of theorems, the role of visual and analytic thinking, students' techniques for coping with the high level of abstraction, and the proof production. My aim is to explore how the literature identifies the reasons for students' difficulty with Group Theory, locates the specific cognitive difficulties linked with certain group theoretic concepts, and determines other factors that cause difficulty.

In the last section, 2.3, I explore the literature in order to discuss issues the secondary-tertiary mathematics education transition. Moreover, I discuss literature that focuses on how successful transition allows students to cope effectively with advanced mathematical thinking and reasoning, the abstract nature of Pure Mathematics and Group Theory in particular, the use of examples, visual images and formal proof production, as well as the manifold issue of communication.

2.1 Acquisitionist and Participationist Theoretical Frameworks in Mathematics Education Research

The use of theoretical frameworks in Mathematics Education research is essential, since they seek to address the gap between "the logic of the mathematical edifice and the logic of cognitive processes" (Artigue et al, 2007, p1013). These frameworks allow us to investigate the different ways of conceptualising mathematical learning. Davis (1990) suggests that without theoretical frameworks, researchers would not be able to notice patterns of behaviours or thoughts, since they would not have the vocabulary to describe them and the lens through which they can be approached.

Here I aim to explain the reasoning that led me to use the particular theory, namely the Commognitive Theoretical Framework (CTF), as well as define its main constructs and how these have been used in the context of my study. In doing so I also give a short definition of other two theoretical frameworks, namely the *Concept Image/Concept Definition* and the *APOS Theory* for two reasons: first, because these two theories have been extensively used, within the field of Undergraduate Mathematics Education, and in particular for the studies of Group Theory that will be mentioned later in this chapter; and in order to highlight better the contrast between the acquisitionist approach that these theories are grounded on, and the participationist approach of CTF.

2.1.1 ***Concept Image/Concept Definition*** is an important and widely used theoretical framework, mostly in cognitive studies. It was introduced by Tall and Vinner (1981) as a tool used to analyse the distinction between the formal concept definition of mathematical concepts, defined in the context of Mathematics, a field of study characterised by its great precision and accuracy, and the individuals' image of these concepts. The need to make this distinction arose from the common observation that students were able to produce formal definitions on certain mathematical notions accurately, yet had difficulty to apply them in exercises or use them in proofs (Selden and Selden, 2001).

Concept Image is defined as "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (Tall and Vinner, 1981, p152). The creation and completion of a concept image is an evolving process that develops through various experiences and changes as the individual matures and meets new stimuli. The part of the concept image that is activated at a specific time and in a specific context is called *evoked concept image*.

Concept definition is defined as "a form of words used to specify that concept" (Tall and Vinner, 1981, p152). Individuals can either learn it in a mechanical way or in a meaningful way, more or less related to the concept as a whole, or it may be a personal reconstruction of the definition. The concept definition will offer the individuals the vocabulary that they will use in their explanation of their concept image. The ***concept definition image*** is the unique concept image generated by a specific concept definition and is part of the concept image (Tall and Viner, 1981).

The ***potential conflict factor*** is defined as "a part of concept image or concept definition which may conflict with another part of the concept image or concept definition" (Tall and Vinner, 1981, p153). If such a factor actually occurs in specific settings causing cognitive conflict, then it is referred to as a *cognitive conflict factor*. Tall and Vinner (1981) exemplify the above using the definition of a complex number $x + iy$ as an ordered pair (x, y), where $x, y \in \mathbb{R}$, and the identification of $x + i0 = (x, 0)$ as the real number x is a potential conflict factor in the concept of complex numbers. Cognitive conflict factors can emerge when these two expressions arise simultaneously (Tall and Vinner, 1981).

Based on this conflict of the concept image and the concept definition, Tall and Vinner (1981) developed this framework in order to study several phenomena, such as students' difficulty to adopt the formal approach that is required for undergraduate mathematics, which was often interpreted by teachers as a lack of student experience or intelligence. This theoretical framework takes into account the fact that mathematical knowledge grows in relation to students' maturity, considering the gradual development of the thinking process and mathematical knowledge and thus allowing mathematics educators to gain a better understanding of the complicated mental mechanisms of knowledge and thinking structure (Tall, 1991).

2.1.2 *APOS (Action, Process, Object, Schema) Theory* is based on the supposition that "mathematical knowledge consists in an individual's tendency to deal, in a social context, with perceived mathematical problem situations by constructing mental actions, processes, and objects and organizing them in schemas to make sense of the situations and solve the problems" (Dubinsky and McDonald, 2001, p276).

Action is defined as "a transformation of objects, which is perceived by the individual as being at least somewhat external. That is, an individual whose understanding of a transformation is limited to an action conception can carry out the transformation only by reacting to external cues that give precise details on what steps to take" (Asiala et al, 1996, p10).

Process emerges when an action is repeated and the individual is able to reflect upon it and the action has therefore been *interiorized*, namely it is not directed by external stimuli. When a process is encapsulated into an ***object***, individuals can reflect on a transformation, or occasionally reverse the steps of transformation, without necessarily performing them. An individual becomes holistically aware of the particular process, realises that transformations can act on it and be able to construct such transformations (Asiala et al, 1996).

Finally, objects and processes can be linked in various ways; they are "related by virtue of the fact that the former act on the latter" (Asiala et al, 1996, p12). A collection of processes and objects can be organised in a structured way to form a ***schema***, which can be treated as objects and be included in other schemas of 'higher level', moreover becoming *thematized* to an object (Asiala et al, 1996).

A further refinement of this framework occurred about a decade after it was first developed, concerning the analysis of schemata. Dubinsky and McDonald (2001) introduced the 'triad mechanism' which consists of three stages of schema development: the ***Intra*** stage, which "is characterized by the focus on individual actions, processes and objects in isolation from other cognitive items of a similar nature" (Dubinsky and McDonald, 2001, p280), the ***Inter*** stage that "is characterized by the construction of relationships and transformations among these cognitive entities" (Dubinsky and McDonald, 2001, p280), and the ***Trans*** stage at which "the individual constructs an implicit or explicit underlying structure through which the relationships developed in the Inter stage are understood and which give the schema a coherence by which the individual can decide what is in the scope of the schema and what is not" (Dubinsky and McDonald, 2001, p280).

APOS is representing knowledge in two directions: the *vertical organization* of knowledge through the action, process, object and schema analysis, and the *horizontal organization* of knowledge, which is achieved by the further schema analysis based on the three stages outlined above (Artigue, 2007).

Concept Image/Concept Definition and APOS theoretical frameworks are based on common philosophical tenets, and stem from Piaget's theories on cognition and constructivism, according to which individuals construct their own knowledge. As discussed above, Concept Image/Concept Definition focuses on the construction of mathematical knowledge triggered by the mismatch between the definition of the concept and the individual's personal mental image (Selden and Selden, 2001), whereas *APOS Theory* is an adaptation of Piagetian theory of reflective abstraction, modeling the mental constructions (Artigue, 2001). Both have been used extensively in studies of GT as evident in 2.2. They are both widely seen as examples of the acquisitionist perspective on mathematical learning.

In some contrast to the acquisitionist perspective, Commognitive Theoretical Framework considers mathematical learning as participation in a well-defined mathematical discourse (participationist perspective). According to Sfard (2002), "there is more to discourse than meets the ears, putting communication in the heart of mathematics education is likely to change not only the way we teach but also the way we think about learning and about what is being learned" (Sfard, 2002, p13). In addition, it considers the “metaphorical nature” of mathematics (Yackel, 2009, p91), particularly apparent in the context of my study.

Yackel (2009, p91) has identified several important characteristics that distinguish the Commognitive framework. First, this theoretical approach “meets accepted standards of scientific rigour”. Moreover, it provides effective definitions of terms such as thinking, communication, discourse and mathematical object. A second distinct element of the framework is its dialogical approach to discourse, according to which the objects of discourse belong to the discourse itself and narratives have human authors, as opposed to monological discourse where the ‘world’ is considered independent of discourse and objects merely exist within this ‘world’. Third, it has a participationist view towards learning. Sfard adopts Vygotsky’s view according to which collective forms are prior to individual forms of human activity.

The commognitive theoretical framework describes in wider way the process of learning, compared to the acquisition-based theories. According to Sfard (2002), the acquisitioned-based theories “‘distill’ cognitive activities from their context and thus tell us only a restricted part of the story of learning. The elements that they leave out of sight are often indispensable for the kind of understanding that should underlie any sensible practical decision” (Sfard, 2002, p22).

The communicational approach for studying human cognition is based on the principle that “*thinking may be conceptualized as a case of communication*” (Sfard, 2002, p26). In the commognitive context, communication is used in its broader sense and also refers to non-verbal interactions. Therefore communication may be defined as a person’s attempt to make an interlocutor act, think or feel according to his or her intentions. In addition, “when one is looking at cognition as a form of communication, an individual becomes automatically a nexus in the web of social relations – both a reason for, and a result of, these relations” (Sfard, 2002, p27).

The Commognitive framework focuses on the study of *discourse*

> [A]ny specific instance of communicating, whether diachronic or synchronic, whether with others or with oneself, whether predominantly verbal or with the help of any other symbolic system. The particularly broad meaning of the term in the present context implies inclusion of instances that would probably by excluded from the category of discourse by everyday users of the term. For example, the production of a written or spoken text, often considered as a defining feature of discourse, is not a necessary ingredient of what will counts for us as 'discursive'" (Sfard, 2002, p28).

In my study, the commognitive theoretical framework has been used for the analysis of the strictly *mathematical discursive* data, as represented by the written data (examination scripts and coursework). It has also been used to analyse other types of data, considering issues related to the broader *learning discourse,* for instance the exchanges between mathematical interlocutors – even though the focus of the thesis is mostly on the former.

An important reason for using this theoretical framework to discuss mathematical discourse and learning is the central role of language in the context of 'Thinking as Communicating'. Language has the features of *generativity*, *recursivity* and the possibility to produce *multilevel utterances* (Sfard, 2008). Consequently, according to CTF, discourse analysis is not restricted only to the strictly mathematical narratives. Following recursivity one can further analyse utterances *about* mathematical discourse and metautterances about mathematical learning.

Concluding, the aim of this study is not to adopt an acquisitionist perspective towards learning. It was designed in such a way that it would also examine from a wider, more participationist viewpoint the overall student experience, addressing pedagogical issues as these have emerged in the context of social interaction within the community of students and staff. The Commognitive Theoretical Framework is a multidimensional framework, which allows the researcher not only to apply it for discourse analysis of the students' cognitive issues, but also to discuss issues of communication, thinking, reasoning, teaching and learning. Consequently, this theoretical framework has proved to be particularly suitable.

2.1.3 Characteristics of Commognitive Theoretical Framework Relevant to this Study

In what follows, I briefly describe the ***Commognitive Theoretical Framework***, defining its major theoretical constructs and how they are used within the context of my study.

Commognitive Theoretical Framework (CTF) is a coherent and rigorous theory for thinking about thinking, grounded in classical Discourse Analysis (Yackel, 2009). It involves a number of different constructs such as *metaphor*, *thinking*, *communication*, and *commognition*, as a result of the link between interpersonal communication and cognitive processes, with commognition's five properties *reasoning*, *abstracting*, *objectifying*, *subjectifying* and *consciousness*.

Specifically for the mathematical discourse, CTF defines its discursive characteristics as the *word use*, *visual mediators*, *narratives*, and *routines* with their associated metarules, namely the *how* and the *when* of the routine. In addition, it involves the various objects of mathematical discourse such as the *signifiers*, *realisation trees*, *realisations*, *primary objects* and *discursive objects*. It also involves the constructs of *object-level* and *metadiscursive level* (or metalevel) *rules*, along with their characteristics *variability, tacitness, normativeness, flexibility* and *contingency.* Finally, in the context of CTF, they are defined the *object-level* and *metalevel mathematical learning*, and the often-occurring *commognitive conflict*.

In the context of this study, CTF will be used in the analysis of students' mathematical thinking and learning. In chapter 4, CTF is used in order to analyse students' mathematical thinking and conceptual difficulties as these have been identified in the written data, i.e. coursework and final examination, supporting my claims with data from other sources such as interviews and seminars. In this chapter, I predominantly use constructs that are directly linked with the mathematical discourse.

In chapter 5, CTF is used in order to analyse students' object-level and metalevel learning. The constructs that are predominantly used for the purposes of this chapter are the object-level and metalevel rules along with their characteristics as well as the various routines along with their metarules. In chapter 6, I use CTF in order to analyse students' views regarding the effectiveness of communication between the lecturer and the students, within the context of lecture and through the lecture notes. For the purposes of this chapter, I have predominantly used constructs that are related to the foundations of CTF, namely, communication, commognition and its five characteristics.

In what follows I will present CTF, defining the constructs that will be used later on in this study, and explaining how these constructs will be used giving illustrative examples.

Thinking, according to the commognitive framework, "is an individualized version of (interpersonal) communicating" (Sfard, 2008, p81). Contrary to the acquisitionist approaches, participationists tend to consider thinking as an act (not necessarily interpersonal) of communication, rather than a step primary to communication. In the context of CTF, interpersonal communication processes and cognitive processes are (different) manifestations of the same phenomenon, and Sfard therefore combines the terms cognition and communication producing the new terms ***commognition*** and ***commognitive***.

In the context of CTF, and according to the participationist approach that adopts, the primary and defining feature of communication is its community-coordinating role. Moreover, communication is described as an activity of a collective. Consequently, the researcher, who adopts a collective perspective, takes an outsider's viewpoint that allows him/her to refer to phenomena invisible when the focus shifts to specific individuals. ***Communication*** thus is defined as a "collectively performed patterned activity in which action A of an individual is followed by action B of another individual so that: (1) A belongs to certain well-defined repertoire of actions known as communicational; (2) Action B belongs to a repertoire of re-actions that fit A, that is, actions recurrently observed in conjunction with A. This latter repertoire is not exclusively a function of A, and it depends, among others, on factors such as the history of A (what happened prior to A), the situation in which A and B are performed, and the identities of the actor and re-actor" (Sfard, 2008, p86-87).

In the context of this study, thinking and communication are specified within the mathematical discourse, namely as advanced mathematical thinking, and communication of advanced mathematical ideas among mathematicians of various levels of expertise.

Sfard, based on Vygotsky's remark that "the specifically human capacity for language enables children to provide for auxiliary tools in the solution of difficult tasks, to overcome impulsive action, to plan a solution to a problem prior to its execution, and to master their own behavior", identifies and further discusses the ***commognitive capacities*** that depend on the human ability to rise to higher commognitive levels and involve an "incessant interplay between utterances and utterances-on-former-utterances" (Sfard, 2008, p110). These capacities fall into two distinct categories: those related to *commognitive objects* (i.e. reasoning, abstracting and objectifying), and those who consider the thinkers or speakers, namely the *commognitive subjects* (i.e. subjectifying and consciousness). Sfard claims that it is impossible for these activities to be conceivable without the "unbounded recursivity of human commognition" (Sfard, 2008, p110). In what follows, I give Sfard's definitions of the five capacities, illustrating with examples from the Group Theory context.

Reasoning is defined as “the art of systematic derivation of utterances from other utterances, its metadiscursive nature is implied in its very definition” (Sfard, 2008, p110). Reasoning is an activity of exploring the relations between sentences and moreover requires a metadiscursive approach of going beyond the sentences themselves. In the context of this study, reasoning can be observed in the various proofs that students are required to produce for instance proof that a map is a group homomorphism. Reasoning is intertwined with the metadiscursive rules involved in such as proof.

Abstracting refers to “the activity of creating concepts that do not refer to tangible concrete objects.” Concept is defined as “a symbol together with its uses” (Sfard, 2008, p111). The conceptualisation of the abstract and generalised notion of group is, for instance, an act of abstraction. Unlike the tangible objects of a cube or a dihedral triangle with their symmetries, the conceptualisation of abstract group as this is defined in the general definition of group does not rely on any tangible, concrete object.

Objectifying is defined as the “process in which a noun begins to be used as if it signifies an extradiscursive, self-sustained entity (object), independent of human agency”. Objectification consists of two sub-processes: *reification* (i.e. the process of replacing the talk about processes with talk about objects) and *alienation* (i.e. the use of “discursive forms that present phenomena in an impersonal way, as if they were occurring of themselves, without the participation of human beings”) (Sfard, 2008, p300, 301, 295). Sfard exemplifies reification as follows: “he *cannot cope* with even the simplest arithmetic problems” is reified to “he *has* a learning disability” (Sfard, 2008, p44). An example of alienation can be seen in the following sentence: “number is conserved as nothing is added to or taken way from a set” (Sfard, 2008, p50), where there is use of passive voice and where the noun has the role of grammatical subject. In the context of my study, objectification is often detected in the interviews where students turn the focus on the discussion on the involved concepts and their definitions, rather than on the processes that have applied for the solution of a relevant task.

Subjectifying is "a special case of the activity of objectification in which the discursive focus is reallocated from actions and their objects to the performers of the actions" (Sfard, 2008, p113). Sfard, exemplifies this capacity by the following example, where the second sentence about *being* is the subjectified version of the first sentence about *doing*: "Ludwig writes philosophical books", and "Ludwig *is* a philosopher". In the context of my study, subjectification is often identified when the discussion about certain mathematical concepts turns into a discussion about how a student should operate with these concepts, often in the form of instruction. For instance, instead of talking about the cube symmetries and how these can be identified and listed, the student is giving instructions to his/her classmates about how to achieve identification of these symmetries. Moreover, I intend to investigate the conditions under which objectification shifts to subjectification.

Consciousness is directly interlinked with subjectivity. Sfard does not give a complete and final definition of consciousness but rather discusses and tries to ameliorate the following: "*consciousness* […] is defined as involving "an organism's awareness of its own self and surrounding"" (Sfard, 2008, p114). Moreover, "consciousness is the unbounded human ability to communicate about communication" (Sfard, 2008, p124). In the context of my study, evidence of students' consciousness at work is when students appear fully aware of their mathematical reasoning. At this initial stage of students' learning, this is possibly apparent when the level of abstraction is rather low, for instance when students are asked to find the symmetries of a concrete object, e.g. cube, or when they are engaged with a mathematical task that have seen before, perhaps in a different circumstance.

In mathematical discourse, unlike other scientific discourses, objects are discursive constructs and form part of the discourse. Mathematics is an *autopoietic system* of discourse, i.e. "a system that contains the objects of talk along with the talk itself and that grows incessantly 'from inside' when new objects are added one after another" (Sfard, 2008, p129). Moreover, there are certain features that characterise mathematical discourse, namely, the *use of words*, *visual mediators*, *narratives*, and *routines*.

Word use: In a mathematical discourse, words represent, not necessarily exclusively, quantities and shapes. “Whereas many number-related words may appear in nonspecialized, colloquial discourses, mathematical discourses as practiced in schools or in academia dictate their own, more disciplined uses of these words. Word use is an all-important matter because, being tantamount to what others call ‘word meaning,’ it is responsible for what the user is able to say about (and thus to see in) the world” (Sfard, 2008, p133). In the context of this study, I aim to investigate the extent and quality of the use of the involved mathematical vocabulary and moreover evaluate students understanding of definitions of the involved group-theoretic concepts. In addition, I intend to investigate how the use of words influences the effectiveness of communication between mathematicians.

Visual mediators: These are visible objects that operate as a supportive part of communication. Moreover, unlike in a colloquial discourse, visual mediators are symbolic artifacts, created purposefully for the sake of this particular form of communication. The use of such visual mediators is often automated and embodied. Sfard exemplifies this characteristic of the use of visual mediators by mentioning the procedures of scanning the mediator with one’s eyes in a well-defined way. In the context of this study I intend to evaluate the extension of quality of the use of visual mediators and the role these play in the learning of Group Theory. In the case of my study, obvious visual mediators could be considered the mathematical algebraic notation or several types of diagrams, such as Argand diagrams or other illustrations representing certain objects.

Narrative is a “sequence of utterances framed as a description of objects, or relations between objects, or of processes with or by objects, that is subject to endorsement or rejection with the help of discourse-specific substantiation procedures” (Sfard, 2008, p134). In the case of mathematical discourse, “the consensually endorsed narratives are known as mathematical theories, and this includes such discursive constructs as definitions, proofs, and theorems” (Sfard, 2008, p134). Examples of narratives, in the context of this study could be the given definitions of the concepts of group, subgroup, coset etc. the First Isomorphism theorem, the Lagrange's Theorem and the consequent lemmas, or their proofs etc. My study intends to investigate how effectively students use these narratives, especially in solving related mathematical tasks, or proving given mathematical problems.

Routine is defined as a "set of metarules defining a discursive pattern that repeats itself in certain types of situations" (Sfard, 2008, p301). Moreover, routines are "repetitive patterns characteristic of the given discourse. Specifically, mathematical regularities can be noticed whether one is watching the use of mathematical words and mediators or following the process of creating and substantiating narratives about numbers or geometrical shapes" (Sfard, 2008, p134). Routines are governed by two distinct subsets of metarules, namely the *how* and the *when* the routine. The *how* defined as "a set of metarules that determine, or just constrain, the course of the patterned discursive performance (the *course of action* or *procedure*)" (Sfard, 2008, p202). The *when* of a routine, is defined as "a collection of metarules that determine, or just constrain, those situations in which the discursant would deem this performance as appropriate" (Sfard, 2008, p202). The task of *how* the pattern works is usually straightforward. Presenting the *when* of routines though, i.e. constructing exhaustive lists of conditions according to which given patterns tend to appear in a discourse of a given individual or group, is a more complicated task. Furthermore, the *when* of a routine can be subdivided into two condition categories: *Applicability conditions* are rules that outline the circumstances in which the routine course of action is likely to be evoked by a person. The rules specify the routine *prompts* i.e. the elements of situations whose presence increases the possibility the routine's performance; *Closure conditions* are the set of metarules that define circumstances that the performer is likely to interpret as signaling a successful completion of the performance. An example of a routine in the group-theoretic discourse is the test for a set to be a subgroup, according to which one has to prove that the set is non-empty, and then that it is closed under the operation and closed under inverses. Another example is the proof that a map is an isomorphism, where students need to prove that the given homomorphism is a bijection (both one-to-one and onto). Moreover, routines are the result of students' mathematical activity in their effort to produce something, whereas narratives are given to them as tools. My study aims to: identify students' difficulties in the application of the various routines, both from a object-level and a metalevel perspective; investigate the effectiveness of the application of the *how* and *when* sets of metarules, within the context of Group Theory; and, scrutinize how students cope with the application of applicability and closure conditions in their decision-making regarding the strategy followed in the problem solving and proof production.

Mathematical discourse involves certain objects of different categories and characteristics. ***Primary object*** **(*p-object*)** is defined as "any perceptually accessible entity existing independently of human discourses, and this includes the things we can see and touch (material objects, pictures) as well as those that can only be heard (sounds)" (Sfard, 2008, p169). An example of a p-object in the Group Theory discourse could be the *cube*, which represents the group G of the rotational symmetries of the cube or the *dihedral square*, which represents the symmetric group D_4.

Another category of mathematical objects, the ***simple discursive objects*** *(simple d-objects)* "arise in the process of *proper naming (baptizing)*: assigning a noun or other noun-like symbolic artefact to a specific primary object. In this process, a pair <noun or pronoun, specific primary object> is created. The first element of the pair, the *signifier*, can now be used in communication about the other object in the pair, which counts as the signifier's only *realization*. For example, assigning my dog the noun Rexie (or the words my dog, for that matter) is an act of creation of the discursive object Rexie (my dog)" (Sfard, 2008, p169). In the context of Group Theory, D_4 can be considered the symbolic artefact that has the role of noun (signifier), whereas the dihedral square is the realization of this signifier. Together the two comprise the simple d-object of the dihedral group D_4.

Compound discursive objects (d-objects) arise by "according a noun or pronoun to extant objects, either discursive or primary." For instance, the d-object of symmetric group is a compound d-object, which represents a set of different simple d-objects such as the dihedral groups D_4 or S_3 and the respective p-objects of the dihedral square and the dihedral equilateral triangle. In the context of my study, the compound d-objects that are considered are the group, subgroup, coset, normal subgroup, quotient group, kernel, image, group homomorphism and group isomorphism.

The (*discursive*) *object* signified by S in a given discourse is defined as "the realization tree of S within this discourse." (Sfard, 2008, p166) The ***realization tree*** is a "hierarchically organized set of all the realizations of the given signifier, together with the realizations of these realizations, as well as the realizations of these latter realizations and so forth" (Sfard, 2008, p300). For instance, the realization tree of the signifier "the solution of the equation $7x + 4 = 5x + 8$" would include three branches involving the following: the algebraic solution of the equation; the 2-dimensional graph representing; and the table of values of $x, 7x + 4$ and $5x + 8$ respectively. In the context of Group Theory, an example of a realisation tree can be the set of all realisations of the group G, namely, the symbolic representation of $G = (\circ, S)$, with its axioms, the table of group elements, its symmetric visual representation as a canonical polygon or polyhedron, and/or the set-theoretic visual representation.

Realisation trees and consequently mathematical objects are personal constructs, although they emerge from public discourses that support certain types of such trees. Additionally, realisation trees offer valuable information regarding the given individual's discourse. Moving with dexterity from one realisation to another is the essence of mathematical problem solving. Realisation trees are a personal construction, which may be exceptionally 'situated' and easily influenced by external influences such as the interlocutors. Finally, signifiers can be realised by different interlocutors in different ways, according to their own specific needs.

Human communication, as defined above, is a rule-regulated activity, with discourses being of a repetitive and patterned nature. Any discursive pattern can be considered as a result of rule-governed processes. In the context of CTF, there exist two distinct kinds of rules: ***Object-level rules***, namely rules regarding properties of the objects of certain discourse, taking the form of narratives of these objects; and, ***Metadiscursive rules***, namely, rules that are involved when we examine the patterned activity of formulation and substantiation of the object-level rules. In mathematical discourse, the relevant *metarules are those that govern the activity of proving*. More generally, "object-level rules are narratives about regularities in the behavior of objects of the discourse, whereas metarules define patterns in the activity of the discursants trying to produce and substantiate object-level narratives" (Sfard, 2008, p202).

In mathematics, what is considered as a metarule in one mathematical discourse will give rise to an object-level rule as soon as the present metadiscourse "turns into a full-fledged part of the mathematics itself" (Sfard, 2008, p202). For instance, the utterance "'To multiply a sum of two numbers by a third number one can first multiply each addend and then add the products,' which is a metarule of arithmetic, turns within algebraic discourse into the object-level rule '$a(b+c) = ab + ac$', expressing the relation among three algebraic objects, the variables a, b, c (the variable is the product of saming of all the numbers in a certain domain, in this case, in the domain of all real numbers)" (Sfard, 2008, p202).

An example from the context of Group Theory that distinguishes the object-level and metalevel rules is when students are asked to prove that a map is an isomorphism. For instance: Let G be any group, $h \in G$ and $\varphi: G \to G$ is given by $\varphi(g) = ghg^{-1}$. In order to prove this, students need to apply certain rules that are not only directly related to the d-object of map and its definition, but they rather follow a certain reasoning, namely first to prove that this map is a homomorphism, by examining that the group operation is preserved, and then prove that this homomorphism is a bijection by examining its kernel and image, and proving that it is onto and one-to-one.

Metarules in one mathematical discourse will trigger the rise of an object-level rule as soon as the present metadiscourse turns into an independent branch of mathematics itself. These rules have the following five characteristics: variability, tacitness, normativeness, flexibility, and contingency. In what follows, I give illustrative examples and further information of the five characteristics.

Variability: Metarules regulating various mathematical activities have been evolving, often substantially, through ages. In fact mathematical learning involves gradual modification of the metarules that govern students' mathematical discourse. In the example in the previous paragraph, both object-level and metalevel rules vary according to which mathematical discourse is used, arithmetic or algebra.

Tacitness (Interpretive Nature): metarules are not a subject of conscious analysis that would be followed by mathematicians or students in an intentional way. In the above example, students are not expected to interpret the rules amenable to arithmetic or algebra. These rules are just "retroactively written into interlocutors' past activities and expected to reappear" (Sfard, 2008, p203).

Normativeness (Value-Ladedness): Well-established rules among the entire discourse community are considered norms. In order for metarules to be considered as norms, there should fulfill two conditions: these metarules should be widely enacted within the discourse community; the great majority of the members of the community, especially the experts, should endorse these rules. For instance, the rules of addition and multiplication of algebraic terms as seen in the above example are considered norms since they are widely enacted amongst mathematicians of all levels.

Flexibility: Rules in mathematical discourse, although the word might suggest so, do not imply stringent control. Metarules do not impose a specific modus operandi but they rather eliminate "an infinity of possible discursive moves and leave the interlocutors with only a manageable number of reasonable options" (Sfard, 2008, p206).   For instance if the teacher asks the students to 'investigate the function $f(x) = 3x^3 - 2x + 5$', they might choose to either make a graph or find the turning points or try to find what this function models in the 'real-life' world.

Contingency: This characteristic implies that metarules are the result of 'custom-sanctioned' associations rather than an externally imposed necessity. "Discursive regularities, and thus metadiscursive rules, are the result of custom-sanctioned associations rather than a matter of externally imposed necessity" (Sfard, 2008, p206). Wittgenstein (1978) was the first to identify this characteristic of the rules in the process of mathematical proof. In the case of proof, contingency does not refer to the proof as such, but to the metarules that govern its construction.

Learning Mathematics is defined as a *change of mathematical discourse*. There are two distinct types of learning: object-level learning, and metalevel learning. "***Object-level learning*** that expresses itself in the expansion of the existing discourse attained through extending a vocabulary, constructing new routines, and producing new endorsed narratives; this learning, therefore results in endogenous expansion of the discourse" (Sfard, 2008, p253); "***Metalevel learning,*** which involves changes in metarules of the discourse and is usually related to exogenous change in discourse. This change means that some familiar tasks, such as, say, defining a word or identifying geometric figures, will now be done in a different, unfamiliar way and that certain familiar words will change their uses" (Sfard, 2008, p254). In the context of this study, an example of object-level learning would be the introduction of the various d-objects of group, subgroup, coset, etc. and an example of metalevel learning would be required for proving that a certain map is an isomorphism where novice mathematicians are required to approach this task differently from what they have learned in Calculus, and in accordance to the new metarules of Group Theory,

Metalevel learning, according to Sfard (2008), originates, most likely, in the individual's direct engagement with the new discourse. Since the new discourse has different metarules to the ones he or she was acting so far, such an engagement entails ***commognitive conflict***. This is caused when different discursants are acting according to different metarules. Moreover, such conflict emerges when familiar routines meet with other people's ways of employing the same discursive tasks, based on different metarules. When the learner is exposed to commognitive conflict, he or she has the opportunity for metalevel learning, i.e. for the evolution of metarules.

2.2 Research into the Learning of Group Theory

Group Theory is a particularly demanding module for mathematics students, since they are required to successfully cope with its abstract and rigorous nature and invent new learning approaches. It is the first module in which students must go "beyond learning 'imitative behavior patterns' for mimicking the solution of a large number of variations on a small number of themes (problems)" (Dubinsky et al, 1994). The construction of the newly introduced mental object of group is often an arduous task for novice students and causes serious difficulties in the transition from the informal secondary education Mathematics to the formalism of undergraduate Mathematics (Nardi, 2000). Students' difficulty with the construction of the Group Theory concepts is partly grounded on historical and epistemological factors: "the problems from which these concepts arose in an essential manner are not accessible to students who are beginning to study (expected to understand) the concepts today" (Robert and Schwarzenberger, 1991). Nowadays, the presentation of the 'fundamental concepts' of Group Theory, namely group, subgroup, coset, quotient group, etc. is "historically decontextualized" (Nardi, 2000, p169), since historically the fundamental concepts of Group Theory were permutation and symmetry (Carspecken, 1996).

Research into the learning of Group Theory was mainly initiated in the early 1990's. Several studies, following a constructivist approach, and within the Piagetian tradition of studying the cognitive processes, examined students' cognitive development and analysed the emerging difficulties in the process of learning certain group-theoretic concepts. My study aims to examine students' learning of Group Theory using a discursive perspective, which would not focus only on the development of genetic decomposition of the various concepts, but it would rather inform how students conceptual understanding develops in relation to their applied study skills and the effectiveness of communication amongst students and instructors. In what follows, I discuss issues of learning in the context of Group Theory.

Students' understanding of the concept of group is often primitive at the beginning, predominantly based on their conception of a set. An important step in the development of the understanding of the concept of group is when the student "singles out the binary operation and focuses on its function aspect" (Dubinsky et al, 1994, p292). Students often have the tendency to consider group as a 'special set', ignoring the role of binary operation. Iannone and Nardi (2002) suggest that this conceptualisation of group has two implications: the students' occasional disregard for checking associativity and their neglect of the inner structure of a group. These last conclusions were based on students' encounter with groups presented in the form of group tables. My study aims to check whether this claim holds in a more general, algebraic context. This aim is grounded on the hypothesis that group tables may be considered as a more 'concretized' version of a more algebraic mathematical task. In fact, students when using group tables adopt various methods for reducing the level of abstraction, by retreating to familiar mathematical structure, by using canonical procedure, and by adopting a local perspective (Hazzan, 2001).

An often-occurring confusion amongst novice students is related to the order of the group G and the order of its element g. This is partly based on student inexperience, their problematic perception of the symbolisation used, namely $|g|, \langle g \rangle$, and of the group operation. The use of semantic abbreviations and symbolisation can be particularly problematic at the beginning of their study. Nardi (2000) suggests that there are both linguistic and conceptual interpretations of students' difficulty with the concept of order of an element of the group. The role of symbolisation is particularly important in the learning of Group Theory, and problematic conception of the symbols used probably causes confusion in other instances. I aim to investigate this issue and in particular examine the reasons and consequences of the possibly crucial confusion between groups, sets, their elements and their properties.

Dubinsky et al (1994) suggest that understanding of groups and subgroups may progress simultaneously, whereas Brown et al (1997) suggest that the concept of group is already present when the development of the concept of subgroup begins. My study aims to investigate further how the development of these two concepts is connected, examining also whether this connection is context-sensitive.

The introduction of the concepts leading to quotient groups seems to be an important milestone in the learning of Group Theory. At this point the majority of students face serious difficulties. Dubinsky et al (1994) conjecture that lack of understanding of normality prevents students from understanding the coset operations and quotient groups. A common misunderstanding is possibly the confusion of normality and commutativity (Dubinsky et al, 1994).

Regarding cosets, there were somewhat contradictory conclusions regarding students' understanding. Dubinsky et al (1994) suggest that the introduction of cosets is an important step that causes difficulties to novice students, followed by an attempt to introduce the concept of the binary operation on the set of cosets in order to obtain a new group, namely the quotient group G/H. In contrast, in Asiala (1997) students' performance on tasks involving cosets was satisfactory indicating good understanding. Although Dubinsky's et al (1994) claim is more reasonable and in accordance with the overall argumentation about the related concepts, my study aims to investigate further students' understanding of the notion of coset, as well as its role in the understanding of quotient groups. Furthermore, I aim to investigate how students' understanding of the aforementioned concepts is influenced by the effective use of symbolisation and visual images.

In fact, recent studies have further scrutinised the first step of introducing cosets, linking the issue with engagement and the ability to visualise cosets (Ioannou, 2010) and students' responses to their difficulty to visualise cosets (Ioannou and Iannone, 2011), and concluded that students' expressed desire for having visual images, contradicts to the lack of such use, especially as the level of abstraction increases. The study of visual images, particularly in relation with cosets (Nardi, 2000) contributes to the meaning-bestowing process as well as the specification of its raison-d-être. There are indications of student tendency to use images of familiar regular geometric shapes in order to construct a mental image of new concepts. In addition, these geometric images are often interpreted literally by novice students. Problematic interpretation advances the issue of a 'potential cognitive danger' regarding their use, in spite of their pedagogical significance. These claims are going to be investigated considering all the involved group-theoretic concepts.

The use of visual images and its role in learning Group Theory is an important issue in this study, particularly in the analysis of conceptual difficulties and students' skills applied to overcome these difficulties. The use of visual images and thinking cannot be distinguished from the analytic thinking required in learning mathematics. In fact, according to Piaget (1975, 1977) and Presmeg (1986a, 2006) there is very little visualisation, which does not contain some analysis, and consequently, there is very little analysis without some use of visualisation (Zazkis et al, 1996). In their study, within the context of Group theory, Zazkis et al (1996), using the dihedral group D_4 as a paradigm, concluded that students who can mix, synthesize and harmonize visual and analytic thinking and strategies, have used more mature understanding of the given mathematical problem. It would be interesting to investigate from a wider perspective the use of visualisation as a strategy for solving often-inaccessible mathematical tasks, examining how students use visual images in their solutions. I addition, I aim to focus of how and how extensively novice students use visual images which are not so easily grasped, for instance the image of coset or quotient group, as a meaning-bestowing technique.

Another important milestone in the learning of Group Theory is the introduction of the rich and multifaceted notion of group isomorphism. Literature suggests that for novice students, the concept of isomorphism is a 'complex and compound concept, composed of and connected to many other concepts which in themselves may be only partially understood" (Leron et al, 1995, p153). Leron et al (1995) also suggest that understanding of group isomorphism requires understanding of the concepts of group, function and quantifier, and conversely, learning about group isomorphism may substantiate the understanding of the aforementioned concepts. Moreover, there is connection of the isomorphism tasks to the concepts of the order of group, the order of the elements of the group, commutativity and others. In addition to these claims, my study hypothesizes and aims to prove that successful performance in tasks involving the concept of isomorphism, requires in addition, and most vitally, good understanding of the notions of kernel and image.

This last hypothesis is grounded on the conclusion that students have difficulties in the conceptualisation of properties associated to the notion of mapping, the varying degrees of abstraction involved in the definition of a mapping between elements of a group or the cosets of a subgroup and the elements of the group. The high level of abstraction and the conceptual difficulties are interconnected with the students' cognitive perplexity in Group Theory, which, climaxes at the introduction and proof of the First Isomorphism Theorem (Nardi, 2000). Indeed in a typical introductory module in Group Theory, First Isomorphism Theorem is the last and most crucial result introduced, and is probably a "container of compressed conceptual difficulties" (Nardi, 2000, p179), since it involves all the preceding concepts. I aim to verify this as well investigate it from a metadiscursive level of understanding.

Regarding the concept of homomorphism, I aim to investigate how good understanding of the more general concept homomorphism contributes to the thorough understanding of the more specific notion of isomorphism. I hypothesize that since students show a "strong need for 'canonical', step-by-step procedures and tend to get stuck of having to deal with some degrees of freedom in their choices" (Leron et al, 1995), their performance in proving that a given map is a homomorphism, will be significantly better, since it is grounded on a step-by-step procedure. Proving that a map is an isomorphism is a much less 'canonical' procedure, involving a wider spectrum of concepts as well as higher level of resourcefulness.

In an abstract and rigorous advanced mathematical module such as Group Theory, students often have difficulty with the linguistic condensation of meaning. Moreover, there is contrast between deep and naïve meaning of the various definitions and theorems, for instance Lagrange's Theorem (Hazzan and Leron, 1996) and First Isomorphism Theorem (Nardi, 2000). Deep meaning is not always in agreement with the surface meaning. In particular, the deep meaning of Lagrange's theorem is acquired "from accumulated experience in group theory, mainly the way the theorem is most commonly used in applications and proofs" (Hazzan and Leron, 1996, p24). Students often tend to use theorems as slogans, for instance apply Lagrange's Theorem or some version of its converse where not appropriate or relevant to the problem, or use the theorem and its converse indistinguishably. In addition, students often cannot conceptualise the deep meaning of theorems. My study aims to verify these claims in a more general context, examining the use of theorems as part of the student metalevel learning. In addition, I aim to investigate how this use develops throughout the module as well as in the final stage of the examination, and trace any signs of improvement.

The abstract nature of Group Theory is often an impediment for novice students. In order to successfully cope with learning Group Theory, students often tend to adopt techniques that would reduce the level of abstraction. Reducing the level of abstraction is an "effective mental strategy, which enables students to mentally cope with the new, abstract kind of mathematical objects” (Hazzan, 1999, p73). A significant number of students seem to have the tendency to work on a lower level of abstraction than the one in which concepts are introduced. By reducing the level of abstraction, enables students to “base their understanding on their current knowledge, and proceed towards mental construction of mathematical concepts conceived on higher level of abstraction” (Hazzan, 1999, p84).

Hazzan’s (1999) study examines the reduction of the level of abstraction based on three interpretations for levels of abstraction: the first interpretation refers to the abstraction level as the quality of the relationships between the object of thought and the thinking person, and suggests that each individual for each concept may observe different level of abstraction; the second interpretation refers to the abstraction level as reflection of the process-object duality, and suggests that the more developed the reflective abstraction of a concept is, the less abstract is considered by the individual; and, the third interpretation refers to the abstraction level as the degree of complexity of the concept of thought, and suggests that the more compound a concept is, the more abstract is considered. My study aims to investigate what skills students may use in order to achieve reducing the level of abstraction in the limited duration of the module and how effective these efforts are. I hypothesize that the use of examples, metaphors and visual mediators contribute significantly in reducing abstraction.

2.3 Secondary-Tertiary Transition

In this section I aim to discuss the compound issue of secondary-tertiary Mathematics transition and how students cope with the new demands of university education and the characteristics of university Mathematics. Given the abstract nature of Group Theory and the difficulty many students have in their first encounter with it (Dubinsky et al, 1994; Nardi, 2000), I examine the issue of transition in terms of: the *formal proof production* required mostly in pure mathematics; the *abstract nature* of several mathematical modules one of which is Abstract Algebra; the quality of *reasoning* in the solution of a mathematical task; and, the students' ability to *visualise* and effectively *use the examples*. Finally, I examine transition from a communicational perspective.

Secondary-tertiary transition is a multilayered issue that lasts much longer than the 'transitional period' from the end of secondary school to the beginning of university. It involves a series of adjustments in an individual level and in a sociocultural level. Various studies have examined transition from different perspectives, for instance epistemological, cognitive, sociocultural and didactical (De Guzman et al, 1998; Gueudet, 2008). My study intends to focus predominantly on the transition as an invitation for coping with the new cognitive challenges and metacognitive adjustments, in the abstract context of Group Theory. In particular, I intend to investigate the hypothesis that the introductory module in Group Theory is a significant transitional challenge for students and invitation for various adjustments in their approach to learning and study techniques, because of its abstract nature.

A distinctive characteristic of advanced Mathematics in the university is the ***production of rigorous and consistent proofs***. The often arduous, for the majority of students (Moore, 1994; Segal, 2000), task of successfully producing and communicating their proofs is a significant obstacle in the smooth transition from secondary to university Mathematics. Proof production is far from a straightforward task to analyse and identify the difficulties students face. From a pedagogical perspective, a possible contributing factor to the students' difficulty with proof is the teaching they receive both in high schools and in universities, since "most students have not been enculturated into the practice of proving, or even justifying the mathematical processes they use" (Dreyfus, 1999, p94). In addition, from a communicational perspective, there is a chasm between the professional mathematicians and students regarding their views about issues like conviction or validity of proof (Segal, 2000; Harel and Sowder, 1998), the adequacy of an explanation or justification (Sierpinska, 1994), or the ability to distinguish between different forms of reasoning (Dreyfus, 1999). Teachers often do not aim to give their students the means to learn how to construct proofs and judge their validity. This is a task left to students (Dreyfus, 1999).

Difficulties with proof production have been extensively investigated for various levels of student expertise (from novice undergraduates to experienced doctoral students). Moore (1994) classifies novice students' difficulties with proving in three wide categories referring to: the mathematical language and notation as such; the concept understanding; and, getting started with the proof. This categorisation is in conformity with the CTF where one is able to examine the students' use of words, visual mediators, understanding of the definitions of the related d-objects and the related theorems and lemmas, as well as the routines with their applicability and closure conditions. Weber (2001) categorises student difficulties with proofs into two classes: the first is related to the students' difficulty to have an accurate and clear conception of what comprises a mathematical proof, and the second is related to students' difficulty to understand a mathematical proposition or a concept and therefore systematically misuse it. In his study, an examination of the performance of undergraduate (novice) and doctoral (expert) students in proof-production, Weber (2001) indicated three types of strategic knowledge that the latter applied and the former lacked, in particular, referring to the knowledge of domain's proof techniques, the knowledge of which theorems are important and when they will be useful and the knowledge of when and when not to use 'syntactic' strategies.

My study aims to investigate further, and localise novice students' difficulties with proofs, within the context of Group Theory from two different perspectives: examine what difficulties novice students face in the abstract discursive environment of Group Theory in practice, scrutinizing their coursework and exam papers; and, investigate students' perspectives and views about the process of proof production. I also aim to examine the interaction between the object-level and metalevel understanding. My study aims to consider issues of mathematical communication and its significance in the overall mathematical learning (and proof production as an important activity) (Gueudet, 2008; Epp, 2003).

The transition from secondary to tertiary mathematics requires students to justify their mathematical arguments with strong and clear ***mathematical reasoning***. Reasoning is defined as "the line of thought, the way of thinking" (Lithner, 2000, p166) and, regarding advanced mathematical concepts, it requires the interaction between rigorous and intuitive thought, and the use the formal definitions of these concepts (Weber and Alcock, 2004). Many distinguished mathematicians have reported the interchange between rigor and intuition in advanced mathematical reasoning, as well as the importance of the existence of both in proof production (Thurston, 1994; Poincaré, 1913; Hadamard, 1945).

The solution of a mathematical task or the proof of a mathematical narrative, is a set of subtasks "of different grain size and character" (Lithner, 2000), and the reasoning can be described as a manifold structure. In fact, Lithner (2000) suggests that mathematical reasoning involves four stages: the initial stage where the student faces a *problematic situation*; the *strategy choice*; the *strategy implementation*; and the *conclusion*. Moreover, there are two types of mathematical reasoning, namely, *plausible reasoning*, which is defined as "an extended and 'looser' version of proof reasoning, but still based on mathematical properties of the involved components", and *reasoning based on established experiences* from the learning environment, "which might be mathematically superficial" (Lithner, 2000, p165). Novice students' solving strategies fall mostly into the second type of mathematical reasoning, based on familiarity with the task. Both Lithner (2003) and Sierpinska (2000) demonstrate that students do not have the ability of the experienced mathematicians to develop two kinds of reasoning. Novice students' thinking is limited to practical reasoning based on established experiences. Lithner's (2000) four stage reasoning is in accordance, with CTF and the how and when of a routine, yet without refining the rules that refer to object-level and metalevel reasoning. Furthermore, my study aims to investigate students' reasoning in the various group-theoretic tasks, making the refined distinction between object-level and metalevel reasoning and learning in general in the abstract environment of Group Theory.

I also intend to investigate how students approach mathematical task solution in the examination and what are the differences with their approach in the coursework. I hypothesize that students' solution approaches in the examination possibly involve memorizing. Bergqvist (2007) examines the types of reasoning that students perform when solving mathematical tasks in exams and to which extent it is possible for students to address these tasks by applying imitative reasoning. *Memorized reasoning* has the following two characteristics: "the strategy choice is founded on recalling a complete answer by memory" and "the strategy implementation consists only of writing down (or saying) the answer" (Bergqvist, 2007, p352). *Algorithmic reasoning* fulfills the following conditions: "the strategy choice is founded on recalling by memory a set of rules that will guarantee that a correct solution can be reached" and "the strategy implementation consists of carrying out trivial (to the reasoner) calculations or actions by following the set of rules" (Bergqvist, 2007, p352). I aim to investigate further, the application of memorized and algorithmic reasoning and examine the conditions under which memorized and algorithmic reasoning are used and what causes such use.

A challenging issue that mathematics students need to face in the transition from secondary to university Mathematics is the development of skills for coping with the level of ***abstraction***. The abstract nature of many modules of university Mathematics is a characteristic that makes proof production and mathematical communication even more problematic, and it is something that students need to adjust to. Abstraction is a concept that has been studied from many different perspectives and in the context of different disciplines, e.g. Psychology, Mathematics, and Mathematics Education. There is not accepted variety of definitions of what abstraction is, but it can be examined from various viewpoints, as a characteristic of various mathematical concepts, as the ability 'to abstract' (Hazzan, 1999), or as a consolidated construction used to create new constructions (Hershkowitz et al, 2001; Monaghan and

Ozmantar, 2006; Kidron, 2008). In the context of this discussion, abstraction is an activity towards consolidating a mathematical construction, in accordance with Hershkowitz et al (2001).

Studying Group Theory, and Abstract Algebra in general, requires students to adjust their study skills and learning approaches to cope with the high level of abstraction, as part of the transition from secondary to tertiary education. These adjustments involve the activity of abstraction as an endeavor towards consolidating a mathematical concept (Hershkowitz et al 2001). Hershkowitz et al (2001) define abstraction as "an activity of vertically[1] reorganizing previously constructed mathematics into new mathematical structure" (Hershkowitz et al, 2001, p202). The process of abstraction depends on an individual's learning history (Hershkowitz et al, 2001; Dreyfus and Tsamir, 2004), since novel mathematical structures are based on previously constructed structures, and involves three epistemic actions, namely *recognizing* (recognition of a mathematical structure and realization that this structure is inherited from a previous mathematical experience), *building-with* (when students try to achieve to solve a mathematical task, understand and explain a situation or reflect on a process, by using a given theorem, set of rules or appropriate strategy), and *constructing* (similar to building-with, but here the final goal should be to construct a *new* structure). These actions are not linearly sequenced, but are rather undertaken in a nested way, i.e. construction does not necessarily follow the other two, but it rather requires and involves recognition of and building-with already constructed structures. My study aims to investigate how students cope with abstraction, what tools do they use in order to reduce the level of abstraction as well as verify how previous knowledge and student learning history contributes to this effort. I hypothesize that previous knowledge, in the form of a metaphor, is indispensable in coping with the abstract nature of Group Theory.

An important means for coping with the level of abstraction is possibly the use of visual images. ***Visualisation*** involves "processes of constructing and

[1] Vertical mathematizing is defined as 'an activity in which mathematical elements are put together, structured, organized, developed, etc., into other elements, often in more abstract or formal form than the originals' (Hershkowitz, Parzysz, and van Dormolen, 1996, p177).

transforming both visual mental imagery and all of the inscriptions of the spatial nature that may be implicated in doing mathematics" (Presmeg, 2006). This definition is based on Piaget and Inhelder's (1971) suggestion that when an individual creates a spatial inscription (e.g. mathematical), there is a visual image in the individual's mind directing this creation.

The use of visual mediators, an important characteristic of mathematical discourse (Sfard, 2008), allows students to cope with mathematical difficulties related to problems of generalisation. These problems according to Presmeg (1985, 1986a, 1997) can be overcome by pattern imagery and the use of metaphors via an image, since it allows a static image to carry, for the visualiser, generalised mathematical information. Presmeg also suggests that concrete imagery, in order to be effectively used in the context of mathematics, must be accompanied by rigorous analytical thinking.

In spite of the students' need to use visual images in their learning, many studies have reported on their reluctance to do so (Dreyfus, 1991; Healy and Hoyles, 1996; Ioannou and Nardi, 2010; Stylianou, 2001). In fact, according to Healy and Hoyles (1996, p67) "students of mathematics, unlike mathematicians, rarely exploit the considerable potential of visual approaches to support meaningful learning… Where the mathematical agenda is identified with symbolic representation, students are reluctant to engage with visual modes of reasoning."

Students' reluctance to make use of the available visual images and furthermore use visual thinking is most probably a rather complex issue, influenced by several issues, including pedagogical and sociocultural. This hypothesis is in agreement with Presmeg and Bergsten (1995) and Dreyfus (1991). My study aims to investigate further the use of visual images and the reported reluctance of using them, linking this issue with the students' need for coping with the high level of abstraction and the successful transition towards university mathematics. In addition, I aim to investigate further how and when students use visual images in the process of proving.

Finally, as mentioned in section 2.1, my study intends to investigate the issue of ***communication*** amongst students, in the various communicational contexts, namely lecture, seminar, tutorial, and collaboration with peers. In addition, I aim to examine both oral and written communication. The issue of communication is a manifold one, and embraces different mathematical activities, apart from the strictly teaching and pedagogical context and the use of language (Iannone and Nardi, 2005) and tutoring (Jaworski, 2003). Mathematical proof, for instance, has an important role in mathematical understanding and communication (Hanna and Sidoli, 2007). Clear reasoning of mathematical arguments is an important factor of communication (Lithner, 2000, 2003; Bergqvist, 2007). The use of visual images is also a way for communicating mathematical ideas (Presmeg 1985, 1986a, 1997). My study aims to investigate further the issue of communication from a participationist perspective, examining the effectiveness of communication in the various teaching contexts (lecture, seminar, tutorial), formats (written, oral) as well as investigate the students' perceptions regarding these communicational aspects.

Chapter 3 Methodology and Research Design

In this chapter I discuss the methodological approach that I applied during the various stages of this study, namely the research design, data collection and data analysis, justifying my decision-making along the way. Apart from the theoretical aspects, I give a factual description of the process of data collection and analysis. In addition, I report on ethical issues relating to my research and how these have been tackled.

3.1 Context of Study

My doctoral study is a close examination of Year 2 mathematics students' conceptual difficulties and the emerging learning and communicational aspects in their first encounter with Group Theory. The module was taught in a research-intensive mathematics department in the United Kingdom, in the spring semester of a recent academic year, i.e. from January to April.

The Abstract Algebra (Group Theory and Ring Theory) module is mandatory for Year 2 mathematics undergraduate students, and a total of 78 students attended it. The module was spread over 10 weeks, with 20 one-hour lectures and three cycles of seminars in weeks 3, 6 and 10 of the semester. The role of the seminars was mainly to support the students with their coursework. There were 4 seminar groups, and the sessions were each facilitated by a seminar leader, a full-time faculty member of the school, and a seminar assistant, who was a PhD student in the mathematics department. All members of the teaching team were pure mathematicians.

The lectures consisted largely of exposition by the lecturer LCR, a very experienced pure mathematician, and there was not much interaction between the lecturer and the students. During the lecture, LCR wrote self-contained notes on the blackboard, while commenting orally at the same time. Usually, he wrote on the blackboard without looking at his handwritten notes.

In the seminars, the students were supposed to work on problem sheets, which were usually distributed to the students a week before the seminars. The students had the opportunity to ask the seminar leaders and assistants about anything they had a problem with and to receive help. The module assessment was predominantly exam-based (80%). In addition, the students had to hand in a threefold piece of coursework (20%) by the end of Week 12.

In this module, students were introduced to the notions of group and ring. The first half of the module was dedicated to the concept of group and to some of its applications and related concepts. The description of the Group Theory section of the course given on the mathematics department's website is as follows:

> At the heart of group theory is the study of geometric transformations and symmetry. The module remains close to such fundamental mathematical notions whilst also introducing a theory with enough generality to be applied elsewhere. This includes structural results such as isomorphism theorems as well as results on integer congruences and intriguing methods of counting developed using group actions on sets.

The second half of the module was dedicated to the concept of ring, and according to the description given in the department's website:

> The module starts by introducing rings, using the Integers as a model. The subsequent theory is developed with a variety of examples, giving new insight into familiar concepts such as substitution and factorisation. In contrast to the first section of the module, a commutative setting is soon adopted. Important examples of commutative rings are fields and domains. New constructions of fields are introduced using quotients of rings modulo maximal ideals. The concepts of divisibility and factorisation are tackled in domains. The related notion of a valuation on a domain exposes some fascinating links with other areas of mathematics typically studied at this level.

3.2 Methodological Approach

This study is a qualitative naturalistic inquiry, which aims to define the characteristics, the causes and the consequences of a social phenomenon (Lofland, 1971), namely the students' multi-layered experience with their introduction to Group Theory. In qualitative inquiries, social phenomena can be studied by observing people and their relationships, behaviour and actions, as well as their psychological stances and histories (Baker, 1994). Accordingly, I set out to examine these experiences from different teaching and learning perspectives and to focus on cognitive and pedagogical aspects of these experiences. In doing so, my study was conducted in its natural setting, namely the lecture theatre and seminar rooms, both for practical and substantive reasons. First, it would be the only way to record the teaching activities, and second because the natural milieu of teaching and learning would be important in the analysis and results (Lincoln and Guba, 1985).

In addition, as a qualitative naturalistic inquiry, my study had no preconceived or fixed initial hypotheses during the process of data collection and primary level analysis (Glaser and Strauss, 1967). The reason was that my aim to adopt a flexible stance towards the themes that might emerge, not limiting the study to only purely cognitive issues. Moreover, for the first-level data analysis, emerging themes were coded and categorised according to the similarity of the issues.

In terms of validity, I adopted the view of Winter (2000), who maintains that validity in qualitative research might be addressed through four distinct elements: the objectivity of the researcher, the participants themselves, the extent of triangulation of data and the quality, honesty and scope of data itself. Moreover, validity in a qualitative inquiry such as this study should be, and it practically was, approached as matter of degree and not as an absolute state (Gronlund, 1981). Regarding the first element, I have put every effort to be objective, during the data collection and analysis, by grounding each and every one of the claims on the data and justifying accordingly. Regarding the objectivity of participants, I am convinced that through the often repetitive questioning throughout the three cycles of interviews as well as the consistency of results with others forms of data, I have achieved high levels of participant objectivity. Regarding the last two elements of validity, the different classes of data and the outcomes have been triangulated with each other, and especially with the written data whose quality and honesty are a given.

In addition, my study adopting a commognitive perspective, has aimed to adopt a comprehensive approach to gathering and analysing data. My aim was to describe and interpret students' first encounter with Group Theory, not only from a cognitive perspective, but rather consider also the element of communication as this appears in pedagogy and thinking. This study aims to interpret students' first learning experience with this module and shift from the plain description of the data, to logical inference, justified explanation and theory generation (LeCompte and Preissle, 1993).

3.3 Data Collection

In qualitative naturalistic inquiry, there are a variety of different instruments for gathering data. The selection of methods in a specific study depends on their 'fitness for purpose' (Cohen et al, 2007). According to Lincoln and Guba (1985), there are two categories of data collection methods: obtrusive (including, for example, observations and interviews) and unobtrusive methods (such as using documents and records). This categorisation depends on whether at least one other human is present during the data collection. For the purpose of this study, I have used both obtrusive and unobtrusive methods, for various reasons. First, the inquiry itself has allowed me to do so, since it involved both document production as well as activities that could be observed and audio recorded. Second, thirteen students responded to my invitation for interviews and therefore this gave me the opportunity to enrich my data with these. Finally, in general the use of both obtrusive and unobtrusive methods, when possible, contributes significantly to the richness and therefore the validity and triangulation of the data. In what follows, I will focus on the obtrusive methods of data collection that I applied in this study. In section 3.4, I will also describe the use of unobtrusive methods, such as the collection of coursework and exam papers.

The key advantage of ***observations*** as a method of data collection is the possibility to gain data from actual, naturally occurring, social situations. This enables the researcher to capture a realistic and direct picture of incidents that occur in situ, rather than use mediated and possibly inferential information about them (Cohen et al, 2007). Observations furthermore allow the researcher to take note of particular situations or behaviours that might otherwise be taken for granted, expected or would go unnoticed (Cooper and Schindler, 2001).

According to Morrison (1993), observations enable the researcher to gather data on the *physical setting*, i.e. the environment and its organisation, the *human setting*, i.e. group organisation, people features etc., the *interactional setting*, i.e. the formal, informal, planned, unplanned verbal and non verbal interaction among the group members, and finally the *programme setting*, i.e. resources and their organisation, pedagogic styles, curricula and their organisation.

In the context of my study, observations were not the method of data collection that would allow me to gather the principal data, but they rather gave me the opportunity to collect data of auxiliary nature which had a supporting or illuminating role. These observations had taken place in two distinct settings, namely the lecture theater and the seminar rooms. My aim, through these observations was to become aware, first of all, of the physical setting where the lectures and seminars had taken place. In addition, I wanted to observe the characteristics of the interactional setting, namely the form and characteristics of interaction in the context of lecture and seminar, between the lecturer, the seminar staff and the students, as well as the collaboration amongst the students. These observations allowed me also to report on the characteristics of the programme setting, namely, the organisation and content of lectures and seminars, the student attendance, any unexpected events, such as cancellation of sessions etc. Finally, attending and observing these teaching activities allowed me to check that the audio recording of these sessions was undisturbed.

Regarding the different levels of researcher participation in observations, LeCompte and Preissle (1993) distinguish between the researcher as a *complete participant,* as a *participant-as-observer,* as an *observer-as-participant* and as a *complete observer*. These different roles of the researcher in the observation process represent a continuum from covert research where the researcher assumes an insider role in a particular group that is studied (complete participant) or is 'invisible' (for example by using a one-way mirror) as an observer (complete observer) to a more detached involvement of the researcher whose role as observer is known to the group (observer-as-participant). For the purpose of this study, I assumed the role of an observer-as-participant. There are various reasons for assuming such a role. My aim was to observe and not participate in any active way, influencing by my intrusion the natural development and activities of teaching. At the same time, my presence should be apparent and therefore remind students of the open invitation for further participation. Finally, considering the programme setting of the teaching in the specific mathematical department, I do not think that any other choice of any other role would be possible.

Naturalistic and participant observations, such as the ones taken place for the purposes of this study, help the researcher to generate data that are particularly 'strong on reality' (Morrison, 1993, p88). According to Carspecken (1996), they involve components of 'thick description'[2] recording, among others, speech acts, non-verbal communication, descriptions of low-inference vocabulary, careful and frequent recording of the time and timing of events, the observer's comments that are placed into categories and detailed contextual data. Indeed, observations have allowed me to record events and

[2] Thick description is a term that was first introduced by Geertz (1973a) and referred to the non-simplistic description of the human behavior by which anthropologists try to give deeper and reasoned interpretation to it, addressing also the complexity of situations.

gather valuable data regarding the ‘atmosphere’ in the lecture theater or seminar room, such as instances of excessive talking when students were possibly not following the lecturer, or the variation of absences from lecture to lecture, and as the module was developing, etc.

According to Patton (1990), observational data should give a clear description of the situation under consideration, enabling the researcher to enter and understand the situation. As with other instruments of data collection, observations can be categorised into three different types based on the degree to which they are structured. *Highly structured observation* is used when the researcher knows in advance what he/she is looking for and applies observation categories that were prepared in advance. *Semi-structured observation* is based on a fixed agenda of issues that are to be addressed, with observational data illuminating these issues in a less systematic or predetermined way. Finally, *unstructured observation* is used when there is no fixed list of issues to be addressed. The significance of particular events to a piece of research is only established after the researcher has observed these events. In conclusion, structured observations are often hypothesis-testing, whereas semi-structured and unstructured observations are mainly hypothesis-generating.

For the purpose of my study, I have chosen to use semi-structured observations, for various reasons. Firstly, I had no strong, preconceived hypotheses that I wanted to test. Although there was a certain agenda of issues I wanted to scrutinize, such as student attendance, incidents of verbal and non-verbal reactions, especially in the lectures, ways of student engagement in the seminars, my intention was to generate my hypotheses and conclusions, triangulating these observations with the other forms of data. Another distinction of observations is made between *overt* versus *covert*. Overt observations require the identity of the observer to be known to the participants whereas in covert observations the identity is concealed or partially concealed, when the researcher is present but not known (Cooper and Schindler, 2001). Naturally, since I have invited the group of participants to give interviews, it was both impossible and undesirable to hide my identity.

Regarding the note-taking process during the observations, there are two practical approaches, according to Emerson et al (1995). The first strategy is called *"the salience hierarchy"*. According to this strategy, researchers take notes on certain events that strike them as important, interesting or telling. These field notes allow the researcher later on to make a description of the events. Moreover, these data encompass salience hierarchy. Deviant cases of events often lead to salient data. Deviant cases are the ones that strike the observer with respect to his or her tacit expectations or the ones that look deviant with respect to the tacit knowledge developed in situ.

The second strategy, the one applied for the purposes of this study, is called *"comprehensive note-taking"*. According to this method, the researcher takes notes systematically and comprehensively on everything that happens during a particular period of time. Spradley (1980) offers a list of issues that should be included in a comprehensive and systematic note-taking process. These include: the physical place/s; the people involved; activities as sets of related acts people do; the physical things that are present; people's single actions; events as sets of related activities that people carry out; time; peoples' goals; the emotions that are felt and expressed. The reason for choosing this note-taking approach is because, at these early stages of this study, I had no strong preconceived hypotheses that I wanted to test. As mentioned above, my intention was to gather as much data as possible and try to generate my hypotheses as the data collection, and later the data analysis, was taking place. Therefore following Spradley (1980), I aimed to record information related to the natural setting, participants, activities, attendance, special events and even the participants emotions, and attitudes, when these were obvious and possible to be recorded. Another reason for following this approach is that I had the time to do so, since the lectures and all conversations during the seminar were audio-recorded, and therefore I could take notes of any peripheral incidents. This approach to note-taking during the observation, proved very important since in the data analysis, much of that information was valuable, even though during the data collection could be considered as immaterial, for instance the moments of relative perturbation among the students, which proved to be linked with their difficulty to follow the lecturer.

The ***interview*** is a powerful tool of data collection, enabling research participants to contribute to the process of generating knowledge. According to Kvale (1996, p14), "it is an interchange of views between two or more people on a topic of mutual interest, sees the centrality of human interaction for knowledge production, and emphasizes the social situatedness of research data". Interviews may be considered as neither subjective nor objective, but rather *intersubjective* (Laing, 1967).

There are several approaches towards categorising different interview types. Distinctions have been made between standardised, semi-structured, in-depth, ethnographic, elite, life history, focus group and exploratory interviews (LeCompte and Preissle, 1993; Bogdan and Biklen, 1992; Oppenheim, 1992). Patton (1980) distinguishes between informal conversational interviews, interview guide approaches, standardised open-ended interviews and closed quantitative interviews. In the case of qualitative studies, the researcher attempts to acquire unique, non-standardised, personalised information about the participants and how they view the world. Open-ended, semi-structured or even unstructured interviews are therefore more appropriate than structured interviews (Cohen et al, 2007).

For the purposes of my study, I have used semi-structured interviews. While structured interviews are based on a formalised, fixed list of questions, semi-structured interviews allowed me to ask further questions following the interviewee's responses. I used to have a framework of themes (interview guide) to be addressed, but in addition I needed more flexibility and freedom than the structured interviews would offer me. Interview guides in the context of semi-structured interviews allowed me to be focused on the topics under consideration without constraining them in a particular format (Lindlof and Taylor, 2002).

For instance, the agenda of the first interview with students included the following list of questions.

1. What made you decide to study mathematics?
2. Was it harder/easier than you thought before entering the university?
3. How did you find the first year of your Bachelor Degree?
4. Compared to your first year, how do you find your second year?
5. Did you have any difficult moments in your studies so far? Can you give me an example? How did you cope with them?
6. Do you feel that you have the support you for your studies, from the university, from your department?
7. Do you personally use opportunities for this support such as the lecturer's office hours? Other?
8. Which are your favourite topics in mathematics?
9. Are you familiar with groups? What is you impression about groups from your previous Algebra course?
10. What is your first impression about this second course of Abstract Algebra?
11. What are your expectations from this course? What do you think this course is about?
12. How does this course compare with other courses that you have done so far?
13. If you don't mind, may I ask you what do you think a group is? I know it's a bit early in the course but I just want to hear your first impressions of what a group is!
14. Can you give me an example of a group?
15. When you think about groups what do you think are the most important elements of this concept? What other bits of mathematics pop into your head when you are trying to define what a group is to you?

Following students' responses though, quite often I needed to ask not only further clarifications but to expand the scope of the discussion into other related subjects. For instance, when some students were discussing the support they get from their department of mathematics, I was giving them the opportunity to expand their views and experiences regarding, for instance, the various opportunities for assistance, in the seminars, tutorials, collaboration with peers etc. Eventually these, initially unplanned, discussions proved to give me substantial data that would allow me to investigate issues such as communication or student study skills. In addition, I often turned the discussion, between the aforementioned themes on issues related to the constructs of CTF, for instance issues of communication, the use of language and visual mediators by the lecturer etc.

Interviews, unlike colloquial conversations, develop in several distinct stages. The interviewer leads the conversation "through these stages, paying attention to how the intensity and emotional and intellectual challenge of the questions matches the depth of the relationship between interviewer and interviewee" (Rubin and Rubin, 1995, p129). These stages, which may not all be achieved in one interview, include: creating a natural environment; encouraging conversational competence; showing understanding; getting facts and basic descriptions; asking difficult questions; toning down the emotional level; closing while maintaining contact.

This study has adopted these steps as much as possible throughout the data collection process. At the beginning of the interview, I was usually asking the interviewees how they are and I was saying something irrelevant to the interview or Group Theory, in order to create a natural environment that would make them feel comfortable. In some instances when a student was not feeling confident to reply to a question or more often to respond to a task I was setting, I was trying to encourage them by saying how good they are. My priority was to make students feel comfortable and confident by convincing them that I fully understand them. My effort for showing understanding was expressed both overtly with my statements and more generally with my whole attitude. This proved to be very beneficial since students would be very open to the discussion of any subject and keep coming to all three cycles of interviews, despite the fact that some of the questions would often make students feel somewhat uncomfortable or even exposed, especially when I was setting a task involving Group Theory d-objects and their exemplification. Finally, I was trying to tone-down the atmosphere of the interview with asking questions that were not challenging or directly related to Group Theory. This allowed me to maintain contact and close the discussion with students having a positive impression about his/her performance.

3.4 Process of Data Collection

In this section, I give an overview of the ***data collection process,*** information about the methods of collection, as well as information about the participants.

The data collection was planned in collaboration with my primary and secondary supervisors in a series of meetings in the autumn of 2008. I also received helpful feedback during the presentation of my MPhil/PhD upgrade paper on the 11^{th} of December 2008, both from the members of the panel (members of the faculty of the School of Education and Lifelong Learning, in UEA) and the audience (fellow doctorate students and research associates).

At the end of the autumn semester and before the Christmas break, we arranged a meeting with two members of staff from the Mathematics department in which the data collection would take place to discuss our proposed plan for the data collection. The Head of the mathematics department course and the Director of Learning and Teaching attended this meeting. Both gave their consent and offered advice on a few minor issues regarding the data collection plan.

The process of data collection covered the 12 weeks of the spring semester and the beginning of the examination period. Below there is a description of the activities and the data that was collected.

Table 3.1: Overview of Data Collection Process

Week	Type of Data Collected
1	lecture recordings, lecture field-notes, lecture material
2	lecture recordings, lecture field-notes, lecture material, lecturer's interview, students' interviews 1
3	lecture recordings, lecture field-notes, lecture material, student interviews 1, seminar recordings, seminar field-notes, seminar material
4	lecture recordings, lecture field-notes, lecture material, seminar leaders' interviews 1, seminar assistants' interviews 1
5	lecture recordings, lecture field-notes, lecture material, student's interview 1
6	lecture recordings, lecture field-notes, lecture material, seminar recordings, seminar field-notes, seminar materials, student's interviews 2
7	lecture recordings, lecture field-notes, lecture material, seminar leaders' interviews 2, seminar assistants' interviews 2
8	lecture recordings, lecture field-notes, lecture material
9	lecture recordings, lecture field-notes, lecture material
10	lecture recordings, lecture field-notes, lecture material, seminar recordings, seminar field notes, seminar material, students' interviews 3
11	seminar leaders' interviews 3, seminar assistants' interviews 3
12	copies of unmarked student coursework
Examination Period	copies of marked student coursework, student examination solutions

The data I have gathered includes the following:

1. **Lecture observation fieldnotes**: I attended all 20 lectures and took notes on the following aspects:
    - record of student attendance;
    - instances of interaction between the students and the lecturer;
    - verbal, body or other evidence of student (dis)engagement and emotional response to the lecture; and,
    - general observations of lecturer and student behaviour.

The lecture observation notes give a general picture of the atmosphere in the lecture theatre. These fieldnotes were handwritten alongside the lecture notes.

2. **Lecture notes**: everything the lecturer was writing on the blackboard.

3. **Audio-recordings of the 20 lectures**: All 20 lectures were recorded and summarised. I audio-recorded the lectures:
    - To capture any interaction between the lecturer and the students;
    - To capture the running oral commentary that supplements the lecturer's writing on the board.

4. **Audio-recordings of 21 seminar sessions**: All of the seminars were recorded and summarised.

    Seminar Group 1: Seminar leader was LCR and his assistant was SAA. In total, there were 6 recordings - using two recorders (one for the seminar leader and one for the seminar assistant).

    Seminar Group 2: Seminar leader was SLA and his assistant was SAB. In total there were 5 recordings[3] for the 3 cycles of seminars.

    Seminar Group 3: Seminar leader was SLB and her assistant was SAA. In total there were 6 recordings for the 3 cycles of seminars.

[3] SLA was unable to attend the last seminar

Seminar Group 4: Seminar leader was SLA and his assistant was SAB. In total there were 4 recordings[4] for the 3 cycles of seminars.

When SLA was unavailable, the seminar groups 3 and 4 were merged to one group.

I have captured all conversations with students during which they predominantly discussed difficulties with certain items in the coursework. Usually each member of staff would have on average 10-12 conversations with different students or groups of students per semester. Sometimes students would ask for assistance on questions from previous problem sheets.

5. **Interviews** with 13 volunteering students out of the 78 students attending the module, the lecturer, the two seminar leaders and two assistants, and an experienced algebraist with whom we discussed, in a 40-minute session, general pedagogical issues related to the module as well as Mathematics Education in general. There were three cycles of interviews, at the beginning, the middle and the end of the module. A total of 39 interviews of approximately 15 minutes each were conducted with students in order to discuss their learning experience in Abstract Algebra. The 4 discussions with the lecturer, who was also one of the seminar leaders, covered learning and teaching issues as well as institutional and administrative issues. The interviews with the seminar leaders and assistants, 3 interviews each, focused mainly on their discussions with the students during the seminars, but also on their general views on pedagogical issues. The staff interviews lasted on average 10 minutes.

6. **Student coursework**: Students were given three problem sheets in Weeks 2, 5 and 9 (See Appendix A). They had to work on these before the seminars in Weeks 3, 6 and 10. While the students had to work on all the

[4] SLA did not attend any seminars in the last cycle of seminars.

questions, they only had to hand in a selection of these in Week 12. The lecturer announced the selection of the assessed questions after each seminar. The students were also given the solutions of the non-assessed questions via the online portal for the module. At the end of the semester and after the students had handed in their coursework, the lecturer provided the solutions to all questions (See Appendix B). The 78 threefold students' coursework was photocopied twice: before marking and immediately after marking.

7. **Marker's comments on student coursework:** The seminar assistants marked the coursework during the Easter Break. The marked sheets were collected for photocopying in order to record the comments on the student solutions. The comments were mainly explanatory notes on the mistakes made by the students or, in some cases, expressions of enthusiasm, approval and encouragement.

8. **Student examination scripts** were collected at the end of the academic year and photocopied. The exam paper consisted of six questions - three focusing on the module on Vector Spaces (taught in the autumn semester) and the other three focusing on groups and rings (two on groups and one on rings). The two lecturers, who convened the modules, were each marking their own part of the examination scripts. There were no comments, just the final mark for each question. (See Appendix C for examination paper section on Abstract Algebra)

The following provides a brief background on the ***participants***, namely the module lecturer, the two other seminar leaders and the two seminar assistants, who were interviewed as part of the research process.

LCR is the lecturer of the module and one of the seminar leaders, being responsible for one of the four seminar groups. He is an experienced mathematician, in his late 40's, who holds the position of full professor. His area of expertise is Model Theory and Set Theory. He has taught several modules throughout his career, including introductory modules for first-year undergraduate students, Linear Algebra, Group Theory, Ring Theory, Set Theory, Model Theory, Computability, and Mathematical Logic. He was also the marker of student examination scripts.

SLA is one of the seminar leaders responsible for two of the four seminar groups. He is also an experienced mathematician and an award-winning lecturer. He is in his early 50's and holds the position of full professor. His area of expertise is Number Theory and in particular the theory of Diophantine equations with focus on elliptic curves. He has taught several modules, including Analytic Number Theory and Arithmetic among others.

SLB, who is in her early 40's, is one of the seminar leaders and responsible for one of the four seminar groups. She is a new lecturer in the Mathematics department with limited lecturing experience at the time of data collection. She previously taught modules such as Discrete Mathematics, Linear Algebra and Group Theory, mainly in other universities. Her area of expertise is Group Theory with particular interest in generation problems and probabilistic methods and asymptotic results related to the study of various types of growth sequences associated with finite and profinite groups.

SAA is one of the seminar assistants, responsible for two of the four seminar groups. He is a PhD student in the Mathematics department with his research focusing on the area of Algebraic Combinatorics and Group Representation Theory. He is in his late 20's and has recently obtained his first degree from the same Mathematics department. He is one of the two markers of student coursework.

SAB is one of the seminar assistants, responsible for two of the four seminar groups. He is also a PhD student in the Mathematics department with his research focusing on the Diophantine Approximation and its applications in the study of Diophantine Equations. He is supervised by SLA and is the assistant in SLA's seminars. He is in his late 20's and has recently obtained his first degree from the same Mathematics department. He is one of the two markers of student coursework.

Table 4.2 provides a brief overview of the 13 interviewed students, mentioning the title of their degree, its duration, their examination and coursework grades, whether they have passed or failed as well as other characteristic information.

Table 3.2: Details of Student Interviewees

Student code name	Course	Origin	Pass/ Fail	Exam (in %)[5]	Course work (in %)[6]	Additional Information
Otello	MMath	Nigeria	Pass	85.25	92	Mature Student
Kostanza	BSc	UK	Pass	40.38	48	Recently overcame depression
Manrico	BSc	UK	Pass	53	48	
Norma	BSc	UK	Pass	70.75	63	
Amelia	BSc	UK	Pass	86	79	
Carmen	Visiting student	France	Pass	71.88	72	
Musetta	BSc	Cyprus	Pass	37.13	34	
Francesca	BSc	Cyprus	Fail	31.63	11	
Dorabella	BSc	UK	Pass	45.63	50	
Tamino	MMath	UK	Pass	57.75	54	Dyslexic
Leonora	BSc	UK	Pass	63.13	63	
Norina	BSc	UK	Pass	68.25	73	
Calaf	BSc	Taiwan	Pass	47.5	50	

In what follows I provide a brief ***factual description of the data collection process***, during the semester and post-semester period.

[5] The final examination grade includes both Linear Algebra and Abstract Algebra.
[6] The coursework mark refers only to the coursework in Abstract Algebra.

Week 1

I sent an email to all the students attending the course introducing myself and describing my research project. Additionally, I attached a participant consent form, asking the students to read, sign and return the form to me. I emphasised that participation was voluntary, that any participant had the right to withdraw at any time and that anonymity and confidentiality rules would be strictly adhered to throughout the study. In Week 1, after the first lecture, I had my first recorded interview with the module lecturer. The interview lasted 15 minutes. By the end of Week 1, 6 out of the 78 students had volunteered for interview.

The first problem sheet (see Appendix A1) was handed out to the students to be discussed in the first seminar in Week 3. Some questions on the sheet were to be submitted at the end of the semester for assessment.

Lecture 1 included the following: definition of binary operation and examples, definition of associativity and examples, definition of group, definition of Abelian group, comments on notation of binary operation, lemma – justification of axioms in the definition of group, proof of lemma, and examples of groups from fields.

Lecture 2 included the following: more examples from fields, demonstration of how the axioms are applied on the elements of groups, definition of permutation, exercise on permutations, remarks on notation of the set of symmetries of X, $Sym\,(X)$, definition/theorem of symmetric group, proof of theorem related to symmetric groups, remarks on notation, examples of symmetric groups mainly S_4, theorem which states that $|S_n| = n!$, and introduction to multiplication tables.

At the end of the lecture, the first problem sheet (see Appendix A1) was handed out to the students to be discussed in the first seminar in Week 3. Some questions on the sheet were to be submitted at the end of the semester for assessment.

Week 2

I interviewed six students, and the interviews lasted, on average, 15-20 minutes each.

Lecture 3 included the following: definition of power of a group element, lemma about the power of a group element, examples related to the power of group elements from S_4, definition of subgroup, theorem – test for a set being a subgroup, further remarks on subgroups and on notation, definition of a subgroup that is generated from subset X,$\langle X \rangle$ lemma about $\langle X \rangle$ with its proof, further remarks about terminology and notation and examples of subgroups.

Lecture 4 included the following: examples of subgroups, non-examples of subgroups, introduction to dihedral groups by using regular pentagon, generalisation of dihedral groups introducing the $n - gon$ definition of Dihedral group.

Week 3

I continued the first cycle of interviews. I interviewed six other students who had volunteered in the meantime. In addition to the student interviews, the first cycle of seminars took place. All of the sessions were recorded, and I was present at two of the four. Recorders were given to the seminar leader 10 minutes prior to the seminar sessions I was not observing myself.

Lecture 5 included the following: theorem about dihedral groups which states that $|D_n| = 2n$ with its proof, introduction to rotational symmetries of the cube, introduction to the order of element, definition of the order of element of finite groups, examples of order of element of finite groups, theorem regarding the order of element and its proof.

Lecture 6 included the following: theorem about finite groups referring to the conditions of H being a subgroup with its proof, definition of cyclic groups, examples of cyclic groups, introduction to cosets and Lagrange's Theorem, Lagrange's Theorem, corollaries of Lagrange's Theorem, examples illustrating the key ideas of Lagrange's Theorem, definition of left and right coset, examples and illustration.

Week 4

I individually interviewed for the first time the three seminar leaders and the two seminar assistants. Our conversations were loosely structured around their discussions with the students during the seminars about specific parts of Problem Sheet 1 and their general views on pedagogical issues.

Lecture 7 included the following: examples of cosets, theorem and about the intersection of cosets and its proof, lemma about counting in cosets and its proof, proof of Lagrange's Theorem, definition of the index of H in G,$|H: G|$, definition of equivalence relations, and examples of equivalence relations.

Lecture 8 included the following: examples of equivalence relations, definition of ~-equivalence classes, theorem about equivalence relations and its proof, corollary about ~-equivalence classes, theorem about properties of equivalence relations and its proof, a second proof of Lagrange's theorem, definitions of homomorphism, image, kernel and isomorphism, lemma about the properties of homomorphisms and its proof.

Week 5

A 13th student expressed his wish to be part of the project and we arranged his first interview on Friday after Lecture 10. The questions were the same as with the other 12 students. The lecturer circulated the second problem sheet (see Appendix A2) on Group Theory and advised the students to start working on it.

Lecture 9 included the following: lemma about the properties of homomorphisms, such as composition and inverse of homomorphism, with its proof, examples of homomorphisms, definition of normal subgroups, remarks following the definition, notation of normal subgroups, lemma about $ker\varphi$ being a normal subgroup of G with its proof.

Lecture 10 included the following: lemma about the characteristics of a normal subgroup with its proof, examples of normal subgroups, definition of a factor group, theorem about factor groups and its proof, examples of factor groups, and the First Isomorphism Theorem (FIT) and its proof.

Lecture 10 was the last one on groups. Since this study focuses only on Group Theory, I do not include the lecture contents of the rest of the lectures or any other information regarding Ring Theory.

Week 6

The second cycle of student interviews took place. I sent the interviewees a reminder email, as agreed in the first interviews, and all of them responded positively. In Week 6, the second cycle of seminars took place, which aimed to help the students with the second problem sheet on Group Theory.

Week 7

I interviewed the three seminar leaders and the two seminar assistants for the second time. Again, our conversations were loosely structured around their impressions of their discussions with the students during the seminars and their general views on pedagogical issues. These second interviews were significantly longer than the first ones, with an average length of around 20 minutes.

Week 9

The lecturer circulated the third, and last, problem sheet (see Appendix A3) to be discussed at the seminar in the following week.

Week 10

The third and final cycle of the student interviews was conducted. I sent an email to the 13 students informing them about this final interview. Most of the students replied immediately, apart from one who replied later and was therefore interviewed in Week 11. In Week 10, the third and final cycle of seminars also took place, which aimed to help the students with the third problem sheet on Ring Theory, although students asked extensively questions related to the second problem sheet.

Week 11

I conducted the third and final set of interviews with the three seminar leaders and the two seminar assistants. Our conversations were again loosely structured around their discussions with the students during the seminars and their general views on students' performance in the seminars.

Following discussion with my primary supervisor, I decided to also interview EPM, one of the most experienced and distinguished algebraists of the mathematics department. This interview aimed to enrich my knowledge beyond the microscopic view of this module to gain a more macroscopic view of the teaching and learning of Algebra in the United Kingdom and abroad. This interview was structured and was also attended by my primary supervisor.

Week 12

In the final week of the semester, I photocopied all student solutions of the coursework. I collected the coursework from the two seminar assistants who were the markers and returned it to them 48 hours later.

By the end of week 12, full coursework solutions (see Appendix B) were added to the website for the students' future use, such as for exams revision.

Post-semester Period

During the Easter break and before the coursework was handed back to the students, I photocopied the papers again, since the markers, apart from correcting and marking the scripts, were writing potentially useful comments.

In early June, just after the examination period, I collected and photocopied students' solutions to the exam papers. The lecturer marked the exam papers with a final mark being awarded to each question with hardly any comments on the scripts. The exam paper consisted of six questions of 20 marks each, and the students had to select and attempt five of these questions. Three of the questions focused on the module on Vector Spaces, taught in the autumn semester, and three questions related to groups and rings as the content of the spring semester module (two on groups and one on rings).

3.5 Data Analysis

Qualitative data analysis is a dynamic, intuitive and creative process of inductive thinking, reasoning and theorising. Data analysis allows researchers to comprehend the context of their data and refine their interpretations. The core objective of this process is to determine the codes, categories, relationships and assumptions related to the topic under study (Cohen et al, 2007). In what follows, I give a detailed factual description of how the data was analysed

Lecture observation notes and ***Lecture notes***: Lecture observation notes were handwritten during the data collection process alongside the lecture note taking. Both of them were summarised in 10 *Lecture Summaries* (See Appendix G). In these summaries, I organised the content and systematised the presentation of lecture observation notes and I included a brief index of content for each lecture. At the end of each lecture summary, I comment on issues of content, presentation, general ambiance and other incidents that interested me, and were connected with the prospective focus of this study, such as communication or any form of interaction between the lecturer and the students.

Seminar audio-recordings: I listened the audio recordings of all seminar sessions several times. In the first cycle of analysis, my intention was to identify the coursework exercise under discussion and the students involved, if one of the thirteen that participated in the interview process. In the second cycle of analysis, I produced documents called *Seminar Vignettes* (See Appendix F, for an example) for each seminar session. In particular, I first listened carefully to the recording of the various discussions, and then I organised these vignettes in linear order following the natural successive order, taking note of the time each discussion commenced. Moreover, I summarised the content of each discussion, including the main questions of the student and the summarising the seminar staff's answer; when possible I was mentioning whether a certain p-object, such as a cube was used in order to assist explanation. I was also reporting on the number of students involved in the discussion and any interaction between the students.

Interviews: Both student and staff interviews were fully transcribed, trying in between the lines to illustrate with comments regarding the mood, voice tone, emotions and attitudes, or incidents of laughter, long pauses etc. The final interview documents were called *Annotated Interview Transcriptions*, since they were an annotated version of the interview transcription. In these annotated versions of interview transcriptions, I highlighted certain phrases or even parts of the dialogues that were related to a particular theme. A certain number represented a different theme. For instance, in the first cycle of student interviews theme 7 represented students' difficulty to use visual imagery. When all interview transcriptions were annotated, I produced six *Interview Theme Tables* in which I summarised all themes emerging in each cycle of interviews with students and staff, listing horizontally all the emerging themes, and vertically the students or staff members. In the cycles of the student interviews, I identified 36, 22 and 29 themes respectively and in the cycles of staff interviews 22, 25 and 25 themes respectively. When available, I was adding a short and representative comment for each theme and each student, giving the flavour or his/her response on the given issue. For an example of Interview Theme Tables see Appendix H). This process of interview analysis allowed me to familiarise myself with the extensive amount of data as well as the rich list of emerging themes. Moreover, this process of data analysis seemed only natural to be adopted, given that they were not strong, preconceived hypotheses to be investigated or tested.

Coursework and ***Examination Scripts***: In the data analysis process, both the coursework solutions and the exam scripts were analysed last. These data were possibly the most substantial and extensively used, particularly for Chapter 4. Coursework student solutions were analysed in detail, mostly focusing on issues such as conceptual difficulties with certain d-objects, the use of mathematical vocabulary and symbolisation, the use of language and the style of language used, the proof production process and the use of visual imagery and external p-objects. Concerning the examination scripts, I followed the same approach as with the coursework, but in addition I was searching for any signs of improvement or regression in the student performance. In parallel, I was considering the comments of the markers, which I was further analysing and commenting on. For the purposes of this study, I have only analysed the documents of the 13 students that were interviewed, since the analysis of the written data and the results are linked with the analysis and results of the data emerging in the interviews. Moreover, I scrutinised students' solutions, identifying the errors and therefore, scanning the bits that were of interest and in which one could detect misconceptions or other important elements requiring further consideration, such as problematic use of visual images, or problematic metaphors from other mathematical fields. In doing so, I produced data analysis documents (that later formed the first part of Threefold Data Analysis Accounts), scanned all the excerpts referring to the aforementioned incidents and I commented and further analysed them, using the Commognitive Theoretical Framework.

Finally, when all forms of data were analysed, I made an effort to combine my results of the different analyses, in order to finalise my conclusions and verify their validity through triangulation. This analysis resulted the 13 *Threefold*

Data Analysis Accounts (TFA's), each for each of the 13 students. They were called Threefold Accounts, since I aimed to address three distinct questions referring to the conceptual difficulties, the affective issues and the pedagogical considerations. In TFA's I revised and extended my report on the scrutinised written data (including excerpts from the Seminar Vignettes and Interviews) and added two more sections with the entire interview excerpts related to the other two issues (See Appendix E, for an example). The creation of TFA's and the more refined and deep analysis involved was crucial, for various reasons: first, it allowed me to reflect on the emerging results in a more holistic way considering all forms of data and the various views both of members of staff and students; second, the analysis was more focused and consistent with respect to CTF, which allowed me to identify the important results that were to be directly connected with the constructs of CTF, and which are not considered before, such as communication and use of language; finally, it allowed to evaluate the central issues that have emerged through the data analysis process, and therefore omit other issues that were not that strong, despite my initial intentions, such as the purely affective issues.

Before writing up the data analysis chapters of this thesis I realised that I needed to further systematise my results, mainly in order to have both a qualitative and quantitative flavour of the certain issues e.g. what observation had occurred to which and how many students. For this reason, I created other data analysis documents, either hand-written or typed. Therefore, for the conceptual difficulties I produced five hand-written *Data Analysis Maps* (DAM's) around the five distinct sections of Chapter 4, namely conceptual difficulties regarding the d-objects of group, subgroup, cube symmetries, equivalence relations, and the various d-objects leading to the FIT. These maps were table-like, where horizontally I listed all the possible conceptual issues that emerged and vertically I listed all students. Then, I would tick on the conceptual issues that had occurred to a certain student. For the analysis in chapters 5 and 6, I produced four *Pedagogical Statement Tables* (PST's) in which I gathered and systematically categorised data according to four emerging themes and various subthemes (See Appendix I for an example).

These themes were: proof production, teaching, student assistance, and assessment. This process was important for two reasons: first, it allowed me to reconsider which themes were more interesting to be highlighted (apparently I did not focus on the initially identified four themes, but rather I analysed my data in accordance to others which were less or not investigated in the literature, and/or for which CTF would be more appropriate, such as communication or student study skills); second, this refinement in the analysis helped me to decide about the structure of chapters 5 and 6, and finally separate them.

3.6 Ethical Issues

During the process of data collection and analysis, I have made an effort to address the following ethical issues:

The issue of power: The Head of the Mathematics department kindly suggested that he would encourage all students to participate in this research project. The email that I sent to the group of students described the purpose and the nature of the project and gave a short description of myself. While this was very welcomed from the point of view of researcher access, it also raised certain issues of power, which required consideration and sensitive handling. Any email or announcement on the virtual learning environment used in this university was encouraging and informing the students about the progress of the project, but without any hint of enforcement/coercion. In addition, at the beginning of the first interview with the students I strongly emphasised the fact that participation in this project was on a completely voluntary basis and not because the lecturer had suggested so.

Equal opportunities: Every student who attended the Abstract Algebra module had the right to participate in this research project. There was not even the slightest discrimination or preference towards any student to be part of this project. This equality was preserved throughout. Additionally, all students were informed before the beginning of the project that the ones who would be interviewed would receive a small compensation at the end. This compensation (£20 in cash) was given to the participating students a few days after the third interview.

Right to withdraw: Extra care was taken to inform the students about their right to withdraw from the research project at any time they wished to. Additionally, it was made clear to the students that the content of their interviews would not be communicated to anyone else in the Mathematics department.

Procedure for complaints: In the participant consent form circulated at the beginning of the semester, it was clearly emphasised that all the students involved in the project would have the right to complain at any time about the research project to the Chair of the School of Education's Ethics Committee. I note that there were no complaints.

Confidentiality: It was clearly communicated to the students and staff that all forms of data, such as interview recordings, copies of coursework, copies of exam papers, other forms of written or oral documentation would be seen only by myself and, if necessary, by my supervisors. No piece of data was to be shared with other colleagues and no information was to be given to any other participant of the module or the project.

Anonymity: The real names of the students and staff were not used during the data analysis and throughout the thesis or in any publications based on this data. Regarding the students I have used pseudonyms taken from several operas, whereas staff code names were created according to the title of their post.

Role of the researcher: I clearly explained to the students that participating in the process of the interviewing would not mean that they were going to receive extra academic assistance or guidance in any form from me. It was also communicated to them that I am a researcher based outside their

Mathematics department and thus do not have the authority to change or influence their coursework or examination marks.

Consent of students: Most of the above information was laid out in simple language in the participant consent form, which was circulated to the students by email in Week 1 of the spring semester. This form clearly stated what students were consenting to, such as the forms of data collection (their problem sheet solutions, examination solutions, any recorded conversations in the seminars, and the recorded conversations from those who would choose to be interviewed by me) and that their participation would help contribute to the field of Undergraduate Mathematics Education, specifically the teaching and learning of Abstract Algebra, which many students find difficult. The students were asked to sign the form and return it to me electronically or to print, sign and hand the form back to me in the lectures. No student, apart from the 13 students who participated in the interviews, signed and returned the consent form, or objected either to participate in the recording of his/her conversations in the seminars, or his/her coursework and examination scripts to be copied. Therefore, data collection proceeded without any veto.

Sensitive interview discussions: As stated before, anonymity was highly important and guaranteed, especially of the student interviews. In the case that the interview, because of its aim of understanding student difficulties, evoked intense emotions of anxiety and despair, I attempted to handle the situation and the student with as much sensitivity as possible. Following guidance from the Ethics Committee and my supervisors and with the consent of the student, in extreme circumstances, I would contact the student's advisor, if the problem had to do with his or her studies, or the student counselling services, if the problem related to any social or psychological aspects of the student's life. Qualitative interviews, by their nature, are likely to give rise to such expressions and extra care was taken in order to ensure that the student's well-being was prioritised. Fortunately, no such episodes occurred.

Chapter 4 Conceptual Difficulties of Undergraduate Students' First Encounter with Group Theory

In this chapter, I aim to analyse the conceptual difficulties students encounter in their first module of Group Theory, as these appear mainly in the written data, as part of the discursive shift towards Group Theory learning. In addition, when possible, I aim to examine the students' performance and how this has changed during the period of study, i.e. whether there is evidence of progress or regression. Finally, whenever this information is available, I intend to discuss the contrast between the actual and the reported achievement.

Moreover, I aim to investigate the following issues, as these have emerged in the discussion in the Literature Review Chapter: the use of symbols and in particular the reasons of problematic use and the consequences this use may have in the learning of Group Theory; the development of the d-objects of group and subgroup, and whether their learning is achieved in parallel or consecutively, or whether their learning is context sensitive; the role of the d-objects of coset, normal subgroup and normality in the learning of Group Theory, as well as in the successful application of FIT; the role of use of visual mediators in the learning of Group Theory; students' difficulties with the d-object of isomorphism and the identification of the d-objects that influence its object-level and metalevel understanding; students' encounter with the d-object of homomorphism and comparison with the learning of isomorphism; and, the application of metarules in the particular routines that appear in this module.

According to the data analysis, as this appears in the Threefold Data Analysis Accounts (TFA's), there have emerged five categories of tasks related to different d-objects. Namely, tasks related to the d-objects of group, subgroup, symmetries of cube, equivalence relations, and the prerequisite concepts for the definition of the First Isomorphism Theorem. Students' conceptual difficulties were predominantly identified in the coursework and examination exercises that involved these d-objects or related routines in the form of errors or incompleteness of the solution.

For the data analysis I use the Commognitive Theoretical Framework by Anna Sfard (2008), placing emphasis on the process of endorsement of d-objects under study in the context of the overall development of students' performance throughout the module, and the development of object-level learning and metalevel learning. Moreover, I also focus on how students use the mathematical vocabulary, visual mediators, narratives and routines, the four characteristic elements of mathematical discourse, and what this use suggests for the discursive development. For the purposes of this task I will predominantly use the coursework and examination solutions of the thirteen students, but also, data from the student and staff interviews (when available), the seminars and the markers' report on coursework. The solutions to the coursework can be found in Appendix B.

This chapter is structured in five sections, with respect to the five categories of tasks related to the d-objects of group, subgroup, symmetries of cube, equivalence relations and the First Isomorphism Theorem, following this order. The reason for structuring the discussion in this order is because I wanted to keep the natural order of teaching the module and therefore be able to keep the natural sense of conceptual and skill development of the students.

In the first section, I investigate students' encounter with the d-object of group. In this section I discuss two common errors in the process of proving that a given set is in a fact a group. The first is directly related to the definition of group and the proof that a group is abelian. The second is the misuse of the metalevel rule, and in particular the assumption that what needs to be proved is true and used within the proof. Finally, I discuss two cases of serious disengagement with the concept of group.

In the second section, I investigate students' first encounter with the d-object of subgroup. I discuss seven errors, as these have appeared in students' solutions of the exercises involving predominantly the d-object of subgroup, namely the absence of clarification about the importance of distinction of the prime numbers p and q, orders of the group elements; errors related to metalevel rules involved in the proof that a set is actually a subgroup, namely, proof of non-emptiness, closure under operation and closure under inverses; errors related to the use of Argand diagrams; confusion regarding the role of the elements of the set and the elements of the group; and finally, erroneous use of notation.

In the third section, I investigate how students cope with the identification of the symmetries of cube and further prove that there exists a subgroup of order 8. Although this task involved the d-object of subgroup, it was analysed separately for various reasons: the lower level of abstraction that the identification of cube symmetries involved, the evidently higher student performance, and the possibility of recording signs of improvement or regression in students' performance, since the same exercise was included in the examination paper.

In the fourth section, I discuss students' difficulties with relation to the d-concepts of equivalence relations. Namely, the errors that have occurred are related to the students' misconception about the elements of group $\mathrm{Sym}(X)$ and elements of the set X, the elements and size of equivalence classes, and the relations of transitivity and symmetry.

Finally, in the fifth section, I investigate students' difficulties regarding the d-objects that lead to the introduction of the First Isomorphism Theorem. Namely, the difficulties students face in the learning of kernel, image and isomorphism, the difficulties related to the d-objects of cosets, normal subgroups and quotient groups, and the finally the application of the First Isomorphism Theorem as a routine in the various mathematical tasks.

For the purposes of this chapter, I have used various types and pieces of data, predominantly, excerpts from the student coursework and examination papers, but also interview excerpts or data from the seminars and other supportive documents, such as the markers' report on the coursework student performance. These excerpts are used as evident for the claims that precede or follow. The criterion of the choice of these excerpts was the following: amongst the number of excerpts expressing the same view, I was choosing the one conveying this view/claim in the best and most descriptive way. In addition, the intention of this chapter is not to focus on particular students, but rather concentrate on the conceptual difficulties and emerging errors.

4.1 D-object of Group

In this section I discuss students' performance regarding the d-object of group, as this has been demonstrated in exercise 1 part (i) of the first coursework (from now on I will use the notation CS1E1i), which students had to solve.

1. (i) Suppose $(G, \cdot)$ is a group with the property that $g^2 = e$ for all $g \in G$. Prove that for all $g_1, g_2 \in G$ we have $g_1 g_2 = g_2 g_1$ (that is, G is *abelian*).
(ii) Suppose $(G, \cdot)$ is a group with 4 elements which has the property that $g^2 = e$ for all $g \in G$. If a, b are distinct non-identity elements of G, show that the other non-identity element of G is ab. Hence write down the multiplication table of G.

Text 4.1: CS1E1

Moreover, this section aims to identify and analyse the students' main performance and conceptual difficulties concerning the d-object of group.

> Definition: A *group* $(G,*)$ consists of a set G and a binary operation $*$ on G which satisfies: Associativity, i.e. $\forall g,h,k \in G, \big((g * h) * k\big) = \big(g * (h * k)\big)$; Existence of an identity element $\exists e \in G: \forall g \in G, e * g = g * e = g$; and Existence of inverses, i.e. $\forall g \in G, \exists g': g * g' = g' * g = e$. In addition, a group is *commutative* (or *Abelian*) if $\forall g,h \in G, g * h = h * g$.

In general, students' encounter with the d-object of group was satisfactory, and their performance was generally in agreement with their impression as this has been revealed in their interviews. Their understanding is generally quite explicit and their mathematical narratives show good use of the mathematical vocabulary and notation as well as the ability to specifically demonstrate their reasoning in the specific routine. Regarding CS1E1, students' performance, in comparison to other exercises, was good, and their perception of this exercise was positive.

The above impression is in agreement with the impression of the seminar leaders and assistants, as this has been given in the seminars. Moreover, their general impression of the student cohort's (78) performance on CS1E1 is shown in the following excerpts of interviews with LCR, SLA and SAB.

> *Well… I think that if they had a problem it was knowing how to start and how to use the information in the question... once they were given a hint most of them could see what was going on...*LCR

> *With question 1 the news was good generally. Some students had found a way of doing it... There was not the printed solution so that was very exciting... And I think there were three separate groups that had come up with solutions that weren't on the solution sheet and one that I had not seen before so that was really nice.* SLA

> *Yeah, I was surprised about this actually. Um... so yeah... they didn't really seemed to have an idea about… yeah… so I had to show this group was commutative um.... and I think it is just a question of being able to spot the trick... And so once you know what the trick is – good question because it is straightforward properties apart from that.* SAB

Five out of thirteen (5/13) students, Calaf, Norina, Norma, Carmen and Otello produced fully correct answers. Their performance was in agreement with their related statements in the second interview.

> *Something that is routine. That's it. And now given a property one has to say that the elements of the group commute…* Otello

Yeah, I thought that was alright, and could do, sort of like it was more like um, just sort of like using equations, like sort of like timesing by things, on – each side, so you could kind of see what you were working, like – it's each steps and stuff, so I thought that was – alright. It was all right, quite basic, sort of concept that we'd covered, so I thought that was all right actually... Norina

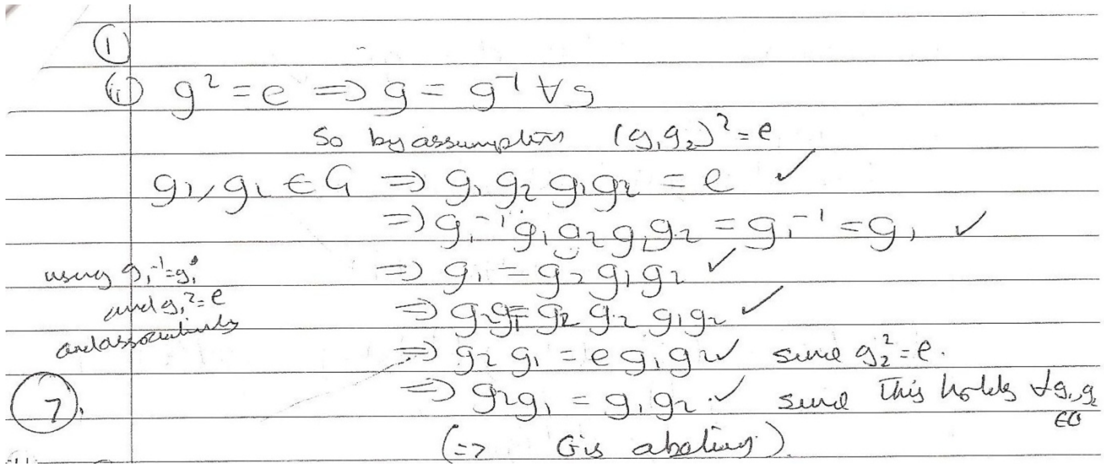
There have emerged three types of errors in the student solutions of CS1E1i: errors in the process of proving that the group is abelian; assumption of what needs to be proved; and serious object-level understanding and use of the definition of group. Below I discuss in detail these errors.

4.1.1 Proving that a Group Is Abelian

Problematic proof that a group is abelian has occurred in two out of thirteen (2/13) students' solutions (Kostanza and Manrico). In particular, these inaccuracies were related to proving that the group is Abelian using the group axioms. These inaccuracies are possibly linked with the incomplete object-level understanding of the definition of group and the involved object-level rules.

In general, Kostanza and Manrico's solutions almost completely lacked clarifications about assumptions at certain stages of the proof. Both students' solutions, according to the TFA's and the following excerpts, more often than not lack explicitness.

As the excerpt below suggests, Kostanza's attempt was good. Yet, she does not clarify, for instance, that she assumes that$(g_1\, g_2)^2 = e$, but she rather takes it for granted. As seen below, she applies all the necessary manipulations of $g \in G$, for instance $g_1^{-1} = g_1$ and $g_1^2 = e$ as well as associativity but with no further explanation. In addition, she does not clearly state that since $g_1 g_2 = g_2 g_1$ therefore G is Abelian. This is probably an indication of an incomplete object-level understanding of the property of commutativity, or an inaccurate application of the governing metarules, resulting deficient presentation of her reasoning.

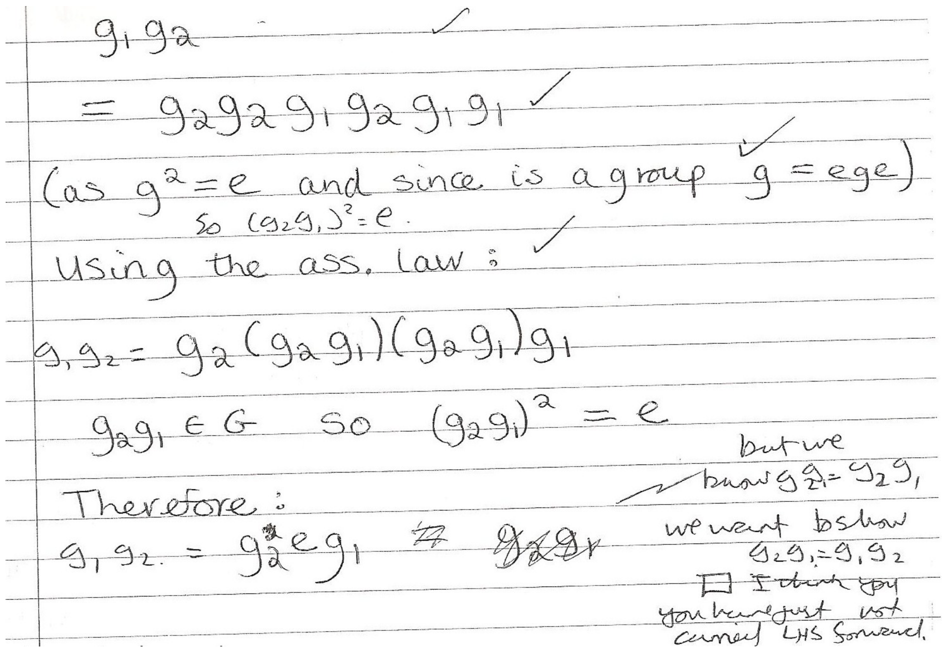


Kostanza's impression of CS1E1i, shared in the second interview, seems to reflect her uncertainty about its solution.

MI: *What did you think of this exercise?*

Kostanza: *Um... hard, it was hard...*

Another inaccuracy regarding the notion of commutativity is related to the actual proof of the expression $g_2g_1 = g_1g_2$. Manrico, as the excerpt below indicates, shows good object-level understanding of the definition of group, and demonstrates the ability to use the group axioms and apply the object-level rules for proving that the group is Abelian, yet he does not always justify his steps. He correctly multiplies both sides of the expression g_1g_2 by $g_2^2\, and\, g_1^2$ respectively and correctly uses associativity. The main problem with his solution appears at the end of the exercise where a problematic understanding of the notion of commutativity becomes apparent. Instead of demonstrating that $g_2g_1 = g_1g_2$, his final bit of narrative shows that $g_2g_1 = g_2g_1$, which puts in doubt his solutions' endeavour as well as his understanding of the definition of Abelian group. This error indicates problematic application of the object-level rules related to commutativity and the manipulation of the group elements.



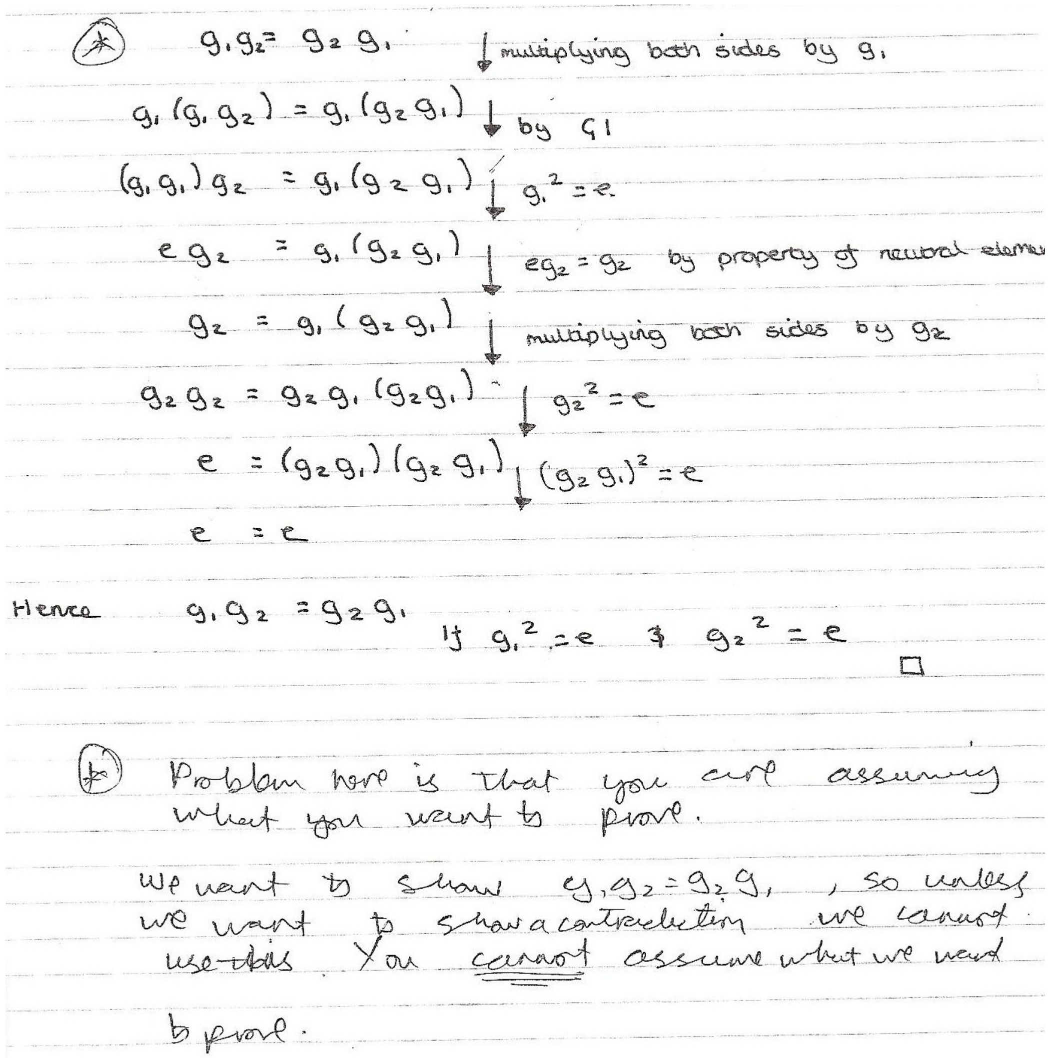
Manrico's impression of this exercise is in some agreement with his solution, since it is obvious that he is aware of the first steps of the solution i.e. multiplying both sides with certain elements of the group, but his description lacks explicitness regarding the final steps of the proof. He gives the impression that he does not know that the solution should end up by proving that $g_2 g_1 = g_1 g_2$.

MI:	*Can we see the first coursework and go question by question? Let's go to question 1, what is your idea about question 1?*
Manrico:	*Err, well basically that's just like... pre-multiplying and post-multiplying, by various elements, eventually... you will prove it. Which I did...*

4.1.2 Assumption of What Needs to Be Proved

A second type of errors regarding the proof that a group is abelian was the assumption of what was supposed to be proved is true and used at a certain stage of the proof. This error was usually part of an overall satisfactory attempt that would suggest an explicit understanding of the object-level rules of the d-object of group, but would highlight a problematic encounter with the metalevel rules and the 'how' of proving, even during the very first step of the module. Moreover, this kind of errors is a typical misapplication of the metalevel rules, since it is directly linked with the 'norms' of proving and not of the d-objects as such. This problematic assumption has occurred in the CS131i solutions of four out of the thirteen (4/13) students, namely Amelia, Dorabella, Leonora and Tamino. In addition, this error has occurred in the solutions of other exercises as well, as the discussion in the following sections will reveal.

As the following excerpt possibly suggests, problematic application of metarules does not require problematic object-level understanding of the d-objects under study. For instance, Amelia's writing style is very analytical with very clear mathematical narratives, good presentation and explicit use of symbolisation, in all her written mathematical narratives. Although her overall performance, both in the coursework as well as the exam, was very good, in this exercise she assumed what was supposed to be proved at the beginning of the solution i.e. $g_1 g_2 = g_2 g_1$. This indicates an unawareness of how to approach a proof of this kind and the required course of action, and consequently leads to a problematic encounter with this type of routine and the amenable metarules. In general, a significant obstacle in the application of metarules, as the following excerpt suggests, is the distinction between the different proving techniques and how the amenable metarules should be used. For instance, assuming that a certain mathematical narrative is valid and used within the proof is only applied in proof by contradiction, which is not the case for CS1E1i.


(*) $g_1 g_2 = g_2 g_1$ ↓ multiplying both sides by g_1

$g_1(g_1 g_2) = g_1(g_2 g_1)$ ↓ by G1

$(g_1 g_1) g_2 = g_1(g_2 g_1)$ ↓ $g_1^2 = e$

$e g_2 = g_1(g_2 g_1)$ ↓ $eg_2 = g_2$ by property of neutral elemen

$g_2 = g_1(g_2 g_1)$ ↓ multiplying both sides by g_2

$g_2 g_2 = g_2 g_1 (g_2 g_1)$ ↓ $g_2^2 = e$

$e = (g_2 g_1)(g_2 g_1)$ ↓ $(g_2 g_1)^2 = e$

$e = e$

Hence $g_1 g_2 = g_2 g_1$ if $g_1^2 = e$ & $g_2^2 = e$ □

(*) Problem here is that you are assuming what you want to prove.

We want to show $g_1 g_2 = g_2 g_1$, so unless we want to show a contradiction we cannot use this. You cannot assume what we need to prove.

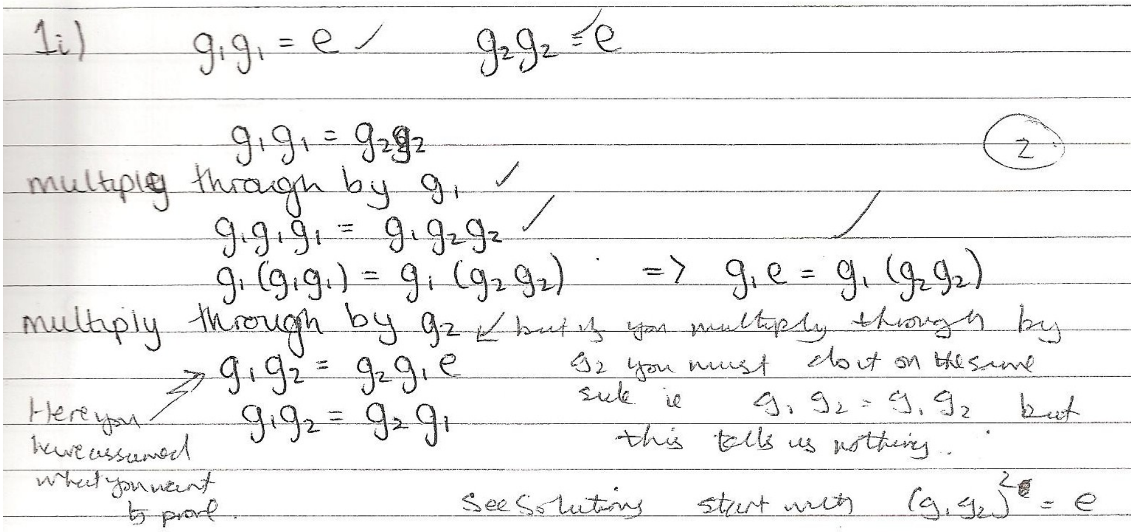
In her second interview, Amelia expressed her confidence regarding CS1E1i. She did not seem to be aware of the above issue, which probably explains why she had not asked for any help in the seminars.

Amelia: *Yeah, I quite liked question 1; I did that before the seminars. I think they're quite interesting as well, like, when you can show stuff like that.*

MI: *Interesting from which aspect?*

Amelia: *I don't know, I just like being able to prove stuff like that, and show that you can do that. I think when you can do proofs like that they're quite interesting to actually see how they work. Cos I understand how you* *get to the next step and it just flows, so I quite like them.*

The following excerpt demonstrates another example of problematic use of metarules, similar to the above.


1i) $g_1g_1 = e$ ✓ $g_2g_2 = e$

$g_1g_1 = g_2g_2$ (2)

multiply through by g_1 ✓

$g_1g_1g_1 = g_1g_2g_2$ ✓

$g_1(g_1g_1) = g_1(g_2g_2)$ => $g_1e = g_1(g_2g_2)$

multiply through by g_2 ← but if you multiply through by g_2 you must do it on the same side ie $g_1g_2 = g_1g_2$ but this tells us nothing.

$g_1g_2 = g_2g_1e$

$g_1g_2 = g_2g_1$

Here you have assumed what you want to prove.

See solutions start with $(g_1g_2)^2 = e$

Similar to Amelia, Leonora has assumed what she needs to prove to be true and she therefore uses it within the proof. Such error is based on problematic understanding of the governing metarules. In addition to this error, she multiplies both sides of $g_1g_1 = g_2g_2$ by g, but on the one side she multiplies from the left and on the other side from the right (as shown below). This indicates problems with the object-level rules regarding the d-object of group and its axioms.

Unlike Amelia, Leonora seems to be aware that something is not correct with her attempt, but she is not sure what this is, as the following excerpt suggests.

MI: *What about the first question?*

Leonora: *Yeah, it was – it wasn't – I just weren't sure whether I'd done it right, whether I was allowed to do it that way, cos I think I – I did something with like timesing it by, like both of them and then timesing g_1 them by g_2 and* *getting like the identity and stuff, but – I weren't sure whether it was the right way to do it? But I came out with like, the answer, but...*

MI: *But you weren't sure whether the process was correct.*

Leonora: *Yeah.*

The above excerpt possibly indicates that she has some object-level understanding of the related narratives, such as the definitions of group, and its axioms, and perhaps some acquaintance with examples she has studied in the literature, but she is not fully capable yet to use any metalevel rules in the process of proving the required statement.

A third example of using what is supposed to be proved within the proof occurred in Tamino's solution. Similar to Amelia and Leonora, he assumes that what he is trying to prove is valid and uses it during the proof, which indicates problems in applying a fundamental metadiscursive rule regarding the role of the 'to-be-proved' mathematical narrative. As in the other two cases, this is possibly directly linked with the metalevel understanding and indicates ignorance of the governing metarules, i.e. the norms of the process of proving. At the same time, the indicated by $*$ step, reveals an object-level error regarding the manipulation of group theoretical expression $(g_2 g_1)^{-1}$. Although he has proven that $(g_2 g_1)^{-1} = g_1 g_2$, at a later stage, instead of writing $(g_2 g_1)^{-1} = g_1^{-1} g_2^{-1}$ he has written $(g_2 g_1)^{-1} = g_2^{-1} g_1^{-1}$, which indicates problematic object-level understanding of the notion of inverse.

Suppose $(G, \cdot)$ is a group with the property that $g^2 = e$ for all $g \in G$

I need to prove that for all $g_1, g_2 \in G$ we have $g_1 g_2 = g_2 g_1$

Thus $g_1 g_1 = e$

$g_1 = e g_1^{-1}$

$g_1 = g_1^{-1}$ ✓

$g_2 g_2 = e$

$g_2 = e g_2^{-1}$

$g_2 = g_2^{-1}$ ✓

So $g_1 g_1 = e$

$g_1 e g_1 = e$

$g_1 g_2 g_2 g_1 = e$ ✓ $\Rightarrow (g_1 g_2)^{-1} = g_2 g_1$ or $(g_2 g_1)^{-1} = g_1 g_2$.

$(g_1 g_2)(g_2 g_1)(g_2 g_1)^{-1} = (g_2 g_1)^{-1}$

$g_1 g_2 = (g_2 g_1)^{-1} = g_2^{-1} g_1^{-1} = g_2 g_1$

(*) ↑ but why is $(g_2 g_1)^{-1} = g_2^{-1} g_1^{-1}$?

So

$g_1 g_2 = g_2 g_1$

Surely above you have $(g_2 g_1)^{-1} = g_1 g_2$ and this is general result $(g_2 g_1)^{-1} = g_1^{-1} g_2^{-1}$

So we get $g_1 g_2 = g_1 g_2$ above.

So you seem to have assumed in (*) what we want to prove. See Solutions

Similar to Leonora, Tamino expressed his uncertainty as to whether his attempt was correct or not, showing, an awareness of something having gone wrong.

MI:	*What is your idea about question 1?*
Tamino:	*Um... mostly done it kind of still not sure how I have proved it is complete and utter provable yea, it is sort of like still trying to look back through some of my old notes and stuff about it but it is not that bad a question,*

I didn't mind it. Shame, I liked the second part of the question more than the first but he only gave us the first part to hand in.

The following excerpt also demonstrates good object-level understanding of the group axioms, as well as the definition of commutativity.

1(i) $(G, \cdot)$ is a group with $g^2 = e$ for all $g \in G$.
Prove for all $g_1, g_2 \in G$, we have $g_1 g_2 = g_2 g_1$

As G is a group and $g_1, g_2 \in G$, $g_1 g_2 \in G$.

So $(g_1 g_2)^2 = (g_1 g_2)(g_1 g_2) = e = g_1 g_1 = g_1 e g_1 = g_1 (g_2 g_2) g_1 = (g_1 g_2)(g_2 g_1)$

$g_1 g_2 = g_1 g_2 (e) = (g_1 g_2)[(g_1 g_2)(g_1 g_2)] = (g_1 g_2)[(g_1 g_2)(g_2 g_1)] = [(g_1 g_2)(g_1 g_2)](g_2 g_1) =$
$= e(g_2 g_1) = g_2 g_1$

Therefore G is abelian.

Here, Dorabella, similar to the other students above, uses the metalevel rules that govern the process of proof inappropriately. She uses what she is supposed to be proved, in the process of proving. This is clearly related to the application of metalevel rules, since this error is not directly related to the d-objects involved in this mathematical task but it rather concerns the 'norms' of proving.

The fact that some students use mathematical narrative that needs to be proved, as a datum that should be used during the proof, possibly indicates a problematic engagement with the *how* of the routine. Namely, they have not a stabilised strategy about the 'course of action' for the given mathematical task. These particular students, probably, have not yet a clear idea of how to approach a proof, what the role of the final statement is and how to use it in order to achieve rigorous and clear proof. Consequently, the closure conditions, the set of metarules that define circumstances interpreted as signaling a successful completion of the proof, are not clear to these students. This is suggested by the fact that these students are generally not aware that the final narrative should not be used in the middle of the proof, even though in some incidents the interview excerpts express students' uncertainty regarding their solution.

The above discussion indicates that, at these early stages of the module, students' object-level understanding is better compared to the metalevel understanding and the application of the required metarules, which, for some students, was problematic from the beginning. This fact is probably inherited by other Pure Mathematics modules, as a result of unresolved problems with the process of proving. As it will be apparent in the following section of this chapter and as the module progresses, object-level understanding gets more problematic, which has an unfavorable impact on the use of the involved metarules.

4.1.3 Fundamental Difficulties with the Definition of Group

While I was doing the analysis of data regarding the conceptual difficulties of students, I identified two special cases of students, with a significant number of common characteristics. These students, namely Musetta and Francesca, had severe difficulties from the very beginning of the module, and distinguished them from the results based on the analysis of the other students' data. Naturally, the observations in the following discussion cannot not be generalised with certainty, but I believe that they give an indication of how very weak students cope with such severe difficulties. The following discussion could be a basis for further investigation.

Francesca and Musetta's attempt indicates serious difficulties with the d-object of group in an object-level understanding, contrary to the discussion in sections 4.1.1 and 4.1.2, where the occurring errors were mostly a result of problematic metalevel understanding.

As the following excerpts will show, the object-level learning of these students is seriously problematic from the very beginning, indicating lack of understanding of the prerequisite fundamental d-objects of associativity, commutativity, identity, inverses as well as operations in a set of a certain level of abstraction. In these cases, the application of any routines and their amenable metarules is impossible.

For instance, Musetta has copied from textbooks, or other sources, the theory that is directly or indirectly related to the exercise, namely the definitions of group, inverse, Abelian group and other mathematical narratives that have no relation to the essence of the exercise. She follows a *'write all theory'* approach that indicates a very poor object-level understanding of the definitions of d-objects, the object-level rules and the metalevel rules that should be applied in the particular proof. The only part that has been considered as solution by the marker, shown below, uses other notation than the one given in the exercise; in general, as noticed below, she uses different notation for the elements of the group, which possibly indicates severe problems in her object-level understanding. What it is considered correct by the marker is merely the statement of the definition of the Abelian group, which is basically not embedded in the context of the solution of this exercise but it is rather stated at the beginning of the attempt, as a guide of what one should do to prove that a group is abelian.

Then, let G be a group. Prove that G is abelian if for every element $a \in G$, $a^2 = e$.

(7) To prove that: Let $a, b \in G$, then $a = a^{-1}$, $b = b^{-1}$ and $(ba)^{-1} = ba$ so that $ab = a^{-1}b^{-1} = (ba)^{-1} = ba$.

by general group theory result

It follows that G is abelian.//

For every $g, h \in G$, $g*h = h*g$ that is the $*$ operation is commutative.
This is not true of all groups. If it's true of $(G,*)$, we say that $(G,*)$ is abelian.

Again This is not relevant to question asked

Continuing, If $(G,*)$ is a group, then each $g \in G$ has exactly one inverse.

To proving that let $g \in G$, and suppose g_1, g_2 are inverses of G, (we know that there is at least one by the definition of a group); that is $g*g_1 = g_1*g = e$ and $g*g_2 = g_2*g = e$

By associativity, $(g_1*g)*g_2 = g_1*(g*g_2)$

Since g_1 is an inverse of g, $(g_1*g)*g_2 = e*g_2 = g_2$

Since g_2 is an inverse of g, $g_1*(g*g_2) = g_1*e = g_1$

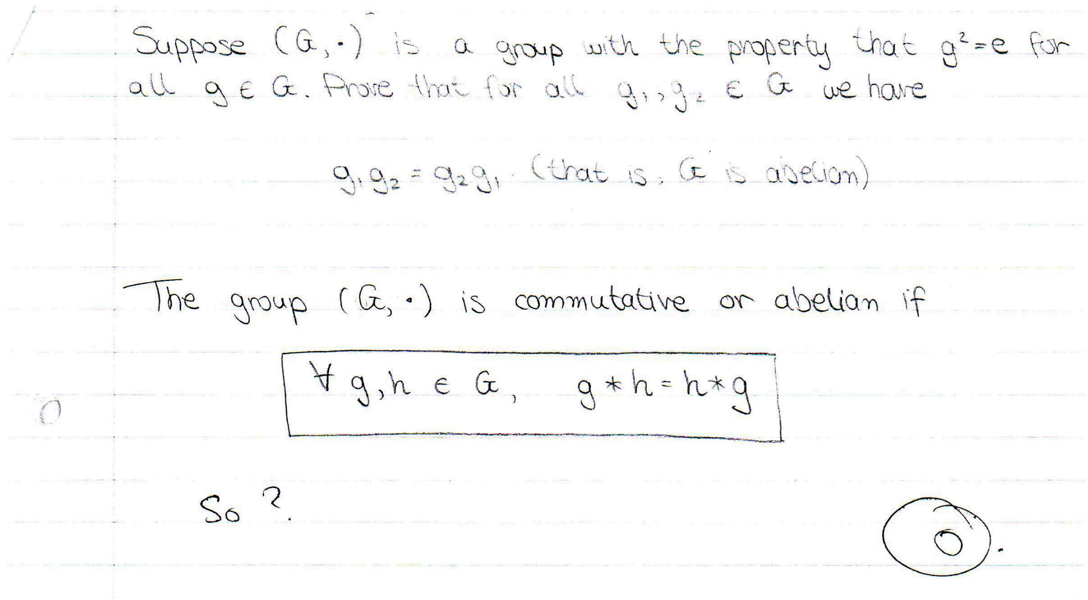
Hence, $g_2 = g_1$//

The following statement is related to the solution of CS1E1i.

> *I searched in the notes and I found something similar… something that would help me… and I was excited… but I did not try anything else to be honest… Just number 1… It was a bit easy as well… and that's why I managed…* Musetta

Her statement, together with her solution above, indicates that her final aim was not to comprehend the theory and produce a substantial solution, but rather to find something *similar* in the notes and consequently use it as a solution. The superficial approach in solving the given mathematical tasks also reflects her overall approach to learning, which, as the module was progressing, indicated further regression in her performance.

Similarly, Francesca, as the following excerpt suggests, merely copied the question and stated the definition of the Abelian group.


Suppose $(G, \cdot)$ is a group with the property that $g^2 = e$ for all $g \in G$. Prove that for all $g_1, g_2 \in G$ we have

$g_1 g_2 = g_2 g_1$ (that is, G is abelian)

The group $(G, \cdot)$ is commutative or abelian if

$\forall g, h \in G, \quad g * h = h * g$

So ?

0

0.

Most probably, her object-level understanding of the d-object of group, including commutativity, was very poor and therefore she was not able to use the involved object-level rules and apply the governing metalevel rules of the group axioms or even make the necessary algebraic manipulations in order to achieve the expected result. Francesca's approach to solving the coursework and, more generally, towards learning, possibly indicates an even more severe disengagement with the learning of Group Theory, where she seems to have abandoned any effort to cope with the level of difficulty of this module. This claim is reinforced by her overall performance, as the discussion in the following sections will indicate.

The narrative below indicates familiarity with, yet not understanding of, the d-object of Abelian group, which probably led her to state the definition as part of her attempt to solve the exercise. In addition, there is an indication that she realises her difficulty to apply 'with precision' what in the lecture appeared 'very simple'.

MI: *How did you find the first exercise?*

Francesca: *Ok… I think we have seen Abelian groups last year so it was easier to be understood… initially, before we study for the coursework everything seems very simple, but then you need to apply it with precision… Some exercises seemed to me ok, for example number five was the only one I had solved easily… It was the easiest, I think… Of course I had help from the seminar assistant… I haven't solved everything… but for some of them I could do more…*

Overall, the data in Francesca's case shows severe problems regarding the object-level understanding of all fundamental concepts of Group Theory. As the module was progressing, Francesca's performance showed obvious signs of regression.

In sum, in this section I have discussed the conceptual difficulties related to the d-object of group. As the above discussion suggests, there have emerged two types of errors. The first one is related to the proof that commutativity holds and therefore the group is Abelian. This error is a result of problematic object-level understanding of the d-object of commutativity and occurred in two students' solutions. The second error occurred when students used the mathematical statement that was supposed to be proved during proof. This error is related to problematic metalevel learning and application of the metarules that govern the particular routine. Finally, I have briefly discussed two cases of very weak students, conjecturing about what characterizes these students' encounter with Group Theory. Namely, they have severe problems with object-level understanding as well as metalevel understanding, being almost impossible to apply any metarules in the process of proving; their approach to solving a mathematical task is by 'copying all theory' that vaguely looks similar to what their impression of how the solution should look like, without emphasizing on the actual learning and understanding of that theory; and finally, their solutions lack any acceptable reasoning whatsoever.

4.2 D-object of Subgroup

In this section, I discuss the conceptual difficulties related to the d-object of subgroup as such and the application of the routine for proving that a given set is in fact a subgroup, as these have occurred in the students' solutions of the exercises CS1E3iii, iv, CS1E4i, CS2E1 and CS2E2i. Here it follows the definition of d-object of subgroup, the theorem describing the routine for proving that a set is a subgroup, and the four exercises mentioned above.

Definition: Suppose $(G,*)$ is a group and $H \subseteq G$. We say that H is a *subgroup* of G if H together with $*$ restricted to H is a group, i.e. $\forall h_1 h_2 \in H, h_1 * h_2 \in H$, and $(H,*)$ satisfied the group G axioms.

Theorem: If $(G,*)$ is a group and $H \subseteq G$, then H is a subgroup of G if and only if: (i) $H \neq \emptyset$; (ii) H is closed under $*$ i.e. $\forall h_1 h_2 \in H, h_1 * h_2 \in H$; (iii) H is closed under inverses i.e. $\forall h \in H, h^{-1} \in H$.

3. Using the usual test for being a subgroup, give proofs of the following:
(i) For any $n \in \mathbb{N}$, the set $n\mathbb{Z}$ of integers divisible by n is a subgroup of $(\mathbb{Z}, +)$.
(ii) If A is an $m \times n$ matrix over $\mathbb{R}$, then $\{x \in \mathbb{R}^n : Ax = 0\}$ is a subgroup of $(\mathbb{R}^n, +)$.
(iii) $\{z \in \mathbb{C}^\times : |z| = 1\}$ and $\{e^{(1+i)t} : t \in \mathbb{R}\}$ are subgroups of $(\mathbb{C}^\times, \cdot)$ (where $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$). Draw these sets on an argand diagram.
(iv) For any $n \in \mathbb{N}$, the sets $\{g \in \mathrm{GL}(n, \mathbb{R}) : \mathrm{Det}(g) = 1\}$ and $\{g \in \mathrm{GL}(n, \mathbb{R}) : gg^T = I_n\}$ are subgroups of $\mathrm{GL}(n, \mathbb{R})$ (where g^T denotes the transpose of g and I_n is the $n \times n$ identity matrix).

Text 4.2: CS1E3

4. Suppose $(G, \cdot)$ is a group and H, K are subgroups of G.
(i) Show that $H \cap K$ is a subgroup of G.
(ii) Show that if $H \cup K$ is a subgroup of G then either $H \subseteq K$ or $K \subseteq H$. [Hint: if not, then take $h \in H \setminus K$ and $k \in K \setminus H$ and consider hk.]

Text 4.3: CS1E4

1. Suppose p, q are distinct prime numbers and G is a group with $|G| = pq$. Suppose $\alpha \in G$ has order p and $\beta \in G$ has order q. Prove that $\langle \alpha, \beta \rangle = G$. [Hint: Use the same idea as in Example 2.3 of the notes.]

Text 4.4: CS2E1

2. Suppose X is a non-empty set and $G \leq \mathrm{Sym}(X)$.
Let $a \in X$ and $H = \{g \in G : g(a) = a\}$.
(i) Prove that H is a subgroup of G.
(ii) Prove that $g_1H = g_2H \Leftrightarrow g_1(a) = g_2(a)$ (for all $g_1, g_2 \in G$).
(iii) Now suppose X is finite and let $A = \{g(a) : g \in G\}$.
Prove that $|A||H| = |G|$.
(Remark: H is called the *stabiliser* of a in G and A is the G-*orbit* containing a. The results are a version of the *Orbit-Stabiliser Theorem.*)

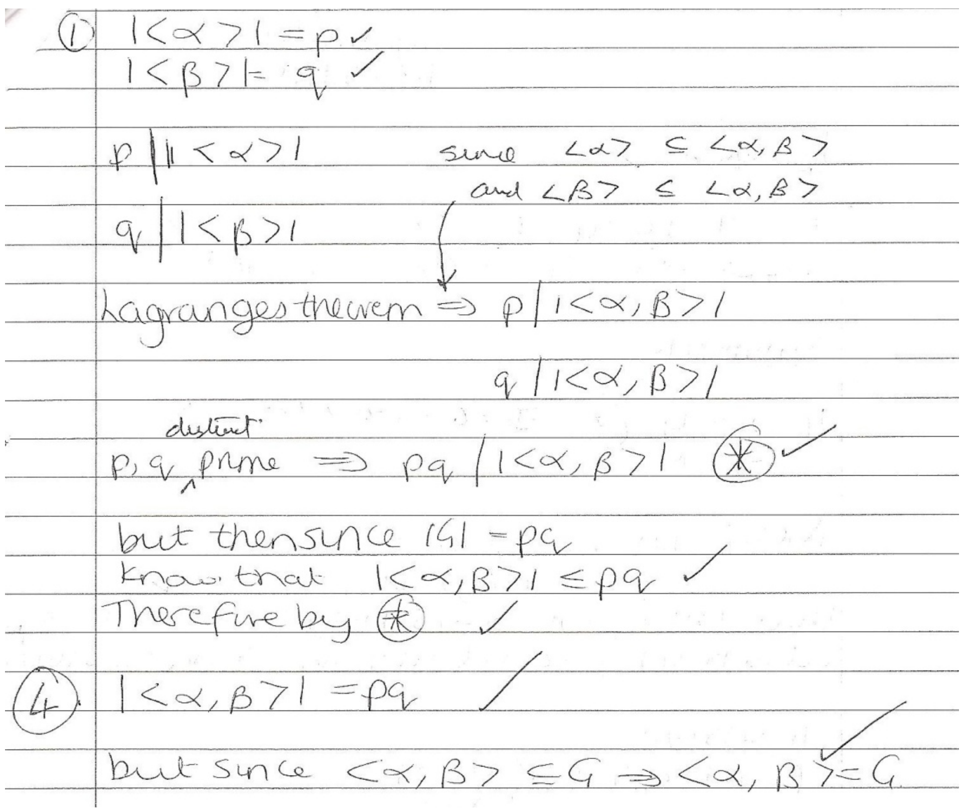
Text 4.5: CS2E2

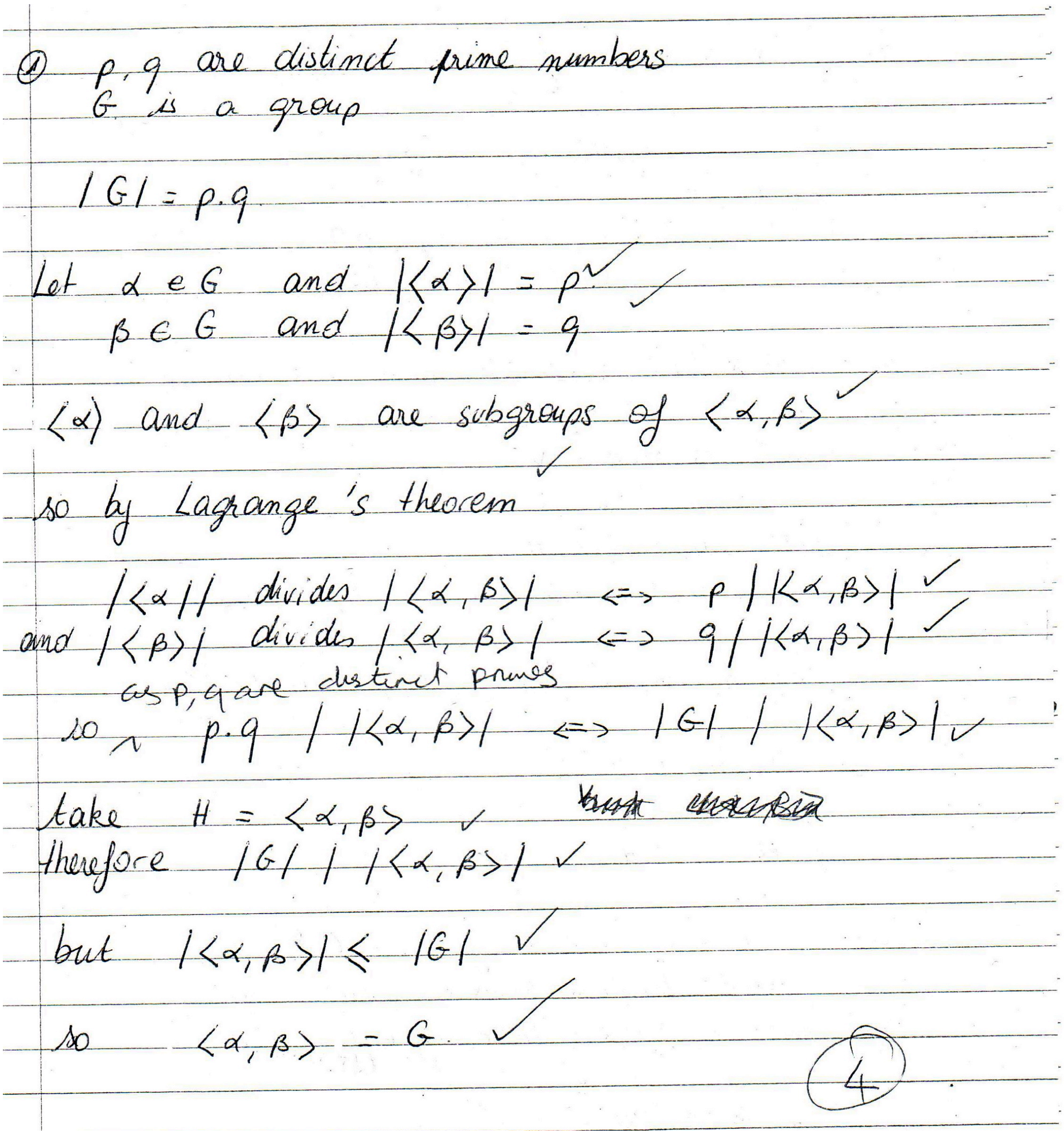
The aim of this section is to identify the main difficulties students have regarding the d-object of subgroup, to evaluate the use of mathematical symbols, and visual mediators (such as Argand Diagrams), and investigate further how the development of the d-objects of group and subgroup is connected.

Below it follows the analysis of the main errors that have occurred in students' solutions, in a descending order of frequency. Namely, the absence of clarifications about the importance of distinction of the prime numbers p and q in CS2E1; the absence or problematic proof regarding the first condition of the theorem, namely non-emptiness; problems in the proof of the third condition of the theorem, namely closure under inverses; problems in the use of Argand diagrams in CS1E3iii; errors regarding the role of the elements of the set and the elements of the group; problems in the proof of the second condition of the theorem, regarding closure under operation; and finally, erroneous use of notation.

4.2.1 Distinct Prime Numbers p and q

In CS2E1, students' performance was generally very good apart from one detail that was noted in ten (10/13) solutions (excluding Francesca, who did not attempt the exercise, and Otello and Amelia whose solutions were correct). These students did not state explicitly that p and q, representing the orders of two elements of the group, are *distinct* primes. If $p/|\langle a, b\rangle|$ and $q/|\langle a, b\rangle|$, then $pq/|\langle a, b\rangle|$ since p and q are distinct prime numbers. Otherwise it may not be the case. Here they follow two typical examples of this error, taken from the coursework of Kostanza and Carmen respectively.





① p, q are distinct prime numbers
G is a group

$|G| = p.q$

Let $\alpha \in G$ and $|\langle\alpha\rangle| = p$ ✓ ✓
$\beta \in G$ and $|\langle\beta\rangle| = q$

$\langle\alpha\rangle$ and $\langle\beta\rangle$ are subgroups of $\langle\alpha,\beta\rangle$ ✓

so by Lagrange's theorem ✓

$|\langle\alpha\rangle|$ divides $|\langle\alpha,\beta\rangle| \Leftrightarrow p \mid |\langle\alpha,\beta\rangle|$ ✓
and $|\langle\beta\rangle|$ divides $|\langle\alpha,\beta\rangle| \Leftrightarrow q \mid |\langle\alpha,\beta\rangle|$ ✓
so (as p, q are distinct primes) $p.q \mid |\langle\alpha,\beta\rangle| \Leftrightarrow |G| \mid |\langle\alpha,\beta\rangle|$ ✓

take $H = \langle\alpha,\beta\rangle$ ✓
therefore $|G| \mid |\langle\alpha,\beta\rangle|$ ✓

but $|\langle\alpha,\beta\rangle| \leq |G|$ ✓

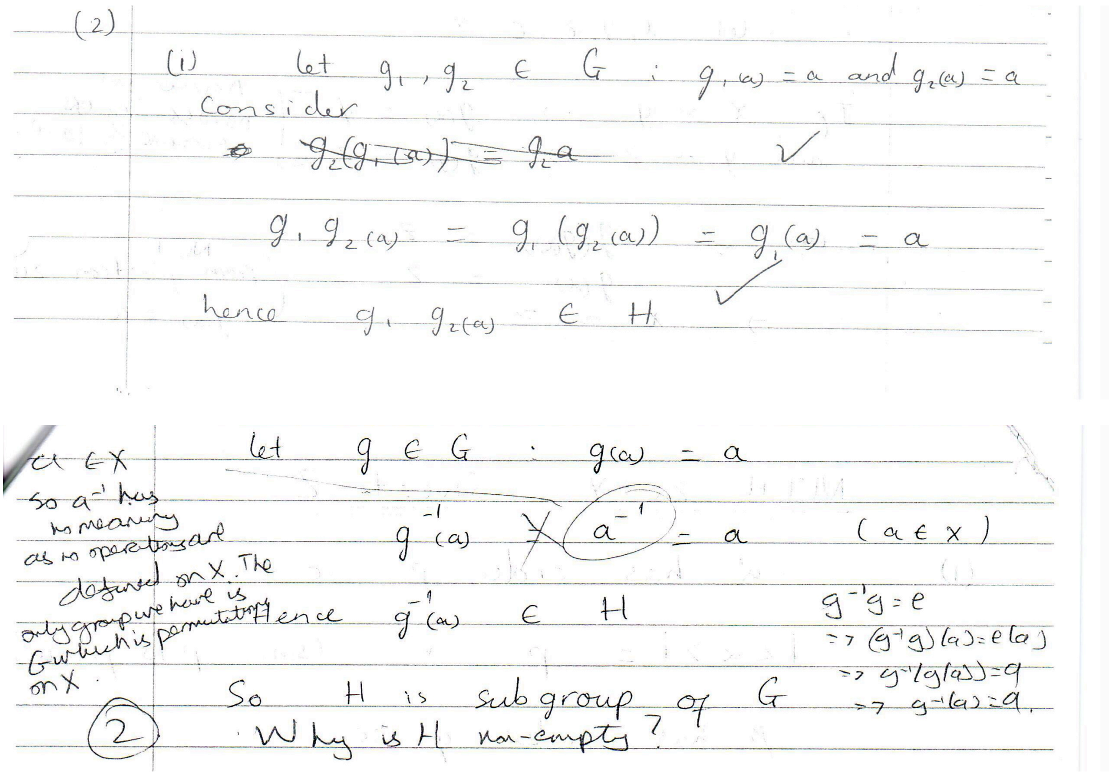
so $\langle\alpha,\beta\rangle = G$ ✓

4

These ten students do not seem to be aware of either the fact that the two primes need to be distinct or its importance. Analysing the student interviews, students who have made this error did not seem to be aware of either the fact that the two primes need to be distinct, since they did not refer to the importance of this distinction, at all. Most of the students' solutions of this exercise indicate a relatively good object-level understanding of the notion of subgroup, but their attempts lack completeness and precision of their reasoning, which would lead them to address all the required details. At this level, it seems that students are not fully aware of every theoretical aspect involved in the exercises, possibly because of their inexperience with the rigour and clarity that Pure Mathematics proofs require.

4.2.2 Non-emptiness of the Subgroup

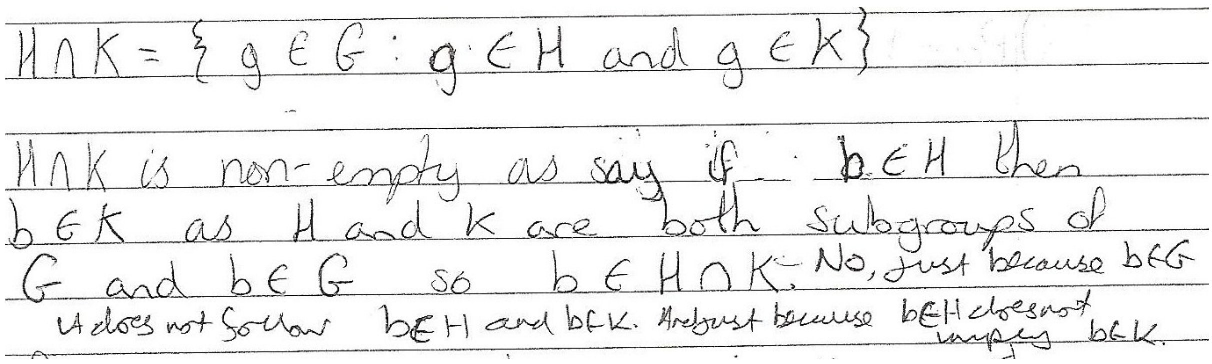
Another omission that occurred in the seven out of thirteen (7/13) students' solutions, with indications of students' unawareness of its importance, is the proof of non-emptiness of the prospective subgroup. In order to prove that a set is a subgroup, students should apply the related routine and prove the validity of all three conditions, namely that the set is non-empty, that it is closed under operation and closed under inverses. These students were: Calaf, Francesca, Dorabella, Manrico, Musetta, Norma and Tamino. Below I give an illustrative example, taken from Calaf's solution, of an overall successful application of the routine for a set to be a subgroup, yet lacking the proof of non-emptiness.



Regarding the proof of non-emptiness, as the excerpt above demonstrates, even though these students showed good object-level understanding of the definition of subgroup and capability to successfully apply the routine for proving that a set is indeed a subgroup, they simply ignored the first condition and considered it valid without any attempt to prove it. The reason for this omission is possibly based on problematic understanding of the governing metarules of the particular routine. Metalevel understanding, namely the understanding of the governing metarules or, in other words, the established 'norms' of the process of proving (Sfard, 2008) within the mathematical community, seem not to be fully adopted by novice students yet. The lack of precision and the omission of the first condition in the particular routine, possibly indicate inadequate evaluation of the importance of proving non-emptiness, or considering that non-emptiness is obvious. The following excerpt is the only example in the student interviews in which the student (Tamino in this case) reveals the confusion that this group of students probably face, regarding the application of the routine for proving that a set is in fact a subgroup.

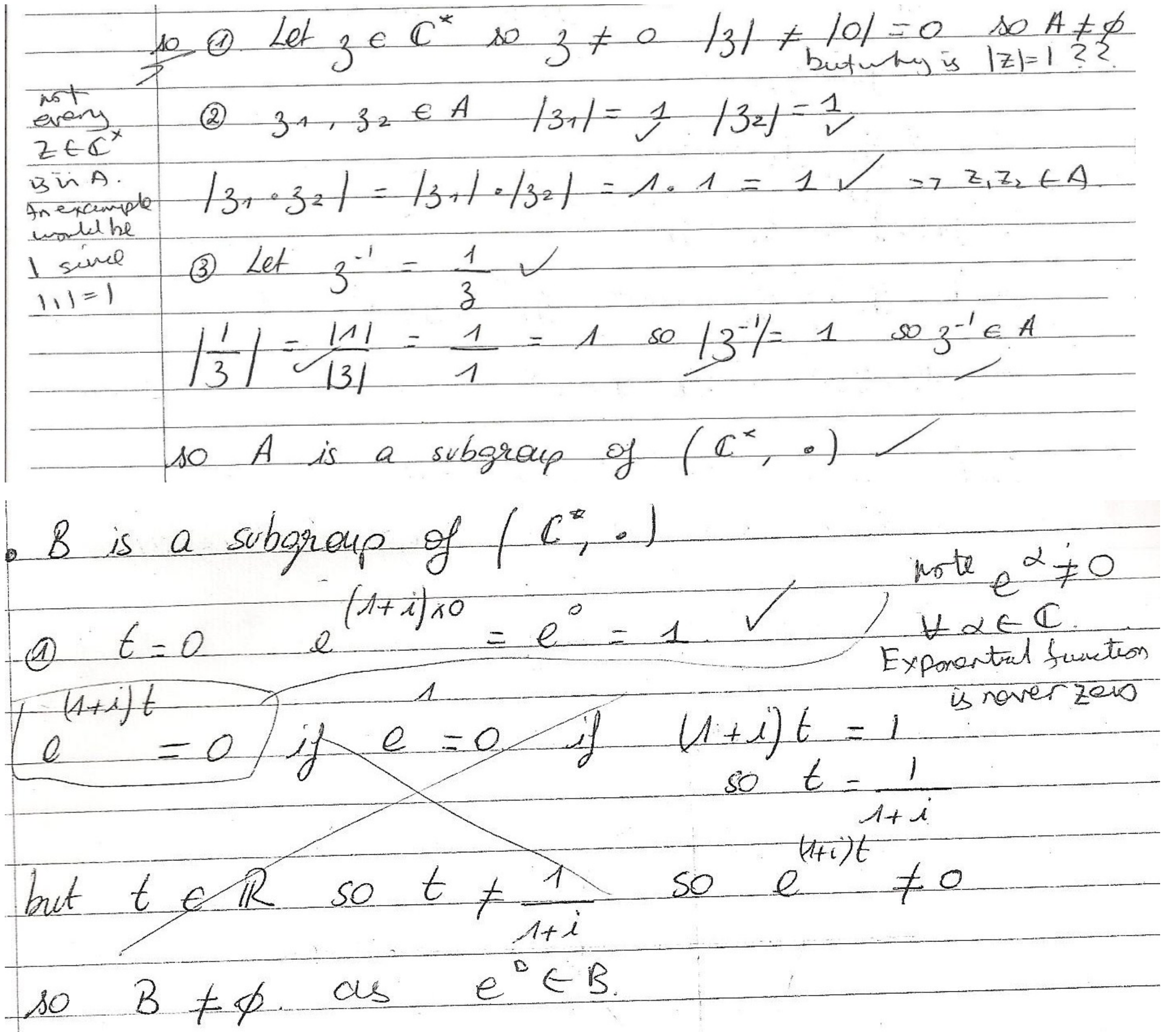
> *Tamino: Question 3 was a bit more... of a struggle, I think... and – it was just – it's – the problem with that is, getting a – kind of vaguely got it right, but it's just a bit proving you need to really make it crystal-clear, and I'm not too sure if I've... done that, made it really clear, what I'm – definitely proving.* [...] *I mean, say like – the – I'm thinking some of the proofs are probably... proving them being – going through tests for being subgroup, I mean, like those, I mean I'll get some marks, but I don't think I'll get the whole...*
>
> *MI: Where do you think you are going to lose marks?*
>
> *Tamino: I don't know, when I just write down... well I think I've done the first two parts, don't think were too bad... it's more... say I drew that out, I haven't actually got into testing yet…*

Apart from the seven students who did not attempt to prove non- emptiness, three students tried to prove it yet unsuccessfully, with indication of problematic metaphor from Set Theory. In particular, Norina, in her answer to CS1E4i, correctly proves closure under operation and closure under inverses showing efficiency in the application of metalevel rules regarding the proof for a set to be a subgroup. Yet, her attempt to prove non-emptiness indicates problematic object-level understanding of the idea of non-emptiness, which resulted inexplicitness in her mathematical narratives and overall reasoning. She proposes that $H \cap K$ is non-empty since if $b \in H$ then $b \in K$ as $H$and K are both subgroups of G and therefore $b \in G$ so $b \in H \cap K$, which is wrong. She was expected to state that the group identity belongs to the subgroup and therefore it is non-empty. Her answer indicates problems with the closure of elements of the different subgroups, possibly an erroneous metaphor from Set Theory.

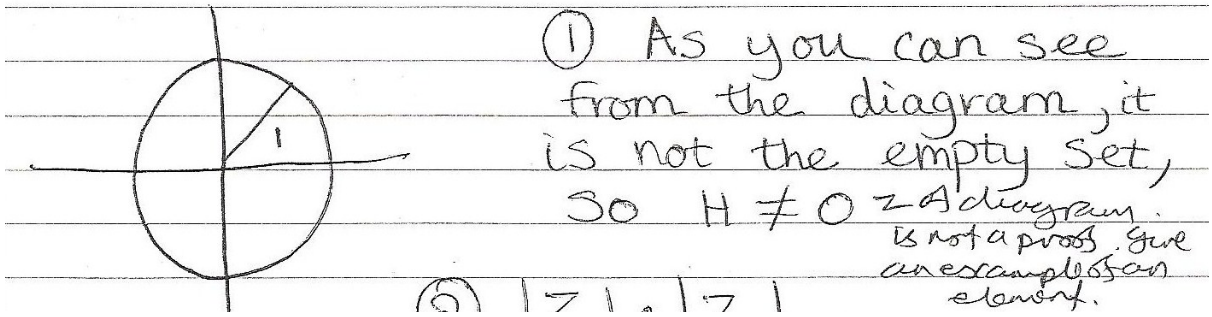

$H \cap K = \{ g \in G : g \in H \text{ and } g \in K \}$

$H \cap K$ is non-empty as say if $b \in H$ then $b \in K$ as H and K are both subgroups of G and $b \in G$ so $b \in H \cap K$. No, just because $b \in G$ it does not follow $b \in H$ and $b \in K$. And just because $b \in H$ does not imply $b \in K$.

Similarly to the above, Carmen in CS1E3iii correctly proved closure under operation and closure under inverses. In both examples though, her proof for non-emptiness is problematic. There are several inaccuracies: First of all she considers every $z \in \boldsymbol{C}^x$ to satisfy the condition$|z| = 1$, without proving that there is such z belonging to the group which satisfies this condition. Another inaccuracy occurs in the second example while trying to prove non-emptiness. The inaccuracy is trivial and is inherited as a problematic metaphor from arithmetic. She concluded that the exponential expression is equal to zero, which is not possible. Both these errors indicate problematic object-level understanding, since these errors are not related to the process of proving and the governing metalevel rules, but with the very d-object of subgroup and the ideas of closure of elements in a set. Carmen shows that she knows what steps she should follow to prove non-emptiness yet her understanding of the d-object of subgroup does not allow her to do so.



Although the lack of proving non-emptiness may possibly indicate inaccuracy in the application of metarules, the three cases of students who did it unsuccessfully indicates problematic object-level understanding. In addition to Norina and Carmen, Manrico's case is interesting regarding the application of this particular routine, since he is the only student who used visualisation, namely visual metaphors from Complex Analysis and Set Theory, in substitution to a formal 'algebraic' answer. Although he is aware of the test for a set to be a subgroup, he attempts to prove non-emptiness by offering on two occasions visual mediators (an Argand diagram for CS1E3iii and Venn diagrams for CS1E4i). The marker considers this as inadequate reasoning, as seen below. Instead, the student should have given an example of an element.



Interestingly, Manrico seems aware that using only visual mediators is not adequate for solving the mathematical problem.

MI: *Question 4?*

Manrico: *Err... see, there – this thing – I mean – the – statement, makes sense... I drew a little picture and like – I was just like – I mean – course that's going to be in it, but – how you prove that by actual kind of – prove that mathematically rather than just drawing a picture and just saying, it is true, it's just the actual showing that...*

Concluding, the proof or non-proof of non-emptiness has possibly revealed for different students, different problems in their understanding in both an object-level and metadiscursive level. At this early stage of the discussion, I speculate that often students need to use visual representation as strong metaphors from Set Theory and Complex Analysis. It seems that well-objectified concepts, from other directly and not directly related mathematical theories, are considered useful tools and students tend to apply them in (for them) new mathematical discourses such as Group Theory. Elements of metaphors that are perceived as relatively concrete and 'accessible', such as Venn diagrams and Argand diagrams are more likely to appear as part of the solution.

From a more macroscopic viewpoint, looking at all 78 students on the module, a problematic deal with proving non-emptiness is also highlighted by the seminar assistants' comments in their report on the students' coursework results. On two occasions, CS1E3 and CS1E4, their comments were as follows:

- Some lost marks by forgetting to check that the set is non-empty. Giving a concrete example is sufficient e.g $1 \in \{z \in \mathbb{C}^{\times} \ : \ |z| = 1\}$.

Text 4.6: Markers' Comment 1

Q4 - Again most people correctly applied the subgroup criteria. Some lost marks on showing that $H \cap K$ is non-empty. The fact that H and K are non-empty is not sufficient, you need to justify that they share at least one element (namely the identity element).

Text 4.7: Markers' comment 2

4.2.3 Closure Under Inverses

Errors or inaccuracies in the proof of closure under inverses appeared in eight out of the thirteen (8/13) students' coursework solutions, namely Kostanza, Musetta, Norina, Norma, Tamino, Amelia, Dorabella and Manrico. These errors and inaccuracies predominantly indicate problematic object-level understanding rather than the application of metarules, since these errors are connected with the definition of the involved mathematical d-objects such as the identity element, and its characteristics, or the idea of closure.

As the following excerpts will reveal, although the reasoning of the solution of these eight students would suggest good object-level understanding, the solution would often lack precision regarding, for instance, the uniqueness of the inverse. In other instances there were indications of problematic metaphors from other pure mathematical discourses such as Linear Algebra. Often, the routine of finding inverse was confused with the routine for finding the transposed matrix, when the context of the exercise involved matrices.

The most common inaccuracy, that occurred in seven out of the thirteen (7/13) students, was to clarify the uniqueness of the inverse. For instance, as the following example of excerpt suggests, Amelia, in CS1E3iv, successfully applies the routine for a set to be a subgroup for the first set, i.e. $X = \{g \in GL(n, R): Det(g) = 1\}$. Her solution indicates good object-level understanding of the d-objects involved, successful application of the governing metarules, as well as good connectivity across different mathematical discourses such as Linear Algebra and, later on, Complex Analysis. For the second set, $W = \{g \in GL(n, R): gg^T = I_n\}$, she successfully applies the routine and proves non-emptiness and closure under operation, and for the closure under inverses she correctly states that the inverse in this case is the transpose. Yet she has omitted to clarify the *uniqueness of inverse* taken both from the right and the left as shown below. Without this clarification the algebraic manipulations would be invalid.

3iv) $W = \{g \in GL(n, \mathbb{R}) : gg^T = I_n\}$

Using Theorem 1·16

Empty set?

$I_n, I_n^T \in GL(n,\mathbb{R})$

$$I_n . I_n^T = I_n . I_n = I_n$$ ✓

Hence $I_n \in W$ ✓

$\Rightarrow W \neq \emptyset$ ✓

Exist Closure?

Let $\omega_1, \omega_2 \in W$

st $\omega_1 \omega_1^T = I_n$ ✓

$\omega_2 \omega_2^T = I_n$ ✓

$\omega_1 \omega_2 \in GL(n, \mathbb{R})$

$$(\omega_1\omega_2)(\omega_1\omega_2)^T = (\omega_1\omega_2)\omega_2^T\omega_1^T$$ ✓

$$= \omega_1(\omega_2\omega_2^T)\omega_1^T$$ ✓

$$= \omega_1 I_n \omega_1^T$$ ✓

$$= \omega_1 \omega_1^T$$ ✓

$$= I_n$$ ✓

So for $\omega_1, \omega_2 \in W$ then $\omega_1 . \omega_2 \in W$

Hence $\omega_1\omega_2 \in W$

W has closure ✓

Is the Inverse an Element?

For $\omega \in W$

ω^T is the inverse, — of ω so, using that matrices have unique inverse

because $\omega\omega^T = I_n$

4

$\omega^T(\omega^T)^T = \omega^T\omega$ so both left/right inverse $W^TW = I_n = W^TW$

$= I_n$ $\Rightarrow W^T(W^T)^T = I_n$

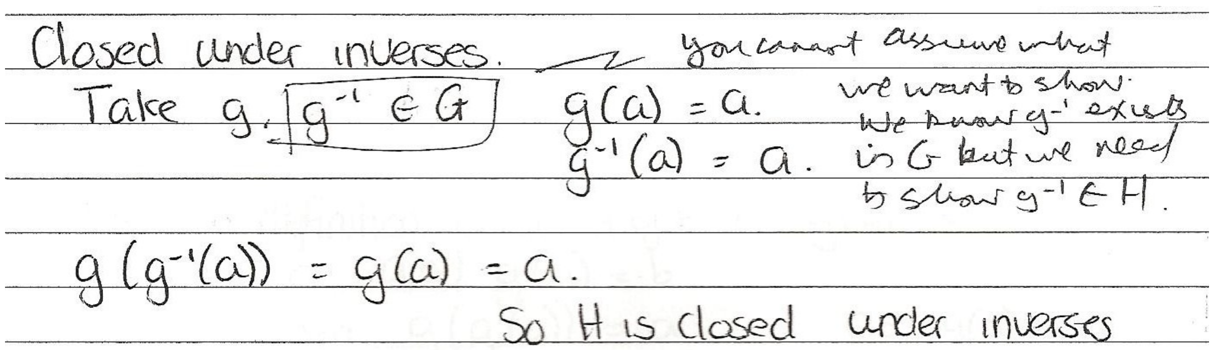
Since $(W^T)^T = W$ $\Rightarrow W^T \in W$.

Hence $\omega^T \in W$ such that ω^T is the inverse of $\omega \in W$.

Therefore $W \leq GL(n, \mathbb{R})$ ✓

(8)

A second type or errors regarding the proof of closure under inverses appeared in Norma’s attempt to solve CS2E2i. Norma successfully applies the routine in order to prove that the particular set is a subgroup, showing relatively good object-level understanding, of all d-objects. In addition, she successfully uses the datum given in the exercise i.e. that $\mathrm{a} \in \mathrm{X}$ and $\mathrm{H} = \{\mathrm{g} \in \mathrm{G}: \mathrm{g}(\mathrm{a}) = \mathrm{a}\}$ to prove that H is non-empty and that the closure under operation holds. Nevertheless, her attempt to prove that H is closed under inverses is problematic, due to incomplete application of the metalevel rules. She assumed that since $\mathrm{g}^{-1} \in \mathrm{G}$ it is granted that it belongs to H as well, instead of showing it. Instead she should apply g^{-1} in both sides of $\mathrm{g}(\mathrm{a}) = \mathrm{a}$ and get that $\mathrm{a} = \mathrm{g}^{-1}(\mathrm{a})$. This suggests incomplete metalevel understanding and consequently inaccurate application of metarules, particularly regarding the precision and rigor that mathematical reasoning in this advanced context requires.



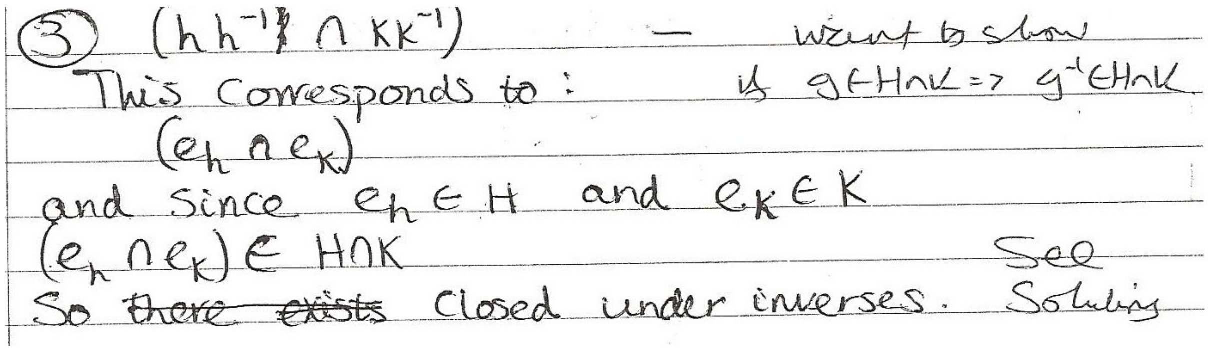
Apart from Norma’s problematic application of the governing metarules, in this particular exercise her performance seems to be unfavorably influenced by problematic object-level understanding, particularly at the initial stages of her attempt to solve it, as the following interview excerpt suggests.

> *Um... but I – I did manage to sort it out eventually – I just think – I found it hard cos – I was going between X and G and H and A, there was just a lot of – different groups, that I was trying to get my head round, but um, I did manage to sort that out eventually.* Norma

Norma's object-level difficulties at this initial stage are related to the identification of the difference between the various sets and the groups, which would allow her to apply with facility the routine for a set to be a subgroup.

Similar to the last claim above, problematic object-level understanding seems to have negative impact in the application of a specific set of metarules governing a certain routine. This is obvious in Manrico's attempt to prove closure under inverses. In the following excerpt, one can detect problematic object-level understanding of the involved d-objects. In particular, there are indications of problematic understanding regarding the d-object of subgroup as such and its elements. These indications are particularly obvious in the notation used in the narrative $(hh^{-1} \cap kk^{-1})$, since Manrico possibly does not realise what hh^{-1} and kk^{-1} are, and the circumstances under which the operation of intersection can be used.

In addition to the aforementioned problems with object-level understanding, there are also indications of problematic metalevel understanding and precise application of the governing metarules. In particular, Manrico seems to not have a clear idea of how and where the proof needs to be further developed, indicating some difficulty with the applicability conditions of a routine, as well as the *how* of the routine and the 'course of action'. This is obvious in his attempt to prove closure under inverses, since he does not seem to be fully aware that he has to prove that $g^{-1} \in H \cap K$ if $g \in H \cap K$.


③ $(hh^{-1} \cap kk^{-1})$ — want to show
This corresponds to: if $g \in H \cap K \Rightarrow g^{-1} \in H \cap K$
$(e_h \cap e_k)$
and since $e_h \in H$ and $e_k \in K$
$(e_h \cap e_k) \in H \cap K$ See
So ~~there exists~~ closed under inverses. Solution

Manrico expressed his concern about applying the particular routine and connected it with his ability to communicate the proof in a way that was comprehensible to others.

> *But I – yeah, again, it might be – me not – it makes perfect sense, but I might not... make it – it's just like you know – I can understand it, but it's trying to, I mean because proof is really trying to make someone else understand it, and I say, possibly I do struggle at – giving, you know, making someone else understand it by writing it down, but, so it's where I might lose some marks, but...*Manrico

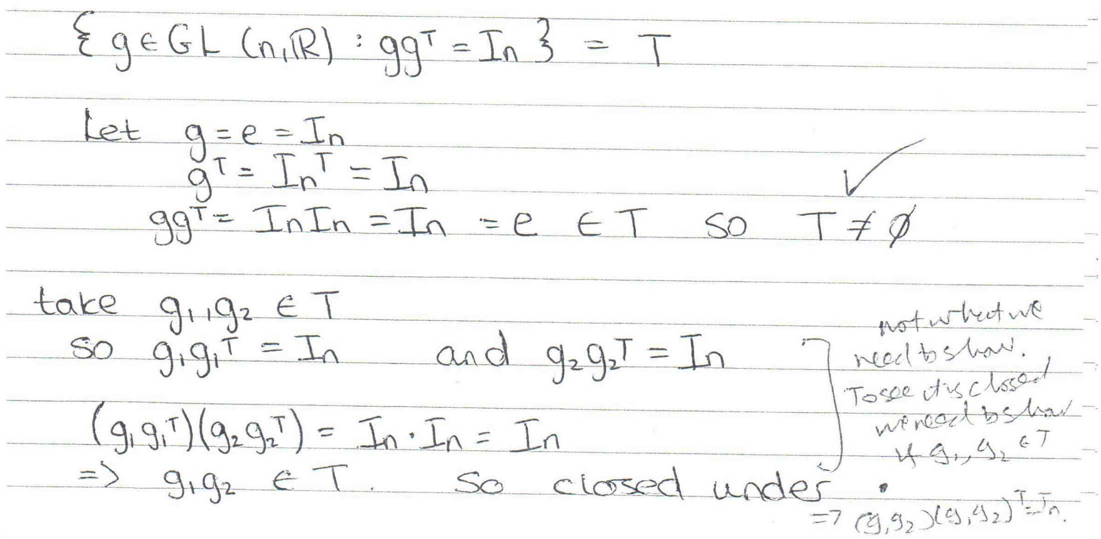
Manrico's writing as seen in his scripts is personalised with signs of tentativeness on many occasions. Tentative writing occurs when his understanding is not clear. In these instances, his solutions are nonlinear and messy. On other occasions, when the tasks are more concrete, such as finding the symmetries of cubes, his solutions are very well structured and his reasoning very explicit.

4.2.4 Closure Under Group Operation

The application of routine's second condition for a set to be a subgroup, namely closure under operation, was problematic in six (6/13) students' coursework solutions, namely, Dorabella, Leonora, Manrico, Musetta, Francesca and Tamino. The errors and inaccuracies were less serious and appeared less frequently than the errors discussed in section 4.2.3.

The first error was related to a problematic object-level understanding of the distinction between the element of a group and a subgroup. This possibly indicates unresolved problems regarding the definition of the group and its axioms and, moreover, the properties of the elements of a group. There are indications of a problematic perception of the notation used in the exercises, for instance the subgroup $Sym(X)$ and the set X. Similar to the discussion is section 4.2.3, the unfavorable effect of the problematic metaphors from Linear Algebra regarding the inverse and the transpose of the matrices and the influence they have in the application of the routine and the solution of the exercises become again obvious.

For instance, as the following excerpt suggests, Leonora shows a good metalevel understanding since she has applied the test appropriately in CS1E3iv, showing that she has a clear understanding of the applicability and closure conditions of this particular routine as well as its course of action (prove the three conditions to be a subgroup).

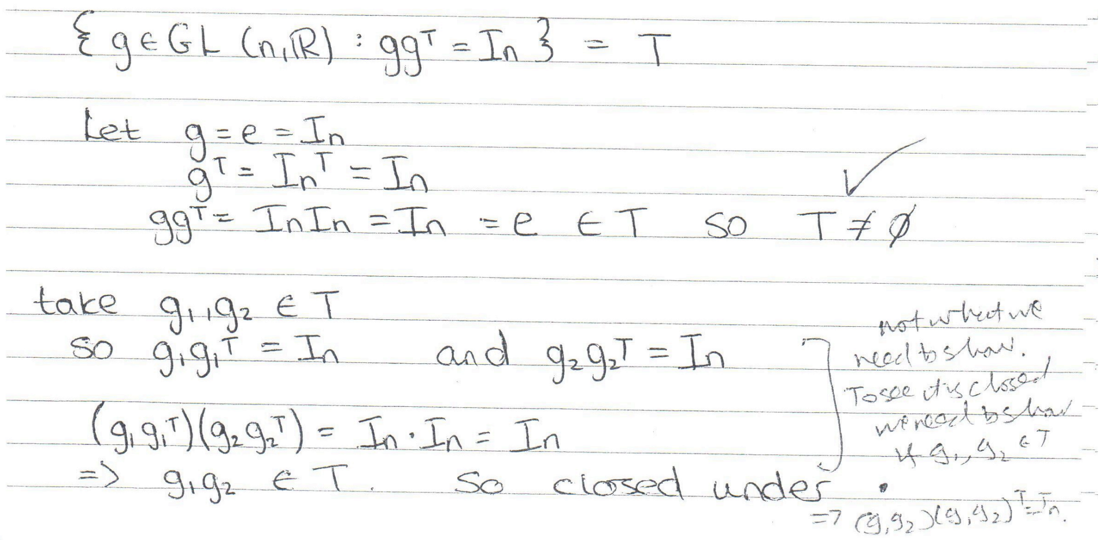


$\{g \in GL(n,\mathbb{R}) : gg^T = I_n\} = T$

Let $g = e = I_n$
$g^T = I_n^T = I_n$
$gg^T = I_nI_n = I_n = e \in T$ so $T \neq \emptyset$ ✓

take $g_1, g_2 \in T$
so $g_1g_1^T = I_n$ and $g_2g_2^T = I_n$
$(g_1g_1^T)(g_2g_2^T) = I_n \cdot I_n = I_n$
$\Rightarrow g_1g_2 \in T$. So closed under •

not what we need to show. To see it's closed we need to show if $g_1, g_2 \in T$ $\Rightarrow (g_1g_2)(g_1g_2)^T = I_n$.



$\{g \in GL(n,\mathbb{R}) : gg^T = I_n\} = T$

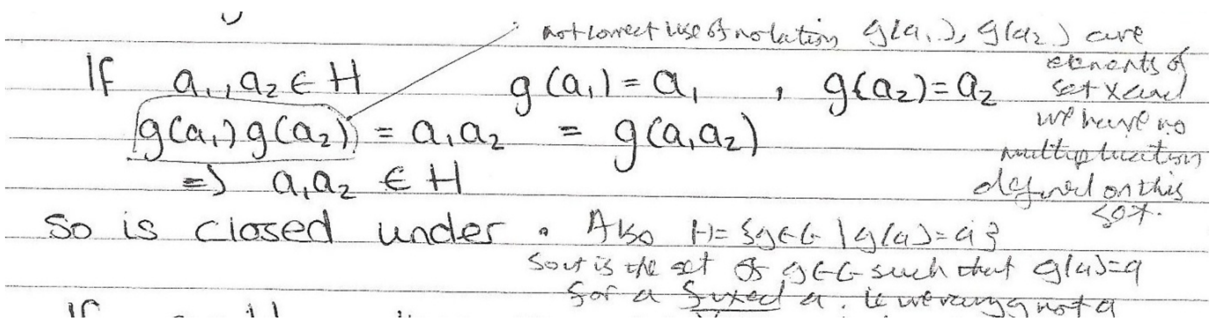
Let $g = e = I_n$
$g^T = I_n^T = I_n$
$gg^T = I_nI_n = I_n = e \in T$ so $T \neq \emptyset$ ✓

take $g_1, g_2 \in T$
so $g_1g_1^T = I_n$ and $g_2g_2^T = I_n$
$(g_1g_1^T)(g_2g_2^T) = I_n \cdot I_n = I_n$
$\Rightarrow g_1g_2 \in T$. So closed under •

not what we need to show. To see it's closed we need to show if $g_1, g_2 \in T$ $\Rightarrow (g_1g_2)(g_1g_2)^T = I_n$.

She presents her solution in a comprehensive way, using verbal explanations on many occasions. The only inaccuracy occurred in the use of symbolisation in the second example, while proving closure under operation. Instead of proving that $(g_1 g_2)(g_1 g_2)^T = I_n$, she proved that $(g_1 g_1^T)(g_2 g_2^T) = I_n$. This possibly suggests problematic object-level understanding of transposition as well as problematic object-level understanding of the definition of group and the group axioms in particular. In addition, this probably suggests that she may not have realised that $g_1 g_2$ is another element of the group and not a subgroup, and that it has to be considered as such.

Furthermore, in Leonora's CS2E2i solution as seen below, there are also indications of a problematic object-level understanding. The first one is revealed in the use of notation, which may have deeper roots relating to the essence of understanding the d-objects of the elements of the group and their properties. Additionally, she finds it difficult to define the different operations in the different groups. For example, she writes the expression $g(a_1)g(a_2)$, which uses elements of the set X but under operation, which does not operate in X. She has a vague perspective of what is H and what is X, i.e. that X is a non-empty set and that H is a subgroup of G with a certain condition. At some point she also writes $a \in H$, which is not true since a is an element of X.


If $a_1, a_2 \in H$ $g(a_1) = a_1$, $g(a_2) = a_2$
$g(a_1)g(a_2) = a_1 a_2 = g(a_1 a_2)$
$\Rightarrow a_1 a_2 \in H$
So is closed under . Also $H = \{g \in G \mid g(a) = a\}$
not correct use of notation $g(a_1)$, $g(a_2)$ are elements of set X and we have no multiplication defined on this set.
So it is the set of $g \in G$ such that $g(a) = a$ for a fixed a, ie we vary g not a

Her problematic object-level understanding regarding the concepts involved in this exercise is clearly expressed in the following interview excerpt.

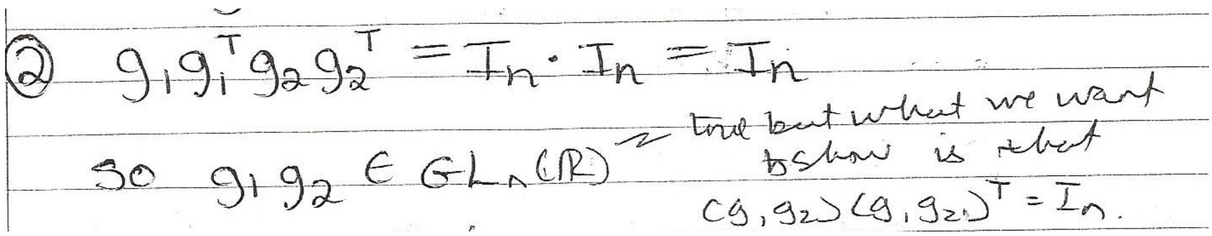
MI: *Question two?*

Leonora: *Question two, I found quite hard, because... I got a bit confused with this um –*

MI: *Sym (X)*

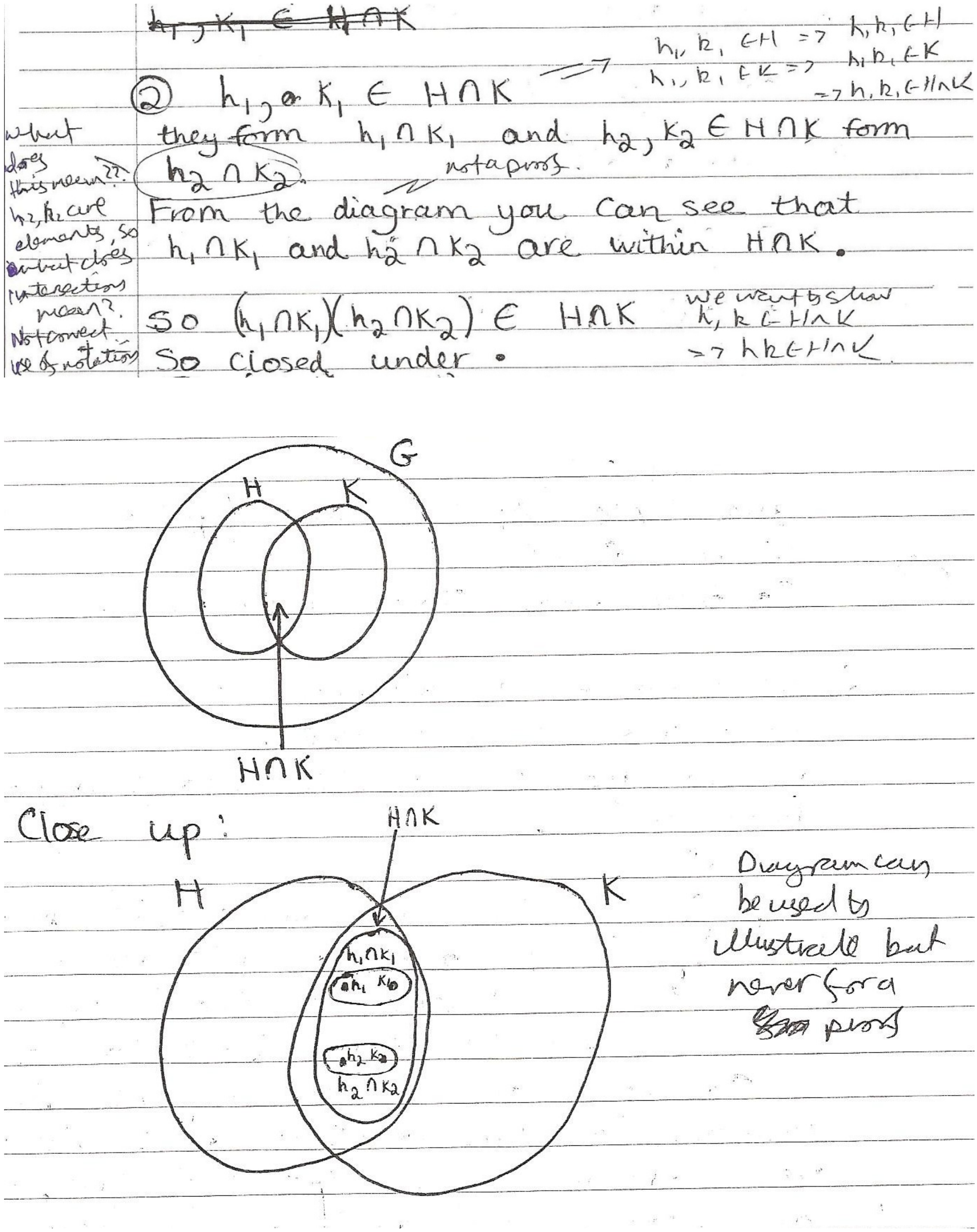
Leonora: *Sym (X) and stuff, but – so I don't - I started it but then I weren't sure, whether I was doing it right, so I kind of have stopped, and I'm gonna go ask for help. To like – because I – I don't like, if I'm doing something and I'm not sure if it's right, I don't like to carry on because I don't want to do it all wrong.*

A second example of problematic proof of closure under operation occurred in Manrico's CS1E3iv solution. In the second example for closure under operation he does not prove what he is supposed to prove. He rather concludes that $g_1g_2 \in GL$, instead of proving $(gh)^T = (gh)^{-1}$.



In contrast to Leonora's case, the above excerpt reveals problematic metalevel understanding, since the error is not a result of object-level misunderstanding. The algebraic manipulations are correct in general, yet inappropriate for the context of this exercise. I would suggest that this error is grounded on problematic metalevel understanding, since it is a result of inaccurate consideration of the applicability conditions of the particular routine as well as the 'course of action'.

Moreover, in exercise CS1E4i, Manrico's work also shows some inaccuracies. The first relates to the expression $h_1 \cap k_1$ and $h_2 \cap k_2$. There is indication of problematic object-level understanding of the d-object of subgroup as well as the elements of the subgroup. In addition, there are problems with the application of metarules (the well-defined and established, among the mathematical community norms of proving), regarding the use of visual mediators. Here Manrico, for the second time (see also section 4.2.3), has based his proof entirely on visual mediators (in this case Venn diagrams, used as metaphor from Set Theory), a tactic that is not acceptable by the markers.



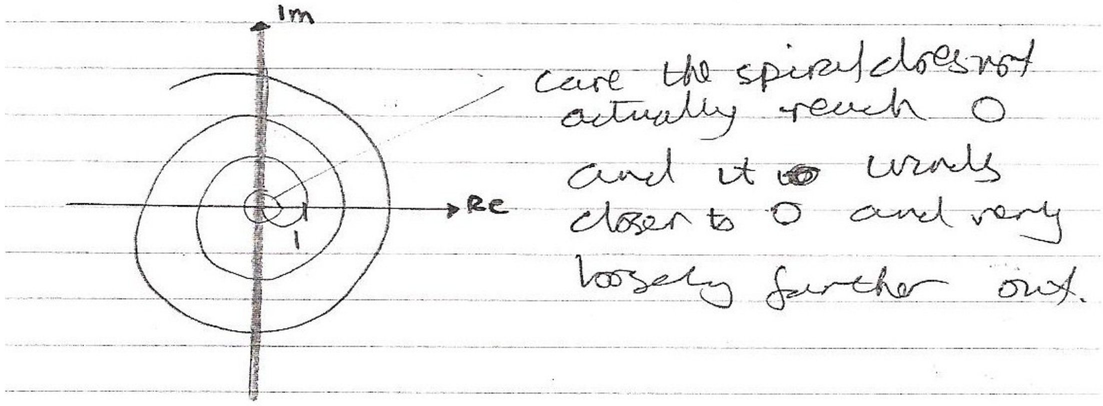
Regarding the use of visual mediators as core part of the solutions, there are indications that such use is often linked with lack of confidence or certainty about the quality of the algebraic reasoning. In three students' cases, namely Manrico, Calaf and Tamino, they make use of visual means of representation, such as Venn diagrams. The use of such visual mediators is not supportive; instead when such approach to solution is applied, these students tend to base the core of their solution on them.

4.2.5 Use of Visual Images

In this section, contrary to the above, I do not focus on the use or no use of visual images as part of the solution, but rather on the problematic use of such images and the errors that have occurred. In particular, I examine the errors that have appeared in the solutions of CS1E3 of eight out of the thirteen students (8/13), namely Amelia, Calaf, Dorabella, Kostanza, Carmen, Tamino, Musetta and Francesca.

As the following discussion suggests, the problematic use of visual mediators is often irrelevant from the students' object-level understanding of the d-objects of Group Theory. The problematic use was a result of inherited problems, in the form of metaphors from other mathematical discourses such as Complex Analysis. Therefore, more often than not the errors were located only on the Argand Diagram, in the context of an overall correct proof, both from object-level and metalevel perspective. Moreover, the algebraic reasoning and application of the particular routine was often correct, whereas the Argand diagram was problematic. Below I list some typical examples.

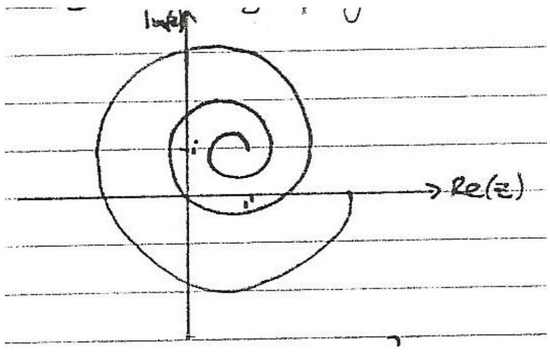
Amelia applied the routine and the governing metarules correctly in the second task in CS1E3iii, i.e. $\{e^{(1+i)t}: t \in R\}$, showing an overall good understanding of the definitions of the involved d-objects and the ability applicability and closure conditions as well as apply the required course of action for that particular test. The only minor error was on the Argand diagram. Amelia presented it as if the spiral started from the origin rather than approaching but never reaching it. This does not indicate a problematic object-level understanding with the Group Theory discourse, but possibly indicates a problematic inherited metaphor from Complex Analysis.

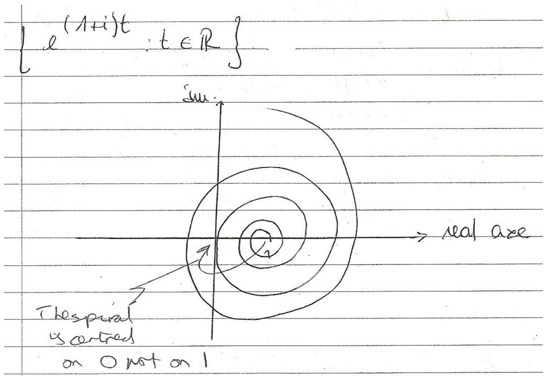


Her difficulty in drawing the Argand Diagram is obvious in the following excerpt, which reinforces the claim that her main problem was not to understand the d-object of subgroup and apply the metarules that govern the routine, but rather represent the subgroup on an Argand Diagram.

> *Um yeah, they were alright, I find like – visualizing sometimes the actual sets of them, quite difficult, to work out actually what you're talking about, and then – cos like they're quite big sets aren't they, like if you're doing an Argand Diagram, it's quite hard to prove it because you've got to do it for all of them, you can't just show the little ones, and then for part 3, we couldn't do them at all, cos we couldn't really show what they were… before we went to the seminars. And... Yeah, and I still can't draw that properly, you have to draw them don't you.*
> Amelia

Dorabella produced a spiral in the same exercise, but with its centre misplaced instead of the origin in $(1,1)$. Similarly, Carmen misplaced the centre in $(1,0)$.





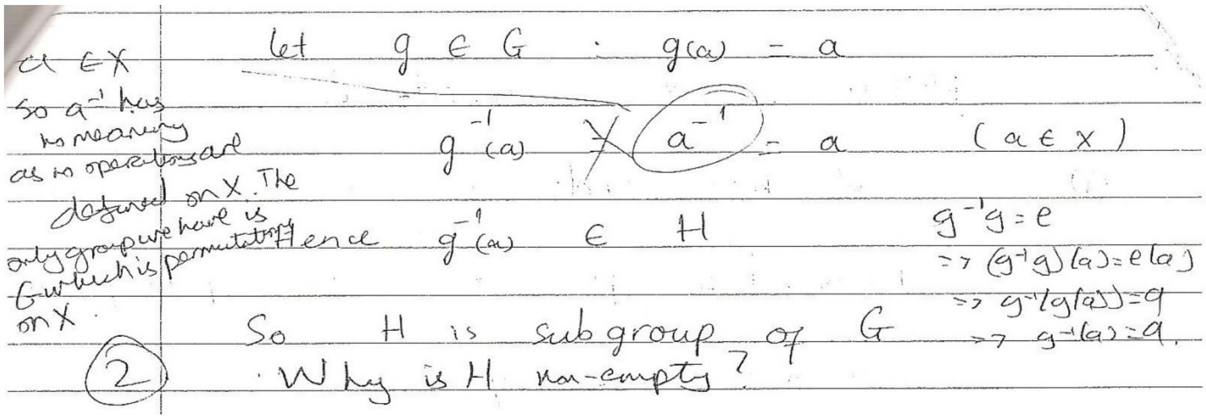
In both diagrams, there is no indication that the spiral approaches but never touches the centre. Again, this indicates problematic metaphors from Complex Analysis and consequently an undeveloped connectivity across the discourses of Group Theory and Complex Analysis.

4.2.6 Distinction Between Group, Subgroup, Set and their Elements

This last category of errors was related to the problematic object-level understanding of the d-objects of groups, subgroups, sets, and their elements, in the case of five students, namely Calaf, Dorabella, Kostanza, Leonora and Manrico. As the following discussion will reveal, on several occasions, these students confused the elements of the groups and the elements of the sets. Consequently, they applied group operations and axioms to elements of sets, for instance in CS2E2i.

Interestingly, students, despite their problematic object-level understanding of the involved d-objects, were able to successfully apply the metarules of the routine correctly, following all the steps of the test for a set to be a subgroup. Nevertheless, there were moments when they were not able to distinguish the elements of the set from the elements of the group. This possibly indicates that object-level learning does not always precede metalevel learning. Successful application of metalevel rules does not necessarily imply that all the involved mathematical d-objects have been fully objectified. Below they follow three representative examples, showing the particular type of errors and supporting the aforementioned claim.

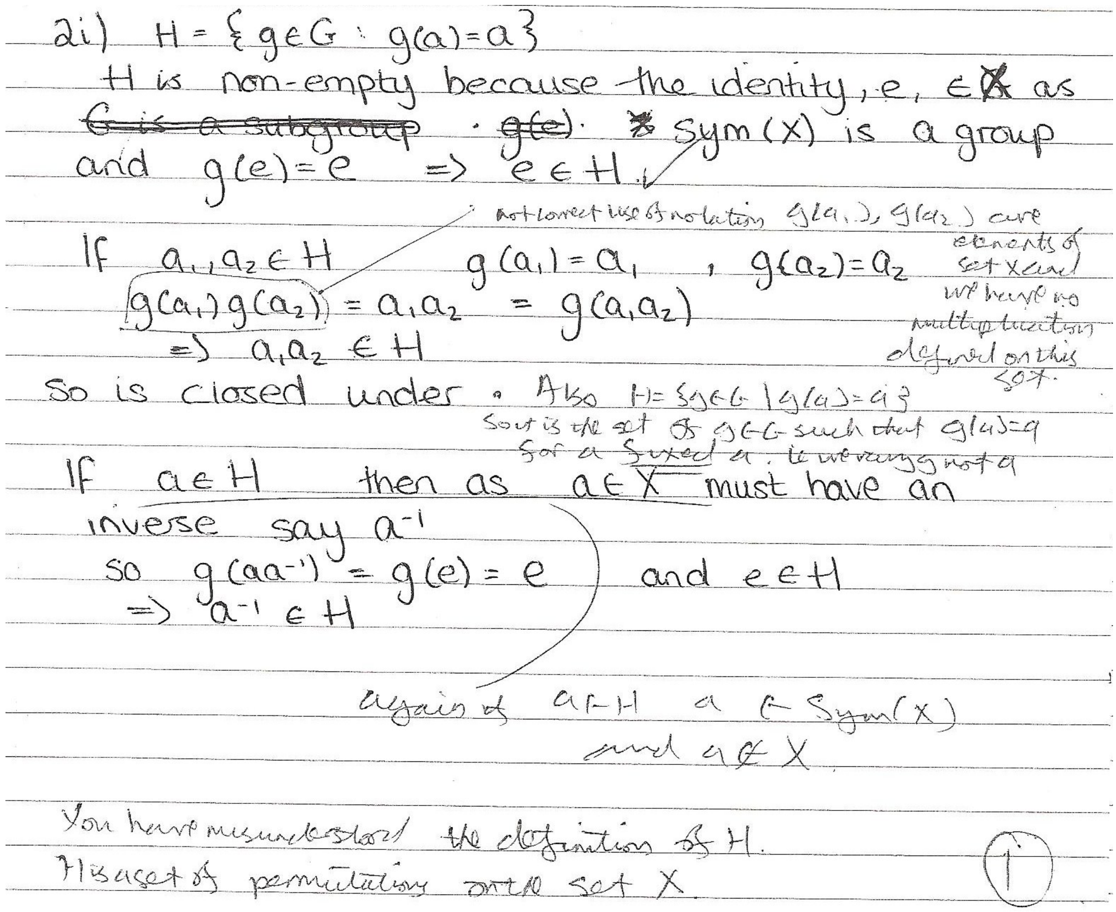
An illustrative example of confusion between the elements of the set X and the elements of G and the elements of the group G appeared in Calaf's coursework. Although Calaf seems to have a good object-level understanding of the condition given in the exercise of CS2E2i, i.e. $H = \{g \in G: g(a) = a\}$, and the application of the metarules has no particular problems, there are indications of a problematic object-level understanding regarding the elements of X and the elements of G, as shown below. He has not clearly located the group operation in G, but he seems to have tried to apply it to $a \in X$ in an effort to prove inverses. He has not realised that X is a set and not a group, and therefore there is not defined binary operation in X. This clearly reveals problematic object-level understanding of the distinction between the d-objects of group and set.



Calaf's problematic object-level understanding is also revealed in the following excerpt from his second interview, in which he was asked to discuss CS2E2.

> *Um, using the definition of the left coset, using, prove that, so that h – $g_1H = g_1h$, $\forall h \in H$, so it gives us $g_1h_1 + g_2h_2$ in there and you – because um… – because $H$is subgroup of that, and then h is in the – big H there so do what $ga = a$ so you what – as – I did it yeah, using that definition there, that part that I'm struggling with…* Calaf

Another example of problematic object-level understanding of the d-objects of group, set and their elements occurred in Leonora's solution of CS2E2i. Although she has shown good metalevel understanding, applying correct the routine for proving that a set A is a subgroup, there is a problematic application of the definitions of the involved d-objects, and therefore several inaccuracies occurred in her work. The first one is related to the use of notation, which is possibly linked with the object-level understanding of the elements of the group and their properties. In addition, she does not seem to be able to define the different operations in the different groups. For example, she writes the expression $g(a_1)g(a_2)$, which used elements of the set X, but under operation that does not operate in X. She does not have a clear view of what is H and what is X, i.e. that X is a non-empty set and that H is a subgroup of G with a certain condition. At some point she also writes $a \in H$, which is not true since a is an element of X.



2i) $H = \{g \in G : g(a) = a\}$

H is non-empty because the identity, e, ∈ ~~X~~ as ~~G is a subgroup~~ · ~~g(e)~~ · Sym(X) is a group and $g(e) = e \Rightarrow e \in H$. ✓

(not correct use of notation $g(a_1)$, $g(a_2)$ are elements of set X and we have no multiplication defined on this set.)

If $a_1, a_2 \in H$ $g(a_1) = a_1$, $g(a_2) = a_2$

$g(a_1)g(a_2) = a_1a_2 = g(a_1a_2)$

$\Rightarrow a_1a_2 \in H$

So is closed under ·. Also $H = \{g \in G \mid g(a) = a\}$ so it is the set of $g \in G$ such that $g(a) = a$ for a fixed a, ie we vary g not a

If $a \in H$ then as $a \in X$ must have an inverse say a^{-1}

so $g(aa^{-1}) = g(e) = e$ and $e \in H$

$\Rightarrow a^{-1} \in H$

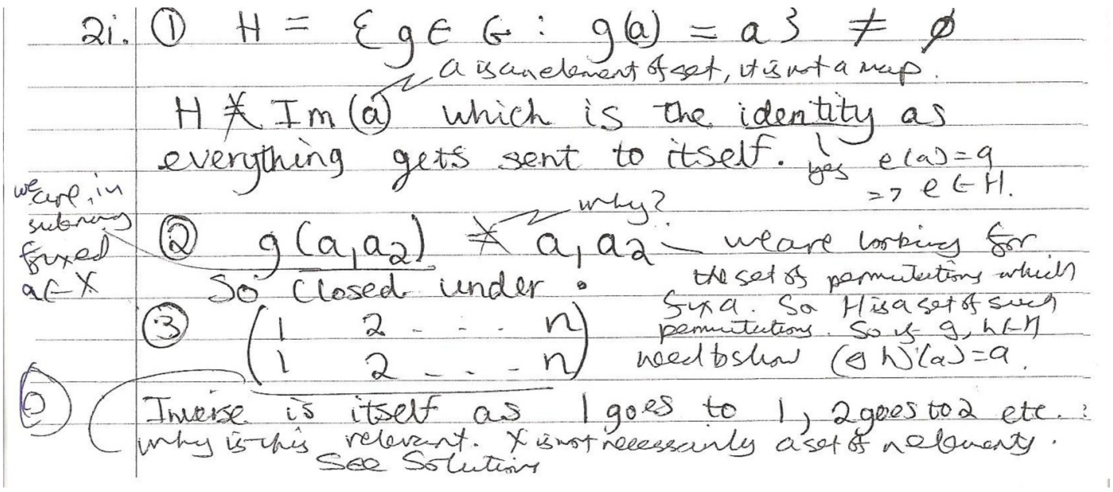
again of $a \in H$ a ∈ Sym(X) and a ∉ X

You have misunderstood the definition of H. H is a set of permutations on the set X

1

Leonora's problematic object-level understanding in CS2E2i is also revealed in her statement as this can be read in section 4.2.4, when she was asked to discuss this exercise.

A third example of problematic object-level understanding, this time, regarding the elements of the set and the d-object of a map was revealed in Manrico's solution of CS2E2, by writing $H = im(a)$. As the following excerpt shows, on several occasions, he states definitions of various d-objects, e.g. of the image, but apparently his object-level understanding of these definition is rather naïve, since he does not really show knowledge of how and when to properly use these definitions in the context of the exercise. In addition to problematic object-level understanding, this possibly indicates that he is not yet able to apply the governing metarules appropriately and in particular the applicability and closure conditions of a routine. His object-level understanding of the d-object of image is still problematic. In addition, his attempt reveals uncertainty regarding the 'course of action' that should be followed in this particular proof. He has probably not realised that he is looking for a set of permutations that fix a. His use of mathematical vocabulary is also problematic, which affects the narratives he produces. For instance, the expression $g(a_1 a_2) = a_1 a_2$ is irrelevant. What he needs to prove is that $gh(a) = a$.


2i. ① $H = \{g \in G : g(a) = a\} \neq \emptyset$
a is an element of set, it is not a map.
$H \neq Im(a)$ which is the identity as everything gets sent to itself.
yes $e(a) = a \Rightarrow e \in H$.
we are in subgroup fixed $a \in X$
why?
② $g(a_1 a_2) \neq a_1 a_2$ — we are looking for the set of permutations which fix a. So H is a set of such permutations. So if $g, h \in H$ need to show $(g h)(a) = a$.
So closed under ∘
③ $\begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 2 & \dots & n \end{pmatrix}$
④ (Inverse is itself as 1 goes to 1, 2 goes to 2 etc.
why is this relevant. X is not necessarily a set of n elements. See Solution

The claim that a number of students had a problematic object-level understanding of the elements of groups and sets was reinforced by LCR based on his seminar experience.

MI: *So mainly students had a problem with question 2?*

LCR: Yea, I think so. I mean it is interesting, you know you have got a set in the group and somehow separating out in their minds the different roles of the elements of the setting up around and the group which is acting is something that, you know, somehow they don't have a picture in their mind of – so they – you know writing a string of symbols round like $g_1 g_2(a)$ its – the sort of - the distinction between the elements of the group and the elements of the set is something that is not necessarily clear.

Additionally, SAA and SAB highlighted this kind of error in their report on the coursework of all 78 students, particularly identifying it in CS2E2, which is in agreement with the data analysis.

Q2(i) - At times a poorly done question. Many people were confused by the definition of H. The thing to note is that X and $Sym(X)$ (the permutations of the set X) are two separate entities and elements of one are not elements of the other. So here H is a subset of $G \leqslant Sym(X)$ and in particular H is the subset of those $g \in G$ which leave some fixed element a in X unmoved. So we are fixing a and looking for the permutations that do not move a, not a set of elements of X. Importantly we have no operations defined on the elements of X only on the elements of G, so $g_1(a) \cdot g_2(a)$ as some wrote is undefined since $g_1(a)$ and $g_2(a)$ are elements of X an arbitrary set.

Text 4.8: Markers' comment 3

In sum, as revealed in the above discussion the thirteen students' encounter with the d-object of subgroup was overall more problematic compared to the d-object of group. There have occurred in total six categories of errors. The first was the lack of clarification about the distinction of the prime numbers p and q, mostly due to problematic metalevel understanding. The second error was related to the absence of proof of non-emptiness, again due to imprecise application of metarules. The third was related to the proof closure under inverses. This error was mostly due to problematic object-level understanding of the d-object of inverse which had, however, negative consequences to the application of metarules. The fourth error was related to closure under operation and was due to problematic object-level understanding of the definition of subgroup and the idea of closure. The fifth category of errors was related to the use of visual mediators, such as Argand Diagrams, due to erroneous metaphors from other mathematical discourses and finally the sixth category of errors was related to the problematic object-level understanding of the d-objects of group, subgroup, set and their elements.

4.3 Symmetries of Cube

Listing the symmetries of the cube and proving that the set of rotational symmetries of the cube that send one of the three pairs of opposite faces to itself forms a subgroup of order eight, was differentiated from the other mathematical tasks because students' results were much better, as a consequence of the lower level of abstraction. In this case, the use of concrete visual mediators was possible, and it significantly contributed to the more consistent solutions, compared to other exercises involving subgroups.

In particular, in CS1E6 and FEE4i, students had to list the 24 rotational symmetries of the cube and prove that the set of rotational symmetries of the cube that send one of the three pairs of opposite faces to itself forms a subgroup of order eight.

> **6.** Describe the group G of rotational symmetries of a cube, saying what the possible axes of rotation are and what the possible angles of rotation are. Hence show that there 24 such rotational symmetries. Consider one of the three pairs of opposite faces of the cube. Show that the set of rotational symmetries of the cube which send this pair of faces to itself forms a subgroup of G of order 8.

Text 4.9: CS1E6

4. (i) Describe the group S of rotational symmetries of a solid cube in $\mathbb{R}^3$. List the possible axes of rotation and angles of rotation, and hence show that $|S| = 24$. Let ℓ be an axis passing though the centres of a pair of opposite faces of the cube and T be the set of rotations in S which send ℓ to itself. Prove that T is a subgroup of S and $|T| = 8$. [10 marks]

Text 4.10: FEE4i

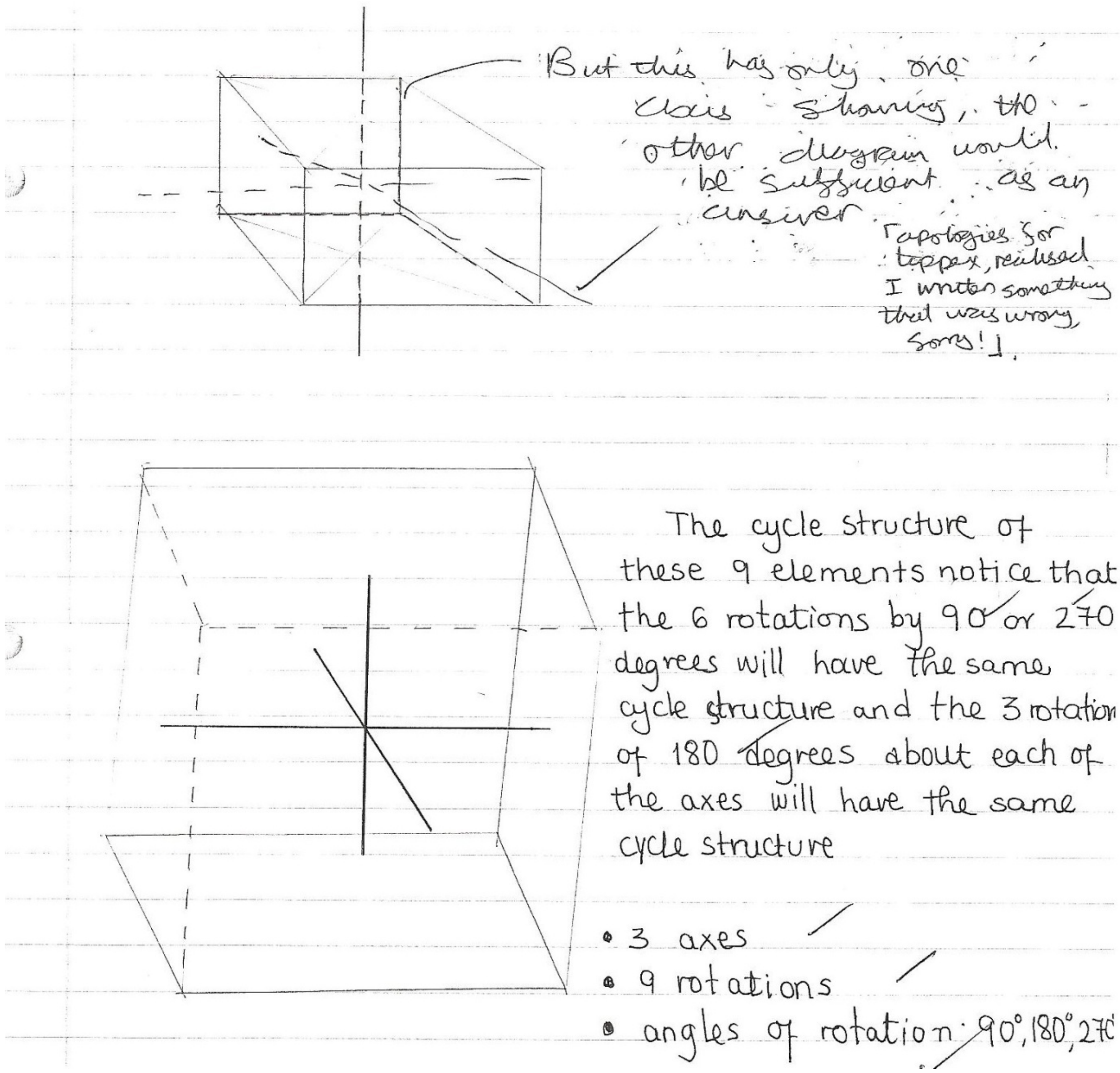
Eleven out of the thirteen students (11/13) were able to flawlessly and explicitly list all 24 cube symmetries in the coursework, demonstrating a good object-level understanding and, in most cases, the ability to clearly represent these symmetries using visual images to support their narratives. The use of images possibly suggests the students' inclination to have visual representations of the involved mathematical d-objects and p-objects, whenever possible. According to the discussion in this section, use of visual mediators, one of the four characteristics of the mathematical discourse, according to Sfard (2008) and Commognitive Theoretical Framework, is favoured by the novice students, since it not only allows them to reveal their reasoning more explicitly, but it is also an important means for reducing the level of abstraction. This claim is justified by both the extensive use of the visual images and by the efficiency of this use.

In what follows, I discuss the listing of the 24 symmetries of the cube, both in the coursework and in the examination, identifying, the minor errors that occurred, which in their majority were inaccuracies and were disconnected with the object-level understanding of the d-objects of Group Theory. In addition, I discuss the more serious errors that occurred in the second part of both exercises, CS1E6 and FEE4i, in which students had to prove that there is subgroup, of the group of symmetries, of order 8. Moreover, since this is the first occasion in which the same, or similar, mathematical task is set both in the coursework and the examination, I report, when possible, on the progress or regression of student performance.

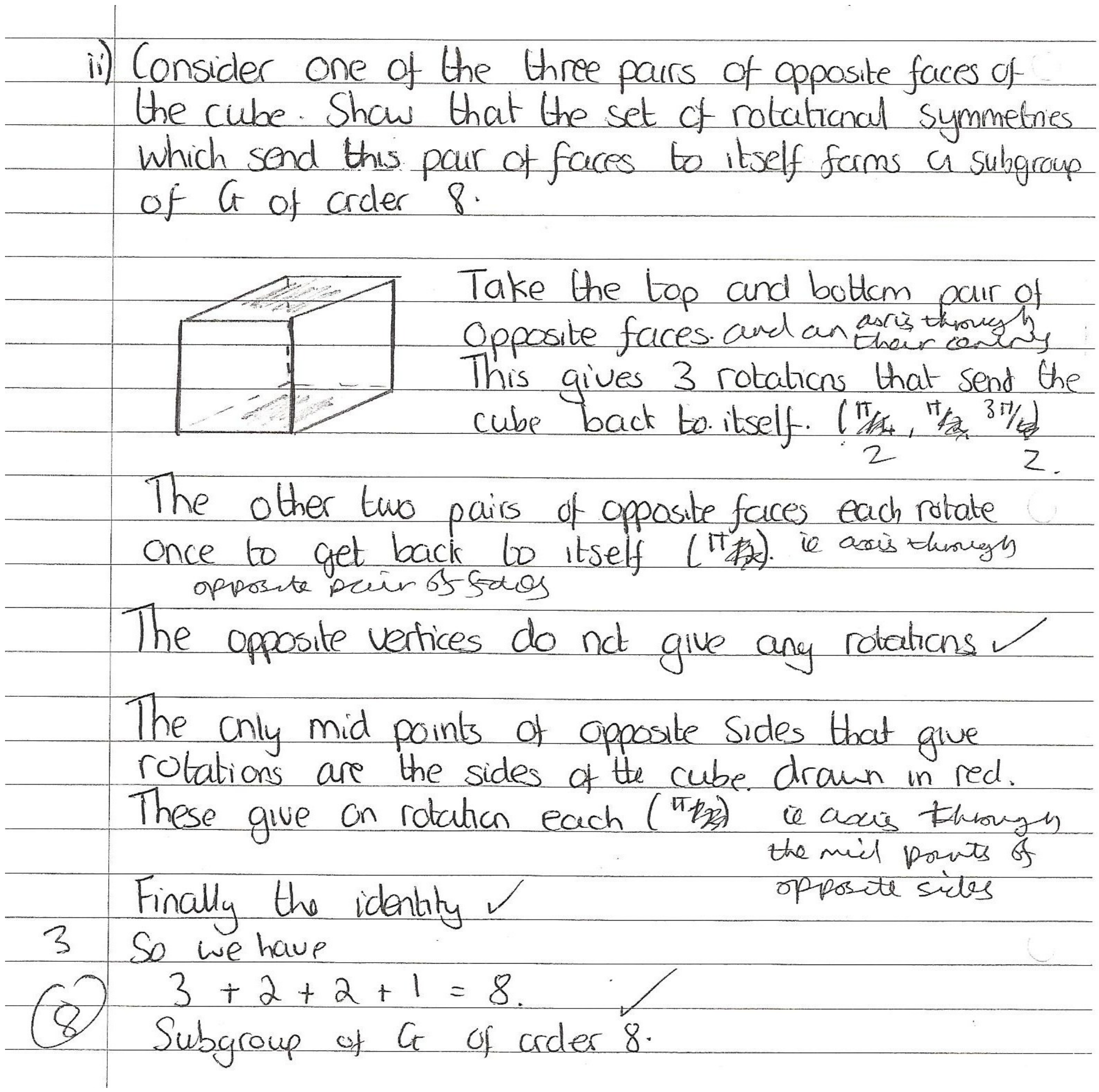
4.3.1 Listing of the 24 Symmetries of Cube

Eleven out of the thirteen (11/13) students answered the first part of the CS1E6 flawlessly, indicating excellent object-level understanding of the cube symmetries. In the first part of exercise CS1E6, regarding the list of cube symmetries, some minor inaccuracies occurred, as a result of a problematic metaphor from trigonometry and elementary geometry, regarding the notation and some problematic use of the terminology. These errors did not occur in the examination, indicating that these problems had been resolved and highlighting progress in the students' performance.

In the solutions of two students, namely Musetta and Norma, they occurred more serious errors, connected, in particular, with the use of visual images. For instance, Musetta correctly lists the 24 symmetries, but the use of the visual mediator is problematic. The first diagram does not explicitly represent the nine rotations of the cube about the three axes that pass through the midpoints of the edges of the cubes, giving the impression of difficulty in visualizing these symmetries. Then she produces another diagram that is much more appropriate, yet without cancelling the first.



Norma has used very clear and understandable visual images in order to present the different axes of rotation. However, as seen in the following excerpt, there is some problematic word use, possibly caused by a problematic metaphor of d-objects from elementary geometry. Instead of *edges* she says *sides*. Another issue is the problematic use of π terms. She repeatedly writes wrong angles, i.e. for example, instead of $\frac{\pi}{2}$, she writes $\frac{\pi}{4}$. These errors do not occur in the examination, suggesting improvement in the production of narratives and consequently in her mathematical skills and learning. Overall Norma has shown signs of a good capacity to verbalise, i.e. to express her mathematical thinking in words. These errors cannot be connected with problematic object-level understanding of the d-object of symmetry or the p-object of cube, as such, but rather they are inaccuracies inherited in the form of metaphors from other mathematical discourses.


ii) Consider one of the three pairs of opposite faces of the cube. Show that the set of rotational symmetries which send this pair of faces to itself forms a subgroup of G of order 8.

Take the top and bottom pair of opposite faces and an axis through their centres. This gives 3 rotations that send the cube back to itself. (π/2, π, 3π/2)

The other two pairs of opposite faces each rotate once to get back to itself (π). ie axis through opposite pair of sides

The opposite vertices do not give any rotations ✓

The only mid points of opposite sides that give rotations are the sides of the cube drawn in red. These give on rotation each (π) ie axis through the mid points of opposite sides

Finally the identity ✓

So we have

3 + 2 + 2 + 1 = 8. ✓

Subgroup of G of order 8.

Norma's problematic encounter with this exercise was also obvious in her second interview, as revealed in the following excerpt, locating her difficulty in the act of visualizing the cube, which possibly led to make errors on the angles of rotations.

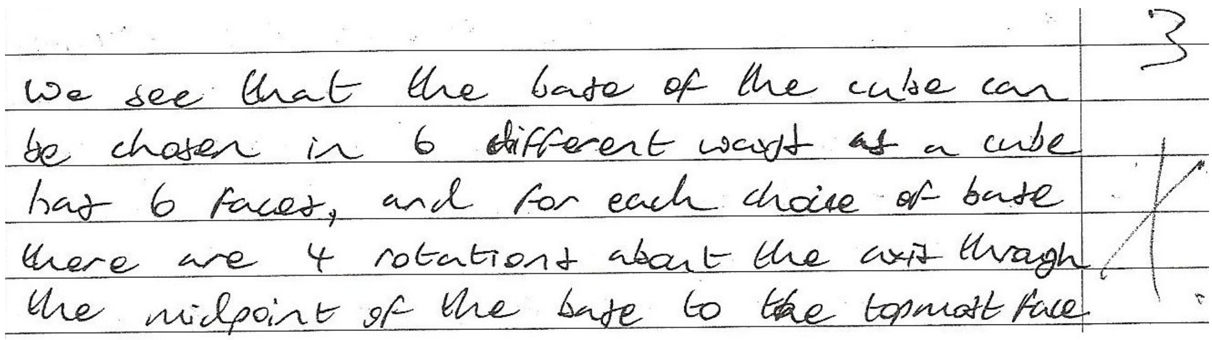
Norma: I'm a little – I'm still getting cards to make a cube, to um, actually help me, cos I was finding it hard … it's hard when you *do it in your head to like – visualise turning it around, so I have actually made a cube, from a piece of paper, and I'm using that to help me go through it… […] I tried finding something in my room, but I couldn't and so I just – made one! Laughs… So um, yeah, I've got a cube at home and I'm using that to help so...*

MI: Why do you think question 6 was the most difficult?

Norma: I think it might be because of visualization problems? (MI: Mm hmm) Just cos... I mean I know a cube is, like, an object, but when you're trying to – to do it. […] You're only trying to like do the question without actually seeing it in front of you, I think you can get a bit confused as to how things work, so... um, I think that might be why, just visualization probably, which is obviously why I made my cube, to help me.

Regarding the listing of the 24 symmetries of the cube, there are two incidents indicating a regression in the performance of two students, namely Otello and Kostanza. In particular, their list was correct in the coursework, in the exam their answer was problematic for different reasons. Although, regression in the performance cannot certainly be connected with problematic object-level understanding of the involved d-objects, I think it would be interesting to briefly present these inaccuracies.

Otello's description of some classes of symmetry was not clear and therefore unaccepted by the marker, as seen below.



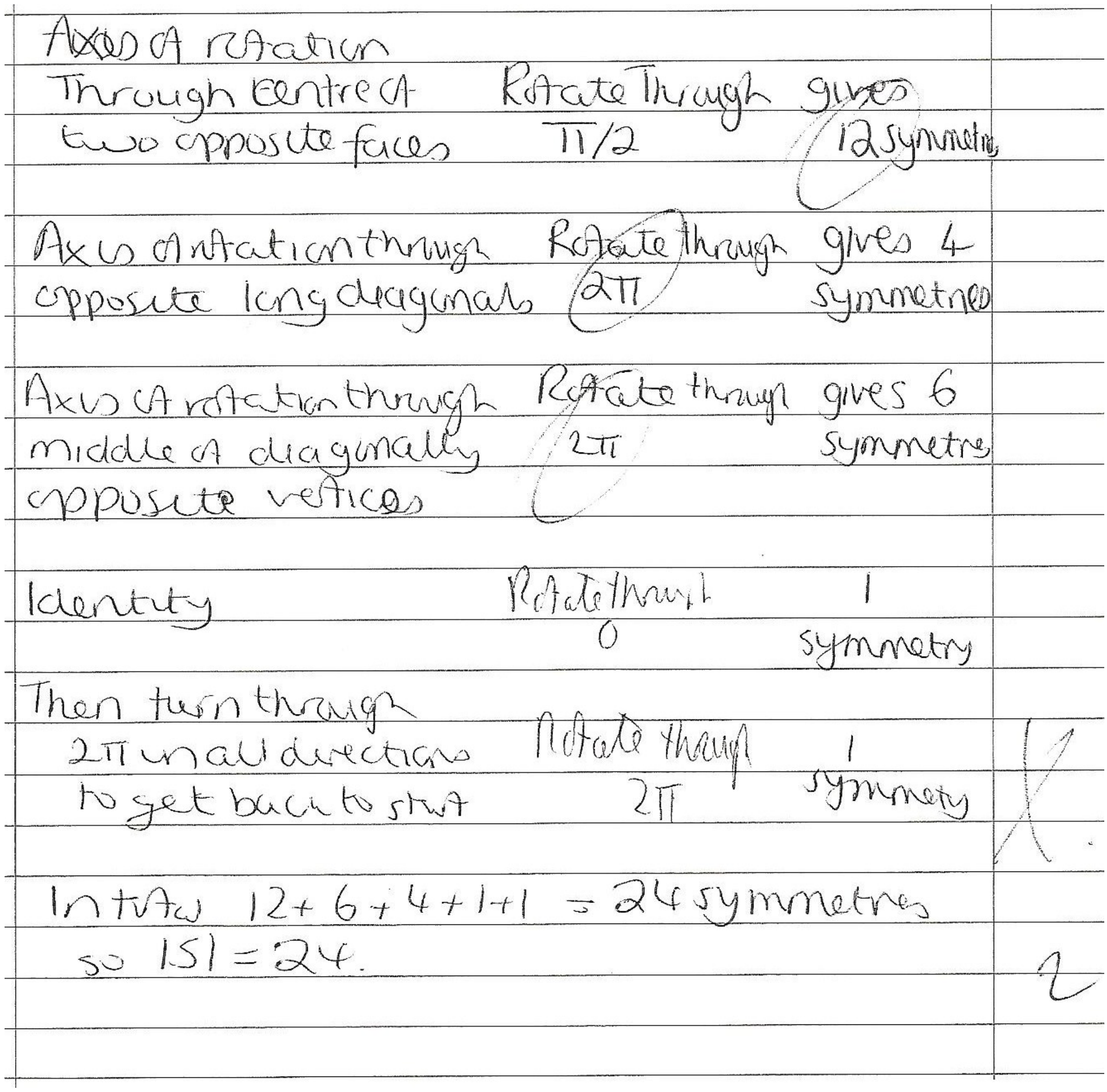
We see that the base of the cube can be chosen in 6 different ways as a cube has 6 faces, and for each choice of base there are 4 rotations about the axis through the midpoint of the base to the topmost face

3

> Thus there are 6 × 4 = 24 rigid motions
> which preserve symmetry
> Hence |S| = 24
>
> Similarly we can choose a vertex in
> 8 different ways and for each choice
> of vertex there are 3 rotations about
> the axis joining this vertex to the
> vertex directly opposite
> Thus there are 6 × 4 = 24
> rotational symmetries

Otello has not followed the expected way of presenting the 24 symmetries by using visual images, so as to achieve clear listing and convincing reasoning of his answer. This error can perhaps be considered as of metalevel nature, since he has not followed the norms of how such a mathematical task should be proved and presented, leaving the marker unsatisfied.

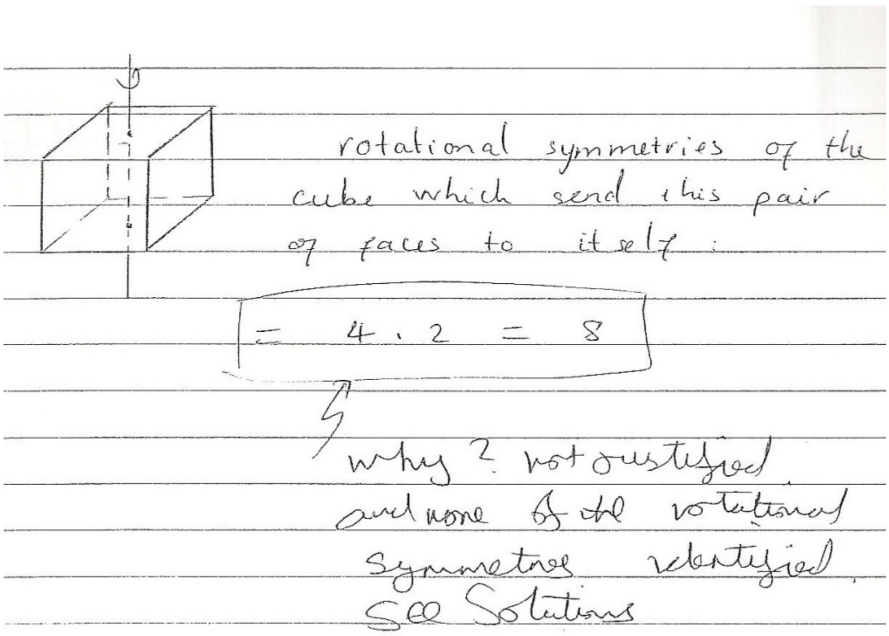
Kostanza's FEE4i solution, contrary to her attempt in the coursework, was seriously problematic regarding the list of symmetries, indicating unstable object-level understanding of the notion of symmetry, as well as unresolved problems in other mathematical discourses, inherited as erroneous metaphors. Her narratives are not accurate and the use of π terms is problematic. In addition, the use of visual mediators is weak. In the coursework, although her narratives were not always complete, she had managed to correctly list the 24 symmetries. There is a clear indication of regression in her performance, as illustrated by the following excerpt from FEE4i, possibly, in the case of this student, suggesting problematic object-level understanding.


Axes of rotation
Through centre of two opposite faces — Rotate through π/2 — gives 12 symmetries
Axis of rotation through opposite long diagonals — Rotate through 2π — gives 4 symmetries
Axis of rotation through middle of diagonally opposite vertices — Rotate through 2π — gives 6 symmetries
Identity — Rotate through 0 — 1 symmetry
Then turn through 2π in all directions to get back to start — Rotate through 2π — 1 symmetry
In total 12 + 6 + 4 + 1 + 1 = 24 symmetries so |S| = 24.
2

4.3.2 Proof of the Subgroup of Order 8

Proving that there is a subgroup of cube symmetries of order 8, was the part of the both CS1E6 and FEE4i that challenged students the most. In particular in CS1E6, six of the thirteen (6/13), namely Musetta, Carmen, Calaf, Kostanza, Manrico and Norma did not manage to solve this question correctly. In addition, in FEE4i six out of the eleven (6/11) students' solutions that attempt this exercise, namely, Calaf, Dorabella, Kostanza, Carmen, Tamino, and Musetta, there were also errors that revealed problematic object-level understanding of the d-object of subgroup and its order. Below they follow examples of all errors that occurred.

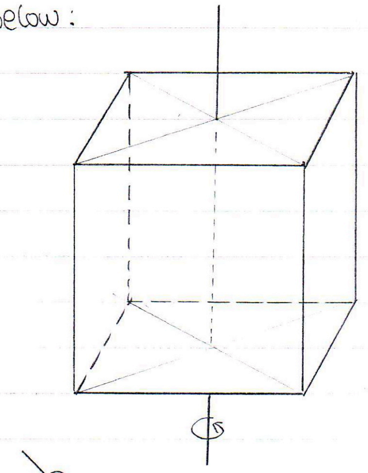
In the student solutions of this particular mathematical task it is more obvious the significance of the interaction between the object-level understanding and the application of the governing metarules. For instance, Calaf could not explicitly justify his answer regarding the second part of the exercise, both in the coursework and exam. This error is obviously of metalevel nature, yet it is probably grounded on problematic understanding of the d-object of subgroup and the application of the routine for proving that a given set is a subgroup. In particular, Calaf has neither tried to find the 8 symmetries of the cube that satisfy the condition nor did he justify his answer regarding the order. His coursework attempt is shown below, whereas in the examination he did not attempt to do this part, at all.



In the exam, Calaf shows no further progress in his understanding, compared to CS1E6. He successfully lists the 24 rotational symmetries of cube, but he does not answer the second question. This may suggest that he has either not yet objectified the metarules that govern the test for proving that a set is a subgroup or he cannot apply this routine to the specific example. Again these claims are, most probably, grounded on unresolved object-level understanding problems.

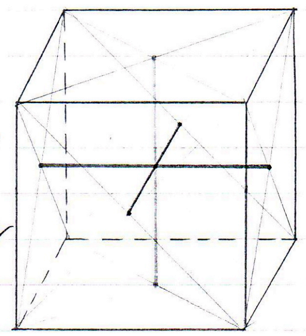
Similar to Calaf, Francesca is still not able to apply successfully the governing metarules for proving that there is a subgroup of symmetries of cube of order 8. In particular, although she is relatively successful in coping with more concrete tasks, such as the first part of the exercise, she has failed to prove that there was such a subgroup. Her very brief attempt to prove this last bit indicates problematic object-level understanding of the d-object of group, as well as uncertainty as to how to apply the metalevel rules regarding the determination of the order of the subgroup. Additionally, some of her narratives indicate a poor object-level understanding of the d-objects of groups, rotations, symmetries, order and identity.

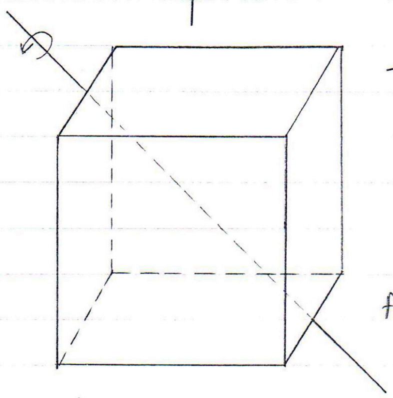
The possible axes of rotation are 3 and they are shown below:



Through opposite force centres

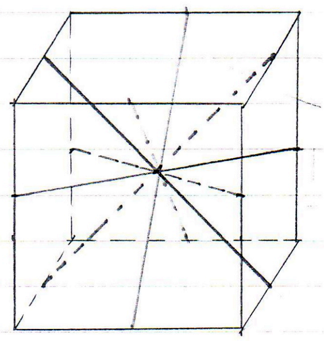
3 possible axes ✓
9 possible rotations ✓
Angles of rotation:
90°, 180°, 270°
$\pi/2$, π, $3\pi/2$





Through mid points of opposite sides

6 possible axes ✓
6 possible rotations
Angles of rotation: ✓
$180° = \pi$



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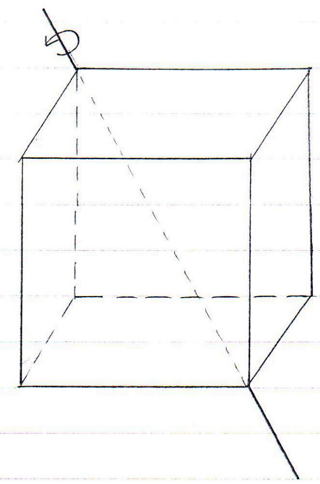
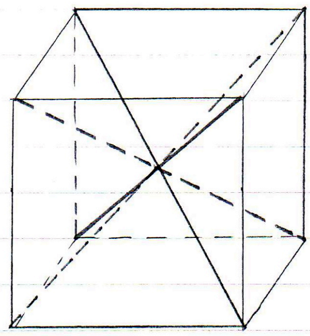


Through opposite vertices

✓
4 possible axes
8 possible rotations
Angles of rotation:
120°, 240° ✓
$\frac{2\pi}{3}$, $\frac{4\pi}{3}$

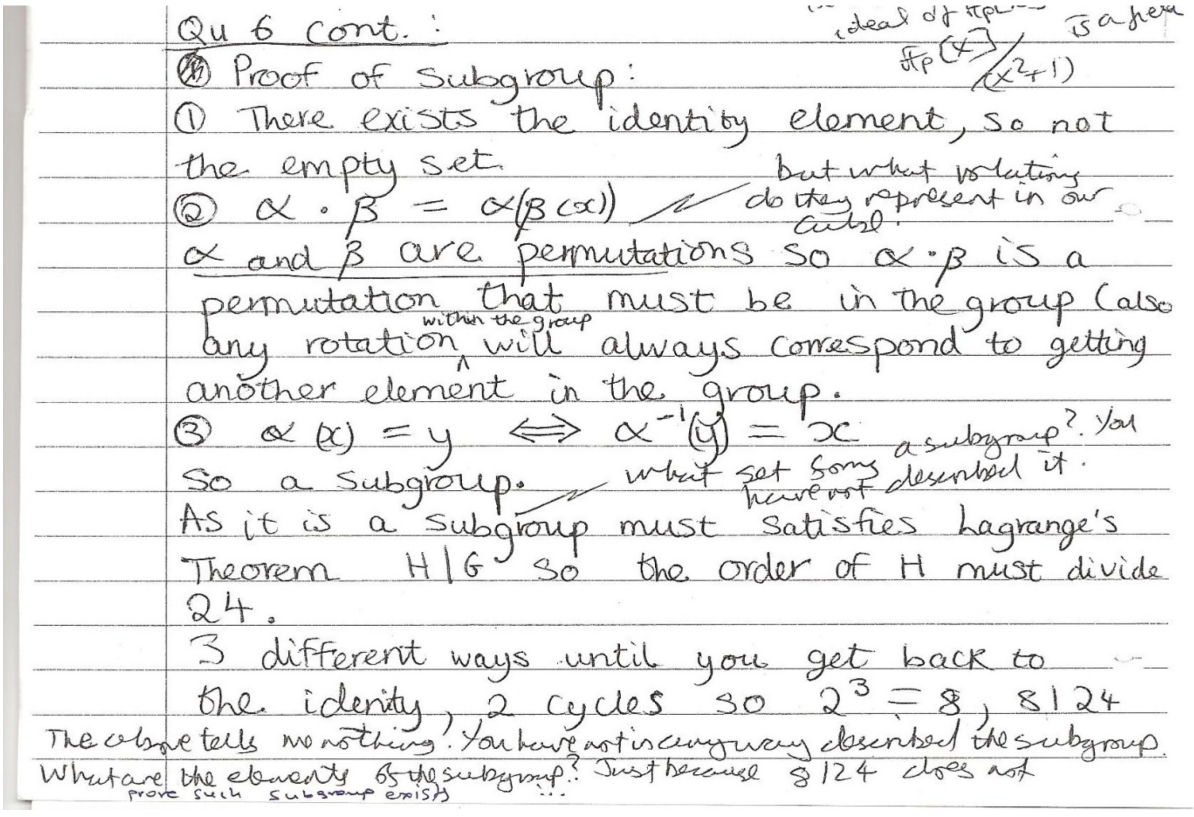


Hence, Total number of rotations = 9 + 6 + 8 = 23 + 1 (Identity) = 24 ✓

(*) Also: In the rotation group of the cube has order 2 : 3axes * 2 = 6 + 2 (Identity) = 8 (-3-)

(*) This does not really explain what this subgroup of order 8 contains and how is identity now 2 rotations? See solutions

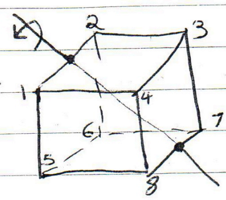
Another example of problematic object-level understanding regarding the d-object of subgroup of symmetries of cube is apparent in Manrico's solution of CS1E6. The second part of the question, as shown below, is problematic. There appear to be problems both in his object-level understanding as well as the application of the governing metarules required in this situation. For example, he states that there must be a permutation $a \cdot b = a\big(b(x)\big)$ in the group, but he does not explicitly show or suggest practically what this permutation is in the context of the exercise. An indication of problematic object-level understanding regarding the d-object of subgroup is the fact that for closure under inverses he does not mention what the subgroup is, but he rather assumes that since it has order 8 that divides 24 it is a subgroup. His thinking is not linear, i.e. his line of argument is not straightforward.


Qu 6 cont.:
(b) Proof of subgroup:
① There exists the identity element, so not the empty set.
② $\alpha \cdot \beta = \alpha(\beta(x))$ — but what rotations do they represent in our cube?
α and β are permutations so $\alpha \cdot \beta$ is a permutation that must be in the group (also any rotation within the group will always correspond to getting another element in the group.
③ $\alpha(x) = y \iff \alpha^{-1}(y) = x$
So a subgroup. — what set? a subgroup? You have not described it.
As it is a subgroup must satisfies Lagrange's Theorem H|G so the order of H must divide 24.
3 different ways until you get back to the idenity, 2 cycles so $2^3 = 8$, 8|24
The above tells me nothing! You have not in anyway described the subgroup. What are the elements of the subgroup? Just because 8|24 does not prove such subgroup exists...

Unlike the coursework, there are indications of improvement in the exam, as he successfully proves that there is subgroup of order eight that sends l to itself. He does not clarify though the reasons for the order of T to be eight.

Manrico is an example of improved performance. For the first part of CS1E6, unlike any other exercise in the coursework, his reasoning is explicit and the narratives are well presented and clear, as corroborated by the high mark. He is much more descriptive and his narratives are authoritative.

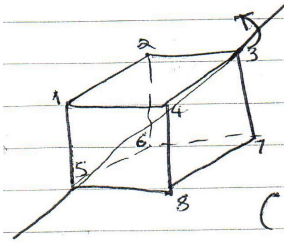
Opposite face centres this generates 9 new elements.



You can rotate through mid points of opposite sides. If you rotate it by an angle of π you will get a new element, if you rotate it again you'll end up at the identity again. You can pick 6 axes from the following:

Mid point	opp. mid point
(1, 2)	(7, 8)
(2, 3)	(5, 8)
(3, 4)	(5, 6)
(1, 4)	(6, 7)
(1, 5)	(3, 7)
(2, 6)	(4, 8)

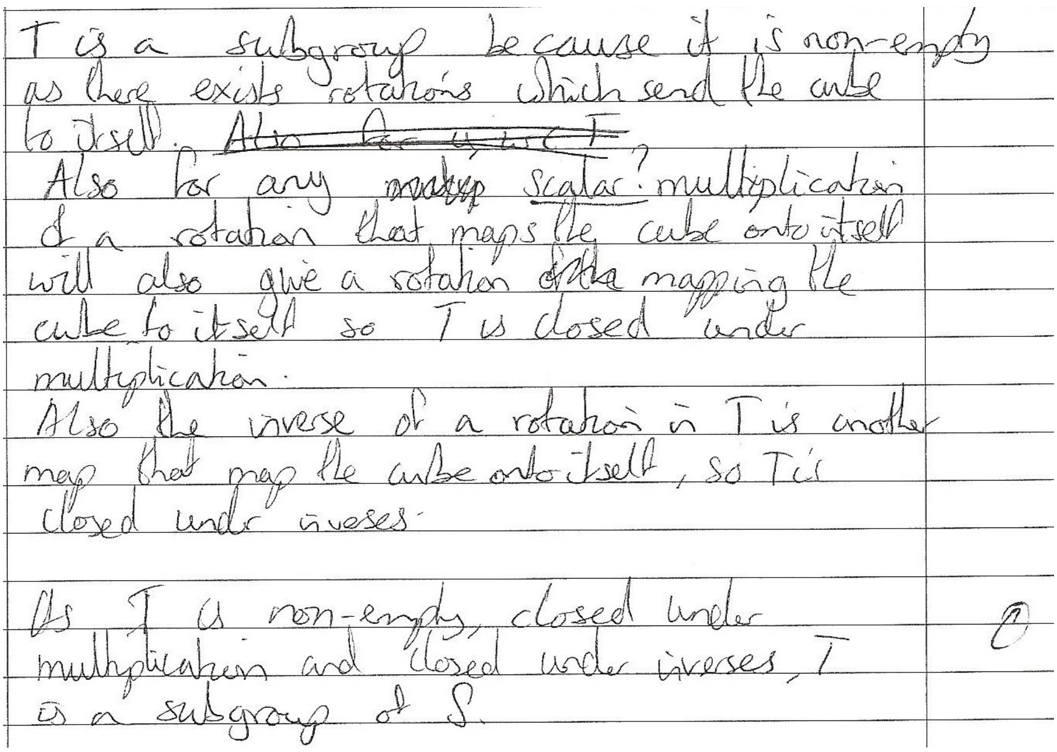
So 6 of these ~~generates~~ generating one new element, will mean rotating through mid points of opp. sides will generate 6 new elements.



You can also rotate through ~~mid points of~~ opposite vertices. You can rotate on the following: (1 and 7), (2 an 8), (3 and 5), (4 and 6). You can rotate on a angle of $\frac{2}{3}\pi$ ~~before coming~~ and you will generate 2 new elements this way, so in all you will get 8 new elements.

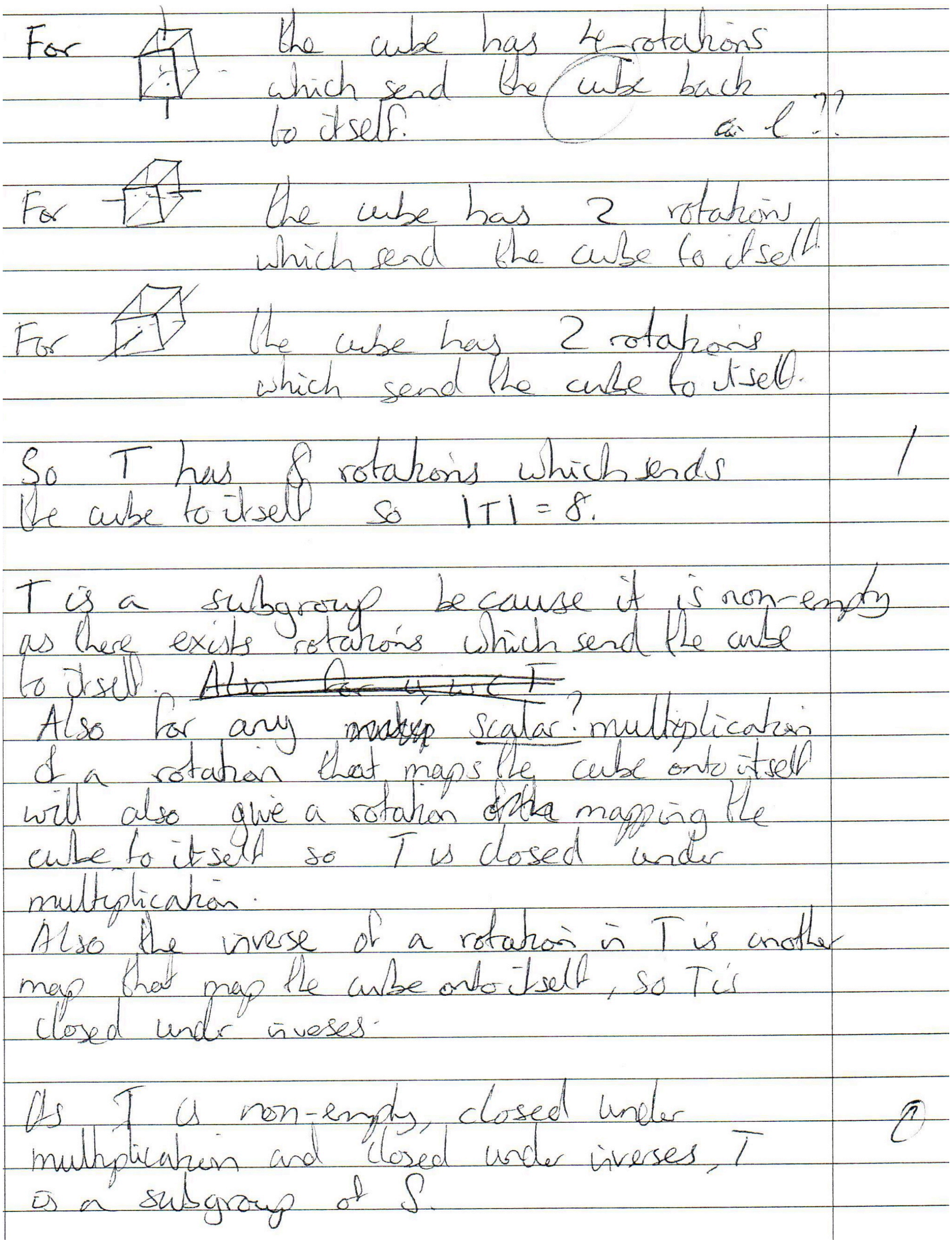
As seen above, many sentences start with “you can...”, almost as if instructing someone to follow his suggestions. He uses two different kinds of visual mediators with success: illustrations of cubes and table of permutations. At the end of the first part he summarised all the findings in a table. This linguistic syntax suggests that in this specific exercise Manrico has fully objectified the d-objects and has also achieved subjectification. The latter involves a change of the discursive focus from the object to the performer and his/her actions.

Apart from the more serious errors, described so far, there have occurred certain inaccuracies, similar to the ones described in section 4.2. For instance, Norina’s solution of CS1E6 is overall correct apart from certain instances of problematic word use. Instead of saying ‘edges’ she writes ‘sides’. As seen in the excerpt below, Norina particularly uses verbalisation in order to express her reasoning, a characteristic apparent in all her scripts. This characteristic possibly expresses her need to communicate clearly her reasoning, indicating possibly confidence and security in her object-level understanding, yet perhaps less rigorously than the established metarules would entail.


T is a subgroup because it is non-empty as there exists rotations which send the cube to itself. ~~Also for x, y ∈ T~~
Also for any ~~matrix~~ scalar? multiplication of a rotation that maps the cube onto itself will also give a rotation ~~of the~~ mapping the cube to itself so T is closed under multiplication.
Also the inverse of a rotation in T is another map that map the cube onto itself, so T is closed under inverses.

As T is non-empty, closed under multiplication and closed under inverses, T is a subgroup of S.

Unlike her coursework, in the exam Norina fails to apply the routine of proving that the given set is a subgroup. As seen below, she made an effort to apply the routine, yet her reasoning is not explicit. This might suggest unstable object-level understanding regarding the involved d-object, although this is merely a hypothesis. Additionally there are indications of problematic word use, a result of problematic metaphors from Linear Algebra, e.g. scalar multiplication of a rotation.



For [cube] the cube has 4 rotations which send the cube back to itself. all ??

For [cube] the cube has 2 rotations which send the cube to itself.

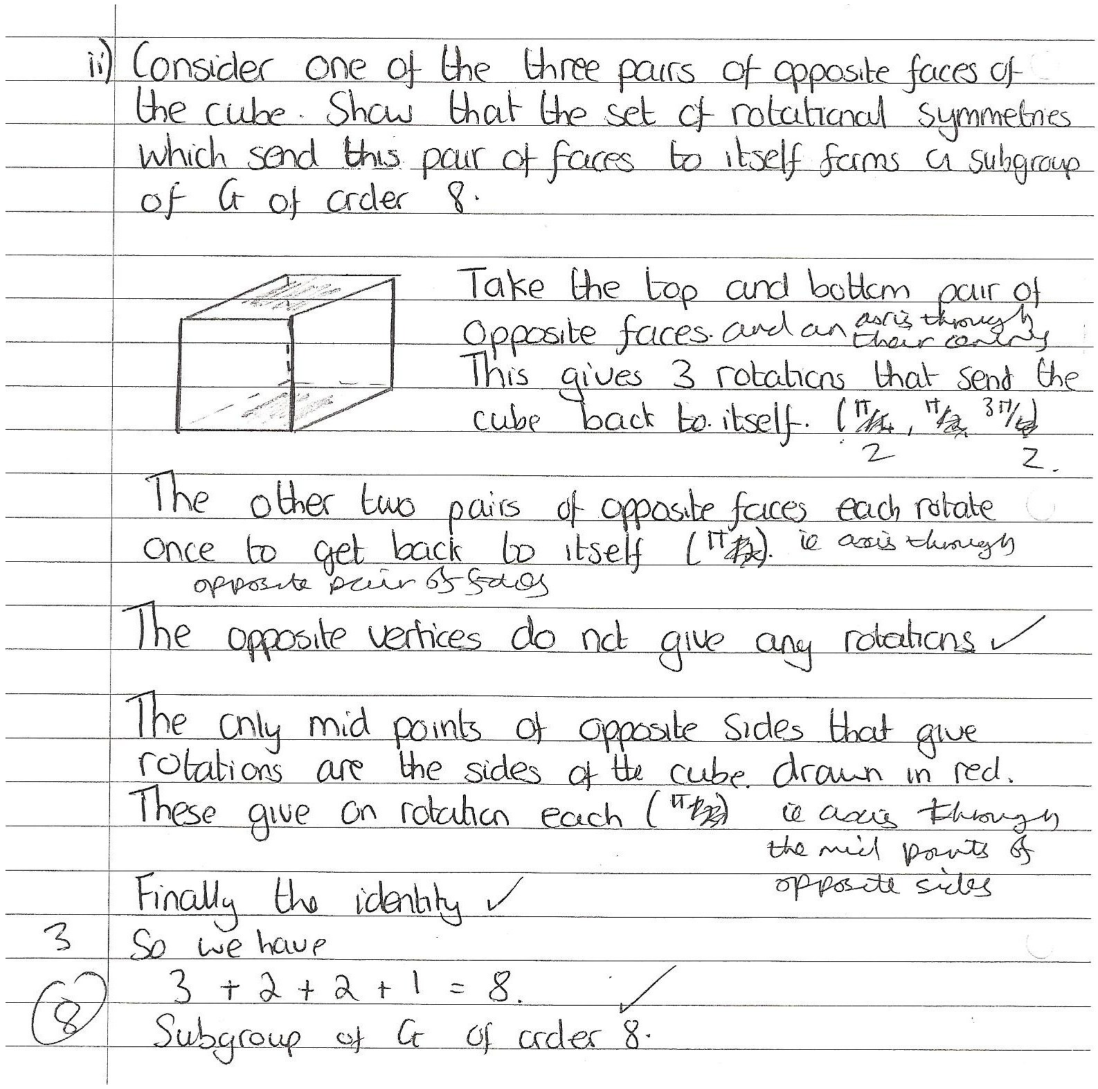
For [cube] the cube has 2 rotations which send the cube to itself.

So T has 8 rotations which sends the cube to itself so |T| = 8.

T is a subgroup because it is non-empty as there exists rotations which send the cube to itself. ~~Also for u, v ∈ T~~
Also for any ~~matrix~~ scalar? multiplication of a rotation that maps the cube onto itself will also give a rotation of the mapping the cube to itself so T is closed under multiplication.
Also the inverse of a rotation in T is another map that map the cube onto itself, so T is closed under inverses.

As T is non-empty, closed under multiplication and closed under inverses, T is a subgroup of S.

Another instance of problematic use of notation regarding the π terms and consequently the description of rotations appeared in Norma's solution of the second part of CS1E6. Similar to her solution of the first part of the same exercise, Norma used an efficient illustration to support her solution, showing the ability to use one of the characteristics of mathematical discourse, i.e. visual mediators. Yet, the use of notation regarding the π terms is wrong due to problematic metaphors from geometry and trigonometry. In addition, her solution indicates problematic encounter with the application of the required routine and the application of the governing metarules, since she does not follow all the steps of the test for proving that a set is a subgroup closely and with precision. In this case, I believe that not being explicit does not necessarily imply that her understanding is flawed; as the rest of her answers suggests.


ii) Consider one of the three pairs of opposite faces of the cube. Show that the set of rotational symmetries which send this pair of faces to itself forms a subgroup of G of order 8.

Take the top and bottom pair of opposite faces and an axis through their centres. This gives 3 rotations that send the cube back to itself. ($\pi/2$, π, $3\pi/2$)

The other two pairs of opposite faces each rotate once to get back to itself (π). ie axis through opposite pair of sides

The opposite vertices do not give any rotations ✓

The only mid points of opposite sides that give rotations are the sides of the cube drawn in red. These give on rotation each (π) ie axis through the mid points of opposite sides

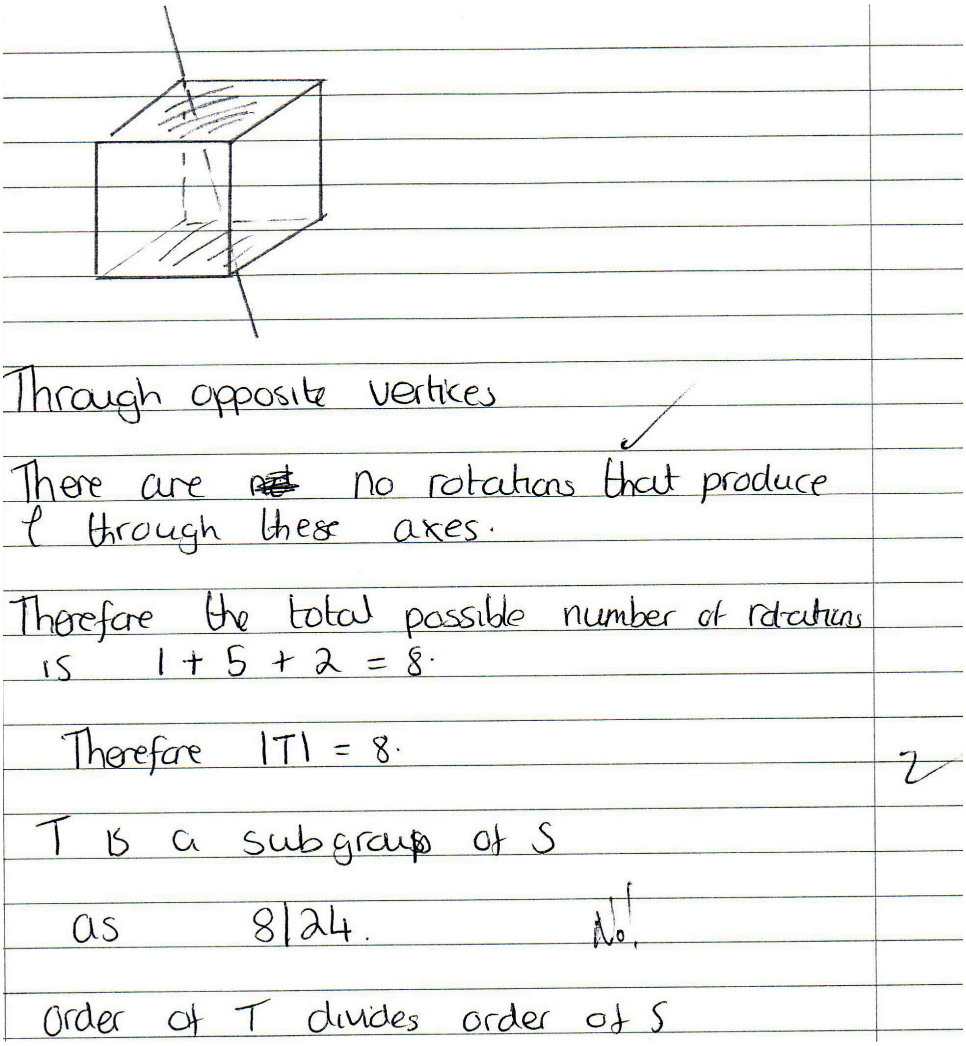
Finally the identity ✓

3 So we have

(8) 3 + 2 + 2 + 1 = 8. ✓

Subgroup of G of order 8.

In the examination, Norma seems to have improved her object-level and metalevel understanding, and has overcome the errors that occurred above, such as the problem with the π factors and more accurate verbal description of the rotations. She used excellent visual mediators. The only objection the marker had related to the statement that "T is a subgroup of S as 8/24", as seen below.



Through opposite vertices

There are ~~not~~ no rotations that produce e through these axes.

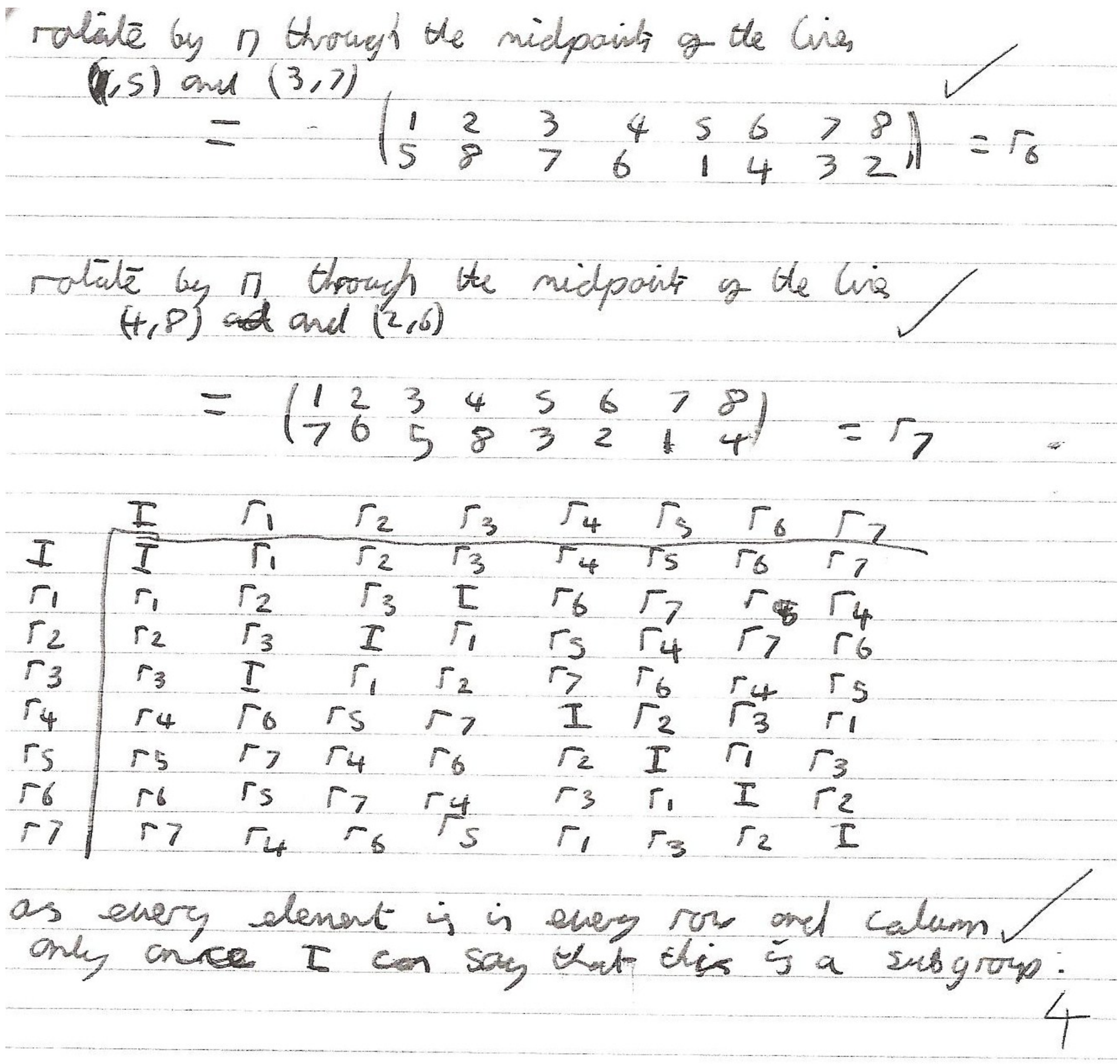
Therefore the total possible number of rotations is 1 + 5 + 2 = 8.

Therefore |T| = 8.

T is a subgroup of S as 8|24. No!

Order of T divides order of S

Apart from discussing and analysing the errors, it would be interesting, particularly from a metalevel viewpoint, to mention an innovative approach to solving this mathematical task. Tamino produced an excellent and innovative answer in the coursework, with excellent use of mathematical vocabulary and notation, including excellent permutation presentation and a unique group table representing the symmetries of the subgroup. (Only Leonora did something similar.) His solutions indicate good object-level understanding of the multiplication tables of groups, as well as metalevel understanding since by using this approach of solving this task was still convincing and his solution was accepted as accurate by the marker. As the excerpt below also demonstrates, there is a good use of diagrams and metaphors from other mathematical discourses and a tendency to reduce the level of abstraction by relying on concrete examples of groups.



rotate by π through the midpoint of the lines (1,5) and (3,7)

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 8 & 7 & 6 & 1 & 4 & 3 & 2 \end{pmatrix} = r_6$$

rotate by π through the midpoint of the line (4,8) and (2,6)

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 6 & 5 & 8 & 3 & 2 & 1 & 4 \end{pmatrix} = r_7$$

	I	r_1	r_2	r_3	r_4	r_5	r_6	r_7
I	I	r_1	r_2	r_3	r_4	r_5	r_6	r_7
r_1	r_1	r_2	r_3	I	r_6	r_7	r_5	r_4
r_2	r_2	r_3	I	r_1	r_5	r_4	r_7	r_6
r_3	r_3	I	r_1	r_2	r_7	r_6	r_4	r_5
r_4	r_4	r_6	r_5	r_7	I	r_2	r_3	r_1
r_5	r_5	r_7	r_4	r_6	r_2	I	r_1	r_3
r_6	r_6	r_5	r_7	r_4	r_3	r_1	I	r_2
r_7	r_7	r_4	r_6	r_5	r_1	r_3	r_2	I

as every element is in every row and column only once I can say that this is a subgroup.

4

Unlike the coursework, he did not manage to prove the second question about the subgroup of order eight in the exam. This is perplexing since in the coursework he demonstrated a solution of excellent standards. This is possibly an indication of regression of performance in the final examination.

Finally, when analysing the interview excerpts regarding CS1E6, two interesting trends emerged. Firstly, students who got high marks in the coursework, such as Norina, Norma and Amelia, expressed how difficult they found this exercise. The other students though considered this as the most accessible and tangible one.

> *So I didn't look at question 6 before the seminar, and didn't ask about it, and then we kind of did a bit in the lecture… I still didn't understand, and then I went through it with him, there was about four of us, and we were sitting there with a cube, trying to spin it round on the corners, and I just can't do stuff like that.* Amelia

Secondly, students expressed their need to visualise as well as their difficulty to do so. Most of the students had created a cube in order to be able to find the rotations.

> *Yeah, we were trying to do the last bit yesterday with a fag packet, and it just didn't work because it was rectangular, and I was like, can't do it, don't understand! [Laughs] Yeah, so the first bit was alright, and then you had to work out... the identity thing confused me quite a bit to begin with, but now I've worked out that you only include it once, I understand, um... cos I thought you included it once for every – like they were separate identities? So you know that if you're spinning it with the axes like through the middle, that was one identity, and then you had one through the corner, so I thought that was a different identity? Just kind of silly moment... Um – yeah, this subgroup thing confused a lot of us, though, I think we were – I don't really know if it's just – like this question, I just don't like, because it's got so much writing? It's just all words. I don't know, that just puts me off because I just can't always read it all. Yeah, and I'm not sure if I finished it, because we've written what the three pair – no, we've written like, the eight elements, but we haven't really shown it as a subgroup, and I'm not sure if we have to do that for that question. So I'm not sure if I finished that one either.* Amelia

> *Yeah, it's understand – I – it's just visualizing, is a bit of a – just try and think in your mind, just like, rotating this cube and doing all the sorts of things with it, it's a bit hard to get your head round but once you – I mean I know what to aim for cos I mean it was 24, um...* Manrico
>
> *It – cos he went over it in the lecture, and I thought like – when he was doing it I was like – that makes sense, I can do that, went home and I just can't get my head around it? Like I've like made myself a little cube, and I can't work out, to get it to have like 24 um, (MI: Symmetries) symmetries. I just – I can't do it, I just can't get my head around it, I can't see, how it's meant to actually work...* [...] *So yeah, so that's why I made myself a little cube, so I could actually like, play around with it, and I'm still completely confused, by it, that's like – like I know what I'm doing, I just can't get it to add up, to what it's meant to do! [Both laugh]… So, yeah, I'm struggling with that one. But um...* Norma

The observations and conclusions seem to be in agreement with the markers' more macroscopic report on this exercise regarding the entire group of 78 students.

> Q6 - Most were able to find the 24 rotational symmetries, with marks only lost through insufficient explanation of the angles of rotations and axes (Note: use edges instead of sides, since a side is usually another name for a face). The idea for the subgroup of order 8 was to find a subset of size 8 from the list of 24 you have shown in the earlier part. Some people described flipping the cube, but without saying what this flip was. Normally a flip would describe a reflection. Here we are dealing only with rotations, so this flip must be a rotation and then we need to describe its axis of rotation and angle of rotation.

Text 4.11: Markers' comment 4

In sum, regarding the proof of a subgroup of order eight, certain students' solutions were more problematic than the listing of the 24 symmetries, showing problematic object-level and metalevel understanding in various instances. The fact that some students were not able to find the eight symmetries that satisfy the required conditions or justify that the order is indeed eight, suggests a problematic object-level understanding of the definition of subgroup. It may also indicate desultoriness in the application of the required routine, and the governing metarules, for proving that a structure is indeed a subgroup, when the context of the exercise is different. In fact, as the analysis in sections 4.2 and 4.3 suggests, certain students who had successfully applied the related routine in a more 'algebraic' context in other exercises were not successful in applying the same principles in a more 'geometric' context. This observation possibly suggests that the mathematical learning (both object-level and metalevel) requires extensive experience of solving tasks in different contexts. I conjecture that these contexts should vary with regard to their *nature* (arithmetic, algebraic or geometric), the *field* (Linear Algebra, Complex Analysis, etc.) and the *level of abstraction*. Good object-level or metalevel understanding in one context does not naturally imply good performance or understanding in a different context.

4.4 Equivalence Relations

The fourth category of errors as these occurred in the solution of coursework and final examination was related to the notion of equivalence relations and the application of the routine for proving that a given relation ~ is indeed equivalence, namely reflective, symmetric and transitive.

> Definition: Suppose A is any set. A *relation* ~ on A is specified by a subset of A^2, we write (for $\mathrm{a}, \mathrm{b} \in \mathrm{A}$) $\mathrm{a} \sim \mathrm{b}$ to mean (a, b) in this subset and $\mathrm{a} \nsim \mathrm{b}$ if not. The relation ~ on A is: *reflexive* if $\forall \mathrm{a} \in \mathrm{A}, \mathrm{a} \sim \mathrm{a}$; *symmetric* if $\forall \mathrm{a}, \mathrm{b} \in \mathrm{A}, \mathrm{a} \sim \mathrm{b}$ then $\mathrm{b} \sim \mathrm{a}$; and *transitive* if $\forall \mathrm{a}, \mathrm{b}, \mathrm{c} \in \mathrm{A}$, if $\mathrm{a} \sim \mathrm{b}$ and $\mathrm{b} \sim \mathrm{c}$, then $\mathrm{a} \sim \mathrm{c}$. A relation ~ on a set A is called an *equivalence relation* on A, if it is reflexive, symmetric and transitive. If ~ is an equivalence relation on A, then a subset of A of the $[b]^{\sim} = \{a \in A : b \sim a\}$ (for $b \in A$) is called a ~-*equivalence class*.

In CS2E3 students had to define a relation ~ on a set and prove that this is an equivalence relation.

3. Suppose X is a non-empty set and $G \leq \mathrm{Sym(X)}$. Define a relation $\sim$ on X by:

$$x \sim y \Leftrightarrow \text{ there exists some } g \in G \text{ with } g(x) = y.$$

Prove that $\sim$ is an equivalence relation on X.
(Remark: The equivalence classes are the G-orbits, as in question 2.)

Text 4.12: CS2E3

Similarly, in FEE4ii students had to state the definition of equivalence relation, prove that the given relation is equivalent as well as prove that the size of equivalence classes is $|H|$, where H is a subgroup, in the case the group G is finite.

(ii) Suppose G is a group and H a subgroup of G. Prove that the relation $\sim$ on G given by

$$g_1 \sim g_2 \text{ if and only if } g_1^{-1}g_2 \in H$$

is an *equivalence relation*, saying carefully what this means. In the case where G is a finite group, prove that all equivalence classes have $|H|$ elements. **[7 marks]**

Text 4.13: FEE4ii

There have emerged three categories of errors regarding *the difference between the group, the set and their elements,* the *size of equivalence classes,* and the proof of *transitivity and symmetry*. As in section 4.3, since there have been set similar tasks both in the coursework and the final examination, it is possible to pursue signs of regression or improvement in the student performance. In this case, the analysis of the student solutions indicates regression of performance in the examination.

4.4.1 Elements of the Group $\mathbf{Sym(X)}$ Versus Elements of the Set $\mathbf{X}$

The first major error the occurred in the student solutions regarding the equivalence relations is related to the distinction between the elements of the set X and the elements of the group $Sym(X)$, when these coexist in the same context. Six out of the eleven students' solutions (6/11), (Musetta and Francesca did not attempt the exercise at all neither in the coursework nor the exam), indicated serious problems in their object-level understanding of the d-objects of group and set, since they applied the group axioms on the elements of the set. For instance they were trying to define and use the inverse of a set element. Naturally, this is impossible since there is no defined operation in the set. This type of errors suggests that students have not yet objectified the d-object of group and have fully identified the differences between the set and the group in the object-level learning. Furthermore, they have possibly not yet fully objectified the dual character of the d-object of group. As the following analysis will suggest, there are indications that the role of binary operation is not central to their object-level understanding. Below I discuss all the emerging errors giving representative examples.

An example of problematic object-level understanding regarding the elements of the set X and the elements of the group $Sym(X)$ was apparent in Calaf's solution of CS2E3.

a ∈ X so a^{-1} has no meaning as no operations are defined on X. The only group we have is G which is permutations on X.

let $g \in G : g(a) = a$

$g^{-1}(a)$ ~~$\times a^{-1} = a$~~ $(a \in X)$

Hence $g^{-1}(a) \in H$

$g^{-1}g = e$
$\Rightarrow (g^{-1}g)(a) = e(a)$
$\Rightarrow g^{-1}(g(a)) = a$
$\Rightarrow g^{-1}(a) = a$.

So H is subgroup of G

(2) Why is H non-empty?

(3) • let $g \in G$ with $g(x) = y$

Consider $g(y) = g(g(x)) = g(x)$ why? (No! Question 2 part i)

No this does not tell us this

$= y$

But what we want to show is $\exists \bar{g} \in G$ such that $\bar{g}(x) = x$

Hence the ~ is reflexive. Since then $x \sim x$ so why bring in y??

• If $x \sim y \Rightarrow g(x) = y$

$g(y) = g(g(x)) = g(x) = x$ (question 2 part i No!)

~~Hence~~ $\Rightarrow y \sim x$

No this is not necessarily g which gives $g(y) = x$. In fact it is $g^{-1}(y) = x$ so only time is $g^{-1} = g$.

So the ~ is symmetric

• let $x, y, z \in X$

If $x \sim y \Rightarrow g(x) = y$
and $y \sim z \Rightarrow g(y) = z$

Again not necessarily the same g here.

~~So~~ $\Rightarrow g(g(x)) = z$
$g(x) = z$
$\Rightarrow x \sim z$

No! (From question 2(i) $g(a) = a$

Have a look at solutions - you seem to have misunderstood the idea.

So the ~ is transitive

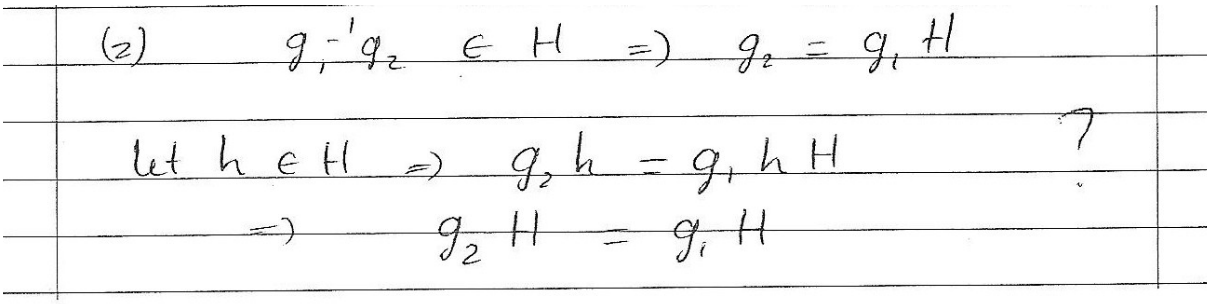
Hence ~ is an equivalence relation on X by definition

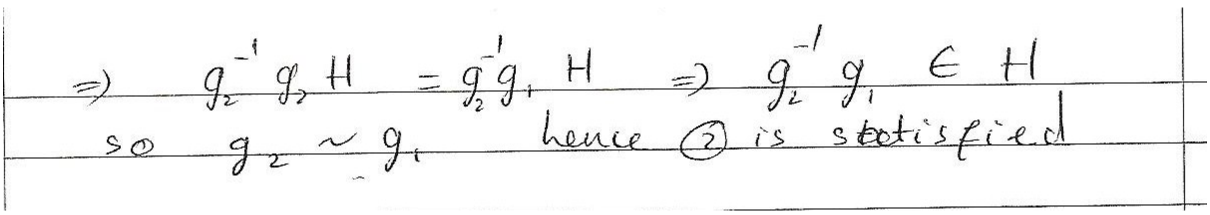
As seen in the excerpt above, Calaf's solution indicates problematic object-level understanding of the d-object of group and its axioms. For instance, he uses the inverse of an element of the set. This is not valid, since in the d-object of a set, there is no operation defined. In addition, another indication of problematic object-level understanding occurs when he writes "Let $g \in G$ with $g(x) = y$, consider $g(y) = g\big(g(x)\big) = g(x)$ ". This contradicts the initial impression given in CS2E2i, according to which Calaf had strong object-level understanding of the initial condition, namely that $g(a) = a$. Although, he seems aware of the definition of equivalence relation and the routine for proving that a relation is in fact equivalent (since he attempts to prove symmetry, transitivity and reflexivity), it seems that he has difficulty of applying it in practice. This indicates that he still is not capable of applying the metarules governing the routine for proving equivalence relations. Possibly one of his major obstacles is that he has not yet objectified the notion of group operation, including several aspects, i.e. how inverses work, and how elements as such can be manipulated.

A problematic objectification of the group operation has possibly an unfavourable effect on the application of metarules regarding the routine for proving equivalence relations, since often does not allow students to use the group axioms efficiently. Similar to Calaf's solution, in another five students' solutions of CS2E3, namely Amelia, Tamino, Dorabella, Norma and Norina, there were problems in the application of the group axioms, despite the fact that overall they had shown good object-level understanding of the three characteristics of equivalence relations. This difficulty does not necessarily imply that students have not fully grasped the definition of the d-object of group, as such. I rather conjecture that when, in the context of the same mathematical task, the two d-objects, namely a set and a group coexist, students are often unable, at this initial stage of their learning, to fully distinguish them and treat them accordingly.

Due to the aforementioned conjecture, some students had problems in applying the metarules governing equivalence relations, especially in the proof of transitivity and symmetry, as will be exemplified below. There are indications of a problematic encounter with the *how* of the routine for proving that a relation is equivalent. In fact this is the first exercise in which the level of performance was significantly lower, as the coursework and the examination results suggest. Comparing the results of the two assessments, there were indications of progress in one student, five students remained at the same level, and five students showed signs of regression.

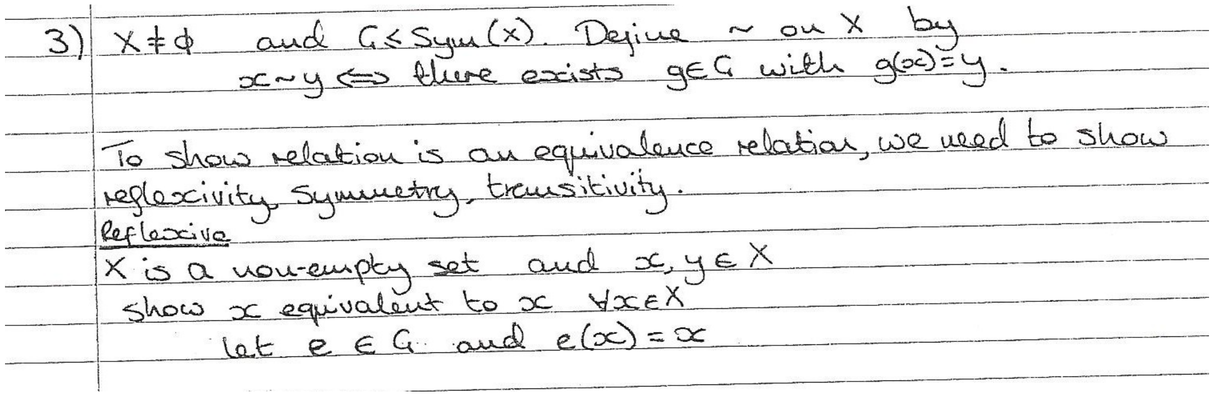
A second type of error that emerged in proving that a relation is equivalent, yet it probably has a more general appeal, is related to the role of group and its elements. According to the following representative example, for the second part of FEE4ii, Calaf has successfully stated the definition of equivalence relations, with one error: instead of writing $\forall a \in G$he writes$\forall a \in g$. Apparently the use of notation, even in this later stage in the Group Theory module is problematic. This possibly indicates problematic object-level understanding of the d-object of group and its elements and moreover he is not yet in a position to distinguish them. Regarding reflexivity, instead of writing that $g_1 \sim g_2$ is reflexive since $g_1^{-1} g_1 = e \in H$ he writes that $e(g_1) = g_1$. He treats g_1 as a variable of a function. He did not attempt to prove transitivity; in symmetry though he predominantly uses the expression gH, instead of concentrating on the proof of the relation, i.e. if $g_1 \sim g_2$ then $g_1^{-1} g_2 \in H$ so as $H \leq G, (g_1^{-1} g_2)^{-1} = g_2^{-1} g_1 \in H$, i.e. $g_2 \sim g_1$.


(2) $g_1^{-1} g_2 \in H \Rightarrow g_2 = g_1 H$
let $h \in H \Rightarrow g_2 h = g_1 h H$?
$\Rightarrow g_2 H = g_1 H$


$\Rightarrow g_2^{-1} g_2 H = g_2^{-1} g_1 H \Rightarrow g_2^{-1} g_1 \in H$
so $g_2 \sim g_1$ hence (2) is satisfied

Similar problems, related to the role of group and its elements, appeared in six students' solutions, namely, apart from Calaf, in Kostanza, Dorabella's, Leonora's, Manrico's and Norma's.

Another example of errors that indicates combination of both problematic metalevel and object-level understanding was seen in Dorabella's solution of CS2E3. Although initially her solution indicates good object-level understanding by stating the three characteristics of equivalence, and that she has well objectified the definition of equivalence relations, yet her attempt later on indicates problematic object-level understanding of the d-object of map and certain issues regarding the application of metalevel rules. She successfully proves reflexivity using the identity element. Regarding symmetry she does not use the group element correctly and her notation shows a problematic object-level understanding of the elements of X and $Sym(X)$. Similar problems occur in her attempt to prove transitivity. Similar errors have occurred in two other students namely Manrico and Kostanza.


3) $X \neq \phi$ and $G \leq Sym(x)$. Define $\sim$ on X by

$x \sim y \iff$ there exists $g \in G$ with $g(x) = y$.

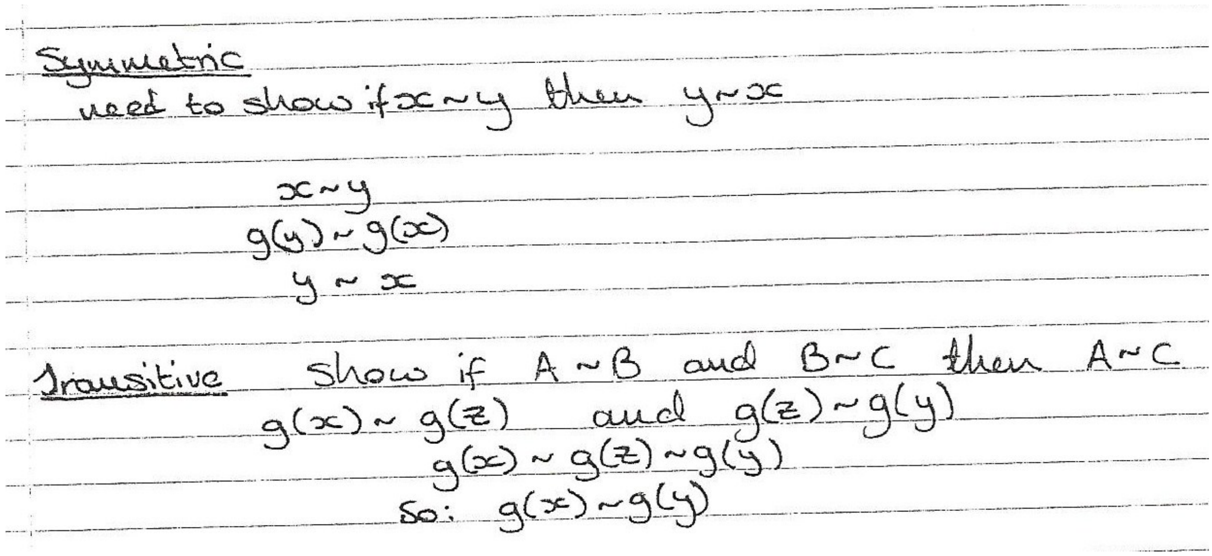
To show relation is an equivalence relation, we need to show reflexivity, symmetry, transitivity.

Reflexive

X is a non-empty set and $x, y \in X$

show x equivalent to x $\forall x \in X$

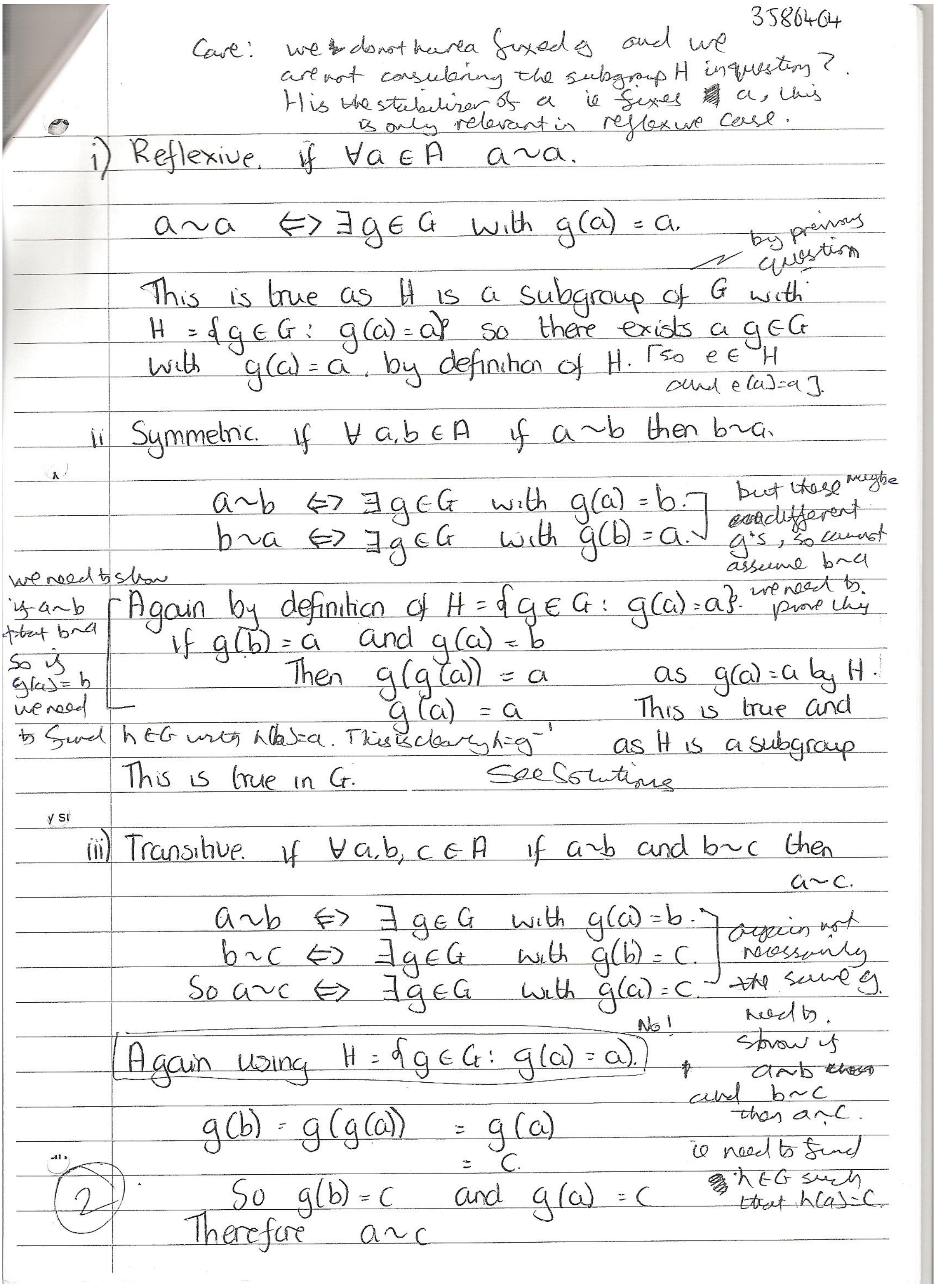
let $e \in G$ and $e(x) = x$


Symmetric
need to show if $x \sim y$ then $y \sim x$

$x \sim y$
$g(y) \sim g(x)$
$y \sim x$

Transitive show if $A \sim B$ and $B \sim C$ then $A \sim C$
$g(x) \sim g(z)$ and $g(z) \sim g(y)$
$g(x) \sim g(z) \sim g(y)$
So: $g(x) \sim g(y)$

Another example of problematic encounter with equivalence relations appeared in Norma's solution of CS2E3. Her object-level understanding of equivalence relations appears to be problematic, with negative consequence in the application of the governing metarules for proving that a relation is equivalent. Her object-level understanding seems to be problematic from various aspects. For instance she does not seem to have objectified the Orbit-Stabiliser Theorem, given as a hint in the problem sheet. The reason of this problematic objectification is that she has possibly not understood the definitions of the involved d-objects, namely what the stabiliser and orbit means, as well as the d- object of subgroup, as her solution later reveals. Her use of notation, in particular her use of $g's$, and the fact that she has omitted to distinguish the different $g's$, it probably indicates that her object-level understanding of the elements of the group is not yet clear. This misunderstanding is obvious in the entire solution. Problematic object-level understanding is linked in this case with problematic application of the amenable metarules. In particular, in order to prove reflexivity, she used the data in the previous exercise i.e. $a \in X \, and \, H = \{g \in G: g(a) = a\}$ as suggested by LCR, but she tried to use it to prove symmetry and transitivity as well, which is not acceptable by the marker. Again, Norma has still not a clear view of the *how* of the routine, which involves the use of group and set elements in order to prove that $\sim$ is an equivalence relation on X.


3586404

Care: we do not have a fixed g and we are not considering the subgroup H in question? H is the stabilizer of a ie fixes a, this is only relevant in reflexive case.

i) Reflexive. if $\forall a \in A$ $a \sim a$.

$a \sim a \iff \exists g \in G$ with $g(a) = a$. — by previous question

This is true as H is a subgroup of G with $H = \{g \in G : g(a) = a\}$ so there exists a $g \in G$ with $g(a) = a$, by definition of H. [so $e \in H$ and $e(a) = a$].

ii Symmetric. if $\forall a, b \in A$ if $a \sim b$ then $b \sim a$.

$a \sim b \iff \exists g \in G$ with $g(b) = b$.
$b \sim a \iff \exists g \in G$ with $g(b) = a$. — but these maybe different g's, so cannot assume $b \sim a$

we need to show if $a \sim b$ that $b \sim a$ So if $g(a) = b$ we need to find $h \in G$ with $h(b) = a$. This is clearly $h = g^{-1}$

Again by definition of $H = \{g \in G : g(a) = a\}$. — we need to prove this
if $g(b) = a$ and $g(a) = b$
Then $g(g(a)) = a$ as $g(a) = a$ by H.
$g(a) = a$ This is true and as H is a subgroup
This is true in G. See Solutions

iii) Transitive. if $\forall a, b, c \in A$ if $a \sim b$ and $b \sim c$ then $a \sim c$.

$a \sim b \iff \exists g \in G$ with $g(a) = b$.
$b \sim c \iff \exists g \in G$ with $g(b) = c$.
So $a \sim c \iff \exists g \in G$ with $g(a) = c$. — again not necessarily the same g.

No!

Again using $H = \{g \in G : g(a) = a\}$. — need to show if $a \sim b$ and $b \sim c$ then $a \sim c$. ie need to find $h \in G$ such that $h(a) = c$.

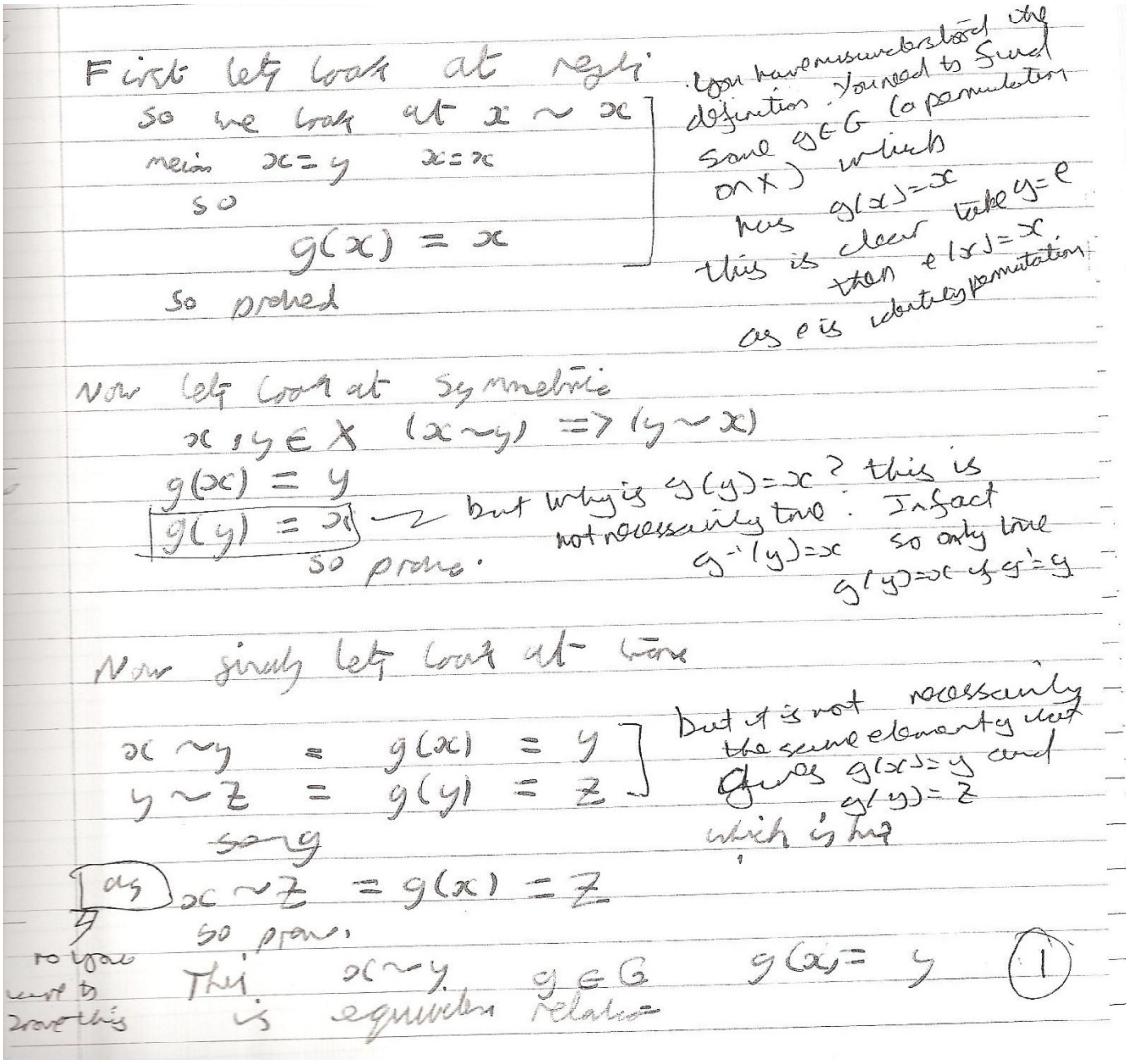
$g(b) = g(g(a)) = g(a) = c$.

So $g(b) = c$ and $g(a) = c$
Therefore $a \sim c$

(2)

It would be interesting to mention that Norma was probably the only student that regarding equivalence relations has shown progress in her performance in the exams, having resolved all problems in her object-level understanding and the application of the metarules that occurred in the coursework. The rest of the students either remained at the same level of performance (Otello, Calaf, Carmen, Tamino, Amelia) or showed signs of regression (Dorabella, Kostanza, Leonora, Norina, Manrico).

Problematic understanding with negative consequences in the application of the governing metarules appeared also in Tamino's coursework and examination solutions. In CS2E3 there are indications of a problematic object-level understanding regarding the definition of equivalence relations as well as the difficulty to distinguish the elements of X and the elements of $Sym\ (X)$.


First let look at refl
so we look at x ~ x
mean x = y x = x
so
g(x) = x
so proved

You have misunderstood the definition. You need to find some g ∈ G (a permutation on X) which has g(x) = x this is clear take y = e then e(x) = x, as e is identity permutation

Now let look at symmetric
x, y ∈ X (x ~ y) => (y ~ x)
g(x) = y
g(y) = x
so prove.

but why is g(y) = x? this is not necessarily true. In fact g⁻¹(y) = x so only true g(y) = x if g⁻¹ = g

Now finally let look at trans
x ~ y = g(x) = y
y ~ z = g(y) = z
so g
as x ~ z = g(x) = z
so prove.

But it is not necessarily the same element g that gives g(x) = y and g(y) = z which is key

This x ~ y g ∈ G g(x) = y is equivalence relation

so you want to prove this

1

As the excerpt above suggests, although he seems to know the various steps of the routine to be applied, namely prove reflexivity, symmetry and transitivity, there are particular problems with the d-object of permutation g

and how this acts on x. Moreover, there are problems on a metadiscursive level resulting problematic application of the governing metarules. For instance, throughout his proof he assumes certain overgeneralized claims that are not necessarily true in certain instances. In addition, regarding the application of metarules, he assumes the truth of statements without adequately proving them. In general his proving strategy seems to be partial since, according to the marker, there were steps missing in his proof. For instance he assumes that $x \sim y = g(x) = z$, a narrative which should be proven. These errors indicate that Tamino has not yet achieved metalevel learning regarding equivalence relations. In fact his regressive performance in the final examination indicates that Tamino has not overcome these difficulties.

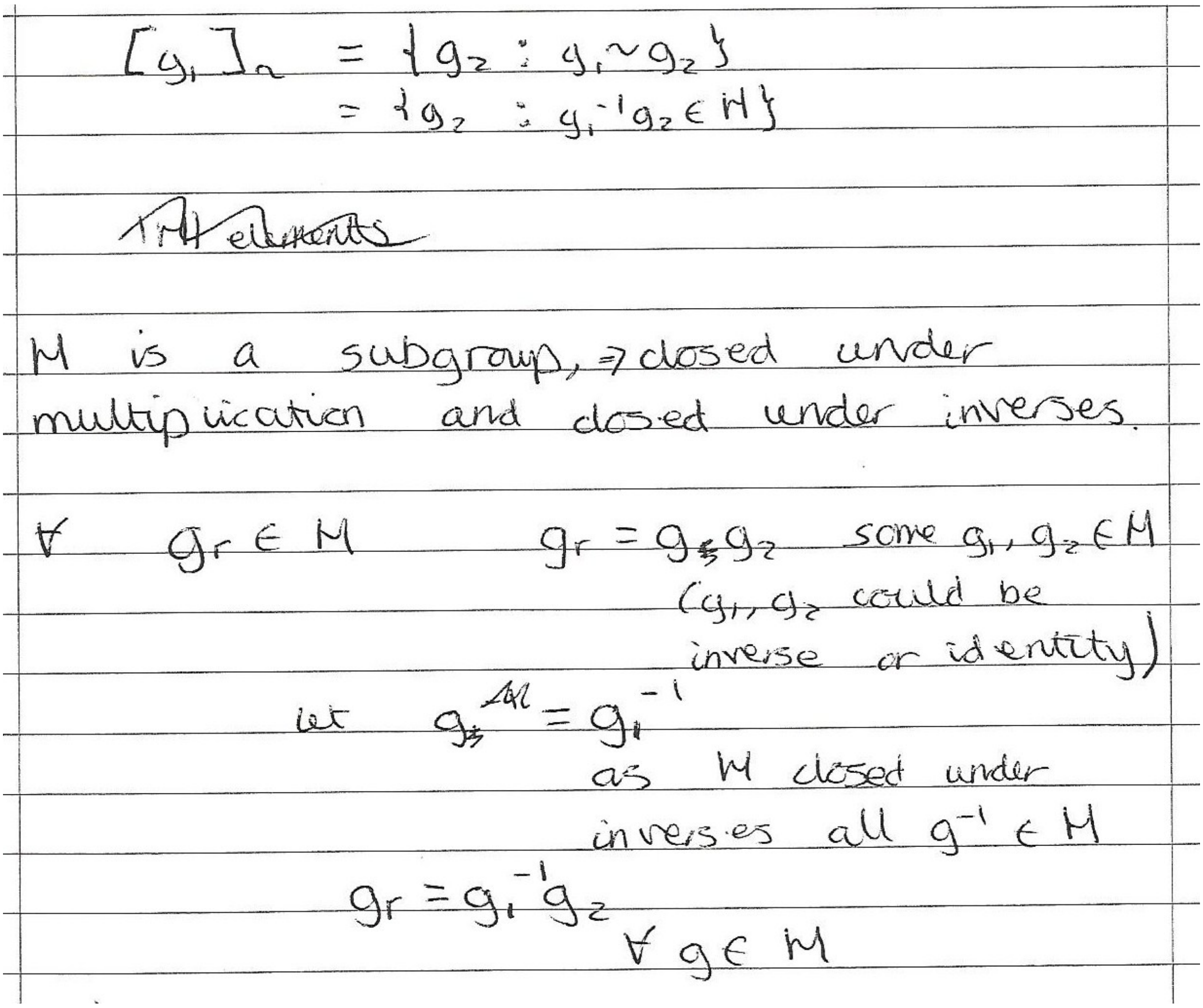
The above analysis possibly suggests that metalevel understanding is a dynamic and multilevel process. For instance, Tamino's metalevel understanding allows him to structure his proof correctly, by attempting to prove the three characteristics of equivalence relations, a more microscopic analysis reveals problematic application of metarules involving overgeneralized claims, missing steps, and ungrounded assumptions.

4.4.2 Elements of Equivalence Classes

A second category of errors regarding equivalence relations is related to the last part of FEE4ii asking the students to prove that if G is a group and Ha subgroup, all equivalence classes have $|H|$ elements. An equivalence class is of the form $\{g_2: g_1 \sim g_2\}$ for some$g_1 \in G$. By definition, this is $\{g_1 h: h \in H\}$. The map H→ $\{g_1 h: h \in H\}$, i.e. $h \mapsto g_1 h$ is a bijection, therefore the number of elements in the set is$|H|$. Since, the following discussion is based on the students' solution of an examination exercise, I had no interview data to strengthen my claims.

Of the eleven students that chose to do FEE4, four did not attempt this part at all, and only seven (7/11) attempted to solve it, with only one, Otello, producing a flawless solution. Namely, they attempted to solve it, Amelia, Leonora, Manrico, Norma, Norina, Carmen and Otello. The main problem with this part of the exercise is related to students' object-level understanding of the structure and form of equivalence classes. Consequently, these six students, whose solution was problematic, seemed not to be able to define a bijection that would prove that the number of elements in each equivalence class is $|H|$. In order to be able to solve this part of the exercise, one should have already concrete object-level understanding of the structure of equivalence classes, objectified all the previously given definitions of the involved d-objects, and overcome any problematic object-level understanding regarding the object of equivalence relation, bijection, order of group as well as group operations. As the analysis below possibly suggests, an important reason of students' difficulty with this exercise is the incomplete object-level understanding of the definition of equivalence classes. Again, this error results from the problematic object-level understanding of the distinction between the set and their group and consequently the distinction of their elements. Below, I discuss through examples the errors that occurred in relation to this issue.

A common error in the proof of the size of equivalence classes is the omission of defining a bijection in order to prove that the size of equivalence classes is $|H|$. For instance, although Amelia has successfully solved the first part of FEE4ii, regarding the proof of equivalence relation, showing very good object-level and metalevel understanding, and consciousness in the application of the related routine, yet she is still not able to prove that all equivalence classes have $|H|$ elements. There is no indication of attempting to define a bijection that would lead her to prove it, albeit the fact that her use of mathematical vocabulary and notation is excellent. She is the only student that uses the more 'sophisticated' notation, used by the lecturer, i.e. $[g_1]_\sim = \{g_2 : g_1 \sim g_2\}$. This suggests that Amelia has read the lecture notes thoroughly and has understood and adopted the Group Theory discursive vocabulary used by the lecturer. The above analysis indicates that Amelia has the potential to endorse all the involved object-level rules regarding the equivalence relations as well as the routine's governing metarules, yet she is still in the process of doing so.



$[g_1]_\sim = \{g_2 : g_1 \sim g_2\}$
$= \{g_2 : g_1^{-1} g_2 \in H\}$

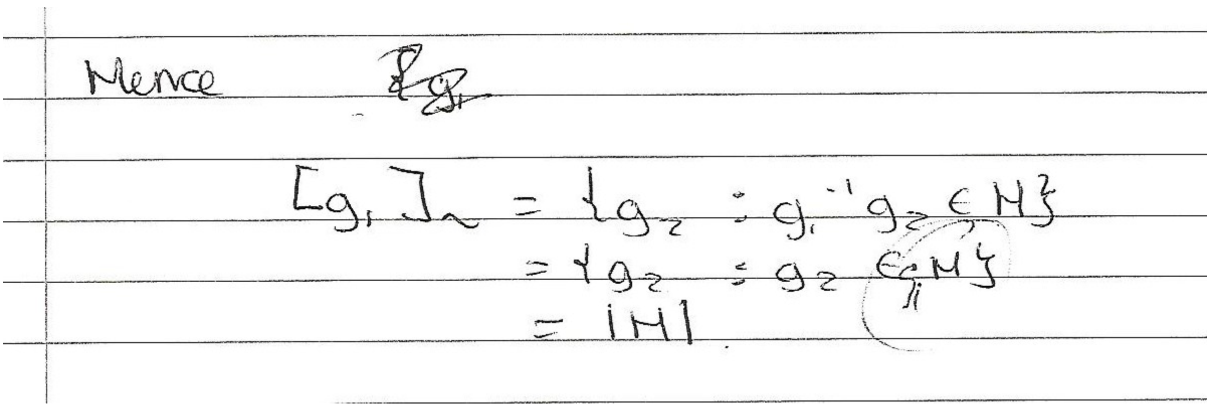
~~TAH elements~~

H is a subgroup, ⇒ closed under multiplication and closed under inverses.

$\forall \; g_r \in H \qquad g_r = g_3 g_2$ some $g_1, g_2 \in H$
(g_1, g_2 could be inverse or identity)

let $g_3 = g_1^{-1}$
as H closed under inverses all $g^{-1} \in H$

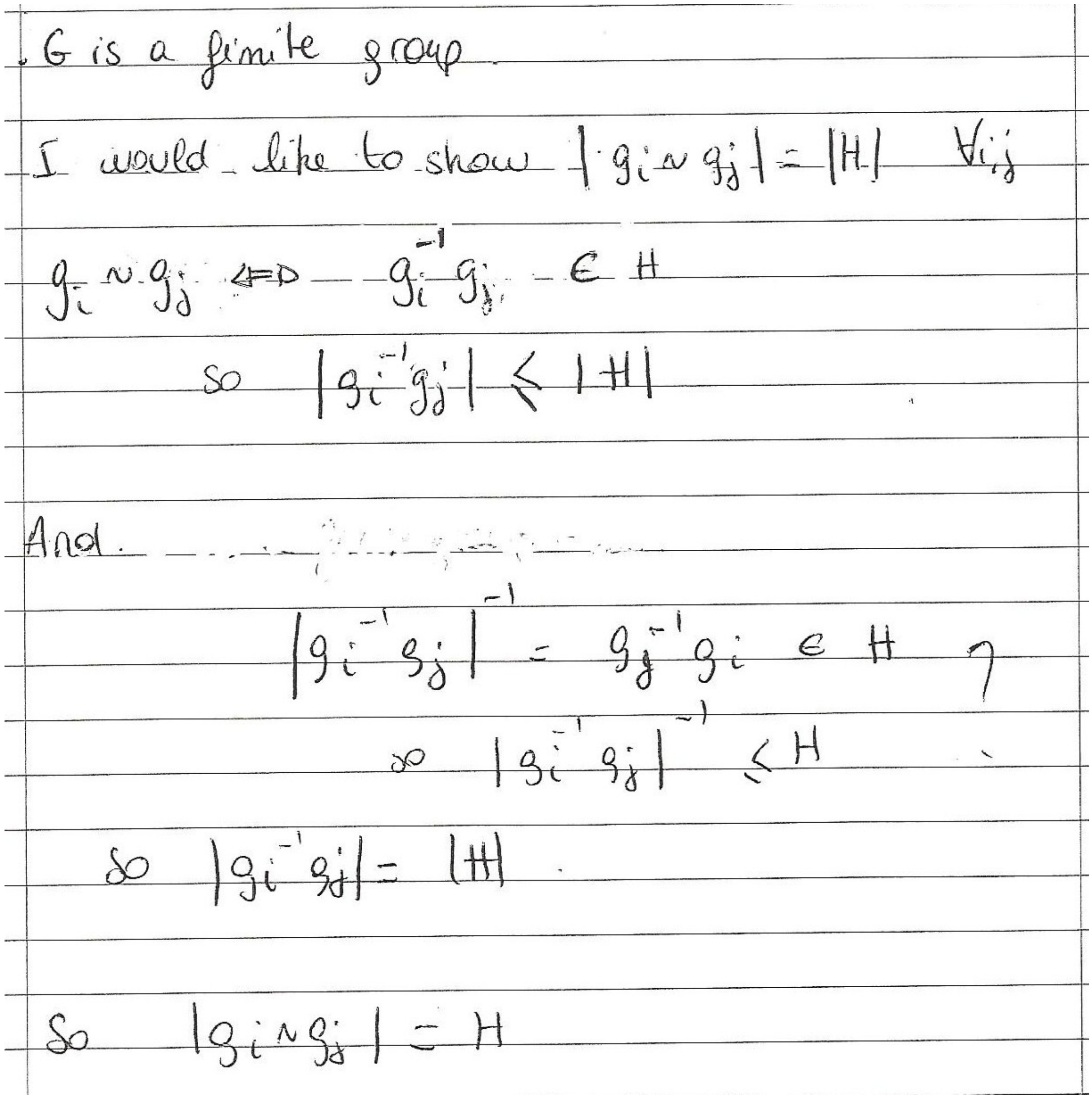
$g_r = g_1^{-1} g_2$
$\forall g \in H$



Hence

$[g_1]_\sim = \{g_2 : g_1^{-1} g_2 \in H\}$
$= \{g_2 : g_2 \in H\}$
$= |H|$

Another example, in which object-level and metalevel understanding are problematic, appeared in Carmen's solution of FEE4. Carmen has not managed to prove that equivalence classes are of size $|H|$. There are indications of problematic object-level understanding regarding the order of the group and the group as such, as the notation in the narratives suggests. For instance she treats $|g_i^{-1}g_j|$ as an element of the group, trying to find its inverse. This indicates incomplete objectification of the definition of the d-object of inverse of an element. In addition, her attempt to solve this exercise lacks explicitness and perspective, as well as clarity in her solving approach. Her proof indicates problematic understanding of *how* she is going to apply the required routine. Similar to Amelia, Carmen is still not able to define a bijection that would allow her to achieve the expected outcome.


G is a finite group

I would like to show $|g_i \sim g_j| = |H|$ $\forall i,j$

$g_i \sim g_j \iff g_i^{-1} g_j \in H$

so $|g_i^{-1} g_j| \leq |H|$

And.

$|g_i^{-1} g_j|^{-1} = g_j^{-1} g_i \in H$?

so $|g_i^{-1} g_j|^{-1} \leq H$

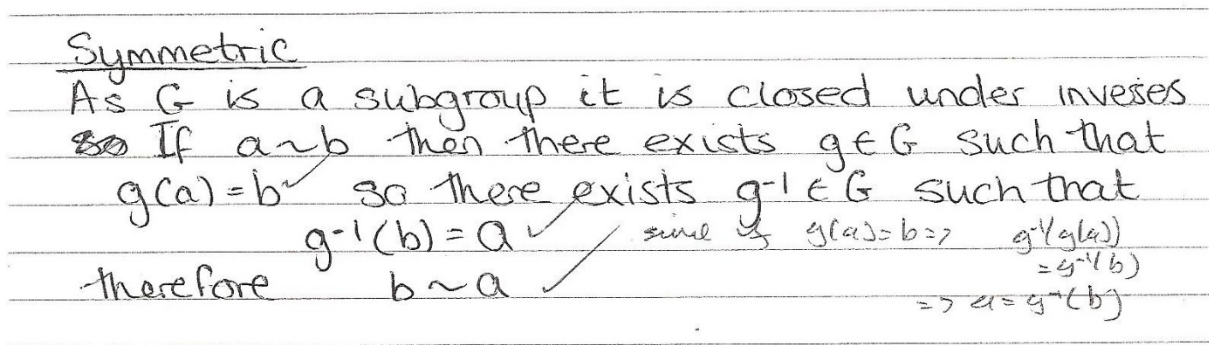
so $|g_i^{-1} g_j| = |H|$

So $|g_i \sim g_j| = H$

4.4.3 Symmetry and Transitivity

The third category is related to minor errors regarding symmetry and transitivity, but not in relation to the previously discussed problems with the set and group elements. Such minor inaccuracies occurred in three out of the thirteen (3/13) students' coursework. I give two representative examples below.

Leonora has successfully proven that ~ is an equivalence relation in CS2E3. There are indications of a good object-level understanding of the definition of equivalence relations as well as some efficiency in applying the metarules in order to prove reflexivity, symmetry and transitivity. Her answer was correct, however her narratives regarding symmetry were not particularly explicit. She was expected to include more detailed mathematical narratives as the markers' comments suggest. These omissions were related to her metalevel understanding of the routine, and the necessity to justify in a precise and rigorous way her mathematical claims. These errors indicate signs of immaturity in the application of the metalevel rules, namely the 'norms' of proving, established amongst the mathematical community.


Symmetric
As G is a subgroup it is closed under inveses
If a~b then there exists g∈G such that
g(a)=b so there exists g⁻¹∈G such that
g⁻¹(b)=a since g(a)=b => g⁻¹(g(a)) = g⁻¹(b) => a=g⁻¹(b)
therefore b~a

Her initial problematic encounter with proving equivalence relations was also reported in her interview, as the following excerpt suggests.

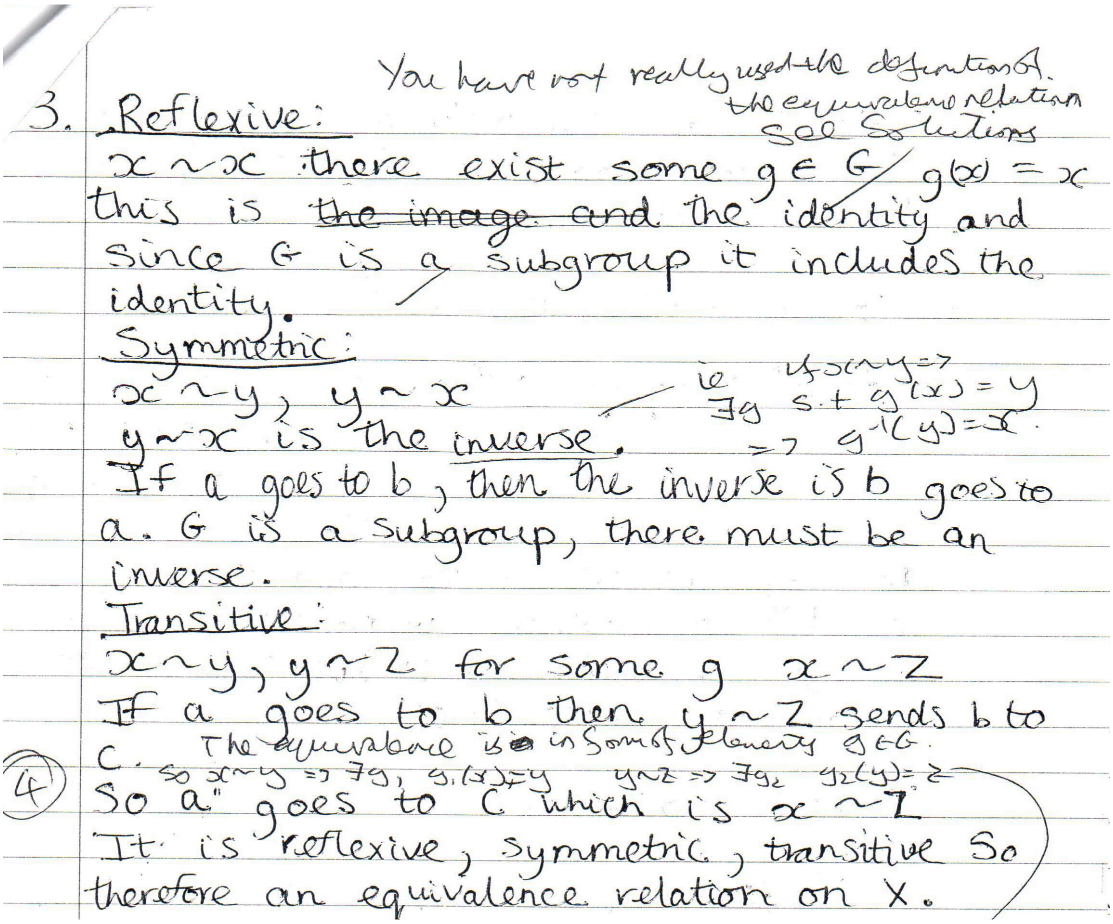
MI: Which concept for you was the hardest, which part of the course was the hardest?

Leonora: Um... the equivalence – stuff – like –

MI: Equivalence relations.

Leonora: It's not – what it is, like… like that, it's just like proving it, like when they ask on the new question sheet that we've got, when it's asking to prove that, I find that – I found that quite hard…

Another representative example of problematic proof of symmetry occurred in Manrico's solution of CS2E3.


3. Reflexive:
$x \sim x$ there exist some $g \in G$ / $g(x) = x$
this is ~~the image and~~ the identity and
since G is a subgroup it includes the
identity.
Symmetric:
$x \sim y$, $y \sim x$
$y \sim x$ is the inverse.
If a goes to b, then the inverse is b goes to
a. G is a subgroup, there must be an
inverse.
Transitive:
$x \sim y$, $y \sim z$ for some g $x \sim z$
If a goes to b then $y \sim z$ sends b to
c.
So a goes to c which is $x \sim z$
It is reflexive, symmetric, transitive So
therefore an equivalence relation on X.

You have not really used the definition of the equivalence relation. See Solutions
ie if $x \sim y$ => $\exists g$ s.t $g(x) = y$ => $g^{-1}(y) = x$.
The equivalence is in some of elements $g \in G$.
So $x \sim y$ => $\exists g_1$ $g_1(x) = y$ $y \sim z$ => $\exists g_2$ $g_2(y) = z$
4

As seen above, he proves reflexivity correctly, demonstrating excellent use of words, syntax and quantifiers, e.g. $\exists g \in G \; such \; that \; g(x) = x$, as well as metalevel awareness of the routine to be applied. However, regarding symmetry, although he correctly states the end result, he does not prove using mathematical narratives i.e. $if \; x \sim y, \exists g \in G: g(x) = y \; \Rightarrow g^{-1}(y) = x$. His narratives are not complete, possibly showing some level of uncertainty. Manrico's narratives regarding equivalence relations indicate a partial object-level and metalevel understanding at this stage. His attempt lacks explicitness, suggesting uncertainty regarding the *how* of the routine. This claim is reinforced by the fact that he asked SLB for guidance in the seminar. As the following vignette shows he had not objectified the definition given in the lecture and needed further explanation.

Seminar B5 - Vignette IV 20:30
Manrico is asking SLB about CS2E3 and the equivalence relation. He says that he is not entirely sure what is reflexive, symmetric and transitive. SLB did not understand what the student asked and Manrico repeated the question. SLB said that what he has to understand is what is the equivalence relation between x and y. SLB says that there is some g that sends x to y. SLB says that he should consider the relation stated in the exercise and that this relation exists because of some element g, which sends from x to y. SLB says that he should check the reflexive equivalence relation and asked him to find such an element. Manrico replies the identity and follows a short discussion. *Manrico sounds confident*. SLB tries to explain to Manrico the three elements of equivalence relations using quite formal language.

Manrico's performance in the examination regarding the equivalence relations in FEE4 indicates regression since he only states the definition of equivalence relation but does not apply the routine to prove the given relation $\sim$ is an equivalence relation on G.

The above issues are in accordance with the comments of SAA and SAB about the coursework of the 78 students.

Q3 - Again at times a poorly done question. Most recognized the need to check that $\sim$ is reflexive, symmetric and transitive. The main problem again was the confusion between the set X and the elements of $Sym(X)$. Elements of X are not elements of $Sym(X)$ and thus if we have $x, y \in X$ to say x has an inverse is not defined and neither is the composition xy, since no operations are defined on the set. We only have operations defined for elements of $G \leqslant Sym(X)$. So for each property (reflexive, symmetric and transitive), the idea was to show that there was always an element of G that fits the bill.

Text 4.14: Markers' comment 5

In sum, there have emerged three categories of errors regarding equivalence relations. The first one is related to the distinction between the elements of sets and elements of groups, and emerged due to problematic object-level understanding of the involved d-objects. The second category is related to the proof of the size of equivalence classes is predominantly due to problematic object-level understanding of the form and structure of the d-object of equivalence classes as well as the d-object of bijection. The third category includes minor errors regarding the proof of symmetry and transitivity.

4.5 First Isomorphism Theorem and Related D-objects

The most serious conceptual difficulties, in the context of this introductory module in Group Theory, have emerged in the last part, in which students were introduced to the First Isomorphism Theorem (FIT) and before that to the prerequisite d-objects, namely the cosets, normal subgroups, quotient groups, isomorphisms, etc.

> First Isomorphism Theorem: Suppose G and H are groups and $\varphi\colon \mathrm{G} \to \mathrm{H}$ is a homomorphism. Then $\frac{\mathrm{G}}{\mathrm{Ker}\varphi} \cong \mathrm{Im}\varphi$.

FIT can be considered as the pinnacle of this introductory module in Group Theory and, consequently, a good object-level understanding of definitions, theorems and routines related to previously encountered d-objects, as well as successful and stabilised application of the amenable metarules is necessary to allow students to successfully cope with it. As the following discussion will show, there have emerged three distinct classes of errors. The first is related to the definition and use of the d-objects of *kernel, image and isomorphism*, the second is related to the d-objects of *coset, normal subgroups and quotient groups* and the third is related to the application of the metarules governing the *First Isomorphism Theorem* as a routine. These errors are the most troublesome and difficult to overcome, as the literature (e.g. Nardi, 2000) and the data analysis that follows suggest.

4.5.1 Kernel, Image and Isomorphisms

Regarding the first category of errors, all thirteen students have shown some degree of problematic object-level understanding with the d-objects of kernel, image and isomorphism.

Definition: Suppose G and H are groups. A function $\varphi: \mathrm{G} \rightarrow \mathrm{H}$ is a *homomorphism*, if $\forall \mathrm{g}_1 \mathrm{g}_2 \in \mathrm{G}$, $\varphi(\mathrm{g}_1 \mathrm{g}_2) = \varphi(\mathrm{g}_1)\varphi(\mathrm{g}_2)$. The *image* of φ is $\mathrm{im}\varphi = \{\varphi(\mathrm{g}): \mathrm{g} \in \mathrm{G}\}$. The *kernel* of φ is $\ker\varphi = \{\mathrm{g} \in \mathrm{G}: \varphi(\mathrm{g}) = \mathrm{e_H}\}$. If the homomorphism $\varphi: \mathrm{G} \rightarrow \mathrm{H}$ is a bijection, then we say it is an *isomorphism*. If there is an isomorphism $\varphi: \mathrm{G} \rightarrow \mathrm{H}$, then we say that G, H are isomorphic and write $\mathrm{G} \cong \mathrm{H}$.

Analysing the thirteen students' solution of CS2E5, as the following discussion demonstrates, ten of the thirteen students produced flawless answers regarding the first part of the definition above in relation to the d-object of homomorphism.

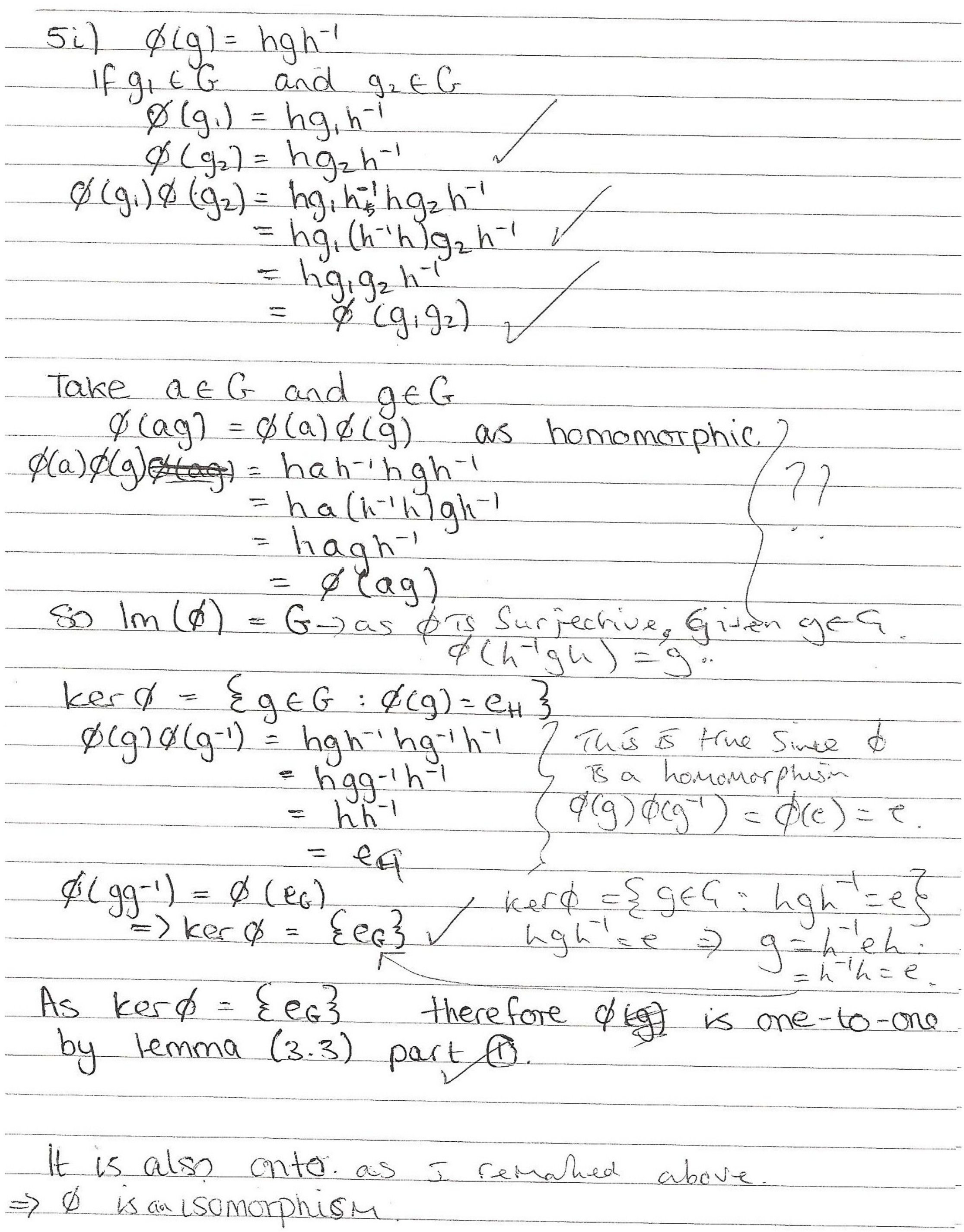
5. Prove that the following are homomorphisms:
(i) G is any group, $h \in G$ and $\phi : G \rightarrow G$ is given by $\phi(g) = hgh^{-1}$.
(ii) $G = \mathrm{GL}(n, \mathbb{R})$ and $\phi : G \rightarrow G$ is given by $\phi(g) = (g^{-1})^T$.
(Here $\mathrm{GL}(n, \mathbb{R})$ is the group of invertible $n \times n$-matrices over $\mathbb{R}$ and the ^{T} denotes transpose.)
(iii) G is any abelian group and $\phi : G \rightarrow G$ is given by $\phi(g) = g^{-1}$.
(iv) $\phi : (\mathbb{R}, +) \rightarrow (\mathbb{C}^\times, \cdot)$ given by $\phi(x) = \cos(x) + \mathrm{i}\sin(x)$.
(v) G is any group, N is a normal subgroup of G and $\phi : G \rightarrow G/N$ is defined by $\phi(g) = gN$.
In each case say what is the kernel and the image of ϕ. In which cases is ϕ an isomorphism?

Text 4.15: CS2E5

This rather successful encounter with the d-object of homomorphism is possibly due to the fact that proving the existence of a homomorphism $\varphi: \mathrm{G} \rightarrow \mathrm{H}$, does not entirely depend on the thorough comprehension of the definition of the homomorphism, but it is mostly a result of the application of a clear and concrete routine involving a simple metalevel process, namely to prove that $\forall \mathrm{g}_1 \mathrm{g}_2 \in \mathrm{G}$, $\varphi(\mathrm{g}_1 \mathrm{g}_2) = \varphi(\mathrm{g}_1)\varphi(\mathrm{g}_2)$.

Regarding the d-object of isomorphism, there were four occasions in the student solutions, in which the errors were independent of the d-objects of kernel and image and will be presented last. In what immediately follows, I present and analyse some typical examples of the errors involving problematic object-level understanding of the d-objects of kernel and image, in the process of proving that a given map is in fact an isomorphism.

A typical example of problematic object-level understanding of the d-objects of kernel and image could be found in Leonora's solution of CS2E5.


5i) $\phi(g) = hgh^{-1}$
If $g_1 \in G$ and $g_2 \in G$
$\phi(g_1) = hg_1h^{-1}$
$\phi(g_2) = hg_2h^{-1}$ ✓
$\phi(g_1)\phi(g_2) = hg_1h^{-1}hg_2h^{-1}$
$= hg_1(h^{-1}h)g_2h^{-1}$ ✓
$= hg_1g_2h^{-1}$
$= \phi(g_1g_2)$ ✓

Take $a \in G$ and $g \in G$
$\phi(ag) = \phi(a)\phi(g)$ as homomorphic
$\phi(a)\phi(g)$ ~~$\phi(ag)$~~ $= hah^{-1}hgh^{-1}$
$= ha(h^{-1}h)gh^{-1}$
$= hagh^{-1}$
$= \phi(ag)$??
So $\text{Im}(\phi) = G \rightarrow$ as ϕ is surjective, given $g \in G$. $\phi(h^{-1}gh) = g$.

$\ker\phi = \{g \in G : \phi(g) = e_H\}$
$\phi(g)\phi(g^{-1}) = hgh^{-1}hg^{-1}h^{-1}$
$= hgg^{-1}h^{-1}$
$= hh^{-1}$
$= e_G$
This is true since ϕ is a homomorphism $\phi(g)\phi(g^{-1}) = \phi(e) = e$.
$\phi(gg^{-1}) = \phi(e_G)$
$\Rightarrow \ker\phi = \{e_G\}$ ✓
$\ker\phi = \{g \in G : hgh^{-1} = e\}$
$hgh^{-1} = e \Rightarrow g = h^{-1}eh = h^{-1}h = e$.
As $\ker\phi = \{e_G\}$ therefore ϕ ~~(g)~~ is one-to-one by lemma (3.3) part (i). ✓

It is also onto. as I remarked above.
$\Rightarrow \phi$ is an isomorphism.

The first task for proving that a map is a homomorphism was proved successfully, indicating successful application of the governing metarules of the given routine. Furthermore, her attempt to prove that this homomorphism is in fact bijective and therefore an isomorphism reveals several inaccuracies, suggesting problematic object-level understanding of kernel and image.

Her reasoning is occasionally not explicit. She does not produce full narratives for proving what is the kernel and image of the homomorphism, suggesting probably some difficulty with the definitions of these two objects, as well as understanding of the connection between image and surjectivity, and kernel and injectivity. As the markers' additions suggest, her narratives need to be more detailed, predominantly by referring to the definitions of image and kernel. Since her object-level understanding is rather seriously problematic, the application of metarules and therefore her metalevel understanding in the context of this mathematical task suffers equally seriously. Her reasoning is incomplete with inadequate justification and her claims are rather vague, indicating some disconnection with the more solid reasoning of the first part of her solution.

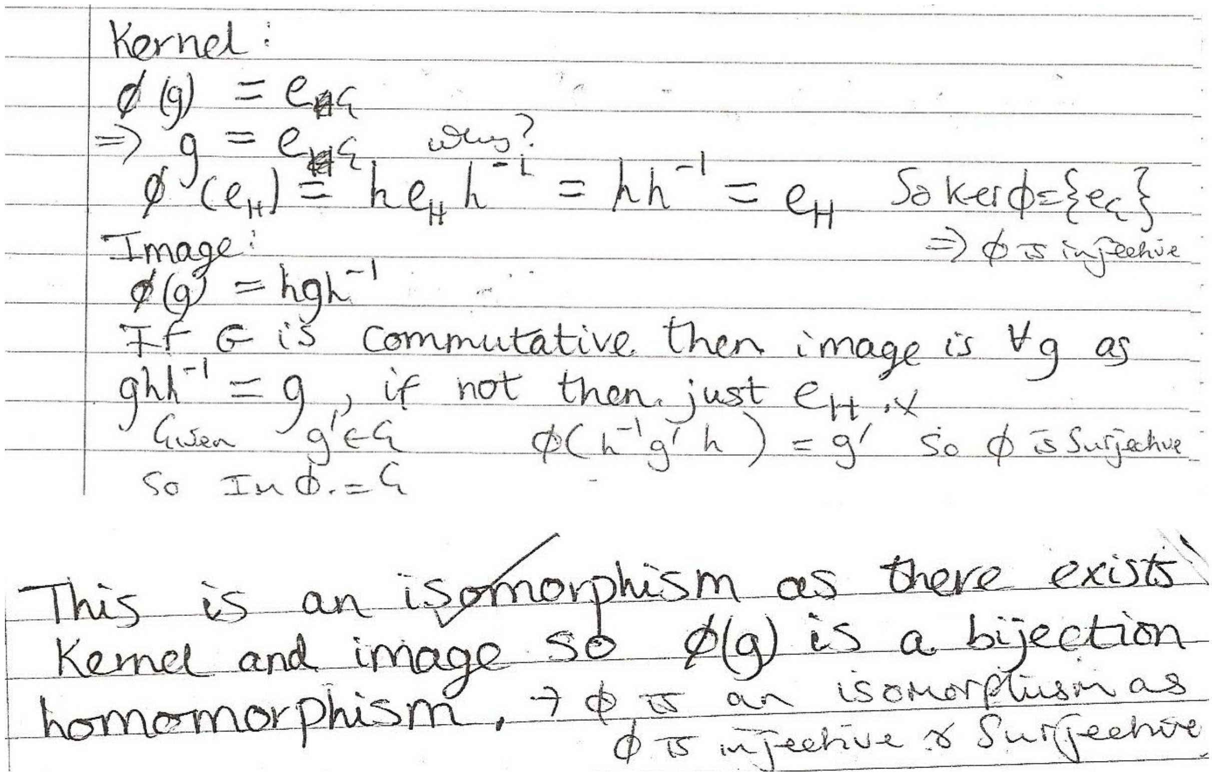
Her word use becomes rather problematic. Instead of stating that φ is an *isomorphism*, she mentions that φ is *isomorphic*. The use of the word 'isomorphic' instead of 'isomorphism' suggests that Leonora has not yet objectified the concept of isomorphism. This concept may not have been reified yet, since Leonora refers to the concept of isomorphism as being a procedural activity and not an object.

In general, the above analysis is in agreement with Leonora's initial perception about CS2E5 and the d-objects of homomorphism, kernel and image.

> *Question 5, I found it easy enough to show that they're homomorphisms. The kernel's… it's just the image… that I'm having problems with at the moment, yeah. So I know someone in my course that – he did it and he got help in his seminars, so he said he'd*

try and explain it to me? So hopefully, I'll understand it then, if not I'll have to go ask him. Leonora

Other typical inaccuracies regarding the d-objects of kernel and image were apparent in Manrico's solution of CS2E5i. In particular, there were occasion of problematic use of notation, for instance instead of writing e_G he writes e_H. This inaccuracy is rather important because it is probably a result of problematic object-level understanding of the d-object of group and the identity element in particular and furthermore it possibly suggests that he is not yet aware of the fact that the identity element of $G$is the same for every subgroup. In addition, there are problems with his metalevel understanding resulting inaccuracies in the process of proving. For instance, he does not justify why $g = e$ when he concludes that $ker(\varphi) = \{e_G\}$. There is no explicit explanation saying that since $ker(\varphi) = \{e_G\}$, therefore φ is injective. Manrico has probably not yet objectified the d-objects of φ being *injective* or *surjective*, therefore an *isomorphism*, in relation to the kernel and image.


Kernel:
$\phi(g) = e_G$
$\Rightarrow g = e_G$ why?
$\phi(e_H) = h e_H h^{-1} = hh^{-1} = e_H$ So $\ker\phi = \{e_G\}$
$\Rightarrow \phi$ is injective
Image:
$\phi(g) = hgh^{-1}$
If G is commutative then image is $\forall g$ as $ghh^{-1} = g$, if not then just e_H. ✗
Given $g' \in G$ $\phi(h^{-1}g'h) = g'$ So ϕ is Surjective
So $\mathrm{Im}\,\phi = G$

This is an isomorphism as there exists kernel and image so $\phi(g)$ is a bijection homomorphism, $\therefore \phi$ is an isomorphism as ϕ is injective & Surjective

His performance is relatively mirrored in his somewhat optimist impression regarding CS2E5, as discussed in the second interview.

> *Yeah, 5, I mean, proving the homomorphisms, wasn't too much of a problem, sometimes saying what the kernel, the image was, was a bit – harder, and – which were isomorphisms, basically though, that was fine, once I'd kind of – went over the definition of an isomorphism, I mean, it just kind of pretty much relies on what you've – get in your kernel image, but um... cos I mean it's probably the whole visualizing, I was just a bit kind of – it's kind of – this is abstract, it's the whole abstract concepts of kernels and images and – not all – you know, not kind of meeting them everyday, things... but – yeah, I think I'm getting there with 5.* Manrico

Manrico expresses his need to have a visual image of the d-objects of kernel and image and links his difficult to cope with them with the level of abstraction. He emphasises that the new d-objects are different from the 'usual' mathematical d-objects ('not kind of meeting them everyday'), and indirectly suggests that his approach to objectify them should be different.

Another representative example of problematic object-level understanding of the d-object of image was apparent in Norma's solution of CS2E5i. In particular, in this exercise, $im(\varphi) = G$, since φ is surjective. Instead, Norma wrote that $im(\varphi) = hgh^{-1}$, without explicitly stating that the image of the homomorphism in this case is the group G itself indicating also problematic application of the governing metarules, showing lack of precision, clarity and rigor. In addition, her attempt to prove that the homomorphism is bijective and therefore an isomorphism is problematic because of her insufficient object-level understanding of the kernel. In particular, Norma was not yet able to conclude that when $ker(\varphi) = \{e_G\}$ then φ is one-to-one and therefore isomorphism (since it is already proven that it is surjective as well).

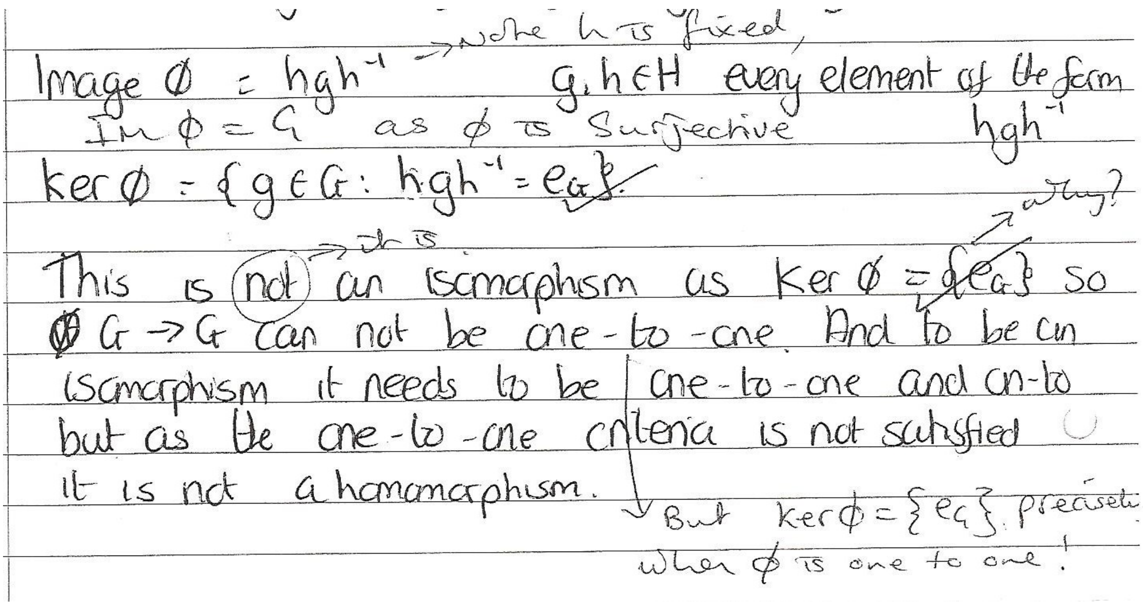


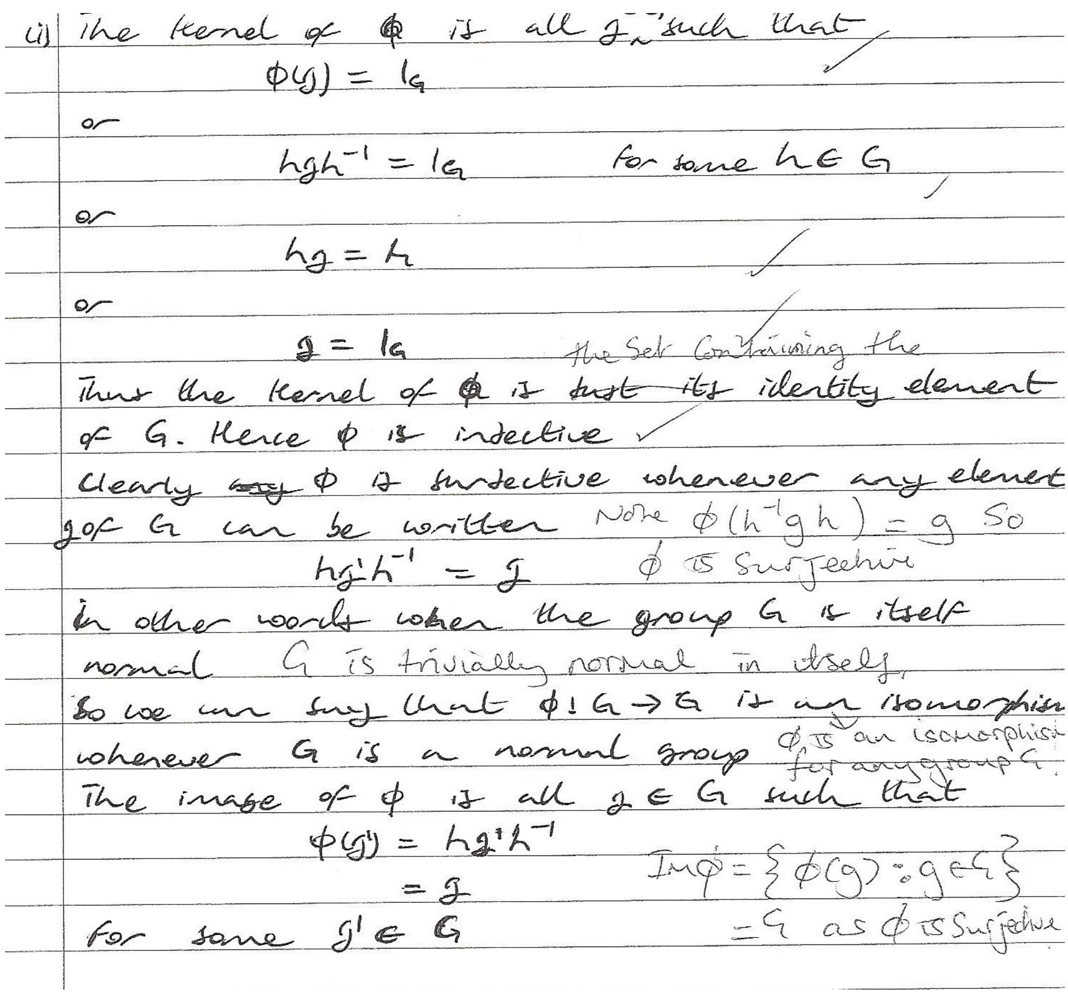
Image ϕ = hgh^{-1} → note h is fixed, g, h ∈ H every element of the form hgh^{-1}
Im ϕ = G as ϕ is Surjective
ker ϕ = {g ∈ G : hgh^{-1} = e_G}. → why?
This is (not) → this is an isomorphism as Ker ϕ = {e_G} so ϕ G → G can not be one-to-one. And to be an isomorphism it needs to be one-to-one and on-to but as the one-to-one criteria is not satisfied it is not a homomorphism.
→ But ker ϕ = {e_G} precisely when ϕ is one to one!

Norma's impression, as this has been expressed in her interviews is in agreement with her performance, expressing her initial difficulty with the d-objects of kernel, image and isomorphism.

> *Um... did question 5 like the first couple of parts, but obviously didn't get in – again, they were getting a bit harder towards the end because they were getting more difficult examples… so yeah... question 6, I think I found quite hard, as well... and again, it's just trying to get your head round all like the concepts, I think, you just need to keep going over them and then – obviously the questions will become easier to do. And then... because obviously we've only just done this one as well, so I think I need to go through it a couple more times, just to see whether I can um, do it.* Norma

The above analysis possibly suggests that problematic object-level understanding affects unfavorably the metalevel understanding and application of metarules in the given context. Moreover, application of metarules is possibly context sensitive, and even if metalevel understanding is high in a certain mathematical discourse, the application of the same metarules might be problematic in a different one.

Another typical error, one that has occurred in the analysis in the previous sections of this chapter, is the problematic object-level understanding (or occasionally the imprecise use of notation) regarding the sets and their elements. For instance, Otello seems to have problematic object-level understanding of the d-objects of kernel and image in CS2E5i. These problems were revealed when he considered kernel to be the identity element of G, instead of the set containing the identity, as the excerpt below suggests.


ii) The kernel of ϕ is all g such that ✓
$\phi(g) = 1_G$
or
$hgh^{-1} = 1_G$ for some $h \in G$ ✓
or
$hg = h$ ✓
or
$g = 1_G$ ✓
Thus the kernel of ϕ is ~~just its~~ the set containing the identity element of G. Hence ϕ is injective ✓
Clearly ϕ is surjective whenever any element g of G can be written
$hg'h^{-1} = g$
Note $\phi(h^{-1}gh) = g$ So ϕ is surjective
In other words when the group G is itself normal
G is trivially normal in itself.
So we can say that $\phi : G \to G$ is an isomorphism whenever G is a normal group
ϕ is an isomorphism for any group G.
The image of ϕ is all $g \in G$ such that
$\phi(g') = hg'h^{-1}$
$= g$
for some $g' \in G$
$\text{Im}\,\phi = \{\phi(g) : g \in G\}$
$= G$ as ϕ is surjective

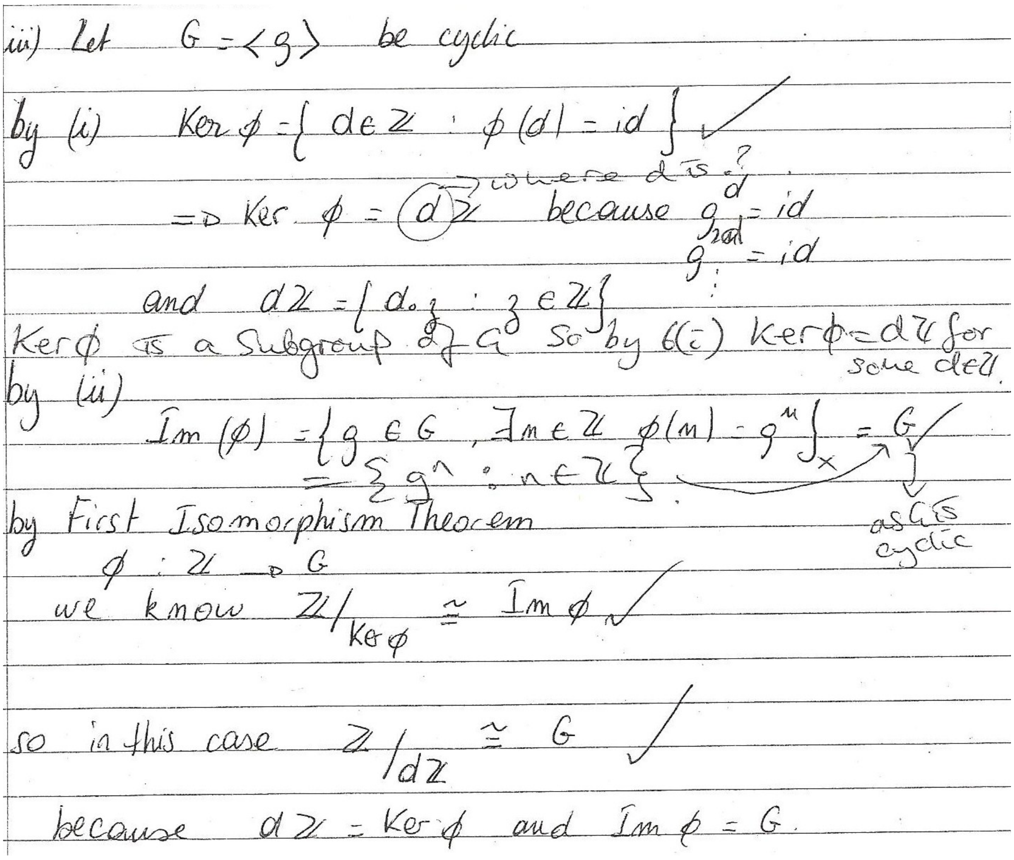
Instead of writing $\varphi(h^{-1}gh) = g$, Otello stated that φ is surjective whenever any element $g \in G$ can be written as $h^{-1}gh = g$, without considering the role of homomorphism. Finally, regarding the image of φ, he failed to explicitly state what this is. These object-level inaccuracies indicate negative impact in the application of metarules.

Regarding CS2E6, Carmen's solution reveals problematic object-level understanding with the d-objects of kernel and image.

6. (i) Suppose H is a non-trivial subgroup of $\mathbb{Z}$, the group of integers under addition, and let d be the smallest natural number in H (- why is there such a thing?). Prove that $H = d\mathbb{Z}$.
(ii) Suppose $G = \langle g \rangle$ is a cyclic group (written multiplicatively). Define $\phi : \mathbb{Z} \to G$ by $\phi(n) = g^n$ (for $n \in \mathbb{Z}$). Prove that this is a homomorphism.
(iii) Using (i) and (ii) and the First Isomorphism Theorem, deduce that if G is a cyclic group then there exists an integer d such that $G \cong \mathbb{Z}/d\mathbb{Z}$.

Text 4.16: CS2E6

In her solution of CS2E6, as seen below, Carmen seems to have not yet fully understood the object-level rules of kernel and image, since she does not seem to know that kernel is a subgroup of G and that in this case it is $ker\varphi = d\boldsymbol{Z}$. There is no sign of clear objectification of the fact that $im\varphi = G$ because of the cyclicity of the group G. Finally, there are indications of a good metalevel understanding of the First Isomorphism Theorem, since it has been successfully applied in the context of this exercise. Carmen's narratives are, more often than not, explicit, her reasoning is linear and her use of mathematical symbolisation is efficient, elements that indicate a successful discursive shift towards Group Theory and therefore, according to Sfard (2008), effective mathematical learning.



iii) Let $G = \langle g \rangle$ be cyclic

by (i) $\text{Ker}\,\phi = \{ d \in \mathbb{Z} : \phi(d) = id \}$ ✓

where d is ?

$\Rightarrow \text{Ker}\,\phi = (d)\mathbb{Z}$ because $g^{d} = id$

$g^{2d} = id$

⋮

and $d\mathbb{Z} = \{ d \cdot z : z \in \mathbb{Z} \}$

Kerϕ is a Subgroup of G So by 6(i) Ker$\phi = d\mathbb{Z}$ for some $d \in \mathbb{Z}$.

by (ii)

$\text{Im}(\phi) = \{ g \in G, \exists n \in \mathbb{Z}\ \phi(n) = g^n \} = G$ ✓

$= \{ g^n : n \in \mathbb{Z} \}$ × as G is cyclic

by First Isomorphism Theorem

$\phi : \mathbb{Z} \to G$

we know $\mathbb{Z}/_{\text{Ker}\,\phi} \cong \text{Im}\,\phi$ ✓

so in this case $\mathbb{Z}/_{d\mathbb{Z}} \cong G$ ✓

because $d\mathbb{Z} = \text{Ker}\,\phi$ and $\text{Im}\,\phi = G$.

Finally regarding the problematic engagement with the notion of isomorphism independent from kernel and image, I present a representative example, taken from Norina's attempt to solve CS2E5.

$Im\,\phi = \{\phi(g) : g \in G\} = \{hgh^{-1} : g \in G\} = G$ ✓ as ϕ is surjective.

$ker\,\phi = \{g \in G : hgh^{-1} = e_G\}$
$= \{g \in G : g = h^{-1}e_G h\} = \{e_G\}$.

If the homomorphism is a bijection then we say it is an isomorphism. ✓

To be a bijection the homomorphism needs to be injective and surjective.

ϕ is injective as $ker\,\phi = \{e_G\}$ ✓
? { ϕ is surjective as there is at least one x in G such that $\phi(x_1 x_2) = y$ where $y = \phi(x_1)\phi(x_2)$.
As ϕ is injective and surjective it is a bijection and therefore an isomorphism.
Given $g \in G$ $\exists x \in G$ such that $\phi(x) = g$ namely $x = h^{-1}gh$. So ϕ is surjective and therefore $Im\,\phi = G$.

She successfully identifies the kernel and the image of φ. Consequently, she proves that since $ker\varphi = \{e_G\}$ then the homomorphism is injective. The only error relates to her inability to infer from that $im\varphi = G$ that the homomorphism is also surjective and therefore an isomorphism. This error does not occur anywhere else in her solutions, so it can be considered to be incidental and not persistent. It is probably based on problematic application of the governing metarules that result inaccuracy in the mathematical reasoning and consequently, lack of the necessary precision.

In conclusion, the first major signs of problematic object-level understanding for the majority of students have occurred when the notions of kernel and image are introduced. There were also indications of problematic metalevel understanding, which were revealed through the lack of explicitness in the thirteen students' mathematical narratives, and by the absence of important steps in the application of the routines involving these d-objects. Another indication of a problematic encounter with these d-objects was the increasingly problematic use of vocabulary and notation.

The d-objects of kernel and image were problematic to the majority of students, mainly because these novice students were not able yet to objectify them properly. Kernel and Image have not been fully objectified as algebraic structures linked directly with the notion of homomorphism and giving valuable information about the injectivity and surjectivity, respectively, of this homomorphism. Moreover, students have possibly not realised that $ker(\varphi)$ is an element of G and $im(\varphi)$ is an element of H.

Full object-level understanding of the d-objects of kernel and image will be an indispensable requirement for the objectification of isomorphism and the application of FIT. Therefore, the conceptual disengagement with the d-object of isomorphism is not so much based on the level of object-level understanding of homomorphisms and the application of the routine for proving homomorphisms, but rather on the objectification of kernel and image.

The above analysis suggests that the majority of students seem to know the steps of the routine for proving that a homomorphism is indeed an isomorphism. Usually, the problems occur in the application of the different subroutines, namely the proof of injectivity and surjectivity. This again indicates a problematic metalevel understanding of the notions of kernel and image.

4.5.2 Cosets, Normal Subgroups and Quotient Groups

The second category of errors is related to the d-objects of cosets, normal subgroups and quotient groups as these have occurred in FEE5ia. Since these errors occurred in the solution of an examination exercise, there are no interview data that could be used to reinforce the claims that follow, as in the section above.

Definition: Suppose G is a group and H is a subgroup of G. Let $g \in G$. Then the subset $gH = \{gh: h \in H\}$ of G is called a *left coset* of H in G (or H-coset). An element of gH is called a *representative* of gH.

Definition: Suppose $(G,*)$ is a group. A subgroup $N \trianglelefteq G$ is called a *normal subgroup* if $\forall k \in N, \forall g \in G\ gkg^{-1} \in N$.

Definition: Suppose G is a group and $N \trianglelefteq G$. Consider $G\,/\,N$ the set of left cosets of N in G, i.e. $G/N = \{gN: g \in G\}$.

5. (i) Suppose G is a group.

(a) What does it mean to say that a subgroup N of G is a *normal* subgroup? If N is a normal subgroup of G, explain how to make the set G/N of left cosets of N in G into a group. **[3 marks]**

(b) State the First Isomorphism Theorem for groups, defining the terms *kernel* and *image* in your statement. **[4 marks]**

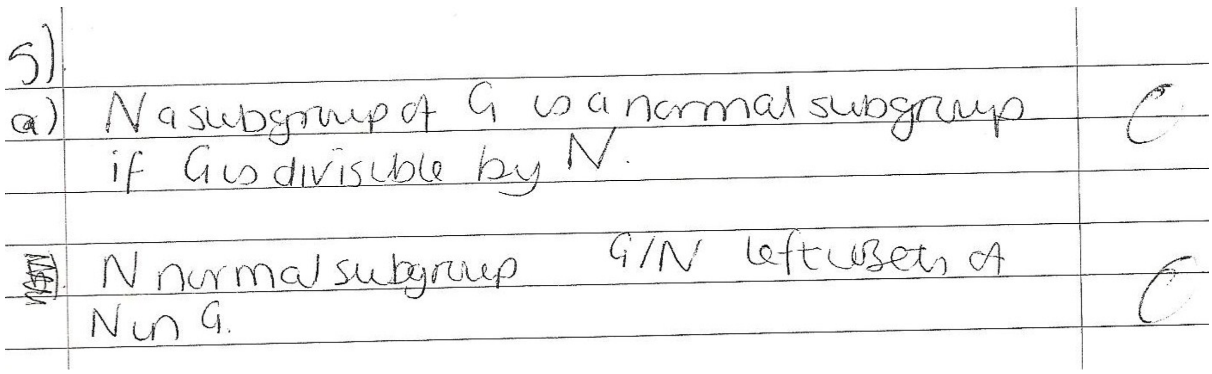
(c) Suppose H is a cyclic group. By defining a suitable homomorphism $\phi : (\mathbb{Z},+) \to H$, or otherwise, prove that $H \cong \mathbb{Z}/m\mathbb{Z}$ for some $m \in \mathbb{Z}$. **[3 marks]**

Text 4.17: FEE5

In general, all thirteen students' engagement with these d-objects, as these were used in their attempts to solve FEE5ia, indicated seriously problematic object-level understanding. Scrutinising the thirteen students' answers of FEE5ia, there have emerged a number of errors in the application of both the object-level and metalevel rules. Eight out of the thirteen (8/13) students, namely Calaf, Francesca, Dorabella, Kostanza, Leonora, Manrico, Tamino failed to state the definition of normality. This possibly suggests problematic object-level understanding of the d-object of normal subgroup.

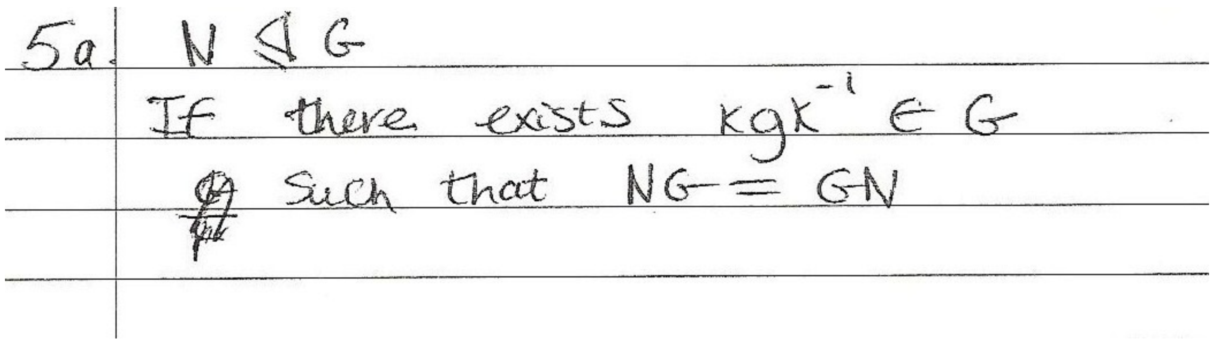
There were also indications of problematic understanding of the notation of the quotient group $G\ /\ N$ and what this represents in six out of these eight (6/8) students' solutions. These students do not seem to have realised that the elements of a quotient group are cosets. Moreover, having not objectified the d-objects of coset and normal subgroup, it is almost impossible to understand, both in object-level and metalevel, the d-object of quotient group. The weak object-level understanding regarding the d-objects of normality and quotient groups does not allow these students to move to the metalevel understanding and the application of the routine in order to answer the question 'how to make the set $G\ /\ N$ of left cosets of N in G into a group'. Below I discuss some representative examples.

Kostanza's performance in exercise FEE5ia is particularly poor. She fails to state the definition of normal subgroup, possibly indicating problematic object-level understanding of the definition of normal subgroup.



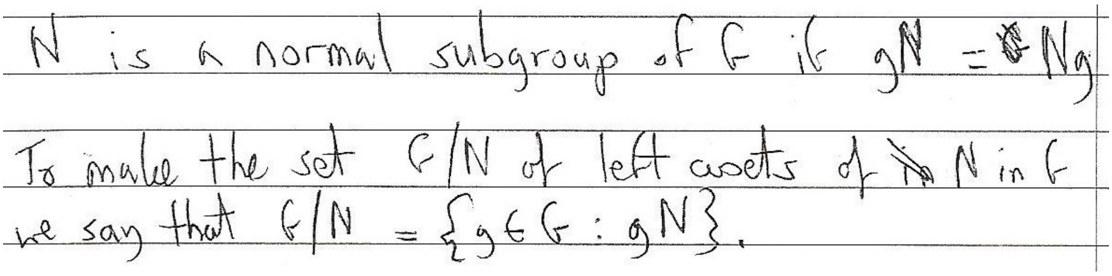
There are certain errors related to the meaning of the notation G/N. Her solution indicates a problematic engagement with the object-level rules related to the objects of group, subgroup, normality, and order. The word use, an essential element of mathematical discourse, is erroneous. She gives no substantial answer to the question of 'how to make the set G/N of left cosets of N in G into a group'. The expected answer should be that $(g_1N)(g_2N) = g_1g_2N$ on left cosets is well defined and makes G/N into a group. Answering this particular question requires, apart from good object-level understanding of the d-object of normal subgroup, thorough metalevel understanding of the routine for proving that a subgroup is in fact normal according to which one should prove that either $\forall g \in G, gNg^{-1} = N$, or $\forall g \in G, gN = Ng$. This is another typical routine in the discourse of Group Theory that caused problems to these students.

Similar to above, Manrico has not stated the definition of normal subgroup. In the following excerpt, one can notice a number of errors that reveal problematic object-level understanding as well as ambiguous metalevel perception of the metarules that should be applied in the routine for proving that a subgroup is normal.



From the excerpt above and the very short attempt of Manrico to solve FEE5ia one can deduce that both his object-level understanding of the d-objects of group and normal subgroup as well as the application of the metarules that govern the routine for proving normality are problematic. First, instead of writing $\forall k \in N, \forall g \in G, gkg^{-1} \in N$ he has written $kgk^{-1} \in G$. His object-level understanding regarding the elements of the group and the elements of the normal subgroup does not appear to be clear yet. In addition, he does not use the appropriate notation, suggested in the lecture notes, indicating an inconsistency in the discursive shift towards learning Group Theory. Moreover he has neither fully objectified the idea of normality nor he is capable at this stage to use appropriately the metarules of normal subgroups.

An example of problematic metalevel understanding, yet complete object-level understanding was demonstrated in Tamino's solution of FEE5ia.

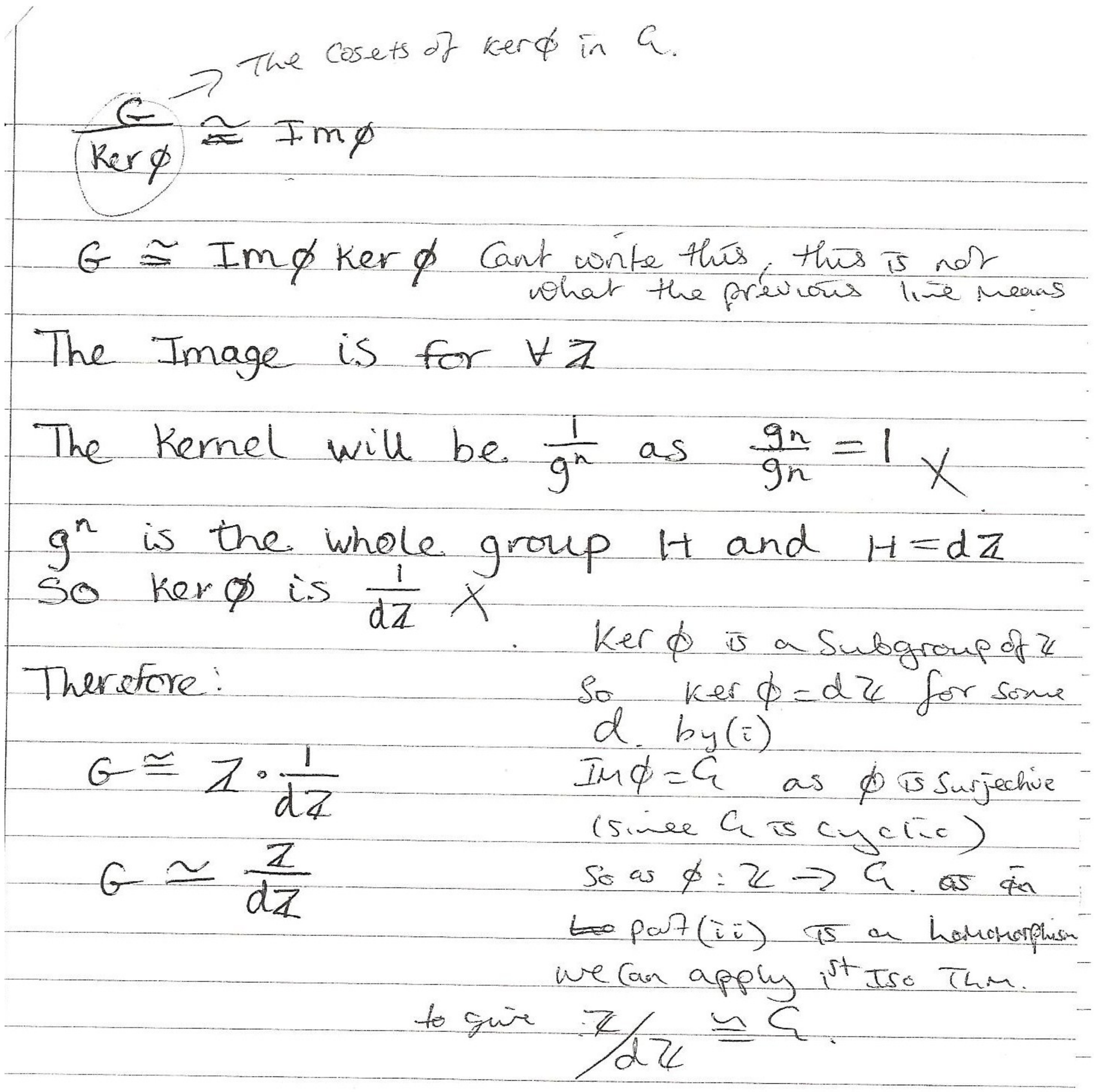


Tamino has successfully managed to explicate what it means for a subgroup N of G to be normal, yet his attempt to explain *how* to make the set G/N of left cosets of N in G into a group is problematic. There are indications of a problematic metalevel engagement with the definition of quotient group and coset. While there appear to be signs of successful object-level learning up to a certain level, there is little indication of metalevel learning, which would allow Tamino to successfully apply the involved routines. Consequently, his solution lacks perspective in the sense that there is no clear indication of what he is trying to prove. He seems to be unaware that he is supposed to show a well-defined operation on left-cosets. In this case, it should be $(g_1N)(g_2N) = g_1g_2N$.

4.5.3 First Isomorphism Theorem

The third category of errors in this section is related to the application of FIT as a routine. Twelve out of thirteen (12/13) students' solutions indicated problematic engagement with FIT. There are strong indications of problematic application of the governing metarules of this routine by the majority of students. FIT is the pinnacle of this module (Nardi, 2000) and students are required to resolve any problems with their object-level understanding of the prerequisite d-objects as well as achieve metalevel understanding of the metarules that are required in the application of FIT as a routine. Below I discuss some representative examples of the typical errors that occurred.

Good object-level understanding of the involved d-objects, such as kernel, image and isomorphism, it is naturally of vital importance for objectifying FIT and applying it successfully. For instance, as the excerpt below indicates, Manrico seems to have problematic understanding of the d-object of isomorphism and therefore he is unable, at this stage of his learning to apply it successfully.


The cosets of $ker\phi$ in G.

$\frac{G}{Ker\phi} \cong Im\phi$

$G \cong Im\phi \, Ker\phi$ Cant write this, this is not what the previous line means

The Image is ~~for~~ $\forall \mathbb{Z}$

The Kernel will be $\frac{1}{g^n}$ as $\frac{gn}{gn} = 1$ X

g^n is the whole group H and $H = d\mathbb{Z}$

So $Ker\phi$ is $\frac{1}{d\mathbb{Z}}$ X

Therefore:

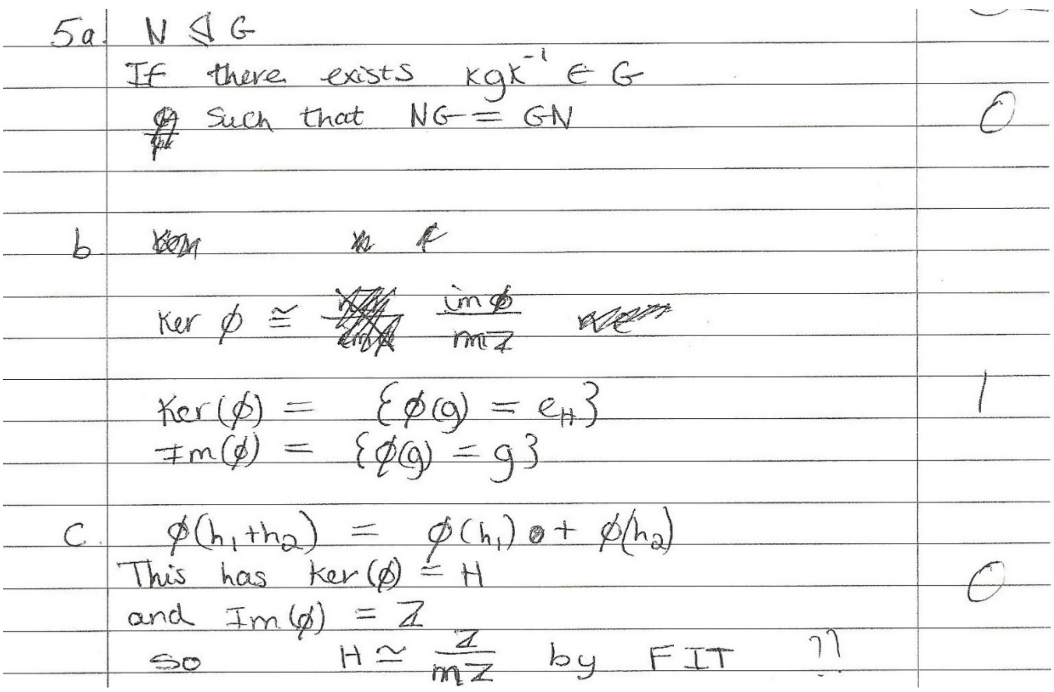
$G \cong \mathbb{Z} \cdot \frac{1}{d\mathbb{Z}}$

$G \simeq \frac{\mathbb{Z}}{d\mathbb{Z}}$

$Ker\phi$ is a subgroup of $\mathbb{Z}$ so $Ker\phi = d\mathbb{Z}$ for some d. by (i)
$Im\phi = G$ as ϕ is surjective (since G is cyclic)
So as $\phi: \mathbb{Z} \to G$, as ϕ ~~to~~ part (ii) is a homomorphism we can apply 1st Iso Thm. to give $\mathbb{Z}/d\mathbb{Z} \cong G$.

For instance, the notation $\frac{G}{ker\varphi} \cong Im\varphi$ is treated as an algebraic equation in which Manrico has applied cross-multiplication, i.e. $G \cong Im\varphi Ker\varphi$. This is an erroneous metaphor from elementary algebra that indicates a problematic object-level understanding of the d-object of isomorphism, as well as kernel and image and therefore it reveals an inconsistent discursive shift, from elementary to abstract algebra. Manrico has not realised that $G/ker\varphi$ is a mathematical structure and the symbol ≅ does not refer to equality relation but to bijective relation. His object-level and metalevel understanding of the d-objects of kernel and image is also either problematic or partial and consequently he is not still able to use FIT efficiently. He has not realised that $im\varphi$ is a subgroup of H and that $ker\varphi$ is a normal subgroup of G. FIT is a narrative introduced at the very end of the introductory module of Group Theory and its efficient application requires resolution of all conceptual difficulties and misunderstandings of all antecedent d-objects. It is obvious at this late stage of students' encounter with Group Theory that all unresolved problems of both object-level and metalevel understanding probably become obstacles for further learning.

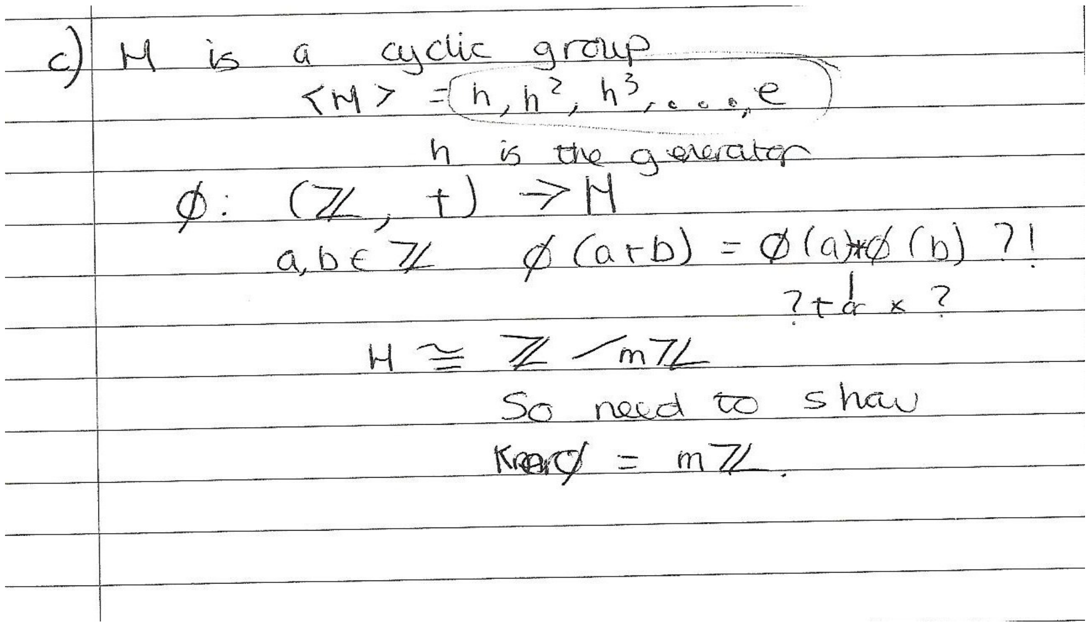
In the second part of FEE5, as seen below, Manrico does not state FIT. He only writes the definition of the involved d-objects such as kernel and image, but his solution is fragmental, indicating an incomplete metalevel understanding and inability to apply the theorem. His narratives are partial and his reasoning is not linear. Compared to the coursework, Manrico's performance regarding FIT and the preceding d-objects shows signs of regression in the exam.

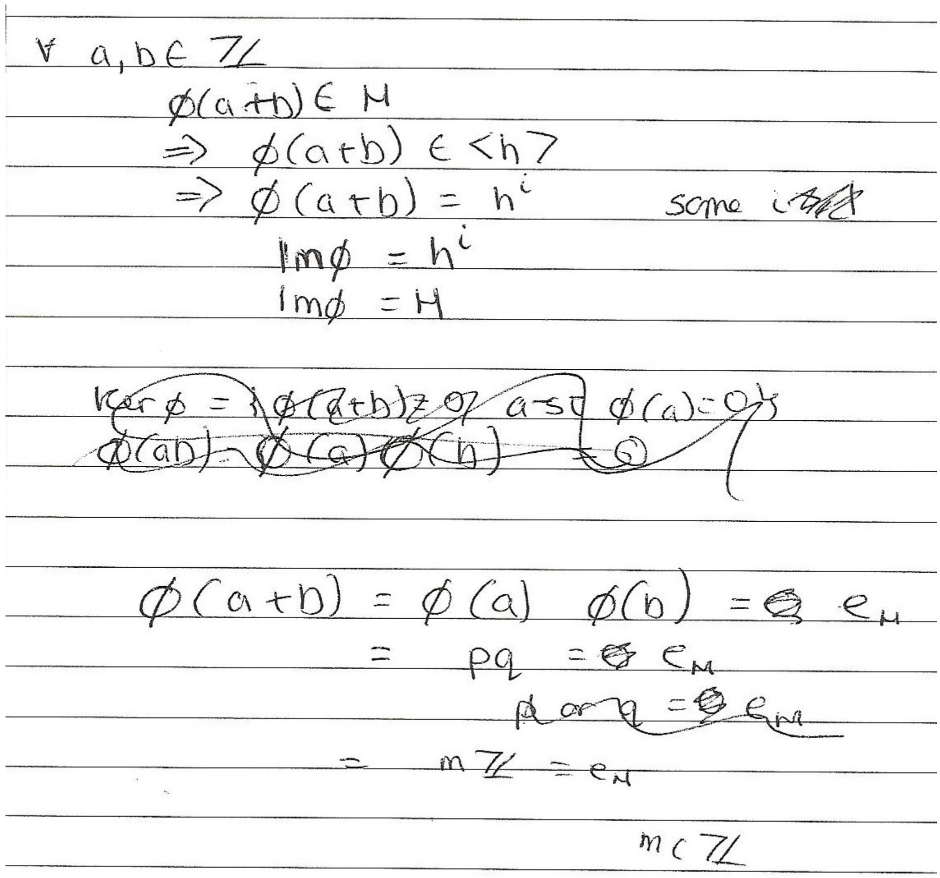

5a $N \trianglelefteq G$
If there exists $kgk^{-1} \in G$
such that $NG = GN$
0

b $Ker\ \phi \cong \frac{im\ \phi}{m\mathbb{Z}}$
$Ker(\phi) = \{\phi(g) = e_H\}$
$Im(\phi) = \{\phi(g) = g\}$
1

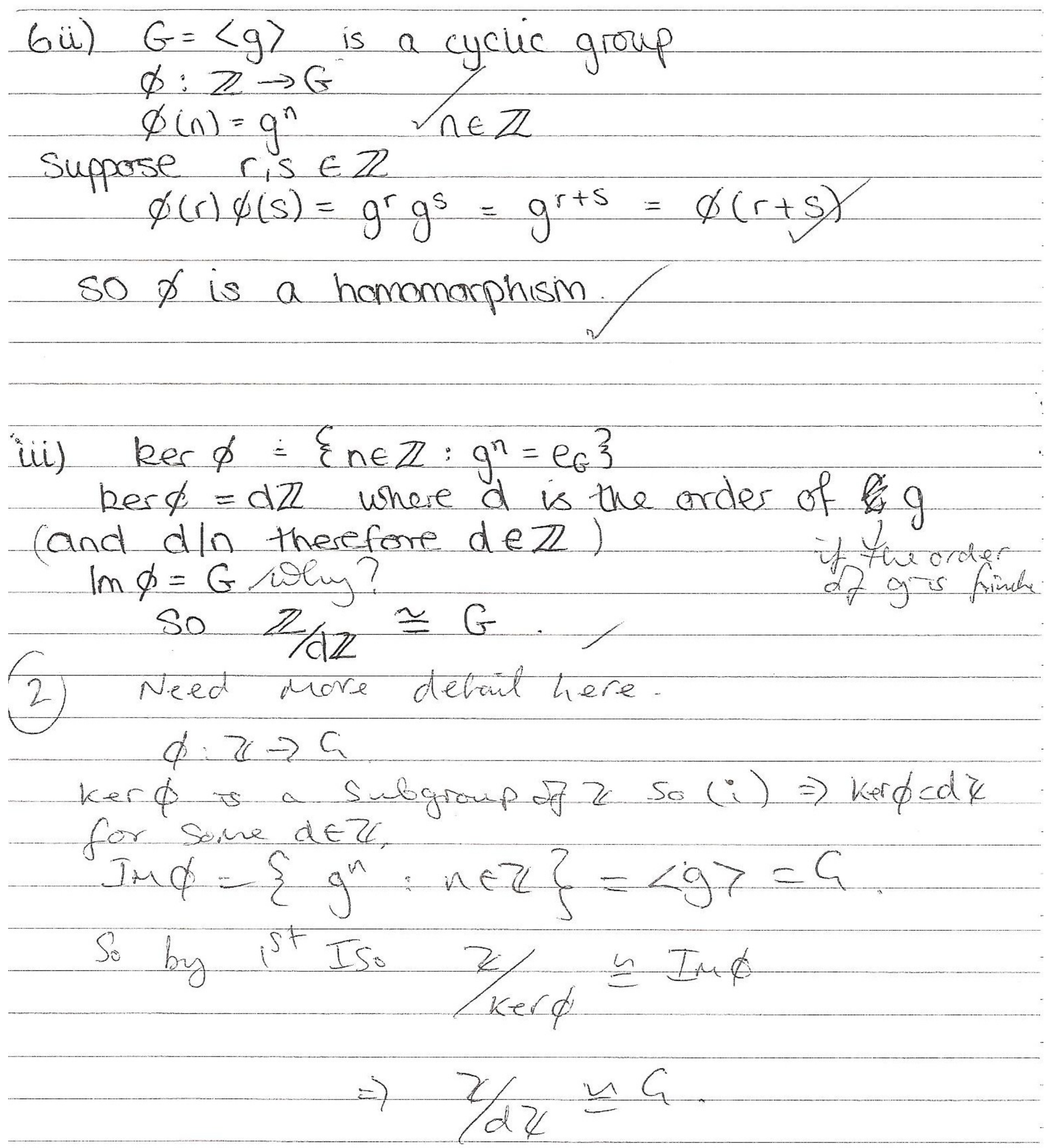
c. $\phi(h_1 + h_2) = \phi(h_1) + \phi(h_2)$
This has $Ker(\phi) = H$
and $Im(\phi) = \mathbb{Z}$
So $H \cong \frac{\mathbb{Z}}{m\mathbb{Z}}$ by FIT ??
0

Similar to Manrico, Amelia failed to state FIT in FEE5, since she did not provide the definitions for image and kernel. Possibly, not stating kernel, image and FIT suggests that her metalevel understanding regarding the kernel and image of a group is problematic when applied in a more advanced context and combined with the routine of FIT. The third part of her solution, involving the definition of a suitable homomorphism, reinforces the claim that she is still not able to efficiently use FIT.   Her attempt to solve the last bit is fragmental and lacks linear reasoning and structure. Amelia's data suggests that her objectification of the d-objects of homomorphisms, kernel and image has yet to be fully established.





A third example of errors regarding the application of FIT occurred in Leonora's solution of CS2E6. Her attempt to solve this exercise has several errors, especially in the third part regarding the FIT. She correctly states that the image of the homomorphism is the group G itself, and therefore it is an isomorphism, but she does not mention anything about the kernel. This possibly suggests a problematic object-level understanding of the d-object of isomorphism, and in particular relating to the fact that one has to prove that a homomorphism needs to be both injective and surjective in order to be an isomorphism. Moreover, her solution indicates that she is not aware of the importance of the fact that the order of g is finite. Her narratives are not explicit, possibly indicating a partial metalevel understanding of the involved routine as well as problematic application of the governing metarules. She does not seem to realise that the kernel in this case is $d\boldsymbol{Z}$ and the reasoning behind that.



6ii) $G = \langle g \rangle$ is a cyclic group
$\phi : \mathbb{Z} \to G$
$\phi(n) = g^n$ $n \in \mathbb{Z}$
Suppose $r, s \in \mathbb{Z}$
$\phi(r)\phi(s) = g^r g^s = g^{r+s} = \phi(r+s)$

So ϕ is a homomorphism.

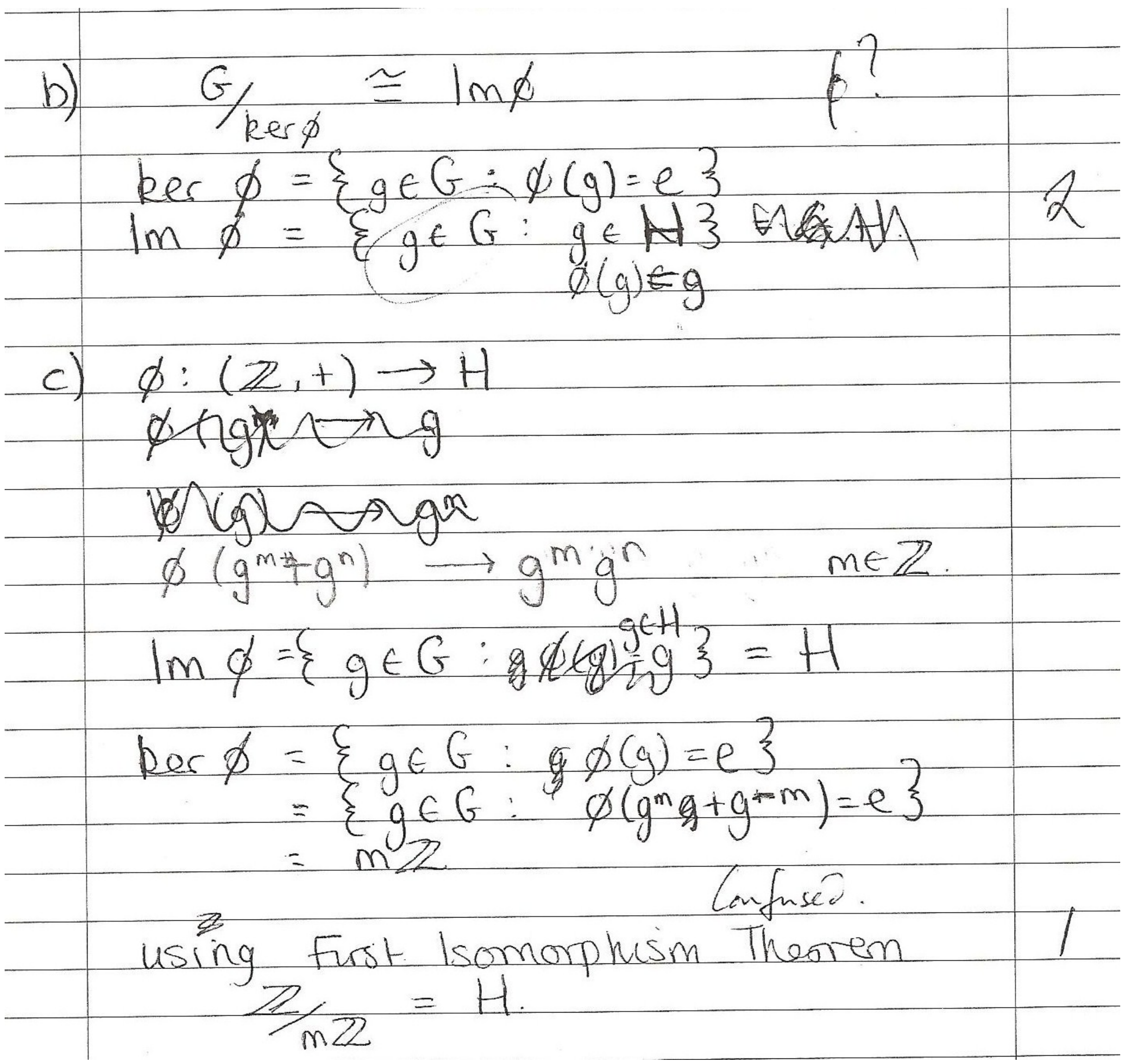
iii) $\ker \phi = \{n \in \mathbb{Z} : g^n = e_G\}$
$\ker\phi = d\mathbb{Z}$ where d is the order of g
(and $d|n$ therefore $d \in \mathbb{Z}$)
$\mathrm{Im}\,\phi = G$ Why?
So $\mathbb{Z}/d\mathbb{Z} \cong G$.

if the order of g is finite

2) Need more detail here.
$\phi : \mathbb{Z} \to G$
$\ker\phi$ is a subgroup of $\mathbb{Z}$ so (i) $\Rightarrow$ $\ker\phi = d\mathbb{Z}$ for some $d \in \mathbb{Z}$,
$\mathrm{Im}\,\phi = \{g^n : n \in \mathbb{Z}\} = \langle g \rangle = G$.
So by 1st Iso $\mathbb{Z}/\ker\phi \cong \mathrm{Im}\,\phi$

$\Rightarrow \mathbb{Z}/d\mathbb{Z} \cong G$.

In part FEE5ii one can notice several problems. First of all, Leonora failed to state the First Isomorphism Theorem, but instead she wrote the mathematical expression $\frac{G}{Ker\varphi} \cong Im\varphi$ without any further explanation. The definition of image is problematic, since instead of writing $Im\varphi = \{\varphi(g) : g \in G\}$ she stated $Im\varphi = \{g \in G : g \in H\}$. This shows that she has a problematic object-level understanding of the definition of image and its elements. In addition, her narratives suggest that she is not fully aware yet of what she is writing mathematically. In part (iii), her solution lacks explicitness, reflecting her problematic understanding of FIT, indicating incomplete metalevel understanding of the particular routine, as well as imprecise application of 'norms' required for proving in this advanced mathematics module. The marker wrote the comment "Confused", which he rarely does. Leonora's performance in the exams indicates regression regarding the understanding of FIT.



In sum, regarding the application of FIT theorem and the prerequisite d-objects there have emerged three categories of errors: the first category includes errors due to problematic object-level understanding of the d-objects of kernel, image and isomorphism; the second category includes errors due to problematic object-level understanding of the d-objects of coset, normal subgroup and quotient group; and the third category of errors is due to problematic application of metarules involved in the application of FIT.

4.6 Epilogue

4.6.1 Summary

This chapter's intention was to investigate students' conceptual difficulties with the main d-objects of Group Theory, analysing these difficulties both from an object-level as well as metalevel perspective.

Regarding the d-object of group, students' performance was satisfactory suggesting good object-level understanding and successful application of the routine and governing metarules for proving that a set is in fact a group. There have emerged two types of errors regarding the d-object of group. The first one occurred to two students' solutions and it was related with the proof of commutativity and therefore the group to be Abelian. This error was due to incomplete object-level understanding of the d-object of Abelian group and the axiom of commutativity. The second error was grounded on problematic metalevel understanding and erroneous application of the metarules, namely students were using the final statement that was supposed to be proved as part of the proving process. This error occurred in four students' solutions.

Regarding the d-object of group there have emerged six different categories of errors. The first category of errors is related to the absence of clarification about distinction of the prime numbers p and q, representing the orders of group elements. This error occurred in ten students' solutions and it is a result of lack of precision due to incomplete application of metarules. The second category of errors is related to the absence of proof of non-emptiness of the prospective subgroup. Similar to the previous one, for seven students, this error was a result of inaccuracy due to incomplete metalevel understanding and application of metarules. In addition, for three students it was a result of problematic metaphors from Set Theory, Complex Analysis and Arithmetic.

The third category of errors is related to the proof of closure under inverses. This error occurred in eight students' solutions and was due to object-level understanding of the d-object of inverse and its object-level rules as well as the idea of closure. These object-level problems would often have negative impact on the application of metarules.

The fourth category of errors regarding the d-object of subgroup occurred in six students' solutions and is related to the closure under operation. These errors resulted from the problematic object-level understanding and distinction between the elements of group and a subgroup. The fifth category of errors is related to the use of visual mediators and in particular with the use of Argand Diagrams. These errors are, most probably, irrelevant to the object-level understanding of the group-theoretic d-objects, but they are rather due to problematic metaphors from other mathematical discourses such as Complex Analysis and lack of connectivity with Group Theory. These errors appeared in eight students' solutions. The sixth category of errors is related to the d-objects of group, subgroup, set and their elements and it is a result of problematic object-level understanding. This last category of errors occurred in five students' solutions.

Regarding the listing of the symmetries of cube, the great majority of students showed very good object-level understanding and overall successful performance in the coursework and examination. This was probably due to the lower level of abstraction that this mathematical task involved as well as the possibility of the use of visual mediators and therefore the assistance of visualisation. Nevertheless, there have emerged two categories of errors. The first was directly related to the listing of the 24 symmetries of the cube and was due to problematic metaphors from Trigonometry and Elementary Geometry and use of problematic visual mediators and mathematical vocabulary. This type of error occurred in two students' coursework solutions. The second category of errors occurred in six students' attempt to prove that there is a subgroup of order 8, in the group of cube symmetries. The errors in this category were due to problematic object-level understanding of the d-object of subgroup. Their difficulty was probably reinforced by the different than the typical algebraic context in which students needed to apply the routine for proving that a given set of elements is a subgroup.

Regarding equivalence relations there have emerged there have emerged three categories of errors. The first category is related to the distinction between the elements of the group $\mathrm{Sym}(X)$ and the elements of the set X and occurred in six students' solutions. This error was predominantly due to problematic object-level understanding of the definition of group and the difficulty to distinguish the role of the set X and of the group $\mathrm{Sym}(X)$, when these coexisted in the same context. Problematic object-level understanding of these d-objects appears to have an unfavorable impact of limited extend, to the application of metarules governing the routine for proving equivalence relations. The second category of errors is related to the proof of the size of equivalence classes. This error appeared in seven students' solutions and was due to problematic object-level understanding of the structure and form of equivalence classes as well as the d-object of bijection. The third category includes minor errors, related to the proof of symmetry and transitivity, as these appeared in three students' solutions.

Finally, regarding the First Isomorphism Theorem and the prerequisite d-objects, there have emerged three categories of errors. The first category is related to the d-objects of kernel, image and isomorphism and emerged due to problematic object-level understanding of kernel and image, which furthermore resulted problematic proof of isomorphism and application of the governing metarules for proving bijection. The second category of errors is due to problematic object-level understanding of the d-objects of coset, normal subgroup and quotient groups. These errors have occurred in all thirteen students. The third category of errors was due to the problematic metalevel understanding and the application of FIT's governing metarules. These errors occurred in twelve out of the thirteen students' solutions.

4.6.2 Discussion

In what follows I intend to answer the research questions that were set at the introduction of this chapter, synthesize the results and embed them into the literature discussed in Chapter 2. For better presentation, the research questions will precede.

How object-level and metalevel learning develops in the context of Group Theory?

Students' object-level understanding in the context of this introductory module in Group Theory seems to be good at the beginning for the majority of students, yet it becomes more problematic as the module progresses. In certain cases, as suggested by Dubinsky et al (1994), Brown et al (1997) and Iannone and Nardi (2002), the first signs of a problematic object-level understanding can occur as early as in the introduction of the d-object of group. This study suggests that the very first signs of minor problems in object-level understanding for two students occurred when they had to prove that a certain group is Abelian. There were signs of an inherited problematic object-level understanding with the notion of commutativity.

Mathematical learning, in the context of Group Theory, cannot be totally disconnected from the object-level learning of other Pure Mathematics discourses. In fact, the creation of effective mathematical learning requires the creation of well-structured realization trees that may involve various d-objects belonging to different mathematical discourses. This claim has been proved in many instances where problematic object-level understanding of d-objects in other modules would affect the quality of students' solutions. This conclusion is in agreement with other studies, such as Dubinsky et al (1994) and Asiala et al (1997, 1998), which point out the importance of well-established schemata. Moreover, according to Sfard (2008), successful problem solving requires moving from one realization to another with dexterity and agility. The findings of this study suggest that the process of moving between interdiscursive and intradiscursive realization trees is not without obstacles.

For instance, one of the overarching conceptual problems is the confusion between the d-objects of set and group and between their elements, especially when the two structures coexisted in the same mathematical task. This suggests lack of fluency of movement between the realization trees of sets, groups, and their substructures, and the amenable metarules. A slightly contradictory conclusion is that in many instances and in various contexts many students were able to successfully apply a certain routine, but on other occasions they were not able to distinguish the elements of the different structures. This suggests that object-level learning does not always precede metalevel learning. Successful application of metalevel rules does not necessarily imply that all the involved mathematical d-objects have been fully objectified.

What is the general impression about the students' encounter with the binary operation? What was the impact on their object-level understanding of group?

The majority of students do not seem to have fully objectified the dual character of the d-object of group, despite the generally good performance in CS1E1. Similar to what Iannone and Nardi (2002) suggest, group is often considered as a special, more sophisticated type of set, whose attached feature, namely the binary operation, is of secondary importance. Moreover, in many students' perception, the binary operation is not central, and therefore they are not able to realise that the group axioms emerge from the binary operations.

The problematic objectification of the d-object of group and the role of binary operation has a negative effect on the object-level understanding of the d-objects that follow. As the discussion is section 4.4 suggests, the first major difficulties occurred when equivalence relations were introduced. One of the major obstacles was that students had not yet objectified the d-object of group together with the binary operation, including several aspects, i.e. how inverses work and how elements as such can be manipulated. Moreover, a problematic objectification of the group operation has an unfavorable effect on the object-level understanding of equivalence relations and consequently on the application of metarules for proving that a relation is equivalent. On several occasions, there was a contradiction between students' seemingly good object-level understanding of the three characteristics of equivalence relations and their problematic attempts to apply the group axioms.

How students evaluate their solutions? How problematic understanding influences the quality of their solutions?

The discussion throughout this chapter suggests that there is possibly a connection between the quality of students' object-level learning and their capability to accurately evaluate their performance. In addition, there seems to be some connection between the quality of their understanding and the quality of their solution's narrative presentation and syntax. A problematic mathematical understanding is reflected in the lack of explicitness and completeness of their reasoning, a tentative and messy writing that does not follow deductive logic, a decrease of students' awareness of how to approach a certain mathematical task and what tactics to apply, and the inconsistency of their narratives.

In certain cases, it was possible to track the different levels of quality of the mathematical narratives within the same coursework script. When the object-level understanding was very good, for instance in more 'concrete' tasks, such as the listing of cube symmetries, narratives were relatively more descriptive and the language more authoritative. In addition, there was a more extensive and effective use of visual mediators and the summary of the results was presented in different formats, such by using tables. On these occasions, it was possible to track the transition from objectification to subjectification. That was expressed in the form of instructions; students' narratives resembled instructions that should be followed by another mathematician in order to solve or understand the task.

What is the relation between object-level and metalevel understanding?

Students' object-level understanding and metalevel understanding are often independent from each other. There were instances in the discussion in this chapter that problems in one level of understanding did not affect student understanding in the other level. For example, although for the great majority of students, applying the routine and the governing metarules for proving that map is a homomorphism, was a successful task, the overall analysis suggests that their object-level understanding was problematic. The opposite also occurred, in many students' cases they had showed good object-level understanding of the d-object of subgroup, yet, they were not able to apply the routine and metarules for proving that the given set is in fact a subgroup. This difficulty often occurred when the context of the exercise was different from what they had experienced.

Moreover, problematic encounter with the metalevel rules is partly dependent on the inherited unresolved problems from other Pure Mathematics discourses. As the module progresses, object-level learning becomes more problematic, which has an unfavorable impact on the use of the involved metarules and metalevel learning in general. Similarly, Moore (1994) suggests that a problematic object-level understanding is a source of problematic proof production. Furthermore, very weak students differ from others in that their object-level learning is problematic from the very beginning, with indications of disengagement with the core structure of the group and its axioms. In these cases, application of any metarules in any kind of routines was impossible.

What are the students' main problems in the application of metarules?

Two distinct problems emerged in the application of certain routines and their governing metarules. First, students' application of metalevel rules was often erroneous at certain stages of the proof, and second, many students were underestimating and moreover omitting certain steps.

Several problems occurred with the application of the metalevel rules in the context of several routines. These problems were identified in accordance with the actual definition of the routine as a set of pattern-defined metarules, divided into the 'how' of the routine, namely the *course of action*, and the 'when' of the routine, namely the *applicability* and *closure conditions*. The level of performance in the various routines followed a regressive development, mostly regarding the *course of action*.

The second problem occurred, for instance, in the application of the routine/test for a set to be a subgroup. Many students would usually overlook the proof of the first condition, i.e. non-emptiness, despite their good understanding of the metalevel rules and their capability to apply them successfully to the other two conditions. This suggests that in the very first stages of metalevel learning in a new mathematical discourse, students have an undefined tendency to evaluate the importance of several steps, as a result of their undeveloped metalevel understanding.

How different contexts of the various mathematical tasks influence metalevel understanding?

Successful application of metarules of a certain routine in a specific context does not automatically imply that there will be the same success with a different one. For instance, in the case of the subgroup test, students who successfully applied a routine in a more 'algebraic' context were not as successful in applying it to a more 'geometric' context. This observation suggests that metalevel learning, in particular, requires extensive involvement with mathematical tasks in different contexts. These contexts should vary in three characteristics: in *nature* (arithmetic, algebraic, geometric, etc.), in *field* (Linear Algebra, Complex Analysis, Calculus etc.) and in the *level of abstraction*. Good performance or understanding in one context does not necessarily imply good performance or understanding in a different context.

What this study suggests to be the general requirements for successful mathematical learning in Group Theory?

Successful mathematical learning requires a relative *ontological understanding* of the particular discourse. This conclusion is in agreement with Nardi's (2000) more notion-specific suggestion that students need to know the raison-d-être of the concepts they use. In the context of this study, Algebra is mainly the study of several structures, namely groups, rings, fields etc., and the corresponding substructures, for instance, in the case of Group Theory, subgroups, cosets, equivalence classes, and quotient groups. The discussion in chapter suggests that novice students' first major difficulty is to understand these less concrete substructures, the amenable metarules, and the central role that they play in the context of Group Theory. When new substructures are introduced and the prerequisite objectification of other structures is problematic, for many students, their engagement develops unfavorably. For instance, for understanding equivalence classes, it is required to resolve, at least, any object-level misunderstandings regarding the sets, groups, subgroups and their elements, and the notions of bijection and equivalence relations. If students do not objectify the prerequisite d-objects and realise that an equivalence class is a structure within a structure having certain characteristics, it will be impossible to achieve stable object-level and metalevel understanding.

What this study suggests about the students' encounter with the FIT?

A fully developed object-level understanding of all the aforementioned d-objects and effective understanding and application of the metalevel rules are essential requirements for the use of the First Isomorphism Theorem, which, as Nardi (2000) also suggests, is considered the pinnacle of this introductory module to Group Theory. FIT is a very demanding mathematical theorem for novice students, since it requires a thorough understanding of all the relevant d-objects, such as kernel, image, coset, normal subgroup, quotient group and the relations of homomorphisms and isomorphisms with the related routines. In addition, even if full object-level understanding is achieved, the application of FIT is still a serious challenge since it requires good metalevel understanding.

What this study suggests about the students' encounter with homomorphisms and isomorphisms?

Students' encounter with the notion of homomorphism is generally successful. The discussion in section 4.5.1 suggests that this is due to the fact that proving that a map φ is a homomorphism does not usually require the full object-level understanding of the notion of homomorphism, but rather a successful application of a relatively 'concrete' and explicit routine with its governing metarules.

Contrary to the above, and in agreement with Leron et al (1995), for the majority of students, their encounter with the notion of isomorphism was problematic. This occurs partly since the majority of students have not properly objectified the d-objects of kernel and image. In fact, the second major conceptual crisis occurs when the d-objects of kernel and image are introduced, since many students, as discussed above, do not automatically realise that kernel and image are substructures of the group with certain characteristics. The apparent conceptual disengagement with the d-object of isomorphism is not predominantly based on the objectification of homomorphism, but rather on the objectification of the d- objects of kernel and image, and the metalevel rules regarding injectivity and surjectivity.

What this study suggests about students' encounter with the d-objects of coset, normal subgroup and quotient group?

Difficulties in the object-level learning occurred, for the majority of the students, especially regarding the d-objects of cosets, normal subgroups and quotient groups, as other studies such as Asiala et al (1997), Iannone and Ioannou (2011) and Ioannou (2010) suggest. Conceptual challenges and obstacles included, among other issues, the understanding of the notation of the quotient group G / N, the fact that cosets are elements of the quotient group, and the distinction between the elements of the group and the elements of the normal subgroup. At a metadiscursive level, students had difficulty to prove normality and to show how to make the set G / N of left cosets of N in G into a group. The significance of a good understanding and effective use of notation, particularly in proof production, is in agreement with Moore (1994), Weber and Alcock (2004), and Iannone and Nardi (2007).

How the students use of mathematical vocabulary and symbols developed throughout the module?

As the module was progressing, and the object-level and metalevel understanding generally became more problematic, for the majority of students, the quality of use of mathematical vocabulary and symbols was decreasing. This was particularly reflected in the increasingly more problematic use of vocabulary regarding the d-objects introduced at the end of the module. Word use often indicated a particular difficulty in the shift towards objectification of certain d-objects. For instance, a homomorphism φ is characterized as 'isomorphic' instead of isomorphism. In particular, this suggests that the d-object of isomorphism has not been reified, since students refer to it as a procedural instead of an object.

Chapter 5 Students' Study Skills and Perspectives on Learning and Proof

In this chapter I discuss the students' perspectives and applied study skills in the context of proof production, preparation of the coursework and revision for the final examination, as these have emerged and discussed in the student interviews. Analysis of students' interviews, described in Chapter 4, has formed these three categories of issues according to which this chapter is structured. Namely, in section 5.1, I discuss students' perspectives about *proofs* and analyse the expressed difficulties that students face in the process of proving, their difficulty to initiate a proof and their difficulty to clearly record their reasoning on paper. In section 5.2, I discuss the students' perspectives about the preparation for the *final examination* and the techniques that students apply for the revision of the learning material and the preparation for the actual exam. Finally, in section 5.3, I discuss the students' perspective regarding the *coursework* and the applied skills and techniques for its preparation and completion.

Moreover, the research questions I aim to answer in the following discussion are: What are the students' perspectives about the process of proof production? What difficulties do novice students face in the proof production in the context of Group Theory? What are the students' study skills, applied for the preparation of the coursework? How students prepare for the coursework in a module such as this?

The discussion that follows suggests that proof is considered a difficult task mainly because of problematic object-level understanding of the involved d-objects as well as problems in the application of metarules. This difficulty has often a negative impact on students' engagement with Pure Mathematics resulting their intention not to study other Pure Mathematics modules. An often-expressed difficulty with proofs refers to the *when* of the applied routines and it is related to the applicability conditions and closure conditions of the routine.

The revision for the final examination involves, firstly, the review of the lecture notes, followed by the solution of the coursework together with the use of model solutions and the solution of the past papers. The order of the last two activities varies. An often-occurring revision technique involves, instead of a linear succession of the aforementioned activities, a spiral approach towards revision, with the three activities interchanging until the students who apply such approach feel that they have achieved adequate object-level and metalevel understanding.

Finally, for the preparation of the coursework, students often summarise the lecture notes and highlight the important mathematical narratives, namely definitions, lemmas, and theorems. This activity, according to students that use it, contributes favorably in their object-level understanding and facilitates the solution of the coursework. Furthermore, many students have expressed their belief about the importance of attending the seminars in the process of the solution of the coursework, and how this activity contributes to object-level and metalevel learning. A small number of students wished to know the assessed questions before going to the seminar, although they expressed their concern about the possibly negative effect this might have in their mathematical learning.

5.1 Proof Production

The production of proofs is considered by eleven of the thirteen (11/13) students as an arduous task, and expressed their difficulty with the process of proving. Nevertheless, perceptions about proofs as such and their role vary among the thirteen students. For instance, proof, an important part of mathematical learning at the university level, seems to be often considered as a form of *communicational interaction* with others. Three of the thirteen (3/13) students seem to realise the importance of effective communication, and the significance of bridging the gap between their possibly immature ability to express their reasoning and make an experienced mathematician understand their proof. This claim is obvious in the following excerpt.

> *But I – yeah, again, it might be – me not – it makes perfect sense, but I might not... make it – it's just like you know – I can understand it, but it's trying to, I mean because proof is really trying to make someone else understand it, and I say, possibly I do struggle at – giving, you know, making someone else understand it by writing it down, but, so it's where I might lose some marks, but...* Manrico

In this case, Manrico shows particular sensitivity in effectively communicating his proof and making his reasoning clear. It is his priority not just to apply the appropriate routine and produce a correct proof, but also a comprehensible proof, which according to the above excerpt possibly increases the level of difficulty. In addition, the excerpt possibly suggests that Manrico is putting into action the human capacities for language i.e. adequate reasoning and effective abstracting, since Group Theory is a module of a higher level of abstraction for novice students (capacities referring to commognitive objects).

Manrico's effort to achieve successfully communicable proof possibly indicates two things: realisation of the significance of proving as an essential part of the learning of Group Theory and development of his mathematical maturity. Moreover, there is an indication of awareness of the need not only to understand or objectify the newly introduced d-objects, but also to improve his metalevel understanding abilities, to apply the appropriate metarules, and moreover produce clear and valid proofs.

Four out of the thirteen (4/13) students overtly and explicitly expressed their difficulty with the concept of proof as such, not only in the context of Group Theory, and the negative impact it has in their engagement with Pure Mathematics and their intention not to study any other Pure Mathematics modules. The following is a representative example of students' perception of difficulty regarding proofs, and which also indicates future disengagement with Pure Mathematics, because of this difficulty.

> *Yeah, definitely, I think – it's quite a different style of Maths as well, isn't it, when like – especially the Pure side where, you suddenly have to do proofs, I think that was what first shocked me, about this degree, cos I'd never done it. […] Um, yeah, this is something I'm doing at the minute, and it worries me for next year – I prefer Pure, as like – interesting, but I do worse in it (laughs) so I prefer Applied, in the fact that I get better marks, cos I suppose you know when you've got the right answer don't you, whereas Pure is quite fiddly and you can – do a proof but not write it quite how they want […] I suppose... in applied, it's more like – what you've done all your life, there's a method, and you just follow the steps, and you should – hopefully, get the answer. And like, you fiddle with it, don't you, when you're doing integration and stuff, but you follow the steps? Whereas in pure, I often find, especially when it's the proofs, it's like, just completely random, like, it doesn't follow something that you've done already?* Amelia

For Amelia, proving was considered a shock, as she did not have any experience of proof production in her secondary Mathematics Education. Amelia's shock is located in the way in which she should approach proofs. Successful proof production requires two elements: successful objectification of the involved d-objects and familiarisation with the governing metarules *in the context* of Group Theory. The results in Chapter 4, as well as Sfard (2008), suggest that familiarity of metarules in a mathematical discourse, even if it is part of the same general field, does not imply immediate command of metalevel rules in a different mathematical discourse.

Even though Amelia had some preliminary Group Theory experience in the first year of her degree at university, the familiar routines she has used are now not adequate for the level of rigour and abstraction of this module. This suggests that variability of metarules is not achieved automatically, but with experience and familiarity with the newly introduced mathematical discourse, in this case Group Theory.

Moreover proof, in the context of this module, has apparently caused commognitive conflict, since the metarules that the lecturer is using are different from the ones that Amelia used up to that point. Therefore the reported shock is for Amelia a challenge related to metalevel learning. One of the major obstacles that novice students face in their transition from secondary to tertiary Mathematics is possibly to adjust their learning approach so they are able to address the metalevel learning demands that the rigorous university teaching approach, the nature of mathematics and the overall university education require. This last claim emerges from Amelia's reference to her success in Applied Mathematics, because of the similarity of "what you've done all your life". Yet new demands on metadiscursive level require the transition from the 'method' and the 'just follow the steps', to the analytic and creative thinking and the ability to follow develop metalevel understanding together with the object-level understanding. What is described here is an indication of commognitive conflict and therefore an invitation to achieve the discursive shift, which is the essence of mathematical learning.

Certain students were able to locate, in specific, what difficulties they faced in the process of proving, making the distinction of the *how* and *when* characteristics of the routines. As discussed in Chapter 2, the *when* characteristic of the routine is subdivided into two condition categories: the *applicability conditions* and *closure conditions*. An example of problematic encounter with the applicability conditions was apparent in the following excerpt from Calaf's interview.

> *The problem with pure maths… there's a lot of theorems to learn and to – you know, to prove something, you have to use all the basic theorems, to prove that problem.* […] *I think pure maths is more – your brain have to think, a certain way, rather than sort of – applied way, you have to think different way, to applied maths, that's why I think applied maths is easier because you can use the basic of the pure maths, what the pure maths people created, and you apply it for real-life situations, sort of thing, and pure maths is I think, something totally original… sort of like new theory coming out, so that's what*

> *they want to do… so that's why it's harder to think of something new, rather than just – apply it, and have a little on top of that.* Calaf

As the above excerpt suggests, some novice students face difficulty in combining the related theory with the new metarules in order to produce correct proofs, in the context of Pure Mathematics. The *how* of a specific routine is a set of metarules that determine the 'course of action'. Similar to other novice students, Calaf seems not to be confident about the way he should approach a mathematical task that involves proving, and moreover which mathematical narratives, namely, theorems or lemmas he should involve. This possibly implies that these students are not yet in a position to successfully prove a given mathematical task. In addition, Calaf locates the difficulty with the *how* of routines in the characteristics of abstraction and generalisation of Pure Mathematics, contrary to the more concrete nature of Applied Mathematics, as he has experienced them.

Proofs, as routines, do not impose predesigned modus operandi. Therefore, the novice student is required to demonstrate agency, creativity and inventiveness, qualities which require a thorough understanding of the theory and therefore the ability to distinguish the *how* and *when* characteristics of the routines, in each case. As the results of the thirteen students, listed in Chapter 3, and the analysis in Chapter 4 suggest, the majority of students are not able to fully develop these qualities in an academic semester during which they experience their first encounter with Group Theory. University Mathematics, unlike primary and secondary Mathematics, involve complicated routines with complicated pattern mechanisms, making therefore the *how* characteristic convoluted and sometimes inaccessible for many students.

Another difficulty with the process of proving emerged in the interviews with Kostanza. Similar to Calaf, she locates her main problem with proofs in the *when* characteristic of the routines, but this time focusing on the closure conditions that define the successful completion of the given mathematical task.

> *Oh sorry, yeah, cos you need to know what it is – it's terms – but yeah, but it's just um, yeah, I've always had problems with proofs, and there's a lot of proof in groups, isn't there? […] I always think that – I never quite know if I've proved it enough, like- I never know when I've got to the end of a proof, and I keep trying to carry on and then I'm like well actually that's the end and I'm like – I never quite know, when what I've got matches up with what I was trying to show? Yeah, just like – now I never know when it's quite – proven... completely, so...* Kostanza

> *Yeah, yeah, it's never quite – and then – I never know if I've quite got to the end of the proof and I never know if that fully shows you and – like even when it gives you examples in the notes, I'm like – so why does it finish there, why – why is that enough? It's really – that's – my main problem with pure maths is that I never – it's the proof of – when it's got to the end, yeah?* Kostanza

Kostanza's statements suggest that she has yet to acquire the metarules that will allow her to identify the circumstances under which the proof is completed. This statement highlights the commognitive conflict, which is located in the tacitness of metarules. Kostanza is aware of the discursive shift that is required in learning Pure Mathematics. She seems to realise that in some way she needs to adjust her 'solution skills' to this new mathematical discourse. She is not able to interpret what is required to successfully prove the mathematical task. This indicates that metarules should be understood tacitly with experience. This is a characteristic of University Mathematics Education, which requires a more personal effort to eventually achieve metalevel learning.

An often-reported difficulty that seven out of the thirteen (7/13) students faced was how to start a proof, i.e. the initial step. For many students, getting help on how to start a proof is a main incentive for attending the seminars, as the following excerpt suggests.

> […] *Most of the time I can't do them, and I need a little... sort of, starting point, so I – I – but I'm – I have a good kind of understanding about what the question's asking me to do, um, before I go, so I can ask the right questions... cos it's a lot more abstract, like you don't have anything to compare it to… I mean there's a lot of imaginary stuff in it…It's like nothing you've ever experienced before* […] *I mean I've done groups before, so I know kind of – the basics of it umm, just to get a greater understanding really.* Dorabella

The above suggests that normativeness of metarules is not automatically established among all members of the mathematical community. Normativeness of metarules in a certain discursive context is objective, since it requires experience. Well-established metarules that would allow experienced mathematicians to successfully solve a mathematical task would not automatically be obvious to the majority of novice students.

Moreover, difficulty with starting a proof is possibly an indication of problematic metalevel understanding regarding the *how* of a routine as well as its applicability conditions. Application of metalevel rules governs the formulation and substantiation of the object-level rules. Incomplete object level understanding, as Chapter 4 suggests, may hinder the successful application of the involved metadiscursive rules and the construction of the required proof, although this is not always the case.

As the above excerpt suggests, Dorabella, similarly to other students, seems to have neither fully developed a thorough metalevel understanding of the related metarules nor does she have a clear perspective of what she wants to prove and how she is going to prove it. This is not an unexpected or rare event among novice students in university Mathematics in general (Moore, 1994), and in Group Theory in particular, since students need to achieve a transition towards a more sophisticated level of metarules and abstraction, even between different university Mathematics modules.

Similar to above, the starting point of a proof is a particularly difficult step, especially when students have seen nothing similar to the proof under discussion. The following excerpt of Leonora highlights the importance of previous experience for the development of metalevel learning.

> *I don't know, I mean I think once I've done it, and been told – like – having – so I've got like an example basically, of how to do it, then – it will be in my mind, so it'll be hopefully, something I can keep repeating, but just initially starting it off and – it's – I find quite hard, I found that quite hard with like a lot of things, it's just initially start…* Leonora

The above excerpt possibly suggests the necessity of some students to be guided in the first steps of the learning of a new mathematical discourse of guidance and examples. For these students, examples possibly have a twofold role, first to improve their object-level understanding regarding the involved d-objects, and second to enrich their experience of how metarules should be applied.

The primary importance of the starting point in proof is also in agreement with Norma's perception about proofs, considering it, as the major obstacle to be overcome. After overcoming this threshold, it is much easier both to gain perspective and to move on.

> *I do try and do something, but it – it may not be like a specific question, cos sometimes, err, I just need help in getting started, and then once I've started I'm ok, it's just – finding like the thing to do first… Norma*

This is again an example of problematic encounter with the applicability conditions in the process of proving, as possibly a result of lack of experience in the majority of novice students. Deciding about the *how* and the *when* of a particular routine, namely what course of action to follow and how to initiate and finish a proof, is an essential part of metalevel learning of a particular mathematical discourse.

Discussions with students have revealed their perception about their general approach to required object-level and metalevel learning. It was apparent in the case of five of the thirteen (5/13) students that instead of trying to understand the related d-objects and the amenable metarules in proof production process, these students, at least at the initial stage of their learning, would excessively depend on similar examples, Internet and book use or other exogenous factors, and mechanically imitate them. This approach to proving, and learning in general is clearly expressed in the following excerpt.

> *I googled for one of the proofs, to see if it was on there, but it wasn't, but there was something similar, that then I worked out like – that you're just meant to go through and then times it by the inverses and stuff, but um – I think, some of our algebra nowadays is so... specific, that like there aren't proofs and stuff out there that's – that's easy to find now.* Amelia

Overdependence on exogenous sources for understanding instead of focusing on the endogenous change of discourse (i.e. metalevel learning) is obvious in the above excerpt. Apparently, some students need to see similar routines to the ones they have to produce in the context of an exercise. Studying similar routines possibly helps them to understand the metarules that need to be applied in certain situations and to be able, at the first stages of their learning, to imitate. In addition, Amelia seems to realise the level of specificity and rigour that proofs in Abstract Algebra require through her effort to find similar proofs. Nevertheless she uses the 'similar' proof to her benefit in order to learn the metarules related to notions such as inverses.

Regarding the issue of self-evaluation of students' produced proofs, many students, based on their experience so far, show an awareness of the quality of their proof as well as whether their proofs are correct or not, even when they closely follow a routine similar to the one they are supposed to produce. In the excerpt below, Francesca discusses the importance of a logical order of steps of a proof, without missing any.

> *You should follow the correct order though… because I remember once I was jumping steps in a proof and I was considering some steps as granted… I shouldn't though… I had to prove every step and then go to the next step… That's why I lose so many points… because I arrive at the correct result but I was missing some things…* Francesca

She seems to be aware of the normative nature of metarules and that she is expected to meet the required characteristics of explicitness and rigour. She realises that proof production is in a way a commognitive activity within a certain discourse, which has well-established metarules. Even though she is willing to apply a routine that will lead her to an endorsable and valid mathematical narrative, she does not yet possess the required capability to apply the involved metarules that will allow her to achieve this, in this specific mathematical discourse. This claim is in accordance with the data analysis in Chapter 4, where in many occasions, students' reported intention of action is in disagreement with the action taken, even if they are aware of the inappropriateness of their actions.

Students' own evaluation of their proofs is an interesting issue from the secondary-tertiary level Mathematics transition. According to Gueudet (2008), novice students do not have the experience that will allow them to decide whether a proof is valid or not. Nevertheless the above excerpt and the ones that follow shows that students, although they may not have the capability to precisely evaluating their proofs, can say however, whether it is problematic or adequate, based on their previous experience.

A representative example of contradiction between the reported intention of proof strategy and the applied action can be seen in the following excerpt. Manrico, although he eventually does so in practice, is aware that a proof over-dependent on visual mediators is not rigorous enough and acceptable in the context of Group Theory and not in accordance with the metalevel norms.

> *Hmm. see, there – this thing – I mean – the – statement, makes sense... I drew a little picture and like – I was just like – I mean – course that's going to be in it, but – how you prove that by actual kind of – prove that mathematically rather than just drawing a picture and just saying, it is true, it's just the actual showing that...* Manrico

Even though the use of visual mediators is an important aspect of mathematical discourse and often an indication of object-level understanding, extensive use is not an adequate approach for proving a mathematical narrative, especially in the context of Group Theory, at least as it is taught in the specific Mathematics department[7]. Excessive use of illustrations and lack of algebraic reasoning in the proofs often indicates problematic metalevel understanding.

Proof production also depends on the thorough understanding of d-objects and their realization trees, as well as the good interaction between the object-level and metalevel rules. The excerpt below reveals the negative consequences on the interaction between the different realisation trees as these have been developed possibly in different modules.

> *I cannot understand many things that… for example… one of the things I cannot accept is… I am given an exercise in which I have to prove something and in the notes we are not given something that will help us or guide us to solve the exercise… or the fact that something that we see now it is related to something that we have seen several months ago…* Musetta

The above suggests that Musetta's learning lacks connectivity between the different modules in her degree. Although she is aware of this lack, her approach towards overcoming this issue is rather passive, and as her written

[7] As discussed in Chapter 4, in many instances in the marked coursework, markers would not welcome excessive use of visual images and it would have a negative effect on the marking.

data suggests, she has not effectively achieved it. Moreover, if the d-objects and the corresponding realization trees involved in a particular discourse or other related ones have not been encapsulated, then discursive expansion has yet to be achieved. If the student is not able to construct usable and accessible realization trees, then proof production is very difficult, if not impossible to be achieved. As the discussion in chapter 4 reveals, mathematical learning in Group Theory requires realization trees that involve compound d-objects emerging by reification, namely regarding the shift of the focus from processes on the group-theoretic objects towards discussions of the group-theoretic objects as such and their relations.

In sum, this section focuses on the students' perceptions about proofs and the difficulties they face in the process of proving. Eleven out of the thirteen students believe that proof is an arduous task. Three students have emphasized on the role of proof as a means of mathematical communication between mathematicians. Four students expressed their difficulty for proofs and the negative impact this has on their engagement with Pure Mathematics and their intention not to study further Pure Mathematics modules. Moreover according to certain students' perceptions as these have been revealed in their interviews, the main problems with proofs are related to the *when* of the applied routines. In particular these difficulties are related to the applicability conditions, with special focus on the initial step of the proof and also related to the closure conditions, namely what signals the end of a proof. Signs of immature approach to proving were obvious in the case of five students. These students' reported priority, in the initial stages of their learning, was not to achieve object-level understanding but to find similar examples to the given task, and copy them.

5.2 Final Examination

In this section I focus on the students' skills and techniques of preparation and revision for the final examination, highlighting also their perceptions about it. Twelve of the thirteen (12/13), discussed their approach towards the revision for the final examination in some detail. Carmen did not talk in detail about her revision techniques, mostly because of her poor English language skills.

Eleven out of the thirteen (11/13) students study the lecture notes, solve the coursework using the model solutions given by the lecturer and solve a various number of past papers. Otello was the exception. Preparation for the final examination is the final stage in the students' learning process for most of the mathematical modules that form part of their university education. At this stage students are invited to resolve any preexisting conceptual gaps and overcome any commognitive conflicts. This period and eventually the final examination will be the last act of their mathematical learning, which will ideally lead them to full objectification of Group Theory and moreover to subjectification and mathematical consciousness.

As the following discussion will reveal, usually, the first step for revising is the study of the lecture notes. Students' approaches vary, but their predominant aim is to go through the definitions and theorems, both to improve their object-level understanding but also to memorise the ones that will possibly be asked to state. In addition, five of the thirteen students (5/13) students produce their own revision notes, which help them to improve their object-level understanding and assist them in memorizing easier, as the following excerpt suggests.

> *I normally write out my notes, a lot... Hmm, yeah like I make revision notes, and I do revision cards. And I normally just sit and rewrite out the definitions a million times and the theorems a million times, and just like – do the revision cards and get people to test me and I'll write them down, and then I'll work through past papers and all the problem sheets.* Amelia

A significant part of the exam paper each year (See Appendix C) is to state various mathematical narratives, such as definitions and theorems and reproduce important routines such as the proof of Lagrange's Theorem or First Isomorphism Theorem. The majority of students, as their exam solutions suggest, memorise these routines from the lecture notes. This approach to mathematical learning is possibly limiting because it does not encourage the students to comprehend and endorse these narratives and routines. This is confirmed by the fact that students were often not able to use these narratives for solving a related mathematical task in a subsequent question (as the data analysis in Chapter 4 suggests).

The remaining seven of the thirteen (7/13) students study their lecture notes as part of the process of revision without producing revision notes. This is usually the first step of their revision. Studying the lecture notes for the final exam requires a different, all-inclusive, approach from the preparation of the coursework. Naturally, students are invited to develop further their object-level and metalevel understanding of a well-organized mathematical discourse and achieve successful construction, if not yet achieved, of several realization trees that will allow the student to move with 'dexterity' from one realization to the other.

Studying the lecture notes for the exam is a 'renewed task' leading to improved understanding of the theory. As the following excerpt suggests, having a holistic picture of the entire theory, and consequently having already, up to a certain extent, created realizations of the involved d-objects and realization trees, makes the task of revision and objectification a different experience.

> *Usually, like the coursework… we start from the lecture notes…and usually I am trying to understand everything… not like when we prepare a coursework. . For the coursework we do not have much time so we are going for the exercises… I believe that if you do not understand something, then you cannot understand what it follows as well… In the past, I used to make my own notes, but since it was time consuming, I decided to stop that… I study the notes and I highlight the important things… Something that I need to see again… I study only from the notes…* Musetta

The above excerpt is a representative example of all thirteen (13/13) students' awareness regarding the different approach that students should follow for the examination revision. Musetta expresses her desire to change her study approach and wishes to improve her understanding of the material. She realises both her problematic approach towards the material as well as her limited understanding. She identifies that solving a mathematical task without studying the related narratives and routines is a faulty approach. For her, studying the lecture notes as part of the final revision is a task that has to be faced anew. Experience has led her to prioritise efficiency in her study skills and approaches, as well as the awareness of the demands of examination revision and the ways to cope with it more effectively.

The next step in the twelve of the thirteen (12/13) students' revision is usually the solution of the coursework and past exam papers. There are two distinct categories of students according to which of the two tasks they undertake first: six of the thirteen (6/13) students are studying the coursework first and six of the thirteen (6/13) students start with past exam papers.

Studying the coursework together with the given model solution, as occurred in the twelve of the thirteen (12/13) students' cases, is an important step in the learning process. As the following excerpt suggests, this revision approach allows students to have the chance to exactly locate their weakness and improve their object-level understanding of the definitions of certain d-objects, at this final stage, mainly by reifying. Consequently this process will allow them to successfully cope with the level of abstraction, improve the structure of the realization trees of these d-objects and consequently objectify them, something that it will permit them to achieve better metalevel understanding.

> *Um, probably with the questions that we've been given, and with the solutions, I'm hoping to like – help teach myself how to do it… and then I learn by doing past exam papers, mainly, so I intend on doing like – a lot of them, until like – I tend to do like quite a few years back, like do all of them, and once I've done them, go back, and like the*

questions that I have – like, didn't – weren't able to do before, I try and do it again, cos I've hoped that I've taught myself. Leonora

In the above excerpt, Leonora considers working with the coursework and the model solution as a means to 'teach herself' the *how* and *when* of the routines involved. It is a chance to correct and/or improve her object-level and metalevel learning, application of metarules and solving techniques, and consequently overcome any commognitive conflicts resulted by the learning of this new mathematical discourse. Using the solutions, the particular students will be able to see the metalevel rules of Group Theory in practice and understand them in more depth and how they should be applied. For these students, model solutions are apparently an indispensable tool that can be used in order to resolve any preexisting commognitive conflicts and improve the realization trees. These students will possibly have the chance to realize not only the metadiscursive level rules, but it will also allow them to understand how they should approach a mathematical task in general, namely, specifying the *routine prompts*[8], applying the decided *course of action*, and successfully *completing* the task.

Another positive impact of working with the model solutions while revising the coursework exercises is the improvement of self-confidence. Although this perception was clearly expressed by only Tamino, I believe that is important to be highlighted, since other students have implied it as well. In particular, according to Tamino, working with the coursework helps him to increase his level of confidence, which is a key element of success in exams. When this task is completed, he then works with the past papers.

> *I don't generally look at the exam papers… only slightly towards the end – only because they can freak you out if you – I like spending a few days building up your confidence just reading through lectures notes and that sort of thing – examples of the course sheets I like looking through them for a while then go....* […] *You need confidence.*

[8] Elements of situations whose presence increases the possibility of the routine's performance.

If I have confidence I am quite good. I can actually generally breeze through even if I don't actually know the answers entirely. Tamino

Working with the coursework and the model solution in parallel is a learning technique that most probably allows these students to improve further their object-level understanding of the d-objects and revise the routines and the governing metarules involved in the exercises. Working with their marked coursework, they are possibly able to easily locate their errors, since the markers highlight them, study the problematic narratives in the applied routines and overcome any commognitive gaps that have occurred. This process of learning will help them to gain confidence and independence, to some extent, as students and as mathematicians; an important element for their revision and performance, according to Tamino.

Another approach to revision for the final examination is by studying the past papers first and then the coursework. For instance, Manrico is planning to start by working with the past papers. This will allow him to identify the demands of the examination as well as his weaknesses.

Um, well I'll definitely be looking at past papers. […] Then go to lecturers and just get feedback on what I've done, and then they'll help me say like oh no don't do this, or yeah, you're doing all right in this bit. So any kind of gaps in my knowledge hopefully they'll – help fill in. […] *I kind of look at what would come up on the exam, and um, then I kind of find out where the gaps were, have a little look at lecture notes, maybe a few problem sheets, then maybe get and attempt another one, with a bit more knowledge. So it's like… actually I do one, that I kind of do with my lecture notes open really, then try – as I'm getting a little bit better, try and do it without the lecture notes, cos obviously that's gonna be what's happening in the exam.* Manrico

This approach allows the students who adopt it to identify the difficulties that will possibly face in their examination, identify the expected types of questions that will probably need to solve and therefore adjust their revision in order to overcome the new object-level demands in their understanding and revise the routines and mathematical tasks in general. In the particular case of Manrico, he is the only student that is willing to ask assistance from the lecturer in solving past papers. Regarding the overall process of revision for the final examination, as Manrico has suggested above, five of the thirteen (5/13) students have clearly stated the interchange in three activities, namely studying the lecture notes, reworking the coursework and solving the past papers. This approach can be described as a *3-dimensional, spiral approach towards revising*.

The five students that follow this revision approach, work with the lecture notes, coursework and past papers in a interchangeable way until they think they have overcome any commognitive conflicts caused by the nature of Group Theory; securely shifted towards the metalevel learning; and finally reached adequate object-level and metalevel understanding. In each spiral cycle of revision their level of comprehension improves. This process probably helps them face any commognitive conflicts that are in abeyance. This possibly helps students to cope with the level of abstraction. There are no substantial data, though, that would allow us to claim that this form of revision has better results than other approaches. Nevertheless, it has the possibility to improve mathematical learning, and it is preferred amongst a number of students.

Unlike Manrico, Calaf does not require any assistance from the lecturer but instead he is marking his solutions to the past papers by himself.

> *And then start past exam papers, and get the solution and see what they're looking for in the exam questions… So do a few of that, and do a proper exam conditions, and…* […] *Mark it – no – I done it first, mark it, and then look at the marks scheme, yeah, mark myself and see how much I get? And then, after a few days, redo the paper again, to see how much I improve, or which area I still don't understand or something.* Calaf

Calaf's approach towards revision has some very useful and interesting elements, such as the solution of past papers under exam conditions, or the repetitive, spiral like, approach already encountered in Manrico's case. His active approach to revision is manifold with repetitive cycles that possibly allow him to better objectify the material and enrich his experience. Relying solely on his marking, though, without asking for any external control might jeopardise his learning. Calaf's examination results (50%) do not show that his promising revision scheme has led to the expected outcome.

The weakest students, Francesca and Musetta, expressed a similar perspective regarding what makes a good examiner according to which good examiner is the one whose papers are the same every year.

> *Last year we had a very good lecturer… his papers were exactly the same every year, but with different numbers…?* Francesca

The above statement suggests that these students adopt a 'utilitarian' perspective of learning Mathematics at the university level. This statement, as well as her performance, suggests a difficulty in the transition from secondary education towards university Mathematics education and its demands. It indicates that her mathematical thinking is, according to Sierpinska (2000), only practical based on *prototypical* examples that she needs to have seen in the past papers, and not *theoretical*.

Finally, Otello is the only student who does not revise by using the three elements of revision, namely the revisit of lecture notes, the solution of coursework making use of the model solutions, and the solution of past papers. He rather uses only the first two. In particular, Otello makes use of books, in parallel with the lecture notes and coursework, and on many occasions places special emphasis on the way he reads the books and the relaxed pace of his reading.

> *It's a matter of revising what you have done and revising your coursework answers, going through books just being relaxed. I am*

> *sure most other people would have a different approach would be do past questions, but I prefer to be more relaxed more laid back about it.* […] *I don't want to go through it at top speed, just go through it normally. Hopefully it will sink in. But then I read it and then close the book and try to reproduce what they have....* Otello

Otello's perception is quite distinct, indicating his effort to not only approach revision in a superficial way and get a good mark, but rather as a chance to improve his object-level and metalevel understanding in the discourse of Group Theory. He considers exams as an opportunity to widen his object-level and metalevel knowledge, overcome any possible commognitive conflicts that occurred in the coursework and hopefully achieve endogenous discursive expansion. The last is encapsulated in his phrase "sink in".

The above excerpt possibly indicates maturity in his way of reading a mathematical text. Otello has realised that reading a mathematical text is fruitful only if the pace of reading is not fast, but rather compatible with the difficulty of the test and the speed in which an individual understands the discourse. His approach is overall mature, indicating successful transition towards university Mathematics and its norms, showing also awareness of the amenable affective issues by trying to keep himself relaxed.

In sum, regarding the revision for the final examination, the majority of students (12/13) revisits the lecture notes, rework on the coursework using the model solutions, and solve past papers. There have emerged two different trends regarding the order of these activities. All students start with the lecture notes, yet six students work with the coursework first and six students work with the past papers first. Five of the thirteen students apply a spiral approach towards revision, with interchange of the three elements until they achieve adequate object-level and metalevel understanding. Weaker students, namely Musetta and Francesca, have a utilitarian perspective towards mathematical learning, according to which good examiner is the one whose papers are the same each year. Finally, mature students, such as

Otello, have a responsible perspective towards mathematical learning, and whose priority is learning as such.

5.3 Coursework

In this section I mainly discuss the students' methods and applied skills for the preparation and completion of the coursework, highlighting also their perceptions about it.

Condensing the lecture notes, by making record cards, shorter notes or highlighted notes is a technique applied before working with each piece of coursework. Amelia and Tamino have adopted this approach in a similar way.

> *I always write up my coursework notes, like I've got them on big A4 plain sheets and then with blocks of color, cos I just can see them better and then I can find the definitions quicker, so I wrote all – wrote up all my lecture notes like that, and then I used them to go through and find... the definitions, or – whatever in the question and try and pick the question apart. And then mark down if we've got any hints, I try and use that, and then try and work it through.* Amelia

> *Well I read through the lecture notes a few times then I generally shorten them – which is a weird thing – I actually cut them down to notes to note and then I sometimes cut them down again so then I have got a really short key points and then I will basically learn them and then I will look at it…* Tamino

These students' need to downsize the lecture notes and given hints to make them easily accessible and usable for coursework and examination revision purposes is quite apparent. Firstly, it is an effective way to highlight the definitions of d-objects, identify, if possible, their realizations, and help them to construct the emerging realization trees. In addition, they list and use theorems in a more effective way during the process of solving mathematical tasks both for the coursework and examination. Secondly, in this way they pinpoint the definitions, theorems and proofs that they are likely to be asked to state or produce in the examination. Finally, coloring different blocks of narratives and routines assists students in achieving embodiment and automation of realizing procedures (physical and mental actions respectively contributing to realization of certain mathematical objects).

An important element for the preparation of the coursework, overtly expressed by six out of the thirteen (6/13) students and recognised as significant for the good preparation of the coursework, is to try out the coursework questions before going to the seminar.

> *Yeah, cos if you haven't prepared, and you go in there, then you're wasting time, you can't ask them anything, cos you haven't got anything… You haven't done anything…* Norina

Working out the coursework questions before the seminar is essential for identifying their weaknesses and problems in object-level and metalevel understanding, locating their difficulties and realising the emerging commognitive conflicts. Otherwise, students cannot take full advantage of the opportunity to share their thinking with experienced mathematicians in the seminars, expose their difficulties with certain d-objects and the governing metarules, improve their mathematical knowledge and capabilities, and be exposed to the practical demonstration of routines. The practice of the seminar is another characteristic of university Mathematics, which students need to accommodate and make use of. Failure to do so demonstrates difficulty in transiting to the university education norms. Figure 6.1 represents a typical process for the preparation of the coursework based on the data analysis.

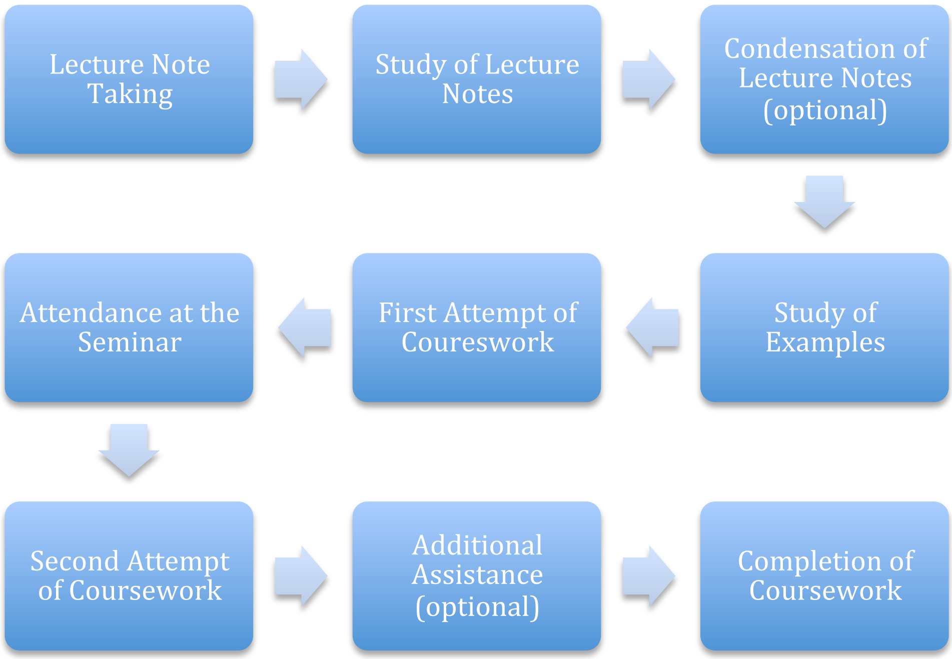


Figure 5.1: Coursework Preparation Process

Four of the thirteen (4/13) students have similarly expressed their wish to know the assessed questions before going to the seminar, as the following excerpt from an interview with Kostanza reveals.

> *Oh, this one stresses me out! There's a lot of questions to do, and I always tend to find – because obviously you go to seminars and they launch questions and it's – and then – I always find that like I end up doing the ones that aren't the actual coursework questions, and it's so annoying, because it's like oh, I've put all this effort into them... and it's not wasted, because it's good practice anyway, but – sometimes I prefer to know which my questions are, and then I can save the other ones for nearer the exams, to practice?* Kostanza

Although Kostanza realises the benefit of attempting all the questions of the coursework she prefers to work on the assessed ones. Not knowing the assessed exercises possibly encourages students to attempt all the questions without exception and therefore widens their experience in different tasks, achieving in this way broader object-level understanding and variety in the contexts in which metarules are applied. Accepting this learning technique would require a mature approach towards learning from the students' viewpoint. Knowing though the exact questions beforehand would possibly allow students to take more advantage of the seminar assistance and focus, at this stage of their learning, on the 'creditable' ones.

Finally, Tamino has raised the role of language and the assistance that this might offer to novice students in the process of solving a mathematical task. In particular he expresses an interesting view about the phrasing of the coursework exercises and how a good phrasing of the question helps the student in its solution.

> *I think the coursework puts you off really […] this subject has been vicious and horrible, but how they phrase the coursework you are more inclined to go into it and start adapt or you just look at it and you can try and don't want to start trying it as such… you have to force yourself to go and try… but some of the other ones which I haven't liked I have quite happily started trying them even if I don't like them, because of how they phrase the question or they break it up more so the more they break it up I feel like it's more bite. Obviously the questions further down the page they do it all in one bit but at the beginning they sort of bite size it which is kind of I find really useful because it gets your brain into the mode...* Tamino

The above excerpt leads to the conjecture that well phrased questions, from the students' perspective, are considered to be the ones that are constructed in a way that each step of the solution is linked to one subsection of the question, guiding the students towards the application of the particular object-level and metalevel rules and moreover the correct direction for solving the question, but also encouraging them to attempt it. This suggests that students wish to have exercises in the format and structure of the A Level examination questions, indicating incomplete transition towards university Mathematics. Tamino seems to realise that detailed, multi-divided questions are part of the initial stages of the coursework and their purpose is to guide students. Nevertheless this kind of structured questions are appreciated and well received.

In sum, in this section I have discussed certain applied methods and techniques that students have adopted for the preparation of the coursework, as well as some interesting perspectives. Summarising the lecture notes and highlighting the important mathematical narratives, namely definitions, lemmas, and theorems, is a technique that occurred often in the student interviews. In addition, and according to students that use it, contributes favorably in their object-level understanding and facilitates the solution of the coursework. Many students have expressed their belief about the importance of attending the seminars in the process of solution of the coursework. Four students wished to know the assessed questions before going to the seminar, although they expressed their concern about the possibly negative effect this might have in their mathematical learning. Finally, the way coursework questions are phrased is conjectured to affect students' performance.

5.4 Discussion

In what follows, I intend to answer the general research questions that were set at the introduction of this chapter, synthesize the results and embed them into the literature discussed in Chapter 2, when such literature exists. For better presentation, similar to what I did in Chapter 4, the research questions will precede.

What are the students' perspectives regarding the process of proving?

The majority of students consider proof production a difficult milestone in the process of learning Group Theory and Pure Mathematics in general. The discussion in this chapter suggests that some students consider proofs as a means of mathematical communication, demonstrating sensitivity and consciousness about its importance. These students are aware about the level of rigour and explicitness that their narratives should demonstrate in their attempt to produce proofs. Proof production often appears to be a 'shocking' experience for novice students. Successful proof production requires good object-level understanding and successful objectification of the involved d-objects as well as familiarisation with the governing metarules.

What difficulties students face in the process of proving?

In an abstract mathematical discourse such as Group Theory, students are invited to produce proofs for several mathematical problems, both in the coursework and the examination. As the literature (Moore, 1994; Harel and Sowder, 1998; Weber, 2001; Weber and Alcock, 2004, etc.) and the discussion both in this chapter and in chapter 4 suggests, proof production, especially in Group Theory, represents a particular challenge, because students have to develop several indispensable skills, such as their ability to cope with the abstract nature of this module and a certain flexibility in the application of metarules. It cannot be assumed that the majority of the students can develop these skills instantly or easily.

In addition, proof production is a new element in the students' learning experience, requiring successful application of both object-level and metalevel rules, and therefore a challenge in the secondary-tertiary transition that needs to be confronted. As the discussion in this chapter suggests, many novice students often have difficulty with the 'how' and 'when' of the required routines. Successful proof production depends on the thorough object-level understanding of involved d-objects and their realization trees, as well as the successful and precise application of the governing metalevel rules, in the particular context.

Evidently, students face various difficulties with the three steps of the procedure for developing a certain routine, namely, the applicability conditions, the course of action and the closure conditions. In particular, some students often face difficulties initiating a proof. It is a difficulty that frequently occurred in the context of this study and was also identified by Moore (1994). In addition, some students often have difficulty in recognising the signs that would signal the end of the proof, leaving them with a feeling of doubt.

What techniques and skills are applied in the revision for the final examination?

The discussion in this chapter suggests that the revision for the final examination is a ‘renewed contract’ for mathematical learning, during which students need to develop and/or apply certain study skills. They are invited to revisit what they have been taught, localise the conceptual gaps and overcome the remaining commognitive conflicts. Ideally, successful revision will help students to achieve full objectification of the introductory d-objects of Group Theory and moreover obtain mathematical consciousness.

The discussion in this chapter indicates that the majority of students adopt a similar approach towards revision for the final examination. These students usually commence the revision process by rereading the lecture notes. There are several approaches in this step, but their predominant aim is to engage again, after having acquired more experience, with the various mathematical narratives, namely definitions, theorems, lemmas and proofs, both to improve their object-level understanding and memorise the ones that are most likely to appear in the examination paper. This process will help them to fully and properly construct the involved realization trees that will allow them to move with more dexterity from one realization tree to the other and consequently be more efficient in the problem solving and production of proofs.

The second step of revision is either to the study of the coursework questions in parallel to the given model solution or attempt to solve past papers. Regarding the solution of coursework using the model solutions, the discussion above indicates that for many students it is an important step in their learning process. Students have the opportunity to compare their solutions with the model solutions and precisely localise their errors. This will enable them to resolve any commognitive conflicts related to these errors, by improving their object-level understanding regarding the involved d-objects and will also help them to resolve problems with the governing metalevel rules and, more generally, with proof production. This process requires *autodidactical skills* (self-teaching) that will enable them to teach themselves, among other things, the ‘how’ and the ‘when’ of the involved routines, and to correct and/or improve their learning and solving techniques. It is suggested by a particular student and implied by others that this process contributes favorably in their self-confidence.

Regarding the solution of past papers, many students at this stage try to specifically identify the definitions, theorems and proofs that are likely to be included in the examination paper, to pinpoint possible mathematical tasks that they may be asked to prove or solve, to extend their experience by solving the past papers as such, and, moreover, to have an opportunity to apply their solving skills, knowledge and understanding to a variety of tasks.

The revision process is often nonlinear, but rather involves interchange between the study of lecture notes and literature, revisiting of the coursework with the model solutions and the solution of past papers. Students that follow this revision approach work with the three elements interchangeably until they feel that they have overcome any commognitive conflicts and have achieved adequate object-level and metalevel understanding.

What specific skills and techniques have the students in this study applied for the preparation of coursework?

For the preparation of the coursework, it has emerged that summarizing the lecture notes and highlighting the important mathematical narratives such as the definitions of the involved d-objects and the related theorems and lemmas who describe the respective object-level rules, is an important first step for many students in the preparation of the coursework. This technique possibly allows the students who apply it to improve their object-level understanding as well as make more practical and efficient notes that allows them easier access to the mathematical tools that need to use.

Other study skills, especially for the preparation of the coursework, involve self-discipline and good study planning, as well as the ability to adjust ones’ schedule to the programme of the department time table. This will allow students to take full advantage of the opportunities for assistance that are offered. For instance, students should, and many do, attempt the coursework questions before going to the seminar. This helps them to identify their weaknesses, locate their difficulties, realise commognitive conflicts and make a list of the issues they would like to discuss with the seminar staff. Not doing so, students still have the opportunity to get enough help, expose their difficulties, and improve their mathematical knowledge and capabilities. A small group of students wished to know the assessed questions before going to the seminar, although they expressed their concern about the possibly negative effect this might have in their mathematical learning.

Chapter 6 Students' Perceptions Regarding the Effectiveness of Communication

In this chapter, I will discuss students' perceptions about the issue of communication through teaching, tutoring and working with peers, in different communicational environments and communicational formats (oral or written). The first refers to the *lecture* and *lecturer*, and the *lecture notes*; and the second to the *seminar*, the *tutorial*, and *collaboration among peers*, all means of assistance for the solution of the coursework. For the purposes of the discussion in this chapter, I have included evidence from the students' interviews, which justifies the claims in the analysis before or after the data, aiming to identify what are the qualities of good teaching and communication, according to students' viewpoint.

In particular, in section 6.1, I discuss students' perceptions regarding the quality and effectiveness of communication and teaching in the context of the lecture, and the characteristics of good lecture notes. In the section 6.2, I discuss students' perceptions regarding the tutorial with the lecturer, the seminar sessions in which students work on their coursework, and collaboration among students. Moreover, I aim to identify the characteristics of communication in these contexts that may influence favorably or unfavorably the students' learning experience.

The research questions that this chapter aims to answer are the following: What is the impact of good communication in the context of teaching, according to students' perception? What are the students' perceptions regarding the lecture and the produced lecture notes? What are the students' perceptions regarding the one-to-one tutorials with the lecturer? What are the students' views regarding the seminars as they operate now and how this operation could be improved? Why collaboration with peers is the favorite communicational context, according to students' perception? What is the role of visual images and how it contributes to the effectiveness of communication?

6.1 Lectures and Lecture Notes

In this section, I discuss the students' perceptions regarding a number of different characteristics of teaching in the lectures, namely their views about the lecturer and his teaching practices and issues related to the lecture notes. For the analysis and interpretation of this data, I have used the Commognitive Theoretical Framework, focusing on the concept of communication and, when possible, its connection with the activity of thinking, namely commognition. In addition, I discuss students' perception about the effectiveness of mathematical communication with the lecturer in the context of lecture, and the produced lecture notes. When possible, I analyse the impact on the object-level and metalevel learning.

6.1.1 Students' Perceptions about the Lecture

LCR, the lecturer of this module, is an experienced pure mathematician with more than twenty years of teaching experience. All thirteen (13/13) students have clearly expressed their positive view about LCR as a lecturer. In what follows, I present data that covers all the reasons for which LCR and his lecturing practices are well received by students. A representative example of these statements is the following by Kostanza who, in addition, highlights the importance of previous teaching and how the overall academic experience is an important factor concerning the quality of one's teaching practices.

> *He sees – he's learnt from everyone else's mistakes because he gets told about them all, I'd imagine, um – I like it, I mean he breaks down with lots of examples, which is really good, for looking back for the coursework and the exams. I um... no, I like his teaching style. Think he's a good lecturer. I'm even nearly tempted by his third-year course, even though it's pure, that's how...!* Kostanza

The above excerpt possibly suggests that, from the students' perspective, good teaching needs to have certain characteristics, including systematic exemplification of the introduced d-objects and routines, which will be connected to the mathematical tasks in the exams and in the coursework. Another characteristic that a good lecturer should have, according to the above excerpt, is to reflect on his or her teaching practices and constantly adjust them. In addition, Kostanza's statement leads to the conjecture that good lecturing has positive effects on the attitudes and emotions of students towards Pure Mathematics. The apparently well-received teaching practices of LCR have favorable effect on students' choice for Pure Mathematics modules in their final year.

Dorabella expressed an interesting viewpoint regarding the characteristics of good teaching as a form of communication between the lecturer and the students. As the following excerpt reveals, LCR is 'very good', since he reminds her of her high school teachers.

> *Um... I think he's – doing a very good job actually, he's a lot better than a lot of lecturers I've had, and um – I really liked my A-level teachers, both – I mean I had the same ones for both years, and he's a very similar teacher to both of them, um...* Dorabella

The above excerpt suggests Dorabella's undefined preference for practices that remind her of the ones used in secondary education, possibly indicating some hesitation to fully accept the teaching norms of the tertiary Mathematics Education and therefore to adjust to a complete transition. In another instance, Dorabella discusses the characteristics of a good lecturer, most of which she can identify in the lecturer of the module.

> *Um, clear notes, I think he's got – you know, specific sections, um, which I like, numbered sections, um... he speaks very clearly and writes very clearly, which helps enormously um, and I think he's very approachable, I feel like you can speak to him. Um... I just like the whole way he's done it really. Um, and I think the best thing is the ordered notes.* Dorabella

Teaching is a form of communication through which the new material is presented and explained by the lecturer, and consequently has significant contribution to students' object-level and metalevel learning. As a result the lecture notes emerge, which are the main source of study as well as an explicit guide of the material students are expected to know for the examination and coursework. Therefore the quality of teaching and the emerging notes, both in content and in presentation, is possibly an important contributing factor to the successful communication between LCR and students, which is essential for effective mathematical learning.

Another interesting view regarding good teaching and communication in the context of lecture is, according to Carmen, the attitude of the lecturer in the lecture. She suggests that a good lecturer should be evidently happy to teach.

> *MI:* *How do find the way the lecturer teaches the course? Are you happy?*
>
> *Carmen:* *Yeah. But in fact here it is – I don't know if the teacher is happy to do the lecture. All the teachers – because in fact the teacher speak and how his lecture and there are – they don't have a relation with the students.... Yeah, so speak, speak, speak… and after I finish but for me it's okay I went to lecture. In France we have more relation with the student... I prefer that.*

According to the above excerpt, good communication needs to have several characteristics. Good teaching is probably not enough to guarantee good communication between the student and the lecturer. Carmen's view about communication embraces communication not only as a means for achieving good mathematical learning, but also to have perhaps good communication in a social level. Namely, in order to establish a comfortable relationship between the lecturer and the students, the lecturer should not teach in a mechanical and unstoppable way but should ideally allow more freedom for some interaction between the two parties.

In fact, six out of the thirteen (6/13) students believe that for achieving better object-level and metalevel learning, teaching should be more interactive. For instance, Otello repeatedly expressed his desire to be part of an interactive lecture in which students would not just been taught but actively participate, also making use of modern technology gadgets such as interactive whiteboards, computing devices, and other elements of information technology.

> *Um.... That's very you know – it is important for a lecturer to engage the students so at the end of the day it is not a matter of just copying, but I would prefer more – maybe that would not be a lecture – but at the end of the day what you should have is more interactive session which lecturer does throw questions on members of the audience and expects answers…[…] in the lecture. Definitely would expect in this modern day and age more use of interactive whiteboards, computers yeah and… also since we are dealing with groups… the lecturer should be able to juxtapose his lectures with real life examples… cause as you know symmetry does play a big part in lecture.* Otello

The expressed desire for more engagement and participation is mainly a need for better and more effective communication. Although different students express this desire for different reasons, some for satisfying their desire to go beyond the expected level and curriculum or others to achieve better object-level and metalevel learning, the need for better mathematical communication is the main point.

Another interesting view regarding the characteristics of good teaching and good communication in the context of lecture is related to the way lecturing should inspire students. Otello needs the lectures to be not just an initial step towards mathematical learning of a new theory, but also a source of inspiration.

> *I prefer it more relaxed conversation and approach something that would really inspire - you know – there are some lecturers here that tend to go into too much detail which I find not too good but there are other lecturers which really challenge you and make you think for yourself.* Otello

According to the above excerpt, Otello wishes to be inspired and challenged, apart for merely given new information. Being inspired is not only a result of the content of the lecture but can also be caused by a series of communicational actions that the lecturer's behaviour, through the agency that communication norms allow him, can trigger.

Two of the thirteen (2/13) students commented on another characteristic of good teaching and mathematical communication regarding the extent of further explanation regarding the introduced d-objects and related examples. According to these students, the lecturer sometimes was not giving enough explanation of the new theory. For instance, Norina although her overall impression about LCR is very good, she suggests that sometimes the lecturer is not giving any additional explanation or clarification because he probably thinks that something is 'obvious'.

> *Well sometimes the examples, they feel like um, complete like he'll go like um, oh well this bit's obvious, and then not carry on, and I'm kind of like it may be obvious to you, but it would be nice if like, you had like – like a complete example, where every stat was put down, so then you could be able to see what was going on.* Norina

The above excerpt may point to an unintentional *commognitive gap* between the lecturer and the students, as a result of the lecturer's extensive experience and the inexperience of the novice students. This often leads to some details in the process of teaching being considered as known, a fact that may appear both in secondary and in tertiary Mathematics Education. Norina expresses her concern regarding the exhaustiveness of communication about the new material, wishing for a more detailed and explanatory approach. In addition, the above excerpt does not necessarily suggest incomplete or hasty teaching but it may indicate the lecturer's intention to encourage students to become independent learners. This is a characteristic of university level education and there are indications in the above excerpt that Norina has not yet fully adjusted. This can possibly be interpreted as part of a, so far, unfinished transition from secondary to tertiary level education.

Another perception expressed by the students regarding the teaching practice, and its impact on object-level and metalevel learning, was related to the use of visual mediators. Regarding the first issue, eight of the thirteen (8/13) students asked for more pictures and expressed their need to visualise. In fact, the lecturer had used eighteen pictures in his lectures, but the majority of students seem to not have used the visual mediators as expected. Nevertheless Leonora says:

> *I don't... mainly – like, obviously, because I'm a visual learner, but I don't know if there's an – is anything visual within it, that you – that they could actually – like that he could actually put – on the board, I'm not sure if he could do that or not, but I think it would help me, personally, a lot, but...* […] *More pictures and stuff, like... explaining what's happening and sort of thing, so that – maybe then I could picture it in my head, I'd find it easier. Cos if I'm trying to picture, as he's going along, then sometimes if he's going too fast, I don't have time to try and think about it, when I'm trying to think about something else. So... maybe stop and then draw a picture, or...* Leonora

The illustrations in the lecture notes were of two categories: six *representations of external p-objects*, namely, rotations of a square (twice), multiplication table of a group, rotations of a regular pentagon and a regular hexagon as examples of dihedral groups and symmetries of a cube (See Table 6.1); and eight *representations of mental objects,* namely, an image of coset, example of a coset involving complex numbers and Argand diagrams, a Venn diagram as part of the proof of Lagrange's Theorem, an illustration of equivalence classes, a representation of homomorphism, a representation of homomorphism including the kernel and image and an image of a quotient group (See Table 6.2).

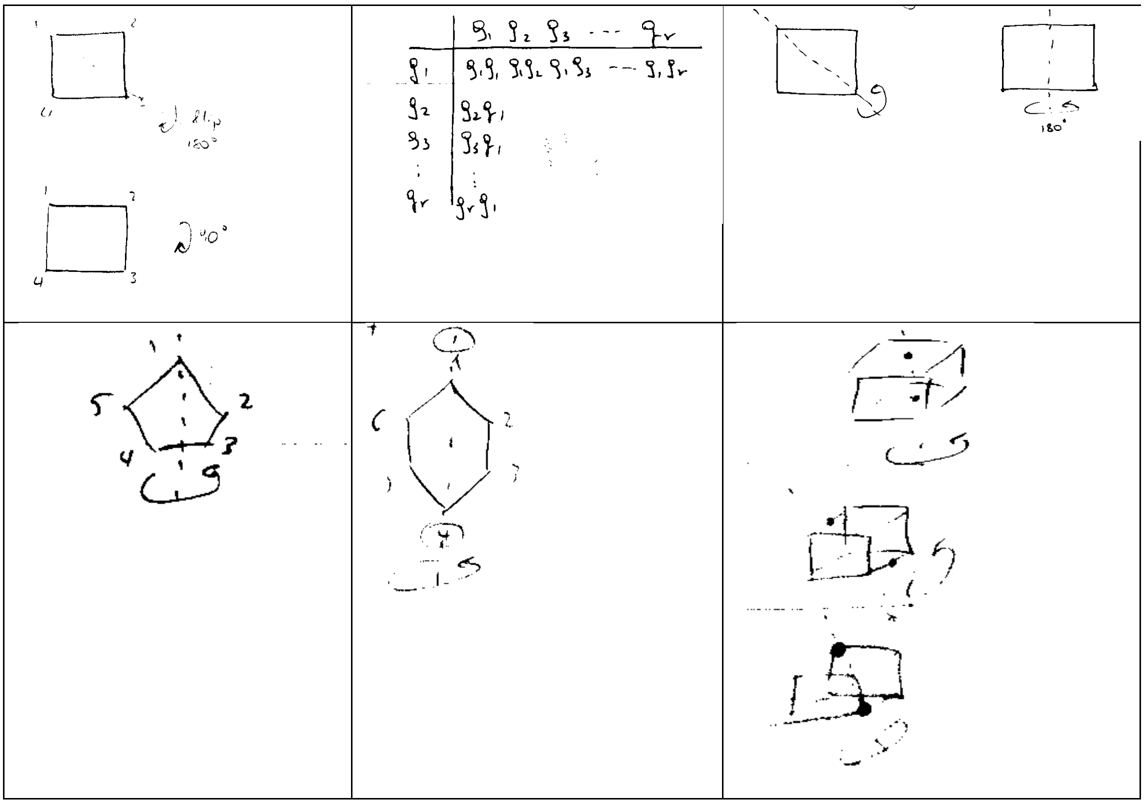


Table 6.1: Representations of External p-objects

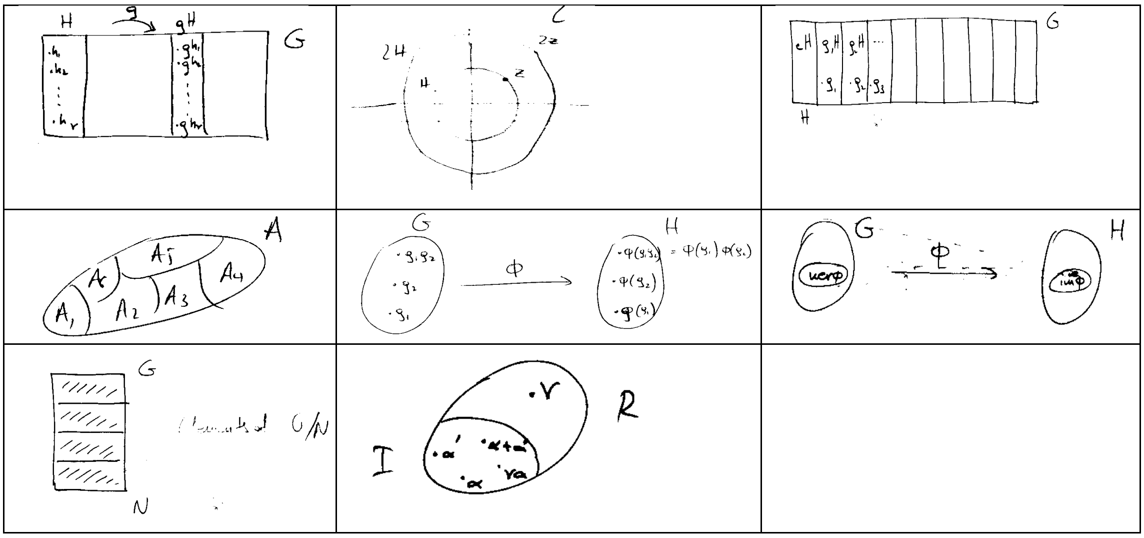


Table 6.2: Representation of Mental d-objects

The majority of students in their interviews expressed more often than not their preference towards Applied Mathematics, partly because of the less abstract nature and the more direct visual representation that it involves. The following excerpt is a typical example of this fact.

MI:	*Why do you think you prefer applied to pure?*
Leonora:	*I think cause it easier cause I think you can apply to things like I like to learn my visually more like so I can see what is happening.*

Visual mediators are one of the four main characteristics of the mathematical language. Moreover, problematic use of visual mediators may indicate problematic use of the mathematical language and, henceforth, problematic mathematical learning. The role of illustrations, as these were used by LCR, was supportive, assisting students to achieve object-level understanding of the various group-theoretic p-objects and d-objects, which were visually represented. Effective use of such images though requires at least some primitive object-level understanding of nature of the d-objects they represent, as Zazkis et al (1996) suggest. This possibly implies that visual mediators are an intermediate mean of the shift from object-level learning to the metalevel learning, along with the necessary familiarity with the routines of the mathematical discourse under study. At the very early stages of their encounter with Group Theory, students are in the process of object-level understanding and consequently visual images are not yet objectified.

6.1.2 Students' Perceptions about the Lecture Notes

In this section, I discuss students' perceptions regarding the lecture notes as a means of learning and communication between the lecturer and his audience. Analysing the interviews of the thirteen students it emerges that six out of thirteen (6/13) students do not consider the lecture notes as always complete, adequate or efficient enough to be the only source of object-level and metalevel learning, necessary for the solution of the coursework and the revision for the exams. This view is clearly expressed in the following excerpt from an interview of Amelia, according to which she would prefer to use electronic notes.

> *You can sit with your lecture notes… I often find it quite useful when I do coursework, um, we get online notes sometimes, I don't know if you have those…* […] *Because you can just do your control F and type in what you're looking for, and it brings it up, rather than having to read all the lecture notes…* Amelia

The above excerpt suggests that Amelia prefers to use lecture notes in electronic pdf format for reasons of efficiency that technology offers. By using Control F, she can easily locate the narratives or mathematical expressions that she needs. Lecture notes in an electronic format may contribute to the effectiveness of mathematical communication, and consequently object-level and metalevel learning, although there is a risk of partial and less thorough studying of the material.

Five of the thirteen (5/13) students discussed the structure and the content of the lecture notes. Although students consider LCR as a good lecturer for many reasons, one of which is the rich repertoire of examples (among other characteristics), they express their need for even more examples. They suggest that lecture notes in this particular module should include more examples. Students consider examples as an essential element of mathematical communication and indispensable part of their object-level and metalevel learning. For instance, the importance of examples, according to these students' perception, is expressed in the following excerpt from Manrico's interview. Manrico highlights well the importance of the example in the context of mathematical teaching and its significance in the discursive shift, which results successful object-level and metalevel learning.

> *Um – I'd probably throw in maybe a few more examples, now and then, just because that is – that I mean – different people learn different ways, but I find it just kind of actually doing it and just you*

know, actually seeing examples – I mean there are quite good, there are quite good few examples on the course, but may – yeah, that's probably how I best learn, just by seeing how it actually works, and then to see how you get there you can then go back and look at the proof that – to just see it working first. It's kind of a good starting point for me, personally… Manrico

Examples are, according to the above view, an indispensable element of mathematical learning since they are elements, which assist students to achieve objectification of the involved d-objects as well as see in practice how the governing metarules are applied, in a particular task. Manrico, while reporting his need for more examples, indirectly suggests that the list of examples should offer a variety of different metalevel rules that demonstrate how "it actually works".

Another student perception regarding lecture notes was expressed by four of the thirteen (4/13) students and is related to their format. In particular, these students expressed their preference for printed notes instead of handwritten notes, even though all of them stressed the fact that notes should not be circulated freely among the students before the end of the module; otherwise the attendance in the lectures would decrease. A representative example of this view is revealed in the following interview excerpt by Norina, according to which she prefers to have more detailed printed notes, which would help her to engage more effectively with the solution of the exercises.

Um, yeah, um, a student asked if he'd maybe put um, more detailed notes on the Internet, like to go through […] *Um... I wouldn't say before, like, it means no-one would go to a lecture, but um, no, I think like, after the lecture, because then you've got something that you can work through for coursework as well… Because um, sometimes when you're in a lecture and you're just copying it down, it's always like you're not – you're maybe rushing or something, you may not write it down, like 100%, so it would be nice to have some detailed notes you can go through the coursework.* Norina

According to Norina, handing out printed notes would increase the effectiveness of mathematical communication, since instead of copying the main results from the blackboard students would be able to record the *peripheral explanatory comments* that the lecturer would add orally. Moreover, printed lecture notes possibly improve mathematical communication, since they would probably enable students to record a wider spectrum of communicational actions. It would turn the lecture into a more student-engaging activity, since students would not mechanically copy what is written in the blackboard but instead they would be more involved in thinking about what is currently taught.

Four of the thirteen (4/13) students have expressed their disapproval of the fast pace, which according to them may influence negatively their object-level and metalevel learning. Moreover, efficiency of communication may be negatively affected by the fast pace required for copying the lecture notes. A representative example of this view is expressed in the following statement by Tamino, according to which he suggests that the focus of students shifts from attempting to objectify the material to the act of copying explicitly and exhaustively. He is particularly sensitive to the effectiveness of communication between the lecturer and the students.

> *The speed also doesn't help because you have got to write so fast cause you are having to write so fast to concentrate you can't concentrate on what you are actually learning as such so I think alright he doesn't like it but what would probably be better for him is if he did printed lecture notes and then he kind of went through them and we could annotate them whilst – cause he says things as well which are very useful to write down but you're too busy writing everything down to actually concentrate on what he's actually saying...* Tamino

Despite the fact that the use of blackboard as the primary tool of teaching this module might occasionally cause some haste in the presentation of the new material, twelve out of the thirteen (12/13) students, have expressed their preference in the use of this teaching method. For instance Norma prefers this teaching approach to the use of electronic devices and programmes such as PowerPoint for instance, as adopted in the teaching of other disciplines. She is one of the students that prefer to give priority to the *norms* of teaching of Mathematics rather than the *efficiency* that, according to others, technology can offer.

> *I do find that the writing on the blackboard is a really good way to do it, cos my business module, um, it's all on PowerPoint presentation… And although like, I print off the slides before I go, I do find that they say more than what's written on the slide… And you don't have time to write down before they've gone on to the next thing…* […] *I do quite like that most of the maths is um, like written on the blackboard. I think that is a really good way to teach it. Um... yeah!* Norma

Finally, four out of thirteen (4/13) students focused on the structure of the lecture notes, emphasizing on the significance of well organised and presented notes. Complete, clearly presented, well-structured and coherently numbered notes play an important role in the effectiveness of learning and communication. Amelia, in particular, focuses on the importance of correctly numbered notes and the effect it has on her object-level learning and the construction of realisation trees.

> *I think it's been taught quite well,* […] *it's just really stupid, but I find it really difficult cos he – didn't keep to his numbering? And it just confuses you and it makes you feel like you've got a gap in your knowledge? And it – it – puts my back up, cos I get nervous that it's like – oh, why have we skipped a bit, and then we haven't got section 4, and it muddles your notes up a bit more. Which I don't like, being very organized and logic, I think a lot of us are like that, and really um... I don't really know, cos I don't think I know the course well enough to know how else you'd be able to teach it, do you know what I mean?* Amelia

Even though the lecture notes were well numbered and structured, the lecturer decided to omit a chapter without adjusting the numbering. This not only caused confusion at the level of note organisation, but also led Amelia to perceive a 'gap of knowledge'. Well-structured notes prevent any confusion and disorder in the process of object-level learning and teaching. According to Amelia, any technical and organisational problem in the teaching would represent an extra burden in the already difficult process of object-level learning.

In sum, in section 6.1, I have discussed students' perceptions regarding the lecture and the lecture notes. Students' view about the lecturer is overall positive, considering him as a very experienced teacher. Students also discussed their views about the characteristics of a good lecturer, namely to have: experience in teaching, the ability to reflect on his teaching activities, positive attitude towards teaching, mathematics and his students, and close relationship with his students. Good lecturing should involve, according to the students' perspectives, more interaction between the lecturer and the audience, further explanation of metarules governing the demonstrated examples, systematic use of visual mediators, and finally lectures should be a source of inspiration of students. Regarding the lecture notes, they are generally well received although some students have expressed suggestions regarding the format of these notes. For instance, lecture notes should be offered both in hand written format as well as electronic format, in order to allow students to identify the various definitions and theorems more easily, as well as give students the time to record any additional information given orally in the lecture. Regarding the content of lecture notes, there should be even more examples, in which the governing object-level and metalevel rules should be demonstrated and clearly explained. Finally regarding the often fast pace of lecturing, students expressed their disapproval and concern about the negative impact that this may have on their learning.

6.2 Seminars, Tutorial, and Collaboration with Peers

In this section, I discuss the students' perceptions regarding the seminar, personal tutorial with the lecturer in his office hours and the collaboration with their peers. The analysis below suggests that the majority of students' favorite means of assistance for the preparation of the coursework is the collaboration with peers, expressing also their unwillingness to attend the personal tutorials. Regarding seminars, the overall impression is that they are helpful, despite various suggestions for operational improvement.

6.2.1 Students' Perceptions about Tutorials with the Lecturer

In the mathematics department in which this study was realised, students had the opportunity to meet the lecturer of each of their modules at a fixed hour, on a weekly basis. The analysis below suggests that students make very little or no use of this source of assistance. When interviewing the lecturer and the seminar leaders and assistants it emerged that only one seminar leader had a visit from a student seeking help with an exercise of this module's coursework during the semester.

Although eleven of the thirteen (9/13) students believe that visiting the lecturer during office hours is beneficial, nine of them consider this as their very last solution, whereas two stated that they do not go at all.

> *We normally try and help each other in a group, so if one of us can't do it, then either someone in the group helps us, if they've done it, or if none of us can do it, we can't work it out, then we do go and see the lecturers.* Dorabella

Instead of going to the tutorials, these students seem to prefer working with their peers and attending the seminars, in order to improve their object-level and metalevel learning and therefore be able to overcome any commognitive conflicts that emerged from the lectures. Seminars offer a direct way of communication, yet less committing than the one-to-one tutorial with the lecturer, where many students may not always be in a position to be fully engaged with a discussion. One-to-one tutorials do not offer the safety of a group discussion between a small group of people that share the same difficulties and challenges, and a seminar leader or assistant.

Visiting the lecturer for a personal tutorial appears to be a difficult experience for students, as described by Calaf.

> *I think most of the people... I don't know, just cannot bother to make an appointment to go up and ask I think, that's – but if you've got a seminar there, and then you be – in front of everyone, like a small group you can talk to them, if you – you see either got different idea to yours, so you sort of like chatting about it first, before you sort of like asking some lecturer? Because the lecturer, you sort of like – quite scary, you think they are quite a scary – to go up and ask them somehow? And like most of the people, in the seminars, are not like – put their hand up to ask any questions, if they don't understand it.*
> Calaf

The communication with the lecturer in the context of a tutorial seems to be a commognitive challenge, for some students. According to Calaf it is 'scary'. By asking questions there is the possibility of being exposed. There is not the communicational safety of a group discussion during which he is not obliged to answer something that he is not sure about. Group discussion is also a source of information through 'chatting about it first'. The discussion possibly helps the students to specify and clarify what they want to ask. In addition the answers to each students' question addresses any gaps in the object-level and perhaps metalevel understanding and learning.

For these novice students, tutorials seem to be a very uncomfortable commognitive experience, since they have not yet developed all the characteristics of commognition in the specific discourse. Furthermore, I conjecture that the students who expressed this view do not yet have the *agency* that the mature participants of a certain discursive community should have. They probably do not have the certainty that their reactions to certain communicational actions of the lecturer would be appropriate and in agreement with the expected repertoire of reactions. For instance, in the process of explaining the d-object of isomorphism the lecturer asked the student to tell him what its kernel is. Many students are not able to say what exactly the kernel is at this early stage, indicating incomplete object-level learning. This is suggested by the analysis of the thirteen student interviews, since when during the student interviews I would ask them to give me a definition, or description, or representation of any kind of certain d-objects, none of them was able to respond apart from three students who drew Venn diagrams that represented the groups and subgroups. Here is a typical conversation and written excerpt that illustrates the above.

MI: How would you represent the group?

Amelia: No, I'm just thinking about how I actually don't really know, I know it by the axioms, rather than actually - the visual picture, I suppose just as a circle or something... like that. A group, and then when you've got subgroups, it's like – a bit of it. It's not a very exciting picture, but I don't really know how else I'd do it…

MI: Could you give me an example of a group? When you are talking about groups do you have a specific example that pops into your mind?

Amelia: Not really... I don't – I can't – um... we've done so much now, I more know the definition of a subgroup than a group, it's really bad, um... this is the way to start the morning... laughs

EPM, an experienced pure mathematician of the department in which this study has taken place, points out the interactive nature of these tutorials.

Now, how do your office hours work? Does ever anybody show up? No. Why don't they show up? I can tell you. Because there's a commitment, student comes into the office and then you say yes, can

I help you? This person has to say, can you please explain this to me. And then the student would have had to do something. If you asked them to have a nice cup of coffee, and a biscuit, they do come! Yeah, but the many - there is any serious amount of, you know, engagement, they will shy away from this. EPM

According to the above excerpt, engagement in these meetings is more often than not a discouraging factor for students, whose aim is solely to gather the information and clarifications they need without taking any interactive communicational action.

Incomplete object-level understanding often causes students' hesitance to ask further questions as a result of affective issues.

Um, I'll just see how I get along, cos like I said again, I still feel sometimes that my questions are too... too basic almost, but – I mean if I – I – I'm not afraid to ask questions, I'll email or whatever and – but generally I find that if I just work through it I'll get there in the end, well as far as I'm every going to get in the end, so I might as well just – yeah... Kostanza

The quality and accuracy of the questions, according to their perception, often influences unfavorably their intention to ask for clarifications and further explanation, as the excerpt of Kostanza suggests. This most probably leads her to pedagogical isolation in which she depends more on herself than being more interactive with the other members of the Group Theory discourse community. The feeling of inferiority about the 'silly' questions may have a more general impact on a students' attitude towards mathematical learning. For instance, Kostanza did not ask for assistance in the seminars, as the data analysis suggests.

6.2.2 Students' Perceptions about Collaborating with Peers

Twelve out of thirteen students find the collaboration with their fellow students very helpful, for several reasons. According to Leonora, in the excerpt below, it is more convenient working with friends, since she feels comfortable to discuss something repeatedly and without adjusting her schedule according to the lecturer's office hours. The context of the communicational interaction between peers has different characteristics than a discussion with an experienced mathematician. Linguistic commognition among peers does not require the precise use of the group-theoretic terminology and has a wider spectrum of acceptable reactions (the repertoire of acceptable reactions may include erroneous or not so complete and precise responses), whereas a discussion with the lecturer would imply, from the students' perspective, that their reaction should be more 'appropriate'.

> *Really helpful… Cos if I don't have to spend of time figuring out when my lecturers have office hours and I can go see them, instead I can just – ask like, say we have an hour off or something, I can just ask one of my friends to just explain it to me? […] If I don't understand something, and I know they do, I often get them to try and explain it in a different way, like a different point of view and – see if I understand it better, so, with - especially with the coursework, I'll – like – talk to other people like – cos if – it makes me feel better sometimes if I know they're finding it hard as well.* Leonora

As the above excerpt suggests, working with peers allows students to express themselves more freely, using perhaps a simpler mathematical vocabulary, not hesitating to ask something which might sound naïve, repeat their questions or even argue, as in a colloquial discussion among equals, and in this way achieving, perhaps, better object-level understanding. Apparently, students feel comfortable to insist when they do not understand something and require an explanation from a different viewpoint; something that they do not do so easily in the seminars or meetings with the lecturer. It seems that the freedom of expression and comfort among the peers improves the effectiveness of communication and consequently students' object-level and metalevel learning.

While peers cannot substitute the lecturer or the seminar staff, they significantly contribute to joint mathematical learning and consequently object-level and metalevel understanding, both in the first and second attempt to solve the coursework.

> *I normally meet up with some friends and go through the sheet and write down the problems that I had...* […] *If I am unsure or unclear and don't really understand then I will ask the seminar leader I just ask what I don't understand!* […] *Always try out the exercises then go to the seminar, seminars, and then after the seminars, I work with some friends, like, to go over it a bit more, so...* Amelia

According to the above excerpt, working with peers has a dual role. It possibly helps the students to precisely locate their metalevel weaknesses, misconceptions, points in a proof that they cannot overcome by themselves or cognitive conflicts, but also to establish the newly objectified knowledge, both in the form of d-objects and consequently the corresponding realization trees, or object-level rules. Working with peers contributes to the development of a mathematical learning, since the sessions can be repeated several times, without following a certain schedule or having a fixed duration.

6.2.3 Students' Perceptions about Seminars

In the Mathematics department in which this study was realised, seminars are considered the main source of assistance, on an institutional level, helping students to solve the coursework exercises. Students have the opportunity to ask questions, discuss their object-level and metalevel difficulties in the process of solving the coursework, and reveal the emerged commognitive conflicts, which may be addressed through communication with the seminar staff. All thirteen students consider seminars helpful. Amelia has described the benefits of attending seminars in detail.

> *I find the seminars really useful, because it's a chance – obviously, to speak to people – and the lecturers are really friendly.* […] *And a lot*

of them – cos – I think sometimes, you can just be told the answer in maths, and a lot of them make you think about it, and just ask kind of the questions, to get you on the right track? Rather than just saying, oh, you just do this! Amelia

The quality of the communication between the seminar staff and the student is, according to the above, an important factor in the process of mathematical learning. While, ideally, the students are able to resolve any object-level and metalevel difficulties through an act of communication by exposing their thinking, the communication with experienced mathematicians also enables them to learn how to utilise the language of a certain mathematical discourse, use visual mediators or be obliged to produce illustrations to clarify their reasoning – in general learning the norms of a particular mathematical discourse. All these improve the effectiveness of communication between the novices and the experienced mathematicians for the benefit of the former, and consequently their object-level and metalevel learning is favorably influenced.

Four of the thirteen (4/13) students expressed their need for more frequent opportunities for communicating with the seminar staff, as the following excerpt suggests.

Yeah. And then give you feedback, instead of – like instead of the – three – we were three coursework, and three seminars at the moment, instead of that, say no, three coursework and six seminars? So one – one seminars is for the coursework and one seminar is for, no it's not – discuss and if anyone have any questions, ask them there, and sort of like – small test, like, 10 minutes test and see – summarize whether you understand everything you done from – previous section or something? Calaf

In the above except, Calaf suggests that there should be six, instead of three seminars during the semester, three of which should enable students to work on the coursework problems and three should facilitate discussions about the theory taught in the lectures, providing students with the opportunity to ask specific questions and resolve any commognitive conflicts or gaps. In addition, according to this view there should be tests during these seminars, through which the lecturer should record students' object-level and metalevel understanding and adjust his teaching accordingly. These suggestions demonstrate students' desire for more opportunities, for effective communication and help, within the context of an organised seminar.

A second student perception about the role of seminars is revealed in the following excerpt.

> *Yes I think the seminar system helps…and I think it has this support… you can discuss the coursework questions which is what it is about… although it is not my own ideal thing… I would be interested in comparing the seminar system with other maths departments…with a seminar system that has us a part of it … if this is not too far fetched… A lot of students go to the seminars to ask about the seminar questions and not to expand their mind on what the subject is about* […] *Yeah… I mean the seminar system just perpetuates the… it should give encouragement to students who are too lazy to do their work… but it doesn't give any advantage to the ones who have done their work before hand…laughs…a seminar should be a discussion among peers… not just asking the lecturer how I am supposed to solve this…* Otello

The role of the seminar, according to Otello, should be manifold. First, it should not only give students the chance to resolve any object-level or metalevel difficulties and improve their mathematical learning, but also to encourage them to 'expand their minds' through a more advanced level of commognitive activities triggered by the discussion with peers and inspiring conversations with staff members. Seminars should expand students' spectrum of experience through exposure to new routines, applications, and examples that would help them embody the realisation procedures and routine applications seen in the core-examinable part of the course. For Otello, *communication* is a challenge for *thinking*. Seminars, moreover, should be involving commognitive activities through which students would improve their object-level and metalevel mathematical learning.

In sum, in section 6.2, I have discussed students' perceptions regarding the tutorial, the seminar and the collaboration with peers. Regarding tutorials, the majority of students do not favor them, since this kind of mathematical communication might develop into a rather challenging and exposing experience, in which the majority of these novice students do not still feel confident to do so. The great majority of students have expressed their preference to working with their peers instead, and having the solution of tutorial as their very final option. Working with peers allows students to communicate more freely, using a simpler vocabulary in a significantly more comfortable communicational environment, which allows them to discuss repeatedly their own object-level and metalevel problems and through this communication to resolve any commognitive conflicts. Finally, seminars are considered really helpful by the majority of students, yet there have emerged certain objections about the way seminars operate and the upgraded role that these should have in the students' mathematical learning.

6.3 Discussion

In what follows, I aim to answer the research questions that were set at the introduction of this chapter, synthesize the results and embed them into the literature discussed in Chapter 2, when such literature exists. For better presentation, similar to what I did in Chapters 4 and 5, the research questions will precede.

What is the impact of good communication in teaching, according to students' perception?

Good teaching, according to the students' perception, requires effective communication between the lecturer and the students. In addition, teaching is a manifold form of communication (oral, visual and demonstrative) through which the new material is presented and explained by the lecturer, which ideally leads to successful object-level and metalevel mathematical learning.

Good teaching requires good communication and, according to the data analysis, it has favorable effects on students' attitude towards Group Theory and other Pure Mathematics discourses. The effectiveness of *communication* between the lecturer and the students contributes favorably to the students' learning. Therefore, the quality of teaching and the emerging notes, both in content and in presentation, significantly contribute to the successful communication between the lecturer and the students that is essential for effective mathematical learning. Moreover, good teaching should be a source of inspiration and challenge, triggered both by the content of the lecture and by the communicational actions of the lecturer.

What are the students' perceptions regarding the lecture?

The discussion in this chapter suggests that students have, in general, a positive perception about the lectures. According to their views, good lecturing should involve good communicational interaction between the lecturer and the students. In the context of this communication, the lecturer should explain in detail, at the initial stages of the students' mathematical learning, the metarules governing the routines applied in the demonstrated examples, and he should make a systematic use of visual mediators.

Good communication in the context of lecture, according to the students' perspectives, must have certain characteristics with respect to the *atmosphere* and the *approach*: regarding the first, teaching should involve a comfortable and relatively friendly relationship between the lecturer and the students; and regarding the second, the lecture should not involve uninterrupted teaching, but ideally should allow more freedom and more opportunities for student engagement and interaction.

What are the students' perceptions regarding the produced lecture notes?

Lecture notes are generally well received although some students have expressed suggestions regarding the format of these notes. For instance, lecture notes should be offered both in hand written format as well as electronic format, in order to allow students to identify the various definitions and theorems more easily, as well as give students the time to record any additional information given orally in the lecture. They should include many examples, in which the governing object-level and metalevel rules should be demonstrated and clearly explained. Finally, regarding the often fast pace of lecturing, students expressed their disapproval and concern about the negative impact that this may have on their learning.

In fact, the effectiveness of communication seems to often be negatively affected by the fast pace of the lecture and the time required for copying the lecture notes. Some students reported that their focus shifted from trying to comprehend the material to the actual act of copying explicitly and exhaustively. On several occasions, students stated that lecture note taking should be an interactive process of communication between the lecturer and the students. Moreover, these students suggest that they should be able to follow the reasoning of the several narratives, examples and proofs, and have the opportunity to record any significant peripheral remarks.

What are the students' perceptions regarding the personal tutorial with the lecturer?

The majority of students do not favor personal tutorials with the lecturer, since this kind of mathematical communication might develop into a rather challenging and exposing experience, in which the majority of these novice students do not still feel confident to do so. Tutorials do not offer the safety of a group discussion between a small group of people that share the same difficulties and challenges and a seminar leader or assistant. In agreement with Jaworski (2003), this study suggests that tutorials involve a significant 'mathematical challenge'.

For the majority of novice students, tutorials seem to be a rather uncomfortable learning experience, since they have neither fully-developed all the characteristics of commognition in the specific discourse, for instance, *agency,* in the context of Group Theory, nor the required certainty about their reactions to certain commognitive actions of the lecturer. The discussion in this chapter suggests that the level of expected engagement in tutorials is a discouraging factor for students, whose aim is solely to gather information and clarifications without taking any substantially interactive communicational action.

What are the students' perceptions about seminars?

Seminars are considered helpful by the majority of students, since they offer a direct, yet more flexible way of communication compared to the communicational norms of the lecture. Despite the generally positive perception of students, there have been expressed certain objections about the way seminars operate and the upgraded role that these should have in the students' mathematical learning.

Students in their interviews would often focus on the quality of their communication with seminar staff and the importance of such communicational activity. Such communicational interaction seems to allow students to resolve any problems regarding the emerged commognitive conflicts, expose their thinking and reasoning, learn how to utilise the language of a certain mathematical discourse, use visual mediators or produce illustrations to clarify their reasoning, and, in general, learn the norms of a particular mathematical discourse. This improves the effectiveness of communication between the novice and the experienced mathematicians for the benefit of the former.

Why collaboration with peers is the favorite communicational context, according to students' perception?

Collaborating with peers is the preferred way of getting assistance. Working with peers allows students to communicate more freely, using a simpler vocabulary in a significantly more comfortable communicational environment, which allows them to discuss repeatedly their own object-level and metalevel problems and through this communication to resolve any commognitive conflicts.

The characteristics of communication amongst peers contribute to this preference. The discussion in this chapter suggests that students feel more comfortable to discuss something repeatedly, as many times as they need, with other students. Communication among peers does not require precise use of the group-theoretic terminology, whereas a discussion with the lecturer would be more demanding regarding this aspect. This conclusion is in agreement with Iannone and Nardi (2005) who discuss the informal language used among peers based on individual intuition.

What is the role of visual images and how it contributes to the effectiveness of communication?

The use of visual mediators, as also suggested by Ioannou and Nardi (2010), seems to be an indispensable communicational tool for achieving object-level understanding of the defined d-objects, as well as coping with the level of abstraction in Group Theory. The majority of students have demonstrated a natural inclination for using illustrations in their solutions, whenever possible, according, also, to the data in Chapter 4. In the discussion in this chapter, many students expressed their desire for a more extensive use of such illustrations by the lecturer.

Yet, although students repeatedly expressed their need to use visual mediators as a means to cope with the high level of abstraction, and improve mathematical communication in the context of teaching Group Theory, they have not extensively used them in practice. This conclusion is in agreement with Presmeg and Bergsten (1995), Dreyfus (1991) and Healy and Hoyles (1996), who report on the reluctance of students to use visual images. As pointed out by Presmeg (1985, 1997) and Zazkis et al (1996), this phenomenon may occur because concrete imagery, in order to be effectively used in the context of Mathematics, must be accompanied by rigorous analytical thinking.

The discussion in this chapter, as well as in chapter 4, suggests that illustrations given in the lecture notes do not necessarily have the intended impact, since they are rarely noticed or adopted by the students, both in their scripts and in the interview tasks. Students expressed a need for more visual images but often failed to use them when available. Moreover, the ability to visualise plays a significant role in comprehending the metalevel rules of a certain routine and therefore constructing proofs. Yet, students' mental images are often far from being detailed and workable. Images are subjective constructions, as is the choice to use them for coping with the level of abstraction, and not all students are willing to do so. Furthermore, being unable to draw a picture does not imply that one does not have an adequate image of the d-object.

Chapter 7 Conclusion

In this chapter, I discuss this study's *contribution to knowledge*, some *pedagogical implications* that have emerged from the discussion in the analysis chapters, the *methodological limitations* that have occurred in the data collection and analysis process, *further ideas for research*, and a *personal statement* in which I express some thoughts and emotions about my overall experience with this doctorate study.

7.1 Contribution to Knowledge

This study's contribution to literature is located in two different fields of study. Namely, the conceptual difficulties novice students have in their first encounter with Group Theory, and second the emerged pedagogical issues, and, in particular, the students applied study skills as well as their perception regarding the proofs and communication. In what follows, I summarise the results and claims that have emerged in the literature-embedded discussion sections in the analysis chapters, and which they comprise this study's contribution to literature.

Regarding the ***conceptual difficulties*** undergraduate students have in their first encounter with Group Theory, this study suggests that problems with the object-level understanding can occur as early as the introduction of the d-object of group, and the application of the axioms, for instance the proof that a group is Abelian. Object-level and metalevel learning in Group Theory is not disconnected by the object-level learning of other Pure Mathematics modules, since problems can be caused in the learning of Group Theory through problematic metaphors. The creation of effective mathematical learning requires the creation of well-structured realization trees that may involve various d-objects belonging to different mathematical discourses. At the early stages of their learning many students do not move with the required dexterity from one realisation tree to the other.

This study suggests that object-level learning does not always precede metalevel learning. Successful application of metalevel rules does not necessarily imply that all the involved mathematical d-objects have been fully objectified. This claim was repeatedly proved in occasions when many students had revealed problematic object-level understanding yet they were able to apply the routine and its metarules with success. In addition students' object-level understanding and metalevel understanding are often independent from each other. There were instances in the discussion in this chapter that problems in one level of understanding did not affect student understanding in the other level.

Regarding metarules, this study suggests that their successful application in one context does not imply successful application of the same metarules in a different context. This leads to the conclusion that students' metalevel understanding is often context-sensitive, and therefore requires extensive involvement with mathematical tasks in different contexts.

The first major conceptual crisis occurred when equivalence relations were introduced. One of the major obstacles was that many students had not yet objectified the d-object of group together with the binary operation. Moreover, a problematic objectification of the group operation has an unfavorable effect on the object-level understanding of equivalence relations and consequently on the application of metarules for proving that a relation is equivalent.

The second major crisis occurred with students were introduced the d-objects preceding the First Isomorphism Theorem, namely cosets, quotient groups, kernel, image etc. Problematic object-level understanding of these d-objects will not allow students to successfully cope with FIT, since this study suggests that it is a very demanding mathematical theorem for novice students, as it requires a thorough understanding of all the relevant d-objects.

Regarding the students' ability to evaluate their own solutions, this study suggests that there is an analogy between the quality of students' object-level learning and their capability to accurately evaluate their performance, as well as connection between the quality of their understanding and the quality of their solution's narrative presentation and syntax.

The d-objects of homomorphism and isomorphism seem to develop differently, according to this study. This is due to the fact that proving that a map φ is a homomorphism does not usually require fully developed object-level understanding of the d-object of homomorphism, but rather a successful application of a relatively 'concrete' and explicit routine with its governing metarules. Contrary to this, students' object-level and metalevel understanding of the d-object of isomorphism is problematic, since the majority of students have not properly objectified the d-objects of kernel and image. The apparent conceptual disengagement with the d-object of isomorphism is not predominantly based on the objectification of homomorphism, but rather on the objectification of the d- objects of kernel and image, and the metalevel rules regarding injectivity and surjectivity.

Regarding the emerging ***pedagogical issues***, this study suggests that proof production is, according to the majority of students' perception, an arduous task. Proof is often considered as a means of mathematical communication, and the majority of students seem to realise the rigor and precision that the activity of proving requires.

Proof production is more often than not considered a challenge, because students have to develop several indispensable skills, such as their ability to cope with the abstract nature of this module and a certain flexibility in the application of metarules, skills that require time and experience to be gained. Novice students often have difficulty with the 'how' and 'when' of the required routines. Successful proof production depends on the thorough object-level understanding of involved d-objects and their realization trees, as well as the successful and precise application of the governing metalevel rules, in the particular context. Two difficulties with proof production that often occurred in the data analysis were related to the applicability and the closure conditions of a certain proof. In particular, some students often face difficulties initiating a proof, and often have difficulty in recognising the signs that would signal the end of the proof.

Regarding the study skills applied in the revision for the final examination, this study suggests that students consider revision as a new contract for mathematical learning. They revisit what they have been taught, localise the conceptual gaps and overcome the remaining commognitive conflicts achieving, ideally, full objectification of the introductory d-objects of Group Theory and moreover obtain mathematical consciousness. The revision process, for the majority of students, involves three steps, namely study of the lecture notes, solution of the coursework alongside the model solutions, and solution of the past papers. The revision process is often nonlinear, but rather involves interchange between the study of lecture notes and literature, revisiting of the coursework with the model solutions, and the solution of past papers. Students that follow this revision approach work with the three elements interchangeably until they feel that they have overcome any commognitive conflicts and have achieved adequate object-level and metalevel understanding.

Regarding the study skills applied for the preparation of the coursework, this study suggests that students, at the beginning of the process, summarize their lecture notes and highlight the important mathematical narratives such as the definitions of the involved d-objects and the related theorems and lemmas who describe the respective object-level rules. This technique seems to contribute favorably in their object-level and metalevel understanding. Other study skills involve self-discipline and good study planning, as well as the ability to adjust ones' schedule to the programme of the department time table, so they are able to take full advantage of the opportunities for assistance that are offered.

Regarding the students' perspectives on communication, this study suggests that good teaching requires, according to students, effective communication between the lecturer and the students. Teaching is a manifold form of communication, through which the new material is presented and explained by the lecturer, and ideally leads to successful object-level and metalevel mathematical learning. The effectiveness of *communication* between the lecturer and the students contributes favorably to the students' commognition.

Regarding lectures, students suggest that they should have certain characteristics with respect to the *atmosphere* and the *approach*. Namely, teaching should involve a comfortable and relatively friendly relationship between the lecturer and the students, and uninterrupted teaching, which ideally would allow more freedom and more opportunities for student engagement and interaction between the two parties.

Lecture notes are generally well received although some students have suggested that lecture notes should be offered both in hand written format as well as electronic format, for practical reasons, as well as give students the time to record any additional information given orally in the lecture. Regarding the often fast pace of lecturing, students expressed their disapproval and concern about the negative impact that this may have on their learning.

This study suggests that the majority of students do not favor personal tutorials with the lecturer, since this kind of mathematical communication might develop into a rather challenging and exposing experience, in which the majority of these novice students do not still feel confident to do so. Regarding seminars, students consider them as significantly helpful in their process of learning, since they offer a direct, yet more flexible way of communication compared to the communicational norms of the lecture. Despite the generally positive perception of students, there have been expressed certain objections about the way seminars operate and the upgraded role that these should have in the students' mathematical learning. Collaborating with peers is the preferred way of getting assistance, due to the characteristics of communication amongst peers. Students feel more comfortable with discussing something repeatedly, as many times as they need, without being obliged to use very precisely the group-theoretic terminology, whereas a discussion with the lecturer would be more demanding regarding this aspect.

The use of visual mediators seems to be an indispensable communicational tool for achieving object-level understanding of the defined d-objects, as well as coping with the level of abstraction in Group Theory. The majority of students have demonstrated a natural inclination for using illustrations in their solutions. This study suggests that illustrations given in the lecture notes do not necessarily have the intended impact, since they are rarely noticed or adopted. Although students have expressed their need for more visual images, they often failed to use them when available. Moreover, the ability to visualise plays a significant role in comprehending the metalevel rules of a certain routine and therefore constructing proofs.

7.2 Pedagogical Implications

In this section, I discuss some pedagogical implications which are grounded on the discussion in the data analysis chapters and which would possibly improve undergraduate students' first encounter with Group Theory. This section could not be considered as a part of this study's contribution to literature, but rather as a result of reflection on the main core of the discussion in the previous chapters. In addition, my study's purpose was not to give substantial pedagogical recommendations and therefore the following discussion does not include any strong theoretical claims.

7.2.1 Teaching Group Theory

The traditional 'chalk and blackboard' method of teaching appears to be appropriate and well received in Mathematics, and in Group Theory in particular. Good communication between the lecturer and the students seems to be a contributing factor in the quality of teaching.

Although not always possible in the context of a lecture, students expressed their need for interaction, up to a certain level, between the lecturer and the audience. The purpose of this interaction is to solve any misunderstandings on the spot and to increase the level of student engagement. This may not be possible during the lecture, but, as will be discussed below, it would be a significant pedagogical suggestion in the context of the seminars.

Students' expressed need to have more examples has triggered concerns regarding the efficiency with which they take advantage of them. Although the lecturer has integrated several examples into his lecture notes, they do not always seem to be comprehensible to the students. It would be therefore more efficient if the lecturer was not only using examples to achieve object-level learning, i.e. to clarify the definition of certain mathematical concepts, but also as an opportunity to demonstrate how certain routines are applied, advancing in this way the metalevel mathematical learning, which is necessary in the solution of the coursework.

Regarding the coursework, it would be more beneficial for the students to hand in three smaller pieces of coursework after each cycle of seminars instead of submitting one long piece of coursework at the end of the semester. In this way, students could get their solutions (and the model solutions) for each coursework before handing in the next one. In this way they would have the chance to reflect on the previous coursework, pinpoint their errors and localise and resolve their problematic understanding.

7.2.2 Assisting Students in Learning Group Theory

Students should be encouraged to attend the one-to-one tutorials with the lecturer during his office hours. One of the reasons for students to avoid the tutorials is that they are expected to interact with the lecturer and reveal their level of understanding in a personal way, without the safety that a group of students might offer. Instead of having tutorials with the lecturer, it would therefore be beneficial for students to be able to meet the seminar assistants (research students) in an unofficial yet scheduled context, in which, as the data analysis suggests, they feel more comfortable to reveal their difficulties. Moreover, seminar assistants should assume this role of personal tutor for whoever requires it.

Seminars are the primary source of assistance that the department can offer to the students. The students expressed clear views regarding the role of seminars and how they should operate. There should be two different types of seminars spread over the semester. The first type should give students the opportunity to discuss their difficulties with the coursework as currently practised, but with further attention to the number of students attending the sessions as well as the time allocated to each student for discussion with the seminar staff. There should be a stricter control of student attendance so that students attend the sessions for their seminar groups.

The second type of seminar should be structured in a way that gives students the opportunity to discuss and reflect on the material given in the lectures. In this type of seminars students should be able to participate in a more general group discussion led by the seminar staff, having the opportunity to ask about issues they did not understand during the lectures. These sessions should not only concentrate on the object-level learning, but should also provide the opportunity for further expansion on the taught material, perhaps giving historical information about the development of the mathematical discourse under study or pointing to further applications and contemporary developments. Therefore the role of this type of seminar should be dual. First, to give the opportunity to discuss gaps that might have occurred in the lectures, and second, to serve as a think tank that would give students the opportunity for further challenge and inspiration.

7.3 Methodological Limitations

While appraising my study, I identified a number of methodological limitations with regard to the data collection process, in particular the interviews, seminars and observation field notes, the data analysis, as this was conducted and presented in the Threefold Accounts (TFA's), and finally the nature of the study in general.

7.3.1 Data Collection

Regarding the data collection, several methodological limitations occurred, especially during interviews, partly as a result of my inexperience. On some occasions during the first cycle of interviews, I was overtly surprised by the interviewees' responses about finding the module easy, revealing in this way my expectation of difficulty. After this observation, I made every effort to avoid repeating this mistake in the following circles of interviews.

In addition, there were some rare instances of listening comprehension difficulty, between the interviewees and myself, as well as several misunderstandings of what the student was trying to say. These problems occurred especially when I was interviewing Calaf and Carmen, both non-native speakers of English.

On certain occasions, the way I was asking the questions was rather leading, for instance: 'Which part of Group Theory did you find the hardest, factor groups, cosets or isomorphisms?' Without realising it, I was insisting on extracting statements of difficulty from the interviewees regarding certain issues that I had studied in the literature or discussed with the members of staff. I was often leading the interviewees towards acknowledging difficulty with particular concepts, instead of allowing the interviewees to identify them by themselves. This oversight was addressed during the process of interviewing.

During the interviews, I discussed a significant number of issues. As a result, the interview data, as indicated in the analysis that followed, was rich in variety and width, but not always in depth. At that time I did not have the experience or the research focus to ask further questions on issues that students had risen, which would have been beneficial.

Regarding the questions on affective issues, I noticed during the data analysis that I was quite leading by asking for instance: *'what are your emotions, what do you feel about this course?',* or '*Did you have any moments of stress?',* or

'What makes you feel angry or disappointed?'. The questions were rather leading towards acknowledging particular kinds of emotions, for instance stress, anger or disappointment, based on my experience with other students. Instead I should have allowed the students to identify their emotions by themselves. In any case this study's focus was eventually not on affective issues and therefore these data were not used for the purposes of this study.

In certain instances the question that followed the ones mentioned above was about the ways in which students dealt with the emotionally difficult moments. On a few occasions, especially in the first cycle of student interviews, instead of allowing the students to identify these ways by themselves, I was making suggestions, e.g. *'how do you deal with these moments, are you giving up, are you... working harder?'* This issue was addressed in the following cycles of interviews.

Since I am a non-native English speaker, my command of the English language, especially in oral conversations, is not always of the highest level. I noted this weakness in several instances during the interviews and it was also verified during the data analysis. In certain moments my questions were not completely comprehensible to the interviewees. There were also moments in which, although I wanted to ask further questions on certain issues, I was not able to do it exactly as I wished, because this would require a more advanced language use or description of more delicate issues such as the affective aspects of students' experience. In other instances I would understand the opposite of what the interviewee was trying to say, revealing in a way what I was expected to hear.

In some instances, I was slightly 'rude', in order for the data to be complete and consistent. For instance, at times students were referring to an exercise from the coursework, indicating it to me on paper without stating its number, and therefore I usually emphasised *'you are talking about exercise 2'*, for instance, in order to be able to record the number and do not loose track of the recorded discussion. In other instances we were discussing an interesting issue before the recorder was switched on, and usually I would say *'Can you repeat it because I am now recording you.'* Finally, on rare occasions, I was asking questions in an unnecessarily tactless way, making assumptions such as: '*I assume you were a good student in Mathematics at high school, eh?*'

During the part of the interview in which students had to produce an illustration, give an example or algebraically describe the concepts of Group Theory, the data I collected was not as rich as I had hoped for. The reason is that I was not insisting enough to get an answer, easily accepting the students' unwillingness to do so. I should have returned later on in the interviews, when possible, although this may have prevented students from attending a second or third interview. This would probably result in some inconsistency of the data.

In certain instances during the interviews I felt that I was more overtly compassionate or agreeable than I should be. Sometimes my approach was unnecessarily fatherly, trying to comfort students or in other instances my approach to questioning was adjusted according to students' reactions, resulting in partial information on certain issues, mostly the conceptual difficulties. In addition, on other occasions I unnecessarily discussed personal information related to my studies and experience, making unnecessary comparisons. These methodological limitations mostly occurred in the first cycle of interviews and were tackled in the following ones.

In certain interviews, especially the very first ones, I was not very persistent in keeping the natural continuity of the discussion as this was set by the informal question list, but I was rather shuffling up the subjects. I was often jumping from subject to subject without trying to achieve uniformity and consistency during the interview. In addition, sometimes I was not letting the students finish what they were trying to say but would assume what they were about to say and finish their sentence. These mistakes were addressed before the first cycle of interviews was finished.

In other moments I was using very specific terminology from Mathematics education, namely terms like abstraction, cognition, perception, visualisation, cognitive and affective aspects of learning. The use of these terms was inappropriate during the interviews, since the interviewees were not accustomed to them.

Concerning the seminar data and in particular the recording of the discussion between the students and members of the staff, the quality of the recording was not the best in a few instances, since the members of staff would often not place the recorder near enough or they would put it in their pocket causing excessive noise. For some of these discussions it was very difficult or impossible to produce descriptive ‘seminar vignettes’.

Finally, regarding the observations and field notes I identified an important methodological limitation. I did not keep a record of the number of students present in the lectures. This would have allowed me to have a clearer view of the fluctuation of presence and I would have been able to link it to the concept of engagement with the lectures and the module in general. In a few lectures when the number of students was low I took notice of it, but in general I did not keep a consistent record of student presence.

7.3.2 Data Analysis

Regarding the data analysis process, as this was prepared and presented in the Threefold Accounts (TFA’s), I identified a number of methodological limitations, which nevertheless have been addressed and rectified in the thesis.

In some instances during the data analysis, I was deducing certain claims that were not fully justified or obvious. For instance, I would deduce the inability to use visual or symbolic language from the students’ use of verbal language. Moreover, I was grounding my claim not on the evidence that I had, but rather on the absence of the evidence. In other words, in some instances my claims were based on a non-answer.

In a few instances, my analysis was slightly biased according to the general impression I had about some students and according to their results. For instance, in Musetta's case I claimed that some of her answers were copied from textbooks, judging the language, the use of symbols and the quality of narratives, but without being able to explicitly prove it.

My analysis was sometimes speculative about the origin of students' responses or reasons for success or failure. In many instances, I was overusing expressions like 'possibly', 'most probably', 'my impression', without giving a substantial interpretation of an error, although in several instances this would be considered as unjustified interpretation or absolutistic conclusion.

On other occasions, I drew conclusions based on my experience, claiming things that could not be extracted by analysing certain categories of data. For instance, based on the coursework data, I claimed that Manrico's writing lacks confidence. Although this objective claim sounds valid to me, it is not possible to detect evidence of confidence in a written piece of data and demonstrate it adequately.

Sometimes my analysis was too positive, expressing my enthusiasm on certain occasions, and other times it was too negative. For instance, Francesca and Musetta's writing was very difficult to understand, decode and analyse, which occasionally influenced my analysis. As a result, I attributed the difficulty they experienced to a lack of mathematical ability, when analysing the written data produced by these two students. Another example is related to Norma's verbalisation. On certain occasions during the data analysis I was slightly distraught and harsh, 'accusing' some of her attempts to verbalise of being excessive.

7.3.3 Nature of the Study

According to Winter (2000), validity in qualitative research can be addressed, among other elements, through the objectivity of the researcher and the participants themselves. Regarding the issue of researcher objectivity I made every effort to be objective, but my previous experience as a student of Group Theory under similar conditions might have influenced the interpretation of the data. The objectivity of the participants is difficult to judge. The general impression is that participants were mostly objective in their interviews, since there is generally consistency between their statements and other sources of data, such as their performance in certain exercises. There were moments though in student interviews, especially when discussing emotional aspects of their experience, when objectivity could be questioned.

Regarding the issue of generalisability, the sample of 13 out of the 78 students might not be considered as significant enough to make confidents claims relating to the applicability of the findings to other similar contexts. Nevertheless the conclusions of this study are generally in agreement with the existing literature on the field, extending the boundaries of knowledge and understanding of conceptual and learning issues regarding Group Theory and providing a new perspective on these issues.

7.4 Further Ideas for Research

During data analysis several interesting ideas for research have emerged that should be further investigated. These issues relate to three broad categories: conceptual, learning and affective issues related both to Group Theory and Ring Theory. Given the scope of this study, I did not discuss affective issues or issues related to Ring Theory. However, the data analysis suggests that there are important aspects that should be further investigated in order to systematically evaluate the effect of affective issues on the understanding of abstract algebra concepts and how these could be addressed by pedagogy. Finally, there are no exhaustive studies on Ring Theory education and it would be enthralling to start a systematic research in this field.

7.4.1 Further Investigation on Conceptual Issues in Group Theory

Although there are several studies regarding students' understanding of the concepts of Group Theory, it would be enlightening to further investigate specific areas on which there has not yet been much research or which appear to be of particular interest. For instance, it would be interesting to further investigate, by using all 78 students' scripts, the concept of equivalence relations and the application of the routine for a relation to be equivalent. Another area of further investigation relates to students' understanding of each object-level rule regarding reflexivity, symmetry and transitivity as well as the involved metalevel rules as they occur in the application of the routine. It would also be interesting to analyse the data in its entirety, to look for any possible patterns regarding the application of the routine and try to link this with previous misunderstandings of the involved prerequisite concepts. For all these issues I intend to use Commognitive Theoretical Framework, following the approach I have adopted for the purposes of this study.

Another important issue that, according to the data analysis, should be further investigated is students' very problematic engagement with the First Isomorphism Theorem. FIT is undoubtedly the capstone of a typical first module in Group Theory. Students' disengagement, as identified in this study, points to difficulties with the understanding of the prerequisite d-objects of kernel, image, cosets, normality, isomorphism, quotient group that prohibit students to achieve metadiscursive level learning of the FIT and to apply it successfully. Again, using all 78 students' data, it would be possible to further systematise and therefore be able to extract deeper and more generalisable conclusions regarding the apparent conceptual difficulties, for each of the aforementioned prerequisite d-objects and finally with FIT.

A third important issue regarding the conceptual fold of this study is the connection of the conceptual difficulties with the notion of visualisation, especially regarding the metalevel learning of Group Theory; the effect that successful use of visual mediators offered by the lecturer has on the metalevel learning of group theoretic d-objects such as coset, kernel, image, homomorphisms and quotient groups. The analysis of the data for the 13 students involved in this study suggests that they do not effectively use the illustrations offered in the lecture notes, yet in the interviews they express their need to have a mental picture. There are indications of mechanical use without any indication of objectification of these illustrations. Again, scrutinising all 78 student scripts, it would be interesting to identify any patterns in the use of the illustrations and to further investigate the ways in which students use them in order to achieve the required discursive shift. As above I intend to use Commognitive Theoretical Framework, focusing on the use of characteristics of Mathematical Discourse.

Finally, the relationship between object-level and metalevel understanding of students in their first encounter with Group Theory should be linked with the notion of abstraction. Undoubtedly, Group Theory, as the literature suggests, is considered one of the most abstract modules that a novice student is expected to face. The abstract nature of Group Theory and the effect it has on mathematical learning should be further investigated using the Commognitive Theoretical framework, which would allow us to separately study the object-level and the metalevel learning.

7.4.2 Further Investigation on Pedagogical Issues in Group Theory

Several issues regarding pedagogy, both learning and teaching, in Group Theory have emerged. It would be interesting to further investigate, by scrutinising all 78 students' scripts, their difficulty with proofs in this specific field, using the Commognitive Theoretical Framework, which would enable the distinction to be made between the object-level difficulties and the metalevel difficulties regarding the process of proving. It would also be enlightening to further explore the students' ability to apply the 'how' and the 'when' of the routine as this develops in the several stages of the module. In addition it is important to investigate the applicability conditions and the closure conditions of a proof and how students respond to these.

Another category of issues that should be further investigated is directly linked with the teaching of Group Theory. It would be interesting to explore the traditional approach that the lecturer has used and to analyse the lecture notes and the seminar and lecture recordings in order to identify his pedagogical intentions and techniques. Moreover, using the student interviews and scripts, it could be explored whether the lecturer's pedagogical intentions have been adopted by the students and, if not, to investigate the reasons for this failure. In addition, it would be interesting to further examine the role of seminars, how these should be designed, considering the need for more interaction expressed by the students.

A third category of issues is related to students' skills regarding the preparation of the coursework and the revision and taking of the final examination. It would be interesting to further investigate in a holistic way the progress or regression of the 78 students by analysing both their coursework and their examination scripts and trying to identify the reasons for these phenomena. It would also be interesting to explore ways in which teaching could assist students to improve their study skills and consequently their performance in Group Theory.

Another interesting issue that should be further investigated is related to the transition from secondary to tertiary Mathematics education. For instance, Lithner (2000) is referring to plausible reasoning versus the reasoning on established experiences and Sierpinska (2000) discusses theoretical thinking versus practical thinking. There are indications that students tend to use reasoning connected to established experiences and their thinking is practical. It would be interesting to investigate how this fact is connected to students' performance in Group theory and how students would be able to make the transition towards plausible reasoning and theoretical thinking.

Finally, it would be interesting to further examine, by analysing the data for all 78 students, the role of metaphors, how problematic metaphors influence student learning in Group Theory and what means should be adopted to prevent interference of such problematic metaphors in the process of mathematical learning. Examples of metaphors in Group Theory may be considered the following: fields → groups, inverse in arithmetic → inverse in groups, bijection as defined in introductory courses → isomorphism, symmetries in elementary mathematics → symmetries of permutation groups, multiplication tables in arithmetic → multiplication tables in group theory, powers in arithmetic → order of g, the concept of order → the order of group, and symbolic metaphor of order $|G|$ versus absolute value of an integer $|x|$.

7.4.3 Investigation on Affective Issues

When scrutinising the data in relation to affective issues several patterned phenomena emerged that favorably or unfavorably influence students' first encounter with Group Theory. These issues have been summarised and the most important are discussed below.

Regarding the *nature of Group Theory* as such, six students expressed their dislike and four students stated their preference towards this module. *Working on the coursework without any immediate results* is probably the most serious factor in the development of an unfavourable attitude. Nine students stated that this fact causes frustration and a decrease of confidence and motivation. The process of *preparation for the coursework* as such is also a factor that influences students' attitudes. Seven students directly stated that they are anxious in their effort to prepare for the coursework whereas one student overtly expressed his enjoyment to do so.

The *advisers and the department* are very supportive and helpful according to six students, contributing to the positive student attitude, whereas two students believe that they do not get the support they need. Regarding the *lecturer,* five students stated that the lecturer of the module is very good, expressing also their preference regarding his methods and how his enthusiasm helps them to have a more positive attitude towards the module, whereas two students do not like his methods considering his teaching rather formal and hasty. Three students talked positively about their *previous* *encounter with Group Theory*, and how this has favourably influenced their level of confidence in the current module.

Possibly the strongest positive factor influencing students' experience was *working with peers*. Eight students discussed the importance of working with their peers and how this gives them confidence and encouragement. Two students stated that they did not like to work with others; one because he feels that it is time consuming and the other because she simply does not want to. Working with peers is a student learning method that, from the affective viewpoint, seems to be one of the most important elements in the learning process.

Regarding *visualisation*, four students discussed their difficulty to visualise in this module and how this influences their confidence and attitude towards the module. Three students expressed their frustration about the fact that they did not know whether their solutions were adequate or complete (closure and applicability conditions of metalevel learning).

Finally, it is interesting to note that the three foreign students whose command of English language was poor had declared three issues that were in common and influenced their learning experience unfavourably. The *language* as such was considered as a serious obstacle, but consequently this difficulty had led them to state and believe that their *level of intelligence is not adequate* for this module and therefore their *lack of motivation,* as expressed in the interviews, was severe.

Affective issues were not discussed in this thesis and their analysis has not been interpretive with the systematic use of a particular theoretical framework. Moreover, in a future attempt to investigate these issues I intent to use the language of Goldin (2000), which was further developed by Weber (2008), and, in addition, it would be interesting to make a systematic and thorough analysis of these issues using the Cooling off/Cooling out theoretical framework by Daskalogianni and Simpson (2002), which would allow me to investigate the progressive engagement or disengagement of students as it is influenced by affective issues.

7.5 Personal Statement

My personal encounter with Mathematics Education as an academic discipline commenced four years ago, with the initiation of this doctorate study. It was an overall rewarding journey, coloured with many different emotions, experiences and alterations in attitude and inducement. There were moments of fulfillment and joy, especially during the period of data collection, my participation in national and international conferences, the presentation of my research in front of colleagues and during the final stages of data analysis and writing up of my thesis. As with every path that can lead us to a major achievement, I have experienced some difficult moments that were mostly a result of the continuous process of adjustment to a new field of study with its new corresponding requirements and a new level of philosophical and academic sophistication.

During these four years, I feel that I have taken some steps forward in the non-ending journey of my development, both academic and personal. Regarding my academic development, this doctorate study has contributed substantially. I engaged with a significant number of studies in the field of general Mathematics Education, undergraduate Mathematics Education and methodology, which introduced me to this discipline and broadened my horizons as an academic. I attended the Mathematics Education seminar series offered by my school and reflected on the discussed material. I also participated in the two seminar cycles on philosophical and methodological issues in social sciences and Mathematics Education in particular and reflected on the discussed material, giving me the opportunity to appreciate the importance of methodology and epistemology in the context of a social science research project. In addition my doctorate study per se was a continuous source of learning and new experiences in all its phases, namely, how to approach a substantial sector of scientific literature, how to plan a social science, naturalistic inquiry, how to proceed with the data collection and data analysis, how to execute the analysis of a large quantity of data, how to systematise and interpret the results and finally how to write a thesis of this magnitude. All these elements are indispensable tools for my future career as an academic, mathematician, mathematics educator and active researcher.

This study has also significantly contributed to my personal development. I have experienced the difficulties that an academic career entails and the persistence, patience, self-motivation, love for one's field of study, endurance and self-discipline that is required. A doctorate student needs to overcome many challenges, keep the initial spark alive inside him/her despite the difficulties he/she will face, try to sharpen his/her mind and deepen his/her perception, increase his/her stamina and courage, try to find the truth in everything he/she is engaged with, honestly appreciate help of any form and from anybody, and strengthen his/her relationships with his/her loved ones for his/her own good. My experience so far in life has taught me that it is during our difficulties and challenging moments that we develop our character and personality, leaving in each of these moments an ε of our bad self.

APPENDIX A: Coursework Problem Sheets

A1 – Problem Sheet 1

1. (i) Suppose $(G, \cdot)$ is a group with the property that $g^2 = e$ for all $g \in G$. Prove that for all $g_1, g_2 \in G$ we have $g_1 g_2 = g_2 g_1$ (that is, G is *abelian*).
(ii) Suppose $(G, \cdot)$ is a group with 4 elements which has the property that $g^2 = e$ for all $g \in G$. If a, b are distinct non-identity elements of G, show that the other non-identity element of G is ab. Hence write down the multiplication table of G.

2. (i) Suppose X is a non-empty set and α, β are permutations of X with the property that any element of X moved by α is fixed by β and any element of X moved by β is fixed by α, i.e. for all $x \in X$:

$$(\alpha(x) \neq x \Rightarrow \beta(x) = x) \text{ and } (\beta(x) \neq x \Rightarrow \alpha(x) = x).$$

Prove that $\alpha \circ \beta = \beta \circ \alpha$. [Hint: consider $\alpha(\beta(x))$ in the cases where x is moved by β and where it is moved by α; note that if x is moved by β, then so is $\beta(x)$.]
(ii) Suppose $(G, \cdot)$ is a group and $g, h \in G$ are such that $gh = hg$. Show that for all $r \in \mathbb{N}$ we have $(gh)^r = g^r h^r$. Give an example of $g, h \in S_3$ (the symmetric group on $\{1, 2, 3\}$) where $(gh)^2 \neq g^2 h^2$.

3. Using the usual test for being a subgroup, give proofs of the following:
(i) For any $n \in \mathbb{N}$, the set $n\mathbb{Z}$ of integers divisible by n is a subgroup of $(\mathbb{Z}, +)$.
(ii) If A is an $m \times n$ matrix over $\mathbb{R}$, then $\{x \in \mathbb{R}^n : Ax = 0\}$ is a subgroup of $(\mathbb{R}^n, +)$.
(iii) $\{z \in \mathbb{C}^\times : |z| = 1\}$ and $\{e^{(1+i)t} : t \in \mathbb{R}\}$ are subgroups of $(\mathbb{C}^\times, \cdot)$ (where $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$). Draw these sets on an argand diagram.
(iv) For any $n \in \mathbb{N}$, the sets $\{g \in \mathrm{GL}(n, \mathbb{R}) : \mathrm{Det}(g) = 1\}$ and $\{g \in \mathrm{GL}(n, \mathbb{R}) : gg^T = I_n\}$ are subgroups of $\mathrm{GL}(n, \mathbb{R})$ (where g^T denotes the transpose of g and I_n is the $n \times n$ identity matrix).

4. Suppose $(G, \cdot)$ is a group and H, K are subgroups of G.
(i) Show that $H \cap K$ is a subgroup of G.
(ii) Show that if $H \cup K$ is a subgroup of G then either $H \subseteq K$ or $K \subseteq H$. [Hint: if not, then take $h \in H \setminus K$ and $k \in K \setminus H$ and consider hk.]

5. Consider the following permutations in S_5:

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 4 & 3 & 2 \end{pmatrix}.$$

Prove that $\langle \alpha, \beta \rangle$ is a subgroup of S_5 with 10 elements which can be identified with the symmetries of a regular pentagon with vertices labelled 1,2,3,4,5. [Hint: This is in your notes; you just need to fill in the details.]

6. Describe the group G of rotational symmetries of a cube, saying what the possible axes of rotation are and what the possible angles of rotation are. Hence show that there 24 such rotational symmetries. Consider one of the three pairs of opposite faces of the cube. Show that the set of rotational symmetries of the cube which send this pair of faces to itself forms a subgroup of G of order 8.

A2 – Problem Sheet 2

1. Suppose p, q are distinct prime numbers and G is a group with $|G| = pq$. Suppose $\alpha \in G$ has order p and $\beta \in G$ has order q. Prove that $\langle \alpha, \beta \rangle = G$. [Hint: Use the same idea as in Example 2.3 of the notes.]

2. Suppose X is a non-empty set and $G \leq \mathrm{Sym(X)}$.
Let $a \in X$ and $H = \{g \in G : g(a) = a\}$.
(i) Prove that H is a subgroup of G.
(ii) Prove that $g_1 H = g_2 H \Leftrightarrow g_1(a) = g_2(a)$ (for all $g_1, g_2 \in G$).
(iii) Now suppose X is finite and let $A = \{g(a) : g \in G\}$.
Prove that $|A||H| = |G|$.
(Remark: H is called the *stabiliser* of a in G and A is the G-*orbit* containing a. The results are a version of the *Orbit-Stabiliser Theorem.*)

3. Suppose X is a non-empty set and $G \leq \mathrm{Sym(X)}$. Define a relation $\sim$ on X by:

$$x \sim y \Leftrightarrow \text{ there exists some } g \in G \text{ with } g(x) = y.$$

Prove that $\sim$ is an equivalence relation on X.
(Remark: The equivalence classes are the G-orbits, as in question 2.)

4. Suppose G is a group of order 4 and G is not cyclic. Prove that $g^2 = e$ for every $g \in G$. Deduce that any group of order 4 is abelian.

5. Prove that the following are homomorphisms:
(i) G is any group, $h \in G$ and $\phi : G \to G$ is given by $\phi(g) = hgh^{-1}$.
(ii) $G = \mathrm{GL}(n, \mathbb{R})$ and $\phi : G \to G$ is given by $\phi(g) = (g^{-1})^T$.
(Here $\mathrm{GL}(n, \mathbb{R})$ is the group of invertible $n \times n$-matrices over $\mathbb{R}$ and the ^{T} denotes transpose.)
(iii) G is any abelian group and $\phi : G \to G$ is given by $\phi(g) = g^{-1}$.
(iv) $\phi : (\mathbb{R}, +) \to (\mathbb{C}^\times, \cdot)$ given by $\phi(x) = \cos(x) + \mathrm{i}\sin(x)$.
(v) G is any group, N is a normal subgroup of G and $\phi : G \to G/N$ is defined by $\phi(g) = gN$.
In each case say what is the kernel and the image of ϕ. In which cases is ϕ an isomorphism?

6. (i) Suppose H is a non-trivial subgroup of $\mathbb{Z}$, the group of integers under addition, and let d be the smallest natural number in H (- why is there such a thing?). Prove that $H = d\mathbb{Z}$.
(ii) Suppose $G = \langle g \rangle$ is a cyclic group (written multiplicatively). Define $\phi : \mathbb{Z} \to G$ by $\phi(n) = g^n$ (for $n \in \mathbb{Z}$). Prove that this is a homomorphism.
(iii) Using (i) and (ii) and the First Isomorphism Theorem, deduce that if G is a cyclic group then there exists an integer d such that $G \cong \mathbb{Z}/d\mathbb{Z}$.

A3 – Problem Sheet 3

1. (i) Suppose R is an integral domain with only finitely many elements. Prove that R is a field. (Hint: you have to show that every non-zero $x \in R$ has a multiplicative inverse. To do this, show that there is some $n \in \mathbb{N}$ with $x^n = 1$.)
(ii) For $m \in \mathbb{N}$ (and $m \geq 2$) let $\mathbb{Z}_m$ denote the ring of 'integers modulo m'. For which m is $\mathbb{Z}_m$ an integral domain? Justify your answer.

2. Let R be the ring $\mathbb{Z}[\sqrt{3}] = \{m + n\sqrt{3} : m, n \in \mathbb{Z}\}$.
(i) Prove that the function $N : R \to \mathbb{Z}$ given by $N(m + n\sqrt{3}) = m^2 - 3n^2$ satisfies $N(xy) = N(x)N(y)$ for all $x, y \in R$.
(ii) Prove that if $u \in R$ is a unit, then $N(u) = \pm 1$.
(iii) Find a unit $u \neq \pm 1$ in R. By considering powers of u (or otherwise) show that there are infinitely many units in R.

3. Prove that $x^2 + 1$ is irreducible over $\mathbb{F}_7$. Prove that $x^3 - 9$ is irreducible over $\mathbb{F}_{19}$ but reducible over $\mathbb{F}_{11}$.

4. (i) Complete the proof in the notes (3.10) that $\mathbb{Z}[\sqrt{-5}]$ is not a unique factorization domain by showing that 2, 3, $1 + \sqrt{-5}$ and $1 - \sqrt{-5}$ are irreducible in $\mathbb{Z}[\sqrt{-5}]$.
(ii) Find the units in $\mathbb{Z}[\sqrt{-7}] = \{x + y\sqrt{-7} : x, y \in \mathbb{Z}\}$, and show that it is not a unique factorization domain. (Hint: Use the norm function $N(x + y\sqrt{-7} = x^2 + 7y^2$ and the same method as in 3.10.)

5. Suppose R is a ring and I is an ideal of R. Prove that if there is an element $u \in I$ which is a unit in R, then $I = R$.

6. Suppose R is a ring and $I, J \subseteq R$ are ideals of R. Prove that $I + J = \{a + b : a \in I, b \in J\}$ is an ideal of R. Prove that $I \cap J$ is an ideal of R.

7. In following, find all ideals in the principal ideal domain R which contain the given ideal I.
(i) $R = \mathbb{Z}$ and $I = 6R$;
(ii) $R = \mathbb{Q}[x]$ and $I = (x^2 + 1)R$;
(iii) $R = \mathbb{F}_3[x]$ and $I = (x^3 - x + 1)R$.
In which cases is I a maximal ideal of R?

8. (i) Suppose $p > 2$ is a prime and x^2+1 is reducible in $\mathbb{F}_p[x]$. Prove that the multiplicative group of $\mathbb{F}_p$ contains an element of order 4. Deduce that p is congruent to 1 modulo 4.
(ii) For which prime numbers p is the quotient ring

$$\mathbb{F}_p[x]/(x^2 + 1)$$

a field?

APPENDIX B: Model Solutions to Coursework

B1 – Model Solutions to Problem Sheet 1

1. (i) By assumption, $(g_1g_2)^2 = e$, so $g_1g_2g_1g_2 = e$. Multiply both sides on the left by g_1 and on the right by g_2 to get $g_1^2g_2g_1g_2^2 = g_1g_2$. As $g_1^2 = g_2^2 = e$ this gives what we want.
(ii) Let $c = ab$. As $a^2 = e$ we have $a^{-1} = a$, thus as $a \neq b$, we have $c \neq e$. Also $c \neq a, b$ (for example, $c = b$ would imply $a = e$). So the elements of G are a, b, c, e. By (i) G is abelian (so the multiplication table is symmetric). We know $aa = e$ etc, so the only things left to compute are ac and bc. But $ac = aab = b$ and $bc = cb = cbb = c$.

2. (i) Consider various cases, as suggested by the Hint.
Case 1: x is moved by β. Then $\beta(x)$ is also moved by β (otherwise β is not one-to-one!), so both x and $\beta(x)$ are fixed by α. Thus $\alpha(\beta(x)) = \beta(x) = \beta(\alpha(x))$.
Case 2: x is moved by α. This is similar (by the symmetry of the argument).
Case 3: x is fixed by both α and β. Then $\alpha(\beta(x)) = x = \beta(\alpha(x))$.
So in all possible cases $\alpha(\beta(x)) = \beta(\alpha(x))$.
(ii) Informally: $(gh)^r = ghghgh \ldots gh$ (r times) and $gh = hg$ means we can rearrange the g's and h's collecting them together to obtain g^rh^r.
More formally, do this by induction on r. The inductive step is:

$$(gh)^{r+1} = (gh)^r(gh) = g^rh^rgh = g^rgh^rh = g^{r+1}h^{r+1}.$$

For the example, you could take

$$g = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \text{ and } h = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

(In fact it's worth noting that in general $(gh)^2 = g^2h^2$ iff $hg = gh$.)

3. In each case we should observe that the sets are non-empty (I'll skip this part), closed under the group opertation and closed under taking inverses. In each case, call the subset under consideration H.
(i) $n\mathbb{Z}$ is the set of integers of the form na for some $a \in \mathbb{Z}$. As $na + nb = n(a + b)$ and $-(na) = n(-a)$ (for any $a, b \in \mathbb{Z}$), the test is satified.
(ii) If $x, y \in H$ then $A(x+y) = Ax+Ay = 0+0 = 0$ so $x+y \in H$. Also $A(-x) = -Ax = 0$ so $-x \in H$.
(iii) The first part follows from the fact that $|z_1z_2| = |z_1||z_2|$. For the second, note that $e^{(1+i)t}e^{(1+i)s} = e^{(1+i)(s+t)}$ and $(e^{(1+i)t})^{-1} = e^{-(1+i)t}$. On an argand diagram, the first set is the unit circle. The second is harder to draw, but is a spiral which winds very tightly near to 0, and very loosely further out.
(iv) The first follows from the fact that $\mathrm{Det}(gh) = \mathrm{Det}(g)\mathrm{Det}(h)$ (and so $\mathrm{Det}(g^{-1}) = 1/\mathrm{Det}(g)$). For the second, note that an equivalent condition for being in the subset H is $g^{-1} = g^T$. If $g, h \in H$ then $(gh)^T = h^Tg^T = h^{-1}g^{-1} = (gh)^{-1}$, so $gh \in H$. Also $(g^{-1})^{-1} = g = (g^T)^T = (g^{-1})^T$ so $g^{-1} \in H$.

4. (i) Use the test from the notes. As $e \in H \cap K$ we have $H \cap K \neq \emptyset$. If $g, h \in H \cap K$ then $g, h \in H$, so $gh \in H$ as H is a subgroup. Similarly $gh \in K$, so $gh \in H \cap K$. Also $g^{-1} \in H$ as H is a subgroup and $g \in H$; similarly $g^{-1} \in K$. So $g^{-1} \in H \cap K$.
(ii) With the notation as in the hint, we have $hk \in H \cup K$, so $hk \in H$ or $hk \in K$. In the first case we have $hk = h'$ for some $h' \in H$. Rearranging, we obtain $k = h'h^{-1}$. As $h, h' \in H$ and H is a subgroup, this means $k \in H$ contradicting how it was chosen. But also the case $hk \in K$ leads to a similar contradiction. Thus no such choice of h, k is possible: we have either $H \subseteq K$ or $K \subseteq H$.

5. The first thing to do is to calculate that $\alpha^5 = e = \beta^2$ and $\beta\alpha\beta = \alpha^{-1}$, and this is an explicit calculation with permutations. We can therefore write any element of $\langle \alpha, \beta \rangle$ in

the form $\alpha^r\beta^s$ for some $r = 0, 1, 2, 3$ or 4 and $s = 1$ or 2 (use the relation $\beta\alpha = \alpha^{-1}\beta$ to collect all the α's together in any product of powers of α and β). Thus $\langle\alpha, \beta\rangle$ has at most 10 elements. The expressions $\alpha^r\beta^s$ above are all distinct: otherwise we obtain a relation $\alpha^{r-r'} = \beta^{s'-s}$ for some $r, r' \in \{0, \ldots, 4\}$ and $s, s' \in \{0, 1\}$. The only way a power of α can equal a power of β is if they are both the identity (check by explicit calculation, or something cleverer). This implies that $r = r'$ and $s = s'$.
Label the vertices of a regular pentagon as 1,2,3,4,5 (clockwise around the pentagon). Then the above permutations can be induced by symmetries: clockwise rotation through $2\pi/5$ for α and a flip about an axis through vertex 1 and the centre for β. The product of two symmetries is a symmetry, so any permutation generated by α, β is also induced by a symmetry of the pentagon. But there are only 10 of these, so α, β generate all of them.

6. You should draw some pictures for this.
The possibilities are:
(i) Axis of rotation through centres of a pair of opposite faces; angle of rotation π. There are 3 symmetries of this type.
(ii) Axis of rotation through centres of a pair of opposite faces; angle of rotation $\pm\pi/2$. There are 6 symmetries of this type.
(iii) Axis of rotation through mid-points of a pair of (diametrically) opposite edges; angle of rotation π. There are 12 edges, so 6 such pairs of edges, and therefore 6 symmetries of this type.
(iv) Axis of rotation through mid-points of (diametrically) opposite vertices; angle of rotation $\pm 2\pi/3$. There are 4 such axes, so 8 symmetries of this type.
Adding in the identity element, we get $1 + 3 + 6 + 6 + 8 = 24$ rotational symmetries.

For the final part, note that, by inspection, the rotations which send the pair of faces to itself are the rotations about an axis through the centres of the faces (4 of these, type (i), (ii) above); rotations through π about axis through centres of pairs of opposite edges between the faces (2 of these, type (iii)); rotation through π about an axis through one of the other pairs of opposite faces (2 of these, type (i) above). This gives 8 symmetries in total.

B2 – Model Solutions to Problem Sheet 2

1. The subgroup $\langle \alpha, \beta \rangle$ has subgroups $\langle \alpha \rangle$ and $\langle \beta \rangle$ of orders p and q respectively. So by Lagrange's Theorem, $|\langle \alpha, \beta \rangle|$ is divisible by p and q. As p, q are distinct primes it follows that $|\langle \alpha, \beta \rangle|$ is divisible by pq. But $\langle \alpha, \beta \rangle \leq G$ and G has only pq elements. So $\langle \alpha, \beta \rangle = G$.

2. (i) Use the test in 1.16 of the notes. First, $H \neq \emptyset$ as H contains the identity permutation. Suppose $g_1, g_2 \in H$. Then $g_1 g_2(a) = g_1(a) = a$, so $g_1 g_2 \in H$. Also as $g_1(a) = a$ we obtain (applying g_1^{-1} to both sides) that $a = g_1^{-1}(a)$. Thus H is closed under the group operation and under taking inverses, so is a subgroup.
(ii) For $g_1, g_2 \in G$ we have:

$$g_1 H = g_2 H \Leftrightarrow g_2^{-1} g_1 H = H \Leftrightarrow g_2^{-1} g_1 \in H$$

(for example, by 2.7 in the notes), and

$$g_2^{-1} g_1 \in H \Leftrightarrow g_2^{-1} g_1(a) = a \Leftrightarrow g_1(a) = g_2(a).$$

(iii) Let $a_1 \in A$. So there is $g_1 \in G$ such that $g_1(a) = a_1$. By (ii), the set of $g \in G$ with $g(a) = a_1$ is the left coset $g_1 H$. Thus there are exactly $|H|$ elements of G which send a to a_1 (by 2.8 in the notes). It follows that $|G| = |H||A|$ (because, for example, the function $f : G \to A$ given by $f(g) = g(a)$ is $|H|$-to- one and is onto).

3. Just check that $\sim$ is reflexive, symmetric and transitive. For $x, y, z \in X$ we have:
(i) The identity e is in G and $e(x) = x$, so $x \sim x$.
(ii) If $x \sim y$ there is $g \in G$ with $g(x) = y$, so $x = g^{-1}(y)$. As $g^{-1} \in G$, this shows $y \sim x$.
(iii) If $x \sim y$ and $y \sim z$ there exist $g, h \in G$ with $g(x) = y$ and $h(y) = z$. Then $hg(x) = z$ and $hg \in G$, so $x \sim z$.

4. As a Corollary to Lagrange's Theorem (see 2.2), the possible orders of an element in a group of order 4 are 1, 2 or 4. If G is not cyclic it has no element of order 4, so all $g \in G$ satisfy $g^2 = e$. Such a group is abelian (by 1(i) on Sheet 1). On the other hand, the only other possibility for a group of order 4 is that it is cyclic: as cyclic groups are abelian, we're done.

5. (i) $\phi(g_1)\phi(g_2) = hg_1h^{-1}hg_2h^{-1} = hg_1g_2h^{-1} = \phi(g_1g_2)$, so ϕ is a homomorphism. As $\phi(g) = e \Leftrightarrow hgh^{-1} = e \Leftrightarrow g = e$, the kernel of ϕ is the trivial subgroup $\{e\}$. As $\phi(h^{-1}gh) = g$, ϕ is onto. (Thus ϕ is an isomorphism.)
(ii) $\phi(g_1g_2) = ((g_1g_2)^{-1})^T = (g_2^{-1}g_1^{-1})^T = (g_1^{-1})^T(g_2^{-1})^T = \phi(g_1)\phi(g_2)$ (which properties of matrices are being used here?). Note that $\phi(g) = h$ iff $g = (h^{-1})^T$ so ϕ is a bijection: the kernel is $\{e\}$, and ϕ is onto.
(iii) As G is abelian, $\phi(g_1g_2) = g_2^{-1}g_1^{-1} = g_1^{-1}g_2^{-1} = \phi(g_1)\phi(g_2)$. Again, ϕ is an isomorphism.
(iv) To see that ϕ is a homomorphism, note that $\phi(x) = \exp(ix)$ and use the fact that $\exp(i(x+y)) = \exp(ix)\exp(iy)$ (or write it out in full and use trig formulae). The kernel is $\{2\pi n : n \in \mathbb{Z}\}$ and ϕ is onto.
(v) As N is a normal subgroup of G, $\phi(g_1)\phi(g_2) = g_1Ng_2N = g_1g_2N = \phi(g_1g_2)$. The kernel of ϕ is N and ϕ is surjective.

6. (i) Note that as H is a non-trivial subgroup of $\mathbb{Z}$ it contains a non-zero element, and therefore a positive element. So we can consider the smallest positive element d of H. Suppose x is any other element of H. By the division algorithm in $\mathbb{Z}$, there exist $q, r \in \mathbb{Z}$ with $x = qd + r$ and $0 \leq r < d$. Note that as $x, d \in H$, also $r = x - qd \in H$. So by choice of d we must have $r = 0$ i.e. $x = qd$. So $H \subseteq d\mathbb{Z}$. But as $d \in H$ we have $d\mathbb{Z} = \langle d \rangle \subseteq H$. So $H = d\mathbb{Z}$, as required.
(ii) We note that $\phi(n+m) = g^{n+m} = g^n g^m = \phi(n)\phi(m)$, so ϕ is a homomorphism.
(iii) Let g be a generator of G and let ϕ be as in part (ii). By part (i), if H is the kernel of the ϕ then either $H = \{0\} = 0\mathbb{Z}$, or there is a natural number d with $H = d\mathbb{Z}$. Allowing $d = 0$, there is therefore some $d \in \mathbb{Z}$ with $H = d\mathbb{Z}$. As g is a generator of G, $im(\phi) = G$. Thus by the Isomorphism Theorem, $\mathbb{Z}/H \cong G$. So $G \cong \mathbb{Z}/d\mathbb{Z}$.

B3 – Model Solutions to Problem Sheet 3

1. (i) Suppose $x \in R$ and $x \neq 0$. By considering powers of x there exist natural numbers $n < m$ with $x^n = x^m$ (as R is finite). So $x^m(x^{m-n} - 1) = 0$. As R is an ID, either $x^m = 0$ or $x^{m-n} = 1$. The first is impossible as $x \neq 0$ (and R is an ID). So $x^r = 1$, for some $r > 0$. Thus x has a multiplicative inverse $x^{r-1} \in R$. So R is a field.
(ii) $\mathbb{Z}_m$ is an ID iff m is a prime. Indeed, if $m = ab$ where $1 < a, b < m$ then in $\mathbb{Z}_m$ we have $a \neq 0$ and $b \neq 0$ and $ab = 0$, so $\mathbb{Z}_m$ is not an ID. Conversely, if m is a prime and a, b are integers with $ab = 0$ (modulo m) then m divides ab so (as m is a prime) m divides a or m divides b. Thus (in $\mathbb{Z}_m$), $a = 0$ or $b = 0$.

2. (i) Write $x = m + n\sqrt{3}$ and $y = a + b\sqrt{3}$ and simply compute $N(xy)$ and $N(x)N(y)$.
(ii) If there is $v \in R$ with $uv = 1$ then $N(u)N(v) = 1$ (by (i)). So $N(u)$ is a unit in $\mathbb{Z}$, and so must equal ± 1.
(iii) Note (taking a hint from (ii)) that $(2 + \sqrt{3})(2 - \sqrt{3}) = 2^2 - 3 = 1$. So $u = 2 + \sqrt{3}$ is a unit. The powers u^n of u are also units and as $u > 1$, we have $u^m < u^n$ for $0 < m < n$, so they are all distinct.

3. Note that a (non-constant) polynomial f over a field F has degree ≤ 3 then it is irreducible iff it has no root in F. So in each case we just need to check whether the given poynomial has a root in the given finite field. One checks:
(a) $x^2 + 1$ has no root in $\mathbb{F}_7$;
(b) $x^3 - 9$ has no root in $\mathbb{F}_{19}$ (the cubes in $\mathbb{F}_{19}$ are $0, 1, 8, 7, 18, 11, 12$).
(c) Either find $x \in \mathbb{F}_{11}$ with $x^3 = 9$, or argue as follows. The map $x \mapsto x^3$ is a homomorphism α from the multiplicative group $\mathbb{F}_{11}^{\times}$ to itself. This group has order 10, and so has no elements of order 3 (by Lagrange's theorem). Thus the kernel of the homomorphism is just the identity: so the homomorphism is one-to-one. The image therefore has 10 elements: so α is surjective. In particular, there is $x \in \mathbb{F}_{11}$ with $\alpha(x) = 9$.

4. (i) Note that the norms are (respectively) 4, 9, 6, 6. So if one of them factorizes as ab where a, b are not units, then because $N(ab) = N(a)N(b)$ and the only things with norm 1 are units, we must have $N(a) = 2$ or $N(a) = 3$. But there are no elements of norm 2 or 3 as there is no solution to $x^2 + 5y^2 = 2$ (or $= 3$) with $x, y \in \mathbb{Z}$.
(ii) As in the above, consider the function $N(x + y\sqrt{-7}) = x^2 + 7y^2$. This is just the square of the modulus, so satifies $N(ab) = N(a)N(b)$. If $ab = 1$ then $N(a) = 1$ so $a = \pm 1$.
To see that the ring is not a UFD, note that $2.2.2 = 8 = (1 + \sqrt{-7})(1 - \sqrt{-7})$. As 2 is not an associate of either factor on the right hand side, it will suffice to observe that 2 and $(1 \pm \sqrt{-7})$ are irreducibles. This is done using the same argument as in (i), noting that the norms of these are 4 and 8 (respectively). In more detail, if $2 = ab$ where a, b are non-units then $N(a) = N(b) = 2$: but there are no elements with norm 2. Similarly, if $(1 \pm \sqrt{-7}) = ab$ where a, b are non-units, then $8 = N(a)N(b)$ so one of $N(a), N(b)$ is 2, and this is not possible.

5. Suppose $uv = 1 = vu$ and $u \in I$. Then for any $y \in R$ we have $y = y1 = (yv)u \in I$. So $I = R$.

6. It's enough to check that the given sets are closed under $+$ and multiplication (on left and right) by elements of R.
For $I + J$: If $a, a' \in I$ and $b, b' \in J$ then $(a + b) = (a' + b') = (a + a') + (b + b')$. As I, J are closed under $+$, so is $I + J$. If $r \in R$ then $r(a + b) = ra + rb$; as I, J are ideals $ra \in I$ and $rb \in J$ so $r(a + b) \in I + J$. Similarly $(a + b)r \in I + J$ (Note: 'ideal' means 'left and right ideal').
For $I \cap J$: If $a, a' \in I \cap J$ then $a + a' \in I$ and $a + a' \in J$ so $a + a' \in I \cap J$. Also if $r \in R$ then $ra \in I$ and $ra \in J$ so $ra \in I \cap J$; similarly with ar.

7. Use the facts from the notes that $aR \subseteq bR$ iff b divides a and we have equality iff a, b are associates.
(i) $R = \mathbb{Z}$ and $I = 6R$. The ideals containing I are R, $2R$, $3R$ and I.
(ii) $R = \mathbb{Q}[x]$ and $I = (x^2 + 1)R$. The polynomial $x^2 + 1$ is irreducible over $\mathbb{Q}$ (it is of degree 2 and has no roots in $\mathbb{Q}$); so the only ideals containing I are I and R.
(iii) $R = \mathbb{F}_3[x]$ and $I = (x^3 - x + 1)R$. Again, the polynomial $x^3 - x + 1$ has no roots in $\mathbb{F}_3$, so as it's a cubic, it is irreducible over $\mathbb{F}_3$. So the only ideals containing I are I and R. In cases (ii) and (iii), I is a maximal ideal of R.

8. (i) If $x^2 + 1$ is reducible it has a root in $\mathbb{F}_p$. So there is $a \in \mathbb{F}_p$ with $a^2 = -1$. But then a has order 4: we know that $a^4 = 1$ so the only other possibility for its order would be 2, and $a^2 \neq 1$. So as a Corollary to Lagrange's theorem, 4 divides $p - 1$, the order of the multiplicative group. So p is congruent to 1 mod 4.
(ii) As in question 7, the issue is for which values of p is $x^2 + 1$ irreducible. If $p = 2$ then $x^2 + 1 = (x + 1)^2$. If $p \equiv 1 \pmod 4$ then there is an integer T with $T^2 \equiv -1 \pmod p$ (– in section 4 of notes), so $x^2 + 1$ is reducible over $\mathbb{F}_p$. On the other hand, if $p \equiv 3 \pmod 4$ then -1 is not a square mod p (by (i)), so $x^2 + 1$ is irreducible in $\mathbb{F}_p[x]$. Thus $\mathbb{F}_p[x]/(x^2 + 1)$ is a field iff $(x^2 + 1)$ is a maximal ideal in $\mathbb{F}_p[x]$ iff f is an irreducible in $\mathbb{F}_p[x]$ iff p is congruent to 3 modulo 4.

APPENDIX C: Examination Paper (Abstract Algebra Section)

C1 – Examination Question 4

4. (i) Describe the group S of rotational symmetries of a solid cube in $\mathbb{R}^3$. List the possible axes of rotation and angles of rotation, and hence show that $|S| = 24$. Let ℓ be an axis passing though the centres of a pair of opposite faces of the cube and T be the set of rotations in S which send ℓ to itself. Prove that T is a subgroup of S and $|T| = 8$. **[10 marks]**

(ii) Suppose G is a group and H a subgroup of G. Prove that the relation $\sim$ on G given by

$$g_1 \sim g_2 \text{ if and only if } g_1^{-1}g_2 \in H$$

is an *equivalence relation*, saying carefully what this means. In the case where G is a finite group, prove that all equivalence classes have $|H|$ elements. **[7 marks]**

(iii) State Lagrange's Theorem, and use (ii) to give a proof of this. **[3 marks]**

C2 – Examination Question 5

5. (i) Suppose G is a group.

(a) What does it mean to say that a subgroup N of G is a *normal* subgroup? If N is a normal subgroup of G, explain how to make the set G/N of left cosets of N in G into a group. [3 marks]

(b) State the First Isomorphism Theorem for groups, defining the terms *kernel* and *image* in your statement. [4 marks]

(c) Suppose H is a cyclic group. By defining a suitable homomorphism $\phi : (\mathbb{Z}, +) \to H$, or otherwise, prove that $H \cong \mathbb{Z}/m\mathbb{Z}$ for some $m \in \mathbb{Z}$. [3 marks]

(ii) Let R be the ring $\mathbb{Z}[\sqrt{-7}] = \{m + n\sqrt{-7} : m, n \in \mathbb{Z}\}$. In the following you may use the fact that the function $N : R \to \mathbb{Z}$ given by $N(m + n\sqrt{-7}) = m^2 + 7n^2$ satisfies $N(ab) = N(a)N(b)$ for all $a, b \in R$.

(a) Prove that the only units in R are ± 1. [4 marks]

(b) Give two different factorizations of 8 into irreducibles in R and deduce that R is not a unique factorization domain. You should justify carefully any assertions you make about irreducibility of various elements of R. [6 marks]

C3 – Examination Question 6

6. (i) Suppose R is a commutative ring. What is meant by saying that a subset $I \subseteq R$ is an *ideal* of R? Suppose $f \in R$ and let $fR = \{fr : r \in R\}$ be the *principal ideal* generated by f. Prove that this *is* an ideal of R. Prove that, for $f, g \in R$

$$fR \subseteq gR \Leftrightarrow g \text{ divides } f \text{ in } R.$$

[10 marks]

(ii) Prove that the element $f = x^3 + x^2 + x - 1$ in the polynomial ring $R = \mathbb{F}_3[x]$ is an irreducible (where $\mathbb{F}_3$ denotes the field with 3 elements). Explain why the ideal $I = fR$ is a maximal ideal in R. What does this imply about the quotient ring R/I? Let $h = x^4 - x^2 + 1 \in R$. Is hR a maximal ideal in R? Justify your answer. [10 marks]

APPENDIX D: Markers' Comments on 78 Students' Coursework Performance

D1 – Comments on Problem Sheet 1

Sheet 1

Q1 - A generally well done question. Some tried to prove the required result by assuming it is true, you can only do this if you wish to prove that the statement is false and so obtain a contradiction. Some also wrote $g^2 = e$ implies $g = e^{\frac{1}{2}}$: this has no meaning in an arbitrary group setting.

Q3 - A generally well done question, with people correctly applying the subgroup check. Some specific comments:

- Some lost marks by forgetting to check that the set is non-empty. Giving a concrete example is sufficient e.g $1 \in \{z \in \mathbb{C}^\times : |z| = 1\}$.
- Some felt that checking a set was closed under inverses meant checking that each element has an inverse, this is not the case. We already know these sets are subsets of a group so the inverses already exist in the larger group, so we need to show that for any element of a set its inverse is also in the set i.e. it meets the conditions of the subset.
- On the argand diagram for the set $\{e^{(1+i)t} : t \in \mathbb{R}\}$ many did not include the part of the spiral for $t < 0$, remember here $t \in \mathbb{R}$ so takes positive and negative values.
- When the question is about subsets of $GL(n, \mathbb{R})$, we need to show this holds for all $n \in \mathbb{N}$. Showing it for the $n = 2$ case is not sufficient.

Q4 - Again most people correctly applied the subgroup criteria. Some lost marks on showing that $H \cap K$ is non-empty. The fact that H and K are non-empty is not sufficient, you need to justify that they share at least one element (namely the identity element).

Q6 - Most were able to find the 24 rotational symmetries, with marks only lost through insufficient explanation of the angles of rotations and axes (Note: use edges instead of sides, since a side is usually another name for a face). The idea for the subgroup of order 8 was to find a subset of size 8 from the list of 24 you have shown in the earlier part. Some people described flipping the cube, but without saying what this flip was. Normally a flip would describe a reflection. Here we are dealing only with rotations, so this flip must be a rotation and then we need to describe its axis of rotation and angle of rotation.

D2 – Comments on Problem Sheet 2

Sheet 2

Q1 - Most people got the idea for the question but lost a mark for not pointing out where they use that p and q are distinct primes. If you have $p\big||\langle\alpha,\beta\rangle|$ and $q\big||\langle\alpha,\beta\rangle|$ then you can say $pq\big||\langle\alpha,\beta\rangle|$ since p, q are distinct prime numbers (otherwise it may not be the case).

Q2(i) - At times a poorly done question. Many people were confused by the definition of H. The thing to note is that X and $Sym(X)$ (the permutations of the set X) are two separate entities and elements of one are not elements of the other. So here H is a subset of $G \leqslant Sym(X)$ and in particular H is the subset of those $g \in G$ which leave some fixed element a in X unmoved. So we are fixing a and looking for the permutations that do not move a, not a set of elements of X. Importantly we have no operations defined on the elements of X only on the elements of G, so $g_1(a) \cdot g_2(a)$ as some wrote is undefined since $g_1(a)$ and $g_2(a)$ are elements of X an arbitrary set.

Q3 - Again at times a poorly done question. Most recognized the need to check that $\sim$ is reflexive, symmetric and transitive. The main problem again was the confusion between the set X and the elements of $Sym(X)$. Elements of X are not elements of $Sym(X)$ and thus if we have $x, y \in X$ to say x has an inverse is not defined and neither is the composition xy, since no operations are defined on the set. We only have operations defined for elements of $G \leqslant Sym(X)$. So for each property (reflexive, symmetric and transitive), the idea was to show that there was always an element of G that fits the bill.

Q5 - Note that unless the group G is abelian you cannot rearrange the order of terms in the multiplication. So the elements h, g, h^{-1} may not commute and so ϕ is not necessarily the identity map, $\phi(g) = g \ \ \forall g \in G$. Quite a few people neglected to say what the kernel and image of ϕ were and consequently whether ϕ is an isomorphism, often a definition of Ker, Im was given but no attempt to describe the sets. In the cases where this was attempted it was often not correct. Also you should note that $Ker(\phi), Im(\phi)$ are sets not elements. A number of people said in part (i) that since $Ker(\phi) = \{e\}$ then ϕ is surjective, this is not the case, but instead is equivalent to ϕ being injective. The statement about surjectivity is the following, $\phi : G \longrightarrow H$ is surjective iff $Im(\phi) = H$. In part (ii) there was some confusion over the operations, note that $\mathbb{R}$ is an additive group and $\mathbb{C}^\times$ is a multiplicative group, therefore we add in $\mathbb{R}$ and multiply in $\mathbb{C}^\times$ so to show ϕ is a homomorphism you need to show that $\phi(x + y) = \phi(x) \cdot \phi(y)$ for all $x, y \in \mathbb{R}$.

Q6 - (ii) Again to show ϕ is a homomorphism you need to show that $\phi(x + y) = \phi(x) \cdot \phi(y)$ for all $x, y \in \mathbb{Z}$. (iii) Again it was often stated what Ker, Im were without explanation. Several students said that for $k \in \mathbb{N}$ equal to the order of g then $Ker(\phi) = k\mathbb{Z}$ which although true was not very well explained, also if g has infinite order then the k as defined does not exist. When g has infinite order, $g^n = e$ iff $n = 0$, so in this case $Ker(\phi) = \{0\} = 0\mathbb{Z}$. However the fact that $Ker(\phi) = d\mathbb{Z}$ for some d can be deduced much easier by noting that $Ker(\phi)$ is a subgroup of $\mathbb{Z}$ so by 6(i) $Ker(\phi) = d\mathbb{Z}$ for some $d \in \mathbb{N} \cup \{0\}$.

D3 – Comments on Problem Sheet 3

Sheet 3

Q2 - (i) Some people did not really compute $N(xy)$ properly for example some said that if $x = a + b\sqrt{3}, y = c + d\sqrt{3}$, then $N(xy) = (a + b\sqrt{3})(c + d\sqrt{3})(a - b\sqrt{3})(c - d\sqrt{3})$ which cannot really be said straight away. To work out $N(xy)$ compute $xy = ac + 3bd + (bc + ad)\sqrt{3}$ then applying the definition of N gives $N(xy) = (ac + 3bd)^2 - 3(bc + ad)^2$. (ii) A lot of confusion in this part. Remember you want to make a statement about the norm of u, so once you get to the stage $N(u)N(v) = 1$ for some $v \in R$ then you are almost done as then all you have to do is note that $N(u), N(v) \in \mathbb{Z}$ so this forces $N(u) = N(v) = \pm 1$. (iii) Several students tried to claim that 2,7 are units of R, this of course cannot be true as it would imply that $\frac{1}{2}, \frac{1}{7}$ are elements of R which of course they are not. Alternatively $N(2) = 2^2 = 4$, but if 2 were a unit then $N(2) = \pm 1$ by part (ii), so 2 is not a unit.

Q4 - The main problems were that few people seemed to take account of the fact that the norm is non-negative and so in the equation

$$N(a)N(b) = 4$$

this rules out the possibility that $N(a) = N(b) = -2$. Also most people stated without explanation that the equations $x^2 + 5y^2 = 2, 3$ are not soluble for $x, y \in \mathbb{Z}$. Note that if $|y| \geq 1$ then $x^2 + 5y^2 \geq 5$ so cannot be equal to 2,3. So this forces $y = 0$, which in turn says $x^2 = 2, 3$ but this is not possible for $x \in \mathbb{Z}$. Some people also slipped up when computing the norm of $1 + \sqrt{-5}$ claiming it was $6 + 2\sqrt{-5}, -4$,... etc. Remember the definition $N(x + y\sqrt{-5}) = x^2 + 5y^2$ so $N(1 + 1\sqrt{-5}) = 1^2 + 5(1^2) = 6$.

Q7 - Most people got the first part correct although some missed the obvious containments $6R \subseteq 6R, 6R \subset R$. Most students concluded that $6R$ was not maximal but did not say why (i.e. because $6R \subset 2R \subset R$, or $6R \subset 3R \subset R$). The results you needed to use in parts (ii),(iii) are that for $R = K[x]$ (K a field), R is a principal ideal domain and so the given ideal fR is maximal iff f is irreducible over K and that a degree 2 or 3 polynomial is irreducible over K iff it has no root in K. Some people thought that because $x^2 + 1 = (x + i)(x - i)$ then $x^2 + 1$ is reducible, but this is a factorisation in $\mathbb{C}[x]$ (not $\mathbb{Q}[x]$), so $x^2 + 1$ is reducible over $\mathbb{C}$, but as $x^2 + 1$ has no roots in $\mathbb{Q}$ it is irreducible in $\mathbb{Q}[x]$. Hence $(x^2 + 1)$ is a maximal ideal of $\mathbb{Q}[x]$ and this implies that the only ideals of R containing I are I, R. (iii) Similar things here, most people stated that $x^3 - x + 1$ is irreducible over $\mathbb{F}_3$ but gave no proof.

Q8 - A wide variety of answers to this question. Most common causes for lost marks were for not considering what is true when $p = 2$, misunderstanding the implication of $8(i)$. Note that 8(i) implies that for $p > 2, p \not\equiv 1 \pmod 4$ (i.e. for $p \equiv 3 \pmod 4$), $x^2 + 1$ is irreducible over $\mathbb{F}_p$. 8(i) does not imply that for $p \equiv 1 \pmod 4$, $x^2 + 1$ is reducible over $\mathbb{F}_p$, this is implied by the fact from your notes that for such p there exists $t \in \mathbb{F}_p$ such that $t^2 = -1$. Also note that $i \notin \mathbb{F}_p$ so you cannot say that $x^2 + 1 = (x - i)(x + i)$ in $\mathbb{F}_p[x]$. Rather an analogue of i may exist in $\mathbb{F}_p$ for some p. For example if $p = 5$ then

$$x^2 + 1 = (x - 3)(x - 2) \quad \text{in } \mathbb{F}_5[x].$$

So $2^2 = 3^2 = -1$ in $\mathbb{F}_5$.

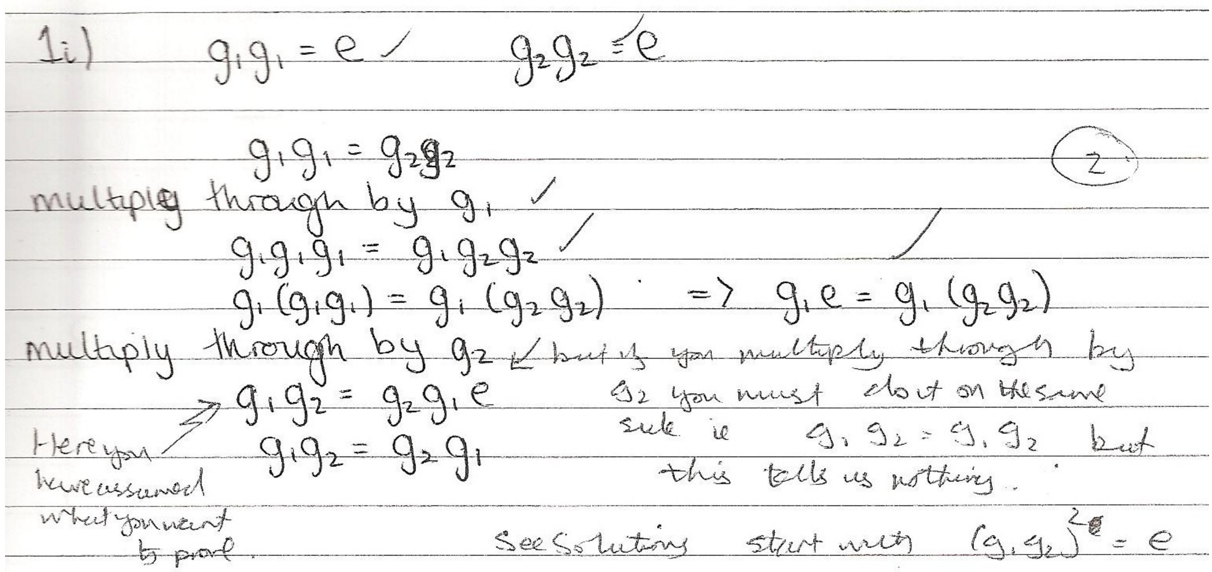
APPENDIX E: Example of a Threefold Data Analysis Account

Threefold Data Analysis Account of Leonora

- **Cognitive Difficulties – Difference between actual and reported achievement *(coursework, exams, interviews, seminars)***
- **Evolution of students' emotions about Group Theory *(interviews, observation notes, seminars)***
- **Teaching and Learning Issues – Excellent lecturer vs. Poor learning outcome *(coursework, exams, interviews, seminars)***

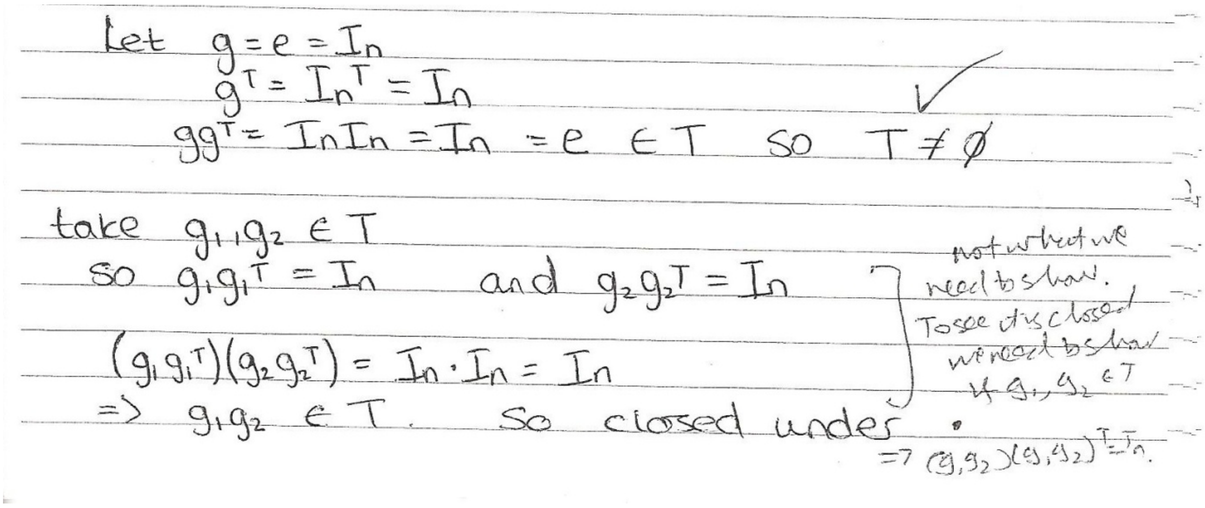
Coursework/Examination/Seminars

CS1E1: There are several misconceptions that occur in Leonora's solution. First of all she multiplies both sides of a mathematical g term, but on the one side she multiplies from the left and on the other side from the right (as shown below). This shows a lack of knowledge about the elements of the group and how they work/ are manipulated. Another misconception, more philosophical and related to the idea of proof in general, occurs when Leonora assumes that the statement she has to prove is true and she uses it during the proof. She got 2/8.



CS1E3iii: Leonora shows a good understanding of the d-object of subgroup, its relationship with the group and consequently the three elements she has to check, i.e. non-emptiness, closure under operation and closure under inverses. She applied the test correctly in both examples. The use of notation is flawless as well as the narratives she has used. The routines were also very well applied and presented. The only deficiencies that occurred and, according to the marker, should not appear have to do with some additional clarifications that would justify certain elements of groups. For instance she should state that $t_1 + t_2 \in \boldsymbol{R}$ and that $-t \in \boldsymbol{R}$, otherwise the proof would not be valid. The use of visual mediators was correct and the Argand Diagrams clear.

CS1E3iv: Again Leonora shows a good understanding and applied the test well. Everything is correct; the use of mathematical vocabulary, the produced narratives, the application of the routine etc. The only misconception occurred in the second example, while proving closure under operation. Instead of proving that $(g_1g_2)(g_1g_2)^T = I_n$, she proved that$(g_1g_1^T)(g_2g_2^T) = I_n$. This probably shows that she has not realised that $g_1g_2$is another element of the group by its own and that one has to treat it as such.



CS1E4: Excellent answer, showing very good and thorough understanding of the axioms of groups as well as the concept of subgroup and its characteristics. Also here one can notice a very strong and secure metaphor from Set Theory, about the properties of sets, i.e. intersection etc. The narratives in this case were more verbal and there was not much use of mathematical narratives and notation. Nevertheless, the reasoning is flawless. She got full marks 7/7.

CS1E6: Leonora has produced a perfect solution here. The marker agrees with this opinion as reflected in his comments. The verbal description of the rotations of cube is flawless and in perfect harmony with the illustrative images. Her narratives demonstrate “humble confidence”. All the results were summarised in a table. Moreover for the second part of the question she correctly proves that there is a subgroup formed by the set of rotational symmetries, which send the pair of faces to itself, and consequently she proved that its order is 8. Again, her understanding was excellent, relating to how the rotations work, the angles of rotations, the resulting symmetries etc. What is fantastic about Leonora’s solution is the additional table that she produced, unique among the students of this project, demonstrating all the combinations of elements. She got full marks 10/10.

the combination of both will be within the subgroup.

	a	b	c	d	e	f	g	h
a	a	b	c	d	e	f	g	h
b	b	c	d	a	h	g	e	f
c	c	d	a	b	f	e	h	g
d	d	a	b	c	g	h	f	e
e	e	g	f	h	a	c	b	d
f	f	h	e	g	c	a	d	b
g	g	f	h	e	d	b	a	c
h	h	e	g	f	b	d	c	a

Very good (but it's not really needed or expected).

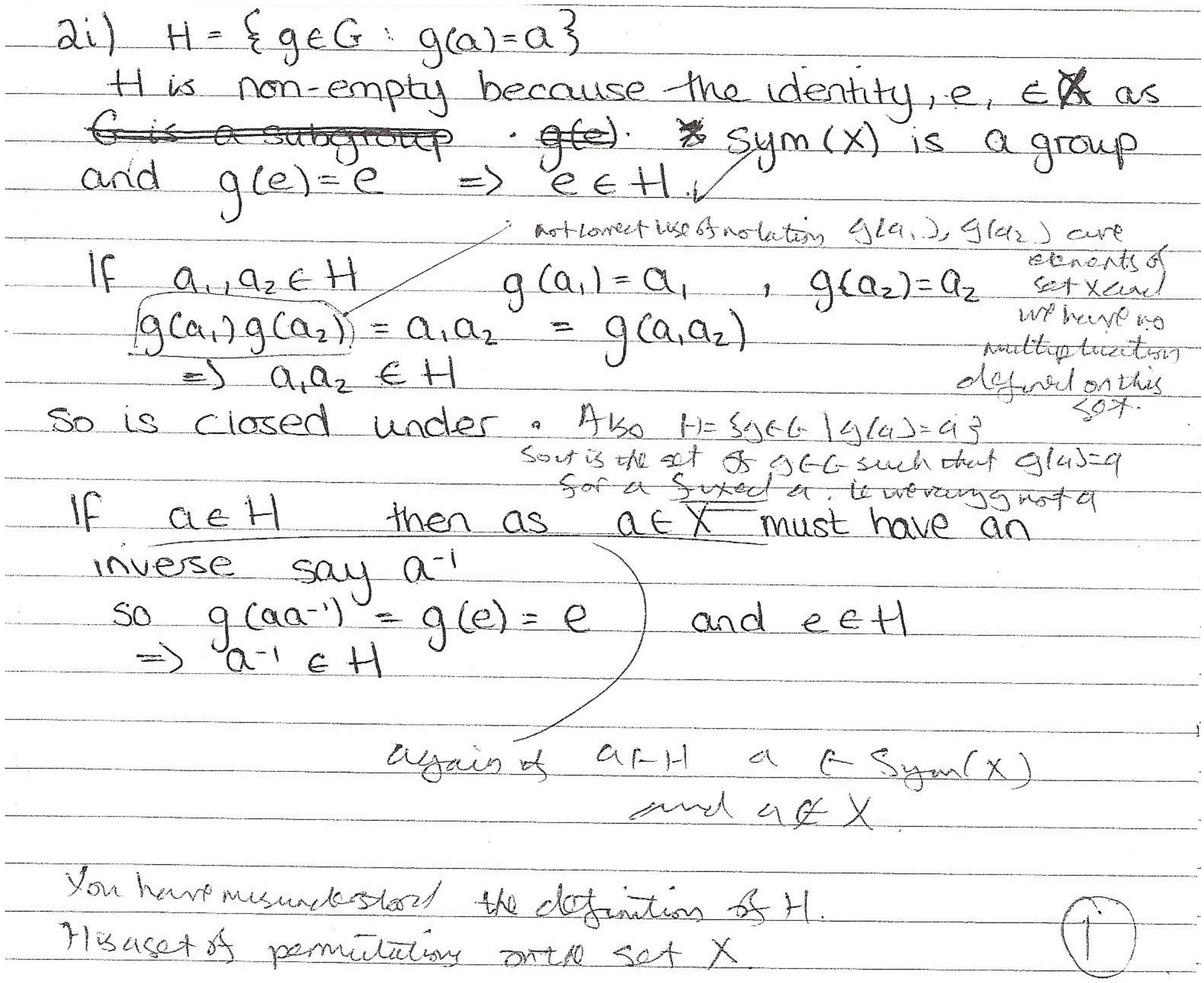
✓ 4

Therefore it has all the axioms that satisfy a subgroup.

10

CS2E1: There are two points that show some misunderstanding. First of all, as many other students, Leonora does not seem to take into account the fact that p and q are distinct primes, and therefore the written narratives are incorrect. The second misconception has to do with the use of symbol $\in$. Leonora used the expression $\langle a,b\rangle \in G$ which is wrong since this symbol is used for elements and not groups. She should have written $\langle a,b\rangle \leq G$ which is used for groups. Nevertheless, she got 4/5.

CS2E2: Leonora shows a good understanding of the routine for proving that a set is a subgroup. There is a problematic application though, contrary to the previous exercises, since several misconceptions have occurred that did not seem to exist before. The first one is related to the use of notation, which in my opinion has deeper roots into the essence of understanding the do objects of the elements of the group and their properties. Additionally she does not seem to be able to define the different operations in the different groups. For example she writes the expression $g(a_1)g(a_2)$ which used elements of the set X but under operation which does not work in X. She does not have a clear view of what is H and what is X, i.e. that X is a non-empty set and that H is a subgroup of G with a certain condition. Also at some point she writes $a \in H$, which is not true since a is an element of X. She got 1/4.


2i) $H = \{g \in G : g(a) = a\}$

H is non-empty because the identity, e, ∈ ~~X~~ as ~~G is a subgroup~~ . ~~g(e)~~ · Sym(X) is a group and $g(e) = e \Rightarrow e \in H$.

not correct use of notation $g(a_1)$, $g(a_2)$ are elements of set X and we have no multiplication defined on this set.

If $a_1, a_2 \in H$ $g(a_1) = a_1$, $g(a_2) = a_2$

$g(a_1)g(a_2) = a_1 a_2 = g(a_1 a_2)$

$\Rightarrow a_1 a_2 \in H$

So is closed under. Also $H = \{g \in G \mid g(a) = a\}$ So it is the set of $g \in G$ such that $g(a) = a$ for a fixed a, ie we vary g not a

If $a \in H$ then as $a \in X$ must have an inverse say a^{-1}

so $g(aa^{-1}) = g(e) = e$ and $e \in H$

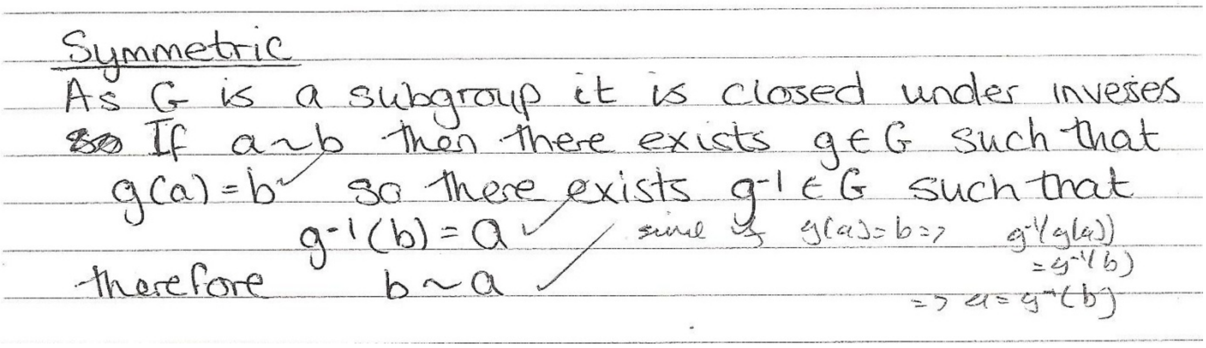
$\Rightarrow a^{-1} \in H$

again $a \notin H$ $a \notin Sym(X)$ and $a \in X$

You have misunderstood the definition of H. H is a set of permutations on the set X.

1

CS2E3: Leonora has successfully proven that this is an equivalence relation. She shows a good understanding of the definition of equivalence relations as well as how to apply it in order to prove reflexivity, symmetry and transitivity. Her answer was flawless, but the marker needed more detailed narratives regarding symmetry. He expected Leonora to demonstrate, using mathematical expressions, how she ended up with $b{\sim}a$, as shown below. She got 7/8.



CS2E5: As in the case of other students, Leonora seems to have some problems with the notions of image and kernel. She does not produce full narratives for proving what is the kernel and image of the homomorphism, although the first part in which she had to prove that this is a homomorphism was done successfully. There is also some problem with the word use, which also occurred in Amelia's work. Instead of saying that φ is an isomorphism, she says that φ is isomorphic. (According to Sfard, using nouns and not adjectives that characterise an object, is called *objectification*, composed by two processes, reification and alienation, and shows commognitive development.) In this case though it also indicates that Leonora has not realised what an isomorphism really is.

5i) $\phi(g) = hgh^{-1}$

If $g_1 \in G$ and $g_2 \in G$

$\phi(g_1) = hg_1h^{-1}$

$\phi(g_2) = hg_2h^{-1}$ ✓

$\phi(g_1)\phi(g_2) = hg_1h^{-1}hg_2h^{-1}$

$= hg_1(h^{-1}h)g_2h^{-1}$ ✓

$= hg_1g_2h^{-1}$

$= \phi(g_1g_2)$ ✓

Take $a \in G$ and $g \in G$

$\phi(ag) = \phi(a)\phi(g)$ as homomorphic

$\phi(a)\phi(g)$ ~~$\phi(ag)$~~ $= hah^{-1}hgh^{-1}$

$= ha(h^{-1}h)gh^{-1}$

$= hagh^{-1}$

$= \phi(ag)$

??

So $\mathrm{Im}(\phi) = G$ → as ϕ is surjective, given $g \in G$, $\phi(h^{-1}gh) = g$.

$\ker\phi = \{g \in G : \phi(g) = e_H\}$

$\phi(g)\phi(g^{-1}) = hgh^{-1}hg^{-1}h^{-1}$

$= hgg^{-1}h^{-1}$

$= hh^{-1}$

$= e_G$

This is true since ϕ is a homomorphism $\phi(g)\phi(g^{-1}) = \phi(e) = e$.

$\phi(gg^{-1}) = \phi(e_G)$

$\Rightarrow \ker\phi = \{e_G\}$ ✓

$\ker\phi = \{g \in G : hgh^{-1} = e\}$

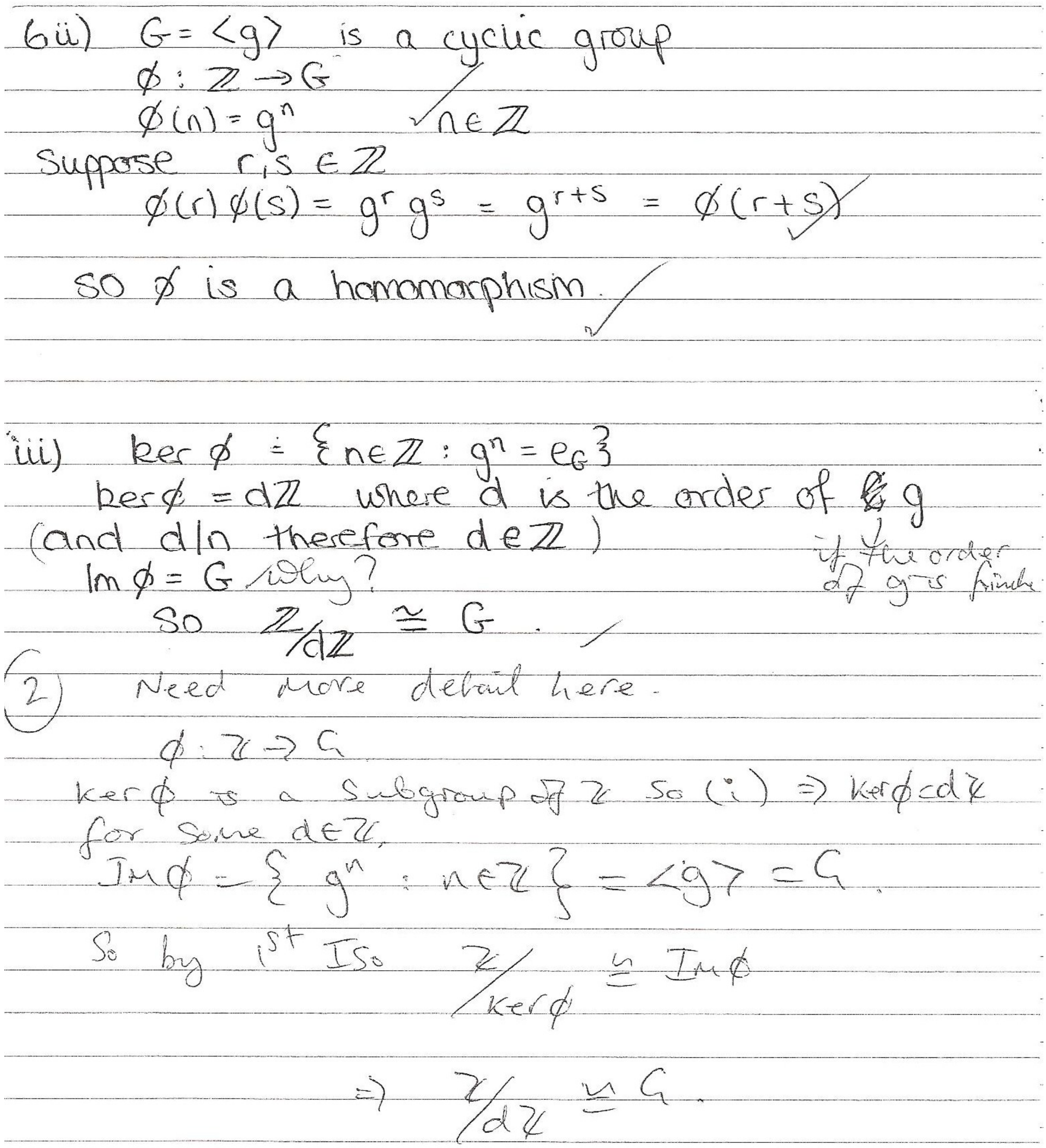
$hgh^{-1} = e \Rightarrow g = h^{-1}eh = h^{-1}h = e$.

As $\ker\phi = \{e_G\}$ therefore ϕ ~~(g)~~ is one-to-one by lemma (3.3) part (1). ✓

It is also onto. as I remarked above.

$\Rightarrow \phi$ is an isomorphism

CS2E6: The second part of the exercise in which Leonora had to prove that it is a homomorphism was done correctly. There are several weaknesses in the third part though, which are not obvious since, although the answer given is correct, there are no narratives showing the reasoning behind the answer. I think there are serious misconceptions regarding the First Isomorphism theorem, which appears to be the most troublesome routine. She correctly states that the image of the homomorphism is G and therefore the isomorphism holds, but she does not mention anything about the kernel and she does not seem to pay attention to the fact that the order of g is finite. She got 2/5.

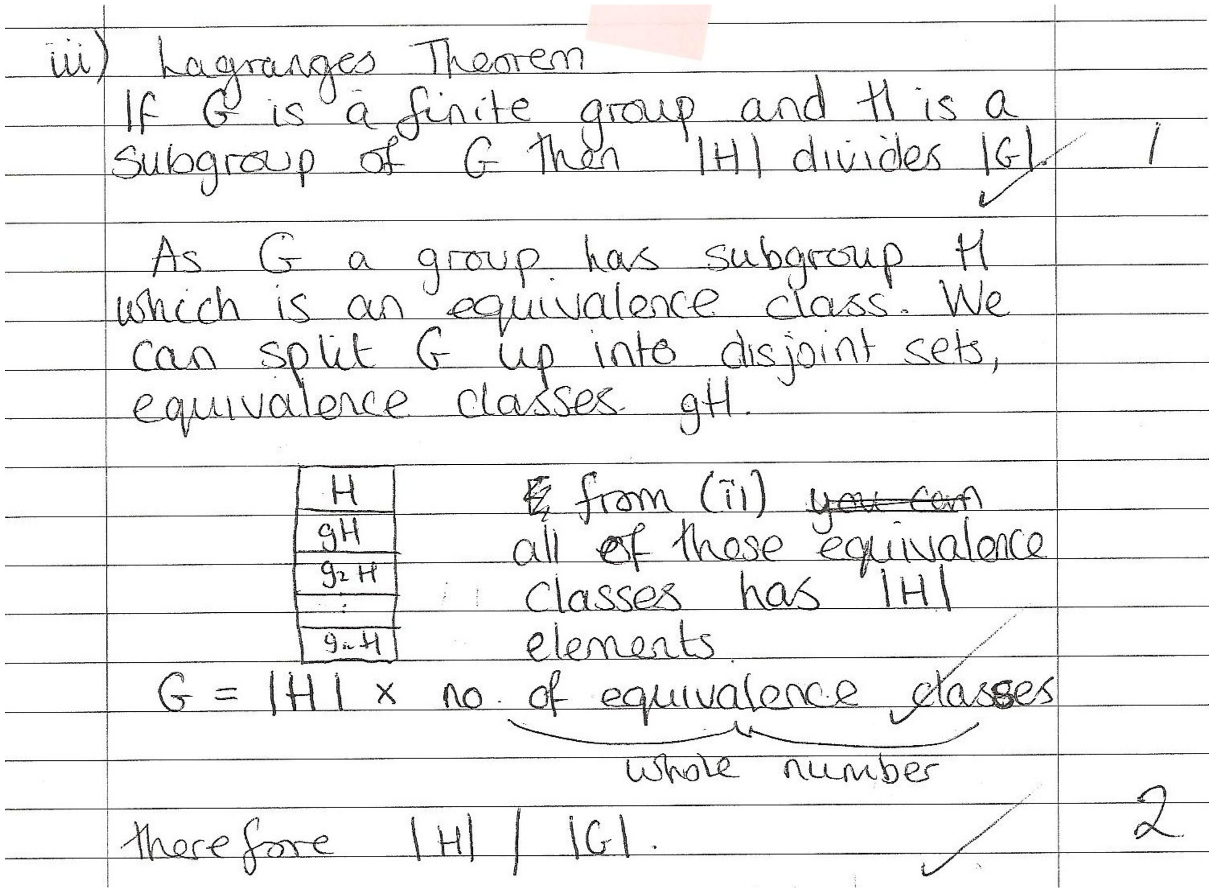

6ii) $G = \langle g \rangle$ is a cyclic group
$\phi : \mathbb{Z} \to G$
$\phi(n) = g^n$ ✓ $n \in \mathbb{Z}$
Suppose $r, s \in \mathbb{Z}$
$\phi(r)\phi(s) = g^r g^s = g^{r+s} = \phi(r+s)$ ✓
so ϕ is a homomorphism. ✓

iii) $\ker \phi \doteq \{n \in \mathbb{Z} : g^n = e_G\}$
$\ker\phi = d\mathbb{Z}$ where d is the order of g
(and $d|n$ therefore $d \in \mathbb{Z}$) — if the order of g is finite
$\operatorname{Im}\phi = G$ Why?
So $\mathbb{Z}/d\mathbb{Z} \cong G$. ✓
(2) Need more detail here.
$\phi : \mathbb{Z} \to G$
$\ker\phi$ is a subgroup of $\mathbb{Z}$ so (i) $\Rightarrow \ker\phi = d\mathbb{Z}$ for some $d \in \mathbb{Z}$,
$\operatorname{Im}\phi = \{g^n : n \in \mathbb{Z}\} = \langle g \rangle = G$.
So by 1st Iso $\mathbb{Z}/\ker\phi \cong \operatorname{Im}\phi$
$\Rightarrow \mathbb{Z}/d\mathbb{Z} \cong G$.

<u>FEE4</u>: The first part of (i) about listing the 24 symmetries was done correctly in the same style as in the coursework. Unlike the first part, the second was totally wrong and Leonora got no marks. She does not show the correct reasoning that she demonstrated in the coursework, although the exercise was exactly the same. She shows several misconceptions about how the

rotations are achieved as well as how the test for a set to be a subgroup is applied in the case of a cube.

For part (ii) she mentioned that an equivalence relation must be transitive, symmetric and reflexive, but she did not give a correct definition and she did not manage to prove that the given example is indeed an equivalence relation. The solution is extremely messy. Unlike the similar exercise in the coursework, in which she got full marks, in the exams her performance with regard to the same material is significantly lower. For part (iii) she manages to state and prove correctly Lagrange's theorem, also using a visual mediator to support her answer.



iii) Lagranges Theorem

If G is a finite group and H is a subgroup of G then |H| divides |G|. 1

As G a group has subgroup H which is an equivalence class. We can split G up into disjoint sets, equivalence classes gH.

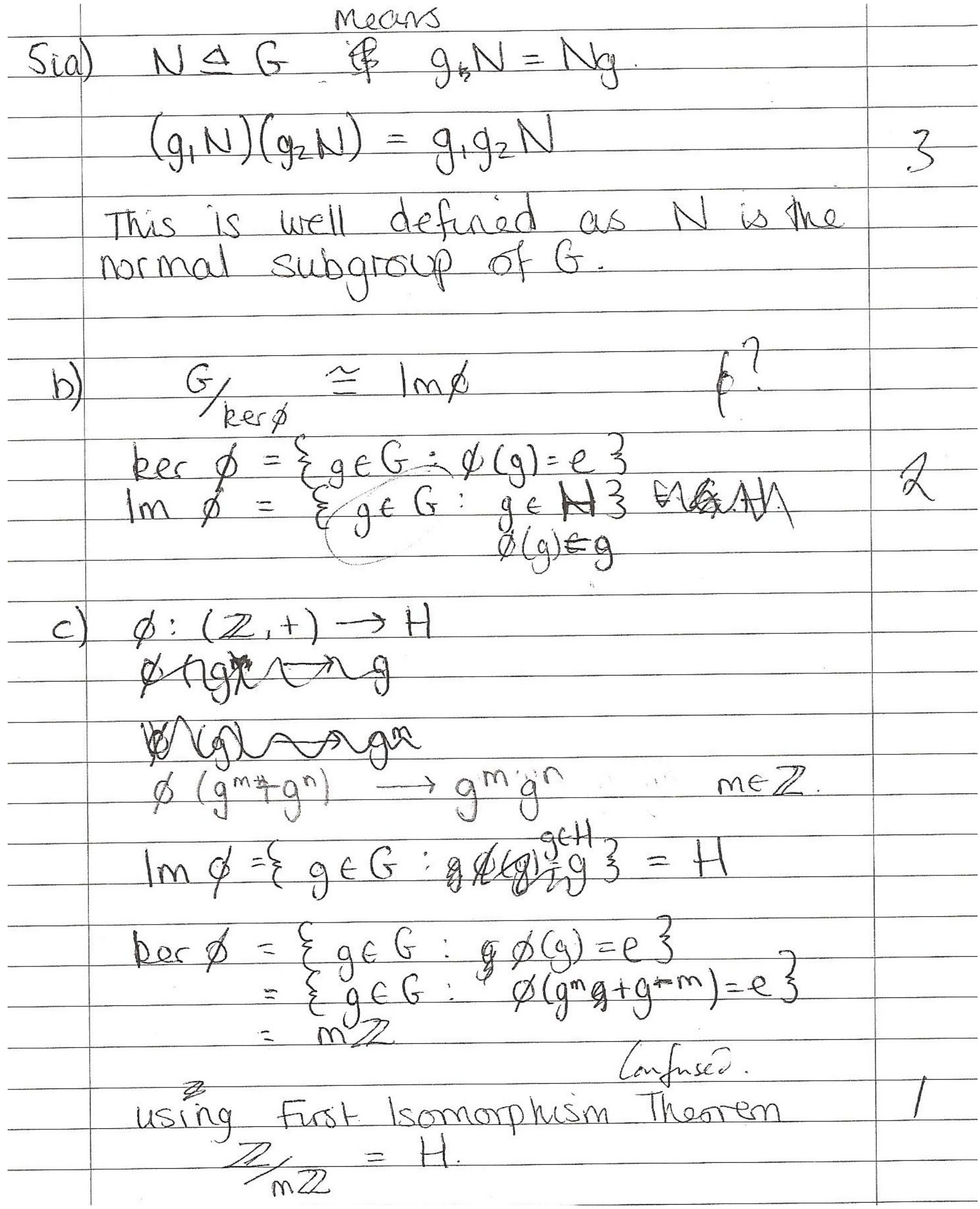
H
gH
g_2H
⋮
g_nH

from (ii) ~~you can~~ all ~~of~~ those equivalence classes has |H| elements

G = |H| × no. of equivalence classes (whole number)

therefore |H| \| |G|. 2

FEE5: For part (i) she has stated correctly the definition of normal subgroup, exactly copied from the lecture notes. In part (ii) one can notice several misconceptions. First of all she has not stated the First Isomorphism Theorem, apparently a very troublesome routine in the group theory subdiscourse, but she rather wrote the mathematical expression $\frac{G}{Ker\varphi} \cong Im\varphi$ without any further explanation. Another misconception occurs in the definition of Image where instead of stating $Im\varphi = \{\varphi(g): g \in G\}$ she wrote what occurs below. This shows that she does not understand the elements of the image and also fails to fully understand what she is writing mathematically. In part (iii), she offers a very messy answer, which mirrors her confusion about the first isomorphism theorem as well as her unsuccessful effort to reproduce CS2E6ii. The marker wrote the comment "Confused", which he rarely does.


5i a) $N \trianglelefteq G$ ~~iff~~ means $g_1N = Ng$.

$(g_1N)(g_2N) = g_1g_2N$ — 3

This is well defined as N is the normal subgroup of G.

b) $G/\ker\phi \cong \text{Im}\,\phi$ — ϕ? — 2

$\ker\phi = \{g \in G : \phi(g) = e\}$

$\text{Im}\,\phi = \{g \in G : g \in H\}$ ~~$\in G \in H$~~

$\phi(g) \in g$

c) $\phi : (\mathbb{Z}, +) \to H$

~~$\phi(g) \mapsto g$~~

~~$\phi(g) \mapsto g^n$~~

$\phi(g^m + g^n) \to g^m g^n$ $\quad m \in \mathbb{Z}$.

$\text{Im}\,\phi = \{g \in G : \phi(g) = g, g \in H\} = H$

$\ker\phi = \{g \in G : \phi(g) = e\}$

$= \{g \in G : \phi(g^m g + g^{-m}) = e\}$

$= m\mathbb{Z}$

Confused.

using First Isomorphism Theorem — 1

$\mathbb{Z}/m\mathbb{Z} = H$.

Seminar A4 **Vignette III** Leonora is asking how one can prove that there is an inverse in the group. SAB asks what the identity of the group is and Leonora replies 0. Moreover, SAB explains the group properties and then leads the student towards the answer. Leonora sounds confused. **Vignette V** Leonora is asking something inaudible. SAB further explains what she should assume in the method by contradiction. SAB explains the steps and the inclusion of certain elements in the $H \cup K$.

EVOLUTION OF ATTITUDES AND EMOTIONS

1st interview

Leonora: Yeah, and I think knowing that it is going towards my actual degree... like last year I just thought I had to pass so it was... I don't put too much pressure on myself so it was okay....
Leonora: Yeah, both actually. I... with revision last year it was if I didn't understand everything then I was scared. This year with coursework if I especially when we got closer and closer to the deadlines last term, if I don't understand it and I can't find anything to help me to understand it then I get a bit angry and get frustrated and stuff.
Leonora: And I find it like sometimes just really hard that if I don't understand it to nothing but why and I just really get angry and I have to.... most of the time I will ask someone else in my course to explain it to me and like someone else explaining it sometimes helps.
Leonora: Yeah, so I find that is really helpful that to have people that can help

me out, especially if I panic and they like calm me down as well and I find that nice so.... I don't know that many people on the course but I know enough to ….

2nd interview

Leonora: But – most of the time I – I try like – I probably won't try them all, I'll try a few of them, I'll like look over them all and then see which ones – I think I can try and start, try and start them and then ask about the other ones? That I don't have a clue about really?
Leonora: Um... I get a bit stressed out… cos I think when – in like, the lectures, he's pretty good, and I think I understand it, and then when I go back and try these and I can't do it, I get a bit stressed out that – I think – but I understood it, but it's harder than I thought, and I was like – I get a bit... angry, and I get a bit – stressed, but other than that, I think... pretty much ok, which I think it's – once I know what I'm doing, I'm ok.
Leonora: Because – I think I should know, I think I should be able to do it, and I think... why can't I do it? So it's pretty much – if err, also if like other people can do it, I think why can't I? But I know it's – for some – like obviously different people struggle in different areas, so they can – obviously start it, and then they get help, whereas I just can't start it.

3rd interview

Leonora: Um – well, to start off with it wasn't too bad, cos obviously I thought – well, when I first started, I thought well, I don't really understand it, but then, once I'd been to a seminar, and like – basically like – they explained the group thing to me, and like how you know it's a group and that, and like – the rest of it just kind of – followed? So I found it a lot easier after that, so I was like, quite happy with – it makes me happy when I know what I'm doing, and that I can do something by myself, like when I've finished something I think – oh, I did that by myself, I'm quite happy but – since the rings, it's been like – when I look at the question sheet, since then I thought, oh dear... that this does make no – like I really don't know where this

has come from, sort of thing, I don't – MI: So, there was – I guess from – there – there was a quite – change in your emotions, when you started rings? Leonora: Yeah. MI: Yeah. You are a bit more disappointed, or... Leonora: Yeah, a bit. MI: Less motivated a bit. Leonora: Yeah, a bit… stressed out a bit. Worried about it, sort of thing, cos like – I don't think I – but hopefully, like over Easter, mainly, that I can find some way of like just maybe going through the notes again, doing some more like – trying to – like once they give the solutions out to the questions, I like to like try it, if I can't do it then just see where they've come from? (MI: Mm hmm) And then maybe a week or two later, then go back and see if I've remembered it, and know – then know how to do it, or then do a different question and see if I can do it. (MI: I see) So, I'm hoping that's gonna help, during Easter.
MI: Are you confident? Leonora: At the moment, I'm not too confident, but I would – I'd like to think that I'm going to do well, like – I intend on doing a lot of revision and a lot of hard work. MI: Mm hmm. That's necessary. Leonora: Yeah, definitely, I am really – hoping, that it's going to go well. MI: From my experience, when I was a student, you know, I was struggling as everybody else, but um, everything was getting clearer during the revision for the exams. Leonora: Oh yeah.

MI: It's really – yeah, it's really important.

Leonora: I do find that quite often that, I'll think oh, I don't understand any of it, like during the whole year, but then when it comes to revision it's just all – kind of falls into place...

PEDAGOGICAL ISSUES

1st interview

Leonora: Yeah but it is hard to do that with like some analysis stuff and I just don't get some of it so it is really hard…

MI: It is harder, yeah. Definitely. That is probably the reason why we call it applied. So you say that you like the course of fluids and solids?

Leonora: Yeah… fluids and solids…

MI: Who was the lecturer?

Leonora: Last term it was Dr X and this term its … oh can't think who he is...

MI: Never mind. But you had some demonstrations....

Leonora: Yeah, last term. We had some... I can't remember what the room is but it is downstairs and there was like some big tanks and stuff and sort of like the water flow and did some experiments.

MI: Ah interesting. Very interesting. So it was clear to you, because you had visualised what is going on…

Leonora: Umm, well at the time it was hard when I first came. I did all the work and stuff because it was like pretty much as soon as I came there were loads of work it wasn't like they built you up to it or anything. So I found that hard but once I adjusted to it, once I got into it was okay.... it was better.

Leonora: Yeah, probably, and yeah. If I am studying to help me like remember things I will do like past exam papers and stuff to so that if I then don't get the first one but I can then do another one and then I will learn it and like if, especially if I have got solutions to pass exam papers then sometimes they are a bit differently to how you do it. I can work out what it means and understand it more if I have got...

Leonora: Yeah. Especially the seminars. I think they are really helpful because you have obviously got the PhD student doing them as well so if I ever need help that I know I can get it in my seminars and even if I can't get help I know they are really approachable like all the lecturers seem really approachable to get help with and that. If I am still stuck on something so....

MI: Umm.

Leonora: So it is quite good

MI: So you prefer... I mean you find that the PhD students are more helpful....

Leonora: I don't know... it is just like it is because they have just learnt it quite recently that they have got a different perspective on it sometimes and it is nice to hear a different way of....

Leonora: Yeah… like some of the group stuff are so it's like... so it is easier to understand and in the lectures he makes it really… quite easy to understand like… he will keep going through it and then it is like… oh Yeah.... cause I normally find in a lot of lectures that I don't understand what is going on in them but until I get outside and read my notes and do examples but with this… it is easy to understand in the lecture like I understand more of it in this lectures than my other lectures so.....

Leonora: Yes, definitely. When I like look at the questions before I go. I attempt them most of the time I attempt them but a lot of the time I find it hard to start, to know where I got to start and then when I ask it is always they will start talking and then I will be like oh yeah and then do you do this and that and then they are really helpful. They seem to go round.... you just put your hand up and they will go round and talk to everyone that needs like help and they will even go round and ask like if anyone needs help and … so you say explain it to them and like a lot of times like there is people that keep putting their hand up for the same question but

they will still go back and explain it in different ways....

<u>2^{nd} interview</u>

MI: I see. Um – which part, which, you know, which concept for you was the hardest, which part of the course was the hardest? Leonora: Um... the equivalence – stuff – like – MI: Equivalence relations. Leonora: It's not – what it is, like… like that, it's just like proving it, like when they ask on the new question sheet that we've got, when it's asking to prove that, I find that – I found that quite –
Leonora: I don't know, I mean I think once I've done it, and been told – like – having – so I've got like an example basically, of how to do it, then – it will be in my mind, so it'll be hopefully, something I can keep repeating, but just initially starting it off and – it's – I find quite hard, I found that quite hard with like a lot of things, it's just initially start –
MI: Why do you think is that? What is the difficult – what is the difficulty, basically, err, about question 6? Leonora: Err... because I don't think... it's been done a lot, in… we've obviously been given a hint, (MI: Mm hmm) for how to do it, but it's not something that – we've just done examples of and then can – like take that from it, it's just we've basically done this bit on it, (MI: Exactly) and then – it's kind of – like you've got nothing to go off from, like from before to – with like examples you can basically... see where it's come from?
Leonora: Yeah, it's like – drawing it is fine, but then you still have to see in your head like how it's – rotating, (MI: Rotating...) and how it's going – one of um, the people that I sit near, in the course, he actually said that he was gonna think about making a cube, and like put string on it and just – rotating it so he could see it better, and –

Leonora: Question 2, I found quite hard, because... I got a bit confused with this um – MI: $Sym\ (X)$ – Leonora: $Sym\ (X)$ and stuff, but – so I don't - I started it but then I weren't sure, whether I was doing it right, so I kind of have stopped, and I'm gonna go ask for help. To like – because I – I don't like, if I'm doing something and I'm not sure if it's right, I don't like to carry on because I don't want to do it all wrong,
Leonora: That I'm having problems with at the moment, yeah. So I... I know someone in my course that – he did it and he got help in his seminars, so he said he'll try and explain it to me? So hopefully, I'll understand it then, if not I'll have to go ask him. MI: Mm, good. It seems that you cooperate very well with your classmates. Leonora: Yes. MI: Yes. And do you find this helpful? Leonora: Yeah, definitely. Really helpful. Cos if I – then I don't have to spend like a lot – a lot of time figuring out when my lecturers have office hours and I can go see them, instead I can just – ask like, say we have an hour off or something, I can just ask one of – like my friends to – just explain it to me? Cos sometimes it's better to also get it from a different –
Leonora: Well yeah, I – I attempt – I often look at them, but I find it really hard to know what to do, straight away, like, and I'll attempt them but most of the time I get a bit... MI: Halfway... Leonora: Yeah, and I just can't do it and I just like – give up and I just think, oh I'll just have to ask about them like – cos – it's not that I can't do them, I really – most

of the time it's just I need to get start – started off with – doing it, and then most of the time I'll be ok.

Leonora: Um – sometimes I'll use books, if I'm particularly in the library doing it, or something, and we've been given, if we've been given certain books that – might help and stuff, it's not – normally I don't use that until like, nearer the end of the year when like the coursework's due in and if I'm still struggling, then I'll go get a book out, sort of thing, and help and – I'll look – look on the internet for certain things, not... everything, like, certain things that may help me with it, not the actual –

MI: Solution.

Leonora: Solution, or anything, it'll be like – certain things like formulae, and stuff like that, if I can't remember it.

Leonora: I don't know, really... I suppose with a group, it's just more like – I just – be like – an enclosed thing with certain – like say certain elements in it that... will relate to each other? Don't... so they'll... have things in common, like – you could do certain things to them and... So there'll be like a halfway point, say you times them and then, they'd make another element – sort of – certain things like that?

MI: Yeah, that's very interesting. Err – about the subgroup, how would you...

Leonora: A subgroup? Basically just like – a part of it, that's – that has the same...

MI: Yes.

Leonora: Basic things on it.

MI: Er... coset?

Leonora: A coset... I don't really know... I don't – I'm not too sure in my head at the moment, about a coset, it's the thing I struggle most with, like, with um... it's cosets, when I – I don't think I can picture them, and I think – cos obviously I'm a visual learner, I learn – I'm better if I can picture in my head, but a coset, at the moment, I'm just... a bit...

MI: Mm hmm... did you find – I think he has given... he has given a... picture... but I haven't seen it... you know... before... I think this is it...

Leonora: Oh yeah. Yeah. It's still...

MI: It's rectangular, you know, group...

Leonora: Yeah... and I get that obviously they have to be...

MI: Mutually exclusive, yeah.

Leonora: Exclusive yeah, I just don't... that does help, but it still doesn't... register in my... in my head at the moment, but it definitely did help.

MI: Yes. Err... homomorphism? How would you picture a homomorphism?

Leonora: Um...

MI: Or how would you symbolize homomorphism?

Leonora: Well it's just... if you – I suppose – I don't really know how to picture a function, but it's like– I know that it's – say it's – g_1, g_2 and that will be the same as g_1g_2? (MI: Mm hmm) But... I don't know how I'd picture it, because I wouldn't know how to do the function, but... I suppose if you have the function that you do to one element and go to another; it's the same as doing it to them both.

MI: Mm hmm, mm hmm. I think he has given... another… coset... but anyway – um... what about, a normal group, how could you... describe a normal group?

Leonora: I don't... I really don't really know –

MI: Don't worry.

Leonora: Because it's so... I dunno, it's – it's not something that I've had a lot of practice with, like – once I've done examples and stuff, I – find it easier to picture in my head, like – with the exercises and stuff, but cos I haven't actually – done examples, on the normal group like, by myself, obviously done it in lectures and that, but I haven't really got a picture of it in my head, at the moment, no... So...

3rd interview

MI: Yes. So you find it more difficult than, than groups.

Leonora: Yeah, a lot more difficult.

MI: Um... in what sense more difficult?

Leonora: I find it hard to picture, I find it hard to... understand what, what's really happening, sort of thing, what's – what's meant by certain things. I don't – I thought when it first started, oh this is going to be ok, but then it just was like – with ideals and stuff, it was just really confusing, I was like, oh...

MI: Hopefully, hopefully. Um... how do you find the – the pace of the – of the lecturer, and the pace of the teaching?

Leonora: Um – sometimes I think it's a bit fast, like, sometimes it's ok like, err, it's fine, cos, if I understand what's going on in the lecture, then – I find it ok, but if I'm a bit confused, then sometimes I think – oh, he's just moved straight on to that, like - don't know where he's gone to that, because I still don't understand. Still don't understand the first bit, so I don't – not – I'm not going to be able to understand like – the next bit, so... sometimes it's a bit fast.

MI: Mm hmm... um – if you were the lecturer, would you use any other ways of... teaching, or... could you do any – could you give me any... suggestions, teaching suggestions?

Leonora: Err, I don't... mainly – like, obviously, because I'm a visual learner, but I don't know if there's an – is anything visual within it, that you – that they could actually – like that he could actually put – on the board, I'm not sure if he could do

that or not, but I think it would help me, personally, a lot, but... MI: Using more illustrations, more.... yeah. Leonora: Yeah. More pictures and stuff, like... explaining what's happening and sort of thing, so that – maybe then I could picture it in my head, I'd find it easier. Cos if I'm trying to picture, as he's going along, then sometimes if he's going too fast, I don't have time to try and think about it, when I'm trying to think about something else. So... maybe stop and then draw a picture, or... MI: Yeah... Leonora: Or use a – some – some kind of visual aid or something, maybe.
Leonora: Um, probably with – obviously the – the questions that we've been given, and with the solutions, I'm hoping to like – help teach myself how to do it, (MI: Mm hmm) and then I will – I learn by doing past exam papers, mainly, so I intend on doing like – a lot of them, until like – I tend to do like quite a few years back, like do all of them, and once I've done them, go back, and like the questions that I have – like, didn't – weren't able to do before, I try and do it again, cos I've hoped that I've taught myself. At some point, how to do it. But, the only thing is, because they're not allowed to give out the answers, (MI: No, they are not, yeah) sometimes I just – like I don't know if – (MI: Yeah) I'm – don't know if it's right or wrong, it might be right but then it might – sometimes I think – I could have made like a stupid little mistake, and it – well it's made the whole thing wrong. So I'd rather have the answers, that's why I use the questions, rather than – I've probably used the questions more than the exam papers this year, because... (MI: I see) won't obviously have the answers like – I know I can like – bring it in, afterward – like after Easter, and show it like – he'll mark it for me, so I might do that, to – hopefully, then I'll know how – whether I'm ready or not, so it's – MI: I see... um... in the case you get stuck in the past exam papers, are you going to the, to the lecturers for assistance? Leonora: Oh, well – depends, how – how I find that I'm doing myself, like if I need the help like, I'll probably email them and then see them like after Easter, like, if

there's something in particular that I don't understand, then ask them, or ask someone else, like one of the other – maybe I need it from a different point of view… like… But I'll probably go to the lecturer first, and see if he explaining it to me again is – will help.

MI: I see... um... so, when you study for the exams, or for the third coursework, err, have you used any other source of information apart from the, from the lecture notes? Have you worked with any classmates of yours, or...

Leonora: Oh yeah, my – a lot – um, I find it really helpful with – I work quite a lot with my – like classmates, if I don't understand something, and I know they do, I often get them to try and explain it in a different way, like a different point of view and – see if I understand it better, so, with - especially with the coursework, I'll – like – talk to other people like – cos if – it makes me feel better sometimes if I know they're finding it hard as well.

APPENDIX F: Example of a Seminar Vignette

Seminar Vignettes A1: First Cycle H1 with LCR

Vignette I 01:40

Calaf is referring to Q1S1. He has managed to prove the first part, but he asks LCR about the use of the inverse since he does not understand and he does not know what to do.
LCR approves the first part. For the second part LCR tries to clarify the four elements of the group and the fact that the group is abelian. LCR mentions the element a, b and he asks Calaf to give him another element.
Calaf mentions c, d but LCR suggests that he should stick to the letters he has used.
Then Calaf says $a^{-1}b^{-1}$, but LCR says that it's too complicated and says ab, mentioning that it's the same as the aforementioned by Calaf and explains. Then LCR says that we have to prove why, for instance, a is different from ab. There is an issue about symbolisation for Calaf since he insists on naming the element c as ab.
LCR mentions and then uses multiplication table.
Calaf sounds happy now.

Vignette II 05:30

Two girls are asking LCR about Q3iiiS1. They basically ask LCR whether their approach is correct. He sounds relatively satisfied but a bit hesitant. The girls used different approaches one involving trigonometry.
The other is using an equation involving real and imaginary parts from complex analysis. LCR sounds happy with that stating that he is happy to mark this equation.
The first girl has a question about Q1S1. She mentions that she has written $g_1 g_2 g^{-1}{}_2 g^{-1}{}_1 = e$ and asks whether it is correct. A discussion follows about what information is needed in order to prove what the exercise asks for.

Vignette III 08:50

Not clear at the beginning which exercise is discussed between the girl and LCR. What the girl has done is not correct.
The girl has missed a step in the exercise. From what is audible, there is some difficulty with the use of inverse and the nature of the group as such. At some point LCR mentions an equation.
Most probably they are talking about Q1S1. LCR is suggesting another method of approaching the question mentioning the use of inverses and the product of elements.
The student sounds a bit hesitant to follow his suggestion but LCR says that although it is a bit risky she should give a bit more of reasoning.

Vignette IV 11:20

A student is asking LCR about Q1S1. The student has difficulty to identify the four elements of the group. LCR says that we naturally know the identity element and adds that the fourth element we can call c or we can try to find which one is it (*I have noticed that he adjusts his tactics based on previous discussions with other students within the group*).

LCR works using the multiplication table and mentions the element ab and tries to investigate how it has emerged and how the multiplication works.
He tries to emphasise the distinction of the elements. Then he gives a hint of how the student can work further on the exercise.

Vignette V 14:40

Two female students ask about Q2S1. They sound a bit stressed and LCR says that this question is hard. LCR asks whether the girls had read the question through to the end because there are several hints in the question.
The first girl says that she had used the hint for her attempt but basically it did not help them (*There is a gap between how much the lecturer believes that hints help and how much the students find them helpful*).

The girls say that they had tried to give an example as well, but LCR said 'don't do that!' The girls tried to find an example where the object of proving does not work.

They ask LCR whether $gh = hg$. LCR says that $(gh)^2 = ghgh$, and makes the necessary manipulations. The first girl sounds happier now.

After that they go back to the first part of the exercise. The girls have not done anything significant on that. LCR is giving some hints working first on the permutations. He mentions that x is fixed by $\beta(x)$ and $\beta(x)$ is fixed by α. Students have difficulty to see that the second part of the main statement of the question is the negation of the first part of the main statement.

Vignette VI 18:50

A male student asks about question 4 and whether there are similar examples in the lecture notes. LCR says no and he suggests that he should use the test for being a subgroup.

LCR suggests he should take two elements and then use a bit of an argument. The student says that the idea of an argument does not make much sense. LCR says that it should do. LCR suggests that he should read carefully the notes on the subgroup test.

They go back to the notes and LCR says that the notes give the instruction of how to do it. First LCR talks about the non-emptiness and closure. The student takes a specific element of the group, but LCR emphasises that the student should have in mind that this should hold for an arbitrary element.

LCR explains product closure. LCR uses pictures. The student seems to follow.

LCR is explaining the closure in H. The student has difficulty to follow the idea of $h_1 h_2$ being in the subgroup. There is also a confusion of the role of K.

Vignette VII 25:20

A student has a question of Q1iS1. He asks whether his attempt is correct. He also mentions that he has used associativity to prove it.

LCR disagrees with that and explains to the student what associativity is. He has seen what associativity means in his notes but he cannot transfer this knowledge to the exercises. Apparently the student confuses associativity with commutativity.

LCR says that this exercise is a special example since the square of g is the identity element. The student apparently has generalised this property for every group.

LCR says that the student cannot just use the axioms without having in mind the property that holds in this specific group.

LCR emphasises that it's this extra information that $g^2 = e$ that make things work here.

That is the property that the student has to use in order to prove commutativity.

LCR gives a name to this element and then tries to lead the student to find the inverses of elements.

Vignette VIII 29:15

Calaf asks about Q2S1. LCR disagrees with what Calaf has done because he is working with one side of the equation. LCR suggests that he should work on both sides of the equation. LCR suggests that he should examine different cases: $a\ (x) \neq x$ is the one case so $\beta\ (x) = x$ and according to the calculation of Calaf $a\ (\beta\ (x)) = a\ (x)$ and he says that $a\ (x) \neq x$.

Calaf does not understand why he has to work on two cases and he asks which the second case is.

The second case according to LCR is when x is fixed a. LCR suggests that he should use the case when x is moved by b.

LCR suggests that he should prove that $a\ (x)$ is fixed by b.

Vignette IX 34:00

A student is asking LCR about Q2S1. He is confused and thinks that x is permutation.

LCR corrects him saying that a and b are the permutations. LCR gives an example of the permutations of 1,2,3,4,5 and x is 1 or 2 or etc…. LCR describes the idea of composition of functions and he explains we used them.
LCR clarifies the hint saying that the students have to examine two different cases. x Is moved by b so is fixed by a. The student has some difficulty to understand the idea of elements being moved or fixed. LCR tries to partly solve the exercise by using the main statement of the exercise. He tries to explain that when b moves x we have $b\ (x)$ and the other way round.

Vignette X 38:25

The student is asking LCR to check whether her answer on Q2S1 is correct. LCR says that the first part is fine, but then he says that the student needs to check the second part of the equation and try to solve the exercise in two different cases.
LCR says that the student has to check what is happening when we are doing it the other way round. The student is muttering her thoughts, basically repeating the main statement.
LCR says that she is on the right track. They are checking whether the statement holds. They are trying to establish the fact that when an element x is moved by a then that means that it is fixed by b.
Students have difficulty to see that. (*I have the impression that the notation used in this exercises does not help the students to reveal the relatively simple idea of permutations that they have seen and possibly understood in the notes*).
LCR is using a picture to help the students. The students are much more positive after the picture is used. They said that they got it now!

Vignette XI 44:10

The student is trying to understand Q4S1. LCR is initially using Venn Diagrams. For the second part of the question, LCR suggests that the student should use the method by contradiction.

LCR suggests that the student should use the hint. The student replied that he understood neither the hint nor the notation. Then LCR suggests to the student that he should consider hk.
Student asks whether element belongs to either solely in H or K or in the intersection of the two.
LCR said that it might be in the intersection of the two. Then he emphasised that they are trying to achieve a contradiction here. LCR names an element g and he is trying to examine where it belongs. He is analysing the fact that $H \subseteq K \text{ or } K \subseteq H$.
The student is trying to apply the subgroup test. LCR is correcting his writing e.g. it is not 'if' is 'therefore'…

Vignette XII 50:20

The student is asking LCR about Q3ivS1. He shows to LCR what he has done and he LCR agrees. Then student says that he is not sure about the inverse of the matrix.
LCR names two elements and encourages the student to work with them. Students say that the matrix plus its inverse should give the identity element.
Then the students ask about the Argand Diagrams and LCR reminds them that they had seen them in complex analysis. They combine real and imaginary numbers.
Students have forgotten the idea of modulus and they sound a bit confused. LCR calms them down and he explains the relationship between the increase of t and of modulus. Students ask whether the Argand Diagram is a spiral and ask how come. LCR explains.

Vignette XIII 54:30

Calaf is asking about Q4iiS1 whether he should prove that HK is a subgroup of G. LCR suggests that Calaf should apply contradiction (see above). LCR mainly explains the hint given in that exercise.

APPENDIX G: Reflections on the Lectures in Group Theory

Lecture 1

The lecturer made a small general introduction about several academic and administrative issues, such as what students have studied in the Algebra course last semester (Vector Spaces) and what they are going to study this semester (Groups and Rings). He stated that there are going to be 20 lectures in this course; 10 on groups and 10 on rings. He additionally said that the detailed syllabus of the course is on Blackboard.

Additionally he suggested two books:

- J. Fraleigh, “A first course in Abstract Algebra”.
- J. Rotman, “A first course in Abstract Algebra”.

He announced that his office hours are on Friday at 12-13 and 14-15. He also announced that there are no printed notes. All students must come to the lectures in which detailed notes will be given on the blackboard.

After this introduction, he introduced me writing my name and email on the blackboard and he described my project giving the title and a small description. He emphasised that anonymity and confidentiality are guaranteed, but he said at the end that in his case it is very easy to figure out to whom I am referring.

Then he started the lecture introducing the following:

- Definition of binary operation
- Examples of binary operations
- Definition of associativity
- Examples of associativity
- Definition of Group

- Definition of Abelian group
- Comments on notation of binary operation
- Lemma – justification of claims in the definition of group
- Proof of lemma
- Examples of groups from fields

At some point during the lecture, LCR said that it is important for the students to remember the modulo arithmetic idea e.g. $17 \equiv 2mod3$. To his question "*who is familiar to this idea*" before writing the $17 \equiv 2mod3$ on the blackboard, only one student responded by rising his hand. After writing this on the blackboard, 4-5 students raised their hands.

He made a smooth introduction to the course by giving many accessible examples of binary operations and groups, which were covering a variety of subjects such as linear Algebra and operations over fields such as $\mathbb{R}$ and $\mathbb{R}^n$. He was referring on many occasions to material that the students had studied recently, mainly Vector Spaces and Matrices.

Additionally, when he was using a more advanced notation he was explaining it to the students. The overall impression of the lecture and the lecture notes is that they were very well structured and organised. The numbering of the lecture notes was very precise and my impression is that it was appreciated and followed by the students.

Lecture 2

At the beginning of this lecture the lecturer continued with the examples introduced in the previous one, and introduced two additional examples:

- The first was of a group with $2{\times}2$ matrices over $\mathbb{R}$ where the operation is the usual multiplication.

- The second was a generalisation of the first about the $n \times n$ matrices with entries from any field where the operation is the usual matrix multiplication. He introduced here the notation $GL\ (n, \mathbb{R})$.

He continued by giving four general facts about groups and their elements.

He then introduced the notion of symmetric groups. He first stated the definition of the permutation and gave some illustrative examples on the composition of permutations. Moreover, he gave a theorem/definition of the symmetric group by defining first the set $Sym\ (x)$ of all symmetries.

He offered some remarks on notation and then he gave some illustrative examples mainly using S_4. Additionally, he stated and proved that $|S_n| = n!$

Finally he briefly explained the multiplication tables, which are used to display a group, but he mentioned that in his lectures he does not like to use this kind of presentation, because in general it is not very useful.

At the end of the lecture he handed out the first problem sheet, which will be discussed in the first seminar and from which some questions will be handed in at the end of the semester for assessment.

Briefly he introduced the following:

- More examples from fields
- Basic facts about the elements of groups
- Definition of permutation
- Exercise on permutations
- Remarks on notation of the set of symmetries of X
- Definition/theorem of symmetric group
- Proof of symmetric group
- Remarks on notation
- Examples of symmetric groups mainly S_4

- Theorem which states that $\left|S_4\right| = n!$
- Introduction to multiplication tables

Most of his examples were trivial and not compatible with the level of the exercises given in the problem sheets. More diverse examples in the lectures would be an asset to the students. The basic facts and mechanisms of the group were very well presented, as well as the concept of symmetric groups. Students also, according to the interviews, seemed to grasp these ideas from the beginning.

The use of illustrations on the blackboard in the symmetric group examples was very helpful. Using objects such as squares or regular pentagons, however, would be much better for the students, since they would be visualising them better.

Presenting the same idea in different ways is beneficial to the students, both because they get accustomed to different ways of notation but also because it helps them to approach this idea from a different point of view, possibly more intelligible to them.

Lecture 3

This lecture started with the introduction of the notion of power of the element of group. The lecturer gave a more naïve and then a more precise definition, as he called it. Basically the second was a more formal and general definition.

Then he offered a lemma, which was about the index operations of the powers and added a hint for the proof of this lemma, which is by induction. This was followed by illustrative examples on the notion of powers, which involved the symmetric group S_4.

Then followed the definition of a subgroup. Here he made a comment about the analogy of the subgroup with the subspace, which the students had encountered in the first part of the Algebra module in the autumn semester.

What followed was the Theorem 1.16, which was used extensively later in the lecture and it was about testing whether a subset H is a subgroup or not, giving three criteria:

- H must be non-empty;
- H is closed under the binary operation for every h_1, h_2 elements of H;
- H is closed under inverses: for every $h \in H, h^{-1} \in H$.

He did not prove this theorem, but he advised the students to see the similar proof that was given in a first-year module.

He added some remarks about subgroups i.e. the trivial subgroup and the group itself as subgroup and a final remark about the subgroup notation $H \leq G$. At that point he asked for any questions, but there was no response.

Then he introduced the subset X and the subgroup generated by the subset X, i.e. $\langle X \rangle$, and stated a related lemma with its proof. At this point, the lecturer made another comment on the analogy of the subgroup $\langle X \rangle$, and the linear span, which students had seen in the autumn-semester algebra module.

It was obvious that the students were puzzled. They started talking to each other and the lecturer made a small pause and then he asked whether everything was clear and asked whether there were any questions.

He finished the lecture with some examples of subgroups of the group of real numbers $\mathbb{R}$ with addition as the binary operation.

The lecture briefly included the following:

- Definition of power of a group element
- Lemma about the power of a group element
- Example power of group element from S_4
- Definition of subgroup
- Theorem-test for being a subgroup
- Remarks on subgroups and on notation
- Definition of a subgroup that is generated from subset X
- Lemma about the subgroup generated by X
- Proof of the above lemma
- Remark about terminology
- Examples of subgroups

In my opinion the lecturer should prove Theorem 1.16 which was about testing whether a subset is a subgroup or not. Referring to the lecture notes from the Vector Space lecture notes of last semester is risky both because he was not the lecturer and therefore does not know how well that proof was understood and also because the students (this is shown in the interviews) could not see the analogy between Linear Algebra and Abstract Algebra. Since this theorem is important, it should be proved in the lecture.

I got the feeling that students had difficulty in understanding the notation $\langle X \rangle$, since they looked puzzled and started talking after its introduction. The lecturer realised that and asked them whether they had understood everything and invited them to interrupt him if they did not understand something.

In my opinion, notation should be explained not just stated as a remark. Behind each symbol there is a logical explanation. It is a result of a long process and I feel that explaining the history behind each symbol is both exciting and helpful to the students, enabling them to understand it and therefore use it.

Lecture 4

Before starting the lecture, the lecturer made an announcement about the coursework problems. After the seminars the students would find out which exercises will be assessed. The rest of the exercise solutions would be put on Blackboard. Students were strongly encouraged to be prepared for the seminar by attempting all the exercises.

After that announcement, the lecturer continued with examples of subgroups and how to test a set to be a subgroup of a group G. Additionally he gave a non-example, which involved vector spaces. The examples were involving permutations, rotational symmetries, matrices, real numbers and subgroups generated by certain elements.

Furthermore, he introduced the Dihedral groups by giving a first example using a regular pentagon. He stated the rotational symmetries of the pentagon and then made a generalisation for the $ngon$. He examined the two cases for n being odd or even. After these examples he gave the definition of the dihedral group. Finally he gave some hints for question 5 of the first problem sheet, which involved symmetries of Dihedral groups.

This lecture included:

- Examples of subgroups
- Non-examples of subgroups
- Introduction to dihedral groups by using regular pentagon
- Generalisation of dihedral groups introducing the $n - gon$
- Definition of Dihedral group.

This lecture predominantly focused on examples of Dihedral groups. Students looked very interested. After the more trivial examples, which were introduced first, the lecturer gave some more general examples from S_n.

Again, since most of his time was spent on examples involving regular pentagons, it would be very beneficial for the students to see these symmetries as a paper regular pentagon or on the computer. The use of technology would be definitely advantageous both for the course and the students.

Lecture 5

At the beginning of the lecture, the lecturer announced which questions would be included in the final coursework. These questions were: Q1i, 3iii, 3iv, 4i and 6. These questions were evaluated as follows: Q1: 8; Q3: 10; Q4: 7; Q6: 10. The answers for the rest of the questions of problem sheet 1 were made available on Blackboard.

He continued the lecture by reminding the students what a dihedral group is and of the idea of rotation and flip of a $n - gon$. He then introduced the theorem which states that $|D_n| = 2n$ and proved it.

Furthermore, he introduced the rotational symmetries of the cube. He discussed a bit the possible rotations of the cube and made a special comment the idea of composition of rotations is not obvious. He then gave a long hint for exercise 6 of the problem sheet, which asks students to find out how many rotations of cube exist.

He mentioned at the end that after describing the classes of rotations they should find 23 rotations. There was no reaction apart from one of Otello, which very discretely said that there should be 24 (That is what the exercise in the problem sheet says.) The lecturer smiled and asked which one is missing and very few students reacted saying the identity.

Moreover, the lecturer introduced the idea of the order of the elements. He stated the definition of the finite order of the element g and then gave some examples involving rotations of cube and permutations of S_3. He then stated and proved a theorem about the order of the group element g.

This lecture briefly included:

- Theorem of dihedral groups which stated that $|D_n| = 2n$.
- Proof of the theorem
- Introduction to rotational symmetries of cube.
- Introduction to the order of elements
- Definition of the order of elements of finite groups.
- Examples of order
- Theorem about the order of elements
- Proof of the theorem

Illustrating the examples with 2-dimensional or 3-dimensional objects is very beneficial for the novice students. Again in this lecture there were opportunities to use regular polygons and a cube. Visualisation is much appreciated by the students. Most of the students I have interviewed mentioned as their favorite module an applied mathematics module, which involved some laboratory demonstration of the mathematical ideas discussed in the lecture.

Moreover, much time was spent on the hint for question 6 about the cube symmetries. Additionally, the hint that was given was almost the whole solution of the exercise. I find it unreasonable. There were other exercises, probably much less straightforward for which no hint or any other guidance of this magnitude was provided.

Lecture 6

At the beginning of the lecture he continued a proof of a theorem that was stated at the end of the previous lecture and gave an example related to the theorem.

Moreover the lecturer stated another theorem about the properties of finite groups and their subgroups and gave the proof as well.

Furthermore, he introduced the notion of cyclic groups and stated the definition of the cyclic groups and gave some examples of the concept. The examples were the additive group of integers, the multiplicative group of complex numbers without the zero and a more general example of a cyclic group with order n.

Then the lecturer moved to the second chapter according to his notes, which is about cosets and Lagrange's theorem. First of all he stated the Lagrange's Theorem and gave some examples and corollaries of that theorem. He did not prove the Lagrange's in that lecture.

Moreover he stated some other theorem about the cyclic groups and the order of the cyclic groups and proved it. Moreover he provided the definition of the coset and also some examples.

The lecture included:

- Theorem about finite groups
- Proof of the above theorem
- Definition of cyclic groups
- Examples of cyclic groups
- Introduction to cosets and Lagrange's Theorem
- Lagrange's theorem
- Corollaries of Lagrange's Theorem

- Examples illustrating the idea of Lagrange's Theorem
- Definition of left and right coset
- Examples and picture representation

The lecturer offered a very smooth and well-developed introduction of the second chapter of Group Theory. He first introduced the idea of cyclic groups, he stated the properties of finite groups and he reminded the students of the idea of an order of a group element. He supported the theory with examples. I find that this introduction at the beginning of the lecture helps the students to smoothly make the transition to the more sophisticated material involving cosets and Lagrange's theorem.

A really good idea was not to prove the Lagrange's Theorem immediately after the theorem was stated. Giving illustrative examples before the proof helps the students understand what the theorem states and then be able to better follow the proof.

The lecturer offered a picture of how he visualises the coset. Although students like to visualise new concepts, the interviews highlighted the phenomenon of students not using this illustration at all in the solution of the problem sheet.

Lecture 7

The lecturer reminded the students where they had stopped in the last lecture. Moreover, he stated and explained some diverse examples of cosets.

He stated and proved a theorem about the intersections of cosets giving a very long and detailed proof and great emphasis was given to every step of the proof.

He then gave a lemma that was naturally linked with the aforementioned result, which was about counting in cosets. He proved this result as well.

Then he gave a modern version of the proof of Lagrange's theorem, which was stated in the previous lecture.

Moreover he gave a definition/corollary of the index and described it by drawing a Venn diagram. He then gave some historical comments about Lagrange's theorem and said that Lagrange's era there was not an established definition of the notion of group so he had to use terms like "symmetries of solutions of polynomial equations" in his very long proof.

Then he gave the definition of the equivalence relations and some non-mathematical examples. There was a comment of a student at that point, which is reported in the lecture notes.

This lecture briefly included:

- Examples of cosets
- Theorem about the intersection of cosets
- Proof of the above theorem
- Lemma about counting in cosets
- Proof of the lemma
- Proof of Lagrange's Theorem
- Definition/corollary of the index of H in G
- Definition of the equivalence relations
- Examples of the equivalence relations

Cosets appear to be one of the most problematic notions for students and, according to the interviews; it is at this point of the module that most of them feel lost. Although several examples of cosets were given, I have the feeling that the students did not comprehend in depth the concept of coset. There is some confusion with the concept of subgroup. A very thorough proof was given to the theorem concerning intersections of cosets. This is very helpful since emphasis is placed on the distinctive nature of cosets.

Only after this long introduction the Lagrange's theorem is proved. I have the feeling that this helps the students understand the proof much better.

Finally the lecturer introduced the idea of equivalence relations using a non-mathematical example i.e. "50% of the western literature is based on the symmetric equivalence relation". Every attempt to give non-mathematical examples and analogies to 'real life' makes the lecture more attractive.

Lecture 8

The lecturer gave a small reminder about the equivalence relations: reflexive, symmetric and transitive. Moreover, he provided examples from arithmetic and geometry.

He also gave a definition of the ~-equivalence class and gave an illustrative example. A theorem followed as well as its proof.

Then, he gave a corollary about the partition of a set A which is formed by ~-equivalence classes. He offered an illustration of this idea.

Then he stated a theorem about some characteristics of equivalence relations on G. He proved it step by step considering the three kinds of equivalence relations.

Moreover, he provided a second proof of Lagrange's Theorem, using this time the idea of equivalence relations. This bit was the last one of Section 2.

He then introduced section 3, the last of Group Theory, which was about homomorphisms, normal subgroups and quotient groups. He gave a definition of the homomorphism φ as well as an illustration.

Moreover, he gave the definitions of the image, the kernel of φ, as well as the definition of an isomorphism. Then he added a lemma about the properties of isomorphism and he proved it.

This lecture included:

- Examples of equivalence relations
- Definition of ~-equivalence classes
- Theorem about equivalence relations
- Proof of the above theorem
- Corollary about ~-equivalence classes
- Theorem about properties of equivalence relations
- Proof of the above theorem
- Second proof of Lagrange's theorem
- Definitions of homomorphism, image, kernel, isomorphism
- Lemma about the properties of homomorphisms
- Proof of the above lemma.

This lecturer sometimes tends to give two different versions of the same result or proof at different stages of the module. For instance in this case, after introducing the idea of equivalence relations, he provided a second proof of Lagrange's theorem, which involved equivalence relations. The second proof was much more rigorous.

In my opinion this teaching habit is very beneficial since presenting the same idea in a different way helps the students to widen their understanding. Additionally, in this case the level of rigour and abstraction of the two approaches is different, which makes the students understand and appreciate the rigorous and abstract nature of advanced mathematics and learn how to produce more rigorous results.

Lecture 9

The lecturer started by drawing a very illustrative picture of homomorphism including the image and the kernel. He then gave a triple lemma about some properties of homomorphisms and gave the proof as well for each bit.

He then provided 6 examples of homomorphism in which the first one was called trivial and was about the identity function. The examples were involving matrices, integers, exponential functions, complex numbers, and rotations of cubes. One of these examples was a hint for an exercise in question sheet 2.

He then introduced the normal subgroups. He gave a definition of a normal subgroup and also made two additional remarks: one was about the Abelian groups and the other was about the G itself and the trivial group.

He then introduced the notation about normal groups. Finally, he gave a lemma, which stated that $ker\varphi$ is a normal subgroup of G. He gave a proof as well.

This lecture included:

- Lemma about the properties of homomorphisms
- Proof of the above lemma
- Examples of homomorphisms
- Definition of normal subgroups
- Remarks following the definition
- Notation of normal subgroups
- Lemma about $ker\varphi$ being a normal subgroup of G
- Proof of the above lemma.

I find that the picture of the homomorphism given at the beginning of the lecture was very good, although I am not sure whether the novice students have understood the idea that the lecturer wanted to give.

The examples were good and involved different mathematical concepts such as the exponential function, the linear group, the integers, complex analysis and permutations. Giving diverse examples is very useful since it offers a holistic picture of a concept and helps the students to better understand previously studied mathematical concepts and to make the connection between different theories.

Again, Group Theory offers a lot of possibilities for using information technology in order to represent some ideas such as permutations of cube. One should use as much technological means as possible because students are attracted by these teaching methods.

Lecture 10

First he reminded the students about the notation and definition of a normal subgroup. Then he gave a lemma about the properties of a normal subgroup and how one can test whether a subgroup is normal. He clarified the notation a bit. He then proved the lemma.

Moreover, he provided some examples of normal subgroups; the one was involving symmetries. He also informally defined the concept of index reminding the students of Lagrange's Theorem.

He then introduced the concept of factor group (quotient group) and made some additional remarks. Later on, he gave a theorem/ definition of a quotient group and how one can check the group axioms.

Then he provided some examples using complex numbers. Furthermore, he introduced the First Isomorphism Theorem, giving examples and then he proved it.

In general the students were talking a lot more than usual. According to the lecturer, this happens when the students cannot follow the lecture. I noticed that students were gradually talking and being less motivated and the number of students attending the lectures has been reduced a bit.

This lecture included:

- Lemma about the characteristics of a normal subgroup
- Proof of the above lemma
- Examples of normal subgroups
- Definition of a factor group
- Remark about the factor groups
- Theorem about factor groups
- Proof of the above theorem
- Examples of factor groups
- First Isomorphism Theorem
- Proof of FIT

The definition and the lemma about the normal subgroups were given very clearly and were well explained. The students though did not look very satisfied but rather puzzled and unmotivated. This of course is the most difficult part of the module and the lecturer was aware of that. He offered some examples of normal subgroups but at that stage the students were not able to make the necessary generalisations.

In my opinion there is an issue of notation concerning quotient groups. Students do not comprehend G/N*. As a result of this there is great difficulty in understanding the first isomorphism theorem. I have the feeling that students have not grasped the previous notions that are necessary for this theory. Perhaps it would be helpful if group and ring isomorphisms and factor groups and rings were taught at a different stage of the degree or in a different semester from the rest of the Algebra module.*

Lecture 11

In this lecture, the lecturer finished the part of Group Theory during the first ten minutes by giving several examples of modular arithmetic. Additionally, he announced the assessed problems of problem sheet 2: Q1, 2 (i), 3, 5 (i) (iv) and 6 (ii) (iii). Respectively, they are marked as follows: 5, 4,8,8,5.

APPENDIX H: Staff Interview Theme Table

Theme	Description	LCR	SLA	SLB	SAA	SAB
1.	**Member of Staff's impression about the second seminar.**	It was ok. Variation in the level of student preparation.	The main problem the students had was question 2.		Most of the people had done the first bulk of the questions but not the second.	
2.	**Discussion on question 1**			They just wanted to check whether their answer was correct.	I was really surprised because some people hadn't grasped the idea of Lagrange's theorem.	I didn't get a lot of questions on this one.
3.	**Discussion on question 2**	This was the main problem.			Part three was really difficult.	The third part was really difficult.
4.	**Discussion on question 3**					They seemed to cope really well with this one
5.	**Discussion on question 4**	No inquiries for this one			Students had problem in this question because they hadn't grasped Lagrange's theorem which is quite important.	I was surprised by the number of students that struggled with this one.
6.	**Discussion on question 5**	This was sort of routine			People were generally ok with this.	They couldn't really understand how to show that it is a homomorphism in the fifth part of the exercise.
7.	**Discussion on question 6**				The students did not have the trick with division.	
8.	**The role of examples**	They had an example similar to question 1 and they sort of knew what to do.			They need to see some examples of Lagrange's theorem.	If you see the definition in the examples then it is easier to understand it.
9.	**Student's perception of group theory concepts – and the difficulty to deal with them.**	The distinction between the elements of the group and the elements of the set is something that is not necessarily clear	Cosets is a particularly formidable barrier	Students had problem to understand what the equivalence relation is. Students shouldn't treat the notions in	A serious problem is that they haven't really grasped the idea of Lagrange's theorem and what it means.	Coset is a new thing for them.

				group theory as objects but as concepts.		
10.	**Visualisation**	They don't have a picture of a group and its elements in their mind.			For cosets I draw groups of players.	Visualising cosets it is really something new to the students.
11.	**Students' difficulty with proofs**	It's interesting they had difficulty to prove the statement in question 3				
12.	**Staff's teaching strategies**	We have tried to explain them in different ways by using pictures as well	Last year I was teaching this group discrete mathematics and I tried to encourage more engagement and creativity from their part and they responded wonderfully.		Sometimes I explain it them again and again.	
13.	**Member of staff's impression about the course so far.**	Changing the topic halfway through is good I think.		Groups should be taught first and rings afterwards.		May be students have found vector spaces more difficult because it is the first abstract notion they see.
14.	**Student's presence in lectures and seminars**	At lecture 8-9 the students were somewhere less than 60 and he thinks that it is fairly ok.			A person had missed the lecture and he had problem with question 3	
15.	**Last year's algebra course**	The exam paper last year was too hard.	The difference of students from last year's course is quite profound.			
16.	**Student's response to the course – staff's impression.**	I am not displeased with the way the students respond to the course.	This year the students have willingness to be involved with the course and they are possibly brighter than last year. Creativity in pure mathematics happens within a very constraint environment – like music!	Not many students try the exercises before the seminar although they should. Students should be more involved and engaged in the seminars.	Some students are ok with abstraction but some come up with concrete examples	My impression is that they all seem quite ok, working together well.
17.	**Member of staff about the course.**	The courses we give are self contained but this is	I wonder whether it is better to teach rings first.			

		not always good.				
18.	**Use of other sources of knowledge by students**	Books are expensive and the students don't buy them.				
19.	**Student's difficulty with the exercises and the course overall.**		They were clueless and they did not know how to start with question 2.	There is a Malaysian girl that has serious problems. She is much weaker than the others and also she does not speak in English. Language is an issue.	Sometimes we need to explain several times and go through again for the student to understand it.	The main problem with exercise 4 was to interpret correctly what the exercise says.
20.	**Student's difficulty with the 'if and only if' statement.**		The if and only if statement needed more thought and the students had trouble with it.			
21.	**The role of definition in pure mathematics.**		I needed to know whether the definition LCR used was of my understanding. The role of definition is fundamentally important.			Definitions are very important in pure mathematics although they don't do a lot in understanding.
22.	**Staff's ideas about the emotional aspects of learning.**		Self confidence is a big issue in mathematics			The looks of their faces is just sheer confusion.
23.	**What makes a student good in pure maths?**		Students must be able to discipline their minds. Pure mathematical argument is similar to a legal argument – you don't just have to understand it, you have to resolve it.			
24.	**Symbolisation**			In the example of the equivalence relation the symbol x~y should be substituted by xRy so to refer to the relation better.		
25.	**Abstraction**				Quotient groups are really abstract. I mean you start from	

					rotations and symmetries but then the course builds up to more abstract things.	

APPENDIX J: Pedagogical Statements' Table (teaching)

Amelia	I think – because the coursework, you can ask for help as well, and you can sit with your lecture notes, I mean – I often find it quite useful when I do coursework, um, we get online notes sometimes, I don't know if you have those...? MI: But there are no printed notes for this course, are they? A: No, but I like that, because you can just do your control F and type in what you're looking for, and it brings it up, rather than having to read all the lecture notes… A: Um, there – coursework I do find easier cos you can work with the notes, and kind of find the section in your notes and then just work through an example or something, (MI: Yes, yes, that's true) whereas for an exam I find it very much – I just need to memorize everything, (MI: Mm) cos you don't have the notes, so you can't A: And I looked at – for the second problem sheet I was looking at what the Sym (X) meant and trying to find a different definition, (MI: Mm hmm) cos I didn't like the one in our notes, I don't really understand it. Um... and I think I looked at the answer – have you got the second problem sheet? (MI: Yes I have) I think I looked a lot more up for the second one, because the first one is kind of the easiest, (MI: Mm hmm) I looked up... oh, and I tried to look at this – all but stabilizer theorem, just to see where it was coming from, (MI: Mm hmm) this bit. MI: Usually Wikipedia is quite good I think. A: Sometimes, I can't always find the proofs on there, and I don't always – I think sometimes it goes into a bit too deep, and I get a bit confused as to where the – MI: Yeah, too theoretical, yeah... A: Um – I found – huh! I found Manchester's University lecture notes for algebra online there? (MI: Oh…) which were quite – useful, cos they had some um, examples, of... question 4, I think, they had an example of it, so I was using that to kind of – cos they'd actually done a whole one and completed it, so I was using that to work through that, and then understand it? So it was really helpful. But yeah, I often just type it in, to Google, and hope that something comes up. (MI: laughs) MI: Err, now – the last question – do you have any, any – suggestions to make, about the teaching of the course, I mean if you were the lecturer, how would you teach the course? A: Um – no not really, I think it's been taught quite well, I found his err, for rings, it's just really stupid, but I find it really difficult cos he – didn't keep to his numbering? And it just confuses you and it makes you feel like you've got a gap in your knowledge? And it – it – puts my back up, cos I get nervous that it's like – oh, why have we skipped a bit, and then we haven't got section 4, and it muddles your

	notes up a bit more. Which I don't like, being very organized and logic, I think a lot of us are like that, and really um... I don't really know, cos I don't think I know the course well enough to know how else you'd be able to teach it, do you know what I mean? Um... yeah, I think examples are needed, a lot, but he seems quite good at giving us examples... but yeah, I've really struggled to have done a lot of that work without examples
Francesca	The first lectures look really simple and easy… many things that we have seen so far are really comprehensible… for example the properties, the axioms are things that we have seen before and they are easy… of course he explains them a bit further… They are ok… but when he goes a bit deeper then things get difficult. F: I will gather all the notes, and usually I study the printed notes not the lecture notes… I always go to the lectures for the notes and I end up studying from the printed notes… MI: Why do you prefer the printed notes? F: They are more tidy… or I a may miss a lecture… Of course when I go to a lecture and I am fully concentrated or I don't feel sleepy, I take extra notes, which might be helpful… MI: Do you believe that the notes on the blackboard are enough or the oral explanation is helpful as well? F: Yes, yes… then I move to the coursework… I check the solutions and compare them with my solutions… and if I have time I see the past papers… MI: Past papers are very important… F: Yes, but you don't have the solutions of the past papers… MI: Yes, that's true… they don't give solutions… F: Then, what can I do with it? MI: Look… first you do the coursework, and you will notice that many of the thinks in the coursework are in the papers as well… if you have any questions you should go and see the lecturer… F: Last year we had a very good lecturer… his papers were exactly the same every year, but with different numbers…? MI: At the end I will probably write a report and I will give it to the school… The last question… If you were the lecturer, what would you change in the way this course was taught? F: First of all I would organise sessions on a voluntary basis, for the people who want to come and ask questions… although we have 5 seminars in total, we may miss one or two… but even in the case we go there we do not have the time to ask questions… most probably just one… or two…It's just one hour… it's not enough… Whereas, if we had some sessions with some postgraduate students

	or third year students this would be very helpful… I would also be more illustrative in the lectures… MI: What do you mean by that? F: I think algebra is a matter of imagination… so you can understand things… I do not think that it is very helpful for the students to go to the lecture and see someone writing on the blackboard and simply copy the notes in rush, doesn't help a lot… Because when I copy the notes I do not think about them… sometimes I cannot figure out the handwriting of the lecturers… I don't say that they don't do their job well. This way of teaching might be the best possible, but I thing it would be more useful if for example they were using cubes for example in order to show how thinks are done… Or for the rings, they should give us something to see, in order to understand… Through your interviews, they should try to understand how students comprehend these notions and try to find ways to explain to us how to think about these ideas… To offer us alternative ways of dealing with algebra… Every student thinks in a different way… Of course I don't say that having printed notes before the lecture is good but, in any case… Most probably, the way it is now taught is the best…
Dorabella	MI: What did you feel? D: With the course, I think, it's ok, um, I mean the lecturer's good, I can understand him and um, and his writing as well, which helps, um, MI: Ah, there are lecturers that don't write, on the blackboard? D: Um, no they do, it's just their handwriting's quite difficult to – decipher! Um, but no, his is good. With the coursework, err, I find it quite difficult, I get like quite worked up before the seminars, and err – during the seminars it's fine, cos I can ask um, but afterwards I find it a bit difficult, like, (MI: frustrating) yeah. And frustrating. And also working out when the office hours are, so I can go and speak to them. Um, afterwards, it's kind of – like during the seminars they're there to hand, but afterwards – more time. D: Um, the rings, yes – it's – I find it very... difficult to understand the concepts, I think. Um... I'm getting there, but I was ill for a little bit, so I missed... a fair content of it! So I'm just trying to copy that up as well, um, as well as doing the coursework. So... I kind of understand it but it's so hard, when you don't go to the lectures. To – cos they obviously explain it as they write it, whereas just copying someone else's is a bit difficult. Um, yeah. It's ok. I'm getting there! MI: I see. When you say you tried did you use lecture notes or examples? D: Lecture notes, um, lecture notes and I've tried looking in books for examples, um, but – it's quite difficult to find ones that really match up, very similarly, (MI: Exactly, yeah) um, and I find it difficult to get an example that um, looks different, and then apply it to another one! I find that difficult.

	D: Um... I think he's – doing a very good job actually, he's a lot better than a lot of lecturers I've had, and um – I really liked my A-level teachers, both – I mean I had the same ones for both years, and he's a very similar teacher to both of them, um... MI: What are the characteristics of this lecturer that you like? D: Um, clear notes, I think he's got – you know, specific sections, um, which I like, numbered sections, um... he speaks very clearly and writes very clearly, which helps enormously um, and I think he's very approachable, I feel like you can speak to him. Um... I just like the whole way he's done it really. Um, and I think the best thing is the ordered notes. MI: Ordered notes. D: Yeah. Yeah. MI: Um... any other suggestions – anything additional that you would like to see in his way of teaching? D: Um... MI: Or, in the whole curriculum, if you want? D: Um, I think it goes very fast, um, but in a way you kind of have to do that to get all the content in, so that's quite a difficult one to change, um, more examples probably, I like a lot of examples, um, and yeah... yeah, more examples I think.
Kostanza	K: Yeah, so like when he was giving the examples, like, when he was just – oh and like – that made a bit more sense then, like, when he tried to put it into real terms, (MI: Mm, mm) but a lot of groups you can't really do that, there's not really a valid example is there, in the real world, so... K: Yeah, that's probably because it's quite like the example in the notes, isn't it, where you've group 7, so I could look like exactly the same almost, apart from – apart from slightly different things, but – I – I understood that one a bit better because – I had such a close example in the notes. MI: No, that's ok. Um, and the last question, if you were the lecturer, (K: Yeah) how would you teach this course? K: Pretty much the same as he does. I think he's good, yeah. MI: Ah hah. Any suggestions, any other err, additional... K: Bless you. Um – no I think I've – I've – he teaches it well, probably of all our lecturers, I mean he's the head of department and he knows what he's doing, doesn't he. He sees – he's learnt from everyone else's mistakes because he gets told about them all, I'd imagine, um – I like it, I mean he breaks down with lots of examples,

	which is really good, for looking back for the coursework and the exams. I um... no, I like his teaching style. Think he's a good lecturer. I'm even nearly tempted by his third-year course, even though it's pure, that's how...!
Leonora	L: Yeah… like some of the group stuff are so it's like... so it is easier to understand and in the lectures he makes it really… quite easy to understand like… he will keep going through it and then it is like… oh Yeah.... cause I normally find in a lot of lectures that I don't understand what is going on in them but until I get outside and read my notes and do examples but with this… it is easy to understand in the lecture like I understand more of it in this lectures than my other lectures so..... L: Err... because I don't think... it's been done a lot, in… we've obviously been given a hint, (MI: Mm hmm) for how to do it, but it's not something that – we've just done examples of and then can – like take that from it, it's just we've basically done this bit on it, (MI: Exactly) and then – it's kind of – like you've got nothing to go off from, like from before to – with like examples you can basically... see where it's come from? MI: Hopefully, hopefully. Um... how do you find the – the pace of the – of the lecturer, and the pace of the teaching? L: Um – sometimes I think it's a bit fast, like, sometimes it's ok like, err, it's fine, cos, if I understand what's going on in the lecture, then – I find it ok, but if I'm a bit confused, then sometimes I think – oh, he's just moved straight on to that, like - don't know where he's gone to that, because I still don't understand. Still don't understand the first bit, so I don't – not – I'm not going to be able to understand like – the next bit, so... sometimes it's a bit fast. MI: Mm hmm... um – if you were the lecturer, would you use any other ways of... teaching, or... could you do any – could you give me any... suggestions, teaching suggestions? L: Err, I don't... mainly – like, obviously, because I'm a visual learner, but I don't know if there's an – is anything visual within it, that you – that they could actually – like that he could actually put – on the board, I'm not sure if he could do that or not, but I think it would help me, personally, a lot, but... MI: Using more illustrations, more.... yeah. L: Yeah. More pictures and stuff, like... explaining what's happening and sort of thing, so that – maybe then I could picture it in my head, I'd find it easier. Cos if I'm trying to picture, as he's going along, then sometimes if he's going too fast, I don't have time to try and think about it, when I'm trying to think about something else. So... maybe stop and then draw a picture, or... MI: Yeah... L: Or use a – some – some kind of visual aid or something,

	maybe.
Manrico	MI: Do you think that the notes are enough for the solution of the coursework... I mean when you solve the coursework do you stick only to the lecture notes or do you search in the literature? M: I will just use the lecture notes unless I am struggling then I will try and look.... I usually just try and use the lecture notes cause I think as usually at the end of the day, you know, they test you what they have taught us so.... it should all be in the lecture notes but sometimes if I find it is good to get a second sort of opinion so I go to the library sometimes and get the help so I double check it. M: Err yeah, I mean that was another thing that – because it was in the note – of a similar sort of example was in the notes; it was very easy to kind of understand where this comes from. So that was, no problem really. But I – yeah, again, it might be – me not – it makes perfect sense, but I might not... make it – it's just like you know – I can understand it, but it's trying to, I mean because proof is really trying to make someone else understand it, and I say, possibly I do struggle at – giving, you know, making someone else understand it by writing it down, but, so it's where I might lose some marks, but... M: Err yeah, I err, heavily use the lecture notes, cos I mean... say there's just not enough, it's just the lectures do go quite quickly and you do need to then sit down and read your lecture notes again, and it's kind of helpful to do it, a bit like a questionnaire, when you've got some sort of goal to achieve, and you just err – look how to achieve that with the lecture notes. Cos I mean lectures do – you can't take – take in everything, all in the one hour of the lecture, you do need to go over it in your own time. MI: Basically, I want you to tell me you know, teaching suggestions or any... you know your own personal approach. M: Um – I'd probably throw in maybe a few more examples, now and then, just because that is – that I mean – different people learn different ways, but I find it just kind of actually doing it and just you know, actually seeing examples – I mean there are quite good, there are quite good few examples on the course, but may – yeah, that's probably how I best learn, just by seeing how it actually works, and then to see how you get there you can then go back and look at the proof that – to just see it working first. It's kind of a good starting point for me, personally… MI: I see. Um... so – how – how do you use the examples, um... are you trying to imitate, you know, the method in the example, or are you trying to understand it and apply it? M: In a way at first, it sort of just kind of made me copy it, but then you kind of then learn the technique and the process of doing it, so then you just might have to refer to it every now and then, just to see what step you do next, you think oh right, I do that step next because of this and then – just apply to your one, so you take that step because you need to do this.

Musetta	MI: How are you planning to get prepared for the coursework? Are you planning to use the notes? M: Yes. Basically I use only the lecture notes when I study… For example, I rarely go and ask the lecturer in his office hours… or if we do not have a seminar I will rarely go and ask the lecturer… We only used a book in the last semester's algebra and we did that because the book was exactly the same as the lecture notes… whatever I searched for about certain exercises, it was what I was exactly looking for… But this is not the case for other books… that's why I do not use books… I use a lot of internet… Usually, like the coursework… we start from the lecture notes…and usually I am trying to understand everything… not like when we prepare a coursework. For the coursework we do not have much time so we are going for the exercises… I believe that if you do not understand something, then you cannot understand what it follows as well… In the past, I used to make my own notes, but since it was time consuming, I decided to stop that… I study the notes and I highlight the important things… Something that I need to see again… I study only from the notes… M: I will be honest. I have no idea! *Laughs…* Can I explain to you what I am doing? When I am studying something, especially when I have time pressure, I just read the notes with all the theorems and the definitions… and then I read the exercise… if something looks relevant I try to use that and solve the exercise… MI: Your way of studying moves around the coursework… the lecture notes are secondary to you. M: Yes… only in the exams I study in depth the lecture notes… and usually when I study for the exams I say "Ah, that's what it means… that's how is done…" Now I may see this exercise and also check the notes and try to take things from the notes and put them in the solution of the exercise if it's relevant… It's really funny…
Norina	N: Well sometimes the examples, they feel like um, complete like he'll go like um, oh well this bit's obvious, and then not carry on, and I'm kind of like it may be obvious to you, but it would be nice if like, you had like – like a complete example, where every stat was put down, so then you could be able to see what was going on. N: I'm not saying like every example, he needs to go through it step by step, but it would just be nice like after he's introduced a – an idea, to have one example, that covered every step? So... N: Yeah, because it's sort of like – he is pretty good at explaining, what everything means, like – but it's just - you need to know how to apply it… it isn't like – you – you can have all the information as well…like cos it's not gonna be helpful, so I think it's just um – a student point of view… MI: Um, any other suggestions, any other –

	N: No, I think that's the only problem that I've really had, yeah. MI: What about the lecture notes, are you satisfied with the lecture notes? N: Um, yeah, um, a student asked if he'd maybe put um, more detailed notes on blackboard, (MI: More detailed notes) like to go through, cos um – MI: So you would prefer more detailed, written notes, you know, before or after the course? N: Um... I wouldn't say before, like, it means no-one would go to a lecture, but um, no, I think like, after the lecture, because then you've got something that you can work through for coursework as well… (MI: Mm hmm) Because um, sometimes when you're in a lecture and you're just copying it down, it's always like you're not – you're maybe rushing or something, you may not write it down, like 100%, so it would be nice to have some detailed notes you can go through the coursework.
Norma	MI: Um, something else, how do you find the teaching – approach, the teaching method? N: Um, yeah, I find that quite good, um, obviously, like cos it – write on the blackboard and they were like – a lot of other courses, that - like – computer-based, and err, I don't think I'd be able to take it in as much if I wasn't actually reading it and writing it down as well, so I was... like the writing on the blackboard is a good thing, (MI: Yes) um... yeah... it's quite good. MI: I see. So how did you prepare for this coursework, did you use your err, lecture notes, I assume, N: Yeah, yeah, I used them to help me, yeah. MI: Did you use any other source of information? N: Not really... MI: Internet, books? N: No, just like my lecture notes and help from the lecturer and postgraduate in the seminar, and that was it, so... yeah. MI: Um, that's ok. Um... the last questions, is the following: if you were the lecturer, how would you teach this course? N: That's tricky...! Um... I don't – I do find that the writing on the blackboard is a really good way to do it, cos my business module, um, it's all on PowerPoint presentation… And although like, I print off the slides before I go, I do find that they say more than what's written on the slide… And you don't have time to write down before they've gone on to the next thing… And I – I really do find – I don't know whether that's because I'm used to the maths, all being written down and writing it down (MI: Probably) but it's really fast-paced, and I

	can't keep up? Um… so I do quite like that most of the maths is um, like written on the blackboard. I think that is a really good way to teach it. Um... yeah. MI: Or, let me say something else: Some students said that, um, apart from the lecture notes he's trying to explain something, you know, around this concept that was written on the – on the blackboard, which are quite important, and they do have the time to write down this additional explanations – N: Oh right, yeah, I see what you mean, yeah, sometimes, um – I do try to make little – like little extra notes, but... yeah... that can be a little bit annoying when it's not written down, but then I sort of think well, you are spoon-fed most of it, so maybe like, it – odd little bits here and there you do have to write them down, that aren't being written down, and maybe – like we should do it because – you are actually given a lot in our notes so, (MI: Yeah) I think it's just using initiative.
Otello	MI: So you think that the lecturers do not explain a lot of things…? O: Yeah… they don't really… because if they do they will spend the whole lecture explaining and they will not get things done… So they make assumption that this was taught in year 1…you are supposed to know this, you are supposed to know that… MI: How do you find the course so far? There have been two lectures. Do you feel that … the style and the approach … do you like the approach of this course? Do you think that the approach is a strictly formal mathematical approach? O: Yes, yes, yes… Although I have said that students have problems transiting … because may be in year 1 they were giving a lot of explanation, but now there is a step up in class, sort of saying… MI: Do you think that the notes… the material the lecturer is giving in the lectures is enough for the solution of the first sheet? …Because I think there are parts of the exercises that one has to search for… To investigate by himself… to discuss and discover… O: Hmm… I would direct this question to you…Are you one of those students that would rely on the notes? MI: Certainly it is not enough… especially in courses like this… one has to search by himself … go to the suggested literature… but what about you? What is you approach to the exercises? O: I consult internet…and there are so many recommended books in the library…that is the way I did… I would have thought that the notes are sufficient. There are like a guide. MI: Hmm… O: There are just to guide you… I mean… a lot of students do not have this approach as I said… they just wait for the seminars and ask questions…

	MI: Do you find the examples given by the lecturer in the lecture notes to be illustrative enough? Do you think that students refer to these examples and help them to understand the material? O: Yes, I must say that the examples given in this class are comprehensive. Of course this is the case for the entire maths department…you know? In some departments in other universities they may only solve the easy questions, the easy examples…and leave the difficult ones in the exam or the test…you know? O: *Laughs…* More informal approach. MI: More informal? O: Yes, Not just of talking and writing on the blackboard is the best way of – the way of teaching. I think it is important for every lecturer to compose himself – composed and really taking the time to explain informally what exactly these concepts like your teaching mean so maybe that is why if we have not known enough background reading you know of the course then when you now come to a lecture you find that if that was presented at 100 miles per hour you would have difficulty in keeping up. But there is nothing one can do about that you just need to you know put more personal effort in and do your reading. I don't what was your own experience? MI: But don't you think that presenting mathematics in a formal way makes a student think, you know, in a more rigorous way do you think? O: Ur.... maybe not. As I said I prefer it more relaxed conversation and approach something that would really inspire - you know – there are some lecturers here that tend to go into too much detail which I find not too good but there are other lecturers which really challenge you and make you think for yourself. MI: Thank you so much, that you so much for this. And yeah… I would like to discuss some teaching issues. If you are in a lecturer – put yourself in the position of a lecturer - what would you do different. How would you approach the teaching of this course? O: Um.... That's very you know – it is important for a lecturer to engage the students so at the end of the day it is not a matter of just copying, but I would prefer more – maybe that would not be a lecture – but at the end of the day what you should have is more interactive session which lecturer does throw questions on members of the audience and expects answers. MI: You mean in the lecture not the seminar? O: Yea, in the lecture. Definitely would expect in this modern day and age more use of interactive whiteboards, computers yeah and… also since we are dealing with groups… the lecturer should be able to juxtapose his lectures with real life examples… cause as you know symmetry does play a big part in lecture
Carmen	MI: It's okay.... do you think that the examples help you? I mean the examples that the lecturer gives?

	C: Yeah...because after I look just example so okay so... MI: so... you imitate the pattern of the examples. C: Yeah MI: Um.... how do find the way LCR teaches the course? Are you happy? C: Yeah. But in fact here it is – I don't know if the teacher is happy to do the lecture. MI: Uh huh. C: All the teachers – because in fact the teacher speak and how his lecture and there are – they don't have a relation with the students.... MI: They don't have a relationship with the students? C: Yea, so speak, speak, speak… and after I finish but for me it's okay I went to lecture. In France we have more relation with the student... I prefer that. MI: Closer relationship. If you were the lecturer how would you teach the course? C: Yea, I would teach all of the notes on the blackboard… it's difficult for me – so.... MI: So you are going to teach in France, no? You are going to be a teacher in France. C: Yea. MI: If you are going to teach something similar course to student but to under graduate student in University would you use any other method? C: No.... MI: Any other method, any other way of teaching all? Would you give suggestions? C: But in fact I think in the University it is difficult to – because it is {XXX} and in fact the students are not warned for that because the teacher is a student that knows the theorem before so it is possible to have an idea before the session so... C: And the teacher for example it is a lesson in geometry about the circle and the angle and at the first - the teacher ask the pupils to go to the board and to draw the presentation about that and so there relation and after the teacher do the lecture and the abstract of that so I think it is possible to do that after – its just...
Tamino	T: That went kind of cause we did not have many examples in it as such so it was kind of in places. You gave the proof and then you had to work out how to do an example via the proof and it was like... not by the theory realistically so sometimes you would just look at it and go mmm this might take a while... *Laughs* T: Um... It doesn't look that bad. I haven't actually finished it – I have a rough idea of how to do it but it doesn't look that bad really, that question. It sort of like once you have – cause I mean you have got two of them you have got to do – but and he has also given us proofs showing us how to do the other form as well so you can just look at them and sort of use them as a guide – I mean I have got a rough proof of what I have already done but I will wait until he has put his notes up and see if that looks right with the others and then 6 just looks – in some ways 6 looks nicer but in some ways looks completely and utterly....

	T: I did look through my previous – cause I mean obviously I have had these notes – last year's notes I have had a look at them a little bit and some of the notes I had from – I still have a few of my notes I had when I was at my high – Sixth Form, sorry... They came in useful a little bit to do cycle groups and that cause we looked at them. T: Um not nice at first. I now understand the first a bit more but I don't the second bit I think I will understand it more in depth a bit more but… and when I actually put the lecture notes up for it because I find his {xxx} I mean I don't like reading my own handwriting back its weird. I quite like his handwriting it's not the neatest in the world but it's still easier I find! T: Indenting them both into one with linear algebra in the first semester, probably not the wisest idea in the world! Cause linear algebra is something completely almost different to these two really. Putting all three of them together – you can see why they are together because they are the whole algebra bit but it's still a rush to do rings and groups in one semester really isn't it T: I mean and the speed also doesn't help because you have got to write so fast cause you are having to write so fast to concentrate you can't concentrate on what you are actually learning as such so I think alright he doesn't like it but what would probably be better for him is if he did printed lecture notes and then he kind of went through them and we could annotate them whilst – cause he says things as well which are very useful to write down but you're too busy writing everything down to actually concentrate on what he's actually saying... MI: Um.... T: Really.... so for him I think it would be better for him to actually to do printed or even just hand written notes and then we can just annotate his notes with other things which would give us a lot better chance of staying with him at the speed he needs to go at for this part of the module. T: Well I read through the lecture notes a few times then I generally shorten them – which is a weird thing – I actually cut them down to notes to note and then I sometimes cut them down again so then I have got a really short key points and then I will basically learn them and then I will look at it… MI: Now the last question. If you were the lecturer how would you do different? How would you teach different this course? T: Well I'd obviously do it with lecture notes and put them up for every lecture. I know – and to deal with the idea of people not coming because obviously you are giving out lecture notes take a register every lecture basically or something. Or the easiest option – or another option that's actually easier – instead of putting the lecture notes on blackboard you have the lecture notes for that lecture you have the register of who turns up and you just email it to only them people so they are the only ones who have that file okay other people could get it by other people emailing it to someone else but it still

	requires effort from someone else to do it whilst if it's on the blackboard they can do it just for themselves but they would have to ask someone else to do it if they have to get it emailed to get into them. MI: So it is very important for the lecture notes. T: Very. MI: In the way of teaching in the lecture or.... T: Well I would try and explain things a little bit more and go a little bit slower so that – I would go slightly slower so we have more time to write and listen or do those printed notes for the lectures maybe like in one of our lectures he gives us printed lectures notes but he leaves key words out so you obviously have to pay attention because he says oh this is a key word so then you have to fill in the words and there is a lot less to write but you still have to pay attention because otherwise you are going to miss key words phrases or definitions or something.

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