CAYLEY GRAPHS AND RECONSTRUCTION PROBLEMS

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Abstract

In this thesis we study new kinds of reconstruction problems introduced by V.I. Levenshtein. These problems are concerned with finding the minimum number of distorted objects that are needed to restore or identify the original object. Our main goal is to find these numbers in Cayley graphs on the symmetric and alternating groups generated by a conjugacy class of permutations of order two. We found that the numbers are closely related to a well-known class of numbers in combinatorics, the Stirling numbers.

Acknowledgements

I would like to thank Johannes Siemons not only for his supervision but also for his support. Since the first day I met him I have been given many good ideas and useful comments from him. This certainly includes his introduction of reconstruction problems.

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Chapter 1

Introduction

In this thesis we are interested in reconstruction problems. These were first introduced by V.I. Levenshtein [16, 17] and have been studied under the name *efficient reconstruction*. They are also known as *trace reconstruction* in the area of computational biology concerning evolutionary processes [1, 12, 21]. These problems have no connection to the 'classical problem' that refers to the problem of reconstructing a graph from its multiset of subgraphs obtained by deleting one vertex from the original graph.

1.1 Reconstruction Problems

Problems of our interest are quite similar to other problems in coding theory and information theory. However, there is a small difference between efficient reconstruction and the error correction which is studied in coding theory. Normally, in coding theory when a piece of information (or a message) encoded as a codeword is transmitted through a channel from the sender S to the receiver R, the information is likely to be distorted by noise in the channel. Here one needs a code in order to manage these distortions and errors.

In coding theory, researchers are interested in codes which allow them to reconstruct an object, i.e. a codeword, from one distorted object. It is as if we had a key book to show us the method to encode and decode the objects. As is shown in Figure 1.1, the distorted information will be corrected by an efficient algorithm with respect to each code used.

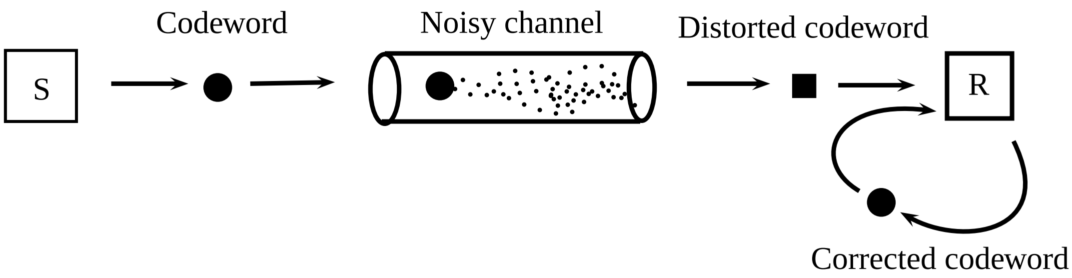

Figure 1.1: Error-correction

In the problems of efficient reconstruction (or trace reconstruction), as demonstrated in Figure 1.2, a number of distorted pieces of information (maybe considered as samples) are needed to identify the original message. It is essential that sufficiently many distorted messages are present. Here, we need to find the minimum number of samples that are required to help us find or identify the original message.

Of particular interest in real-world applications are ancestor DNA reconstruction and genome rearrangement [1, 8, 12, 13, 21, 22]. For example, studies of this type of reconstruction aim to identify the DNA sequences of a common ascestor, provided we have sufficiently many samples of DNA from his descendants.

In order to study reconstruction problems of this kind Levenshtein [16, 17] introduced *error graphs*. This allows us to formulate the efficient reconstruction problem as a problem about graphs. Let us demonstrate the kind of reconstructions we are interested in. Suppose that the problem consists of transmitting strings of length four made from the letters in $\{1, 2, 3, 4, 5, 6, 7\}$. Here we are given that errors occur in the form of a single *positional interchange*, that is, some two letters swap positions in the string, with no other changes. For example, when the string 1453 is transmitted through this channel it may be distorted in $\binom{4}{2}$ ways to become 1354,

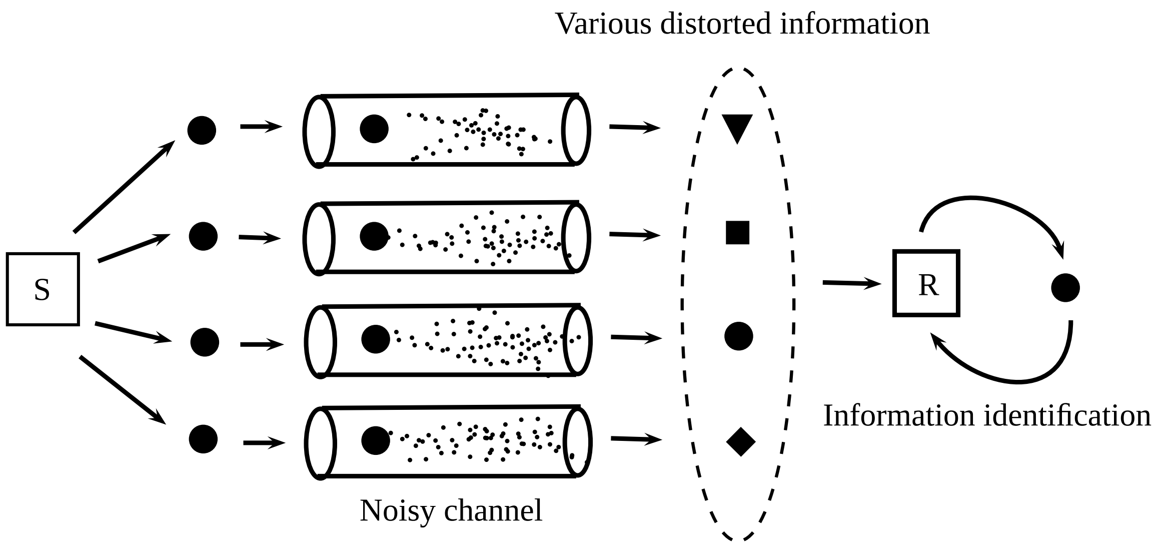

Figure 1.2: An example of efficient reconstruction problems

4153, 3451, etc., or may be not distorted at all. Such errors will be called *single errors*. In addition, we allow the possibility that several errors occur in succession. For instance $1453 \to 1534$ is a distortion that could occur as a sequence of the single errors $1453 \to 1543 \to 1534$.

The notion of an error graph is now quite easy to explain: we view the strings as the vertex set of the graph, and join two vertices by an edge if one is obtained from the other by a single error. The precise definition of error graphs and single errors is given in the next chapter. Here instead is what we are interested in: what is the least number of different distortions of a string one needs to identify the original string? Note it is possible that the original string may be included in these distorted strings.

In Figure 1.3 we show that one needs at least four different distorted strings to reconstruct the original string that is distorted by single errors of transpositional interchanges. Suppose that the original is distorted by three different single errors to 1547, 1754 and 1475. To find the original string one needs to swap two letters in these distorted strings. Clearly, the original string is one of the strings obtained by swapping two letters in 1547, 1754 and 1475. However, as shown below, there are three candidates left, namely 1574, 1457 and 1745.

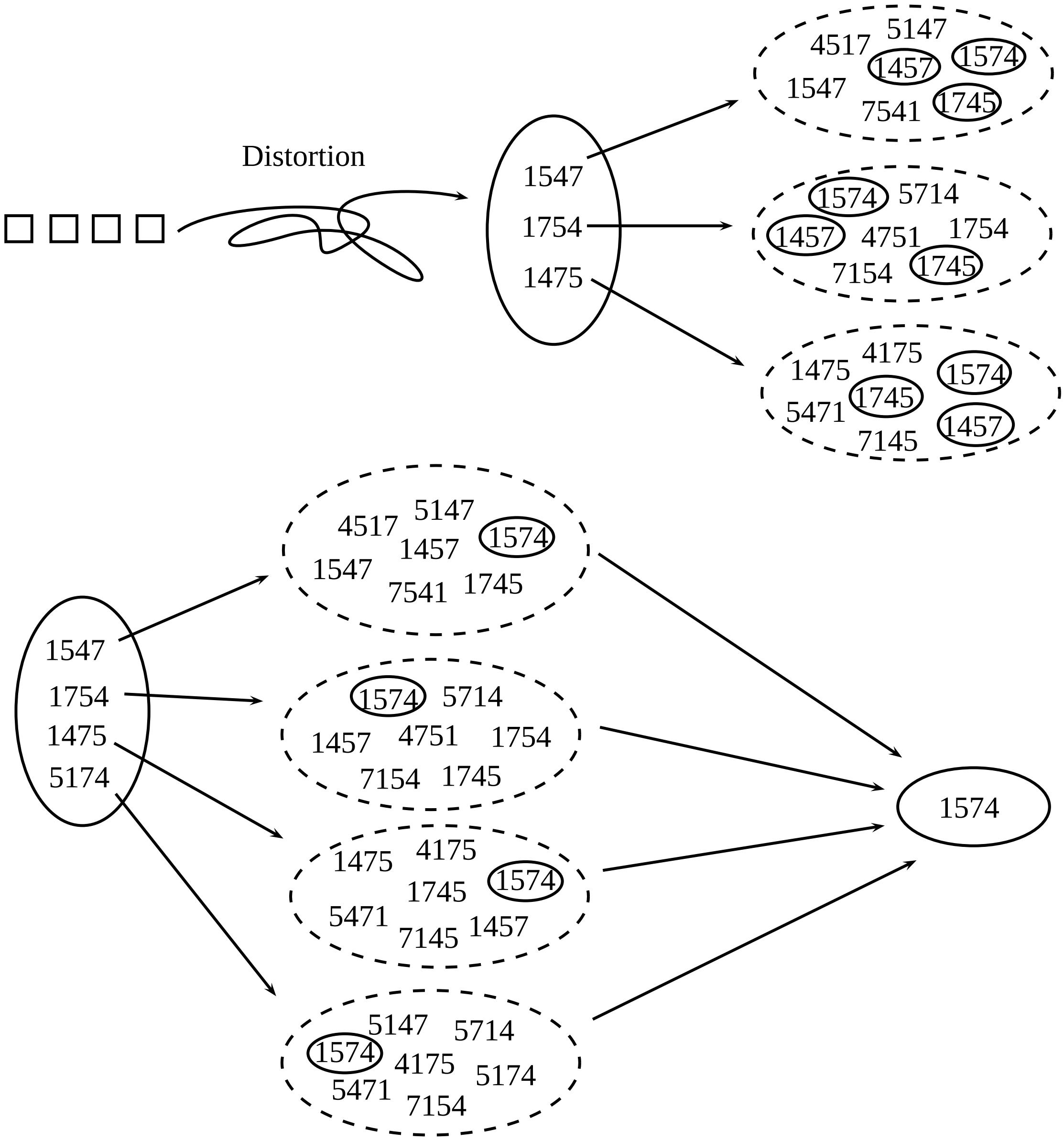

Figure 1.3: Reconstructing a string distorted by single positional interchanges

To reduce the number of candidates we need to be provided with more clues. As soon as another distortion from the original string is added, here 5174, we are able to identify or reconstruct the original string, which is 1574. That is, in this case, one needs at least four distorted strings for tracking down the original string. This is the typical problem of efficient reconstruction: Given units of information −here strings− and specified errors −here single positional interchanges− what is the least number $N+1$ of distorted information units required to restore or reconstruct the original information? Here this number is 4, and Theorem 3.1.1 proves that this is correct for any string and single positional errors.

In real-world applications we cannot exclude that errors are accumulated, as discussed before. So if one specifies an integer $r \geq 1$, the efficient reconstruction problem asks for the least number $N+1$ of strings required to uniquely reconstruct an original string where the strings are obtained by up to r single errors from the original.

With our so far informal definition of error graphs it is now clear that the efficient reconstruction problem can be phrased entirely in the language of graph theory, as follows: Let Γ be a graph. For each $r \geq 0$, we denote by $B_r(\Gamma, u)$, or in brief $B_r(u)$, the *ball of radius* r about the vertex u. That is,

$$B_r(u) = \{v \in \Gamma : d(u, v) \leq r\} \tag{1.1}$$

where $d(u, v)$ is the distance between u and v (see [7] for details). The elements in $B_r(u)$ are the *r-neighbours* of u. Given $r > 0$ we let

$$N(\Gamma, r) := \max_{u \neq v} \{ \, |B_r(u) \cap B_r(v)| \, \}. \tag{1.2}$$

Hence, the number $N = N(\Gamma, r)$ is the largest number of $r-$neighbours that two distinct vertices can have in common. This then is exactly the number N we require for the reconstruction problem: as soon as $N + 1$ or more distinct $r-$neighbours are available, a unique vertex will be restored from these $r-$neighbours. Note the number $N = N(\Gamma, r)$ in (1.2) is called the *intersection number*.

The problem of ball intersection can be studied in any graph, and we will give some references to the literature later. In this thesis we solve this problem when Γ is a Cayley graph on the symmetric and alternating groups where the generators are elements of order two. From the viewpoint of error correction these are precisely the positional interchanges mentioned before.

In our solution of intersection numbers in these Cayley graphs we found that these parameters are very closely related to the Stirling numbers. In fact, one could say that this thesis is also a study of the Stirling recursion which is quite independent of the work on reconstruction. Once again it shows that interesting problems lead to the deeper connections in mathematics.

1.2 The Structure of the Thesis and Results

We begin in Chapter 2 by providing the standard terminology of graphs needed here, including error graphs, Cayley graphs and Stirling numbers. Most the standard materials can be found in [7]. Our main interest will in particular be the Cayley graphs of the symmetric and alternating groups. Also, Stirling numbers of the first and the second kind will be reviewed. This includes their generalizations, r-Stirling numbers. Moreover, the basic background in representation theory as needed here will be provided.

In Chapter 3, as the core motivation of this thesis, the joint work [15] of Levensthein and Siemons on reconstruction numbers in the *transposition Cayley graph* is introduced. All significant results concerning our work will be reviewed thoroughly. In addition, we suggest another point of view to consider the transposition Cayley graph. With this new method we found that there is a tight relation between transposition Cayley graphs and Stirling numbers. Also, an important concept of *ascent* and *descent pattern* will be introduced. This idea of pattern yields us a simple but effective lemma, called the *Cancellation Lemma*. This lemma is the key to our study of the transposition Cayley graph over the symmetric groups and its generalisations in the next chapters. At the end we will discuss the relation between intersection numbers and the representation theory of the symmetric groups.

In Chapter 4 we introduce the *double-transposition Cayley graph* $G'_n(2^2)$, which is the Cayley graph on the alternating group Alt_n generated by all double-transpositions $(\alpha\ \beta)(\gamma\ \delta)$ where all $\alpha, \beta, \gamma, \delta$ are distinct. This graph is considered as a generalisation of the transposition Cayley graph in Chapter 3. Originally, the idea we use in Chapter 3 is a modification of the idea we created for the double-transposition Cayley graphs. Not surprisingly, we found that Stirling numbers, in particular those of the first kind, still play a crucial role in our study of this type of graphs. Here is the main theorem in this chapter:

Theorem 4.2.7 (page 69). Let $r \geq 2$ and let $\Gamma_n = G'_n(2^2)$. We have

$$N(\Gamma_n, r) = \left[\begin{matrix} n \\ n-2r \end{matrix} \right]_{(1\ 2\ 3)} \tag{1.3}$$

if n is sufficiently large.

We define the numbers $\left[\begin{smallmatrix} n \\ n-2r \end{smallmatrix} \right]_{(1\ 2\ 3)}$ of (1.3) in Section 4.2.2. They are closely related to the Stirling numbers of the first kind. With some help from the computer programming **GAP** we have another main result of the case when $r = 2$.

Theorem 4.3.1 (page 74). Let $\Gamma_n = G'_n(2^2)$. For $n \geq 5$ we have

$$N(\Gamma_n, 2) = \left[\begin{matrix} n \\ n-4 \end{matrix} \right]_{(1\ 2\ 3)} = \frac{1}{16}(n^6 - 7n^5 + 5n^4 + 23n^3 + 90n^2 - 112n - 480).$$

In Chapter 5 the *k-transposition Cayley graphs* $G_n(2^k)$ and $G'_n(2^k)$ are introduced. These are the graphs on Sym_n and Alt_n generated by all permutations of the shape $(\alpha_1\ \beta_1)(\alpha_2\ \beta_2)\ldots(\alpha_k\ \beta_k)$ with k odd and even, respectively, where all α's and β's are distinct. The pattern of study will be the same as in Chapter 3 and Chapter 4. Most results concern the asymptotic behaviour of the intersection numbers. However, the larger the graph becomes, the more disciplined it becomes. It seems that when the graph is small, the metric buried inside the graph is quite disorganised. Due to the parity of k, we need to consider the k-transposition Cayley graph separately in two cases. The first is the case of k odd. The main theorem is the following.

Theorem 5.2.6 (page 87). Let $r \geq 3$ and let $k \geq 3$ be an odd integer. Let $\Gamma_n = G_n(2^k)$. Then we have

$$N(\Gamma_n, r) = \begin{bmatrix} n \\ n - rk \end{bmatrix}_{(1\ 2\ 3)} + \begin{bmatrix} n \\ n-(r-1)k \end{bmatrix}_{(1\ 2\ 3)} \tag{1.4}$$

if n is sufficiently large.

For the case of k even, here is the main result.

Theorem 5.2.7 (page 90). Let $\Gamma_n = G'_n(2^k)$ with k even and let $r \geq 2$. We have

$$N(\Gamma_n, r) = N(G'_n(2^2), \frac{rk}{2}) \tag{1.5}$$

if n is sufficiently large.

In the last chapter, we introduce another Cayley graph on the alternating group, namely the 3-cycle Cayley graph where the generating set is the set of all 3-cycles. The graph turns out to be quite different from the previous kinds of generalisation of the transposition Cayley graph that we considered earlier on. Some results about the 3-cycle Cayley graph will be discussed.

Chapter 2

Preliminaries

In this chapter we first give some terminology on graph theory and then provide the definition of error graphs and Cayley graphs. The material concerning graph theory can be found in [7]. Also, some basic topics on group theory we need are introduced here. Later we review Stirling numbers. These numbers play a significant role in this thesis. At the end of this chapter we provide some background knowledge on the representation theory of symmetric groups, especially concerning the class algebra constants.

2.1 Basic Notation on Graphs

A *graph* Γ is a system consisting of a set V of vertices and a set E of edges. Usually, such a graph is denoted by $\Gamma = (V, E)$. We call V the *vertex set* and E the *edge set* of Γ. The edge set E is a set of unordered pairs $\{u, v\}$ of distinct vertices from V. In particular, for us graphs are *simple*, i.e. they are undirected, have no loops and multi-edges. The *order* of a graph, denoted by $|\Gamma|$, is the number of vertices. Two distinct vertices u and v are *adjacent* (or *neighbours of each other*), written $u \sim v$, if $\{u, v\}$ is in E. A vertex u and an edge e are incident if $e = \{u, v\}$ for some v. The *degree* of a vertex u is the number of vertices adjacent to u. A *path* of length n

is a sequence of $n+1$ distinct vertices $u_0, u_1, \ldots, u_n$ of V with $\{u_i, u_{i+1}\}$ in E for all $i = 0, \ldots, n-1$. A graph is *connected* if for every pair u, v of vertices there is a path starting at u and ending at v. A *cycle* of length n is a connected graph of order n such that every vertex has degree two. The *distance* between the vertices u and v in Γ, denoted by $d_\Gamma(u,v)$ or briefly $d(u,v)$, is the length of a shortest path joining u and v. The *diameter* of Γ is the maximum distance of two vertices in Γ. For a non-negative integer r the *sphere* $S_r(\Gamma, u)$ of radius r centred at the vertex u of the graph Γ is the set of vertices v with $d(u,v) = r$, that is

$$S_r(\Gamma, u) = \{v \in V : d(u,v) = r\}. \tag{2.1}$$

Similarly, we let

$$B_r(\Gamma, u) = \{v \in V : d(u,v) \leq r\} \tag{2.2}$$

be the *ball* of radius r centred at u of the graph Γ. For instance, considering Figure 2.1, we have $B_1(K_{3,3}, a) = \{a, b, c, d\}$ and $S_2(K_{3,3}, e) = \{a, f\}$.

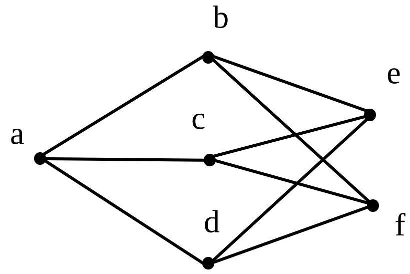

Figure 2.1: The complete bipartite graph $K_{3,3}$.

When there is no ambiguity we sometimes use $B_r(u)$ and $S_r(u)$ to stand for $B_r(\Gamma, u)$ and $S_r(\Gamma, u)$, respectively. An *automorphism* f of a graph $\Gamma = (V, E)$ is a bijection on V satisfying that $\{u, v\}$ is an edge in E if and only if $\{f(u), f(v)\}$ is an edge in E for all edges. A graph is called *vertex-transitive* if for every pair of distinct vertices u and v there is an automorphism f such that $v = f(u)$. It is then clear that every vertex of a vertex-transitive graph has the same degree. A graph whose vertices all have the same degree k is called *k-regular* or briefly *regular*. A vertex-transitive graph is therefore regular. A connected graph Γ is *distance-transitive* if for any ordered pairs (u_1, v_1) and (u_2, v_2) in $V \times V$ with $d(u_1, v_1) = d(u_2, v_2)$ there is an automorphism f of Γ such that $(u_2, v_2) = (f(u_1), f(v_1))$. More generally, a connected graph is *distance-regular* if for each $i \geq 0$ there are constants c_i, a_i, b_i such that for all u, v with $d(u, v) = i$ the number of neighbours of v at distances $i-1, i, i+1$ from u are c_i, a_i, b_i, respectively. By the definition of distance-transitivity, the automorphism group of a distance-transitive graph acts transitively on the set of ordered pairs of vertices. It follows that any distance-transitive graph is distance-regular, but the converse is not true. Another well known class of graphs is the class of strongly regular graphs. A graph is *strongly regular* with parameters (k, λ, μ) if it is k-regular satisfying that pairs of adjacent vertices have exactly λ adjacent vertices in common and pairs of non-adjacent vertices have exactly μ adjacent vertices in common. It becomes clear that any connected strongly regular has diameter two. The Petersen graph is an example of strongly regular graphs with parameters (3,0,1).

2.2 The Notion of Error Graphs

As we discussed in the previous chapter, error graphs appear for a class of reconstruction problems related to symmetrical errors. Here we give a precise definition of error graphs, including single errors and error sets. This material can be found in [15, 17].

Let V be a countable non-empty set and let H be a set of partial one-to-one functions on V whose domain is non-empty. That is, for each element h in H we have that $h : V_h \to V$ is an injective map with non-empty domain $V_h \subseteq V$. The set H is a *single error set*, or briefly an *error set* if

(1) $h(v) \neq v$ for all h in H and v in V_H and

(2) if h belongs to H then the inverse h^{-1} of h belongs to H.

Any element in an error set is called *a single error*.

Definition 1. The graph $\Gamma = (V, E_H)$ is an *error graph* if there is a single error set H such that

$$E_H = \{\{v, h(v)\} : v \in V \text{ and } h \in H\}.$$

Remark: By Condition (1) above, every error graph has no loops, and by Condition (2) they can be considered as undirected graphs.

From the definition, an error graph may not be connected. However, in this thesis we are only interested in connected error graphs, that is, any two vertices u, v can be transformed by a series of single errors, one into the other. The error graph $\Gamma = (V, E_H)$ will therefore be equipped with the metric $d : V \times V \to \mathbb{Z}$ where $d(u, v)$ is the minimum number of single errors used to transform u to v.

Next, we consider some interesting examples of error graphs. The *Hamming space*, written F_q^n, is the set of all n-tuple vectors over the alphabet $F_q = \{0, 1, 2, \ldots, q-1\}$. For convenience, one may think of vectors in F_q^n as words of length n. That is

$$v = (v_1, v_2, \ldots, v_n) = v_1 v_2 \ldots v_n$$

with v_i in F_q for all i. The *Hamming distance* between two vectors u, v is the number of positions that u and v differ in. It can be considered as an error graph by letting the vertex set be the set F_q^n. Two different vertices u and v are adjacent if u and v exactly differ in one position. Then the edge between u and v is referred to as an error distorting one to the other. If we let

$$N(F_q^n, r) := \max_{u \neq v} |B_r(u) \cap B_r(v)|$$

then from [16, 17] it is known that

$$N(F_q^n, r) = q \sum_{i=0}^{r-1} \binom{n-1}{i} (q-1)^i.$$

Hence, it is straightforward that any unknown vertex u can be reconstructed from $N(F_q^n, r)+1$ vertices at distance at most r from u, that is, any word u in F_q^n can be identified from $N(F_q^n, r)+1$ words in F_q^n that differ from u in at most r positions.

Another well known example is the Johnson space. For any $1 \leq w \leq n-1$ the *Johnson space* J_w^n is the set of binary vectors of length n with Hamming weight w. The *Hamming weight* is the sum of unities that appear in the binary vector. The *Johnson distance* is half the Hamming distance. For example, 1111000 and 1001101 belong to J_4^7, and the distance between these vectors is two. It is clear that the Hamming distance between two vectors in J_w^n is even. Again, from [16, 17] we have that

$$N(J_w^n, r) = n\sum_{i=0}^{r-1}\binom{w-1}{i}\binom{n-w-1}{i}\frac{1}{i+1}$$

is the maximum number of vectors at distance at most r that any two vectors can have in common. That is, any vector u in J_w^n can be identified from its $N(J_w^n, r)+1$ neighbours at distance at most r.

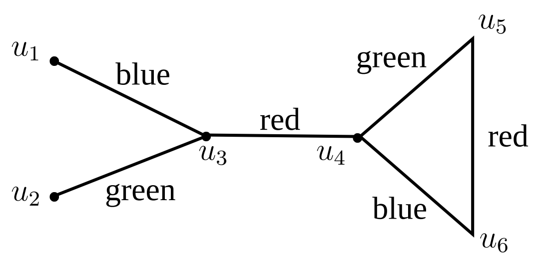

Figure 2.2: A 3-edge-colouring of a graph

For anyone who is interested in efficient reconstruction problem, more material can be found in [18, 19]. We now introduce the edge colouring. Let Γ be a graph and let K be a set. A function $\kappa : E(\Gamma) \to K$ is an *edge colouring* if $\kappa(e_1) \neq \kappa(e_2)$ whenever e_1 and e_2 are adjacent. For any positive integer k, a k*-edge-colouring* is an edge colouring $\kappa : E(\Gamma) \to \{1, 2, \ldots, k\}$. Also, the elements in the image set are considered as *colours*. Here we discuss the connection between errors and edge colourings. In Figure 2.2 the graph accompanied with its colouring corresponds one-to-one to an error graph Γ in a sense that each colour is considered as a single error. For example, u_3 is the vertex distorted from the vertex u_4 by the single error 'red', and vice versa. This implies that error graphs and edge colourings actually are the same.

2.3 Cayley Graphs

Cayley graphs occur as an important class of error graphs. We survey these here.

Let G be a non-trivial group and let H be a generating set of G satisfying the following conditions:

(1) $H = H^{-1}$ with $H^{-1} = \{h^{-1} : h \in H\}$ and

(2) H does not contain the identity element e of G.

The *Cayley graph* $\Gamma_H := (G, E_H)$ on the group G with the generating set H is the graph whose vertex set is G and the edge set E_H is defined as follows: Two vertices g and g' are adjacent in Γ_H if and only if $g' = gh$ for some h in H. With Condition (1) throughout this thesis we then consider the graph Γ_H as an undirected graph with the edge set E_H defined by

$$E_H := \{\{g, g'\} : g^{-1}g' \in H\}.$$

Also, from Condition (2) we have that the graph Γ_H has no loops. In the next proposition we state well-known results on Cayley graphs.

Proposition 2.3.1. *Let Γ_H be the Cayley graph on a group G with the generating set H. Then Γ_H is a connected regular graph of degree $|H|$. In particular, it is vertex-transitive.*

2.3.1 Basic Definitions from Permutation Groups, and Symmetric Groups in particular

For any positive integer n we let $[n] := \{1, 2, \ldots, n\}$. Without loss of generality a *permutation* g of n letters is a bijection on the set $[n]$. We denote by Sym_n the set of all permutations g of n letters. As is well known, the set is actually a group, called the *symmetric group* Sym_n of *degree* n. Note that throughout this thesis, the permutations act on the right. That is, $\alpha(gg') = (\alpha g)g'$ or more customarily $\alpha^{gg'} = (\alpha^g)^{g'}$ for all α in $[n]$ and g, g' in Sym_n. Hence, the product of $(1\ 2)$ and $(1\ 3)$ is $(1\ 2)(1\ 3) = (1\ 2\ 3)$, not $(1\ 3\ 2)$. There are quite a few ways to represent permutations. The first method is by representing the permutations g as arrays:

$$g := \begin{pmatrix} 1 & 2 & \ldots & n \\ 1^g & 2^g & \ldots & n^g \end{pmatrix}. \tag{2.3}$$

The permutation g in (2.3) maps i to i^g for all $i = 1, \ldots, n$. Another common representation of permutations is the *disjoint cycle decomposition.* Throughout this thesis, we represent permutations as the disjoint cycle decomposition in the following way: For each permutation g, put 1 in the first cycle followed by 1^g, 1^{g^2} and so on until we get 1 again. The first cycle of g then is $(1\ 1^g\ 1^{g^2}\ \ldots)$. If there is a number not belonging to the first cycle we then put the minimum of the remainder in a new cycle, and then repeat the process as we did before. Continuing in this fashion, each cycle in the cycle decomposition of g will start with the smallest number appearing in its. Also, the leading numbers of all cycles will be arranged in ascending order.

For example,

$$g := \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 3 & 8 & 5 & 4 & 1 & 2 & 7 \end{pmatrix} = (1\ 6)(2\ 3\ 8\ 7)(4\ 5). \tag{2.4}$$

The permutation g in (2.4) consists exactly of cycles of length two and four. Let g be a permutation in Sym_n. The *support* of g, written $Supp(g)$, is the set of elements in

$[n]$ that are not fixed by g. The $Fix(g)$ is the set of elements in $[n]$ fixed by g. Also, we let $supp(g)$ and $fix(g)$ be the number of non-fixed elements and fixed elements of g, respectively.

Definition 2. Given a permutation g with h_i cycles of length i for all $i = 1, \ldots, n$, the *cycle type* of g is

$$ct(g) := 1^{h_1} 2^{h_2} \cdots n^{h_n}.$$

A *transposition* is a permutation g containing a cycle of length two where the other cycles are all of length one. That is, g is a transposition if and only if $ct(g) = 1^{n-2}2^1$. Also, we denote by $|\,g|$ the number of cycles appearing in the usual cycle decomposition of g, including the cycles of length one. Hence

$$|\,g| = \sum_{i=1}^{n} h_i \tag{2.5}$$

and also

$$n = \sum_{i=1}^{n} i h_i. \tag{2.6}$$

Two permutations g and g' in Sym_n are *conjugate*, written $g \sim g'$, if $g = t^{-1} g' t$ for some permutation t in Sym_n. It is well known that the set of elements in Sym_n conjugate to g is the set of elements in Sym_n having the same cycle type as g. If $ct(g) = 1^{h_1} 2^{h_2} \cdots n^{h_n}$ we let $g^{\mathrm{Sym}_n} = (1^{h_1} 2^{h_2} \cdots n^{h_n})^{\mathrm{Sym}_n}$ be the conjugacy class of Sym_n to which g belongs. As is well known, any permutation g can be expressed as a product of transpositions. A permutation g is *even* if g is a product of even number of transpositions, otherwise g is *odd*. The set Alt_n of all even permutations of n letters is a group, called the *alternating group*.

Theorem 2.3.2 ([23], p. 18). *The number of permutations of cycle type* $1^{h_1} 2^{h_2} \cdots n^{h_n}$ *is*

$$\frac{n!}{h_1! h_2! \cdots h_n! 1^{h_1} 2^{h_2} \cdots n^{h_n}}.$$

2.3.2 Cayley Graphs on the Symmetric Group

Here we introduce Cayley graphs on the symmetric group. Of particular interest are the Cayley graphs generated by the set of all transpositions. We start with a lemma.

Lemma 2.3.3. *Let H be a union of conjugacy classes on $G_n = Sym_n$ and let $r \geq 0$. Suppose that C is the group of inner automorphisms of Sym_n, i.e. $C = \{\sigma_g : g \in G_n\}$ where $u\sigma_g = g^{-1}ug$ for all u in G_n. If Γ_H is the Cayley graph of Sym_n with generating set H, then the sphere $S_r(\Gamma_H, e)$ is a union of C-orbits.*

Proof. Let u_r be an element in $S_r(\Gamma_H, e)$. Suppose that $P := e, u_1, u_2, \ldots, u_r$ is a shortest path from e to u_r with u_k in $S_k(\Gamma_H, e)$ for all $1 \leq k \leq r$. Then, for each g in G_n, the path

$$P_g := e\sigma_g = e, u_1\sigma_g, \ldots, u_r\sigma_g$$

must be a shortest path from e to $u_r\sigma_g$. This implies that $S_r(e)$ is a union of C-orbits. □

From the above lemma one can see that if H is a union of Sym_n– conjugacy classes then every sphere is a union of conjugacy classes of Sym_n. That is, it consists exactly of all permutations having the same cycle type. Note that Lemma 2.3.3 can be generalised to any group G and any automorphism group C of Γ_H.

Remark: In any group G, the trivial conjugacy class is $\{e\}$ where e is the identity element.

Let $H = \bigcup C_i$ be a union of conjugacy classes of Sym_n. If $h = h_1 \cdots h_j$ with h_k in C_i for some i then $g^{-1}hg = (g^{-1}h_1g)\cdots(g^{-1}h_jg)$. Clearly, $g^{-1}h_kg$ and h_k have the same cycle type, i.e. they are in the same conjugacy class. Hence $g^{-1}hg$ is in $\langle H\rangle$. It follows that $\langle H\rangle$ is normal in Sym_n. It is well known that Alt_n is simple when $n \geq 5$, that is, Alt_n has no non-trivial normal subgroup. We then have that:

Proposition 2.3.4. *Let $n \geq 5$ and let $H \neq \{e\}$ be a union of conjugacy classes of Sym_n. If H contains an odd permutation then $\langle H \rangle = Sym_n$. Otherwise, $\langle H \rangle = Alt_n$.*

In order to generate the symmetric group Sym_n by a set of transpositions, a smallest generating set must be of size $n-1$ as the longest cycles we need to extract are those conjugate to $(1\ 2\ \ldots n)$. Of collections of $n-1$ permutations that generate the whole symmetric group Sym_n, the set $H_b = \{(1\ 2), (2\ 3), \ldots, (n-1\ n)\}$ of *bubble-sort transpositions* and the set $H_{st} = \{(1\ 2), (1\ 3), \ldots, (1\ n)\}$ of *prefix-transpositions* are well known. On the other extremal case, the set $(2^1)^{G_n}$ of all transpositions clearly is the biggest collection of transpositions generating the symmetric group. The Cayley graph Γ_{H_b} is called the *bubble-sort Cayley graph* and the Cayley graph $\Gamma_{H_{st}}$ is the *star Cayley graph*. The details of these Cayley graphs can be found in [14].

In the remainder our work will be devoted to the Cayley graph generated by conjugacy classes of involutions – permutations of order two. Basically, all the graphs we are interested in are based on the *transposition Cayley graph*, written $G_n(2^1)$, on Sym_n. More precisely, $G_n(2^1)$ is the Cayley graph $\Gamma_H := (G_n, E_H)$ where H is the set of all transpositions and $G_n = \mathrm{Sym}_n$. The symbols '2^1' and 'G_n' refer to the set of transpositions and the symmetric group Sym_n, respectively. Figure 2.3 shows the transposition Cayley graph $G_4(2^1)$ on Sym_4. It is clear that the graph is not distance-regular and therefore not distance-transitive. Both $(1\ 2\ 3)$ and $(1\ 2)(3\ 4)$ are in $S_2(e)$. The former is adjacent to three elements in $S_1(e)$ while the latter only has two neighbours in $S_1(e)$.

Proposition 2.3.5 ([15], p.806). *Let Γ_n be the transposition Cayley graph $G_n(2^1)$ on Sym_n with identity element e. Then the sphere $S_i(\Gamma_n, e)$ consists exactly of all permutations having $n-i$ cycles.*

Proof. Suppose that $x = (\alpha_1\ \alpha_2 \ldots \alpha_i)(\beta_1\ \beta_2 \ldots \beta_j)$ is a permutation consisting of

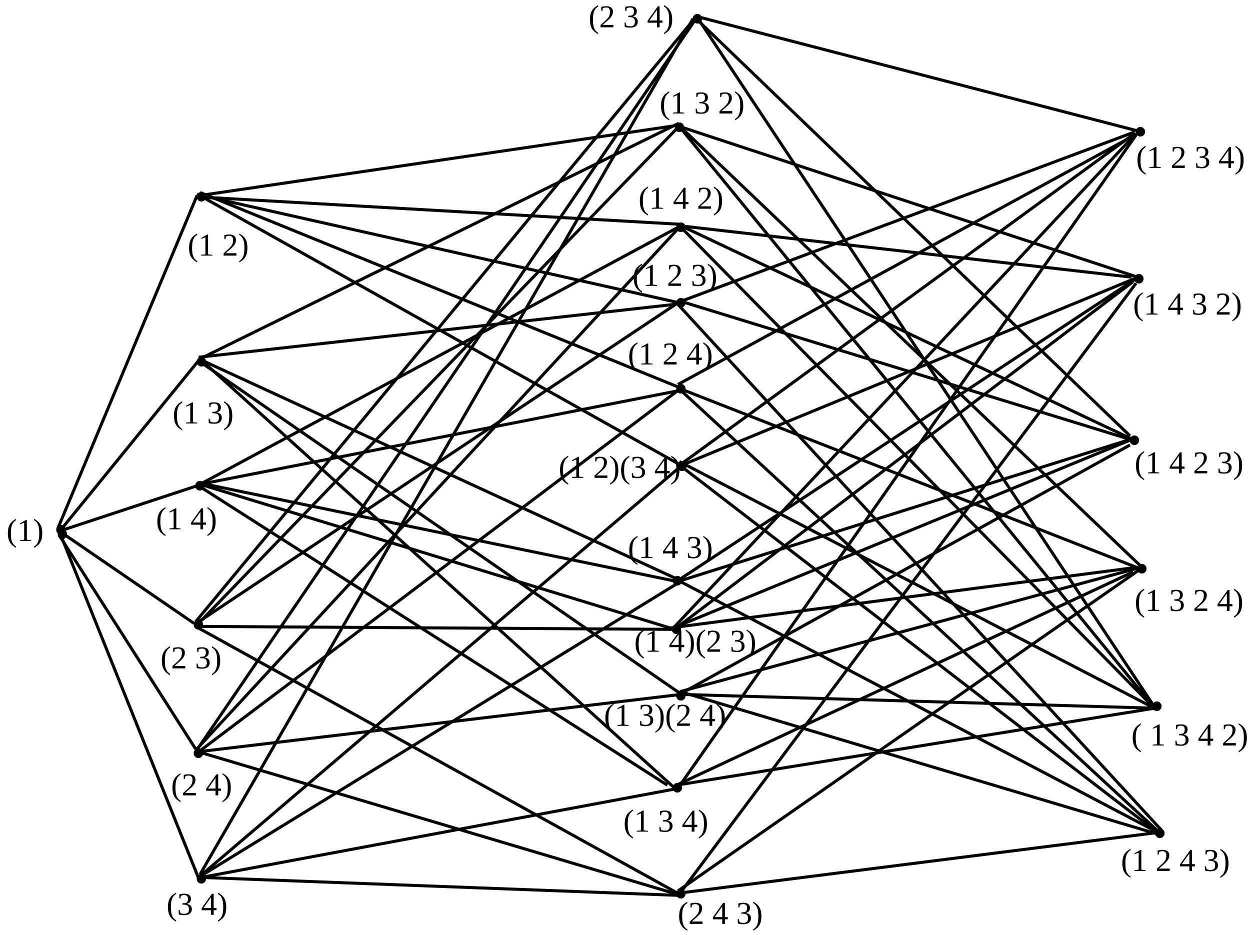

Figure 2.3: The transposition Cayley graph on Sym_4

two disjoint cycles. Let $h = (\alpha_1\ \beta_1)$ be a transposition. Multiplying x by h is just gluing the two disjoint cycles of x together, namely,

$$xh = (\alpha_1\ \alpha_2 \dots \alpha_i\ \beta_1\ \beta_2 \dots \beta_j) =: y \tag{2.7}$$

On the other hand, when y is multiplied by h then $yh = xhh = x$, that is,

$$yh = (\alpha_1\ \alpha_2 \dots \alpha_i\ \beta_1\ \beta_2 \dots \beta_j)(\alpha_1\ \beta_1) = x. \tag{2.8}$$

Literally, when a permutation is multiplied by a transposition this amounts to either gluing two disjoint cycles together or separating a cycle into two cycles. The proof is now complete by induction as we start at the identity, for which the number of cycles is n. $\square$

From the above proposition we have that for any permutation g the distance $d(e,g)$ is at most $n-1$. Hence by Proposition 2.3.1 we can conclude that:

Proposition 2.3.6 ([15], p. 805). *For any $n \geq 3$ the transposition Cayley graph $G_n(2^1)$ on Sym_n is a connected $\binom{n}{2}$-regular graph of order $n!$ with diameter $n-1$.*

2.3.3 k-Transposition Cayley Graphs

In the previous section we have discussed the transposition Cayley graph $G_n(2^1)$, which is the Cayley graph on the symmetric group $G_n = \mathrm{Sym}_n$ generated by the set $(2^1)^{G_n}$ of all transpositions. Here we introduce other kinds of Cayley graphs on the symmetric and alternating groups. A permutation g whose cycle type is $ct(g) = 1^{n-2k}2^k$ is called a *k-transposition*. For instance, a 2-transposition is a double-transposition $(\alpha\ \beta)(\gamma\ \delta)$ where $\alpha, \beta, \gamma, \delta$ are distinct. From Proposition 2.3.4 one can see that the conjugacy class $(2^k)^{G_n}$ of all k-transpositions will generate the symmetric group if k is odd. On the other hand, it generates the alternating group if k is even and greater than four. These Cayley graphs on Sym_n and Alt_n generated by $(2^k)^{G_n}$ are called *k-transposition Cayley graphs*.

2.4 Stirling Numbers

In this section the well known (signless) Stirling numbers are introduced. There are two types of these numbers, the first and the second kind. Most of this thesis is in fact devoted to the Stirling numbers of the first kind. The numbers provide a framework of this thesis as they deeply relate to the transposition Cayley graph, which is the starting point in our research. We will discuss their recursion and a generalisation of them, the r-Stirling numbers. Note that the signed types of these numbers will be omitted.

We first introduce Stirling numbers of the first kind. Let n and k be positive integers. The *Stirling number* $\left[{n \atop k} \right]$ *of the first kind* is the number of permutations of $[\,n\,]$ consisting of k cycles, including fixed points. That is,

$$\left[{n \atop k} \right] = |\, \{\; g \in \mathrm{Sym}_n : \;\; |\, g\,| = k \;\}\, | \,. \tag{2.9}$$

Note that in some literature the authors may use another notation for Stirling numbers of the first kind, for instance $c(n,k)$ and $(-1)^k s(n,k)$. We can express them as the coefficients of y^k in the *rising factorial function*

$$y^{[n]} := y(y+1)(y+2)\ldots(y+n-1). \tag{2.10}$$

That is, given a positive integer n, we have the generating function

$$\sum_{k=1}^{n} \left[{n \atop k} \right] y^k = y(y+1)(y+2)\ldots(y+n-1) \tag{2.11}$$

We next show that the numbers $\left[{n \atop k} \right]$ are endowed with a recurrence which eventually becomes a common recurrence for new families of numbers we invent in the next chapters.

Proposition 2.4.1. *For positive integers n and k with $2 \leq k \leq n$ the Stirling numbers $\left[{n \atop k} \right]$ of the first kind satisfy the recurrence*

$$\left[{n \atop k} \right] = \left[{n-1 \atop k-1} \right] + (n-1) \left[{n-1 \atop k} \right] \tag{2.12}$$

with the initial conditions $\left[{n \atop k} \right] = 0$ if $k > n$ and $\left[{n \atop 1} \right] = (n-1)!$.

The recurrence (2.12) is well known but we use it many times. It would then be nice to prove it here. Before we do this, let us introduce a simple, but beautiful idea of embedding Sym_n into Sym_{n+1}. For each $j = 0, 1, \ldots, n$ we have an *insert operation* $i_j : \mathrm{Sym}_n \to \mathrm{Sym}_{n+1}$ which puts '$n+1$' after the number j in our standard cycle decomposition of the permutations in Sym_n. The operation i_0 will put '$n+1$'

in a new cycle. For example, let $g = (1\ 6\ 3)(2\ 5)(4)$ belong to Sym_6. For each j, considering $i_j : \mathrm{Sym}_6 \to \mathrm{Sym}_7$, we have

$$
\begin{aligned}
i_0(g) &= (1\ 6\ 3)(2\ 5)(4)(7), & i_4(g) &= (1\ 6\ 3)(2\ 5)(4\ 7), \\
i_1(g) &= (1\ 7\ 6\ 3)(2\ 5)(4), & i_5(g) &= (1\ 6\ 3)(2\ 5\ 7)(4), \\
i_2(g) &= (1\ 6\ 3)(2\ 7\ 5)(4), & i_6(g) &= (1\ 6\ 7\ 3)(2\ 5)(4), \\
i_3(g) &= (1\ 6\ 3\ 7)(2\ 5)(4).
\end{aligned}
$$

In the other way around we have the converse operation $i^* : \mathrm{Sym}_{n+1} \to \mathrm{Sym}_n$ which deletes '$n+1$' from the permutations in Sym_{n+1}. These corresponding operations are highly significant for our work and the idea will be thoroughly discussed in Section 3.2.3. Throughout this thesis, these insert operations i_j will act in the following sense: They map a permutation g to $i_j(g)$ in Γ_{n+1} if g is in Γ_n. In the other way around, $i^*(g)$ is in Γ_{n-1} if g belongs to Γ_n. For example, if g is in Γ_5 then $i_j(g)$ is in Γ_6 while $i^*(g)$ is in Γ_4. Now we can prove Proposition 2.4.1.

Proof. Let n and k be positive integers. The number of permutations in Sym_n consisting exactly of one cycle is equal to $(n-1)!$. This accounts for $\left[{n \atop 1} \right]$. Clearly, $\left[{n \atop k} \right] = 0$ if $k > n$. For the rest we suppose that $2 \leq k \leq n$. Let Z be the set counted by $\left[{n \atop k} \right]$. We can partition Z into two disjoint subsets. The first, say X, contains all permutations in Z fixing n. The other set Y consists of permutations in Z not fixing n. Every permutation g in X corresponds one-to-one to $i_0^{-1}(g)$. This accounts for $\left[{n-1 \atop k-1} \right]$. In the other case, if we let $T = i^*(Y)$ be the set obtained from Y by deleting n from all elements in Y then Y can be partitioned into $n-1$ mutually disjoint parts, namely $i_1(T), i_2(T), \ldots, i_{n-1}(T)$. This accounts for $(n-1)\left[{n-1 \atop k} \right]$ since $|T| = \left[{n-1 \atop k} \right]$. The proof is then complete. $\square$

In Table 2.1 we show some first Stirling numbers of the first kind. Next we introduce another kind of Stirling numbers, those of the *second kind*. They are the

n	k				
	1	2	3	4	5
1	1	0	0	0	0
2	1	1	0	0	0
3	2	3	1	0	0
4	6	11	6	1	0
5	24	50	35	10	1
$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$

Table 2.1: Stirling numbers of the first kind

number of ways that the set $[n]$ can be partitioned into k parts, written $\left\{ {n \atop k} \right\}$. Recall that $[n] = \{1, 2, \dots, n\}$. One can see that the numbers $\left\{ {n \atop k} \right\}$ are equal to the number of conjugacy classes of Sym_n whose elements have k cycles. As is well known, they satisfy the recursion (2.13). Some of these numbers are shown in Table 2.2.

Proposition 2.4.2. *For positive integers n and k with $2 \leq k \leq n$ the Stirling numbers $\left\{ {n \atop k} \right\}$ of the second kind satisfy the recurrence*

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\} \tag{2.13}$$

with the initial conditions $\left\{ {n \atop k} \right\} = 0$ if $k > n$, and $\left\{ {n \atop 1} \right\} = 1$.

Proof. Let $2 \leq k \leq n$ and let Z be the set of partitions of n with k parts. We can divide Z into two piles, say X and Y. The former consists of all partitions whose class containing n has size one. The latter contains the remainder, that is, $Y = Z \setminus X$. Each partition λ in X corresponds to a partition of $n-1$ obtained from λ by deleting the class of n, and vice versa. This accounts for $\left\{ {n-1 \atop k-1} \right\}$. Let γ belong to Y. Then after deleting n from γ the number of classes of γ is the same. There are k partitions of n that yields the same partition of $n-1$ after deleting n, and this accounts for $k\left\{ {n-1 \atop k} \right\}$. The initial conditions are clear. $\square$

Now we introduce a generalisation of Stirling numbers of the first kind, r-Stirling numbers of the first kind. Recall that $[r] = \{1, 2, \dots, r\}$. Let $r \geq 1$. The r*-Stirling*

n	k				
	1	2	3	4	5
1	1	0	0	0	0
2	1	1	0	0	0
3	1	3	1	0	0
4	1	7	6	1	0
5	1	15	25	10	1
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$

Table 2.2: Stirling numbers of the second kind

number $\left[{n \atop k} \right]_r$ *of the first kind* is the number of permutations in Sym_n having k cycles such that all elements in $[r]$ are in different cycles. Similarly, the r*-Stirling number* $\left\{ {n \atop k} \right\}_r$ *of the second kind* is the number of ways to partition $[n]$ into k parts where elements in $[r]$ are in different parts. Clearly, $\left[{n \atop k} \right]_1 = \left[{n \atop k} \right]$ and $\left\{ {n \atop k} \right\}_1 = \left\{ {n \atop k} \right\}$. Not surprisingly, these r-Stirling numbers have the same recursion as the ordinary Stirling numbers but with different initial conditions. That is, for any $r \leq k \leq n$ we have

$$\left[{n \atop k} \right]_r = \left[{n-1 \atop k-1} \right]_r + (n-1) \left[{n-1 \atop k} \right]_r \tag{2.14}$$

with initial conditions $\left[{n \atop k} \right]_r = 0$ if $k > n$ or $k < r$, $\left[{r \atop r} \right]_r = 1$. Further,

$$\left\{ {n \atop k} \right\}_r = \left\{ {n-1 \atop k-1} \right\}_r + k \left\{ {n-1 \atop k} \right\}_r \tag{2.15}$$

with initial conditions $\left\{ {n \atop k} \right\}_r = 0$ if $k > n$ or $k < r$, $\left\{ {r \atop r} \right\}_r = 1$. Good introductory papers on these numbers are [2, 3, 4, 20]. Tables 2.3 and 2.4 show some of those numbers collected from [4]. From the definition one needs to start at $n = 2$ for 2-Stirling numbers, and at $n = 3$ for 3-Stirling numbers.

In the next chapter we will introduce a family of numbers related to the ordinary Stirling numbers and the r-Stirling numbers.

n	k				
	2	3	4	5	6
2	1	0	0	0	0
3	2	1	0	0	0
4	6	5	1	0	0
5	24	26	9	1	0
6	120	154	71	14	1
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$

Table 2.3: 2-Stirling numbers of the first kind

n	k				
	3	4	5	6	7
3	1	0	0	0	0
4	3	1	0	0	0
5	9	7	1	0	0
6	27	37	12	1	0
7	81	175	97	18	1
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$

Table 2.4: 3-Stirling numbers of the second kind

2.5 Poincaré Polynomials

For a given graph Γ and a vertex v in Γ we let

$$\Pi_{\Gamma,v}(y) := \sum_{i\geq 0} s_i y^i \tag{2.16}$$

be the *Poincaré polynomial* of Γ. Here s_i is the size of the sphere $S_i(v)$. When the polynomial in (2.16) is independent of v we simply write

$$\Pi_{\Gamma}(y) := \Pi_{\Gamma,v}(y). \tag{2.17}$$

For instance, in the transposition Cayley graph $\Gamma_n = G_n(2^1)$, if we let

$$g(y) = y^{[n]} := y(y+1)\cdots(y+(n-1)),$$

then from Propositions 2.3.5 and 2.3.6 we have

$$\Pi_{\Gamma_n}(y) = \sum_{i=0}^{n-1} s_i y^i = \sum_{i=0}^{n-1} \begin{bmatrix} n \\ n-i \end{bmatrix} y^i. \tag{2.18}$$

Recall from (2.11) that

$$\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} y^k = y(y+1)(y+2)\dots(y+n-1).$$

Therefore, substituting $n-i$ with k in (2.18), we have

$$\Pi_{\Gamma_n}(y) = \sum_{i=0}^{n-1} \begin{bmatrix} n \\ n-i \end{bmatrix} y^i = \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} y^{n-k} = y^n g(y^{-1}).$$

Now one can see that the Poincaré polynomial and the rising factorial function are associated with Stirling numbers.

2.6 Ball Intersection Numbers

Let $r \geq 1$. Given a graph Γ the *ball intersection number*, or in brief, *intersection number* is

$$N(\Gamma, r) := \max_{u \neq v} |B_r(u) \cap B_r(v)|. \tag{2.19}$$

When the graphs of interest are Cayley graphs, we make use of the vertex transitivity of these graphs to reduce the index set $u \neq v$ in (2.19). Fix $r \geq 1$ and let G be a group with generating set H. Let $\Gamma = \Gamma_H$. Denote $B_r := B_r(e)$ and $S_r := S_r(e)$ where e is the identity element of G. By the vertex transitivity one can reduce (2.19) to

$$N(\Gamma, r) = \max_{u \neq e} |B_r \cap B_r(u)|. \tag{2.20}$$

Further, it is easy to see that

$$N(\Gamma, r) = \max_{1 \leq i \leq 2r} N_i(\Gamma, r) \tag{2.21}$$

where $N_i(\Gamma, r) = \max_{u \in S_i} |B_r \cap B_r(u)|$. Note that if $d(e, u) > 2r$ then $B_r \cap B_r(u) = \emptyset$.

Next, suppose that $\{v, vh\}$ is an edge in Γ. Then

$$u\{v, vh\} := \{uv, u(vh)\} = \{uv, (uv)h\} \tag{2.22}$$

clearly is an edge in Γ. Hence,

$$uB_r := \{u\} \cdot B_r = B_r(u) \qquad \text{and} \qquad uS_r := \{u\} \cdot S_r = S_r(u). \tag{2.23}$$

On the other hand, if we suppose that H is a union of conjugacy classes of G then

$$B_r u := B_r \cdot \{u\} = B_r(u) \qquad \text{and} \qquad S_r u := S_r \cdot \{u\} = S_r(u). \tag{2.24}$$

This is because

$$\{v, vh\}u := \{vu, (vh)u\} = \{vu, vu(u^{-1}hu)\} \tag{2.25}$$

is an edge in Γ. In addition, for any permutation g we have

$$g^{-1}(B_r \cap uB_r)g = (g^{-1}B_r g) \cap (g^{-1}uB_r g) = B_r \cap (g^{-1}ug)B_r$$

and

$$g^{-1}(B_r \cap B_r u)g = (g^{-1}B_r g) \cap (g^{-1}B_r ug) = B_r \cap B_r(g^{-1}ug).$$

This shows that if H is a union of conjugacy classes, then the functions $f^r, f_r : G \to \mathbb{C}$ defined by

$$f^r(u) = |B_r \cap uB_r| \qquad \text{and} \qquad f_r(u) = |B_r \cap B_r u| \tag{2.26}$$

are class functions for any fixed $r \geq 1$. Consequently, for any g in G we have

$$f^r(u) = f^r(g^{-1}ug) = |B_r \cap B_r(u)| = f_r(u) = f_r(g^{-1}ug). \tag{2.27}$$

Now from (2.26) and (2.27) we can reduce (2.20) to

$$N(\Gamma, r) = \max_{u \in R} |B_r \cap B_r(u)| \tag{2.28}$$

where R is a set of representatives of all non-trivial cycle types in Sym_n.

As we have seen in Proposition 2.3.1, all Cayley graphs are vertex-transitive, and hence they are regular. Before we move to the next section it would be nice to have a glance on some bounds of intersection numbers of general regular graphs. The results are based on a generalisation of the parameters λ and μ that are studied in strongly regular graphs. Recall that a strongly regular graph Γ is a regular graph such that any two adjacent vertices have λ neighbours in common and any two non-adjacent vertices have μ neighbours in common. Here we introduce a generalisation of this property of strongly regular graphs to any regular graph by letting

$$\lambda := \max_{v \sim v'} |\{u : d(u,v) = d(u,v') = 1\}|, \tag{2.29}$$

$$\mu := \max_{v \not\sim v'} |\{u : d(u,v) = d(u,v') = 1\}|. \tag{2.30}$$

Then we have

$$N_1(\Gamma,1) = \lambda + 2 \quad \text{and} \quad N_2(\Gamma,1) = \mu.$$

Hence, from (2.21) one can see that

$$N(\Gamma,1) = \max\{\lambda+2, \mu\}. \tag{2.31}$$

In [15], in order to bound intersection numbers of the graphs of interest, Levenshtein and Siemons studied the relation between these parameters and the intersection numbers, and gave the results shown in the following theorems.

Theorem 2.6.1 ([15], p. 800). *For any k-regular graph Γ with $k \leq |\Gamma| - 2$ we have*

$$N(\Gamma,1) \leq \frac{1}{2}(|\Gamma| + \lambda). \tag{2.32}$$

Theorem 2.6.2 ([15], p. 801). (Linear Programming Bound) *For any strongly k-regular graph Γ with $k \geq 2$ we have*

$$N_2(\Gamma,2) \geq \mu\left(k - 1 - \frac{1}{2}(\mu-1)\left(N(\Gamma,1)-2\right)\right) + 2. \tag{2.33}$$

Remark: In Inequation (2.33) above, $N_2(\Gamma, 2) = \max_{d(u,v)=2} |B_2(u) \cap B_2(v)|$. This is the general definition of $N_i(\Gamma, r)$ which is defined to be $\max_{d(u,v)=i} |B_r(u) \cap B_r(v)|$.

2.7 Representation Theory

According to the definition of Cayley graphs, for each $r \geq 1$ we have

$$B_r = \{e\} \cup H \cup H^2 \cup H^3 \cup \ldots \cup H^r.$$

Then if H is a union of conjugacy classes of the symmetric group, then B_r will concern the product of conjugacy classes of the symmetric group. To this study, we need to deploy the representation theory.

Let G be a finite group and let $F = \mathbb{R}$ or $\mathbb{C}$. We denote by $GL(n, F)$ the group of invertible matrices with entries in F. A homomorphism $\rho : G \to GL(n, F)$ is called *a representation* of G over F with *degree* n. Further, the *character* χ of a representation ρ is the function from G into F defined by $\chi(g) = tr(g\rho)$ for all g in G. That is, the value $\chi(g)$ is the sum of all entries in the NW-diagonal line of $g\rho$. Note that we write characters as functions acting on the left. Clearly, if e is the identity of G then $\chi(e)$ is equal to the degree of the representation. In the rest of this thesis we are interested in the case $F = \mathbb{C}$.

Two representations ρ_1 and ρ_2 are *equivalent* if there is an invertible matrix P such that $g\rho_1 = P^{-1}(g\rho_2)P$ for all g in G. A representation ρ is *reducible* if there is an invertible matrix P such that for each g in G,

$$g\rho = P^{-1} \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} P$$

for some square matrices A and B; otherwise ρ is *irreducible*. Also, if χ is the character of a representation ρ we say that χ is *reducible* if ρ is reducible; otherwise we say that χ is *irreducible.* In representation theory, characters play a significant role as we need to keep track only of one number instead of the n^2 numbers in each $g\rho$. In the remainder of this section we provide some known results of the characters, collecting from [11].

Proposition 2.7.1. (1) *Any equivalent representations have the same character.* (2) *If g_1 and g_2 are elements in G belonging to the same conjugacy class then $\chi(g_1) = \chi(g_2)$ for all characters χ of G.*

Proposition 2.7.2. *The number of all irreducible characters is equal to the number of all conjugacy classes.*

Let ϑ and ϕ be functions from G to $\mathbb{C}$. The inner product of ϑ and ϕ is defined by

$$\langle \vartheta, \phi \rangle = \frac{1}{|G|} \sum_{g \in G} \vartheta(g) \overline{\phi(g)}. \tag{2.34}$$

Proposition 2.7.3. *Let $\chi_1, \ldots, \chi_m$ be the irreducible characters of G. If ϑ is a character then*

$$\vartheta = d_1 \chi_1 + \ldots + d_m \chi_m$$

where $d_i = \langle \vartheta, \chi_i \rangle$ are non-negative integers. In addition, ϑ is irreducible if and only if $\langle \vartheta, \vartheta \rangle = 1$.

Now we introduce systems in group theory that are called *group algebras.* First we define the vector space $\mathbb{C}G$ over $\mathbb{C}$ that has all elements g in G as its basis. The addition and scalar product are defined naturally, that is, if $g_1, \ldots, g_n$ are all elements of G, and $x = \sum \lambda_i g_i$ and $y = \sum \mu_i g_i$ then

$$x + y = \sum (\lambda_i + \mu_i) g_i$$

and

$$\lambda x = \sum (\lambda \lambda_i) g_i$$

for all λ in $\mathbb{C}$. The *group algebra* $\mathbb{C}G$ is the vector space $\mathbb{C}G$ equipped with multiplication defined by

$$(\sum \lambda_g g)(\sum \mu_h h) = \sum (\lambda_g \mu_h) gh$$

where λ_g, μ_h are in $\mathbb{C}$.

Next let $C_1, \ldots, C_m$ be all the conjugacy classes of G. For each $1 \leq i \leq m$, we let

$$\overline{C}_i := \sum_{g \in C_i} g.$$

The element $\overline{C}_i$ then is an element in the group algebra $\mathbb{C}G$, and it is called the *class sum* of C_i. Note that $\langle \overline{C}_i : i \leq m \rangle \subseteq \mathbb{C}G$ is the centre of the group algebra.

Proposition 2.7.4. *There exist non-negative integers a_{ijk} such that*

$$\overline{C}_i \overline{C}_j = \sum_{k=1}^{m} a_{ijk} \overline{C}_k \tag{2.35}$$

for all $1 \leq i \leq m$ and $1 \leq j \leq m$.

The integers a_{ijk} in (2.35) are called the *class algebra constants* of G.

Proposition 2.7.5. *Let G be a finite group and let $\{C_i\}_{i=1}^m$ be the collection of all conjugacy classes of G. For each $1 \leq i \leq m$, we let g_i be an element in C_i. Then we have*

$$a_{ijk} = \frac{|G|}{|C_G(g_i)|\,|C_G(g_j)|} \sum_{\chi} \frac{\chi(g_i)\chi(g_j)\overline{\chi(g_k)}}{\chi(1)} \tag{2.36}$$

where $C_G(g)$ is the centraliser of g in G, and the sum is over the irreducible characters χ of G.

Note that the *centraliser* $C_G(g)$ of g in G is the set of all elements in G that commute with g, that is,

$$C_G(g) = \{x \in G : xg = gx\}.$$

In Section 3.5 we will discuss this material in the transposition Cayley graph $G_n(2^1)$. Especially, we are interested in the connection between the class function $f_r(u) := |B_r \cap B_r u|$ and the characters. Note that the characters span the space of all class functions.

Chapter 3

Variation on Ball Intersection Numbers in the Transposition Cayley Graph

The transposition Cayley graph and its intersection numbers were thoroughly studied in [15]. In this chapter we show that there is another point of view in studying both the graph and its intersection numbers. Here we study in depth the results proved by Levenshtein and Siemons. In this chapter we let Γ_n stand for the transposition Cayley graph $G_n(2^1)$ of Sym_n, so this is the Cayley graph on Sym_n generated by all transpositions. Also, we let $S_r := S_r(\Gamma_n, e)$ and $B_r := B_r(\Gamma_n, e)$.

3.1 Current Results on Transposition Cayley Graphs

We start with some theorems concerning $N(\Gamma_n, r)$ for all $r \leq 3$.

Theorem 3.1.1 ([15], p. 809). *Let Γ_n be the transposition Cayley graph on Sym_n.*

For all $n \geq 3$ *we have*

$$N(\Gamma_n, 1) = 3, \tag{3.1}$$

and for all $n \geq 5$ *we have*

$$N(\Gamma_n, 2) = \frac{3}{2}(n+1)(n-2). \tag{3.2}$$

Theorem 3.1.2 ([15], p. 810). *Let* $n \geq 4$. *Then we have*

(1) $N_1(\Gamma_n, 3) = 2\,|S_0| + 2\,|S_2|$,

(2) $N_2(\Gamma_n, 3) = \sum_{i=0}^{2} |S_i| + (n+2)(n-3) + 24\binom{n-3}{2} + 22\binom{n-3}{3} + 6\binom{n-3}{4}$ *and*

(3) $N(\Gamma_n, 3) = N_2(\Gamma, 3)$ *for all* $n \geq 16$.

Here is the main theorem in [15] concerning the asymptotic behaviour of $N(\Gamma_n, r)$.

Theorem 3.1.3 ([15], p. 815). *Let* $r \geq 1$ *and let* Γ_n *be the transposition Cayley graph* $G_n(2^1)$ *on* Sym_n. *If* n *is sufficiently large then*

$$N(\Gamma_n, r) = N_2(\Gamma_n, r) = |B_{r-1}| + c_{3^1}(n, n-r) + c_{3^1}(n, n-(r+1)) \tag{3.3}$$

where $c_{3^1}(n, m)$ *is the number of permutations in* Γ_n *having* m *cycles such that* $1, 2, 3$ *are in the same cycle.*

Remark: Be aware that the numbers $c_{3^1}(n, m)$ are not the 3-Stirling numbers we introduced before. The theorem above is our main motivation in order to find the corresponding results for any k-transposition Cayley graph, defined in the previous chapter.

3.2 Some Facts about Transposition Cayley Graphs

In this section we slowly consider the behaviour of the transposition Cayley graph $G_n(2^1)$. We also recall some results so that one can understand the canonical properties of these graphs.

3.2.1 Edge Labelling

Any edge in the transposition Cayley graph Γ_n can be represented in the following way. Suppose that $\{u, v\}$ is the edge linking vertices u and v. Then there is a transposition h such that $v = uh$, and on the other hand as $h = h^{-1}$ we have $u = vh$. It then makes sense to label the edge $\{u, v\}$ with $\{h\} = \{u^{-1}v, v^{-1}u\}$.

Let g be a vertex (permutation) in Γ_n. The *left multiplication* $\sigma_g : v \mapsto g^{-1}v$ for all v in Γ_n is an automorphism on Γ_n. Obviously, σ_g maps the edge $\{v, vh\}$ to $\{g^{-1}v, g^{-1}vh\}$. Hence $\{v, vh\}$ and $\{g^{-1}v, g^{-1}vh\}$ have the same label $\{h\}$. Note that the map σ_g requires the inverse so that $v\sigma_{gg'} = v\sigma_g\sigma_{g'}$ for all v, g, g' in Γ_n. Also, in general, the left multiplication can be extended to any other Cayley graph.

Next, taking advantage of being generated by the conjugacy class $H = (2^1)^{G_n}$ of Γ_n we have another automorphism, the *right multiplication* $\rho_g : v \mapsto vg$. One can see that $\{v, vh\} \mapsto \{vg, vhg\} = \{vg, (vg)(g^{-1}hg)\}$. Clearly, $g^{-1}hg$ is a transposition. Hence, in this case, the right multiplication still preserves edges, but changes the labels.

3.2.2 Distance Statistics in Transposition Cayley Graphs

To evaluate the intersection number

$$N(\Gamma_n, r) := \max_{u \neq e} |B_r \cap B_r u| \tag{3.4}$$

with $r > 3$ one may try to guess intelligently the value $N(\Gamma_n, r)$ in (3.4) by the number of permutations $g := uh_1 \dots h_{r*}$ where $r^* \leq r$ and h's are errors belonging to H. Recall that in the transposition Cayley graph, errors are transpositions.

Before we do this we give some useful terminology. Let Γ_H be a Cayley graph on a group G accompanied by the (distance) metric d induced by the generating set

H of G, and let P be a path starting at the identity e, say

$$P := e, v_1, v_2, \ldots, v_k.$$

We say that P has a *descent* at step k if $d(e, v_k) = d(e, v_{k-1}) - 1$. On the other hand, P has an *ascent* at step k if $d(e, v_k) = d(e, v_{k-1}) + 1$. Hence, given a path in $\Gamma_n = G_n(2^1)$, it must have either descent or ascent at each step. This fact follows directly from the parity of permutations. In other words, for any vertex v in S_i, its neighbours must belong to either S_{i-1} or S_{i+1}.

Fix $i \geq 0$. Let v be a vertex in S_i. The number of the neighbours of v in S_{i-1}, denoted by $c(v)$, is equal to the number of transpositions h that split a cycle in v to two. If $ct(v) = 1^{h_1} 2^{h_2} \ldots n^{h_n}$ then we have that

$$c(v) = \sum_{j=1}^{n} \binom{j}{2} h_j = \frac{1}{2} \left(\sum_{j=1}^{n} j^2 h_j - n \right)$$

since $\sum_j j h_j = n$. As we discussed in the preceding chapter, any vertex v in Γ_n can be embedded into Γ_{n+1} by fixing $n+1$. It follows that the value $c(v)$ is the same. That is, the value $c(v)$ actually does not depend on n, but is a constant. Hence, as v has degree $|H| = \left|(2^1)^{G_n}\right| = \dfrac{n(n-1)}{2}$ we have that the number $b(v)$ of neighbours of v belonging to S_{i+1} is

$$b(v) = \frac{n(n-1)}{2} - c(v). \tag{3.5}$$

Since $c(v)$ is a constant we have $b(v) = O(n^2)$. Therefore, it follows directly that:

Proposition 3.2.1 ([15], p. 810). *In Γ_n the number of vertices that are reachable from a vertex v with a ascent steps is $O(n^{2a})$.*

Remark: The parameters $c(v)$ and $b(v)$ introduced above may be regarded as the *downward* and *upward* degree for the vertex v. The letters 'c' and 'b' refer to the parameters c_i and b_i in distance-regular graphs, as defined earlier (p. 11).

Generally, in some graphs we may have to use 'a' referring to the parameter a_i in distance-regular graphs. Note that $a(v)$ will be defined to be the number of neighbours of v belonging to the same sphere as v. By this definition and the parity of permutations, it is clear that $a(v) = 0$ for any permutation v in Γ_n. In the next proposition we provide parameters satisfying results very similar to those of distance-regular graphs. Recall that in any distance-regular graph, for any $i \geq 0$ there are constants c_i, a_i, b_i such that for any vertices u, v with $d(u, v) = i$, the number of neighbours of v at distance $i-1, i, i+1$ are c_i, a_i and b_i, respectively. In [5, p.72], the author shows that for a given distance-regular graph,

$$c_1 \leq c_2 \leq \ldots \leq c_{d^*} \quad \text{and} \quad b_0 \geq b_1 \geq \ldots \geq b_{d^*-1}$$

where d^* is the diameter of the graph.

Proposition 3.2.2. *Let $\Gamma_n = G_n(2^1)$, and let $c_{max}(i) = \max\{c(g) : g \in S_i\}$ and $b_{min}(i) = \min\{b(g) : g \in S_i\}$. Then*

$$c_{max}(0) \leq c_{max}(1) \leq c_{max}(2) \leq \ldots \leq c_{max}(n-2) \leq c_{max}(n-1) \tag{3.6}$$

and

$$b_{min}(0) \geq b_{min}(1) \geq b_{min}(2) \geq \ldots \geq b_{min}(n-2) \geq b_{min}(n-1). \tag{3.7}$$

In addition, we have

$$c_{max}(i) = c((1\ 2\ 3\ \ldots\ i+1)) \quad \textit{and} \quad b_{min}(i) = b((1\ 2\ 3\ \ldots\ i+1))$$

for all $0 \leq i \leq n-1$.

Proof. We first observe that $c_{max}(0) = 0$ and $c_{max}(1) = 1$, and $c_{max}(n-1) = \frac{n(n-1)}{2}$. Let Ω be a collection of subsets of $[n]$. We let

$$C(\Omega) := \{\ \{\alpha_1, \alpha_2\} \ :\ \alpha_1 \neq \alpha_2 \ \text{ and } \ \alpha_1, \alpha_2 \in A \ \text{ for some } \ A \in \Omega\ \}.$$

For each g in Γ_n we let Ω_g be the collection of subsets of $[\,n\,]$ defined by

$$\Omega_g := \{ \quad \bigcup_{m\geq 0}\{\alpha g^m\} \ : \quad \alpha\text{'s are representatives of all cycles in } g \quad \}.$$

Then $\{\alpha_1, \alpha_2\}$ is in $C(\Omega_g)$ if and only if α_1 and α_2 belong to the same cycle in the decomposition of g. Clearly, $C(\Omega_g)$ is the set counted by $c(g)$, the downward degree of g.

Now fix $2 \leq i \leq n-2$. Let $g_0 = (1\ 2\ 3\ \ldots i\ i+1)$ and let g' belong to S_i so that g' and g_0 are not conjugate. Then g' has at least two cycles of length greater than one in its cycle decomposition. Suppose that

$$g' = (\alpha_{11}\ \alpha_{12}\ldots\ \alpha_{1k_1})(\alpha_{21}\ \alpha_{22}\ldots\ \alpha_{2k_2})\ldots(\alpha_{j1}\ \alpha_{j2}\ldots\ \alpha_{jk_j}),$$

suppressing all cycles of length one. Then

$$\begin{aligned}
|C(\Omega_{g'})| &= \Big|C\big(\ \{\{\alpha_{t1}\ \alpha_{t2}\ldots\ \alpha_{tk_t}\}\}_{t=1}^{j}\ \big)\Big| \\
&= \Big|C\big(\ \{\{\alpha_{11}\ \alpha_{t2}\ldots\ \alpha_{tk_t}\}\}_{t=1}^{j}\ \big)\Big| \\
&< \big|C\big(\ \{\{\alpha_{11}\ \alpha_{12}\ldots\ \alpha_{1k_1}\alpha_{22}\ \alpha_{23}\ \ldots\ \ \alpha_{j-1,k_{j-1}}\alpha_{j2}\ \alpha_{j3}\ \ldots\ \alpha_{jk_j}\}\}\ \big)\big|.
\end{aligned}$$

Note that the last inequality holds as $\{\alpha_{12}, \alpha_{22}\}$ is counted by the latter, but not by the former. Recall g' now belongs to S_i. Then $|g'| = n - i$ and therefore, we have

$$k_1 + k_2 + \ldots + k_j - i = j.$$

Hence,

$$\begin{aligned}
\big|\{\alpha_{11}\ \alpha_{12}\ldots\ \alpha_{1k_1}\alpha_{22}\ \alpha_{23}\ \ldots\ \ \alpha_{j-1,k_{j-1}}\alpha_{j2}\ \alpha_{j3}\ \ldots\ \alpha_{jk_j}\}\big| &= k_1 + \ldots + k_j - (j-1) \\
&= i+1 \\
&= \big|\{1\ 2\ 3\ \ldots i\ i+1\}\big|.
\end{aligned}$$

This follows that

$$\begin{aligned}
|C(\Omega_{g'})| &< \big|C\big(\ \{\{\alpha_{11}\ \alpha_{12}\ldots\ \alpha_{1k_1}\alpha_{22}\ \alpha_{23}\ \ldots\ \ \alpha_{j-1,k_{j-1}}\alpha_{j2}\ \alpha_{j3}\ \ldots\ \alpha_{jk_j}\}\}\ \big)\big| \\
&= \big|C\big(\ \{\{1\ 2\ 3\ \ldots i\ i+1\}\}\ \big)\big| \\
&= \big|C\big(\Omega_{g_0}\big)\big|.
\end{aligned}$$

Hence, $c(g') < c(g_0)$, as required. In addition, it is clear that (3.7) follows directly from (3.5). The proof is then complete. □

Next, we give a bound of $N_i(\Gamma_n, r)$. It is an extended version of a proof in [15, p. 815].

Corollary 3.2.3. *Let $r \geq 1$ and let u belong to S_m in Γ_n with $2j-1 \leq m \leq 2j$ for some j. The number of vertices in B_r reachable by a path of length $r' \leq r$ starting at u is at most $O(n^{2(r-j)})$.*

Proof. Let P be a path of length $r' \leq r$ from u to a vertex v in B_r. Suppose that P has a ascents and d descents, and v belongs to $S_{m'} \subseteq B_r$. We then have $a + d = r' \leq r$ and $d - a = m - m'$. Since $m' \leq r$ we have $2a \leq 2r - m$. It follows that $a \leq r - j + \frac{1}{2}$. Since a is an integer we have $a \leq r - j$. The proof follows directly from Proposition 3.2.1. □

Remark: From Corollary 3.2.3 one can see that if u belongs to S_i with $i \geq 3$, then the number of vertices reachable from u in no more than r steps is at most $O(n^{2(r-2)})$. In addition, if u is in either S_1 or S_2 then there are at most $O(n^{2(r-1)})$ vertices reachable from u in at most r steps.

3.2.3 The Ascent-Descent Pattern

Let u and g belong to Sym_n. We want to understand the paths from u to ug. For this we first suppose that g is a single cycle, say $g = (\alpha_1\ \alpha_2\ \ldots \alpha_m)$. Now let $t_i = (\alpha_1\ \alpha_i)$ for all $2 \leq i \leq m$. Then $g = t_2 t_3 \cdots t_m$. Next, consider the path

$$P := u, ut_2, ut_2t_3, \ldots, ug, \tag{3.8}$$

which is a path of length $m-1$ starting at u and ending at ug. One can see that any factorisation of g into $m-1$ transpositions gives such a path, and every path of length $m-1$ from u to ug in Γ_n provides a factorization of g, mappping u and ug by the left multiplication σ_u and walking along the path from $e = u^{-1}u$ to $g = u^{-1}ug$.

The idea now is to *embed* the path P into Γ_{n+1}, the transposition Cayley graph on Sym_{n+1}. For this we view g and the t_i's as elements in Sym_{n+1} that fix $n+1$ and move all other points as before. For any $j = 0, \ldots, n$ let i_j be the insert operation in Section 2.4. Now the path P in Γ_n can be embedded into Γ_{n+1} via $P \mapsto P_j$ where

$$P_j := i_j(u), i_j(u)t_2, i_j(u)t_2t_3, \ldots, i_j(u)g. \tag{3.9}$$

Note that in (3.9) we consider g and t_i's as $i_0(g)$ and $i_0(t_i)$'s. In general, if g is any permutation in Γ_n with $|g| = n-k$ for some k, then g can be expressed as a product of k transpositions, and therefore we will get the path P and P_j as in (3.8) and (3.9). The function $P \mapsto P_j$ has an invariant property that we call its *ascent-descent pattern.*

Definition 3. *For a path $P := x_1, \ldots, x_j$ of length $j-1$, the ascent-descent pattern of P, written $AD(P)$, is the $j-1$ tuple*

$$AD(P) = (*_1, *_2, \ldots, *_{j-1})$$

where $*_i = a$ if $d(e, x_i) = d(e, x_{i-1}) + 1$, otherwise $*_i = d$.

Let v be a vertex in Γ_n and let h be a transposition in Γ_n. Then $\{v, vh\}$ is an edge in Γ_n. We claim that for any j in $\{0, \ldots, n\}$

$$i_j(vh) = i_j(v)i_0(h). \tag{3.10}$$

From the definition of i_j one can see that, for any α in [n+1]

$$\alpha^{i_0(v)} = \begin{cases} \alpha & \text{if } \alpha = n+1, \\ \alpha^v & \text{otherwise.} \end{cases}$$

Also, for any j in $[n]$ and α in [n+1]

$$\alpha^{i_j(v)} = \begin{cases} n+1 & \text{if } \alpha = j, \\ j^v & \text{if } \alpha = n+1, \\ \alpha^v & \text{otherwise.} \end{cases}$$

Recall that i^v is the image of i under v. To show that Equation (3.10) holds, we first prove that $i_0(vh) = i_0(v)i_0(h)$. Let α belong to $[n]$. Then

$$\alpha^{i_0(vh)} = \alpha^{vh} = (\alpha^v)^h = (\alpha^v)^{i_0(h)} = (\alpha^{i_0(v)})^{i_0(h)} = \alpha^{i_0(v)i_0(h)}.$$

Also, we have

$$(n+1)^{i_0(vh)} = n+1 = (n+1)^{i_0(h)} = ((n+1)^{i_0(v)})^{i_0(h)} = (n+1)^{i_0(v)i_0(h)}.$$

Therefore, $i_0(vh) = i_0(v)i_0(h)$. We next show that (3.10) holds for any j in [n] too. Clearly,

$$(n+1)^{i_j(vh)} = j^{vh} = (j^v)^h = (j^v)^{i_0(h)} = ((n+1)^{i_j(v)})^{i_0(h)} = (n+1)^{i_j(v)i_0(h)},$$

and

$$j^{i_j(vh)} = n+1 = (n+1)^{i_0(h)} = (j^{i_j(v)})^{i_0(h)} = j^{i_j(v)i_0(h)}.$$

Moreover, if α does not belong to $\{j, n+1\}$, then

$$\alpha^{i_j(vh)} = \alpha^{vh} = (\alpha^v)^h = (\alpha^v)^{i_0(h)} = (\alpha^{i_j(v)})^{i_0(h)}.$$

That is,

$$i_j(vh) = i_j(v)i_0(h)$$

for any j in $[n]$. This follows that for any j in {0,1,2,...,n}

$$i_j(vh) = i_j(v)i_0(h) \tag{3.11}$$

and hence, we have

$$i_j(\{v, vh\}) = \{i_j(v), i_j(vh)\} = \{i_j(v), i_j(v)i_0(h)\}.$$

Note from the definition of i_j that

$$d_{\Gamma_{n+1}}(e, i_j(v)) \;=\; \begin{cases} d_{\Gamma_n}(e, v) & \text{if } j = 0, \\ d_{\Gamma_n}(e, v) + 1 & \text{otherwise,} \end{cases} \tag{3.12}$$

since adding $n+1$ in a cycle of v in Γ_n is the reduction of the number of cycles of $i_j(v)$ in Γ_{n+1} by one. Hence, from Proposition 2.3.5 we have that (3.12) holds. Recall that $d_\Gamma(u, v)$ is the distance between vertices u and v in Γ. Hence, considering $i_0(h) = h$, since each i_j preserves edges and increases the distance $d_{\Gamma_{n+1}}(e, i_j(v)h)$ from $d_{\Gamma_n}(e, vh)$ by at most one, we have that both of paths $P := v, vh$ and $P_j := i_j(v), i_j(v)h$ have either ascent or descent at step one, that is, $AD(P) = AD(P_j)$. Recall that in the transposition Cayley graphs, there is no edge linking two vertices in the same sphere.

Proposition 3.2.4. *Let $g = t_1 t_2 \dots t_m$ be a product of m transpositions and let v be a vertex in Γ_n. Suppose that*

$$P := v, vt_1, vt_1t_2, vt_1t_2t_3, \dots, vg$$

be a path from v to vg in Γ_n. For each $j = 0, 1 \dots, n$, let

$$P_j := i_j(v), i_j(v)t_1, i_j(v)t_1t_2, \dots, i_j(v)g.$$

Then $AD(P) = AD(P_j)$.

Proof. This follows directly by comparing each step of P and P_j. $\square$

In Figure 3.1 we illustrate this manoeuvre. Let $u = (1\,5)(2\,4)(3)$ and $g = (1\,2\,3\,4)$ belong to $\Gamma_5 = G_5(2^1)$. Then $u_1 := i_1(u) = (1\ 6\ 5)(2\ 4)(3)$ is the permutation

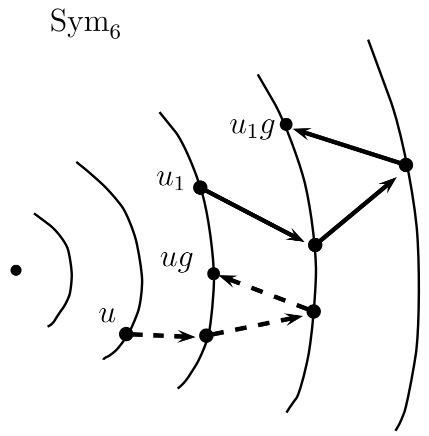

Figure 3.1: The $A-D$ pattern of paths and their embedding

in $\Gamma_6 = G_6(2^1)$ obtained from u by adding 6 after 1. Also, suppose that g is decomposed as $g = (1\ 2)(1\ 3)(1\ 4)$. By Proposition 2.3.5 we have that u is in $S_2(\Gamma_5, e)$ and u_1 is in $S_3(\Gamma_6, e)$. Let

$$P := u, u(1\ 2), u(1\ 2)(1\ 3), u(1\ 2)(1\ 3)(1\ 4)$$

and

$$P_1 := u_1, u_1(1\ 2), u_1(1\ 2)(1\ 3), u_1(1\ 2)(1\ 3)(1\ 4)$$

be paths in Γ_5 and Γ_6, respectively. One can see that

$$AD(P) = AD(P_1) = (a, a, d)$$

as shown in Figure 3.1.

3.3 The Stirling Recursion in Transposition Cayley Graphs

In the preceding section we have seen how to embed a path P in Γ_n into Γ_{n+1}. Using this method, the ascent-descent pattern is an invariant under the embeddings $i_j : P \mapsto P_j$ we defined before. Conversely, a given path P in Γ_{n+1} might be expected to provide a path in Γ_n having the same ascent-descent pattern as P. One strategy would be to delete 'n+1' from the vertices in P. But this does not always work. For instance, suppose that $v = (1\ 2\ 3)(4)$ and $t = (1\ 4)$. Then $vt = (1\ 2\ 3\ 4)$ and $P := v, vt$ is a path of length one in Γ_4. Deleting 4 from v and vt, we get the same permutation, namely $(1\ 2\ 3)$. That is, in general, the method seems not to be a good one. In fact, this is because 4 belongs to the support of t.

To distill a path in Γ_{n+1} into Γ_n one needs to choose the paths whose adjacent vertices u, v satisfy $(n+1)^{u^{-1}v} = n+1$, that is, $n+1$ is fixed by $u^{-1}v$. Note that $u^{-1}v$ is now considered as the edge between u and v. Under this condition we need to find a path P satisfying that all vertices in P either fix $n+1$ or move $n+1$.

Before going to the next lemma let us recall that the i_j's are the insert operations. Also, for any positive integers m and k, the spheres S_m and S_k in (3.13) are considered as spheres in Γ_{n+1} while S_{m-t} and S_{k-t} refer to spheres in Γ_n.

Lemma 3.3.1 (Cancellation Lemma). *Let g be a permutation in Γ_{n+1} fixing $n+1$. If $v = i_j(u)$ for some u in Γ_n and j in $\{0, 1, 2, \ldots, n\}$ then*

$$(v, vg) \in S_m \times S_k \quad \textit{if and only if} \quad (i_j^{-1}(v), i_j^{-1}(vg)) \in S_{m-t} \times S_{k-t} \tag{3.13}$$

for some $t = 0, 1$.

Proof. Suppose that $g = t_1 t_2 \cdots t_p$ is expressed as a product of p transpositions. Let $v_0 = v$ and let $v_f = v_{f-1}t_f$ for all $f = 1 \ldots, p$. Assume that (v, vg) is in $S_m \times S_k$. From the above discussion, $n+1$ is either fixed by the v_f's or moved by the v_f's. From Proposition 3.2.4, we have that the paths $P := v, v_1, \ldots, v_f$ and $i_j^{-1}(P)$ have the same ascent-descent pattern, i.e. $AD(P) = AD(i_j^{-1}(P))$ for all j. Due to the construction of i_j we have $t = 0$ or 1. The converse is clear by applying the insert function i_j. □

3.3.1 Ball Intersection Numbers for Permutations

A function f has the *Stirling recursion* if there are n_0 and k_0 such that

$$f(n,k) = f(n-1,k-1) + (n-1)f(n-1,k) \tag{3.14}$$

for all $n \geq n_0$ and $k \geq k_0$. Then it is clear that the ordinary Stirling numbers of the first kind and the $r-$Stirling numbers of the first kind have the Stirling recursion. We now turn our attention to the ball intersection number $N(\Gamma_n, r)$, and then show that these numbers have the Stirling recursion too.

Let $r \geq 0$. For a permutation g in $\Gamma_n = G_n(2^1)$ we let

$$I_g(n,r) := |B_r \cap B_r g| . \tag{3.15}$$

Recall that $B_r g := B_r \cdot \{g\} = \{g\} \cdot B_r = B_r(\Gamma_n, g)$ as shown in (2.24). Obviously, we have that $I_g(n,0) = 0$ if g is not the identity and $I_g(n,r) = n!$ if $r \geq n-1$. Also, from (2.26), fixing n and r, the function $g \mapsto I_g(n,r)$ is a class function. Then, throughout this thesis, we can pay our attention to permutations g with $Supp(g) = [\, supp(g)\,]$. Recall that $[\, n\,] = \{1, 2, \ldots, n\}$.

For any positive integers n and r, suppose that g is a vertex in Γ_n fixing n. Let $B_r := B_r(\Gamma_n)$. The set $Z := B_r \cap B_r g$ can be divided to two subsets, say X and Y. The set X consists of permutations in Z fixing n, and Y is the set of those not fixing n. Since every permutation in X fixes n, and since i^* is the function from Γ_n to Γ_{n-1} that deletes n from the disjoint cycle decomposition of permutations in Γ_n, we have $|\, i^*(X)| = |\, X|$. Further, if we let $T = i^*(Y)$ then $\{i_j(T)\}_{j=1}^{n-1}$ is a partition of Y. Therefore, from the Cancellation Lemma, for each v in Z we have that $i^*(vg^{-1})$ and $i^*(v)$ belong to $B_r(\Gamma_{n-1})$ if v is in X; otherwise, they are in $B_{r-1}(\Gamma_{n-1})$. Note that $i^*(vg^{-1}) = i^*(v)i^*(g^{-1}) = i^*(v)i^*(g)^{-1}$. Hence,

$$I_g(n,r) = |X| + |Y| = |X| + (n-1)\,|T| = I_g(n-1,r) + (n-1)I_g(n-1,r-1).$$

Here we collect these facts.

Proposition 3.3.2. *Let n and s be positive integers with $n > s$. If g is a permutation in Γ_n such that $Supp(g) = [\,s\,]$, then*

$$I_g(n,r) = I_g(n-1,r) + (n-1)I_g(n-1,r-1) \tag{3.16}$$

for any $r \geq 1$.

Remark: The permutation g on the right-hand side of the equation in (3.16), and also in (3.18), is a permutation in Γ_{n-1}.

Having a glance at (3.16), it is almost the same as the Stirling recursion (3.14) with only a small difference. What we may say is that they are defined in reverse to each other: the reason is that the sphere S_i consists exactly of permutations having $n-i$ cycles (not i cycles). Therefore, for all $r \geq 0$ if we instead let

$$|B_r \cap B_r g| =: \begin{bmatrix} n \\ n-r \end{bmatrix}_g \tag{3.17}$$

then the recurrence in (3.16) should become the Stirling recursion.

Theorem 3.3.3. *Let n and s be positive integers with $n > s$. If g is a permutation in Γ_n such that $Supp(g) = [\,s\,]$, then*

$$\begin{bmatrix} n \\ k \end{bmatrix}_g = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_g + (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix}_g . \tag{3.18}$$

for any integer $1 \leq k \leq n$.

Remark: If we let $s = supp(g)$ in Theorem 3.3.3 then $\begin{bmatrix} n \\ k \end{bmatrix}_g$ in (3.18) is fully determined by s initial values, namely $\begin{bmatrix} s \\ 1 \end{bmatrix}_g, \begin{bmatrix} s \\ 2 \end{bmatrix}_g, \ldots, \begin{bmatrix} s \\ s-1 \end{bmatrix}_g, \begin{bmatrix} s \\ s \end{bmatrix}_g$. From Definition (3.17) and Proposition 2.3.5 we have $\begin{bmatrix} s \\ s \end{bmatrix}_g = 1$ if $g = e$; otherwise $\begin{bmatrix} s \\ s \end{bmatrix}_g = 0$. Moreover, $\begin{bmatrix} s \\ m \end{bmatrix}_g = |\Gamma_n| = n!$ for all $m \leq 1$.

Proof. By Proposition 3.3.2 and Definitions (3.15) and (3.17) we have that

$$\begin{aligned}
\begin{bmatrix} n \\ n-r \end{bmatrix}_g &= I_g(n,r) \\
&= I_g(n-1,r) + (n-1)I_g(n-1,r-1) \\
&= \begin{bmatrix} n-1 \\ n-1-r \end{bmatrix}_g + (n-1)\begin{bmatrix} n-1 \\ (n-1)-(r-1) \end{bmatrix}_g \\
&= \begin{bmatrix} n-1 \\ n-r-1 \end{bmatrix}_g + (n-1)\begin{bmatrix} n-1 \\ n-r \end{bmatrix}_g,
\end{aligned}$$

and then substituting $n-r$ by k, we have

$$\begin{bmatrix} n \\ k \end{bmatrix}_g = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_g + (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix}_g$$

as required. ☐

Corollary 3.3.4. *Let g_1 and g_2 be non-identity permutations. If there exist positive integers m and t such that $\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]_{g_1} \geq \left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]_{g_2}$ for all $t \leq k \leq m$ then $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{g_1} \geq \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{g_2}$ for all $m \leq n$ and $t+(n-m) \leq k \leq n$. Moreover, if $t=1$ then $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{g_1} \geq \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{g_2}$ for all $n \geq m$ and all k.*

Proof. This follows from Theorem 3.3.3 and the fact that, fixing n and k, the function $g \mapsto \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_g$ is a class function. ☐

3.3.2 Generating Functions

From Theorem 3.3.3 and Proposition 2.4.1 we see that $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_g$ and $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ satisfy the Stirling recursion, with different initial conditions. Recall that in $\Gamma_n = G_n(2^1)$ we have $|S_i| = \left[\begin{smallmatrix} n \\ n-i \end{smallmatrix}\right]$ and

$$\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} y^k = y(y+1)(y+2)\cdots(y+n-1). \tag{3.19}$$

It makes sense to think of $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_g$ having a generating fuction that would be similar to that of $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$.

Let n and s be integers with $n > s$ and let g be a non-identity permutation in Γ_n such that $Supp(g) = [\,s\,]$. For any $k > n$ we define $\left[{n \atop k} \right]_g$ by letting $\left[{n \atop k} \right]_g = 0$. Let $\Psi(\Gamma_n, g; y) := \sum_{k=-\infty}^{\infty} \left[{n \atop k} \right]_g y^k$. Note that the case $k \leq 0$ means that we consider the ball whose radius is larger than the diameter of the graph Γ_n. Recall that $diam(G_n(2^1)) = n - 1$. This implies that all vertices are counted in this case. That is, $\left[{n \atop k} \right]_g = n!$ if $k \leq 0$. Multiplying both sides with y^k and summing over k in (3.18) we get

$$\sum_{k=-\infty}^{\infty} \left[{n \atop k} \right]_g y^k = \sum_{k=-\infty}^{\infty} \left[{n-1 \atop k-1} \right]_g y^k + (n-1) \sum_{k=-\infty}^{\infty} \left[{n-1 \atop k} \right]_g y^k$$

and hence

$$\begin{aligned} \Psi(\Gamma_n, g; y) &= y\Psi(\Gamma_{n-1}, g; y) + (n-1)\Psi(\Gamma_{n-1}, g; y) \\ &= (y + (n-1))\Psi(\Gamma_{n-1}, g; y). \end{aligned} \tag{3.20}$$

For instance, let $g = (1\ 2\ 3)$. One can get that $\left[{3 \atop 1} \right]_{(1\ 2\ 3)} = \left[{3 \atop 3-2} \right]_{(1\ 2\ 3)} = 6$ which counts all elements in Sym_3. Also, by (3.1) we have $\left[{3 \atop 2} \right]_{(1\ 2\ 3)} = \left[{3 \atop 3-1} \right]_{(1\ 2\ 3)} = 3$. Hence

$$\Psi(\Gamma_3, (1\ 2\ 3); y) = 3y^2 + 6y + 6 + \frac{6}{y} + \frac{6}{y^2} + \cdots .$$

Recall that we let $y^{[n]} := y(y+1)\cdots(y+(n-1))$. By (3.20) it follows that:

Theorem 3.3.5. *Let $g = (1\ 2\ 3)$ and $n \geq 3$. Then*

$$\Psi(\Gamma_n, (1\ 2\ 3); y) := (y+3)^{[n-3]}(3y^2 + 6y + 6 + \frac{6}{y} + \frac{6}{y^2} + \cdots)$$

is the generating function of $\left[{n \atop k} \right]_g$.

Remark: In the same way, $\Psi(\Gamma_n, g; y)$ can be defined from (3.20) for all g.

3.3.3 Intersection Tables

For a fixed m and a permutation g in Γ_m, let $s = supp(g)$. One can construct a table listing the values of $\left[{n \atop k} \right]_g$ for all k, n with $k \leq n$ and $n \geq s$ as in Table 3.1.

This table is called the *intersection table* IN_g of g. It stores the information about the initial conditions of $\left[{n \atop k}\right]_g$. Note that we ignore the value $\left[{n \atop k}\right]_g$ for all $k \leq 0$ since it is equal to $\left[{n \atop 1}\right]_g = n!$.

n	k								
	1	2	$\dots$	$s-1$	s	$s+1$	$s+2$	$s+3$	$\cdots$
s	$\left[{s \atop 1}\right]_g$	$\left[{s \atop 2}\right]_g$	$\cdots$	$\left[{s \atop s-1}\right]_g$	$\left[{s \atop s}\right]_g$	0	0	0	$\cdots$
$s+1$	$\left[{s+1 \atop 1}\right]_g$	$\left[{s+1 \atop 2}\right]_g$	$\cdots$	$\left[{s+1 \atop s-1}\right]_g$	$\left[{s+1 \atop s}\right]_g$	$\left[{s+1 \atop s+1}\right]_g$	0	0	$\cdots$
$s+2$	$\left[{s+2 \atop 1}\right]_g$	$\left[{s+2 \atop 2}\right]_g$	$\cdots$	$\left[{s+2 \atop s-1}\right]_g$	$\left[{s+2 \atop s}\right]_g$	$\left[{s+2 \atop s+1}\right]_g$	$\left[{s+2 \atop s+2}\right]_g$	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	

Table 3.1: IN_g

It is clear that permutations with the same cycle type produce the same table. Also, if g does not belong to $B_{2t}(\Gamma_s)$ we have $\left[{n \atop k}\right]_g = 0$ for all $s-t \leq k \leq s$. On the other hand, if g belongs to $B_{2j}(\Gamma_s)$ for some j then $\left[{n \atop k}\right]_g > 0$ for all $k \leq s-j$. From (3.18) we see that the value of $\left[{n \atop n-r}\right]_g$ in IN_g is obtained from two directions. The first is by summing the values in vertical lines, which is equal to $O(n^{2(r-j-1)+1}) = O(n^{2r-2j-1})$ by induction on $m := r-j$, and the second is summing those of the first in $n - c_k$ times for some constant c_k. Hence

Lemma 3.3.6. *Let g be a permutation in Γ_n. Suppose that j is the smallest integer such that g belongs to B_{2j}. Then $\left[{n \atop n-r}\right]_g = O(n^{2(r-j)})$.*

Remark: Translating Corollary 3.2.3 into the language of intersection numbers, we have that $O(n^{2(r-j)})$ is an upper bound for $\left[{n \atop n-r}\right]_g$ if g belongs to $S_{2j} \cup S_{2j-1}$, and by Lemma 3.3.6 this bound is sharp.

3.3.4 Domination from Cycles of Length Three

By Corollary 3.3.4, for any two permutations g_1, g_2 whose cycle types are different, we may find some useful information linking the intersection numbers $\left[{n \atop k}\right]_{g_1}$ and $\left[{n \atop k}\right]_{g_2}$

by comparing certain rows in their intersection tables. For example, let $\Gamma_4 = G_4(2^1)$ on Sym_4 and let $g_1 = (1\ 2\ 3)$, $g_2 = (1\ 2)(3\ 4)$ and $g_3 = (1\ 2)$. We have that in the 4th rows of IN_{g_1}, IN_{g_2} and IN_{g_3} we have the entries

$$\begin{bmatrix} 4 \\ 1 \end{bmatrix}_{g_1} = 24 \qquad \begin{bmatrix} 4 \\ 2 \end{bmatrix}_{g_1} = 15 \qquad \begin{bmatrix} 4 \\ 3 \end{bmatrix}_{g_1} = 3 \qquad \begin{bmatrix} 4 \\ 4 \end{bmatrix}_{g_1} = 0$$

and

$$\begin{bmatrix} 4 \\ 1 \end{bmatrix}_{g_2} = 24 \qquad \begin{bmatrix} 4 \\ 2 \end{bmatrix}_{g_2} = 14 \qquad \begin{bmatrix} 4 \\ 3 \end{bmatrix}_{g_2} = 2 \qquad \begin{bmatrix} 4 \\ 4 \end{bmatrix}_{g_2} = 0,$$

and also

$$\begin{bmatrix} 4 \\ 1 \end{bmatrix}_{g_3} = 24 \qquad \begin{bmatrix} 4 \\ 2 \end{bmatrix}_{g_3} = 12 \qquad \begin{bmatrix} 4 \\ 3 \end{bmatrix}_{g_3} = 2 \qquad \begin{bmatrix} 4 \\ 4 \end{bmatrix}_{g_3} = 0.$$

It then follows by Corollary 3.3.4 that $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{g_1} \geq \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{g_2} \geq \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{g_3}$ for all $n \geq 4$ and $1 \leq k \leq n$ as shown for example in Tables 3.2, 3.3 and 3.4.

In [15, p. 813] the authors showed that $N_2(\Gamma_n, r) > N_1(\Gamma_n, r)$ when n is sufficiently large. Comparing Table 3.2 with Table 3.4 we have by Corollary 3.3.4 that

Corollary 3.3.7. *Let Γ_n be the transposition Cayley graph on Sym_n. We have*

$$N_2(\Gamma_n, r) > N_1(\Gamma_n, r)$$

for all $n \geq 3$ where $N_i(\Gamma, r) = \max_{g \in S_i} |B_r(\Gamma_n, e) \cap B_r(\Gamma_n, g)|$.

3.3.5 The Closed Formula for Intersection Numbers

Now we take advantage of the computer programming language **GAP** to figure out $N(\Gamma_n, 2)$ for $\Gamma_n = G_n(2^1)$. From (2.28) we have

$$N(\Gamma_n, 2) = \max_{g \in R \cap B_4} \begin{bmatrix} n \\ n-2 \end{bmatrix}_g,$$

where R is a collection of representatives of non-trivial cycle types in Sym_n. For ease of computing, since the mapping $g \mapsto \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_g$ is a class function, when fixing n and k, we choose R to be a collection of permutations g such that $Supp(g) = [\,supp(g)\,]$.

n	k								
	1	2	3	4	5	6	7	8	$\cdots$
3	6	3	0	0	0	0	0	0	$\cdots$
4	24	15	3	0	0	0	0	0	$\cdots$
5	120	84	27	3	0	0	0	0	$\cdots$
6	720	540	219	42	3	0	0	0	$\cdots$
7	5040	3960	1854	471	60	3	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$	

Table 3.2: The table IN_g with $g = (1\ 2\ 3)$

n	k								
	1	2	3	4	5	6	7	8	$\cdots$
4	24	14	2	0	0	0	0	0	$\cdots$
5	120	80	22	2	0	0	0	0	$\cdots$
6	720	520	190	32	2	0	0	0	$\cdots$
7	5040	3840	1660	382	44	2	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$	

Table 3.3: The table IN_g with $g = (1\ 2)(3\ 4)$

Note for any permutation g in $S_3 \cup S_4$ we have, by Lemma 3.3.6, that $\left[{n \atop n-2} \right]_g$ is a constant for any $n \geq supp(g)$. Using **GAP**, we have that $\left[{n \atop n-2} \right]_g \leq 20$ for any g in $R \cap (S_3 \cup S_4)$ and any $n \geq 5$. Recall that we have $\left[{n \atop n-2} \right]_{(1\ 2\ 3)} \geq \left[{n \atop n-2} \right]_{(1\ 2)(3\ 4)} \geq \left[{n \atop n-2} \right]_{(1\ 2)}$ for all $n \geq 4$. Hence, since $\left[{n \atop n-2} \right]_{(1\ 2\ 3)} \geq \left[{5 \atop 3} \right]_{(1\ 2\ 3)} = 27$ we have $N(\Gamma_n, 2) = \left[{n \atop k} \right]_{(1\ 2\ 3)}$ for all $n \geq 5$. Further, by induction, for any $n \geq 3$ we have

$$\left[{n \atop n-2} \right]_{(1\ 2\ 3)} = 6 + 3(3 + 4 + \cdots + (n-1)) = \frac{3}{2}(n+1)(n-2),$$

which is the closed formula shown in Theorem 3.1.1. Also, using the Stirling recursion for $\left[{n \atop k} \right]_g$ with $g = (1\ 2\ 3)$, where $r = 3$ one can find the closed formula for $\left[{n \atop n-3} \right]_g$ that is eventually equal to $N(\Gamma_n, 3)$.

Let $a_t := \left[{t \atop t-2} \right]_g$ with $g = (1\ 2\ 3)$. The sequence $(a_t)_{t \geq 3}$ lies on the second diagonal line of non-zero entries in IN_g. By the induction and the Stirling recursion

n	k								
	1	2	3	4	5	6	7	8	$\cdots$
2	2	0	0	0	0	0	0	0	$\cdots$
3	6	2	0	0	0	0	0	0	$\cdots$
4	24	12	2	0	0	0	0	0	$\cdots$
5	120	72	20	2	0	0	0	0	$\cdots$
6	720	480	172	30	2	0	0	0	$\cdots$
7	5040	3600	1512	352	42	2	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$	

Table 3.4: The Table IN_g with $g = (1\ 2)$

we have that for all $n \geq 4$,

$$\begin{bmatrix} n \\ n-3 \end{bmatrix}_g = 6 + 3a_3 + 4a_4 + \cdots + (n-1)a_{(n-1)} \tag{3.21}$$

since $\begin{bmatrix} 3 \\ 0 \end{bmatrix}_g = \begin{bmatrix} 3 \\ 1 \end{bmatrix}_g = a_3 = 6$. Recall that $a_t = \begin{bmatrix} t \\ t-2 \end{bmatrix}_g = \frac{3}{2}(t+1)(t-2)$ for all $t \geq 3$. Then

$$\begin{aligned} \begin{bmatrix} n \\ n-3 \end{bmatrix}_g &= 6 + \sum_{t=3}^{n-1} t a_t & (3.22) \\ &= 6 + \sum_{t=3}^{n-1} \frac{3}{2} t(t+1)(t-2) & (3.23) \\ &= 6 + \sum_{t=3}^{n-1} \frac{3}{2} t(t^2 - t - 2) & (3.24) \\ &= 6 + \frac{3}{2} \left[\sum_{t=3}^{n-1} t^3 - \sum_{t=3}^{n-1} t^2 - 2 \sum_{t=3}^{n-1} t \right]. & (3.25) \end{aligned}$$

Simplifying (3.25) gives us the closed formula for $\begin{bmatrix} n \\ n-3 \end{bmatrix}_g$. That is, for all $n \geq 3$,

$$\begin{bmatrix} n \\ n-3 \end{bmatrix}_g = \frac{1}{8}(3n^4 - 10n^3 - 3n^2 + 10n + 72) \tag{3.26}$$

which is the closed formula for (2) in Theorem 3.1.2.

Now, by induction, one can extend the idea to the case $r \geq 4$ to get the closed formula of $\begin{bmatrix} n \\ n-r \end{bmatrix}_g$. Recall that $g = (1\ 2\ 3)$ and that $\begin{bmatrix} 3 \\ 3-k \end{bmatrix}_g = 6$ for all $k \geq 2$.

Then, by the Stirling recursion (3.18) and the induction on r we get

$$\begin{bmatrix} n \\ n-r \end{bmatrix}_g = 6 + \sum_{t=3}^{n-1} t \begin{bmatrix} t \\ t-(r-1) \end{bmatrix}_g . \tag{3.27}$$

In the last step, simplifying (3.27) we have

Theorem 3.3.8. *Fix $r \geq 1$. There exists a polynomial $F_r(x) := \sum_{t=0}^{2r-2} f_t x^t$ so that $F_r(n) = N(G_n(2^1), r)$ for all sufficiently large n.*

The above theorem holds as when n is large enough we have

$$\begin{bmatrix} n \\ n-r \end{bmatrix}_{(1\ 2\ 3)} = N(G_n(2^1), r) \tag{3.28}$$

and also by the induction on n in (3.27), we have that $\begin{bmatrix} n \\ n-r \end{bmatrix}_{(1\ 2\ 3)}$ is a polynomial of n. Here are some examples. Let $g = (1\ 2\ 3)$. From (3.27) we have

$$\begin{bmatrix} n \\ n-4 \end{bmatrix}_g = 6 + \sum_{t=3}^{n-1} t \begin{bmatrix} t \\ t-3 \end{bmatrix}_g \tag{3.29}$$

and then by (3.26) we get

$$\begin{aligned} \begin{bmatrix} n \\ n-4 \end{bmatrix}_g &= 6 + \frac{1}{8} \sum_{t=3}^{n-1} (3t^5 - 10t^4 - 3t^3 + 10t^2 + 72t) \\ &= 6 + \frac{1}{8}(3\sum_{t=3}^{n-1} t^5 - 10\sum_{t=3}^{n-1} t^4 - 3\sum_{t=3}^{n-1} t^3 + 10\sum_{t=3}^{n-1} t^2 + 72\sum_{t=3}^{n-1} t). \end{aligned} \tag{3.30}$$

Simplifying (3.30), we have

$$\begin{bmatrix} n \\ n-4 \end{bmatrix}_g = \frac{1}{16}(n^6 - 7n^5 + 11n^4 + 3n^3 + 60n^2 - 68n - 240).$$

From the recurrence (3.27) and using **Mathematica** we have for example

$$\begin{aligned}
\left[{n \atop n-5} \right]_g &= \frac{1}{1920}(15n^8 - 180n^7 + 710n^6 - 1008n^5 + 2135n^4 - 6060n^3 - 8620n^2 \\
&\quad + 13008n + 63360), \\
\left[{n \atop n-6} \right]_g &= \frac{1}{3840}(3n^{10} - 55n^9 + 380n^8 - 1238n^7 + 2527n^6 - 5399n^5 + 3130n^4 \\
&\quad + 13508n^3 + 45800n^2 - 58656n - 241920), \\
\left[{n \atop n-7} \right]_g &= \frac{1}{967680}(63n^{12} - 1638n^{11} + 17199n^{10} - 94094n^9 + 306369n^8 - 706890n^7 \\
&\quad + 1117557n^6 - 262122n^5 + 1373148n^4 - 9398872n^3 - 20232576n^2 \\
&\quad + 27881856n + 124830720), \\
\left[{n \atop n-8} \right]_g &= \frac{1}{1935360}(9n^{14} - 315n^{13} + 4641n^{12} - 37583n^{11} + 186599n^{10} \\
&\quad - 614273n^9 + 1399179n^8 - 1957077n^7 + 1656900n^6 - 4912432n^5 \\
&\quad + 34720n^4 + 32416400n^3 + 86712192n^2 - 114888960n - 493516800).
\end{aligned}$$

3.4 Connection between Ball and Sphere Intersection

In this section we show that for a given permutation g in Γ_n the number $\left[{n \atop k} \right]_g$ can be considered as a sum of functions satisfying the Stirling recursion. These functions will provide us the numbers later called *sphere intersection numbers.*

As we have seen before, the Stirling number $\left[{n \atop k} \right]$ of the first kind is the number of permutations in Sym_n having k cycles in their disjoint cycle decomposition. Furthermore, the sphere S_{n-k} is the set counted by $\left[{n \atop k} \right]$, that is, $|S_{n-k}| = \left[{n \atop k} \right]$ or the other way around, $|S_k| = \left[{n \atop n-k} \right]$. Suppose that g is a permutation in Sym_n with $|g| = n-k$ for some $0 \leq k \leq n-1$. Recall that $|g|$ is the number of cycles of g. Since $|g| = n-k$ we have that g is in S_k and therefore can be expressed as a product of k transpositions. It then follows that

$$S_i g \subseteq \bigcup_{i-k \leq j \leq i+k} S_j.$$

for all $0 \leq i \leq n-1$. Let

$$Z(n,i,j;g) = S_j(\Gamma_n) \cap S_i(\Gamma_n)g$$

and let $z(n,i,j;g) = |Z(n,i,j;g)|$. Given a fixed integer t, the numbers

$$z(n,i) := z(n,i,i+t;g)$$

satisfy the Stirling recursion. Using the Cancellation Lemma, the proof is straightforward as before. Clearly, $z(n,i,i;e) = \left[{n \atop n-i} \right]$, the Stirling number of the first kind. In addition, for any fixed integer $r \geq 1$ we have

$$\sum_{i,j} z(n,i,j;g) = \left[\begin{matrix} n \\ n-r \end{matrix} \right]_g \tag{3.31}$$

where $0 \leq i \leq r$ and $0 \leq j \leq r$. Therefore, the number $\left[{n \atop n-r} \right]_g$ has the $z(n,i,j;g)$'s as its building blocks, and the latter depend on the initial conditions $z(s,i,i+t;g)$ with $s = supp(g)$ and $-k \leq t \leq k$. Also, we have $z(n,i,i+t;g) = 0$ if k and t do not have the same parity. Recall that $k = n - |g|$, and this means that g belongs to $S_k(\Gamma_n)$. Note from the definition of r-Stirling numbers, one can see that

$$\left[\begin{matrix} n \\ k \end{matrix} \right]_r = z(n, n-k, n-k+r-1; (1\ 2\ 3\ \ldots\ r)).$$

Next, we observe a relation between $z(n,r,r;g)$ and $\left[{n \atop n-r} \right]_g$ for any integer $r \geq 1$. Suppose that v is in $Z(n,r,r;g)$. Then

$$v = \underbrace{\ldots(\ldots\alpha\ldots)\ldots}_{\text{exactly } n-r \text{ cycles}} \qquad \text{and} \qquad v \cdot g = \underbrace{\ldots(\ldots\alpha\ldots)\ldots \cdot g}_{\text{exactly } n-r \text{ cycles}}. \tag{3.32}$$

In particular, if $g = e$ then $Z(n,r,r;g) = S_r(\Gamma_n)$. Moreover, if we let $0 \leq i \leq r$ and $0 \leq j \leq r$ then

$$\bigcup_{i,j=0}^{r} Z(n,i,j;g) = B_r(\Gamma_n) \cap B_r(\Gamma_n)g, \tag{3.33}$$

and hence we have (3.34) obtained from (3.32) by changing the word 'exactly' to 'at least'. That is, if v belongs to $B_r \cap B_r g$ then

$$v = \underbrace{\ldots(\ldots\alpha\ldots)\ldots}_{\text{at least } n-r \text{ cycles}} \qquad \text{and} \qquad v \cdot g = \underbrace{\ldots(\ldots\alpha\ldots)\ldots \cdot g}_{\text{at least } n-r \text{ cycles}}. \tag{3.34}$$

In Figure 3.2, the single lines refer to the connection between the ball (or sphere) and its multiplication with a permutation, while the double lines refer to swapping between the words *exaxtly* and *at least*. Further, if we choose $g = e$ then $B_i \cdot g = B_i$ and $S_i \cdot g = S_i$, which means that there is nothing being moved.

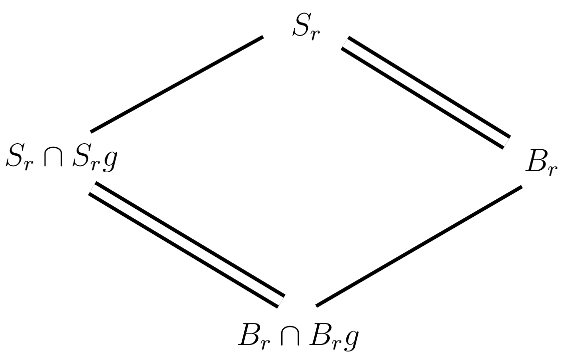

Figure 3.2: Relation between ball and sphere intersection

3.5 Comments from the Point of View of Representation Theory

In this section we look at a property of the intersection numbers of the transposition Cayley graph $\Gamma_n = G_n(2^1)$.

Fix $r \geq 0$ and $n \geq 3$. Suppose that g is a permutation in $G_n := \mathrm{Sym}_n$. Then the function $f_{n,r} : G_n \to \mathbb{C}$ defined by

$$f_{n,r}(g) := \begin{bmatrix} n \\ n-r \end{bmatrix}_g = |B_r \cap B_r g|$$

g	(1)	(1 2)	(1 2)(3 4)	(1 2 3)	(1 2 3 4)
$f_{4,0}$	1	0	0	0	0
$f_{4,1}$	7	2	2	3	0
$f_{4,2}$	18	12	14	15	12
$f_{4,3}$	24	24	24	24	24

Table 3.5: The value of $f_{n,r}$ on each conjugacy class

g	(1)	(1 2)	(1 2)(3 4)	(1 2 3)	(1 2 3 4)
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	-1	2	0
χ_4	3	1	0	-1	-1
χ_5	3	-1	0	-1	1

Table 3.6: The character table of Sym_4

is a class function, that is, $\left[\begin{smallmatrix} n \\ n-r \end{smallmatrix}\right]_g = \left[\begin{smallmatrix} n \\ n-r \end{smallmatrix}\right]_{g'}$ for every g' in the conjugacy class containing g. It is known in representation theory that any class function is a linear combination of the irreducible characters. Here an interesting problem arises: Is the class function $f_{n,r}$ a character? More generally, what is the connection between $f_{n,r}$ and the irreducible characters of Sym_n? Computing by **GAP**, we have the value of $f_{4,r}$ as shown in Table 3.5. Note that if $k \geq n$ then $f_{n,k}(g) = |G_n| = n!$ for all g in G_n. To this end, let us recall Proposition 2.7.3. It states that:

Proposition 3.5.1. *Let $\chi_1, \ldots, \chi_m$ be the irreducible characters of G. If ϑ is a character then*

$$\vartheta = d_1\chi_1 + \ldots + d_m\chi_m$$

where $d_i = \langle \vartheta, \chi_i \rangle$ are non-negative integers.

Consider the character table of the symmetric group Sym_4, shown in Figure 3.6. The number of all irreducible characters is equal to the number of all conjugacy classes of Sym_n. This is also equal to the number of ways to partition a set of size n, which is known as the n^{th} Bell number (see [25] for details). This is equal to $\sum_{i=1}^{n} \left\{ {n \atop i} \right\}$ where $\left\{ {n \atop i} \right\}$'s are Stirling numbers of the second kind.

Recall that for any group G and functions $\vartheta, \phi : G \to \mathbb{C}$ the inner product of ϑ and ϕ is defined by

$$\langle \vartheta, \phi \rangle = \frac{1}{|G|} \sum_{g \in G} \vartheta(g) \overline{\phi(g)}. \tag{3.35}$$

From Tables 3.5 and 3.6 and (3.35) we have

$$\begin{aligned}
f_{4,0} &= \frac{1}{24}\chi_1 + \frac{1}{24}\chi_2 + \frac{1}{12}\chi_3 + \frac{1}{8}\chi_4 + \frac{1}{8}\chi_5,\\
f_{4,1} &= \frac{49}{24}\chi_1 + \frac{25}{24}\chi_2 + \frac{1}{12}\chi_3 + \frac{9}{8}\chi_4 + \frac{1}{8}\chi_5,\\
f_{4,2} &= \frac{27}{2}\chi_1 + \frac{3}{2}\chi_2 \qquad\quad + \frac{1}{2}\chi_4 + \frac{1}{2}\chi_5,\\
f_{4,3} &= 24\chi_1
\end{aligned}$$

By Proposition 3.5.1, since the coefficient of χ_1 is not a non-negative integer, none of $f_{4,0}, f_{4,1}$ and $f_{4,2}$ is a character. Note that $f_{4,3}$ is a character of the representation $g \mapsto I_{24} \in GL(24, \mathbb{C})$ where I_{24} is the identity matrix in $GL(24, \mathbb{C})$. Also, by **GAP**, we have that for all $3 \leq n \leq 7$ and $1 \leq r \leq n-2$, none of the $f_{n,r}$'s is a character. We conjecture that this is true for any $n \geq 3$. Nevertheless, since all coefficients are positive, we have that $24 \cdot f_{4,r}$ is a character for all r. In addition, it is clear that if $k \geq n-1$ then $f_{n,k}$ is the character of the representation $g \mapsto I \in GL(n!, \mathbb{C})$ where I is the identity of $GL(n!, \mathbb{C})$. This is because the diameter of Γ_n equals $n!$.

In Section 2.7 we introduced the class sums of conjugacy classes of groups. We next show how these class sums in the symmetric group G_n relate to the transposition Cayley graph $\Gamma_n = G_n(2^1)$. Suppose that u is a vertex in S_i for some i. Then the downward degree $c(u)$ is the number of edges $\{v, u\}$ incident to the vertex u with v in S_{i-1}, or in other words, it is equal to the number of vertices v in S_{i-1} that are adjacent to u. A question arises here: Given vertices v in S_{i-1} and u in S_i, what is the number of vertices in the conjugacy class v^{G_n} adjacent to u?

g	(1)	(1 2)	(1 2)(3 4)	(1 2 3)	(1 2 3 4)	(1 2 3)(4 5)	(1 2 3 4 5)
χ_1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	-1	-1	1
χ_3	4	2	1	0	0	-1	-1
χ_4	4	-2	1	0	0	1	-1
χ_5	6	0	0	-2	0	0	1
χ_6	5	1	-1	1	-1	1	0
χ_7	5	-1	-1	1	1	-1	0

Table 3.7: The character table of Sym_5

EXAMPLE: Let $G_5 = \mathrm{Sym}_5$. Suppose that $C_0 = H = (1\ 2)^{G_5}, C_1 = (1\ 2\ 3\ 4)^{G_5}, C_2 = (1\ 2\ 3)^{G_5}$ and $C_3 = (1\ 2)(3\ 4)^{G_5}$. From (2.36) we have that the number of vertices in C_2 adjacent to $(1\ 2\ 3\ 4)$ is equal to the class algebra constant a_{201} and we know that

$$a_{201} = \frac{|G|}{|C_G(g_2)|\,|C_G(g_0)|} \sum_{\chi} \frac{\chi(g_2)\chi(g_0)\overline{\chi(g_1)}}{\chi(1)}.$$

with $g_0 = (1\ 2), g_1 = (1\ 2\ 3\ 4)$ and $g_2 = (1\ 2\ 3)$. From the character table of Sym_5 shown in Table 3.7, we have

$$a_{201} = \frac{5!}{6 \cdot 12}\left[1 + 1 + 0 + 0 + 0 + \frac{1}{5} + \frac{1}{5}\right] = 4.$$

Similarly, we have that the number of vertices in $g_3^{G_5}$ adjacent to $(1\ 2\ 3\ 4)$, with $g_3 = (1\ 2)(3\ 4)$, is equal to $a_{301} = 2$. Further, we know that $c((1\ 2\ 3\ 4)) = \binom{4}{2} = 6 = 4 + 2$. In the graph's point of view,

$$c(g_1) = |S_{i-1} \cap S_1 g_1| = \left|g_2^{G_5} \cap S_1 g_1\right| + \left|g_3^{G_5} \cap S_1 g_1\right| = a_{201} + a_{301}, \tag{3.36}$$

and in general (3.36) holds for any $n \geq 4$ since the downward degree is independent of n.

Chapter 4

Double-Transposition Cayley Graphs

Throughout this chapter we devote our attention to Cayley graphs on the alternating group that are generated by the conjugacy class of all double-transpositions, that is, all elements of the shape $(\alpha\ \beta)(\gamma\ \delta)$ where $\alpha, \beta, \gamma, \delta$ are mutually different. In the remainder, G_n is referred to as the symmetric group Sym_n, and G'_n is referred to as the derived subgroup of G_n, the alternating group Alt_n. In $G_4 = \mathrm{Sym}_4$ the subgroup of G_4 generated by the conjugacy class $H = (2^2)^{G_4}$ of all double-transposition is the Klein four-group V. For $n \geq 5$ we have already shown in Proposition 2.3.4 that the subgroup $\langle H \rangle$ is the alternating group Alt_n. Therefore, for $n \geq 5$ we let $G'_n(2^2)$ be the Cayley graph on Alt_n generated by the set of all double-transpositions. We call $G'_n(2^2)$ the *double-transposition Cayley graph* of Alt_n. Unless stated otherwise, we let Γ_n be $G'_n(2^2)$ and let $H = (2^2)^{G_n}$ be the set of all double-transpositions.

4.1 Sphere Classification

We start with a short survey of this new graph. Obviously, $|H| = \frac{1}{2}\binom{n}{2}\binom{n-2}{2}$. Then, by its construction, Γ_n is a $\frac{1}{2}\binom{n}{2}\binom{n-2}{2}$-regular graph. Also, it contains triangles, for instance $(1) \to (1\ 2)(3\ 4) \to (2\ 3)(1\ 4) = (1\ 2)(3\ 4) \cdot (1\ 3)(2\ 4) \to (1)$. Hence Γ_n cannot be embedded into any transposition Cayley graph since the latter contains no triangle. However, on their own the vertices of $G'_n(2^2)$ can be viewed as group elements in $G_n(2^1)$. This gives us an opportunity to use our powerful equipment from $G_n(2^1)$, the Cancellation Lemma. As customary we let $S_r = S_r(\Gamma_n, e)$ and $B_r = B_r(\Gamma_n, e)$ with $\Gamma_n = G'_n(2^2)$. Recall that $|g|$ is the number of cycles in the cycle decomposition of g, including cycles of length one. Also, a *k-cycle* is a permutation whose cycle type is $1^{n-k}k^1$.

Here we give a proposition that allows one to determine which sphere a given vertex belongs to.

Proposition 4.1.1. *Let $n \geq 5$. We have that g belongs to S_2 if and only if either $|g| = n-4$ or g is a 3-cycle. If $r \geq 3$ then g belongs to S_r if and only if $|g| = n-2r$.*

Proof. Clearly, it suffices to show that this holds for a representative of each cycle type as permutations with the same cycle type must be in the same sphere. This is an earlier result. Let g belong to Γ_n. If $g = (1\ 2\ 3)$ then $g = (1\ 2)(4\ 5) \cdot (1\ 3)(4\ 5)$. Hence g belongs to S_2. Suppose that $|g| = n-4$. If $g = (1\ 2\ 3\ 4)(5\ 6)$ then $g = a \cdot b$ with $a = (1\ 2)(3\ 4)$ and $b = (1\ 3)(5\ 6)$. There are four other cycle types with $n-4$ cycles, namely $1^{n-8}2^4, 1^{n-5}5^1, 1^{n-6}3^2$ and $1^{n-7}2^23^1$. The following shows how to express such permutations as a product of two double-transpositions.

$$(1\ 2)(3\ 4) \cdot (5\ 6)(7\ 8) = (1\ 2)(3\ 4)(5\ 6)(7\ 8), \qquad (1\ 2)(3\ 4) \cdot (1\ 3)(2\ 5) = (1\ 5\ 2\ 3\ 4),$$

$$(1\ 2)(3\ 4) \cdot (1\ 5)(3\ 6) = (1\ 2\ 5)(3\ 4\ 6), \qquad (1\ 2)(3\ 4) \cdot (1\ 5)(6\ 7) = (1\ 2\ 5)(3\ 4)(6\ 7).$$

Verifying through these representatives, we have proved that if g is a 3-cycle or $|g| = n-4$ then g must be in S_2. Next, suppose that g belongs to S_2. Considering g as a group element in $G_n(2^1)$ we have that g is contained in the ball of radius four of $G_n(2^1)$, as g is a product of two double-transpositions. Since g is an even permutation we have $|g| = n-2i$ with $i = 0, 1, 2$. As $H = S_1$ and $\{e\} = S_0$, we have proved the first part.

In general, any permutation g in B_r is a product of at most $2r$ transpositions, by the definition. Therefore, every permutation g in B_r has at least $n-2r$ cycles. Since $S_r = B_r \setminus B_{r-1}$ we have that for $r \geq 3$, if g is in S_r, then $|g| = n-2r$. Next assume that g is in S_3. Then there exist h in S_1 and x in S_2 such that $g = xh$. Since multiplication by a transposition is either gluing or splitting two disjoint cycles together, multiplying by a double-transposition in Γ_n either increases or decreases the number of cycles by two, or leaves the number of cycles constant. Since x belongs to S_2 and g belongs to S_3 we have $|x| = n-4$. Hence we get $|g| = n-6, n-4$ or $n-2$. Since g is not in B_2 we have $|g| = n-6$. We leave the proof now as the inductive step can be proved similarly, considering any permutation g in S_r as a product of some permutations x in S_{r-1} and h in S_1. $\square$

The above proposition provides us a way to determine the diameter of a given double-transposition Cayley graph.

Corollary 4.1.2. *For $n \geq 5$ the graph $G'_n(2^2)$ has diameter $\left\lfloor \frac{n-1}{2} \right\rfloor$.*

Proof. Let $n \geq 5$ and let g be a vertex in $G'_n(2^2)$. From Proposition 4.1.1, we have that for all $i \geq 2$ if $|g| = n-2i$ then g belongs to S_i. Suppose that $n = 2k$ for some k. Then the value of $|g|$ is at least two since any permutation having only one cycle is not an even permutation. Hence the distance $d(e,g)$ is at most $\frac{2k-2}{2} = k-1$. On the other hand, if $n = 2k+1$ then $|g| \geq 1$. Therefore $d(e,g)$ is at most $\frac{2k+1-1}{2} = k$. $\square$

4.2 Intersection Numbers of Double-Transposition Cayley Graphs

As we said before in Chapter 1, the idea of considering the intersection number $N(G_n(2^1), r)$ originally came up when we began to study the double-transposition

Cayley graphs $G'_n(2^2)$ and their intersection numbers. In the rest of this chapter, the pattern of studying the graphs and the intersection numbers will be similar to the preceding chapter. Recall that in this chapter we let $\Gamma_n = G'_n(2^2)$ be the double-transposition Cayley graph on Alt_n.

4.2.1 The Case of Radius One

Here we find the value of $N(\Gamma_n, 1)$ for all $n \geq 5$.

Theorem 4.2.1. *We have $N(\Gamma_5, 1) = 5$ and $N(\Gamma_n, 1) = \frac{3}{2}(n^2 - 7n + 12)$ for $n \geq 6$.*

Proof. To determine $N(\Gamma_5, 1)$ it suffices to consider only three cycle types, namely $1^{n-4}2^2, 1^{n-3}3^1$ and $1^{n-5}5^1$. Let $N(g) = |B_1 \cap B_1 g|$. Then $N(\Gamma_5, 1) = \max\limits_{g=g_1,g_2,g_3} N(g)$ with $g_1 = (1\ 2)(3\ 4)$, $g_2 = (1\ 2\ 3)$ and $g_3 = (1\ 2\ 3\ 4\ 5)$. Using **GAP** we have that $N(g_1) = 4$, $N(g_2) = 3$ and $N(g_3) = 5$. Hence $N(\Gamma_5, 1) = 5$.

Next, we let $n \geq 6$. Suppose that g belongs to S_2 with $|g| = n - 4$. Then there are three possible cycle types of g, namely $1^{n-6}3^2, 1^{n-6}2^1 4^1$ and $1^{n-5}5^1$. It is not hard to see that $N((1\ 2\ 3)(4\ 5\ 6)) = \binom{3}{2}\binom{3}{2} = 9$ and $N((1\ 2\ 3\ 4\ 5)) = 5$, and $N((1\ 2)(3\ 4\ 5\ 6) = 4$. Clearly, these numbers are independent of n. It then remains to compare $N((1\ 2\ 3)(4\ 5\ 6))$ with $N(g_1)$ and $N(g_2)$ where $g_1 = (1\ 2)(3\ 4)$ and $g_2 = (1\ 2\ 3)$. Suppose that there is $h = (\alpha\ \beta)(\gamma\ \delta)$ such that $g_1 \cdot h$ is in S_1. As $(\alpha\ \beta)$ and $(\gamma\ \delta)$ commute, we may suppose that $g_1(\alpha\ \beta)$ is a transposition and $g_1(\alpha\ \beta)(\gamma\ \delta)$ becomes a double-transpostion again. Then $(\alpha\ \beta)$ must be one of $(1\ 2)$ or $(3\ 4)$ and after that we have $\binom{n-4}{2}$ choices for $(\gamma\ \delta)$. Hence, including e and g_1 itself, we have that

$$N(g_1) = 2\binom{n-4}{2} + 2 = n^2 - 9n + 22.$$

Similarly, we have that

$$N(g_2) = 3\binom{n-3}{2} = \frac{3}{2}(n^2 - 7n + 12).$$

Hence for all $n \geq 6$ we have $N(g_2) \geq \max\{N(g_1), 9\}$. Therefore,

$$N(\Gamma_n, 1) = \frac{3}{2}(n^2 - 7n + 12).$$

$\square$

4.2.2 The Stirling Recursion in Double-Transposition Cayley Graphs

We first introduce notation to enable us to link the graphs $G_n(2^1)$ and $G'_n(2^2)$, and then to apply (3.15) in a very natural way to determine the value $N(\Gamma_n, r)$ of $\Gamma_n = G'_n(2^2)$. In Proposition 4.2.3 we show that the numbers $N(\Gamma_n, r)$, considered in another form, satisfy the Stirling recursion too.

Let $\overline{S}_r$ and $\overline{B}_r$ be the sphere $S_r(G_n(2^1), e)$ and the ball $B_r(G_n(2^1), e)$ in the transposition Cayley graph $G_n(2^1)$ on Sym_n, respectively. We let $Z_r := Z_{n,r}$ be the set of vertices in $G_n(2^1)$ of Sym_n defined by

$$Z_r := \begin{cases} \overline{B}_r \cap \mathrm{Alt}_n & \text{if } r \text{ is even,} \\ \overline{B}_r \cap (\mathrm{Sym}_n \setminus \mathrm{Alt}_n) & \text{if } r \text{ is odd.} \end{cases} \tag{4.1}$$

That is, Z_r can be obtained from the ball $\overline{B}_r$ of radius r in $G_n(2^1)$ by omitting $\overline{S}_i$ for all $i \not\equiv r \pmod 2$.

For each g in Γ_n and $r \geq 0$ we let

$$I_g(n, r; 2^2) := \left|\overline{B}_r \cap Z_r g\right|. \tag{4.2}$$

Obviously, if $r \geq 2$ then, by Proposition 4.1.1, we have

$$I_g(n, 2r; 2^2) = \left|\overline{B}_{2r} \cap B_r g\right| = |B_r \cap B_r g| \tag{4.3}$$

with $B_r = B_r(G'_n(2^2), e)$.

Remark: (1) $I_g(n, 2; 2^2) > |B_1 \cap B_1 g|$ since $(3^1)^{G_n} \not\subseteq B_1$.

(2) Instead of studying the double-transposition Cayley graph $G'_n(2^2)$, we now think of the vertices in $G'_n(2^2)$ as vertices in $G_n(2^1)$ so that we can use our facilities provided for the transposition Cayley graphs, for instance the Cancellation Lemma.

(3) Note that the symbol '2^2' is added so that there will be no ambiguity.

Next we show that the numbers $I_g(n, r; 2^2)$ satisfy a familiar recursion.

Proposition 4.2.2. *Let n and s be positive integers with $n > s$ and let g be a permutation in Γ_n such that $Supp(g) = [\,s\,]$. Then*

$$I_g(n, r; 2^2) = I_g(n-1, r; 2^2) + (n-1) I_g(n-1, r-1; 2^2). \tag{4.4}$$

Note that the permutation g on the right-hand side of the equation in (4.4), and also in (4.6), is a permutation in Γ_{n-1}.

Proof. Let Z be the set counted by $I_g(n, r; 2^2)$. As we proved in Proposition 3.3.2, Z is divided into two sets, one of which, say X, consists exactly of those permutations fixing n. The other set Y is composed of those moving n. Using the same arguments as before, by the Cancellation Lemma we have

$$|X| = I_g(n-1, r; 2^2) \quad \text{and} \quad |Y| = (n-r) I_g(n-1, r-1; 2^2).$$

The proof is complete as X and Y are disjoint. □

Here we provide a function defined analogously to (3.17) in the preceding chapter. For each $r \geq 0$ we let

$$\begin{bmatrix} n \\ n-r \end{bmatrix}_g := \left|\overline{B}_r \cap Z_r g\right|. \tag{4.5}$$

Remark: In this chapter $\left[\begin{smallmatrix} n \\ n-r \end{smallmatrix}\right]_g$ is defined for the graph $G'_n(2^2)$, not for $G_n(2^1)$.

With the same arguments we used in Proposition 3.3.3 we have:

Proposition 4.2.3. *Let n and s be positive integers with $n > s$. If g is a permutation in Γ_n such that $Supp(g) = [\,s\,]$, then*

$$\left[\begin{array}{c} n \\ k \end{array}\right]_g = \left[\begin{array}{c} n-1 \\ k-1 \end{array}\right]_g + (n-1)\left[\begin{array}{c} n-1 \\ k \end{array}\right]_g \tag{4.6}$$

for any integer $1 \leq k \leq n$.

Remark: As in Theorem 3.3.3, one can determine the value $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_g$ by the s initial conditions $\left[\begin{smallmatrix} s \\ 1 \end{smallmatrix}\right]_g$, $\left[\begin{smallmatrix} s \\ 2 \end{smallmatrix}\right]_g, \ldots,$ $\left[\begin{smallmatrix} s \\ s-1 \end{smallmatrix}\right]_g, \left[\begin{smallmatrix} s \\ s \end{smallmatrix}\right]_g$. Also, from Definition (4.5) and Proposition 4.1.1 we have $\left[\begin{smallmatrix} s \\ s \end{smallmatrix}\right]_g = 1$ if $g = e$; otherwise $\left[\begin{smallmatrix} s \\ s \end{smallmatrix}\right]_g = 0$. Moreover, $\left[\begin{smallmatrix} s \\ m \end{smallmatrix}\right]_g = \left|\Gamma_n\right| = \dfrac{n!}{2}$ for all $m \leq 1$.

4.2.3 Ball Intersection Numbers for Vertices in the Ball of Radius Two

In this section we show in Theorem 4.2.5 that the vertex (1 2 3) still dominates (1 2)(3 4) in the same way as it does in the transposition Cayley graph.

We now consider $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_g$. It is clear that $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_g$ is a class function. From (4.5), given an element g in Γ_n, one can construct Table $IN_g^{2^2}$ listing the values of $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_g$ as we did in Chapter 3. These were computed by **GAP**.

n	k: 1	2	3	4	5	6	7	8	$\cdots$
5	60	60	24	3	0	–	–	–	$\cdots$
6	360	360	180	39	3	0	–	–	$\cdots$
7	2520	2520	1440	414	57	3	0	–	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\ddots$

Table 4.1: $IN_g^{2^2}$ with $g = (1\ 2\ 3)$

From (4.6), we have:

Corollary 4.2.4. *Let g_1 and g_2 be non-identity vertices in Γ_n. If there exist positive intergers m and t such that $\left[\begin{smallmatrix} m \\ k \end{smallmatrix} \right]_{g_1} \geq \left[\begin{smallmatrix} m \\ k \end{smallmatrix} \right]_{g_2}$ for all $t \leq k \leq m$ then $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_{g_1} \geq \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_{g_2}$ for all $n \geq m$ and $t + (n-m) \leq k \leq n$. Moreover, if $t = 1$ then $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_{g_1} \geq \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_{g_2}$ for all $n \geq m$ and $k \geq 1$.*

Theorem 4.2.5. *For any $n \geq 5$ and $r \geq 2$ we have $N_2(\Gamma_n, r) \geq N_1(\Gamma_n, r)$. Further, if $r < \left\lfloor \frac{n-1}{2} \right\rfloor$ then $N_2(\Gamma_n, r) > N_1(\Gamma_n, r)$.*

Proof. Let $g_1 = (1\ 2\ 3)$ and $g_2 = (1\ 2)(3\ 4)$. Using **GAP**, the initial values of $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_{g_1}$ and $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_{g_2}$ are provided below.

$$\begin{bmatrix} 5 \\ 1 \end{bmatrix}_{g_1} = 60 \qquad \begin{bmatrix} 5 \\ 2 \end{bmatrix}_{g_1} = 60 \qquad \begin{bmatrix} 5 \\ 3 \end{bmatrix}_{g_1} = 24 \qquad \begin{bmatrix} 5 \\ 4 \end{bmatrix}_{g_1} = 3 \qquad \begin{bmatrix} 5 \\ 5 \end{bmatrix}_{g_1} = 0$$

and

$$\begin{bmatrix} 5 \\ 1 \end{bmatrix}_{g_2} = 60 \qquad \begin{bmatrix} 5 \\ 2 \end{bmatrix}_{g_2} = 60 \qquad \begin{bmatrix} 5 \\ 3 \end{bmatrix}_{g_2} = 20 \qquad \begin{bmatrix} 5 \\ 4 \end{bmatrix}_{g_2} = 2 \qquad \begin{bmatrix} 5 \\ 5 \end{bmatrix}_{g_2} = 0.$$

From Corollary 4.2.4 we have

$$\begin{bmatrix} n \\ n-m \end{bmatrix}_{g_1} \geq \begin{bmatrix} n \\ n-m \end{bmatrix}_{g_2}$$

for all $n \geq 5$ and $m \geq 0$. Hence,

$$N_2(\Gamma_n, r) \geq \begin{bmatrix} n \\ n-2r \end{bmatrix}_{g_1} \geq \begin{bmatrix} n \\ n-2r \end{bmatrix}_{g_2} \geq N_1(\Gamma_n, r)$$

since g_1 belongs to S_2 and since $S_1 = (2^2)^{G_n}$, not $(2^2)^{G_n} \cup (3^1)^{G_n}$.

Recall that

$$N_i(\Gamma_n, r) = \max \left\{ \begin{bmatrix} n \\ n-2r \end{bmatrix}_g : g \in S_i \right\} \tag{4.7}$$

for any $r \geq 2$ and $i \geq 2$. From the initial conditions shown above one can see, by Corollary 4.2.4, that

$$\begin{bmatrix} n \\ k \end{bmatrix}_{g_1} > \begin{bmatrix} n \\ k \end{bmatrix}_{g_2} \tag{4.8}$$

for any $k \geq 3$. Clearly, $\left[{n \atop k} \right]_{g_1} = \left[{n \atop k} \right]_{g_2}$ for any $k = 1, 2$ as they count all vertices in Γ_n. The last inequality follows from (4.7) and (4.8), and also from the fact that the diameter of Γ_n is $\left\lfloor \frac{n-1}{2} \right\rfloor$. □

4.2.4 The Asymptotic Behaviour of Ball Intersection Numbers

We first recall that $\overline{B}_i$ and $\overline{S}_i$ are the ball $B_i(G_n(2^1), e)$ and the sphere $S_i(G_n(2^1), e)$, respectively. Also, in the remainder of this thesis $\left[{n \atop n-r} \right]_g$ is as defined in (4.5). From Lemma 3.3.6 in the previous chapter we have that for any permutation g in $\overline{S}_{2j} \cup \overline{S}_{2j-1}$,

$$\left|\overline{B}_r \cap \overline{B}_r g\right| = O(n^{2(r-j)}). \tag{4.9}$$

Here, we claim that $\left[{n \atop n-r} \right]_g = O(n^{2(r-j)})$ too, when g is in $\overline{S}_{2j}$.

Lemma 4.2.6. *Let $\Gamma_n = G'_n(2^2)$ and let g belong to $\overline{S}_{2j}$. Then*

$$\left[\begin{matrix} n \\ n-r \end{matrix} \right]_g = O(n^{2(r-j)}). \tag{4.10}$$

Proof. Let $r \geq 1$. From the definitions of $\overline{B}_r$ and Z_r one can see that

$$\overline{B}_r = Z_r \cup Z_{r-1}$$

where Z_r is as defined in (4.1). Since g is an even permutation we get

$$\overline{B}_r \cap \overline{B}_r g = (Z_r \cap Z_r g) \mathbin{\dot\cup} (Z_{r-1} \cap Z_{r-1} g),$$

and then

$$\left|\overline{B}_r \cap \overline{B}_r g\right| = |Z_r \cap Z_r g| + |Z_{r-1} \cap Z_{r-1} g|. \tag{4.11}$$

Since $Z_{r-1} \subseteq \overline{B}_{r-1}$, by (4.9) we have

$$|Z_{r-1} \cap Z_{r-1} g| \leq \left|\overline{B}_{r-1} \cap \overline{B}_{r-1} g\right| = O(n^{2(r-1-j)}).$$

From (4.11), it follows that

$$\begin{bmatrix} n \\ n-r \end{bmatrix}_g = \left|\overline{B}_r \cap Z_r g\right| = |Z_r \cap Z_r g| = O(n^{2(r-j)}),$$

as $\left|\overline{B}_r \cap \overline{B}_r g\right| = O(n^{2(r-j)})$. $\square$

Now by Theorem 4.2.5, Proposition 4.1.1 and Lemma 4.2.6 we have that for all $r \geq 2$

$$N(\Gamma_n, r) = N_2(\Gamma_n, r) = \begin{bmatrix} n \\ n-2r \end{bmatrix}_{(1\ 2\ 3)} \tag{4.12}$$

if n is sufficiently large. Hence we have

Theorem 4.2.7. *Let $r \geq 2$ and let $\Gamma_n = G'_n(2^2)$. We have*

$$N(\Gamma_n, r) = \begin{bmatrix} n \\ n-2r \end{bmatrix}_{(1\ 2\ 3)} \tag{4.13}$$

if n is sufficiently large.

We believe that the above theorem holds for any $n \geq 5$. We try to assert this by showing that it is true for $r = 2$ in Section 4.3.

Conjecture 4.2.8. *Let $r \geq 2$ and let $\Gamma_n = G'_n(2^2)$. Then*

$$N(\Gamma_n, r) = \begin{bmatrix} n \\ n-2r \end{bmatrix}_{(1\ 2\ 3)}$$

for all $n \geq 5$.

4.2.5 Generating Functions in Double-Transposition Cayley Graphs

From the previous theorem we know that when n gets bigger, the vertex $g = (1\,2\,3)$ is likely to give us the ball intersection number. Actually, this means that any vertices u, v satisfying that $u^{-1}v$ is a 3-cycle will give us the maximum size of the ball intersection between any two vertices, when n is large enough. Here we provide the generating function for the size of ball intersection of those vertices.

Theorem 4.2.9. *Let* $g = (1\ 2\ 3)$ *and* $n \geq 5$. *Then*

$$\Psi(\Gamma_n, g; y) := (y+5)^{[n-5]}(3y^4 + 24y^3 + 60y^2 + 60y + 60 + \frac{60}{y} + \frac{60}{y^2} + \cdots)$$

is the generating function for $\left[{n \atop k} \right]_g$ *where* $y^{[n]} := y(y+1)\ldots(y+(n-1))$ *is the ascending factorial.*

Proof. Let $g = (1\,2\,3)$ and let $\Psi(\Gamma_n, g; y) = \sum_{-\infty}^{\infty} \left[{n \atop k} \right]_g y^k$ be the generating function of $\left[{n \atop k} \right]_g$ where $\left[{n \atop k} \right]_g = 0$ if $k > n$. Then, by Proposition 4.2.3, we have

$$\begin{aligned} \Psi(\Gamma_n, g; y) &= y\Psi(\Gamma_{n-1}, g; y) + (n-1)\Psi(\Gamma_{n-1}, g; y) \\ &= (y + (n-1))\Psi(\Gamma_{n-1}, g; y). \end{aligned}$$

By Table 4.1 we have that $\Psi(\Gamma_5, g; y) = 3y^4 + 24y^3 + 60y^2 + 60y + 60 + \frac{60}{y} + \frac{60}{y^2} + \cdots$. The proof is then complete. $\square$

4.3 The Case of Radius Two

In this section our main purpose is to determine $N(\Gamma_n, r)$ when $r = 2$. To do so we need to find the value of $\left[{n \atop n-4} \right]_g$ for all g in $B_4 = B_4(G'_n(2^2))$.

4.3.1 Computational Results from the Spheres of Radius One, Two and Three

Using **GAP**, we know the value of $\left[{8 \atop m} \right]$ for all g in $B_2(\Gamma_n)$ and all $4 \leq m \leq 8$ as listed in Table 4.2. Therefore, from Corollary 4.2.4 it follows that $\left[{n \atop n-4} \right]_{(1\ 2\ 3)} \geq \left[{n \atop n-4} \right]_g$ for all $n \geq 8$ and g in $B_2(\Gamma_n)$. Note that $n = 8$ is needed as it is the smallest number such that the graph $\Gamma_n = G'_n(2^2)$ contains all vertices g satisfying $|g| = n-4$. Recall that for all $i \geq 2$ any permutation g with $|g| = n - 2i$ belongs to S_i. It is not hard to see that if $|g| = n - 2i$ then $supp(g)$ is at most $4i$ and only the permutations g with $\mathrm{ct}(g) = 1^{n-4i}2^{2i}$ have $supp(g) = 4i$. Similarly, we need $n = 12$ to get all possible permutation types that will exist in S_3. Again, using **GAP** we list in Table 4.3 the value of $\left[{12 \atop m} \right]_g$ for all $8 \leq m \leq 12$ and all g in S_3, including $g = (1\ 2\ 3)$.

g	k				
	4	5	6	7	8
(1 2 3)	4338	813	78	3	0
(1 2)(3 4)	3700	634	56	2	0
(1 2 3 4 5)	3280	420	20	0	0
(1 2 3)(4 5 6)	2850	299	11	0	0
(1 2 3 4)(5 6)	2840	308	12	0	0
(1 2 3)(4 5)(6 7)	2438	226	8	0	0
(1 2)(3 4)(5 6)(7 8)	2052	176	6	0	0

Table 4.2: The valule of $\left[{8 \atop k} \right]_g$ with $|g| = n - 4$

g	k				
	8	9	10	11	12
(1 2 3)	87420	5394	192	3	0
(1 2 3 4 5 6 7)	9135	175	0	0	0
(1 2 3 4 5 6)(7 8)	5930	100	0	0	0
(1 2 3 4 5)(6 7 8)	5187	80	0	0	0
(1 2 3 4)(5 6 7 8)	4956	74	0	0	0
(1 2 3 4 5)(6 7)(8 9)	3872	60	0	0	0
(1 2 3 4)(5 6 7)(8 9)	3400	50	0	0	0
(1 2 3)(4 5 6)(7 8 9)	3117	45	0	0	0
(1 2 3 4)(5 6)(7 8)(9 10)	2586	38	0	0	0
(1 2 3)(4 5 6)(7 8)(9 10)	2364	34	0	0	0
(1 2 3)(4 5)(6 7)(8 9)(10 11)	1810	26	0	0	0
(1 2)(3 4)(5 6)(7 8)(9 10)(11 12)	1410	20	0	0	0

Table 4.3: The value of $\left[{12 \atop k} \right]_g$ with $|g| = n - 6$

Now from Tables 4.2 and 4.3, Theorem 4.2.5 and Corollary 4.2.4 we have

$$\begin{bmatrix} n \\ n-4 \end{bmatrix}_{(1\ 2\ 3)} \geq N_i(\Gamma_n, 2) \tag{4.14}$$

for all $i = 1, 2, 3$.

4.3.2 Results from the Sphere of Radius Four

To determine $N_4(\Gamma_n, 2)$ we need to find the value of $\left[\begin{smallmatrix} n \\ n-4 \end{smallmatrix}\right]_g$ for every g in S_4. Unfortunately, we have a problem in using **GAP**, since to get all permutations in S_4 we need to compute at $n = 16$ and this is too big. Instead, for any g in S_4 we estimate $\left[\begin{smallmatrix} n \\ n-4 \end{smallmatrix}\right]_g$ by computing the downward degree of g. Recall that if g belongs to S_r then the downward degree $c(g)$ of g is the number of permutations g' in S_{r-1} that are adjacent to g. This is the number of double-transpositions h such that $g \cdot h$ is in S_{r-1}.

Next we show how to find in general the downward degree $c(g)$ for any g in S_r when $r \geq 3$. Let us suppose that $\mathrm{ct}(g) = 1^{h_1} 2^{h_2} \ldots n^{h_n}$. The first case to consider occurs by choosing any two letters a and b from a single cycle of g, and by choosing another two letters c and d from a different cycle. For this choice $g \cdot h$ is in the sphere S_{r-1}. There are

$$\sum_{2 \leq i \leq n} \binom{h_i}{2} \binom{i}{2} \binom{i}{2} + \sum_{2 \leq i < j \leq n} h_i h_j \binom{i}{2} \binom{j}{2} \tag{4.15}$$

choices to do so. The first term in (4.15) is the number of choices when we choose a, b and c, d from cycles of the same length while the second is when we choose them from cycles whose length are different.

The only other way for $g \cdot h$ to be in S_{r-1} is to choose four letters a, b, c and d from a single cycle of g. Suppose that $(a \cdots b \cdots c \cdots d \cdots)$ is such a cycle. Then there are two ways to get h, namely $(a\ b)(c\ d)$ and $(a\ d)(c\ b)$. We cannot let h be

$(a\ c)(b\ d)$, because we will have $|g \cdot (a\ c)(b\ d)| = |g|$, that is, they are in the same sphere. Hence there are

$$\sum_{4 \le i \le n} 2h_i \binom{i}{4} \tag{4.16}$$

choices for the second case. In conclusion, for each g in S_r and $r \geq 3$, the downward degree $c(g)$ of g can be computed as

$$c(g) = \sum_{2 \le i} \binom{h_i}{2}\binom{i}{2} + \sum_{2 \le i < j \le n} h_i h_j \binom{i}{2}\binom{j}{2} + \sum_{4 \le i \le n} 2h_i \binom{i}{4} \tag{4.17}$$

with $ct(g) = 1^{h_1} \ldots n^{h_n}$.

From Equation (4.17) we have that

$$\max\{c(g) : g \in S_3\} = 70 \tag{4.18}$$

and

$$\max\{c(g) : g \in S_4\} = 252. \tag{4.19}$$

This gives directly that

$$\begin{bmatrix} n \\ n-4 \end{bmatrix}_g \leq 70 \cdot 252 = 17640 \tag{4.20}$$

for all $n \geq 9$ and g in S_4. Note that the number $c(g)$ does not depend on n and that $n = 9$ is the smallest number such that the sphere S_4 is not empty. Fortunately, using **GAP** we computed that $\left[\begin{smallmatrix} 10 \\ 6 \end{smallmatrix}\right]_{(1\ 2\ 3)} = 23775$ (we cannot use $\left[\begin{smallmatrix} 9 \\ 6 \end{smallmatrix}\right]_{(1\ 2\ 3)}$ as it is less than 17640). Hence, we have

$$\begin{bmatrix} n \\ n-4 \end{bmatrix}_{(1\ 2\ 3)} > N_4(\Gamma_n, 2) \tag{4.21}$$

for all $n \geq 10$.

4.3.3 The Conclusion for Double-Transposition Cayley Graphs

Here we show that the maximum of the ball intersection numbers occurs on the 3-cycles.

From (4.14) and (4.21) we have

$$N(\Gamma_n, 2) = \begin{bmatrix} n \\ n-4 \end{bmatrix}_{(1\ 2\ 3)} \tag{4.22}$$

for all $n \geq 12$. Again, computing by **GAP** we obtain $N(\Gamma_n, 2) = \left[\begin{smallmatrix} n \\ n-4 \end{smallmatrix}\right]_{(1\ 2\ 3)}$ for all $5 \leq n \leq 11$. Hence we can conclude that

Theorem 4.3.1. *Let* $\Gamma_n = G'_n(2^2)$. *For* $n \geq 5$ *we have*

$$N(\Gamma_n, 2) = \begin{bmatrix} n \\ n-4 \end{bmatrix}_{(1\ 2\ 3)} = \frac{1}{16}(n^6 - 7n^5 + 5n^4 + 23n^3 + 90n^2 - 112n - 480).$$

Proof. It remains to verify the second equality. Let $g = (1\ 2\ 3)$. By the Stirling recursion (4.6) we have that for a fixed r,

$$\begin{bmatrix} n \\ n-r \end{bmatrix}_g = \begin{bmatrix} 5 \\ 5-r \end{bmatrix}_g + \sum_{t=5}^{n-1} t \begin{bmatrix} t \\ t-(r-1) \end{bmatrix}_g \tag{4.23}$$

with $\left[\begin{smallmatrix} k \\ k \end{smallmatrix}\right]_g = 0$ for all $k \geq 5$. The proof is complete by substituting the initial conditions. $\square$

Recall that $N(\Gamma_n, 1) \neq \left[\begin{smallmatrix} n \\ n-2 \end{smallmatrix}\right]_{(1\ 2\ 3)}$ since no 3-cycle is in B_1. Then we cannot apply (4.23) to get the closed formula for $N(\Gamma_n, 1)$.

Lastly, we list for example the closed formulas of $\left[\begin{smallmatrix} n \\ n-2r \end{smallmatrix}\right]_g$ with $g = (1\ 2\ 3)$ for $r = 3, 4$. Recall that from Theorem 4.2.7 they agree with $N(\Gamma_n, r)$ when n is sufficiently large. From (4.23), using mathematical induction we have

$$\begin{bmatrix} n \\ n-6 \end{bmatrix}_g = \frac{1}{3840}(3n^{10} - 55n^9 + 350n^8 - 878n^7 + 1347n^6$$

$$- 5063n^5 + 60n^4 - 31148n^3 + 84640n^2 - 111552n - 483840),$$

$$\begin{bmatrix} n \\ n-8 \end{bmatrix}_g = \frac{1}{1935360}(9n^{14} - 315n^{13} + 4515n^{12} - 34307n^{11} + 153713n^{10}$$

$$- 453805n^9 + 962841n^8 - 985809n^7 + 100674n^6 - 6939940n^5 - 2681336n^4$$

$$+ 66912736n^3 + 169835904n^2 - 226874880n - 987033600).$$

Chapter 5

The Cayley Graphs Generated by k-Transpositions

In this chapter, we study the class of Cayley graphs on Sym_n and Alt_n whose generating set is the set of k-transpositions. Recall that a permutation g in Sym_n is a k-transposition if g has the cycle type $1^{n-2k}2^k$. For example, any ordinary transposition is a 1-transposition, and a double-transposition is a 2-transposition. Given positive integers n and k, we let $H(n,k)$, or in brief H, be the set of all $k-$transpositions in Sym_n. If $n \neq 4$ then the subset of G_n generated by $H(n,k)$ is either the symmetric group $G_n = \mathrm{Sym}_n$ or the alternating group $G'_n = \mathrm{Alt}_n$. We denote by $G_n(2^k)$ and $G'_n(2^k)$ the *k-transposition Cayley graph* with k odd and even, respectively. As is usual, throughout this chapter we let $B_r = B_r(\Gamma_n, e)$ and $S_r = S_r(\Gamma_n, e)$ where $\Gamma_n = G_n(2^k)$ or $G'_n(2^k)$. We also use $\overline{B}_r$ and $\overline{S}_r$ to refer to the ball and the sphere at distance r about e in the transposition Cayley graph $G_n(2^1)$.

5.1 Spheres in k-Transposition Cayley Graphs

As we have seen in the transposition Cayley graph $G_n(2^1)$, the distance between e and a vertex u in $G_n(2^1)$ is equal to $n - |u|$ where $|u|$ is the number of cycles in the cycle decomposition of u. With the natural embedding i_0 defined in Chapter 3 this distance is preserved when we embed $G_n(2^1)$ to $G_{n+1}(2^1)$. This phenomenon is the same in the double-transposition Cayley graphs. However, in general the situation becomes quite different in the $k-$transposition Cayley graph when k is greater than two.

For a fixed k the graphs $G_n(2^k)$ and $G_{n+1}(2^k)$ may contain a vertex (permutation) u in different spheres. For instance, using **GAP**, we have that $(1\ 2\ 3)$ and $(1\ 2\ 3\ 4\ 5)$ belong to $S_4(G_6(2^3), e)$, and that $(1\ 2\ 3)(4\ 5)$ belongs to $S_5(G_6(2^3), e)$. However, for any $n \geq 7$, we have that $(1\ 2\ 3)$ and $(1\ 2\ 3\ 4\ 5)$ belong to $S_2(G_n(2^3), e)$, and that $(1\ 2\ 3)(4\ 5)$ belongs to $S_3(G_n(2^3), e)$ instead. In Proposition 5.1.3 and Proposition 5.1.4 we show that if $n \geq 4k$ every conjugacy class will be held in a certain distance from the identity e in the graphs $G_n(2^k)$ and $G'_n(2^k)$.

Here, we start with a little lemma.

Lemma 5.1.1. *Let $k \geq 1$. Any permutation g in Sym_n with $|g| = n - 2k$ can be expressed as a product of two permutations whose cycle type is $1^{n-2k}2^k$.*

Proof. Let g be a permutation with $|g| = n - 2k$. Suppose that

$$g = a_1 a_2 a_3 \cdots a_{2t-1} a_{2t} d_1 d_2 \cdots d_{j-1} d_j$$

where

$$a_i = \ (\alpha_{i1}\ \alpha_{i2} \ldots \alpha_{i,2m_i}) \text{ and } d_i = \ (\beta_{i1}\ \beta_{i2} \ldots \beta_{i,2q_i+1})$$

for all i. Obviously, a's are the cycles of even length, and d's are the cycles of odd length. The number of even cycles must be even since $|g| = n - 2k$. Here, we have

$$2k = \sum_{i=1}^{2t}(2m_i - 1) + \sum_{i=1}^{j} 2q_i$$

and therefore

$$k = \sum_{i=1}^{2t} m_i + \sum_{i=1}^{j} q_i - t. \tag{5.1}$$

Then for each i, we have $a_i = b_i \cdot c_i$ and $d_i = e_i \cdot f_i$ with

$$\begin{aligned}
b_i &= (\alpha_{i1}\ \alpha_{i,2m_i-1})(\alpha_{i2}\ \alpha_{i,2m_i-2})\dots(\alpha_{i,m_i-1}\ \alpha_{i,m_i+1}),\\
c_i &= (\alpha_{i1}\ \alpha_{i,2m_i})(\alpha_{i2}\ \alpha_{i,2m_i-1})\dots(\alpha_{i,m_i}\ \alpha_{i,m_i+1}),\\
e_i &= (\beta_{i1}\ \beta_{i,2q_i})(\beta_{i2}\ \beta_{i,2q_i-1})\dots(\beta_{i,q_i}\ \beta_{i,q_i+1}),\\
f_i &= (\beta_{i1}\ \beta_{i,2q_i+1})(\beta_{i2}\ \beta_{i,2q_i})\dots(\beta_{i,q_i}\ \beta_{i,q_i+2}).
\end{aligned}$$

If we let

$$\begin{aligned}
x &= b_1 b_3 \dots b_{2t-1} c_2 c_4 \dots c_{2t} e_1 e_2 \dots e_j \quad \text{and}\\
y &= b_2 b_4 \dots b_{2t} c_1 c_3 \dots c_{2t-1} f_1 f_2 \dots f_j
\end{aligned}$$

then from (5.1) we have

$$\begin{aligned}
|x| &= (m_1 - 1) + (m_3 - 1) + \dots + (m_{2t} - 1)\\
&\quad + m_2 + m_4 + \dots + m_{2t} + q_1 + \dots + q_j\\
&= k.
\end{aligned}$$

Also, we have $|y| = k$. Since x and y are products of transpositions we have that x and y are of cycle type $1^{n-2k}2^k$ and $g = x \cdot y$, as required. □

Proposition 5.1.2. *For any $k \geq 3$ and $n \geq 4k$, let $\Gamma_n = G_n(2^k)$ or $G'_n(2^k)$. Then $S_2 = \{\, g : |g| = n - 2t$ for some $t = 1, 2 \dots, k \,\}$.*

Proof. Let $H = (2^k)^{G_n}$ be the generating set of Γ_n. Any product g of two elements in H has at least $n - 2k$ cycles. Also, $|g|$ and n must have the same parity. The case $|g| = n - 2k$ is done by Lemma 5.1.1. Let g be an element having $|g| = n - 2t$ for some $t < k$. Suppose that $k = t + p$ for some p. By Lemma 5.1.1 we have that $g = g_1g_2$ for some g_1, g_2 of cycle type $1^{n-2t}2^t$. Since $n \geq 4k$ there exist disjoint transpositions $t_1, t_2, \ldots, t_p$ such that $Supp(t_i)$ does not intersect $Supp(g_1) \cup Supp(g_2)$ for all $i = 1, \ldots, p$. As g_1 and g_2 are of cycle type $1^{n-2t}2^t$ and $k = t + p$, we have

$$g = (g_1t_1t_2 \ldots t_p) \cdot (g_2t_1t_2 \ldots t_p)$$

is a product of permutations of cycle type 2^k The proof is then complete. $\square$

Extending the above proposition to other spheres, we have

Proposition 5.1.3. *Let $k \geq 3$ be an odd number and let $n \geq 4k$. Let S_i be the sphere of radius i centred at e in $G_n(2^k)$. Then*

$$S_0 = \{(1)\}, \quad S_1 = H = (2^k)^{G_n},$$

$$S_2 = \{g \colon\ |g| \equiv n (mod\ 2)\ and\ n-2 \geq |g| \geq n-2k\},$$

$$S_3 = \{g \colon g \notin (2^k)^{G_n},\ |g| \not\equiv n\ (mod\ 2)\ and\ n-1 \geq |g| \geq n-3k\},$$

$$S_{2(i+1)} = \{g \colon\ |g| \equiv n\ (mod\ 2)\ and\ n-2ik > |g| \geq n-2(i+1)k\},$$

$$S_{2i+3} = \{g \colon\ |g| \not\equiv n\ (mod\ 2)\ and\ n-(2i+1)k > |g| \geq n-(2i+3)k\},$$

for all $i \geq 1$.

Proof. The statement about S_2 is true by Proposition 5.1.2. Considering each permutation as a group element in $G_n(2^1)$, for each element g in B_r we have $|g| \geq n-rk$ since H is the set of k-transpositions. We then have a lower bound for each sphere S_r. Since H is a set of odd permutations we have that for each g in B_r, the parities of $|g|$ and n are the same if and only if r is even. Further,

$$(1\ 2) = (1\ 2)(3\ 4)(5\ 6) * (1\ 2)(3\ 5)(4\ 6) * (1\ 2)(4\ 5)(3\ 6).$$

Hence S_3 certainly contains the set of transpositions.

Next, let g be an odd permutation not belonging to H with $n-3 \geq |g| \geq n-3k$. We claim that g is in S_3. Clearly $|g| \not\equiv n$ (mod 2). Suppose first that $|g| \leq n-2k-1$.

Then there exist g_1 in S_2 and $t \leq k$ such that $g = g_1 x$ with $|x| = n - t$ and $|x| \not\equiv n \pmod 2$. Since g_1 belongs to S_2 we have $g = h_1 h_2 x$ with h_1, h_2 in H. Moreover, $|h_2 x| = n - (k + t) \leq n - 2k$ and $|h_2 x| \equiv n \pmod 2$. Hence $h_2 x$ is in S_2 and therefore $h_2 x = h_3 h_4$ for some h_3, h_4 in H. Then we have $g = h_1 h_2 x = h_1 h_3 h_4$ is in S_3.

On the other hand, if $n - 3 \geq |g| \geq n - 2k + 1$ then g has a cycle of length at least two in its cycle decomposition, say $(\alpha_1 \dots \alpha_s)$. Therefore $g = g_1(\alpha_1 \ \alpha_s)$ for some g_1 with $n - 2 \geq |g_1| \geq n - 2k + 2$. This shows that g_1 belongs to S_2 and hence $g_1 = h_1 h_2$ for some h_1, h_2 in H. It follows that $g = h_1 h_2(\alpha_1 \ \alpha_s)$ and $n - 2 \leq |h_2(\alpha_1 \ \alpha_s)| \leq n - 2k$. Hence $h_2(\alpha_1 \ \alpha_s) = h_3 h_4$ for some h_3, h_4 in H. Therefore, $g = h_1 h_2(\alpha_1 \ \alpha_s) = h_1 h_3 h_4$, therefore g belongs to B_3. Clearly, g does not belong to B_2. Then we have

$$S_3 = \{g \colon g \notin (2^k)^{G_n},\ |g| \not\equiv n (\bmod 2) \text{ and } n - 1 \geq |g| \geq n - 3k\}.$$

For the case of $S_{2(i+1)}$ with $i \geq 1$, it is easy to see that any permutation in $S_{2(i+1)}$ has at least $n - 2(i+1)k$ cycles. It then remains to show that

$$A(i+1) := \{g \ : \ |g| \equiv n \ (\bmod 2) \text{ and } n - 2ik > |g| \geq n - 2(i+1)k\} \subseteq S_{2(i+1)}.$$

We prove this by the induction on $i \geq 0$. The basic step $i = 0$ is done. Let g be in $A(i+1)$. Then $|g| = n - 2ik - 2t$ for some $t = 1, \dots, k$. By the induction hypothesis, we have $g = g_1 x$ where $|x| = n - 2t$ and g_1 belongs to $A(i)$. Hence g_1 belongs to S_{2i} and therefore $g = h_1 h_2 \cdots h_{2i} x$. Also, we have that $n - 1 \geq |h_{2i} x| \geq n - 3k$, which implies that $h_{2i} x = h_1' h_2'$ for some h_1', h_2' in $H = (2^k)$. Hence, g belongs to $S_{2(i+1)}$.

The case S_{2i+3} for any $i \geq 1$ can be proved similarly. Note that the parities of $|g|$ and n agree automatically. □

Proposition 5.1.4. *Let $k \geq 2$ be an even number and let $n \geq 4k$. Let $\Gamma_n = G'_n(2^k)$. We have that*

$$S_0 = \{(1)\}, \quad S_1 = (2^k)^{G_n},$$

$$S_2 = \{g \in Alt_{\mathrm{n}} \colon g \notin (2^k)^{G_n} \text{ and } n-2 \geq |g| \geq n-2k\},$$

$$S_i = \{g \in Alt_{\mathrm{n}} \colon n-(i-1)k > |g| \geq n-ik\}.$$

Proof. For each $i \geq 3$, it is clear that

$$S_i \subseteq B_i \subseteq \ \{g \in \mathrm{Alt}_n \ : \ |g| \geq n-ik\}.$$

It then suffices to show that any even permutation g with $|g| \geq n-ik$ belongs to B_i for all $i \geq 2$. We prove this by induction on $i \geq 2$. The case when $i = 2$ is true by Proposition 5.1.2. Suppose that $i \geq 3$. Let g be a permutation with $|g| = n-(i-1)k-m$ with $2 \leq m \leq k$ and $m \equiv 0 \pmod 2$. Hence there must be permutations g_1 and x such that $g = g_1 x$ with $|g_1| = n-(i-1)k$ and $|x| = n-m$. By the induction hypothesis we have that g_1 belongs to B_{i-1}. Hence, $g = h_1 h_2 \dots h_{i-2} h_{i-1}$ for some $h_1, h_2, \dots, h_{i-1}$ in H. Since $|h_{i-1}x| \geq n-2k$, again by the induction hypothesis, there exist h'_1, h'_2 such that $h_{i-1}x = h'_1 h'_2$. Therefore, $g = g_1 x = h_1 h_2 \dots h_{i-2} h_{i-1} x = h_1 h_2 \dots h_{i-2} h'_1 h'_2$, so g belongs to B_i. □

5.2 Intersection Numbers in k-Transposition Cayley Graphs

This section is devoted to the intersection numbers in the $k-$transposition Cayley graphs. There are two cases to consider. The first is the case when k is an odd number, and the other is the situation when k is an even number. Before moving to the next lemmas, let us recall that for any permutation g in G_n and any subset

A of G_n we let $Ag = A \cdot \{g\}$. That is, in the next lemmas, $H \cap Hg$ will stand for $H \cap (H \cdot \{g\})$.

Lemma 5.2.1. *Let n and k be positive integers with $k \geq 2$ and $n \geq 2k$ and let $H = (2^k)^{G_n}$. Then*

$$\left| H \cap H(1\ 2\ 3) \right| = \begin{cases} \dfrac{3 \cdot (n-3)!}{(n-2k-1)!\,(k-1)!\,2^{k-1}} & \text{if } n > 2k, \\ 0 & \text{if } n = 2k. \end{cases} \tag{5.2}$$

Proof. Suppose that h_1 and h_2 are elements in H such that $h_2 = h_1(1\ 2\ 3)$. Let

$$P := h_1, h_1(1\ 2), h_1(1\ 2)(1\ 3)$$

be a path in $G_n(2^1)$ starting at h_1 and ending at h_2. Then $AD(P) = (a,d)$ or (d,a) where $AD(P)$ is the ascent-descent pattern of P. If $AD(P) = (a,d)$ then 1 and 2 must belong to different cycles in h_1, and therefore to have a descent at the next step, 3 must belong to the cycle containing either 1 or 2 in h_1. Hence h_1 contains $(1\ 3)(2), (1)(2\ 3), (1\ 3)(2\ m)$ or $(1\ m)(2\ 3)$ for some $m \neq 1,2,3$. However, the last two cases do not exist as $h_1(1\ 2\ 3)$ will be a 3-cycle, not a $k-$transposition. If $AD(P) = (d,a)$ then 1 and 2 must belong to the same cycle, and to get an ascent step, 3 and 1 must be in different cycles of h_1. Hence, h_1 contains $(1\ 2)(3)$ or $(1\ 2)(3\ m)$ for some $m \neq 1,2,3$. But clearly the latter case does not exist. This forces that h_1 has either $(1\ 2)(3), (1\ 3)(2)$ or $(2\ 3)(1)$ as part of its cycle decomposition. This is impossible when $n = 2k$. Hence $|\, H \cap H(1\ 2\ 3)\,| = 0$ if $n = 2k$. We next suppose that $n > 2k$ and that h_1 contains either $(1\ 2)(3), (1\ 3)(2)$ or $(2\ 3)(1)$. Assume that h_1 contains $(1\ 2)(3)$, say

$$h_1 = (1\ 2)(3)(\alpha_{11}\ \alpha_{12}) \cdots (\alpha_{k-1,1}\ \alpha_{k-1,2})(\alpha_1)(\alpha_2) \cdots (\alpha_t)$$

with $t = n - 2k - 1$. The number of ways to proportion $4, 5, 6, \ldots, n$ to those α's is equal to the number of permutations of $[\, n-3\,]$ whose cycle type is $1^{n-2k-1}2^{k-1}$.

From Theorem 2.3.2, this number is equal to

$$\frac{(n-3)!}{(n-2k-1)!(k-1)!2^{k-1}} \, .$$

The proof is now complete. $\square$

Remark: From Proposition 2.3.5 and Lemma 5.2.1, we have that every 3-cycle belongs to $S_2(\Gamma_n, e)$ with $\Gamma_n = G_n(2^k)$ or $G'_n(2^k)$ for any $k \geq 1$ and $n > 2k$.

Lemma 5.2.2. *Let n and k be positive numbers with $k \geq 2$ and $n \geq 2k$ and let $H = (2^k)^{G_n}$. Then*

$$\Big| H \cap H(1\ 2)(3\ 4) \Big| = \begin{cases} \dfrac{2\cdot(n-4)![(n-2k)^2-(n-4k+2)]}{(n-2k)!\,(k-1)!\,2^{k-1}} & \textit{if } \ n \geq 2k+2, \\ \dfrac{2\cdot(n-4)!}{(n-2k)!\,(k-2)!\,2^{k-2}} & \textit{if } \ n = 2k, 2k+1. \end{cases} \tag{5.3}$$

Proof. Let h belong to $H \cap H(1\ 2)(3\ 4)$. Suppose that $P := h, h(1\ 2), h(1\ 2)(3\ 4)$ is a path in $G_n(2^1)$ with $h, h(1\ 2)(3\ 4)$ belonging to H. Then we have $AD(P) = (a, d)$ or (d, a), and this implies that h has either $(1\ 2)(3)(4), (1)(2)(3\ 4), (1\ 3)(2\ 4)$ or $(1\ 4)(2\ 3)$ as part of its cycle decomposition. Then the set $H \cap H(1\ 2)(3\ 4)$ can be divided into two classes, say X and Y. The set X consists of elements in H having $(1\ 2)(3)(4)$ or $(1)(2)(3\ 4)$ in their cycle decompositions. Clearly, this set exists only if $n \geq 2k+2$. The other set Y is composed of those in H having $(1\ 3)(2\ 4)$ or $(1\ 4)(2\ 3)$ in their cycle decompositions. Using the same arguments as in the previous lemma, we have

$$|X| = \frac{2\cdot(n-4)!}{(n-2k-2)!\,(k-1)!\,2^{k-1}} \, ,$$

and

$$|Y| = \frac{2\cdot(n-4)!}{(n-2k)!\,(k-2)!\,2^{k-2}} \, .$$

Hence, if $n \geq 2k+2$ then

$$|H \cap H(1\ 2)(3\ 4)| = |\,X| + |\,Y| \, ,$$

and therefore, the proof is complete, simplifying $|X| + |Y|$. $\square$

Remark: From Lemma 5.2.1 and Lemma 5.2.2 one can see that if $g = (1\ 2\ 3)$ or $(1\ 2)(3\ 4)$ and $H = (2^k)^{G_n}$ for a fixed positive integer k then we have that $|H \cap Hg|$ asymtotically is a polynomial of degree $2k-2$.

Proposition 5.2.3. *Let n and k be positive integers with $n > 2k$. If $H = (2^k)^{G_n}$ then*

$$|H \cap H(1\ 2)(3\ 4)| \leq \frac{2}{3}\, |H \cap H(1\ 2\ 3)|.$$

Proof. The case when $k = 1$ is shown at Table 3.2 and Table 3.3 in Chapter 3. Then we suppose that $k \geq 2$. Let $m_1 = |H \cap H(1\ 2\ 3)|$ and $m_2 = |H \cap H(1\ 2)(3\ 4)|$. If $n = 2k+1$ then we have

$$\begin{aligned} m_1 &= \frac{3(n-3)!}{(n-2k-1)!(k-1)!2^{k-1}} \\ &= \frac{3 \cdot 2 \cdot (k-1)(2k-3)!}{2(k-1)\cdot(k-2)!2^{k-2}} \\ &= \frac{3}{2}\, m_2. \end{aligned}$$

Hence, $m_2 = \frac{2}{3}\, m_1$. Next, suppose that $n \geq 2k+2$. Then

$$\begin{aligned} m_2 &= \frac{2m_1}{3(n-3)(n-2k)} \Big[\, (n-2k)^2 - (n-2k) + (2k-2)\, \Big] \\ &= \frac{2m_1}{3(n-3)} \Big[\, (n-2k) - 1 + \frac{2k-2}{n-2k}\, \Big] \\ &< \frac{2m_1}{3(n-3)} \Big[\, (n-2k) - 1 + (2k-2)\, \Big] \\ &= \frac{2m_1(n-3)}{3(n-3)} \\ &= \frac{2}{3}\, m_1. \end{aligned}$$

Hence, we can conclude that if $n > 2k$ then $m_2 \leq \dfrac{2}{3}\, m_1$. $\square$

5.2.1 The Case of k Odd

Here the intersection number $N(\Gamma_n, r)$ of the $k-$transposition Cayley graph $\Gamma_n = G_n(2^k)$, with k odd, are given. We first start with the case when $r = 1$.

Theorem 5.2.4. *Let $\Gamma_n = G_n(2^k)$ with k odd. Then*

$$N(\Gamma_n, 1) = \frac{3 \cdot (n-3)!}{(n-2k-1)!\,(k-1)!\,2^{k-1}} \tag{5.4}$$

when n is sufficiently large.

Proof. Because k is an odd number, $S_1 = H = (2^k)^{G_n}$ is a set of odd permutations. Then the product $S_1 g$ with g in S_1 is a set of even permutations. Since there are no edges between vertices in S_1 and since the identity e is the only even permutation in B_1, we have $B_1 \cap B_1 g$ = {e,g} for any g in H. Next, suppose that g is in S_2. It is clear that g is an even permutation, and hence

$$B_1 \cap B_1 g = S_1 \cap S_1 g \subseteq \overline{B}_k \cap \overline{B}_k g.$$

From Corollary 3.2.3, $\left|\overline{B}_k \cap \overline{B}_k g\right|$ is a polynomial of degree $2(k-j)$ if g is in $\overline{S}_{2j-1} \cup \overline{S}_{2j}$. Hence, from Proposition 5.2.3, if n is large enough then

$$\begin{aligned}\left|\overline{B}_k \cap \overline{B}_g\right| &= \max\{\, |S_1 \cap S_1 g| \;:\; g = (1\ 2\ 3), (1\ 2)(3\ 4)\,\} \\ &= |\, S_1 \cap S_1 (1\ 2\ 3)|\,.\end{aligned}$$

Therefore, the proof is finished. □

Next we consider for the case when $r = 2$.

Theorem 5.2.5. *Let $\Gamma_n = G_n(2^k)$ with k odd and greater than one. Then we have*

$$N(\Gamma_n, 2) = N(\Gamma_n, 1) + N(G'_n(2^2), k) \tag{5.5}$$

for any n sufficiently large.

Proof. Let $\overline{B}_r = B_r(G_n(2^1), e)$ and $\overline{S}_r = S_r(G_n(2^1), e)$. If g is an odd permutation then, by Proposition 5.1.3, we have that

$$|B_2 \cap B_2 g| \leq 2\,|H| \tag{5.6}$$

since $(\{e\} \cup S_2)g \cap B_2 \subseteq S_1$ and $(S_1 g \cap B_2)g^{-1} \subseteq S_1 = H$. Recall that $H = (2^k)^{G_n}$. Then from Theorem 2.3.2 we have

$$|H| = \frac{n!}{k! \cdot 2^k \cdot (n-2k)! \cdot 1^{n-2k}}\ . \tag{5.7}$$

Hence, by (5.6) and (5.7) one can conclude that $|B_2 \cap B_2 g|$ is bounded by $2\,|H|$, which is a polynomial of degree at most $2k$ when g is an odd permutation.

Next, we let g be a non-identity even permutation. Then $S_1 g \cap B_2 \subseteq S_1$ and $(S_0 \cup S_2)g \cap B_2 \subseteq (S_0 \cup S_2)$. Hence we have

$$B_2 \cap B_2 g = (S_1 \cap S_1 g)\ \dot{\cup}\ ((S_0 \cup S_2) \cap (S_0 \cup S_2)g).$$

Therefore

$$|B_2 \cap B_2 g| = |S_1 \cap S_1 g| + |(S_0 \cup S_2) \cap (S_0 \cup S_2)g| \tag{5.8}$$

since $S_1 \cap S_1 g$ and $(S_0 \cup S_2) \cap (S_0 \cup S_2)g$ are disjoint.

From Proposition 5.1.3, we have

$$S_0 \cup S_2 = \bigcup_{i=0}^{2k} \overline{S}_i = B_k(G'_n(2^2))$$

for every $n \geq 4k$. Recall that $B_k(G'_n(2^2)) = B_k(G'_n(2^2), e)$. Further, from the previous chapter we know that

$$|(S_0 \cup S_2) \cap (S_0 \cup S_2)g| = \left|B_k(G'_n(2^2)) \cap (B_k(G'_n(2^2)) \cdot g)\right|. \tag{5.9}$$

Therefore, from (5.8) and (5.9) we can conclude that if n is sufficiently large, then

$$N(\Gamma_n, 2) = N(\Gamma_n, 1) + N(G'_n(2^2), k). \tag{5.10}$$

$\square$

Remark: From the proof shown above it is clear that the 3-cycles still provide us the intersection number $N(\Gamma_n, 2)$ in (5.10).

Next, we compute $N(G_n(2^k), r)$ with k odd and $r \geq 3$. Here, we let $\left[{n \atop n-m} \right]_g$ be as defined in (4.5) for $G'_n(2^2)$, not for $G_n(2^1)$.

Theorem 5.2.6. *Let $r \geq 3$ and let $k \geq 3$ be an odd integer. Let $\Gamma_n = G_n(2^k)$. If n is sufficiently large then we have*

$$N(\Gamma_n, r) = \left[{n \atop n-rk} \right]_{(1\ 2\ 3)} + \left[{n \atop n-(r-1)k} \right]_{(1\ 2\ 3)}. \tag{5.11}$$

Proof. From Proposition 5.1.3, the ball B_r in Γ_n can be divided into two disjoint sets. The first set X comprises all even permutations in B_r and the other Y consists of the remaining odd permutations, that is,

$$X = B_r \cap \mathrm{Alt}_n \quad \text{and} \quad Y = B_r \cap (\mathrm{Sym}_n \setminus \mathrm{Alt}_n).$$

We claim that if g is an even permutation, then

$$\Big| B_r \cap B_r g \Big| = \left[{n \atop n-rk} \right]_g + \left[{n \atop n-(r-1)k} \right]_g \leq O(n^{2rk-2}), \tag{5.12}$$

and that if g is an odd permutation, then

$$\Big| B_r \cap B_r g \Big| \leq O(n^{2rk-2k}). \tag{5.13}$$

Note from (4.10) we know that $\left[{n \atop n-m} \right]_g = O(n^{2(m-j)})$ if g belongs to $\overline{S}_{2j}$, where $\overline{S}_m = S_m(G_n(2^1), e)$. Hence, if both of (5.12) and (5.13) hold, then, since $k \geq 3$ we have

$$N(\Gamma_n, r) = max \left\{ \left[{n \atop n-rk} \right]_g + \left[{n \atop n-(r-1)k} \right]_g : g = (1\ 2\ 3) \ \text{ or } (1\ 2)(3\ 4) \right\}.$$

Recall that $(1\ 2\ 3)$ and $(1\ 2)(3\ 4)$ belong to $\overline{S}_2 = S_2(G_n(2^1), e)$. Moreover, by considering the initial values of $\left[{n \atop n-m} \right]_{(1\ 2\ 3)}$ and $\left[{n \atop n-m} \right]_{(1\ 2)(3\ 4)}$ shown in Theorem 4.2.5 we have that

$$N(\Gamma_n, r) = \left[{n \atop n-rk} \right]_{(1\ 2\ 3)} + \left[{n \atop n-(r-1)k} \right]_{(1\ 2\ 3)}, \tag{5.14}$$

which is a polynomial of degree $2(rk-1)=2rk-2$. Hence, it remains to show that (5.12) and (5.13) hold.

First, we suppose that g is an even permutation. By the parity of permutations one can see that

$$X\cap Yg=Y\cap Xg=\emptyset$$

and

$$B_r\cap B_rg=(X\cap Xg)\,\dot{\cup}\,(Y\cap Yg). \tag{5.15}$$

If r is an even number, then by Proposition 5.1.3 we have

$$X=\bigcup\{\overline{S}_{2i}\}_{i=0}^{\frac{rk}{2}} \tag{5.16}$$

and

$$Y=\bigcup\{\overline{S}_{2i-1}\}_{i=1}^{\frac{(r-1)k+1}{2}}. \tag{5.17}$$

Similarly, if r is odd, then

$$X=\bigcup\{\overline{S}_{2i}\}_{i=0}^{\frac{(r-1)k}{2}} \tag{5.18}$$

and

$$Y=\bigcup\{\overline{S}_{2i-1}\}_{i=1}^{\frac{rk+1}{2}}. \tag{5.19}$$

Note that (5.17) and (5.19) will not hold if $r\le 2$. Also, recall that $\overline{S}_i$ is the sphere $S_i(G_n(2^1),e)$. Therefore, from (5.16)–(5.19), it follows that

$$\Big|(X\cap Xg)\,\dot{\cup}\,(Y\cap Yg)\Big|=\begin{bmatrix} n \\ n-rk\end{bmatrix}_g+\begin{bmatrix} n \\ n-(r-1)k\end{bmatrix}_g.$$

Hence, from (5.15) we have

$$\Big|B_r\cap B_rg\Big|=\begin{bmatrix} n \\ n-rk\end{bmatrix}_g+\begin{bmatrix} n \\ n-(r-1)k\end{bmatrix}_g.$$

Recall that $\left[\begin{smallmatrix} n \\ n-m\end{smallmatrix}\right]_g$ is the function we defined in (4.5).

Next, suppose that g is an odd permutation. Then by the parity of permutations we have

$$X \cap Xg = Y \cap Yg = \emptyset,$$

and then

$$B_r \cap B_r g = (X \cap Yg) \mathbin{\dot\cup} (Y \cap Xg).$$

Suppose that r is an even number. Then

$$(X \cap Yg) \cdot g^{-1} \subseteq Y \qquad \text{and} \qquad Y \cap Xg \subseteq Y. \tag{5.20}$$

Since r is even and $Y = B_r \cap (\mathrm{Sym}_n \setminus \mathrm{Alt}_n)$ we have

$$Y = \bigcup \{\overline{S}_i\}_{i=1}^{(r-1)k}. \tag{5.21}$$

Similarly, if r is an odd number, then

$$(Y \cap Xg) \cdot g^{-1} \subseteq X \qquad \text{and} \qquad X \cap Yg \subseteq X, \tag{5.22}$$

and also, since r is odd we have

$$X = \bigcup \{\overline{S}_i\}_{i=0}^{(r-1)k}. \tag{5.23}$$

Note that (5.21) and (5.23) will not hold if $r \leq 2$. Also, recall that for any $m \geq 1$,

$$\left|\overline{B}_m\right| = O(n^{2m}), \tag{5.24}$$

since the degree of any vertex in the transposition Cayley graph $G_n(2^1)$ is equal to $\dfrac{n(n-1)}{2}$ and since $(2^m)^{G_n} \subseteq \overline{S}_m \subseteq \overline{B}_m$. In addition, by Theorem 2.3.2 we know that $\left|(2^m)^{G_n}\right| = \dfrac{n!}{m!2^m(n-2m)!} = O(n^{2m})$.

Hence, from (5.20)–(5.24), it follows that for any r greater than or equal to three,

$$|B_r \cap B_r g| \leq 2\left|\overline{B}_{(r-1)k}\right| = O(n^{2(r-1)k}),$$

when g is an odd permutation.

Therefore, we can now conclude that if n is sufficiently large then

$$N(\Gamma_n, r) = \Big| B_r \cap B_r(1\ 2\ 3) \Big| = \begin{bmatrix} n \\ n - rk \end{bmatrix}_{(1\ 2\ 3)} + \begin{bmatrix} n \\ n - (r-1)k \end{bmatrix}_{(1\ 2\ 3)}.$$

$\square$

5.2.2 The Case of k Even

If k is an even positive number then the Cayley graph Γ_H with $H = (2^k)^{G_n}$ has the alternating group as its vertex set. As in $G'_n(2^2)$, the graph $G'_n(2^k)$ contains triangles. For instance,

$$(1) \to (1\ 2)(3\ 4)(5\ 6)(7\ 8) \to (1\ 3)(2\ 4)(5\ 7)(6\ 8) \to (1)$$

is a triangle in $G'_8(2^4)$. In the remainder of this chapter we let $\Gamma_n = G'_n(2^k)$.

Comparing to the case of k odd, it is far easier to get the ball intersection number $N(G'_n(2^k), r)$ with k even.

Theorem 5.2.7. *Let $\Gamma_n = G'_n(2^k)$ with k even and let $r \geq 2$. We have*

$$N(\Gamma_n, r) = N(G'_n(2^2), \frac{rk}{2}) \tag{5.25}$$

and

$$N(\Gamma_n, 1) = \frac{3 \cdot (n-3)!}{(n-2k-1)!\,(k-1)!\,2^{k-1}} \tag{5.26}$$

if n is sufficiently large.

Proof. Let $r \geq 2$. From Proposition 5.1.4 we have

$$B_r := B_r(\Gamma_n, e) = \{g : |g| \geq n - rk\},$$

and in another sense it is equal to $B_{\frac{rk}{2}}(G'_n(2^2), e)$. Then $N(\Gamma_n, r) = N(G'_n(2^2), \frac{rk}{2})$ when n is sufficiently large. Next, consider $N(\Gamma_n, 1)$. As usual, we have

$$B_1 \cap B_1 g \subset \overline{B}_k \cap \overline{B}_k g.$$

This implies that $|B_1 \cap B_1 g|$ is a polynomial of degree at most $2(k-1)$. Again, when n is sufficiently large, we have $|B_1 \cap B_1 g| \leq |B_1 \cap B_1(1\ 2\ 3)|$ for all non-identity g. Hence, by Proposition 5.2.3 and Corollary 3.2.3, the proof is now complete. $\square$

Chapter 6

Three-Cycle Cayley Graphs

In this chapter we try to generalise our results for the transposition Cayley graph $G_n(2^1)$ to other Cayley graphs on the alternating group $G'_n = \mathrm{Alt}_n$. We would like to look at the Cayley graph that is generated by all 3-cycles. A permutation g in the symmetric group $G_n = \mathrm{Sym}_n$ is a 3-*cycle* if it has the cycle type $1^{n-3}3^1$. That is, it belongs to the the conjugacy class $(3^1)^{G_n}$. Throughout this chapter we let $H = (3^1)^{G_n}$.

In the remainder we pay attention to this Cayley graph, denoted by $G'_n(3^1)$. This graph has the set $H = (3^1)^{G_n}$ as generating set. As is well known, the set H generates the alternating group Alt_n for all $n \geq 3$. This means that the vertex set is the alternating group $G'_n = \mathrm{Alt}_n$. We call this graph the 3-*cycle Cayley graph*. Its edge set is the set of unordered pairs $\{u, v\}$ such that $v = uh$ (or $u^{-1}v = h$) for some h in $H = (3^1)^{G_n}$. That is,

$$E_H := \{\{u, v\} : v = uh \text{ for some } \ h \in H = (3^1)^{G_n}\}. \tag{6.1}$$

In general, we call any permutation g or a cycle in its decomposition *k-cycle* if it is of the form $(\alpha_1\ \alpha_2\ \ldots\ \alpha_k)$. For example, there are only two 3-cycles in Alt_3, namely $(1\ 2\ 3)$ and $(1\ 3\ 2)$. Also, we consider $(1\ 2\ 3\ 4\ 5)$ in $(1\ 2\ 3\ 4\ 5)(6\ \ 7\ 8)(9\ 10)$ as a 5-cycle. Furthermore, g is an *odd cycle* if g is a k-cycle with k odd. Similarly, g

is an *even cyle* if g is a k-cycle and k is even. As is usual, throughout this chapter we let S_i and B_i be the sphere $S_i(G'_n(3^1), e)$ and the ball $B_i(G'_n(3^1), e)$ of radius i about the identity $e := (1)$, respectively.

6.1 Spheres in Three-Cycle Cayley Graphs

Here we start with some basic properties of the graph $G'_n(3^1)$.

Proposition 6.1.1. *Let $\Gamma_n = G'_n(3^1)$ be the 3-cycle Cayley graph on Alt_{n}. We have that*

(1) Γ_n *contains triangles, and*

(2) Γ_n *is a* $2\binom{n}{3}$*-regular graph.*

Recall that for any permutation g in Sym_n we let $|g|$ be the number of cycles in the disjoint cycle notation of g, including cycles of length one, and $supp(g)$ is the number of elements in $[\,n\,]$ moved by g. Now we denote by $|g|_o$ the number of odd cycles in g. Similarly, we let $|g|_e$ be the number of even cycles in g. It is clear that

$$|g| = |g|_o + |g|_e \tag{6.2}$$

for any g in Sym_n. Also, we denote by $|g|_o^*$ the number of odd cycles of length greater than one in g.

Proposition 6.1.2. *Let $n \geq 3$ and let $\Gamma_n = G'_n(3^1)$. For any $i \geq 0$, if g belongs to Γ_n then*

$$g \in S_i \ \text{ if and only if } \ supp(g) - |g|_o^* = 2i. \tag{6.3}$$

Proof. For convenience, throughout this proof we let $s_g = supp(g)$ and $m_g = |g|_o^*$. Recall that $H = (3^1)^{G_n}$ is the generating set of Γ_n. Fix $n \geq 3$. We prove this by induction on i. Note that it is clear when $i = 0, 1$. Let $t \geq 1$. Assume that (6.3) holds for all $i = 0, \ldots, t$. Then by the induction hypothesis we have that for any g in Γ_n,

$$g \in S_i \text{ if and only if } s_g - m_g = 2i \tag{6.4}$$

whenever $0 \leq i \leq t$.

We claim that

$$g \in S_{t+1} \text{ if and only if } s_g - m_g = 2(t+1). \tag{6.5}$$

First, we assume that g belongs to S_{t+1}. To prove the necessary condition, since $S_{t+1} \subseteq S_t H$, by (6.4) it suffices to show that $s_g - m_g$ is equal to either $2(t-1), 2t$ or $2(t+1)$ for any element g in $S_t H$. Let g belong to $S_t H$ and suppose that $g = g' \cdot (\alpha\ \beta\ \gamma)$ with g' in S_t and $(\alpha\ \beta\ \gamma)$ in H.

Case 1: $Supp(g') \cap (\alpha\ \beta\ \gamma) = \emptyset$.

In this case we have $s_g = s_{g'} + 3$ and $m_g = m_{g'} + 1$. Then $s_g - m_{g'} = s_{g'} + 2 - m_{g'} = 2(t+1)$.

Case 2: Only one of α, β, γ belongs to $Supp(g')$.

Then $s_g = s_{g'} + 2$ and $m_g = m_{g'}$. Hence $s_g - m_{g'} = s_{g'} + 2 - m_{g'} = 2(t+1)$.

Case 3: Two of α, β, γ are in $Supp(g')$. Without loss of generality we assume that α, β are in $Supp(g')$.

Case 3.1: α, β are in the same cycle of g'.

If $\beta g' = \alpha$ then g' and g have the same cycle type and therefore g is in S_t. Suppose not. Then we have $s_g = s_{g'} + 1$ and $m_g = m_{g'} \pm 1$. So $s_g - m_g = 2t$ or $2(t+1)$.

Case 3.2: α, β are in different cycles.

We have $s_g = s_{g'} + 1$ and $m_g = m_{g'} \pm 1$. Hence, $s_g - m_g = 2t$ or $2(t+1)$.

Case 4: All α, β, γ are in $Supp(g')$.

Case 4.1: All α, β, γ are in the same cycle of g'.

If γ appears in the cycle containing β in $g'(\alpha\ \beta)$ then g' and g have the same cycle type. This means that g belongs to S_t. Suppose not. Then γ is in the cycle containing α in $g'(\alpha\ \beta)$. Hence, there exist positive integers k_1, k_2, k_3 such that

$$g' = (\alpha\ \alpha_2\ \ldots\ \alpha_{k_1}\ \gamma\ \gamma_2\ \ldots\ \gamma_{k_2}\ \beta\ \beta_2\ \ldots\ \beta_{k_3})\, g''$$

for some permutation g'' with $\alpha_1 = \alpha, \beta_1 = \beta, \gamma_1 = \gamma$. If $k_1 = k_2 = k_3 = 1$ then $s_g = s_{g'} - 3$ and $m_g = m_{g'} - 1$, so $s_g - m_g = s_{g'} - m_{g'} - 2 = 2(t-1)$. If only one of k_1, k_2, k_3 is 1 then $s_g = s_{g'} - 1$ and $m_g = m_{g'} \pm 1$. Hence we have $s_g - m_g = 2t$ or $2(t-1)$. If only two of k_1, k_2, k_3 are 1 then $s_g = s_{g'} - 2$ and $m_g = m_{g'}$. Then we get $s_g - m_g = 2(t-1)$. If none of k_1, k_2, k_3 is 1 then $s_g = s_{g'}$ and m_g is either $m_{g'}$ or $m_{g'} - 2$. Therefore, $s_g - m_g = 2t$ or $2(t+1)$.

Case 4.2: Only two of α, β, γ are in the same cycle of g'. Without loss of generality, assume that α, β are in the same cycle. If $\beta g' = \alpha$ then $s_g = s_k - 1$ and $m_g = m_{g'} \pm 1$. So we have $s_g - m_g = 2(t-1)$ or $2t$. If not, then $s_g = s_{g'}$ and $m_g = m_{g'}$ or $m_{g'} \pm 2$. Hence, $s_g - m_g = 2(t-1), 2t$ or $2(t+1)$.

Case 4.3: None of α, β, γ appears in the same cycle. Then $s_g = s_{g'}$ and $m_g = m_{g'}$ or $m_{g'} - 2$, and hence $s_g - m_g = 2t$ or $2(t+1)$.

Then from these arguments we have proved that if g belongs to S_{t+1} then $s_g - m_g = 2(t+1)$.

Conversely, assume that $s_g - m_g = 2(t+1)$. Since g is an even permutation we have that g contains

$$(\alpha_1\ \ldots\ \alpha_{2j+1}) \qquad \text{or} \qquad (\alpha_1\ \ldots\ \alpha_{2j})(\beta_1\ \beta_2)$$

with $j \geq 1$ in its disjoint cycle decomposition. First, if g contains $(\alpha_1\ \ldots\ \alpha_{2j+1})$ then $g = g'(\alpha_1\ \alpha_{2j}\ \alpha_{2j+1})$ for some g' in Γ_n with $g' \neq e$. Hence, $s_{g'} = s_g - 2$ and $m_{g'} = m_g$, and therefore $s_{g'} - m_{g'} = s_g - 2 - m_g = 2(t+1) - 2 = 2t$. Then g' belongs to S_t, by (6.4). This forces, again by (6.4), that g belongs to S_{t+1}. Similarly, if g

contains $(\alpha_1 \ \ldots \ \alpha_{2j})(\beta_1 \ \beta_2)$ then

$$g = g'(\alpha_1 \ \alpha_{2j})(\beta_1 \ \beta_2) = g'(\alpha_1 \ \beta_1 \ \alpha_2)(\alpha_2 \ \beta_2 \ \beta_1)$$

for some g'. Let $g_0 = g'(\alpha_1 \ \beta_1 \ \alpha_2)$. Then $g = g_0(\alpha_2 \ \beta_2 \ \beta_1)$, and therefore $s_{g_0} = s_g - 1$ and $m_{g_0} = m_g + 1$. Hence, $s_{g_0} - m_{g_0} = s_g - m_g - 2 = 2(t+1) - 2 = 2t$. From (6.4) it follows that g belongs to S_{t+1}. $\square$

From (6.3), when one includes those cycles of length one in g we have the following result.

Proposition 6.1.3 ([24], p. 162). *Let $\Gamma_n = G'_n(3^1)$ and let $S_i = S_i(\Gamma_n, e)$ be the sphere of radius i. We have*

$$S_i = \{g \in Alt_{\mathrm{n}} : |g|_o = n - 2i\} \tag{6.6}$$

where $|g|_o$ is the number odd cycles in g.

Remark: At first when we tried to find parameters to determine which kind of (even) permutations belongs to S_i, we came up with the condition in (6.3). After that we realised that this problem was proposed by Bogdan Suceavă and solved by Richard Stong in [24]. For the rest of this chapter we will use (6.6) to determine the vertices that belong to S_i as this form relates to the parameters n and i; these are likely to be more practical than the parameters appearing in (6.3) in our work.

Before moving to the next Corollary, let us recall that, for any permutation g in Sym_n we define $|g|_o$ to be the number of odd cycles in the disjoint cycle decomposition of g.

Corollary 6.1.4. *Let $n \geq 3$ and let $\Gamma_n = G'_n(3^1)$. We have*

$$diam(\Gamma_n) = \begin{cases} \dfrac{n-1}{2} & \text{if } n \text{ is odd,} \\ \dfrac{n}{2} & \text{if } n \text{ is even.} \end{cases} \tag{6.7}$$

Proof. Suppose that n is an even number. Then Γ_n contains a permutation g whose cycle type $\mathrm{ct}(g)$ is either $2^{\frac{n}{2}}$ or $2^{\frac{n-4}{2}}4^1$ and therefore $|g|_o = 0$. Hence, from (6.6) the diameter of Γ_n is $\dfrac{n}{2}$ if n is even.

For the other case, suppose that n is an odd number. If there is a permutation g with $|g|_o = 0$ then g has only cycles of even length in its disjoint cycle decomposition, and therefore n must be even, a contradiction. Hence, for any v in Γ_n we have $|v| \geq 1$ and it is clear that this is the greatest lower bound as Γ_n contains $(1\ 2\ \ldots\ n)$. This shows that if n is an odd number then the diameter of Γ_n is $\dfrac{n-1}{2}$. The proof is now complete. ☐

Theorem 6.1.5. *Let* $\Gamma_n = G'_n(3^1)$*. We have* $N(\Gamma_4, 1) = 8$*. If* $n \geq 5$ *then* $N(\Gamma_n, 1) = 3(n-2)$.

Proof. From Proposition 6.1.3 we have that any permutation in S_2 has one of the following cycle types, namely $(2^2)^{G_n}, (5^1)^{G_n}$ and $(3^2)^{G_n}$. It is obvious that such a permutation exists when n is greater than or equal to four, five and six, respectively.

For each non-identity element g in B_2, we let $N(g) = |B_1 \cap B_1 g|$. Clearly, there are only five 3-cycles h such that $(1\ 2\ 3\ 4\ 5) \cdot h$ is in B_1, namely $(1\ 5\ 4)$, $(2\ 1\ 5)$, $(3\ 2\ 1)$, $(4\ 3\ 2)$ and $(5\ 4\ 3)$. Hence $N((1\ 2\ 3\ 4\ 5)) = 5$. Further, it is easy to see that $N((1\ 2\ 3)(4\ 5\ 6)) = 2$ and $N((1\ 2)(3\ 4)) = 8$.

Next we determine $N((1\ 2\ 3))$. Let h be a 3-cycle such that $(1\ 2\ 3) \cdot h$ belongs to $B_1 = \{e\} \cup (3^1)^{G_n}$. Clearly, $|Supp(h) \cap \{1,2,3\}| \leq 3$. If $|Supp(h) \cap \{1,2,3\}| = 0{,}1$ then $(1\ 2\ 3) \cdot h$ has cycle type $1^{n-6}3^2$ or $1^{n-5}5^1$. If $|Supp(h) \cap \{1,2,3\}| = 3$, then $(1\ 2\ 3) \cdot h$ is either e or $(1\ 3\ 2)$. Suppose that $|Supp(h) \cap \{1,2,3\}| = 2$ and that $h = (l_0\ l_1\ l_2)$ where only l_0 does not belong to $\{1,2,3\}$. There are $n-3$ possible choices of l_0. Since $(1\ 2\ 3) \cdot h$ belongs to $(3^1)^{G_n}$ we have $l_1(1\ 3\ 2) = l_2$ (if not, then $(1\ 2\ 3) \cdot h$ will belong to $(2^2)^{G_n}$). Hence there are $3(n-3)$ possible choices to choose h. This shows that

$$N((1\ 2\ 3)) = 3(n-3) + 2 + 1 = 3(n-2), \tag{6.8}$$

where '1' in (6.8) accounts for the vertex $(1\,2\,3)$ itself. The proof is now complete. $\square$

6.2 Further Research on Three-Cycle Cayley Graphs

At this moment, the intersection number $N(G'_n(3^1), r)$ with $r \geq 2$ is far beyond our understanding. Comparing to k-transposition Cayley graphs, the 3-cycle Cayley graph seems to have its own character that is different from other Cayley graphs we have considered before. Not only are its intersection numbers complicated to figure out, but also the size of each sphere is not simple to determine.

What we could say is that one may take advantage of *generatingfunctionology* (see [25] for details) to find the size of spheres in the graph; however, we now do not know any connection between spheres' sizes (or even balls' sizes) and intersection numbers. We believe that in the 3-cycle Cayley graph the recursion of intersection numbers, if it has, should be composed of three terms, not just two terms as it occurs in the transposition Cayley graph. Moreover, we conjectured that for any positive integer n and r,

$$N(G_n(3^1), r) = |B_r \cap B_r(1\ 2\ 3)|$$

when n is large enough.

Appendix A

Glossary

Notation		Page
$AD(P)$	The ascent-descent pattern of the path P	40
a_{ijk}	A class algebra constant	31
Alt_n	The alternating group of $[\,n\,]$	16
$B_r(\Gamma, u)$	The ball of radius r centred at u in Γ	10
$\overline{C}$	The class sum of the conjugacy class C	31
$ct(g)$	The cycle type of the permutation g	16
$d_\Gamma(u, v)$	The distance between u and v in the graph Γ	10
$Fix(g)$	The set of letters fixed by g	16
$fix(g)$	The number of letters fixed by g	16

Notation		Page(s)
$\mid g \mid$	The number of cycles in the disjoint cycle decomposition of g	16
Γ_H	The Cayley graph generated by H	14
$G_n(2^1)$	A transposition Cayley graph	18
$G'_n(2^2)$	A double-transposition Cayley graph	60
$G_n(2^k)$	A $k-$transposition Cayley graph, with k odd	76
$G'_n(2^k)$	A $k-$transposition Cayley graph, with k even	76
i_j	An insert operation	21
i^*	The converse operation of i_j	22
$[\,n\,]$	The set $\{1, 2, 3, \ldots, n\}$	15
$N(\Gamma, r)$	An intersection number of the graph Γ	26
$\begin{bmatrix} n \\ k \end{bmatrix}$	A Stirling number of the first kind	21
$\begin{bmatrix} n \\ k \end{bmatrix}_r$	An $r-$Stirling number of the first kind	24
$\begin{bmatrix} n \\ k \end{bmatrix}_g$	A Stirling-type number, defined differently in two ways for the permutation g	$46, 65$
$\begin{Bmatrix} n \\ k \end{Bmatrix}$	A Stirling number of the second kind	23
$\begin{Bmatrix} n \\ k \end{Bmatrix}_r$	An $r-$Stirling number of the second kind	24

Notation		Page
$Supp(g)$	The set of letters moved by g	15
$supp(g)$	The number of letters moved by g	16
$S_r(\Gamma, u)$	The sphere of radius r centred at u in Γ	10
Sym_n	The symmetric group of $[\,n\,]$	15
$y^{[n]}$	The rising factorial $y(y+1)(y+2)\dots(y+(n-1))$	21

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